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Indian Forest Service Main Exam

Mathematics

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**Union Public Service
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Popular Master Guide

IFS

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Main Examination

MATHEMATICS

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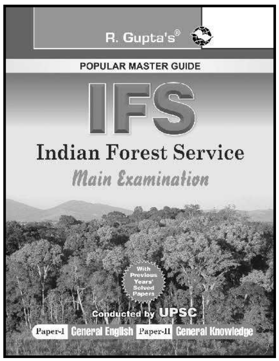


SCHEME OF THE MAIN EXAMINATION

The main examination for the Indian Forest Service comprises:

(A) The written examination consisting of the following papers:-

Paper I	General English	300 marks
Paper II	General Knowledge	300 marks
Paper III **Paper IV** **Paper V** **Paper VI**	Any two subjects to be selected from the list of the optional subjects. Each subject will have two papers.	200 Marks for each paper

(B) Interview for Personality Test —Maximum marks: 300

LIST OF OPTIONAL SUBJECTS

1. Agriculture
2. Agricultural Engineering
3. Animal Husbandry & Veterinary Science
4. Botany
5. Chemistry
6. Chemical Engineering
7. Civil Engineering
8. Forestry
9. Geology
10. Mathematics
11. Mechanical Engineering
12. Physics
13. Statistics
14. Zoology

Provided that the candidate will not be allowed to offer the following combination of subjects:

(a) Agriculture and Agricultural Engineering
(b) Agriculture and Animal Husbandry & Veterinary Science
(c) Agriculture and Forestry
(d) Chemistry and Chemical Engineering
(e) Mathematics and Statistics
(f) Of the Engineering subjects viz. Agricultural Engineering, Chemical Engineering, Civil Engineering and Mechanical Engineering not more than one subject.

GENERAL INSTRUCTIONS

1. All the question papers for the examination will be of conventional (Essay) type.

2. All question papers must be answered in English. Question papers will be set in English only.

3. The duration of each of the papers referred to above will be 3 hours.

Contents

Previous Years' Paper (Solved)

IFS Main Examination, 2017

MATHEMATICS

PAPER-I

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos.* ***1*** *and* ***5*** *are compulsory. Out of the remaining six questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections* ***A*** *and* ***B.*** *Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1. If $B = bA$ is orthogonal, determine the values of a and b. (8)

(*b*) Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where

(*i*) W contains all 2×2 matrices with zero determinant.

(*ii*) W consists of all 2×2 matrices A such that $A^2 = A$. (8)

(*c*) Using the Mean Value Theorem, show that

(*i*) $f(x)$ is constant in $[a, b]$, if $f'(x) = 0$ in $[a, b]$.

(*ii*) $f(x)$ is a decreasing function in (a, b), if $f'(x)$ exists and is < 0 everywhere in (a, b). (8)

(*d*) Let $u(x, y) = ax^2 + 2hxy + by^2$ and $v(x, y) = Ax^2 + 2Hxy + By^2$. Find the Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)}$, and hence show that u, v are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$. (8)

(*e*) Find the equations of the planes parallel to the plane $3x - 2y + 6z + 8 = 0$ and at a distance 2 from it. (8)

2. (*a*) State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Hence find A^{-1}. (10)

(*b*) Show that

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p,q > -1.$$

Hence evaluate the following integrals:

(*i*) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

(*ii*) $\int_0^1 x^3 (1 - x^2)^{5/2} dx$

(*iii*) $\int_0^1 x^4 (1 - x)^3 dx$ (10)

(*c*) Find the maxima and minima for the function

$f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

Also find the saddle points (if any) for the function. (10)

(*d*) Show that the angles between the planes given by the equation

$2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz$

$= 0$ is $\tan^{-1}\dfrac{\sqrt{50}}{4}$. (10)

3. (*a*) Reduce the following matrix to a row-reduced echelon form and hence find its rank : (10)

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

(*b*) Given that the set $\{u, v, w\}$ is linearly independent, examine the sets

(*i*) $\{u + v, v + w, w + u\}$

(*ii*) $\{u + v, u - v, u - 2v + 2w\}$

for linear independence. (10)

(*c*) Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$, by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$. (10)

(*d*) Find the angle between the lines whose direction cosines are given by the relations $l + m + n = 0$ and $2lm + 2ln - mn = 0$. (10)

4. (*a*) Find the eigenvalues and the corresponding eigenvectors for the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $D = P^{-1} AP$. (10)

(*b*) A function $f(x, y)$ is defined as follows:

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $f_{xy}(0, 0) = f_{yx}(0, 0)$. (10)

(*c*) Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios (1, – 2, 2). (10)

(*d*) Find the shortest distance and the equation of the line of the shortest distance between the lines $\dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1}$ and $\dfrac{x+3}{-3} = \dfrac{y+7}{2} = \dfrac{z-6}{4}$. (10)

SECTION–B

5. (*a*) Solve

$(2D^3 - 7D^2 + 7D - 2)\, y = e^{-8x}$

where $D = \dfrac{d}{dx}$. (8)

(*b*) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4. \quad (8)$$

(*c*) A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position P (OP = b) with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\dfrac{T}{\pi}\tan^{-1}\left(\dfrac{vT}{2\pi b}\right)$. (8)

(*d*) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\dfrac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is 90°. (8)

(*e*) Prove that

$$\nabla^2 r^n = n(n + 1)\, r^{n-2}$$

and that $r^n\vec{r}$ is irrotational, where

$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. (8)

6. (*a*) Solve the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2.\frac{dy}{dx}.y\cot x = y^2. \quad (15)$$

(*b*) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\dfrac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$. (10)

(*c*) Using Stokes' theorem, evaluate

$$\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz],$$

where C is the boundary of the triangle with vertices at (2,0,0), (0,3,0) and (0,0,6). (15)

7. (*a*) Solve the differential equation

$$e^{3x}\left(\frac{dy}{dx} - 1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0. \quad (10)$$

(*b*) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\dfrac{2\pi\, ae}{T\sqrt{1-e^2}}$ where $2a$ is the major axis, e is the eccentricity and T is the periodic time. (10)

(*c*) A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth. If the minor axis lies on the surface, then find the eccentricity in order that the focus may be the centre of pressure. (10)

(*d*) Evaluate

$$\iint_S \left(\nabla \times \vec{f}\right).\hat{n}\, dS,$$

where S is the surface of the cone, $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane and $\vec{f} = (x-z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$. (10)

8. (*a*) Solve $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ by using the method of variation of parameter. (10)

(*b*) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(\dfrac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\dfrac{8}{15}\sqrt{\dfrac{6}{\mu}}$. (10)

(*c*) Find the curvature and torsion of the circular helix

$$\vec{r} = a(\cos\theta, \sin\theta, \theta\cot\beta)$$

β is the constant angle at which it cuts its generators. (10)

(*d*) If the tangent to a curve makes a constant angle α, with a fixed line, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$.

Conversely, if $\dfrac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction. (10)

PAPER-II

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos.* ***1*** *and* ***5*** *are compulsory. Out of the remaining six questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections* ***A*** *and* ***B****. Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Prove that every group of order four is Abelian. (8)

(*b*) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as below:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. (10)

(*c*) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ and $u + 2v = x^3 - 2y^3 + 3xy\,(2x - y)$ then find $f(z)$ in terms of z. (8)

(*d*) Solve by simplex method the following LPP:

Minimize $Z = x_1 - 3x_2 + 2x_3$

subject to the constraints

$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 0$$

and $x_1, x_2, x_3 \geq 0$ (14)

2. (*a*) Let G be the set of all real numbers except -1 and define $a*b = a + b + ab \;\forall\; a, b \in G$. Examine if G is an Abelian group under $*$. (10)

(*b*) Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh \;\forall h \in H$ and $k \in K$. (10)

(*c*) Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A$ for some positive integer $n\}$. Is B an ideal of R? Justify your answer. (10)

(*d*) Prove that the ring

$\mathbb{Z}(i) = \left\{a + ib : a, b \in \mathbb{Z}, i = \sqrt{-1}\right\}$ of Gaussian integers is a Euclidean domain. (10)

3. (*a*) Evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ given that

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} & \text{if } xy \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

(10)

(*b*) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1.$$ (10)

(*c*) Prove that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent but not absolutely convergent. (12)

(*d*) Find the volume of the region common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. (8)

4. (*a*) Prove by the method of contour integration that $\int_0^\infty \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\,d\theta = 0.$ (12)

(*b*) Find the sum of residues of $f(z) = \frac{\sin z}{\cos z}$ at its poles inside the circle $|z| = 2$. (8)

(*c*) Evaluate $\int_{x=0}^{\infty} \int_{y=0}^{x} x e^{-x^2/y}\,dy\,dx$. (8)

(*d*) A computer centre has four expert programmers. The centre needs four application programs to be developed. The head of the centre after studying carefully the programs to be developed, estimates the computer times in hours required by the experts to the application programs as follows:

		Programs			
		A	B	C	D
Programmer	P_1	5	3	2	8
	P_2	7	9	2	6
	P_3	6	4	5	7
	P_4	5	7	7	8

Assign the programs to the programmers in such a way that total computer time is least. (12)

SECTION–B

5. (*a*) Form the partial differential equation by eliminating arbitrary functions φ and ψ from the relation $z = \varphi(x^2 - y) + \psi(x^2 + y)$. (8)

(*b*) Write a BASIC program to compute the multiplicative inverse of a non-singular square matrix. (12)

(*c*) A uniform rectangular parallelopiped of mass M has edges of lengths $2a$, $2b$, $2c$. Find the moment of inertia of this rectangular parallelopiped about the line through its centre parallel to the edge of length $2a$. (10)

(*d*) Evaluate $\int_0^1 e^{-x^2}\,dx$ using the composite trapezoidal rule with four decimal precision, *i.e.*, with the absolute value of the error not exceeding 5×10^{-5}. (10)

6. (*a*) Solve the partial differential equation:

$$(x-y)\frac{\partial z}{\partial x} + (x+y)\frac{\partial z}{\partial y} = 2xz \quad (8)$$

(*b*) Find the surface which is orthogonal to the family of surfaces $z(x + y) = c(3z + 1)$ and which passes through the circle $x^2 + y^2 = 1$, $z = 1$. (8)

(*c*) Find complete integral of $xp - yq = xqf(z - px - qy)$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. (12)

(*d*) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement $y(x, t)$. (12)

7. (*a*) Find the real root of the equation $x^3 + x^2 + 3x + 4 = 0$ correct up to five places of decimal using Newton-Raphson method. (10)

(*b*) A river is 80 metre wide, the depth y, in metre, of the river at a distance x from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the area of cross-section of the river using Simpson's $\frac{1}{3}$rd rule. (10)

(*c*) Find y for $x = 0.2$ taking $h = 0.1$ by modified Euler's method and compute the error, given that: $\frac{dy}{dx} = x + y,\ y(0) = 1.$ (10)

(*d*) Assuming a 32 bit computer representation of signed integers using 2's complement representation, add the two numbers –1 and –1024 and give the answer in 2's complement representation. (10)

8. (*a*) Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume that the mass can only move up and down in the vertical direction. Show that

$$L = \frac{1}{2}my'^2 - \frac{1}{2}k(y-l)^2 + mgy$$

Also determine and solve the corresponding Euler-Lagrange equations of motion. (12)

(*b*) Find the streamlines and pathlines of the two dimensional velocity field:

$$u = \frac{x}{1+t},\ v = y,\ w = 0. \quad (8)$$

(*c*) The velocity vector in the flow field is given by

$$\vec{q} = (az - by)\hat{i} + (bx - cz)\hat{j} + (cy - ax)\hat{k}$$

where a, b, c are non-zero constants. Determine the equations of vortex lines. (8)

(*d*) Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\left(\frac{n\pi x}{l}\right)$. (12)

ANSWERS

PAPER-I

1. (***a***) Given that $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$

$$B = bA = \begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix}$$

As matrix B is orthogonal matrix.

$$B.B^T = I$$

$$\begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix}\begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2+2b^2 & 2ab^2+b^2 & 2ab^2+b^2 \\ 2ab^2+b^2 & a^2b^2+2b^2 & 2ab^2+b^2 \\ 2ab^2+b^2 & 2ab^2+b^2 & a^2b^2+2b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^2b^2 + 2b^2 = 1$ and $2ab^2 + b^2 = 0$

From $\quad (2a + 1)b^2 = 0$

as $\quad b \neq 0,$

$\therefore \quad 2a + 1 = 0$

$$a = -\frac{1}{2}$$

Again, $\quad b^2(a^2 + 2) = 1$

$$b^2\left(\left(-\frac{1}{2}\right)^2 + 2\right) = 1$$

$$b^2\left(\frac{1}{4} + 2\right) = 1$$

$$b^2\left(\frac{9}{4}\right) = 1$$

$$b = \sqrt{\frac{4}{9}}$$

$$b = \pm\,\frac{2}{3}$$

$$\therefore \quad a = -\frac{1}{2}, \; b = \pm\frac{2}{3}$$

1. (***b***) (***i***) Let $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, where a, $b \in R$ and $a \neq 0$, $b \neq 0$.

We have $|A| = 0$, $|B| = 0$ so A, B $\in$ W.

Now, $A + B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. We have $|A + B| = ab \neq 0$.

$\therefore$ A + B $\notin$ W.

Thus, A $\in$ W, B $\in$ W but A + B $\notin$ W.

Hence, W is not a subspace of V.

(***ii***) We have $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Now, $\quad 2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

$$(2I)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq 2I.$$

Thus, I $\in$ W, but 2I $\notin$ W.

Hence, W is not a subspace of V.

1. (***c***)(***i***) $f(x)$ is continuous in (a, b) and also differentiable in (a, b). Then by mean value theorem, for $a < x < b$,

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

As $f'(x) = 0$

$$\therefore \frac{f(b)-f(a)}{b-a} = 0$$

$$\therefore f(b) = f(a)$$

Thus, $f(x)$ is constant in (a, b).

(*ii*) Again, given that $f'(x) < 0$ in (a, b)

$$\therefore f'(x) = \frac{f(b)-f(a)}{b-a} < 0$$

$$f(b) - f(a) < 0$$

$$\therefore f(b) < f(a)$$

Hence, $f(x)$ is a decreasing function in (a, b).

1. (*d*) $u(x, y) = ax^2 + 2hxy + by^2$

$$\frac{\partial u}{\partial x} = 2ax + 2hy$$

$$\frac{\partial u}{\partial y} = 2by + 2hx$$

$$V(x, y) = Ax^2 + 2Hxy + By^2$$

$$\frac{\partial v}{\partial x} = 2Ax + 2Hy$$

$$\frac{\partial v}{\partial y} = 2By + 2Hx$$

$$\text{Jacobian J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2ax+2hy & 2by+2hx \\ 2Ax+2Hy & 2By+2Hx \end{vmatrix}$$

$$= (2ax + 2hy)(2By + 2Hx) - (2Ax + 2Hy)(2by + 2hx)$$

$$= 4[(ax + hy)(By + Hx) - (by + hx)(Ax + Hy)]$$

Now, Let $\frac{a}{A} = \frac{b}{B} = \frac{h}{H} = k$ (say)

Then, $a = k.A$, $b = k.B$, $h = k.H$.

So, $$\frac{\partial(u,v)}{\partial(x,y)} = 4[(kAx + kHy)(By + Hx) - (kBy + kHx)(Ax + Hy)]$$

$$= 4k[(Ax + Hy)(By + Hx) - (By + Hx)(Ax + Hy)] = 0$$

Hence, Jacobian are independent of u, v for $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$.

1. (*e*) Equation of any plane parallel to the given plane $3x - 2y + 6z + 8 = 0$ is

$$3x - 2y + 6z + k = 0$$

Distance between two planes are 2 unit. So,

$$\left|\frac{3(0)-2(0)+6(0)+8}{\sqrt{(3)^2+(-2)^2+(6)^2}} - \frac{3(0)-2(0)+6(0)+k}{\sqrt{(3)^2+(-2)^2+(6)^2}}\right| = 2$$

$$\left|\frac{8-k}{\sqrt{49}}\right| = 2$$

$$\frac{8-k}{7} = \pm 2$$

$$8 - k = \pm 14$$

$$k = 22 \text{ or } -6$$

Hence, required equation of plane is

$$3x - 2y + 6z + 22 = 0$$

or $3x - 2y + 6z - 6 = 0$

2. (*a*) From Cayley-Hamilton theorem:

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= (1 - \lambda)((1 + \lambda)(\lambda) - 1)$$

$$= (1 - \lambda)(\lambda^2 + \lambda - 1)$$

$$= - \lambda^3 + 2\lambda - 1)$$

According to Cayley-Hamilton theorem

$$P(A) = 0$$

Or, $-A^3 + 2A - I = 0$

$$A^3 - 2A + I = 0$$

As $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\text{And, } A^3 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

Now, $A^3 - 2A + I$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Again, from $A^3 - 2A + I = 0$

$$-A^3 + 2A = I$$

$$-A^2 + 2I = A^{-1}$$

$$A^{-1} = -\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. (*b*) Let us connect $\int \sin^p x \cos^q x\, dx$ with $\int \sin^{p-2} x \cos^q x\, dx$

(*i*) $P = \sin^{p-2+1} x \cos^{q+1} x$

[Rule of "smaller index + 1" (Art. 68)]

$= \sin^{p-1} x \cos^{q+1} x.$

(*ii*)
$$\frac{dP}{dx} = (p-1)\sin^{p-2} x \cos x . \cos^{q+1} x + \sin^{p-1} x.(q+1)\cos^q x(-\sin x)$$

$$= (p-1)\sin^{p-2} x \cos^{q+2} x - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x(1-\sin^2 x) - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x - (p-1)\sin^p x \cos^q x - (q+1)\sin^p x \cos^q x$$

$$= (p-1)\sin^{p-2} x \cos^q x - (p+q)\sin^p x \cos^q x.$$

(*iii*) Integrating both sides w.r.t. x,

$$P = (p-1)\int \sin^{p-2} x \cos^q x\, dx - (p+q)\int \sin^p x \cos^q x\, dx$$

Transposing,

$$(p+q)\int \sin^p x \cos^q x\, dx = -P + (p-1)\int \sin^{p-2} x \cos^q x\, dx$$

or
$$= \int \sin^p x \cos^q x\, dx = -\frac{P}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx$$

[Substitute the value of P]

$$= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \left[-\frac{\sin^{p-1} x \cos^{q+1} x}{p+q}\right]_0^{\frac{\pi}{2}} + \frac{p-1}{p+q}\int_0^{\frac{\pi}{2}} \sin^{p-2} x \cos^q x\, dx$$

[But when $x = \frac{\pi}{2}$, $\sin^{p-1} x \cos^{q+1} x = 0$, and when $x = 0$, $\sin^{p-1} x \cos^{q+1} x = 0$, if p is *a* + ve integer > 1]

$$= \frac{p-1}{p+q}\int_0^{\frac{\pi}{2}} \sin^{p-2} x \cos^q x\, dx \qquad ...(i)$$

Changing p to $p - 2$,

$$\int_0^{\frac{\pi}{2}} \sin^{p-2} x \cos^q x\, dx$$

$$= \frac{p-3}{p+q-2}\int_0^{\frac{\pi}{2}} \sin^{p-4} x \cos^q x\, dx$$

Substituting this value of

$$\int_0^{\frac{\pi}{2}} \sin^{p-2} x \cos^q x\ dx \text{ in (1)}$$

$$\int_0^{\frac{\pi}{2}} \sin^{p} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)}{(p+q)(p+q-2)}\int_0^{\frac{\pi}{2}} \sin^{p-4} x \cos^q x\ dx$$

$$...(ii)$$

Generalizing from (*i*) and (*ii*),

Case I. If p is a +ve odd integer,

$$\int_0^{\frac{\pi}{2}} \sin^{p} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)...2}{(p+q)(p+q-2)...(q+3)}\int_0^{\frac{\pi}{2}} \sin^{1} x \cos^q x\ dx$$

$$\left[\text{But } \int_0^{\frac{\pi}{2}} \sin^1 x \cos^q dx = -\int_0^{\frac{\pi}{2}} \cos^q x(-\sin x) dx\right.$$

$$\left. = \left[-\frac{\cos^{q+1} x}{q+1}\right]_0^{\frac{\pi}{2}} = -\frac{1}{q+1}[0-1] = \frac{1}{q+1}\right]$$

$$= \frac{(p-1)(p-3)...2}{(p+q)(p+q-2)...(q+3)(q+1)}$$

Case II. If p is a +ve even integer,

$$\int_0^{\frac{\pi}{2}} \sin^{p} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)...1}{(p+q)(p+q-2)...(q+2)}\int_0^{\frac{\pi}{2}} \sin^{0} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)...1}{(p+q)(p+q-2)...(q+2)}\int_0^{\frac{\pi}{2}} \cos^q x\ dx \quad ...(iii)$$

Sub-case (*i*), If q is a +ve odd integer, then from (*iii*),

$$\int_0^{\frac{\pi}{2}} \sin^{p} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)...1}{(p+q)(p+q-2)...(q+2)} \cdot \frac{(q-1)(q-3)...2}{q(q-2)...3}$$

[Art, 70, Case I]

Sub-case (*ii*). If q is a +ve even integer, then from (*iii*),

$$\int_0^{\frac{\pi}{2}} \sin^{p} x \cos^q x\ dx$$

$$= \frac{(p-1)(p-3)...1}{(p+q)(p+q-2)...(q+2)} \cdot \frac{(q-1)(q-3)...1}{q(q-2)...2} \cdot \frac{\pi}{2}.$$

[Art. 70, Case II]

(*i*) $\int_0^{\frac{\pi}{2}} \sin^4 x \cdot \cos^5 x dx = \frac{4.2 \times 5 \times 3.1}{9 \times 7 \times 5 \times 3 \times 1} = \frac{8}{63}$

(*ii*) $\int_0^1 x^3 (1-x^2)^{\frac{5}{2}} \cdot dx$

Let $x = \sin\theta$, then $dx = \cos\theta . d\theta$.

$$\int_0^1 x^3 \cdot (1-x^2)^{\frac{5}{2}} \cdot dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot (1-\sin^2\theta)^{\frac{5}{2}} \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot \cos^6\theta . d\theta$$

$$= \frac{2 \cdot 5 \cdot 3 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{2}{63}$$

(*iii*) $\int_0^1 x^4 (1-x)^3 . dx$

Let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta . \cos\theta . d\theta$

$$\int_0^1 x^4 (1-x)^3 . dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^8\theta \cdot (1-\sin^2\theta)^3 \cdot 2\sin\theta \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2 \cdot \sin^9\theta \cdot \cos^7\theta \cdot d\theta$$

$$= 2\left(\frac{(8 \cdot 6 \cdot 4 \cdot 2) \times (6 \cdot 4 \cdot 2)}{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}\right) = \frac{1}{280}$$

2. (*c*) $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

$f(x) = 3x^2 - 3$ and $f_{xx} = 6x$

When, $f_x = 0 \Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow \quad x = \pm 1$

Again, $f_y = 3y^2 - 12$ and $f_{yy} = 6y$

When, $f_y = 0 \Rightarrow 3(y^2 - 4) = 0$

$\Rightarrow \quad y = \pm 2$

Thus, critical points = (1, 2), (1, –2), (–1, 2), (–1, –2)

$$D = f_{xx} \times f_{yy} - [f_{xy}]^2$$
$$= 6x \times 6y - 0$$
$$D = 36xy$$

At $(x, y) = (1, 2)$, $D = 72 > 0$

And $f_{xx} = 6 > 0$

$\therefore \quad f_{(1,\,2)} = 1 + 8 - 3 - 24 + 20 = 2$

is the local minimum.

