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UPSC CSE 2022 Mathematics Paper 2 – Solutions

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1	Paper 1-5c	Analytic Geometry	Full Length Test 7 -2b
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3	Paper 2-4b	Real Analysis	Full Length Test 4-3c

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1	Paper 1-1d	Calculus	SC B10 Qn 52
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1a) Show that the multiplicative group $G = \{1, -1, i, -i\}$, where $i = \sqrt{(-1)}$, is isomorphic to the group $G' = \{\{0, 1, 2, 3\}, +_4)$.

Let $G = \{1, -1, i, -i\}$ and $G' = \{[0], [1], [2], [3]\}$

To show that $(G', *) \cong (G, +)$, where f denote a map

$f: G \to G'$ such that $o(a) = o[f(a)], \forall a \in G$

If we show that f is an isomorphism, then result will follow, 1 and [0] are identities in G and G' respectively. Also, order of identiy element in every group is one.

Hence, $o(1) = 1 = o([0])$

For elements of G: $o(a) = n \Rightarrow a^n = e = 1$

$$\begin{aligned}
(-1)^1 &= -1, (-1)^2 = 1 = e && \Rightarrow o(-1) = 2 \\
(i)^4 &= 1 = e && \Rightarrow o(i) = 4 \\
(-i)^4 &= 1 = e && \Rightarrow o(-i) = 4
\end{aligned}$$

Now for element of G', $o([1]) = n \Rightarrow n[a] = [0]$

$$1[1] = [1], 2[1] = [2], 3[1] = [3], 4[1] = [0] = e \Rightarrow o[(1)] = 4$$

Similarly,

$$o([2]) = 2 \text{ and } o([3]) = 4$$

Therefore, order of elements $1, -1, i, -i \in G$ are respectively $1,2,4,4$ and order of elements $[0], [1], [2], [3] \in G'$ are respectively given by $1,4,2,4$. Clearly

$$o(1) = 1 = o([1]), o(-1) = 2 = o([2])$$
$$o(i) = 4 = o([1]), o(-i) = 4 = o([3])$$

This implies

$$f(1) = [0], \mathrm{f}(-1) = [2], f(i) = [1], f(-i) = [3]$$

$\Rightarrow f$ is one-one onto.

Also, $f(1 \cdot i) = f(i) = [1] = [0] + [1] = f(1) + f(i)$

Similarly,

$$f\{i(-i)\} = f(1) = [0] = [1] + [3] = f(i) + f(-i)$$

$\Rightarrow f$ preserves composition in G and G'.

Hence, f is an isomorphism.

1b) If $f(z) = u + iv$ is an analytic function of z, and

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}},$$ **then find $f(z)$ subject to $f\left(\frac{\pi}{2}\right) = 0$.**

SuccessClap Question Bank: Complex Analysis-SC-H 01- Qn 43

$$\begin{aligned}
u - v &= \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}} \\
&= \frac{1}{2}\left[1 + \frac{2\cos x + 2\sin x - 2e^{-y}}{2\cos x - e^{y} - e^{-y}} - 1\right] \\
&= \frac{1}{2}\left[1 + \frac{2\sin x + e^{y} - e^{-y}}{2\cos x - e^{y} - e^{-y}}\right] = \frac{1}{2}\left[1 + \frac{\sin x + \sinh y}{\cos x - \cosh y}\right].
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} &= \frac{1}{2}\left[\frac{\cos x(\cos x - \cosh y) + (\sin x + \sinh y)\sin x}{(\cos x - \cosh y)^2}\right] \\
&= \frac{1}{2}\left[\frac{1 - \cos x\cosh y + \sin x\sinh y}{(\cos x - \cosh y)^2}\right] \\
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} &= \frac{1}{2}\left[\frac{\cosh y(\cos x - \cosh y) + \sinh y(\sin x + \sinh y)}{(\cos x - \cosh y)^2}\right] \\
&= \frac{1}{2}\left[\frac{\cosh y\cos x + \sinh y\sin x - 1}{(\cos x - \cosh y)^2}\right]
\end{aligned}$$

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{\cosh y\cos x + \sinh y\sin x - 1}{(\cos x - \cosh y)^2}\right],$$

Solving we get

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\left[\frac{1 - \cos x\cosh y}{(\cos x - \cosh y)^2}\right] = \phi_1(x, y) \\
\frac{\partial v}{\partial x} &= -\frac{\sin x\sinh y}{2(\cos x - \cosh y)^2} = \phi_2(x, y) \\
\therefore\ f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \phi_1(z, 0) + i\phi_2(z, 0) \\
&= \frac{1}{2}\frac{1 - \cos z}{(\cos z - 1)^2} = \frac{1}{2(1 - \cos z)} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}.
\end{aligned}$$

$$\therefore\ f(z) = \frac{1}{4}\int \operatorname{cosec}^2\frac{1}{2}z\,dz + c = -\frac{1}{2}\cot\frac{1}{2}z + c.$$

At $z = \frac{\pi}{2}, f(z) = 0,\ \therefore\ c = f\left(\frac{\pi}{2}\right) + \frac{1}{2}\cot\frac{\pi}{4} = \frac{1}{2}.$

$$\therefore\ f(z) = \frac{1}{2}\left(1 - \cot\frac{1}{2}z\right).$$

1c) Test the convergence of $\int_0^{\infty} \frac{\cos x}{1+x^2} dx$.

SuccessClap - Question Bank SC-B-10 - Qn 55

$x=0$: Not pt of discontinuity

$$\underset{x\to 0}{Lt} \frac{\cos x}{1+x^2} = \frac{1}{1} = 1$$

$x=\infty$:
$$\int_c^{\infty} \frac{\cos x}{1+x^2} dx \leq \int_c^{\infty} \frac{dx}{1+x^2} \leq \int_c^{\infty} \frac{dx}{x^2}$$

$$\cos x \leq 1 \qquad \frac{1}{1+x^2} \leq \frac{1}{x^2}$$

If $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

& If $\int_a^b g(x)\,dx$ converge $\Rightarrow \int_a^b f(x)dx$ converge

Comparision test

$\int_c^{\infty} \frac{dx}{x^2}$ converge

$\hookrightarrow$ Why $\int_c^{\infty} \frac{dx}{x^{\mu}}$ converge if $\mu > 1$

$\mu = 2$ So converge

So $\int_c^{\infty} \frac{\cos x\, dx}{1+x^2}$ converge

1d) Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for the regions (i) $0 < |z-1| < 2$ and (ii) $0 < |z-3| < 2$.

