11 Years

Previous Years Solved Papers

Civil Services Main Examination

(2009-2019)

Mathematics

Paper-II

Topicwise Presentation

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Civil Services Main Examination Previous Solved Papers : Mathematics Paper-II

First Edition: 2020

Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the first edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes

B. Singh (Ex. IES)

CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-II

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4 Linear Programming Problems

1. Basic Feasible Solutions

1.1 By the method of Vogel, determine an initial basic feasible solution for the following transportation problem :

Product P_1, P_2, P_3 and P_4 have to be sent to destinations D_1, D_2 and D_3. The cost of sending product P_i to destinations D_j is C_{ij}, where the matrix

$$[C_{ij}] = \begin{bmatrix} 10 & 0 & 15 & 5 \\ 7 & 3 & 6 & 15 \\ 0 & 11 & 9 & 13 \end{bmatrix}$$

The total requirements of destinations D_1, D_2 and D_3 are given by 45, 45, 95 respectively and the availability of the products P_1, P_2, P_3 and P_4 are respectively 25, 35, 55 and 70.

(2012 : 20 Marks)

Solution:

The above problem can be expressed as

	D_1	D_2	D_3	Supply
P_1	$\boxed{10}$	$\boxed{7}$	$\boxed{0}$ 25	(25) (7) (3)
P_2	$\boxed{0}$	$\boxed{3}$ 20	$\boxed{11}$ 15	(35) (3) (3)
P_3	$\boxed{15}$	$\boxed{6}$	$\boxed{9}$ 55	(55) (3) (9)
P_4	$\boxed{5}$ 45	$\boxed{15}$ 25	$\boxed{13}$	(70) (8) (10)
Demand	(45) (5)	(45) (3)	(95) (9)	

$\therefore$ By Vogel's method, an initial basic feasible solution of the given problem is given by the above table.

$$\text{Total transportation cost} = 3 \times 20 + 11 \times 15 + 9 \times 55 + 5 \times 45 + 15 \times 25$$
$$= 60 + 165 + 495 + 225 + 375 = \text{Rs. } 1320$$

Explanation of Vogel's Method :

1. The given values C_{ij} (given cost) are written in the upper left corner.
2. Find the difference between the smallest and next to the smallest C_{ij}'s across rows and columns and write at the bottom and extreme right of the boxes.
3. Select the row/column having the largest difference.
4. Allocate the maximum possible to the smallest cost and write in the middle.
5. Cross-out the columns/rows whose cost/demand gets completely filled.

1.2 How many basic solutions are there in following linearly independent set of equations? Find all of them.

$$2x_1 - x_2 + 3x_3 + x_4 = 6$$
$$4x_1 - 2x_2 - x_3 + 2x_4 = 10$$

(2018 : 15 Marks)

Solution:

Form the table from given set of equations :

Basic Variables	Non - Basic Variables	Solution	Is it possible?
x_1, x_2	$x_3 = x_4 = 0$	Inconsistent	No
x_1, x_3	$x_2 = x_1 = 0$	$x_1 = 2.57$ $x_3 = 0.236$	Yes
x_1, x_4	$x_2 = x_3 = 0$	Inconsistent equation	No
x_2, x_3	$x_1 = x_4 = 0$	$x_2 = -5.14$ $x_3 = 0.286$	Yes
x_2, x_4	$x_1 = x_3 = 0$	Inconsistent equation	No
x_3, x_4	$x_1 = x_2 = 0$	$x_3 = 0.286$ $x_1 = 5.14$	Yes

∴ There are 3 basic solutions.

(i) $x_1 = 2.57, x_2 = 0, x_3 = 0.286, x_4 = 0$

(ii) $x_1 = 0, x_2 = -5.14, x_3 = 0.286, x_4 = 0$

(iii) $x_1 = 0, x_2 = 0, x_3 = 0.286, x_4 = 5.14$

2. Optimal Solutions

2.1 A paint factory produces both interior and exterior paint from two raw materials M_1 and M_2. The basic data is as follows :

	Tons of raw material per ton of		
	Exterior Paint	Interior Paint	Max. Daily Availability
Raw Material, M_1	6	4	24
Raw Material, M_2	1	2	6
Profit per ton (Rs. 1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The maximum daily demand of interior paint is 2 tons. The factory wants to determine the optimum product mix of interior and exterior paint that maximizes daily profits. Formulate the LP problem for this situation.

(2009 : 12 Marks)

Solution:

Let the required exterior paint be x_1 tons and interior paint be x_2 tons.

Now as per the given problem table,

$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$

Also given that

$$x_2 - x_1 \le 1$$

and $$x_2 \le 2$$

Also, we need to determine the optimum product mix of both exterior and interior paint that maximizes the problem. So, the LPP will be :

Maximize : $z = 5000x_1 + 4000x_2$

Subject to :
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$x_2 - x_1 \le 1$$
$$x_2 \le 2$$
where $x_1, x_2 \ge 0$

2.2 Consider the following linear programming problem :

$$\text{Maximize, } z = x_1 + 2x_2 - 3x_3 + 4x_4$$

subject to
$$x_1 + 2x_2 + 2x_3 + 3x_4 = 12$$
$$x_2 + 2x_3 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \ge 0$$

(i) Using the definition, find all its basic solutions. Which of these are degenerate basic feasible solutions and which are non-degenerate basic feasible solutions.

(ii) Without solving the problem, show that it has an optimal solution. Which of the basic feasible solution(s) is/are optimal.

(2015 : 20 Marks)

Solution:

(i) For given set of equation we make the following table :

S.No.	Basic Variables	Non-basic Variables	Solution	Is solution degenerate	....
1.	x_1, x_2	$x_3 = x_4 = 0$	$x_1 = 4$ $x_2 = 8$	No	
2.	x_1, x_3	$x_2 = x_4 = 0$	$x_1 = 4$ $x_3 = 4$	No	.....
3.	x_1, x_4	$x_2 = x_3 = 0$	$x_1 = -12$ $x_4 = 8$	Yes	.....
4.	x_2, x_3	$x_1 = x_1 = 0$	No solution	–	–
5.	x_2, x_4	$x_1 = x_3 = 0$	$x_2 = 6$ $x_4 = 2$	No	.....
6.	x_3, x_4	$x_1 = x_2 = 0$	$x_3 = 3$ $x_4 = 2$	No	.....

Table gives all basic solutions $(x_1 = -12, x_4 = 8)$ is a degenerate solution.

Non-degenerate solutions are :
$$(x_1, x_2) \equiv (4, 8)$$
$$(x_1, x_3) \equiv (4, 4)$$
$$(x_2, x_4) \equiv (6, 2)$$
$$(x_3, x_4) \equiv (3, 2)$$

(ii) z is optimal (from table - 1)

at
$$(x_1, x_2) \equiv (4, 8)$$
$$(x_2, x_4) \equiv (6, 2)$$

and value of z is 20.

2.3 An agricultural firm has 180 tons of nitrogen fertilizers, 250 tons of phosphate and 220 tons of potash. It will be able to sell a mixture of these substances in their respective ratio 3 : 3 : 1 at a profit of Rs. 1500 per ton and a mixture in the ratio 2 : 4 : 2 at a profit of Rs. 1200 per ton. Pose a linear programming problem to show how many tons of these two mixture should be prepared to obtain the maximum profit.

(2018 : 10 Marks)

Solution:

Let x and y tons of mixture is brought.

$\therefore$ Quantity of nitrogen is $\frac{3x}{10}+\frac{2y}{8}$.

Quantity of phosphate is $\frac{3x}{10}+\frac{4y}{8}$

Quantity of potash is $\frac{4x}{10}+\frac{2y}{8}$

Also, since quantity of fertilizers is limited.

$\therefore$ Respective equations can be written as

$$\text{Nitrogen}: \quad \frac{3x}{10}+\frac{2y}{8} \le 180 \Rightarrow \frac{3x}{10}+\frac{y}{4} \le 180$$

$$\text{Phosphate}: \quad \frac{3x}{10}+\frac{47}{8} \le 250 \Rightarrow \frac{3x}{10}+\frac{y}{2} \le 250$$

$$\text{Potash}: \quad \frac{4x}{10}+\frac{2y}{8} \le 220 \Rightarrow \frac{2x}{5}+\frac{y}{4} \le 220$$

The profit, $z = 1500x + 1200y$

$\therefore$ The linear programming problem is

$$\text{Max. } z = 1500x + 1200y,\ x,\ y \ge 0$$

subject to

$$\frac{3x}{10}+\frac{y}{4} \le 180$$

$$\frac{3x}{10}+\frac{y}{2} \le 250$$

$$\frac{2x}{5}+\frac{y}{4} \le 220$$

3. Graphical Method of Solution

3.1 Write down the dual of the following LP problem and hence solve it by graphical method :

Minimize, $Z = 6x_1 + 4x_2$

Constraints,

$$2x_1 + x_2 \ge 1$$
$$3x_1 + 4x_2 \ge 1.5$$
$$x_1, x_2 \ge 0$$

(2011 : 20 Marks)

Solution:

The given linear programming problem is

Minimize, $z = 6x_1 + 4x_2$

Constraints,

$$2x_1 + x_2 \ge 1$$
$$3x_1 + 4x_2 \ge 1.5$$
$$x_1, x_2 \ge 0$$

The given L.P.P. is in the standard primal form (all the constraints involve the sign ≥ if it is a problem of minimization).

