2011 CSE

If $f(z) = u+iv$ is an analytic function of $z = x+iy$ and
$$u-v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$
find $f(z)$ subject to the condition, $f(\frac{\pi}{2}) = \frac{3-i}{2}$

Let $f(z) = u+iv$

$$i f(z) = iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

ie. $F(z) = U + iV$

where, $F(z) = (1+i) f(z)$ —— (1)

$U = u-v$ and $V = u+v$

$$U = u-v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

By Milne' Thompson Method

$$\phi_1(z,0) = \left.\frac{\partial U}{\partial x}\right|_{\substack{x=z\\y=0}} , \quad \phi_2(z,0) = \left.\frac{\partial U}{\partial y}\right|_{\substack{x=z\\y=0}}.$$

$$\frac{\partial U}{\partial x} = \frac{1}{(\cosh y - \cos x)^2}\left[(\sin x + \cos x)(\cosh y - \cos x) - (\cancel{\sin x} + \sin x)(e^y - \cos x + \sin x)\right]$$

$$\phi_1(z,0) = \frac{(\sin z + \cos z)(1-\cos z) - \sin z(1 - \cos z + \sin z)}{(1-\cos z)^2}$$

$$= \frac{\cos z(1-\cos z) - \sin^2 z}{(1-\cos z)^2}$$

$$= \frac{\cos z - 1}{(1-\cos z)^2} = \frac{-1}{1-\cos z}$$

$$\frac{\partial v}{\partial y} = \frac{e^y(\cosh y - \cos x) - \sinh y\,(e^y - \cos x + \sin x)}{(\cosh y - \cos x)^2}$$

$$\left.\frac{\partial v}{\partial y}\right|_{\substack{x=z\\ y=0}} = \frac{(1-\cos z) - 0}{(1-\cos z)^2} = \frac{1}{1-\cos z}$$

$$= \phi_2(z,0)$$

$$\therefore \; F(z) = \int \left(\phi_1(z,0) - i\,\phi_2(z,0)\right) dz + C$$

$$= \int \left(\frac{-1}{1-\cos z} - i\,\frac{1}{1-\cos z}\right) dz + C$$

$$= C - (1+i)\int \frac{1}{1-\cos z}\, dz$$

$$= C - (1+i)\int \frac{1}{2}\operatorname{cosec}^2\frac{z}{2}\, dz$$

$$= C + (1+i)\cot\frac{z}{2}.$$

$\therefore$ from ①

$$f(z) = \frac{1}{1+i}F(z) = \frac{C}{1+i} + \cot\frac{z}{2}$$

$$f(z) = +\cot\frac{z}{2} + C_1 \quad , \text{ where } C_1 = \frac{C}{1+i}$$

Also, $f\left(\frac{\pi}{2}\right) = +\cot\frac{\pi}{4} + C_1 = +1 + C_1$

$$= \frac{3-i}{2}$$

$$\therefore \; C_1 = \frac{3-i}{2} - 1 = \frac{1-i}{2}$$

Hence, $\boxed{f(z) = \cot\left(\frac{z}{2}\right) + \frac{1-i}{2}}$

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2(c) If the function $f(z)$ is analytic and one valued in $|z-a|<R$, prove that for $0<r<R$,

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta,$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

Since $f(z)$ is analytic in $|z-a|<R$ and $r<R$, it follows that $f(z)$ is also analytic inside the circle $C$ defined by $|z-a| = r$.

