CSE - 2017

Q. Find the differential eqn representing all the circles in xy-plane.

sol: Eqn of circle in xy-plane

$$x^2+y^2+2gx+2fy+C=0$$

On differentiating w.r.t. to x

$$2x+2yy'+2g+2fy'=0$$

$$\Rightarrow \quad yy'+fy' = -(x+g)$$

Again differentiating

$$(y')^2+yy''+fy''=-1$$

$$\Rightarrow \quad (y')^2+yy''+1 = -fy''$$

$$\Rightarrow \quad \frac{(y')^2+yy''+1}{y''} = -f$$

Again differentiating

$$\frac{(2y'y''+y''y'+yy''')y''-y'''((y')^2+yy''+1)}{(y'')^2}=0$$

$$\Rightarrow \quad \frac{(3y'y''+yy''')y''-y'''((y')^2+yy''+1)}{(y'')^2}=0$$

$$\Rightarrow \quad \cancel{3y'(y'')^2+yy''y'''-y'''(y')^2-yy''y'''-y'''=0}$$

$$\cancel{3y'(y'')^2+yy''y'''-y'''(y')^2}$$

$$\Rightarrow \quad 3y'(y'')^2 = y'''(1+(y')^2)$$

$$y''' = \frac{3y'(y'')^2}{1+(y')^2}$$

$$\Rightarrow \quad \boxed{\frac{d^3y}{dx^3} = \frac{3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2}{1+(dy/dx)^2}}$$ is the required equation

Q 2 Suppose streamlines are given by family of curves $xy = c$. Find the equipotential lines that is orthogonal to the family of curves representing the streamlines.

sol. Given trajectory of streamlines $xy = c$

$$\therefore \quad x\frac{dy}{dx} + y = 0 \qquad \text{(By differentiating w.r.t. } x\text{)}$$

Replacing $\frac{dy}{dx}$ by $-1/(dy/dx)$ to obtain the orthogonal trajectories.

$$-x\frac{dx}{dy} + y = 0$$

$$\Rightarrow \quad y = \frac{x\,dx}{dy}$$

$$\Rightarrow \quad y\,dy = x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$\Rightarrow \quad \boxed{y^2 - x^2 = C_1'} \qquad \text{where } C_1' = 2C_1$$

is the required orthogonal trajectory which represents equipotential lines.

Q5 Given simultaneous linear differential equations:
$(D+1)y = z+e^x$ and $(D+1)z = y+e^x$ when y and z are functions of independent variable x and $D \equiv \frac{d}{dx}$

Sol. Given

$$(D+1)y = z+e^x \quad -(1)$$
$$(D+1)z = y+e^x \quad -(2)$$

By $(D+1)\times(2)$, we get

$$(D+1)^2 z = (D+1)y + (D+1)e^x$$
$$\Rightarrow (D+1)^2 z = (D+1)y + D(e^x) + e^x$$

Using (1)

$$(D+1)^2 z = z + e^x + e^x + e^x$$
$$\Rightarrow (D+1)^2 z - z = 3e^x$$
$$\Rightarrow (D^2+2D)z = 3e^x$$

Auxiliary eqn: $m^2+2m = 0$
$m(m+2) = 0$
$m = -2, 0$

$$\therefore \text{C.F.} = C_1 e^{-2x} + C_2 e^{0x} = C_1 e^{-2x} + C_2$$

$$\text{P.I.} = \frac{1}{D^2+2D} e^{3x} = e^{3x}\frac{1}{3^2+2(3)} = \frac{e^{3x}}{15}$$

$$\therefore Z = \text{C.F.} + \text{P.I} = (C_1 e^{-2x} + C_2) + \frac{e^{3x}}{15}$$

Putting value of z in (1).

$$(D+1)y = C_1 e^{-2x} + C_2 + \frac{e^{3x}}{15} + e^x$$
$$\frac{dy}{dx} + y = C_1 e^{-2x} + C_2 + \frac{e^{3x}}{15} + e^x$$

I.F. $= e^{\int 1 dx} = e^x$

$\therefore$ solution of y is $y(e^x) = \int e^x\left(C_1 e^{-2x} + C_2 + \frac{e^{3x}}{15} + e^x\right) dx$

$$\Rightarrow y(e^x) = \int C_1 e^{-x} + C_2 e^x + \frac{e^{4x}}{15} + e^{x}\, dx$$

$$\Rightarrow y(e^x) = -C_1 e^{-x} + C_2 e^x + \frac{e^{4x}}{60} + e^x + C_3$$

$\therefore \quad y = -c_1 e^{-2x} + c_2 + \frac{e^{3x}}{60} + c_3 e^{-x} + 1$

and

$Z = c_1 e^{-2x} + c_2 + \frac{e^{3x}}{15}$

are solutions of the given equations.

