ANALYTIC GEOMETRY

CSE - 2012:

1(e) Prove that two of the straight lines represented by the equation $x^3+bx^2y+cxy^2+y^3=0$ will be at right angles if $b+c=-2$.

$\Rightarrow$ $x^3+bx^2y+cxy^2+y^3=0$. —— (1)

If these two lines in this equation are ⊥ar, then the coeffs of x^2 and y^2 summed is zero.

Eqn of a pair of ⊥ar straight lines is given by

$x^2+pxy-y^2=0$.

Making this equation cubic by multiplying with suitable expression:

$$(x^2+pxy-y^2)(x-y)=0$$

$$\Rightarrow x^3+(p-1)x^2y-(p+1)xy^2+y^3=0 \quad \text{—— (2)}$$

If (1) and (2) are the same, then, we can equate the coeff of x^2y and xy^2.

$\therefore b=p-1$ and $c=-(p+1)$

$\Rightarrow p=b+1$ and $p=-c-1$

$\therefore$ $b+1=-c-1 \Rightarrow \boxed{b+c=-2}$ which is the reqd condn

4(b) A variable plane parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$

$\rightarrow$ Any plane parallel to the given plane is given by

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$ where k is a parameter. —— (1)

its intercepts are $A(ak,0,0)$, $B(0,bk,0)$ & $C(0,0,ck)$.

Let us find the equation of a sphere passing through the origin $O(0,0,0)$ and the points A, B and C.

Let the eqn of sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

It passes through the origin $O(0,0,0)$ $\Rightarrow d=0$.

$\therefore$ It reduces to $x^2+y^2+z^2+2ux+2vy+2wz=0$ —— (2)

It passes through:

$A(ak,0,0) \equiv a^2k^2+2uak=0 \Rightarrow u=-\frac{(ak)}{2}$

$B(0,bk,0) \equiv b^2k^2+2vbk=0 \Rightarrow v=-\frac{(bk)}{2}$

$C(0,0,ck) \equiv c^2k^2+2wck=0 \Rightarrow w=-\frac{(ck)}{2}$.

Putting in (2): $x^2+y^2+z^2-(akx+bky+ckz)=0$

$\Rightarrow x^2+y^2+z^2-k(ax+by+cz)=0$ —— (3)

The plane (1) and the sphere (3) gives the equation of circle. Then, the locus of circle is found by eliminating k b/w (1) and (3).

Putting $k=\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ from (1) into (3)

$x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$

$\Rightarrow \cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\cancel{x^2}-xy\frac{b}{a}-xz\frac{c}{a}-xy\frac{a}{b}-\cancel{y^2}-yz\frac{c}{b}-zx\frac{a}{c}$

$-zy\frac{b}{c}-\cancel{z^2}=0$

$\Rightarrow \boxed{yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0}$

4(c) Show that the locus of a point from where three mutually perpendicular lines can be drawn to the paraboloid $x^2+y^2+2z=0$ is $x^2+y^2+4z=1$.

$\rightarrow$ Let $S \equiv x^2+y^2+2z=0$ —— (1)

Let (α,β,γ) be the required point from where three mutually lar tangent lines can be drawn to S.

Let $S_1 \equiv \alpha^2+\beta^2+2\gamma$ —— (2)

Tangent plane to S at (α,β,γ) is given by $\alpha.x+\beta.y+\gamma+z=0$.

Let $T \equiv (\alpha x+\beta y+z+\gamma)$. —— (3)

The enveloping cone of given paraboloid S is given by $T^2=SS_1$

$\Rightarrow (\alpha x+\beta y+z+\gamma)^2=(\alpha^2+\beta^2+2\gamma)(x^2+y^2+2z)$ —— (4)

Which has vertex at (α,β,γ).

Since the point (α,β,γ) is the point from where three mutually perpendicular tangent lines are drawn to paraboloid S, these lines are the three mutually ⊥ar generators of the enveloping cone of S for which the condition is that the sum of coeff. of x^2, y^2 and z^2 is equal to zero in the equation of enveloping cone given by (4). In (4):

Coeff of $x^2 \equiv \beta^2 + 2\gamma$

coeff of $y^2 \equiv \alpha^2 + 2\gamma$

coeff of $z^2 \equiv -1$

$\therefore$ Sum of coeff of x^2, y^2 & z^2 is zero.

$\therefore \beta^2 + 2\gamma + \alpha^2 + 2\gamma - 1 = 0$

$\Rightarrow \alpha^2 + \beta^2 + 4\gamma = 1$

Then, the locus of (α, β, γ) is $\boxed{x^2 + y^2 + 4z = 1}$