WORKSHEET - 3

Riemann Integration

1. If f(x) be defined on [0, 1] as follows—
f(x) = 1, when x is rational
= –1, when x is irrational
then prove that f is not Riemann integrable over [0, 1].

2. Let $f(x)=\frac{n}{n+2}$ if $\frac{1}{n+2}\le x\le\frac{1}{n}$, where n = 1, 2, 3,..... and f(0)=0. Prove that f is Riemann integrable in [0, 1].

3. Show that the function f defined on [0, 1] as $f(x)=2rx$ if $\frac{1}{r+1}<x<\frac{1}{r}, r\in N$ is integrable over [0, 1] and $\int_0^1 f(x)dx=\frac{\pi^2}{6}$

4. A function f is defined in [0, 1] by the condition that if, r, is a positive integer,
$F(x)=\frac{1}{a^{r-1}}$ when $\frac{1}{a^r}<x\le\frac{1}{a^{r-1}}$ for r=1,2,3...
Where a is an integer greater than 2. Show that $\int_0^1 f(x)dx$ exists and is equal to $\frac{a}{a+1}$.

5. Show that $\int_0^2 f(x)dx=2$,
where f(x) = 0, when x = n/(n + 1), (n + 1)/n, (n = 1, 2, 3, ...)
f(x) = 1, elsewhere.
Examine for continuity the function f so defined at the point x = 1.

6. Find the upper and lower Riemann integral for the function defined in the interval (0, 1) as follows:
$$f(x)=\begin{cases}\sqrt{1-x^2} & \text{when } x \text{ is rational}\\ 1-x & \text{when } x \text{ is irrational}\end{cases}$$
and show that f is not Riemann integrable in (0, 1).

7. (i) A Function of is defined on [0 , 1] by f (x) = $\frac{1}{n}$ for
$\frac{1}{n+1}<x\le\frac{1}{n}$, n = 1,2,3, and f (0) = 0.
Prove that f ∈ R[0, 1] and evaluate $\int_0^1 f(x)dx$

(ii) Prove that for every x > 0
$\frac{x}{1+x^2}<\tan^{-1}x<x$

8. If f (x) = 2
when x = $\frac{n}{n+1},\frac{n+1}{n}; n=1,2,3........$ = 1
else where,
then f(x) ∈ ℜ [0 , 2] and $\int_0^2 f(x)dx=2$

9. Prove that the function f defined on [–1, 1] by
f(x) = x sin $\frac{1}{x^2}-\frac{1}{x}\cos\frac{1}{x^2}$ when x ≠ 0,
= 0 when x = 0
admits a primitve $\frac{1}{2}x^2\sin\frac{1}{x^2}$ but $\int_{-1}^{1} f$ does not exist.
Give a different example of discontinuous function which admits primitive but not the integral on a closed interval.

10. If f is bounded, defined on [0,1] and
f(x) = $(-1)^{n-1}$ when $\frac{1}{n+1}<x<\frac{1}{n}; n\in N$,
then prove that $f\in R[0,1]$ $\int_0^1 f=2\log 2-1$

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11. Show that f defined on [0, 1] by

$$f(x)=\begin{cases} 1/n, & \frac{1}{n+1}<x\le\frac{1}{n}, (n=1,2,3,.....) \\ 0, & x=0 \end{cases}$$

is integrable on [0, 1]. Also show that

$$\int_0^1 f(x)dx=\frac{\pi^2}{6}-1.$$

12. Show by an example that every bounded function need not be Riemann integrable.

13. Prove that the sequence $\{x_n\}$ defined by $x_1=\sqrt{2}, x_{n+1}=\sqrt{2+x_n}$ converges to the positive root of the equation $x^2-x-2=0$.

14. If f(x) be defined on [0, 1] as follows—
$f(x)=1$, when x is rational
$=-1$, when x is irrational then prove that f is not Riemann integrable over [0, 1]

15. The function f is defined on (0, 1] by

$\int_0^1 f(x) = \sum_0^\infty (-1)^{n-1}$

$$f(x)=(-1)^{n+1}n(n+1), \frac{1}{n+1}\le x\le\frac{1}{n}, n\in N.$$

NOT convergent
⇒ NOT Riemann

Show that $\int_0^1 f(x)dx$ does not converge.

16. Let f be a real valued function defined on [0, 1] as follows:

$$f(x)=\begin{cases} \frac{1}{a^{r-1}}, & \frac{1}{a^r}<x\le\frac{1}{a^{r-1}}, r=1,2,3..... \\ 0, & x=0 \end{cases}$$

where a is an integer greater than 2.

Show that $\int_0^1 f(x)dx$ exists and is equal to $\frac{a}{a+1}$.

17. Prove that $\frac{1}{\pi}\le\int_0^1\frac{\sin\pi x}{1+x^2}dx\le\frac{2}{\pi}$.

18. Show that the function $f(x)=\sin x$ is Riemann integrable in any interval $[0, t]$ by taking the partition

$$P=\left\{0,\frac{t}{n},\frac{2t}{n},\frac{3t}{n},....,\frac{nt}{n}\right\} \text{ and }$$

$$\int_0^t \sin x dx=1-\cos t.$$

19. Show that the function [x], where [x] denotes the greatest integer not greater than x, is integrable in

[0, 3]. Also evaluate $\int_0^3 [x]dx$. = 3

20. Discuss the fucntion defined by

$$f(x)=\begin{cases} 1/q, \text{ when } x \text{ is a rational } p/q \ne \text{ in its lowest terms} \\ 0, \text{ when } x \text{ is irrational and } x=0. \end{cases}$$

as regards its continuity

21. Show that the function [x] where [x] denotes the greatest integer not greater than "x" is integrable in [0, 3] and $\int_0^3 [x]dx=3$.