And at (–1, –2), $D = 36(-1)(-2) = 72 > 0$

$$f_{yy} = 6(-2) = -12 < 0$$
$$f_{(-1,\,-2)} = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 = 38$$

is the local maximum.

At (1, –2), $D = 36(1)(-2) = -72 < 0$

and (–1, 2), $D = 36(-1)(2) = -72 < 0$

Thus, (1, –2) and (–1, 2) are two saddle points of the given function.

2. (*d*) Equation of plane

$2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$

$(2x + y + z)(x - y + 3z) = 0$

Thus, two equation of planes are

$2x + y + z = 0$ and $x - y + 3z = 0$

Angle between two planes

$$\cos\alpha = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{2 \times 1 + 1(-1) + 1(3)}{\sqrt{2^2 + 1^2 + 1^2}.\sqrt{1^2 + (-1)^2 + (3)^2}}$$

$$\cos\alpha = \frac{4}{\sqrt{66}}$$

$$\tan\alpha = \frac{\sqrt{\left(\sqrt{66}\right)^2 - (4)^2}}{4}$$

$$\tan\alpha = \frac{\sqrt{50}}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{\sqrt{50}}{4}\right).$$

3. (*a*)

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

[Applying $R_2 = R_2 + 2R_1$]

$$= \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

[Applying $R_4 = R_4 + R_1$]

$$= \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

[Applying $R_1 \to (-1)R_1$]

$$= \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_4 = R_4 - R_2$]

$$= \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_2 \to \frac{R_2}{8}$]

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_1 = R_1 + 2R_2$]

$$= \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_1 = R_1 - \frac{3}{2}R_3$]

$$= \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Applying $R_2 = R_2 - \frac{R_3}{4}$]

∴ Hence, Rank of the matrix = 3.

3. (*b*) (u, v, w) are linearly independent and further suppose that a, b, c are three constant ($a, b, c \in F$) such that

$$a(u + v) + b(v + w) + c(w + u) = 0$$

After distributing, we see that

$$au + av + bv + bw + cw + cu = 0$$

on collecting like terms, we get

$$(a + b)u + (a + c)v + (b + c)w = 0$$

Since, u, v and w are linearly independent

$\therefore \quad a + b = b + c = (c + a) = 0$

$\Rightarrow \quad a = b = c = 0$

So, $(u + v, v + w, w + u)$ is linearly independent.

3. (*c*) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}.dx.dy$

Let, $\quad x = r\cos\theta$ and $y = r\sin\theta$

Then, $\quad x^2 + y^2 = r^2$

and $\quad dx.dy = r.dr.d\theta$

This is double integral over the first quadrant. The first quadrant is

$$(r, \theta) : r \geq 0 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\therefore \quad I^2 = \int_0^\infty e^{-x^2}.dx.\int_0^\infty e^{-y^2}.dy$$

$$= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}.dx.dy$$

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}.r.dr.d\theta$$

$$= \int_0^\infty e^{-r^2}.r.dr.\int_0^{\frac{\pi}{2}} d\theta$$

$$= -\frac{1}{2}e^{-r^2}\Big|_0^\infty . \frac{\pi}{2}$$

$$= \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}$$

$$\therefore \quad I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

Again, we have seen that

$$\left(\int_0^\infty e^{-x^2}.dx\right)^2 = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}.dx.dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}.r.dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} . d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}. d\theta = \frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-x^2}. dx = \frac{\sqrt{\pi}}{2}$$

3. (*d*) Given lines are

$$l + m + n = 0$$

$$l = -(m+n) \qquad ...(i)$$

and $2lm + 2ln - mn = 0$ $\qquad ...(ii)$

Substituting '*l*' into (*ii*), we get

$$-2(m+n)(m+n) - mn = 0$$

$$2m^2 + 2n^2 + 5mn = 0$$

$$(2m+n)(m+2n) = 0$$

$$m = -2n \text{ and } n = -2m$$

$$\frac{m}{-2} = \frac{n}{1} \text{ and } \frac{n}{-2} = \frac{m}{1} \qquad ...(iii)$$

Substituting '*m*' into equation (*i*), we get

$$l - 2n + n = 0 \Rightarrow l = n$$

and $\quad l - \frac{n}{2} + n = 0$

$\Rightarrow \quad l = -\frac{n}{2} \qquad ...(iv)$

From (*iii*) and (*iv*), we get

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

and $\quad \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

$$l : m : n = 1 : -2 : 1$$

and $\quad l : m : n = 1 : 1 : -2$

i.e., D.r's (1, –2, 1) and (1, 1, –2)

Angle between these lines

$$\cos\theta = \frac{1\times1+(-2)1+1(-2)}{\sqrt{1^2+(-2)^2+1^2}\,\sqrt{(1)^2+1^2+(-2)^2}}$$

$$\theta = \cos^{-1}\left(-\frac{3}{6}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{2\pi}{3}.$$

4. (*a*) $\quad A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}.$

Eigenvalue : $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$$-\lambda(3-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1) = 0$$

$\therefore \qquad \lambda = 1, 2$

Thus, the eigenvalue of matrix A is $\lambda = 1,2$.

Eigenvector corresponding to $\lambda = 1$.

$$(A - \lambda I)V = 0$$

$$\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$-V_1 - 2V_2 = 0$$

$$V_1 + 2V_2 = 0$$

If we let $V_2 = t$, then $V_1 = -2t$

All eigenvector corresponding to $\lambda = 1$ are multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and thus eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Similarly eigenvector corresponding to $\lambda = 2$

$$\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2(V_1 + V_2) = 0$

and $\quad V_1 + V_2 = 0$

Thus eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Now, $\quad P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

and $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Now, $\quad AP = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}$

and $\quad PD = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}$

As $\quad AP = PD$

Hence, matrix A is diagonalizable

and $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$

4. (*b*) When $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{2xy^4}{(x^2 + y^2)^2}$$

and $$f_y(x, y) = \frac{2x^4 y}{(x^2 + y^2)^2}$$

$$f_{yx}(x, y) = \frac{8x^3 y^3}{(x^2 + y^2)^2}$$

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0.$$

$\therefore f_x(0, 0) = 0.$

Similarly, $f_y(0, 0) = 0.$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist in any *nbd* of (0, 0).

Also, $\lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$ (k being fixed)

$$= \lim_{h\to 0} \frac{\frac{h^2k^2}{h^2+k^2} - 0}{h} = \lim_{h\to 0} 0 = 0$$

$\therefore\ f_x(0, k) = 0$

Further, $\lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$ (h being fixed)

$$= \lim_{k\to 0} \frac{\frac{h^2k^2}{h^2+k^2} - 0}{k} = \lim_{k\to 0} \frac{h^2 k}{h^2 + k^2} = 0$$

$\therefore\quad f_y(h, 0) = 0$

Now, $\lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$\therefore f_{xy}(0, 0) = 0$

Also, $\lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

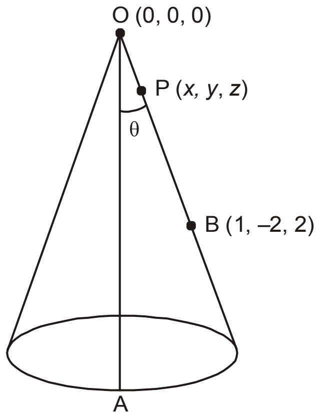
$\therefore f_{yx}(0, 0) = 0, = f_{xy}(0, 0)$

4. (*c*) Since the axis OA makes equal angles with the coordinate axes, the direction ratios of the line OA are 1, 1, 1.

The right circular cone passes through line OB whose direction cosines are proportional to (1, –2, 2). Hence, direction ratios are 1, –2, 2. Let θ be the semi-vertical angle.

$$\cos\theta = \frac{1(1) + 1(-2) + 1(2)}{\sqrt{1+1+1}\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}}$$

Let P(x, y, z) be any point on the right circular cone. The direction ratios of the line OP are x, y, z. The line OP makes an angle θ with the axis whose direction ratios are 1,1,1.



$$\cos\theta = \frac{x(1) + y(1) + z(1)}{\sqrt{x^2 + y^2 + z^2}\sqrt{1+1+1}}$$

$$\frac{1}{3\sqrt{3}} = \frac{x + y + z}{\sqrt{x^2 + y^2 + z^2}\sqrt{3}}$$

$$\sqrt{x^2 + y^2 + z^2} = 3(x + y + z)$$

$$x^2 + y^2 + z^2 = 9(x + y + z)^2$$

$$x^2 + y^2 + z^2 = 9(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)$$

$$8x^2 + 8y^2 + 8z^2 + 18xy + 18yz + 18zx = 0$$

This is the equation of the right circular cone.

4. (*d*) The given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda(\text{say}) \quad ...(i)$$

and $$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu(\text{say}) \quad ...(ii)$$

Suppose the line of the shortest distance between (*i*) and (*ii*) meets the line (*i*) at the point P and the line (*ii*) at the point Q.

Let the coordinates of P be ($3\lambda + 3, -\lambda + 8, \lambda + 3$) and the coordinates of Q be ($-3\mu - 3, 2\mu - 7, 4\mu + 6$). Direction ratios of PQ are

$-3\mu - 3\lambda - 6,\ 2\mu + \lambda - 15,\ 4\mu - \lambda + 3$

Also, direction ratios of the line (*i*) are 3, –1, 1 and those of the line (*ii*) are –3, 2, 4. Since PQ is perpendicular to both the lines (*i*) and (*ii*), we have

$$3(-3\mu - 3\lambda - 6) - 1(2\mu + \lambda - 15) + 1(4\mu - \lambda + 3) = 0$$

and $$-3(-3\mu - 3\lambda - 6) + 2(2\mu + \lambda - 15) + 4(4\mu - \lambda + 3) = 0$$

These equations on simplification reduce to

$-11\lambda + 7\mu = 0$ and $7\lambda + 29\mu = 0$

Solving these equations, we get $\lambda = 0$ and $\mu = 0$. Putting $\lambda = 0$ and $\mu = 0$ in the coordinates of P and Q respectively, the required points where the line of shortest distance meets the lines (*i*) and (*ii*) are given by P(3, 8, 3) and Q(–3, –7, 6).

∴ Shortest distance betwen the lines (*i*) and (*ii*)

$$= \text{PQ} = \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$$

$$= \sqrt{36+225+9} = \sqrt{270} = 3\sqrt{30}$$

Also, direction ratios of PQ are –3 –3, – 7 – 8, 6 – 3.

i.e., –6, –15, 3, or 2, 5, –1 dividing each ratio by –3.

The line of the shortest distance between (*i*) and (*ii*) is the line PQ.

i.e., the line passing through P(3, 8, 3) and having direction ratios 2, 5, –1. So its equations are $\dfrac{x-3}{2} = \dfrac{y-8}{5} = \dfrac{z-3}{-1}$.

5. (*a*) A.E. $= 2\text{D}^3 - 7\text{D}^2 + 7\text{D} - 2 = 0$

$2\text{D}^2(\text{D} - 1) - 5\text{D}^2 + 5\text{D} + 2\text{D} - 2 = 0$

$(\text{D} - 1)(2\text{D}^2 - 5\text{D} + 2)$

$= (\text{D} - 1)(\text{D} - 2)(2\text{D} - 1) = 0$

$\therefore \quad \text{D} = 1, 2, \dfrac{1}{2}$

$$\text{C.F.} = ae^x + be^{2x} + ce^{\frac{x}{2}}$$

$$\text{P.I.} = \frac{1}{(2\text{D}^3 - 7\text{D}^2 + 7\text{D} - 2)}e^{-8x}$$

$$= \frac{x}{(6\text{D}^2 - 14\text{D} + 7)}e^{-8x}$$

$$= \frac{x.e^{-8x}}{3}$$

$$\therefore \text{ Solution } y = ae^x + be^{2x} + ce^{\frac{x}{2}} + \frac{x.e^{-8x}}{3}$$

5. (*b*) Given equation is:

$$\frac{x^2.d^2y}{dx^2} - 2x.\frac{dy}{dx} - 4y = x^4$$

Putting $x = e^t$, the given equation reduces to $[\text{D}(\text{D} - 1) - 2\text{D} - 4]y = e^{4t}$,

Where, $\text{D} = \dfrac{d}{dx}$

$$[\text{D}^2 - 3\text{D} - 4]y = e^{4t} \qquad ...(i)$$

Auxilliary equation is:

$m^2 - 3m - 4 = 0$

$m = -1, 4$

$\text{C.F.} = c_1e^{4t} + c_2e^{-t}$

where, c_1 and c_2 are constant

And, $\text{P.I.} = \dfrac{1}{(\text{D}-4)(\text{D}+1)}e^{4t}$

The R.H.S. of equation is $b(t) = e^{4t}$; which has the finite family $s = \{e^{4t}\}$.

Then, $\text{T} = t.e^{4t}$

$y(t) = \text{A}te^{4t}$

On differentiating, we get

$\text{D}y = \text{A}(4t + 1)e^{4t}$

and $\text{D}^2y = 8\text{A}(2t + 1).e^{4t}$

From equation (*i*)

$$8A(2t+1)e^{4t} - 3A(4t+1)e^{4t} - 4Ate^{4t} = e^{4t}$$

$$8A - 3A = 1 \Rightarrow A = \frac{1}{5}$$

$$\text{P.I.} = y(t) = \frac{t.e^{4t}}{5}$$

∴ General solution of the equation

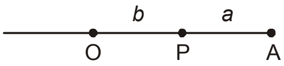
$$y = c_1 e^{4t} + c_2 e^{-t} + \frac{t.e^{4t}}{5}$$

Putting $t = \log x$

$$y = c_1.x^4 + \frac{c_2}{x} + \frac{x^4 \log x}{5}$$

5. (*c*) Let the equation of the S.H.M. with centre O-as origin be $d^2x/dt^2 = -\mu x$, where $\mu = \omega^2$

The time period $T = \dfrac{2\pi}{\sqrt{\mu}}$



Let the amplitude be a.

Then, $(dx/dt)^2 = \mu(a^2 - x^2)$ ...(*i*)

When the particle passes through P its velocity is given to be v in the direction OP. Also OP = b. So putting $x = b$ and $dx/dt = v$ in (*i*), wet get

$$v^2 = \mu(a^2 - b^2). \qquad ...(ii)$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$$

or

$$dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2 - x^2)}} \qquad ...(iii)$$

Let t_1 be the time from A to P. Then at A, $t = a$ and at P, $t = t_1$ and $x = b$. Therefore integrating (*iii*), we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{(a^2 - x^2)}}$$

or

$$t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}1\right]$$

$$= \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

Hence the required time

$$= 2t_1 = \frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{\sqrt{(a^2 - b^2)}}{b}\right)$$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

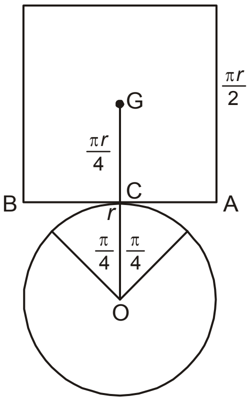
$$\left[\because \text{ from } (ii), \sqrt{(a^2 - b^2)} = \frac{v}{\sqrt{\mu}}\right]$$

$$= \frac{2}{2\pi/T}\tan^{-1}\left(\frac{v}{b(2\pi/T)}\right)$$

$$\left[\because T = \frac{2\pi}{\sqrt{\mu}} \text{ so that } \sqrt{\mu} = \frac{2\pi}{T}\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

5. (*d*) A heavy uniform cube balances on the highest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical. In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.



First we shall show that the equilibrium of the cube is stable.

Here ρ_1 = the radius of curvature of the upper body at the point of contact C = ∞,

And, ρ_2 = the radius of curvature of the lower body at the point of contact = r.

Also h = the height of the centre of gravity G of the upper body above the point of contact C

= half the edge of the cube

= $\pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2}$$

i.e., $$\frac{1}{\pi r/4} > \frac{1}{\infty}+\frac{1}{r}$$

i.e., $$\frac{4}{\pi r} > \frac{1}{r} \quad i.e., \quad \frac{4}{\pi} > 1$$

i.e., $$4 > \pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. So if the cube is slightly displaced, it will tend to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere, we have

$$r\theta = \text{half the edge of the cube} = \pi r/4,$$

so that $\theta = \pi/4$

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2,\frac{1}{4}\pi$ *i.e.*, $\frac{1}{2}\pi$.

5. (*e*)
$$\nabla^2(r^n) = \nabla.(\nabla r^n) = \nabla.\left(nr^{n-1}\frac{\vec{r}}{r}\right)$$

$$= n\nabla.(r^{n-2}\vec{r})$$

$$= n\left[(\nabla r^{n-2}).\vec{r} + r^{n-2}(\nabla.\vec{r})\right]$$

$$\left[\because \nabla.(\phi\vec{\text{A}}) = (\nabla\phi).\vec{\text{A}} + \phi(\nabla.\vec{\text{A}})\right]$$

$$= n\left[(n-2)r^{n-3}\frac{\vec{r}}{r}.\vec{r} + r^{n-2}(3)\right]$$

$$\left[\because \nabla.\vec{r} = 3\right]$$

$$= n[(n-2)r^{n-4}(r^2) + 3r^{n-2}]$$

$$\left[\because \vec{r}.\vec{r} = r^2\right]$$

$$= n(n+1)r^{n-2}$$

(Second Method)

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)r^n$$

$$= \sum\frac{\partial^2}{\partial x^2}(r^n) = \sum\frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) = \sum\frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right)$$

$$= \sum n\frac{\partial}{\partial x}(r^{n-2}x)$$

$$= n\sum\left[(n-2)r^{n-3}\frac{\partial r}{\partial x},x + r^{n-2}\right]$$

$$= n\sum\left[(n-2)r^{n-3}\frac{x}{r},x+r^{n-2}\right]$$

$$= n\sum\left[(n-2)r^{n-4}x^2+r^{n-2}\right]$$

$$= n[(n-2)r^{n-4}(x^2+y^2+z^2)+3r^{n-2}]$$

$$= n[(n-2)r^{n-4}(r^2)+3r^{n-2}]$$

$$= n(n+1)r^{n-2}$$

Again,

$$\nabla \times (r^n\ r) = \nabla r^n \times r + r^n\ \nabla \times r$$

$$= nr^{n-2}\ r \times r + 0$$

$$= 0$$

Hence $r^n\ r$ is irrotational vector for any value of n.

