Date : 22/7/20

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2020

(JULY *to* DEC.-2020)

IAS/IFoS

MATHEMATICS

192/250

Under the guidance *of* K. Venkanna

ALGEBRA, REAL ANALYSIS AND COMPLEX ANALYSIS & LPP

TEST CODE: TEST-2: IAS(M)/19-JULY.-2020

Time: 3 Hours | Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 54 pages and has 38 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Prashant Kumar Sharma

Roll No.

Test Centre Delhi

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

Prashant

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, all parts/ sub-parts of the previous question attempted. This is to be strictly followed

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			07	
	(b)			08	
	(c)			08	
	(d)			08	
	(e)			08	39
2	(a)			16	
	(b)			05	
	(c)			08	
	(d)			08	37
3	(a)				
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4	(a)			12	
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5	(a)			08	
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7	(a)			13	
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8	(a)				
	(b)				
	(c)				
	(d)				
			Total Marks		192

SECTION – A

1. (a) Which of the following multiplication tables defined on the set G = {a, b, c, d} form a group ? Support your answer in each case.

(i)

o	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(ii)

o	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(ii)

o	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(iv)

o	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

[10]

Sol (i) from table for $x, y \in \{a, b, c, d\} (= G)$

$x \circ y \in G$

Since $a \circ a = a$; $b \circ a = b$; $c \circ a = c$; $d \circ a = a$ (first column)

so a is identity

$a \circ a = a$; but b does not have inverse

(2nd row does not contain a)

so (i) is not group.

(ii) It is group

Since (a is identity element; first column) and every row contain element a, so inverse exist

(iii) It is group

Since a is identity element (1st row) and every element has inverse (every column contain a)

(iv) It is not group
since inverse of d does not exist
since (neither column nor row related
with d does not contain a)

1. (b) Show that the ring $\mathbf{Z}_p$ of integers modulo p is a field if and only if p is prime. [10]

Sol (a) Z_p is field
since Z_p is field then Z_p must be ID
also
and let P is not prime
let $P = ab$ ($\because 0 < a, b < P$)
but $ab \equiv 0 \pmod{P}$
but neither $a = 0$ nor $b = 0$ but
$ab = 0$
so contradication since Z_p is ID (given)
so our assumption is that P is not prime
$\therefore$ P must be prime

(b) If P is prime show Z_p is field

p is prime

and let $a \in Z_p$, and $b \in Z_p$

and let $ab = 0 \pmod{p} \Rightarrow p \mid ab$

Since P is prime then $p \mid a$ or $p \mid b$

$\therefore a = 0 \pmod{p}$ or $b = 0 \pmod{p}$

$\therefore Z_p$ is I.D

Since Z_p is finite I.D then Z_p must be field

$\therefore Z_p$ is field also

1. (c) Show that the function f(x) = 1/x, x > 0 is continuous in (0, 1) but not uniformly continuous. **[10]**

Sol $f(x) = \frac{1}{x}$; $x > 0$

the only point of discontinuity of $f(x) = \frac{1}{x}$ is 0 but it is not included (0, 1)

-08

$\therefore f(x) = \frac{1}{x}$ is continous in $(0,1)$

Uniform continuity

on domain $(0,1)$

let $x_n = \frac{1}{2n}$; $y_n = \frac{1}{2n+1}$

$$|x_n - y_n| = \left|\frac{1}{2n} - \frac{1}{2n+1}\right| = \left|\frac{1}{(2n)(2n+1)}\right|$$

$$|x_n - y_n| \to 0$$

$$|f(x_n) - f(y_n)| = |2n - 2n - 1| = |1| \not\to 0$$

so $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \not\to 0$

so by Cauchy formula $f(x) = \frac{1}{x}$ is not uniform continous.

1. (d) Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, z \neq 0 \\ 0, z = 0 \end{cases}$$ is not differentiable at z = 0 **[10]**

Sol $f(z) = \begin{cases} \frac{z^5}{|z|^4}, z \neq 0 \\ 0, z = 0 \end{cases}$

08

$f(z) = \begin{cases} \frac{(x+iy)^5}{(x^2+y^2)} ; & x; y \neq 0 \\ 0 , & \cancel{x}y = 0 \end{cases}$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

where h is complex number

$h = (a+ib)$

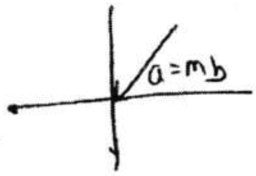
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^5}{|h|^4} - 0}{h} = \lim_{h \to 0} \frac{h^5}{h|h|^4}$$

$$= \lim_{h \to 0} \frac{h^4}{|h|^4} = \lim_{a, b \to 0} \frac{(a+ib)^4}{(a^2+b^2)^2}$$

let $a = mb$ route

$$= \lim_{a, b \to 0} \frac{b^4(1+im)^4}{b^4(1+m^2)^2} = \frac{(1+im)^4}{(1+m^2)^2}$$