(i) $0<|z-1|<2 \qquad z-1=u \qquad 0<u<2 \Rightarrow \frac{u}{2}<1$

$$f(z) = \frac{1}{u^2(u-2)} = \frac{1}{-2u^2\left(1-\frac{u}{2}\right)}$$

$$= \left(\frac{-1}{2u^2}\right)\left(1+\frac{u}{2}+\frac{u^2}{4}+\frac{u^3}{8}+\cdots\right)$$

$$= -\frac{1}{2u^2} - \frac{1}{4u} - \frac{1}{8} - \frac{u}{16} - \frac{u^2}{32}$$

$$= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{(z-1)}{16} - \frac{(z-1)^2}{32}$$

(ii) $0<|z-3|<2 \qquad z-3=u \qquad 0<u<2 \Rightarrow \frac{u}{2}<1$

$$f(z) = \frac{1}{(u+2)^2u} = \frac{1}{4u\left(1+\frac{u}{2}\right)^2} = \frac{1}{4u}\left(1+\frac{u}{2}\right)^{-2}$$

$$= \frac{1}{4u}\left[1-2\left(\frac{u}{2}\right)+3\left(\frac{u}{2}\right)^2-4\left(\frac{u}{2}\right)^3+5\left(\frac{u}{2}\right)^4+\cdots\right]$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4\ldots$$

$$f(z) = \frac{1}{4u}\left[1-u+\frac{3}{4}u^2-\frac{u^3}{2}+\frac{5u^4}{16}+\cdots\right]$$

$$= \frac{1}{4u}-\frac{1}{4}+\frac{3}{16}u-\frac{u^2}{8}+\frac{5u^3}{64}-\cdots$$

$$= \frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{(z-3)^2}{8}+\frac{5}{64}(z-3)^3+\cdots$$

1e) **Use two-phase method to solve the following linear programming problem : Minimize** $Z = x_1 + x_2$

subject to

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Converting to max problem

$\text{Max } Z^* = -Z = -x_1 - x_2 + 0.S_1 + 0.S_2 - A_1 - A_2$

$2x_1 + x_2 + A_1 - S_1 = 4$ $\quad A_1 = 4$ Basic

$x_1 + 7x_2 + A_2 - S_2 = 7$ $\quad A_2 = 7$ variables

phase I $\quad Z^* = -A_1 - A_2$

	C_j	0	0	0	0	-1	-1		
Coeff	B.V	x_1	x_2	S_1	S_2	A_1	A_2	Soln	Ratio
-1	A_1	2	1	-1	0	1	0	4	4/1=4
-1	A_2	1	7	0	-1	0	1	7	7/7=1 →
	z_j	-3	-8	1	1	-1	-1		
	$c_j - z_j$	3	$\boxed{8}$	-1	-1	1	1		

-1	A_1	13/7	0	-1	1/7	1	-	3	21/13 →
0	x_2	1/7	1	0	-1/7	0	-	1	7
	z_j	-13/7	0	1	-1/7	-1			
	$c_j - z_j$	13/7	0	-1	1/7	1			

	C_j	0	0	0	0	-1	-1		
Coeff of BV	BV	x_1	x_2	S_1	S_2	A_1	A_2	Soln	Ratio
0	x_1	1	0	$-7/13$	$1/13$	–	–	$21/13$	
0	x_2	0	1	$7/13$	$-2/13$	–	–	$10/13$	
	z_j	0	0	0	0				
	$c_j - z_j$	0	0	0	0				

As all $c_j = 0$ & optimality of phase I is reached
bcoz $c_j - z_j \le 0$

Phase II : $Z^* = -x_1 - x_2 + 0.S_1 + 0.S_2$

	C_j	-1	-1	0	0		
Coeff of BV	BV	x_1	x_2	S_1	S_2	Soln	Ratio
-1	x_1	1	0	$-7/13$	$1/13$	$21/13$	
-1	x_2	0	1	$1/13$	$-2/13$	$10/13$	
	z_j	-1	-1	$6/13$	$1/13$		
	$c_j - z_j$	0	0	$-6/13$	$-1/13$		

As $c_j - z_j \le 0 \rightarrow$ optimal reached

$x_1 = \frac{21}{13} \quad x_2 = \frac{10}{13}$

$$\text{Max } Z^* = -\frac{21}{13} - \frac{10}{13} = -\frac{31}{13}$$

$$\text{Min } Z = -Z^* = \frac{31}{13}$$

2a) Let $f(x) = x^2$ on $[0, k], k > 0$. Show that f is Riemann integrable on the closed interval $[0, k]$ and $\int_0^k f\,dx = \frac{k^3}{3}$

Similar question in SuccessClap Question Bank SC-B26 Qn 3 for $f(x)=x^3$

Let $P=\{0, \frac{k}{n}, \frac{2k}{n}, \frac{3k}{n} \dots \frac{r-1}{n}k, \frac{nk}{n}=k\}$ Partition in $[0\,k]$

$\rightarrow f(x)=x^2$ increasing monotonic

In $I_r = \left[\frac{(r-1)k}{n}, \frac{rk}{n}\right] \longrightarrow \delta_r = \text{Width} = \frac{k}{n}$

$\hookrightarrow m_r = \left(\frac{(r-1)k}{n}\right)^2 \rightarrow M_r = \left(\frac{rk}{n}\right)^2$

$\rightarrow U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \frac{k^3}{n^3}\sum r^2 = \frac{k^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{k^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$\overline{\int_a^b} f(x)\,dx = \underset{n\to\infty}{Lt}\, U(P,f) = \underset{n\to\infty}{Lt}\, \frac{k^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{k^3}{3}$

$\rightarrow L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{(r-1)^2 k^2}{n^2} \cdot \frac{k}{n} =$

$= \frac{k^3}{n^3}\left[0 + 1^2 + 2^2 + 3^2 + \dots (n-1)^2\right]$

$= \frac{k^3}{n^3}\left[\underline{1^2+2^2+3^2+\cdot(n-1)^2+n^2} - n^2\right]$

$= \frac{k^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - n^2\right] = k^3\left(\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n}\right)$

$\underline{\int_a^b} f(x)\,dx = \underset{n\to\infty}{Lt}\, L(P,f) = \frac{k^3}{3}$

Since $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx \Rightarrow f$ is R.I $\Rightarrow$

$\int_0^k f(x)\,dx = \frac{k^3}{3}$

2b) Prove that every homomorphic image of a group G is isomorphic to some quotient group of G.

SuccessClap Question Bank: Algebra -SC-G 07- Qn 17

Proof: This is Fundamental Theorem of Homomorphism. This proof is present in ANY GRADUATION TEXT BOOK

By definition, $K = \{x \in G: f(x) = e'\}$. ...(1)
We know that K is a normal subgroup of G and so the quotient group G/K is defined, where $\frac{G}{K} = \{Kg: g \in G\}$.

It is given that the mapping
$f: G \to G'$ is homomorphism and onto.

$$\Rightarrow f(g) \in G' \text{ for all } g \in G$$

We define a mapping

$$\phi \quad : \frac{G}{K} \to G' \text{ as}$$
$$\phi(Kg) \quad = f(g) \forall g \in G$$

Firstly, we show that ϕ is well-defined i.e., we have to show that

$$Kg_1 = Kg_2 \Rightarrow \phi(Kg_1) = \phi(Kg_2); g_1, g_2 \in G.$$

Now

$$Kg_1 = Kg_2 \Rightarrow g_1 g_2^{-1} \in K$$

$\Rightarrow f(g_1 g_2^{-1}) = e'$, by (1)
$\Rightarrow f(g_1) f(g_2^{-1}) = e'$, since f is a homomorphism $\Rightarrow f(g_1)[f(g_2)]^{-1} = e'$, since f is a homomorphism $\Rightarrow f(g_1) = f(g_2) \Rightarrow \phi(Kg_1) = \phi(Kg_2)$, by (2).
Thus ϕ is well-defined.

Next we show that ϕ is a homomorphism.
Let $X, Y \in G/K$. Then $X = Kx, Y = Ky$ for some $x, y \in G$. We have

$$\begin{aligned} XY &= KxKy = Kxy, \text{ as } K \triangleleft G \\ \phi(XY) &= \phi(Kxy) \\ &= f(xy), \text{ by } (2) \\ &= f(x)f(y), \text{ since } f \text{ is a homomorphism} \\ &= \phi(Kx)\phi(Ky), \text{ by } (2) \\ &= \phi(X)\phi(Y) \end{aligned}$$

Thus ϕ is a homomorphism.

