* Functions of Several Variables *

Introduction:-

Most measurable quantities in the real world donot depend on one single factor but on many factors. This indicates that functions of several variables are natural entities in the world of mathematics.

So far we have studied the concepts of limits, continuity, differentiability etc. for functions of a single variable.

Now we introduce the concept of limit, continuity and differentiability of functions of several variables. Mainly we study these concepts for real valued functions of two variables which can be generalised to functions of several variables.

* Euclidean Space:

For a fixed $n \in \mathbb{N}$, let $\mathbb{R}^n$ be the set of all ordered n-tuples $x = (x_1, x_2, \ldots x_n)$ where $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are called the coordinates of x.

→ The elements of $\mathbb{R}^n$ are called points or vectors and denoted by x, y, z etc.

→ We define the addition of vectors and multiplication of a vector by real numbers (called scalar) as follows:

Let $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots y_n) \in \mathbb{R}^n$; $\alpha \in \mathbb{R}$

then $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n$

and $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \in \mathbb{R}^n$

these two operations make $\mathbb{R}^n$ a vector space over the real field $\mathbb{R}$.

— The zero elements of $\mathbb{R}^n$ (sometimes called the origin or null vector) is the point $0 = (0, 0, \ldots 0)$.

— We define the scalar product (or inner product) of two vectors x and y by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

and the norm of x by $\|x\| = (x \cdot x)^{1/2}$

$$= \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

The vector space $\mathbb{R}^n$ with the above inner product and norm is called n-dimensional Euclidean space.

— In particular, we get $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ for $n = 1, 2, 3$ respectively. We write

$x = (x_1, x_2)$ if $x \in \mathbb{R}^2$

$x = (x_1, x_2, x_3)$ if $x \in \mathbb{R}^3$.

Functions of Several Variables:

Let $f: X \to \mathbb{R}$; if $X \subset \mathbb{R}^n$ then f is called a function of 'n' variables.

→ f is a function of several variables if $n > 1$

→ $f: X \to \mathbb{R}$ is a function of two variables if $X \subset \mathbb{R}^2$.

$f: X \to \mathbb{R}$ is a function of three variables if $X \subset \mathbb{R}^3$.

* Neighbourhood of a point:

→ Spherical neighbourhood of a point:

Let $\mathbb{R}^n$ be the Euclidean space and $a \in \mathbb{R}^n$ (i.e. $a = (a_1, a_2 \cdots a_n)$).

If δ is any +ve real number, then the set $\{x \in \mathbb{R}^n / \|x - a\| < \delta\}$ is called an open sphere.

(Since $\|x - a\| < \delta \Rightarrow \left[\sum_{i=1}^{n} (x_i - a_i)^2\right]^{1/2} < \delta$

$$\sum_{i=1}^{n} (x_i - a_i)^2 < \delta^2$$

$$\Rightarrow (x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 < \delta^2$$

The point 'a' is called the centre and 'δ' the radius of the sphere.

This open sphere is denoted by $S(a, \delta)$

— A closed sphere is denoted by $S[a, \delta]$ and is defined by

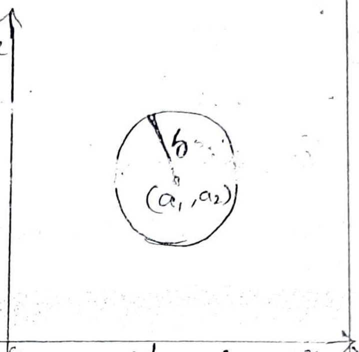
$$S[a, \delta] = \{x \in \mathbb{R}^n / \|x - a\| \le \delta\}.$$

— Any open sphere with 'a' as its centre is called a spherical neighbourhood of the point 'a'.

The open sphere with centre $a = (a_1, a_2) \in \mathbb{R}^2$ and radius δ is

$$S(a, \delta) = \{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}.$$

∴ The open sphere in this case consists of all points of the Cartesian plane which lie within the circle $(x_1 - a_1)^2 + (x_2 - a_2)^2 = \delta^2$.



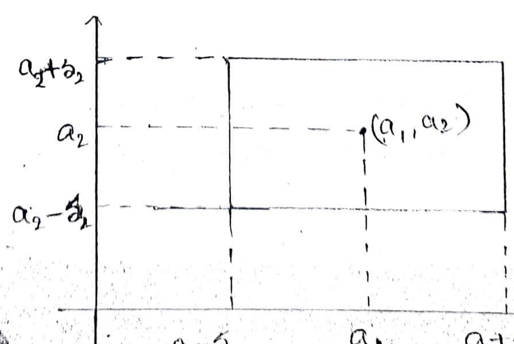
$\{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}$

* Rectangular Neighbourhood of a point:

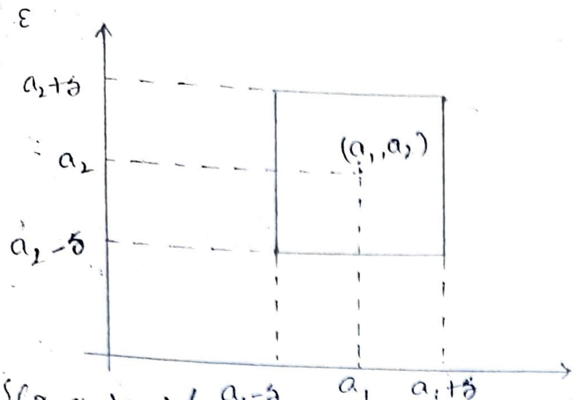
Rectangular neighbourhood of a point $a = (a_1, a_2 \cdots a_n) \in \mathbb{R}^n$ is defined to be the set $\{x \in \mathbb{R}^n / |x_i - a_i| < \delta_i\}, i = 1, 2 \cdots n$

— The rectangular neighbourhood of (a_1, a_2) is $\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1$ and $|x_2 - a_2| < \delta_2\}$

If in particular, $\delta_1 = \delta_2 = \delta$ then such a neighbourhood is referred to as a square neighbourhood of side 2δ.



$\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2\}$



$\{(x_1,x_2)\in\mathbb{R}^2 / |x_1-a_1|<\delta$ and $|x_2-a_2|<\delta\}$

Note:(1) Every spherical neighbourhood of a point in $\mathbb{R}^n$ contains a rectangular neighbourhood of a point and viceversa.

* δ – neighbourhood of a point:

Let $a=(a_1,a_2,\dots a_n)\in\mathbb{R}^n$ and δ be a +ve real number. The set of points $x=(x_1,x_2\dots x_n)\in\mathbb{R}^n$ where $a_r-\delta<x_r<a_r+\delta$

i.e. $|x_r-a_r|<\delta$; $r=1,2\dots n$ is called a δ-neighbourhood of the point 'a' and is denoted by $N(a,\delta)$.

– If we exclude the point 'a' from $N(a,\delta)$ then it is called a deleted neighbourhood of 'a' and is denoted by $N'(a,\delta)$.

* Limit of a function:-

Let $f:X\to\mathbb{R}$, $X\subset\mathbb{R}^n$. Then f is said to tend to limit l as 'x' approaches 'a'.

i.e. $\lim_{x\to a} f(x)=l$

i.e. for a given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x)-l|<\epsilon$ whenever

$0<\|x-a\|<\delta$ (or) $0<|x_r-a_r|<\delta$.

Note:- (1) The limit of a function $f(x)$ as $x\to a$, if it exists at all, is unique.

(2) If $x\to a$ where $n\geq2$ then x approaches 'a' along infinitely many ways unlike the case of $n=1$ when x approaches 'a' along two ways only. (i.e $x\to a-$ & $x\to a+$)

Further for $n\geq2$, x may approaches 'a' along straight lines (or) along different curves.

In the case of $n=1$, the existence of limit of $f(x)$ as $x\to a$ is independent of the two approaches.

In the case $n\geq2$ also, the existence of limit is independent of infinitely many approaches.

(3) For a function of two variables i.e. in the case of $n=2$, two approaches are of special importance. These are

(i) x approaches 'a' first along a line ||lel to the first axis, then along a line ||lel to the second axis.

(ii) x approaches 'a' first along a line ||lel to the second axis and then along a line ||lel to the first axis.

The limits, if they exist, in these approaches are called repeated limits (or) iterated limits.

[i.e. the function $f(x,y)$ of two variables x and y and (x_0, y_0) is the limiting point of a set of values in two dimensional space

If $\lim_{y\to y_0} f(x,y) = g(x)$ then $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$ is defined by $\lim_{x\to x_0} g(x)$.

If $\lim_{x\to x_0} f(x,y) = h(y)$ then $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$ is defined by $\lim_{y\to y_0} h(y)$.

The limits (if exist) $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$ and $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$ are the repeated limits.

— The limit defined above independent of the different approaches is referred to as the double limit (or) Simultaneous limit to distinguish the two approaches.

[i.e. we say that the simultaneous limit exists and is equal to l as $(x,y) \to (x_0, y_0)$.

Symbolically written as

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$ if for given $\epsilon > 0$ (however small) $\exists$ a $\delta > 0$ (depending on ϵ) such that $|f(x,y) - l| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$|x - x_0| < \delta,\ |y - y_0| < \delta$.]

Note!

(i) The double (Simultaneous) limit may exist but the repeated limits may not exist, but if they exist they must be equal to the double limit.

(ii) The repeated limits may exist but the double limit may not exist.

(iii) If the repeated limits are not equal, the simultaneous limit cannot exist.

Non - Existence of Simultaneous limit:

For the existence of simultaneous limit, not only must we have same limiting value if the variable point (x,y) approaches the limiting point (x_0, y_0) through any set of values dense at the point, but we must also have the same limiting value as the variable point approaches its limiting position along any curve what so ever.

Thus, if we can find two methods of approach to the limiting point, which give different limiting values then we can conclude that the simultaneous limit does not exist.

Problems:

→ Show that the simultaneous limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$ doesnot exist.

Sol^n: Let $f(x,y) = \frac{xy^3}{x^2+y^6}$; $(x,y)\neq(0,0)$

If we approach the origin along any axis then $\lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} f(0,y)$

$f(x,y)\to 0$ as $(x,y)\to(0,0)$ along coordinate axes.

If we approach $(0,0)$ along a straight line path $y=mx$.

$$f(x,mx) = \frac{xm^3x^3}{x^2+m^6x^3} = \frac{m^3x^2}{1+m^6x}$$

$$\lim_{x\to 0} f(x,mx) = 0$$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along a straightline path.

If we approach $(0,0)$ along the curve $x = my^3$

$$\therefore f(my^3,y) = \frac{my^3y^3}{m^2y^6+y^6} = \frac{m}{1+m^2}$$

$$\therefore \lim_{y\to 0} f(my^3,y) = \frac{m}{1+m^2} \neq 0.$$

Since the limit dependence upon the value of m.

$\therefore f(x,y)$ approaches different values along the different curves.

$\therefore$ The limit at the origin doesnot exist

Note: The existence of the simultaneous limit

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \Rightarrow$ the single limits

$\lim_{x\to x_0} f(x,y_0)$, $\lim_{y\to y_0} f(x_0,y)$

also exist.

However, it doesnot follow the single limits $\lim_{x\to x_0} f(x,y)$, $\lim_{y\to y_0} f(x,y)$ exist for $y\neq y_0$, $x\neq x_0$ respectively.

→ Show that the simultaneous limit $\lim_{(x,y)\to(0,0)} y\sin\left(\frac{1}{x}\right)$ exists and equal to 0 but the single limit $\lim_{x\to 0} y\sin\frac{1}{x}$ $(y\neq 0)$ doesnot exist.

Sol^n:- Let $\epsilon>0$ be given,

Now we have $\left|y\sin\frac{1}{x} - 0\right| = \left|y\sin\frac{1}{x}\right|$

$$= |y|\left|\sin\frac{1}{x}\right|$$

$$= |y| \quad \left(\because \left|\sin\frac{1}{x}\right|\leq 1\right)$$

$$< \epsilon$$

whenever $0<|y|<\frac{\epsilon}{1} = \delta$ (choosing)

$\therefore \left|y\sin\frac{1}{x} - 0\right| < \epsilon$ whenever $0<|x|<\delta$, $0<|y|<\delta$

$$\therefore \lim_{(x,y)\to(0,0)} y\sin\frac{1}{x} = 0.$$

but for any constant value of $\frac{1}{?}=y_1 \neq 0$, we get

$\lim_{x\to 0} y_1 \sin\frac{1}{x} = y_1 \lim_{x\to 0} \sin\frac{1}{x}$ which doesnot exist.

→ show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ doesnot exist.

Sol'n: If we put $x=my^2$ and let $y\to 0$

$\therefore \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4} = \frac{2m}{(m^2+1)}$ doesnot exist

(∵ the limit dependence upon the value of m).

→ show that $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}=0$.

Sol'n:- put $x=r\cos\theta$; $y=r\sin\theta$

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right| = |r^2\cos\theta\sin\theta(\cos 2\theta)|$$

$$=\left|\frac{r^2}{2}\sin 2\theta\cos 2\theta\right|$$

$$=\left|\frac{r^2}{4} 2\sin 2\theta\cos 2\theta\right|$$

$$=\left|\frac{r^2}{4}\sin(4\theta)\right|$$

$$\leq \frac{r^2}{4} = \frac{x^2+y^2}{4}$$

$$=\frac{x^2}{4}+\frac{y^2}{4}$$

$$<\epsilon$$

if $\frac{x^2}{4}<\frac{\epsilon}{2}$, $\frac{y^2}{4}<\frac{\epsilon}{2}$

i.e. if $|x|<\sqrt{2\epsilon}=\delta$, $|y|<\sqrt{2\epsilon}=\delta$

∴ for $\epsilon>0$, $\exists$ a $\delta>0$.

Such that $\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right|<\epsilon$

whenever $0<|x|<\delta$

$0<|y|<\delta$

$\therefore \lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}=0$

→ show that repeated limits exists when $(x,y)\to(0,0)$

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & ; (x,y)\neq(0,0); \\ 0 & ; (x,y)=(0,0)\end{cases}$$

Sol'n: $\lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0}(0)=0$

and $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{x\to 0}(0)=0$.

∴ the repeated limit exists and are equal. But the simultaneous limit does not exist by putting $y=mx$.

→ $f(x,y)=\frac{y-x}{y+x}\cdot\frac{1+x}{1+y}$ then

$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\left[(-1)\left(\frac{1+x}{1}\right)\right]=-1$

$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\left(\frac{1}{1+y}\right)=1$

∴ The repeated limits exist ~~but~~ are not equal.

∴ The simultaneous limit doesnot exist.

→ Show that the simultaneous limit exists at the origin

$$f(x,y)=\begin{cases}x\left(\sin\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right) & ; xy\neq 0 \\ 0 & ; xy=0\end{cases}$$

Sol'n: Here $\lim_{y\to 0} f(x,y)$, $\lim_{x\to 0} f(x,y)$ donot exist.

$\therefore \lim_{x\to 0}\lim_{y\to 0} f(x,y)$; $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ do not exist.

Now $|f(x,y)-0| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$\leq |x|+|y|<\epsilon$

whenever $0<|x|<\epsilon/2$; $0<|y|<\epsilon/2$

Choosing $\delta = \epsilon/2$

$\therefore |f(x,y)-0|<\epsilon$ whenever $0<|x|<\delta$; $0<|y|<\delta$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

→ Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist,

where $f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$

sol'n: $\lim_{y\to 0} f(x,y) = 1 = \lim_{x\to 0} f(x,y)$

$\therefore \lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1 = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$

$\therefore$ Repeated limits exist and are equal.

Let $(x,y)\to(0,0)$ along the coordinate axes.

$\lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} (0,y)$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the coordinate axes.

Let $(x,y)\to(0,0)$ along any other path.

$\lim_{(x,y)\to(0,0)} f(x,y) = 1$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

→ show that $\lim_{(x,y)\to(0,0)} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exist but $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ doesnot exist.

where $f(x,y) = \begin{cases} y + x\sin(1/y) & \text{if } y\neq 0 \\ 0 & \text{if } y=0 \end{cases}$

sol'n:- Here $\lim_{y\to 0} f(x,y)$ does not exist.

$\therefore \lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Now $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} y$

$= 0.$

$\therefore \lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exists and is equal to 0.

and now $|f(x,y)-0| = \left|y + x\sin\frac{1}{x}\right|$

$\leq |y|+|x| \quad (\because |\sin\frac{1}{y}|\leq 1)$

$<\epsilon$ whenever $0<|x|<\epsilon/2$; $0<|y|<\epsilon/2$

Choosing $\epsilon/2 = \delta$

$\therefore |f(x,y)-0|<\epsilon$ whenever $0<|x|<\delta$; $0<|y|<\delta$.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

* Algebra of Limits :-

— If f, g are two functions defined on some neighbourhood of a point (a,b) such that $\lim_{(x,y)\to(a,b)} f(x,y) = l$.

$\lim_{(x,y)\to(a,b)} g(x,y) = m$ then (i) $\lim f\pm g = \lim f \pm \lim g$

$= l\pm g.$

2). $\lim (f.g) = \lim f . \lim g$

$= l.m$

3) $\lim (f/g) = \frac{\lim f}{\lim g} = \frac{l}{m}$ Provided $m \neq 0$

when $(x,y) \to (a,b)$.

Problems

$\rightarrow \lim\limits_{(x,y)\to(1,2)} (x^2+2y) = \lim\limits_{(x,y)\to(1,2)} x^2 + \lim\limits_{(x,y)\to(1,2)} 2y$

$= 1+2(2)$

$= 1+4$

$= 5$

$\rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{x \sin(x^2+y^2)}{x^2+y^2}$

$= \lim\limits_{(x,y)\to(0,0)} x \lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)}$

$= 0 \cdot 1 = 0$

$\rightarrow \lim\limits_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$

$= \lim\limits_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(3t)}$ [Put $xy-2=t$ and $(x,y)\to(2,1)$ $\Rightarrow t\to 0$]

$= \lim\limits_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{3}{1+9t^2}}$

$= \lim\limits_{t\to 0} \frac{1}{3} \cdot \frac{1+9t^2}{\sqrt{1-t^2}}$

$= \frac{1}{3}$

Continuity:-

$\rightarrow$ A function $f(x,y)$ is said to be continuous at a point (a,b) if

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e. if for given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-f(a,b)|<\epsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$

$\rightarrow$ If f is not continuous at $(a,b) \in D \subset \mathbb{R}^2$ then f is said to be discontinuous at (a,b).