Since $f'(0)$ does not exist because limit depends on route taken to origin (depends on m)

$\therefore f(z)$ is not differentiable

1. (e) A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at the most 12 hours, whereas machine III must be operated at least for 5 hours a day. He produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of the items M and N on the three machines are given in the following table :

Item	Number of hours required on machines		
	I	II	II
M	1	2	1
N	2	1	1.25

He makes a profit of Rs. 600 and Rs. 400 on item M and N respectively. How many of each item should he produce so as to maximize his profit assuming that he can sell all the items that he produces ? What will be the maximum profit ? **[10]**

Sol Let ~~he p~~ manufacturer produce x unit of item M and y unit of item N

08

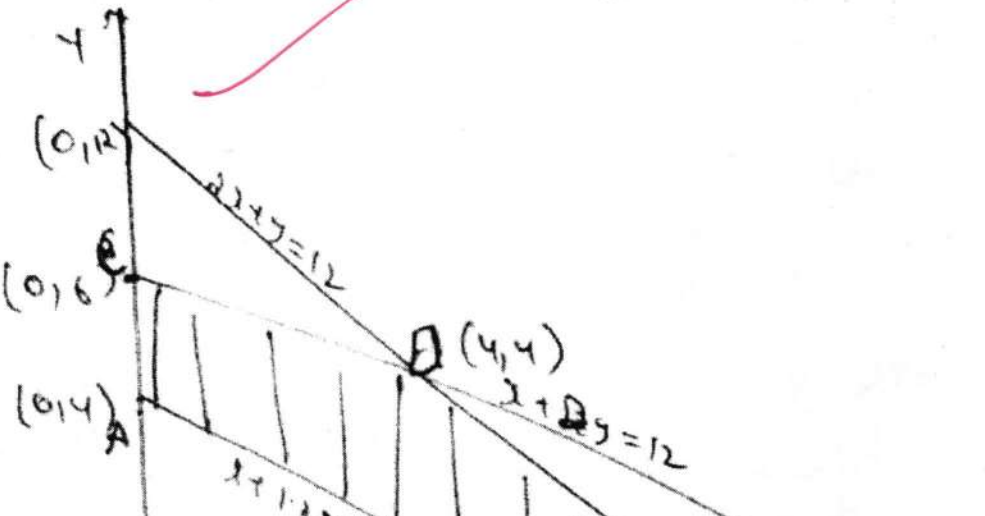
Then according to constraints

$x + 2y \leq 12$ (time constraint of machine I)

$2x + y \leq 12$ (time constraint of machine II)

$x + 1.25y \geq 5$ (time constraint of machine III)

Maximize $z = 600x + 400y$ (objective



Shadow region is the feasible region

Then z at $(0,6) = 400 \times 6 = 2400$

at $(0,4) = 400 \times 4 = 1600$

at $(5,0) = 600 \times 5 = 3000$

at $(6,0) = 600 \times 6 = 3600$

at $(4,4) = 600 \times 4 + 400 \times 4 = 4000$

$\therefore$ z will be maximum at D $(4,4)$

so manufacture must produce 4 unit of M unit and 4 unit of N unit

Maximum Profit $= \boxed{4000}$

2. (a) (i) In S_3 give an example of two elements x, y such that $(x.y)^2 \neq x^2.y^2$.

(ii) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field? Is it an integral domain? **[18]**

Sol (i) $S_3 = \{I, (12), (13), (23), (123), (132)\}$

Let $x = (12)$ and $y = (23)$

$x \cdot y = (12)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$

$(x \cdot y)^2 = (123)^2 = (132)$

and $x^2 = (12)(12) = I$

$y^2 = (23)(23) = I$

$\therefore x^2 \cdot y^2 = I$

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$\therefore (x \cdot y)^2 = (132)$ but $x^2 \cdot y^2 = I$

$\therefore \boxed{(x \cdot y)^2 \neq x^2 \cdot y^2}$

where $x = (12)$ and $y = (23)$

(ii)

$\cdot$	0	1	i	$1+i$
0	0	0	0	0
1	0	1	i	$1+i$
i	0	i	$+1$	$1+i$
$1+i$	0	$1+i$	$i+1$	0