Now we show that ϕ is one-to-one.

$$\text{Let } \phi(X) = \phi(Y); X = Kx, Y = Ky \in G/K$$

$$\begin{aligned}
\Rightarrow \quad \phi(Kx) &= \phi(Ky) \\
\Rightarrow \quad f(x) &= f(y), \text{ by } (2) \\
\Rightarrow \quad f(x)[f(y)]^{-1} &= e^{'} \\
\Rightarrow \quad f(x)f(y^{-1}) &= e^{'}, \text{ since } f \text{ is a homomorphism} \\
\Rightarrow \quad f(xy^{-1}) &= e^{'}, \text{ since } f \text{ is a homomorphism} \\
\Rightarrow \quad xy^{-1} &\in K, \text{ by } (1) \\
\Rightarrow \quad Kx &= Ky \Rightarrow X = Y.
\end{aligned}$$

Thus ϕ is one-to-one.

Lastly, we show that- ϕ is onto.

Let $g^{'} \in G^{'}$ be arbitrary. Since $f: G \to G^{'}$ is onto, there exists some $g \in G$ such that $f(g) = g^{'}$

$\Rightarrow \ \phi(Kg) = g^{'}$, by (2). Here $Kg \in G/K$.

Thus ϕ is onto.

We have now shown that $\phi: G/K \to G^{'}$ is a homomorphism, onto and one-to-one. Hence

$$\frac{G}{K} \approx G^{'} \text{ or } G^{'} \approx \frac{G}{K}.$$

2c) Apply the calculus of residues to evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)},\ a > b > 0.$$

SuccessClap Question Bank: Complex Analysis -SC- H 05- Qn 43

Consider $\int_C f(z)dz = \int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz$, where C is the same contour

By residue theorem, we have

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz = 2\pi i \sum R^+$$

By Jordan's lemma, we have $\lim_{R\to\infty} \int_\Gamma f(z)dz = 0.$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum R^+ \text{or} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$$

$z = \pm ai, \pm bi$ are the simple poles of $f(z)$.

Out of these only $z = ai, bi$ lie in the upper half plane.

Residue at $z = ai$ is $\lim_{z\to ai}(z-ai)f(z) = \lim_{z\to ai} \frac{e^{iz}}{(z+ai)(z^2+b^2)} = \frac{e^{-a}}{2ai(b^2-a^2)}$.

Residue at $z = ib$ is $\lim_{z\to bi}(z-bi)f(z) = \lim_{z\to bi} \frac{e^{iz}}{(z^2+a^2)(z+bi)} = \frac{e^{-b}}{2bi(a^2-b^2)}$.

Sum of the residues $= \frac{1}{2i(a^2-b^2)}\left\{\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right\}$.

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = 2\pi i \frac{1}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

3a) Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i|=2$

SuccessClap Question Bank: Complex Anaysis-SC-H 02- Qn 33

$$\frac{z+4}{z^2+2z+5}=\frac{z+4}{(z+1)^2-(2i)^2}=\frac{z+4}{(z+1+2i)(z+1-2i)}$$

Hence the singularities of the function $\frac{z+4}{z^2+2z+5}$ are $-1-2i$ and $-1+2i$. So the function is not analytic at the points $(-1,-2)$ and $(-1,2)$.

When C is the circle $|z+1-i|=2$ i.e. $(x+1)^2+(y-1)^2=4$

The point $(-1,-2)$ i.e. $z=-1-2i$ does not lie within c whereas the point $(-1,2)$ i.e. $z=-1+2i$ lies inside.

$$\therefore \int_c \frac{z+4}{(z+1+2i)(z+1-2i)} dz=\int_c \frac{\frac{z+4}{z+1+2i}}{z+1-2i} dz$$

$$=\int_{cz} \frac{f(z)}{z-a} dz \text{ where } f(z)=\frac{z+4}{z+1+2i} \text{ and } a=-1+2i$$

By Cauchy's integral formula $\int_C \frac{f(z)}{z-a} dz=2\pi i f(a)$

But

$$f(z)=\frac{z+4}{z+1+2i}, a=-1+2i$$

$$f(a)=f(-1+2i)=\frac{-1+2i+4}{-1+2i+1+2i}=\frac{3+2i}{4i}$$

$$\int_c \frac{z+4}{z^2+2z+5} dz=2\pi i\left(\frac{3+2i}{4i}\right)=\frac{\pi}{2}(3+2i)$$

3b) Find the maximum and minimum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, when

$lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Interpret the result geometrically.

SuccessClap Question Bank: Calculus &Real Analysis-SC-B 07- Qn 4

Let $u=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$; then

$$du=\frac{2x}{a^4}dx+\frac{2y}{b^4}dy+\frac{2z}{c^4}dz=0 \quad ..(1)$$

$$ldx+mdy+ndz=0 \quad ..(2)$$

$$\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0 \quad ..(3)$$

Multiplying (1), (2) and (3) by $1,\lambda_1,\lambda_2$ respectively and adding, and then equating the coefficients of dx, dy and dz to zero, we get

$$\frac{x}{a^4}+\lambda_1 l+\lambda_2\frac{x}{z^2}=0 \quad ...(4)$$

$$\frac{y}{b^4}+\lambda_1 m+\lambda_2\cdot\frac{y}{b^2}=0 \quad ...(5)$$

and $\frac{z}{c^4}+\lambda_1 n+\lambda_2\cdot\frac{z}{c^2}=0$ ...(6)

Multiplying (4) by x, (5) by y, and (6) by z and adding, we get

$$u+\lambda_2=0 \text{ or } \lambda_2=-u,$$

$\therefore\ \frac{x}{a^4}+\lambda_1 l-\frac{x}{a^2}\cdot u=0$ etc.

Hence $x=-\frac{\lambda_1 l a^4}{1-a^2u}$

Similarly, $y=\frac{-x_1 m b^4}{1-b^2u}$ and $z=\frac{-\lambda_1 n c^4}{1-c^2u}$

Substituting these values of x,y,z in $lx+my+nz=0$, we get

$$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0 \quad ...(7)$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical Interpretation. The tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at (x',y',z') is $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$. Perpendicular distance to the tangent plane from origin, i.e., p is given by

$$p^2 = \frac{1}{\frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}}$$

If the point (x', y', z') lies on the given plane $lx + my + nz = 0$, the given problem consists of finding out the maximum or minimum value of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the, points common to the plane $lx + my + nz = 0$ and the ellipsoid.