6. (*a*) The given equation can be written as

$$(p + y\cot x)^2 = y^2(1 + \cot^2 x)$$

(Adding $y^2\cot^2 x$ to both sides)

or $p + y\cot x = \pm\, y\operatorname{cosec} x$

$\therefore$ The component equations are

$$p = y(-\cot x + \operatorname{cosec} x) \quad ...(i)$$

and $$p = y(-\cot x - \operatorname{cosec} x) \quad ...(ii)$$

$$(i) \Rightarrow \quad p = y\left(\frac{1-\cos x}{\sin x}\right) \Rightarrow \frac{dy}{dx} = y\tan\frac{1}{2}x$$

$$\Rightarrow \quad \frac{dy}{y} = \tan\frac{1}{2}x\,dx$$

Integrating,

$$\log y = 2\log\sec\frac{1}{2}x+\log c$$

$$\Rightarrow \quad y = c\sec^2\frac{1}{2}x \Rightarrow y\cos^2\frac{1}{2}x-c=0$$

$$(ii) \Rightarrow \quad p = -y\left(\frac{1+\cos x}{\sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -y\cot\frac{1}{2}x$$

$$\Rightarrow \quad \frac{dy}{y} = -\cot\frac{1}{2}x\,dx$$

Integrating,

$$\log y = -2\log\sin\frac{1}{2}x+c$$

$$\Rightarrow \quad y = \frac{c}{\sin^2\frac{1}{2}x} \Rightarrow y\sin^2\frac{1}{2}x-c = 0$$

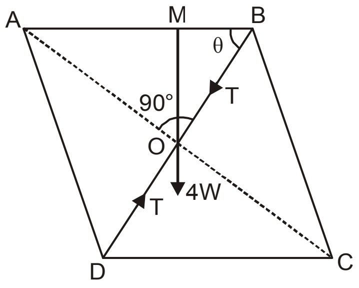
$\therefore$ The general solution of the given equation is

$$\left(y\cos^2\frac{x}{2}-c\right)\left(y\sin^2\frac{x}{2}-c\right) = 0.$$

6. (*b*) ABCD is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight 4W of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have ∠AOB = 90°.

Let ∠ABO = θ. Draw OM perpendicular to AB.



Give the system a small symmetrical displacement in which θ changes to θ + δθ. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The ∠AOB will remain 90°.

We have BD = 2BO = 2AB cosθ = 2b cosθ.

[Note that in the position of equilibrium BD = a. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line

$$AB = MO$$
$$= BO \sin\theta = (AB \cos\theta) \sin\theta$$
$$= b \sin\theta \cos\theta.$$

By the principle of virtual work, we have

$$-T\delta(2b \cos\theta) + 4W\ \delta(b \sin\theta \cos\theta) = 0$$

or $2bT \sin\theta\ \delta\theta + 4bW (\cos^2\theta - \sin^2\theta)\ \delta\theta = 0$

or $2b\ [T \sin\theta - 2W (\sin^2\theta - \cos^2\theta)]\ \delta\theta = 0$

or $T\sin\theta - 2W (\sin^2\theta - \cos^2\theta) = 0$

$[\because\ \delta\theta \neq 0]$

or $$T = \frac{2W(\sin^2\theta - \cos^2\theta)}{\sin\theta}$$

$$= \frac{2W(1-2\cos^2\theta)}{\sqrt{(1-\cos^2\theta}}$$

In the position of equilibrium, BD = a or BO = $\frac{1}{2}a$. So in the position of equilibrium,

$$\cos\theta = \frac{BO}{AB} = \frac{\frac{1}{2}a}{b} = \frac{a}{2b}$$

$$\therefore \quad T = \frac{2W\{1-2(a^2/4b^2)\}}{\sqrt{\{1-(a^2/4b^2)\}}}$$

$$= \frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

6. (*c*) $\int_c \left[(x+y)dx + (2x-z).dy + (y+z)dz\right]$

$$= \int_c \left[(x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}\right].\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

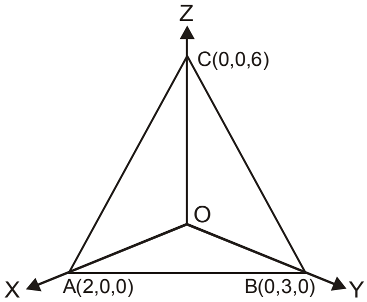
$$= \int_c \left[(x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}\right].dr$$

$$= \iint_s \text{curl}\left\{(x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}\right\}.\hat{n}\, ds \quad ...(i)$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (2x-z) & (y+z) \end{vmatrix}$$

$$= (1+1)\hat{i} + (2-1)\hat{k}$$
$$= 2\hat{i} + \hat{k}$$

Equation of plane passes through the given points A(2,0,0), B(0,3,0) and C(0,0,6) is



$$z = 6 - 3x - 2y$$
$$3x + 2y + z = 6$$

$$\nabla(3x + 2y + z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$
$$(3x + 2y + z)$$
$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

Unit normal at any point of the ΔABC, is

$$\hat{n} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{3^2 + 2^2 + 1^2}}$$

$$= \frac{1}{\sqrt{14}}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

Putting $\hat{n}$ values in equation (*i*), we get

$$\oint\left[(x+y)dx + (2x-z)dy + (y+z)dz\right]$$

$$= \iint_S \left(2\hat{i} + \hat{k}\right)\frac{1}{\sqrt{14}}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)ds$$

$$= \iint_R \frac{7}{\sqrt{14}}\ \frac{dx.dy}{|\hat{n}.k|}$$

Where R is the projection of S on the *xy*-plane

$$= \iint_R \frac{7}{\sqrt{14}}.\frac{dx.dy}{\left(\frac{1}{\sqrt{14}}\right)} = 7\iint_R dx.dy$$

$$= 7\ (\text{Area of } \Delta\text{DAB}) = 7\times\frac{1}{2}\times 2\times 3$$

$$= 21.$$

7. (*a*) From question $e^{3x}\left(\frac{dy}{dx}-1\right)+\left(\frac{dy}{dx}\right)^3 e^{2y}=0$

Let $\frac{dy}{dx} = p,$

then, $e^{3x}(p-1) + p^3e^{2y} = 0$ ...(*i*)

Re-writing (*i*),

$$1 - p = p^3\, e^{2y-3x}$$

or $e^y(1 - p) = (p\, e^{y-x})^3$

which is of the form II. Note that here

$$a = 1,\ b = 1.$$

Putting $e^x = u$ and $e^y = v$

so that $e^x dx = du$ and $e^y dy = dv$

We get,

$$\frac{e^y dy}{e^x dx} = \frac{dv}{du} \text{ or } \frac{v}{u}p = p$$

or $p = \frac{u\text{P}}{v},$

where $\text{P} = \frac{dv}{du}$

Putting $e^x = u$, $e^y = v$ and $p = u\text{P}/v$ in (*i*) We have

$$u^3\left(\frac{u\text{P}}{v}-1\right)+\frac{u^3\text{P}^3}{v^3}\times v^2 = 0$$

or $u\text{P} - v + \text{P}^3 = 0$

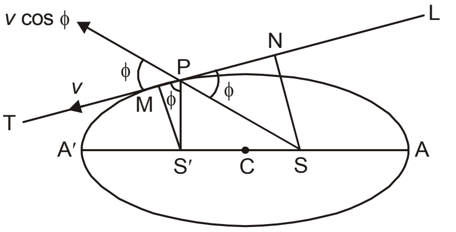
or $v = u\text{P} + \text{P}^3$

which is in Clairaut' is form. So replacing P by c, the required solution is

$$v = uc + c^3 \text{ or } e^y = ce^x + c^3$$

c → being an arbitrary constant.

7. (*b*) Let the sun be at the focus S of the ellipse and S′ be the other focus (Figure). Let v be the velocity of the planet acting along the tangent LT to the ellipse at P. From S and S′ draw SN and S′M perpendicular to LT. Join SP and S′P.



We know that the tangent at any point on an ellipse is equally inclined to the focal radii of the point of contact; therefore ∠SPN = ∠S′PM = ϕ(say).

From the right triangle SNP, we have

$$\sin\phi = \frac{\text{SN}}{\text{SP}} \qquad ...(i)$$

Again from the right triangle S′MP, we have

$$\sin\phi = \frac{\text{S}'\text{M}}{\text{S}'\text{P}} \qquad ...(ii)$$

Multiplying (*i*) and (*ii*), we get

$$\sin^2\phi = \frac{\text{SN}}{\text{SP}}\cdot\frac{\text{S}'\text{M}}{\text{S}'\text{P}} \qquad ...(iii)$$

From the property of ellipse we know that the product of perpendiculars drawn from the foci on any tangent to an ellipse is constant and equal to the square of the length of the semi-minor axis of the ellipse.

$$\therefore \quad \text{SN}\cdot\text{S}'\text{M} = b^2 \qquad ...(iv)$$

Where $2b$ = Length of the minor axis of the ellipse.

Also we know that the sum of the focal distances of a point on an ellipse is equal to the length of the major axis of the ellipse.

∴ SP + S′P = $2a$, $2a$ being the length of the major axis.

Let, SP = r.

Then, S′P = $2a-r$ ...(*v*)

From (*iii*), (*iv*) and (*v*), we get

$$\sin^2\phi = \frac{b^2}{r(2a-r)}$$

or $$\cos^2\phi = 1-\frac{b^2}{r(2a-r)}$$

Also the path of the planet being an ellipse, its velocity at any point is given by

$$v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right) = \mu\left(\frac{2a-r}{ar}\right)$$

Now the velocity of the planet away from the sun

$$= \text{Resolved part of } v \text{ along SP}$$

$$= v\cos\phi$$

$$= \mu\sqrt{\left(\frac{2a-r}{ar}\right)}\sqrt{\left[1-\frac{b^2}{r(2a-r)}\right]}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left(\frac{2ar-r^2-b^2}{r^2}\right)}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left(\frac{2a}{r}-1-\frac{b^2}{r^2}\right)}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left[\frac{a^2}{b^2}-1-\left(\frac{b}{r}-\frac{a}{b}\right)^2\right]}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left[\left(\frac{a^2-b^2}{b^2}\right)-\left(\frac{b}{r}-\frac{a}{b}\right)^2\right]}$$

Now its value will be maximum when $\left(\frac{b}{r}-\frac{a}{b}\right)^2$ is minimum. The minimum value of $\left(\frac{b}{r}-\frac{a}{b}\right)^2$ is zero.

$\therefore\ \frac{b}{r}-\frac{a}{b}=0$ or $r=\frac{b^2}{a}$ or SP $=\frac{b^2}{a}=l$, the semi-latus rectum.

This means that the radius vector SP is perpendicular to the major axis AA′.

And the greatest velocity away from the sun

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left(\frac{a^2-b^2}{b^2}\right)}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left[\frac{a^2-a^2(1-e^2)}{a^2(1-e^2)}\right]}$$

$$[\because b^2 = a^2(1-e^2)]$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\frac{e^2}{1-e^2}}$$

Also the periodic time T is given by

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} \text{ or } \sqrt{\frac{\mu}{a}} = \frac{2\pi a}{T}$$

∴ The greatest velocity away from the sun

$$= \frac{2\pi ae}{T\sqrt{1-e^2}}.$$

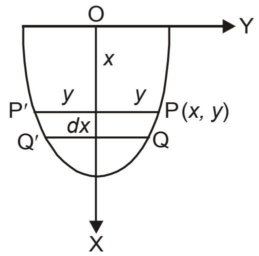
7. (*c*) Taking the major and minor axes of the ellipse as axes of x and y respectively, the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad ...(i)$$

Here minor axis is in the surface.

By symmetry, it is clear that the C.P. lies on OX.

Take an elementary strip PQQ′P′ at a depth x below O, the centre of the ellipse and of width dx.



Then, dS = area of the elementary strip

$= 2y\ dx$

p = the intensity of pressure at any point of the strip

$= \rho g\ x = (kx)\,gx$

where k is some constant.

∴ $\bar{x}$ = depth of the C.P. of the semi-ellipse below O

$$= \frac{\int xp\ dS}{\int p\ dS} = \frac{\int_0^a x\,gk\,x^2.2y\,dx}{\int_0^a gk\,x^2.2y\,dx} = \frac{\int_0^a x^3\,y\,dx}{\int_0^a x^2\,y\,dx}$$

The parametric equations of the ellipse (*i*) are $x = a \cos t$, $y = b \sin t$.

$\therefore \qquad dx = -\, a \sin t \; dt.$

When, $\quad x = 0, \cos t = 0 \Rightarrow t = \frac{\pi}{2}$

and when $\; x = a, \cos t = 1 \Rightarrow t = 0.$

$$\therefore \qquad \bar{x} = \frac{\int_{\pi/2}^{0} a^3 \cos^3 t \cdot b \sin t(-a \sin t\, dt)}{\int_{\pi/2}^{0} a^2 \cos^2 t \cdot b \sin t(-a \sin t\, dt)}$$

$$= \frac{a\int_{0}^{\pi/2} \cos^3 t \sin^2 t\, dt}{\int_{0}^{\pi/2} \cos^2 t \sin^2 t\, dt}$$

$$= \frac{a\left(\frac{2.1}{5.3.1}\right)}{\frac{1.1}{4.2}\cdot\frac{\pi}{2}} = \frac{32a}{15\pi}$$

If C.P. coincides with the focus, then $\bar{x} = ae$.

$$\therefore \qquad \frac{32a}{15\pi} = ae \text{ or } e = \frac{32a}{15\pi}.$$

7. (*d*) Here S is not a closed surface. The surface $z = 2 - \sqrt{x^2 + y^2}$ meets the *xy*-plane in a circle C given by $x^2 + y^2 = 4$, $z = 0$. Let S_1 be the plane region bounded by the circle C. If S′ is the surface consisting of the surface S and S_1, then S′ is a closed surface. By the divergence theorem, we have

$$\iint_{S'} \text{curl } \vec{A}.\hat{n}\, dS = 0$$

$$\Rightarrow \iint_{S} \text{curl } \vec{A}.\hat{n}\, dS + \iint_{S_1} \text{curl } \vec{A}.\hat{n}\, dS = 0$$

[∵ S′ consists of S and S_1]

$$\Rightarrow \iint_{S} \text{curl } \vec{A}.\hat{n}\, dS - \iint_{S_1} \text{curl } \vec{A}.\hat{k}\, dS = 0$$

$\left[\because \text{ on } S_1,\ \hat{n} = -\hat{k}\right]$

$$\Rightarrow \iint_{S} \text{curl } \vec{A}.\hat{n}\, dS = \iint_{S_1} \text{curl } \vec{A}.\hat{k}\, dS$$

Now, $\quad \text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - z & x^3 + yz & -3xy^2 \end{vmatrix}$

$$= \hat{i}(-6xy - y) + \hat{j}(-1 + 3y^2) + \hat{k}(3x^2 - 0)$$

$\therefore \qquad \text{curl } \vec{A}.\hat{k} = 3x^2$

$$\therefore \iint_{S} \text{curl } \vec{A}.\hat{n}\, dS = \iint_{S} 3x^2\, dx$$

$$= \int_0^{2\pi}\int_0^{2} 3(r^2 \cos^2\theta) r\, dr\, d\theta,$$

changing to polars

$$= 3\int_0^{2\pi}\int_0^{2} r^3 \cos^2\theta\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 \cos^2\theta\, d\theta$$

$$= 12\int_0^{2\pi} \cos^2\theta\, d\theta = 12\pi.$$

8.(*a*) The symbolic form of the given differential equation is:

$$(D^2 + 4)y = \tan 2x$$

Its auxiliary equation is $m^2 + 4 = 0$, which yields $m = \pm 2i$. Thus,

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

To find P.I., let

$$y_1 = \cos 2x \text{ and } y_2 = \sin 2x$$

Then Wronskian W is

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Hence,

$$\text{P.I.} = -\,y_1\int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx$$

$$= -\frac{\cos 2x}{2}\int \sin 2x \tan 2x\, dx$$

$$+ \frac{\cos 2x}{2}\int \cos 2x \tan 2x\, dx$$

$$= -\frac{\cos 2x}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$+ \frac{\sin 2x}{2}\int \cos 2x \tan 2x\, dx$$

$$= -\frac{\cos 2x}{2}\int \frac{1 - \cos^2 2x}{\cos 2x} dx$$

$$+ \frac{\sin 2x}{2}\int \cos 2x \tan 2x\, dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx$$

$$+\frac{1}{2}\sin 2x \int \sin 2x\ dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x]$$

$$-\frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x$$

$$-\frac{1}{4}\ \cos 2x \log (\sec 2x + \tan 2x).$$

8. (*b*) Take the centre of force O as origin. Suppose a particle starts from rest at A, where OA = a. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where OP = x. The acceleration of the particle at P is $\mu a^{5/3}\ x^{-2/3}$ directed towards O. Therefore, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\ \mu a^{5/3}\ x^{-2/3} \qquad ...(i)$$

Multiplying both sides of (i) by $2(dx/dt)$ and integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu^{5/3} a\, x^{1/3}}{1/3} + k$$

$$= -\ 6\mu a^{5/3}\ x^{1/3} + k,$$

where k is a constant.

At A, $x = a$ and $dx/dt = 0$

So, that

$-\ 6\mu a^{5/3}\ a^{1/3} + k = 0$ or $k = 6\mu a^2$

$\therefore\ (dx/dt)^2 = -\ 6\mu a^{5/3}\ x^{1/3} + 6\mu a^2$

$= 6\mu a^{5/3}\ (a^{1/3} - x^{1/3}) \qquad ...(ii)$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x = 0$ and $(dx/dt)^2 = v_1^2$. So from (ii), we have

$$v_1^2 = 6\mu a^{5/3}\ (a^{1/3} - 0) = 6\mu a^2$$

or $\qquad v_1 = a\sqrt{(6\mu)}$

Now taking square root of (ii), we get

$$dx/dt = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3} - x^{1/3})}$$

where the –ive sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$dt = -\ \frac{1}{\sqrt{(6\mu a^{5/3})}}\cdot\frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}} \qquad ...(iii)$$

Let t_1 be the time from A to O. Then integrating (iii) from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}}\int_a^0 \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

$$= \frac{1}{\sqrt{(6\mu a^{5/3})}}\int_0^a \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

Put $x = a \sin^6 \theta$, so that $dx = 6a \sin^5 \theta \cos \theta\ d\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad t_1 = \frac{1}{\sqrt{(6\mu a^{5/3})}}\int_0^{\pi/2}\frac{6a \sin^5\theta\cos\theta\ d\theta}{a^{1/6}\cos\theta}$$

$$= \sqrt{\left(\frac{6}{\mu}\right)}\int_0^{\pi/2}\sin^5\theta\, d\theta$$

$$= \sqrt{\left(\frac{6}{\mu}\right)}\cdot\frac{4.2}{5.3.1} = \frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}.$$

8. (*c*) Any curve drawn on circular cylinder and cutting its generators at a constant angle is known as a circular helix.

The vector equation of the curve is given by

$$\bar{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + \theta a\cot\beta\hat{k}$$

$$\therefore \quad \frac{d\bar{r}}{d\theta} = -a\sin\theta\hat{i} + a\cos\theta\hat{j} + a\cot\beta\hat{k}$$

its arclength from $P_0(\theta = 0)$ to any point $P(\theta)$ is

$$S = \int_0^\theta \left|\frac{dr}{d\theta}\right| . d\theta = \sqrt{a^2(1+\cot^2\beta)\theta}$$

$$= a\ \text{cosec}\beta.\theta$$

$$\therefore \quad \frac{ds}{d\theta} = \text{a cosec } \beta$$

Also, $$\hat{\theta} = \frac{dr}{ds} = \frac{\frac{dr}{d\theta}}{\frac{ds}{d\theta}}$$

$$= \frac{a\left(-\sin\theta\hat{i} + \cos\theta\hat{j} + \cot\beta\hat{k}\right)}{a\text{cosec}\beta}$$

$$\hat{\theta} = \frac{\left(-\sin\theta\hat{i} + \cos\theta\hat{j} + \cot\beta\hat{k}\right)}{\text{cosec}\beta}$$

and $$\frac{d\hat{\theta}}{ds} = \frac{\frac{d\hat{\theta}}{d\theta}}{\frac{ds}{d\theta}} = \frac{-\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)}{a.\text{cosec}^2\beta}$$

$$k = \left|\frac{d\hat{\theta}}{ds}\right| = \frac{1}{a \cdot \text{cosec}^2\beta}$$

Also, $$\hat{n} = -\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

and $$\hat{b} = \hat{\theta} \times \hat{n}$$

$$= \frac{\left(\cot\beta.\sin\theta\hat{i} - \cot\beta.\cos\theta\,\hat{j} + \hat{k}\right)}{\text{cosec }\beta}$$

$$\Rightarrow \quad \frac{d\hat{b}}{ds} = \frac{\frac{d\hat{b}}{d\theta}}{\frac{ds}{d\theta}} = \frac{\cot\beta\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)}{a.\text{cosec}^2\beta}$$

$$= -\tau.\hat{n} = \tau\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

$$\tau = \frac{\cot\beta}{a \cdot \text{cosec}^2\beta}.$$

8. (*d*) A cylindrical helix is a space curve which lines on a cylinder and cuts the generators at a constant angle. Its tangent makes a constant angle α with a fixed line known as the axis of the helix.