$Z_2(i)$ is not field since inverse of $(1+i)$ does not exist (row ~~and~~ or column contain $(1+i)$ does not contain 1)

$Z_2(i)$ is not integral domain because $(1+i)(1+i) = 0$ and $1+i \neq 0$

$\therefore$ $Z_2(i)$ is not ID

2 (b) Find three elements σ in S_9 with the property that $\sigma^3 = (157)(283)(469)$. **[06]**

Sol $\sigma^3 = (157)(283)(469)$

$O(\sigma^3) = 3 \Rightarrow (\sigma^3)^3 = I$

$\Rightarrow \boxed{\sigma^9 = I}$

order of σ cannot be 1, 3

(other wise $\sigma^3 = I$ which is not the case)

05

$\therefore \sigma^9 = I \Rightarrow \sigma = (a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9)$

$\sigma^3 = (a_1 a_4 a_7)(a_2 a_5 a_8)(a_3 a_6 a_9)$

or $(a_2 a_5 a_8)(a_3 a_6 a_9)(a_1 a_4 a_7)$

σ is or $(a_3 a_6 a_9)(a_1 a_4 a_7)(a_2 a_5 a_8)$

for $\sigma^3 = (a_1 a_4 a_7)(a_2 a_5 a_8)(a_3 a_6 a_9)$

$\boxed{\sigma = (1\,2\,4\,5\,8\,6\,7\,3\,9)}$ (by comparing)

for $\sigma^3 = (a_2 a_5 a_8)(a_3 a_6 a_9)(a_1 a_4 a_7)$

$\boxed{\sigma = (4\,1\,2\,6\,5\,8\,9\,7\,3)}$ (by comparing)

for $\sigma^3 = (a_3 a_6 a_9)(a_1 a_4 a_7)(a_2 a_5 a_8)$

$\boxed{\sigma = (2\,4\,1\,8\,6\,5\,3\,9\,7)}$ (by comparing)

2. (c) Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$. **[10]**

Sol $\Sigma f(n) = \Sigma(-1)^{n+1}\frac{n}{n^2+1}$

for convergence

$\Sigma f(n)$ is alternative series

with $u_n = \frac{n}{n^2+1} \Rightarrow \boxed{\lim_{n\to\infty} u_n = 0}$

08

and u_n is decreasing also

So by Leibnitz test $(-1)^{n+1}u_n$ is convergence for absolute converge

$\sum \left|(-1)^{n+1} u_n\right| = \sum \frac{n}{n^2+1}$

$x_n = \frac{n}{n^2+1}$ and $y_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \boxed{1}$ finite

so x_n and y_n converge and diverge together but $y_n = \frac{1}{n}$ diverges

so x_n also ~~conver~~ diverges

$\sum (-1)^{n+1} u_n$ is not absolutely convergent

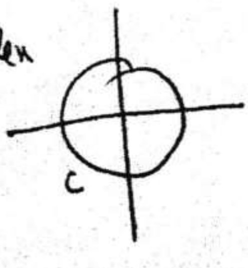
[16]

2. (d) Evaluate $\int_0^{2\pi} \frac{d\theta}{\left(a+b\cos^2\theta\right)^2}$, where a > b > 0.

sol $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$

$= \int_0^{2\pi} \frac{d\theta}{\left(a + b\left(\frac{1+\cos 2\theta}{2}\right)\right)^2}$

transform above integral into complex form



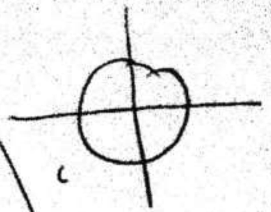
$I = R.P \int_0^{2\pi} \frac{d\theta}{\left(a+\frac{b}{2}\left(1+e^{2i\theta}\right)\right)^2}$

08

Let $e^{2i\theta} = z \Rightarrow d\theta (e^{i\theta}) i = dz$

$\boxed{d\theta = \frac{dz}{2zi}}$

$$I = \int_c \frac{dz}{\left(a + \frac{b}{2}\left(1 + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)\right)\right)^2} \times \frac{1}{2zi}$$



$$I = \int_c \frac{dz}{\left(a + \frac{b}{4}\left(2 + z^2 + \frac{1}{z^2}\right)\right)^2} \times \frac{1}{2zi} = \frac{16z^3\,dz}{\left(z^4 + (4a+2b)z^2 + b\right)^2}$$

$$I = \frac{16z^3\,dz}{\left(z^4 + (4a+2b)z^2 + b\right)^2} \times \frac{1}{i}$$

$I = 2\pi i \times \frac{1}{i}(\Sigma\,\text{residues}) = 2\pi\,(\text{residue})$