$\rightarrow$ f is said to be continuous on the domain D, if f is continuous at each point of D.

Note:-

Let $D \subset \mathbb{R}^2$ and $f: D \to \mathbb{R}$ be continuous function at $(a,b) \in D$.

Let $f_1(x) = f(x,b)$ then f is a function of single variable 'x'.

Since $f(x,y)$ is continuous at (a,b)

$\Rightarrow \lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e. given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-f(a,b)|<\epsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$; $(x,y)\in D$.

$\Rightarrow |f(x,b)-f(a,b)|<\epsilon$ whenever $|x-a|<\delta$; $(x,b)\in D$.

$\rightarrow |f_1(x)-f_1(a)|<\epsilon$ whenever $|x-a|<\delta$; $(x,b)\in D$.

$\Rightarrow f_1$ is continuous at 'a'.

Similarly, we show that $f_2(y) = f(a,y)$ is continuous at 'b'.

If $f(x,y)$ is continuous at (a,b) then

(i) $f(x,b)$ is continuous at $x=a$ and

(ii) $f(a,y)$ is continuous at $y=b$.

But the converse of above is not true.

i.e. if $f(x,b)$ is continuous at $x=a$ and $f(a,y)$ is continuous at $y=b$.

then $f(x,y)$ need not be continuous at (a,b).

Problem :-

(1) Examine the continuity at $(1,2)$ of the function

$$f(x,y) = \begin{cases} x^2+4y & \text{when } (x,y)\neq(1,2) \\ 0 & \text{when } (x,y)=(1,2) \end{cases}$$

Sol'n: $\lim\limits_{(x,y)\to(1,2)} f(x,y) = \lim\limits_{(x,y)\to(1,2)} (x^2+4y)$

$= \lim\limits_{(x,y)\to(1,2)} x^2 + \lim\limits_{(x,y)\to(1,2)} 4y$

$= 1^2+4(2)$

$= 1+8 = 9$ and

$f(1,2)=0$

$\therefore \lim\limits_{(x,y)\to(1,2)} f(x,y) \neq f(1,2)$.

$\therefore f(x,y)$ is not continuous at $(1,2)$

→ Show that the function $f: \mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{(y^2-x^2)yx}{x^2+y^2} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0) \end{cases}$$

is continuous at $(0,0)$.

Sol'n: Let $\epsilon>0$ given, Now we have

$$|f(x,y)-f(0,0)| = \left|\frac{(y^2-x^2)yx}{x^2+y^2} - 0\right|$$

$$= \left|\frac{(y^2-x^2)}{x^2+y^2}\, yx\right|$$

$$= \left|\frac{y^2-x^2}{x^2+y^2}\right| |xy|$$

$$\leq |xy| \quad \left[\because \left|\frac{y^2-x^2}{x^2+y^2}\right| \leq 1 \text{ for } f(x,y)\neq(0,0)\right]$$

$= |x|\,|y|$

$< \epsilon$ whenever $|x|<\sqrt{\epsilon}$ & $|y|<\sqrt{\epsilon}$

Choosing $\sqrt{\epsilon}=\delta$

$\therefore |f(x,y)-f(0,0)|<\epsilon$ whenever $|x|<\delta$, $|y|<\delta$.

$\therefore f(x,y)$ is continuous at $(0,0)$

→ Discuss the continuous of the function

(2)
$$f(x,y) = \begin{cases} \dfrac{2xy^2}{x^3+3y^3} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Sol'n: $f(0,0)=0$

Let $(x,y)\to(0,0)$ along the coordinate axes then $\lim\limits_{x\to0} f(x,0) = 0 = \lim\limits_{y\to0} f(0,y)$.

$\therefore f(x,y)\to0$ as $(x,y)\to(0,0)$ along the coordinate axes.

Let $(x,y)\to(0,0)$ along straight line $y=x$ then $\lim\limits_{x\to0} f(x,x) = \lim\limits_{x\to0} \dfrac{2x\cdot x^2}{x^3+3x^3}$

$= \lim\limits_{x\to0} \dfrac{2}{4} = \dfrac{1}{2}$.

$\therefore f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along the straight line path.

Since the two methods of approach to the limiting points give different limiting values.

$\therefore$ the simultaneous limits do not exist.

i.e, $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ does not exist

$\therefore f(x,y)$ is not continuous at $(0,0)$.

$\rightarrow$ Show that function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ;(x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0) \end{cases}$$ is

continuous at $(0,0)$.

Sol'n: Let $\epsilon>0$ be given

Now we have $|f(x,y)-f(0,0)| = \left|\dfrac{xy}{\sqrt{x^2+y^2}}-0\right|$

$= \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|$

$= |xy| \cdot \dfrac{1}{\sqrt{x^2+y^2}}$

$= \left[\dfrac{xy}{x^2+y^2}\right] \sqrt{x^2+y^2}$

$\leq \frac{1}{2}\sqrt{x^2+y^2}$ $\left[\because 2|xy| \leq x^2+y^2 \Rightarrow \left|\dfrac{xy}{x^2+y^2}\right| \leq \frac{1}{2} \text{ if } (x,y)\neq(0,0)\right]$

$< \sqrt{x^2+y^2}$

$< \epsilon$

whenever $x^2+y^2 < \epsilon^2 = \delta$ (choosing)

$\therefore |f(x,y)-f(0,0)| < \epsilon$ whenever $x^2+y^2<\epsilon^2$

$\therefore f(x,y)$ is continuous at $(0,0)$

(or)

Let $x = r\cos\theta$; $y = r\sin\theta$

$|f(x,y)-f(0,0)| = \dfrac{xy}{\sqrt{x^2+y^2}}$

$= \dfrac{|r^2\sin\theta\cos\theta|}{r}$

$= r|\sin\theta||\cos\theta|$

$\leq r$ $(\because |\sin\theta|\leq 1 \;\&\; |\cos\theta|\leq 1)$

$= \sqrt{x^2+y^2}$

$< \epsilon$ whenever $x^2+y^2<\epsilon^2=\delta$.

$\therefore |f(x,y)-f(0,0)|<\epsilon$ whenever $x^2+y^2<\delta$

$\therefore f(x,y)$ is continuous at $(0,0)$.

H.W. $\rightarrow$ Show that the following functions are discontinuous at $(0,0)$

(i) $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

(ii) $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^4+y^4} & ; (x,y)\neq 0 \\ 0 & ; (x,y)=(0,0) \end{cases}$

(iii) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)\neq(0,0) \end{cases}$

$\rightarrow$ Show that the following functions are continuous at the origin.

(i) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

(ii) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

(iii) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

$\rightarrow$ Discuss the following function for continuity at $(0,0)$

$f(x,y) = \begin{cases} \dfrac{x^2y}{x^3+y^3} & ; x^3+y^3\neq 0 \text{ i.e. } (x,y)\neq(0,0) \\ 0 & ; x=y=0 \text{ i.e., } (x,y)=(0,0) \end{cases}$

6

Partial Derivatives

→ The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables constant is called the partial derivative of the function w.r.t the variable.

→ Partial derivative of $f(x,y)$ w.r.t x denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x,y)$, while those w.r.t y denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x,y)$.

$$\therefore \frac{\partial f}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\text{and } \frac{\partial f}{\partial y} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$ when these limits exist.

→ The partial derivatives at a particular point (a,b) are often denoted by $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}$, $\frac{\partial f(a,b)}{\partial x}$ or $f_x(a,b)$ and

$$\left(\frac{\partial f}{\partial y}\right)_{(a,b)}, \frac{\partial f(a,b)}{\partial y}, f_y(a,b)$$

$$\therefore f_x(a,b) = \underset{h \to 0}{Lt} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\text{and } f_y(a,b) = \underset{k \to 0}{Lt} \frac{f(a,b+k) - f(a,b)}{k}$$

Note: we have, by definition

$$f_x(a,b) = \underset{h \to 0}{Lt} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \underset{x \to a}{Lt} \frac{f(x,b) - f(a,b)}{x-a}$$

and

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$= \lim_{y\to b} \frac{f(a,y) - f(a,b)}{y-b}$$

Problem:

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$f(x,y) = x^2 + xy + y^3$.

Find $f_x(x,y)$ and $f_y(x,y)$.

Soln: By definition,

$$f_x(x,y) = \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + (x+h)y + y^3 - x^2 - xy - y^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{xy} + hy + \cancel{y^3} - \cancel{x^2} - \cancel{xy} - \cancel{y^3}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + hy}{h}$$

$$= \lim_{h\to 0} 2x + h + y.$$

$$= 2x + y.$$

Similarly

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{x^2 + x(y+k) + (y+k)^3 - x^2 - xy - y^3}{k}$$

$$= \lim_{k\to 0} \frac{xk + 3y^2k + 3k^2y + k^3}{k}$$

$$= \lim_{k\to 0} x + 3y^2 + 3ky + k^2$$

$$= x + 3y^2$$

→ Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$f(x,y,z) = xy+yz+zx$.

Find the partial derivatives f_x, f_y, f_z at (a,b,c).

Sol^n: Given $f(x,y,z) = xy+yz+zx$.

By definition

$$f_x(a,b,c) = \underset{h\to 0}{Lt} \frac{f(a+h,b,c)-f(a,b,c)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(a+h)b+bc+c(a+h)-ab-bc-ca}{h}$$

$$= \underset{h\to 0}{Lt} \frac{bh+ch}{h}$$

$$= b+c$$

$$f_y(a,b,c) = \underset{k\to 0}{Lt} \frac{f(a,b+k,c)-f(a,b,c)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{a(b+k)+(b+k)c+ca-ab-bc-ca}{k}$$

$$= a+c$$

$$f_z(a,b,c) = \underset{l\to 0}{Lt} \frac{f(a,b,c+l)-f(a,b,c)}{l}$$

$$= \underset{l\to 0}{Lt} \frac{ab+b(c+l)+(c+l)a-ab-bc-ca}{l}$$

$$= b+a$$

→ Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function defined by

$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots\ldots + x_n^2$

Find the f_{x_i} at the point $(a_1, a_2, \ldots\ldots a_n)$

Sol^n: To find the partial derivative of f w.r.t x_i at the point $(a_1, a_2, \ldots a_n)$;

we write

$$f_{x_i}(a_1, a_2, \dots a_n) = \lim_{h\to 0} \frac{f(a_1, a_2, \dots a_i+h, a_{i+1} \dots a_n) - f(a_1, a_2, \dots a_n)}{h}$$

$$= \lim_{h\to 0} \frac{a_1^2 + a_2^2 + \dots + a_{i-1}^2 + (a_i+h)^2 + a_{i+1}^2 + \dots + a_n^2 - (a_1^2 + a_2^2 + \dots a_n^2)}{h}$$

$$= \lim_{h\to 0} \frac{2a_i h + h^2}{h} = 2a_i$$

Note : $f_x(x,y)$ is nothing but the derivative of $f(x,y)$ considered as a function of a single variable x, treating y as a constant.

Similarly $f_y(x,y)$ is nothing but the derivative of $f(x,y)$ considering it as a function of the single variable y, and treating x as a constant.

In general, $f_{x_i}(x_1, x_2, \dots x_n)$ is the derivative of $f(x_1, x_2 \dots x_n)$ w.r.t x_i treating all the other variables as constants.

→ Let us find the partial derivatives of the following functions.

(i) $f = x^3 - 4x^2y^2 + 3y^2$

(ii) $f = x \sin y + y \cos x$

(iii) $f = xe^y + ye^x$.

Sol: In all the three cases, the functions involved are either polynomials or trigonometric or exponential functions.

This ensures that the partial derivatives exist (∵ the polynomial, trigonometric and exponential functions of single variable are differentiable).

∴ By direct differentiation, we get

(i) $\frac{\partial f}{\partial x} = 3x^2 - 8xy^2$

and $\frac{\partial f}{\partial y} = -8x^2y + 16y$

(ii) $\frac{\partial f}{\partial x} = \sin y - y\sin x$

and $\frac{\partial f}{\partial y} = x\cos y + \cos x$.

(iii) $\frac{\partial f}{\partial x} = e^y + ye^x$.

and $\frac{\partial f}{\partial y} = xe^y + e^x$.

→ If $f(x,y) = 2x^2 - xy + 2y^2$.
Then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(1,2)$.

Sol^n: $\frac{\partial f}{\partial x} = 4x - y = 2$ at $(1,2)$

$\frac{\partial f}{\partial y} = -x + 4y = 7$ at $(1,2)$.

Note: The calculation of partial derivatives is not always as simple as in these examples.

In some exceptional cases, we have to use the limiting process.

→ Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Find the two partial derivatives at the points $(0,0)$, $(a,0)$, $(0,b)$ and (a,b) where $a \neq 0$, $b \neq 0$.

Soln: By definition

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$f_x(a,0) = \lim_{h\to 0} \frac{f(a+h,0) - f(a,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(a,0) = \lim_{k\to 0} \frac{f(a,0+k) - f(a,0)}{k} = \lim_{k\to 0} \frac{f(a,k) - f(a,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{ak}{a^4+k^4} - 0}{k} = \lim_{k\to 0} \frac{a}{a^4+k^4}$$

$$= \frac{a}{a^4} = \frac{1}{a^3}$$

$$f_x(0,b) = \lim_{h\to 0} \frac{f(0+h,b) - f(0,b)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{bh}{b^4+h^4} - 0}{h} = \frac{1}{b^3}$$

17

$$f_y(0,b) = \lim_{k \to 0} \frac{f(0,b+k) - f(0,b)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(a+h)b}{(a+h)^4+b^4} - \frac{ab}{a^4+b^4}}{h}$$

$$= \lim_{h \to 0} \frac{(ab+hb)(a^4+b^4) - (ab)\left[(a+h)^4+b^4\right]}{h(a^4+b^4)\left[(a+h)^4+b^4\right]}$$

$$= \frac{b^5 - 3a^4b}{(a^4+b^4)^2}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{a(b+k)}{a^4+(b+k)^4} - \frac{ab}{a^4+b^4}}{k}$$

$$= \frac{a^5 - 3ab^4}{(a^4+b^4)^2}$$

Note: In the above problem, by direct differentiation, we could have obtained $f_x(a,b)$ and $f_y(a,b)$ correctly, $(a,b) \neq (0,0)$, but not $f_x(0,0)$ or $f_y(0,0)$.

because f is defined as a quotient of two polynomial functions for all $(x,y) \neq (0,0)$, we can use direct differentiation to calculate the partial derivatives at these points. But to calculate $f_x(0,0)$ or $f_y(0,0)$ we need to use $f(0,0)$, which is not defined.

→ If $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{x}{y} + \frac{y}{x}, & y \neq 0, x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then show that $f_x(0,1)$ and $f_y(1,0)$ do not exist.

Solⁿ: By definition

$$f_x(0,1) = \lim_{h\to 0} \frac{f(0+h,1) - f(0,1)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,1) - f(0,1)}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h\left[1 + \frac{1}{h^2}\right]}{h}$$

$$= \lim_{h\to 0} \left(1 + \frac{1}{h^2}\right) = \infty$$

and $$f_y(1,0) = \lim_{k\to 0} \frac{f(1,0+k) - f(1,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{1}{k} + k - 0}{k}$$

$$= \lim_{k\to 0} \frac{1}{k^2} + 1$$

$$= \infty$$

$\therefore f_x(0,1)$ and $f_y(1,0)$ do not exist.

→ **Note**:

The existence of partial derivatives at a point need not imply continuity at that point.

For example:

2004. If $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$

Show that both the partial derivatives exist at (0,0) but the function is not continuous at (0,0).

Soln:

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to0}\frac{0-0}{h}$$

$$=0$$

$$f_y(0,0)=\lim_{k\to0}\frac{f(0,0+k)-f(0,0)}{k}$$

$$=\lim_{k\to0}\frac{0-0}{k}$$

$$=0$$

$\therefore f$ possesses both the partial derivatives at (0,0).

Now let $(x,y)\to(0,0)$ along the straight line $y=mx$.

$$\lim_{x\to0} f(x,mx)=\lim_{x\to0}\frac{mx^2}{x^2(1+m^2)}$$

$$=\frac{m}{1+m^2}$$

which depends upon m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

$\therefore f(x,y)$ is not continuous at (0,0)

HW Find $f_x(0,0)$ and $f_y(0,0)$ & $f_y(0,0)$

if $f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

→ If $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & ; x \neq y \\ 0 & ; x = y. \end{cases}$

then show that f is discontinuous at the origin but the partial derivatives exist at the origin.

Soln: Let $(x,y) \to (0,0)$ along the curve $y = x - mx^3$.

2005 → Show that the given below is not continuous at $(0,0)$

$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$

Soln: Let $(x,y) \to (0,0)$ along the co-ordinate axes.

$\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} 0 = 0$

and $\lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} 0 = 0$

$\therefore \lim\limits_{x\to 0} f(x,0) = 0 = \lim\limits_{y\to 0} f(0,y)$

$\therefore f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the co-ordinate axes.

Let $(x,y) \to (0,0)$ along any other path

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$.

Since the two methods of approach to the limiting point give different limiting values.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$.

2006. P-II. → Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

(i) is continuous at $(0,0)$

(ii) possesses partial derivatives $f_x(0,0)$ & $f_y(0,0)$

2007 P-II. → Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$ but partial derivatives f_x & f_y exist at $(0,0)$.

→ Examine the continuity of the function $f(x,y) = \sqrt{|xy|}$ at the origin.

Note: We know that a real valued continuous function of a real variable need not be differentiable. The same is true for functions of several variables.

of several variables which, i.e., function of several ~~variables~~ variables which is continuous at a point need not have any of the partial derivatives at the point.

For example:

→ Let $f: \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|.$$

Show that f is continuous at $(0, 0, 0)$ but doesnot possess any of the three first order partial derivatives at $(0, 0, 0)$.

Sol^n:

Now at the point $(0,0,0)$,

we have
$$\frac{f(0+h,0,0)-f(0,0,0)}{h}$$

$$= \frac{|h|}{h} = f_1(h) \text{ (say)}$$

$$\therefore \underset{h\to 0^+}{Lt} f_1(h) = \underset{h\to 0^+}{Lt} \frac{|h|}{h}$$

$$= 1$$

but $\underset{h\to 0^-}{Lt} f_1(h) = \underset{h\to 0^-}{Lt} \frac{|h|}{h} = -1$

Hence $\underset{h\to 0}{Lt} f_1(h)$ does not exist.

Similarly,

$\underset{h\to 0}{Lt} f_2(h)$ and $\underset{h\to 0}{Lt} f_3(h)$ also do not exist.

