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6(a) Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$ by changing the independent variable. (10)

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - (4x^2)y = 8x^2\sin x^2$$

Comparing it with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$P = \frac{-1}{x}, \quad Q = -4x^2, \quad R = 8x^2\sin x^2$$

Let $\left(\frac{dz}{dx}\right)^2 = \pm a^2 Q = 4x^2$ (for $a=1$)

$$\frac{dz}{dx} = 2x \Rightarrow \boxed{z = x^2}$$

(Note that $\frac{dz}{dx} = e^{-\int P dx}$ is not working here)

Now, $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \dfrac{2 + \left(\frac{-1}{x}\right)2x}{4x^2}$

$= 0$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2\sin x^2}{4x^2} = 2\sin x^2$$

$= 2\sin z$

Transformed Eqn is :

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

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$$\frac{d^2y}{dz^2} - 1(y) = 2\sin z$$

i.e. $(D'^2 - 1)y = 2\sin z$

Auxiliary Eqn: $D'^2 - 1 = 0$

$D' = 1, -1$

$$C.F. = c_1 e^{z} + c_2 e^{-z}$$
$$= c_1 e^{x^2} + c_2 e^{-x^2}$$

$$P.I. = \frac{1}{D'^2 - 1} 2\sin z$$

$$= \frac{2}{(-1^2) - 1} \sin z = -\sin z$$

$$= -\sin x^2$$

Hence Complete Solution is:

$$y = CF + PI$$

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$

Example 3. Forces X, Y, Z act along the three straight lines $y = b$, $z = -c$; $z = c$, $x = -a$; and $x = a$, $y = -b$ respectively. Show that they will have a single resultant if $\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0$ and that the equations of its line of action are any two of the three

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \quad \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \quad \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0.$$

Solution. The forces X, Y, Z act along the lines

$$y = b, \ z = -c; \quad z = c, \ x = -a; \quad x = a, \ y = -b.$$

The equations of these lines are

$$\frac{x-0}{1} = \frac{y-b}{0} = \frac{z+c}{0}, \quad \frac{x+a}{0} = \frac{y-0}{1} = \frac{z-c}{0}, \quad \frac{x-a}{0} = \frac{y+b}{0} = \frac{z-0}{1}$$

The forces acting on the body are as follows :

(*i*) A force X acting at the point $(0, b, -c)$ along the line whose d.c.'s are $< 1, 0, 0 >$

(*ii*) A force Y acting at the point $(-a, 0, c)$ along the line whose d.c.'s are $< 0, 1, 0 >$

(*iii*) A force Z acting at the point $(a, -b, 0)$ along the line whose d.c.'s are $< 0. 0, 1 >$.

$\therefore$ The components of the forces parallel to the axes are

$$X_1 = X \, . \, 1 = X, \quad X_2 = Y \, . \, 0 = 0, \quad X_3 = Z \, . \, 0 = 0$$

$$Y_1 = X \, . \, 0 = 0, \quad Y_2 = Y \, . \, 1 = Y, \quad Y_3 = Z \, . \, 0 = 0$$

$$Z_1 = X \, . \, 0 = 0, \quad Z_2 = Y \, . \, 0 = 0, \quad Z_3 = Z \, . \, 1 = Z$$

If the system reduces to a single force $R = (X, Y, Z)$ acting at O and a couple $G = (L, M, N)$, then

$$X = \Sigma X_1 = X_1 + X_2 + X_3 = X + 0 + 0 = X$$

$$Y = \Sigma Y_1 = 0 + Y + 0 = Y$$

and $$Z = \Sigma Z_1 = 0 + 0 + Z = Z$$

To find L, M, N

(*i*) Consider

$$\hat{i}L_1 + \hat{j}M_1 + \hat{k}N_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & -c \\ X & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(cX) + \hat{k}(-bX)$$

$\therefore$ $$L_1 = 0, \quad M_1 = -cX, \quad N_1 = -bX$$

(*ii*) $$\hat{i}L_2 + \hat{j}M_2 + \hat{k}N_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & 0 & c \\ 0 & Y & 0 \end{vmatrix}$$

$$= \hat{i}(-cY) - \hat{j}(0) + \hat{k}(-aY)$$

$\therefore \quad L_2 = -cY, \ M_2 = 0; \ N_2 = -aY$

(*iii*) $\hat{i}L_3 + \hat{j}M_3 + \hat{k}N_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_3 & y_3 & z_3 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & -b & 0 \\ 0 & 0 & Z \end{vmatrix}$$

$$= \hat{i}(-bZ) - \hat{j}(aZ) + \hat{k}(0)$$

$$\therefore \quad L_3 = -bZ, \ M_3 = -aZ, \ N_3 = 0$$

Here $L = \Sigma L_1 = -(bZ + cY), \ M = \Sigma M_1 = -(cX + aZ)$

and $$N = \Sigma N_1 = -(bX + aY)$$

The system is equivalent to a single force if

$$LX + MY + NZ = 0 \quad ...(1)$$

Substituting the values of L, M, N in (1), we have

$$-[(bZ + cY)X + (cX + aZ)Y + (bX + aY)Z] = 0$$

or $$2[aYZ + bZX + cXY] = 0$$

or $$\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0 \quad ...(2)$$

which is the required condition.