I has twice pole at $\alpha =$

$$I = \int_c \frac{dz}{\left(a + \frac{b}{2}\left(1 + \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)^2} \times \frac{1}{2zi} = \int_c \frac{8z\,dz}{\left(z^2 + (4a+2b)z + b\right)^2}$$

I has twice pole at $\alpha = \frac{-(4a+2b) \pm \sqrt{(4a+2b)^2 - 4b}}{2}$

$\alpha = -(2a+b) + \sqrt{(2a+b)^2 - b}$, $\beta = -(2a+b) - \sqrt{(2a+b)^2 - b}$

Residue at $\alpha = \lim_{z\to\alpha} \frac{d}{dz}\left(\frac{z}{\cancel{(z-\alpha)^2}(z-\beta)^2}\right) = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$

$$= \frac{-(4a+2b)}{(8)\left(\sqrt{(2a+b)^2-b}\right)^3}$$

$$I = \pi i\left(\frac{-(4a+2b)\times 2}{8\left(\sqrt{(2a+b)^2-b}\right)^3} \times \frac{1}{i}\right)$$

$$\boxed{I = \frac{-2\pi(2a+b)}{\left((2a+b)^2-b\right)^{3/2}}}$$

3\. (a) Let $z = \cos\theta + i\sin\theta$ be in T where $\theta \in Q$. Prove that the order of z is infinite. **[10]**

3. (b) Examine the convergence of

$$\int \frac{\log x}{\sqrt{2 \quad x}} dx$$

[08]

3. (c) f (x) is defined as follows:

$$f(x)=\begin{cases}\frac{1}{2}(b^2-a^2) & for \quad 0<x<a \\ \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x} & for \quad a<x<b \\ \frac{1}{3}\frac{b^3-a^3}{x} & for \quad x>b\end{cases}$$

Prove that f (x) and $f'(x)$ are continuous but $f''(x)$ is discontinuous. **[14]**

3. (d) Determine an optimal transportation programme so that the transportation cost of 340 tons of a certain type of material from three factories F_1, F_2, F_3 to five warehouses W_1, W_2, W_3, W_4, W_5 is minimized. The five warehouses must receive 40 tons, 50 tons, 70 tons, 90 tons and 90 tons respectively. The availability of the material at F_1, F_2, F_3 is 100 tons, 120 tons, 120 tons respectively. The transportation costs per ton from factories to warehouses are given in the table below:

	W_1	W_2	W_3	W_4	W_5
F_1	4	1	2	6	9
F_2	6	4	3	5	7
F_3	.5	2	6	4	8

Use Vogel's approximation method to obtain the initial basic feasible solution. **[18]**

(a) Let $R=\left\{\begin{bmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{bmatrix}\in M_2(\mathbb{C})\mid \bar{\alpha},\bar{\beta} \text{ denote the conjugates of } \alpha,\beta\right\}$.

Define addition + and multiplication • in R by usual matrix addition and matrix multiplication. Show that R is a division ring but not a field. **[14]**

Sol $R=\begin{bmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{bmatrix}$

Since R is (+) abelian group and (R, ·) is semi-ring and satisfy distributive property

∴ R is ring

for division ring

I is identity of any element of R

$$\therefore \begin{bmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{bmatrix}$$

Let $x=\begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix}$ and $y=\begin{bmatrix}c & d\\ -\bar{d} & \bar{c}\end{bmatrix}$ where $x\neq 0$ and $y\neq 0$

$$x\cdot y=\begin{bmatrix}a & b\\ -\bar{b} & \bar{a}\end{bmatrix}\cdot\begin{bmatrix}c & d\\ -\bar{d} & \bar{c}\end{bmatrix}=\begin{bmatrix}ac-b\bar{d} & ad+b\bar{c}\\ -\bar{b}c-\bar{a}\bar{d} & -\bar{b}d+\bar{a}\bar{c}\end{bmatrix}$$

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$$\begin{bmatrix}ac-b\bar{d} & ad+b\bar{c}\\ -\bar{b}c-\bar{a}\bar{d} & -\bar{b}d+\bar{a}\bar{c}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$ad+b\bar{c}=0$ and $\bar{b}c+\bar{a}\bar{d}=0$

and $ac-b\bar{d}=1$ and $-\bar{b}d+\bar{a}\bar{c}=1$

$c=\dfrac{\bar{a}}{a^2+b^2}$, $d=\dfrac{-b}{a^2+b^2}$ } solving above equation

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$\therefore\ x \cdot y = I$ where y exist

for $x = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ and $y = \frac{1}{a^2+b^2}\begin{bmatrix} \bar{a} & -b \\ +\bar{b} & \bar{a} \end{bmatrix}$

