IAS-2008 -P-II

Ques: 3(b)) Show that any maximal ideal in the commutative ring $F[x]$ of polynomials over a field F is the principal ideal generated by an irreducible polynomial.

Solution:-

F is a field $\Rightarrow F[x]$ is an Integral Domain
$\Rightarrow F[x]$ is a Euclidean Ring
$\Rightarrow F[x]$ is a Principal Ideal Ring/Domain

Let $f, g \in F[x]$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_m x^m$$

$$f(x) + g(x) = \sum_{r=0}^{\text{Max}(m,n)} (a_r + b_r) x^r.$$

$$f(x) \cdot g(x) = a_0 b_0 + (a_1 b_2 + b_1 a_2) x + \ldots\ldots + a_n b_m x^{n+m}$$

$\therefore$ All Ideal of $F[x]$ are generated by single element, say $\langle f(x) \rangle$

$\therefore$ maximal ideal is also a principal Ideal.

If F is a field then in $F[x]$, irreducible polynomials are irreducible elements.

$\therefore$ If $f(x)$ is irreducible element

$f(x) = g(x) \cdot h(x) \Rightarrow$ either $g(x)$ or $h(x)$ is a unit.

Condition is necessary:

Let M be a Maximal Ideal of $F[x]$, we have to prove that is $M = \langle a \rangle$;
'a' is irreducible element.

(i) $a \neq 0$; because if $\langle a \rangle = \langle 0 \rangle$ then some $\langle p \rangle$ be an ideal has '0' as its element.

$\langle 0 \rangle \subset \langle p \rangle \subset \langle R \rangle \Rightarrow \langle p \rangle \neq \langle 0 \rangle$

$\boxed{\therefore a \neq 0}$

(ii) 'a' cannot be unit.
because if 'a' is a unit, $\exists\, a^{-1} \in F[x]$, such that
$a^{-1}a \in M \Rightarrow I \in M$; $M = R$.

(iii) 'a' is non-zero; non-unit
$\therefore a = bc \quad / b, c \in R$

Let $B = \langle b \rangle$, let $r \in M = \langle a \rangle$

$r = ag = b \cdot c \cdot g \quad / g \in R$

$r = b(cg)$.

$\therefore r \in \langle b \rangle$

$\boxed{M \subseteq B \subseteq F[x]}$ since, M is a maximal ideal

$\therefore M = B$ or $B = F[x]$

If $M = B$

$b = ah = bch \quad / h \in F[x]$

$ch = f(x) = 1$

$\therefore$ c is a unit element

$\therefore$ a is irreducable.

If $B = F[x]$

$f(x) = 1 \in B$ $\quad f(x) = 1 = bl \;/\; l \in F[x]$

$\therefore b$ is a unit $\Rightarrow$ a is irreducable.

<u>Condition is sufficient:</u>

If 'a' is irreducible polynomial in $F[x]$, the Ideal $\langle a \rangle = A$ is a Maximal Ideal.

To prove if $\langle a \rangle \subseteq I \subseteq F[x]$

either $I = \langle a \rangle$ or $I = F[x]$.

Since; $F[x]$ is a PID, therefore $I = \langle d \rangle$.

(i) <u>Case 1</u> $\Rightarrow$ If $d \in A \Rightarrow d = af \;/\; f \in F[x]$

Let $r \in \langle d \rangle \Rightarrow r = dg = a(fg)$

$\therefore r \in A \Rightarrow I \subseteq A$.

Also; $A \subseteq I$ $\quad \therefore \boxed{A = I}$

(ii) <u>Case 2</u> $\Rightarrow$ If $d \notin A$

$A \subset I$

$\Rightarrow$ so $a = df$

Since, a is irreducible

$\therefore$ Either d or 'f' is a unit.

If f is a unit $\exists f^{-1} \Rightarrow d = af^{-1}$

$\Rightarrow d \in A$

contradictory.

If d is a unit $\exists d^{-1}$ in $F[x]$.

$\therefore 1 = d(d^{-1}) \in I$.

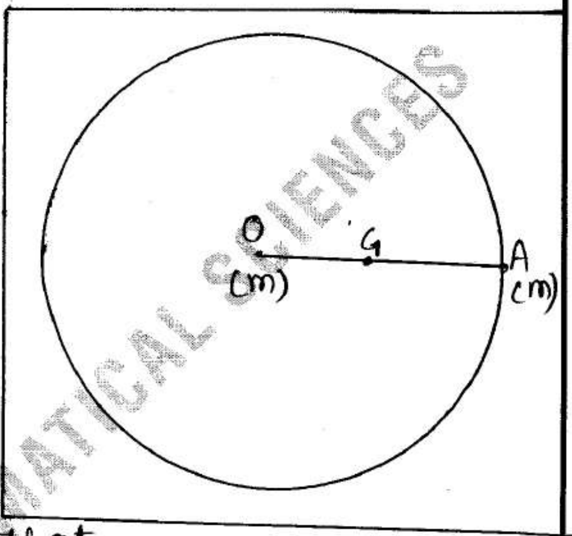
$\Rightarrow I = F[x]$

$\therefore A = \langle a \rangle$ is a Maximal Ideal.

IAS 2008 P-II

Ques: 5(e)) A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board? Explain what happens if the mass of boy and board are equal?

Solution:-



Let M be the mass and O the centre of the board. If initially the boy is at 'A' on the edge of the board then the C.G. 'G' of the system will be on the radius OA, such that

$$\boxed{OG = \frac{M\cdot 0 + m\cdot a}{M+m} = \frac{ma}{M+m}}$$

Since the external forces, weight of the board and the boy act vertically downwards and the reaction of the smooth horizontal plane act vertically upwards, therefore there is no external force in the horizontal direction during the motion. Thus by D'Alembert's principle the C.G. 'G' of the system will remain at rest. Hence as the boy runs round the edge of the board with uniform spread, the centre 'O' of the board will describe a circle of radius $OG = ma/(M+m)$ round the centre at G.

If mass of the board and boy are equal

i.e $\boxed{M = m}$

Then; motion of board $= OG = \dfrac{ma}{M+m}$

$$\Rightarrow OG = \frac{\cancel{m}a}{2\cancel{m}}$$

$$\Rightarrow \boxed{OG = a/2}$$

The motion of board depends only on the position of boy. i.e. $\boxed{OG \propto a}$

2008 · P-II ·

Ques: 8(b)) Let the fluid fills the region $x \geq 0$ (right half of 2d plane). Let a source α be at $(0, y_1)$ and equal sink at $(0, y_2)$, $y_1 > y_2$. Let the pressure be same as pressure at infinity i.e. P_0. Show that the resultant pressure on the boundary (y-axis) is $\pi\rho\alpha^2(y_1-y_2)^2/2y_1y_2(y_1+y_2)$, ρ being the density of the fluid.

Solution :- Here the image system w.r.t x-axis in z-plane consists of

i) a source m at $(0, y_1)$ i.e, at $z = y_1 i$

ii) a sink $-m$ at $(0, y_2)$ i.e at $z = y_2 i$

iii) a source m at $(0, -y_1)$ i.e at $z = -y_1 i$

iv) a sink $-m$ at $(0, -y_2)$ i.e at $z = -y_2 i$

Clearly the image system does away with the boundary $y=0$ (i.e, x-axis). Thus, the complex potential of this entire system is given by -

$$\therefore\ w = -m\log(z-y_1 i) + m\log(z-y_2 i) - m\log(z+y_1 i) + m\log(z+y_2 i)$$

$$w = -m\log(z^2+y_1^2) + m\log(z^2+y_2^2)$$

$$\therefore\ \text{velocity} = \left|\frac{dw}{dz}\right| = \left|-\frac{2zm}{z^2+y_1^2} + \frac{2zm}{z^2+y_2^2}\right|$$

The velocity q at a point on the boundary (i.e. $y=0$) is given by [setting $z = x+iy = x$ as $y=0$]

$$q = \left|\frac{-2xm}{x^2+y_1^2} + \frac{2xm}{x^2+y_2^2}\right| = \frac{2xm(y_1^2-y_2^2)}{(x^2+y_1^2)(x^2+y_2^2)}$$

Let P_0 be the pressure at infinity. Then by Bernoulli's theorem, the pressure p at any point is given by -

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \frac{1}{2}\times 0^2 + \frac{P_0}{\rho}$$

