11 Improper Integrals

1. INTRODUCTION

The concept of Riemann integrals as developed in Chapter 9 requires that the range of integration is finite and the integrand remains bounded in that domain. If either (or both) of these assumptions is not satisfied, it is necessary to attach a new interpretation to the integral.

In case the integrand f becomes infinite in the interval $a \leq x \leq b$, *i.e.*, f has points of *infinite discontinuity* (*singular points*) in $[a, b]$, or the limits of integration a or b (or both) become infinite, the symbol $\int_a^b f\,dx$ is called an *improper* (or *infinite* or *generalised*) integral. Thus

$$\int_1^{\infty} \frac{dx}{x^2}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \int_0^1 \frac{dx}{x(1-x)}, \int_{-1}^{\infty} \frac{dx}{x^2}$$

are examples of improper integrals.

For the sake of distinction, the integrals (of Chapter 9) which are not improper are called *proper* integrals. Thus $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral.

It will be assumed throughout that the number of singular points in any interval is *finite* and therefore, when the range of integration is infinite, that all the singular points can be included in a finite interval. The restriction on the number is not necessary for the existence of the improper integral, but consideration of the discussion is beyond our limits.

Further, it is assumed once for all that *in a finite interval which encloses no point of infinite discontinuity* (*singular point*) *the integrand is bounded and integrable.*

2. INTEGRATION OF UNBOUNDED FUNCTIONS WITH FINITE LIMITS OF INTEGRATION

Definitions. Let a function f be defined in an interval $[a, b]$ everywhere except possibly at a finite number of points.

(*i*) **Convergence at the left-end.** Let a be the only point of infinite discontinuity of f so that according to the assumption made in the last section, the integral $\int_{a+\lambda}^b f\,dx$ exists for every $\lambda, 0 < \lambda < b-a$.

The improper integral $\int_a^b f\,dx$ is defined as the limit of $\int_{a+\lambda}^b f\,dx$ when $\lambda \to 0+$, so that

$$\int_a^b f\,dx = \lim_{\lambda \to 0+} \int_{a+\lambda}^b f\,dx \qquad ...(1)$$

If this limit exists and is finite, the improper integral $\int_a^b f\,dx$ is said to *exist* or *converge* (at a), if otherwise, it is called *divergent*.

Note: For any value c between a and b, $a < c < b$,

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

$\int_c^b f\,dx$ is a proper integral, so that the two integrals $\int_a^b f\,dx$ and $\int_a^c f\,dx$ converge and diverge together. Thus while testing the integral $\int_a^b f\,dx$ for convergence at a, it may be replaced by $\int_a^c f\,dx$ for any convenient c such that $a < c < b$.

(ii) Convergence at the right-end. Let b be the only point of infinite discontinuity of f, the improper integral is then defined by the relation

$$\int_a^b f\,dx = \lim_{\mu \to 0+} \int_a^{b-\mu} f\,dx,\ 0 < \mu < b - a \qquad ...(2)$$

If the limit exists, the improper integral is said to be *convergent* (at b), otherwise it is called *divergent*.

Note: For the same reason as above, when testing the integral $\int_a^b f\,dx$ for convergence at b, it may be replaced by $\int_c^b f\,dx$ for any convenient c between a and b.

(iii) Convergence at both the end-points. If the end-points a and b are the only points of infinite discontinuity of f, then for any point c within the interval $[a, b]$, the improper integral $\int_a^b f\,dx$ is understood to mean

$$\int_a^c f\,dx + \int_c^b f\,dx \qquad ...(3)$$

If *both* the integrals in (3) exist in accordance with the definitions given above, the improper integral *converges*; otherwise it is *divergent*.

The improper integral is also defined as

$$\int_a^b f\,dx = \lim_{\substack{\lambda \to 0+ \\ \mu \to 0+}} \int_{a+\lambda}^{b-\mu} f\,dx \qquad ...(3)$$

The improper integral exists if the limit exists.

(*iv*) **Convergence at interior points.** If an interior point c, $a < c < b$, is the only point of infinite discontinuity of f, we put

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx \qquad ...(4)$$

The improper integral $\int_a^b f\,dx$ is convergent if both the integrals on the R.H.S. exist in accordance with the definitions given above.

Similarly if the function has a finite number of points of infinite discontinuity, $c_1, c_2, ..., c_m$ within $[a, b]$, where

$$a \le c_1 < c_2 < ... < c_m \le b$$

the improper integral $\int_a^b f\,dx$ is defined as

$$\int_a^b f\,dx = \int_a^{c_1} f\,dx + \int_{c_1}^{c_2} f\,dx + ... + \int_{c_m}^b f\,dx \qquad ...(5)$$

and is said to be convergent if *all* the integrals on the R.H.S. of equation (5) are convergent, otherwise it is divergent.

Example 1. Examine the convergence of

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^2}$ (*ii*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$ (*iii*) $\displaystyle\int_0^2 \frac{dx}{(2x - x^2)}$

■ (*i*) 0 is the only point of infinite discontinuity of the integrand in [0, 1]. Thus,

$$\int_0^1 \frac{dx}{x^2} = \lim_{\lambda \to 0+} \int_\lambda^1 \frac{dx}{x^2},\ 0 < \lambda < 1$$

$$= \lim_{\lambda \to 0+} \left(\frac{1}{\lambda} - 1\right) = \infty$$

Thus the improper integral is divergent.

(*ii*) Since 1 is the only point of infinite discontinuity of the integrand in [0, 1], we put

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\mu \to 0+} \int_0^{1-\mu} \frac{dx}{\sqrt{1-x}}, 0 < \mu < 1$$

$$= \lim_{\mu \to 0+} \left[-2\sqrt{1-x}\right]_0^{1-\mu}$$

$$= -\lim_{\mu \to 0+} 2\left(\sqrt{\mu} - 1\right) = 2$$

Thus the improper integral exists and is equal to 2.

(*iii*) Both the end-points 0, 2 are the points of infinite discontinuity of the integrand and are in fact the only such points in [0, 2].

Thus for any point, say 1, within [0, 2], we put

$$\int_0^2 \frac{dx}{2x - x^2} = \lim_{\lambda \to 0+} \int_\lambda^1 \frac{dx}{x(2-x)} + \lim_{\mu \to 0+} \int_1^{2-\mu} \frac{dx}{x(2-x)}$$

$$= \frac{1}{2} \lim_{\lambda \to 0+} \left[\log \frac{x}{2-x}\right]_\lambda^1 + \frac{1}{2} \lim_{\mu \to 0+} \left[\log \frac{x}{2-x}\right]_1^{2-\mu}$$

$$= -\frac{1}{2} \lim_{\lambda \to 0+} \log \frac{\lambda}{2-\lambda} + \frac{1}{2} \lim_{\mu \to 0+} \log \frac{2-\mu}{\mu}$$

$$= \infty$$

Thus, the given integral diverges.

EXERCISE

1. Test for convergence the improper integrals:

(*i*) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ (*ii*) $\int_1^2 \frac{x\,dx}{\sqrt{x-1}}$

(*iii*) $\int_{-2}^2 \frac{dx}{(1-x)x^2}$ (*iv*) $\int_0^\pi \frac{dx}{\sin x}$

(*v*) $\int_0^1 \log x\,dx$ (*vi*) $\int_a^b \frac{dx}{(b-x)^n}$

2. Compute the integrals or prove their divergence:

(*i*) $\int_0^2 \frac{dx}{x^2 - 4x + 3}$ (*ii*) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$

(iii) $\int_1^2 \frac{dx}{x \log x}$ (iv) $\int_3^5 \frac{x^2\, dx}{\sqrt{(x-3)(5-x)}}$

(v) $\int_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$ (vi) $\int_{-1}^1 \frac{x-1}{x^{5/3}}\, dx$

ANSWERS

1. (i) C to $\pi/2$ (ii) C to $8/3$ (iii) D (iv) D (v) C to -1 (vi) C for only $n<1$.
2. (i) D (ii) C to 1 (iii) D (iv) $33\pi/2$ (v) $\pi/\sqrt{3}$ (vi) D.

3. COMPARISON TESTS FOR CONVERGENCE AT *a* OF $\int_a^b f dx$

(*Integrand retaining its sign*)

Let a, the left end of the interval, be the only point of infinite discontinuity of f in $[a, b]$. The case when b is the only point of infinite discontinuity can be dealt within the same way.

When the integrand keeps the same sign, positive or negative, in a small neighbourhood of a, we may suppose that f is non-negative therein, for, if negative it can be replaced by $(-f)$, for testing the convergence of $\int_a^b f\, dx$. The case $f = 0$ being trivial, there is no loss of generality to suppose that f is *positive* throughout.

The case when f does not necessarily keep the same sign, will be considered in § 3.5 wherein a *general test for convergence* is considered, which holds whether or not f retains its sign.

Theorem 1. *A necessary and sufficient condition for the convergence of the improper integral* $\int_a^b f\, dx$ *at a, where f is positive in* $[a, b]$ *is that there exists a positive number M, independent of* λ, *such that*

$$\int_{a+\lambda}^b f\, dx < M, \; 0 < \lambda < b - a$$

We know that the improper integral $\int_a^b f\, dx$ converges at a if for $0 < \lambda < b - a$, $\int_{a+\lambda}^b f\, dx$ tends to a finite limit as $\lambda \to 0+$.

Since f is positive in $[a, b]$, the positive function of λ, $\int_{a+\lambda}^{b} f\,dx$ is monotone increasing as λ decreases and will therefore tend to a finite limit if and only if it is bounded above, *i.e.*, there exists a positive number M independent of λ such that

$$\int_{a+\lambda}^{b} f\,dx < M,\ 0 < \lambda < b - a$$

Hence, the proof.

Note: If no such number M exists, the monotonic increasing function $\int_{a+\lambda}^{b} f\,dx$ is not bounded above, and therefore tends to $+\infty$ as $\lambda \to 0+$, and so the improper integral diverges to $+\infty$.

3.1 Comparison Test I (Comparison of two integrals)

If f and g be two positive functions such that $f(x) \le g(x)$, for all x in $[a, b]$, then

(i) $\int_a^b f\,dx$ *converges, if* $\int_a^b g\,dx$ *converges, and*

(ii) $\int_a^b g\,dx$ *diverges, if* $\int_a^b f\,dx$ *diverges.*

Let f and g be both bounded and integrable in $[a + \lambda, b], 0 < \lambda < b - a$ and a is the only point of infinite discontinuity in $[a, b]$.

Since f and g are positive and

$$f(x) \le g(x),\ \forall\, x \in [a, b]$$

$$\therefore \quad \int_{a+\lambda}^{b} f\,dx \le \int_{a+\lambda}^{b} g\,dx \qquad ...(1)$$

(i) Let $\int_a^b g\,dx$ be convergent, so that there exists a positive number M such that

$$\int_{a+\lambda}^{b} g\,dx < M,\ \text{for } 0 < \lambda < b - a$$

Thus, from equation (1)

$$\int_{a+\lambda}^{b} f\,dx < M\text{, for } 0 < \lambda < b-a$$

Hence, $\int_{a}^{b} f\,dx$ converges at a.

(ii) Again, if $\int_{a}^{b} f\,dx$ is divergent at a, then the positive function $\int_{a+\lambda}^{b} f\,dx$ is not bounded above and therefore from equation (1), $\int_{a+\lambda}^{b} g\,dx$ is also not bounded above. Hence, $\int_{a}^{b} g\,dx$ is divergent at a.

