CSE-2017 — Paper II

5)(b) Explain the main steps of the Gauss-Jordan method and apply this method to find the inverse of the matrix, $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$

⇒ In Gauss-Jordan method, we reduce the matrix (let A) to the form of I and in the process the matrix I is transformed to A^{-1}.

i.e, $[A \mid I^{-1}] \longrightarrow [I \mid A^{-1}]$

$$\therefore [A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 6 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 4 & -3 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_1' \to R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & -3 & -3 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_1' \to R_1 - 3R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & -3/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} R_1' \to R_1/2 \\ R_2' \to R_2/2 \\ R_3' \to R_3/2 \end{array}$$

$$= [I \mid A^{-1}]$$

∴ Inverse of the matrix $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$ is,

$$\begin{bmatrix} 7/2 & -3/2 & -3/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

6) (b) For given equidistant values u_{-1}, u_0, u_1 and u_2, a value is interpolated by Lagrange's formula. Show that it may be written in the form,

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0,$$

where $x+y=1$

$\Rightarrow$ R.H.S

$$yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

$$= (1-x)u_0 + xu_1 + \frac{(1-x)\{(1-x)^2-1\}}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

$$= u_0 - xu_0 + xu_1 + \frac{(1-x)(x^2-2x)}{3!}\Delta(\Delta u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(\Delta u_0)$$

$$= u_0 - xu_0 + xu_1 + \frac{x(1-x)(x-2)}{3!}\Delta(u_0 - u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(u_1 - u_0)$$

$$= u_0 - xu_0 + xu_1 + \frac{x(1-x)(x-2)}{3!}\{(u_1-u_0)-(u_0-u_{-1})\}$$
$$+ \frac{x(x^2-1)}{3!}\{(u_2-u_1)-(u_1-u_0)\}$$

$$= u_0 - xu_0 + xu_1 + \frac{x(1-x)(x-2)}{3!}(u_1 - 2u_0 + u_{-1})$$
$$+ \frac{x(x^2-1)}{3!}(u_2 - 2u_1 + u_0)$$

$$= u_0\left\{1 - x - \frac{x(x-2)(1-x)}{3} + \frac{x(x^2-1)}{6}\right\} + u_1\left\{x + \frac{x(1-x)(x-2)}{6} - \frac{x(x^2-1)}{3}\right\}$$
$$+ u_2\left\{\frac{x(x^2-1)}{3!}\right\} + u_{-1}\left\{\frac{x(1-x)(x-2)}{6}\right\}$$

$$= \frac{x(1-x)(x-2)}{6}u_{-1} + \frac{(1-x)(2+x-x^2)}{2}u_0$$
$$+ \frac{x(2+x-x^2)}{2}u_1 + \frac{x(x^2-1)}{6}u_2$$

$$= u_x$$

7)(b) Derive the formula

$$\int_a^b y\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\Big]$$

Is there any restriction on n? state that condition. what is the error bound in the case of Simpson $3/8$ rule?

⇒ we know that, by the a general quadrature for equidistant ordinates

$$I = \int_a^b f(x)dx = h\Big[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\Big(\frac{n^3}{2}-\frac{n^2}{2}\Big)\Delta^2 y_0 + \frac{1}{6}\Big(\frac{n^4}{4}-n^3+n^2\Big)\Delta^3 y_0 + \cdots\Big]$$

Putting $n=3$ in the quadrature formula and taking the curve through (x_i, y_i); $i=0,1,2,3$ as a polynomial of third order so that differences of the third order vanish.

we get $\int_{x_0}^{x_3} f(x)dx = \int_{x_0}^{x_0+3h} f(x)\,dx$

$$= 3h\Big[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\Big].$$

$$= \frac{3h}{8}\Big[y_0 + 3y_1 + 3y_2 + y_3\Big]$$

Similarly, $\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\Big[y_3 + 3y_4 + 3y_5 + y_6\Big]$

Adding all these integrals from x_0 to x_0+nh where 'n' is a multiple of 3. and we obtain,

$$\int_{x_0}^{x_0+nh} = \int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3h}{8}(y_0+3y_1+3y_2+y_3) + \cdots\cdots + \frac{3h}{8}(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)$$

$$= \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\Big]$$

— is known as Simpson $3/8$ rule.

□ Yes, there is a restriction of n. While using Simpson 3/8 rule, the number of subintervals always should be taken as multiple of ~~3~~ three.

□ The error bound in the case of Simpson's 3/8 rule is,

$$E = -\frac{3h^5}{80} y^{iv}(\bar{\xi}),$$ where $y^{iv}(\bar{\xi})$ is the largest value of the fourth order derivation.