(or) $$\boxed{\frac{P_0 - p}{\rho} = \frac{1}{2}q^2}$$

$\therefore$ The resultant pressure on the boundary

$$= \int_0^\infty (P_0 - p)\,dx = \frac{1}{2}\rho\int_0^\infty q^2\,dx = 2\rho m^2 \int_0^\infty \frac{x^2(y_1^2-y_2^2)^2\,dx}{(x^2+y_1^2)^2(x^2+y_2^2)^2}$$

$$= 2\rho m^2 \int_0^\infty \left[-\frac{y_1^2+y_2^2}{y_1^2-y_2^2}\left[\frac{1}{x^2+y_1^2} - \frac{1}{x^2+y_2^2}\right] - \frac{y_1^2}{(x^2+y_1^2)^2} - \frac{y_2^2}{(x^2+y_2^2)^2}\right] dx$$

[on resolving into partial fraction]

$$= 2\rho m^2 \left\{\frac{y_1^2+y_2^2}{y_1^2-y_2^2}\left[\frac{\pi}{2y_1} - \frac{\pi}{2y_2}\right] - \frac{\pi}{4y_1} - \frac{\pi}{4y_2}\right\}.$$

[on simplification]

$$= \frac{\pi\rho m^2}{2y_1y_2}\left[\frac{2(y_1^2+y_2^2) - (y_1+y_2)^2}{(y_1+y_2)}\right]$$

$$= \frac{\pi\rho m^2(y_1-y_2)^2}{2y_1y_2(y_1+y_2)}$$ Hence proved

$\therefore$ Resultant pressure on the boundary (y-axis) $= \dfrac{\pi\rho m^2(y_1-y_2)^2}{2y_1y_2(y_1+y_2)}$

2009-P.I Ques: 1(c) Suppose that f'' is continuous on $[1,2]$ and that f has three zeroes in the interval $(1,2)$. Show that f'' has at least one zero in the interval $(1,2)$.

Solution:-

Suppose $f(x)=0$ has three zeroes in the interval $(1,2)$.

To show that: $f''(x)=0$ has at least one zero in the interval $(1,2)$

Let the three zeroes be a_1, a_2 and a_3 and suppose that $1<a_1<a_2<a_3<2$ (three distinct zeroes).

$$\Rightarrow \quad f(a_1)=f(a_2)=f(a_3)=0$$

Since f is infinitely differentiable, it is continuous on $[a_1,a_2]$ and differentiable on (a_1,a_2)

$\therefore$ By Rolle's theorem, there exists b_1 in (a_1,a_2) such that $f'(b_1)=0$

likewise f is continuous on $[a_2,a_3]$ and differentiable on (a_2,a_3).

So, by Rolle's theorem, there exists b_2 in (a_2,a_3) such that $f'(b_2)=0$.

Since, $1<b_1<a_2<b_2<2$, they cannot coincide and, infact, $b_1<b_2$.

Since, f is infinitely differentiable, f' is continuous on $[b_1,b_2]$ and differentiable on (b_1,b_2).

Also, $f'(b_1) = f'(b_2) = 0$

Therefore, by Rolle's theorem, there exists c in (b_1, b_2) such that $f''(c) = 0$.

Thus, $f''(x) = 0$ has atleast one zero in the interval $(1, 2)$.

Hence, the result.

2009-P-I

Ques: 1(d) If 'f' is the derivative of some function defined on $[a,b]$, prove that there exists a number $\eta \in [a,b]$, such that

$$\int_a^b f(t)\,dt = f(\eta)(b-a).$$

Solution:-

Given that 'f' is the derivative of some function defined on $[a,b]$.

$\Rightarrow$ $f(t)$ is continuous on $[a,b]$

$\Rightarrow$ $f(t)$ must have a maximum 'M' (say) and a minimum 'm' (say) somewhere in that interval. — [By Extreme value theorem]

i.e. $m \leq f(t) \leq M$

This leads to the inequality for areas

$$m(b-a) = \int_a^b m\,dt \leq \int_a^b f(t)\,dt \leq \int_a^b M\,dt = M(b-a)$$

$$\Rightarrow \quad m \leq \frac{\int_a^b f(t)\,dt}{(b-a)} \leq M$$

Since, f is continuous, by the Intermediate Value Theorem, it takes every positive value between 'm' and 'M'.

In particular, there exists a number $\eta \in [a,b]$ at which the function $f(t)$ has a value equal to

$$f(\eta) = \frac{\int_a^b f(t)\,dt}{(b-a)}$$

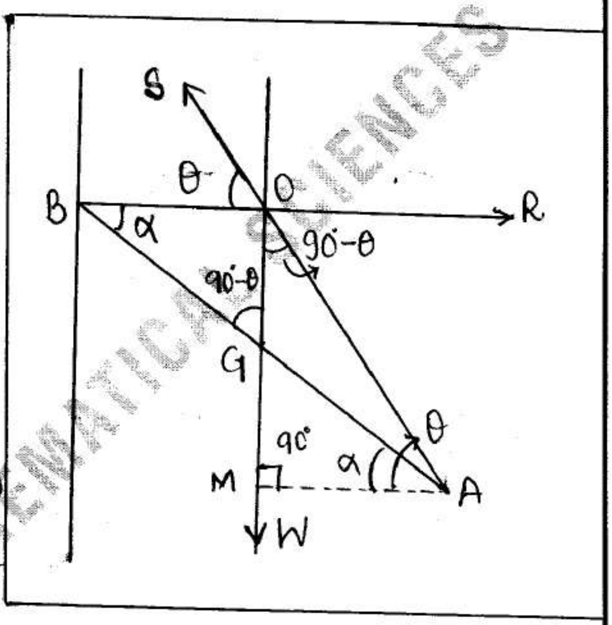
i.e. $$\boxed{\int_a^b f(t)\,dt = f(\eta)(b-a)}$$

Hence proved

2009 P-I

Ques: 5(c)) A uniform rod AB is movable about a hinge at A and rests with one end in contact with a smooth vertical wall. If the rod is inclined at an angle of 30° with the horizontal, find the reaction at the hinge in magnitude and direction?

Solution :-



Uniform Rod AB, length=2a.
movable about the end A and rest with end B.
W - weight of Rod
G - middle point of AB.

The rod is in equilibrium under the action of following three forces only:

(i) R, the reaction of the wall at B acting at right angles to the wall.

(ii) S, the reaction of the hinge at A, and

(iii) W, the weight of the rod acting vertically downwards at its middle point G.

Let, the rod AB and the reaction S makes angles α and θ respectively with the horizontal; ie

$\angle ABO = \alpha$ and $\angle OAM = \theta$ and given $\alpha = 30°$

$\therefore \angle OGB = 90° - \alpha = 60°$ and $\angle AOM = 90° - \theta$

In ΔOAB, by the trigonometric theorem, we have

$(AG + BG)\cot OGB = AG \cot AOG - BG \cot BOG$

$(a+a)\cot(90° - \alpha) = a \cot(90° - \theta) - a \cot 90°$

$$2a \tan\alpha = a\tan\theta$$

$$\tan\theta = 2\tan\alpha \quad \text{——— (1)}$$

$\therefore$ The reaction S at the hinge makes an angle $\theta = \tan^{-1}(2\tan 30^\circ)$

$\boxed{\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)}$ with the horizontal.

