MAINS TEST SERIES-2021

TEST-8

FULL SYLLABUS (PAPER-II)

ANSWER KEY

1(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ in S_5.

Find a permutation γ in S_5 such that $\alpha\gamma = \beta$.

Sol'n: Here $\alpha, \beta, \gamma \in S_5$

Given $\alpha\gamma = \beta$

$$\alpha^{-1}\alpha\gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

$$\therefore \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

1152 → Prove that every field is an integral domain, but every integral domain is not a field.
Give an example of an integral domain which is also a field.

Sol^n: Let F be a field then by definition F is a commutative ring with unity and every non-zero element is invertible w.r.t $\times$.

In order to prove that a field is an I.D. we have to prove that a field F has no zero divisors.

Let $a, b \in F$ and $a \neq 0$.

Since F is a field.

For $a \neq 0 \in F \Rightarrow a^{-1}$ exists in F.

$\therefore aa^{-1} = a^{-1}a = 1$

Now we have

$$ab = 0$$
$$\Rightarrow a^{-1}(ab) = a^{-1}0$$
$$\Rightarrow (a^{-1}a)b = 0$$
$$\Rightarrow 1b = 0$$
$$\Rightarrow b = 0$$

Similarly we can prove that $a, b \in F$

$b \neq 0$ and $ab = 0 \Rightarrow a = 0$

$\therefore a, b \in F$ and $ab = 0$

$\Rightarrow$ either $a = 0$ or $b = 0$

$\therefore$ F has no zero divisors.

$\therefore$ A field is an I D

The converse of the above need not be true
i.e., every ID need not be a field.
For example, the set of integers $\mathbb{Z}$ is an integral domain which is not a field.
Since $a \neq 0 \in \mathbb{Z}$ doesnot have the multiplicative inverse in $\mathbb{Z}$.

The ring $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ of integers modulo 5 is both an integral domain and a field.

1(c) Two sequences $\{x_n\}$, $\{y_n\}$ are defined by $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0, y_1 > 0$

Prove that both the sequences converge to a common limit.

Sol^n Case - 1 Let $x_1 \neq y_1$

$$x_2 = \frac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} = y_2.$$

Let us assume that $x_k > y_k$

Then $x_{k+1} = \frac{1}{2}(x_k + y_k) > \sqrt{x_k y_k} = y_{k+1}$

$x_k > y_k$ implies $x_{k+1} > y_{k+1}$ and $x_2 > y_2$.

By the principle of induction, $x_n > y_n$ for all $n \geq 2$.

$$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n, \text{ for all } n \geq 2.$$

$$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n y_n} = y_n \text{ for all } n \geq 2.$$

So we have. $y_2 < y_3 < y_4 < \cdots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim x_n = l$, $\lim y_n = m$.

$$x_{n+1} = \frac{1}{2}(x_n + y_n) \text{ for all } n \in \mathbb{N}.$$

Proceeding to limit as $n \to \infty$, we have.

$$l = \frac{1}{2}(l + m) \quad \text{i.e. } l = m.$$

Therefore the sequence $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

Case 2:- Let $x_1 = y_1$

In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$

Therefore $\{x_n\}$ and $\{y_n\}$ both converge to to the limit x_1.

2.(a) (i) Let $|G| = 33$. What are the possible orders for the elements of G? Show that G must ~~ha~~ have an element of order '3'.

(ii) prove that group $\frac{4Z}{12Z} \cong Z_3$.

(iii) Give an example of an infinite integral domain that has characteristic 3.

Sol (i) The possible orders are : 1, 3, 11, 33

let $|x| = 33$; $x \in G$

~~then~~ $|x^{11}| = 3$.

we may assume that there is no element of order 33.

Since we know that

No. of elements of order d in a ~~finite group~~. In a finite group, the no. of elements of order d is a multiple of $\phi(d)$.

$\therefore$ The number of elements of order 11 is a multiple 10 so they account for 0, 10, 20 or 30 elements of the group.

The identity accounts for one more. So, at most we have accounted for 31 elements.

By Lagrange's theorem, the elements unaccounted for have order '3'.

(ii) Define $f: 4Z \longrightarrow Z_3$ by $f(4n) = [n]$ $\forall\, n \in Z$.

Clearly f is onto. and

$$f(4n+4m) = f(4(n+m)) = [n+m]$$
$$= [n] + [m]$$
$$= f(4n) + f(4m)$$

$\therefore$ f is homomorphism and onto

$\therefore$ By the fundamental theorem of homomorphism,

$$\frac{4Z}{\ker f} \cong Z_3.$$

Now we have $\ker f = \{4n \in 4Z \,/\, f(4n) = [0] \text{ in } Z_3\}$

$= \{4n \in 4Z \,/\, [n] = [0] \text{ in } Z_3\}$

$= \{4n \in 4Z \,/\, n \text{ is a multiple of } 3\}$

$= \{4n \in 4Z \,/\, n = 3k,\ k \in Z\}$

$= 12Z.$

$$\therefore \frac{4Z}{12Z} \cong Z_3.$$

(iii). Let $Z_3[x]$ be the ring of all polynomials over the ring $(Z_3 = \{0,1,2\}, +_3, \times_3)$

clearly it is an infinite integral domain.

$\because$ Char of $Z_3 = 3$.

$\therefore$ Char of $Z_3[x] = 3$

2(b) (i) Test for convergence of the series

$$\sum \frac{1^2.3^2 \cdots (2n-1)^2}{2^2.4^2 \cdots (2n)^2} x^{n-1}, x>0$$

(ii) Show that the function f defined by

$f(x)=\frac{1}{x}$, $x \in [1,\infty)$ is uniformly continuous on $[1,\infty)$.

Soln: Here $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} = \frac{1}{x}$.

Hence by Ratio test, the series converges if $x<1$ and diverges if $x>1$. Now for $x=1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} \frac{4n^2+3n}{(2n+1)^2} = 1$$

Hence Raabe's Test fails.

Let us now apply Gauss Test.

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2}+\cdots\right)$$

$= 1 + \frac{1}{n} - \frac{1}{4n^2} + \dots$ higher powers of $1/n$

So that by Gauss Test, the series diverges.

Hence, the series converges for $x < 1$ and diverges for $x \geq 1$.

(ii) let $c \geq 1$. Then for all $x \geq 1$

$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{x-c}{cx}\right| \leq |x-c|$,

since $|cx| \geq 1$.

let us choose $\epsilon > 0$. Then for all $x \geq 1$, satisfying $|x-c| < \epsilon$, $|f(x) - f(c)| < \epsilon$, what ever $c (\geq 1)$ may be.

This shows that f is uniformly continuous on $[1, \infty)$.

2(C) By using contour integration

Evaluate $\int_0^{\pi} \frac{\sin^4\theta}{a+b\cos\theta}\,d\theta$, where $a>b>0$.