3.2 Comparison Test II (*Limit form*)

If f and g are two positive functions in $[a, b]$ *such that* $\lim\limits_{x\to a+0} \dfrac{f(x)}{g(x)} = l$, *where* l *is a non-zero finite number, then the two integrals* $\int_{a}^{b} f\,dx$ *and* $\int_{a}^{b} g\,dx$ *converge and diverge together at a.*

Evidently $l > 0$.

Let ε be a positive number such that $l - \varepsilon > 0$.

Since $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = l$, therefore there exists a neighbourhood $]a, c[$, $a < c < b$, such that for all $x \in\,]a, c[$

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$$

or

$$(l-\varepsilon)g(x) < f(x) < (l+\varepsilon)g(x) \quad \text{...(2)}$$

Now,

$$(l-\varepsilon)g(x) < f(x),\ \forall\, x \in\,]a, c[$$

So if $\int_{a}^{b} f\,dx$, *i.e.*, if $\int_{a}^{c} f\,dx$ converges at a then by comparison test I, $(l-\varepsilon)\int_{a}^{c} g\,dx$ and consequently $\int_{a}^{b} g\,dx$ converges at a.

Again from equation (2),

$$f(x) < (l+\varepsilon)\, g(x), \ \forall\, x \in\,]a, c[\,.$$

So if $\int_a^b f\,dx$ i.e., if $\int_a^c f\,dx$ diverges at a, then by comparison test, $(l+\varepsilon)\int_a^c g\,dx$ and therefore, $\int_a^b g\,dx$ diverges at a.

It may similarly be shown that $\int_a^b f\,dx$ converges and diverges with $\int_a^b g\,dx$.

Hence, the two integrals behave alike.

Note: It can be easily shown that:

(i) if $f/g \to 0$ and $\int_a^b g\,dx$ converges, then $\int_a^b f\,dx$ also converges, and

(ii) if $f/g \to \infty$ and $\int_a^b g\,dx$ diverges, then $\int_a^b f\,dx$ also diverges.

3.3 A Useful Comparison Integral

The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ *converges if and only if* $n < 1$.

It is a proper integral if $n \le 0$, and improper for other values of n; a being the only point of infinite discontinuity of the integrand.

Now for $n \neq 1$,

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\lambda \to 0+} \int_{a+\lambda}^b \frac{dx}{(x-a)^n}, \quad 0 < \lambda < b-a$$

$$= \lim_{\lambda \to 0+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\lambda^{n-1}}\right]$$

$$= \begin{cases} 1/(1-n)(b-a)^{n-1}, & \text{if } n < 1 \\ \infty, & \text{if } n > 1 \end{cases}$$

Again, for $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\lambda \to 0+} \int_{a+\lambda}^b \frac{dx}{x-a}$$

$$= \lim_{\lambda \to 0+} \{\log(b-a) - \log \lambda\} = \infty$$

Thus, $\int_a^b \frac{dx}{(x-a)^n}$ converges only for $n < 1$

The integral is widely used when applying the comparison tests for testing the convergence of improper integrals.

Note: A similar result holds for convergence of $\int_a^b \frac{dx}{(b-x)^n}$ at b.

Example 2. Test the convergence of

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ (ii) $\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx$

■ (i) Let $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{(1-x)^{1/2}} \frac{1}{(1+x+x^2)^{1/2}}$.

Clearly, $\frac{1}{(1+x+x^2)^{1/2}}$ is a bounded function and let M be its upper bound.

$\therefore$ $f(x) \le \frac{M}{(1-x)^{1/2}}$

Also $\int_0^1 \frac{dx}{(1-x)^{1/2}}$ is convergent.

Therefore by comparison test, $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ is convergent.

(ii) For $p \le 1$, it is a proper integral. For $p > 1$, it is an improper integral, 0 being the point of infinite discontinuity.

Now

$$\frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x}$$

The function $\frac{\sin x}{x}$ is bounded and $\frac{\sin x}{x} \le 1$.

$$\therefore \qquad \frac{\sin x}{x^p} \le \frac{1}{x^{p-1}}$$

Also $\int_0^{\pi/2} \frac{dx}{x^p - 1}$ converges only if $p - 1 < 1$ or $p < 2$.

Therefore by comparison test, $\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx$ converges for $p < 2$ and diverges for $p \ge 2$.

3.4 With the help of § 3.1, 3.2 and 3.3, we deduce two comparison tests which are of much practical utility.

I. *If f is positive in a nbd of a, then the integral $\int_a^b f\, dx$ converges at a if there exists a positive number n less than 1 and a fixed positive number M such that $f(x) \le M/(x-a)^n$ for all x in $]a, b]$.*

Also, $\int_a^b f\, dx$ diverges if there exists a number $n \ge 1$ and a fixed positive number G such that

$$f(x) \ge G/(x-a)^n \text{ in }]a, b].$$

II. *If $\lim_{x \to a+} \left[(x-a)^n f(x)\right]$ exists and is non-zero finite, then the integral $\int_a^b f\, dx$ converges if and only if $n < 1$.*

Example 3. Find the values of m and n for which the following integrals converge:

(*i*) $\int_0^1 e^{-mx} x^n\, dx$

(*ii*) $\int_0^1 (\log 1/x)^m\, dx$

▪ (*i*) Let k be a number greater than 1 and e^{-m}, for all m.

In $[0, 1]$, $e^{-mx} x^n \leq kx^n$, for all m, and $\int_0^1 x^n \, dx = \int_0^1 \frac{dx}{x^{-n}}$ converges for $-n < 1$ or $n > -1$ only. Thus, $\int_0^1 e^{-mx} x^n \, dx$ converges only for $n > -1$, irrespective of the values of m.

(*ii*) Putting $\int_0^1 (\log 1/x)^m \, dx = \int_0^{1/2} (\log 1/x)^m \, dx + \int_{1/2}^1 (\log 1/x)^m \, dx$, 0 and 1 respectively are the points of infinite discontinuity of the integrals on the right.

Let $f(x) = (\log 1/x)^m$.

Convergence at 0.

$\int_0^{1/2} (\log 1/x)^m \, dx$ is a proper integral if $m \leq 0$, for the integrand tends to a finite limit as $x \to 0$ (1 if $m = 0$; 0 if $m < 0$). 0 is the only point of infinite discontinuity if $m > 0$.

For $m > 0$, take

$$g(x) = \frac{1}{x^p}, 0 < p < 1$$

so that

$$\frac{f(x)}{g(x)} = x^p (\log 1/x)^m \to 0 \text{ as } x \to 0.$$

Also $\int_0^{1/2} g \, dx$ converges, therefore $\int_0^{1/2} (\log 1/x)^m \, dx$ converges. Thus, $\int_0^{1/2} (\log 1/x)^m \, dx$ converges for all m.

Convergence at 1.

$\int_{1/2}^1 (\log 1/x)^m \, dx$ is a proper integral if $m \geq 0$, and 1 is the only point of infinite discontinuity if $m < 0$.

For $m < 0$, let

$$g(x) = \frac{1}{(1-x)^{-m}}$$

so that

$$\frac{f(x)}{g(x)} = \left(\frac{\log 1/x}{1-x} \right)^m \to 1 \text{ as } x \to 1.$$

Hence, the two integrals $\int_{1/2}^{1} f\,dx$ and $\int_{1/2}^{1} g\,dx$ behave alike.

$\int_{1/2}^{1} \frac{dx}{(1-x)^{-m}}$ converges if $-m < 1$ or $m > -1$ therefore $\int_{1/2}^{1} (\log 1/x)^m\,dx$ also converges if $0 > m > -1$.

Consequently $\int_0^1 (\log 1/x)^m\,dx$ is convergent when $0 > m > -1$.

Example 4. Show that

(i) $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ is convergent, but

(ii) $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ is divergent.

(i) Since $\log x/\sqrt{x}$ is negative in [0, 1], we take

$$f(x) = -\frac{\log x}{\sqrt{x}} = \frac{\log 1/x}{\sqrt{x}}$$

0 is the only point of infinite discontinuity.

Let $g(x) = \frac{1}{x^{3/4}}$ so that

$$\frac{f(x)}{g(x)} = x^{1/4}(\log 1/x) \to 0 \text{ as } x \to 0+$$

The integral $\int_0^1 g\,dx = \int_0^1 \frac{dx}{x^{3/4}}$ is convergent at 0; therefore $\int_0^1 \frac{\log 1/x}{\sqrt{x}}\,dx$ and so $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ is also convergent.

(ii) Here 1 is the only point of infinite discontinuity.

Let $f(x) = \frac{\sqrt{x}}{\log x}$ and $g(x) = \frac{1}{x-1}$, so that

$$\lim_{x\to 1+} \frac{f(x)}{g(x)} = \lim_{x\to 1+} \frac{(x-1)\sqrt{x}}{\log x} = \lim_{x\to 1+} \frac{\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1/x} = 1$$

The integral $\int_1^2 \frac{dx}{x-1}$ diverges and therefore $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ also diverges.

Example 5. Show that the integral $\int_0^{\pi/2} \left(\frac{\sin^m x}{x^n}\right) dx$ exists if and only if $n < m+1$.

- Let $f(x) = \frac{\sin^m x}{x^n} = \frac{1}{x^{n-m}} \cdot \left(\frac{\sin x}{x}\right)^m$

Here as $x \to 0+$, $f(x) \to 0$ if $n - m < 0$, and $f(x) \to \infty$ if $n - m > 0$.

Thus, it is a proper integral if $n \le m$, and improper if $n > m$, 0 being the only point of infinite discontinuity of f.

When $n > m$, let

$$g(x) = \frac{1}{x^{n-m}}$$

so that

$$\frac{f(x)}{g(x)} = \left(\frac{\sin x}{x}\right)^m \to 1 \text{ as } x \to 0+$$

Also $\int_0^{\pi/2} g\,dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ converges if and only if $n - m < 1$ or $n < m+1$, therefore $\int_0^{\pi/2} \frac{\sin^m x}{x^n}\,dx$ also converges if and only if $n < m+1$ which includes the case $n \le m$ when the integral is proper.

Example 6. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists if and only if m, n are both positive.

- It is a proper integral for $m \ge 1, n \ge 1$, 0 and 1 are the only points of infinite discontinuity; 0 when $m < 1$, and 1 when $n < 1$.

For $m < 1$ and $n < 1$.

Taking a number, say $\frac{1}{2}$, between 0 and 1, we put

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$$

and examine the two integrals on the right for convergence at 0 and 1 respectively.

Convergence at 0, when $m < 1$.

Let

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}}$$

$$g(x)=\frac{1}{x^{1-m}}$$

and

$$\frac{f(x)}{g(x)}=(1-x)^{n-1}\to 1 \text{ as } x\to 0$$

Also $\int_0^{1/2} g\,dx\ \int_0^{1/2}\frac{dx}{x^{1-m}}$ converges at 0 if and only if $1-m<1$, *i.e.*, $m>0$

Thus $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ converges at zero if and only if $m>0$

Convergence at 1, when $n<1$

$$f(x)=x^{m-1}(1-x)^{n-1}=\frac{x^{m-1}}{(1-x)^{1-n}}$$

Let

$$g(x)=\frac{1}{(1-x)^{1-n}}$$

then

$$\frac{f(x)}{g(x)}=x^{m-1}\to 1 \text{ as } x\to 1$$

Also $\int_{1/2}^{1} g\,dx=\int_{1/2}^{1}\frac{dx}{(1-x)^{1-n}}$ converges at 1, if and only if $1-n<1$, or $n>0$.