$\therefore f$ doesnot possess any of the first order partial derivatives at the point $(0,0,0)$.

But this function is continuous at $(0,0,0)$.

→ We have seen that the existence of partial derivatives doesnot imply continuity. However, if the partial derivatives satisfy some more conditions, then we can ensure continuity. In order to prove this theorem we need a simple result which follows easily from Lagrange's mean value theorem.

Mean value theorem:

If f_x exists throughout a nbd of a point (a,b) and $f_y(a,b)$ exists then for any point $(a+h, b+k)$ of this nbd,

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k\left(f_y(a,b) + \eta\right)$$

where $0<\theta<1$ and η is a function of k, which tends to '0' as $k \to 0$

Note: we can see that this is an extension of Lagrange's mean value theorem to functions from $\mathbb{R}^2 \to \mathbb{R}$.

- Interchanging x and y in the above theorem, the theorem can be written as

" If f_y exists throughout a nbd of a point (a,b) and $f_x(a,b)$ exists then for any point $(a+h, b+k)$ of this nbd,

$$f(a+h, b+k) - f(a,b) = k f_y(a+h, b+\theta' k) + h\left(f_x(a,b) + \eta'\right)$$

where $0<\theta'<1$ and η' is a function tends zero as $h \to 0$. "

→ A Sufficient Condition for continuity:

A sufficient condition that a function

f be continuous at (a,b) is that one of the partial derivatives exists and is bounded in a nbd of (a,b) and that the other exists at (a,b).

Proof:

Let f_x exist and be bounded in a nbd of (a,b). and let $f_y(a,b)$ exist.

then for any point $(a+h, b+k)$ of this nbd

we have

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta] \quad (1)$$

where $0<\theta<1$, and $\eta \to 0$ as $k \to 0$.
proceeding to limits as $(h,k) \to (0,0)$,
Since f_x bounded in a nbd of N,

it follows that

$$\underset{(h,k)\to(0,0)}{Lt} h f_x(a+\theta h, b+k) = 0$$

Consequently, from (1), we get

$$\underset{(h,k)\to(0,0)}{Lt} f(a+h, b+k) = f(a,b).$$

$\Rightarrow f$ is continuous at (a,b).

Note: A sufficient condition that a function be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

Proof of the mean value theorem

Now $f(a+h, b+k) - f(a,b) = f(a+h, b+k)$
$- f(a, b+k) + f(a, b+k)$
$- f(a,b)$. —— (1)

Since f_x exists throughout a nbd of a point (a,b),
therefore by Lagrange's mean value theorem,

$$f_x(a+\theta h, b+k) = \frac{f(a+h, b+k) - f(a, b+k)}{h}, \quad 0<\theta<1$$

$$\Rightarrow f(a+h, b+k) - f(a, b+k) = h\, f_x(a+\theta h, b+k), \quad 0<\theta<1 \quad \text{—— (2)}$$

Also $f_y(a,b)$ exists, so that

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k} = f_y(a,b).$$

$$\Rightarrow \lim_{k\to 0}\left[f(a, b+k) - f(a,b)\right] = \lim_{k\to 0} k\left[f_y(a,b) + \eta(k)\right]$$

where $\eta(k)$ is a function of k

$$\therefore f(a, b+k) - f(a,b) = k f_y(a,b) + k\eta \quad \text{—— (3)}$$

where η is a function of k and tends to zero as $k \to 0$.

$\therefore$ From (1), (2) and (3), we have —

$$f(a+h, b+k) - f(a,b) = h\, f_x(a+\theta h, b+k) + k\left[f_y(a,b) + \eta\right].$$

which is the required result.

Differentiability:

Let f be a real valued function defined in a nbd N of a point (a,b).

We say that the function f is differentiable at (a,b), if

$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$.

where

- h and k are real numbers such that $(a+h, b+k) \in N$
- A and B are constants independent of h and k but dependent on the function f and the point (a,b)
- ϕ and ψ are two functions tending to zero as $(h,k) \to (0,0)$.

(or)

Let f be a real a real-valued function defined in a nbd N of a point (a,b). Then the function f is said to be differentiable at the point (a,b), if there exist two constants A and B (depending on f and the point (a,b) only) such that

$f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$;

where $\phi(h,k)$ is real valued function such that $\phi(h,k) \to 0$. as $(h,k) \to (0,0)$.

→ Let $f(x,y) = x^2+y^2$. S.T f is differentiable at any point (a,b).

Sol^n: For any two real numbers h and k. we have

$f(a+h, b+k) - f(a,b) = (a+h)^2 + (b+k)^2 - (a^2+b^2)$
$= 2ah + h^2 + 2bk + k^2$

$= 2ah + 2bk + hh + kk.$

If we let $A = 2a$, $B = 2b$, $\phi(h,k) = h$ and $\psi(h,k) = k$.

Then

$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$

where A and B are constants independent of h & k.

$\because \phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\therefore f$ is differentiable at the point (a,b).

→ Let $f(x,y) = \frac{x}{y}$. Then show that f is differentiable at all points (a,b) in the domain of definition of the function.

Sol^n. Given $f(x,y) = \frac{x}{y}$.

Since f is not defined for $y = 0$.

We take $b \neq 0$.

Let h and k be two real numbers such that $(a+h, b+k)$ is a point in a nbd N of (a,b) which is contained in the domain of f.

Then $b+k \neq 0$.

and $f(a+h, b+k) - f(a,b) = \frac{a+h}{b+k} - \frac{a}{b}$

$= \frac{a}{b+k} - \frac{a}{b} + \frac{h}{b+k}$

$= \frac{-ak}{b(b+k)} + \frac{h}{b+k}$

$= -\frac{ak}{b^2}\left[1 - \frac{k}{b+k}\right] + \frac{h}{b}\left[1 - \frac{h}{b+k}\right]$

$= \frac{1}{b}h - \frac{a}{b^2}k + h\left[\frac{-k}{b(b+k)}\right] + k\left(\frac{ak}{b^2(a+k)}\right)$

Then let $A = \frac{1}{b}$, $B = -\frac{a}{b^2}$, $\phi(h,k) = \frac{-k}{b(b+k)}$ and $\psi(h,k) = \frac{ak}{b^2(b+k)}$

Then

$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$

where A and B are constants independent of h and k, $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Hence f is differentiable at (a,b).

→ Prove that the function given by $f(x,y) = |x| + |y|$ is not differentiable at $(0,0)$.

Sol^n: Suppose, if possible, that f is differentiable at $(0,0)$

Then

$f(0+h, 0+k) - f(0,0) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$

where A and B are constants

$\phi(h,k)$, and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\therefore |h| + |k| = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ —— (1)

Let $h=0$ and $k>0$, then from (1),

$k = Bk + k\psi(0,k) \Rightarrow 1 = B + \psi(0,k)$

Taking limits on both sides as $(h,k) \to (0,0)$

we get $B = 1$, because $\psi(0,k) \to 0$

Now let $h=0$ and $k<0$, then $\left(|x| = x \text{ if } x>0, = -x \text{ if } x<0\right)$

$-k = Bk + k\psi(0,k)$

$-1 = B + \psi(0,k)$.

Taking limits on both sides as $(h,k) \to (0,0)$

we get $B = -1$, because $\psi(0,k) \to 0$

$\therefore$ The assumption that the given function is differentiable at $(0,0)$ leads us to the contradiction $B = 1 = -1$.

$\therefore |x| + |y|$ is not differentiable at $(0,0)$

Theorem $\Rightarrow$ Let f be real valued function defined in a nbd N of a point (a,b). if f is differentiable at (a,b), then f is continuous at (a,b).

The above theorem shows that continuity in the two variables is a necessary condition for differentiability. However, it is not a sufficient condition.

For example.

$\rightarrow$ Show that the function defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

is continuous at the origin but is not differentiable.

Sol^n: we verified it is continuous in the pg. no. 5 (back pages)

Now we prove that $f(x,y)$ is not differentiable at the origin.

we have

$$f(0+h, 0+k) = \frac{hk}{\sqrt{h^2+k^2}}$$

$$\therefore f(0+h,0+k) - f(0,0) = 0\cdot h + 0\cdot k + \sqrt{h^2+k^2}\cdot \frac{hk}{h^2+k^2}$$

So that $A=0$, $B=0$ and $\phi(h,k) = \frac{hk}{h^2+k^2}$.

i.e, A and B are independent of h, k.

If we put $k = mh$, then we have

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{h^2+m^2h^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{h^2(1+m^2)} = \frac{m}{1+m^2}$$

$\therefore$ This Limit does not exist since it depends upon m.

$\therefore$ $\not\to \phi(h,k) \neq 0$ as $(h,k) \to (0,0)$.

It follows that the given function is not differentiable at $(0,0)$.

Note: To show that the function is not differentiable, it enough to show that f is not continuous.

For example,

The function f,

where $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$

is not differentiable at the origin because it is discontinuous there at.

→ Show that the function f,

where $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$

is not differentiable

→ Show that the following functions are not differentiable at $(0,0)$ by showing that they are discontinuous at $(0,0)$

(1) $f(x,y) = \begin{cases} \frac{x^2+y^2}{x-y} & x \neq y \\ 1 & x=y \end{cases}$

(2) $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 2 & (x,y)=(0,0) \end{cases}$

(3) $f(x,y) = \begin{cases} \frac{x^5}{x^4+y^4} & (x,y) \neq (0,0) \\ 3 & (x,y) = (0,0) \end{cases}$

[Hint: $x^4+y^4 \geq x^4 \Rightarrow |f(x,y)| = \left|\frac{x^5}{x^4+y^4}\right| \leq \left|\frac{x^5}{x^4}\right| = |x|$]

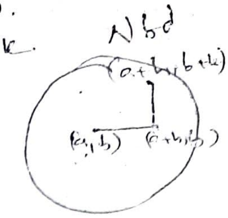
→ If f is differentiable at (a,b) then f possesses both the partial derivatives at (a,b).

Sol since f is differentiable at (a,b)

$\therefore f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ — (1)

where A and B are constants depend on f & the point (a,b).

and independent of h & k



$\phi(h,k) \to 0, \; \psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

If $(a+h, b+k)$ belonging to the nbd of (a,b)

then $(a+h, b)$ and $(a, b+k)$ also belong to nbd of (a,b)

$\therefore$ If we take $k=0$ in (1) then, we have

$f(a+h, b+0) - f(a,b) = Ah + h\,\phi(h,0)$

$\Rightarrow \dfrac{f(a+h,b) - f(a,b)}{h} = A + \phi(h,0)$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h} = A$ $\left(\because \lim\limits_{(h,k)\to(0,0)} \phi(h,0) = 0\right)$

$\Rightarrow \boxed{f_x(a,b) = A} \Rightarrow A = \left(\dfrac{\partial f}{\partial x}\right)_{(a,b)}$.

Similarly we can prove that $B = \left(\dfrac{\partial f}{\partial y}\right)_{(a,b)} = f_y(a,b)$.

from this we see that for small values of h and k we can approximate $f(a+h, b+k) - f(a,b)$ by the expression $h f_x(a,b) + k f_y(a,b)$

i.e $f(a+h, b+k) - f(a,b) \approx h f_x(a,b) + k f_y(a,b)$.

<u>Def^n</u>. Let $f(x,y)$ be a real-valued function defined in a nbd of the point (a,b). then If $f(x,y)$ is differentiable at (a,b) then the linear function $T: \mathbb{R}^2 \to \mathbb{R}$ defined by

$T(h,k) = h f_x(a,b) + k f_y(a,b)$, is called the differential of f at (a,b) and is denoted by $df(a,b)$.

→ Note!

The converse of the above need not be true. i.e. f is continuous and possesses partial derivative at a point then f need not be differentiable at that point.

For example:-

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$ is continuous

and possesses partial derivatives but is not differentiable at the origin.

Sol

put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \left| r(\cos^3\theta - \sin^3\theta) \right|$$

$$\le |r| \left[|\cos^3\theta| + |\sin^3\theta| \right]$$

$$\le |r| [1+1]$$

$$= 2|r|.$$

$$= 2\sqrt{x^2+y^2} < \epsilon$$

whenever $x^2 < \frac{\epsilon^2}{8}$, $y^2 < \frac{\epsilon^2}{8}$

(or)

$|x| < \frac{\epsilon}{2\sqrt{2}}$, $|y| < \frac{\epsilon}{2\sqrt{2}}$

$\therefore \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| < \epsilon$ whenever $|x| < \frac{\epsilon}{2\sqrt{2}}$, $|y| < \frac{\epsilon}{2\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

$\therefore f(x,y)$ is conti. at $(0,0)$.

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Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

and $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{-k^3}{k^2} - 0}{k} = \lim_{k\to 0} \frac{-k^3}{k^3} = -1$$

$\therefore$ f possesses partial derivatives at $(0,0)$.

Now we prove that f is not differentiable at $(0,0)$.

If possible suppose that f is differentiable at $(0,0)$.

Then $f(0+h, 0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + \sqrt{h^2+k^2}\,\phi(h,k)$

where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

where $A = f_x(0,0) = 1$
and $B = f_y(0,0) = -1$.

$\Rightarrow f(h,k) = h - k + \sqrt{h^2+k^2}\,\phi(h,k)$

$\Rightarrow \phi(h,k) = \dfrac{f(h,k) - h + k}{\sqrt{h^2+k^2}}$ where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

i.e, $\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \dfrac{f(h,k)-h+k}{\sqrt{h^2+k^2}} = 0$

Now if $h = r\cos\theta$, $k = r\sin\theta$.

then

$$\frac{f(h,k)-h+k}{\sqrt{h^2+k^2}} = \frac{\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2(\cos^2\theta+\sin^2\theta)} - r\cos\theta + r\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta.$$

$$\therefore 0 = \lim_{(h,k)\to(0,0)} \frac{f(h,k) - h + k}{\sqrt{h^2+k^2}} = \lim_{r\to 0} (\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta) \quad \text{—— (1)}$$

Since the expression $\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$ is independent of r and (1) implies that

$\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta = 0 \quad \forall \theta.$

which is impossible for arbitrary θ.

$\therefore$ our assumption that f is differentiable is wrong.

f is not differentiable at $(0,0)$

Sufficient condition for differentiability:

Theorem

→ If (a,b) be a point of the domain of definition of a function f such that

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b)

then f is differentiable at (a,b).

Similarly, the statement that f is differentiable at (a,b) if f_x exists at (a,b) and f_y is continuous at (a,b) is true.

i.e, the continuity of one of partial derivatives and the existence of other guarantees the differentiability of the function under consideration.

Note: The conditions of the theorem are not necessary for differentiability. i.e, a function can be differentiable at a point even when none of the partial derivatives is continuous at that point.

However if the function is not differentiable at a point the partial derivatives cannot be continuous thereat.

for example:

Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & ; \text{ if } xy \neq 0 \\ x^2 \sin\frac{1}{x} & ; \text{ if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin\frac{1}{y} & ; \text{ if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{ if } x = y = 0. \end{cases}$$

Prove that f is differentiable at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$.

Sol^n: Here the partial derivatives at $(0,0)$ are given by

$$f_x(x,y) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

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and $$f_y(x,y) = \begin{cases} 2y \sin\frac{1}{y} - \cos\frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Since $\underset{x \to 0}{Lt} \cos\frac{1}{x}$ does not exist and $\underset{t \to 0}{Lt}\, t \sin\frac{1}{t} = 0.$

$\therefore \underset{(x,y)\to(0,0)}{Lt} f_x(x,y)$ and $\underset{(x,y)\to(0,0)}{Lt} f_y(x,y)$ do not exist.

i.e., f_x and f_y are discontinuous at $(0,0)$.

$\therefore$ Both the partial derivatives exist at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$

Now show that the function is differentiable at $(0,0)$.

Here
$$f(h,k) - f(0,0) = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k}$$
$$= 0h + 0k + h \cdot h \sin\frac{1}{h} + k \cdot \sin\frac{1}{k}$$

Now $\underset{(h,k)\to(0,0)}{Lt} h \sin\frac{1}{h} = 0$ and $\underset{(h,k)\to(0,0)}{Lt} k \sin\frac{1}{k} = 0$

$\therefore f$ is differentiable at $(0,0)$.

Note:

→ A real valued function f of two variables is said to be continuously at a point (a,b) if both the first order partial derivatives exist in a nbd of (a,b) and are continuous at the point (a,b).

→ A function, which is continuously differentiable at a point is differentiable at that point.

→ Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

Show that f is differentiable at $(0,0)$.

Sol:

$$\text{Now } f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= 0$$

Similarly $f_y(0,0) = 0$

and for $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

Using polar co-ordinates:

$x = r\cos\theta$; $y = r\sin\theta$

we get $|f_x(x,y) - f_x(0,0)| = r|\cos^4\theta\sin\theta + 4\cos^2\theta\sin^3\theta - \sin^5\theta|$

$$\leq r\left[|\cos^4\theta\sin\theta| + 4|\cos^2\theta\sin^3\theta| + |\sin^5\theta|\right]$$

$$= 6r \quad (\because \sin\theta \leq 1 \text{ \& } \cos\theta \leq 1)$$

$$= 6\sqrt{x^2+y^2}.$$

$$< \epsilon \text{ if } |x| < \frac{\epsilon}{\sqrt{72}} \text{ and } |y| < \frac{\epsilon}{\sqrt{72}}.$$

$$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

∴ we have f_x is continuous at $(0,0)$ and f_y exists at $(0,0)$.

∴ f is differentiable at $(0,0)$.

→ S.T. $f(x,y) = e^{x+y}\sin x + x^2 + 2xy$ is differentiable everywhere.

Sol: Since $f_x(x,y) = e^{x+y}\sin x + e^{x+y}\cos x + 2x + 2y$ and $f_y(x,y) = e^{x+y}\sin x + 2x$ are continuous every where. ∴ f is differentiable everywhere.

PARTIAL DERIVATIVES OF HIGHER ORDER

we studied partial derivatives of first order and differentiability. we must have seen often, that partial derivatives of first order again define functions.

For example:

if $f(x,y) = 3x^3 + 2xy^2 + 5y^2 + 6$.

then $f_x(x,y) = 9x^2 + 2y^2$

and $f_y(x,y) = 4xy + 10y$ are again real valued functions of two variables.

with the domain $\mathbb{R}^2$.