In the matrix form, the above problem can be written as

$$\text{Min. } Z = (6, 4)[x_1, x_2] = CX$$

$$\text{S.T.} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$$

or $$AX \geq B$$

∴ **The dual of the given problem is**

$$\text{Max. } Z_D = B'W = (1, 1.5)[W_1, W_2]$$
$$= W_1 + 1.5W_2$$
$$\text{S.T.} = A'W \leq C'$$

i.e., $$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

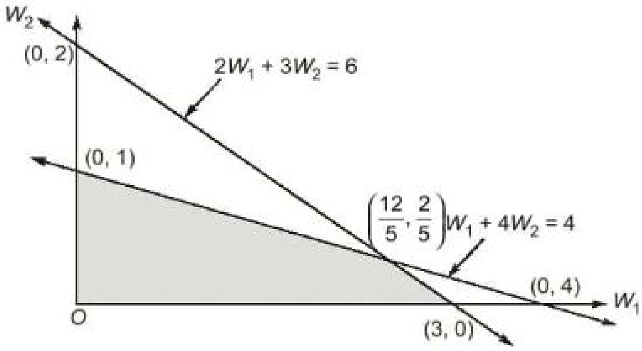
⇒ $$2W_1 + 3W_2 \leq 6$$
$$W_1 + 4W_2 \leq 4$$
$$W_1, W_2 \geq 0$$

Solution by Graphica Method :

Draw the lines

$$2W_1 + 3W_2 = 6$$

and $W_1 + 4W_2 = 4$ on the graph paper.



The shaded region is the permissible region. We can see that maximum Z_D is obtained at $\left(\frac{12}{5}, \frac{2}{5}\right)$ and (3, 0).

∴ Z_D is maximum at all points of the line joining $\left(\frac{12}{5}, \frac{2}{5}\right)$ and (3, 0).

3.2 For each hour per day that Ashok studies mathematics, it yields him 10 marks and for each hour that he studies physics, it yields him 5 marks. He can study at most 14 hours a day and he must get at least 40 marks in each. Determine graphically how many hours a day he should study mathematics and physics each, in order to maximize his marks?

(2012 : 12 Marks)

Solution:

Let Ashok studies Maths x hours per day and Physics y hours per day.

∴ $$\text{Total marks} = 10x + 5y$$

He can study at the most 14 hours a day.

∴ $$x + y \leq 14$$

He must get at least 40 marks in each paper.

$\therefore \quad 10x \geq 40$ and $5y \geq 40$

i.e., $\quad x \geq 4$ and $y \geq 8$

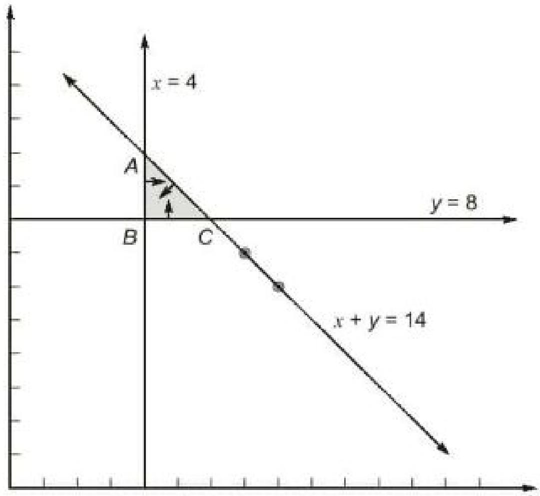
Hence, the linear programming problem is :

$$\text{Max. } z = 10x + 5y$$

Subject to the constraints

$$x + y \leq 14$$
$$x \geq 4$$
$$y \geq 8$$

Draw the lines, $x + y = 14$, $x = 4$, $y = 8$.



The shaded region ABC is the permissible region.

Here, $\quad A = (4, 10)$, $B = (4, 8)$, $C = (6, 8)$

$$Z \text{ at } A(4, 10) = 40 + 50 = 90$$
$$Z \text{ at } B(4, 8) = 40 + 40 = 80$$
$$Z \text{ at } C(6, 8) = 60 + 40 = 100$$

$\therefore$ For maximum Z, $\quad x = 6$, $y = 8$

3.3 Maximize $z = 2x_1 + 3x_2 - 5x_3$ subject to $x_1 + x_2 + x_3 = 7$ and $2x_1 - 5x_2 + x_3 \geq 10$; $x_i \geq 0$.

(2013 : 10 Marks)

Solution:

Approach : A graphical solution is possible only for two variables. We use the first condition of equality to convert it into a problem of 2 variables.

$$x_1 + x_2 + x_3 = 7 \quad \Rightarrow \quad x_3 = 7 - x_1 - x_2$$

Again $x_3 \geq 0 \;\Rightarrow \quad 7 - x_1 - x_2 \geq 0$

$\Rightarrow \quad x_1 + x_2 \leq 7$

So, the problem becomes

$$\text{Max. } z = 2x_1 + 3x_2 - 5(7 - x_1 - x_2) = 7x_1 + 8x_2 - 35$$

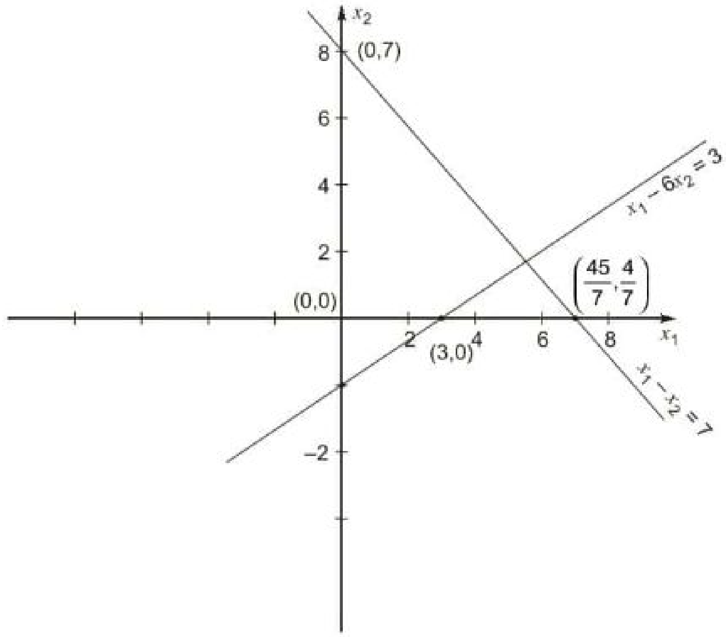
subject to $\quad x_1 + x_2 \leq 7$

$$2x_1 - 5x_2 + (7 - x_1 - x_2) \geq 10$$

or $\quad x_1 - 6x_2 \geq 3$

and $\quad x_1, x_2 \geq 0$

We use the graphical method to solve the given problem.



Maxima will occur at a corner point value of z at various corner points :

Point	$Z = 7x_1 + 8x_2 - 35$
(0, 7)	21
(0, 0)	−35
(3, 0)	−14
$\left(\frac{45}{7}, \frac{4}{7}\right)$	$14\frac{4}{7}$

∴ Maxima is at (0, 7) and maximum value is 21.

∴ Maxima is 21 at $x_1 = 0, x_2 = 7, x_3 = 0$.

3.4 Solve graphically :

Maximize : $$Z = 6x_1 + 5x_2$$

subject to :
$$2x_1 + x_2 \le 16$$
$$x_1 + x_2 \le 11$$
$$x_1 + 2x_2 \ge 6$$
$$5x_1 + 6x_2 \le 90$$
$$x_1, x_2 \ge 0$$

(2014 : 10 Marks)

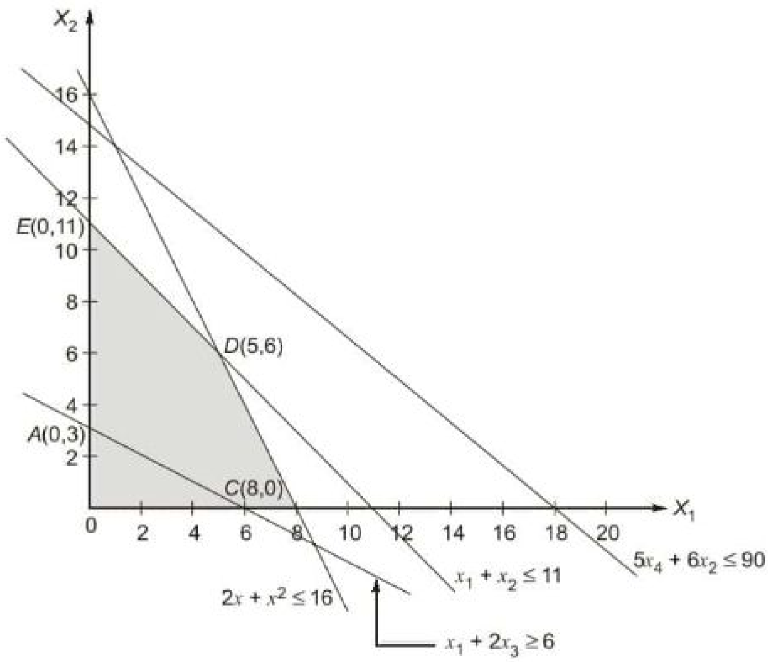
Solution:

Since every point which satisfies the condition $x_1 \ge 0, x_2 \ge 0$ lies in the first quadrant only.