Thus $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$ converges at 1, if and only if $n>0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists for positive values of m and n only.

This integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for $m,n>0$ is called *Beta function* and is denoted by $\beta(m,n)$.

Example 7. For what values of m and n is the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$$

convergent?

- The integrand is negative in [0, 1], therefore we shall test for convergence the integral

$$\int_0^1 -x^{m-1}(1-x)^{n-1}\log x\,dx, \text{ i.e., } \int_0^1 x^{m-1}(1-x)^{n-1}\log\frac{1}{x}\,dx$$

Since 0 and 1 are the possible points of infinite discontinuity, therefore taking a number, say $\frac{1}{2}$ between 0 and 1, we examine the integrals

$$\int_0^{1/2} x^{m-1}(1-x)^{n-1}\log\frac{1}{x}\,dx, \quad \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\log\frac{1}{x}\,dx$$

at 0 and 1 respectively.

Convergence at 0.

It is a proper integral for $m-1>0$, and improper for $m \le 1$, 0 being the only point of infinite discontinuity then.

For $m \le 1$, let

$$f(x) = x^{m-1}(1-x)^{n-1}\log\frac{1}{x} = \frac{(1-x)^{n-1}\log 1/x}{x^{1-m}}$$

and

$$g(x) = \frac{1}{x^p}$$

$\int_0^{1/2} g\,dx$ is convergent if and only if $p<1$.

Also $\dfrac{f(x)}{g(x)} = x^{p+m-1}(1-x)^{n-1}\log\dfrac{1}{x} \to 0$ as $x \to 0+$ if $p+m-1>0$ or $m>1-p>0$.

So by comparison test, $\int_0^{1/2} f\,dx$ converges if and only if $m>0$.

Convergence at 1.

Since $\lim\limits_{x\to 1-}\dfrac{\log 1/x}{(1-x)^{1-n}} = \lim\limits_{x\to 1}\dfrac{(1-x)^n}{(1-n)^x}$ exists finitely when $n \ge 0$, therefore the integral is proper for $n \ge 0$ and improper for $n<0$, 1 being the only singular point.

For $n<0$, let

$$f(x) = x^{m-1}(1-x)^{n-1}\log\frac{1}{x} = \frac{x^{m-1}\log 1/x}{(1-x)^{1-n}}$$

and

$$g(x)=\frac{1}{(1-x)^q}$$

$\int_{1/2}^{1} g\,dx$ is convergent if and only if $q<1$.

Also $\dfrac{f(x)}{g(x)}=\dfrac{x^{m-1}\log 1/x}{(1-x)^{1-n-q}}$, tends to a finite limit as $x\to 1$, if $1-n-q\le 1$, *i.e.*, if $n\ge -q>-1$.

Thus, $\int_{1/2}^{1} f\,dx$ converges at 1 if $n>-1$.

Hence, the given integral is convergent when $m>0, n>-1$.

Example 8. Show that the integral $\int_0^{\pi/2} \log\ \sin x\,dx$ is convergent and hence evaluate it.

Let $f(x)=\log\ \sin x$.

As f is non-positive in $[0,\pi/2]$, we consider the function $(-f)$ for testing convergence of the integral.

0 is the only point of infinite discontinuity of f.

Let $g(x)=\dfrac{1}{x^m}$, $m<1$, so that

$$\frac{-f(x)}{g(x)}=-x^m\log\ \sin\ x\to 0 \text{ as } x\to 0$$

Also $\int_0^{\pi/2}\dfrac{dx}{x^m}$ is convergent for $m<1$.

Therefore, $\int_0^{\pi/2} -\log\ \sin x\,dx$ and so $\int_0^{\pi/2} \log\ \sin x\,dx$ is convergent.

To evaluate the integral, let $I=\int_0^{\pi/2}\log\ \sin x\,dx$.

We know that

$$\sin 2x = 2\sin x\cos x$$

$$\therefore\quad \log\ \sin 2x=\log 2+\log\ \sin x+\log\ \cos x$$

$$\Rightarrow\quad \int_0^{\pi/2}\log\ \sin 2x\,dx=\int_0^{\pi/2}(\log 2)\,dx+\int_0^{\pi/2}\log\ \sin x\,dx+\int_0^{\pi/2}\log\ \cos x\,dx$$

$$= \frac{\pi}{2}\log 2 + I + \int_0^{\pi/2} \log \cos x \, dx$$

Putting $2x = t$ in the first and $x = \frac{\pi}{2} - y$ in the last integral, we get

$$I = \frac{\pi}{2}\log 2 + I + I$$

$$\therefore \qquad I = -\frac{\pi}{2}\log 2.$$

EXERCISE

1. Test for convergence:

(i) $\int_0^1 \frac{x^n}{1-x}\,dx$

(ii) $\int_0^1 \frac{x^n}{1+x}\,dx$

(iii) $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$

(iv) $\int_2^3 \frac{x^2+1}{x^2-4}\,dx$

(v) $\int_0^1 x^{m-1} e^{-x}\,dx$

(vi) $\int_0^{\pi/2} \sin^{p-1} x \cos^{q-1} x\,dx$

(vii) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x}\,dx$

(viii) $\int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$

2. Show that the integral $\int_0^{\pi/2} \sin x \log \sin x\,dx$ converges to the value $\log 2 - 1$.

[*Hint*: Integrate by parts the integral $\int_\lambda^{\pi/2} \sin x \log \sin x\,dx$ and take the limit.]

3. Show that $\int_0^{\pi/2} \cos 2nx \log \sin x\,dx$, $n \geq 1$ converges to the value $-\pi/4n$.

4. Compute, if possible, the integrals

(i) $\int_0^{\pi} x \log \sin x\,dx$

(ii) $\int_0^{\pi/2} x \cos x\,dx$

(iii) $\int_0^1 \frac{\sin^{-1} x}{x}\,dx$

(iv) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$

[*Hint*: (*i*) Put $x = \pi - t$ (*ii*) Integrate by parts

(*iii*) Put $x = \sin t$ (*iv*) Put $x = \sin t$.]

ANSWERS

Div.	(ii) Conv. for $n > -1$	(iii) Conv.
Div.	(x) Conv. for $m > 0$	(xi) Conv. for $p > 0$ and $q > 0$
Div.	(xiii) Conv. for $n > -1$.	

3. (i) $-\frac{\pi^2}{2}\log 2$ (ii) $\frac{\pi}{2}\log 2$

(iii) $\frac{\pi}{2}\log 2$ (iv) $-\frac{\pi}{2}\log 2$

11.5 General Test for Convergence (*Integrand may change sign*)

We now discuss a general test for convergence of an improper integral (finite limits of integration but discontinuous integrand) which holds whether or not the integrand keeps the same sign.

Cauchy's Test. *The improper integral* $\int_a^b f\,dx$ *converges at* a *if and only if to every* $\varepsilon > 0$ *there corresponds* $\delta > 0$ *such that*

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f\,dx\right| < \varepsilon,\ 0 < \lambda_1, \lambda_2 < \delta$$

[Notice that $\int_{a+\lambda_1}^{a+\lambda_2} f\,dx$ tends to 0 as $\lambda_1, \lambda_2 \to 0$.]

The improper integral $\int_a^b f\,dx$ is said to exist when $\lim_{\lambda \to 0+} \int_{a+\lambda}^{b} f\,dx$ exists finitely.

Let $F(\lambda) = \int_{a+\lambda}^{b} f\,dx$

so that $F(\lambda)$ is a function of λ.

According to Cauchy's criterion for finite limits (§ 1.3, Ch. 5) $F(\lambda)$ tends to a finite limit as $\lambda \to 0$ if and only if for every $\varepsilon > 0$ there corresponds $\delta > 0$ such that for all positive $\lambda_1, \lambda_2 < \delta$.

$$|F(\lambda_1) - F(\lambda_2)| < \varepsilon$$

i.e.,

$$\left|\int_{a+\lambda_1}^{b} f\,dx - \int_{a+\lambda_2}^{b} f\,dx\right| < \varepsilon$$

or

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f\,dx\right| < \varepsilon$$

3.6 Absolute Convergence

Definition. The improper integral $\int_a^b f\,dx$ is said to be absolutely convergent if $\int_a^b |f|\,dx$ is convergent.

With the help of Cauchy's test, we now deduce a sufficient condition for the convergence of an improper integral.

Theorem 2. *Every absolutely convergent integral is convergent, or*

$$\int_a^b f\,dx \text{ exists if } \int_a^b |f|\,dx \text{ exists.}$$

Since $\int_a^b |f|\,dx$ exists, therefore by Cauchy's test, for $\varepsilon > 0$ there corresponds $\delta > 0$ such that

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} |f|\,dx\right| < \varepsilon, 0 < \lambda_1, \lambda_2 < \delta \qquad ...(1)$$

Also, we know (Th. 10, Ch. 9) that

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f\,dx\right| \le \int_{a+\lambda_1}^{a+\lambda_2} |f|\,dx \qquad ...(2)$$

Hence, from equations (1) and (2)

$$\left|\int_{a+\lambda_1}^{a+\lambda_2} f\,dx\right| < \varepsilon, 0 < \lambda_1, \lambda_2 < \delta$$

$$\Rightarrow \qquad \int_a^b f\,dx \text{ exists.}$$

Aliter. Since

$$0 \le |f| - f \le 2|f|,$$

therefore by comparison test

$$\int_a^b \{|f| - f\}\,dx \text{ converges}$$

Hence, $\int_a^b f\,dx = \int_a^b \{f - |f|\}\,dx + \int_a^b |f|\,dx$ converges.

Notes:

1. Since $|f|$ is always positive, comparison tests of § 3 are applicable for examining the convergence of $\int_a^b |f|\,dx$, i.e., absolute convergence of $\int_a^b f\,dx$.

2. Every convergent integral is not absolutely convergent. For this reason, a convergent integral which is not absolutely convergent is called a *conditionally convergent* integral.

Example 9. Show that $\int_0^1 \frac{\sin 1/x}{x^p}\,dx$, $p>0$, converges absolutely for $p<1$.

Let

$$f(x) = \frac{\sin 1/x}{x^p},\ p>0$$

0 is the only point of infinite discontinuity, and f does not keep the same sign in any neighbourhood of 0.

In [0, 1]

$$|f(x)| = \left|\frac{\sin 1/x}{x^p}\right| = \frac{|\sin 1/x|}{x^p} < \frac{1}{x^p}$$

Also $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p<1$.

Hence by comparison test, the integral $\int_0^1 \left|\frac{\sin 1/x}{x^p}\right| dx$ converges and so $\int_0^1 \frac{\sin 1/x}{x^p}\,dx$ converges absolutely if and only if $p<1$.

Conditional convergence of improper integrals of this type is generally tested by reducing them to the other type, with infinite range of integration. See examples § 11.5.

4. INFINITE RANGE OF INTEGRATION

We shall now consider the convergence of improper integrals of bounded* integrable function with infinite range of integration (a or b or both infinite).

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* The word bounded seems to be redundant. But some authors extend the class of functions integrable in the Riemann sense to include those unbounded functions whose improper integrals exists. The word bounded is inserted here to exclude the possibility of interpreting the term integrable functions in the extended sense.

Definitions.

(i) **Convergence at ∞.** The symbol

$$\int_a^\infty f\,dx,\ x \geq a \quad \text{...(1)}$$

is defined as the limit of $\int_a^X f\,dx$, when $X \to \infty$, so that

$$\int_a^\infty f\,dx = \lim_{X \to \infty} \int_a^X f\,dx \quad \text{...(2)}$$

If the limit exists and is finite, then the improper integral (1) is said to be *convergent*, otherwise it is said to be *divergent*.