Now by Lami's theorem at the point O, we have

$$\frac{S}{\sin 90^\circ} = \frac{W}{\sin(180^\circ-\theta)} = \frac{R}{\sin(90^\circ+\theta)}$$

$$\therefore S = \frac{W}{\sin\theta} = W\operatorname{cosec}\theta$$

$$S = W\sqrt{1+\cot^2\theta}$$

$$S = W\sqrt{1+\frac{1}{4}\cot^2\alpha} \quad \left[\because \cot\theta = \frac{1}{2}\cot\alpha\right] \text{ from (1)}$$

$$S = \frac{1}{2}W\sqrt{4+1\cot^2\alpha}$$

$$S = \frac{1}{2}W\sqrt{4+\cot^2 30^\circ} \quad [\because \alpha = 30^\circ]$$

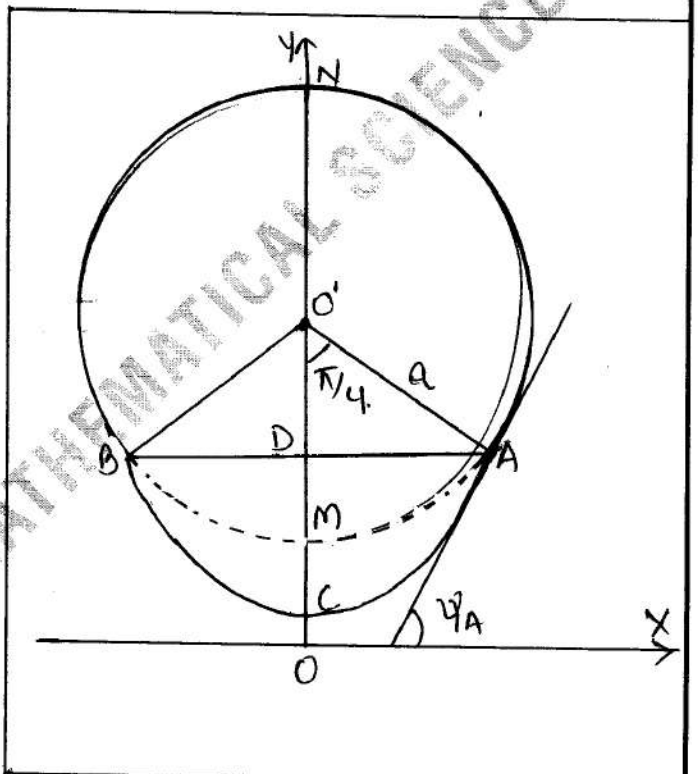
$$S = \frac{1}{2}W\sqrt{4+(\sqrt{3})^2} = \frac{1}{2}W\sqrt{7}$$

$\therefore \boxed{S = \frac{\sqrt{7}}{2}W}$ is the magnitude and with the horizontal.

2009 P-I

Ques: 7(d) Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with three-fourth of the circumference of the pulley?

Solution:- Let ANBMA be the circular pulley of radius 'a' and ANBCA the endless chain hanging over it.



Since, the chain is in contact with the three-fourth of the circumference of the pulley, hence the length of this portion ANB of the chain

$= \frac{3}{4}$ (Circumference of the pulley)

$= \frac{3}{4}(2\pi a)$.

$= \frac{3}{2}\pi a$.

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal C is the lowest point i.e, the vertex, CO'N the axis and OX the directrix of this catenary.

Let $OC = c$ = the parameter of the catenary.

The tangent at A will be perpendicular to the radius O'A.

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal; then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left(\frac{1}{4}\times 2\pi\right) = \frac{\pi}{4}.$$

From the triangle AO'D, we have

$$DA = O'A \sin\frac{\pi}{4} = a\cdot\frac{1}{\sqrt{2}} = \frac{a}{\sqrt{2}}.$$

$\therefore$ from $x = c\log(\tan\psi + \sec\psi)$, for the point A we have; $x = DA = c\log(\tan\pi/4 + \sec\pi/4)$.

$$x = DA = c\log(1+\sqrt{2})$$

$$\Rightarrow \quad \frac{a}{\sqrt{2}} = c\log(1+\sqrt{2})$$

$$\Rightarrow \therefore \boxed{c = \frac{a}{\sqrt{2}\log(1+\sqrt{2})}}$$

From, $s = c\tan\psi$ applied for the point A, we have; $\text{arc } CA = c\tan\psi = c\cdot 1 = \dfrac{a}{\sqrt{2}(\log(1+\sqrt{2}))}$

Hence; the total length of the chain

= arc ABC + length of the chain in contact with the pulley.

$= 2\cdot \text{arc } CA + \frac{3}{2}\pi a.$

$$= \frac{2a}{\sqrt{2}(\log(1+\sqrt{2}))} + \frac{3}{2}\pi a = a\left[\frac{3}{2}\pi + \frac{\sqrt{2}\,a}{\log(1+\sqrt{2})}\right]$$

$$\therefore \boxed{\text{Total length of chain} = a\left[\frac{3}{2}\pi + \frac{\sqrt{2}\,a}{\log(1+\sqrt{2})}\right]}$$

Required result.

2009-P-II⟩

Ques: 2(d)⟩ Show that if $f:[a,b]\to\mathbb{R}$ is a continuous function then $f([a,b])=[c,d]$ for some real numbers c and d, $c\le d$.

Solution:- Let $f:[a,b]\to\mathbb{R}$ be a continuous function.

To prove that :- $f([a,b])=[c,d]$ for some real numbers c, d where, $c\le d$.

i.e to prove that: continuous image of a closed bounded interval is closed and bounded.

Now, if f were not bounded above, we could find a point x_1 with $f(x_1)>1$, a point x_2 with $f(x_2)>2, \ldots$

Now, by Bolzano Weierstrass theorem, the sequence (x_n) has a subsequence (x_{ij}) which converges to a point $\alpha\in[a,b]$.

By our construction the sequence $(f(x_{ij}))$ is unbounded, but by the continuity of f, this sequence should converge to $f(\alpha)$, which is a contradiction.

$\therefore$ The sequence $(f(x_{ij}))$ is bounded above. Similarly, we can show that sequence $(f(x_{ij}))$ is bounded below.

$\Rightarrow$ A continuous function f on a closed bounded interval is bounded.

Now, To show that 'f' attains its bounds.

Let 'd' be the lub of the set $X = \{f(x) \mid x \in [a,b]\}$

Now, we need to find a point $\beta \in [a,b]$ s.t

$$f(\beta) = d$$

Consider a sequence in the following way:

For each $n \in \mathbb{N}$, let x_n be a point for which

$$|d - f(x_n)| < 1/n$$

such a point must exist otherwise $d - 1/n$ would be an upper bound of X.

Now, Some subsequence of $(x_1, x_2, x_3 \ldots)$ converges to β (say)

and $(f(x_1), f(x_2), \ldots) \longrightarrow d$ and by continuity $f(\beta) = d$ which is as required.

Similarly, we show that 'f' attains its lower bound.

Thus, A continuous function 'f' on a closed bounded interval is bounded and attains its bound.

$\Rightarrow$ The image of an Interval $I = [a,b]$ is bounded and is a subset of $[c,d]$ (say) where c, d are the lub and glb of the image.

$$\Rightarrow c \leq d \quad ; \; c, d \in \mathbb{R}$$

Since, the function attains its bounds, $c, d \in f[a,b]$ and so the image is $[c,d]$

Hence, the theorem.

IAS-2009-P-II

Ques: 3(b)) How many elements does the quotient ring $\frac{\mathbb{Z}_5[x]}{[x^2+1]}$ have? Is it an integral domain? Justify your answers.

Solution:-

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

$$|\mathbb{Z}_5| = 5$$

$\frac{\mathbb{Z}_5[x]}{x^2+1}$ has $|\mathbb{Z}_5|^{\deg(x^2+1)}$ elements

$$= 5^2 = 25$$

(x^2+1) is an irreducible polynomial in $\mathbb{Z}_5[x]$.

$\therefore (x^2+1) = f \times g \;/\; f, g \in \mathbb{Z}_5[x]$

where either $\deg f = 0$
or $\deg g = 0$

Let $\deg(g) = 0 \quad \therefore g = a_0$ [constant]
$[a_0 \in \mathbb{Z}_5]$

$\mathbb{Z}_5$ is a field.

$\therefore \exists\ a_0^{-1} \Rightarrow a_0$ is a unit

$\therefore (x^2+1) = f \times a_0 = f \times$ (unit).

$\therefore (x^2+1)$ is an irreducible element in $\mathbb{Z}_5[x]$.

Since, F is a field, F(x) is a P.I.D.

$\therefore \langle x^2+1 \rangle$ is a maximal ideal.

This implied $\frac{\mathbb{Z}_5[x]}{(x^2+1)}$ is a field.

Therefore Integral Domain.

2009-P-II

Ques: 3(c) Show that

$$\lim_{x\to 1} \sum_{n=1}^{\infty} \frac{n^2x^2}{n^4+x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$$

Justify all steps of your answer by quoting the theorems you are using.

Solution:- Let $f_n(x) = \dfrac{n^2x^2}{n^4+x^4}$ for $n = 1, 2, \ldots$

Then, $|f_n(x)| \leq \dfrac{1}{n^4}$ for $n = 1, 2, 3, \ldots$

and for all $x \in \mathbb{R}$

Let $M_n = \dfrac{1}{n^4}$ for $n = 1, 2, 3 \ldots$

Then $|f_n(x)| \leq M_n$ for $n = 1, 2, \ldots$

$\sum M_n$ is a convergent series of positive real number (by p-test) and therefore

By Weierstrass' M-test,

$\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent for all real x.

