CSE 2019 / 5(a) / 10m

5(a) Solve the D.E.

$$(2y\sin x + 3y^4 \sin x \cos x)\,dx - (4y^3\cos^2 x + \cos x)\,dy = 0$$

$$M\,dx + N\,dy = 0.$$

$$\frac{\partial M}{\partial y} = 2\sin x + 12y^3 \sin x \cos x$$

$$\frac{\partial N}{\partial x} = +(8y^3\cos x \sin x + \sin x)$$

As $M_y \neq N_x$, hence given D.E. is NOT Exact.

Let us try to find its integral factor

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\sin x + 4y^3 \sin x \cos x}{-(\cos x + 4y^3\cos^2 x)} = -\tan x$$

$$\text{I.F.} = e^{\int -\tan x\,dx} = e^{\log \cos x} = \cos x$$

Mutiply the D.E. with this I.F.

$$(2y\sin x\cos x + 3y^4 \sin x \cos^2 x)\,dx - (4y^3\cos^3 x + \cos^2 x)\,dy = 0$$

$$\text{Complete Solution} = \int_{y=\text{constant}} M\,dx + \int_{\substack{\text{Excluding terms}\\ \text{containing } x}} N\,dy$$

$$\int (2y\sin x\cos x + 3y^4 \sin x \cos^2 x)\,dx = C$$

$$y\sin^2 x - y^4\cos^3 x = C$$

5 (b) Determine the Complete Solution of DE

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x.$$

$$y'' - 4y' + 4y = 3x^2 e^{2x} \sin 2x$$

Auxiliary Eqn: $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

$$\therefore y_c = (c_1 + x c_2) e^{2x}$$

$$P.I. = \frac{1}{D^2 - 4D + 4}(3x^2 e^{2x} \sin 2x)$$

$$= \frac{1}{(D-2)^2}(3e^{2x} \cdot x^2 \sin 2x)$$

$$= 3e^{2x} \cdot \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 3e^{2x} \cdot \frac{1}{D^2} x^2 \sin 2x.$$

$$\left[\because \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V\right]$$

Let $I = \frac{1}{D^2} x^2 \sin 2x = \frac{1}{D^2} x^2 (\text{Imaginary part of } e^{i2x})$

$$= \text{I.P of } \frac{1}{D^2}(x^2 \cdot e^{2xi})$$

$$I = \text{I.P. of } e^{2xi}\frac{1}{(D+2i)^2}(x^2)$$

$$= \text{I.P of } e^{2xi}\frac{1}{-4\left(1+\frac{D}{2i}\right)^2}(x^2)$$

$$= \text{I.P. of } \frac{-e^{2xi}}{4}\left[1+\frac{D}{2i}\right]^{-2}(x^2)$$

$$= \text{I.P. of } \frac{-e^{2xi}}{4}\left(1-\frac{D}{i}+3\frac{D^2}{(2i)^2}-\cdots\right)x^2$$

$$= \text{I.P. of } \frac{-(\cos 2x + i\sin 2x)}{4}\left[x^2-\frac{2x}{i}-\frac{3}{2}\right]$$

$$= -\frac{1}{4}\sin 2x\left(x^2-\frac{3}{2}\right)+\frac{\cos 2x}{4}\cdot\frac{2x}{i}$$

$$\therefore \quad P.I. = \frac{3}{4}e^{2x}\left[-x^2\cdot\sin 2x+\frac{3}{2}\sin 2x-2x\cos 2x\right]$$

Complete solution, $\quad y = y_c + y_p$

$$y = (c_1+c_2x)e^{2x} - \frac{3}{4}e^{2x}\left[x^2\sin 2x-\frac{3}{2}\sin 2x + 2x\cos 2x\right]$$