Thus we can talk of first order partial derivatives of these new functions

- If we consider a function of two variables, there are two first order partivatives; which may give rise to four more partial derivatives, which might again turn out to be function. If this chain continues, then we obtain higher order partial derivatives.

- In general, let $D \subset \mathbb{R}^2$ and let $f: D \to \mathbb{R}$ have a first order partial derivative $f_x(x,y)$ at every point of D. This new function f_x, which is defined on D may or may not possess first order partial derivatives. In case it does, then f_{xx} and f_{xy} are called

the second order partial derivatives of f.
Similarly, if the function f has a first order partial derivative f_y at every point of D, then f_y defines a new function and if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{xy} and f_{yy}.

If $f(x,y)$ is a real-valued function defined in a nbd of (a,b) having both the partial derivatives at all the points of the nbd, then

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{(say)}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{(say)}$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{(say)}$$

$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{(say)}$$

provided each one of these limits exists.

The second order partial derivatives of f are also denoted by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}; \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}; \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, \quad \frac{\partial^2 f(a,b)}{\partial x^2}, \quad f_{xx}(a,b) \text{ or } f_{x^2}(a,b);$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}, \quad \frac{\partial^2 f(a,b)}{\partial x \partial y} \text{ or } f_{xy}(a,b) \text{ and so on}$$

In a similar manner partial derivatives of order higher than two are defined.

for example

$$\frac{\partial^3 f}{\partial x \partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= f_{xxy} \text{ or } f_{x^2 y}$$

i.e $\frac{\partial^3 f}{\partial x \partial x \partial y}$ stands for the partial derivative of $\frac{\partial^2 f}{\partial x \partial y}$ with respect to x.

→ Find all the second order partial derivatives of the following function.

(i) $f(x,y) = x^3 + y^3 + 3axy$, 'a' is constant

(ii) $f(x,y,z) = x^2 + yz + xz^3$.

Sol (i) $f(x,y) = x^3 + y^3 + 3axy$

$\frac{\partial f}{\partial x} = 3x^2 + 3ay$ and $\frac{\partial f}{\partial y} = 3y^2 + 3ax$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 3ay) = 6x$$

and $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 + 3ay) = 3a$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 + 3ax) = 3a$$

and $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y.$$

(iii) For $f(x,y,z) = x^2 + yz + xz^3$.

$$\frac{\partial f}{\partial x} = 2x + z^3 \; ; \; \frac{\partial f}{\partial y} = z \; ; \; \frac{\partial f}{\partial z} = y + 3xz^2.$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2 \; ; \; \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0 \; ;$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 3z^2.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0 \; ; \; \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 1$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 3z^2 \; ; \; \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 1$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 6xz.$$

→ If $f(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, $x \neq 0, y \neq 0$

Show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$

Sol^n: $f(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Now, $$\frac{\partial f}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

HW If $f(x,y,z) = e^{xyz}$, then show that

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}.$$

→ Find all the second order partial derivatives of the following functions:

(a) $f(x,y) = \cos\frac{y}{x}$ (b) $f(x,y) = x^5 + y^4 \sin x^6$

(c) $f(x,y,z) = \sin xy + \sin yz + \cos xz$

(d) $f(x,y,z) = xyz^2 + xyz + x^3y$

→ If $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

→ verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for each of the following functions

(a) $f(x,y) = x^3y + e^{xy^2}$ (b) $f(x,y) = \tan(xy^3)$

(c) $f(x,y) = \frac{2x-y}{x+y}$ (d) $f(x,y) = x\tan xy$.

→ Already we have seen that it is not always possible to find first order partial derivatives by direct differentiation. The same is true for higher order partial derivatives of some functions.

(i) for example:

Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

Find the second order partial derivatives of f at $(0,0)$.

Solⁿ:

Since $f_{xx}(0,0) = \lim_{h\to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h}$ ——— ①

we first evaluate $f_x(h,0)$ and $f_x(0,0)$.

Now $f_x(0,0) = \lim_{t\to 0} \dfrac{f(t,0) - f(0,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

$f_x(h,0) = \lim_{t\to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

Substitute the values of $f_x(0,0)$ & $f_x(h,0)$ in ①

$\therefore f_{xx}(0,0) = \lim_{h\to 0} \dfrac{0-0}{h} = 0$

Since $f_{xy}(0,0) = \lim_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h}$ ——— ②

Now let $f_y(h,0) = \lim_{t\to 0} \dfrac{f(h,t) - f(h,0)}{t}$

$$= \lim_{t \to 0} \frac{f(h,t) - f(h,0)}{t}$$

$$= \lim_{t \to 0} \frac{ht(h^2 - t^2)}{t(h^2+t^2)} = h$$

Now let $f_y(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t}$

$$= \lim_{t \to 0} \frac{0-0}{t} = 0$$

$\therefore f_{xy}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = 1$

Since $f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0, 0+k) - f_x(0,0)}{k}$

now let $f_x(0,k) = \lim_{t \to 0} \frac{f(t,k) - f(0,k)}{t}$

$$= \lim_{t \to 0} \frac{\frac{tk(t^2-k^2)}{t^2+k^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{k(t^2-k^2)}{t^2+k^2} = -k$$

and $f_x(0,0) = 0$

$\therefore f_{yx}(0,0) = \lim_{k \to 0} \frac{-k-0}{k} = -1$

Since $f_{yy}(0,0) = \lim_{k \to 0} \frac{f_y(0,k) - f_y(0,0)}{k}$

Now let $f_y(0,k) = \lim_{t \to 0} \frac{f(0,k+t) - f(0,k)}{t}$

$$= \lim_{t \to 0} \frac{0-0}{t} = 0$$

and $f_y(0,0) = 0$

$\therefore f_{yy}(0,0) = \lim_{k \to 0} \frac{0-0}{t} = 0$

H.13 Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$, for the function given by

$$f(x,y) = \begin{cases} (x^4+y^4)\tan^{-1}(y^2/x^2), & x \neq 0 \\ \dfrac{\pi y^4}{2}, & x = 0 \end{cases}$$

Note: Now we will given an example of a function whose first order partial derivatives exist, but higher order ones do not exist and also we see that the existence of a partial derivative of a particular order doesnot imply the existence of other partial derivatives of the same order.

(II) For example:

Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Examine whether the second order partial derivatives of f at $(0,0)$ exist or not.

Sol^n: Now $f_{xx}(0,0) = \lim_{h\to 0} \dfrac{f_x(0+h,0) - f_x(0,0)}{h}$

Let $f_x(h,0) = \lim_{t\to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

and $f_x(0,0) = \lim_{t\to 0} \dfrac{f(0+t,0) - f(0,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

$\therefore f_{xx}(0,0) = \lim_{h\to 0} \dfrac{0-0}{h} = 0$

Now $f_{yx}(0,0) = \lim_{k\to 0} \dfrac{f_x(0,0+k) - f_x(0,0)}{k}$

Let $f_x(0,k) = \lim_{t\to 0} \dfrac{f(t,k) - f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{\frac{tK^2}{\sqrt{t^2+K^2}} - 0}{t}$$

$$= \lim_{t\to 0} \frac{K^2}{\sqrt{t^2+K^2}} = \frac{K^2}{|K|} = \pm K \text{ (or) } |K|$$

and $f_x(0,0) = 0$

Now,

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{|K|}{K}$$

which doesnot exist.

$\therefore f_{yx}$ doesnot exist at $(0,0)$

Now

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

Let $f_y(h,0) = \lim_{t\to 0} \dfrac{f(h,t) - f(h,0)}{t}$

$$= \lim_{t\to 0} \frac{\frac{ht^2}{\sqrt{h^2+t^2}} - 0}{t}$$

$$= \lim_{t\to 0} \frac{ht}{\sqrt{h^2+t^2}} = 0$$

and $f_y(0,0) = \lim_{t\to 0} \dfrac{f(0,t) - f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Now $f_{yy}(0,0) = \lim_{k\to 0} \dfrac{f_y(0,k) - f_y(0,0)}{k}$.

let $f_y(0,k) = \lim_{t\to 0} \dfrac{f(0,k+t) - f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

and $f_y(0,0) = 0$.

$$\therefore f_{yy}(0,0) = \lim_{k\to 0} \frac{0-0}{K} = 0$$

$\therefore f_{xx}$, f_{xy} and f_{yy} exist at $(0,0)$ and all equal to zero, while $f_{yx}(0,0)$ doesnot exist

The study of above examples must have convinced that we have to be careful about the order of variables w.r.t which higher order derivatives are taken. For instance, from example (I) it is clear that f_{xy} need not be equal to f_{yx}.

Example (II) goes a step further, where f_{xy} exist at $(0,0)$, while f_{yx} doesnot, showing that the question of their equality doesnot arise at all.

If we look at the definitions of f_{xy} and f_{yx} at a point (a,b) more carefully, we would see why the expectation of the equality $f_{xy}(a,b) = f_{yx}(a,b)$ is farfetched (very difficult to believe)

Now by definition

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(a+h,b+k)-f(a+h,b)}{k} - \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}\right]$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk}$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{\phi(h,k)}{hk}$$

where $\phi(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$

Similarly

$$f_{yx}(a,b) = \lim_{k\to 0} \lim_{h\to 0} \frac{\phi(h,k)}{hk}.$$

The expressions $f_{xy}(a,b)$ and $f_{yx}(a,b)$ are the repeated limits of the same expression taken in different orders. and we have already seen that repeated limits are not equal, in general.

Now we will give the conditions under which these mixed partial derivatives become equal.

Sufficient conditions for the equality of f_{xy} and f_{yx}:

Theorem(I): Let $f(x,y)$ be a real valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a,b).

Then $f_{xy}(a,b) = f_{yx}(a,b)$

Schwarz's theorem:

Let $f(x,y)$ be a real valued function defined in a nbd of (a,b) such that

(i) f_x exists on a certain nbd of (a,b)

(ii) f_{xy} is continuous at (a,b).

Then f_{yx} exists at (a,b)

and $f_{yx}(a,b) = f_{xy}(a,b)$.

Note: The conditions in Schwarz's theorem are less restrictive than those in Theorem (I).

→ Evaluate f_{xy} at a point (x,y) for the function f defined by $f(x,y) = x^4 + x^2y^2 + y^6$. Use Schwarz's theorem to evaluate f_{yx} at the point (x,y).

Sol^n: By direct differentiation

$f_y(x,y) = 2x^2y + 6y^5$.

$\Rightarrow f_{xy}(x,y) = 4xy$.

Since $4xy$ is a polynomial,

$\therefore f_{xy} = 4xy$ is a continuous function.

Further $f_x = 4x^3 + 2xy^2$ exist.

$\therefore f$ satisfies the conditions of Schwarz's theorem.

Note: In theorem (i) we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous and that f_x exists.

But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives.

In otherwords, we can have functions whose mixed partial derivatives at some point are equal, but which donot satisfy the requirements of Schwarz's theorem.

For example:

Consider the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$, even though f doesnot fulfill the requirements of Schwarz's theorem.

Soln: Since $f_{xy}(0,0) = \lim\limits_{h \to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h}$

Now, $f_y(h,0) = \lim\limits_{t \to 0} \dfrac{f(h,t) - f(h,0)}{t}$

$$= \lim_{t \to 0} \frac{\frac{h^2t^2}{h^2+t^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{h^2t}{h^2+t^2} = 0.$$

and $f_y(0,0) = \lim\limits_{t \to 0} \dfrac{f(0,t) - f(0,0)}{t}$

$$= \lim \frac{0-0}{t} = 0$$

$\therefore f_{xy}(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0.$

Since $f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$

Now, let $f_x(0,k) = \lim_{t \to 0} \frac{f(t,k) - f(0,k)}{t}$

$$= \lim_{t \to 0} \frac{\frac{t^2k^2}{t^2+k^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{tk^2}{t^2+k^2} = 0$$

and $f_x(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t}$

$$= \lim_{t \to 0} \frac{0-0}{t} = 0$$

$\therefore f_{yx}(0,0) = 0.$

$\therefore f_{xy}(0,0) = f_{yx}(0,0).$

Now we show that the conditions of Schwarz's theorem are not satisfied.

For $(x,y) \neq (0,0)$, we can find the partial derivatives of f at (x,y) by differentiating directly.

$\therefore f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x^2+y^2}\right)$

$$= \frac{2(x^2+y^2)x^2y - 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{2x^4y}{(x^2+y^2)^2}$$

further $f_{xy} = \frac{\partial}{\partial x}\left[\frac{2x^4y}{(x^2+y^2)^2}\right]$

$$= \frac{8x^3y(x^2+y^2)^2 - 8x^5y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(8x^3y(x^2+y^2) - 8x^5y)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{8x^3y\,(x^2+y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}.$$

$\underset{(x,y)\to(0,0)}{Lt} \dfrac{8x^3y^3}{(x^2+y^2)^3}$ doesnot exist.

for, if (x,y) approach $(0,0)$ along the line $y=mx$.

we get

$$\underset{(x,y)\to(0,0)}{Lt} \frac{8x^3y^3}{(x^2+y^2)^3} = \underset{x\to 0}{Lt} \frac{8\cdot x^3 m^3 x^3}{(x^2+m^2x^2)^3}$$

$$= \underset{x\to 0}{Lt} \frac{8m^3}{(1+m^2)^3}$$

The limit is different for different values of m. i.e, the limit doesnot exist.

$\therefore \underset{(x,y)\to(0,0)}{Lt} f_{xy}(x,y) \neq f_{xy}(0,0) = 0.$

which implies that f_{xy} is not continuous at $(0,0)$.

Young's theorem:

Let $f(x,y)$ be a real-valued function defined in a nbd of a point (a,b) such that both the first order partial derivatives f_x and f_y are differentiable at (a,b).

Then $f_{xy}(a,b) = f_{yx}(a,b)$.

Note: As in the case of Schwarz's theorem the conditions stated in young's theorem are less strict than in theorem (I).

However, these are not necessary for the equality of mixed partial derivatives.

→ Consider the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x=y=0 \end{cases}$$

S.T $f_{xy}(0,0) = f_{yx}(0,0)$, even though the conditions of young's theorem are not satisfied.

Sol: Already we have seen that

$$f_{xy}(0,0) = f_{yx}(0,0) = 0$$

Let us now show that the conditions of young's theorem are not satisfied.

Now we prove that f_x is not differentiable at $(0,0)$.

~~the~~ For this, assume that f_x is differentiable at $(0,0)$.

Then there exist functions $\phi(h,k)$ and $\psi(h,k)$

such that

$$f_x(h,k) - f_x(0,0) = h f_{xx}(0,0) + k f_{yx}(0,0) + h\,\phi(h,k) + k\,\psi(h,k) \quad \text{—(1)}$$

and $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

$\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Now let $f_{xx}(0,0) = \lim\limits_{h\to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h}$ —(2)

Let $f_x(h,0) = \lim\limits_{t\to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$= \lim\limits_{t\to 0} \dfrac{0-0}{t} = 0$

and $f_x(0,0) = \lim\limits_{t\to 0} \dfrac{f(t,0) - f(0,0)}{t}$

$= \lim\limits_{t\to 0} \dfrac{0-0}{t} = 0$

(2):- $f_{xx}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$

$\therefore$ eqn (1) becomes

$f_x(h,k) - 0 = h\,\phi(h,k) + k\,\psi(h,k)$ —— (3)

$(\because f_{yx}(0,0) = 0 \;\&\; f_{xx}(0,0) = 0)$

for $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{2xy^2(x^2+y^2) - 2x(x^2y^2)}{(x^2+y^2)^2}$$

$$= \frac{2xy^4}{(x^2+y^2)^2}$$

$$\Rightarrow f_x(h,k) = \frac{2hk^4}{(h^2+k^2)^2}$$

$\therefore$ (3) $\equiv$

$$\frac{2hk^4}{(h^2+k^2)^2} = h\,\phi(h,k) + k\,\psi(h,k).$$

Putting $h = r\cos\theta$, $k = r\sin\theta$.

we get

$$\frac{2r^5\cos\theta\,\sin^4\theta}{r^4} = r\cos\theta\,\phi(r\cos\theta, r\sin\theta) + r\sin\theta\,\psi(r\cos\theta, r\sin\theta)$$

$2\cos\theta\,\sin^4\theta = \cos\theta\,\phi(r\cos\theta, r\sin\theta) + r\sin\theta\,\psi(r\cos\theta, r\sin\theta)$ —— (4)

For arbitrary θ, $(h,k) = (r\cos\theta, r\sin\theta) \to (0,0)$ as $r \to 0$.

and $\therefore$ $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$.

$\therefore$ Taking the limit of (4) as $r \to 0$, we get

$2\cos\theta\,\sin^4\theta = 0$.

which is impossible for arbitrary θ.

$\therefore$ f_x is not differentiable at $(0,0)$.

Similarly, we can show that f_y is not differentiable at $(0,0)$.

$\therefore$ The function f doesnot satisfy the conditions of young's theorem, even though we have $f_{xy}(0,0) = f_{yx}(0,0)$

Differentials of higher order :

Let $z = f(x,y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x,y) of the domain. The first differential of z at (x,y), denoted by dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{--- (1)}$$

— If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x,y) then dz is a function of x and y and itself is differentiable at (x,y). Then the differential of dz, called the second differential of z, is denoted by d^2z.

$$\therefore d^2z = d(dz)$$
$$= d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$
$$= d\left(\frac{\partial z}{\partial x}\right)dx + d\left(\frac{\partial z}{\partial y}\right)dy \quad \text{--- (2)} \qquad (\because dx \,\&\, dy \text{ are constants})$$

Now let $d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)dy$ (from (1))

$$= \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy$$

Similarly $d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$ $\Bigg\}$ --- (3)

Also by young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable

we have $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ --- (4)

from (2), (3) & (4)

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \quad \text{--- (5)}$$

where $dx^2 = dx \cdot dx = (dx)^2$

$dy^2 = (dy)^2$.

Eqn (5), can be written as

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \text{ ------ } (6)$$

Again d^2z is differentiable at (x,y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ are differentiable at (x,y).

$$\therefore d^3z = d(d^2z) = d\left[\frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2\right]$$

$$= d\left(\frac{\partial^2 z}{\partial x^2}\right)dx^2 + 2\,d\left(\frac{\partial^2 z}{\partial x \partial y}\right)dx\,dy + d\left(\frac{\partial^2 z}{\partial y^2}\right)dy^2$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)dy\right]dx^2$$

$$+ 2\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right)dy\right]dx\,dy$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)dy\right]dy^2$$

$$= \frac{\partial^3 z}{\partial x^3}dx^3 + \frac{\partial^3 z}{\partial y \partial x^2}dy\,dx^2 + 2\frac{\partial^3 z}{\partial x^2 \partial y}dx^2 dy$$

$$+ 2\frac{\partial^3 z}{\partial y\,\partial x \partial y}dy^2 dx + \frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$$

$$= \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$$

$$(\because z_{yxy} = z_{yyx} \;\&\; z_{xyy})$$

$$= \left[\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right]^3 z.$$

In general, $d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z$ exists if $d^{n-1}z$ is differentiable.