Since, the series is uniformly convergent for all real x;

$$\lim_{x\to 1} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to 1} f_n(x).$$

$$\lim_{x\to 1} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to 1} \frac{n^2x^2}{n^4+x^4}$$

$$\boxed{\lim_{x\to 1} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{n^2}{n^2+1}}$$

Hence the result.

IAS-2009-P-II

Ques: 3(d) Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point.

Solution:-

This statement represents the Bolzano-Weierstrass theorem:

Proof: Let S be a bounded infinite subset of $\mathbb{R}$.
Since, S is a non-empty bounded subset of $\mathbb{R}$, sup S and inf S both exist.

Let $S^* = \sup S$ and $S_* = \inf S$.

Then $x \in S \Rightarrow S_* \leq x \leq S^*$

Let H be a subset of $\mathbb{R}$ defined by

$$H = \{x \in \mathbb{R} : x \text{ is greater than infinitely many elements of } S\}$$

Then; $S^* \in H$ and so H is a non-empty subset of $\mathbb{R}$.
Let $h \in H$. Then h is greater than infinitely many elements of S and therefore $h > S_*$, because no element $\leq S_*$ exceeds infinitely many elements of S.

Thus, H is a non-empty subset of $\mathbb{R}$, bounded below, S_* being a lower bound.

So ; inf H exists.

Let; $\inf H = \xi$.
We now show that ξ is a limit point of S.
Let us choose $\epsilon > 0$
Since; $\inf H = \xi$, there exists an element y in H such that $\xi \leq y \leq \xi + \epsilon$.

Since $y \in H$, y exceeds infinitely many elements of S and consequently $\xi + \epsilon$ exceeds infinitely many elements of S.

Since, ξ is the infimum of H, $\xi - \epsilon$ does not belong to H ; i.e $\xi - \epsilon \notin H$

and so $\xi - \epsilon$ can exceed at most a finite numbers of elements of S. Thus, the neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many elements of S.

This holds for each $\epsilon > 0$.

Therefore, ξ is a limit point of S.

Hence proved

2009 P-II Ques: 5(c)(i) The equation $x^2+ax+b=0$ has two real roots α and β. Show that the iterative method given by:

$$x_{k+1} = -(ax_k+b)/x_k \quad , \quad k=0,1,2,\ldots$$

is convergent near $x=\alpha$, if $|\alpha|>|\beta|$.

Solution:- The iterations are given by-

$$x_{k+1} = \frac{-(ax_k+b)}{x_k} = g(x_k) \quad (\text{say})$$

$$k=0,1,2,3\ldots$$

But the known theorem,

If $g(x)$ and $g'(x)$ are continuous in an interval about a root α of the equation $x=g(x)$ and if $|g'(x)|<1$ for all x in the interval, then the successive approximations, $x_1, x_2, \ldots$ given by

$$x_k = g(x_{k-1}) \quad ; \quad k=1,2,3\ldots$$

converges to the root α provided that the initial approximation x_0 is chosen in the interval.

$\therefore$ These iterations converges to α if

$$|g'(x)| < 1 \quad \text{near } \alpha$$

ie. $|g'(x)| = \left|-\frac{b}{x^2}\right| < 1.$

Note that $g(x)$ is continuous near α.

If the iterations converges to $x=\alpha$, then we require

$$|g'(\alpha)| = \left|\frac{-b}{\alpha^2}\right| < 1$$

Thus; $|b| < |\alpha|^2$

i.e $|\alpha|^2 > |b|$ —— (1)

Given that α and β are roots of the equation

$$x^2 + ax + b = 0$$

then $\alpha + \beta = -a$ and $\alpha\beta = b$

$\Rightarrow |b| = |\alpha||\beta|$. —— (2).

Substituting (2) in (1), we get.

$$|\alpha|^2 > |b| = |\alpha||\beta|$$

$$|\alpha|^2 > |\alpha||\beta|$$

$$\boxed{|\alpha| > |\beta|}$$ Hence proved.

2010 P-I

Q.5(a) Consider the differential equation
$$y' = \alpha x \quad , x > 0$$
where α is a constant. show that –

(i) If $\phi(x)$ is any solution and $\psi(x) = \phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant;

(ii) If $\alpha < 0$ then every solution tends to zero as $x \to \infty$.

[There's a correction in the question.]

Consider the differential Equation:
$$y'(x) = \alpha\, y(x) \quad , \quad x > 0$$
where α is a constant. show that –

(i) If $\phi(x)$ is any solution and $\psi(x) = \phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant;

(ii) If $\alpha < 0$ then every solution tends to zero as $x \to \infty$.

Solution:- Given that
$$y'(x) = \alpha\, y(x) \quad , \quad 0 < x < \infty \quad \text{——} \ (1)$$
where α is a real constant.

(i) Let $\phi(x)$ be any solution of (1) and
$$\psi(x) = \phi(x)e^{-\alpha x}$$

To show that: $\psi(x)$ is a constant
$$\because \ \psi(x) = \phi(x)e^{-\alpha x}$$
$$\Rightarrow \ \psi'(x) = e^{-\alpha x}\left(\phi'(x) - \alpha\phi(x)\right)$$

$$\left[\begin{array}{l} \because \ \phi(x) \text{ is a solution of } (1) \\ \Rightarrow \phi(x) \text{ satisfies } (1) \\ \Rightarrow \phi'(x) = \alpha\,\phi(x) \\ \Rightarrow \phi'(x) - \alpha\,\phi(x) = 0 \end{array}\right]$$

$\Rightarrow \quad \varphi'(x) = e^{-\alpha x}[0] = 0$

$\Rightarrow \quad \varphi(x) = \text{constant}$ [on Integrating]

(ii) We know that $y(x)$ of the form $Ce^{\alpha t}$ are solutions of (1)

We shall also show that there are no other solutions.

Let $y(x)$ be an arbitrary solution to (1)

We compare this unknown solution $y(x)$ to the known solution $e^{\alpha x}$ by defining $\rightarrow$

$$Z(x) = e^{-\alpha x} y(x)$$

Then
$$z'(x) = -\alpha e^{-\alpha x} y(x) + e^{-\alpha x} y'(x)$$
$$= -\alpha e^{-\alpha x} y(x) + e^{-\alpha x} \cdot \alpha y(x)$$
$$= 0 \qquad [\because y'(x) = \alpha y(x) \text{ from (1)}]$$

$\Rightarrow Z(x)$ has to be a constant.

Denoting constant by c, we have that

$$c = e^{-\alpha x} y(x)$$

$\Rightarrow \quad y(x) = ce^{\alpha x} \quad \forall x$, where c is an arbitrary constant ——(2)

from (2), for $\alpha < 0$

$y(x) \longrightarrow 0$ as $x \rightarrow \infty$

Hence the result.

2010 P-I

Ques: 7(a)) Show that the set of solutions of the homogeneous linear differential equation

$$y' + P(x)y = 0$$

on an interval $I = [a, b]$ forms a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

Solution:-

(i) Let 'W' be the set of solutions of the homogeneous linear differential equation

$$y' + p(x)y = 0 \quad \text{———} (1)$$

clearly; W is a subset of set of once differentiable function on $I = [a, b]$ which is a real vector space of continuous functions.

Thus, it is sufficient to show that W is subspace of the above, vector space of once differentiable function.

Now $0(x) = 0$ satisfies (1)

and hence $0 \in W$

$\Rightarrow W \neq \phi$

Also if u, v both satisfies (1),

then $\alpha u(x) + v(x)$ is also a solution of (1)

$\Rightarrow \alpha u + v \in W$

$\Rightarrow W$ is a subspace of I.

Hence the proof!

(ii) Obtain Dimension of W – yourself.

IAS-2010-P-II

Ques: 1(b) Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify?

Solution:

Let $A = \langle a \rangle$, $B = \langle b \rangle$.