3c) **Solve the following linear programming problem by the simplex method. Write its dual. Also, write the optimal solution of the dual from the optimal table of the given problem :**

Maximize $Z = x_1 + x_2 + x_3$

subject to

$$2x_1 + x_2 + x_3 \le 2$$
$$4x_1 + 2x_2 + x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

Max $Z = x_1 + x_2 + x_3 + 0S_1 + 0 \cdot S_2$

$2x_1 + x_2 + x_3 + S_1 = 2$

$4x_1 + 2x_2 + x_3 + S_2 = 2$

$S_1 = 2, S_2 = 2$ Basis variable

Coeff of BV	C_j	1	1	1	0	0		
	BV	x_1	x_2	x_3	S_1	S_2	Soln	Ratio
0	S_1	2	1	1	1	0	2	1
0	S_2	4	2	1	0	1	2	$1/2 \to$
	Z_j	0	0	0	0	0		All equal: choose any
	$C_j - Z_j$	$\boxed{1}$	1	1	0	0		
0	S_1	0	0	1/2	1	-1/2	1	$2 \to$
1	x_1	1	1/2	1/4	0	1/4	1/2	2
	Z_j	0	1/2	1/4	0	1/4		
	$C_j - Z_j$	0	1/2	$\boxed{3/4}$	0	-1/4		
1	x_3	0	0	1	2	-1	2	N.D
1	x_1	1	1/2	0	-1/2	1/2	0	0
	Z_j	1	1/2	1	3/2	-1/2		
	$C_j - Z_j$	0	$\boxed{1/2}$	0	-3/2	1/2		
~~1~~	~~x_3~~	~~0~~	~~0~~	~~1~~	~~2~~	~~-1~~		
1	x_3	0	0	1	2	-1	2	
1	x_2	2	1	0	-1	1	0	
	Z_j	2	1	1	1	0		
	$C_j - Z_j$	-1	0	0	-1	0		

$c_j - z_j \leq 0$ optimal reached

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 0$

$\text{Max } Z = 0 + 0 + 2 = 2$

<u>Dual</u>:

$\text{Min } Z^* = 2w_1 + 2w_2$

s.t $\quad 2w_1 + 4w_1 \geq 1$

$w_1 + 2w_2 \geq 1$

$w_1 + w_2 \geq 1$

$w_1, w_2 \geq 0$

IBFS $(w_1\ w_2) = (1, 0)$

$\text{Min } Z^* = 2$

4a) Let R be a field of real numbers and S, the field of all those polynomials $f(x) \in R[x]$ such that $f(0) = 0 = f(1)$. Prove that S is an ideal of $R[x]$. Is the residue class ring $R[x]/S$ an integral domain? Give justification for your answer.

Given $S = \{f(x) \in R[x] : f(0) = 0 = f(1)]$

Let $f(x), g(x) \in S$

$\Rightarrow f(0) = 0 = f(1)$ & $g(0) = 0 = g(1)$

Now $f(0) - g(0) = 0 - 0 = 0$ and $g(0) = 0 = g(1)$

$f(0) - g(0) = 0 = f(1) - g(1)$

$\Rightarrow f(x) - g(x) \in S$

$r(x) \in R[x]$

$r(0) f(0) = r(0).0 = 0$ and $r(1) f(1) = r(1).0 = 0$

$r(x) f(x) \in S$

Also $f(x) r(x) \in A$ (why $R[x]$ is commutative)

Given : S, we got S is ideal of $R[x]$

<u>To show</u> $\frac{R[x]}{S}$ is Not integral domain

$\Rightarrow$ we have to show S is not prime ideal of $R[x]$

Let $f(x) = x$ $g(x) = 1 - x$

$f(x) g(x) = x(1-x) \in R[x]$

$f(0) g(0) = 0(1-0)$ and $f(1) g(1) = 1(1-1) = 0$

$f(x) g(x) \in S$

Also $f(1) = 1 \neq 0$ $f(x) \notin S$

$g(0) = 1 - 0 = 1 \neq 0$ $g(x) \notin S$

$\therefore$ So S is not a prime ideal of $R[x]$

$\frac{R[x]}{S}$ is not an integral domain.

(3c) Test Convergence or Divergence of

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots \quad (x > 0)$$ [15]

→ This question is present in SuccessClap Full Length Test Series 4, Qn 3c asked for 10 marks.
UPSC asked for 15 marks

$$u_n = \frac{n^n x^n}{n!} \qquad u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{(n+1)^n x} = \frac{1}{x\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

→ D-Alembert : Converge if $\frac{1}{ex} > 1 \Rightarrow x < \frac{1}{e}$

Diverge if $x > \frac{1}{e}$

→ when $x = \frac{1}{e}$: use Raabe's Test

$$\ln \frac{u_n}{u_{n+1}} = \ln e - n\log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \cdots$$

$$\lim_{n\to\infty} n \ln \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n} + \cdots\right) = \frac{1}{2} < 1$$

Diverge

Ans: Converge $x < \frac{1}{e}$

Diverge $x \geq \frac{1}{e}$

4c) **Find the initial basic feasible solution of the following transportation problem by Vogel's approximation method and use it to find the optimal solution and the transportation cost of the problem :**

		Destination					
		A	B	C	D		
Source	S1	21	16	25	13	11	Availability
	~~S3~~ S2	17	18	14	23	13	
	S3	32	27	18	41	19	
		6	10	12	15	43	
Requirement							

	A	B	C	D						
S_1	21	16	25	$\boxed{11}$ 13	~~11~~ 0	3	3	3	4	18
S_2	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	~~13~~ ~~9~~ 3	3	3	3	4	18
S_3	32	$\boxed{7}$ 27	$\boxed{12}$ 18	41	~~19~~ ~~7~~ 0	9	9	9	$\boxed{9}$	27
	0 ~~6~~	~~10~~ 3	~~12~~ 0	~~15~~ 4 0	43					
	4	2	4	$\boxed{10}$						
	15	9	4	$\boxed{18}$						
	$\boxed{15}$	8	4							
		9	4							
		9								

	$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$
$u_1=-10$	21	16	25	$\boxed{11}$ 13
$u_2=0$	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23
$u_3=9$	32	$\boxed{7}$ 27	$\boxed{12}$ 18	41

$m+n-1=3+4-1=6$

No of allocation $=6$

Basic feasible non-degenerate

Optimizing

Calculating u_i & v_j s.t $c_{ij} = u_i + v_j$ for allocated cells

evaluating $d_{ij} = c_{ij} - (u_i + v_j)$ for non allocated cells

$d_{11} = 21 - (7) = 14$

$d_{12} = 16 - 8 = 8$

$d_{13} = 25 - (-1) = 26$

$d_{23} = 14 - 9 = 5$

$d_{31} = 32 - 26 = 6$

$d_{34} = 41 - 32 = 9$

As $d_{ij} > 0$ for all non-allocated cell

it ~~rest~~ optimal soln

IBFS = optimal

Optimal case $= 11 \times 13 + 6 \times 17 + 3 \times 23 + 7 \times 27 + 12 \times 18$

$= 796$

5a) It is given that the equation of any cone with vertex at (a, b, c) is $f\left(\frac{x-a}{z-a}, \frac{y-b}{z-c}\right) = 0$. Find the differential equation of the cone.

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Sol. Let $(x-a)/(z-c) = u$ and $(y-b)/(z-c) = v$ ... (1)

Then, the equation of the given cone becomes $f(u, v) = 0$ ...(2)

Differentiating (2) partially with respect to ' x ', we have

$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0$ or $\frac{\partial f}{\partial u}\left(\frac{1-0}{z-c} - \frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right) = 0$, using (1)

$\frac{\partial f}{\partial u}\left(\frac{1}{z-c} - p\frac{x-a}{(z-c)^2}\right) - \frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right) = 0$, where $p = \frac{\partial z}{\partial x}$ ... (3)

Differentiating (2) partially with respect to ' y ', we have $\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0$ or

$\frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1-0}{z-c} - \frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right) = 0$, using (1)

or $-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right) + \frac{\partial f}{\partial v}\left(\frac{1}{z-c} - q\frac{y-b}{(z-c)^2}\right) = 0$, where $q = \frac{\partial z}{\partial y}$ .....(4)