Q4 Growth rate of population of bacteria at any time t is proportional to the amount present at that time and population doubles in one week, then how much bacteria can be expected after 4 weeks? (8 marks)

sol. Let N denote the population of bacteria at any time t. and N_0 be initial population.

then given

$$\frac{dN}{dt} \propto N$$

$$\Rightarrow \quad \frac{dN}{dt} = kN \qquad (k \text{ is constant})$$

$$\Rightarrow \quad \frac{dN}{N} = k\,dt$$

$$\Rightarrow \quad \ln N = kt + \ln C$$

$$\Rightarrow \quad \boxed{N = Ce^{kt}}$$

N_0 = Initial population = Population when $t = 0$

$$\therefore \quad N_0 = Ce^{k(0)} = C$$

$$\Rightarrow \quad N = N_0 e^{kt}$$

As population doubles in 1 week

$$\therefore \quad 2N_0 = N_0 e^{k(1)}$$

$$\Rightarrow \quad e^k = 2$$

$$\boxed{k = \ln 2}$$

$\therefore$ Population after 4 weeks $= N = N_0 e^{(\ln 2)4} = N_0 e^{\ln(2^4)}$

$$\therefore \quad \boxed{N = N_0(2^4) = 16 N_0}$$

$\therefore$ 16 times the initial population of bacteria will be present after 4 weeks.

Q) Consider the differential equation $xyp^2 - (x^2+y^2-1)p + xy = 0$ where $p \equiv \frac{dy}{dx}$. Substituting $u = x^2$ and $v = y^2$ reduce the equation to Clairaut's form in terms of u, v and $p' = \frac{dv}{du}$.
[Hence or otherwise solve the equation]

Sol. Given $xyp^2 - (x^2+y^2-1)p + xy = 0$
multiply the equation by $\frac{y}{x^3}$

$$\Rightarrow \frac{y^2p^2}{x^2} - (x^2+y^2-1)\frac{py}{x^3} + \frac{y^2}{x^2} = 0$$

Let $u = x^2 \Rightarrow du = 2x\,dx$
$v = y^2 \Rightarrow dv = 2y\,dy$

$$\therefore \frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \left(\frac{y}{x}\right)\frac{dy}{dx} = \frac{py}{x}$$

$\therefore \boxed{p' = \frac{py}{x}}$, Putting this in equation.

$$\left(\frac{py}{x}\right)^2 - \left(1 + \frac{y^2}{x^2} - \frac{1}{x^2}\right)\left(\frac{py}{x}\right) + \frac{y^2}{x^2} = 0$$

$$(p')^2 - \left(1 + \frac{v}{u} - \frac{1}{u}\right)p' + \frac{v}{u} = 0$$

$$(p')^2 - \left(\frac{u+v-1}{u}\right)p' + \frac{v}{u} = 0$$

$$u(p')^2 - up' + p' + p'v + v = 0$$

$$\cancel{u(p')^2 + p'(1-u) = -p'(1}$$

$$\cancel{v(p'+1) = p'u - p' - up'^2}$$

$$\cancel{v(p'+1) = p'u(1-p') - p'}$$

$$up'^2 - up' - vp' + p' + v = 0$$

$$up'(p'-1) + p' = vp' - v$$

$$\Rightarrow (p'-1)v = p'u(p'-1) + p'$$

$$\Rightarrow \boxed{v = p'u + \frac{p'}{p'-1}}$$ Clairaut's form.