Note: For $a_1 > a$,

$$\int_a^X f\,dx = \int_a^{a_1} f\,dx + \int_{a_1}^X f\,dx$$

which implies that the integrals $\int_a^\infty f\,dx$ and $\int_{a_1}^\infty f\,dx$ are either both convergent or both divergent. Thus when testing the integral $\int_a^\infty f\,dx$ for convergence, we can replace it by the integral $\int_{a_1}^\infty f\,dx$ for any convenient $a_1 > a$.

Ex. Examine for convergence

$$\int_0^\infty \frac{x\,dx}{1+x^2},\ \int_1^\infty \frac{dx}{\sqrt{x}},\ \int_a^\infty \sin x\,dx$$

(ii) **Convergence at $-\infty$**

$$\int_{-\infty}^b f\,dx,\ x \leq b \quad \text{...(3)}$$

is defined by the equation

$$\int_{-\infty}^b f\,dx = \lim_{X \to -\infty} \int_X^b f\,dx \quad \text{...(4)}$$

If the limit exists and is finite then the integral (3) *converges* (or *exists*). Otherwise it *diverges* (or *does not exist*).

(iii) **Convergence at both ends**

$$\int_{-\infty}^\infty f\,dx\ \ \forall\ x \quad \text{...(5)}$$

is understood to mean

$$\int_{-\infty}^{c} f\,dx + \int_{c}^{\infty} f\,dx \qquad ...(6)$$

where c is any real number.

If *both* the integrals in (6) exist in accordance with the definition given above, then the integral equation (5) converges, otherwise is divergent.

Integral, (5) can also be defined by the relation

$$\int_{-\infty}^{\infty} f\,dx = \lim_{\substack{X \to \infty \\ Y \to -\infty}} \int_{Y}^{X} f\,dx \qquad ...(7)$$

Integrals of unbounded functions with infinite limits of integration.

When the infinite range of integration includes a finite number of points of infinite discontinuity of f, consider an interval $[a, b]$ which contains all the points of discontinuity. The integral $\int_{-\infty}^{\infty} f\,dx$ is then understood to mean

$$\int_{-\infty}^{a} f\,dx + \int_{a}^{b} f\,dx + \int_{b}^{\infty} f\,dx \qquad ...(8)$$

In case *all* the integrals in (8) exist in accordance with the definitions given above, the integral $\int_{-\infty}^{\infty} f\,dx$ converges, otherwise it is divergent.

Example 10. Examine for convergence the integrals:

(*i*) $\int_{0}^{\infty} \sin x\,dx$ (*ii*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

(*iii*) $\int_{2}^{\infty} \frac{2x^2\,dx}{x^4-1}$ (*iv*) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

(*v*) $\int_{0}^{\infty} x^3 e^{-x^2}\,dx$

(*i*) By definition

$$\int_{0}^{\infty} \sin x\,dx = \lim_{X \to \infty} \int_{0}^{X} \sin x\,dx = \lim_{X \to \infty} (1 - \cos X)$$

Thus the improper integral does not exist since $\cos X$ has no limit when $X \to \infty$.

(ii) $$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\lim_{\substack{X\to\infty\\ Y\to-\infty}}\int_{Y}^{X}\frac{dx}{1+x^2}$$

$$=\lim_{\substack{X\to\infty\\ Y\to-\infty}}\left(\tan^{-1}X-\tan^{-1}Y\right)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$$

Thus, the integral exists and is equal to π.

(iii) $$\int_{2}^{\infty}\frac{2x^2}{x^4-1}dx=\lim_{X\to\infty}\int_{2}^{X}\frac{2x^2}{x^4-1}dx$$

$$=\lim_{X\to\infty}\left[\tan^{-1}X-\tan^{-1}2+\frac{1}{2}\log\frac{X-1}{X+1}+\frac{1}{2}\log 3\right]$$

$$=\frac{\pi}{2}-\tan^{-1}2+\frac{1}{2}\log 3$$

Thus, the integral converges.

(iv) $$\int_{-\infty}^{\infty}\frac{dx}{\left(1+x^2\right)^2}=\int_{-\infty}^{0}\frac{dx}{\left(1+x^2\right)^2}+\int_{0}^{\infty}\frac{dx}{\left(1+x^2\right)^2}$$

$$=2\int_{0}^{\infty}\frac{dx}{\left(1+x^2\right)^2}=\lim_{X\to\infty}2\int_{0}^{X}\frac{dx}{\left(1+x^2\right)^2}$$

$$=\lim_{X\to\infty}2\left[\tan^{-1}X+\frac{X}{1+X^2}\right]\text{ (by putting } x=\tan\theta\text{)}$$

$$=\frac{\pi}{2},\text{ so that the integral converges.}$$

(v) $$\int_{0}^{\infty}x^3e^{-x^2}\,dx=\lim_{X\to\infty}\int_{0}^{X}x^3e^{-x^2}\,dx$$

$$=\lim_{X\to\infty}\left[\frac{1}{2}-\frac{1}{2}\left(X^2+1\right)e^{-X^2}\right]$$

$$=\frac{1}{2},\text{ converges.}$$

Ex. Compute the following integrals or prove their divergence:

(i) $\displaystyle\int_{1}^{\infty}\frac{dx}{x^2(x+1)}$ (ii) $\displaystyle\int_{\sqrt{2}}^{\infty}\frac{dx}{x\sqrt{x^2-1}}$

(iii) $\int_a^{\infty} \frac{dx}{x\sqrt{1+x^2}}$ (iv) $\int_0^{\infty} x \sin x \, dx$

(v) $\int_0^{\infty} e^{-\sqrt{x}} dx$ (vi) $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$

(vii) $\int_0^{\infty} e^{-ax} \cos bx \, dx$ (viii) $\int_0^{\infty} \frac{dx}{1+x^3}$

(ix) $\int_0^{\infty} x^2 e^{-x} dx$ (x) $\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx$

ANSWERS

(i) $1-\log 2$ (ii) $\pi/4$ (iii) $\log \frac{\sqrt{a^2+1}+1}{a^2}$ (iv) Div.

(v) 2 (vi) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (vii) $\frac{a}{a^2+b^2}$ if $a>0$, div. if $a \le 0$

(viii) $\frac{2\pi}{3\sqrt{3}}$ (ix) 2 (x) $\frac{1}{2}+\frac{\pi}{4}$.

4.1 Comparison Tests for Convergence at ∞ (*Integrand retaining its sign*)

By definition

$$\int_a^{\infty} f \, dx = \lim_{X \to \infty} \int_a^X f \, dx$$

where f is assumed to be bounded and integrable in $[a, X]$ for every $X \ge a$.

As in the case of unbounded functions with finite limits of integration, there is no loss of generality in supposing that the integrand f is positive in $[a, X]$, for, if negative it can be replaced by $(-f)$.

Theorem 3. *A necessary and sufficient condition for the convergence of* $\int_a^{\infty} f \, dx$, *where f is positive in $[a, X]$ is that there exists a positive number M, independent of X, such that*

$$\int_a^X f \, dx < M, \text{ for every } X \ge a$$

The integral $\int_a^\infty f\,dx$ is said to be convergent if $\int_a^X f\,dx$ tends to a finite limit as $X \to \infty$.

Since f is positive in $[a, X]$, the positive function of X, $\int_a^X f\,dx$, is monotone increasing as X increases and will therefore tend to a finite limit if and only if it is bounded above, *i.e.*, there exists a positive number M, independent of X such that

$$\int_a^X f\,dx < M, \text{ for every } X \geq a$$

Hence, the proof.

Note: If no such number M exists, the monotonic increasing function $\int_a^X f\,dx$ is non-bounded above and therefore tends to ∞, as $X \to \infty$ and so $\int_a^\infty f\,dx$ diverges to ∞.

4.2 Comparison Test I (Comparison of two integrals)

If f and g are positive and $f(x) \leq g(x)$, for all x in $[a, X]$, then

(*i*) $\int_a^\infty f\,dx$ *converges, if* $\int_a^\infty g\,dx$ *converges, and*

(*ii*) $\int_a^\infty g\,dx$ *diverges, if* $\int_a^\infty f\,dx$ *diverges.*

Let f and g be both bounded and integrable in $[a, X]$, $X \geq a$.

Since f and g are both positive, and

$$f(x) \leq g(x), \ \forall\, x \in [a, X]$$

$$\therefore \quad \int_a^X f\,dx \leq \int_a^X g\,dx \qquad \text{...(1)}$$

(*i*) Let $\int_a^\infty g\,dx$ be convergent so that there exists a positive number M such that

$$\int_a^X g\,dx < M, \ X \geq a$$

Hence, from equation (1), $\int_a^X f\,dx < M,\ X \geq a$

Hence, $\int_a^\infty f\,dx$ is convergent.

(ii) If $\int_a^\infty f\,dx$ is divergent then the positive function $\int_a^X f\,dx$ is not bounded above and therefore in view of (1), $\int_a^X g\,dx$ is also not bounded above.

Hence, $\int_a^\infty g\,dx$ diverges.

Comparison Test II (*Limit form*)

If f and g are positive in [a, x] and $\lim_{x\to\infty} \frac{f}{g} = l$, where l is a non-zero finite number, then the two integrals $\int_a^\infty f\,dx$ and $\int_a^\infty g\,dx$ converge or diverge together. Also if $f/g \to 0$ and $\int_a^\infty g\,dx$ converges then $\int_a^\infty f\,dx$ converges, and if $f/g \to \infty$ and $\int_a^\infty g\,dx$ diverges, then $\int_a^\infty f\,dx$ diverges.

Evidently $l > 0$.

Let ε be a positive number such that $l - \varepsilon > 0$.

Since $\lim_{x\to\infty} \frac{f}{g} = l$, therefore there exists a number $k\ (> a)$, however large, such that for all $x > k$,

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$$

or

$$(l-\varepsilon)\,g(x) < f(x) < (l+\varepsilon)\,g(x) \qquad \text{...(2)}$$

Now

$$(l-\varepsilon)\,g(x) < f(x),\ \forall\, x > k > a$$

so that if $\int_k^\infty f\,dx$ converges then by Comparison test I, $\int_k^\infty g\,dx$ and therefore $\int_a^\infty g\,dx$ converges at ∞.

Again from (2),

$$f(x) < (l+\varepsilon)\,g(x),\ \forall\, x > k > a$$

so that if $\int_k^\infty f\,dx$ diverges, then by comparison test, $\int_k^\infty g\,dx$ and therefore $\int_a^\infty g\,dx$ diverges at ∞.

When $f/g \to 0$, we can find k so that

$$\frac{f(x)}{g(x)} < \varepsilon, \ \forall\, x > k$$

i.e.,

$$f(x) < \varepsilon g(x), \ \forall\, x > k$$

so, if $\int_a^\infty g\,dx$ converges, then $\int_a^\infty f\,dx$ also converges.

When $f/g \to \infty$, we can find numbers k, M such that

$$\frac{f(x)}{g(x)} > M \text{ or } f(x) > Mg(x), \ \forall\, x \geq k$$

Hence, if $\int_a^\infty g\,dx$ diverges, then $\int_a^\infty f\,dx$ also diverges.

4.3 A Useful Comparison Integral

Show that the improper integral $\int_a^\infty \frac{C}{x^n}dx$, $a > 0$, *where* C *is a positive constant, converges if and only if* $n > 1$.