Note: In the above discussion, x & y are independent variables and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \delta x$, $dy = \delta y$.

Functions of Functions:

So far we have considered functions of the form
$z = f(x, y, \text{----})$
where the variables $x, y, \text{----}$ are the independent variables.

Now we consider functions

$z = f(x, y, \ldots)$

where $x, y, \ldots$ are not independent variables, but are themselves functions of other independent variables $u, v, \ldots$, so that

$x = g(u, v, \ldots)$ and $y = h(u, v, \ldots)$

Theorem:

If $z = f(x, y)$ is a differentiable function of x, y and $x = g(u, v)$, $y = h(u, v)$ are themselves differentiable functions of the independent variables, u, v, then z is a differentiable function of u, v and

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

just as though x, y ~~were~~ are the independent variables.

Note: (1) The theorem establishes a fact of fundamental importance that the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.

(2) The differential dz is sometimes referred to as the total differential.

Differentials of Higher order of a function of functions

If $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are differentiable functions of x, y, so that they are also differentiable functions of

u, v and dx, dy are differentiable functions of u, v, then from the above theorem we have

$$d^2z = d(dz)$$

$$= d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) \quad \left(\because dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$

$$= d\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial z}{\partial x}d(dx) + d\left(\frac{\partial z}{\partial y}\right)dy + \frac{\partial z}{\partial y}d(dy)$$

$$= \left[\frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy\right]dx + \frac{\partial z}{\partial x}d^2x + \left[\frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy\right]dy + \frac{\partial z}{\partial y}d^2y.$$

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 + \frac{\partial z}{\partial x}d^2x + \frac{\partial z}{\partial y}d^2y.$$

$$= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z + \frac{\partial z}{\partial x}d^2x + \frac{\partial z}{\partial y}d^2y.$$

→ The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given.

→ The introduction of more than two variables, which are functions of independent variables causes no difficulty.

Thus, when $z = f(x_1, x_2, x_3)$, and x_1, x_2, x_3 are not the independent variables.

$$d^2z = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \frac{\partial}{\partial x_3}dx_3\right)^2 z + \frac{\partial z}{\partial x_1}d^2x_1 + \frac{\partial z}{\partial x_2}d^2x_2 + \frac{\partial z}{\partial x_3}d^2x_3.$$

<u>Note</u>: If x, y are linear functions of independent variables u and v, i.e., x and y of the form $x = a + bu + cv$, $y = a' + b'u + c'v$ then dx and dy are constants, and so d^2x, d^2y and all higher differentials of x and y are zero, and

$$\therefore d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z.$$

the form being same as for independent x & y.

→ The Derivation of Composite Functions (The Chain Rule):

Let $z = f(x,y)$ possess continuous first order partial derivatives.

Let $x = \phi(t)$, $y = \psi(t)$ possesses continuous derivatives.

Then the composite function

$z = f(\phi(t), \psi(t))$ has derivative given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

($\frac{dz}{dt}$ is called the total derivative)

Because:

Since x, y are differentiable functions of single variable 't'

∴ $dx = \frac{dx}{dt} \cdot dt$ and $dy = \frac{dy}{dt} \cdot dt$

Since z is differentiable function of x and y and x, y are differentiable functions of t

∴ z is a differentiable function of t.

∴ $dz = \frac{dz}{dt} \cdot dt$ ——— (1)

Also $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$\Rightarrow dz = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} dt + \frac{\partial z}{\partial y}\frac{dy}{dt} dt$ ——— (2)

From (1) & (2) we have

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

***Corollary:-** If $z = f(x,y)$ possesses nth order partial derivatives and x, y are linear functions of single variable 't', i.e. $x = a + ht$, $y = b + kt$ where a, b, h, k are constants then

$$\frac{d^n z}{dt^n} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n z$$

Sol

Now $\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$

$= h \dfrac{\partial z}{\partial x} + k \dfrac{\partial z}{\partial y}$ ——— (1) $\quad$ where $\dfrac{dx}{dt} = h$, $\dfrac{dy}{dt} = k$

$= \left(h \dfrac{\partial}{\partial x} + k \dfrac{\partial}{\partial y}\right) z$ ——— (2)

Now $\quad \dfrac{d^2 z}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dz}{dt}\right)$

$= \dfrac{d}{dt}\left(h \dfrac{\partial z}{\partial x} + k \dfrac{\partial z}{\partial y}\right)$

$= h \dfrac{\partial}{\partial x}\left(h \dfrac{\partial z}{\partial x} + k \dfrac{\partial z}{\partial y}\right) + k \dfrac{\partial}{\partial y}\left(h \dfrac{\partial z}{\partial x} + k \dfrac{\partial z}{\partial y}\right)$

$= h^2 \dfrac{\partial^2 z}{\partial x^2} + 2hk \dfrac{\partial^2 z}{\partial x \partial y} + k^2 \dfrac{\partial^2 z}{\partial y^2}$

$= \left(h \dfrac{\partial}{\partial x} + k \dfrac{\partial}{\partial y}\right)^2 z$

In general,

$$\frac{d^n z}{dt^n} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n z$$

→ Let $z = f(x, y)$ be possess continuous first order partial derivatives.

Let $x = \phi(u, v)$
$\quad y = \psi(u, v)$ possess continuous first order partial derivatives

Then $\quad \dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \dfrac{\partial y}{\partial u}$ and

$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \dfrac{\partial y}{\partial v}$

Because:

Since x, y are differentiable functions of two independent variables u and v

$$\therefore dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \quad \text{——— (1)}$$

Since z is differentiable function of x and y and x, y are differentiable functions of u and v

$\therefore z$ is a differentiable fn of u & v

$$\therefore dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \quad \text{——— (2)}$$

Also

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \quad \text{(from (1))}$$

$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right) dv \quad \text{——— (3)}$$

Hence from (2) & (3),

$$\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right) dv$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

problems:

→ If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$. compute $\frac{dz}{dt}$ at $t = \pi/2$.

Sol $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= (y^2 e^{xy^2})(\cos t - t\sin t) + (2xy\, e^{xy^2})(\sin t + t\cos t)$.

At $t = \frac{\pi}{2}$, $x = 0$, $y = \pi/2$

$\therefore \left[\frac{dz}{dt}\right]_{t=\pi/2} = \frac{\pi^2}{4}\left(-\pi/2\right) = \frac{-\pi^3}{8}$.

→ If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$.

find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$.

Sol $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$

$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

$= (3x^2 - y)(-r\sin\theta) + (3y^2 - x)\, r\cos\theta$

→ Show that $z = f(x^2 y)$, where f is differentiable, satisfies $x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right)$

Sol Let $x^2 y = u$ then $z = f(u)$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u)\cdot 2xy$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u)\cdot x^2$

$\therefore x\frac{\partial z}{\partial x} = f'(u)\, 2x^2 y = 2y\frac{\partial z}{\partial y}$.

Aliter:- $dz = f'(u)du = f'(x^2y)(2xy\,dx + x^2\,dy)$ —(1)

Also $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ —(2).

from (1) & (2), we have $\frac{\partial z}{\partial x} = 2xy f'(x^2y)$, $\frac{\partial z}{\partial y} = x^2 f'(x^2y)$

$\therefore$ The result now follows as above.

* Derivative of an Implicit function:

Let y be a function of x defined implicitly by the equation $f(x,y)=0$

By the above

$$\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

$$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

provided $f_y \neq 0$.

Problems

Let $f(x,y) = ax^2+2hxy+by^2-1$

→ then find $\frac{dy}{dx}$.

Ans: $\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(ax+hy)}{hx+by}$.

→ find $\frac{du}{dx}$ for $u=\sin(x^2+y^2)$ where x and y satisfy the eqn

$a^2x^2+b^2y^2=c^2$.

Sol

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2)+2y\cos(x^2+y^2)\frac{dy}{dx}.$$

$$= \cos(x^2+y^2)\left[2x+2y\frac{dy}{dx}\right] \quad \text{(i)}$$

Now let $\phi(x,y) = a^2x^2+b^2y^2-c^2=0$

then $\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$

$$= \frac{-2a^2x}{2b^2y}$$

$\therefore$ ① $\equiv \frac{du}{dx} = \cos(x^2 - ey^2)\left[2x + 2y/\left(-\frac{cx}{by}\right)\right]$

$= 2x\cos(x^2 - ey^2)\left[1 - \frac{c^2}{b^2}\right]$.

→ Find $\frac{du}{dx}$, for each of the following problems.

(a) $u = x^2 - xy + y^2$, $y = 3x + 1$

(b) $u = x^2 - y^3$; $y = \ln x$

(c) $u = x \ln xy$ where $x^3 + y^3 + 3xy = 1$.

* **Homogeneous functions:-**

A function $Z = f(x,y)$ is called a homogeneous function of degree 'n' if it is expressible as $Z = x^n g\left(\frac{y}{x}\right)$.

Ex:- $ax^2 + 2hxy + by^2 = x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$

$= x^2 g\left(\frac{y}{x}\right)$.

$\therefore$ It is a homo. function degree 2.

→ The following functions are homogeneous functions:

(i) $f(x,y) = \tan\left(\frac{y}{x}\right)$; degree '0'

(ii) $f(x,y) = \sqrt[3]{x^4 + 7y^4}$; degree $\frac{4}{3}$

(iii) $f(x,y) = \dfrac{\sin\left(\frac{x^2 y}{x^3 + y^3}\right)}{\ln\left(\frac{x+y}{x}\right)}$ degree '0'.

65 * Euler Theorem on Homogenous functions:- (63)

→ If $Z = f(x,y)$ is homogenous function of x and y of degree 'n', then

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = nZ.$$

Cor: If $z = f(x,y)$ is a homogenous function of x, y of degree 'n'

then $x^2\frac{\partial^2 Z}{\partial x^2} + 2xy\frac{\partial^2 Z}{\partial x\partial y} + y^2\frac{\partial^2 Z}{\partial y^2} = n(n-1)Z$

Problems:-

→ If $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{4}\sin 2u = 0.$$

Sol: Let $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

then $\cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ $(= Z)$ say ——— (1)

Clearly Z is a homo. function of x and y of degree $\frac{1}{2}$

∴ By Eulers theorem

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = \frac{1}{2}(Z) \quad \text{——— (2)}$$

from (1), $\frac{\partial Z}{\partial x} = -\csc^2 u\frac{\partial u}{\partial x}$

$\frac{\partial Z}{\partial y} = -\csc^2 u\frac{\partial u}{\partial y}$

∴ (2) ⇒ $x\left(-\csc^2 u\frac{\partial u}{\partial x}\right) + y\left(-\csc^2 u\right)\frac{\partial u}{\partial y} = \frac{1}{2}(\cot u)$

⇒ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin^2 u\left(\frac{1}{2}\frac{\cos u}{\sin u}\right)$

$= -\frac{1}{4}\sin 2u.$

→ If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, $x \neq y$ show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u.$

Sol^n:

(i) Here $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ is not a homogeneous function.

However, we write

$$\tan u = \frac{x^3+y^3}{x-y} \; (=z) \text{ say} \quad \text{——(1)}$$

$$\Rightarrow z = x^2\left[\frac{1+(y/x)^3}{1-(y/x)}\right]$$

z is a homogeneous function of x, y of degree 2.

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{——(2)}$$

But from (1) $\frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$ & $\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$ ——(3)

$\therefore$ (2) $\equiv$

$$x\left[\sec^2 u \frac{\partial u}{\partial x}\right] + y\sec^2 u\frac{\partial u}{\partial y} = 2z = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$$= \sin 2u. \quad \text{——(4)}$$

(ii) From (3)

$$\frac{\partial^2 z}{\partial x^2} = \sec^2 u\frac{\partial^2 u}{\partial x^2} + 2\sec^2 u\tan u\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \sec^2 u\frac{\partial^2 u}{\partial y^2} + 2\sec^2 u\tan u\left(\frac{\partial u}{\partial y}\right)^2$$

and $\frac{\partial^2 z}{\partial x\partial y} = \sec^2 u\frac{\partial^2 u}{\partial x\partial y} + 2\sec^2 u\tan u\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}.$

By corollary of Euler's theorem,

we have

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

$$\Rightarrow \sec^2 u \left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right)$$

$$+ 2\sec^2 u \tan u \left[x^2 \left(\frac{\partial u}{\partial x}\right)^2 + 2xy \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + y^2 \left(\frac{\partial u}{\partial y}\right)^2\right] = 2\tan u.$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \cancel{2\tan u}$$

$$+ 2\tan u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)^2 = 2\sin u \cos u$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 2u - 2\tan u \sin^2 2u \quad (by (i))$$

$$= (1 - 2\tan u \sin 2u) \sin 2u$$

$$= (1 - 4\sin^2 u) \sin 2u.$$

→ If $z = (x+y)\,\phi(y/x)$, where ϕ is any arbitrary function prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Solⁿ: $\frac{\partial z}{\partial x} = \phi(y/x) + (x+y)\,\phi'(y/x)\cdot(-y/x^2)$

$$\Rightarrow x\frac{\partial z}{\partial x} = x\,\phi(y/x) - \frac{y}{x}(x+y)\,\phi'(y/x) \quad \text{—— (1)}$$

Also $\frac{\partial z}{\partial y} = \phi(y/x) + (x+y)\,\phi'(y/x)\left(\frac{1}{x}\right)$

$$\Rightarrow y\frac{\partial z}{\partial y} = y\,\phi(y/x) + \frac{y}{x}(x+y)\,\phi'(y/x) \quad \text{—— (2)}$$

Adding (1) & (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x+y)\,\phi(y/x)$$

$$= z.$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

→ If $u = z e^{ax+by}$, where z is a homogeneous function of x and y of degree n, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (ax+by+n)u.$$

→ If $z = x^m f(y/x) + x^n g(y/x)$. prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

→ If $z = \log\left(\frac{x^4+y^4}{x+y}\right)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3$

→ If $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$

→ If $u = f(x+2y) + g(x-2y)$. Show that

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

2017 (15M) → If $u = \phi(x+at) + \varphi(x-at)$. show that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

Q/ → If $z = f\left[\frac{(ny-mz)}{(nx-lz)}\right]$. prove that

$$(nx-lz)\frac{\partial z}{\partial x} + (ny-mz)\frac{\partial z}{\partial y} = 0$$

2006 (15M) If $z = xf(y/x) + g(y/x)$

show that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0.$

→ If $z = x^m f(y/x) + x^n g(x/y)$,

prove that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz$

$$= (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Solⁿ: Let $u = x^m f(y/x)$ and $v = x^n g(x/y)$

Then $z = u + v$ —— ①

Now $u = x^m f(y/x)$ is a homogeneous function in x and y of degree m.

Therefore by Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = m(m-1)u. \quad —— ②$$

Also $v = x^n g(x/y)$ is a homogeneous function in x and y of degree n.

So we have

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v. \quad —— ③$$

Adding ② & ③, we have

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v)$$

$$= m(m-1)u + n(n-1)v.$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = m(m-1)u + n(n-1)v \quad —— ④$$

Now let $m(m-1)u + n(n-1)v$

$$= (m^2u + n^2v) - (mu + nv)$$

$$= m^2u + n^2v - mnu + mnu$$
$$- mnv + mnv - (mu + nv)$$

$$= mu(m+n) + nv(m+n) - mn(u+v) - (mu+nv)$$

$$= (mu+nv)(m+n) - mn(u+v) - (mu+nv)$$

$$= (mu+nv)(m+n-1) - mn(u+v).$$

$$= -mnz + (mu+nv)(m+n-1) \quad \text{---(5)}$$

(from (1) $z = u+v$)

Again from Euler's theorem we have for u & v, which are homogeneous functions in x & y of degree m and n respectively,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu \quad \text{and}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv.$$

Adding these we get,

$$x\frac{\partial(u+v)}{\partial x} + y\frac{\partial(u+v)}{\partial y} = mu+nv.$$

$$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = mu+nv} \quad (\because z = u+v)$$

$\therefore$ from (5) we have

$$m(m-1)u + n(n-1)v = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz \quad \text{---(6)}$$

So $\therefore$ from (4) we have

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

7) If $z = f\left(\frac{ny - mz}{nx - lz}\right)$

prove that $(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$

Sol^n: Given $z = f\left(\frac{ny - mz}{nx - lz}\right)$.

$$\frac{\partial z}{\partial x} = f'\left(\frac{ny - mz}{nx - lz}\right)\frac{\partial}{\partial x}\left(\frac{ny - mz}{nx - lz}\right)$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right)\cdot(ny - mz)\cdot\frac{-n}{(nx - lz)^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{ny - mz}{nx - lz}\right)\frac{\partial}{\partial y}\left(\frac{ny - mz}{nx - lz}\right)$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right)\frac{n}{nx - lz} \quad \text{--- (2)}$$

multiplying (1) by $(nx - lz)$ and (2) by $(ny - mz)$
and adding these, we get

$$(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = (nx - lz)\, f'\left(\frac{ny - mz}{nx - lz}\right)\frac{-n(ny - mz)}{(nx - lz)^2}$$

$$+ (ny - mz)\, f'\left(\frac{ny - mz}{nx - lz}\right)\frac{n}{nx - lz}$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right)\left[\frac{-n(ny - mz)}{nx - lz} + \frac{n(ny - mz)}{nx - lz}\right]$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right)\cdot 0$$

$$= 0$$

$$\therefore (nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$$

→ $z = f(x, y)$ be a homogeneous function of x, y of degree n then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$, $\forall x, y \in$ the domain of the function.

Solⁿ. We have $z = x^n f(y/x)$

$$\Rightarrow \frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2}$$

$$= nx^{n-1} f(y/x) - y x^{n-2} f'(y/x)$$

and $$\frac{\partial z}{\partial y} = x^n f'(y/x) \frac{1}{x}$$

$$= x^{n-1} f'(y/x)$$

Thus, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f(y/x)$$

$$= nz.$$

Hence the result.