Sol'n: $a>b>0 \Rightarrow a^2-b^2>0 = \sqrt{a^2-b^2} = \text{real}$

Let $I = \int_0^{\pi} \frac{\sin^4\theta}{a+b\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\sin^4\theta\,d\theta}{a+b\cos\theta}$ —— ①

Take C as $|z|=1$ or, $z=e^{i\theta}$, $dz = e^{i\theta}\,i\,d\theta$, $\frac{dz}{iz} = d\theta$

By eqs ①,

$$I = \int_C \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4 \frac{d\theta}{2a+b(e^{i\theta}+e^{-i\theta})} = \frac{1}{16}\int_C \left(z-\frac{1}{z}\right)^4 \times \frac{dz}{iz}\,\frac{1}{2a+(z+z^{-1})b}$$

$$\Rightarrow I = \frac{1}{16i}\int_C \frac{(z^2-1)^4\,dz}{z^4[2az+b(z^2+1)]}$$

$$\Rightarrow I = \frac{1}{16ib}\int_C f(z)\,dz \qquad \text{—— ②}$$

where $f(z) = \frac{(z^2-1)^4}{z^4\left[z^2+\frac{2az}{b}+1\right]}$ —— ③

For poles of $f(z)$: $z^4\left[z^2+\frac{2az}{b}+1\right]=0$

$$\Rightarrow bz^2+2az+b=0 \Rightarrow z = \frac{(-a)\pm\sqrt{a^2-b^2}}{b}$$

Take $\alpha = \frac{(-a)+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$.

Then $\alpha\beta=1$ and $\alpha<1$ and so $\beta>1$. Thus $z=0$ (Pole of order 4) and $z=\alpha$ (Pole of order one) lie within C.

$$\text{Res}(z=\alpha) = \lim_{z\to\infty}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)(z^2-1)^4}{z^4(z-\alpha)(z-\beta)}$$

$$= \frac{(\alpha^2-1)^4}{\alpha^4(\alpha-\beta)} = \left(\alpha-\frac{1}{\alpha}\right)^4 \cdot \frac{b}{2\sqrt{a^2-b^2}}$$

$$= (\alpha-\beta)^4 \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \left(\frac{2\sqrt{a^2-b^2}}{b}\right)^4 \cdot \frac{b}{2\sqrt{a^2-b^2}} = \frac{8}{b^3}(a^2-b^2)^{3/2}$$

$$\text{Res}(z=0) = \text{coeff. of } \frac{1}{z} \text{ in expansion of } \frac{(z^2-1)^4}{z^4\left(z^2+\frac{2az}{b}+1\right)}$$

$$= \text{Coeff. of } \frac{1}{z} \text{ in } \frac{(1-z^2)^4}{z^4}\left[1+\left(z^2+\frac{2az}{b}\right)\right]^{-1}$$

$$= \text{coeffi of } \frac{1}{z} \text{ in}$$

$$\frac{1}{z^4}\left(1-4C_1z^2+4C_2z^4\ldots\right)\left[1-\left(z^2+\frac{2az}{b}\right)+\left(z^2+\frac{2az}{b}\right)^2-\left(z^2+\frac{2az}{b}\right)^3+\ldots\right]$$

$$= \left[2\left(\frac{2a}{b}\right)-\left(\frac{2a}{b}\right)^3\right]+4C_1\left(\frac{2a}{b}\right) = -\frac{8a^3}{b^3}+\frac{12a}{b}$$

$$\text{Res}(z=0)+\text{Res}(z=\alpha) = \frac{8}{b^3}(a^2-b^2)^{3/2}+\left(\frac{12a}{b}-\frac{8a^3}{b^3}\right)$$

$$\int_C f(z)\,dz = 2\pi i\left[\text{Res}(z=0)+\text{Res}(z=\alpha)\right]$$

$$= 2\pi i\left[\frac{8}{b^3}(a^2-b^2)^{3/2}+\frac{12a}{b}-\frac{8a^3}{b^3}\right]$$

Putting this in equation ②, we get

$$I = \frac{\pi}{b^4}\left[(a^2-b^2)^{3/2}-a^3+\frac{3}{2}ab^2\right].$$

3.(a)(i) Suppose G is a group that exactly eight elements of order 3. How many subgroups of order 3 does G have?

Sol'n: let $(G, \times)$ be a group and has exactly eight elements of order '3'.

let $a \in G$ such that $o(a) = 3$ then $a^3 = e$ — (1)

$\therefore \Rightarrow a^2 = a^{-1}$ — (2)

Besides every subgroup $H = \{a, a^{-1}, e\} \leq G$ of order '3' is cyclic and is generated both by a and a^{-1}.

This means that we can enumerate the eight elements of order 3 in this way:

a, a^{-1}; b, b^{-1}; c, c^{-1}; d, d^{-1} and there are four subgroups of order 3.

they are : $A = \{a, a^{-1}, e\}$, $B = \{b, b^{-1}, e\}$

$C = \{c, c^{-1}, e\}$ and $D = \{d, d^{-1}, e\}$.

3.(a)(ii) Let $H = \{A \in GL(2,R) / \det A$ is rational$\}$. Prove or disprove that H is a subgroup of $GL(2,R)$. What if "rational" is replaced by an "integer"? Where $GL(2,R) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} / a,b,c,d \in R, ad-bc \neq 0\right\}$.

Sol^n: Given

$$G = GL(2,R) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} / a,b,c,d \in R, ad-bc \neq 0\right\}.$$

To show that H is a subgroup of G, it is enough to show that

$A, B \in H \Rightarrow AB \in H$ and $A \in H \Rightarrow A^{-1} \in H$

The 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is in H because determinant of identity matrix $= 1$ which is a rational.

$\therefore H \neq \phi$.

Let $A, B \in H$

i.e, A and B are 2×2 matrices such that $\det A$ and $\det B$ are rationals.

Now $\det(AB) = (\det A)(\det B)$

= which is a rational number. ($\because$ product of two rational numbers a rational number).

$\Rightarrow AB \in H$

Let $A \in H \Rightarrow \det A$ is a rational

Now $\det(A^{-1}) = (\det A)^{-1}$

= which is a rational

$\Rightarrow A^{-1} \in H$.

$A, B \in H \Rightarrow AB \in H$ & $A^{-1} \in H$.

$\therefore$ H is a subgroup of G.

If rational numbers is replaced by integer.

i.e, $H = \{A \in G \mid \det A \text{ is an integer}\}$

Let $A \in H \Rightarrow \det A$ is an integer.

Now $\det A^{-1} = (\det A)^{-1} = \frac{1}{\det A}$ which is not an integer

$\Rightarrow A^{-1} \notin H$

$\therefore$ H is not a subgroup of G if determinant is not an integer.

3.(b) Show that the series $\sum \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. If $S(x)$ be the sum function verify that $S'(x)$ is obtained by term by term differentiation.

Sol'n: Let $f_n(x) = \frac{1}{n^3+n^4x^2}$, $x \in \mathbb{R}$

For all $x \in \mathbb{R}$, $|f_n(x)| \le \frac{1}{n^3}$ for all $n \in N$

Let $M_n = \frac{1}{n^3}$.

then $\sum M_n$ is a convergent series of +ve terms.