We have

$$\int_a^X \frac{C}{x^n}dx = \begin{cases} C\log\frac{X}{a}, & n = 1 \\ \frac{1}{1-n}\left[\frac{1}{X^{n-1}} - \frac{1}{a^{n-1}}\right], & n \neq 1 \end{cases}$$

and therefore

$$\int_a^\infty \frac{C}{x^n}dx = \lim_{X\to\infty}\int_a^X \frac{C}{x^n}dx = \begin{cases} +\infty & n \leq 1 \\ \frac{C}{(n-1)a^{n-1}} & n > 1 \end{cases}$$

Thus, $\int_a^\infty \frac{C}{x^n}dx$ converges if and only if $n > 1$.

The integral is widely used as a comparison integral when testing the convergence of improper integrals.

11.4.4 With the help of § 11.4.2 and the comparison integral of § 11.4.3, we deduce two comparison tests of much practical utility.

I. *If f is positive in* $[a, x]$ *then the integral* $\int_a^\infty f\,dx$ *converges, if there exists a positive number n greater than 1 and a fixed positive number M such that*

$$f(x) \le M/x^n, \text{ for every } x \ge a$$

Also the integral diverges if there exists a positive number M such that

$$f(x) \ge M/x^n, \text{ for every } x \ge a.$$

II. *If* $\lim_{x\to\infty} x^n f(x)$ *exists and is non-zero finite, then the integral* $\int_a^\infty f\,dx$ *converges if and only if* $n > 1$.

Example 11. Examine the convergence of

(i) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ (ii) $\displaystyle\int_0^\infty \frac{x^2\,dx}{\sqrt{x^5+1}}$

(iii) $\displaystyle\int_0^\infty e^{-x^2}\,dx$ (iv) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx$

(v) $\displaystyle\int_1^\infty x^2 e^{-x}\,dx$ (vi) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

Sol. (i) Let $f(x) = \dfrac{1}{x\sqrt{x^2+1}}$, (behaves like x^{-2} at ∞) and $g(x) = 1/x^2$,

so that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2+1}} = 1, \text{ (non-zero finite)}$$

Hence, the two integrals $\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ behave alike.

As $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, therefore $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ also converges.

(*ii*) Let $f(x)=\frac{x^2}{\sqrt{x^5+1}},\left(\sim x^{-1/2}\right)$ and $g(x)=\frac{1}{\sqrt{x}}$.

$$\frac{f(x)}{g(x)}=\frac{x^{5/2}}{\sqrt{x^5+1}}=\frac{1}{\sqrt{1+x^{-5}}}\to 1 \text{ as } x\to\infty$$

As $\int_0^\infty g\,dx=\int_0^\infty \frac{dx}{\sqrt{x}}$ diverges, therefore by comparison test, $\int_0^\infty \frac{x^2\,dx}{\sqrt{x^5+1}}$ also diverges.

(*iii*) 0 is not a point of infinite discontinuity and so we have to examine convergence at ∞ only.

Let us consider the integral $\int_1^\infty e^{-x^2}\,dx$.

We know

$$e^{x^2}>x^2,\ \forall \text{ real } x$$

$$\therefore \qquad e^{-x^2}<\frac{1}{x^2}$$

As $\int_1^\infty \frac{1}{x^2}\,dx$ converges at ∞, the integral $\int_1^\infty e^{-x^2}\,dx$ and therefore the integral $\int_0^\infty e^{-x^2}\,dx$ converges.

Note: $\int_0^\infty e^{-x^2}\,dx$ is called the *Euler-Poisson* integral, and it will be shown later (Double integrals) that its value is $\sqrt{\pi/2}$.

(*iv*) Here

$$x^{3/2}\frac{\log x}{x^2}=\frac{\log x}{x^{1/2}}\to 0 \text{ as } x\to\infty, \text{ and } \int_1^\infty \frac{dx}{x^{3/2}} \text{ converges}$$

Hence by comparison with $\int_1^\infty \frac{dx}{x^{3/2}}$, the integral $\int_1^\infty \frac{\log x}{x^2}\,dx$ also converges.

(*v*) Now

$$x^2\cdot x^n e^{-x}=x^{n+2}/e^x\to 0 \text{ as } x\to\infty, \text{ for all } n$$

and $\int_1^\infty \frac{dx}{x^2}$ converges.

Hence by comparison test, $\int_1^\infty x^n e^{-x}\,dx$ converges.

(vi) 0 is not a point of infinite discontinuity, so if we put

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx,$$

the first integral on the right being proper, we test the second,

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \text{ for convergence at } \infty.$$

Now $\dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent, therefore the integral $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is also convergent.

Hence, the integral $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Note: Comparison test II (limit form) cannot be used here, for $\lim f(x)/g(x)$, where $g(x) = 1/x^2$ does not exist.

Example 12. Test for convergence the integrals

(i) $\displaystyle\int_0^\infty \frac{x\tan^{-1} x}{\left(1+x^4\right)^{1/3}}\,dx$ (ii) $\displaystyle\int_{e^2}^\infty \frac{dx}{x\log\log x}$

(i) Let $f(x) = \dfrac{x\tan^{-1} x}{\left(1+x^4\right)^{1/3}}$

$$= \frac{\tan^{-1} x}{x^{1/3}\left(1-x^{-4}\right)^{1/3}} \left(\sim x^{-1/3} \text{ at } \infty\right)$$

and $g(x) = \dfrac{1}{x^{1/3}}$,

so that

$$\frac{f(x)}{g(x)} = \frac{\tan^{-1} x}{\left(1+x^{-4}\right)^{1/3}} \to \frac{\pi}{2} \text{ as } x \to \infty$$

Hence, $\displaystyle\int_1^\infty f\,dx$ and $\displaystyle\int_1^\infty g\,dx$ behave alike.

Since $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}}$ diverges, therefore $\displaystyle\int_1^\infty \frac{x\tan^{-1} x\,dx}{\left(1+x^4\right)^{1/3}}$ also diverges.

(*ii*) Putting $\log x = t$ or $x = e^t$, we get

$$\int_{e^2}^{\infty} \frac{dx}{x \log \log x} = \int_{2}^{\infty} \frac{dt}{\log t}$$

which diverges by comparison with the divergent integral $\int_{k}^{\infty} \frac{dt}{t^m}$, $m \le 1$.

Example 13. *Gamma function.* The integral $\int_0^{\infty} x^{m-1} e^{-x}\, dx$ is convergent if and only if $m > 0$.

- Let $f(x) = x^{m-1} e^{-x} = \dfrac{e^{-x}}{x^{1-m}}$.

The integrand f has infinite discontinuity at 0 if $m < 1$. So we have to examine convergence at 0 and ∞ both.

Putting

$$\int_0^{\infty} x^{m-1} e^{-x}\, dx = \int_0^{1} x^{m-1} e^{-x}\, dx + \int_1^{\infty} x^{m-1} e^{-x}\, dx,$$

we test the two integrals on the right for convergence at 0 and ∞ respectively.

Convergence at 0, $m < 1$.

Let $g(x) = \dfrac{1}{x^{1-m}}$ so that

$$\frac{f(x)}{g(x)} = e^{-x} \to 1 \text{ as } x \to 0$$

Also $\int_0^1 g\, dx = \int_0^1 \frac{dx}{x^{1-m}}$, converges if and only if $1 - m < 1$, *i.e.*, $m > 0$.

Hence $\int_0^1 x^{m-1} e^{-x}\, dx$ converges if and only if $m > 0$.

Convergence at ∞ (ref. Example 11.11).

Let $g(x) = \dfrac{1}{x^2}$, so that

$$\frac{f(x)}{g(x)} = \frac{x^{m+1}}{e^x} \to 0 \text{ as } x \to \infty, \text{ for all } m$$

As $\int_1^\infty \frac{dx}{x^2}$ converges, therefore $\int_1^\infty x^{m-1}e^{-x}$ also converges for all m.

Hence $\int_0^\infty x^{m-1}e^{-x}\,dx$ is convergent if and only if $m > 0$.

This integral $\int_0^\infty x^{m-1}e^{-x}\,dx$, $m > 0$ called *Gamma function*, is denoted by $\Gamma(m)$.

Example 14. Examine for convergence

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$$

Let $f(x) = \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\Big/x$

Using L'Hospital rules, we find that $f(x) \to \frac{1}{6}$ as $x \to 0$.

Therefore 0 is not a point of infinite discontinuity of f.

To examine the convergence at ∞, we put

$$f(x) = \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\Big/x = \frac{1}{x^2} - \frac{2e^{-x}}{x\left(1 - e^{-2x}\right)}$$

so that $f(x)$ behaves as $\frac{1}{x^2}$ at ∞.

By comparison with the convergent integral $\int_1^\infty \frac{dx}{x^2}$ we can easily show that $\int_1^\infty f\,dx$ and therefore $\int_0^\infty f\,dx$ is also convergent.

Example 15. Test for convergence the integral

$$\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx$$

Making the substitution $\log\frac{1}{x} = t$ or $x = e^{-t}$, we get

$$\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx = \int_0^\infty t^q e^{-(p+1)t}\,dt$$

The integrand of the last integral has a point of infinite discontinuity 0. So, we put

$$\int_0^\infty t^q e^{-(p+1)t}\,dt = \int_0^1 t^q e^{-(p+1)t}\,dt + \int_1^\infty t^q e^{-(p+1)t}\,dt$$

and examine the integrals on the right for convergence at 0 and ∞ respectively.

Convergence at 0.

Let $f(t) = t^q e^{-(p+1)t} = \dfrac{e^{-(p+1)t}}{t^{-q}}$.

By comparison with $\int_0^1 \dfrac{dt}{t^{-q}}$, we find that $\int_0^1 t^q e^{-(p+1)t}\,dt$ converges at 0 only if $-q < 1$ or $q > -1$, for all values of p.

Convergence at ∞.

Let $f(t) = t^q e^{-(p+1)t} = \dfrac{t^q}{e^{(p+1)t}}$ and $g(t) = \dfrac{1}{t^2}$, so that

$$\frac{f(t)}{g(t)} = \frac{t^{q+2}}{e^{(p+1)t}} \to 0 \text{ as } t \to \infty, \text{ for all } q \text{ and } p+1>0$$

Also $\qquad \int_1^\infty \dfrac{dt}{t^2}$ converges.

Hence $\qquad \int_1^\infty t^q e^{-(p+1)t}\,dt$ converges for all q and only if $p+1>0$.

Hence $\int_1^\infty t^q e^{-(p+1)t}\,dt$ and therefore $\int_0^1 x^p \left(\log \dfrac{1}{x}\right)^q dx$ is convergent for $p > -1$, $q > -1$, and divergent for all other values of p and q.

4.5 Extending methods of evaluating proper integrals to the case of improper integrals. When evaluating improper integrals we can change variables, integrate by parts, etc., *i.e.*, apply all the methods of evaluating proper integrals, provided that all the integrals entering into them are convergent.

Let us consider the integral $\int_3^\infty \dfrac{dx}{x^2 + x - 2}$.

The integral is convergent as can be seen by comparison with $\int_3^\infty \dfrac{dx}{x^2}$.

Again, let us decompose the integrand into partial fraction

$$\frac{1}{x^2+x-2}=\frac{1}{3(x-1)}-\frac{1}{3(x+2)} \quad \ldots(1)$$

It is obvious that the integrals $\int_3^\infty \frac{dx}{x+2}$ and $\int_3^\infty \frac{dx}{x-1}$ are divergent.

