2011 P-I

Ques: 3(c) Find the volume of the solid that lies under the parabola $z = x^2+y^2$ above x-y plane and inside the cylinder $x^2+y^2=2x$.

Solution:-

The desired volume can be obtained by integrating $z = x^2+y^2$. over the circle $x^2+y^2=2x$.

Let us change it into polar co-ordinates

let $x = r\cos\theta$ $y = r\sin\theta$ so that.

Equation of paraboloid $\Rightarrow z = x^2+y^2 = r^2$

Equation of circle $\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2 = 2r\cos\theta$

$$r^{\cancel{2}}(\cos^2\theta+\sin^2\theta) = 2\cancel{r}\cos\theta$$

$$\boxed{r = 2\cos\theta}$$

$$\boxed{dz = r\,dr\,d\theta}$$

Hence, desired volume.

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{2\cos\theta} r^2 . r\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{0}^{2\cos\theta} r^3\,dr\,d\theta$$

$$V = \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi}\left[\frac{2^4\cos^4\theta}{4} - 0\right]d\theta$$

$$V = 4\int_0^{\pi}\cos^4\theta\,d\theta = \cancel{4}\times\frac{3\pi}{\cancel{8}\,2} = \frac{3\pi}{2}.$$

$\therefore$ $\boxed{\text{Desired Volume} = \frac{3\pi}{2}\text{ unit cubic.}}$

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Ques: 7 (a)) A mass of 560 kg. moving with a velocity 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates?

Solution: Given; mass of moving body = m = 560 kg.

Initial velocity of the body = 240 m/sec

Time taken to brought to rest $= \frac{1}{100}$ sec

$= 0.01$ sec.

Final velocity = 0 m/sec.

$\therefore$ The retardation of the body when it hits the fixed target = a.

Hence, by equation of motion.

$$v = u + at$$

$$a = \frac{v-u}{t} = \frac{-240\,m/sec}{0.01\,sec}$$

$$\boxed{a = -24000\,m/s^2.}$$

The impulse of blow = F = ma

$\Rightarrow F = 560\,kg \times -24000\,m/s^2$

$\Rightarrow F = 560000 \times -24000\,gm/s^2$

$\Rightarrow F = -1344 \times 10^7$ N.

$\Rightarrow \boxed{|F| = 1344 \times 10^7\,N}$

Now, the distance through which it penetrates.
$= S$

and with equation motion.

$$S = vt - \frac{1}{2}at^2$$

$$S = 0 \times 0.01 - \frac{1}{2} \times 24000 \times (0.01)^2$$

$$S = 0 + \frac{1}{\cancel{2}} \times \cancel{24000}^{12} \times \frac{1}{10000}$$

$$\boxed{S = 1.2 \text{ meters.}}$$

$\therefore$ $\boxed{\text{Penetrated Distance} = 1.2 \text{ meters}}$

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Ques: 7(c)(i) After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half of its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground.

Solution :-

Initial velocity of ball $= u = 0$ m/sec

Acceleration $= 9.8\,m/s^2$ (falling under gravity)

Time taken $=$ 5 sec.

final velocity before reaches the glass $= V$

$$V = u + at \quad [\because \text{Equation of motion}]$$

$$V = 0 + 9.8 \times 5$$

$$\boxed{V = 49\,m/sec}$$

It reduces half of its velocity when it hit the glass pane i.e $u' = \frac{1}{2}V = 24.5\,m/sec$

Acceleration $= 9.8\,m/sec^2$ (falling under gravity)

(t') Time taken $=$ 1 sec.

By equation of motion

$$S = u't + \frac{1}{2}at'^2$$

$$S = 24.5 \times 1 + \frac{1}{\cancel{2}} \times \cancel{9.8}^{4.9}$$

$$S = 24.5 + 4.9 = 29.4 \text{ metres.}$$

Hence; the height of the glass pane from ground $=$ 29.4 meters.