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6(c) Solve the DE

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\cdot\sin^2 x \quad —①$$

Comparing with, $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

$P = 3\sin x-\cot x$

$Q = 2\sin^2 x$ and $R=e^{-\cos x}\cdot\sin^2 x$

Let $\frac{dz}{dx}=e^{-\int P\,dx} = e^{-\int(3\sin x-\cot x)\,dx}$

$= e^{3\cos x+\log\sin x}$

$= \sin x\cdot e^{3\cos x}$

$\therefore z=\int e^{3\cos x}\cdot\sin x\,dx = \frac{-1}{3}e^{3\cos x}$

Changing the independent variable from x to z in given D.E., we get

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \quad — \cancel{A} \; ②$$

METHOD-I, Take

~~where~~ $P_1=0 \Rightarrow P_1=\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{2\sin^2 x}{\sin^2 x\,(e^{3\cos x})^2}=\frac{2}{9z^2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{-\cos x} \cdot \sin^2 x}{\sin^2 x\,(e^{3\cos x})^2} = e^{-7\cos x}$$

$$= \left(e^{\cos x}\right)^{-7} = (-3z)^{-7/3} = \frac{-z^{-7/3}}{3^{7/3}}$$

Hence (A) becomes

$$\frac{d^2y}{dz^2} + \frac{2}{9} \cdot \frac{1}{z^2} \cdot y = \frac{-1}{3^{7/3}} \cdot z^{-7/3}$$

Multiplying by z^2, we get

$$z^2 \frac{d^2y}{dz^2} + \frac{2}{9} y = -\frac{1}{3^{7/3}} \cdot z^{-1/3} \quad \text{—— (3)}$$

Put $z = e^t \Rightarrow \log z = t$

$\therefore\ z \cdot \frac{d}{dz} = \frac{d}{dt} = D$ and $z^2 \frac{d^2}{dz^2} = D(D-1)$

Eqn (3) becomes,

$$\left[D(D-1) + \frac{2}{9}\right] y = -\frac{1}{3^{7/3}} \cdot e^{-t/3}$$

Auxiliary Eqn : $D^2 - D + \frac{2}{9} = 0 \Rightarrow D = \frac{2}{3}, \frac{1}{3}$

$$\therefore\ C.F. = c_1 e^{2/3\, t} + c_2 e^{t/3} = c_1 z^{2/3} + c_2 z^{1/3}$$

$$C.F. = c_1\left(-\frac{1}{3}e^{3\cos x}\right)^{2/3} + c_2\left(-\frac{1}{3}e^{3\cos x}\right)^{1/3}$$

$$= A_1 e^{2\cos x} + A_2 e^{\cos x}$$

$$P.I. = \frac{1}{D^2 - D + \frac{2}{9}}\left[-\frac{1}{3^{7/3}} \cdot e^{-t/3}\right]$$

$$= \frac{-1}{3^{7/3}} \cdot \frac{1}{\frac{1}{9} + \frac{1}{3} + \frac{2}{9}} e^{-t/3} = \left(\frac{-1}{3^{7/3}}\right)\frac{9}{6} e^{-t/3}$$

$$= \frac{-3}{2(3)^{7/3}} \cdot \left[-\frac{1}{3} e^{3\cos x}\right]^{-1/3}$$

$$= \frac{3}{2(3)^{6/3}} \cdot e^{-\cos x} = \frac{1}{6} e^{-\cos x}$$

$\therefore$ Complete Solution, $\quad y = C.F. + P.I.$

$$y = A_1 e^{2\cos x} + A_2 e^{\cos x} + \frac{1}{6} e^{-\cos x}$$

(Method-2)

Let us take, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{Constant}$

i.e. $\dfrac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2} = 2$ (say) $\Rightarrow \dfrac{dz}{dx} = \sin x$