Therefore, the equality

$$\int_3^\infty \frac{dx}{x^2+x-2}=\frac{1}{3}\int_3^\infty \frac{dx}{x-1}-\frac{1}{3}\int_3^\infty \frac{dx}{x+2}$$

is incorrect.

To use decomposition (1) for evaluating the integral, we can integrate (1) from 0 to X and then transform the R.H.S. of the resulting equality to the form

$$\int_3^X \frac{dx}{x^2+x-2}=\frac{1}{3}\int_3^X \frac{dx}{x-1}-\frac{1}{3}\int_3^X \frac{dx}{x+2}$$

$$=\frac{1}{2}\log\left[\frac{5X-1}{2X+2}\right]$$

Proceeding to limits as $X \to \infty$, we get

$$\int_3^\infty \frac{dx}{x^2+x-2}=\frac{1}{3}\log\frac{5}{2}$$

Example 16. Discuss the convergence of $\int_0^1 \log\Gamma x\,dx$ and hence evaluate it.

- $x=0$ is the only singular point of the integral.

Now

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt,\ x>0, \text{ thus } \Gamma(x+1)=x\Gamma x,\ \forall\, x>0.$$

$$\therefore \quad \log\Gamma(x+1)=\log\left(x\Gamma x\right)=\log x+\log\Gamma x$$

$$\Rightarrow \quad \int_0^1 \log\Gamma x\,dx=\int_0^1 \log\Gamma(x+1)\,dx-\int_0^1 \log x\,dx=I_1-I_2.$$

I_1 is a proper integral and $I_2 = \int_0^1 \log x\, dx$ is convergent

(since $-\int_0^1 \log x\, dx = \int_0^1 \log \frac{1}{x}\, dx < \int_0^1 \frac{\varepsilon}{x^{1/2}}\, dx \left[\because \lim_{x\to 0} x^{1/2} \log \frac{1}{x} = 0\right]$ is convergent.)

Hence, $\int_0^1 \log \Gamma x\, dx$ is convergent.

Again

$$I = \int_0^1 \log \Gamma x\, dx$$

$$I = \int_0^1 \log \Gamma(1-x)\, dx$$

$$\therefore \quad 2I = \int_0^1 \log \Gamma x + \int_0^1 \log \Gamma(1-x)\, dx = \int_0^1 \log \Gamma x\, \Gamma(1-x)\, dx$$

We can prove that (see Appendix 1.)

$$\Gamma p\, \Gamma(1-p) = \frac{\pi}{\sin p\pi}, \text{ for } 0 < p < 1$$

$$\therefore \quad 2I = \int_0^1 \log\left(\frac{\pi}{\sin x\pi}\right) dx, \text{ put } x\pi = z$$

$$= \frac{1}{\pi}\int_0^\pi \log \frac{\pi}{\sin z}\, dz = \frac{1}{\pi}\int_0^\pi \log \pi - \frac{1}{\pi}\int_0^\pi \log \sin z\, dz$$

$$= \log \pi - \frac{2}{\pi}\int_0^{\pi/2} \log \sin z\, dz$$

$$= \log \pi - \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right) = \log 2\pi$$

$$\therefore \quad I = \frac{1}{2}\log 2\pi.$$

EXERCISE

1. Test for convergence of the integrals:

(i) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ (ii) $\displaystyle\int_1^\infty \frac{x^3+1}{x^4}\,dx$

(iii) $\displaystyle\int_0^\infty \sqrt{xe^{-x}}\,dx$ (iv) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$

(v) $\displaystyle\int_{-\infty}^{\infty} e^{-(x-a/x)^2}\,dx$.

2. Examine the convergence of

(i) $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx,\ m,n>0$ (ii) $\displaystyle\int_1^\infty t^m e^{-nt}\,dt$

(iii) $\displaystyle\int_0^\infty x^{2n+1}e^{-x^2}\,dx$, n is a positive integer (iv) $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$.

3. Show that the integral $\displaystyle\int_0^\infty \left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)\frac{e^{-kx}}{x}\,dx$ converges.

4. Prove that $\displaystyle\int_0^\infty \frac{x\log x}{\left(1+x^2\right)^2}\,dx$ converges to 0.

[*Hint*: For evaluating, express as the integral $\displaystyle\int_0^\infty$ as $\displaystyle\int_0^1+\int_1^\infty$ and put $x=1/t$ in the second term.]

5. Show that the improper integral $\displaystyle\int_0^\infty \log\left(1+2\operatorname{sech} x\right)dx$ is convergent.

6. Show that $\displaystyle\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}$ is convergent.

7. Putting $\log\log x = t$, in the integral $\displaystyle\int_e^\infty \frac{dx}{x^p(\log x)^q(\log\log x)^r}$, show that the integral converges for $p>1$ (any q) only if $r<1$. If $p=1$ it converges only if $r<1$ and $q>1$. But if $p<1$, the integral diverges for any r and q.

ANSWERS

1. (i), (iii), (iv), (v) converge, (ii) diverges.

2. All converge for (i) $n-m>\frac{1}{2}$ (ii) $n>0$ (iii) all n (iv) $0<p<1$.

4.6 General Test for Convergence at ∞ (Integrand may change sign)

Cauchy's Test. *The integral $\int_a^{\infty} f\,dx$ converges at ∞ if and only if for every $\varepsilon > 0$ there corresponds a positive number X_0 such that*

$$\left|\int_{X_1}^{X_2} f dx\right| < \varepsilon, \text{ for all } X_1, X_2 > X_0$$

[It implies that $\int_{X_1}^{X_2} f\,dx \to 0$ as $X_1, X_2 \to \infty$.]

The improper integral $\int_a^{\infty} f\,dx$ exists if $\lim_{X\to\infty} \int_a^{X} f\,dx$ exists finitely.

Let $F(X) = \int_a^{X} f\,dx$, a function of X.

According to Cauchy's criterion for finite limits (§ 1.3 Ch. 5) $F(X)$ tends to a finite limit as $X \to \infty$ if and only if for every $\varepsilon > 0$ there corresponds X_0 such that for all $X_1, X_2 > X_0$,

$$|F(X_1) - F(X_2)| < \varepsilon$$

i.e.,

$$\left|\int_a^{X_1} f\,dx - \int_a^{X_2} f\,dx\right| < \varepsilon$$

or

$$\left|\int_{X_1}^{X_2} f\,dx\right| < \varepsilon.$$

Example 17. Show that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is convergent.

- 0 is not a point of infinite discontinuity, for $\frac{\sin x}{x} \to 1$ as $x \to 0$. So, let us put

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \int_0^{1} \frac{\sin x}{x}\,dx + \int_1^{\infty} \frac{\sin x}{x}\,dx$$

and test $\int_0^{\infty} \frac{\sin x}{x} dx$ for convergence at ∞.

For any $\varepsilon > 0$, let X_1, X_2 be two numbers, both greater than $2/\varepsilon$.

Now

$$\int_{X_1}^{X_2} \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_{X_1}^{X_2} - \int_{X_1}^{X_2} \frac{\cos x}{x^2} dx$$

so that

$$\left|\int_{X_1}^{X_2} \frac{\sin x}{x} dx\right| \le \left|\frac{\cos X_1}{X_1} - \frac{\cos X_2}{X_2}\right| + \left|\int_{X_1}^{X_2} \frac{\cos x}{x^2} dx\right|$$

$$\le \frac{1}{X_1} + \frac{1}{X_2} + \int_{X_1}^{X_2} \frac{dx}{x^2}$$

$$< 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

Thus by Cauchy's Test $\int_1^{\infty} \frac{\sin x}{x} dx$ and consequently $\int_0^{\infty} \frac{\sin x}{x} dx$ is convergent.

4.7 Absolute Convergence

Definition. *The improper integral $\int_a^{\infty} f\, dx$ is said to be absolutely convergent if $\int_a^{\infty} |f|\, dx$ is convergent.*

With the help of Cauchy's Test, we can easily prove the following theorem.

Theorem 4. *$\int_a^{\infty} f\, dx$ exists, if $\int_a^{\infty} |f|\, dx$ exists.*

The converse, however, is not true. Integrals which are convergent but not absolutely are called *conditionally convergent* integrals.

Example 18. Show that $\int_1^{\infty} \frac{\sin x}{x^p} dx$ converges absolutely if $p > 1$.

Now

$$\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} \le \frac{1}{x^p} \quad \forall\, x \ge 1$$

Also $\int_1^\infty \frac{dx}{x^p}$ converges only if $p > 1$.

Therefore $\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ converges if $p > 1$.

Hence $\int_1^\infty \frac{\sin x}{x^p} dx$ converges absolutely if $p > 1$.

For $p = 1$ it is not absolutely convergent.

5. INTEGRAND AS A PRODUCT OF FUNCTIONS (Convergence at ∞)

5.1 A Test for Absolute Convergence

A function ϕ is bounded in $[a, \infty[$ and integrable in $[a, X]$ where X is a number $\geq a$. If $\int_a^\infty f\,dx$ is absolutely convergent at ∞, then $\int_a^\infty f\phi\,dx$ is also absolutely convergent at ∞.

Since ϕ is bounded in $[a, \infty[$, there exists a positive K such that

$$\phi(x) \leq K, \ \forall\, x \geq a \quad \text{...(1)}$$

Again, since $|f|$ is positive and $\int_a^\infty |f|\,dx$ is convergent, a number M exists such that

$$\int_a^X |f|\,dx \leq M, \text{ for all } X \geq a \quad \text{...(2)}$$

Using equation (1), we have

$$|f\phi| \leq K|f|, \ \forall\, x \geq a$$

$$\therefore \quad \int_a^X |f\phi|\,dx \leq K \int_a^X |f|\,dx$$

$$\leq KM, \ \text{for all } X \geq a,$$

which implies that the positive function $\int_a^X |f\phi|\,dx$ is bounded above by KM, for $X \geq a$.

Hence $\int_a^\infty |f\phi|\,dx$ converges.

Example 19. Discuss the convergence of the integral

$$\int_1^\infty f(x)\,dx, \text{ where } f(x)=\begin{cases}\dfrac{1}{x^2}, & x \text{ is a rational number}\\[2mm] -\dfrac{1}{x^2}, & x \text{ is an irrational number}\end{cases}$$

Since $\int_1^\infty |f(x)|\,dx = \int_1^\infty \frac{1}{x^2}\,dx$

is convergent, and every absolutely convergent integral is convergent. Therefore the given integral is convergent.

4.2 Tests for Convergence

Abel's Test. *If ϕ is bounded and monotonic in $[a, \infty[$, and $\int_a^\infty f\,dx$ is convergent at ∞, then $\int_a^\infty f\phi\,dx$ is convergent at ∞.*

Or

An infinite integral which converges (not necessarily absolutely) will remain convergent after the insertion of a factor which is bounded and monotonic.

Since ϕ is monotonic in $[a, \infty[$, it is integrable in $[a, X]$, for all $X \geq a$. Also, since f is integrable in $[a, X]$, we have by Second Mean Value Theorem

$$\int_{X_1}^{X_2} f\phi\,dx = \phi(X_1)\int_{X_1}^{\xi} f\,dx + \phi(X_2)\int_{\xi}^{X_2} f\,dx \qquad ...(1)$$

for $a < X_1 \leq \xi \leq X_2$.

Let $\varepsilon > 0$ be arbitrary.