$\therefore$ R is division ring

for field

$x \cdot y = \begin{pmatrix} \cancel{ad+bc}\ ac - b\bar{d} & ad + b\bar{c} \\ -\bar{b}c - \bar{a}\bar{d} & -\bar{b}d + \bar{a}c \end{bmatrix}$

for $y \cdot x = \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} ac - \bar{b}d & bc + \bar{a}d \\ -a\bar{d} + \bar{c}b & -b\bar{d} + \bar{a}\bar{c} \end{bmatrix}$

$\therefore$ clearly $x \cdot y \neq y \cdot x$

$\therefore$ R is not commutative ring

$\therefore$ R is not field

4. (b) (i) Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. Show that the sequence converges to $\frac{1+\sqrt{17}}{2}$

(ii) Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when x is rational} \\ 1 & \text{when x is irrational} \end{cases}$$

on the interval [0, 1]. **[16]**

Sol (i) $x_1 = \cancel{\sqrt{}}5$ and $x_{n+1} = \sqrt{4 + x_n}$

$x_1 = \cancel{\sqrt{}}5$

~~$x_2 = \sqrt{4+\sqrt{5}} > x_1$~~

~~suppose $x_n > x_{n-1}$~~

~~$4 + x_n > 4 + x_{n-1}$~~

~~$\sqrt{4 + x_n} >$~~

$x_2 = \sqrt{4+5} = 3$

$(x_2 < x_1)$

let $x_n < x_{n-1}$

$\sqrt{4 + x_n} < \sqrt{4 + x_{n-1}}$

$\boxed{x_{n+1} < x_n}$

$\therefore$ by Principal of mathematical induction
$\{x_n\}$ is decreasing

for bounded below

$x_1 = 5 > 2$
$x_2 = 3 > 2$
$x_3 = \sqrt{4+3} = \sqrt{7} > 2$

Let $x_n > 2 \Rightarrow \sqrt{4+x_n} > \sqrt{6}$

$\boxed{x_{n+1} > \sqrt{6} > 2}$

$\therefore$ by PMI $\{x_n\} > 2$

$\therefore$ $\{x_n\}$ is monotically decreasing and bounded below

$\therefore$ $\{x_n\}$ is Convergent

$\therefore$ $x_n \to l$ and $x_{n+1} \to l$

$x_{n+1} = \sqrt{4+x_n} \Rightarrow l = \sqrt{4+l}$

$l^2 - l - 4 = 0 \Rightarrow l = \frac{1 \pm \sqrt{17}}{2}$

Since $x_1 = 5 \Rightarrow \therefore$ l can't be negative

$\therefore \boxed{l = \frac{1+\sqrt{17}}{2}}$

(b) $f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$

Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots \frac{n}{n} = 1\}$

$U(P, f) = \Sigma(1) \times \frac{1}{n} = \Sigma \frac{1}{n} = \boxed{1}$ (between any two numbers there must be one irrational number)

$L(P, f) = \Sigma m\delta = \Sigma 0 \frac{1}{n} = 0$ (b/w any 2 number there must be one rational number)

$\therefore \overline{\int_0^1} f(x) = \text{Inf}\,(U(P, f)) = 1$

and $\underline{\int_0^1} f(x) = \text{Sup}\,(L(P, f)) = 0$

and $\overline{\int_0^1} f(x) \neq \underline{\int_0^1} f(x)$

$\therefore$ f is not Reiman integral

4. (c) Expand the function $f(z) = \dfrac{2z^2 + 11z}{(z+1)(z+4)}$ in a Laurent's series valid for $2 < z < 3$.