By Weierstrass M test $\sum f_n$ is uniformly convergent for real x.

$$f_n'(x) = \frac{-2x}{n^2(1+nx^2)^2} = u(x), \text{ say. Then } u'(x) = \frac{2(3nx^2-1)}{n^2(1+nx^2)^3}$$

$u'(x) = 0$ at $x = \pm\frac{1}{\sqrt{3n}}$, $u'(x) < 0$ for $|x| < \frac{1}{\sqrt{3n}}$

$u'(x) > 0$ for $|x| > \frac{1}{\sqrt{3n}}$

Therefore u is a minimum at $\frac{1}{\sqrt{3n}}$ and maximum at $= \frac{-1}{\sqrt{3n}}$

$$u_{max} = \frac{9}{8\sqrt{3}} \cdot \frac{1}{n^{5/2}},$$

$$u_{min} = \frac{-9}{8\sqrt{3}} \cdot \frac{1}{n^{5/2}}$$

$u(0) = 0$, u is decreasing on $(0, \frac{1}{\sqrt{3n}})$, u is a minimum at $\frac{1}{\sqrt{3n}}$

u is increasing for $x > \frac{1}{\sqrt{3n}}$ and $\lim_{x \to \infty} u(x) = 0$.

Since u is an odd function, it follows that for all real x,

$$|f_n'(x)| \le \frac{3\sqrt{3}}{8} \cdot \frac{1}{n^{5/2}}$$

By Weierstrass' M-test, $\sum f_n'$ is uniformly convergent for all real x.

This ensures validity of term by term differentiation of the series $\sum f_n(x)$.

$$\therefore f_1'(x) + f_2'(x) + \cdots = S'(x)$$

4(a) Show that $\mathbb{Z}[\sqrt{-6}]$ is not a U.F.D.

Soln: we see that $10 = 10 + 0\sqrt{-6} \in \mathbb{Z}[\sqrt{-6}]$

$$\left.\begin{aligned} &\text{and } 10 = 2 \cdot 5 \\ &\text{and } 10 = (2+\sqrt{-6})(2-\sqrt{-6}) \end{aligned}\right\} \rightarrow (1)$$

we shall prove that $2, 5, 2 \pm \sqrt{-6}$ are irreducible elements of $\mathbb{Z}[\sqrt{-6}]$

Let $2 = (a+b\sqrt{6}i)(c+d\sqrt{6}i)$ and

so $2 = \bar{2} = (a-b\sqrt{6}i)(c-d\sqrt{6}i)$

Here $a, b, c, d \in \mathbb{Z}$

$\Rightarrow 4 = (a^2+6b^2)(c^2+6d^2)$

which gives the following possibilities

(i) $a^2+6b^2 = 1$ and $c^2+6d^2 = 4$.

(ii) $a^2+6b^2 = 4$ and $c^2+6d^2 = 1$.

(iii) $a^2+6b^2 = 2$ and $c^2+6d^2 = 2$.

It is clear that case (iii) is not possible in $\mathbb{Z}$.

case (i) is possible when $a = \pm 1$, $b = 0$.

$\Rightarrow a + b\sqrt{6}i = \pm 1$, which are units in $\mathbb{Z}[\sqrt{-6}]$

Similarly case (ii) yields that $c + d\sqrt{6}i = \pm 1$

which are units in $\mathbb{Z}[\sqrt{-6}]$.

Hence 2 is an irreducible elements of $Z[\sqrt{-6}]$

Similarly, we can show that 5 is an irreducible element of $Z[\sqrt{-6}]$.

Ans.

Now let $2+\sqrt{-6} = (a+b\sqrt{-6})(c+d\sqrt{-6})$; $a,b,c,d \in Z$.

$$2-\sqrt{-6} = (a-b\sqrt{-6})(c-d\sqrt{-6})$$

on multiplying respective sides of the above equation we get

$10 = (a^2+6b^2)(c^2+6d^2)$ which gives the following possibilities.

$\Rightarrow$ (i) $a^2+6b^2=1$ and $c^2+6d^2=10$

(ii) $a^2+6b^2=10$ and $c^2+6d^2=1$.

(iii) $a^2+6b^2=2$ and $c^2+6d^2=5$.

(iv) $a^2+6b^2=5$ and $c^2+6d^2=2$.

The first possibilities imply

$a=\pm 1, b=0$ (or) $c=\pm 1, d=0$

$a+b\sqrt{-6}=\pm 1$ (or) $c+d\sqrt{-6}=\pm 1$ which are units in $Z[\sqrt{-6}]$.

Hence $2\pm\sqrt{-6}$ are irreducible elements of $Z[\sqrt{-6}]$

$\therefore$ from (1) we see that $10 \in Z[\sqrt{-6}]$ has two distinct expressions as product of irreducible elements of $Z[\sqrt{-6}]$.

Hence $Z[\sqrt{-6}]$ is not a U.F.D.

4.(b) Show that the sequence $\{f_n\}$ where

$$f_n(x) = \begin{cases} n^2x, & 0 \le x \le 1/n \\ -n^2x+2n & 1/n \le x \le 2/n \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$

is not uniformly convergent on $[0,1]$.

Sol^n: The sequence converges to f, where $f(x)=0$, for all $x \in [0,1]$. Each function f_n and f are continuous on $[0,1]$.

Also $$\int_0^1 f_n\,dx = \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n} (-n^2x+2n)\,dx + \int_{2/n}^{1} 0\,dx = 1$$

But $$\int_0^1 f\,dx = 0$$

$$\lim_{n\to\infty} \int_0^1 f_n\,dx \neq \int_0^1 f\,dx$$

So by the theorem

(i) If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable, then f is integrable on $[a,b]$ and the sequence $\left\{\int_a^x f_n\,dt\right\}$ converges.

uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt \quad \forall\, x \in [a,b]$$

(8) If a series Σf_n converges uniformly to f on $[a,b]$ and each term $f_n(x)$ is integrable then 'f' is integrable on $[a,b]$ and the series $\Sigma\left(\int_a^x f_n\,dt\right)$ converges uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right), \forall\, x \in [a,b]$$

$\therefore$ from the above theorem the sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

4(c) (i) Evaluate the following integrals by using Cauchy's integral formula:

(a) $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{e^{zt}}{z^2+1}\,dz \;\forall\, t>0$ where C is $|z|=3$.

(b) $\displaystyle\int_C \dfrac{(z-1)\,dz}{(z+1)^2(z-2)}$, where C is $|z-i|=2$

Sol'n: Here we use two results:

(R_1) If $f(z)$ is analytic within and on a closed contour C, then $\displaystyle\int_C f(z)\,dz=0$.

(R_2) If $z=a$ is a point inside a closed contour C and $f(z)$ is analytic within and on C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}=f(a)$$

(R_3) $f^n(a)=\dfrac{n!}{2\pi i}\displaystyle\int_C \dfrac{f(z)\,dz}{(z-a)^{n+1}}$ for $n=1,2,3\ldots$

where $z=a$ is inside C.