→ If $z = f(x, y)$ is a homogeneous function of x, y of degree n, then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$.

Solⁿ: we have $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ — ①

Differentiating equation ① partially w.r.t x and y seperately, we obtain

$$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x} \quad \text{— ②}$$

$$x\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y} \quad \text{— ③}$$

Multiplying ②, ③ by x, y respectively and adding, we obtain

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

where we have assumed the equality of $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$.

Taylor's theorem for function of two variables

Monomial:

Defn: Let x & y denote two variables. Then an expression of the form $a_{jk}x^j y^k$, where j and k are non-negative integers and $a_{jk} \in \mathbb{R}$, is called a monomial. The integer $j+k$ is called the degree of the monomial.

for examples:

x^2y^3 is a monomial of degree 5.

x^4 is a monomial of degree 4

y^7 is a monomial of degree 7.

A polynomial in two variables in x & y with coefficients in $\mathbb{R}$ is in an expression of the type

$$P(x,y) = a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \cdots + (a_{i0}x^i + a_{(i-1)1}x^{i-1}y + \cdots + a_{0i}y^i) + \cdots + (a_{n0}x^n + \cdots a_{0n}y^n)$$

where a_{ij}'s are real numbers.

In the first bracket each term is a monomial of degree 1.

In the second, each is a monomial of degree 2, and so on.

for example:

$P(x,y) = 1 + 2xy + x^2y$ is a polynomial in two variables.

This polynomial is a sum of three monomials, having degree 0, 2 and 3 respectively.

The number 3, which is the maximum of these numbers is called the degree of this polynomial.

→ The highest degree of the monomials present in a polynomial $p(x,y)$ is called the degree of $p(x,y)$.

→ n^{th} Taylor polynomial of a function of two variables

Defn: Let $f(x,y)$ be a real-valued function of two variables. Assume that it has continuous partial derivatives of all types of orders less than or equal to 'n' in some nbd of a point (x_0, y_0).

Then
$$T_n(x,y) = \sum_{\substack{i,j=0}}^{i+j\le n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_0,y_0)\right](x-x_0)^i(y-y_0)^j$$
is called the n^{th} Taylor polynomial of f at (x_0, y_0)

In particular, if $f(x,y)$ is a polynomial of degree 'n', then all partial derivatives of order m for $m>n$ will be zero.

$\therefore$ $T_m(x,y) = T_n(x,y)$ for all $m \ge n$.

Further, as in the case of one variable, we can see that $T_n(x,y)$ at $(0,0)$ is equal to $f(x,y)$.

From the defn, we can see that
$$T_{n+1}(x,y) = T_n(x,y) + \sum_{\substack{i,j=0\\ i+j=n+1}} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i(y-y_0)^j$$

→ Find the Taylor polynomials of the function $p(x,y) = 1+2xy+x^2y$ at $(1,1)$.

Soln: Given $p(x,y) = 1+2xy+x^2y$

$p(1,1) = 4$.

By Defn
$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} p(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i(y-y_0)^j$$

Put $n=0$ in ①

$$T_0(x,y) = \frac{1}{0!\,0!}\left[\frac{\partial^{0+0} P(x_0,y_0)}{\partial x^0 \partial y^0}\right](x-x_0)^0 (y-y_0)^0$$

$$= P(x_0,y_0)$$

$$= P(1,1) = 4$$

$\therefore T_0(x,y) = 4$

Put $n=1$ in ①

$$T_1(x,y) = \sum_{i,j=0}^{i+j\le 1} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} P(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i (y-y_0)^j$$

$$= P(x_0,y_0) + \frac{1}{1!\,0!}\frac{\partial P(x_0,y_0)}{\partial x}(x-x_0) + \frac{1}{0!\,1!}\frac{\partial P(x_0,y_0)}{\partial y}(y-y_0)$$

$$= T_0(x,y) + \frac{\partial P}{\partial x}(1,1)(x-1) + \frac{\partial P}{\partial y}(1,1)(y-1) \quad \text{②}$$

$(\because P(1,1) = T_0(x,y))$

Now

$$\frac{\partial P}{\partial x} = 2y + 2xy \Rightarrow \left(\frac{\partial P}{\partial x}\right)_{(1,1)} = 4$$

$$\frac{\partial P}{\partial y} = 2x + x^2 \Rightarrow \left(\frac{\partial P}{\partial y}\right)_{(1,1)} = 3$$

$\therefore$ ② $\equiv$

$$T_1(x,y) = 4 + 4(x-1) + 3(y-1)$$

Put $n=2$ in ① & $(x_0,y_0) = (1,1)$

$$T_2(x,y) = \sum_{i,j=0}^{i+j\le 2}\left[\frac{1}{i!\,j!}\frac{\partial^{i+j} P(1,1)}{\partial x^i \partial y^j}\right](x-1)^i (y-1)^j$$

$$= T_1(x,y) + \frac{(x-1)^2}{2!}\frac{\partial^2 P(1,1)}{\partial x^2} + \frac{(x-1)(y-1)}{1!\,1!}\frac{\partial^2 P}{\partial x \partial y}(1,1) + \frac{(y-1)^2}{2!}\frac{\partial^2 P}{\partial y^2}(1,1) \quad \text{③}$$

Now

$$\frac{\partial^2 P}{\partial x^2} = 2y \Rightarrow \left(\frac{\partial^2 P}{\partial x^2}\right)_{(1,1)} = 2$$

$$\frac{\partial^2 P}{\partial x \partial y} = 2 + 2x \Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y}\right)_{(1,1)} = 4$$

$\frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)} = 0$

Substituting these values in (3)

we get

$T_2(x,y) = 4 + 4(x-1) + 3(y-1) + (x-1)^2 + 4(x-1)(y-1)$

Since $\frac{\partial^3 f}{\partial x^3} = 0$, $\frac{\partial^3 f}{\partial x^2 \partial y} = 2$, $\frac{\partial^3 f}{\partial x \partial y^2} = 0$ and $\frac{\partial^3 f}{\partial y^3} = 0$

we get

$T_3(x,y) = T_2(x,y) + (x-1)^2(y-1)$

and $T_r(x,y) = T_3(x,y)$ for all $r \geq 3$.

→ Find the Taylor polynomial $T_3(x,y)$ for the function $\sin(x+y)$ at $(0,0)$.

Sol^n: Let $f(x,y) = \sin(x+y)$.

Clearly f has continuous partial derivatives of all orders.

Also $f(0,0) = 0$.

Now, $\frac{\partial f}{\partial x} = \cos(x+y) = \frac{\partial f}{\partial y}$

$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 1$

$\frac{\partial^2 f}{\partial x^2} = -\sin(x+y) = \frac{\partial^2 f}{\partial y^2}$

$\Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 0$

and $\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$.

$\Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$.

$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = -\cos(x+y)\Big|_{(0,0)} = -1$.

The third Taylor polynomial of $\sin(x+y)$ at $(0,0)$ is

$$T_3(x,y) = \sum_{i,j=0}^{i+j\le 3} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(0,0)}{\partial x^i \partial y^j}\right] x^i y^j$$

$$= \frac{1}{0!\,0!} f(0,0) + \frac{x}{1!\,0!}\frac{\partial f}{\partial x}(0,0) + \frac{y}{0!\,1!}\frac{\partial f}{\partial y}(0,0) + \frac{x^2}{2!\,0!}\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{xy}{1!\,1!}\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{y^2}{0!\,2!}\frac{\partial^2 f}{\partial y^2}(0,0)$$

$$+ \frac{1}{3!\,0!}\left(\frac{\partial^3 f(0,0)}{\partial x^3}\right)x^3 + \frac{1}{2!\,1!}\left(\frac{\partial^3 f(0,0)}{\partial x^2 \partial y}\right)x^2 y + \frac{1}{1!\,2!}\left(\frac{\partial^3 f(0,0)}{\partial x \partial y^2}\right)x y^2 + \frac{1}{3!}\frac{\partial^3 f(0,0)}{\partial y^3} y^3$$

$$= 0 + \frac{x}{1!} + \frac{y}{1!} - 0 - \frac{1}{3!}x^3 - \frac{1}{2!\,1!}x^2 y - \frac{1}{1!\,2!}x y^2 - \frac{1}{3!}y^3$$

$$\Rightarrow T_3(x,y) = (x+y) - \frac{1}{3!}(x^3 + 3x^2 y + 3x y^2 + y^3)$$

$$= (x+y) - \frac{(x+y)^3}{3!}$$

H.W. → Find the second Taylor polynomial of e^{x+y} at $(0,0)$

H.W. → Find the Taylor polynomials of $f(x,y) = 2 + x^3 + y^3$ at $(1,1)$.

→ Now let us consider a function $f(x,y)$ of two variables. Assume that f has continuous partial derivatives of all orders less than or equal to n, for some integer 'n', in a nbd of a point (x_0, y_0).

Then the nth Taylor polynomial

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j$$

has the same value as $f(x,y)$ at (x_0, y_0), and the same partial derivatives of all orders $\le n$ as f at (x_0, y_0).

As in the case of one variable, we would naturally like to know whether we can approximate 'f' by the corresponding Taylor polynomials.

put differently, we would like to have some information about the function

$R_{n+1}(x,y) = f(x,y) - T_n(x,y)$.

An analogue of Taylor's theorem which we state now, provides us some information about the function $R_{n+1}(x,y)$.

Taylor's theorem:

Let 'f' be a real-valued function of two variables x and y with continuous partial derivatives of orders $\leq n+1$ in some nbd of $S(\bar{x}, r)$ of $\bar{x} = (x_0, y_0)$. Then for a given $(x,y) \neq (x_0, y_0)$ in $S(\bar{x}, r)$, there exists a point (c_1, c_2) on the line segment joining (x_0, y_0) and (x,y) such that —— ①

$$f(x,y) = T_n(x,y) + R_{n+1}(x,y)$$

where $T_n(x,y) = \sum_{\substack{i+j \leq n \\ i,j=0}} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j$

and $R_{n+1}(x,y) = \sum_{i+j=n+1} \left(\frac{1}{i!\,j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(c_1,c_2)} (x-x_0)^i (y-y_0)^j$.

i.e, $R_{n+1}(x,y) = \frac{1}{(n+1)!} \left(\frac{\partial^{n+1} f}{\partial x^{n+1}} \right)_{(c_1,c_2)} (x-x_0)^{n+1} +$

$\frac{1}{n!\,1!} \left(\frac{\partial^{n+1} f}{\partial x^n \partial y} \right)_{(c_1,c_2)} (x-x_0)^n (y-y_0) +$

$\frac{1}{(n-1)!\,2!} \left(\frac{\partial^{n+1} f}{\partial x^{n-1} \partial y^2} \right)_{(c_1,c_2)} (x-x_0)^{n-1} (y-y_0)^2 + \cdots$

$\cdots + \frac{1}{(n+1)!} \left(\frac{\partial^{n+1} f}{\partial y^{n+1}} \right)_{(c_1,c_2)} (y-y_0)^{n+1}$.

(38)

i.e., $R_{n+1}(x,y)$ involves all the $(n+1)^{th}$ order partial derivatives of f evaluated at the point (c_1, c_2).

The RHS of (1) is called the n^{th} Taylor expansion of f at (x_0, y_0).

Now we consider only the second Taylor expansions of functions.

If we look at the expression for $R_3(x,y)$, we will see that it contains powers of $(x-x_0)$ and $(y-y_0)$. Now if we take the point (x,y) close enough to (x_0, y_0), then $(x-x_0)$ and $(y-y_0)$ will be very small.

Therefore, we can get a good enough approximation of $f(x,y)$ by a second degree polynomial. Of course, $f(x,y)$ can be approximated as closely as we like by a polynomial by choosing n sufficiently large.

We write the expression for $T_2(x,y)$ and the second Taylor expansion of $f(x,y)$ at (x_0, y_0) explicitly:

$$f(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0)\right]$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0)\right.$$

$$\left. + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y-y_0)^2\right] + R_3(x,y).$$

$$= f(x_0, y_0) + \left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right] f(x_0, y_0)$$

$$+ \frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^2 f(x_0, y_0) + R_3(x,y).$$

→ Find the second Taylor expansion of the function $f(x,y) = \log(1+x+2y)$. for points close to $(2,1)$.

Sol^n: Given $f(x,y) = \log(1+x+2y)$

$$f(2,1) = \log 5$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+2y} \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(2,1)} = \frac{1}{5}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+x+2y}(2) \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(2,1)} = \frac{2}{5}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(2,1)} = \frac{-1}{25}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-4}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(2,1)} = -\frac{4}{25}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(2,1)} = \frac{-2}{25}$$

∴ The second Taylor expansion is given by

$$f(x,y) = \log 5 + \left[\frac{1}{5}(x-x_0) + \frac{2}{5}(y-y_0)\right]$$
$$+ \frac{1}{2}\left[\left(\frac{-1}{25}\right)(x-x_0)^2 + (2)\left(-\frac{2}{5}\right)(x-x_0)(y-y_0)\right.$$
$$\left. + \left(\frac{-4}{25}\right)(y-y_0)^2\right]$$

→ Find the second Taylor expansion for the function $f(x,y) = xy^2 + \cos xy$ about $(1, \pi/2)$.

→ Find an approximation to the function $f(x,y) = e^x \sin y$ by a second degree polynomial near $(0,0)$.

EXTREME VALUES

Maxima and Minima

Defn: A function $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, is said to have maximum at 'a' if there exists a nbd of 'a' for every point x of which $f(x) < f(a)$.

i.e, if $f(x,y)$ be a real valued function of two variables, we say that the function f has a maximum at (a,b) if $f(x,y) < f(a,b)$ for every $(x,y) \in N_\delta(a,b)$ for some $\delta > 0$.

Similarly, $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ is said to have a minimum at 'a' if there exists a nbd of 'a' at every point of which $f(x) > f(a)$.

i.e, if $f(x,y)$ be a real valued function of two variables, we say that the function f has a minimum at (a,b) if $f(x,y) > f(a,b)$ for every $(x,y) \in N_\delta(a,b)$ for some $\delta > 0$.

(OR)

→ Let (a,b) be a point of the domain of definition of function f. Then $f(a,b)$ is an extreme value of f, if for every point (x,y), (other than (a,b)) of some nbd of (a,b), the difference

$f(x,y) - f(a,b)$ ——— ①

keeps the same sign.

The extreme value $f(a,b)$ is called a max or min. value according as the sign of ① is $-$ve or $+$ve.

→ A function f is said to have an extreme value at (a,b), if $f(a,b)$ is either a maximum or minimum value of the function.

→ A necessary condition for $f(x,y)$ to have an extreme value at (a,b) is that $f_x(a,b)=0$, $f_y(a,b)=0$, provided these partial derivatives exist.

(or)

Let f be a function of two variables. Suppose f has an extremum at some point (a,b) and the partial derivatives of f exist at that point. Then

$$f_x(a,b)=0=f_y(a,b).$$

→ To check whether a given function has an extremum at some point or not, we can use above theorem. All we have to do is to see whether its partial derivatives vanish at that point (if they exist).

Ex: Check whether the function given by

$f(x,y) = x^2 - 2x + \frac{y^2}{4}$ has maximum or minimum values.

Sol: The given function $f(x,y) = x^2 - 2x + \frac{y^2}{4}$ is differentiable everywhere.

First we have to find out the points (x,y) such that $f_x(x,y) = 0 = f_y(x,y)$.

Now $f_x(x,y) = 2x-2$.

$f_y(x,y) = \frac{y}{2}$.

$\because$ $f_x(x,y)$ and $f_y(x,y)$ will vanish only when $x=1$ and $y=0$.

$\therefore$ the point $(1,0)$ is the only possible point where f can have a max or min value.

II.

Now, let us see whether $(1,0)$ is a max or min point for f.

$f(x,y) = x^2 - 2x + \frac{y^2}{4}$

$= x^2 - 2x + 1 + \frac{y^2}{4} - 1$

$= (x-1)^2 + \frac{y^2}{4} - 1$

$f(1,0) = 1 - 2 = -1$

$f(x,y) - f(1,0) = x^2 - 2x + \frac{y^2}{4} + 1$

$= (x-1)^2 + \frac{y^2}{4} \geq 0$

$\therefore f(x,y) - f(1,0) \geq 0$

$\therefore f(x,y) \geq f(1,0)$

This shows that $f(x,y) \geq -1 = f(1,0)$. $\forall (x,y)$.

$\therefore$ The function f has minimum at $(1,0)$.

The minimum value is $f(1,0) = -1$.

and the function has no maximum value.

Note: If $f_x \neq 0$ or $f_y \neq 0$ at some point, then we can straightaway say that the function does not have an extremum at that point.

- But if $f_x = 0 = f_y$ at some point, then this does not imply that the function has extremum at that point.

It is possible that all the first order partial derivatives of a function are zero at some point (a,b), but still, that point is not an extremum point for that function.

i.e, the converse of the above theorem is not true.

For example.

Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$f(x,y) = 1 - x^2 + y^2$.

Sol^n: we have $f_x = -2x$ & $f_y = 2y$

$\therefore f_x(0,0) = 0 = f_y(0,0)$.

Now, let us check whether the function f has an extremum at (0,0).

we have $f(0,0) = 1$.

$f(a, 0) < 1$ and $f(0, b) > 1$ for all non-zero 'a' and 'b'.

$\therefore$ In the nbd of (0,0), we can find points of the type (a,0) and (0,b).

$\therefore$ there exists no nbd 'N' of (0,0) for which $f(x,y) < f(0,0)$ or $f(x,y) > f(0,0)$. $\forall (x,y) \in N$.

$\therefore$ (0,0) is neither a maximum nor a minimum point for f, even though ~~both~~ the partial derivatives of f vanish at (0,0).

Ex-2: If $f(x,y) = 0$ if $x=0$ or $y=0$
$= 1$ else where

then both the partial derivatives exist (each equal to zero) at the origin, but $f(0,0)$ is not an extreme value.

Thus the conditions obtained in the above theorem are only necessary and not sufficient.

Some times it may happen that the partial derivatives of a function donot exist at a point, but, still the function has an extremum at that point.

for example :

Consider the function given by

$f(x,y) = 1 + \sqrt{x^2+y^2}$.

Sol^n: Since $f(0,0) = 1$

$f(x,y) = 1 + \sqrt{x^2+y^2} > 1 = f(0,0)$

i.e, $f(x,y) > f(0,0)$ for every point (x,y) in the nbd of $(0,0)$

It follows that f has a minimum at $(0,0)$.