[10]

Sol $f(z) = \dfrac{2z^2 + 11z}{(z+1)(z+4)} = \dfrac{2z^2 + 11z}{3} \left(\dfrac{z+4 - (z+1)}{(z+1)(z+4)} \right)$

$f(z) = \dfrac{2z^2 + 11z}{3} \left(\dfrac{1}{z+1} - \dfrac{1}{z+4} \right)$

$= \dfrac{2z^2 + 11z}{3(z+1)} - \dfrac{2z^2 + 11z}{3(z+4)}$

$= \dfrac{2z^2 + 11z}{3z(1 + \frac{1}{z})} - \dfrac{2z^2 + 11z}{3 \times 4 (1 + \frac{z}{4})}$

(for first fraction $\left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$

and for 2nd fraction $\left|\frac{z}{4}\right| < 1 \Rightarrow |z| < 4$)

08/

$$= \frac{2z^2+11z}{3z}\left(\left(1+\frac{1}{z}\right)^{-1}\right) - \frac{2z^2+11z}{12}\left(1+\frac{z}{4}\right)^{-1}$$

$$= \boxed{\frac{2z^2+11z}{3}\left(\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}\right) - \frac{2z^2+11z}{12}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n}$$

$$= \frac{2z^2+11z}{3}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} + \cdots - \frac{1}{4}\left(1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \left(\frac{z}{4}\right)^3 + \cdots\right)\right)$$

$$= \boxed{\frac{2z^2+11z}{3}\left\{+ \cdots \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z} - \frac{1}{4} + \frac{z}{4} - \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^2 + \cdots\right\}}$$

4. (d) How many basic solutions are there in the following linearly independent set of equations? Find all of them.

$2x_1 - x_2 + 3x_3 + x_4 = 6$

$4x_1 - 2x_2 - x_3 + 2x_4 = 10$ **[10]**

sol Since 2 equations are given and 4 variable are there

$\therefore$ $^4C_2 = \boxed{6}$ basic solutions ~~will~~ may be there

	basic variable	equation	Solution
1)	x_1, x_2	$2x_1 - x_2 = 6$ $4x_1 - 2x_2 = 10$	$(2, -1, 0, 0)$
2)	x_1, x_3	$2x_1 + 3x_3 = 6$ $4x_1 - x_3 = 10$	$\left(\frac{18}{7}, 0, \frac{2}{7}, 0\right)$

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3) x_1, x_4	$2x_1 + x_4 = 6$ $4x_1 + 2x_4 = 10$	Solution does not exist
4) x_2, x_3	$-x_2 + 3x_3 = 6$ $-2x_2 - x_3 = 10$	$\left(0, -\frac{36}{7}, \frac{2}{7}, 0\right)$
5) x_2, x_4	$-x_2 + x_4 = 6$ $-2x_2 + 2x_4 = 10$	Solution does not exist
6) x_3, x_4	$3x_3 + x_4 = 6$ $-x_3 + 2x_4 = 10$	$\left(0, 0, \frac{2}{7}, \frac{36}{7}\right)$

$\therefore$ 4 basic solution exist

SECTION – B

5. (a) Prove that all cyclic groups of infinite order are isomorphic to Z. [10]

Sol Let $G = \langle a \rangle$

Let define a ~~def~~ function from $Z \to G$

such that $f(n) = a^n$

Prove isomorphism $\to$ 1) well-defined

$n_1 = n_2$

$a^{n_1} = a^{n_2}$

$\boxed{f(n_1) = f(n_2)}$

2) Homomorphism

$f(n_1 + n_2) = a^{n_1 + n_2} = (a^{n_1})(a^{n_2})$

$= f(n_1) f(n_2)$

$\therefore \boxed{f(n_1 + n_2) = f(n_1) f(n_2)}$

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3) One- one

$f(n_1) = f(n_2)$

$a^{n_1} = a^{n_2} \Rightarrow a^{n_1 - n_2} = e$

$\therefore$ G is infinite cyclic group the $a^n = e$ for $n = 0$ only

$\therefore n_1 - n_2 = 0 \Rightarrow \boxed{n_1 = n_2}$

4) Onto

for $x = a^n$

$\boxed{x = f(n)}$

So f is onto

$\therefore$ ~~Xim~~ $Z \cong G \Rightarrow \boxed{G \cong Z}$

5. (b) Investigate what derangement of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

will reduce its sum to zero. **[10]**

Sol $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - -$

$s = \log 2$

after deranegment sum must be 0

By appling formula

$K = \log 2 + \frac{1}{2} \log\left(\frac{\alpha}{B}\right)$

wher K is sum of new series

$0 = \log 2 + \frac{1}{2} \log\left(\frac{1}{B}\right)$

08

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$-\log 2 + \frac{1}{2}\log\left(\frac{\alpha}{\beta}\right) \Rightarrow \log\left(\frac{\beta}{\alpha}\right) = \log 4$

$\therefore \quad \frac{\beta}{\alpha} = 4$

$\therefore$ To get sum 0 every positive term must be followed by 4 negative terms

$$\Rightarrow \boxed{1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} + \cdots = 0}$$