(a) $I=\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{e^{zt}\,dz}{(z^2+1)} \;\forall\, t>0$

where C is $|z|=3$. Take $f(z)=e^{zt}$

$$\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

Here $z=i$, $z=-i$ both lie inside C.

$$I=\frac{1}{2\pi i}\int_C \frac{f(z)}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]dz$$

This $\Rightarrow$ $2iI=f(i)-f(-i)$, by (R_2)

$=e^{it}-e^{-it}=2i\sin t$

$\Rightarrow I=\sin t$

(b) $I = \int_C \frac{(z-1)\,dz}{(z+1)^2(z-2)}$.

where C is $|z-i| = 2$

$(z+1)^2(z-2) = 0 \Rightarrow z = -1,\ z = 2$

If $z = -1$, then $|z-i| = |-1-i| = |1+i| = \sqrt{2} < 2 = R$

If $z = 2$ then $|z-i| = |2-i| = \sqrt{5} > 2 = R$

$\therefore$ $z = -1$ lies inside C and $z = 2$ lies out-side C.

Take $f(z) = \frac{(z-1)}{z-2}$, then $I = \int_C \frac{f(z)}{[z-(-1)]^2}$

$\Rightarrow I = 2\pi i \frac{f'(-1)}{1!}$, by $(R_3) = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$

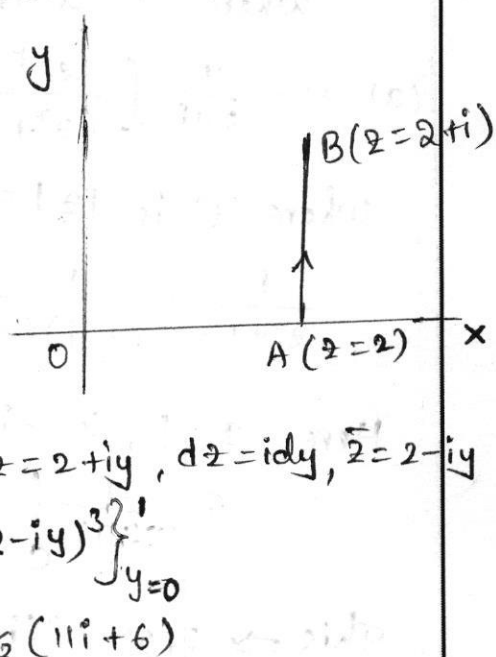
For $f(z) = \frac{z-1}{z-2} = 1 + \frac{1}{z-2} \Rightarrow f'(z) = -\frac{1}{(z-2)^2}$,

$f'(-1) = -\frac{1}{(3)^2}$

(ii) Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along real axis from $z=0$ to $z=2$ and then along a line parallel to y-axis from $z=2$ to $z=2+i$

Sol^n: Let $I = \int_0^{2+i} (\bar{z})^2\,dz$ — ①

Then $I = I_1(\vec{OA}) + I_2(\vec{AB})$ — ②



For I_1: equation of $\vec{OA}$ is $y=0$

$z = x+iy = x,\ dz = dx,\ \bar{z} = x$

$I_1 = \int_0^2 x^2\,dx = \left(\frac{x^3}{3}\right)_{x=0}^{2} = 8/3$

For I_2: equation of $\vec{AB}$ is $x=2$, $z = 2+iy$, $dz = i\,dy$, $\bar{z} = 2-iy$

$I_2 = \int_0^1 (2-iy)^2\,i\,dy = -\frac{1}{3}\left\{(2-iy)^3\right\}_{y=0}^{1}$

$= -\frac{1}{3}\left[(2-i)^3 - (2)^3\right] = \frac{1}{3}(11i+6)$

$I = I_1 + I_2 = \frac{8}{3} + \frac{1}{3}(11i+6) = \frac{1}{3}(14+11i)$

5.(a) Solve the following differential equation:
$(D^3-4D^2D'+5DD'^2-2D'^3)z = e^{y+2x}+(y+x)^{1/2}$.

Sol'n: Here auxiliary equation is $m^3-4m^2+5m-2=0$

$\Rightarrow (m-1)^2(m-2)=0$ So that $m=1,1,2$

$\therefore C.F = \phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+2x)$

Now, P.I corresponding to e^{y+2x}

$$= \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3} e^{y+2x}$$

$$= \frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2} e^{y+2x}\right\}$$

$$= \frac{1}{(D-2D')}\frac{1}{(2-1)^2}\iint e^v dv, \text{ where } v = y+x, \text{ by formula (i)}$$

$$= \frac{1}{D-2D'}\int e^v dv dv = \frac{1}{D-2D'}e^v$$

$$= \frac{1}{(1.D-2D')^1} e^{y+2x} = \frac{x}{1.1!}e^{y+x} = xe^{y+x} \quad \text{--- (2)}$$

[using formula (ii) with $a=2, b=1, m=1$]

finally, P.I corresponding to $(y+x)^{1/2}$

$$= \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}(y+x)^{1/2} = \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$$

$$= \frac{1}{(D-D')^2}\frac{1}{1-2.1}\int v^{1/2}dv, \text{ where } v = y+x,$$
using formula (i)

$$= -\frac{1}{D-D'}\cdot\frac{2}{3}v^{3/2} = -\frac{2}{3}\frac{1}{(D-D')^2}(y+x)^{3/2}$$

$$= -\frac{2}{3}\cdot\frac{x^2}{1^2.2!}(y+x)^{3/2}$$

$$= -\left(\frac{x^2}{3}\right)(y+x)^{3/2} \quad \text{[using formula (ii) with } a=b=1, m=2]$$

from (1), (2) and (3), the required general solution is — (3)

$$z = \phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+2x)+xe^{y+x}-\left(\frac{x^2}{3}\right)(y+x)^{3/2}$$

5(b) → Find the complete integral of

$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1.$

Solution:-

Let X and Y be two new variables such that

$$X^2 = x+y \quad \text{and} \quad Y^2 = x-y \quad \text{——} \; (1)$$

∴ Given equation is $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ —— (2)

Now; $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x}$

$$p = \frac{1}{2X} \frac{\partial z}{\partial x} + \frac{1}{2Y} \cdot \frac{\partial z}{\partial Y} \quad \text{——} \; (3)$$

$[\because$ from (1), $\frac{\partial X}{\partial x} = \frac{1}{2X}$ and $\frac{\partial Y}{\partial x} = \frac{1}{1/2Y}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$

$$q = \frac{1}{2X} \frac{\partial z}{\partial X} - \frac{1}{2Y} \frac{\partial z}{\partial Y} \quad \text{——} \; (4)$$

$[\because$ from (1); $\frac{\partial X}{\partial y} = \frac{1}{2X}$; $\frac{\partial Y}{\partial y} = \frac{-1}{2Y}]$

from (3) & (4) ⇒

$$p+q = \frac{1}{X} \frac{\partial z}{\partial X} \quad \text{and} \quad p-q = \frac{1}{Y} \frac{\partial z}{\partial Y} \quad \text{——} \; (5)$$

from, using (5) and (1), (2) reduces to

$$\cancel{X^2} \cdot \frac{1}{\cancel{X^2}} \left(\frac{\partial z}{\partial X}\right)^2 + \cancel{Y^2} \cdot \frac{1}{\cancel{Y^2}} \left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

$$\boxed{P^2 + Q^2 = 1} \quad \text{——} \; (6)$$

Where; $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$

(4) is of the form $f(P, Q) = 0$

$\therefore$ Solution of (4) is

$$\boxed{z = aX + bY + c} \quad \text{--- (7)}$$

where ; $a^2 + b^2 = 1$

or $b = \sqrt{1-a^2}$

Putting a for P and b for Q in (6)

$\therefore$ from (7), the required complete intergral is

$$z = aX + Y\sqrt{1-a^2} + c$$

$$\boxed{z = a\sqrt{x+y} + \sqrt{x-y}\sqrt{1-a^2} + c} \quad \text{--- by (1)}$$

5.(c) Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ by Gauss-Jordan method.