A characteristic property of helices is that the ratio of the curvature to the torsion is constant at all points. To prove this, let a denote a unit vector in the direction of the axis; then

$$t.a = \cos \alpha.$$

By differentiating this relation, $\kappa n.a = 0$ from which (if the case $\kappa = 0$ is excluded) it follows that the vector a must lie in the rectifying plane. Writing

$$a = t \cos \alpha + b \sin \alpha \qquad ...(i)$$

and differentiating to get

$$a' = (\kappa \cos \alpha — \tau \sin \alpha)n = 0$$

it follows that

$$\kappa/\tau = \tan \alpha = \text{constant} \qquad ...(ii)$$

Conversely, if

$$\kappa/\tau = \text{constant} = \tan \alpha, \text{ then}$$

$$(\kappa \cos \alpha — \tau \sin \alpha)n = 0$$

i.e., $d(t \cos \alpha + b \sin \alpha)/ds = 0$

It follows that $(t \cos\alpha + b \sin\alpha) = a$, a constant unit vector. Hence $a.t = \cos\alpha$, and the curve is thus a helix. This proves that the property κ/τ = constant is characteristic of helices.

PAPER-II

1. (*a*) We know that every group of prime order is cylic and every cyclic group is abelian. Since 2, 3 and 5 are prime numbers, therefore all groups of order 2, 3 and 5 are prime numbers, therefore all groups of order 2, 3 and 5 must be abelian.

Now let G be a finite group of order 4. If every element of G is its own inverse, surely G is abelian. So, let G contains an element, say a, such that $a \neq a^{-1}$. But then $a \neq a^{-1} \Rightarrow aa \neq aa^{-1} \Rightarrow a^2 \neq e$, so that O $(a) > 2$. since in a finite group, the order of an element must be a divisor of the order of the group, therefore O (a) cannot be 3 and so we must have O (a) = 4 = the order of the group G. But the G is cyclic and so abelian. Thus, every group of order four is always abelian. Hence every group of order less than six must be abelian.

1. (*b*) We have $f(x) = x$ if x is rational number

$f(x) = 1 - x$ if x is irrational number.

By definition of continuity: $|x - 1/2| < \delta \Rightarrow |f(x) - 1/2| < e$

(a) If we will pick any rational number x within δ and 1/2: $|x - 1/2| < e = \delta$

(b) If we will pick any irrational number x within δ and 1/2: $|1 - x - 1/2| < e = \delta$

Then we can use the fact, that by continuity of f at x:

$$f(x) = f(\lim x_n) = \lim f(x_n) = x,$$

if x is rational

$$f(x) = f(\lim x_n) = \lim f(x_n) = 1 - x,$$

if x is irrational

the only possibility of choice is 1/2.

1. (*c*) $f(z) = u(x, y) + iv(x, y)$

Here,
$$\begin{aligned} u + 2v &= x^3 - 2y^3 + 3xy\ (2x - y) \\ &= (x^3 - 3xy^2) + (6x^2y - 2y^3) \\ &= (x^3 - 3xy^2) + 2(3x^2y - y^3) \end{aligned}$$

$\therefore$ $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Now,
$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3) \\ &= x^3 - 3xy^2 + 3x^2y.i - y^3.i \\ &= x^3 + i3xy^2 + 3x^2yi + (iy)^3 \\ &= x^3 + (iy)^3 + 3xyi\ (x + iy) \\ &= (x + iy)^3 = z^3 \end{aligned}$$

$\therefore$ $f(z) = z^3.$

1. (*d*) We first convert the given problem to the maximization problem by taking

Maximize

$$z' = -z = -x_1 + 3x_2 - 2x_3$$

Subject to conditions

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$

Introduction slack variables, the standard form of the I.PP in question is

Maximize

$$z' = -x_1 + 3x_2 - 2x_3 + 0r + 0s + 0t$$

subject to constraints

$$3x_1 - x_2 + 2x_3 + r + 0s + 0t = 7$$
$$-2x_1 + 4x_2 + 0x_3 + 0r + s + 0t = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0r + 0s + t = 10$$
$$x_1, x_2, x_3, r, s, t \ge 0$$

The basic feasible solution is

$$x_1 = x_2 = x_3 = 0$$
$$r = 7, s = 12, t = 10$$

Thus, the initial basic feasible solution is shown by the table:

	x_1	x_2	x_3	r	s	t	Solution	
z'	1	–3	2	0	0	0	0	Ratios
r	3	–1	2	1	0	0	7	
s	–2	④	0	0	1	0	12	3 ←
t	–4	3	8	0	0	1	10	$\frac{10}{3}$
		↑						

The most negative value of z' is in x_2-column and the minimum ratio is in s-row. Therefore, pivot element is 4. Making the pivot element 1 by dividing s-row throughout by 4, replacing s by x_2 and applying Gauss elimination to annihilate the element in x_2-column, we get the following table:

	x_1	x_2	x_3	r	t	s	Solution	
z'	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	9	Ratio
r	$\textcircled{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	10	4 ←
x_2	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3	
t	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	1	
	↑							

Now, the pivot element is $\frac{5}{2}$. Making it equal to 1 by dividing throughout by $\frac{5}{2}$, replacing r by x_1 and using Gauss elimination method, we get the table.

	x_1	x_2	x_3	r	s	r	Solution
z'	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	11
x_1	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	4
x_2	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{20}$	3	5
t	0	0	10	$\frac{1}{2}$	$-\frac{1}{2}$	1	11

Since all the entries in Z′-row are non-negative, the solution to the problem is achieved. Therefore, the solution to the problem is

$$x_1 = 4, x_2 = 5, x_3 = 0$$

and $\quad z = -z' = -11.$

2. (*a*) The set G is closed under the operation *.

Associative property: Let $a, b, c \in$ G, then we have

$$\begin{aligned}(a * b) * c &= (a + b + ab) * c \\ &= a + c + ac + b + c + bc \\ &\quad + ab + c + abc \\ a * (b * c) &= a * (b + c + bc) \\ &= a + b + ab + a + c + ac \\ &\quad + a + bc + abc\end{aligned}$$

$$\therefore \quad (a * b) * c = a * (b * c)$$

Closure property: The set G is closed under the operation *. Since $a * b = a + b + ab$ is real number. Hence, belongs to G.

Identity: To find the identity element, Let us assume that e is a +ve real number. Then $e * a = a$, where $a \in$ G.

$$e * a = a \Rightarrow e = 1$$

Similarly, $a * e = a \Rightarrow e = 1$

Inverse : Let us assume that $a \in$ G. If $a^{-1} \in$ Q is an inverse of a, then

$$a * a^{-1} = a + a^{-1} + 1$$

Similarly, $\quad a^{-1} * a = a^{-1} + a + 1$

Commutative: The operation on G is commutative since

$$a * b = a + b + ab = b * a$$

Thus, the algebric system (G, *) is closed, associative identity element, inverse and commutative.

Hence, the system (G, *) is an abelian group.

2. (*b*) Suppose that K is normal in G. If $hk \in$ HK then $hk = (hkh^{-1})h \in$ KH because $hkh^{-1} \in h\text{K}h^{-1} =$ K. Hence HK $\subseteq$ KH. The other inclusion is proved the same way, so Lemma 2 applies. A similar argument works if H $\triangleleft$ G.

2. (*c*) For $br_1 + a_1, br_2 + a_2 \in$ B and $r, r' \in$ R,

$$\begin{aligned}(br_1 + a_1) - (br_2 + a_2) &= b(r_1 - r_2) \\ &\quad + (a_1 - a_2) \in \text{B}\end{aligned}$$

and $\quad r'(br + a) = b(r'r) + r'a \in$ B.

Thus B is an ideal by the ideal test.

2. (*d*) To be a Euclidean domain means that there is a defined application (often called norm) that verifies this two conditions:

- $\forall a, b \in \mathbb{Z}[i] \setminus 0 \; a|b \rightarrow \text{N}(a) \leq \text{N}(b)$
- $\forall a, b \in \mathbb{Z}[i] \; b \neq 0 \rightarrow \exists c, r \in \mathbb{Z}[i]$

So that $\quad a = bc + r$

and $\quad (r = 0 \text{ or } r \neq 0 \; \text{N}(r) < \text{N}(b))$

Let, $\quad a = \alpha_1 + \alpha_2 i, \; b = \beta_1 + \beta_2 i$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$, Then

$$\begin{aligned}\frac{a}{b} &= \frac{\alpha_1 + \alpha_2 i}{\beta_1 + \beta_2 i} = \frac{(\alpha_1 + \alpha_2 i)(\beta_1 - \beta_2 i)}{\text{N}(b)} \\ &= \frac{(\alpha_1\beta_1 + \alpha_2\beta_2) - (\alpha_1\beta_2 - \alpha_2\beta_1)i}{\text{N}(b)}\end{aligned}$$

By a modified form of the division algorithm on the integers, $\exists q_1, q_2, r_1, r_2 \in$ Z such that

$$\alpha_1\beta_1 + \alpha_2\beta_2 = \text{N}(b)q_1 + r_1$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \text{N}(b)q_2 + r_2$$

Where $\quad -\frac{1}{2}\text{N}(b) \leq r_i \leq \frac{1}{2}\text{N}(b)$

Then our quotient is $q = q_1 - q_2 i$ and our remainder is $r = r_1 - r_2 i$.

Then, $$\frac{a}{b} = \frac{\mathrm{N}(b)q+r}{\mathrm{N}(b)}$$

or $$a = bq - \frac{r}{b}$$

By closure, $\frac{r}{b} \in \mathbb{Z}[i]$, so $\frac{r}{b}$ is the remainder.

$$\mathrm{N}\left(\frac{r}{b}\right) = \mathrm{N}\left(\overline{b^{-1}}\right)\mathrm{N}(r) = \mathrm{N}(b)^{-1}\mathrm{N}(r)$$

While $$\mathrm{N}(r) = r_1^2 + r_2^2 \le 2\left(\frac{1}{2}\mathrm{N}(b)\right)^2$$

$$= \frac{1}{2}\mathrm{N}(b)^2$$

Thus the remainder satisfies

$$\mathrm{N}\left(\frac{r}{b}\right) \le \frac{1}{2}\mathrm{N}(b)^{-1}\mathrm{N}(b)^2 = \frac{1}{2}\mathrm{N}(b).$$

3. (*a*) Here $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

$$f_x(x, y) = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right)$$

$$+\tan^{-1}\left(\frac{y}{x}\right) \cdot 2x - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= \frac{-x^2 y}{x^2+y^2} + 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y^3}{x^2+y^2}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial y}(f_x(x,y))$$

$$= -x^2 \cdot \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2} + 2x \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$-\frac{(x^2+y^2)\cdot 3y^2 - y^3 \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2(x^2-y^2)}{(x^2+y^2)^2} + \frac{2x^2}{x^2+y^2} + \frac{x^2+y^4}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^4-x^4+x^2y^2}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)}$$

$= g(x, y)$, say

Now, $\underset{(x,y)\to(0,0)}{\mathrm{Lt}}\ g(x, y)$

$$= \underset{(x,y)\to(0,0)}{\mathrm{Lt}} \frac{x^2+y^4-x^4+x^2y^2}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \left(\frac{x^2+m^4x^4-x^4+x^4m^2}{x^4(1+m^2)^2} + \frac{2x^2}{x^2(1+m^2)}\right)$$

take $y = mx$

$$= \underset{x\to 0}{\mathrm{Lt}} \left(\frac{x^2+(m^4-1+m^2)x^4}{x^4(1+m^2)^2} + \frac{2}{1+m^2}\right)$$

$$= \underset{x\to 0}{\mathrm{Lt}} \left(\frac{1+(m^4-1+m^2)x^2}{x^2(1+m^2)^2} + \frac{2}{1+m^2}\right)$$

$= \infty$, does not exist finitely.

3. (*b*) We must find the extrema of $\mathrm{F} = x^2 + y^2 + z^2$ subject to the constraint conditions

$$\phi_1 = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0$$

and $$\phi_2 = x + y - z = 0$$

In this case we use two Lagrange multipliers $\lambda_1 . \lambda_2$ and consider the function

$$\mathrm{G} = \mathrm{F} + \lambda_1\phi_1 + \lambda_2\phi_2 = x^2 + y^2 + z^2$$

$$+ \lambda_1 \left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x + y - z)$$

Taking the partial derivatives of G with respect to x, y, z and setting them equal to zero, we find

$$\mathrm{G}_x = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0$$

$$\mathrm{G}_y = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0$$

$$\mathrm{G}_z = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \quad ...(i)$$

Solving these equations for x, y, z, we find

$$x = \frac{-2\lambda_2}{\lambda_1+4},\ y = \frac{-5\lambda_2}{2\lambda_1+10}$$

$$z = \frac{25\lambda_2}{2\lambda_1 + 50} \qquad ...(ii)$$

From the second constraint condition, $x + y - z = 0$, we obtain on division by λ_2, assumed different from zero (this is justified since otherwise we would have $x = 0$, $y = 0$, $z = 0$, which would not satisfy the first constraint condition). the result.

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0$$

Multiplying both sides by $2(\lambda_1 + 4)(\lambda_1 + 5)(\lambda_1 + 25)$ and simplifying yields

$$17\lambda^2 + 245\lambda_1 + 750 = 0$$

or $(\lambda_1 + 10)(17\lambda_1 + 75) = 0$

from which $\lambda_1 = -10$ or $-75/17$.

Case I: $\lambda_1 = -10$

From (ii), $x = \frac{1}{3}\lambda_2, y = \frac{1}{2}\lambda_2, z = \frac{5}{6}\lambda_2.$

Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields $\lambda_2^2 = 180/19$

or $\lambda_2 = \pm 6\sqrt{(5/19)}$

This gives the two critical points

$$\left(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}\right)$$

$$\left(-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these critical points is

$$(20 + 45 + 125)/19 = 10$$

Case 2: $\lambda_1 = -75/17$

From (ii), $x = \frac{34}{7}\lambda_2, y = -\frac{17}{4}\lambda_2, \ z = \frac{17}{28}\lambda_2$

Substituting in the first constraint condition, $x^2/4 + y^2/5 + z^2/25 = 1$, yields

$$\lambda_2 = \pm 140/\left(17\sqrt{646}\right)$$

which gives the critical points

$$\left(40/\sqrt{646}, -35\sqrt{646}, 5/\sqrt{646}\right)$$

$$\left(-40/\sqrt{646}, 35/\sqrt{646}, -5/\sqrt{646}\right)$$

The value of $x^2 + y^2 + z^2$ corresponding to these is $(1600 + 1225 + 25)/646 = 75/17$. Thus, the required maximum value is 10 and the minimum value is 75/17.

3. (*c*) We have

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx \qquad ...(i)$$

Since the integral $\to 0$, as $x \to 0$, therefore, 0, is not a point of infinite discontinuity. Hence $\int_0^1 \frac{\sin x}{x}dx$ is a proper integral and therefore it is convergent.

We now test, $\int_1^\infty \sin x \cdot \frac{1}{x}dx$ for its convergence.

Let $\qquad f(x) = \sin x$ and $\varphi(x) = 1/x$

Now, $\varphi(x)$ is monotonic and $\to 0$ as $x \to \infty$. Also,

$$\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \sin x\,dx\right| = \left|\left[-\cos x\right]_1^t\right|$$

$$= \left|\cos 1 - \cos t\right| \le \left|\cos 1\right| + \left|\cos t\right| \le 2$$

showing that $\int_1^t f(x)\,dx$ is bounded $\forall\ t \ge 1$.

Hence, by Dirichlet's test, $\int_1^\infty f(x)\,\varphi(x)\,dx$, *i.e.*, $\int_1^\infty \frac{\sin x}{x}dx$ is convergent. Therefore, from (i), the given integral is convergent. In order to show that the given integral is not absolutely convergent, we must show that $\int_0^\infty \frac{|\sin x|}{x}dx$ is not convergent. Consider the proper integral $\int_0^{n\pi} \frac{|\sin x|}{x}dx$ where n is a positive integer. We have

$$\int_0^{n\pi} \frac{|\sin x|}{x}dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{n\pi} \frac{|\sin x|}{x}dx$$

We put $x = (r - 1)\pi + y$ so that y varies in $[0, \pi]$. We have

$$\left|\sin[(r-1)x + y]\right| = \left|(-1)^{r-1}\sin y\right| = \sin y$$

$$\therefore \int_{(r-1)\pi}^{n\pi} \frac{|\sin x|}{x} dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} dy$$

Since, $r\pi$ is the maximum value of $[(r-1)\pi + y]$ when $y \in [0, \pi]$, we have

$$\int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} dy \geq \frac{1}{r\pi}\int_0^{\pi} \sin y \, dy = \frac{2}{r\pi}$$

$$\therefore \int_0^{n\pi} \frac{|\sin x|}{x} dx \geq \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_1^n \frac{1}{r}$$

Since $\sum_1^n \frac{1}{r} \to \infty$, as $n \to \infty$ we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx \to \infty \text{ as } n \to \infty$$

Let, now, t be a real number. There exists a positive integer n such that $n\pi \leq t < (n+1)\pi$

We have

$$\int_0^{t} \frac{|\sin x|}{x} dx \geq \int_0^{n\pi} \frac{|\sin x|}{x} dx$$

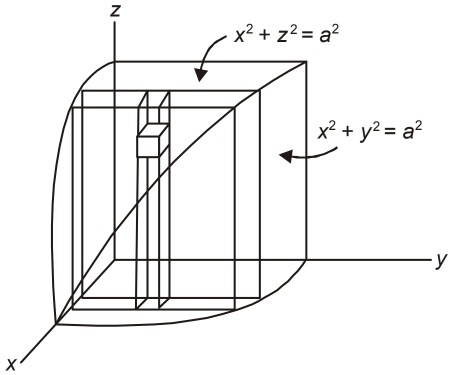
Let $t \to \infty$ so that n also $\to \infty$. Thus we see that

$$\int_0^{t} \frac{|\sin x|}{x} dx \to \infty$$

$$\Rightarrow \int_0^{\infty} \frac{|\sin x|}{x} dx \text{ does not converge.}$$

Hence $\int_0^{\infty} \frac{\sin x}{x} dx$ is not absolutely convergent.

3. (*d*)



Required volume = 8 times volume of region shown in Fig.

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\, dx$$

$$= 8 \int_{x=0}^{a} (a^2 - x^2) dx = \frac{16a^3}{3}.$$

4. (*a*) Here is the desired elementary calculus argument. Note first that

$$\int_0^{2\pi} \frac{1+2\cos x}{5+4\cos x} dx = 2\int_0^{\pi} \frac{1+2\cos x}{5+4\cos x} dx$$

Let $u = \arccos\left(-\frac{4+5\cos x}{5+4\cos x}\right)$. Note that u decreases continuously from π to 0 as x goes from 0 to π. It is easy to check that

$$1 + 2\cos x = -3\frac{1+2\cos u}{5+4\cos u}$$

and $$du = \frac{-3dx}{5+4\cos x}$$

So $$\int_0^{\pi} \frac{1+2\cos x}{5+4\cos x} dx = \int_{\pi}^{0} \frac{1+2\cos u}{5+4\cos u} du$$

$$= -\int_0^{\pi} \frac{1+2\cos u}{5+4\cos u} du,$$

and thus $\int_0^{\pi} \frac{1+2\cos x}{5+4\cos x} dx = 0.$

4. (*b*) $$f(z) = \frac{\tan z}{z} = \frac{\sin z}{z \cos z}$$

has poles at $z = 0, \pm(2n+1)\frac{\pi}{2}$

where $n = 0, 1, 2, ...$

i.e., at $0, \pm\frac{\pi}{2}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$ poles 0

and $\frac{+\pi}{2}$ lies inside $|z| = 2$

$\Rightarrow$ residue of $f(z)$ at $z = 0$ is $= 0$

$\Rightarrow$ residue of $f(z)$

at $\quad z = \frac{-\pi}{2}$ is $= \frac{2}{\pi}$

and $f'(x)$ residue of $f(z)$

at $\quad z = \frac{\pi}{2}$ is $= -\frac{2}{\pi}$

Hence, sum of residues = 0.

4. (*c*) Let I = $\int_0^\infty \int_0^x xe^{-x^2/y}dy\,dx$

Here, x varies from 0 to ∞

and for each x, y varies from 0 to x.

The lower limit of y lies in x-axis (*i.e.,* on $y = 0$)

The upper limit of integration y lies on $y = x$.