∴ The desired pair (x_1, x_2) is restricted to the points of the first quadrant only.

The following are the end points of the straight line in 1st quadrant.

Let
$$2x_1 + x_2 = 16 \Rightarrow (8, 0)(0, 16)$$
$$x_1 + x_2 = 11 \Rightarrow (11, 0)(0, 11)$$
$$x_1 + 2x_2 = 6 \Rightarrow (6, 0)(0, 3)$$
$$5x_1 + 6x_2 = 90 \Rightarrow (18, 0)(0, 15)$$



The shaded region $ABCDE$ is the feasible region corresponding to the given constraints with $A(0, 3)$, $B(6, 0)$, $C(8, 0)$, $D(5, 6)$, $E(0, 11)$ as the extreme points.

The values of the objective function $z = 6x_1 + 5x_2$ at these extreme points are :

$$z(0, 3) = 0 + 15 = 15$$
$$z(6, 0) = 36 + 0 = 36$$
$$z(8, 0) = 48 + 0 = 48$$
$$z(5, 6) = 30 + 30 = 60 \rightarrow$$
$$z(0, 11) = 0 + 55 = 55$$

The value of z is maximum at $D(5, 6)$.

$\therefore$ The maximum value if $z = 60$.

$\therefore$ $$x_1 = 5, x_2 = 6$$

3.5 Find the maximum value of

$$5x + 2y \quad \text{with constraints}$$
$$x + 2y \ge 1, \quad 2x + y \le 1, \quad x \ge 0, y \ge 0$$

by graphcial method.

(2016 : 10 Marks)

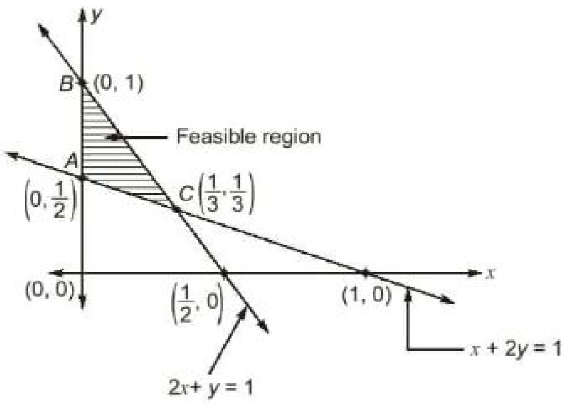
Solution:

Given objective function,

$$\max(z) = 5x + 2y$$

subject to
$$x + 2y \ge 1$$
$$2x + y \le 1, x \ge 0, y \ge 0$$

We use corner-point method.



The feasible region is drawn in figure. The corner of points are

$$A \equiv \left(0,\frac{1}{2}\right)$$

$$B \equiv (0, 1)$$

$$C \equiv \left(\frac{1}{3},\frac{1}{3}\right)$$

$$\text{Value of } z \text{ at } A = 1$$

$$\text{Value of } z \text{ at } B = 2$$

$$\text{Value of } z \text{ at } C = \frac{7}{3}$$

So, maximum value of z is at $C\left(\frac{1}{3},\frac{1}{3}\right)$ is $\frac{7}{3}$.

3.6 Using graphica method, find the maximum value of $2x + y$ subject to

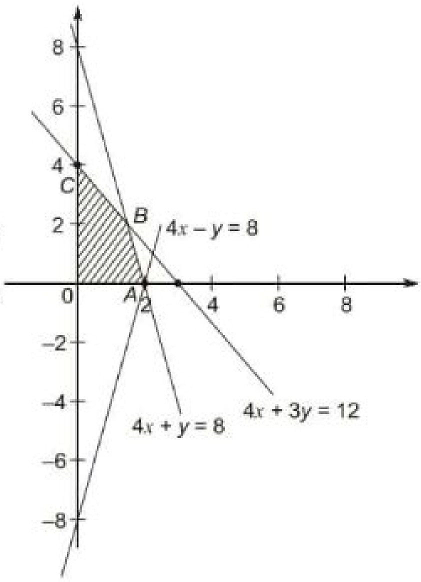
$$4x + 3y \leq 12$$
$$4x + y \leq 8$$
$$4x - y \leq 8,\ x,\ y \geq 0$$

(2017 : 10 Marks)

Solution:

As $x \geq 0$, $y \geq 0$, feasible solutions are restricted in the first quadrant.

Let
$$z = 2x + y$$
$$4x + 3y = 12 \Rightarrow (3, 0), (0, 4)$$
$$4x + y = 8 \Rightarrow (2, 0), (0, 8)$$
$$4x - y = 8 \Rightarrow (2, 0), (0, -8)$$



The shaded region $OABC$ is feasible region, corresponding to the given constraints, with $O(0, 0)$, $A(2, 0)$, $B\left(\frac{3}{2},2\right)$, $C(0, 4)$ as the extreme points.

The value of the objective function at these extreme points are

At $O(0, 0)$, $z = 2(0) + 0 = 0$

At $A(2, 0)$, $z = 4$

At $B\left(\frac{3}{2},2\right)$, $z = 2\cdot\frac{3}{2}+2 = 5$

At $C(0, 4)$, $z = 4$

$\therefore$ z is maximum at $B\left(\frac{3}{2},2\right)$ with value 5.

3.7 Use graphical method to solve the linear programming problem.

Maximize $Z = 3x_1 + 2x_2$

subject to
$$x_1 - x_2 \geq 1,$$
$$x_1 + x_3 \geq 3$$
and $x_1, x_2, x_3 \geq 0$

(2019 : 10 Marks)

Solution:

Given linear programming problem

$$\text{Max. } z = 3x_2 + 2x_2$$

subject to
$$x_1 - x_2 \geq 0 \quad \text{...(1)}$$
$$x_1 + x_3 \geq 3 \quad \text{...(2)}$$
and $x_1, x_2, x_3 \geq 0$

Corresponding to (1), we have
$$x_1 - x_2 = 1$$
If $x_1 = 0; x_2 = 1$

$\therefore$ Point $(x_1, x_2, x_3) = (0, -1, 0)$

Similarly, corresponding to (2), we have
$$x_1 + x_3 = 3$$
If $x_1 = 0, x_3 = 3$

$\therefore$ Point $(x_1, x_2, x_3) = (0, 0, 3)$

If $x_3 = 0; x_1 = 3$

$\therefore$ Point $(x_1, x_2, x_3) = (3, 0, 0)$

Since, the graphical representation shows the intersection of the both planes.

Since, the common region/surface is bounded below.

$\therefore$ There does not exist any optional solution for z_m. Hence, the solution is unbounded.

4. Simplex Method of Solution

4.1 Maximize : $z = 3x_1 + 5x_2 + 4x_3$

Subject to :
$$2x_1 + 3x_2 \leq 8$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$2x_2 + 5x_3 \leq 10$$
$$x_i \geq 0$$

(2009 : 30 Marks)

Solution:

Step 1 : Checking whether objective function is to be maximized and all *b*'s are non-negative.

$\Rightarrow$ These conditions are satisfied. So, proceed to next step.

Step 2 : Express the above problem in standard form. By introducing the slack variables S_1, S_2 and S_3, the problem in standard form becomes :

$$\text{Max. } z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to
$$2x_1 + 3x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$3x_1 + 2x_2 + 4x_3 + 0s_1 + s_2 + 0s_3 = 15$$
$$0x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + s_3 = 10$$

where $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Step 3 : Find initial basic feasible solution. The basic (non-degenerate) feasible solution is
$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}$$
$$s_1 = 8, s_2 = 15, s_3 = 10 \text{ (basic)}$$

$\therefore$ Initial basic feasible solution is given by the table below :

	C_j	3	5	4	0	0	0			
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ	
0	s_1	2	(3)	0	1	0	0	8	$\frac{8}{3}$	$\leftarrow$
0	s_2	3	2	4	0	1	0	15	$\frac{15}{2}$	
0	s_3	0	2	5	0	0	1	10	$\frac{10}{2}$	
$z = \Sigma(Ba_{ij})$		0	0	0	0	0	0			
$C_j = C_j - Z_j$		3	5	4	0	0	0			
			$\uparrow$							

As C_js are positive under the columns. But we choose the largest positive, i.e., '5' under column of 'x_2'. Now $\theta = \frac{8}{3}$ is smallest. So, x_2 is the incoming variable and s_1 is the outgoing variable and (3) is the key element. Again producing :

	C_j	3	5	4	0	0	0			
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ	
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	$\frac{8}{3(0)} = \infty$	
0	s_2	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	1	0	$\frac{29}{3}$	$\frac{29}{3\times4} = \frac{29}{12}$	
0	s_3	$-\frac{4}{3}$	0	(5)	$-\frac{2}{3}$	0	1	$\frac{14}{3}$	$\frac{14}{3\times5} = \frac{14}{15}$	$\leftarrow$
$z = \Sigma(Ba_{ij})$		$\frac{10}{3}$	5	0	$\frac{5}{3}$	0	0			
$C_j = C_j - Z_j$		$-\frac{1}{3}$	0	4	$-\frac{5}{3}$	0	0			
				$\uparrow$						

Again largest positive $C_j = 4$ under column of 'x_3' and now $\theta = \frac{14}{15}$ is smallest. So, x_3 is the incoming variable and s_3 is the outgoing variable and (5) is the key element. Again proceeding :

	C_j	3	5	4	0	0	0			
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ	
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	$\frac{8\times3}{3\times2} = 4$	
0	s_2	$\left(\frac{41}{15}\right)$	0	0	$-\frac{2}{15}$	1	$-\frac{4}{15}$	$\frac{89}{15}$	$\frac{89\times15}{15\times41} = \frac{89}{41}$	$\leftarrow$
4	x_3	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	0	$\frac{1}{5}$	$\frac{14}{15}$	$\frac{14\times15}{15\times(-4)}$	
$z = \Sigma C_B a_{ij}$		$\frac{34}{15}$	5	4	$\frac{17}{15}$	0	$\frac{4}{5}$			
$C_j = C_j - Z_j$		$\frac{11}{15}$	0	0	$-\frac{17}{15}$	0	$-\frac{4}{5}$			
		$\uparrow$								

Again $\frac{11}{15}$ is the only positive C_j under column of 'x_1' and now $\theta = \frac{89}{41}$ is the smallest positive fraction. So, x_1 is the incoming variable and s_2 is the outgoing variable and $\left(\frac{41}{15}\right)$ is the key element.