$\boxed{z = \cos x}$

Now,

$$P_1 = \frac{\frac{d^2z}{dx^2} + P \cdot \frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{-\cos x + (3\sin x - \cot x)(-\sin x)}{(-\sin x)^2}$$

$$= \frac{-3\sin^2 x}{\sin^2 x} = -3$$

$$R_1 = \frac{R(x)}{(dz/dx)^2} = \frac{e^{-\cos x} \cdot \sin^2 x}{(-\sin x)^2} = e^{-\cos x}$$

$$= e^{-z}$$

$\therefore$ New D.E. becomes

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = e^{-z}$$

$$(D^2 - 3D + 2)y = e^{-z} \qquad \Rightarrow m = +1, +2$$

$$y_c(z) = c_1 e^{z} + c_2 e^{2z}, \quad y_p(z) = \frac{1}{D^2 - 3D + 2} e^{-z} = \frac{1}{6} e^{-z}$$

$$\therefore\ y = c_1 e^{\cos x} + c_2 e^{2\cos x} + \frac{1}{6} e^{-\cos x}$$

6C(ii) Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$. Prove that the Laplace transform of $t^{n+1/2}$, where $n \in N$, is

$$\frac{T(n+1+\frac{1}{2})}{s^{n+1+1/2}}.$$

Let $f(t) = t^{-1/2}$

$$Lf(t) = \int_0^\infty e^{-st} \cdot \frac{1}{\sqrt{t}}\, dt$$

Put $st = y \Rightarrow s\,dt = dy$

$$\therefore Lf(t) = \int_0^\infty e^{-y} \cdot \frac{\sqrt{s}}{\sqrt{y}} \cdot \frac{dy}{s}$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-y} \cdot y^{-1/2}\, dy$$

$$= \frac{1}{\sqrt{s}} \cdot T\left(-\frac{1}{2}+1\right) \qquad \left[T(n+1) = \int_0^\infty e^{-x} \cdot x^n\, dx\right]$$

$x > 0$

$$= \frac{1}{\sqrt{s}}\, T\left(\frac{1}{2}\right) = \frac{1}{\sqrt{s}} \cdot \sqrt{\pi}. \qquad \left(T\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

Again let $f(t) = t^{1/2}$

$$Lf(t) = \int_0^\infty e^{-st} \sqrt{t}\, dt \,, \qquad \text{Put } st = y, \; s\,dt = dy$$

$$= \int_0^\infty e^{-y} \cdot \frac{\sqrt{y}}{\sqrt{s}} \cdot \frac{dy}{s} = s^{-3/2} \int_0^\infty e^{-y} \cdot y^{1/2} dy$$

$$= s^{-3/2}\, \Gamma\left(\frac{1}{2}+1\right)$$

$$= s^{-3/2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n+1) = n\,\Gamma(n)\right]$$

$$= \frac{1}{2s^{3/2}} \cdot \sqrt{\pi}$$

Now, Let $f(t) = t^{n+1/2}$

$$Lf(t) = \int_0^\infty e^{-st} \cdot t^{n+1/2} dt \qquad \text{Let } st = y, \; s\,dt = dy$$

$$= \int_0^\infty e^{-y} \cdot \left(\frac{y}{s}\right)^{n+1/2} \frac{dy}{s}$$

$$= \frac{1}{s^{n+1+1/2}} \cdot \int_0^\infty e^{-y} \cdot y^{n+1/2} dy$$

$$= \frac{\Gamma(n+1+1/2)}{s^{n+1+1/2}}$$

8(a). Obtain the singular solution of the DE

$$\left(\frac{dy}{dx}\right)^2\left(\frac{y}{x}\right)^2\cot^2\alpha - 2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\operatorname{cosec}^2\alpha = 1$$

Also find the complete primitive of the given DE. Give the geometrical interpretations of the complete primitive and singular solution.