$|A| = 2$ $\quad$ $|B| = 3$

$A = \{a, a^2\}$ $\quad$ $B = \{b, b^2, b^3\}$

$A \times B = \{ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3\}$

Let $C = \langle c \rangle$

$|C| = 6$

$C = \{c, c^2, c^3, c^4, c^5, c^6\}$

$|c| = 6$ $\quad$ $|c^2| = 3$ $\quad$ $|c^3| = 2$

$|c^4| = 3$ $\quad$ $|c^5| = 6$ $\quad$ $|c^6| = 1$

$A \times B =$

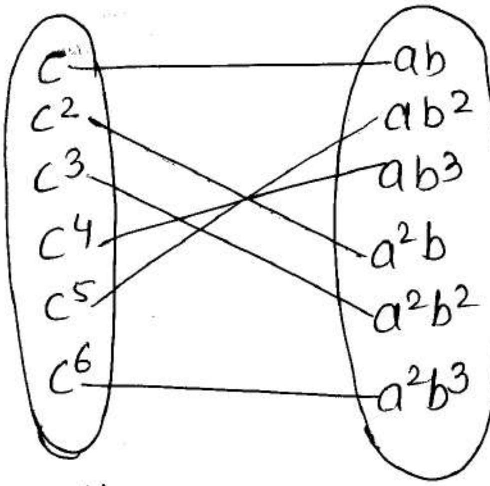
	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3
ab	a^2b^2	a^2b^3	a^2b	ab^2	ab^3	ab
ab^2	a^2b^3	a^2b	a^2b^2	ab^3	ab	ab^2
ab^3	a^2b	a^2b^2	a^2b^3	ab	ab^2	ab^3
a^2b	ab^2	ab^3	ab	a^2b^2	a^2b^3	a^2b
a^2b^2	ab^3	ab	ab^2	a^2b^3	a^2b	a^2b^2
a^2b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3

$a^2b^3 = e$ $\quad$ $\therefore |a^2b^3| = 1$

$|ab| = 6$ $\quad$ $|a^2b| = 3$

$|ab^2| = 6$ $\quad$ $|a^2b^2| = 3$

$|ab^3| = 2$



Let us generalise;

the order of $\langle c \rangle = p = m \times n$.

Let order of $\langle A \rangle = m$, $b = \langle n \rangle$

$$A \times B = \left\{ a^r b^s \,/\, 0 < r \leq m,\ 0 < s \leq n \atop r, s \in I \right\}$$

Order of $|a^r \cdot b^s| = |c^t|$.

$\therefore \langle c \rangle \cong A \times B$.

Therefore, $\langle c \rangle$ (a cyclic group of order 6) is isomorphic to the product of a cyclic group of order 2 i.e. $\langle A \rangle$ and a cyclic group of order 3 i.e $\langle B \rangle$.

2010 P-II. Ques: 5(d)(i) Suppose a computer spends 60% of its time handling a particular type of computation when running a given program and its manufacturers makes a change that improves its performance on that type of computation by a factor of 10. If the program takes 100 seconds to execute. What will its execution time after the change?

Solution:- Given;

Execution time of program before change = 100 sec.

It spends 60% of its time handling a particular type of computation = 60% of 100 sec

= 60 sec.

Remaining time for execution = (100 − 60) sec

= 40 sec.

Manufactures improves its computation by a factor of 10 by some change ⇒ which increase its efficiency by a factor of 10 and hence decrease the computation time by factor of 10

i.e. $\frac{60}{10} = 6$ sec.

Now, Execution time after the change = 40 + 6

= 46 seconds.

Ques: 5(d)ii) If $A \oplus B = AB' + A'B$; find the value of $x \oplus y \oplus z$.

Solution:- Given $A \oplus B = AB' + A'B$.

$$x \oplus y \oplus z = (x \oplus y) \oplus z$$

$$= (x'y + y'x) \oplus z$$

$$= (x'y \oplus z) + (y'x \oplus z) \quad \text{[Distributive law]}$$

$$= x'yz' + (x'y)'z + y'xz' + (y'x)'z$$

$$= x'yz' + (x'' + y')z + y'xz' + (y'' + x')z$$

$$= z'(x'y + xy') + z(x + y' + x' + y)$$

$$= z'(x'y + xy') + z((x + x') + (y + y'))$$

$$= z'(x'y + xy') + z[1 + 1] \quad \text{[since } A + A' = 1]$$

$$= z'(x'y + xy') + z \quad \text{[since } 1 + 1 = 1]$$

$$\therefore \boxed{x \oplus y \oplus z = z + z'(x'y + xy')}$$

Required result.

2010 P-II

Ques: 7(c)(ii) For the given set of data points $(x_1, f(x_1)), (x_2, f(x_2)), \ldots (x_n, f(x_n))$. write an algorithm to find the value of $f(x)$ by using Lagrange's interpolation formula.

Solution:-

1: Start

2: Read x, n

3: for $i = 1$ to $(n+1)$ in steps of 1 do Read x_i, $f(x_i)$ endfor

Remarks: The above statement reads x_i's and the corresponding values of $f(x_i)$'s.

4: sum $\leftarrow 0$

5: for $i = 1$ to $(n+1)$ in steps of 2 do

6: prodfunc $\leftarrow 1$

7: for $j = 1$ to $(n+1)$ in steps of 2 do

8: If $(j \neq i)$ then

prodfunc $\leftarrow$ prodfunc $\times (x - x_j)/(x_i - x_j)$

endfor

9: sum $\leftarrow$ sum + $f(x_i)$ × prodfunc

Remarks: sum is the value of f at x

endfor

10: Write x, sum

11: Stop.

2011 P-I

Ques: 3(c)) Find the volume of the solid that lies under the parabola $z = x^2+y^2$ above x-y plane and inside the cylinder $x^2+y^2=2x$.

Solution:-

The desired volume can be obtained by integrating $z = x^2+y^2$. over the circle $x^2+y^2=2x$.

Let us change it into polar co-ordinates

let $x = r\cos\theta$ $y = r\sin\theta$ so that.

Equation of paraboloid $\Rightarrow z = x^2+y^2 = r^2$

Equation of circle $\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2 = 2r\cos\theta$

$$r^{\cancel{2}}(\cos^2\theta+\sin^2\theta) = 2\cancel{r}\cos\theta$$

$$\boxed{r = 2\cos\theta}$$

$$\boxed{dz = r\,dr\,d\theta}$$

Hence, desired volume.

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{2\cos\theta} r^2.r\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{0}^{2\cos\theta} r^3\,dr\,d\theta$$

$$V = \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi}\left[\frac{2^4\cos^4\theta}{4} - 0\right]d\theta$$

$$V = 4\int_0^{\pi}\cos^4\theta\,d\theta = \cancel{4}\times\frac{3\pi}{\cancel{8}2} = \frac{3\pi}{2}.$$

$$\therefore \boxed{\text{Desired Volume} = \frac{3\pi}{2}\text{ unit cubic.}}$$

2011
P-I

Ques: 7 (a) A mass of 560 kg. moving with a velocity 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates?

Solution: Given; mass of moving body $= m = 560$ kg.

Initial velocity of the body $= 240$ m/sec

Time taken to brought to rest $= \frac{1}{100}$ sec

$= 0.01$ sec.

Final velocity $= 0$ m/sec.

∴ The retardation of the body when it hits the fixed target $= a$.

Hence, by equation of motion.

$$v = u + at$$

$$a = \frac{v-u}{t} = \frac{-240 \text{ m/sec}}{0.01 \text{ sec}}$$

$$\boxed{a = -24000 \text{ m/s}^2.}$$

The impulse of blow $= F = ma$

$\Rightarrow F = 560 \text{ kg} \times -24000 \text{ m/s}^2$

$\Rightarrow F = 560000 \times -24000 \text{ gm/s}^2$

$\Rightarrow F = -1344 \times 10^7$ N.

$$\Rightarrow \boxed{|F| = 1344 \times 10^7 \text{ N}}$$

Now, the distance through which it penetrates.
$= S$

and with equation motion.

$$S = vt - \frac{1}{2}at^2$$

$$S = 0 \times 0.01 - \frac{1}{2} \times 24000 \times (0.01)^2$$

$$S = 0 + \frac{1}{\cancel{2}} \times \cancel{24000}^{12} \times \frac{1}{10\cancel{000}}$$

$$\boxed{S = 1.2 \text{ meters.}}$$

$\therefore$ $\boxed{\text{Penetrated Distance} = 1.2 \text{ meters}}$

2011
P-I

Ques: 7(c)(i) After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half of its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground.