Now, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{1+\sqrt{h^2}-1}{h} = \lim_{h\to 0} \dfrac{\sqrt{|h|^2}}{h}$

$= \lim_{h\to 0} \dfrac{|h|}{h}$

which does not exist.

~~Similarly~~ $\therefore f_x$ doesnot exist at $(0,0)$.

Similarly f_y doesnot exist at $(0,0)$.

Ex 2: The function $f(x,y) = |x| + |y|$ has an extreme value at $(0,0)$ even though the partial derivatives f_x and f_y donot exist at $(0,0)$.

Sol^n: Since $f(0,0) = 0 < |x| + |y| = f(x,y)$

i.e, $f(x,y) > f(0,0)$ for every point (x,y) in the nbd ~~of (0,0)~~.

It follows that f has a minimum at $(0,0)$

Now for the existence of partial derivatives of f at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{|h|-0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

which doesnot exist.

$\therefore f_x$ doesnot exist at $(0,0)$.

Similarly f_y doesnot exist at $(0,0)$.

Defn: Let f be a function of two variables. A point (a,b) is said to be a stationary point of f if both the partial derivatives are zero at (a,b).

Imp. **Sufficient condition for $f(x,y)$ to have an extreme value at (a,b):**

Theorem: If $f(x,y)$ has an extreme value at (a,b) and second order partial derivatives of $f(x,y)$ are continuous at (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$ and $(f_{xx}f_{yy}-f_{xy}^2)(a,b)>0$ then $f(a,b)$ is a max or min according as f_{xx} (or f_{yy}.) is $-ve$ or $+ve$ at (a,b)

i.e, $\rightarrow f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)>0$

and $f_{xx}(a,b)<0$ or $f_{yy}(a,b)<0$

then f has a maximum at (a,b)

$\rightarrow f_{xx}(a,b)\,f_{yy}(a,b)-f_{xy}^2(a,b)>0$

and $f_{xx}(a,b)>0$ or $f_{yy}(a,b)>0$

then f has minimum at (a,b).

Note 1: Further investigation is necessary, if

$f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) = 0$

Note 2: $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) < 0$, then f has neither maximum nor minimum

problems

Q1) Find the maxima and minima of the function

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Solⁿ: $f(x,y) = x^3+y^3-3x-12y+20$

$f_x(x,y) = 3x^2 - 3$

$f_y(x,y) = 3y^2 - 12$.

Equating to zero these f_x and f_y

we get $3x^2 - 3 = 0$

$\Rightarrow x = \pm 1$

and $3y^2 - 12 = 0$

$y = \pm 2$

$\therefore$ The function f has four stationary points

$(1,2), (-1,2), (1,-2), (-1,-2)$.

Now $f_{xx}(x,y) = 6x$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = 6y$

At $(1,2)$ $f_{xx} = 6 > 0$, $f_{yy} = 12 > 0$ and $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = 6 \times 12 - 0$

$= 72 > 0$

Hence $f(1,2)$ is the minimum point of $f(x,y)$.

i.e., $f(x,y)$ has minimum at $(1,2)$.

At $(-1,2)$ $f_{xx} = -6$, $f_{yy} = 12$ and $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = -6 \times 12 - 0$

$= -72 < 0$.

$\therefore$ The f^n $f(x,y)$ has neither max. nor min. at $(-1,2)$

At $(1,-2)$

$f_{xx} = 6$, $f_{xy} = 0$ and $f_{yy} = -12$

and $f_{xx} \cdot f_{yy} - f_{xy}^2 = -72 < 0$

$\therefore$ The function $f(x,y)$ has neither max. nor min at $(1,-2)$.

At $(-1,-2)$

$f_{xx} = -6 < 0$, $f_{yy} = -12 < 0$ and $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = 72 > 0$

$\therefore$ $(x,y) = (-1,-2)$ is the max pt of $f(x,y)$.

i.e, $f(x,y)$ has maximum at $(-1,-2)$.

Note: Stationary points like $(-1,2)$, $(+1,-2)$ which are not extreme (neither max. nor min) points are called the saddle points.

IAS 2012 → Find the all the maximum and minimum of $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$.

Sol^n $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

$f_x(x,y) = 3x^2 - 63 + 12y$

but $f_x(x,y) = 0$

$\Rightarrow 3x^2 - 63 + 12y = 0$

$\Rightarrow 3(x^2 - 21 + 4y) = 0$

$\Rightarrow x^2 + 4y = 21$ — ①

also $f_y(x,y) = 3y^2 - 63 + 12x$

and $f_y(x,y) = 0$

$\Rightarrow 3y^2 - 63 + 12x = 0$

$\Rightarrow y^2+4x=21 \quad — (2)$

(1) − (2) ⇒

$x^2-y^2+4(y-x)=0$

$\Rightarrow (x-y)(x+y)+4(y-x)=0$

$\Rightarrow (x-y)[x+y-4]=0$

$\Rightarrow x-y=0 \;;\; x+y=4$

$\Rightarrow \boxed{x=y} \;;\; \boxed{y=4-x.}$

Now Sub $x=y$ in (1)

$x^2+4x-21=0$

$(x-3)(x+7)=0$

$\Rightarrow x=3,-7$

$\Rightarrow y=3,-7$

∴ $(3,3), (-7,-7)$ are Stationary points.

Now sub $y=4-x$ in (1)

$\Rightarrow x^2+4(4-x)=21$

$\Rightarrow x^2-4x+16=21$

$\Rightarrow x^2-4x-5=0$

$\Rightarrow (x-5)(x+1)=0$

$\Rightarrow x=5,-1$

sub $x=5$ in $y=4-x$; sub $x=-1$ in $y=4-x$

$y=-1.$; $\Rightarrow y=5$

$(x,y)=(5,-1)$; $(x,y)=(-1,5).$

∴ $(3,3), (-7,-7), (-1,5), (5,-1)$ are stationary points.

~~At (3,3)~~

Now $f_{xx}=6x,\; f_{yy}=6y,\; f_{xy}=12$

put $D=f_{xx}f_{yy}-f_{xy}^2=6x\cdot6y-(12)^2$

$=36xy-144$

$\boxed{D=36xy-12\cancel{4}4}$

At $(3,3)$

$f_{xx} = 18 > 0$ and $D = 36 \times 9 - 144$
$= 324 - 144 = 180 > 0$

$\therefore$ the function $f(x,y)$ is minimum at $(3,3)$.

At $(-7,-7)$

$f_{xx} = -42 < 0$ and $D = 36(-7)(-7) - 144$
$= 36 \times 49 - 144 > 0$

$\therefore$ the function is maximum at $(-7,-7)$

At $(-1,5)$

$f_{xx} = -6 < 0$ and $D = 36(-1)(5) - 144 < 0$

$\therefore$ the function $f(x,y)$ has neither max nor min at $(-1,5)$

At $(1,-5)$

$f_{xx} = 6 > 0$ and $D = 36(1)(-5) - 144 < 0$

$\therefore$ the function has neither max nor min at $(-1,5)$.

$\therefore$ $(3,3)$, $(-7,-7)$ are called extreme points.

$\rightarrow$ S.T $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a max nor a minimum at $(0,0)$, where $f_{xx}f_{yy} - f_{xy}^2 = 0$

Sol^n

$f(x,y) = 2x^4 - 3x^2y + y^2$.

$f_x(x,y) = 8x^3 - 6xy$

$f_y(x,y) = -3x^2 + 2y$

also $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

and $f_{xx} = 24x - 6y$ $f_{xy} = -6x$ and $f_{yy} = 2$

At $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0(2) - 0 = 0$.

$\therefore f_{xx}f_{yy} - f_{xy}^2 = 0$.

So that it is a doubtful case, and so requires

61

Again $f(x,y) = 2x^4 - 3x^2y + y^2$

$= 2x^4 - 2x^2y - x^2y + y^2$

$= 2x^2(x^2-y) - y(x^2-y)$

$= (2x^2-y)(x^2-y)$

and $f(0,0) = 0$

now let $f(x,y) - f(0,0) = (x^2-y)(2x^2-y)$

> 0 for $y < 0$. or $x^2 > y > 0$

< 0 for $y > x^2 > \frac{y}{2} > 0$. $\because y > x^2$ and $2x^2 > y \Rightarrow x^2 > \frac{y}{2}$

$\therefore f(x,y) - f(0,0)$ doesnot keep the same sign near the origin.

Hence f has neither maximum nor minimum at the origin.

→ S.T the function $f(x,y) = (y-x)^4 + (x-2)^4$ has a minimum at $(2,2)$.

→ S.T $f(x,y) = y^2 + x^2y + x^4$ has a minimum at $(0,0)$.

Sol: $f(x,y) = y^2 + x^2y + x^4$.

$f_x(x,y) = 2xy + 4x^3$; $f_{xx} = 2y + 12x^2$

$f_y(x,y) = 2y + x^2$; $f_{yy} = 2$.

and $f_x(0,0) = 0$. $f_{xy} = 2x$.

$f_y(0,0) = 0$

At $(0,0)$ $f_{xx} = 0$, $f_{yy} = 2$ and $f_{xy} = 0$

$\therefore f_{xx} f_{yy} - f_{xy}^2 = 0$

So that it is a doubtful case and requires further investigation.

Now $f(x,y) = y^2 + x^2y + x^4$

$= (y + \frac{1}{2}x^2)^2 + \frac{3}{4}x^4$

and $f(x,y) - f(0,0) = (y + \frac{1}{2}x^2)^2 + \frac{3}{4}x^4 > 0 \;\forall (x,y)$ in the nbd of $(0,0)$

$\therefore f$ has a minimum at the origin.

Extreme values of a function of n variables.

12

A point $(a_1, a_2, \ldots a_n)$ is said to be an extreme point, and $f(a_1, a_2, \ldots a_n)$ an extreme ~~point~~ value of a function f, if for every point $(x_1, x_2, \ldots x_n)$, other than $(a_1, a_2, \ldots a_n)$ of some nbd of $(a_1, a_2, \ldots a_n)$, the difference,

$f(x_1, x_2, \ldots x_n) - f(a_1, a_2, \ldots a_n)$ keeps the same sign.

The extreme value is a maximum or a minimum value according as the sign is -ve or +ve.

→ The necessary conditions for $f(a_1, a_2, \ldots a_n)$ to be an extreme value of the function 'f' are that all the partial derivatives $f_{x_1}, f_{x_2}, f_{x_3} \ldots f_{x_n}$, in case they exist, vanish at $(a_1, a_2, \ldots a_n)$.

Since these are only necessary and not sufficient conditions, therefore points which satisfy these conditions may not be extreme points. A point $(a_1, a_2, \ldots a_n)$ is called a stationary point if all the first order partial derivatives of the function vanish at that point. Thus the stationary points are determined by solving the following n equations simultaneously.

$$f_{x_1}(x_1, x_2, \ldots x_n) = 0$$

$$f_{x_2}(x_1, x_2, \ldots x_n) = 0$$

$$\ldots\ldots\ldots\ldots$$

$$f_{x_n}(x_1, x_2, \ldots x_n) = 0$$

for a function of n independent variables $x_1, x_2, \ldots x_n$ the condition can be given in a more compact form.

i.e. if $(a_1, a_2, \ldots a_n)$ is a stationary point, then $df(a_1, a_2, \ldots a_n) = 0$.

$\because df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, if f is f of two variables,
$\Rightarrow df = f_x dx + f_y dy$: if (a, b) is stationary point

i.e, the differential of the function vanishes at a stationary point. For, at the stationary point all the partial derivatives vanish and therefore

$$df(a_1,a_2,\dots a_n) = f_{x_1}(a_1,a_2\dots a_n)dx_1 + f_{x_2}(a_1,a_2\dots a_n)dx_2 + \dots + f_{x_n}(a_1,a_2,\dots a_n)dx_n = 0$$

Conversely, when $df=0$, the coefficients of the differentials $dx_1, dx_2, \dots dx_n$ of independent variables, are seperately equal to zero.

<u>Rule</u> For a function $f(x,y,z)$ of three independent variables, sufficient conditions for (a,b,c) to be an extreme point are that

i) $df(a,b,c) = f_x dx + f_y dy + f_z dz = 0$, so that $f_x = f_y = f_z = 0$

and

ii) $d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx$,

keeps the same sign for arbitrary values of dx, dy, dz, according the extreme point being a maxima or a minima as the sign of d^2f is $-$ve or $+$ve. The point will be neither a maxima nor a minima if d^2f doesnot keep the same sign; and requires further investigation, if d^2f keeps the same sign but vanishes at some points of a nbd of (a,b,c).

$df = f_x dx + f_y dy + f_z dz$.

$d^2f = d(df) = d(f_x dx + f_y dy + f_z dz) = d(f_x dx) + d(f_y dy) + d(f_z dz)$

$= (f_{xx}dx + f_{yx}dy + f_{zx}dz)\,dx + f_x d^2x$
$+ (f_{xy}dx + f_{yy}dy + f_{zy}dz)\,dy + f_y d^2y$
$+ (f_{xz}dx + f_{yz}dy + f_{zz}dz)\,dz + f_z d^2z$.

$\therefore d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 \dots$ but $f_x = f_y = f_z = 0$ at (a,b,c)

The conditions that d^2f keeps the same sign may be stated in terms of matrices, as follows.

Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

d^2f will always be +ve iff. the three principal minors

$$|f_{xx}|, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all +ve.

Then f has minimum at (a, b, c).

and d^2f will always negative iff their signs are alternatively negative and positive.; then f has maximum at (a,b,c)

(OR)

Let $f: D \to R$, $D \subset R^n$ be a function possessing continuous partial derivatives upto second order ~~partial derivatives~~ in a nbd of a point 'a' at which all the first order partial derivatives vanish,

then

i) 'f' has minimum at 'a', if $D_1, D_2, \ldots D_n$ are all +ve

ii) 'f' has maximum at 'a' if $D_1, D_2, \ldots D_n$ are alternatively -ve. and +ve.

where $D_1 = |d_{11}|$, $D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$, $D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$, ------

and $d_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

→ Examine the following function for extreme values.

$f(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z.$

Sol^n Given $f(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$

$f_x = 4x - 3y$

$f_y = 6y - 3x$

$f_z = 8z + 8$

Equating f_x, f_y, f_z to zero.

we get $4x - 3y = 0$

$x - 2y = 0$

$z + 1 = 0$

$\Rightarrow x = 0,\ y = 0$ and $z = -1$

$\therefore$ we get the only stationary point of f as $(0,0,-1)$.

Now at $(0,0,-1)$,

we have

$d_{11} = f_{xx} = 4$; $d_{12} = f_{xy} = -3$; $d_{13} = f_{xz} = 0$

$d_{21} = f_{yx} = -3$; $d_{22} = f_{yy} = 6$; $d_{23} = f_{yz} = 0$

$d_{31} = f_{zx} = 0$; $d_{32} = f_{zy} = 0$; $d_{33} = f_{zz} = 8$

Also $D_1 = d_{11} = 4 > 0$, $D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix}$

$= 24 - 9 = 15 > 0$

$$D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = \begin{vmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix}$$

$= 8(24-9) = 8 \times 15 = 120 > 0.$

Hence $f(x,y,z)$ has a minimum at $(0,0,-1)$

→ S.T the minimum and the maximum values of 47

S2-f-25 $f(x,y,z) = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$ are

$$-\sqrt{\tfrac{1}{2}\left(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2}\right)/e} \quad \text{and} \quad \sqrt{\tfrac{1}{2}\left(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2}\right)/e}$$

Solⁿ:

$$f(x,y,z) = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

$$f_x = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}\cdot(-2\alpha^2x) + a\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

$$= \left[a-(ax+by+cz)(+2x\alpha^2)\right] e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

$$= \left[a-2\alpha^2x\sum ax\right] e^{-\sum\alpha^2x^2}$$

sly $f_y = \left[b-2\beta^2y\sum ax\right] e^{-\sum\alpha^2x^2}$

sly $f_z = \left[c-2\gamma^2z\sum ax\right] e^{-\sum\alpha^2x^2}$

Equating f_x, f_y, f_z to zero.

$f_x = \left(a-2\alpha^2x\sum ax\right) e^{-\sum\alpha^2x^2} = 0$

$\Rightarrow a-2\alpha^2x\sum ax = 0$, since $e^{-\sum\alpha^2x^2} \neq 0$

$f_y = 0$

$\Rightarrow b-2\beta^2y\sum ax = 0$ — (1)

$f_z = 0$

$\Rightarrow c-2\gamma^2z\sum ax = 0$

$\therefore x\sum ax = \dfrac{a}{2\alpha^2}$ — (2)

$y\sum ax = b/2\beta^2$ — (3)

$z\sum ax = c/2\gamma^2$ — (4)

Multiplying (2) by a, (3) by b, (4) by c and adding.

$$(ax+by+cz)\sum ax = \frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)$$

$$\sum ax\sum ax = \frac{1}{2}\sum a^2\alpha^{-2}$$

$$\Rightarrow \left(\sum ax\right)^2 = \frac{1}{2}\sum a^2\alpha^{-2}$$

$$\pm\sqrt{\tfrac{1}{2}\sum a^2\alpha^{-2}} = \pm k \text{ (say)}$$

Hence from (1) the stationary points are

$a - 2\alpha^2 x \sum ax = 0$ putting $\sum ax = +k$ in (1)

$a - 2\alpha^2 x(+k) = 0$

$\Rightarrow 2\alpha^2 x k = a$

$\Rightarrow \quad x = \frac{a}{2\alpha^2 k}$.

$|||^{ly}$ $y = \frac{b}{2\beta^2 k}$, $z = \frac{c}{2\gamma^2 k}$.

putting $\sum ax = -k$ in (1)

$x = -\frac{a}{2\alpha^2 k}$, $y = \frac{-b}{2\beta^2 k}$, $z = \frac{-c}{2\gamma^2 k}$.