(c) Determine whether

$$f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

is Riemann-integrable on [0, 1] and justify your answer **[10]**

Sol $f(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$\left|\sin\frac{1}{x}\right| < 1$ and $\cos\frac{1}{x} < 1$

$\therefore f(x)$ is bounded in $[0, 1]$

0

and $f(x)$ is discontinous at 0 only

$\therefore f(x)$ is bounded and discontinous at finite no. of points is given interval

$\therefore f(x)$ is Riemann-integrable

$\therefore$ there exist an anti-derivative

$g'(x) = f(x)$

$g'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \Rightarrow$

$g(x) = \boxed{x^2 \sin\frac{1}{x}}$

$\therefore \boxed{\int f(x) = x^2 \sin\frac{1}{x}}$

?

5. (d) If f(z) = u + iv is analytic function and u – v = e^x (cos y – sin y), find f(z) in terms of z. **[10]**

Sol $f(z) = u + iv$

$if(z) = iu - v$

$\therefore f(z)$ is analytic $\Rightarrow if(z)$ is also analytic

$F(z) = f(z) + if(z)$ is also analytic

$F(z)$ is also analytic

$F(z) = u - v + i(u + v)$

$F(z) = U + iV$

by milne's formula $F'(z) = \frac{dU(z,0)}{dx} - i\frac{dU(z,0)}{dy}$

$$\frac{du}{dx} = e^x(\cos y - \sin y) \; ; \; \frac{du}{dy} = e^x(-\sin y + \cos y)$$

$$\therefore f'(z) = e^z - i(e^z(-1)) = e^z(1+i)$$

$$f(z) = \int e^z(1+i) = e^z(1+i) + C$$

$$\therefore (1+i)f(z) = e^z(1+i) + C$$

$$\boxed{f(z) = e^z + \frac{C}{1+i}}$$

5. (e) $x_1 = 4, x_2 = 1, x_3 = 3$ is a feasible solution of the system of equations

$2x_1 - 3x_2 + x_3 = 8$

$x_1 + 2x_2 + 3x_3 = 15$

Reduce the feasible solution to two different basic feasible solutions. **[10]**

Sol $x_1 = 4, \; x_2 = 1, \; x_3 = 3$

$$2x_1 - 3x_2 + x_3 = 8$$

$$x_1 + 2x_2 + 3x_3 = 15$$

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A, B, C are not independent

for feasible solution ~~soln~~ x_1, x_2, x_3 must be independent

<u>feasible solution</u>

(i) $2\lambda_1 - 3\lambda_2 + \lambda_3 = 8$

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 15$

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Let $\lambda_1 = 6$

then $\lambda_2 = \frac{21}{11}$, $\lambda_3 = \frac{19}{11}$

$\min \left\{\frac{x_r}{\lambda_r}\right) = \left\{\frac{4}{6}, \frac{11}{21}, \frac{33}{19}\right\} = \frac{11}{21} \left(\frac{x_2}{\lambda_2}\right)$

feasible solution $x_1' = x_1 - \frac{x_2}{\lambda_2} \times \lambda_1 = 4 - \frac{11}{21} \times 6$

$= \frac{18}{21} = \boxed{\frac{6}{7}}$

$x_2' = x_2 - \frac{x_2}{\lambda_2} \times \lambda_2 = 0$

$x_3' = x_3' - \frac{x_2}{\lambda_2} \times \lambda_3 = 3 - \frac{\cancel{11}}{21} \times \frac{19}{\cancel{11}} = \frac{14}{21} = \frac{2}{3}$

feasible solution $= \boxed{\left\{\frac{6}{7}, 0, \frac{2}{3}\right\}}$

another feasible solution let $\lambda_2 = 8$

$\lambda_1 = \frac{31}{15}$, $\lambda_3 = \frac{5}{11}$

$\min \left(\frac{x_r}{\lambda_r}\right) = \left(\cdot\frac{11}{31}, \frac{11}{31}, \frac{33}{5}\right\} = \frac{11}{31} \left(\frac{x_2}{\lambda_2}\right)$

$x_1'' = x_1 - \frac{x_2}{\lambda_2} \times \lambda_1 = \cancel{3 - \frac{8 \times 3}{51} = 11}$

$= 4 - \frac{11 \times 8}{31} = \frac{36}{31}$

$x_2'' = 0$

$x_3'' = 3 - \frac{5}{\cancel{11}} \times \frac{\cancel{11}}{31} = \frac{88}{31}$

another solution $= \boxed{\left\{\frac{36}{31}, 0, \frac{88}{31}\right\}}$

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6. (a) (i) Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β. What are the possibilities for β if $\beta \in S_9$?