Again proceeding :

	C_j	3	5	4	0	0	0		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
5	x_2	0	1	0	$\frac{15}{41}$	$-\frac{10}{41}$	$\frac{8}{41}$	$\frac{50}{41}$	
3	x_1	1	0	0	$-\frac{2}{41}$	$\frac{15}{41}$	$-\frac{12}{41}$	$\frac{89}{41}$	
4	x_3	0	0	1	$-\frac{6}{41}$	$\frac{4}{41}$	$\frac{5}{41}$	$\frac{62}{41}$	
$z = \Sigma C_B a_{ij}$		3	5	4	$\frac{45}{41}$	$\frac{11}{41}$	$\frac{24}{41}$		
$C_j = C_j - Z_j$		0	0	0	$-\frac{45}{41}$	$-\frac{11}{41}$	$-\frac{24}{41}$		

As all C_js are negative. So, the optimal solution is achieved.

Optimal solution is :

$$x_1 = \frac{89}{41}$$

$$x_2 = \frac{50}{41}$$

$$x_3 = \frac{62}{41}$$

and

$$\text{Max. } z = 3\times\frac{89}{41}+5\times\frac{50}{41}+4\times\frac{62}{41}$$

$$= \frac{765}{41}$$

or

$$\text{Max. } z = \frac{765}{41}$$

4.2 Solve by Simplex method, the following LP Problem :

Maximize, $Z = 5x_1 + 3x_2$

Constraints,

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

(2011 : 12 Marks)

Solution:

The given LPP is

Maximize, $z = 5x_1 + 3x_2$

Constraints,

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

Convert the inequalities by the introduction of slack variables x_3 and x_4 as follows :

$$3x_1 + 5x_2 + x_3 = 15$$
$$5x_1 + 2x_2 + x_4 = 10$$

Taking $x_1 = 0, x_2 = 0$, we get, $x_3 = 15, x_4 = 10$ which is the starting basic feasible solution.
Construct the starting simplex table as follows :

		C_j	5	3	0	0	Minimum Ratio $\frac{X_B}{Y_1}$	
B	C_B	X_B	Y_1	Y_2	Y_3	Y_4		
1/3	0	15	3	5	1	0	$\frac{15}{3} = 5$	
1/4	0	10	$\boxed{5}$	2	0	1	$\frac{10}{5} = 2$ (Min)	←
		Δ_j						
			↑ Incoming Vector					

$$\Delta_j = C_j - C_B Y_j$$

$\therefore$
$$\begin{aligned}\Delta_i &= C_1 - C_B Y_1 \\ &= 5 - [(0, 0)(3, 5)] = 5 - (0 \times 3 + 0 \times 5) \\ &= 5 - 0 = 5 \\ \Delta_2 &= C_2 - C_B Y_2 \\ &= 3 - [(0, 0)(5, 2)] = 3\end{aligned}$$

The values of Δ_3 and Δ_4 must be zero as they correspond to unit column vectors.
Since all Δ_j are not less than or equal to zero, therefore the solution is not optimal.
Since $\Delta_1 = 5$ is maximum, $\therefore$ Y_1 is the incoming vector.

Since, $\frac{X_B}{1/1}$ ratio is minimum against Y_4 in the table, therefore Y_4 is the outgoing vector.

The element at the intersection of the minimum ratio arrow and incoming vector arrow is the pivot element.

We mark this element in $\square$.

Divide the second row containing the key element (= 5) by 5 to get unity at this position and add 'minus 3 times' the second row to first row to zero at all other positions of the column containing the pivot element.

		C_j	5	3	0	0	Minimum Ratio $\frac{X_B}{Y_2}$	
B	C_B	X_B	Y_1	Y_2	Y_3	Y_4		
Y_3	0	9	0	$\boxed{\frac{19}{5}}$	1	$\frac{-3}{5}$	$\frac{45}{19}$(min)	←
Y_1	5	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	5	
		Δ_j	0	1	0	–1		
				↑ IV	↓ OV			

Using
$$\begin{aligned}\Delta_j &= C_j - C_B Yj \\ \Delta_1 &= C_1 - C_B Y_1 \\ &= 5 - [(0, 5)(0, 1)] = 5 - 5 = 0\end{aligned}$$

$$\Delta_2 = C_2 - C_B Y_2$$
$$= 3 - \left[(0,5)\left(\frac{19}{5}, \frac{2}{5}\right)\right] = 3 - 2 = 1$$
$$\Delta_3 = 0$$
$$\Delta_4 = C_4 - C_B Y_4$$
$$= 0 - \left[(0,5)\left(-\frac{3}{5}, \frac{1}{5}\right)\right] = 0 - 1 = -1$$

Since all Δ_j are not less than or equal to zero, therefore the solution is not optimal.

Third Simplex Table can be drawn similarly and this process has to be repeated until all Δ_j's become negative to arrive at the optimal solution.

4.3 Minimize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$ subject to the constraints

$$x_1 + 2x_2 - 2x_3 + 4x_4 \le 40$$
$$2x_1 - x_2 + x_3 + 2x_4 \le 8$$
$$4x_1 - 2x_2 + x_3 - x_4 \le 10$$
$$x_i \ge 0$$

(2013 : 20 Marks)

Solution:

We convert it into a maximisation problem with slack variables.

$$\text{Max. } z^* = -5x_1 + 4x_2 - 6x_3 + 8x_4 + 0x_5 + 0x_6 + 0x_7$$

subject to
$$x_1 + 2x_2 - 2x_3 + 4x_4 + x_5 = 40$$
$$2x_1 - x_2 + x_3 + 2x_4 + x_6 = 8$$
$$4x_1 - 2x_2 + x_3 - x_4 + x_7 = 10$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0$$

For initial B.F.S. we have $7 - 4 = 3$

We choose $x_5 = 40$; $x_6 = 8$; $x_7 = 10$

		C_j	–5	4	–6	8	0	0	0	
Y	C_Y	x_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	x_B/Y_4
Y_5	0	40	1	2	–2	4	1	0	0	40/4 = 10
Y_6	0	8	2	–1	1	$\boxed{2}$	0	1	0	4 (min) →
Y_7	0	10	4	–2	1	–1	0	0	1	–
$Z^* = 0$		Δ_j	–5	4	–6	8↑	0	0	0	x_B/Y_2
Y_5	0	24	–3	$\boxed{4}$	–4	0	1	–2	0	6 (min) →
Y_4	8	4	1	–1/2	1/2	1	0	1/2	0	–
Y_7	0	14	5	–5/2	3/2	0	0	1/2	1	–
$Z^* = 32$		Δ_j	–13	8↑	–10	0	0	–4	0	
Y_2	4	6	–3/4	1	–1	0	1/4	–1/2	0	
Y_4	8	7	5/8	0	0	1	1/8	1/4	0	
Y_7	0	29	25/8	0	–1	0	5/8	–3/4	1	
$Z^* = 80$			–7	0	–2	0	–2	0	0	

So, this is the optimum solution.

Optimum value of $Z^* = 80$

$\therefore$ Optimal value of $Z = -Z^* = -80$

at $x_1 = 0$, $x_2 = 6$, $x_3 = 0$, $x_4 = 7$

4.4 Find all optimal solutions of the following linear programming problem by the simplex method :

Maximize $$Z = 30x_1 + 24x_2$$

Subject to
$$5x_1 + 4x_2 \le 200$$
$$x_1 \le 32$$
$$x_2 \le 40$$
$$x_1, x_2 \ge 0$$

(2014 : 20 Marks)

Solution:

The objective function of the given LPP is of maximization type and RHS of all constraints are ≥ 0.
Now we write the given LPP in the standard form.

$$\text{Max. } z = 30x_1 + 24x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to
$$5x_1 + 4x_2 + s_1 = 200$$
$$x_1 + s_2 = 32$$
$$x_2 + s_3 = 40$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

where s_1, s_2, s_3 are slack variables.

Now the initial basic feasible solution is given by setting $x_1 = x_2 = 0$ (non-basic)
$$s_1 = 200, s_2 = 32, s_3 = 40$$

$\therefore$ The initial basic feasible solution is (0, 0, 200, 32, 40) for which $z = 0$.