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \frac{1}{z-c} - p\frac{x-a}{(z-c)^2} & -p\frac{y-b}{(z-c)^2} \\ -q\frac{x-a}{(z-c)^2} & \frac{1}{z-c} - q\frac{y-b}{(z-c)^2} \end{vmatrix} = 0$$

$$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix} = 0$$

$$\{z-c-p(x-a)\}\{z-c-q(y-b)\} - pq(x-a)(y-b) = 0$$

$$(z-c)^2 - p(x-a)(z-c) - q(y-b)(z-c) = 0 \text{ or } (x-a)p + (y-b)q = z-c$$

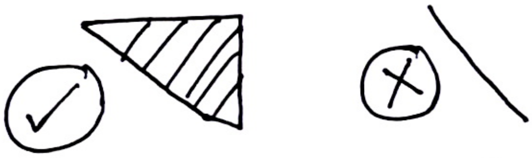
5b) Solve, by Gauss elimination method, the system of equations

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + y + 3z = 14$$

Note : Gauss Elimination

→ No need to check dominant diagonal terms

→ Only Triangular diagonal form



$$[A : I] = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{bmatrix} \quad R_1 \to R_1/2$$

$$= \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

why eliminate A_{32}

$$= \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Final Gauss Elimination Format

Solving : $z = 3$, $y = 2$ $\quad x + y + 2z = 9$
$\Rightarrow x + 2 + 6 = 9 \Rightarrow x = 1$

$$\boxed{\begin{matrix} x = 1 \\ y = 2 \\ z = 3 \end{matrix}}$$

Cross Check Solution.

5c-i) **Convert the number $(1093 \cdot 21875)_{10}$ into octal and the number $(1693.0628)_{10}$ into hexadecimal systems.**

(i) $(1093.21875)_{10}$

$$
\begin{array}{r|l}
8 & 1093 \\
\hline
8 & 136 - 5 \\
\hline
8 & 17 - 0 \\
\hline
8 & 2 - 1 \\
\hline
 & 0 - 2
\end{array}
$$

	Integers
$0.21875 \times 8 = 1.75$	1
$0.75 \times 8 = 6.0$	6

$(2105.16)_8$

(ii) 1693.0628

$$
\begin{array}{r|l}
16 & 1693 \\
\hline
16 & 105 - D \\
\hline
 & 6 - 9
\end{array}
$$

	Integers
$0.0628 \times 16 = 1.00480$	1
$0.00480 \times 16 = 0.07680$	0
$0.07680 \times 16 = 1.22880$	1
$0.22880 \times 16 = 3.66080$	3
$0.66080 \times 16 = 10.57280$	10 = A
$0.57280 \times 16 = 9.16480$	9
$0.16480 \times 16 = 2.63680$	2
$0.63680 \times 16 = 10.18880$	10 = A

$(69D.1013A92A)_{16}$

Calculator value $(69D.1013A92A30553261\text{7}\,\text{C1C})_{16}$

5c-ii) **Express the Boolean function $F(x,y,z) = xy + x'z$ in a product of maxterms form.**

$F(x,y,z) = xy + x'z \quad = x$

$= (xy + x')(xy + z)$

$(x+x')(y+x') \quad (x+z)(y+z) \qquad x+x'=1$

$(x'+y+zz') \quad (x+yy'+z) \quad (xx'+y+z)$

$(x'+y+z)(x'+y+z') \quad (x+y+z)(x+y'+z) \quad (x+y+z)$
$(x'+y+z)$

Same

$F = (x'+y+z)(x'+y+z')(x+y+z)(x+y'+z)$

Repuired Soln.

5d) A particle at a distance r from the centre of force moves under the influence of the central force $F = -\frac{k}{r^2}$, where k is a constant. Obtain the Lagrangian and derive the equations of motion.

Solution : When a particle is moving under central force, then the force is conservative and the motion is in a plane.

Let (r, θ) be the plane coordinates of the particle of mass m.

Kinetic energy $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

Lagrangian $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$

where $V(r)$ is the potential energy in the central force field.

Now, $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$, $\quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $\frac{\partial L}{\partial \theta} = 0, \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

Hence equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or $m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$ and $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

For attractive inverse square law force $F = -\partial V/\partial r = -k/r^2$,

we have equations of motion as

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = \ddot{0}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \text{ or } r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Question Wrong $q_\theta = Mr^{-3}\sin\theta$

5e) The velocity components of an incompressible fluid in spherical polar coordinates (r, θ, ψ) are $(2Mr^{-3}\cos\theta, Mr^{-2}\sin\theta, 0)$, where M is a constant. Show that the velocity is of the potential kind. Find the velocity potential and the equations of the streamlines.

Part 1: To show potential kind $\Rightarrow$ To show $\nabla \times q = 0$

Take $q_\theta = Mr^{-3}\sin\theta$

$$\nabla \times q = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ q_r & r q_\theta & r\sin\theta\, q_\phi \end{vmatrix}$$

$$\nabla \times q = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 2Mr^{-3}\cos\theta & Mr^{-2}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}(0) + r\hat{\theta}(0) + r\sin\phi\,\hat{\phi}\left[\cancel{-2Mr^{-3}\cos\theta + 2Mr^{-3}\cos\theta}\right]\right]$$

$= 0 \rightarrow$ proved

Part 2: Find velocity potential $F(r, \theta, \phi)$

$$-\frac{\partial F}{\partial r} = q_r = 2Mr^{-3}\cos\theta \Rightarrow \frac{\partial F}{\partial r} = -2Mr^{-3}\cos\theta$$

$$-\frac{\partial F}{r\,\partial \theta} = q_\theta = Mr^{-3}\sin\theta \Rightarrow \frac{\partial F}{\partial \theta} = -Mr^{-2}\sin\theta$$

$$\frac{1}{r\sin\theta}\frac{\partial F}{\partial \theta} = q_\phi = 0 \Rightarrow \frac{\partial F}{\partial \phi} = 0$$

$$dF = \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \phi}d\phi = -2Mr^{-3}\cos\theta\, dr - Mr^{-2}\sin\theta\, d\theta = d(Mr^{-2}\cos\theta)$$

$$F = Mr^{-2}\cos\theta + \text{Constant}$$

Part 3: Stream line $\frac{dr}{q_\theta} = \frac{r\,d\theta}{q_\theta} = \frac{r\sin\theta\, d\phi}{q_\phi}$

$$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r\,d\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta\, d\phi}{0} \longrightarrow d\phi = 0 \Rightarrow \underline{\phi = C_1}$$

$$\frac{dr}{r} = 2\cot\theta\, d\theta \Rightarrow \underline{r = C_2\sin^2\theta}$$

ϕ = Constant shows stream line pass through axis of symmetry $\theta = 0$

6a) Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0$ subject to the conditions

$$\boldsymbol{u(0,t) = u(l,t) = 0}$$
$$\boldsymbol{u(x,0) = x(l-x), 0 \le x \le l}$$

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Sol. We can prove that the solution of heat equation
$k(\partial^2 u/\partial x^2) = \partial u/\partial t$
$u(0,t) = u(a,t) = 0$ for all t
subject to the boundary conditions
is given by $u(x,0) = f(x), 0 < x < a$

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin{(n\pi x/a)} e^{-C_n^2 t}$$

where

$$E_n = \frac{2}{a}\int_0^a f(x)\sin{(n\pi x/a)}dx, n = 1,2,3,\ldots$$

And $\quad C_n^2 = (n^2\pi^2 k)/a^2$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k, a = l$ and $f(x) = lx - x^2$. Hence, reduces to

$$\begin{aligned} E_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx \\ &= \frac{2}{l}\left[(lx - x^2)\left\{\frac{-\cos{(n\pi x)/l}}{(n\pi)/l}\right\} - (l - 2x)\left\{\frac{-\sin{(n\pi x)/l}}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos{(n\pi x)/l}}{(n\pi)^3/l^3}\right\}\right]_0^l \end{aligned}$$