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} = f'(a) \quad \text{—— (1)}$$

(Cauchy's formula)

Also, $f(z)$ can be expressed as a Taylor's series about $z=a$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m (z-a)^m$$

Putting $z-a = re^{i\theta}$, we have

$$f(z) = f(a+re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{mi\theta}$$

$$\overline{f(z)} = \sum_{m=0}^{\infty} \overline{a}_m r^m e^{-mi\theta}$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{\overline{f(z)}}{(z-a)^2}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \overline{a}_m r^m e^{-mi\theta}}{r^2 e^{2i\theta}}\, rie^{i\theta} d\theta$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} \overline{a}_m r^{m-1}\int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta$$

$$= 0 \quad \left[\because \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta = 0\right] \quad \text{—— (2)}$$

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Adding ① and ②, we have

$$0 + f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z) + \overline{f(z)}}{(z-a)^2} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2} dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\text{real part of } f(a + re^{i\theta})}{r^2 \cdot e^{i2\theta}} \cdot rie^{i\theta} d\theta$$

$$= \frac{1}{\pi r} \int_0^{2\pi} P(\theta) e^{-i\theta} d\theta$$

Where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

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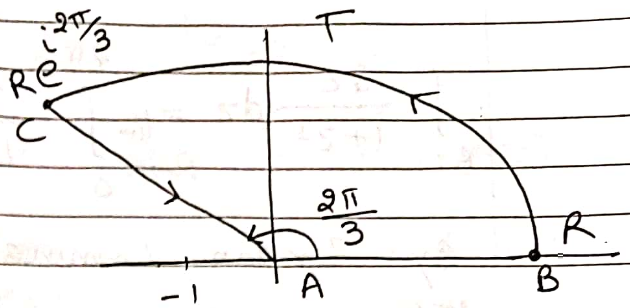
Evaluate, by Contour integration,
$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$. (15)




$$I = \int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$$

$$= \int_0^1 \frac{dx}{x\left(\frac{1}{x}-1\right)^{1/3}}$$

Let $\frac{1}{x}-1 = t^3 \Rightarrow -\frac{1}{x^2}dx = 3t^2dt$

As $x \to 0$ then $t \to \infty$

$x \to 1$ then $t \to 0$

$$I = \int_{\infty}^{0} \frac{-3\cancel{x^2}t^2 dt}{\cancel{x}(t^3)^{1/3}} = \int_0^{\infty} \frac{3t}{1+t^3}dt$$

Poles of $f(t) = \frac{3t}{1+t^3}$ are $-1, e^{i\pi/3}, e^{i2\pi/3}$

Only $e^{i\pi/3}$ lies inside contour

$$Res(z=e^{i\pi/3}) = \lim_{z\to e^{i\pi/3}} (z-e^{i\pi/3}) \cdot \frac{3z}{1+z^3} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{z\to e^{i\pi/3}} \frac{(z-e^{i\pi/3})3 + 3z}{(3z^2)} = \frac{1}{e^{i\pi/3}} = e^{-i\pi/3}$$

$$I = \int_{AB} f(t)dt + \int_{BC} f(t)dt + \int_{CA} f(t)dt \quad \text{— (1)}$$

$$I = \int_T \frac{3z}{1+z^3}dz = 2\pi i\left(e^{-i\pi/3}\right) \quad \text{— (2)}$$

$$\int_{AB} \frac{3z}{1+z^3}\,dz = \lim_{R\to\infty}\int_0^R \frac{3x}{1+x^3}\,dx \quad \text{— (2)}$$

$$\int_{BC} \frac{3z}{1+z^3}\,dz = \lim_{R\to\infty}\int_0^{2\pi/3} \frac{3Re^{i\theta}}{1+(Re^{i\theta})^3}(iRe^{i\theta})\,d\theta \quad \text{— (3)}$$

By Jordan Lemma

$$\int_0^{2\pi/3} \frac{3iR^2e^{i2\theta}\,d\theta}{1+R^3e^{i3\theta}} \le \int_0^{2\pi/3}\left|\frac{3iR^2e^{i2\theta}\,d\theta}{1+R^3e^{i3\theta}}\right|$$

$$\le \int_0^{2\pi}\frac{3R^2\,d\theta}{|1+R^3e^{i3\theta}|} \le \int_0^{2\pi}\frac{3R^2}{R^3}\,d\theta \to 0$$