Let $\frac{dy}{dx} = p$, then given DE can be written as

$$p^2y^2\cot^2\alpha - 2pxy + y^2\operatorname{cosec}^2\alpha = x^2$$

$$p^2y^2 - (2py)\,x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0 \quad (1)$$

$$\therefore\ py = \frac{1}{2}\left[2x\tan^2\alpha \pm \left[4x^2\tan^4\alpha - 4(y^2\sec^2\alpha - x^2\tan^2\alpha)\right]^{1/2}\right]$$

$$= x\tan^2\alpha \pm \sqrt{x^2\tan^2\alpha(\tan^2\alpha+1) - y^2\sec^2\alpha}$$

$$py = x\tan^2\alpha \pm \sec\alpha\sqrt{x^2\tan^2\alpha - y^2}$$

$$\therefore\ y\,dy - x\tan^2\alpha\,dx = \pm\sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}\,dx$$

$$\pm\frac{x\tan^2\alpha\,dx - y\,dy}{\sqrt{x^2\tan^2\alpha - y^2}} = -\sec\alpha\,dx$$

Integrating, $\pm\sqrt{x^2\tan^2\alpha - y^2} = c - x\sec\alpha$

Squaring, $x^2\tan^2\alpha - y^2 = c^2 - 2cx\sec\alpha + x^2\sec^2\alpha$

i.e. $x^2+y^2-2cx\sec x+c^2=0 \quad (\sec^2 x-\tan^2 x=1)$

$-$ (2)

from (1), p-disc relation is

$4x^2y^2\tan^4\alpha - 4y^2(y^2\sec^2\alpha - x^2\tan^2\alpha)=0$

$\frac{4y^2}{\cos^4\alpha}(x^2\sin^4\alpha - \cos^2\alpha(y^2-x^2\sin^2\alpha))]=0.$

$4y^2(x^2\sin^2\alpha - y^2\cos^2\alpha)=0$

ie $4y^2(x^2\tan^2\alpha - y^2)=0.$ $-(3)$

We obtain c-discrimination relation from gen sol (2)

ie. $c^2-2cx\sec\alpha + x^2+y^2=0$

$4x^2\sec^2\alpha - 4(x^2+y^2)=0.$

$x^2\tan^2\alpha - y^2=0$

$\therefore (x\tan\alpha - y)(x\tan\alpha+y)=0$ $-(4)$

$\therefore$ The lines $y=\pm x\tan\alpha$ are singular solution (envelop) and $y=0$ is a

The gen solution (2) represents a family of circles all having centres on x-axis. The family of circles is being touched by $y=\pm x\tan\alpha$, which are equally inclined to the line of centres (x-axis and passes through the origin.

7(a) Find the linearly independent solutions of the corresponding homogeneous DE of the Eqn $x^2y'' - 2xy' + 2y = x^3 \sin x$ and then find the general solution of the given DE by the method of variation of parameters.

Divide the given D.E. by x^2

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x \cdot \sin x \quad \text{--- (1)}$$

Homogeneous Eqn :

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$$

ie $x^2y'' - 2xy' + 2y = 0$.

Put $x = e^z$ and $D_1 = \frac{d}{dz}$

$$[D_1(D_1 - 1) - 2D_1 + 2]y = 0$$

$$(D_1^2 - 3D_1 + 2)y = 0.$$

$$(D_1 - 2)(D_1 - 1) = 0 \qquad \therefore D_1 = 1, 2$$

$$y_c = c_1e^z + c_2e^{2z}$$

$$= c_1x + c_2x^2$$

Now we find P.I. to eqn (1)

Take $u = x$, $v = x^2$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

Hence solutions are independent.

$$P.I. = Au + Bv$$

$$A = -\int \frac{vR}{W}\,dx$$

$$= -\int \frac{\cancel{x^2}\cdot x\sin x}{\cancel{x^2}}\,dx$$

$$= -(-x\cos x + \sin x)$$

$$= x\cos x - \sin x$$

$$B = \int \frac{uR}{W}\,dx$$

$$= \int \frac{x \cdot x\sin x}{x^2}\,dx$$

$$= -\cos x$$

$$\therefore\ P.I. = x(x\cos x - \sin x) + x^2(-\cos x)$$

$$= -x\sin x$$

$$\therefore\ y = y_c + y_p$$

$$= c_1 x + c_2 x^2 - x\sin x$$