Now, we move from the current basic feasible solution to the next better basic feasible solution. Pull the above information in the table form :

	C_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	5	4	1	0	0	200	$\frac{200}{5} = 40$
0	S_2	(1)	0	0	1	0	32	$\frac{32}{1} = 32 \rightarrow$
0	S_3	0	1	0	0	1	40	–
$Z_j = \Sigma a j C_B$		0	0	0	0	1	0	
$C_j = C_j - Z_j$		30	24	0	0	0		
		$\uparrow$						

From the above table, x_1 is the incoming variable as $C_j = 30$ is maximum and the corresponding column is known as key column.

The minimum +ve ratio θ occurs in the second row.

$\therefore$ s_2 is the outgoing variable and the common intersection element (Q) is the key element and convert all other element in its column to zero.

Then, the new iterated simplex table is :

	C_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	0	④	1	–5	0	40	$\frac{40}{4} = 10 \rightarrow$
30	x_1	1	0	0	1	0	32	$\frac{32}{30}$
0	S_3	0	1	0	0	1	40	$\frac{40}{1}$
$Z_j = \Sigma a j C_B$		30	0	0	30	0	960	
$C_j = C_j - Z_j$		0	24	0	–30	0		
			$\uparrow$					

From the above table, x_2 is the increasing variable, s_1 is the outgoing variable and (G) is the key element.
Now convert the key element to unity and all other elements in its column to zero.
Then, we get the new iterated simples table as :

	C_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
24	x_1	0	1	1/4	–5/4	0	10	–
30	x_4	1	0	0	1	0	32	32
0	S_3	0	0	–1/4	5/4	1	30	$\frac{30 \times 5}{5} = 30 \rightarrow$
$Z_j = \Sigma a_j C_B$		30	24	6	0	0	1200	
$C_j = C_j - Z_j$		0	0	–6	0	0		
					$\uparrow$			

Since all $C_j \le 0$, an optimal solution has been reached.

$\therefore$ The optimum basic feasible solution is $x_2 = 10, x_1 = 32$ and $z_{max} = 1200$.

4.5 Maximize $$z = 2x_1 + 3x_2 + 6x_3$$

subject to
$$2x_1 + x_2 + x_3 \le 5$$
$$3x_2 + 2x_3 \le 6$$
$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

Is the optimal solution unique? Justify your answer.

(2016 : 20 Marks)

Solution:

Given, objective function is

$$\max. (z) = 2x_1 + 3x_2 + 6x_3, \qquad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$

Writing the equations in standard form, we get
$$2x_1 + x_2 + x_3 + S_1 = 5$$
$$3x_2 + 2x_3 + S_2 = 6$$

where S_1 and S_2 are slack variables.

$$\max. (z) = 2x_1 + 3x_2 + 6x_3 + 0 \cdot S_1 + 0 \cdot S_2$$

We use simplex method for finding solution.

Initial table is :

C_B	C_j Basis	2 x_1	3 x_2	6 x_3	0 S_1	0 S_2	b	θ	
0	S_1	2	1	1	1	0	5	5	
0	S_2	0	3	2	0	1	6	$\frac{6}{2}=3$	→
$z_j=\Sigma C_B a_{ij}$	0	0	0	0	0				
$C_j=C_j-Z_j$	2	3	6	0	0				
				↑					

$\therefore$ S_2 is outgoing variable and x_3 is incoming variable. The table becomes

C_B	C_j Basis	2 x_1	3 x_2	6 x_3	0 S_1	0 S_2	b	θ	
0	S_1	2	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	2	1	→
6	x_3	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	3	–	
$z_j=\Sigma C_B a_{ij}$	0	9	6	0	3				
$C_j=C_j-Z_j$	2	−6	0	0	−3				
	↑								

$\therefore$ S_1 is outgoing variable and x_1 is incoming variable.
The table becomes

C_B	C_j Basis	2 x_1	3 x_2	6 x_3	0 S_1	0 S_2	b	θ
2	x_1	1	$-\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	1	
6	x_3	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	3	
$z_j=\Sigma C_B a_{ij}$	2	$\frac{17}{2}$	6	1	$\frac{5}{2}$			
$C_j=C_j-Z_j$	0	$-\frac{11}{2}$	0	−1	$-\frac{5}{2}$			

as $C_j = 0$ for only basis variables.
$\therefore$ Optimal solution is unique.
So,

$$x_1 = 1$$
$$x_3 = 3$$
$$x_2 = 0$$

and maximum value of $z = 2 \times 1 + 3 \times 0 + 6 \times 3$
$$= 2 + 0 + 18 = 20$$

4.6 *Solve the following LPP by simplex method :*

Maximize : $z = 3x_1 + 5x_2 + 4x_3$

subject to,

$$2x_1 + 3x_2 \le 8$$
$$2x_1 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0$$

(2017 : 20 Marks)

Solution:
Adding slack variables x_4, x_5, x_6, we convert the LPP in standard form.

$$\text{Max. } z = 3x_1 + 5x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$$

Subject to

$$2x_1 + 3x_2 + x_4 = 8$$
$$2x_1 + 5x_3 + x_5 = 10$$
$$3x_1 + 2x_2 + 4x_3 + x_6 = 15, x_i \ge 0$$

We make the simplex tables :

Basis	C_b	b	x_1	x_2	x_3	x_4	x_5	x_6	$\frac{b}{a}$	
x_4	0	8	2	3	0	1	0	0	$\frac{8}{3}$	→
x_5	0	10	2	0	5	0	1	0	5	
x_6	0	15	3	2	4	0	0	1	$\frac{15}{2}$	
z		0	−3	−5	−4	0	0	0	0	
				↑						

For the next iteration, we enter a variable into the basis for which the value of $\frac{b}{a}$ is minimum. And the values of 'a' are corresponding to that column for which value of z is least.
Here, x_4 is the outgoing variable and x_2 is entering variable. So, we get x_2 as pivot variable.

Basis	C_b	b	x_1	x_2	x_3	x_4	x_5	x_6	$\frac{b}{a}$	
x_2	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	–	
x_5	0	$\frac{14}{3}$	$-\frac{4}{3}$	0	$\boxed{5}$	$-\frac{2}{3}$	1	0	$\frac{14}{15}$	→
x_6	0	$\frac{29}{3}$	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{29}{12}$	
z		$\frac{40}{3}$	$\frac{1}{3}$	0	−4	$\frac{5}{3}$	0	0		
					↑					

Again taking pivot variable as indicated

Basis	C_b	b	x_1	x_2	x_3	x_4	x_5	x_6	$\frac{b}{a}$	
x_2	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	4	
x_3	4	$\frac{14}{15}$	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	$-\frac{7}{2}$	
x_6	0	$\frac{89}{15}$	$\boxed{\frac{41}{15}}$	0	0	$-\frac{2}{15}$	$-\frac{4}{15}$	1	$\frac{89}{41}$	→
z		$\frac{256}{15}$	$-\frac{11}{15}$	0	0	$\frac{17}{15}$	$\frac{4}{15}$	0		
			↑							

Again taking pivot variable as indicated :

Basis	C_b	b	x_1	x_2	x_3	x_4	x_5	x_6	$\frac{b}{a}$
x_2	5	$\frac{50}{41}$	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$-\frac{10}{41}$	–
x_3	4	$\frac{62}{41}$	0	0	1	$-\frac{6}{41}$	$\frac{5}{41}$	$\frac{4}{41}$	–
x_1	3	$\frac{89}{41}$	1	0	0	$-\frac{2}{41}$	$-\frac{12}{41}$	$\frac{15}{41}$	–
z		$\frac{765}{15}$	0	0	0	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	

This gives us the optimal solution as

$$z = \frac{765}{15} \text{ at } x_1 = \frac{89}{41}, x_2 = \frac{62}{41}, x_3 = \frac{62}{41}$$

4.7 Solve the following linear programming problem by Big M-method and show that the problem has finite optimal solutions. Also, find the value of objective function.

(2018 : 20 Marks)

$$\text{Min. } z = 3x_1 + 5x_2$$

subject to

$$x_1 + 2x_2 \geq 8$$
$$3x_1 + 2x_2 \geq 12$$
$$5x_1 + 6x_2 \leq 60$$
$$x_1, x_3 \geq 0$$

Solution:

The objective function of given LPP can be written as max.

$$(z') = \min. (-z) = -3x_1 - 5x_2$$

Now, given equations can be written in standard form as

$$x_1 + 2x_2 - S_1 + A_1 = 8$$
$$3x_1 + 2x_2 - S_2 + A_2 = 12$$
$$5x_1 + 6x_2 + S_3 = 60$$

where S_1, S_2 and S_3 are slack variables; A_1, A_2 are artificial variables.

$$\text{Max. } (z') = -3x_1 - 5x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$$

where M is a very large number.