$\therefore$ solution of the equation is $v = cu + \frac{c}{c-1}$

$$\Rightarrow \boxed{y^2 = cx^2 + \frac{c}{c-1}}$$ Ans.

y eqn $\quad 6x - 48x^2$
is

Solve the initial value differential equations:

$$20y'' + 4y' + y = 0 \quad , \quad y(0) = 3.2 \text{ and } y'(0) = 0$$

sol. Auxiliary eqn of given ODE is

$$20m^2 + 4m + 1 = 0$$

$$\therefore \quad m = \frac{-4 \pm \sqrt{16 - 4(20)}}{2}$$

$$m = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

As roots are $\alpha \pm i\beta$ (imaginary)

$$\therefore \; C.F = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\Rightarrow \quad y = e^{-2x}(C_1 \cos 4x + C_2 \sin 4x)$$

$$y(0) = 3.2 = e^0 (C_1 \cos(0) + C_2 \sin(0))$$

$$\therefore \quad C_1 = 3.2$$

$$y' = e^{-2x}(-4C_1 \sin 4x + 4C_2 \cos 4x) - 2e^{-2x}(C_1 \cos 4x + C_2 \sin 4x)$$

$$y'(0) = 0 = e^0(-4C_1 \sin(0) + 4C_2 \cos(0)) - 2e^{-0}(C_1 \cos(0) + C_2 \sin(0)))$$

$$0 = 4C_2 - 2C_1$$

$$\therefore \quad C_2 = \frac{C_1}{2} = \frac{3.2}{2} = 1.6$$

Hence,

$$\boxed{y = e^{-2x}(3.2 \cos 4x + 1.6 \sin 4x)}$$

Q Solve the differential equation using method of variation of parameters:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 44 - 76x - 48x^2$$

sol. Auxiliary eqn is

$$m^2 - m - 2 = 0$$

$$m^2 - 2m + m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = -1, 2$$

$$\therefore \; C.F. = C_1 e^{-x} + C_2 e^{2x}$$

Hence $u = e^{-x}$ and $v = e^{2x}$ are solutions of homogeneous part of equation.

$$W = \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & -e^{-x} \\ e^{2x} & 2e^{2x} \end{vmatrix} = 2e^{x} + 2e^{x} = 4e^{x} \neq 0$$

$\therefore$ u and v are independent.

Using variation of Parameters

Let $y = Au + Bv$ & $R = 44 - 76x - 48x^2$

then

$$A = -\int \frac{vR}{W} dx = -\int \frac{e^{2x}(44-76x-48x^2)}{4e^{x}} dx$$

$$A = -\int e^{x}(11-19x-12x^2)\,dx = \int e^{x}(12x^2+19x-11)\,dx$$

$$= e^{x}(12x^2+19x-11) - e^{x}(24x+19) + e^{x}(24)$$

$$= e^{x}(12x^2+19x-24x-1-19+24)$$

$$= e^{x}(12x^2-5x+4)$$

$$B = \int \frac{uR}{W} dx = \int \frac{e^{-x}(44-76x-48x^2)}{4e^{x}} dx = \int e^{-2x}(11-19x-12x^2)\,dx$$

$$= \cancel{\left(\frac{-e^{-2x}}{2}\right)(-19-24x) - \left(\frac{e^{-2x}}{4}\right)(-24) - \left(\frac{e^{-2x}}{8}\right)(}$$

$$= \left(\frac{-e^{-2x}}{2}\right)(11-19x-12x^2) - \left(\frac{e^{-2x}}{4}\right)(-19-24x) - \left(\frac{e^{-2x}}{8}\right)(-24)$$

$$= \left(\frac{e^{-2x}}{2}\right)\left[12x^2+19x-11+12x+\frac{19}{2}+3\right] = \left(\frac{e^{-2x}}{2}\right)\left\{12x^2+31x+\frac{3}{2}\right]$$

$\therefore \; B = \frac{e^{-2x}}{4}\left[24x^2 + 62x + 3\right]$

Hence $y_p = Au + Bv$

$y_p = \left[e^{x}(12x^2 - 5x + 4)\right]e^{-x} + \left[\frac{e^{-2x}}{4}(24x^2 + 62x + 3)\right]e^{2x}$

$y_p = 12x^2 - 5x + 4 + \frac{1}{4}(24x^2 + 62x + 3)$

$= 18x^2 - \frac{9}{2}x + \frac{19}{4}$

Hence $y = C.F + P.I$

$$\boxed{y = C_1 e^{-x} + C_2 e^{2x} + 18x^2 - \frac{9}{2}x + \frac{19}{4}}$$

Q Solve the initial value problem using Laplace transform:

$\frac{d^2y}{dx^2} + 9y = r(x) \;,\; y(0) = 0 \;,\; y'(0) = 4$

where $r(x) = \begin{cases} 8\sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x \geq \pi \end{cases}$

sol We know $L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$\therefore L[r(x)] = \int_0^\infty e^{-st} r(t)\,dt = \int_0^\pi e^{-st} r(t)\,dt + \int_\pi^\infty e^{-st} r(t)\,dt$

$= \int_0^\pi e^{-st}(8\sin t)\,dt + \int_\pi^\infty e^{-st}(0)\,dt$

$= \left| \frac{-8e^{-st}(s\sin t + \cos t)}{s^2+1} \right|_0^\pi + 0$

$= \left(\frac{-8e^{-s\pi}(s\sin\pi + \cos\pi)}{s^2+1}\right) - \left(\frac{-8e^{0}(s\sin(0) + \cos(0))}{s^2+1}\right)$

$= \frac{8e^{-s\pi}}{s^2+1} + \frac{8}{s^2+1} = \frac{8(1+e^{-s\pi})}{s^2+1}$

$\therefore \; L[r(x)] = \dfrac{8(1+e^{-s\pi})}{s^2+1}$

Now taking laplace transform of the differential equation.

$L\left[\dfrac{d^2y}{dx^2} + 9y\right] = L[r(x)]$

$\Rightarrow s^2L(y) - sy(0) - y'(0) + 9L(y) = \dfrac{8(1+e^{-s\pi})}{s^2+1}$

$\Rightarrow s^2L(y) - s(0) - 4 + 9L(y) = \dfrac{8(1+e^{-s\pi})}{s^2+1}$

$\Rightarrow (s^2+9)L(y) - 4 = \dfrac{8(1+e^{-s\pi})}{s^2+1}$

$\Rightarrow (s^2+9)L(y) = \dfrac{8(1+e^{-s\pi})}{s^2+1} + 4$

$L(y) = \dfrac{8(1+e^{-s\pi})}{(s^2+1)(s^2+9)} + \dfrac{4}{s^2+9}$

$L(y) = \dfrac{8}{(s^2+1)(s^2+9)} + \dfrac{8e^{-s\pi}}{(s^2+1)(s^2+9)} + \dfrac{4}{s^2+9}$

$\therefore \; y = L^{-1}\left[\dfrac{8}{(s^2+1)(s^2+9)}\right] + L^{-1}\left[\dfrac{8e^{-s\pi}}{(s^2+1)(s^2+9)}\right] + L^{-1}\left[\dfrac{4}{s^2+9}\right]$

Now,

$L^{-1}\left[\dfrac{4}{s^2+9}\right] = 4L^{-1}\left[\dfrac{1}{s^2+3^2}\right] = 4\dfrac{\sin 3x}{3} = \dfrac{4}{3}\sin x$

Let $F(s) = \dfrac{8}{(s^2+1)(s^2+9)}$

then

$L^{-1}\left[\dfrac{8}{(s^2+1)(s^2+9)}\right] = L^{-1}\left[\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9}\right] = L^{-1}\left[\dfrac{1}{s^2+1}\right] - L^{-1}\left[\dfrac{1}{s^2+9}\right]$

$= \sin x - \dfrac{\sin 3x}{3}$

$\therefore \; L^{-1}[F(s)] = \sin x - \dfrac{\sin 3x}{3} = f(x)$ [say]

$$\cancel{= u(x+\pi)} \left(\sin x - \frac{\sin 3x}{3}\right)$$

We know,

$$L^{-1}[e^{as} F(s)] = u(x-a)\, f(x-a)$$

where $u(x)$ is heaveside function.