The equations of the line of action of the single force *i.e.*, of the central axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = 0$$

$\therefore$ The equations of the line of action of the single resultant force are any two of the following three :

$$L - yZ + zY = 0, \ M - zX + xZ = 0, \ N - xY + yX = 0$$

or
$$-(bZ + cY) - yZ + zY = 0$$
$$-(cX + aZ) - zX + xZ = 0$$
$$-(bX + aY) - xY + yX = 0$$

Dividing these equations by YZ, ZX and XY respectively, we get

$$-\left(\frac{b}{Y} + \frac{c}{Z}\right) - \frac{y}{Y} + \frac{z}{Z} = 0, \ -\left(\frac{c}{Z} + \frac{a}{X}\right) - \frac{z}{Z} + \frac{x}{X} = 0;$$
$$-\left(\frac{b}{Y} + \frac{a}{X}\right) - \frac{x}{X} + \frac{y}{Y} = 0$$

Using (2), we have

$$\frac{a}{X} - \frac{y}{Y} + \frac{z}{Z} = 0; \ \frac{b}{Y} - \frac{z}{Z} + \frac{x}{X} = 0, \ \frac{c}{Z} - \frac{x}{X} + \frac{y}{Y} = 0$$

or $$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \ \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \ \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

Hence the equations to its line of action are any two of the three

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \ \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \ \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

6(c) Solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$. (10)

Auxiliary Eqn:

$$D^4+D^2+1=0$$

$$(D^4+2D^2+1)-D^2=0$$

$$(D^2+1)^2-D^2=0$$

$$(D^2+D+1)(D^2-D+1)=0$$

$$\therefore D = \frac{-1\pm i\sqrt{3}}{2}, \frac{1\pm i\sqrt{3}}{2}$$

$$C.F. = e^{x/2}\left(c_1'\cos\frac{\sqrt{3}x}{2} + c_2'\sin\frac{\sqrt{3}x}{2}\right) + e^{-x/2}\left(c_3'\cos\frac{\sqrt{3}x}{2} + c_4'\sin\frac{\sqrt{3}x}{2}\right)$$

$$= c_1 e^{x/2}\cos\left(\frac{\sqrt{3}x}{2}+c_2\right) + c_3 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}+c_4\right)$$

$$P.I. = \frac{1}{D^4+D^2+1} e^{-x/2}\cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^2\left[(D-\frac{1}{2})^2+1\right]+1}\cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2}\frac{1}{(D^2-D+\frac{1}{4})\left[(D^2-D+\frac{1}{4})+1\right]+1}\cos\frac{\sqrt{3}x}{2}$$

Put $f(D^2) = f\left(-\frac{3}{4}\right)$

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$$= e^{-x/2} \cdot \frac{1}{\left(\frac{-3}{4} - D + \frac{1}{4}\right)\left[\frac{-3}{4} - D + \frac{5}{4}\right] + 1} \cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2} \cdot \frac{1}{\left(-D - \frac{1}{2}\right)\left[-D + \frac{1}{2}\right] + 1} \cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2} \cdot \frac{1}{\left(D^2 - \frac{1}{4}\right) + 1} \cos\frac{\sqrt{3}x}{2}$$

$$= e^{-x/2} \cdot \frac{1}{\left(D^2 + \frac{3}{4}\right)} \cos\frac{\sqrt{3}x}{2}$$

$$= x \cdot e^{-x/2} \frac{1}{2D} \cos\frac{\sqrt{3}x}{2} \quad \left(\because f(D^2) = f(-a^2) \text{ is zero}\right)$$

$$= \frac{x \cdot e^{-x/2}}{2} \frac{D}{D^2} \cos\frac{\sqrt{3}x}{2}$$

$$= \frac{x \cdot e^{-x/2}}{\cancel{2}} \cdot \frac{1}{-3/\cancel{4}} \left(-\sin\frac{\sqrt{3}x}{2}\right) \times \frac{\sqrt{3}}{\cancel{2}}$$

$$= \frac{x}{\sqrt{3}} e^{-x/2} \cdot \sin\frac{\sqrt{3}x}{2}$$

Hence, complete solution is

$$y = CF + PI$$

$$y = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2} + c_2\right) + c_3 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2} + c_4\right)$$

$$+ \frac{x}{\sqrt{3}} e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right).$$

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6(d) Examine if the vector field defined by

$$\vec{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$$

is irrotational. If so find the scalar potential ϕ such that $\vec{F} = \text{grad}\,\phi$. (10)

$\vec{F}$ is irrotational if $\text{Curl}\, F = \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= i(3x^2z^2 - 3x^2z^2) + j(6xyz^2 - 6xyz^2) + k(2xz^3 - 2xz^3) = 0$$

$\therefore$ $\vec{F}$ is irrotational.

$$\vec{F} = \text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\left.\begin{aligned} \therefore \frac{\partial\phi}{\partial x} &= 2xyz^3 \\ \frac{\partial\phi}{\partial y} &= x^2z^3 \\ \frac{\partial\phi}{\partial z} &= 3x^2yz^2 \end{aligned}\right\} \Rightarrow \phi = x^2yz^3 + C$$

$\therefore$ Scalar potential, $\phi = x^2yz^3 + C$.