The table can be written as :

	C_j	–3	–5	0	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	A_1	A_2	b	θ
$-M$	A_1	1	2	–1	0	0	1	0	8	8
$-M$	A_2	3	2	0	–1	0	0	1	12	4 →
0	S_3	5	6	0	0	1	0	0	60	12
$Z_j = \Sigma C_B a_{ij}$		$-4M$	$-4M$	M	M	0	$-M$	$-M$		
$C_j = C_j - Z_j$		$-3+4M$	$-5+4M$	$-M$	$-M$	0	0	0		
		↑								

$\therefore$ A_2 is outgoing variable and x_1 is incoming variable, the table now becomes

	C_j	–3	–5	0	0	0	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	A_1	b	θ
$-M$	A_1	0	$\frac{4}{3}$	–1	$\frac{1}{3}$	0	1	4	3 →
–3	x_1	1	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	0	4	6
0	S_3	0	$\frac{8}{3}$	0	$\frac{5}{3}$	1	0	40	15
$Z_j = \Sigma C_B a_{ij}$		–3	$\frac{-4M}{3}-2$	M	$\frac{-M}{3}+1$	0	$-M$		
$C_j = C_j - Z_j$		0	$\frac{4M}{3}-3$	$-M$	$\frac{M}{3}-1$	0	0		
			↑						

So, A_1 is outgoing variable and x_2 is incoming variable. The table becomes

	C_j	–3	–5	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
–5	x_2	0	1	$-\frac{3}{4}$	$\frac{1}{4}$	0	3	
–3	x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	
0	S_3	0	0	2	1	1	32	
$Z_j = \Sigma C_B a_{ij}$		–3	–5	$\frac{9}{4}$	$\frac{1}{4}$	0		
$C_j = C_j - Z_j$		0	0	$-\frac{9}{4}$	$-\frac{1}{4}$	0		

Since all C_js are ≤ 0. This solution is optimal and it is ......

$\therefore$
$$x_1 = 2$$
$$x_2 = 3$$

and
$$\max. (z') = -3 \times 2 - 5 \times 3$$
$$= -21$$

$\therefore$
$$\min. (z) = -z' = 21 = \text{Value of objective function.}$$

4.8 Solve the linear programming problem using Simplex method.

Minimize $$Z = x_1 + 2x_2 - 3x_3 - 2x_4$$

subject to
$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$

and
$$x_1, x_2, x_3, x_4 \geq 0$$

(2019 : 15 Marks)

Solution:

Minimize $$Z = x_1 + 2x_2 - 3x_3 - 2x_4$$

subject to
$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$

and
$$x_1, x_2, x_3, x_4 \geq 0$$

$\therefore$
$$\text{Max } Z = -x_1 - 2x_2 + 3x_3 + 2x_4$$

The problem is converted to canonical form by adding slack; surplus and artificial variables as appropriate.

1. As the constraint –1 is of type '=' we should add artificial variables A_2.
2. As the constraint –2 is of type '=' we should add artificial variable A_2.

After introducing artificial variables.

$$\text{Max. } Z = -x_1 - 2x_2 + 3x_3 + 2x_4 - MA_1 - MA_2$$

subject to
$$x_1 + 2x_2 - 3x_3 + x_4 + A_1 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 + A_2 = 4$$

and
$$x_1, A_j \geq 0$$

	C_i	–1	–2	3	2	–M	–M		
C_B	Basis	n_1	n_2	n_3	n_4	A_1	A_2	b	θ
$-M$	A_1	1	(2)	–3	1	1	0	4	$\frac{4}{2}=2$
$-N$	A_2	1	2	1	2	0	1	4	$\frac{4}{2}=2$
z_i	$\Sigma a_{ij}C_B$	$-2M$	$-4M$	$2M$	$-3M$	$-M$	$-M$	$-8M$	
C_j	$C_j - Z_j$	$-1+2M$	$4M-2$	$3-2M$	$3M+2$	0	0	$-8M$	

↑
$n_2 \rightarrow$ incoming variable (2) $\rightarrow$ key element
$A_1 \rightarrow$ outgoing variable $A_1 \rightarrow$ column can be omitted

	C_i	–1	–2	3	2	–M	–M		
C_B	Basis	n_1	n_2	n_3	n_4	A_1	A_2	b	θ
$-Q$	n_2	1/2	1	–3/2	1/2	1/2	0	2	(–ve)
$-M$	A_2	0	0	(4)	1	–1/2	1	0	0/4 = 0
z_j	$\Sigma a_{ij}C_B$	–1	–2	$3-4M$	$-1-M$	$-1+M/2$	$-M$	–4	
		0	0	$4M$	$M+3$	Omitted	0		

↑
$A_2 \rightarrow$ outgoing variables $A_2 \rightarrow$ Column omitted
$n_3 \rightarrow$ incoming variables 4 $\rightarrow$ Key elements

	C_i	–1	–2	3	2		
C_B	Basis	n_1	n_2	n_3	n_4	b	θ
–2	X_2	1/2	1	0	7/8	2	2/7/8= 16/7 = 2.28
3	X_3	0	0	1	(1/4)	0	0/44 = 0
z_j	$\Sigma a_{ij}C_B$	–1	–2	3	–1		
C_j	$C_j - z_j$	0	0	0	3		

↑
Positive maximum $C_j = 3$ of x_4
Hence $x_4 \rightarrow$ Incoming x_3 = Outgoing
C/4 $\rightarrow$ Key Element
$\Rightarrow R_2 = R_2 \times 4$
$\Rightarrow R_1 = R_1 - 7/8R_2$

	C_j	–1	–2	3	2	
C_B	Basis	n_1	n_2	n_3	n_4	b
–2		1/2	1	–7/2	0	2
2		0	0	4	1	0
z_j	Σa_{ij}	–1	–2	15	2	$z = 4$
C_j	$C_j - z_j$	0	0	–12	0	

$\therefore C_j \leq 0$

Hence, optimal system is assigned with value of variables as :

$$n_1 = 0, n_2 = 2, n_3 = 0, n_4 = 0$$
$$\text{Max. } Z = -n_1 - 2n_2 + 3n_3 + 2n_4$$
$$= 0 - 2 \times 2 + 0 + 0 = -4$$

$\therefore$ Min. $Z = 4$.

5. Duality

5.1 Construct the dual of the problem : Maximize $z = 2x_1 + x_2 + x_3$ subject to the constraints $x_1 + x_2 + x_3 \geq 6$, $3x_1 - 2x_2 + 3x_3 = 3$, $-4x_1 + 3x_2 - 6x_3 = 1$ and $x_1, x_2, x_3 \geq 0$.

(2010 : 12 Marks)

Solution:

The equation is Max. $z = 2x_1 + x_2 + x_3$ such that

$$x_1 + x_2 + x_3 \geq 6 \text{ or } -x_1 - x_2 - x_3 \leq -6 \quad ...(1)$$
$$+3x_1 - 2x_2 + 3x_3 = 3, \text{ which can be written as}$$
$$3x_1 - 2x_2 + 3x_3 \leq 3 \quad ...(2)$$

and
$$3x_1 - 2x_2 + 3x_3 \geq 3 \text{ or } -3x_1 + 2x_2 - 3x_3 \leq -3 \quad ...(3)$$

and
$$-4x_1 + 3x_2 - 6x_3 = 1 \text{ which can be written as}$$
$$-4x_1 + 3x_2 - 6x_3 \leq 1 \quad ...(4)$$

and
$$-4x_1 + 3x_2 - 6x_3 \geq 1 \text{ or } 4x_1 - 3x_2 + 6x_3 \leq -1 \quad ...(5)$$

Let v_1, v_2, v_3, v_4 and v_5 be dual variables.

$\therefore$ from (1), (2), (3), (4) and (5)

Dual of the given primal can be written as

$$\text{Min. } w = -6v_1 + 3v_2 - 3v_3 + v_4 - v_5$$

Subject to

$$-v_1 + 3v_2 - 3v_3 - 4v_4 + 4v_5 \geq 2$$
$$-v_1 - 2v_2 + 2v_3 + 3v_4 - 3v_5 \geq 1$$
$$-v_1 + 3v_2 - 3v_3 - 6v_4 + 6v_5 \geq 1$$

5.2 Solve the following linear programming problem by the simplex method. Write its dual. Also write the optimal solution of the dual from the optimal table of given problem :

Maximize : $z = 2x_1 - 4x_2 + 5x_3$

subject to
$$x_1 + 4x_2 - 2x_3 \leq 2$$
$$-x_1 + 2x_2 + 3x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

(2015 : 20 Marks)

Solution:

Let y_1, y_2 be dual variables and w be the objective function for dual of the given problem.

Dual of the problem is

$$\text{Min. } w = 2y_1 + y_2$$

subject to

$$y_1 - y_2 \geq 2$$
$$4y_1 + 2y_2 \geq -4$$
$$-2y_1 + 3y_2 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

Let $\quad \text{Min. } (w) = \text{Max. } (-w')$

$\therefore \quad w' = -2y_1 - y_2$

Writing given problem in standard form, we get

$$\text{Max. } w' = -2y_1 - y_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$$

subject to

$$y_1 - y_2 - S_1 + A_1 = 2$$
$$-4y_1 - 2y_2 + S_2 = 4$$
$$-2y_1 + 3y_2 - S_3 + A_2 = 5$$

where S_1, S_3 are surplus variables, S_2 is a slack variable. A_1 and A_2 are artificial variables and M is a very large quantity. We follow Big-M method for finding solution.

The simplex table is :

	C_j	-2	-1	0	0	0	$-M$	$-M$		
C_B	Basis	y_1	y_2	S_1	S_2	S_3	A_1	A_2	b	8
$-M$	A_1	1	-1	-1	0	0	1	0	2	–
0	S_2	-4	-2	0	1	0	0	0	4	–
$-M$	A_2	-2	3	0	0	-1	0	1	5	$\frac{5}{3}$
$Z_j = \Sigma C_B a_j$		M	$-2M$	M	0	M	$-M$	$-M$		
$C_j = C_j - Z_j$		$-2-M$	$-1+2M$ ↑	$-M$	0	$-M$	0	0		

$\therefore$ A_2 is outgoing variable and y_2 is incoming variable.