If R denotes the region of integration, then y varies from 0 to ∞. Correspondingly for each y, x varies from 0 to ∞. Hence, we get

$$\text{I} = \int_{x=0}^{\infty}\int_{y=0}^{x} xe^{-x^2/y}dy\,dx$$

$$= \int_{y=0}^{\infty}\int_{x=y}^{\infty} xe^{-x^2/y}dx\,dy$$

$$= \int_{y=0}^{\infty}\left[\int_{x=y}^{\infty} xe^{-x^2/y}dx\right]dy$$

$$= \int_{y=0}^{\infty}\left[\int_{x=y}^{\infty}\left(\frac{-y}{2}\right)e^{-x^2/y}\left(-\frac{2x}{y}dx\right)\right]dy$$

$$= \int_{y=0}^{\infty}\left[-\frac{y}{2}\left[e^{-x^2/y}\right]_{x=y}^{x=\infty}\right]dy$$

$$= \int_{y=0}^{\infty}\left(-\frac{y}{2}\right)(0-e^{-y})dy$$

$$= \frac{1}{2}\int_0^{\infty} ye^{-y}dy = \frac{1}{2}$$

$$= \frac{1}{2}\left\{\left(y\frac{e^{-y}}{-1}\right)_0^{\infty} - \int_0^{\infty}\frac{e^{-y}}{(-1)}dy\right\}$$

$$= \frac{1}{2}\left\{(0-0)+\int_0^{\infty} e^{-y}dy\right\}$$

$$= \frac{1}{2}\left(\frac{e^{-y}}{-1}\right)_0^{\infty} = \frac{-1}{2}(0-1) = \frac{1}{2}.$$

4. (*d*) Given Matrix

Programs

		A	B	C	D
Programmer	P_1	5	3	2	8
	P_2	7	9	2	6
	P_3	6	4	5	7
	P_4	5	7	7	8

Step-I: Substracting the smallest element in each row from every element of the corresponding row, we get the following matrix.

	A	B	C	D
P_1	3	1	0	6
P_2	5	7	0	4
P_3	2	0	1	3
P_4	0	2	2	3

Step-II: Substracting the smallest element in each column of the above matrix from every element of the corresponding column, we get the following matrix:

	A	B	C	D
P_1	3	1	0	3
P_2	5	7	0	1
P_3	2	0	1	0
P_4	0	2	2	0

Since row 3 and row 4 contain exactly two zero. Also, column 3 and column 4 contain exactly two zero, therefore the trial and error method is followed. And we get:

	A	B	C	D
P_1	3	$\boxed{1}$	$\xcancel{0}$	3
P_2	5	7	$\boxed{0}$	1
P_3	2	$\xcancel{0}$	1	$\boxed{0}$
P_4	$\boxed{0}$	2	2	$\xcancel{0}$

Therefore, for least computer time the allotment should be as follows:

Programs:	A	B	C	D
Programmers:	P_4	P_1	P_2	P_3
Program hours:	5	3	2	7

and the total computer time are 5 + 3 + 2 + 7 = 17 hours.

5. (*a*) We have

$$z = \phi(x^2 - y) + \psi(x^2 + y) \qquad ...(i)$$

Differentiating partially w.r.t. x and y, we get

$$p = \phi'(x^2 - y)\frac{\partial}{\partial x}(x^2 - y)$$

$$+\ \psi'(x^2 + y)\frac{\partial}{\partial x}(x^2 + y) \qquad ...(ii)$$

$$q = \phi'(x^2 - y)\frac{\partial}{\partial y}(x^2 - y)$$

$$+\ \psi'(x^2 + y)\frac{\partial}{\partial y}(x^2 + y) \qquad ...(iii)$$

and

$(ii) \Rightarrow\ p = 2x\ \phi'(x^2 - y) + 2x\ \psi'(x^2 + y)$ $\qquad ...(iv)$

$(iii) \Rightarrow q = -\ \phi'(x^2 - y) + 1.\psi'(x^2 + y)$ $\qquad ...(v)$

Differentiating (iv) w.r.t. x, we get

$$r = 2x\ \phi''(x^2 - y)\,.\,2x + 2.1\ \phi''(x^2 - y)$$
$$+\ 2x\ \psi''(x^2 + y).2x + 2.1\ \psi'(x^2 + y)$$

or $\ r = 4x^2\ (\phi''(x^2 - y) + \psi''(x^2 + y))$
$$+\ 2(\phi'(x^2 - y) + \psi'(x^2 + y)) \qquad ...(vi)$$

Differentiating (v) w.r.t. y, we get

$$t = \phi''(x^2 - y).(-1) + \psi''(x^2 + y).1$$

or $\ t = \phi''(x^2 - y) + \psi''(x^2 + y) \qquad ...(vii)$

$\therefore\ (vi) \Rightarrow r = 4x^2 t + 2\left(\dfrac{p}{2x}\right)$

(using (iv) and (vii))

$$\Rightarrow\quad x\frac{d^2z}{dx^2} = 4x^3\frac{d^2z}{dy^2} + \frac{dz}{dx}.$$

5. (*b*) Multiplicative inverse of a square matrix

For every non-singular square matrix A of order n, there exists a non-singular square matrix B of same order, such that AB = BA = I. (Note that I is unit matrix of order n). Here B is called multiplicative inverse of A and is denoted as $A^{-1} \Rightarrow B = A^{-1}$.

Note: If AB = KI, then $A^{-1} = \dfrac{1}{K}B$.

Multiplicative inverse of a 2 × 2 square matrix

For a 2 × 2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can show that

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note:

1. For a singular square matrix |A| = 0, and so its multiplicative inverse doesn't exist. Conversely if a matrix A doesn't have multiplicative inverse, then |A| = 0.
2. If A is a square matrix and K is any scalar, then $(KA)^{-1} = \dfrac{1}{K}A^{-1}$.
3. For any two source matrices A and B of same order $(AB)^{-1} = B^{-1}\ A^{-1}$.

Method for finding inverse of a 2 × 2 square matrix

We know that for a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

From this formula we can find A^{-1} using the following steps.

1. Find whether |A| = 0 or not. If |A| = 0, then the given matrix is singular, so A^{-1} doesn't exist. If |A| ≠ 0, then the matrix has a multiplicative inverse and can be found by the following steps (2), (3) and (4).
2. Interchange the elements of principal diagonal.
3. Multiply the other two elements by –1.
4. Multiply each element of the matrix by $\dfrac{1}{|A|}$.

Example

Find the inverse of the matrix A = $\begin{bmatrix} 2 & -4 \\ 3 & -5 \end{bmatrix}$.

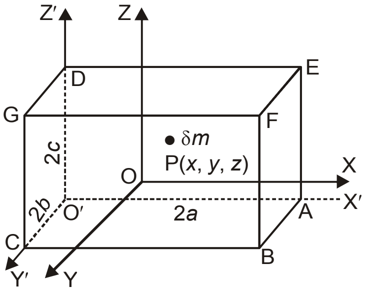
Sol. $|A| = \begin{vmatrix} 2 & -4 \\ 3 & -5 \end{vmatrix} = -10 + 12 = 2 \neq 0$

$\therefore$ A is non singular and A^{-1} exists.

$$\therefore \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & 4 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{4}{2} \\ -\frac{3}{2} & \frac{2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

5. (*c*) Let $2a$, $2b$, $2c$ be the lengths of the sides of a rectangular parallelopiped, and O its centre. Let OX, OY, OZ be the axes through O and parallel to the edges of the rectangular parallelopiped whose mass is M (say). Then ρ = mass per unit volume = $M/8abc$. Consider an element $\delta x\ \delta y\ \delta z$ at the point $P(x, y, z)$. Its mass = $\delta m = \rho\delta x\ \delta y\ \delta z$ and distance of $P(x, y, z)$ from OX is $(y^2 + z^2)^{1/2}$.



$\therefore$ M.I. of δm at P about

$$\text{OX} = \rho(y^2 + z^2)\ \delta x\ \delta y\ \delta z.$$

Hence M.I. of the rectangular parallelopiped about OX (which is parallel to the *edge* $2a$).

$$= \int_{x=a}^{a}\int_{y=b}^{b}\int_{z=c}^{c}\rho(y^2+z^2)dx\,dy\,dz$$

$$= \rho\int_{-a}^{a}\int_{-b}^{b}\left[y^2 z + \frac{1}{3}z^3\right]_{-c}^{c} dx\ dy$$

$$= \rho\int_{-a}^{a}\int_{-b}^{b} 2\left(y^2 c + \frac{1}{3}c^3\right) dx\ dy$$

$$= 2\rho\int_{-a}^{a}\left[\frac{1}{3}y^3 c + \frac{1}{3}c^3 y\right]_{-b}^{b} dx$$

$$= \frac{2\rho}{3}\int_{-a}^{a} 2(b^3 c + c^3 b)\,dx$$

$$= \frac{4\rho}{3}bc(b^2 + c^2)\left[x\right]_{-a}^{a}$$

$$= \frac{8\rho\,abc}{3}(b^2 + c^2) = \frac{M(b^2 + c^2)}{3}$$

as $\quad p = \dfrac{M}{8\,abc}$.

5. (*d*) **Composite Trapezoidal Rule:** We divide the interval $[a, b]$ into n equal subintervals by the points x_0, x_1 .... x_n such that $x_i = x_0 + ih(i = 0, 1, 2, ... n)$ where h is the length of each subinterval and $x_0 = a$, $x_n = b$. Let y_0, y_1, ... y_n be the values of $y = f(x)$ at the points x_0, x_1, ... x_n respectively. If we apply the trapezoidal rule in each of the intervals $[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{n-1}, x_n]$, then we get

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

$$= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + ... + \frac{h}{2}(y_{n-1} + y_n)$$

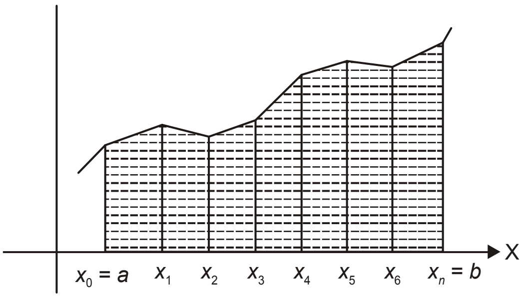
$$= \frac{h}{2}\,[y_0 + y_1 + y_1 + y_2 + y_2 + \ ...\ + y_{n-1}$$
$$+ y_{n-1} + y_n]$$

$$= \frac{h}{2}\,[(y_0 + y_n) + 2(y_1 + y_2 + \ ...\ + y_{n-1})]$$

This formula is called the composite Trapezoidal Rule.

Note: The trapezoidal rule is used for any number of equal subintervals (odd or even) of the interval $[a, b]$.

Geometrical Interpretation of Trapezoidal Rule: In the trapezoidal rule, the curve $y = f(x)$ is approximated by straight lines joining $P(x_i, y_i)$ and $Q(x_{i+1}, y_{i+1})i = 0$, 1, ... n_{-1}. Thus the total area bounded by the curve $y = f(x)$, x-axis and the ordinates at $x_0 = a$, $x_n = b$ is approximated by the sum of the areas of the trapeziums.



6. (*a*) Here the Lagrange's subsidiary equations are

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \qquad ...(i)$$

Taking the first two fractions of (*i*),

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)} \qquad ...(ii)$$

Let, $y/x = v$, *i.e.*, $y = xv$ ...(*iii*)

From (*iii*),

$$(dy/dx) = v + x(dv/dx), \qquad ...(iv)$$

Using (*iii*) and (*iv*), (*ii*) gives

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

or $$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$= \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

or $$\frac{1-v}{1+v^2}dv = \frac{dx}{x}$$

or $$\left(\frac{2}{1+v^2} - \frac{2v}{1+v^2}\right)dv = \frac{2dx}{x}$$

Integrating,

$2\tan^{-1} v - \log(1 + v^2) = 2 \log x - \log c_1$

or $\log x^2 - \log(1 + v^2) - \log c_1 = 2\tan^{-1} v$

or $\log \{x^2(1 + v^2)/c_1\} = 2 \tan^{-1} v$

or $x^2(1 + v^2) = c_1 e^{2\tan^{-1} v}$

or $$x^2\left[1+\left(\frac{y^2}{x^2}\right)\right] = c_1 e^{2\tan^{-1}(y/x)}$$

as $v = y/x$ by (*iii*)

or $(x^2 + y^2)e^{-2\tan^{-1}(y/x)} = c_1$, c_1 being an arbitrary constant. ...(*v*)

Choosing 1, 1, $-1/z$ as multipliers, each fraction of (*i*)

$$= \frac{dx + dy - (1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)}$$

$$= \frac{dx - dy - (1/z)dz}{0}$$

$\Rightarrow \quad dx + dy - (1/z)dz = 0$

so that $x + y - \log z = c_2$ ...(*vi*)

From (*v*) and (*vi*), the required general solution is

$$\phi(x + y - \log z\ (x^2 + y^2)\, e^{-2\tan^{-1}(y/x)}) = 0$$

where ϕ is an arbitrary function.

6. (*b*) In this instance

$$f = \frac{z(x+y)}{3z+1}$$

so that the equations (*v*) take the form

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{(x+y)}$$

which have solutions

$$x - y = c_1$$

$$x^2 + y^2 - 2z^3 - z^2 = c_2$$

Thus, any surface which is orthogonal to the given surfaces has equation of the form

$$x^2 + y^2 - 2z^3 - z^2 = f(x - y)$$

For the particular surface passing through the circle $x^2 + y^2 = 1$, $z = 1$ we must take f to be the constant -2. The required surface is therefore

$$x^2 + y^2 = 2z^3 + z^2 - 2.$$

6. (*c*) Let, $F(x, y, z, p, q)$

$= xp - yq - xq\, f(z - px - qy) = 0$ ...(*i*)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x + p(\partial F/\partial z)} = \frac{dq}{\partial F/\partial y + q(\partial F/\partial z)}$$

$$= \frac{dz}{-p(\partial F/\partial p) - q(\partial F/\partial q)} = \frac{dx}{-(\partial F/\partial p)}$$

$$= \frac{dy}{-(\partial F/\partial q)}$$

or $$\frac{dp}{p - qf + xqpf' - pqxf'}$$

$$= \frac{dq}{-q + xq^2 f' - xq^2 f'} = ... \text{ by } (i) \qquad ...(ii)$$

Each ratio of (*ii*) $= \dfrac{x\,dp + y\,dq}{xp - yq - qxf}$

$$= \frac{x\,dp + y\,dq}{0}, \text{ by } (i)$$

$\Rightarrow \qquad x\,dp + y\,dq = 0$

$\Rightarrow x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow \; dz - d(xp) - d(yq) = 0$

as $\qquad dz = pdx + qdy$

Integrating,

$z - xp - yq = \text{constant}$

$= a$, say $\qquad ...(iii)$

$\therefore \qquad xp + yq = z - a \qquad ...(iv)$

Using (*iii*) becomes

$xp - y\,q = x\,q\,f(a) \qquad ...(v)$

Subtracting (*v*) from (*iv*),

$2yq = z - a - xqf(a)$

$\Rightarrow \qquad q = (z - a)/\{2y + xf(a)\} \;...(vi)$

Using (*vi*), (*iv*)

$$\Rightarrow \qquad p = \frac{(z-a)\{y + xf(a)\}}{x\{2y + xf(a)\}} \quad ...(vii)$$

Using (*vi*) and (*vii*)

$dz = p\,dx + q\,dy$ reduces to

$$dz = (z - a)\left[\frac{\{y + xf(a)\}dx}{x\{2y + xf(a)\}} + \frac{dy}{2y + xf(a)}\right]$$

or $$\frac{2dz}{z-a} = \frac{2y\,dx + 2xf(a)dx + 2x\,dy}{x\{2y + xf(a)\}}$$

$$= \frac{2d(xy) + 2xf(a)dx}{2xy + x^2 f(a)}$$

Integrating

$2\log(z - a) = \log\{2xy + x^2 f(a)\} + \log b$

or $\qquad (z - a)^2 = b\{2xy + x^2 f(a)\}.$

6. (*d*) The displacement y of the particle at a distance x from the end $x = 0$ at time t is governed by

$$\frac{\partial^2 y}{dt^2} = a^2 \frac{\partial^2 y}{dx^2} \qquad ...(1)$$

The boundary, conditions are

$y(0, t) = 0 \qquad$ for all $t \geq 0 \qquad (i)$

$y(l, t) = 0 \qquad$ for all $t \geq 0 \qquad (ii)$

$\left(\dfrac{\partial y}{dt}\right) = 0 \qquad$ for $0 \leq x \leq t \qquad (iii)$

$y(x, 0) = y_0 \sin^3\left(\dfrac{\pi x}{l}\right) \quad$ for $0 \leq x \leq l \qquad (iv)$

Now solving (1) and selecting the proper solution to suit the physical nature of the problem and making use of the boundary conditions (*i*) and (*ii*) as in the previous problem, we get

$$y(x, t) = \text{B}\sin\frac{n\pi x}{l}\left(\text{C}\cos\frac{n\pi\,at}{l} + \text{D}\sin\frac{n\pi\,at}{l}\right) \qquad ...(2)$$

Again using the boundary condition (*iii*),

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \text{B}\sin\frac{n\pi x}{l}\left(\text{D}\cdot\frac{n\pi a}{l}\right)$$

If B = 0, (2) takes the form $y(x, t) = 0$. Hence B cannot be zero.

$\therefore \qquad \text{D} = 0$

Hence (2) becomes,

$$y(x, t) = \text{B}_n \frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

where n is any integer and B_n is any constant.

The most general solution satisfying (1) and the boundary conditions (*i*), (*ii*) and (*iii*) is

$$y(x, t) = \sum_{n=1}^{\infty} \text{B}_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad ...(3)$$

To find B_n use the boundary condition (*iv*)

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$= \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

This is only if $B_1 = \frac{3y_0}{4}$, $B_3 = -\frac{y_0}{4}$

and $B_n = 0$, for $n \neq 1, 3$

Using these values in (3), the solution of the equation is

$$y(x, t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

7. (*a*) We have

$$f(x) = x^3 + x^2 + 3x + 4$$

$$f'(x) = 3x^2 + 2x + 3$$

Since $f(-1) = (-1)^3 + (-1)^2 + 3(-1) + 4$

$= 1 > 0$

and $f(-2) = (-2)^3 + (-2)^2 + 3(-2) + 4$

$= -6 < 0$

Root lies between (–2, –1)

Let, $x_0 = \frac{-2-1}{2} = -1.5$

By Newton-Raphson Method

First approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -1.5 - \frac{(-1.5)^2 + (-1.5)^2 + 3(-1.5) + 4}{3(-1.5)^2 + 2(-1.5) + 3}$$

$$= -1.5 - \frac{(-1.625)}{6.75} = -1.25926$$

Now, $f(x_1) = -0.18888$

$f'(x_1) = 4.7572$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -1.25926 - \frac{(-0.18888)}{4.7572}$$

$= -1.219555$

$f(x_2) = 0.0151$

$f'(x_2) = 5.02254$

$$x_3 = -1.2195 - \frac{0.0151}{5.0225}$$

$= -1.222506$

$f(x_3) = -0.000058,$

$f'(x_3) = 5.038548$

$$x_4 = -1.222506 - \frac{(-0.000058)}{5.038548}$$

$= -1.22251.$

7. (*b*) Here, $h = 10$

$\therefore$ Using Simpson's one-third rule, the approximate area of the cross-section

$$= \int_0^{80} y\,dx$$

$$= \int_0^{0+8b} d\,dx$$

$$= \frac{1}{3}h[y_0 + 4(y_1 + y_3 + y_5 + y_7)$$

$$+ 2(y_2 + y_4 + y_6) + y_8]$$

$$= \frac{1}{3}(10)[0 + 4(4 + 9 + 15 + 8)$$

$+ 2(7 + 12 + 14) + 3]$, (from table)

$$= \frac{1}{3}(10)[4(36) + 2(33) + 3]$$

$$= \frac{1}{3}(10)(144 + 66 + 3)$$

$= 10(48 + 22 + 1)$

$= 710$ square metres.

7. (*c*) Here $x_0 = 0$, $y_0 = 1$ and $f(x, y) = x + y$

Choosing $h = 0.2$, we compute k_1, k_2, k_3 and k_4.

$k_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$= 0.2(0 + 1) = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left(x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\left(0+\frac{0.2}{2}+1\frac{0.2}{2}\right) = 0.24$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= h\left(x_0+\frac{h}{2}+y_0+\frac{k_2}{2}\right)$$

$$= 0.2\left(0+\frac{0.2}{2}+1\frac{0.24}{2}\right) = 0.244$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h(x_0 + y_0 + h + k_3)$$

$$= 0.2(0 + 1 + 0.2 + 0.244) = 0.2888$$

$\therefore$ The solution at $x = 0.2$ is obtained as

$$y(0.2) \simeq y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 1+\frac{1}{6}[0.2 + 2(0.24) + 2(0.244) + 0.2888]$$

$$= 1+\frac{1}{6}(1.4568) = 1.2428.$$

7. (*d*) 32 bit computer representation

$$\begin{array}{rl}
1 = & \text{0000 0000 0000 0000 0000 0000 0000 0001} \\
-1 = & \text{1111 1111 1111 1111 1111 1111 1111 1110 + 1} \\
\hline
 & \text{1111 1111 1111 1111 1111 1111 1111 1111} \\
1024 = & \text{0000 0000 0000 0000 0000 0100 0000 0000} \\
-\ 1024 = & \text{1111 1111 1111 1111 1111 1011 1111 1111 + 1} \\
\hline
 & \text{1111 1111 1111 1111 1111 1100 0000 0000} \\
-1 : & \text{1111 1111 1111 1111 1111 1111 1111 1111} \\
-1024 : & \text{1111 1111 1111 1111 1111 1100 0000 0000} \\
\hline
1025 : & \boxed{1}\ \text{1111 1111 1111 1111 1111 1011 1111 1111}
\end{array}$$

8. (*a*) The kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$.