[Using the chain rule of integration by parts]

$$= (2/l)\{(-2l^3/n^3\pi^3)\cos n\pi + (2l^3/n^3\pi^3)\} = (4l^2/n^3\pi^3)\{1 - (-1)^n\}$$

$$\therefore \quad E_n = \begin{cases} (8l^2)/(2m-1)^3\pi^3, \text{ if } n = 2m-1 \text{ (odd) and } m = 1,2,3,\ldots \\ 0, \text{ if } n = 2m \text{(even) where } m = 1,2,3,\ldots \end{cases}$$

Then, by , $C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$ and so the required solution is given by

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

For given problem k=1 , which gives solution

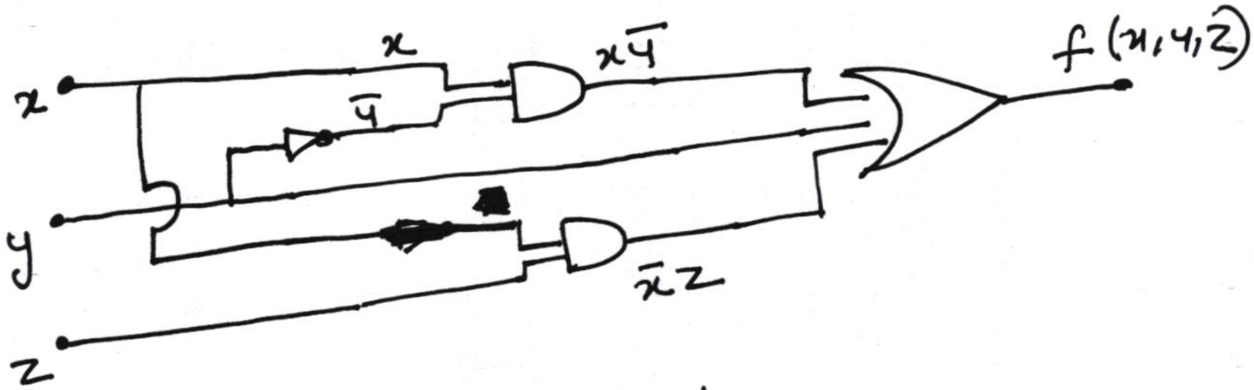
$$\boldsymbol{u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 t\}/l^2}}$$

6b) Find a combinatorial circuit corresponding to the Boolean function

$$f(x, y, z) = [x \cdot (\bar{y} + z)] + y$$

and write the input/output table for the circuit.

$$f(x,y,z) = x\bar{y} + xz + y$$

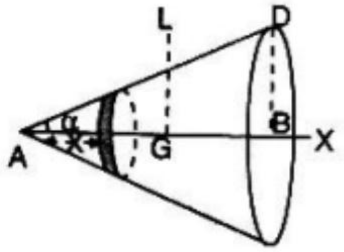


<u>Input-Output table</u>

x	y	z	$\bar{y}$	$x\bar{y}$	xz	$x\bar{y}+xz+y$
0	0	0	1	0	0	0
0	0	1	1	0	0	0
0	1	0	0	0	0	1
0	1	1	0	0	0	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	1	0	0	1	1

6c) Find the moment of inertia of a right circular solid cone about one of its slant sides (generator) in terms of its mass M, height h and the radius of base as a.

Sol. Consider an elementary disc of thickness δx at a distance x from A, then we have radius of the disc $= x\tan\alpha$,



mass of the disc $= \pi x^2\tan^2\alpha\delta x\rho$

From the figure, $\tan\alpha = \frac{a}{h}$

Now M.I. of the disc about $ab = \pi x^2\tan^2\alpha\rho\frac{x^2\tan^2\alpha}{2}\delta x. = \frac{1}{2}\rho\pi\tan^4\alpha x^4\delta x$

Hence M.I. of the cone about $AB = \frac{\rho\pi\tan^4\alpha}{2}\int_0^h x^4 dx = \frac{\pi\tan^4\alpha\rho h^5}{10}$

Now M.I. of the cone about a line through the vertex A and $\perp$ to AB.

$$= \int_0^h \rho\pi x^2\tan^2\alpha dx\left[\frac{x^2\tan^2\alpha}{4} + x^2\right] = \rho\pi\tan^2\alpha\int_0^h\left(\frac{\tan^2\alpha}{4} + 1\right)x^4 dx$$

$$= \rho\pi\tan^2\alpha\left(\frac{\tan^2\alpha}{4} + 1\right)\int_0^h dx = \frac{\rho\pi\tan^2\alpha h^5}{5}\left(\frac{\tan^2\alpha}{4} + 1\right)$$

Now product of inertia of the cone about AB and AK.

$=$ P.I. of the cone about GX and $GL +$ P.I. of the mass of the cone (being concentrated at G) about AG and $AK = 0 + M.0(AG) = 0$.

Again M.I. of the cone about slant side AD

$$= \frac{\pi\tan^4\alpha\rho h^5}{10}\cos^2\alpha + \frac{\pi\tan^2\alpha\rho h^5}{5}\left(\frac{\tan^2\alpha}{4} + 1\right)\sin^2\alpha + 0.$$

$$= \frac{\pi\tan^4\alpha\rho h^5}{20}\left[2\cos^2\alpha + \sin^2\alpha\right] + \frac{\pi\tan^2\alpha\rho h^5}{5}\sin^2\alpha$$

$$= \frac{\pi a^4\rho h}{20}\left[1 + \frac{h^2}{a^2 + h^2}\right] + \frac{\pi a^2\rho h^3}{5}\cdot\frac{a^2}{a^2 + h^2} \quad \text{since } h\tan\alpha = a$$

$$= \frac{\pi a^4\rho h}{20}\left[\frac{a^2 + 2h^2 + 4h^2}{a^2 + h^2}\right] = \boxed{\frac{3Ma^2}{20}\left[\frac{6h^2 + a^2}{a^2 + h^2}\right]}, \text{ since } M = \frac{1}{3}\pi a^2 h\rho.$$

7a) Find the general solution of the partial differential equation

$$\left(D^2 + DD' - 6D'^2\right)z = x^2 \sin(x+y)$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

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Re-writing the given equation is $(D+3D')(D-2D')z = x^2 \sin(x+y)$.
C.F. $= \phi_1(y-3x) + \phi_2(y+2x), \phi_1, \phi_2$ being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+3D')(D-2D')} x^2 \sin(x+y) = \frac{1}{D+3D'} \cdot \left\{\frac{1}{D-2D'}, x^2 \sin(x+y)\right\}$$

$$= \frac{1}{D+3D'} \int x^2 \sin(x+c-2x)dx = \frac{1}{D+3D'} \int x^2 \sin(c-x)dx, \text{ where } c = y+2x$$

$$= \frac{1}{D+3D'}\left[x^2 \cos(c-x) - \int 2x \cos(c-x)dx\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}\left[x^2 \cos(c-x) - \left\{-2x \sin(c-x) + \int 2\sin(c-x)dx\right\}\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}\left[x^2 \cos(c-x) + 2x \sin(c-x) - 2\cos(c-x)\right]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y) + 2x \sin(x+y)\right], \text{ as } c = y+2x$$

$$= \int \left[(x^2-2)\cos\left(x+c'+3x\right) + 2x \sin\left(x+c'+3x\right)\right]dx, \text{ where } c' = y-3x$$

$$= \int (x^2-2)\cos\left(4x+c'\right)dx + 2\int x \sin\left(4x+c'\right)dx$$

$$= (x^2-2)\frac{\sin\left(4x+c'\right)}{4} - \int 2x \frac{\sin\left(4x+c'\right)}{4}dx + 2\int x \sin\left(4x+c'\right)dx$$

$$= \frac{1}{4}(x^2-2)\sin\left(4x+c'\right) + \frac{3}{2}\int x \sin\left(4x+c'\right)dx =$$

$$= \frac{x^2-2}{4}\sin\left(4x+c'\right) + \frac{3}{2}\left[-\frac{x\cos\left(4x+c'\right)}{4} + \int \frac{\cos\left(4x+c'\right)}{4}dx\right]$$

$$= \frac{x^2-2}{4}\sin\left(4x+c'\right) - \frac{3}{8}x\cos\left(4x+c'\right) + \frac{3}{32}\sin\left(4x+c'\right)$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x), \text{ as } c' = y-3x$$

