2015

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(b) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

Suppose $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8}$. What is $\sum_{n=1}^{\infty} a_n$?

Justify your answer. (Majority of marks is for the correct justification). 8

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Q1a
sol. $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{2n} + \sum_{n=0}^{\infty} a_{2n+1}$$

Given $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = -\frac{3}{8}$

$$\therefore \boxed{\sum_{n=1}^{\infty} a_n = \frac{9}{8} - \frac{3}{8} = \frac{3}{4}}$$

Justification — As $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series of real numbers, $\therefore$ Every rearrangement of series will converge absolutely to the same sum.

(b) Let $X = (a, b]$. Construct a continuous function $f : X \to \mathbb{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a + \varepsilon, b]$, $a + \varepsilon < b$? Why ? *14*

Q2 Given $X = (a, b]$.

Let $f: X \to R$ be a continuous function defined by $f(x) = \frac{1}{x-a}$ $\forall x \in (a, b]$

Clearly, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{x-a} = \infty$

$\therefore$ $f(x)$ is unbounded

As $\lim_{x \to a} f(x)$ doesn't exist

$\therefore f(x)$ cannot be uniformly ~~convergent~~ continuous in X.

In the interval $[a+\varepsilon, b]$, $a+\varepsilon < b$

$$\lim_{x \to a+\varepsilon} f(x) = \lim_{x \to a+\varepsilon} \frac{1}{x-a} = \frac{1}{\varepsilon} \text{ (finite) } \cancel{\text{and}}$$

$$\lim_{x \to b} f(x) = \lim_{x \to b} \frac{1}{x-a} = \frac{1}{b-a} \text{ (finite)}$$

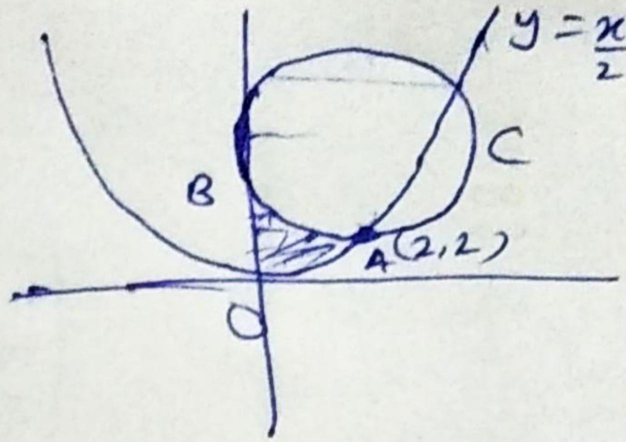
As limits exist & is finite for both the end-points of the given interval and $f(x)$ is continuous at every point in the interval.

$\therefore f(x)$ is uniformly continuous in the interval $[a+\varepsilon, b]$.

(a) Compute the double integral which will give the area of the region between the y-axis, the circle $(x-2)^2 + (y-4)^2 = z^2$ and the parabola $2y = x^2$. Compute the integral and find the area. 15

Q4

Sol. The area bounded by y-axis ($x=0$), parabola $y=\frac{x^2}{2}$ and circle $(x-2)^2+(y-4)^2=4$. is denoted by OAB



Area of OAB (A) $= \iint dx\,dy$

$$= \int_0^2 \int_{x^2/2}^{4-\sqrt{4-(x-2)^2}} dx\,dy$$

$$= \int_0^2 |y|_{x^2/2}^{4-\sqrt{4-(x-2)^2}} dx = \int_0^2 \left(4-\sqrt{4-(x-2)^2}-\frac{x^2}{2}\right) dx$$

$$= \left| 4x - \left(\frac{x-2}{2}\sqrt{4-(x-2)^2}+\frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right) - \frac{x^3}{6} \right|_0^2$$

$$= \left[8-\left(0+0-0-2\sin^{-1}(-1)\right)-\frac{8}{6}\right]$$

$$= \left(8-\frac{4}{3}\right)-\pi \quad = \boxed{\frac{20}{3}-\pi} \text{ units}$$

(b) Let $f_n(x) = \frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f ? Show that for $x \neq 0$, $f_n'(x) \to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$. 13

Q3 Given $f_n(x) = \dfrac{x}{1+nx^2}$
sol.