The potential energy is composed of two parts: the gravitational part is —mgy (the minus sign is included because the coordinate y is measured downwards, the elastic spring part is $\frac{1}{2}k(y-\ell)^2$. Together this gives $V = \frac{1}{2}k(y-\ell)^2 - mgy$.

Thus the Lagrangian is L = T − V

$$= \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\ell)^2 + mgy$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + k(y-\ell) - my = 0$$

This is a inhomogeneous linear constant coefficients second order DE.

$$\ddot{y} + \frac{k}{m}y = g + \frac{k}{m}\ell$$

The general solution is

$$y = \ell + \frac{mg}{k} + A\ \sin\omega t + B\ \cos\omega t$$

where $\omega^2 = k/m$ and the arbitrary constants A and B would be determined from initial conditions. Note that the frequency ω is the same with and without gravity.

8. (*c*) The vorticity components are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = C + C = 2C$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = A + A = 2A$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = B + B = 2B$$

The equations of the vertex lines are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

$$\frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$
$$(i) \qquad (ii) \qquad (iii)$$

From (*i*) and (*ii*), we have

$$Ax - Cy = k_1 \qquad ...(1)$$

From (*ii*) and (*iii*), we have

$$By - Az = k_2$$

where k_1 and k_2 are integration constants...(2)

Hence the vertex lines (1) and (2) are the straightlines.

8. (*d*) As per the given conditions, out of the three solutions of the Laplace equation, the solution

$$u(x, y) = (c_1 \cos px + c_2 \sin px)\,(c_3 e^{py} + c_4 e^{-py})$$

is suitable. The boundary condition

$$u(0, y) = 0$$

implies $c_1 = 0$

Therefore,

$$u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$$

Now using the condition $u(l, y) = 0$ gives

$$0 = c_2 \sin pl(c_3 e^{py} + c_4 e^{-py})$$

and so

$$\sin pl = 0 = \sin n\pi, n = 0, \pm 1, \pm 2, ...$$

giving $p = \frac{n\pi}{l}$. Thus the solution becomes

$$u(x, y) = c_2 \sin\frac{n\pi x}{l}(c_3 e^{n\pi y/l} + c_4 e^{-n\pi y/l})$$

Now the use of condition $u(x, 0) = 0$ gives

$$0 = c_2 \sin\frac{n\pi x}{l}(c_3 + c_4)$$

and so $c_3 + c_4 = 0$ or $c_4 = -c_3$. Thus the solution reduces to

$$u(x, y) = c_2 c_3 \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l})$$
$$= A_n \sin\frac{n\pi x}{l}\sinh^{n\pi y/l}$$

Now the last condition $u(x, a) = \sin\frac{n\pi x}{l}$ yields

$$\sin\frac{n\pi x}{l} = A_n \sin\frac{n\pi x}{l}\sinh\frac{n\pi a}{l}$$

or $$A_n = \frac{1}{\sinh\frac{n\pi a}{l}}$$

Hence,

$$u(x, y) = \frac{\sin n\pi x}{l}\cdot\frac{\sin n\pi y/l}{\sin n\pi a/l}.$$

Previous Years' Paper (Solved)

IFS Mathematics Main Examination, 2016

PAPER-I

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos.* ***1*** *and* ***5*** *are compulsory. Out of the remaining six questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections* ***A*** *and* ***B****. Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Let T : $\mathbb{R}^3 \to \mathbb{R}^4$ be given by $T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$. Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (*i.e.,* (1, 0, 0), (0, 1, 0), etc.) Examine if T is a linear map.

(*b*) Show that $\frac{x}{(1+x)} < \log(1 + x) < x$ for $x > 0$.

(*c*) Examine if the function $f(x, y) = \frac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at (0, 0). Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at points other than origin.

(*d*) If the point (2, 3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord.

(*e*) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$.

2. (*a*) After changing the order of integration of

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy,$$

show that $\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$.

(*b*) A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

(*c*) Using mean value theorem, find a point on the curve $y = \sqrt{x - 2}$, defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve.

(*d*) Let T be a linear map such that $T : V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis B_1[(1, 1, 0) (1, 0, 1) (0, 1, 1)] and B_2[(1, 1) (1, –1)]. Obtain the matrix T_1 relative to B_1 and B_2.

3. (*a*) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ = I. Hence find A^{-1}.

(*b*) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point (1, 0, 0).

(*c*) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$.

(*d*) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has (0, 1, 0) and (3, –5, 2) as the end points of its diameter is a great circle.

4. (*a*) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature.

(*b*) Show that the integral $\int_0^\infty e^{-x}x^{\alpha-1}dx,\ \alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$.

(*c*) Prove that $\lceil\overline{2z} = \frac{2^{2z-1}}{\sqrt{\pi}}\lceil\overline{z}\,\lceil\overline{z+\frac{1}{2}}$.

(*d*) A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C.

SECTION–B

5. (*a*) Obtain the curve which passes through (1, 2) and has a slope = $\frac{-2xy}{x^2+1}$. Obtain one asymptote to the curve.

(*b*) Solve the dE to get the particular integral of $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=x^2\cos x$.

(*c*) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings.

(*d*) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot\beta = 2\tan(\alpha - \beta)$.

(*e*) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \overline{F}.dS$ where S is the surface bounding the volume E and $\overline{F} = (zx\sin yz + x^3)\hat{i} + \cos yz\,\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}.$

6. (*a*) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet?

(*b*) Using the method of variation of parameters, solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x.$$

(*c*) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line.

(*d*) Evaluate $\iint_S (\nabla\times\overline{f}).\hat{n}\,dS$ for $\overline{f} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane.

7. (*a*) State Stokes' theorem. Verify the Stokes' theorem for the function $\overline{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one.

(*b*) A uniform rod of weight W is resting against an equally rough horizon and a well, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that P = W tan (α – 2λ), where λ is the angle of friction. Is there any condition on λ and α?

(*c*) Obtain the singular solution of the differential equation:

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2,\ p = \frac{dy}{dx}.$$

8. (*a*) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio 3 : 2.

(*b*) Solve the differential equation

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x).$$

(*c*) Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$.

(*d*) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary.

PAPER-II

INSTRUCTIONS: *There are **eight** questions in all, out of which **five** are to be attempted. Question Nos. **1** and **5** are compulsory. Out of the remaining six questions, **three** are to be attempted selecting at least **one** question from each of the two Sections **A** and **B**. Answers must be written in **English** only.*

SECTION–A

1. (*a*) Prove that the set of all bijective functions from a non-empty set X onto itself is a group with respect to usual composition of functions.

(*b*) Examine the Uniform Convergence of:

$$f_n(x) = \frac{\sin(nx + n)}{n} \forall x \in \text{IR},\ n = 1, 2, 3, ...$$

(*c*) Find the maxima and minima of the function

$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$

(*d*) Find the analytic function of which the real part is

$$e^{-x}\left\{\left(x^2 - y^2\right)\cos y + 2xy \sin y\right\}.$$

(*e*) Prove that the set of all feasible solutions of a Linear Programming problem is a convex set.

2. (*a*) Show that any non-abelian group of order 6 is isomorphic to the symmetric group S_3.

(*b*) Let G be a group of order pq, where p and q are prime numbers such that $p > q$ and $q \times (p - 1)$. Then prove that G is cyclic.

(*c*) Show that in the ring R = { $a + b\sqrt{-5}$ | a, b are integers}, the elements α = 3 and β = $1 + 2\sqrt{-5}$ are relatively prime, but αγ and βγ have no g.c.d. in R, where $\gamma = 7\left(1 + 2\sqrt{-5}\right)$.

3. (*a*) If $f_n(x) = \dfrac{3}{x+n}, 0 \le x \le 2$, state with reasons whether $\{f_n\}_n$ converges uniformly on [0, 2] or not.

(*b*) Examine the continuity of

$$f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & (x,\ y) \ne (0,\ 0) \\ \dfrac{1}{2} & (x,\ y) = (0,\ 0) \end{cases}$$

at the point (0, 0).

(*c*) If $u(x,y) = \cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, $0 < x < 1$, $0 < y < 1$ then find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

(*d*) Evaluate the integral :

$$\int_0^2 \int_0^{y^2/2} \frac{y}{\left(x^2+y^2+1\right)^{\frac{1}{2}}} dx.dy.$$

4. (*a*) Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$.

(*b*) Find the Laurent series for the function $f(z) = \dfrac{1}{1-z^2}$ with centre $z = 1$.

(*c*) Evaluate by Contour integration

$$\int_0^\pi \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}.$$

(*d*) A company manufacturing air-coolers has two plants located at Bengaluru and Mumbai with a weekly capacity of 200 units and 100 units respectively. The company supplies air-coolers to its 4 showrooms situated at Mangalore, Bengaluru, Delhi and Goa which have a demand of 75, 100, 100 and 25 units respectively. Due to the differences in local taxes, showroom charges, transportation cost and others, the profits differ. The profits (in Rs.) are shown in the following table:

	To			
From	Mangalore	Bengaluru	Delhi	Goa
Bengaluru	90	90	100	100
Mumbai	50	70	130	85

Plan the production program so as to maximize the profit. The company may have its production capacity at both plants partially or wholly unused.

SECTION–B

5. (*a*) Obtain the partial differential equation governing the equations

$$\phi(u,\ v) = 0,\ u = xyz,$$
$$v = x + y + z.$$

(*b*) Find the general solution of the partial differential equation

$$xy^2\frac{\partial z}{\partial x} + y^3\frac{\partial z}{\partial y} = \left(zxy^2 - 4x^3\right)$$

(*c*) Develop an algorithm for Newton-Raphson method to solve $\phi(x) = 0$ starting with initial iterate x_0, n be the number of iterations allowed, eps be the prescribed relative error and delta be the prescribed lower bound for $\phi'(x)$.

(*d*) Apply Lagrange's interpolation formula to find $f(5)$ and $f(6)$ given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(7) = 128$.

(*e*) Calculate the moment of inertia of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(*i*) relative to the x-axis

(*ii*) relative to the y-axis and

(*iii*) relative to the origin.

6. (*a*) Find the general solution of the partial differential equation :

$$xy^2p + y^3q = (zxy^2 - 4x^3)$$

$$\left[p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right].$$

(*b*) Find the particular integral of

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 2x\cos y.$$

(*c*) A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

7. (*a*) Evaluate $\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}$ by Simpson's $\frac{1}{3}$rd rule, by taking 12 equal sub-intervals.

(*b*) Find the cube root of 10 up to 5 significant figures by Newton-Raphson method.

(*c*) Use the Classical Fourth-order Runge-Kutta method with $h = 0.1$ to calculate a solution at $x = 0.2$ for the initial value problem $\frac{dy}{dx} = x + y^2$ with initial condition $y = 1$ when $x = 0$.

8. (*a*) Find the moment of inertia of a right solid cone of mass M, height h and radius of whose base is *a*, about its axis.

(*b*) A bead slides on a wire in the shape of a cycloid described by the equations

$$x = a(\theta - \sin \theta)$$

$$y = a(1 + \cos \theta)$$

where, $0 \le \theta \le 2\pi$ and the friction between the bead and the wire is negligible. Deduce Lagrange's equation of motion.

(*c*) A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b \cos nt$, where $b > a$, then find the pressure at the surface of the sphere at any time.

ANSWERS

PAPER-I

1. (***a***) T: $R^3 \to R^4$ given by

$$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$$

$$T(1, 0, 0) = (2, 2, 1, 1)$$

$$T(0, 1, 0) = (-1, 0, 0, 1)$$

$$T(0, 0, 1) = (0, 1, 2, 1)$$

$\therefore$ matrix of T

$$M_T = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Let, $u, v \in R^3$

Then,

$$\begin{aligned} T(\alpha u + \beta v) &= T(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \alpha u_3 + \beta v_3) \\ &= (2\alpha u_1 + 2\beta v_1 - \alpha u_2 - \beta v_2, 2\alpha u_1 + 2\beta v_1 \\ &\quad + \alpha u_3 + \beta v_3, \alpha u_1 + \beta v_1 + 2\alpha u_3 + 2\beta v_3, \\ &\quad (\alpha u_1 + \beta v_1 + \alpha u_2 + \beta v_2 + \alpha u_3 + \beta v_3)) \\ &= (\alpha(2u_1 - u_2) + \beta(2v_1 - v_2), \alpha(2u_1 + u_3) \\ &\quad + \beta(2v_1 + v_3), \alpha(u_1 + 2u_3) + \beta(v_1 + 2v_3), \\ &\quad \alpha(u_1 + u_2 + u_3) + \beta(v_1 + v_2 + v_3)) \\ &= \alpha[2u_1 - u_2, 2u_1 + u_3, u_1 + 2u_3, u_1 + u_2 + u_3] \\ &\quad + \beta[2v_1 - v_2, 2v_1 + v_3, v_1 + 2v_3, v_1 + v_2 + v_3] \\ &= \alpha T(u) + \beta T(v) \end{aligned}$$

$\Rightarrow$ T is a linear map.

1. (*b*) $$\frac{x}{1+x} < \log(1+x) < x;\ x > 0$$

Consider $f(x) = \log(1+x) - \frac{x}{(1+x)}$

$$f'(x) = \frac{1}{(1+x)} - \frac{(1+x).1 - x.1}{(1+x)^2}$$

$$= \frac{1}{(1+x)} - \frac{1}{(1+x)^2}$$

$$= \frac{(1+x-1)}{(1+x)^2} = \frac{x}{(1+x)^2} > 0$$

$$[\because x > 0 \Rightarrow (1+x)^2 > 0]$$

$\Rightarrow \quad f'(x) > 0 \ \forall \ x > 0$

$\Rightarrow f(x)$ is an increasing function

$$f(0) = \log 1 - \frac{0}{1} = 0$$

$\therefore$ for $x > 0$, $f(x) > 0$

$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$

$\Rightarrow \log(1+x) > \frac{x}{1+x}$ ...(1)

Again,

Let, $\quad g(x) = x - \log(1+x)$

$$g'(x) = 1 - \frac{1}{(1+x)} = \frac{1+x-1}{1+x}$$

$$= \frac{x}{1+x} > 0 \quad \forall x > 0$$

$\Rightarrow g(x)$ is an increasing function and $g(0) = 0$

$\Rightarrow \quad$ for $x > 0$, $g(x) > 0$

$\Rightarrow \ x - \log(1+x) > 0$

$\Rightarrow \quad x > \log(1+x)$ ...(2)

Combine (1) and (2)

$$\frac{x}{1+x} < \log(1+x) < x$$

1. (*c*) $$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 \quad, & (x,y) = (0,0) \end{cases}$$

To check continuity at (0, 0) we will use difinition of continuity

$$|f(x, y) - f(0, 0)| = \left|\frac{xy}{x^2+y^2}\right| \quad ...(1)$$

$(x - y)^2 \geq 0$

$\Rightarrow x^2 + y^2 - 2xy \geq 0$

$\Rightarrow \quad 2xy \leq x^2 + y^2$

$\Rightarrow \quad \frac{xy}{x^2+y^2} \leq \frac{1}{2}$

$\therefore$ if we choose $\in < \frac{1}{2}$ then by eq. (1)

$$|f(x, y) - f(0, 0)| = \left|\frac{xy}{x^2+y^2}\right| \leq \frac{1}{2} > \in$$

$\Rightarrow \quad |f(x, y) - f(0, 0)| = \in$ when $\in < \frac{1}{2}$

$\therefore f(x, y)$ is not continuous at (0, 0)

Now,

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

$\therefore$ at origin

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(0+\delta x, 0) - f(0,0)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{0-0}{\delta x} = 0$$

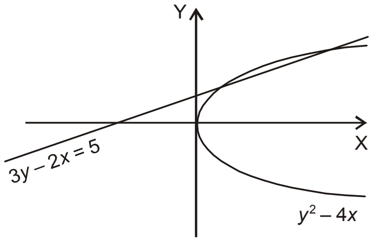
$$\frac{\partial f}{xy} = \lim_{\delta y \to 0} \frac{f(0, 0+\delta y) - f(0,0)}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{0-0}{\delta y} = 0$$

1. (*d*) We know that

Equation of chord of the parabola $y^2 = 4uax$ which is bisected at (x_1, y_1) is given by

$$yy_1 - 2a(x + x_1) = y^2 - 4ax$$



$\therefore$ Here $\quad (x_1, y_1) = (2, 3)$ and $a = 1$

$\therefore \quad 3y - 2(x + 2) = 9 - 4 \times 2$

$$3y - 2x - 4 = 1$$
$$3y - 2x = 5$$

1. (*e*)
$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

for eigen values solve $|A - \lambda I| = 0$

$$\begin{bmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix} = 0$$

$\Rightarrow$ $(-1-\lambda)\,[(-1-\lambda)^2 - 4] - 2[2(-1-\lambda) - 4] + 2[4 - 2\,(-1-\lambda)] = 0$

$\Rightarrow$ $(-1-\lambda)^3 + 4(1 + \lambda) + 4(1 + \lambda) + 8 + 8 + 4(1 + \lambda) = 0$

$$-1-\lambda^3 - 3\lambda^2 - 3\lambda + 12\,(1 + \lambda) + 16 = 0$$
$$-\lambda^3 - 3\lambda^2 + 9\lambda + 27 = 0$$
$$\lambda^3 + 3\lambda^2 - 9\lambda - 27 = 0$$
$$\lambda^2\,(\lambda + 3) - 9(\lambda + 3) = 0$$
$$(\lambda^2 - 9)\,(\lambda + 3) = 0$$

$\Rightarrow$ $\lambda = 3, -3, -3$ are eigen values

By cayley's Hamilton theorem

$$A^3 + 3A^2 - 9A - 27I = 0$$

$\therefore$ $A^4 + 3A^3 - 9A^2 - 27A = 0$

$\Rightarrow$ $A^4 + 3A^3 - 9A^2 = 27A$

$$= \begin{bmatrix} -27 & 54 & 54 \\ 54 & -27 & 54 \\ 54 & 54 & -27 \end{bmatrix}$$

2. (*a*) $\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$

$\because$ $f(x, y)$ is bounded over the limits

$\therefore$ $\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$

$$= \int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dy \, dx$$
$$= \int_0^\infty \sin nx \int_0^\infty e^{-xy} dy \, dx$$
$$= \int_0^\infty \sin nx \left[\frac{e^{-xy}}{-x} \right]_0^\infty dx$$
$$= \int_0^\infty \sin nx \left[0 + \frac{1}{x} \right] dx$$

$\Rightarrow$ $\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy = \int_0^\infty \frac{\sin nx}{x} dx$

Hence, $\int_0^\infty \frac{\sin nx}{x} dx$ gives the imaginary part of the function *i.e.,* $\frac{\pi}{2}$.

2. (*b*) Given equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let the foot of perpendicular be A(p, q) equation of tangent with slope m passing through point A(p, q) is

$$y = mx \pm \sqrt{a^2 m^2 + b^2}$$

Where, $m = -\frac{p}{q}$

$\Rightarrow$ $\sqrt{a^2 \left(-\frac{p}{q} \right)^2 + b^2} = \frac{p^2 + q^2}{q}$

$$\sqrt{\frac{a^2 p^2 + b^2 q^2}{q^2}} = \frac{p^2 + q^2}{q}$$
$$a^2 p^2 + b^2 q^2 = (p^2 + q^2)^2$$

So, required locus is

$$a^2 x^2 + b^2 y^2 = (x^2 + y^2)^2$$

2. (*c*) $y = \sqrt{x - 2}$ on [2, 3]

by M.V.T.

$$y'(c) = \frac{y(b) - y(a)}{(b - a)}$$
$$y'(x) = \frac{1}{2\sqrt{x - 2}}$$

$\therefore$ $y'(c) = \frac{1}{2\sqrt{c - 2}} = \frac{y(3) - y(2)}{(3 - 2)}$

$\Rightarrow$ $\frac{1}{2\sqrt{c - 2}} = \frac{1 - 0}{1}$ $\Rightarrow$ $\sqrt{c - 2} = \frac{1}{2}$

$\Rightarrow$ $c - 2 = \frac{1}{4}$ $\Rightarrow$ $c = 2 + \frac{1}{4} = \frac{9}{4}$

$\therefore$ $c = \frac{9}{4}$ is that point where tangent is II to chord joining the end of curve.