$= (x^2/4 - 13/32)\sin(x+y) - (3x/8) \times \cos(x+y)$,
on simplification The solution is

$$\boldsymbol{z = \phi_1(y-3x) + \phi_2(y+2x) + \left[(x^2/4) - (13/32)\right]\sin(x+y) - (3x/8) \times \cos(x+y).}$$

7b) **The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the velocity in km/ hour :**

t (minutes)	2	4	6	8	10	12	14	16	18	20
v(km/hour)	16	28 · 8	40	46 · 4	51 · 2	32	17 · 6	8	3 · 2	0

Using Simpson's $\frac{1}{3}$rd rule, estimate approximately in km the total distance run in 20 minutes.

Note : starts from rest $\Rightarrow t=0, V=0$ to be considered

$$h = 2 \text{ minutes} = \frac{2}{60} \text{ hrs} = \frac{1}{30} \text{ hrs}$$

$y_0 = 0$ $y_1 = 16$ $y_2 = 28.8$ $y_3 = 40$ $y_4 = 46.4$

$y_5 = 51.2$ $y_6 = 32$ $y_7 = 17.6$ $y_8 = 48$

$y_9 = 3.2$ $y_{10} = 0$

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)\right]$$

$$S = \int v\,dt = I = \frac{(1/30)}{3}(742.4)$$

$$= 8.2488 \text{ km}$$

7c) Two point vortices each of strength k are situated at $(\pm a, 0)$ and a point vortex of strength $-\frac{k}{2}$ is situated at the origin. Show that the fluid motion is stationary and also find the equations of streamlines. If the streamlines, which pass through the stagnation points, meet

the x-axis at $(\pm b, 0)$, then show that $3\sqrt{3}(b^2 - a^2)^2 = 16a^3 b$

Sol. The complex potential of the fluid motion is given by

$$w = (ik/2\pi)\log(z-a) + (ik/2\pi)\log(z+a) - (ik/4\pi)\log z$$

$$\therefore w = (ik/2\pi) \times \left[\log(z^2 - a^2) - (1/2) \times \log z\right] \qquad \text{....(1)}$$

Now, for the motion of the vortex $A(a, 0)$ the complex potential is given by

$$w' = w - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\left[\log(z^2 - a^2) - \frac{1}{2}\log z - \log(z-a)\right]$$

$\therefore$

$$w' = (ik/2\pi)[\log(z+a) - (1/2) \times \log z] \qquad \text{...(2)}$$

The velocity (u_A, v_A) of the vortex K at A is produced solely by the other vertices as it will not move on its own account. Its velocity is therefore given by

$$u_A - iv_A = \left(-\frac{dw'}{2\pi}\right)_{z=a} = -\frac{ik}{2\pi}\left[\frac{1}{z+a} - \frac{1}{2z}\right]_{z=a}, \text{ using (2)}$$

Thus, $u_A - iv_A = 0$. Therefore, $u_A = v_A = 0$, showing that A is stationary. Similarly O and B are stationary. Hence the fluid motion is stationary.

Determination of streamlines.

From (1), we have

$$\phi + i\psi = \frac{ik}{2\pi}\left[\log\{(x+iy)^2 - a^2\} - \frac{1}{2}\log(x+iy)\right] =$$

$$= \frac{ik}{2\pi}\left[\log(x^2 - y^2 - a^2 + 2ixy) - \frac{1}{2}\log(x+iy)\right]$$

$$= (k/2\pi)\left[(1/2) \times \log\{(x^2 - y^2 - a^2)^2 + 4x^2y^2)\} - (1/4) \times \log(x^2 + y^2)\right]$$

Hence stream lines are given by $\psi =$ constant $= (k/4\pi)\log C$

or

$$\frac{k}{4\pi}\log\frac{(x^2 - y^2 - a^2)^2 + 4x^2y^2}{(x^2 + y^2)^{1/2}} = \frac{k}{4\pi}\log C, \text{ using (3)}$$

or

$$(x^2 - y^2 - a^2)^2 + 4x^2y^2 = C(x^2 + y^2)^{1/2}$$

$$(x^2 - y^2)^2 + a^4 - 2a^2(x^2 - y^2) + 4x^2y^2 = C(x^2 + y^2)^{1/2}$$

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = C(x^2 + y^2)^{\frac{1}{2}} \quad .....(4)$$

which are the required streamlines, C being an arbitrary constant.

Determination of stagnation points.

From (1), $\frac{dw}{dz} = \frac{ik}{2\pi}\left[\frac{2z}{z^2 - a^2} - \frac{1}{2z}\right]$. ....(5)

Required stagnation points are given by $dw/dz = 0$

or $\frac{ik}{2\pi}\left[\frac{2z}{z^2-a^2} - \frac{1}{2z}\right] = 0$, using (5)

or $3z^2 + a^2 = 0$ so that $z = \pm\, ia/\sqrt{3}$

Hence $(0, a/\sqrt{3})$ and $(0, -\, a/\sqrt{3})$ are stagnation points. Since the streamlines (4) pass through $(0, \pm\, a/\sqrt{3})$, we have

$(a^2/3)^2 - 2a^2 \times (-\, a^2/3) + a^4 = (Ca)/\sqrt{3}$ giving $C = \left(16\sqrt{3}a^3\right)/9$

Hence the streamlines (4) reduces to

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = \left(6\sqrt{3}a^3/9\right) \times (x^2 + y^2)^{1/2}$$

Again, streamline (6) also passes through $(\pm\, b, 0)$ and hence

$$b^4 - 2a^2b^2 + a^4 = \frac{16\sqrt{3}a^3b}{9} \quad \text{or} \quad (b^2 - a^2)^2 = \frac{16a^3b}{3\sqrt{3}}$$

$$3\sqrt{3}(b^2 - a^2)^2 = 16a^3b.$$

8a) Reduce the following partial differential equation to a canonical form and hence solve it: $y u_{xx} + (x+y) u_{xy} + x u_{yy} = 0$

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Given $yr + (x+y)s + xt = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y, S = (x+y)$ and $T = x$ so that $S^2 - 4RT = (x+y)^2 - 4xy = (x-y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y\lambda^2 + (x+y)\lambda + x = 0$ or $(y\lambda + x)(\lambda + 1) = 0$

so that $\lambda = -1, -\dfrac{x}{y.}$ Then the corresponding characteristic equations are given by

$$(dy/dx) - 1 = 0 \quad \text{and} \quad (dy/dx) - (x/y) = 0$$

$$y - x = c_1 \quad \text{and} \quad y^2/2 - x^2/2 = c_2$$

In order to reduce (1)to its canonical form, we choose

u=x-y , $v = y^2/2 - x^2/2$ ....(2)

$$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right), \text{ using (2)} \quad .....(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}, \text{ using (2)} \quad ....(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\right] = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - \frac{\partial z}{\partial v}$$

$$= -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$$

$$= -\left(-\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u}\right) - x\left(-\frac{\partial^2 z}{\partial u \partial v} - x\frac{\partial^2 z}{\partial v^2}\right) - \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$\therefore\quad r = \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}$$