22. Let f be defined on [0, 1] as

$$f(x)=\begin{cases} \sqrt{1-x^2}, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over [0, 1].

23. f is bounded and integrable in [a, b]; Show that

$$\int_a^b [f(x)]^2 dx=0$$

if, and only if, $f(c)=0$ at every point, c, of continuity of f.

24. Let the function f be defined by

$$f(x)=\frac{1}{2t}, \text{ when } \frac{1}{2^{t+1}}<x\le\frac{1}{2^t}$$

$$(t=0, 1, 2, 3, ...)$$

$$f(0)=0$$

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Is f integrable on [0, 1] ?

If f is integrable, then evaluate $\int_0^1 f\,dx$

25. If f is monotonic on [a, b] and if α is continuous on [a, b], then prove that $\int_a^b f\,dx$ exists.

26. If f and α' are integrable in the sense of Riemann on [a, b], then prove that

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx.$$

27. If f and g are differentiable on [a, b] and f', g' are Riemann integrable over [a, b], then show that

$$\int_a^b f(x)\,dg(x) + \int_a^b g(x)\,df(x) = f(b)g(b) - f(a)g(a)$$

28. Find the upper and lower Riemann integral for the function defined in the interval (0, 1) as follows:

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{when } x \text{ is rational} \\ 1-x & \text{when } x \text{ is irrational} \end{cases}$$

and show that f is not Riemann integrable in (0, 1).

29. Examine for Riemann integrability over [0, 2] of the function defined in [0, 2] by

$$f(x) = \begin{cases} x + x^2, & \text{for rational values of } x \\ x^2 + x^3, & \text{for irrational values of } x \end{cases}$$

30. Let $f(x) = \begin{cases} 0, x & \text{is irrational} \\ 1, x & \text{is rational} \end{cases}$ show that f is not Riemann - integrable on [a, b].

31. Prove that the function f defined on [0, 4] by $f(x) = [x]$, greatest integer $\le x, x \in [0, 4]$ is integrable on [0, 4] and that $\int_0^4 f(x)\,dx = 6$.

32. If f is the derivative of some function defined on [a, b], prove that there exists a number $\eta \in [a, b]$ such that

$$\int_a^b f(t)\,dt = f(\eta)(b-a)$$

33. Show that the function $f(x) = [x^2] + |x-1|$ is Riemann integrable in the interval [0, 2], where $[\alpha]$ denotes the greatest integer less than or equal to α. Can you give an example of a function that is not Riemann integrable on [0, 2]? Compute $\int_0^2 f(x)\,dx$, where $f(x)$ is as above.

34. Evaluate $\int_0^1 \left(2x \sin\frac{1}{x} - \cos\frac{1}{x} \right) dx.$

35. A function f is defined in the interval (a, b) as follows:

$$f(x) = \frac{1}{q^2}, \text{ when } x = \frac{p}{q}$$

$$= \frac{1}{q^3}, \text{ when } x = \sqrt{\frac{p}{q}}$$

where p, q are respectively prime integers. $f(x) = 0$ for all other values of x.

Is f Riemann integrable ? Justify your answer.

36. Show that the function $f(x)$ defined as

$$f(x) = \frac{1}{2^n}; \frac{1}{2^n+1} \le x \le \frac{1}{2^n}, n = 0, 1, 2 \ldots\ldots$$

$$f(0) = 0$$

is integrable in [0, 1], although it has an infinite number of points of discontinuity.

Show that $\int_0^1 f(x)dx = \frac{2}{3}$.

37. Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify.

38. Let $f(x) = \begin{cases} \frac{x^2}{2} + 4 & \text{if } x \geq 0 \\ \frac{-x^2}{2} + 2 & \text{if } x < 0 \end{cases}$

Is f Riemann integrable in the interval [–1, 2]? Why? Does there exist a function g such that $g'(x) = f(x)$?
Justify your answer.

39. Let [x] denote the integer part of the real number x, i.e., if $n \leq x < n+1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in [–1, 2] ? If not, explain why. If it is integrable, compute

$$\int_{-1}^{2} ([x]^2 + 3)dx.$$

40. Integrate $\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \in]0,1] \\ 0 & , x = 0 \end{cases}$$

41. Is the function

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0, & x = 0 \end{cases}$$

Riemann integrable? If yes, obtain the value of $\int_0^1 f(x)dx$.

42. A function f is defined in [0, 1] as

$f(x) = (-1)^{r-1}$; $\frac{1}{r+1} < x < \frac{1}{r}$, where r is a positive integer show that $f(x)$ is Riemann integrable in [0, 1] & find its Riemann integral.

43. Let $f(x) = [x], x \in [0,3]$ where $[x]$ denotes the greatest integer not greater than x, Prove that f is Riemann integrable on [0, 3] and evaluate

$$\int_0^3 f(x)dx$$

44. Evaluate $\int_{-1}^{1} f(x)dx$, where $f(x) = |x|$, by Riemann integration.

45. (i) Check whether or not the following function is Riemann integrable in [0, 1]:

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(ii) Let $f : [-1, 1] \to [0, 1]$ be defined by $f(x) = |x|$. Check whether it is Riemann integrable.

46. If f is continuous on [a, b] and $\int_a^b fg\,dx = 0$ for any continuous function g on [a, b], then show that f = 0 for all $x \in [a,b]$

47. Determine whether

$$f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

is Riemann-integrable on [0, 1] and justify your answer