Solution :-

Initial velocity of ball $= u = 0$ m/sec

Acceleration $= 9.8\, m/s^2$ (falling under gravity)

Time taken $= 5$ sec.

final velocity before reaches the glass $= V$

$$V = u + at \quad [\because \text{Equation of motion}]$$

$$V = 0 + 9.8 \times 5$$

$$\boxed{V = 49\, m/sec}$$

It reduces half of its velocity when it hit the glass pane i.e $u' = \frac{1}{2}V = 24.5\, m/sec$

Acceleration $= 9.8\, m/sec^2$ (falling under gravity)

(t') Time taken $= 1$ sec.

By equation of motion

$$S = u't + \frac{1}{2}at'^2$$

$$S = 24.5 \times 1 + \frac{1}{\cancel{2}} \times \cancel{9.8}^{4.9}$$

$$S = 24.5 + 4.9 = 29.4 \text{ metres.}$$

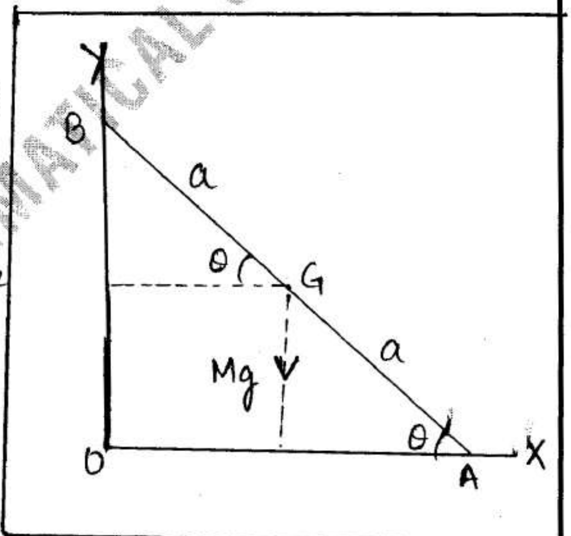
Hence; the height of the glass pane from ground $= 29.4$ meters.

2011 P-II

Ques:- 8(a)) The ends of a heavy rod of length 2a are rigidly attached to two small rings at its ends which can respectively slide on smooth horizontal and vertical wires OX and OY. The rod starts at an angle α to the horizontal with angular velocity $\sqrt{\left\{\frac{3g(1-\sin\alpha)}{2a}\right\}}$ and moves downward. Show that it will strike horizontal wire at the end of time

$$-2\sqrt{\left(\frac{a}{3g}\right)}\log\left\{\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right\}.$$

Solution:-



Let AB be the rod of mass M and length 2a.
At time 't', let the rod be inclined at an angle θ to the horizontal. Referred to OX and OY as axes co-ordinates of C.G. G of AB are $(a\cos\theta, a\sin\theta)$. The velocity of G is given by -

$$v_G^2 = (-a\sin\theta\,\dot{\theta})^2 + (a\cos\theta\,\dot{\theta})^2 = a^2\dot{\theta}^2.$$

$\therefore$ Kinetic Energy of the rod at time t

$$= \frac{1}{2}M\cdot\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}M\cdot v_G^2 = \frac{1}{2}M\left(\frac{1}{3}a^2\dot{\theta}^2 + a^2\dot{\theta}^2\right)$$

$$\boxed{\text{K.E. of the rod at time } t = \frac{2}{3}Ma^2\dot{\theta}^2}$$

But, initially $\dot{\theta} = \sqrt{\frac{3g(1-\sin\alpha)}{2a}}$ (given).

$\therefore$ Initial K.E. of rod $= \frac{\cancel{2}}{\cancel{3}} Ma^{\cancel{2}} \cdot \frac{\cancel{3}g}{\cancel{2a}}(1-\sin\alpha)$

$$\boxed{\text{Initial K.E. of rod} = Mag(1-\sin\alpha)}$$

Hence, every equation gives
Change in K.E. = Work done by gravity

i.e $\frac{2}{3}Ma^2\dot{\theta}^2 - Mga(1-\sin\alpha) = Mg(a\sin\alpha - a\sin\theta)$

(or) $\frac{2}{3}a^2\dot{\theta}^2 = g(1-\sin\theta)$

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\sin\theta)$$

$$\therefore \boxed{\dot{\theta} = \frac{d\theta}{dt} = -\sqrt{\frac{3g}{2a}(1-\sin\theta)}}$$

(-ve sign is taken because motion is towards θ decreasing)

(or) $dt = -\sqrt{\frac{2a}{3g}} \cdot \frac{d\theta}{\sqrt{1-\sin\theta}}$

Integrating from $\theta=\alpha$ to $\theta=0$, the required time is given by—

$$t = -\sqrt{\left(\frac{2a}{3g}\right)} \int_{\theta=\alpha}^{0} \frac{d\theta}{\sqrt{1-\sin\theta}}$$

$$t = -\sqrt{\frac{2a}{3g}} \cdot \int_{\alpha}^{0} \frac{d\theta}{\sqrt{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}}$$

$$t = -\sqrt{\frac{2a}{3g}} \int_{\alpha}^{0} \frac{d\theta}{(\cos\theta/2 - \sin\theta/2)}$$

$$t = -\frac{1}{\sqrt{2}}\sqrt{\frac{2a}{3g}} \int_{\alpha}^{0} \frac{d\theta}{\sin(\pi/4 - \theta/2)}$$

$$= \sqrt{\frac{a}{3g}} \int_0^{\alpha} \operatorname{cosec}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta$$

$$= \sqrt{\frac{a}{3g}} \left[-2 \log \tan\left(\frac{\pi}{8} - \frac{\theta}{4}\right)\right]_0^{\alpha}$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)} \left[-\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) + \log \tan\frac{\pi}{8}\right]$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) - \log \tan\frac{\pi}{8}\right]$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) + \log \cot\frac{\pi}{8}\right]$$

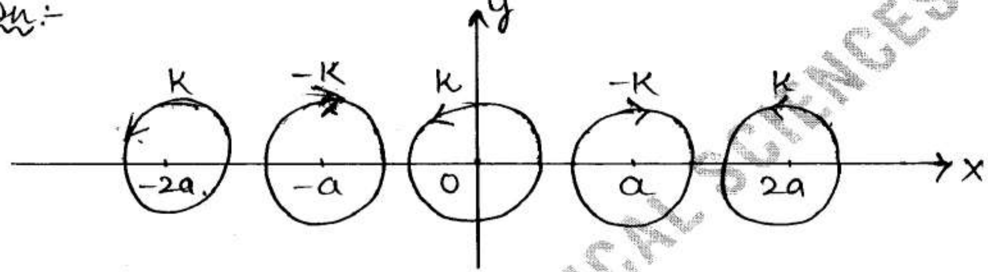
$$\boxed{t = -2\sqrt{\frac{a}{3g}} \left[\log \tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) \cdot \cot\frac{\pi}{8}\right]}$$

Hence the result

2011 P-II.

Ques: 8(b) An infinite row of equidistant rectilinear vortices are at a distance a part. The vortices are of the same numerical strength K but they alternatively of opposite signs. Find the complex function that determines the velocity potential and the stream function.

Solution:-



Let the row of vortices be taken along the x-axis. Let there be vortices of strength K each at the points $(0,0)$, $(\pm 2a, 0)$, $(\pm 4a, 0)$, ... and those of strength $-K$ each at the points $(\pm a, 0)$, $(\pm 3a, 0)$, ...
The complex potential of the entire system is given by-

$$w = (iK/2\pi)\left[\{\log z + \log(z-2a) + \log(z+2a) + \cdots\} - \{\log(z-a) + \log(z+a) + \log(z-3a) + \cdots\}\right]$$

$$w = \frac{iK}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\ldots}{(z^2-a^2)(z^2-3^2a^2)\ldots}$$

$$= \frac{ik}{2\pi}\log\frac{z/2a\left[1-[z/2a]^2\right]\left[1-(z/4a)^2\right]\ldots}{\left[1-(z/a)^2\right]\left[1-(z/3a)^2\right]\ldots} + \text{a Constant}$$

Thus;

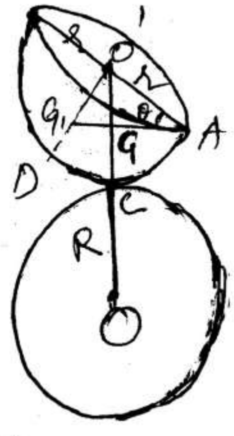
$$w = \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)}$$

$$\therefore \boxed{w = \frac{ik}{2\pi}\log\tan(\pi z/2a)}$$

which is desired potential function that determines the velocity potential and stream function.