Sol'n: we place an identity matrix adjacent to the given matrix as a first step and the resulting augmented matrix is given by

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad \text{——} \; (1)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - 4R_1 \quad \text{——} \; (2)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] R_3 \to R_3 - 3R_1 \quad \text{——} \; (3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array} \quad \text{——} \; (4)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] R_3 \to R_3/-10 \quad \text{——} \; (5)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & -1 & 0 & 3/2 & 0 & -1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 4R_3 \\ R_2 \to R_2 + 5R_3 \end{array} \quad \text{——— (6)}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] R_2 \to -1R_2 \quad \text{——— (7)}$$

Hence we have

$$A^{-1} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \quad \text{——— (8)}$$

It can easily verified that $[A][A^{-1}] = [I]$

5.(d) (i) A NOR gate has three inputs A, B, C. which combination of inputs will give High output?

(ii) Implement the expression $Y = AB + CD$ using only NAND gates.

Sol^n: (i) Given that NOR gate has three inputs A, B, C

the high output (Y) be 1

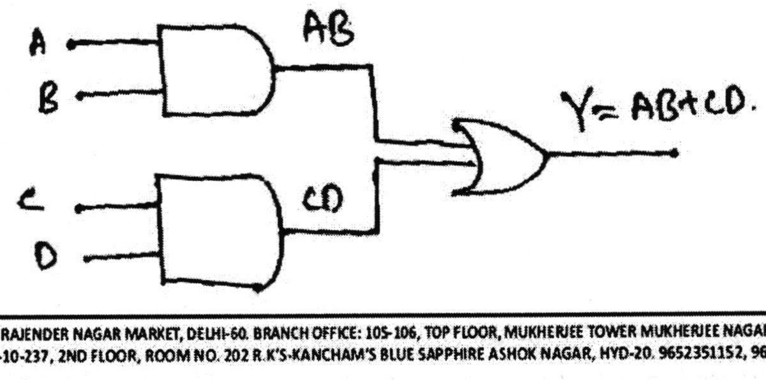
i.e $Y = 1 = \overline{A+B+C}$

$$\text{Since } \overline{A+B+C} = 1$$

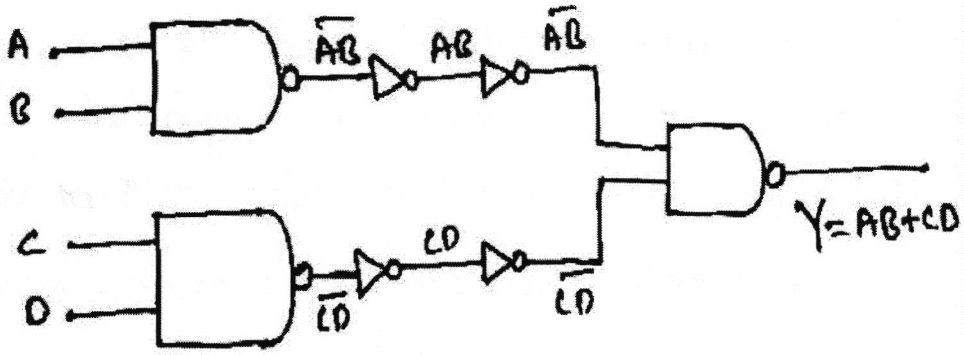
$$\Rightarrow A+B+C = 0$$

This is possible only if $A = B = C = 0$.

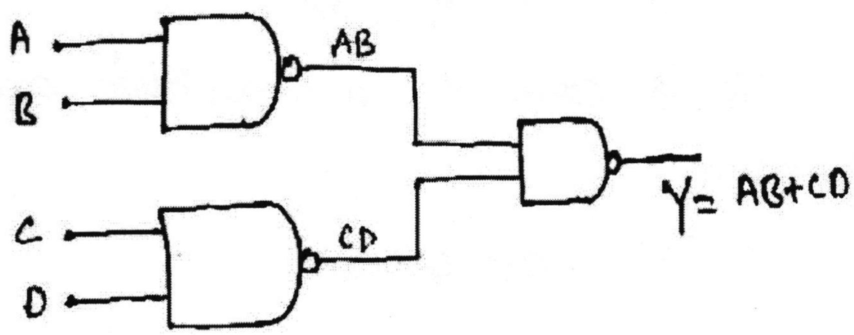
(ii) The straight forward implementation uses two AND gates and one OR gate as shown in fig. 4.51(a). Each AND gate can be replaced by a NAND gate and NOT gate in series. The OR gate can be also replaced by NAND gates. This is shown in fig. 4.51(b) it is seen that NOT gates 1 and 2 are in series and can be eliminated (because $\bar{\bar{A}} = A$). Similarly NOT gates 3 and 4 are in series and can be eliminated. Thus we get the logic circuit shown in fig 4.51(c)

(a)


(b)


A
B
$\overline{AB}$
AB
$\overline{AB}$
C
D
$\overline{CD}$
CD
$\overline{CD}$
Y= AB+CD

(c)


A
B
AB
C
D
CD
Y= AB+CD

5(e). In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of velocity u, v, w are solutions of Laplace's equation.

Sol: Let $W = \xi i + \eta j + \zeta k$,

$$q = ui + vj + wk.$$

vorticity is constant in magnitude and direction

$\Rightarrow \xi, \eta, \zeta$ are constant.

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta = \text{constant}.$$

$$\therefore \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{constant} \quad \text{---} (1)$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{constant} \quad \text{---} (2)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{constant} \quad \text{---} (3)$$

Differentiating of (2) and (3) w.r.t. z and y

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \cdot \partial z},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \, \partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \, \partial x} + \frac{\partial^2 w}{\partial x \, \partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}(0) = 0.$$

$\therefore \nabla^2 u = 0$. Similarly we can prove $\nabla^2 v = 0$, $\nabla^2 w = 0$. It means that components of velocity are solutions of Laplace's equation.