where $R\to\infty$

$$\int_{CA}\frac{3z}{1+z^3}\,dz = \int_R^0 \frac{3re^{i2\pi/3}}{r^3e^{i2\pi}+1}\cdot(e^{i2\pi/3}\,dr)$$

$$= -e^{i4\pi/3}\int_0^R \frac{3r}{1+r^3}\,dr \quad \text{— (4)}$$

∴ from (1), (2), (3) & (4)

$$2\pi i(e^{-i\pi/3}) = \int_0^\infty \frac{3t}{1+t^3}\,dt + 0 - e^{i4\pi/3}\int_0^\infty\frac{3t}{1+t^3}\,dt$$

$$= (1-e^{i4\pi/3})\int_0^\infty \frac{3t}{1+t^3}\,dt$$

$$\therefore\ I = \int_0^\infty \frac{3t}{1+t^3}\,dt = \frac{2\pi i e^{-i\pi/3}}{1-e^{i4\pi/3}} = \frac{2\pi i}{e^{i\pi/3}-e^{i5\pi/3}}$$

$$= \frac{2\pi i}{\frac{1}{2} + i\frac{\sqrt3}{2} - \frac{1}{2} + i\frac{\sqrt3}{2}} = \frac{2\pi i}{i\sqrt3} = \boxed{\frac{2\pi}{\sqrt3}}$$

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find Laurent series of $f(z) = \frac{1}{1-z^2}$ with centre $z=1$.

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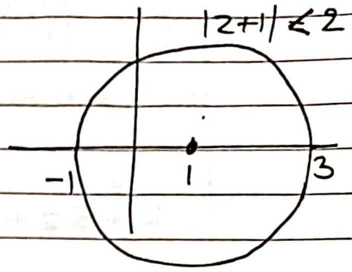
$$f(z) = \frac{1}{1-z} = \frac{+1}{2}\left[\frac{-1}{z-1} + \frac{1}{z+1}\right]$$

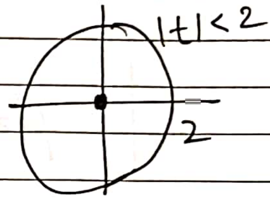



We want to expand around $z=1$.

$\therefore$ Take $z-1 = t$

**Case-1:** $|z-1| = |t| < 2$

$$f(z) = \frac{1}{2}\left[\frac{-1}{t} + \frac{1}{t+2}\right]$$




$$= \frac{1}{2}\left[\frac{-1}{t} + \frac{1}{2\left(1+\frac{t}{2}\right)}\right]$$

$$= \frac{-1}{2t} + \frac{1}{4}\left(1+\frac{t}{2}\right)^{-1}$$ as $|t| < 2 \Rightarrow \left|\frac{t}{2}\right| < 1$

$$= \frac{-1}{2t} + \frac{1}{4}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots\right)$$

$$= \frac{-1}{2}\cdot\frac{1}{(z-1)} + \frac{1}{4}\left(1 - \frac{z-1}{2} + \frac{(z-1)^2}{4} - \frac{(z-1)^3}{8} + \cdots\right)$$

**Case-2:** $|z-1| = |t| > 2$ ie. $\frac{2}{|t|} < 1$

$$f(z) = \frac{1}{2}\left[\frac{-1}{t} + \frac{1}{t+2}\right]$$

$$= \frac{-1}{2t} + \frac{1}{2\cdot t\left(1+\frac{2}{t}\right)} = \frac{-1}{2t} + \frac{1}{2t}\left(1+\frac{2}{t}\right)^{-1}$$

$$= \frac{-1}{2t} + \frac{1}{2t}\left(1 - \frac{2}{t} + \frac{4}{t^2} - \frac{8}{t^3} + \cdots\right)$$

$$= \cancel{\frac{1}{(z-1)}} + \frac{-1}{t^2} + \frac{2}{t^3} - \frac{4}{t^4} + \cdots$$

$$= \frac{-1}{(z-1)^2} + \frac{2}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$