2. (*d*) $T : V_3 \to V_2$ defined by

$$T(e_1) = 2f_1 - f_2$$
$$T(e_2) = f_1 + 2f_2$$
$$T(e_3) = 0f_1 + 0f_2$$

Where e_1, e_2, e_3 and f_1, f_2 are standard basis of V_3 and V_2

$$\therefore \quad M_T = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} = A \text{ (say)}$$

$\therefore$ T is a linear map such that

$$T(x) = Ax$$
$$B_1 = [(1, 1, 0), (1, 0, 1), (0, 1, 1)]$$

And $\quad B_2 = [(1, 1), (1, -1)]$

$$\therefore T(1, 1, 0) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad a + b = 3$$
$$a - b = 1$$
$$\Rightarrow \quad 2a = 4 \Rightarrow a = 2,\ b = 1$$

$$\therefore T(1, 1, 0) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T(1, 0, 1) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad a + b = 2$$
$$a - b = -1$$
$$\Rightarrow \quad 2a = 1 \Rightarrow a = \frac{1}{2},\ b = \frac{3}{2}$$

$$\therefore T(1, 0, 1) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T(0, 1, 1) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad a + b = 1$$
$$a - b = 2$$

$$\Rightarrow \quad 2a = 3 \Rightarrow a = \frac{3}{2},\ b = -\frac{1}{2}$$

$$\therefore T(0, 1, 1) = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \quad T_1 = \begin{bmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

3. (*a*) Given matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

We write A = IAI, *i.e.,*

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each elementary row or column operation of the product can be effected by subjecting the prefactor (past factor) of A to the same.

Operating $R_{1,2}$ (–1), we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_{2,3}$ (1), we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Operating $C_{3,2}$ (–4), we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

Operating $R_{2,1}$ (–2), we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

Thus, $\quad P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

And, $\quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$

Now, Inverse of matrix A *i.e.*, A^{-1}

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$\det A = \left[3(-3+4) - 3(0-2) + 4(-2-0)\right]$$

$$= \left[3(1) - 3(-2) + 4(-2)\right]$$

$$= [3 + 6 - 8) = 1$$

$$Adj\,(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -3\times1+(-4)\times(-1) & 4\times(-1)-(-3)\times1 & (-3)\times4-(-3)\times4 \\ 4\times0-2\times1 & 3\times1-4\times0 & 4\times2-4\times3 \\ 2\times-1-(3)\times0 & -3\times0-3\times(-1) & 3\times(-3)-(2)\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -3+4 & -4+3 & -12+12 \\ 0-2 & 3-0 & 8-12 \\ -2-0 & 0+3 & -9+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

3. (*b*) Let, the required Point is P = (x, y, z)

and Point O = (1, 0, 0) {given}

Thus, the problem ask us to find the minimum value of the function

$$\left|\overrightarrow{OP}\right| = \sqrt{(x-1)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{(x-1)^2 + y^2 + z^2}$$

Subject to the constraint that

$$2x + 3y + 4z - 5 = 0$$

Since $\left|\overrightarrow{OP}\right|$ has a minimum value, wherever the function

$$f(x, y, z) = (x + 1)^2 + y^2 + z^2$$

has a minimum value,

We may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint

$$2x + 3y + 4z - 5 = 0.$$

It we regard x and y as the independent variables in this equation and write z as

$$z = \frac{1}{4}(5 - 2x - 3y)$$

Our problem reduces to one of finding the point (x, y) at which the function

$$h(x, y) = f(x, y, \frac{1}{4}(5 - 2x - 3y)$$

$$= (x-1)^2 + y^2 + \left\{\frac{1}{4}(5 - 2x - 3y)\right\}^2$$

has its minimum value.

Since the domain of h is the entire x,y-plane, the first derivative test fell us that any minima that h might have must occur at points where

$$h_x = 2x + \frac{1}{4} \times 2(5 - 2x - 3y)(-2) = 0$$

$$= 2x - 5 + 2x + 3y = 0$$

$$4x + 3y = 5 \qquad ...(i)$$

$$h_y = 2y + \frac{1}{4} \times 2(5 - 2x - 3y)(-3) = 0$$

$$2y + \frac{3}{2}(2x + 3y - 5) = 0$$

$$4y + 6x + 9y - 15 = 0$$

$$6x + 13y = 15 \qquad ...(ii)$$

and, the solution

$$26y - 9y = 30 - 15$$

$$17y = 15$$

$$y = \frac{15}{17} \text{ and } x = \frac{10}{17}$$

And, $\quad z = \dfrac{1}{4}\left(5 - 2\times\dfrac{10}{17} - 3\times\dfrac{15}{17}\right)$

$$= \frac{1}{4}\left[\frac{85-65}{17}\right] = \frac{5}{17}$$

Thus, Point P = (x, y, z)

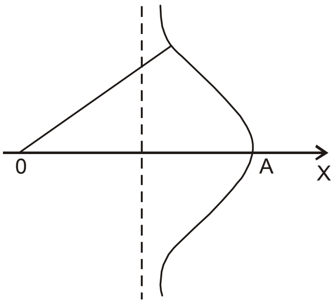
$$= \left(\frac{10}{17}, \frac{15}{17}, \frac{5}{17}\right)$$

3. (*c*) Given curve $r = 3\,(\sec\theta + \cos\theta)$

The asymptote is $r\cos\theta = 3$ *i.e.*, $x = 3$, a line parallel to the y-axis at a distance 3 from it.

$$\text{Area} = 2\int_{0}^{\frac{\pi}{2}} \frac{1}{2}\left(r_1^2 - r_2^2\right).d\theta$$

$$= \int_{0}^{\frac{1}{2}} \left[3^2(\sec\theta + \cos\theta)^2 - 3^2.\sec^2\theta\right].d\theta$$



$$= 3^2\int_{0}^{\frac{\pi}{2}} (2 + \cos^2\theta).d\theta$$

$$= 9\int_{0}^{\frac{\pi}{2}} \left[2 + \left(\frac{1+\cos 2\theta}{2}\right)\right].d\theta$$

$$= \frac{9}{2}\int_{0}^{\frac{\pi}{2}} \left(5 + \cos 2\theta\right).d\theta$$

$$= \frac{9}{2}\left[5.(\theta)\right]_0^{\frac{\pi}{2}} + \frac{9}{4}\left[\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{45}{4}\pi$$

3. (*d*) The equation of the sphere S is

$$\left(\vec{r} - \vec{a}\right).\left(\vec{r} - \vec{b}\right) = 0$$

$$\Rightarrow \quad \vec{r}^2 - \left(\vec{a} + \vec{b}\right).\vec{r} + \vec{a}.\vec{b} = 0$$

Where, $\vec{a} = (0, 1, 0)$, $\vec{b} = (3, -5, 2)$

$\Rightarrow \quad S \equiv x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0 \quad ...(i)$

Plane $U = 5x - 2y + 4z + 7 = 0 \quad ...(ii)$

The equation of the sphere through the intersection of (*i*) and (*ii*)

$S + \lambda U \equiv x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + \lambda(5x - 2y + 4z + 7) = 0 \quad ...(iii)$

Center of the sphere

$$C = \left(\frac{1}{2}(3 - 5\lambda), \lambda - 2, 1 - 2\lambda\right) \quad ...(iv)$$

For great circle, C lies on required circle i.*e.*,

$\Rightarrow \lambda(-25 - 4 - 16) + 15 + 8 + 8 + 14 = 0$

$\Rightarrow \quad \lambda = 1$

Substituting $\lambda = 1$, the equation of the required sphere is found to be

$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

5. (*a*) Given, $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2+1}$

and (*x*y) pass through (1, 2) separate variables

$$\frac{dy}{y} = -\frac{2x}{x^2+1}dx$$

Integrate on both sides

$$\int\frac{dy}{y} = -\int\frac{2x}{x^2+1}dx + c$$

Put, $x^2 + 1 = t$

$2xdx = dt$

$\therefore \quad \log y = -\int\frac{dt}{t} + c$

$\log y = -\log t + c$

$\log y = -\log(x^2 + 1) + c$

Put, $x = 1, y = 2$

$\log 2 = -\log(2) + c$

$\Rightarrow \quad c = 2\log 2 = \log 4$

$\therefore \quad \log y = -\log(x^2 + 1) + \log 4$

$\Rightarrow \quad y = \dfrac{4}{x^2+1}$

5. (*b*) **Hints:** The given differential equation is

$$\frac{d^2y}{dx^4} + 2.\frac{d^2y}{dx^2} + y = x^2.\cos x$$

$$(D^2 + 1)^2 y = x^2.\cos x$$

The auxiliary equation is,

$$(m^2 + 1)^2 = 0$$

Solving for *m*, we get

$(m^2 + 1)(m^2 + 1) = 0$

i.e., $\quad m^2 = -1 = i^2, i^2$

Therefore, $\quad m = \pm i, \pm i$

The roots are pair of complex conjugates

The complementary function is

C.F. = (A + Bx) cos$x$ + (C + Dx) sin$x$

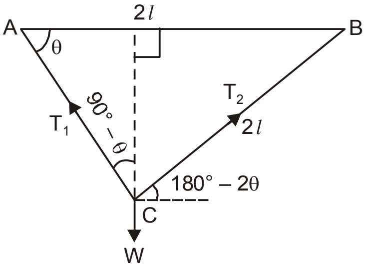
Now, we have to the particular integral

$$\text{P.I.} = \frac{x^2.\cos x}{(D^2+1)^2}$$

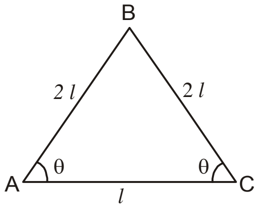
General Solution = C.F. + P.I.

$$= (A + Bx)\cos x + (C + Dx)\sin x + \frac{x^2.\cos x}{(D^2+1)^2}$$

5. (*c*) **Hint:** Let, tension in two string AC of length l and BC of length $2l$ are T_1 and T_2 respectively.



From ΔABC,



$$\angle A = \angle C = \theta \text{ (say)}$$

$$T_1 \sin (90 - \theta) = T_2 \cos (180° - 2\theta)$$

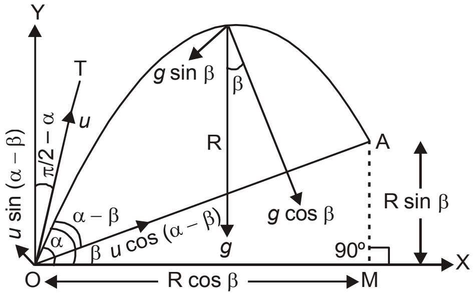
$$T_1 \cos \theta = T_2 \cos 2\theta \qquad ...(i)$$

and $T_1 \cos(90° - \theta) + T_2 \sin(180° - 2\theta) = W$

$$T_1 \sin\theta + T_2 \sin 2\theta = W \qquad ...(ii)$$

From (*i*) and (*ii*), we can get the required values of Tension in two strings.

5. (*d*) Suppose the particle strike the inclined plane at A. Let OA = R. Let T be the time of flight from O to A. As shown in the figure, the components of initial velocity of the particle along and perpendicular to the inclined plane are u cos (α – β) and u sin (α – β) respectively. Again, the component of g along the inclined is g sin β (down the plene) and the component of g perpendicular to the inclined plane is g sin β (along the downward normal to the plane OA).



Let time taken from O to A be T.

While moving from O to A, the displacement of the particle perpendicular to OA is zero. So, considering motion of the particle from O to A perpendicular to OA and using the formula "$s = ut + (1/2) ft^2$",

We have $\quad s = u.t + \frac{1}{2}a.t^2$

$0 = u \sin (\alpha - \beta).T - (1/2)g \cos \beta.T^2$ or $T\{g \cos \beta.T - 2u \sin (\alpha - \beta)\} = 0$

Since T = 0 gives time from O to O, hence time from O to A is given by

∴ T = time of flight up the inclined plane

$$= \frac{2u\sin(\alpha-\beta)}{g\cos\theta} \qquad ...(i)$$

Since the particle strikes the plane OA at right angles at A, hence the direction of velocity of the particle at A is perpendicular to OA and so the component of velocity of the particle at A along OA is zero.

So, considering the motion of the particle from O to A along OA and using the formula.

$$V = u + a.t$$

$$O = u \cos(\alpha - \beta) - g \sin \beta.T$$

$$T = \frac{u}{g}.\frac{\cos(\alpha-\beta)}{\sin\beta} \qquad ...(ii)$$

From (*i*) and (*ii*), we have

$$\frac{2u}{g}.\frac{\sin(\alpha-\beta)}{\cos\beta} = \frac{u}{g}.\frac{\cos(\alpha-\beta)}{\sin\beta}$$

$$2 \tan (\alpha - \beta) = \cot \beta$$

5. (*e*) Given that $\vec{F} = (zx\sin yz + x^3)\hat{i}$

$$+\cos yz\,\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}$$

$$\text{div F} = \frac{\partial}{\partial x}(xz\sin(yz)+x^3)+\frac{\partial}{\partial y}\big(\cos(yz)\big)$$

$$+\frac{\partial}{\partial z}\left(3zy^2 - e^{x^2+y^2}\right)$$

$$= (z\sin(yz) + 3x^2) + (-z\sin(yz))$$

$$+ (3y^2) = 3x^2 + 3y^2$$

Thus, we have from the divergence theorem

$$\iint_S \text{F}.d\text{S} = \iiint_E \text{div F } d\text{V}$$

$$= \iint_D \int_0^{4-x^2-y^2}(3x^2+3y^2)dz\,d\text{A}$$

where D is the disk $x^2 + y^2 \le 4$ in the xy-plane. Thus, we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\iint_S \text{F}.d\text{S} = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r^2)r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2(12r^3-3r^5)dr\,d\theta$$

$$= \int_0^{2\pi}\left[3r^4-\frac{1}{2}r^6\right]_0^2 d\theta$$

$$= \int_0^{2\pi}(48-32)d\theta = 32\pi.$$

6. (*a*) Let u be the initial velocity of projection. Since the greatest height is 40 m, we have

$$0 = u^2 - 2g.40$$

$$\therefore \qquad u = \sqrt{2g\times 40} = 28 \text{ m}$$

Let T be the time after the first stone starts before the two stones meet.

Then the distance traversed by the first stone in time T = distance traversed by the second stone in time (T – 2).

$$\therefore \quad 28\text{T}-\frac{1}{2}g\text{T}^2 = 28(\text{T}-2)-\frac{1}{2}g(\text{T}-2)^2$$

$$= 28\text{T}-56-\frac{1}{2}g(\text{T}^2-4\text{T}+4)$$

$$\therefore \qquad 56 = \frac{1}{2}g(4\text{T}-4) = 4.9(4\text{T}-4)$$

$$\therefore \qquad \text{T} = 3\frac{6}{7} \text{ seconds.}$$

Also the height at which they meet

$$= 28\times\frac{27}{7}-\frac{1}{2}\times 9.8\times\left(\frac{27}{7}\right)^2$$

$$= 108 - 72.9 = 35.1 \text{ m}$$

The first stone will be coming down and the second stone going upwards.

6. (*b*) **Hints:** Let, $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}$$

and $$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Now, $x^2.\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y = 0$

$x^2.m(m-1).x^{m-2} + x.mx^{m-1} - x^m = 0$

$x^m\{m(m-1) + m - 1\} = 0$

$x^m\{m^2 - 1\} = 0$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

The general solution is then

$$y = c_1e^{-x} + c_2.e^x$$

6. (*c*) Given Data:

Diameter of pipe :

d = 80 mm = 0.08 m

Gauge pressure of water:

p = 60 kPa = 60 × 10^3 pa or N/m^2

Mean velocity of water:

V = 2 m/s

Datum head:

z = 7 m

According to Bernoulli's equation:

Total head of water :

$$\text{H} = \frac{p}{\rho g}+\frac{\text{V}^2}{2g}+z$$

$$= \frac{60\times 10^3}{1000\times 9.81}+\frac{(2)^2}{2\times 9.81} + 7$$

$$= 6.11 + 0.20 + 7$$

$$= 13.31 \text{ m of water}$$

6. (*d*) $$\int_C \text{F}\cdot dr = \oint_C(\text{F}_x dx+\text{F}_y dy+\text{F}_z dz)$$

$$= \oint_C\left\{(2x-y)dx - yz^2dy - y^2z\,dz\right\}$$

But the boundary C of S is a circle in the *xy*-plane of radius unity and centre at (0,0,0); Hence the parametric equations of C are $x = \cos\theta$, $y = \sin\theta$, $z = 0$ where θ varies from 0 to 2π.

Thus,

$$\int_C \mathrm{F}\cdot dr = \int_{\theta=0}^{2\pi}\{(2\cos\theta-\sin\theta)(-\sin\theta\, d\theta)-0-0\}$$

$$= \int_0^{2\pi}(2\cos\theta-\sin\theta)\sin\theta\, d\theta$$

$$= \int_0^{2\pi}(\sin 2\theta-\sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi}\left\{\sin 2\theta-\frac{1-\cos 2\theta}{2}\right\}d\theta$$

$$= -\left[\frac{\cos 2\theta}{2}-\frac{\theta}{2}+\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi$$

$$\text{Further } \nabla\times \mathrm{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2z \end{vmatrix} = k$$

Hence, $\iint_S (\nabla\times \mathrm{A}).ds = \iint_S k.ds = \iint_R dx\, dy$

where R is the projection of S on *xy*-plane and $k.ds = dx\,dy$ = projection of *ds* on *xy*-plane.

Thus, R is $x^2 + y^2 = 1$

$$\therefore \iint_R dx\,dy = 4\int_0^1\int_0^1 \sqrt{(1-x^2)}\,dx\,dy$$

$$= 4\int_0^1 \sqrt{(1-x^2)}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{(1-x^2)}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\left[\frac{\pi}{4}\right] = \pi$$

Thus, from above, we have

$\int_C \mathrm{A}.dr = \iint_S (\nabla\times \mathrm{A}).ds$ and hence Stoke's Theorem is verified.

PAPER-II

1. (*a*) Since for bijective, $f : \mathrm{X} \to \mathrm{Y}$, and bijective $g : \mathrm{X} \to$, *gof* is also bijective, the closure property follows. For bijective function $f : \mathrm{X} \to \mathrm{Y}$, $g : \mathrm{X} \to \mathrm{X}$, $h : \mathrm{X} \to \mathrm{X}$, it is easy to observe $ho(gof) = (hog)of$ and hence associativity follows. The identity function $\mathrm{I}_x : \mathrm{X} \to \mathrm{X}$ defined by $\mathrm{I}_x(x) = x$ is a bijective function and $fo\mathrm{I}_x = \mathrm{I}_x of = f$ and hence is the identity element. Further, since every bijective function f possesses an inverse function which is also bijective, the existence of inverse follows. Hence the result.

Note: If X is a finite set having *n* elements, a bijective function from X onto X is called a permutation and the group of all bijections, *i.e.,* permutations is called the *symmetric group of degree n*. Note the order of this group is *n*!

1. (*b*) Here, $f_n(x) = \dfrac{\sin(nx+n)}{n}$

and so $|f_n(x)| = \left|\dfrac{\sin(nx+n)}{n}\right|$

$$= \frac{|\sin(nx+n)|}{n}$$

$$\leq \frac{1}{n} = \mathrm{M}_n(\text{say}) \quad \forall x \in \mathrm{R}$$

Since, $\sum_{n=1}^{\infty}\mathrm{M}_n = \sum_{n=1}^{\infty}\frac{1}{n}$ is convergent

So, by Weierstrass's M test, the given series is converges uniformly for all real values of *x*.

1. (*c*) Given function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

$$f_x(x, y) = 3x^2 - 3$$

$$f_y(x, y) = 3y^2 - 12$$

For stationary points $f_x = 0$ and $f_y = 0$

$\Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad 3(x^2 - 1) = 0 \quad x = \pm 1$

and $\quad 3y^2 - 12 = 0$

$\Rightarrow \quad 3(y^2 - 4) = 0 \quad y = \pm 2$

Thus, stationary points are

$(x, y) = (1, 2), (-1, 2)\ (1, -2)$ and $(-1, -2)$

Now, $\quad f_{xx}(x, y) = 6x, \ f_{yy}(x, y) = 6y$

and $\quad f_{xy}(x, y) = 0$

Let, $\quad D = f_{xx} \cdot f_{yy} - f^2_{x.y}$

$= 6x.6y - 0$

$= 36xy$

Test for maximum, minimum and siddle point

	(1, 2)	(–1, 2)	(1, –2)	(–1, –2)
f_{xx}	6 × 1 = 6	6 × – 1 = – 6	6 × 1 = 6	6 × – 1 = – 6
f_{yy}	6 × 2 = 12	6 × 2 = 12	6 × – 2 = – 12	6 × – 2 = – 12
D	36 × 1 × 2 = 72	36 × – 1 × 2 = – 72	36 × 1 × (–2) = – 72	36 × (–1) × (–2) = 72

Now, for point (1, 2)

$D > 0$ and $f_{xx} > 0, f_{yy} > 0$

Hence, point (1, 2) is minimum

$f(1, 2) = 1^3 + 2^3 - 3 \times 1 - 12 \times 2 + 20$

$= 1 + 8 - 3 - 24 + 20 = 2$

$f_{\min} = 2$

For point (–1, –2),

$D > 0, f_{xx} < 0$ and $f_{yy} < 0$

Hence, point (–1, –2) is maximum

$f(-1, -2) = (-1)^3 + (-2)^3 -3\ (-1)$

$-12\ (-2) + 20$

$= -1 - 8 + 3 + 24 + 20$

$= 38$

$f_{\max} = 38$

For points (–1, 2), and (1, –2)

$D < 0$, thus points (–1, 2) and (1, –2) are siddle points.