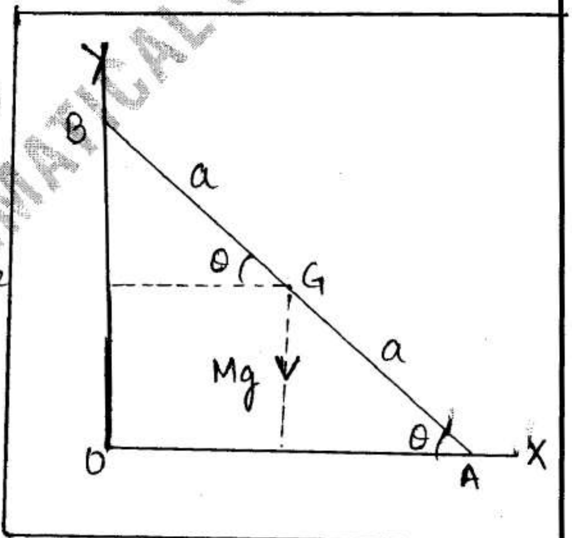
2011 P-II

Ques:- 8(a) The ends of a heavy rod of length 2a are rigidly attached to two small rings at its ends which can respectively slide on smooth horizontal and vertical wires OX and OY. The rod starts at an angle α to the horizontal with angular velocity $\sqrt{\left\{\frac{3g(1-\sin\alpha)}{2a}\right\}}$ and moves downward. Show that it will strike horizontal wire at the end of time

$$-2\sqrt{\left(\frac{a}{3g}\right)}\log\left\{\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right\}.$$

Solution:-

Let AB be the rod of mass M and length 2a.
At time 't', let the rod be inclined at an angle θ to the horizontal. Referred to OX and OY as axes co-ordinates of C.G. G of AB are $(a\cos\theta, a\sin\theta)$. The velocity of G is given by -



$$v_G^2 = (-a\sin\theta\,\dot{\theta})^2 + (a\cos\theta\,\dot{\theta})^2 = a^2\dot{\theta}^2.$$

$\therefore$ Kinetic Energy of the rod at time t

$$= \frac{1}{2}M\cdot\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}M\cdot v_G^2 = \frac{1}{2}M\left(\frac{1}{3}a^2\dot{\theta}^2 + a^2\dot{\theta}^2\right)$$

$$\boxed{\text{K.E. of the rod at time } t = \frac{2}{3}Ma^2\dot{\theta}^2}$$

But, initially $\dot{\theta} = \sqrt{\frac{3g(1-\sin\alpha)}{2a}}$ (given).

$\therefore$ Initial K.E. of rod $= \frac{\cancel{2}}{\cancel{3}} M a^{\cancel{2}} \cdot \frac{\cancel{3}g}{\cancel{2a}} (1-\sin\alpha)$

$$\boxed{\text{Initial K.E. of rod} = Mag(1-\sin\alpha)}$$

Hence, every equation gives
Change in K.E. = Work done by gravity

i.e $\frac{2}{3} M a^2 \dot{\theta}^2 - Mga(1-\sin\alpha) = Mg(a\sin\alpha - a\sin\theta)$

(or) $\frac{2}{3} a^2 \dot{\theta}^2 = g(1-\sin\theta)$

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\sin\theta)$$

$$\therefore \boxed{\dot{\theta} = \frac{d\theta}{dt} = -\sqrt{\frac{3g}{2a}(1-\sin\theta)}}$$

(-ve sign is taken because motion is towards θ decreasing)

(or) $dt = -\sqrt{\frac{2a}{3g}} \cdot \frac{d\theta}{\sqrt{1-\sin\theta}}$

Integrating from $\theta=\alpha$ to $\theta=0$, the required time is given by -

$$t = -\sqrt{\left(\frac{2a}{3g}\right)} \int_{\theta=\alpha}^{0} \frac{d\theta}{\sqrt{1-\sin\theta}}$$

$$t = -\sqrt{\frac{2a}{3g}} \cdot \int_{\alpha}^{0} \frac{d\theta}{\sqrt{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}}$$

$$t = -\sqrt{\frac{2a}{3g}} \int_{\alpha}^{0} \frac{d\theta}{(\cos\theta/2 - \sin\theta/2)}$$

$$t = -\frac{1}{\sqrt{2}} \sqrt{\frac{2a}{3g}} \int_{\alpha}^{0} \frac{d\theta}{\sin(\pi/4 - \theta/2)}$$

$$= \sqrt{\frac{a}{3g}} \int_0^{\alpha} \operatorname{cosec}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta$$

$$= \sqrt{\frac{a}{3g}} \left[-2 \log \tan\left(\frac{\pi}{8} - \frac{\theta}{4}\right)\right]_0^{\alpha}$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)} \left[-\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) + \log \tan \frac{\pi}{8}\right]$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) - \log \tan \frac{\pi}{8}\right]$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) + \log \cot \frac{\pi}{8}\right]$$

$$\boxed{t = -2\sqrt{\frac{a}{3g}} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) \cdot \cot \frac{\pi}{8}\right]}$$