Q.7(b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle

Sol:



Let O' be the centre of the base of the hemispherical shell of radius r. Let weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the spherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre at O at the highest point C.

For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$ i.e., $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e, $\frac{1}{h} > \frac{R+r}{Rr}$

i.e, $h < \frac{rR}{R+r}$ ——— (1)

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is maximum i.e, if $O'G$ is perpendicular to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$.

Then from right angled $\Delta O'G_1$

$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$

$\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ the minimum value of $O'G$

$= O'A \sin\theta = r\left(\frac{1}{\sqrt{5}}\right) = r/\sqrt{5}$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$= r - \frac{r}{\sqrt{5}} = \frac{r(\sqrt{5}-1)}{\sqrt{5}}$.

Hence the equilibrium will be stable, whatever be the weight of the particle at A,

if $\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{r+R}$ i.e, if $\frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$

i.e, if $(\sqrt{5}-1)R + (\sqrt{5}-1)r < R\sqrt{5}$

i.e, if $R/r > \sqrt{5}-1$.

2012 P-I

Ques: 3(b)) Define a sequence S_n of real numbers by

$$S_n = \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{n+i}$$

Does $\lim_{n\to\infty} S_n$ exist? If so, compute the value of this limit and justify your answer.

Solution:-

Given; $S_n = \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{n+i}$

$$\underset{n\to\infty}{\text{lt}} S_n = \underset{n\to\infty}{\text{lt}} \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{(n+i)}$$

$$= \underset{n\to\infty}{\text{lt}} \sum_{i=1}^{n} \frac{\left[\log\left(\frac{n+i}{n}\right)\right]^2}{n(1+i/n]}$$

$$= \underset{n\to\infty}{\text{lt}} \sum_{i=1}^{n} \frac{\left[\log\left[1+i/n\right]\right]^2}{n\left[1+i/n\right]} \quad \text{——} (1)$$

Using Riemann sum

(1) $\underset{n\to\infty}{\text{lt}} \sum_{i=0}^{n} \to \int_0^1$

(2) $\frac{i}{n} \to x$

(3) $\frac{1}{n} \to dx.$

Thus, modifying equation (1)

$$\underset{n\to\infty}{\text{Lt}} S_n = \int_0^1 \frac{[\log(1+x)]^2}{1+x} dx$$

let $\log(1+x) = t$

$$\frac{dx}{1+x} = dt$$

$\log(1+x) = t \quad \rightarrow$ If $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = \log 2.$

$$\therefore \underset{n\to\infty}{Lt} S_n = \int_0^{\log 2} t^2\, dt$$

$$= \left[\frac{t^3}{3}\right]_0^{\log 2}$$

$$\therefore \boxed{\underset{n\to\infty}{Lt} S_n = \frac{(\log 2)^2}{3}}$$ required value of limit.

Hence, $\underset{n\to\infty}{Lt} S_n$ exists.

2012 P-II

Ques: 4(b)) Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify.

Solution:-

Let $f(x)$ be defined on $[a,b]$ as follows

$$f(x) = \begin{cases} -1 & ; \text{when } x \text{ is rational} \\ 1 & ; \text{when } x \text{ is irrational.} \end{cases}$$

T.S.T $f(x)$ is not Reimann integrable on $[a,b]$

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a,b]$ and let its sub interval be

$$I_r = [x_{r-1}, x_r], \text{ for } r = 1, 2, \ldots, n.$$

Here, δ_r = the length of $I_r = x_r - x_{r-1}$

Let M_r and m_r be respectively, the supremum and infimum of the function 'f' in I_r.

Since, rational points and irrational points are everywhere dense so every sub-interval I_r will contain rational and irrational numbers.

Hence, by definition of $f(x)$, it follows that

$m_r = -1$ and $M_r = 1$, for $r = 1, 2, 3, \ldots, n$.

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} 1 \cdot \delta_r$$

$$= \sum_{r=1}^{n} \delta_r$$

$$= \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1})$$

$$= (x_n - x_0)$$

$$= (b-a)$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} (-1)\delta_r$$

$$= \sum_{r=1}^{n} -\delta_r$$

$$= -\left[\sum_{r=1}^{n} (x_r - x_{r-1})\right]$$

$$= -\left[(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})\right]$$

$$= -[x_n - x_0]$$

$$= -(b-a)$$

$$\therefore \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(P,f) = 1$$

and $$\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} L(P,f) = -1$$

Since; $\overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$

$\Rightarrow f(x)$ is not Reimann-integrable.

Now, we consider.

$$|f(x)| = \begin{cases} |-1| & \text{, when } x \text{ is rational} \\ |1| & \text{, when } x \text{ is irrational} \end{cases}$$

i.e. $|f(x)| = 1 \quad \forall\, x \in \mathbb{R}$

Since, $|f(x)|$ is a constant function,

$\Rightarrow |f(x)|$ is a continuous function over $\mathbb{R}$

$\Rightarrow |f(x)|$ is a Reimann integrable.

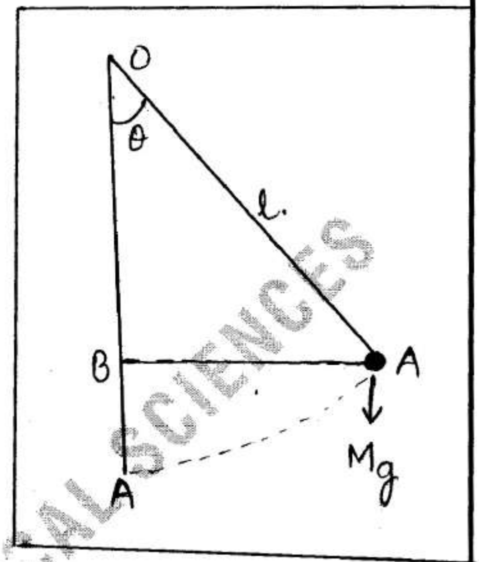
— [By theorem →
Let a function $f : [a,b] \to \mathbb{R}$
be continuous on $[a,b]$.
Then f is Reimann integrable on $[a,b]$]

Hence the result

<u>Alternatively</u>;
$|f(x)|$ is Reimann Integrable can be shown using $U(P,f)$ and $L(P,f)$ similar to I part of the question as well.

2012 P-II. Ques: 5(d)) Obtain the equations governing the motion of a spherical pendulum.

Solve :-.



Let L be the length of the spherical pendulum and θ be the angle made by the string with the vertical at time 't'. Thus, θ is the only generalised co-ordinate. Then the velocity of mass M at A will be $\underline{v = l\dot{\theta}}$

∴ Total Kinetic Energy

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$$

And the potential function

$$V = Mg(A'B) = Mg(l - l\cos\theta) = Mgl(1-\cos\theta)$$

∴ The lagrangian function

$$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$$

Lagrange's θ- equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

i.e. $\frac{d}{dt}\left(Ml^2\dot{\theta}\right) + Mgl\sin\theta = 0$

$$\Rightarrow Ml^2\ddot{\theta} + Mgl\sin\theta = 0$$

$$\ddot{\theta} = \frac{-\cancel{M}g\cancel{l}\sin\theta}{\cancel{M}l^{\cancel{2}}} = -\frac{g\sin\theta}{l}$$

$\boxed{\ddot{\theta} = -\frac{g}{l}\theta}$, since θ is small. which is required equation of motion of spherical pendulum.

2012
P-II

Ques: 7(c)) In a certain examination, a candidate has to appear for one major and two minor subjects. The rules for declaration of results are: marks for major are denoted by M_1 and for minors by M_2 and M_3. If the candidate obtains 75% and above marks in each of the three subjects, the candidate is declared to have passed the examination in first class with distinction. If the candidate obtains 60% and above marks in each of the three subjects, the candidate is declared to have passed the examination in first class. If candidate obtains 50% or more in major, 40% or above in each of the two minors and an average of 50% or above in all the three subjects put together, the candidate is declared to have passed the examination in second class. All those candidates, who have obtained 50% and above in major and 40% or above in minor, are declared to be passed the examination. If the candidate obtain less than 50% in major or less than 40% in any one of the two minors, the candidates is declared to have failed in the examinations. Draw a flow chart to declare the result for the above.