Since ϕ is bounded in $[a, \infty[$, a positive number K exists such that

$$|\phi(x)| \leq K, \ \forall\, x \geq a$$

In particular,

$$|\phi(X_1)| \leq K, |\phi(X_2)| \leq K \qquad ...(2)$$

Again, since $\int_a^\infty f\,dx$ is convergent (by § 4.6), a number X_0 exists such that

$$\left|\int_{X_1}^{X_2} f\,dx\right| < \frac{\varepsilon}{2K}, \text{ for all } X_1, X_2 \geq X_0 \qquad ...(3)$$

Let the numbers X_1, X_2 in (1) be $\geq X_0$ so that the number ξ which lies between X_1 and X_2 is also $\geq X_0$.

Hence from (3),

$$\left|\int_{X_1}^{\xi} f\,dx\right| < \frac{\varepsilon}{2K}, \left|\int_{\xi}^{X_2} f\,dx\right| < \frac{\varepsilon}{2K} \quad ...(4)$$

Thus, from equations (1), (2) and (4), we deduce that a positive number X_0 exists such that for all $X_1, X_2 \geq X_0$,

$$\left|\int_{X_1}^{X_2} f\phi\,dx\right| \leq |\phi(X_1)| \cdot \left|\int_{X_1}^{\xi} f\,dx\right| + |\phi(X_2)| \cdot \left|\int_{\xi}^{X_2} f\,dx\right|$$

$$< K\frac{\varepsilon}{2K} + K\frac{\varepsilon}{2K} = \varepsilon$$

Hence, by Cauchy's test, $\int_a^{\infty} f\phi\,dx$ is convergent at ∞.

Dirichlet's Test. *If ϕ is bounded and monotonic in $[a, \infty[$ and tends to 0 as $x \to \infty$, and $\int_a^X f\,dx$ is bounded for $X \geq a$, then $\int_a^{\infty} f\phi\,dx$ is convergent at ∞.*

Or

An infinite integral which oscillates finitely becomes convergent after the insertion of a bounded monotonic factor which tends to zero as a limit.

Since ϕ is monotonic, it is integrable in $[a, X]$ for all $X \geq a$. Also, since f is integrable in $[a, X]$,

$\therefore$ by Second Mean Value Theorem,

$$\int_{X_1}^{X_2} f\phi\,dx = \phi(X_1)\int_{X_1}^{\xi} f\,dx + \phi(X_2)\int_{\xi}^{X_2} f\,dx \quad ...(1)$$

for $a < X_1 \leq \xi \leq X_2$.

Again, since $\int_a^X f\,dx$ is bounded when $X \geq a$, there exists a number $K > 0$ such that

$$\left|\int_a^X f\,dx\right| \leq K, \ \forall\, X \geq a$$

$$\therefore \quad \left|\int_{X_1}^{\xi} f\,dx\right| = \left|\int_a^{\xi} f\,dx - \int_a^{X_1} f\,dx\right|$$

$$\leq \left|\int_{X_1}^{\xi} f\,dx\right| + \left|\int_{a}^{X_1} f\,dx\right|$$

$$\leq K + K = 2K, \text{ for } X_1, \xi \geq a \quad ...(2)$$

Similarly

$$\left|\int_{\xi}^{X_2} f\,dx\right| \leq 2K, \text{ for } X_2, \xi \geq a \quad ...(3)$$

Let $\varepsilon > 0$ be arbitrary.

Since $\phi \to 0$ as $x \to \infty$, a positive number X_0 exists such that

$$|\phi(X_1)| < \frac{\varepsilon}{4K}, |\phi(X_2)| < \frac{\varepsilon}{4K} \text{ where } X_2 \geq X_1 \geq X_0 \quad ...(4)$$

Let the numbers X_1, X_2 in (1) be $\geq X_0$, so that from (1), (2), (3) and (4), we get

$$\left|\int_{X_1}^{X_2} f\phi\,dx\right| \leq |\phi(X_1)|\left|\int_{X_1}^{\xi} f\,dx\right| + |\phi(X_2)|\left|\int_{\xi}^{X_2} f\,dx\right|$$

$$< \frac{\varepsilon}{4K}2K + \frac{\varepsilon}{4K}2K = \varepsilon$$

Hence, by Cauchy's test, $\int_a^{\infty} f\phi\,dx$ is convergent at ∞.

ILLUSTRATIONS

1. The integral $\int_1^{\infty} \frac{\sin x}{x^p}dx$ is convergent for $p > 0$.

Let $f(x) = \sin x$, and $\phi(x) = \frac{1}{x^p}$.

Here

$$\left|\int_1^X f\,dx\right| = \left|\int_1^X \sin x\,dx\right| = |\cos 1 - \cos X|$$

$$\leq |\cos 1| + |\cos X| \leq 2, \text{ for } 1 \leq X < \infty.$$

Also $\phi(x) = \frac{1}{x^p}$ is a monotone decreasing function tending to 0 as $x \to \infty$ for $p > 0$.

Therefore, by Dirichlet's test, $\int_1^\infty f\phi\,dx = \int_1^\infty \frac{\sin x}{x^p}\,dx$ converges for $p > 0$.

Earlier, in Example 18, it was shown that $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges absolutely for $p > 1$.

Thus we conclude that $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges absolutely for $p > 1$, but only conditionally for $0 < p \le 1$.

2. The integral $\int_1^\infty \frac{\sin x \log x}{x}\,dx$ is convergent.

Let $f(x) = \sin x$, $\phi(x) = \frac{\log x}{x}$.

Now $\left|\int_e^\infty \sin x\,dx\right|$ is bounded above by 2, and ϕ is monotone decreasing to 0 as $x \to \infty$.

Hence, the given integral converges by Dirichlet's test.

EXERCISE

1. Establish the convergence of the integrals

(i) $\int_0^\infty e^{-px} \frac{\sin x}{x}\,dx,\ p \ge 0$

(ii) $\int_1^\infty \left(1 - e^{-x}\right)\frac{\cos x}{x}\,dx$

2. Show the convergence of

(i) $\int_0^\infty \frac{\sin kx}{x}\,dx$

(ii) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$

(iii) $\int_0^\infty e^{-a^2x^2} \cos bx\,dx$

(iv) $\int_0^\infty e^{-a^2x^2} \sin 2bx \frac{dx}{x}$

(v) $\int_0^\infty \frac{\cos x}{\sqrt{x^2 + x}}\,dx$

(vi) $\int_0^\infty \cos x^2\,dx$

Example 20. ***Second method.*** Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent, but not absolutely.

Here, 0 is not a point of infinite discontinuity because $\frac{\sin x}{x} \to 1$ as $x \to 0$. So, let us put

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$$

Now $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral.

To examine the convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$ at ∞, we see that

$$\left|\int_1^X \sin x\,dx\right| = \left|\cos 1 - \cos X\right| \le \left|\cos 1\right| + \left|\cos X\right| < 2$$

so that $\left|\int_1^X \sin x\,dx\right|$ is bounded above for all $X \ge 1$.

Also, $1/x$ is a monotone decreasing function tending to 0 as $x \to \infty$.

Hence by Dirichlet's test, $\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent.

Hence $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

To show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is not absolutely convergent, we proceed as follows:

Consider for $n \ge 1$, the proper integral

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{\left|\sin x\right|}{x}\,dx$$

Now $\forall\, x \in \left[(r-1)\pi, r\pi\right]$

$$\int_{(r-1)\pi}^{r\pi} \frac{\left|\sin x\right|}{x}\,dx \ge \int_{(r-1)\pi}^{r\pi} \frac{\left|\sin x\right|}{r\pi}\,dx$$

Putting $x = (r-1)\pi + y$

$$\int_{(r-1)\pi}^{r\pi} \frac{\left|\sin x\right|\,dx}{r\pi} = \int_0^\pi \frac{\left|\sin\left((r-1)\pi + y\right)\right|\,dy}{r\pi}$$

$$= \frac{1}{r\pi}\int_0^{\pi} \sin y\, dy = \frac{2}{r\pi}$$

Hence,

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|\, dx}{x} \geq \sum_{r=1}^{n} \frac{2}{r\pi}$$

But $\sum_{r=1}^{n} \frac{2}{r\pi}$ is a divergent series.

$$\therefore \quad \lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x} dx \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{2}{r\pi}$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x} dx \text{ is infinite.}$$

Now, let t be a real number. There exist positive integer n such that

$$n\pi \leq t < (n+1)\pi$$

We have

$$\int_0^{t} \frac{|\sin x|}{x} dx \geq \int_0^{n\pi} \frac{|\sin x|}{x} dx$$

Let $t \to \infty$, so that $n \to \infty$, thus we see that

$$\int_0^{t} \frac{|\sin x|}{x} dx \to \infty$$

$$\Rightarrow \quad \int_0^{\infty} \frac{|\sin x|}{x} dx \text{ does not converge.}$$

Ex. Show that $\int_0^{\infty} \frac{\sin x}{x^p} dx,\ 0 < p \leq 1$ is convergent, but not absolutely.

Example 21. Test the convergence of $\int_0^{\infty} \frac{\sin x^m}{x^n} dx$.

- For $m = 0$, the integral reduces to $\int_0^{\infty} \frac{dx}{x^n}$, which converges at 0 for $n < 1$, and converges at ∞ for $n > 1$.

Thus the integral cannot converge at 0 and ∞ both. Hence it diverges for $m = 0$.

Let $m \neq 0$.

Substituting $x^m = t$, we get

$$\int_0^\infty \frac{\sin x^m}{x^n}\,dx = \frac{1}{m}\int_0^\infty \frac{\sin t}{t^{\{(n-1)/m\}+1}}\,dt$$

Let $f(t) = \dfrac{\sin t}{t^{\{(n-1)/m\}+1}} = \dfrac{\sin t}{t}\cdot\dfrac{1}{t^{(n-1)/m}}$.

Taking a number, say 1, greater than 0, we write

$$\int_0^\infty f\,dt = \int_0^1 f\,dt + \int_1^\infty f\,dt$$

and examine the two integrals on the right for convergence at 0 and ∞, respectively.

Convergence at 0

$\int_0^1 f\,dt$ is a proper integral for $\dfrac{n-1}{m} \le 0$ but has infinite discontinuity at 0 for $\dfrac{n-1}{m} > 0$.

For $\dfrac{n-1}{m} > 0$, let $g(t) = \dfrac{1}{t^\alpha}$.

Now

$$\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{t^{\{(n-1)/m\}-\alpha}}$$

$$= \text{a finite quantity, if } \frac{n-1}{m} - \alpha \le 0$$

and $\int_0^1 g\,dt = \int_0^1 \dfrac{dt}{t^\alpha}$ converges if and only if $\alpha < 1$.

Hence $\int_0^1 f\,dt$ converges if $\dfrac{n-1}{m} \le \alpha < 1$.

Thus $\int_0^1 \dfrac{\sin t}{t^{\{(n-1)/m\}+1}}\,dt$ converges at 0 if $\dfrac{n-1}{m} < 1$, which includes the case $\dfrac{n-1}{m} \le 0$ when the integral is proper.

Convergence at ∞

$$\left|\int_0^X \sin t\,dt\right| \le 2, \text{ bounded}$$

and $1/t^{\{(n-1)/m\}+1}$ is a monotone decreasing function, tending to 0 as $t \to \infty$.

when $\frac{n-1}{m}+1>0$ or $\frac{n-1}{m}>-1$.

So by Dirichlet's test, $\int_1^\infty f\,dt$ converges at ∞ if $-1<\frac{n-1}{m}$.