6(a)(i). Form a partial differential equation by eliminating the function 'f' from: $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Sol'n: Given that $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— ①

Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right]\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow -x^2\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ②}$$

and $\frac{\partial z}{\partial y} = 2y + 2f'\left[\frac{1}{x} + \log y\right]\left(\frac{1}{y}\right)$

$$\Rightarrow y\frac{\partial z}{\partial y} - 2y^2 = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ③}$$

from ② & ③ we get

$$-x^2\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y} - 2y^2$$

$$\Rightarrow x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2y^2$$

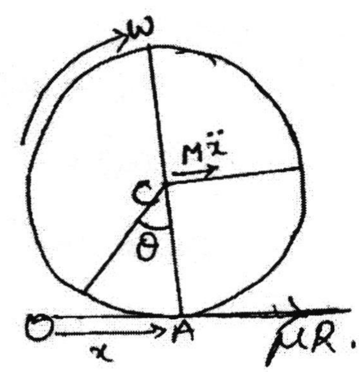
$$\Rightarrow x^2 p + yq = 2y^2$$

which is the required partial differential equation.

6. (C) A homogeneous sphere of radius a, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\frac{2\omega a}{7\mu g}$ and that then the sphere will roll with angular velocity $\frac{2\omega}{7}$.

Sol^n:

As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.



Initial velocity of the point of contact

= Initial velocity of the centre C + initial velocity of the point of contact w.r.t the centre C.

$= 0 + a\omega$ in the direction from right to left,

i.e, the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t

the equations of motion are

$M\ddot{x} = \mu R$, where $R = Mg$ ——— (1)

and $Mk^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu Ra$ ——— (2)

from (1), we have $\ddot{x} = \mu g$ ——— (3)

and from (2), we have

$a\ddot{\theta} = -\frac{5}{2}\mu g$ ——— (4)

Integrating (3) & (4),

we have

$\dot{x} = \mu g t + c_1$ and

$a\dot{\theta} = -\frac{5}{2}\mu g t + c_2$

Since initially when $t=0$, $\dot{x}=0$, $\dot{\theta}=\omega$

$\therefore c_1 = 0$ and $c_2 = a\omega$.

$\therefore \dot{x} = \mu g t$ ——— (5)

and $a\dot{\theta} = -\frac{5}{2}\mu g t + a\omega$ ——— (6)

velocity of the point of contact $= \dot{x} - a\dot{\theta}$

$\therefore$ The point of contact will come to rest

when $\dot{x} - a\dot{\theta} = 0$.

i.e, when $\mu g t - \left(-\frac{5}{2}\mu g t + a\omega\right) = 0$

i.e, when $t = \frac{2a\omega}{7\mu g}$

Therefore after time $\frac{2a\omega}{7\mu g}$ the slipping will stop and pure rolling will commence.

putting this value of t in (6), we get

$\dot{\theta} = \frac{2\omega}{7}$

when rolling commences, let F be the frictional force. Therefore the equations of motion are

$$M\ddot{x} = F, \quad \text{——— (7)}$$

$$M \cdot \frac{2}{5}a^2\ddot{\theta} = -Fa \quad \text{——— (8)}$$

and $\dot{x} - a\dot{\theta} = 0$ ——— (9)

From (9) $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

Now from (7) & (8), we get

$$M\ddot{x} = F = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow M\ddot{x} = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow \ddot{x} = -\frac{2}{5}a\ddot{\theta}$$

$$\Rightarrow a\ddot{\theta} = -\frac{2}{5}a\ddot{\theta} \quad (\because \ddot{x} = a\ddot{\theta})$$

$$\Rightarrow \frac{7}{5}a\ddot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} = 0. \quad (\because \frac{7}{5}a \neq 0)$$

Integrating

$$\dot{\theta} = \text{constant}$$

$$\dot{\theta} = \frac{2\omega}{7}.$$

7(a) Reduce the second-order partial differential equation $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol: Let $x = e^X$, $y = e^Y$

So that $X = \log x$, $Y = \log y$ —(1)

Also let $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$ and $D_1 = \frac{\partial}{\partial X}$, $D_1' = \frac{\partial}{\partial Y}$.

Then the given equation becomes

$$[D_1(D_1-1) - 2D_1D_1' + D_1'(D_1'-1) + D_1 + D_1']u = 0$$

$$(D_1^2 - 2D_1D_1' + D_1'^2)u = 0$$

$$(D_1 - D_1')^2 u = 0$$

Hence the required general solution is

$$C.F = \phi_1(Y+X) + X\phi_2(Y+X)$$

$$= \phi_1(\log y + \log x) + \log x\,\phi_2(\log y + \log x)$$

$$= \phi_1(\log xy) + \log x.\,\phi_2(\log xy)$$

$$= f_1(xy) + \log x\, f_2(xy)$$

7(b) → Using Runge-Kutta method of order 4, find y for $x=0.1, 0.2, 0.3$ given that $dy/dx = xy+y^2$, $y(0)=1$.

sol'n: Given that $f(x,y) = \frac{dy}{dx} = xy + y^2$; $y(0)=1$

To find $y(0.1)$

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$K_1 = hf(x_0, y_0) = (0.1)f(0.1) = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1)f(0.05, 1.05) = 0.1155$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1)f(0.05, 1.0577) = 0.1172$

$K_4 = hf(x_0 + h, y_0 + K_3) = (0.1)f(0.1, 1.1172) = 0.13598$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598)$

$= 0.11687$

$\therefore y(0.1) = y_1 = y_0 + K = 1.1169$

To find $y(0.2)$: Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$

$K_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.1169) = 0.1359$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0.1)f(0.15, 1.1848) = 0.1581$

$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = (0.1)f(0.15, 1.1959) = 0.1609$

$K_4 = hf(x_1 + h, y_1 + K_3) = (0.1)f(0.2, 1.2778) = 0.1888$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1605$

$y(0.2) = y_2 = y_1 + K = 1.2773.$

To find $y(0.3)$: Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$

$K_1 = hf(x_2, y_2) = (0.1)f(0.2, 1.2773) = 0.1887$

$K_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right) = (0.1)f(0.25, 1.3716) = 0.2224$

$K_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right) = (0.1)f(0.25, 1.3885) = 0.2275$

$K_4 = hf(x_2 + h, y_2 + K_3) = (0.1)f(0.3, 1.5048) = 0.2267$

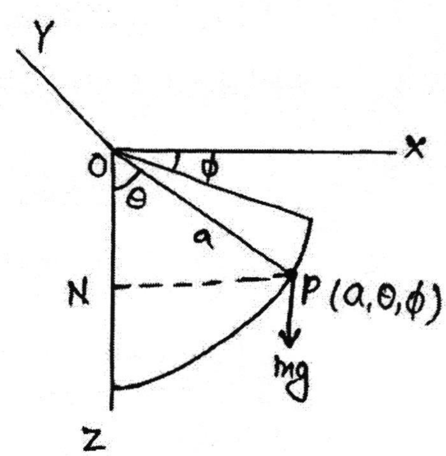
$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.2267$

$\therefore y(0.3) = y_3 = y_2 + K = 1.504$

7.(C) Determine the motion, of a spherical pendulum, by using Hamilton's equations.

Sol'n: Let m be the mass of the bob suspended by a light rod of length a. In a spherical pendulum of length a, the path of the motion of the bob is the surface of a sphere of radius a and centre at the fixed point O.