Hence the result

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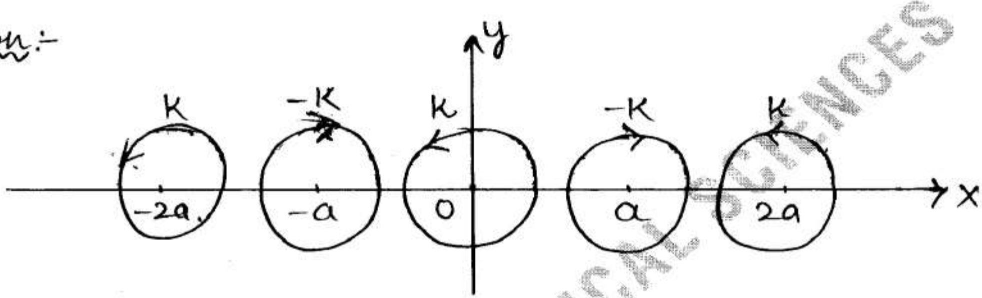
Ques: 8(b) An infinite row of equidistant rectilinear vortices are at a distance a part. The vortices are of the same numerical strength K but they alternatively of opposite signs. Find the complex function that determines the velocity potential and the stream function.

Solution:-



Let the row of vortices be taken along the x-axis. Let there be vortices of strength K each at the points $(0,0)$, $(\pm 2a, 0)$, $(\pm 4a, 0)$, ... and those of strength $-K$ each at the points $(\pm a, 0)$, $(\pm 3a, 0)$, ...

The complex potential of the entire system is given by-

$$w = (ik/2\pi)\left[\{\log z + \log(z-2a) + \log(z+2a) + \cdots\} - \{\log(z-a) + \log(z+a) + \log(z-3a) + \cdots\}\right]$$

$$w = \frac{ik}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\ldots}{(z^2-a^2)(z^2-3^2a^2)\ldots}$$

$$= \frac{ik}{2\pi}\log\frac{z/2a\left[1-(z/2a)^2\right]\left[1-(z/4a)^2\right]\ldots}{\left[1-(z/a)^2\right]\left[1-(z/3a)^2\right]\ldots} + \text{a constant}$$

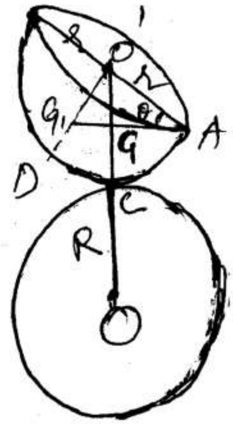
Thus;

$$w = \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)}$$

$$\therefore \boxed{w = \frac{ik}{2\pi}\log\tan(\pi z/2a)}$$

which is desired potential function that determines the velocity potential and stream function.

Q.7(b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.



Sol:

Let O' be the centre of the base of the hemispherical shell of radius r. Let weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the spherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$.

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre at O at the highest point C.

For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$ i.e., $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e., $\frac{1}{h} > \frac{R+r}{Rr}$

i.e., $h < \frac{rR}{R+r}$ ——— (1)

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is maximum i.e, if $O'G$ is perpendicular to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$.

Then from right angled $\Delta O'G_1$

$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$

$\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ the minimum value of $O'G$

$= O'A\sin\theta = r\left(\frac{1}{\sqrt{5}}\right) = r/\sqrt{5}$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$= r - \frac{r}{\sqrt{5}} = \frac{r(\sqrt{5}-1)}{\sqrt{5}}$.

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if $\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{r+R}$ i.e, if $\frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$

i.e, if $(\sqrt{5}-1)R - (\sqrt{5}-1)r < R\sqrt{5}$

i.e, if $R/r > \sqrt{5}-1$