Hence, the given integral converges when $-1<\frac{n-1}{m}<1$ [or equivalently $-1<\frac{n-1}{-m}<1$], i.e., for all m (positive or negative), and $1-m<n<1+m$.

Ex. Discuss the absolute convergence of $\int_0^\infty \frac{\sin x^m}{x^n}\,dx$.

Example 22. Using $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{1}{2}\pi$, show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\frac{1}{2}\pi$$

■ The integral is convergent (Example 11).
To compute it, let us integrate by parts.

$$\therefore \qquad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\left[-\frac{\sin^2 x}{x}\right]_0^\infty+\int_0^\infty \frac{\sin 2x}{x}\,dx$$

$$=\int_0^\infty \frac{\sin t}{t}\,dt=\frac{1}{2}\pi$$

Example 23. Show that $\int_2^\infty \frac{\cos x}{\log x}\,dx$ is conditionally convergent.

■ Let $\phi(x)=\frac{1}{\log x}$, $f(x)=\cos x$

$$\left|\int_2^X \cos x\,dx\right|=|\sin X-\sin 2|\le|\sin X|+|\sin 2|\le 2$$

So that $\int_2^X \cos x\,dx$ is bounded for all $X\ge 2$

Also $\phi(x) = \dfrac{1}{\log x}$ is a monotonic decreasing function tending to 0 as $x \to \infty$.

Hence by Dirichlet's test $\displaystyle\int_2^\infty \frac{\cos x}{\log x}\,dx$ is convergent

For absolute convergence consider

$$I = \int_2^\infty \left|\frac{\cos x}{\log x}\right| dx = \int_2^{3\pi/2} \frac{|\cos x|}{\log x} dx + \int_{3\pi/2}^{5\pi/2} \frac{|\cos x|}{\log x} dx + \ldots + \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \frac{|\cos x|}{\log x} dx + \ldots$$

$$\therefore \quad I = \int_{\pi/2}^{2} \frac{|\cos x|}{\log x} dx + \int_2^{3\pi/2} \frac{|\cos x|}{\log x} dx + \ldots + \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \frac{|\cos x|}{\log x} dx + \ldots - \int_{\pi/2}^{2} \frac{|\cos x|}{\log x} dx$$

$$= \sum_{r=1}^{\infty} \int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \frac{|\cos x|}{\log x} dx - \int_{\pi/2}^{2} \frac{|\cos x|}{\log x}$$

Now

$$\int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \frac{|\cos x|}{\log x} dx \geq \frac{1}{\log(2r+1)\pi/2} \left| \int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \cos x\, dx \right|$$

$$= \frac{1}{\log(2r+1)\pi/2} \left| \sin(2r+1)\pi/2 - \sin(2r-1)\pi/2 \right|$$

$$= \frac{\left|2(-1)^r\right|}{\log(2r+1)\pi/2}$$

$$= \frac{2}{\log(2r+1)\pi/2}$$

$$\therefore \quad I \geq \sum_{r=1}^{\infty} \frac{2}{\log(2r+1)\pi/2} - \int_{\pi/2}^{2} \frac{|\cos x|}{\log x} dx$$

But $\displaystyle\sum_{x=2}^{\infty} \frac{1}{\log x}$ is divergent and $\displaystyle\int_{\pi/2}^{2} \frac{|\cos x|}{\log x} dx$ is a proper integral.

Hence

$$I = \int_2^\infty \frac{|\cos x|}{\log x} dx \text{ is divergent}$$

And so

$$\int_2^{\infty} \frac{\cos x}{\log x} dx \text{ is conditionally convergent.}$$

Example 24. The function f is defined on $[0, \infty[$ by

$$f(x) = (-1)^{n-1}, n-1 \le x < n, n \in \mathbf{N}$$

Show that the integral $\int_0^{\infty} f(x) dx$ does not converge.

■ Now

$$\int_0^{2n} f(x) dx = \int_0^1 (-1)^0 dx + \int_1^2 (-1) dx + \int_2^3 (-1)^2 dx + \ldots + \int_{2n-1}^{2n} (-1)^{2n-1} dx$$
$$= 1 - 1 + 1 - 1 + 1 - \ldots + 1 - 1 = 0$$

and

$$\int_0^{2n+1} f(x) dx = \int_0^1 dx + \int_1^2 (-1) dx + \ldots \int_{2n}^{2n+1} (-1)^{2n} dx$$
$$= 1 - 1 + 1 \ldots - 1 + 1 = 1$$

$$\lim_{n \to \infty} \int_0^{2n} f(x) dx = 0 \text{ and } \lim_{n \to \infty} \int_0^{2n+1} f(x) dx = 1$$

Hence, the integral does not exist and therefore it is not convergent.

Example 25. The function f is defined on $]0, 1]$ by

$$f(x) = (-1)^{n+1} n(n+1), \frac{1}{n+1} \le x < \frac{1}{n}, n \in \mathbf{N}$$

Show that $\int_0^1 f(x) dx$ does not converge.

■ $$\int_{1/2n+1}^{1} f(x) dx = \int_{1/2n+1}^{a/2n} f\, dx + \int_{1/2n}^{1/2n-1} f\, dx + \ldots + \int_{1/3}^{1/2} f\, dx + \int_{1/2}^{1} f\, dx$$

$$= \int_{1/2n+1}^{1/2n} (-1)^{2n+1} 2n(2n+1) dx + \int_{1/2n}^{1/2n-1} (-1)^{2n} (2n-1) 2n\, dx + \ldots + \int_{1/3}^{1/2} (-1)^3 2.3 dx + \int_{1/2}^{1} (-1)^2 1.2 dx$$

$$= -1 + 1 - 1 + 1 - \ldots + 1 = 0$$

And

$$\int_{1/2n}^{1} f\,dx = \int_{1/2n}^{1/2n-1} (-1)^{2n}(2n-1)2n\,dx + \int_{1/2n-1}^{1/2n-2} (-1)^{2n-1}(2n-2)(2n-1)dx + \dots + \int_{1/2}^{1} (-1)^2 1\cdot 2dx$$

$$= 1 - 1 + 1 - 1 + \dots + 1 = 1$$

$\therefore$

$$\lim_{n\to\infty} \int_{1/2n}^{1} f(x)dx = 1$$

and

$$\lim_{n\to\infty} \int_{1/2n+1}^{1} f(x)\,dx = 0$$

Hence, the integral $\int_0^1 f(x)dx$ does not converge.

Example 26. Test the convergence of

(i) $\displaystyle\int_0^\infty \frac{xdx}{1+x^4\cos^2 x}$

(ii) $\displaystyle\int_0^\infty \frac{dx}{1+x^4\cos^2 x}$

(i) The integrand is positive for positive values of x but the tests obtained for the convergence of positive integrands so far, are not applicable. In order to show the integral convergent we proceed as follows:

Consider $\displaystyle\int_0^{n\pi} \frac{xdx}{1+x^4\cos^2 x}$ and write

$\therefore$

$$\int_0^{n\pi} \frac{xdx}{1+x^4\cos^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4\cos^2 x}$$

Now $\forall\, x \in [(r-1)\pi, r\pi]$, we have

$$\frac{x}{1+x^4\cos^2 x} \geq \frac{(r-1)\pi}{1+r^4\pi^4\cos^2 x}$$

$\therefore$

$$\int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4\cos^2 x} \geq \int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi\,dx}{1+x^4\pi^4\cos^2 x}$$

Putting $x = (r-1)\pi + y$, we see that

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi\, dx}{1+r^4\pi^4\cos^2 x} = \int_0^{\pi} \frac{(r-1)\pi\, dy}{1+r^4\pi^4\cos^2\{(r-1)\pi + y\}}$$

$$\int_0^{\pi} \frac{(r-1)\pi\, dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{\sec^2 y\, dy}{1+\tan^2 y + r^4\pi^4}$$

$$= \frac{2(r-1)\pi}{\sqrt{1+r^4\pi^4}} \tan^{-1}\left(\frac{\tan y}{\sqrt{1+r^4\pi^4}}\right)\Bigg|_0^{\pi/2} = \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

$$\therefore \qquad \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^4\cos^2 x} \geq \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

Hence,

$$\lim_{n\to\infty} \int_0^{n\pi} \frac{x\,dx}{1+x^4\cos^2 x} \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

But $\displaystyle\sum_{r=1}^{\infty} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$ is a divergent series $\left(\sim \displaystyle\sum_{r=1}^{\infty}\frac{1}{r}\right)$

$\therefore$ $\displaystyle\int_0^{\infty} \frac{x\,dx}{1+x^4\cos^2 x}$ is divergent.

(ii) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$

Consider

$$\int_0^{n\pi} \frac{dx}{1+x^4\cos^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\cos^2 x}$$

Now $\forall x \in [(r-1)\pi, r\pi]$

$$\frac{1}{1+x^4\cos^2 x} \le \frac{1}{1+(r-1)^4\pi^4\cos^2 x}$$

$$\therefore \quad \int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^2\cos^2 x} \le \int_0^{\pi} \frac{dy}{1+(r-1)^4\pi^4\cos^2 y}$$

where $x = (r-1)\pi + y$

$$= 2\int_0^{\pi/2} \frac{\sec^2 y\, dy}{1+\tan^2 y + (r-1)^4\pi^4}$$

$$= \frac{2}{\sqrt{1+(r-1)^4\pi^4}} \tan^{-1} \frac{\tan y}{\sqrt{1+(r-1)^4\pi^4}} \Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$

Hence

$$\lim_{n\to\infty} \int_0^{n\pi} \frac{dx}{1+x^4\cos^2 x} \le \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$

But $\sum_{r=1}^{\infty} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$ is a convergent series $\left(\sim \sum_{r=1}^{\infty} \frac{1}{r^2}\right)$

$\therefore \int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$ is convergent.

EXERCISE

1. Discuss the convergence or divergence of the following integrals:

(i) $\int_0^{\infty} \frac{\cos\alpha x \cos\beta x}{x} dx$ (ii) $\int_0^{\infty} \frac{x^m \cos ax}{1+x^n} dx$

(iii) $\int_0^{\infty} \frac{x^p \sin^2 x}{1+x^2} dx$ (iv) $\int_1^{\infty} \sin x^p \, dx$

(v) $\int_0^{\infty} \frac{\sin(x+x^2)}{x^n} dx$ (vi) $\int_2^{\infty} \frac{\sin x}{\log x} dx$

2. Using Poisson's integral $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ and Dirichlet's integral $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, show that

(i) $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$

(ii) $\int_0^\infty x^2 e^{-x^2}\,dx = \sqrt{\pi/4}$ [Integrate by parts]

(iii) $\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$ [Use: $4\sin^3 x = 3\sin x - \sin 3x$]

(iv) $\int_0^\infty \frac{\sin^4 x}{x^2}\,dx = \frac{\pi}{4}$.

3. Show that $\int_0^\infty \frac{x\,dx}{1+x^6\sin^2 x}$ converges, but $\int_0^\infty \frac{x\,dx}{1+x^4\sin^2 x}$ does not.

4. Show that $\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$ converges, but $\int_0^\infty \frac{dx}{1+x^2\sin^2 x}$ does not.

ANSWERS

1. (i) Div.
   (ii) Conv. when $n > 0, -1 < m < n$, or $n < 0, 0 > m > n-1$.
   (iii) Conv. for $-3 < p < 1$
   (iv) Conv. for $p > 1$
   (v) Conv. for $-1 < n < 2$
   (vi) Conv. but not absolutely.