1. (*d*) $\quad u = e^{-x}\ [(x^2 - y^2) \cos y + 2xy \sin y]$

$$\frac{\partial u}{\partial x} = e^{-x}\ 2x \cos y - e^{-x}\ (x^2 - y^2) \cos y$$
$$-e^{-x}\ 2xy \sin y + 2e^{-x}\ y \sin y$$

$$\frac{\partial u}{\partial y} = e^{-x}\ (-x^2 \sin y + y^2 \sin y$$
$$+ 2xy \cos y + 2x \sin y)$$

$$\left(\frac{\partial u}{\partial x}\right)_{z,0} = 2ze^{-z} - e^{-z}z^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{z,0} = 0$$

$$f'(z) = 2ze^{-z} - e^{-z}z^2$$

$$f(z) = \int (2z - z^2)e^{-z}dz$$

$$= -2ze^{-z} - 2e^{-z} + z^2e^{-z} + 2ze^{-z} + 2e^{-z}$$

$$= z^2e^{-z} + C$$

1. (*e*) The set of all feasible solutions to the linear programming problem is a convex set.

For the trivial case where the feasible region S is a singleton, the theorem is of course true. For the more general case, we need to show that every convex combination of any two feasible solutions is also feasible. Assume that there are at least two solutions x_1 and x_2 with

$Ax_1 = b, x_1 \geq 0$

and $\quad Ax_2 = b, x_2 \geq 0$

For $0 \leq \alpha \leq 1$,

Let, $x = \alpha x_1 + (1 - \alpha)x_2$ be any convex combination of x_1 and x_2. We note that all elements of the vector x are non-negative; *i.e.*, $x \geq 0$. Substituting for x in the linear equalities gives

$$Ax = A[\alpha x_1 + (1 - \alpha)x_2]$$
$$= \alpha Ax_1 + (1 - \alpha)Ax_2$$
$$= \alpha b + (1 - \alpha)b = b$$

which shows that x is feasible.

2. (*a*) The order of any element other than the identity in G is 2, 3 or 6. If there is an element of order 6, then $G \simeq Z_6$.

Now assume that there is no element or order 6. We show that it is not possible that every element of G other than the identity has order 2.

If $a^2 = e$ for all elements in a group, then the group should be abelian. Now, assume to the contrary that every element of G other than the identity has order 2, and pick elements $a \neq b$ in G such that $a \neq e$ and $b \neq e$. Then $H : = \{e, a, ab\}$ should be a subgroup of G since it is closed under group operation ($a^2 = e$, $b^2 = e$, $(ab)^2 = e$, $ab \in H$, $ba = ab \in H$, $a(ab) = b \in H$, $b(ab) = (ab)b = a \in H$). But this is not possible since a group of order 6 cannot have a subgroup of order 4.

Pick now an element a of order 3 in G and an element

$$b \notin < a > = \{e, a, ab^2\}$$

Then e, a, a^2, b, ba, ba^2 are all distinct, and therefore they should be all the elements of G. To see this, note that clearly e, a, a^2, b are all distinct. $ba \neq b$, $ba \neq a$, $ba \neq e$ (since otherwise b would be the inverse of a which is a^2, but we assumed $b \notin \{e, a, a^2\}$), and $ba \neq a^2$ (since $b \neq a$).

Similarly, ba^2 is not equal to any of the other 5 elements, so

$$G = \{e, a, a^2, b, ba, ab^2\}.$$

We now show that b has order 2. We assumed G does not have any element of order 6. If $b^3 = e$, then

$$b^2 \in \{a, a^2, b, ab, ab^2\}$$

Clearly, b^2 cannot be equal to b, ab, or ab^2. If $b^2 = a$, then

$$e = b^3 = b(b^2) = ba$$

which is not possible. Similarly, if $b^2 = a^2$, then

$$b = be = b(b^3) = b^4 = a^4 = e$$

which is not possible. So the only possibility is that $b^2 = e$, that is b has order 2.

Now ab is either equal to ba or ba^2. This is because $ab \neq e, a, a^2$ or b. If $ab = ba$, then the order of ab should be 6: if $(ab)^2 = e$,

Then, $a^2 = a^2b^2 = (ab)^2 = e$

which is not possible.

If, $(ab)^3 = e$, then

$$e = (ab)^3 = a^3b^3 = eb = b.$$

So, ab has order 6, but we assumed there is no element of order 6, so $ab = ba^2$.

Now, we have the complete multiplication table in G and we can use that to show that G is isomorphic to S_3. We can also give an isomorphism explicitly.

We define

$$\phi : G \rightarrow S_3$$

by $\phi(a) = (1, 2, 3)$, $\phi(a^2) = (1, 3, 2)$, $\phi(b) = (1, 2)$, $\phi(ba) = (3, 2)$,

and $\phi(ba^2) = (1, 3)$

2. (*b*) $|G| = pq$.

Let n_p be the number of Sylow p-subgroups of G. Then $n_p \mid pq$ and $n_p = pr + 1$, $(r = 0, 1, 2, ...)$

$pq = sn_p = s(pr + 1)$ for some positive integers

$$s = pq - spr$$
$$= p(q - sr) = pt \text{ (say)},$$

Where, $t = q - sr < p$ as $q < p$

Hence, $pq = s(pr + 1) = pt\ (pr + 1)$

$$q = t(pr + 1)$$
$$r = 0 \qquad \text{(as } q < p)$$
$$n_p = 1$$

G contains only one Sylow p-subgroups of G H (say) such that $|H| = p \Rightarrow$ H is normal in G.

Again let n_q be the number of Sylow q-subgroups of G.

Then, $n_q|pq$ and $n_q = qr' + 1$, $(r' = 0, 1, 2, ...,)$

$$pq = s'n_q = s'(qr' + 1)$$

For some positive integer s'

$$s' = q\ (p - s'r') = qt' \text{ (say)}$$

Where, $t' = p - s'r'$

Hence, $pq = qt'(qr' + 1)$

$$p = t'(qr' + 1)$$

Since,

$$(q|p - 1),\ p = t'\ (qr' + 1)$$
$$r' = 0 \Rightarrow n_q = 1$$

G contains only one Sylow q-subgroup K (say) such that $|K| = q$

Hence, K is normal in G.

Let, any group G of order 15 is cyclic

Then, $|G| = 15 = 5.3$

Let, n_5 be the number of Sylow 5-subgroups. Then by third Sylow theorem, $n_5|5$ and $n_5 = 5r + 1$, $r = 0, 1, 2, ...$

$$n_5 = 1$$

G contains only one Sylow 5-subgroup H(say) such that $|H| = 5$

H is a normal subgroup.

Similarly, G contains only one Sylow 3-subgroup K(say) such that K is normal in G and $|K| = 3$.

Then, H and K are cyclic as they are of prime orders.

Suppose, $H = <x>$ and $K = <y>$ for some $x \in H$ and $y \in K$ such that $x^5 = 1$ and $y^3 = 1$.

Now $H \cap K$ is a subgroup of both H and K

$$\frac{|H \cap K|}{|H|} \text{ and } \frac{|H \cap K|}{|K|}$$

$$\frac{|H \cap K|}{5} \text{ and } \frac{|H \cap K|}{3}$$

$$|H \cap K) = 1 \Rightarrow H \cap K = \{1\}$$

$hk = kh \; \forall \; k \in K$ and $\forall \; h \in H$

(as both H and K are normal in G)

$$xy = yx$$

$$(xy)^{15} = x^{15} \, y^{15} = (x^5)^3 \, (y^3)^5 = 1$$

$xy \in G$ such that $(xy)^{|G|} = (xy)^{15} = 1$

G is a cyclic group.

2. (*c*) Given that $R = \{a + b\sqrt{-5} \mid a, b \in Z\}$,

$B = 1 + 2\sqrt{-5}, \alpha = 3$. Any common divisor of α and β must have a norm which divides $N(\alpha) = 21$ and $N(\beta) = 9$, *i.e.,* it must have the norm 1 or 3.

Since no element can have the norm 3, the *gcd* has the norm 1, *i.e.,* it must be a unit. Since a unit always divides α and β, the *gcd* of α and β is unit. But there is no element in R with norm 1, Hence α and β have no *gcd* in R.

5. (*a*) $\phi(u, v) = 0 \quad u = x, yz$

$$v = x + y + z$$

$\therefore$ *p.d.* equation is given by

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ yz & xz & xy \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$p(xz - xy) - q(yz - xy) - 1\,(yz - xz) = 0$$

$$xp(z - y) - yq(z - x) - z(y - x) = 0$$

$$px(z - y) + qy(x - z) = z(y - x)$$

5. (*b*) $xy^2 \dfrac{\partial z}{\partial x} + y^3 \dfrac{\partial z}{\partial y} = (zxy^2 - 4x^3)$

$\therefore$ Lagrange's A equation

$$\underset{\text{I}}{\frac{dx}{xy^2}} = \underset{\text{II}}{\frac{dy}{y^3}} = \underset{\text{III}}{\frac{dz}{zxy^2 - 4x^3}}$$

By I and II

$$\frac{dx}{xy^2} = \frac{dy}{y^3} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \quad \log x = \log y + \log c_1$$

$$\Rightarrow \quad x = yc_1$$

Now II and III,

$$\frac{dy}{y^3} = \frac{dz}{zxy^2 - 4x^3}$$

Put $\quad x = yc_1$

$$\therefore \quad \frac{dy}{y^3} = \frac{dz}{zyc_1y^2 - 4c_1^3y^3}$$

$$\Rightarrow \quad \frac{dy}{y^3} = \frac{dz}{y^3(zc_1 - 4c_1^3)}$$

$$\Rightarrow \quad dy = \frac{dz}{(zc_1 - 4c_1^3)}$$

$$\Rightarrow \quad y = \frac{1}{c_1}\log(zc_1 - 4c_1^3) + c_2$$

$$\Rightarrow \quad c_1y = \log(zc_1 - 4c_1^3) + c_2$$

$$\Rightarrow \quad x = \log\left(\frac{zx}{y} - \frac{4x^3}{y^3}\right) + c_2$$

$$\Rightarrow \quad c_2 = x - \log\left(\frac{zx}{y} - \frac{4x^3}{y^3}\right)$$

$\therefore$ General Solution

$$c_2 = \phi(c_1)$$

$$\Rightarrow x - \log\left(\frac{zx}{y} - \frac{4x^3}{y^3}\right) = \phi\left(\frac{x}{y}\right)$$

5. (*c*) Computer Implementation of the Newton-Raphson Procedure

In writing a procedure for the computer to implement the Newton-Raphson algorithm, care must be taken to provide for all possible contingencies. To prevent the program getting stuck in a loop an alternative exit after completing a pre-assigned number of cycles must be provided. Further, if $\phi'(x)$ becomes small anytime during the iterative process it may not converge. Some check should thus be made on this. A computer oriented procedure is described below for implementing the Newton-Raphson technique.

Algorithm: Newton-Raphson Method

```
1. Read $x_0$, epsilon, delta, $n$
   Remarks: $x_0$ is the initial guess, epsilon is the prescribed relative error, delta is the prescribed lower bound for $\phi'$ and $n$ the maximum number of iterations to be allowed. Statements 3 to 8 are repeated until the procedure converges to a root or interations equal $n$.
2. for $i = 1$ to $n$ in steps of 1 do
3.     $\phi_0 \rightarrow \phi(x_0)$
4.     $\phi_0' \rightarrow \phi'(x_0)$
5.     if $|\phi'_0| \leq$ delta then GOTO 11
6.     $x_1 \leftarrow x_0 - (\phi_0/\phi'_0)$
7.     if $|x_1 - x_0)/x_1| <$ epsilon then GOTO 13
8.     $x_0 \leftarrow x_1$ end for
9. Write 'Does not converge in $n$ iterations', $\phi_0$, $\phi'_0$, $x_0$, $x_1$
10. Stop
11. Write 'Slope too small' $x_0$, $\phi_0$, $\phi_0'$
12. Stop
13. Write 'convergent solution', $x_1$ $\phi(x_1)$, $i$
14. Stop
```

5. (*d*) $f(1) = 2, f(2) = 4, f(3) = 8, f(z) = 128$

∴ By Lagrange interpolation

$$f(x) = \frac{(x-2)(x-3)(x-7)}{(1-2)(1-3)(1-7)}f(1)$$

$$+\frac{(x-1)(x-3)(x-7)}{(2-1)(2-3)(2-7)}f(2)$$

$$+\frac{(x-1)(x-2)(x-7)}{(3-1)(3-2)(3-7)}f(3)$$

$$+\frac{(x-1)(x-2)(x-3)}{(7-1)(7-2)(7-3)}f(7)$$

$$= \frac{(x^2-5x+6)(x-7)}{-1\times-2\times-6}(2)$$

$$+\frac{(x^2-4x+3)(x-7)}{1\times(-1)(-5)}4$$

$$+\frac{(x^2-3x+2)(x-7)}{2\times1\times(-4)}8$$

$$+\frac{(x^2-3x+2)(x-3)}{6\times5\times4}128$$

$$= \frac{1}{6}(x^2-5x+6)(x-7)$$

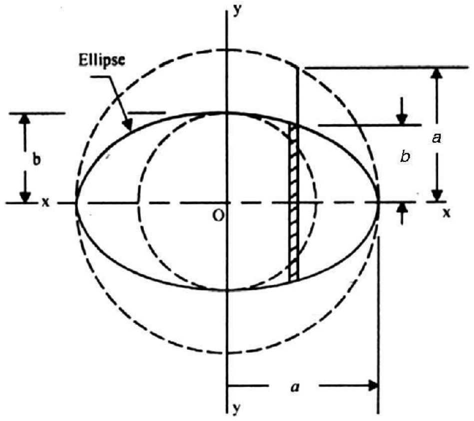
$$+\frac{4}{5}(x^2-4x+3)(x-7)$$

$$-(x^2-3x+2)(x-7)$$

$$+\frac{16}{15}(x^2-3x+2)(x-3)$$

5. (*e*) Figure shows an ellipse with semi major axis = a and semi minor axis = b.

The moment of inertia of circle radius a about x-x-axis = $\frac{\pi a^4}{4}$.



Figure

Now, the radius is reduced to b in the direction y, when we consider the ellipse. Therefore, moment of inertia can be reduced by $\frac{b^3}{a^3}$ as is obvious from the figure $y < y'$.

Moment of inertia of ellipse about axis *xx*,

$$I_{xx} = \frac{\pi a^4}{4}\left(\frac{b^3}{a^3}\right) = \frac{\pi ab^3}{4}$$

Similarly, moment of inertia of ellipse about axis *yy*,

$$I_{yy} = \frac{\pi b^4}{4}\left(\frac{a^3}{b^3}\right) = \frac{\pi a^3 b}{4}$$

Polar moment of inertia of ellipse,

$$I_{00} = I_{xx} + I_{yy}$$

$$I_{00} = \frac{\pi ab}{4}(a^2 + b^2)$$

7. (*a*) Let, $f(x) = \frac{1}{\sqrt{1-x^2}}$ and $n = 12$

$$h = \frac{0.6-0}{12} = \frac{0.6}{12} = 0.05$$

For, $x = 0$

$$f(0) = \frac{1}{\sqrt{1-0}} = 1$$

For, $x = 0.05$

$$f(0.05) = \frac{1}{\sqrt{1-(0.05)^2}}$$

$$= \frac{1}{0.9987}$$

$$= 1.00130$$

Similarly, we can tabulate $f(x)$ for all the values of x as follows.

So, we have

x	$f(x)$
0	1
0.05	1.0013
0.10	1.005
0.15	1.0114
0.20	1.0206
0.25	1.0328
0.30	1.0483
0.35	1.0675
0.40	1.0911
0.45	1.1198
0.50	1.1547
0.55	1.1974
0.60	1.25

From Simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x) = \frac{h}{3}\left[(y_0 + y_{12}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11}) + 2(y_2 + y_4 + y_6 + y_8 + y_{10})\right]$$

$$= \frac{0.05}{3}[(1 + 1.25) + 4\,(1.0013 + 1.0114 + 1.0328 + 1.0675 + 1.1198 + 1.1974) + 2(1.0050 + 1.0206 + 1.0483 + 1.0911 + 1.1547)]$$

$$= \frac{0.05}{3}[2.25 + 4 \times 6.4302 + 2 \times 5.3197]$$

$$= \frac{0.05}{3}[2.25 + 25.7208 + 10.6394]$$

$$= \frac{0.05}{3}[38.6102]$$

$$= 0.6435$$

7. (*b*) Suppose, $f(x) = N - x^3 \Rightarrow f'(x) = -3x^2$,

Where, $N = 10$

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{N - x_n^3}{-3x_n^2}$$

$$= \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

But $8 < 10 < 27 \Rightarrow (8)^{1/3} < (10)^{1/3} < (27)^{1/3}$

$\Rightarrow \quad 2 < (10)^{1/3} < 3$

i.e., cube root of 10 lies between 2 and 3.

Taking $x_0 = 2.5$, we have

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{N}{x_0^2}\right] = \frac{1}{3}\left[2 \times 2.5 + \frac{10}{(2.5)^2}\right] = 2.2$$

$$x_2 = \frac{1}{3}\left[2x_1 + \frac{N}{x_1^2}\right] = \frac{1}{3}\left[2 \times 2.2 + \frac{10}{(2.2)^2}\right] = 2.155$$

$$x_3 = \frac{1}{3}\left[2x_2 + \frac{N}{x_2^2}\right] = \frac{1}{3}\left[2 \times 2.155 + \frac{10}{(2.155)^2}\right]$$

$$= 2.15466$$

So that $\sqrt[3]{10} = 2.15466$ approx.

7. (*c*) Let us choose $h = 0.1$ and calculate various values as follows:

Step I:

$x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\Rightarrow\ k_1 = hf(x_0, y_0) = 0.1\ f(0,\ 1)$

$= 0.1000$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$= 0.1\ f(0.05, 1.1) = 0.1152$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$

$= 0.1 \times f(0.05,\ 1.1152)$

$= 0.1168$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

$= 0.1\ f(0.1,\ 1.1168) = 0.1347$

and, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347)$

$= 0.1165$

So that,

$y(0.1) = y_0 + k = 1.1165$

Step II:

$x_1 = x_0 + h = 0.1,\ y_1 = 1.1165,\ h = 0.1$

$\Rightarrow\ k_1 = hf(x_1, y_1) = 0.1\ f(0.1, 1.1165)$

$= 0.1347$

$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$

$= 0.1\ f(0.15,\ 1.1838) = 0.1551$

$k_3 = hf\left(k_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$

$= 0.1\ f(0.15,\ 1.194) = 0.1576$

$k_4 = hf(x_1 + h,\ y_2 + k_3)$

$= 0.1\ f(0.2,\ 1.1576) = 0.1823$

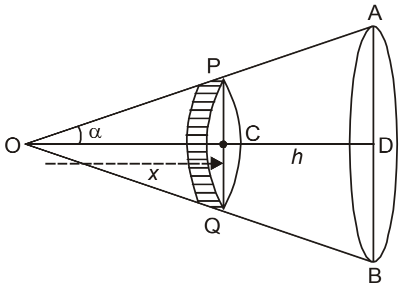
and $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$

So that,

$y(0.2) = y_1 + k = 1.2736$

8. (*a*) Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a.

If α is the semi-vertical angle and ρ the density of the cone, then



$$M = \frac{1}{3}\pi\rho h^3 \tan^2 \alpha \qquad ...(1)$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

Mass δm of this disc $= \rho\pi x^2 \tan^2 \alpha\delta x$.

M.I. of this elementary disc about the axis OD of the cone

$$= \frac{1}{3}\delta m \mathrm{CP}^2 = \frac{1}{2}(\rho\pi x^2 \tan^2 \alpha\delta x)x^2 \tan^2 \alpha$$

$$= \frac{1}{2}\rho\pi x^3 \tan^4 \alpha\delta x.$$

$\therefore$ M.I. of the whole cone about the axis OD

$$= \int_0^h \frac{1}{2}\rho\pi\, x^4 \tan^4 \alpha dx = \frac{\rho\pi}{10}h^5 \tan^4 \alpha$$

$$= \frac{3M}{\pi h^3 \tan^2 \alpha}.\frac{\pi}{10}h^5 \tan^4 \alpha,$$

substituting for ρ from (1)

$$= \frac{3}{10}Mh^3 \tan^2 \alpha = \frac{3}{10}Ma^2,$$

Since $\tan \alpha = a/h$.

8. (*b*) Kinetic energy = T = $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2}ma^2\{[(1-\cos\theta)\dot{\theta}]^2 + [-\sin\theta\dot{\theta}]^2\}$$

$$= ma^2(1-\cos\theta)\dot{\theta}^2$$