Solution:- Brief Algorithm:

Step 1: Start

Step 2: Read candidate's details for ensuring individual specific computation.
Also Read M_1, M_2, M_3 as defined in question.

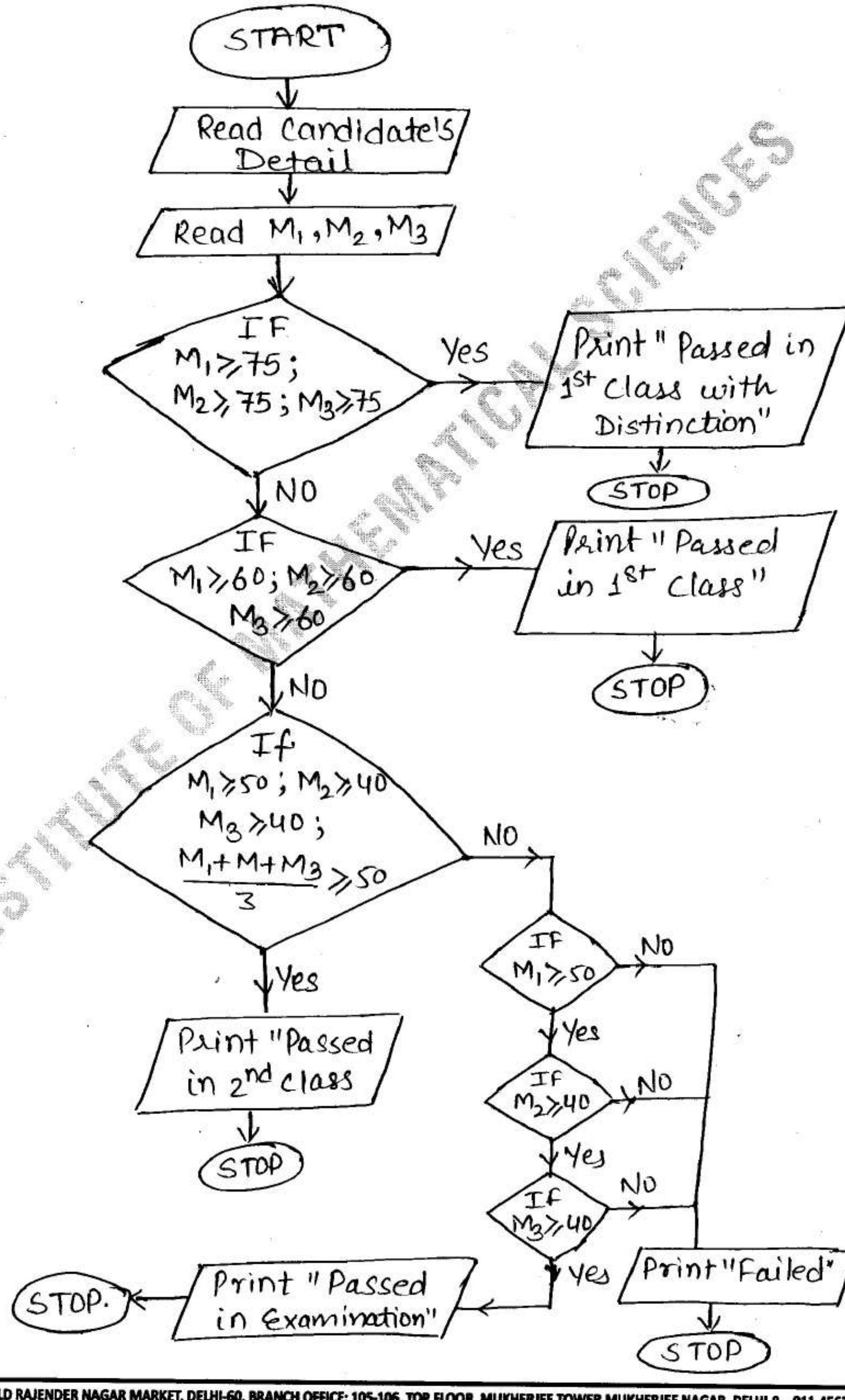
Step 3: Check together for 1st Class with Distinction.

Step 4: Check together for 1st Class

Step 5: Compute together for 2nd Class.

Step -6 : Check separately for the pass class else print failed in the examination

Step 7 : Stop.



2013 P-II

Ques: 2(d) Show that every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals.

Solution:- Let $U \subseteq \mathbb{R}$ be open and let $x \in U$

Either x is rational or irrational.

If x is rational, define

$$I_x = \bigcup_{\substack{I \text{ an open interval} \\ x \in I \subseteq U}} I$$

which, as a union of non-disjoint open intervals (each I contains x), is an open interval subset to U. If x is irrational, by openness of U there is $\epsilon > 0$ such that

$(x-\epsilon, x+\epsilon) \subseteq U$,

And there exists rational $y \in (x-\epsilon, x+\epsilon) \subseteq I_y$ (by definition of I_y)

Hence; $x \in I_y$.

So, any $x \in U$ is in I_q for some $q \in U \cap \mathbb{Q}$, and so

$$U \subseteq \bigcup_{q \in U \cap \mathbb{Q}} I_q.$$

But $I_q \subseteq U$ for each $q \in U \cap \mathbb{Q}$;

Thus,

$$U = \bigcup_{q \in U \cap \mathbb{Q}} I_q,$$

which is a countable union of open intervals.

P-II 2013 Ques:- 7(a)) Develop an algorithm for Newton-Raphson method to solve $f(x)=0$ starting with initial iterate x_0, n be the number of iterations allowed, eps be the prescribed relative error and delta be the prescribed lower bound for $f'(x)$.

Solution:-

1 : Start

2 : Read x_0, eps, delta, n

remark: x_0 is the initial guess, eps is the prescribed relative error, delta is the prescribed lower bound for f' and n the maximum number of iterations to be allowed. Steps 4 to 9 are repeated until the procedure converges to a root or iterations equal n.

3: for $i=1$ to n in steps 2 do

4: $f_0 \leftarrow f(x_0)$

5: $f_0' \leftarrow f'(x_0)$

6: if $|f_0'| \leq$ delta then GOTO 12

7: $x_1 \leftarrow x_0 - (f_0/f'(0))$

8: if $|(x_1 - x_0)/x_1| <$ eps then GOTO 14

9: $x_0 \leftarrow x_1$

end for.

10 : Write 'Does not converge in n iterations', f_0, f_0', x_0, x_1.

11 : Stop

12 : Write "Slope too small" x_0, f_0, f_0', i

13 : Stop

14 : Write 'convergent solution', x_1, $f(x_1)$, i

15 : Stop.

IAS-2013-P-I

Ques: 8(d)) Use stokes theorem to evaluate the integral $\int(-y^3dx+x^3dy-z^3dz)$, where 'C' is the intersection of cylinder $x^2+y^2=1$ and the plane $x+y+z=1$.

Solution:- By stokes theorem

$$\oint_C \vec{F}\cdot dr = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

Here ; $\vec{F} = -y^3\hat{i}+x^3\hat{j}-z^3\hat{k}$

Unit vector normal to the surface is given by normal to the plane

$$\nabla(x+y+z) = \hat{i}+\hat{j}+\hat{k}$$

$$\hat{n} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 3(x^2+y^2)\hat{k}$$

$$\therefore \int(-y^3dx+x^3dy-z^3dz) = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$= \iint_S 3(x^2+y^2)\hat{k}\cdot\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)ds$$

$$= \sqrt{3}\iint_S (x^2+y^2)\,ds$$

$$\left[\because \iint F \cdot ds = \iint F \frac{dx\,dy}{\hat{n}\cdot\hat{k}}\right]$$

$$= \sqrt{3}\iint_R (x^2+y^2)\frac{dx\,dy}{1/\sqrt{3}}$$

R is projection of the surface on xy plane

$$= 3\iint_R (x^2+y^2)\,dx\,dy$$

changing into cylindrical coordinate
$x = r\cos\theta$, $y = r\sin\theta$.

$$= 3\int_0^{2\pi}\int_0^1 \left[(r\cos\theta)^2+(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\,dr.$$

$$= 3\times 2\pi \times \left[\frac{r^4}{4}\right]_0^1 = 3\times \cancel{2}\pi\times\frac{1}{\cancel{4}\,2}$$

$$= 3\pi/2.$$

$$\boxed{\therefore \oint_C (-y^3dx + x^3dy - z^3dz) = \frac{3\pi}{2}}$$

required Solution