Now, $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right) + \dfrac{\partial}{\partial y}\left(y\dfrac{\partial z}{\partial v}\right)$, using (4)

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\therefore t=\frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}$$

Also, $s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$

$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}=-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}-y\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}$, using (2)

$\therefore\ s=-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}$

Using (5) (6) and (7) in (1), we get $y\left(\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}\right)$

$$+(x+y)\left\{-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}\right\}+x\left(\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}\right)=0$$

or $\{4xy-(x+y)^2\}\frac{\partial^2 z}{\partial u\partial v}-y\frac{\partial z}{\partial v}+x\frac{\partial z}{\partial v}=0$ or $(y-x)^2\frac{\partial^2 z}{\partial u\partial v}+(y-x)\frac{\partial z}{\partial v}=0$

or $u^2\frac{\partial^2 z}{\partial u\partial v}+u\frac{\partial z}{\partial v}=0$, by (2) or $\boldsymbol{u\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial z}{\partial v}=0}$, as $u\neq 0$ ...(8)

is the required canonical form of (1).

Solution of (8). Multiplying both sides of (8) by v, we get

$$uv(\partial^2 z/\partial u\partial v)+v(\partial z/\partial v)=0\ \text{ or }\ (uvDD'+vD')z=0$$

where $D\equiv\partial/\partial u$ and $D'\equiv\partial/\partial v$.

To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows.

Let $u=e^X$ and $v=e^Y$ so that $X=\log u$ and $Y=\log v$... (10)

Let $D_1\equiv\partial/\partial X$ and $D_1'\equiv\partial/\partial Y$. Then (9) reduces to $(D_1D_1'+D_1')z=0$ or $D_1'(D_1+1)z=0$

Its general solution is $\boldsymbol{z=e^{-X}\phi_1(Y)+\phi_2(X)=u^{-1}\phi_1(\log v)+\phi_2(\log u)}$

or $\boldsymbol{z=u^{-1}\psi_1(v)+\psi_2(u)=(y-x)^{-1}\psi_1(y^2-x^2)+\psi_2(y-x)}$,

where $\boldsymbol{\psi_1}$ and $\boldsymbol{\psi_2}$ are arbitrary functions.

8b) Using Runge-Kutta method of fourth order, solve the differential equation $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$, at $x = 0 \cdot 2$. Use four decimal places for calculation and step length 0-1.

For y=0.1 :

$x_0=0 \quad y_0=1 \quad h=0.1$

$k_1=hf(x_0,y_0)=0.1$

$k_2=hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)=0.1152$

$k_3=hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)=0.1168$

$k_4=hf(x_0+h, y_0+k_3)=0.1347$

$k=\frac{1}{6}(k_1+2k_2+2k_3+k_4)=0.1165$

$y(0.1)=1.1165$

For y=0.2

$x_1=0.1 \quad y_1=1.1165 \quad h=0.1$

$k_1=hf(x_1,y_1)=0.1347$

$k_2=hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right)=0.1551$

$k_3=hf\left(x_1+\frac{h}{2}, y_0+\frac{k_2}{2}\right)=0.1576$

$k_4=hf(x_1+h, y_1+k_3)=0.1823$

$k=\frac{1}{6}(k_1+2k_2+2k_3+k_4)=0.1571$

$y(0.2)=y_1+k=1.2736$

8c) Verify that $w = ik\log\{(z-ia)/(z+ia)\}$ is the complex potential of a steady flow of fluid about a circular cylinder, where the plane $y=0$ is a rigid boundary. Find also the force exerted by the fluid on unit length of the cylinder.

Sol. We have

$$w = \phi + i\psi = ik\log|(z-ia)/(z+ia)|$$
$$\psi = k\log|(z-ia)/(z+ia)|$$

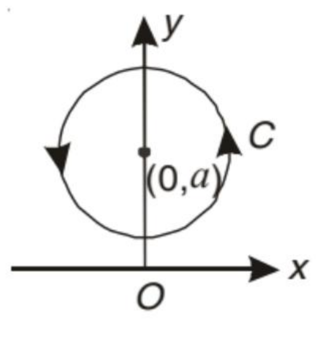
Hence, and so the streamlines are given by $\psi = \text{constant} = k\lambda$, say $ie.$.

$$|(z-ia)/(z+ia)| = \lambda, \text{ or } |z-ia| = \lambda|z+ia|,$$

which are non-intersecting co-axial circles having $z = \pm\, ia$ as the limiting points.

In particular, for $\lambda = 1$,

represents the straight line $|z-ia| = |z+ia|$, i.e., $|x+i(y-a)| = |x+i(y+a)|$, i.e., $x^2 + (y-a)^2 = x^2 + (y+a)^2$, i.e., $y=0$, showing that $y=0$ is a rigid boundary.



$$w = ik\{\log(z-ia) - \log(z+ia)\}$$

so that

$$\frac{dw}{dz} = ik\left(\frac{1}{z-ia} - \frac{1}{z+ia}\right) = \frac{2ka}{z^2+a^2}$$

The adjoining figure shows that circular section C of the cylinder and the rigid plane.

If the pressure thrusts on the given circular disc are represented by (X, Y) then by Blasius theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz = 2k^2 a^2 \rho i \int_C \frac{dz}{\left(z^2 + i^2\right)^2}$$

Again, by Cauchy's residue theorem, we have

$$\int_C \frac{dz}{\left(z^2 + a^2\right)^2} = 2\pi i \times (\text{ sum of the resides })$$

$\therefore$ becomes, $X - iY =- 4k^2 a^2 \rho\pi \times$ (sum of the residues)
where the indicated sum of the residues is calculated at poles of $1/\left(z^2 + a^2\right)^2$ lying within the circular boundary C.

We now proceed to find the residues of $1/\left(z^2 + a^2\right)^2$. The only poles of $1/\left(z^2 + a^2\right)^2$ are at $z = \pm ia$. But only $z = ia$ lies within the boundadry C as shown in the figure. Hence we shall find residue at $z = ia$.

Now, $\frac{1}{z^2+a^2} = \frac{1}{(z-ia)(z+ia)} = \frac{1}{2ia}\left(\frac{1}{z-ia} - \frac{1}{z+ia}\right)$

$$\therefore \frac{1}{\left(z^2 + a^2\right)^2} =- \frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{2}{(z - ia)(z + ia)}\right\}$$

$$=- \frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{1}{2ia}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right)\right\}$$

Hence, Reissue of $1/\left(z^2 + a^2\right)^2$ at $z = ia$ is $1/\left(8ia^3\right)$.

$\therefore$ (5) becomes $X - iY =- \left(4k^2 a^2 \rho\pi\right) \times \left(1/8ia^3\right) = \left\{\left(\pi\rho k^2\right)/2a\right\} i$

$\Rightarrow X = 0$ and $Y =- \left(\pi\rho k^2/2a\right)$

showing that the liquid exerts a downward force on the cylinder of numerical value $\left(\pi\rho k^2/2a\right)$ per unit length.