At $x=0$

$f_n(x) = 0 \quad \therefore \lim_{n\to\infty} f_n(x) = 0$

At $x \neq 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = \lim_{n\to\infty} \frac{1}{n}\left[\frac{x}{1/n + x^2}\right] = 0$$

$\therefore f_n(x)$ is pointwise convergent to $f(x) = 0 \ \forall x$

$$f_n'(x) = \frac{1.(1+nx^2) - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

For maximum $f_n'(x) = 0$

$\Rightarrow \dfrac{1-nx^2}{(1+nx^2)^2} = 0$

$\Rightarrow 1-nx^2 = 0 \qquad (\because (1+nx^2)^2 > 0)$

$\Rightarrow x = \pm \dfrac{1}{\sqrt{n}}$

$$f_n''(x) = \frac{(-2nx)(1+nx^2)^2 - (1-nx^2)(2.(1+nx^2).2nx)}{(1+nx^2)^4}$$

$$= \frac{(2nx(1+nx^2))(-1-nx^2-2+2nx^2)}{(1+nx^2)^4}$$

$$= \frac{(2nx)(nx^2-3)}{(1+nx^2)^3}$$

At $x = 1/\sqrt{n}$

$$= \frac{2\sqrt{n}\,.\,(1-3)}{8}$$

$$= -\frac{\sqrt{n}}{2} \Rightarrow f_n'' < 0$$

$\therefore f_n(x)$ is maximum at $x = \frac{1}{\sqrt{n}}$

At $x = -1/\sqrt{n}$

$$= \frac{-2\sqrt{n}\,(1-3)}{8}$$

$$= \frac{\sqrt{n}}{2} \Rightarrow f_n'' > 0$$

$f_n(x)$ is minimum at $x = \frac{-1}{\sqrt{n}}$

For uniform convergence, $\lim_{n\to\infty} M_n = 0$ where

$$M_n = \text{Sup}\,|f_n(x) - f(x)|$$

$$M_n = \text{Sup}\,|f_n(x) - 0| = \text{Sup}\,|f_n(x)|$$

$$= f_n(1/\sqrt{n}) = \frac{1/\sqrt{n}}{1+n\left(\frac{1}{n}\right)} = \frac{1}{2\sqrt{n}} \quad -(1)$$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$

$\therefore f_n(x)$ is uniformly convergent for all x.

to f. where $f(x) = 0$.

Now, $f_n'(x) = \dfrac{1-nx^2}{(1+nx^2)^2}$ and $f'(x) = 0$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2}$$

Case I: $\cancel{x \neq 0}$

$$\cancel{\lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} = \lim_{n\to\infty} \frac{1/n\,(\frac{1}{n}-x^2)}{1/n^2\,(1/n+x^2)^2}}$$

Case I $\quad x \neq 0$

$$\lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} = \lim_{n\to\infty} \frac{n\,(1/n-x^2)}{n^2\,(1/n+x^2)^2} = 0$$

$\therefore \lim_{n\to\infty} f_n'(x) = f'(x)$ for $x \neq 0$

Case II $\quad x = 0$

$$\lim_{n\to\infty} \frac{1-n(0)^2}{(1+n(0)^2)^2} = 1 \quad \text{but } f'(0) = 0$$

$\therefore \lim_{n\to\infty} f_n'(0) \neq f'(0)$

i.e. $f_n'(0) \nrightarrow f'(0)$

- As already proved $\operatorname{Sup} |f_n(x)| = \dfrac{1}{2\sqrt{n}}$ in (1)

$\therefore |f_n(x)|$ is maximum at $x = \dfrac{1}{\sqrt{n}}$ and maximum value is $\dfrac{1}{2\sqrt{n}}$