$\therefore$ the stationary points are

$\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right)$, $\left(\frac{-a}{2\alpha^2 k}, \frac{-b}{2\beta^2 k}, \frac{-c}{2\gamma^2 k}\right)$

Again, we have

$$f_{xx} = \left[0 - 2\alpha^2 x a - 2\alpha^2 \sum ax\right] e^{-\sum \alpha^2 x^2} + \left[a - 2\alpha^2 x \sum ax\right] - 2\alpha^2 x\, e^{-\sum \alpha^2 x^2}$$

$$= -2\alpha^2 x\left[a - 2\alpha^2 x \sum ax\right] e^{-\sum \alpha^2 x^2} - 2\alpha^2\left[\sum ax + ax\right] e^{-\sum \alpha^2 x^2}$$

$$f_{xy} = \left[b - 2\beta^2 y \sum ax\right] e^{-\sum \alpha^2 x^2}(-2\alpha^2 x) + (-2\beta^2 y a)\, e^{-\sum \alpha^2 x^2}$$

$$= -2\alpha^2 x\left(b - 2\beta^2 y \sum ax\right) e^{-\sum \alpha^2 x^2} - 2\beta^2 y a\, e^{-\sum \alpha^2 x^2}$$

and similar expressions for $f_{yy}, f_{zz}, f_{yz}, f_{zx}$.

At the stationary point $\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right)$

we have $\sum \alpha^2 x^2 = \frac{1}{2}$.

27 $f_{xx} = 0 - 2\alpha^2\left[k + \frac{a^2}{2\alpha^2 k}\right] e^{-1/2}$ 48

$= -\frac{1}{\sqrt{e}}\left[2\alpha^2 k + \frac{a^2}{k}\right] = -\left(\frac{2\alpha^2 k^2 + a^2}{k\sqrt{e}}\right)$

$f_{yy} = -\left(\frac{2\beta^2 k^2 + b^2}{k\sqrt{e}}\right), \quad f_{zz} = -\left(\frac{2\gamma^2 k^2 + c^2}{k\sqrt{e}}\right)$

$f_{xy} = 0 - \frac{ab}{k\sqrt{e}} = -\frac{ab}{k\sqrt{e}}; \quad f_{yz} = -\frac{bc}{k\sqrt{e}}; \quad f_{zx} = -\frac{ca}{k\sqrt{e}}.$

$\therefore \; d^2 f = -\frac{1}{k\sqrt{e}}\left[(2\alpha^2 k^2 + a^2)dx^2 + (2\beta^2 k^2 + b^2)dy^2 + (2\gamma^2 k^2 + c^2)dz^2\right]$

$- \frac{2}{k\sqrt{e}}\left(ab\,dx\,dy + bc\,dy\,dz + ca\,dz\,dx\right)$

Now $f_{xx} = -\left(\frac{2\alpha^2 k^2 + a^2}{k\sqrt{e}}\right) < 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{(2\alpha^2 k^2 + a^2)}{k\sqrt{e}} & -\frac{ab}{k\sqrt{e}} \\ -\frac{ab}{k\sqrt{e}} & -\frac{(2\beta^2 k^2 + b^2)}{k\sqrt{e}} \end{vmatrix}$$

$$= \frac{1}{k^2 e}\begin{vmatrix} 2\alpha^2 k^2 + a^2 & ab \\ ab & 2\beta^2 k^2 + b^2 \end{vmatrix}$$

$$= \frac{1}{k^2 e}\left(4\alpha^2\beta^2 k^4 + 2\alpha^2 k^2 b^2 + 2\beta^2 k^2 a^2 + a^2 b^2 - a^2 b^2\right)$$

$$= \frac{2}{e}\left(2\alpha^2\beta^2 k^2 + \alpha^2 b^2 + \beta^2 a^2\right) > 0$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{k^3 e^{3/2}}\begin{vmatrix} 2\alpha^2 k^2 + a^2 & ab & ac \\ ab & 2\beta^2 k^2 + b^2 & bc \\ ac & bc & 2\gamma^2 k^2 + c^2 \end{vmatrix}$$

$$= \frac{-1}{k^3 e^{3/2}}\Big[(2\gamma^2k^2+c^2)\big[4\alpha^2\beta^2k^4+2\alpha^2k^2b^2+2\beta^2k^2a^2+a^2b^2-a^2b^2\big]$$
$$-bc\big[(bc)(2\alpha^2k^2+a^2)-a^2bc\big]+ac\big[ab^2c-(2\beta^2k^2+b^2)ac\big]\Big]$$

$$= \frac{-1}{k^3e^{3/2}}\Big[(2\gamma^2k^2+c^2)2k^2(\alpha^2\beta^2+\alpha^2b^2+a^2\beta^2)-2b^2c^2\alpha^2k^2$$
$$-2a^2c^2\beta^2k^2\Big]$$

$$= \frac{-4k}{e^{3/2}}\left(2\alpha^2\beta^2\gamma^2k^2+\alpha^2b^2c^2+\alpha^2\beta^2c^2+a^2b^2\gamma^2\right)<0$$

Thus the three principal minors have alternatively $-$ve and $+$ve signs and so d^2f is always $-$ve.

$\therefore \left(\frac{a}{2\alpha^2k}, \frac{b}{2\beta^2k}, \frac{c}{2\gamma^2k}\right)$ is a point of maxima.

and the maximum value $= ke^{-1/2}$

$$= \sqrt{\tfrac{1}{2}\Sigma a^2\alpha^{-2}}\,\sqrt{\tfrac{1}{e}}$$

$$= \sqrt{\tfrac{1}{2}\Sigma a^2\alpha^{-2}/e}$$

At the point $\left(\frac{-a}{2\alpha^2k}, \frac{-b}{2\beta^2k}, \frac{-c}{2\gamma^2k}\right)$, it may be shown as above

that $\Sigma \alpha^2x^2 = \frac{1}{2}$ and

$f_{xx} = \frac{2\alpha^2k^2+a^2}{k\sqrt{e}}$, $f_{yy} = \frac{2\beta^2k^2+b^2}{k\sqrt{e}}$, $f_{zz} = \frac{2\gamma^2k^2+c^2}{k\sqrt{e}}$

$f_{xy} = \frac{ab}{k\sqrt{e}}$, $f_{yz} = \frac{bc}{k\sqrt{e}}$, $f_{zx} = \frac{ca}{k\sqrt{e}}$.

and the three principal minors are of $+$ve signs.

so that d^2f is $+$ve.

$\therefore \left(\frac{-a}{2\alpha^2k}, \frac{-b}{2\beta^2k}, \frac{-c}{2\gamma^2k}\right)$ is a point of minima

and the minimum value of the function $= -ke^{-1/2}$

$$= -\sqrt{\tfrac{1}{2}\Sigma a^2\alpha^{-2}/e}.$$

62 → S.T $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$.
has a minima at $(1,1,1)$ and a maxima at $(-1,-1,-1)$.

Sol^n Given $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$.

$$f_x = 3(x+y+z)^2 - 24yz - 3$$

$$f_y = 3(x+y+z)^2 - 24zx - 3$$

$$f_z = 3(x+y+z)^2 - 24xy - 3$$

∴ the Stationary points are given by

$$(x+y+z)^2 - 8yz - 1 = 0 \quad \text{--- (1)}$$

$$(x+y+z)^2 - 8zx - 1 = 0 \quad \text{--- (2)}$$

$$(x+y+z)^2 - 8xy - 1 = 0 \quad \text{--- (3)}$$

(2)−(1)

$z(x-y) = 0 \Rightarrow z = 0$ or $y = x$

(3)−(2) $x(y-z) = 0 \Rightarrow x = 0$ or $y = z$

(1)−(3) $y(z-x) = 0 \Rightarrow y = 0$ or $z = x$.

i.e, either $x = y = z = 0$ or $x = y = z$.

∴ the Stationary points are $(1,1,1)$ and $(-1,-1,-1)$

Again, we have

$$f_{xx} = 6(x+y+z) = f_{yy} = f_{zz}$$

$$f_{xy} = 6(x+y+z) - 24z = f_{yx}$$

$$f_{yz} = 6(x+y+z) - 24x = f_{zy}$$

$$f_{zx} = 6(x+y+z) - 24y = f_{xz}$$

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At $(1,1,1)$

$$f_{xx} = f_{yy} = f_{zz} = 18$$

$$f_{xy} = f_{yz} = f_{zx} = -6$$

$$\therefore d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

$$d^2f\Big|_{(1,1,1)} = 18(dx^2 + dy^2 + dz^2) - 12(dx\,dy + dy\,dz + dz\,dx)$$

$$= 6\left[3(dx^2+dy^2+dz^2) - 2(dx\,dy + dy\,dz + dz\,dx)\right]$$

$$d^2f = 6\left[(dx^2+dy^2+dz^2) + (dx-dy)^2 + (dy-dz)^2 + (dz-dx)^2\right]$$

which is +ve for all values of dx, dy, dz and does not vanish for $(dx, dy, dz) \neq (0,0,0)$.

$\therefore$ $(1,1,1)$ is a point of minima of the function

i.e., f has minimum at $(1,1,1)$.

At $(-1,-1,-1)$.

$$f_{xx} = f_{yy} = f_{zz} = -18 \; ; \; f_{xy} = f_{yz} = f_{zx} = 6$$

$$\therefore d^2f = -18\left[dx^2+dy^2+dz^2\right] + 12(dx\,dy + dy\,dz + dz\,dx)$$

$$= -6\left[3(dx^2+dy^2+dz^2) - 2(dx\,dy + dy\,dz + dz\,dx)\right]$$

$$= -6\left[(dx^2+dy^2+dz^2) + (dx-dy)^2 + (dy-dz)^2 + (dz-dx)^2\right]$$

which is -ve for all dx, dy, dz and never vanishes.

Hence the function has maximum at $(-1,-1,-1)$.

$\rightarrow$ S.T the following functions have a minima at the points indicated

i) $x^2+y^2+z^2+2xyz$ at $(0,0,0)$

(ii) $x^4+y^4+z^4-4xyz$ at $(1,1,1)$.

$\rightarrow$ S.T the function

$f(x,y,z) = 2xyz - 4zx - 2yz + x^2+y^2+z^2-2x-4y+4z$

has 5 stationary points but has a minimum value only at $(1,2,0)$

$\rightarrow$ S.T the function

$3\log(x^2+y^2+z^2) - 2x^3 - 2y^3 - 2z^3$, $(x,y,z) \neq (0,0,0)$.

has only one extreme value, $\log(3/e^2)$.

IE → Find a point with in a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Sol^n: Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices of the triangle and (x, y) be a point inside the triangle

Let $f(x, y)$ denotes the sum of the squares of the distances of (x, y) from three vertices,

then

$$f(x,y) = \left[(x-x_1)^2 + (y-y_1)^2\right] + \left[(x-x_2)^2 + (y-y_2)^2\right] + \left[(x-x_3)^2 + (y-y_3)^2\right]$$

$\Rightarrow f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3)$ & $f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3)$

for maximum or minimum,

we have

$$f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3) = 0$$

$$\Rightarrow 3x - (x_1 + x_2 + x_3) = 0$$

$$\Rightarrow x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly

$$f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3) = 0$$

$$\Rightarrow 3y - (y_1 + y_2 + y_3) = 0$$

$$\Rightarrow y = \frac{y_1 + y_2 + y_3}{3}$$

Also $f_{xx} = 2+2+2 = 6$

$f_{xy} = 0$ & $f_{yy} = 6$.

$\therefore f_{xx} f_{yy} - (f_{xy})^2 = (6)(6) - 0 = 36 > 0$

and $f_{xx} = 6 > 0$

$\therefore f$ is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, \quad y = \frac{y_1 + y_2 + y_3}{3}$$

$\therefore$ The required point is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

→ Show that the function

$f(x,y,z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

has 5 stationary points but has a minimum value only at $(1,1,0)$.

Sol^n: Given that

$f(x,y,z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

$f_x = 2yz - 4z + 2x - 2$

$f_y = 2xz - 2z + 2y - 4$

$f_z = 2xy - 4x - 2y + 2z + 4$.

∴ The stationary points are given by

$f_x = 0 \Rightarrow 2yz - 4z + 2x - 2 = 0 \Rightarrow yz - 2z + x - 1 = 0$ — (1)

$f_y = 0 \Rightarrow 2xz - 2z + 2y - 4 = 0 \Rightarrow xz - z + y - 2 = 0$ — (2)

$f_z = 0 \Rightarrow 2xy - 4x - 2y + 2z + 4 = 0 \Rightarrow xy - 2x - y + z + 2 = 0$ — (3)

Adding the last two equations, we see that system is equivalent to

$yz - 2z + x - 1 = 0$

$zx - z + y - 2 = 0$

$zx + xy - 2x = 0 \Rightarrow x(z + y - 2) = 0$.

Thus stationary points are given by the two systems of equations

$$\left.\begin{aligned} yz - 2z + x - 1 &= 0 \\ zx - z + y - 2 &= 0 \\ x &= 0. \end{aligned}\right\}$$

&

$$\left.\begin{aligned} yz - 2z + x - 1 &= 0 \\ zx - z + y - 2 &= 0 \\ z + y - 2 &= 0 \end{aligned}\right\}$$

These give $(0,3,1)$, $(0,1,-1)$, $(1,2,0)$, $(2,1,1)$, $(2,3,-1)$ are the stationary points of the function.

Again, we have at any point (x,y,z)

$f_{xx}=2,\ f_{yy}=2,\ f_{zz}=2;$

$f_{xy}=2z,\ f_{yz}=2x-2,\ f_{zx}=2y-4=f_{xz}$
$=f_{yx}\qquad =f_{zy}$

for $(0,3,1)$ the matrix of the quadratic form is

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}.$$

Its principal minors

$$2,\ \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix},\ \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

are 2, 0, and -32 respectively.

Thus the function is neither a maximum nor a minimum at $(0,3,1)$

(∵ f has minimum if principal minors are all +ve
& f has maximum if principal minors are alternatively −ve & +ve)

It may similarly be shown that the function is neither a max nor a minimum at the stationary points $(0,1,-1)$, $(2,1,1)$ and $(2,3,-1)$.

At $(1,2,0)$ the ~~matrix~~ matrix of the quadratic form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The principal minors

$|2|$, $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$ & $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

are 2, 4 & 8 respectively.

i.e., the principal minors are all +ve.

$\therefore$ $f(x, y, z)$ has a minimum at $(1, 2, 6)$

---

7. Show that the function

$3\log(x^2+y^2+z^2) - 2x^3 - 2y^3 - 2z^3$, $(x, y, z) \neq (0, 0, 0)$

has only one extreme value, $\log(3/e^2)$.

Soln:

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37. Find the extreme value of xyz if $x+y+z=a$.

Sol^n: Let $f = xyz$.

$\phi = x+y+z-a$.

Now consider the function F of three independent variables x, y, z such that

$F = xyz + \lambda(x+y+z-a)$.

where λ is a constant

$dF = (yz+\lambda)\,dx + (xz+\lambda)\,dy + (xy+\lambda)\,dz$.

At stationary points $dF = 0$

$\therefore f_x = 0 \Rightarrow yz+\lambda = 0$ —— (1)

$f_y = 0 \Rightarrow xz+\lambda = 0$ —— (2)

$f_z = 0 \Rightarrow xy+\lambda = 0$ —— (3)

multiplying (1) by x, (2) by y & (3) by z and adding,

we get

$3xyz + \lambda(x+y+z) = 0$

$\Rightarrow 3xyz + \lambda(a) = 0$ $\quad (\because x+y+z=a)$

$\Rightarrow \boxed{\lambda = \frac{-3xyz}{a}}$

From (1),

we have

$yz + \lambda = 0 \Rightarrow yz - \frac{3xyz}{a} = 0$

$\Rightarrow yz\left(1 - \frac{3x}{a}\right) = 0$

$\Rightarrow 1 - \frac{3x}{a} = 0$ or $yz = 0$ (ignoring this)

$\Rightarrow x = \frac{a}{3}$.

from (2)

$xz + \lambda = 0 \Rightarrow xz + \left(-\frac{3xyz}{a}\right) = 0$

$\Rightarrow xz\left(1 - \frac{3y}{a}\right) = 0$

$\Rightarrow y = \frac{a}{3}$

Similarly, we get $x = \frac{a}{3}$.

$\therefore$ The stationary point is $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

$\therefore f = xyz$

$= \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)$

$= \frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

Hence $f = xyz = \frac{a^3}{27}$ ——— (4)

Now $d^2F = d(dF)$

$= d\left[(yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz\right]$

$= \left[(yz+\lambda)dx^2 + z\,dy + y\,dz\right]dx +$

$\left[(xz+\lambda)dy^2 + z\,dx + x\,dz\right]dy +$

$\left[(xy+\lambda)dz^2 + (y\,dx + x\,dy)\right]dz.$

$= 2(z\,dx\,dy + x\,dy\,dz + y\,dx\,dz)$,

(as $f_x = 0, f_y = 0, f_z = 0$)

Here $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$

~~$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & f_{xy} \\ f_{yx} & 0 \end{vmatrix}$~~

~~and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & f_{xy} & f_{xz} \\ f_{yx} & 0 & f_{yz} \\ f_{zx} & f_{zy} & 0 \end{vmatrix}$~~

$\therefore$ we require further investigation.

50 (iv)

39

Here $f_{xx} = 0$.

$\therefore$ we require further investigation.

Treating z as function of x and y,

we get from

$$f(x,y,z) = xyz - \frac{a^3}{27} = 0$$

$$yz + xy\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$$

$$\text{Similarly } \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

$$\text{Also } \frac{\partial^2 z}{\partial x^2} = -\frac{\left[x\frac{\partial z}{\partial x} - z\right]}{x^2}$$

$$= -\frac{\left[x\left(-\frac{z}{x}\right) - z\right]}{x^2}$$

$$= \frac{2z}{x^2}$$

$$\text{Similarly } \frac{\partial^2 z}{\partial y^2} = \frac{2z}{y^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{\left(y\frac{\partial z}{\partial x} - z(0)\right)}{y^2}$$

$$= -\frac{\left[y\left(-\frac{z}{x}\right) - 0\right]}{y^2}$$

$$= \frac{z}{xy}.$$

At $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

$$z_{xx} = \frac{2\left(\frac{a}{3}\right)}{\left(\frac{a}{3}\right)^2} = \frac{2\times 3}{a} = \frac{6}{a} > 0 \text{ if } a > 0$$

$$z_{yy} = \frac{6}{a} > 0$$

$$z_{xy} = \frac{a/3}{(a/3)^2} = \frac{3}{a} > 0.$$

$$z_{xx} > 0 \text{ and } z_{xx} z_{yy} - z_{xy}^2 = \left(\frac{6}{a}\right)\left(\frac{6}{a}\right) - \left(\frac{3}{a}\right)^2$$

$$= \frac{36}{a^2} - \frac{9}{a^2}$$

$$= \frac{27}{a^2} > 0.$$

$\therefore$ $f(x,y,z)$ has a minimum value at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ and the minimum value is $3abc$.