	C_j	-2	-1	0	0	0	$-M$		
C_B	Basis	y_1	y_2	S_1	S_2	S_3	A_1	b	θ
$-M$	A_1	$\frac{1}{3}$	0	-1	0	$-\frac{1}{3}$	1	$\frac{11}{3}$	11 →
0	S_2	$-\frac{16}{3}$	0	0	1	$-\frac{2}{3}$	0	$\frac{22}{3}$	–
-1	$\frac{1}{2}$	$-\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	0	$\frac{5}{3}$	–
$Z_j = \Sigma C_B a_{ij}$		$-\frac{M}{3}+\frac{2}{3}$	-1	M	0	$\frac{M+1}{3}$	$-M$		
$C_j = C_j - Z_j$	$-\frac{8}{3}+\frac{M}{3}$ ↑	0	$-M$	0	$-\frac{(M+1)}{3}$	0			

$\therefore$ y_1 is incoming variable and A_1 is outgoing variable.

	C_j	-2	-1	0	0	0		
C_B	Basis	y_1	y_2	S_1	S_2	S_3	b	θ
-2	y_1	1	0	-3	0	-1	11	
0	S_2	0	0	-16	1	-6	66	
-1	y_2	0	1	-2	0	-1	9	
$Z_j = \Sigma C_B a_{ij}$		-2	-1	8	0	3		
$C_j = C_j - Z_j$		0	0	-8	0	-3		

$\because$ All C_j's ≤ 0 $\therefore$ this is the optimal feasible solution.

$\therefore \quad y_1 = 11$

$\quad y_c = 9$

So, $\quad w_{max} = -2 \times 11 - 1 - 9 = -31$

$\therefore \quad w_{min} = 31$

$\therefore$ Maximum value of $z = w_{min} = 31$.

6. Transportation Problems

6.1 Determine an optimal transportation programme so that transportation cost of 340 tons of a certain type of material from three factories F_1, F_2, F_3 to five warehouses w_1, w_2, w_3, w_4, w_5 is minimized. The five warehouses must receive 40 tons, 50 tons, 70 tons, 90 tons and 90 tons respectively. The availability of the material at F_1, F_2, F_3 is 100 tons, 120 tons, 120 tons respectively. The transportation costs per ton from factories to warehouses are given in the table below :

	w_1	w_2	w_3	w_4	w_5
F_1	4	1	2	6	9
F_2	6	4	3	5	7
F_3	5	2	6	4	8

Use Vogel's approximation method to obtain the initial basic feasible solution.

(2010 : 30 Marks)

Solution:

Given :

$$\text{Supply} = 340 \text{ tons}$$
$$\text{Demand} = 40 + 50 + 70 + 90 + 90 = 340 \text{ tons}$$

$\therefore$ Problem is balanced.

Finding initial basic feasible solution by Vogel's approximation method :

	w_1	w_2	w_3	w_4	w_5			
F_1	4	1	2	6	9	100(1)	100(2)	30(2)
F_2	6	4	3	5	7	120(1)	120(2)	120(1)
F_3	5	2	6	4	8	120(2)	70(1)	70(1)
	40	50	70	90	90			
	(1)	(1)	(1)	(1)	(1)			
	10			90	90			
	(1)			(1)	(1)			
	10			20	90			
	(1)			(1)	(1)			

50 at F_3w_2; 70 at F_1w_3; 30 at F_1w_1; 70 at F_3w_4; 10 at F_2w_1; 20 at F_2w_4; 90 at F_2w_5

So, initial basic feasible solution is :

	w_1	w_2	w_3	w_4	w_5
F_1	30		70		
F_2	10			20	90
F_3		50		70	

$$\text{No. of rows} = m = 3$$
$$\text{No. of columns} = n = 5$$
$$\therefore \quad m + n - 1 = 3 + 5 - 1 = 7 = \text{No. of solutions}$$

So, given solution is initial basic feasible solution.

Cost Optimization : For this, we will use u-v method.

Iteration 1 :

	w_1	w_2	w_3	w_4	w_5	4
F_1	30	(0)	70	(–3)	(–4)	–2
F_2	10	(–1)	(1)	20	90	0
F_3	(0)	50	(–3)	70	(–2)	–1
V	6	3	4	5	7	

Now,

$$u_2 + v_7 - 6 = 0 \Rightarrow v_1 = 6$$
$$u_2 + v_4 - 5 = 0 \Rightarrow v_4 = 5$$
$$u_2 + v_5 - 7 = 0 \Rightarrow v_5 = 7$$
$$v_4 + u_3 - 4 = 0 \Rightarrow 5 + u_3 - 4 = 0 \Rightarrow u_3 = -1$$
$$u_3 + v_2 - 2 = 0 \Rightarrow -1 + v_2 - 2 = 0 \Rightarrow v_2 = 3$$
$$v_1 + u_1 - 4 = 0 \Rightarrow 6 + u_1 - u = 0 \Rightarrow u_1 = -2$$
$$u_1 + v_3 - 2 = 0 \Rightarrow -2 + v_3 - 2 = 0 \Rightarrow v_3 = 4$$

Calculate Δ_{ij} for every blank cell where

$$\Delta_{ij} = u_i + v_j - c_{ij}$$

F_2w_3 has positive Δ_{ij} value. Adding t in F_2w_3 and subtracing $\in$ in alternate cells as per u-v method, we get

	w_1	w_2	w_3	w_4	w_5	
F_1	40		60			
F_2			10	20	90	0
F_3		50		70		

Iteration 2 :

	w_1	w_2	w_3	w_4	w_5	4
F_1	40	(1)	60	(–2)	(–3)	–1
F_2	(–1)	(–1)	10	20	90	0
F_3	(–1)	50	(–4)	70	(–2)	–1
V	5	3	3	5	7	

Now,

$$u_2 + v_3 - 3 = 0 \quad \Rightarrow \quad v_3 = 3$$
$$u_2 + v_4 - 5 = 0 \quad \Rightarrow \quad v_4 = 5$$
$$u_2 + v_5 - 7 = 0 \quad \Rightarrow \quad v_5 = 7$$
$$v_3 + u_1 - 2 = 0 \quad \Rightarrow \quad u_1 = -1$$
$$v_4 + u_3 - 4 = 0 \quad \Rightarrow \quad u_3 = -1$$

$$v_1 + u_1 - 4 = 0 \quad \Rightarrow \quad v_1 = 5$$
$$v_2 + u_3 - 2 = 0 \quad \Rightarrow \quad v_2 = 3$$

Calculating Δ_{ij} for blank cells, we get positive value at F_1w_2.
Adding E in this cell as per u-v method, we get

	w_1	w_2	w_3	w_4	w_5	4
F_1	40	20	40	(–3)	(–3)	0
F_2	(–1)	(–2)	30	(–1)	90	1
F_3	(0)	30	(–3)	90	(–1)	1
V	4	1	2	3	6	

Iteration 3 :

Applying same method, we observe that none of the blank cell has $\Delta_{ij} > 0$.
$\therefore$ This is the optimal solution, for minimum

	w_1	w_2	w_3	w_4	w_5
F_1	40	20	40		
F_2			30		90
F_3		30		90	

Cost of transportation.

6.2 **Find the initial basic feasible solution to the following transportation problem by Vogel's approximation method. Also, find its optimal solution and the minimum transportation cost :**

		D_1	D_2	D_3	D_4	Supply
Origins	O_1	6	4	1	5	14
	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
	Demand	6	10	15	4	

(2014 : 20 Marks)

Solution:

Since the total supply and total demand being equal.

$\therefore$ The transportation problem is balanced.

Find the initial basic feasible solution :

Using Vogel's approximation method, the initial basic feasible solution is :

The difference between the smallest and next to the smallest cost in each row and each column are first computed and displayed inside paranthesis against the respective rows and columns.

6	4	1	5	14 (3)
8	9	15 2	7	16 (5)
4	3	6	2	5 (1)
6	10	15	4	
(2)	(1)	(1)	(3)	

The largest of these differences is (5) which is associated with the 2nd row.

Since $C_{23} = 2$, is the minimum cost, we allocate $x_{23} = \min(15, 16) = 15$ in the cell (2, 3).

This exhausts the demand of the 3rd column and therefore we cross it.
Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are as shown below :

6	4	5	14 (1)
8	9	7	1 (1)
4	3	(4) 2	5 (1)
6	10	4	
(2)	(1)	(3)	

6	4	14 (2)
8	9	1 (1)
(1) 4	3	1 (1)
6	10	
↑(2)	(1)	

(4) 6	(10) 4	14 (2)
(1) 8	9	1 (1)
5	10	
(2)	(5)	

∴ The initial basic feasible solution is

(4) 6	(10) 4	1	5
(1) 8	9	(15) 2	7
(1) 4	3	6	(4) 2

Now, find the values of u_i and v_j:
As the maximum number of basic cells exist in the first row.
∴ Let $V_1 = 0$

	(4) 6	(10) 4	1	5
	(1) 8	9	(15) 2	7
	(1) 4	3	6	(4) 2
V_j	0	−2	−6	−2

The net evaluation $\Delta_{ij} = u_i + v_j - C_{ij}$ for all unoccupied cells are less than or equal to zero, i.e., $\Delta_{ij} \leq 0$.
The current basic feasible solution is optimal.
Hence, the optimal allocation is given by :

$$x_{11} = 4, x_{12} = 10, x_{21} = 1, x_{23} = 15, x_{31} = 1, x_{34} = 4$$

∴ The optimal (minimum) transportation cost

$$= 6 \times 4 + 10 \times 4 + 1 \times 8 + 15 \times 2 + 1 \times 4 + 4 \times 2$$
$$= 24 + 40 + 8 + 30 + 4 + 8$$
$$= 114$$

6.3 Find the initial basic feasible solution (bsf) of the following transportation problem using Vogel's approximation method and find the cost.