At time t, let $P(a, \theta, \phi)$ be the position of the bob. If (x, y, z) are the cartesian coordinates of P then



$x = a\sin\theta\cos\phi, \quad y = a\sin\theta\sin\phi$

$z = a\cos\theta$

$\therefore$ K.E., $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$

and potential $V = -mgz = -mga\cos\theta$ (since m is below the fixed point O).

$\therefore L = T - V$

$= \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mga\cos\theta$

$\therefore p_\theta = -\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}$ and

$p_\phi = -\frac{\partial L}{\partial \phi} = ma^2\dot{\phi}\sin^2\theta$ ——— (1).

Since L does not contain t explicitly.

$\therefore H = T + V = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations (1), we get

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \operatorname{cosec}^2\theta\, p_\phi^2\right) - mga\cos\theta.$$

Hence the four Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{ma^2}\operatorname{cosec}^2\theta \cot\theta\, p_\phi^2 - mga\sin\theta \quad \text{—— } (H_1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2} p_\theta \quad \text{—— } (H_2)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{1}{ma^2} p_\theta \quad \text{—— } (H_3)$$

and $$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{ma^2}\operatorname{cosec}^2\theta\, p_\phi \quad \text{—— } (H_4)$$

$\therefore$ from (H_3), Integrating $p_\phi = c$ (constant)

$\therefore$ from (H_4), we have

$$\dot{\phi} = \frac{1}{ma^2} c \operatorname{cosec}^2\theta = A/\sin^2\theta \quad (\text{where } A = c/ma^2)$$

Also from (H_1) & (H_2), we have

$$\ddot{\theta} = \frac{1}{ma^2}\dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2}\frac{\cos\theta}{\sin^3\theta}p_\phi^2 - mga\sin\theta\right]$$

$$= \frac{1}{(ma^2)^2} c^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta, \quad \because p_\phi = c$$

$$= A^2\frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta \quad (\because A = c/ma^2)$$

Multiplying bothsides by $2\dot{\theta}$ and integrating, we get

$$\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B, \quad (B \text{ is Const}). \text{ —— } ③$$

Equations ② and ③ determine the required motion.

8.(a) The deflection of a vibrating string of length l, is governed by the partial differential equation $y_{tt} = c^2 y_{xx}$. The initial velocity is zero. The initial velocity is zero The initial displacement is given by

$$y(x,0) = \begin{cases} x/l, & 0 < x < l/2 \\ (l-x)/l, & l/2 < x < l \end{cases}$$ Here $y_{tt} = \partial^2 y/\partial t^2$ and

$y_{xx} = \partial y^2/\partial x^2$. Find the deflection of the string at any instant of time..

Sol'n: The required deflection $y(x,t)$ of the string is the solution of the one-dimensional wave equation.

$$y_{tt} = c^2 y_{xx} \text{ i.e. } \frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{c^2}\right) \times \frac{\partial^2 y}{\partial t^2} \quad \text{—— (1)}$$

subject to the boundary conditions:

$$y(0,t) = y(l,t) = 0, \text{ for all } t \text{ —— (2)}$$

and the given initial conditions, namely

$$\text{initial velocity} = y(x,0) = f(x) = 0, \ 0 \le x \le l \text{ —— 3(a)}$$

and initial displacement $= y_t(x,t) = g(x) = \begin{cases} x/l, & 0 < x < l/2 \\ (l-x)/l, & l/2 < x < l \end{cases}$ —— 3(b)

Let the solution of (1) be of the form

$$y(x,t) = X(x)\, T(t) \text{ —— (4)}$$

Substituting this value of y in (1) we have

$$X''T = \frac{1}{c^2} X T'' \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu \text{ (say)} \quad (5)$$

$$\Rightarrow X'' - \mu X = 0 \ (6) \quad \text{and} \quad T'' - \mu c^2 T = 0 \ (7)$$

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(l)T(t) = 0$ (8)

$$\Rightarrow X(0) = 0 \ \& \ X(l) = 0 \ \text{—— (9)} \quad (\because T(t) \neq 0)$$

we now solve (6) under B.C. (9).

Three cases arise.

Case (1): Let $\mu = 0$. Then solution of (6) is given by

$$X(x) = Ax + B. \text{ —— (10)}$$

Using B.C. (9), (10) gives $B=0$, and $Al+B=0$

$\Rightarrow A=0=B.$

$\Rightarrow X(x)=0.$

This leads to $y\equiv 0$
which does not satisfy 3(a)&3(b).
So we reject $\mu=0$.

Case(2):

Let $\mu=\lambda^2$, $\lambda\neq 0$.

Then solution of (6) is $X(x)=Ae^{\lambda x}+Be^{-\lambda x}$ —— (11)

Using B.C. (9), (11) gives $A+B=0$ and $Ae^{\lambda l}+Be^{-\lambda l}=0$

$\Rightarrow A=B=0$ so that $X(x)=0$.

This leads to $y=0$
which does not satisfy 3(a) & 3(b).
So we reject $\mu=\lambda^2$.

Case(3):

Let $\mu=-\lambda^2$. $\lambda\neq 0$.

Then solution of (6) is $X(x)=A\cos\lambda x+B\sin\lambda x$ —— (12)

Using B.C. (9), (12) gives $A=0$ and $A\cos\lambda l+B\sin\lambda l=0$

$\Rightarrow A=0$ and $\sin\lambda l=0$ ($B\neq 0$
otherwise $B=0$.
so that $y\equiv 0$)

Now $\sin\lambda l=0 \Rightarrow \lambda l=n\pi$

$\Rightarrow \lambda=\frac{n\pi}{l}$, $n=1,2,\ldots$ —— (13)

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$X_n(x)=B_n\sin\left(\frac{n\pi x}{l}\right)$$

Using (13), (7) reduces to

$$T''+\left(\frac{n^2\pi^2 c^2}{a^2}\right)T=0 \qquad \left(\because \mu=-\lambda^2=-\frac{n^2\pi^2}{l^2}\right)$$

whose general solution is

$$T_n(t)=C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l} \text{ —— (14)}$$

$\therefore y_n(x,t)=X_n(x)T_n(t)$

$$y_n(x,t)=\sum_{n=1}^{\infty}\left(E_n\cos\left(\frac{n\pi ct}{l}\right)+F_n\sin\frac{n\pi ct}{l}\right)\cdot\sin\frac{n\pi x}{l} \text{ —— (15)}$$

where $E_n=B_nC_n$ and $F_n=B_nD_n$.