$$\therefore L^{-1}\left[\frac{8e^{-s\pi}}{(s^2+1)(s^2+9)}\right] = u(x+\pi)\left(\sin(x+\pi) - \frac{\sin 3(x+\pi)}{3}\right)$$

$$= u(x+\pi)\left(-\sin x + \frac{\sin 3x}{3}\right)$$

As

$$y = L^{-1}\left[\frac{8}{(s^2+1)(s^2+9)}\right] + L^{-1}\left[\frac{8e^{-s\pi}}{(s^2+1)(s^2+9)}\right] + L^{-1}\left[\frac{4}{s^2+9}\right]$$

$$= \cancel{8}\left(\sin x - \frac{\sin 3x}{3}\right) + u(x+\pi)\left(-\sin x + \frac{\sin 3x}{3}\right) + \frac{4}{3}\sin 3x$$

$$= (\sin x + \sin 3x) + u(x+\pi)\left(-\sin x + \frac{\sin 3x}{3}\right)$$

$$\therefore y = \sin x + \sin 3x + \begin{cases} -\sin x + \frac{\sin 3x}{3}, & \text{if } x \geq -\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{y = \begin{cases} \frac{4}{3}\sin 3x, & \text{if } x \geq -\pi \\ \sin x + \sin 3x; & \text{otherwise} \end{cases}}$$

Q Solve the differential equation:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$$

Method I: Direct substitution. As sin term has x^2 inside it, $x^2=t$ is taken.
Similar method also worked for 2019 CSE question where $e^{\cos x}$ was in equation and $\cos x = t$ was taken.

sol. Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}(2x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dt}\right) = 2\frac{dy}{dt} + 2x\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}$$

$$= 2\frac{dy}{dt} + 4x^2\frac{d^2y}{dt^2}$$

Putting these in equation.

$$x\left[4x^2\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right] - \left(2x\frac{dy}{dt}\right) - 4x^3y = 8x^3\sin(x^2)$$

$$\Rightarrow 4x^3\frac{d^2y}{dt^2} + 2x\frac{dy}{dt} - 2x\frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$$

$$\Rightarrow 4x^3\frac{d^2y}{dt^2} - 4x^3y = 8x^3\sin(x^2)$$

$$\Rightarrow \frac{d^2y}{dt^2} - y = 2\sin(x^2)$$

$$\Rightarrow \frac{d^2y}{dt^2} - y = 2\sin t \qquad (\because t = x^2)$$

Auxiliary eq^n: $m^2 - 1 = 0$
$\therefore m = \pm 1$

$$C.F. = C_1e^t + C_2e^{-t}$$

$$P.I = \frac{1}{D^2-1}2\sin t = 2\sin t\frac{1}{-1^2-1} = 2\sin t\left(-\frac{1}{2}\right)$$

$$\therefore P.I. = -\sin t$$

Hence $y = C_1e^t + C_2e^{-t} - \sin t$

$$\Rightarrow \boxed{y = C_1e^{x^2} + C_2e^{-x^2} - \sin(x^2)}$$ is the answer.

Method 2 (Normal form)

Given $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$

$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin(x^2)$

Let an independent variable z. such that

$\left(\frac{dz}{dx}\right)^2 = 4x^2 \Rightarrow \frac{dz}{dx} = \pm 2x$

Taking the +ve value & solving.

$\frac{dz}{dx} = 2x \Rightarrow dz = 2x\,dx$.

$\Rightarrow z = \int 2x\,dx = x^2$

$P = \frac{-1}{x} \qquad Q = -4x^2 \qquad R = 8x^2\sin(x^2)$

Transformed eqn is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 = R_1$

where. $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{(2x)^2} = -1$

$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{8x^2\sin(x^2)}{(2x)^2} = 2\sin(x^2)$

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(\frac{-1}{x}\right)(2x)}{(2x)^2} = \frac{2-2}{4x^2} = 0$

Hence, we get

$\frac{d^2y}{dz^2} + (0)\frac{dy}{dz} - y = 2\sin(x^2) = 2\sin(z)$

Auxiliary equation.

$m^2 - 1 = 0$

$m = \pm 1$

$\therefore$ C.F. $= C_1e^{z} + C_2e^{-z}$

P. I. $= \frac{1}{D^2-1}(2\sin z) = 2\frac{1}{D^2-1}\sin z = 2\sin z\frac{1}{-1^2-1}$

$= 2\sin z\left(-\frac{1}{2}\right) = -\sin z$

$\therefore\ y = C.F. + P.I. = C_1 e^{z} + C_2 e^{-z} - \sin z$

$$\boxed{y = C_1 e^{x^2} + C_2 e^{-x^2} - \sin(x^2)}$$ Answer