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	4	7	0	3	6	14
O_2	1	2	−3	3	8	9
O_3	3	−1	4	0	5	17
Demand	8	3	8	13	8	

(2017 : 15 Marks)

Solution:
We take the difference between the least and next least entry.

	D_1	D_2	D_3	D_4	D_5		Diff-I	Diff-II	Diff-III
O_1	[7] 7	10	3	6	[7] 9	~~14~~ ~~7~~	3	1	1
O_2	[1] 4	5	[8] 0	6	11	~~9~~ ~~1~~ 0	(4)	1	2
O_3	6	[3] 2	7	[13] 3	[1] 8	~~17~~ ~~14~~ 1	1	1	(3)
	~~8~~ ~~7~~ 0	~~3~~ 9	~~8~~ 0	13	8				

Diff-1	2	3	3	3	1
Diff-2	2	(3)	–	3	1
Diff-3	2	–	–	3	1
Diff-4	2	–	–	–	1
Diff-5	1				1
					1

We choose the row/column ..... maximum .... and allocate to .... cell with least ....
Using cost given originally

$$\text{Cost} = 7(4) + 1(1) + 3(-1) + 8(-3) + 12(0) + 7(6) + ...$$
$$= 49$$

7. Assignment Problems

7.1 Solve the minimum time assignment problem :

Machines

Joles	M_1	M_2	M_3	M_4
J_1	3	12	5	14
J_2	7	9	8	12
J_3	5	11	10	12
J_4	6	14	4	11

(2013 : 15 Marks)

Solution:
In assignment problem, the solution does not change if we subtract or add same values from a column or row.
So, we create zeros by subtracting minimum of each row and column from that row or column.

	M_1	M_2	M_3	M_4
J_1	0	9	2	11
J_2	0	2	1	5
J_3	0	6	5	7
J_4	2	10	0	7

	M_1	M_2	M_3	M_4
J_1	0	7	2	6
J_2	0	0	1	0
J_3	0	4	5	2
J_4	2	8	0	2

Now, we assign jobs using zero's taking care of choosing only one zero in a row or column.

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	7	2	6
J_2	~~0~~	$\boxed{0}$	1	~~0~~
J_3	~~0~~	4	5	2
J_4	2	8	$\boxed{0}$	2

We have only 3 assignments so this is not optimum.
We draw the minimum lines to cover all zeros.

	M_1	M_2	M_3	M_4	
J_1	0	7	2	6	✓
J_2	0	0	1	0	
J_3	0	4	5	2	✓
J_4	2	8	0	2	
	✓				

Minimum of the unmarked entries i.e., 2.

Subtract this from each unmarked row and add it to marked column.

0	5	0	4
2	0	1	0
0	2	3	0
4	8	0	2

Now, we again assign the zeros.

The assignment is optimal as 4 assignment have been made.

$$\text{Min cost} = 3 + 9 + 12 + 4 = 28$$

(**Note :** In actual exam you will provided less space and may club some of the steps).

7.2 **Solve the following assignment problem to maximize the sales.**

		I	II	III	IV	V
	A	3	4	5	6	7
	B	4	15	13	7	6
Salesman	C	6	13	12	5	11
	D	7	12	15	8	5
	E	8	13	10	6	9

(2015 : 10 Marks)

Solution:

In order to convert it into a minimum problem, we multiply each element by ....

So, table becomes : (We use Hungarian method)

No. of lines = 3

No. of rows = 5

So, solution is degenerate.

Now table becomes

	I	II	III	IV	V
A	0	4	2	0	0
B	6	0	1	6	8
C	2	0	0	6	1
D	4	4	0	6	10
E	0	0	2	5	3

No. of lines = 4

No. of rows = 5

New table becomes

	I	II	III	IV	V
A	0	5	3	0	0
B	5	0	1	5	7
C	1	0	0	5	0
D	3	4	0	5	5
E	0	1	2	5	3

$\therefore$ Optimal arrangement is A = IV, B = II, C = V, D = III, E = I

$$\text{Maximum sales} = 6 + 15 + 11 + 15 + 8 = 55$$

7.3 **In a factory, there are five operators O_1, O_2, O_3, O_4, O_5 and five machines M_1, M_2, M_3, M_4, M_5. The operating costs are given when the O_i operates the M_j machine ($i, j = 1, 2, ... 5$). But there is a restriction that O_3 cannot be allowed to operate the third machine. M_3 and O_2 cannot be allowed to operate the fifth machine. The cost matrix is given above. Find the optimal assignment and optimal assignment also.**

	M_1	M_2	M_3	M_4	M_5
O_1	24	29	18	32	19
O_2	17	26	34	22	21
O_3	27	16	28	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

(2018 : 15 Marks)

Solution:

Since, O_3 cannot operate on M_3 and O_2 cannot operate on M_5, we assign a very large ..... say M to both the corresponding cells. The matrix now becomes

	M_1	M_2	M_3	M_4	M_5
O_1	24	29	18	32	19
O_2	17	26	34	22	M
O_3	27	16	M	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

By row-wise reduction →

	M_1	M_2	M_3	M_4	M_5
O_1	6	11	0	14	1
O_2	0	9	17	5	M
O_3	11	0	14	1	9
O_4	4	0	10	12	6
O_5	12	0	15	8	11

By column-wise reduction ↓

	M_1	M_2	M_3	M_4	M_5
O_1	6	11	0	13	0
O_2	0	9	17	4	14
O_3	11	0	M	0	8
O_4	4	0	10	11	5
O_5	12	0	15	7	10

Since, by covering all zeroes,

$$\text{Number of lines} = 4 < 5.$$

So, solution is degenerate.

New matrix by hungarian method is

	M_1	M_2	M_3	M_4	M_5
O_1	6	15	0	13	0
O_2	0	13	17	4	M
O_3	11	7	M	0	8
O_4	0	0	6	7	1
O_5	8	0	11	3	6

Again, Number of lines = 4 < 5

By following hungarian method, new matrix is

	M_1	M_2	M_3	M_4	M_5
O_1	7	16	0	15	0
O_2	0	13	16	3	M
O_3	12	5	M	0	8
O_4	0	0	5	6	0
O_5	8	0	10	2	5

Number of lines = 5

$\therefore$ Optimal assignment is : O_1 at M_3, O_2 at M_1, O_3 at M_4, O_4 at M_5, O_5 at M_2 and optimal cost is 18 + 17 + 17 + 24 + 16 = 92.

■■■■

5 Partial Differential Equations

1. Formulation of P.D.E.

1.1 Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that this equation is satisfied by the surface $yz + zx + xy = 0$.

(2009 : 12 Marks)

Solution:

The equation cone having vertex at origin

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \quad \text{...(i)}$$

where a, b, c, f, g, h are parameters.

Differentiating w.r.t. x and y, we get

$$2ax + 2hy + 2gz + 2gxp + 2zcp + 2fyp = 0$$

And

$$2by + 2czq + 2hx + 2fyq + 2fz + 2gxq = 0$$

So,

$$ax + hy + qz + p(gx + zc + fy) = 0 \times x$$

$$by + hx + fz + q(cz + fy + gx) = 0 \times y$$

$$\Rightarrow \quad ax^2 + hxy + gzx + p(gx^2 + czx + fyx) = 0$$

$$by^2 + hxy + fzy + q(czy + fy^2 + gxy) = 0$$

On adding,

$$\Rightarrow \quad ax^2 + by^2 + 2hxy + gzx + fzy + px + qy[cz + fy + gx] = 0$$

$$\Rightarrow \quad -(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0$$

$$\Rightarrow \quad (cz + fy + gx)(px + qy - z) = 0$$

Clearly, $px + qy - z = 0$ is required differential equation.

Given surface is $yz + zx + xy = 0$ (*)

Differentiating (*) w.r.t. x and y, we get

$$yp + z + px + y = 0 \quad \text{...(2)}$$

$$z + yq + xq + x = 0 \quad \text{...(3)}$$

So, we get

$$p = \frac{-(z+y)}{(x+y)}, q = \frac{-(x+z)}{(x+y)}$$

$$px + qy - z = \frac{-(z+y)x}{(x+y)} - \frac{(x+z)}{(x+y)}y - z$$

$$= \frac{-(z+y)x - (x+z)y - z(x+y)}{(x+y)}$$

$$= \frac{-xz - xy - xy - zy - zx - zy}{(x+y)}$$

$$= \frac{-2(xy + yz + zx)}{x+y} = \frac{-20}{x+y} = 0$$

1.2 From the partial differential equation by eliminating the arbitrary function f given by :

$$f(x^2 + y^2, z - xy) = 0$$

(2009 : 6 Marks)