Differentiating (15) partially w.r.t t, we get

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\left(-\frac{n\pi cE_n}{l}\left(\sin\frac{n\pi ct}{l}\right)+\frac{n\pi cF_n}{l}\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}=0 \text{ —— (16)}$$

putting $t=0$ in (15) and (16) and using $3(a)$ & $3(b)$

we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{n\pi c F_n}{l} \sin\frac{n\pi x}{l}$$

where $F_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx = 0 \quad (\because f(x)=0)$

and, $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$

$\Rightarrow E_n = 2\int f(x) \sin\frac{n\pi x}{l} dx$

$$\Rightarrow E_n = \frac{2}{l}\int_0^{l/2} g(x) \sin\frac{n\pi x}{l} dx + \int_{l/2}^{l} g(x) \sin\frac{n\pi x}{l} dx$$

(from 3(b))

$$= \frac{2}{l}\int_0^{l/2} \frac{x}{l} \sin\frac{n\pi x}{l} dx + \frac{2}{n\pi c}\int_{l/2}^{l} \frac{l-x}{l} \sin\frac{n\pi x}{l} dx.$$

$$= \frac{2}{l}\left[(x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left[\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]\right]_0^{l/2}$$

$$+ \frac{2}{l}\left[(l-x)\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right] - (-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{2}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{l}\left(\frac{4l}{n^2\pi^2}\right)\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n=2m,\ m=1,2,\dots \\ \frac{(-1)^{m+1}4}{\pi^2(2m-1)^2} & \text{if } n=2m-1 \text{ and } m=1,2,\dots \end{cases}$$

$$= \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2},$$

$\left[\because n=2m-1 \Rightarrow \sin\frac{n\pi}{2} = \sin(2m-1)\frac{\pi}{2} = \sin\left(m\pi - \frac{\pi}{2}\right)\right.$

$\left.= \sin m\pi \cos\frac{\pi}{2} - \cos m\pi \sin\frac{\pi}{2} = 0-(-1)^m = (-1)^{m+1}\right]$

substituting the above value of E_n and F_n in (15),

the required deflection is given by

$$y(x,t) = \frac{4l}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi c t}{l}.$$

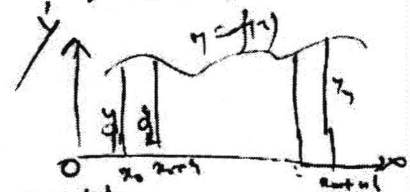
8.(b) Derive the formula

$$\int_a^b y\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots\cdots+y_{n-2})\Big]$$

Is there any restriction on n? State that condition. What is the error bound in the case of Simpson's $\frac{1}{3}$ rule?

Sol^n: Let $I=\int_a^b y\,dx$, where $y=f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x_0, x_1, x_2, \ldots x_n$.

Let us divide the interval (a,b) into n-subintervals of width h so that $x_0=a$, $x_1=x_0+h$, $x_2=x_0+2h$, $\ldots\ldots$ $x_n=x_0+nh=b$.



Then $I=\int_{x_0}^{x_0+nh} y\,dx = h\int_0^n f(x_0+ph)\,dp$

by putting $x=x_0+ph$, $dx = h\,dp$

Approximating y by Newton's forward difference formula, we obtain

$$I = h\int_0^{n}\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+\cdots\right]dp$$

which gives on simplification

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad (1)$$

This is known as Newton-Cotes quadrature formula. From this general formula, we deduce the different integration formulae by putting $n=1,2,3\ldots$ etc.

<u>Simpson's $\frac{1}{3}$-rd rule:</u> putting $n=2$ in (1), and taking the curve through (x_0,y_0), (x_1,y_1)

and (x_2, y_2) as a parabola i.e, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_2=x_0+2h} y\,dx = 2h\left(y_0+\Delta y_0+\frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}\left[y_0+4y_1+y_2\right)$$

Similarly

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left(y_2+4y_3+y_4\right)$$

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and finally

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2}+4y_{n-1}+y_n\right]$$

Summing up we obtain,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\Big[y_0+4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots y_{n-2})+y_n\Big]$$

which is known as Simpson's $\frac{1}{3}$-rule. This rule requires the division of the whole range into an even no. of subintervals of width h.

The error in the Simpsons rule is given by

$$\int_a^b y\,dx - \frac{h}{3}\Big[y_0+4(y_1+y_3+\cdots+\cdots y_{n-1}) +2(y_2+y_4+\cdots-y_{n-2})+y_n\Big]$$

$$= -\frac{(b-a)}{180}h^4 y^{iv}(\bar{x})$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivates.

8.(c) A Source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is

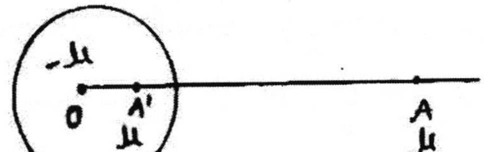
$$2\pi\rho\mu^2 a^2 / r(r^2 - a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure?

Solⁿ: Let X & Y be the components of the required force. then we have to prove that

$$\sqrt{X^2+Y^2} = \frac{2\pi\rho\mu^2 a^2}{r(r^2-a^2)}$$

$\Rightarrow r > a$



By Blasius Theorem,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

where ρ represents the boundary of the disc.

Since $2\pi\rho\mu$ represents the mass of the fluid emitted at A hence strength of the sources is μ.

The image of source $+\mu$ at A $(OA = r)$ is a source $+\mu$ at the inverse point A' such that $OA \cdot OA' = a^2$ and sink $-\mu$ at O.

Then $OA' = \frac{a^2}{r} = r'$, (say)

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to the object system and its image systems with no rigid boundary. Hence.

$w = -\mu \log(z-r) - \mu \log(z-r') + \mu \log(z-0)$

$\frac{dw}{dz} = -\mu\left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z}\right]$

$\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{(z-r)^2} + \frac{1}{(z-r')^2} + \frac{1}{z^2} + \frac{2}{(z-r)(z-r')} - \frac{2}{z(z-r')} - \frac{2}{z(z-r)}$

The function $\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2$ has poles $z=0$ and $z=r'$ within C. Residue at $z=0$ is the sum of coefficients of $\frac{1}{z}$ which is equal to

$\left[-\frac{2}{z-r'} - \frac{2}{z-r}\right]_{z-0} = 2\left(\frac{1}{r'} + \frac{1}{r}\right)$

Residue at $z=r'$ = sum of coefficients of $\frac{1}{z-r'}$

$= \frac{2}{r'-r} - \frac{2}{r'} + \frac{2}{r'} + \frac{2}{r} = \frac{2r}{(r'-r)r} = \frac{2a^2}{(a^2-r^2)r}$

By Cauchy's residue theorem,

$\int_C \frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 dz = 2\pi i \,.$ (sum of residues within C)

$= 2\pi i \frac{2a^2}{(a^2-r^2)r}$

we have seen that

$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$

$= \frac{i\rho}{2} \cdot \frac{2\pi i 2a^2}{(a^2-r^2)} \frac{\mu^2}{r} = \frac{2\pi a^2 \mu^2 \rho}{r(r^2-a^2)}$

$\Rightarrow X = \frac{2a^2 \pi \mu^2 \rho}{r(r^2-a^2)}, \quad Y = 0$

$\Rightarrow \sqrt{X^2+Y^2} = \frac{2\pi a^2 \mu^2 \rho}{r(r^2-a^2)}$

This also declares that the force is purely along $\vec{OA}$, the disc will be urged to move along OA. Also the cylinder is attracted towards the source, and sketch of the streamlines reveals that the pressure is greater on the opposite side of the disc than that of the source.