(ii) Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form.

[7+5=12]

6. (b) Show that the group G of four transformations f_1, f_2, f_3, f_4 defined by $f_1(z) = z$. $f_2(z) = -z$, $f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$ with composite composition is isomorphic to the permutation group G′of degree 4 consisting of the permutation I, (a b), (c d), (a b) (c d). **[15]**

6. (c) Is the ideal $M = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ a maximal ideal of $Z/(12)$, the ring of integers modulo 12? Justify your answer. **[08]**

6. (d) Every Euclidean domain is a principal ideal domain Is a converse true? Justify your Answer.

[15]

7. (a) Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$. Deduce that

$\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} < \log 2$, n being a positive integer. **[15]**

sol $f(x) = \log(1+x)$

Since $f(x)$ is Continous and differentiable for $x > 0$ ~~and~~

$\therefore$ by rolles theorem

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x}$$

$\because 0 < c < x$

$$\frac{1}{1+x} < \frac{1}{1+c} < 1$$

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$\therefore \quad \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$

$$\boxed{\frac{1}{1+x} < \log(1+x) < x}$$

$\therefore$ Let $x = \frac{1}{n+1} \Rightarrow \log\left(\frac{2+n}{n+1}\right) < \frac{1}{n+1}$

$\log(n+2) - \log(n+1) < \frac{1}{n+1}$

Put $x = \frac{1}{n+2} \Rightarrow \log(n+3) - \log(n+2) < \frac{1}{n+2}$

$\vdots$

$x = \frac{1}{2n} \Rightarrow \log(2n+1) - \log(2n) < \frac{1}{2n}$

add above equation

$\log(2n+1) - \log(n+1) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n}$

$$\boxed{\log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n}}$$

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} < \frac{1}{2n} + \frac{1}{3n} + \cdots \frac{1}{2n}$

$\frac{1}{n+1} + \frac{1}{n+2} \nless 1 - \frac{1}{2}$

$\vdots$

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} < 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$\boxed{\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} < \log 2}$$

7. (b) Show that if $f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$, then $<f_n>$ converges non-uniformly on [0, 1]. **[10]**

Sol $\quad f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{\frac{1}{n^2} + n^2x^2} = 0$$

$$|f_n(x) - f(x)| = \left| \frac{n^2x}{1+n^4x^2} \right| = g(x)$$

$$g'(x) = \frac{n^2(1+n^4x^2) - n^2x(2n^4x)}{(1+n^4x^2)^2} = \frac{n^2 - n^6x^2}{(1+n^4x^2)^2}$$

$$g'(x) = 0 \Rightarrow x^2 = \frac{1}{n^4}$$

$$g''(x) = \frac{-6n^6x^2 - 4n^2x + 2n^{10}x^4}{(1+n^4x^2)^2} = \frac{-4n^2 - 4}{(1+n^4x^2)^2} < 0$$

$\therefore$ g(x) is maximum at $x^2 = \frac{1}{n^4}$

$$|f_n(x) - f(x)| \leq \left| \frac{n^2 \times \frac{1}{n^2}}{1 + n^4 \times \frac{1}{n^4}} \right| < \frac{1}{2} \not\to 0$$

$\therefore$ by <u>M_n test</u> $f_n(x)$ <u>is not</u>

<u>converges ~~non~~ uniformly</u>

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7. (c) Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x,y)=\begin{cases}\frac{xy(3x^2-2y^2)}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

Also, discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at (0, 0). **[15]**

Sol $f_x(0,0)=\lim\limits_{h\to 0}\frac{f(h,0)-f(0)}{h}=0$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,k)-f(0)}{k}=0$$

$$\frac{\partial^2 f(0,0)}{\partial y \partial x}=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}$$

and $f_y(h,0)=\lim\limits_{k\to 0}\frac{f(h,k)-f(0,0)}{k}=\lim\limits_{k\to 0}\frac{h\cancel{k}(3h^2-2k^2)}{\cancel{k}(h^2+k^2)}$

$$=\frac{h(3\cancel{h^2})}{\cancel{h^2}}=3h$$

$$\boxed{\frac{\partial^2 f(0,0)}{\partial y \partial x}=\frac{3h-0}{h}=\boxed{3}}$$

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$$\frac{\partial^2 f(0,0)}{\partial x \partial y}=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$$

$$f_x(0,k)=\lim_{h\to 0}\frac{f(h,k)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{hk(3h^2-2k^2)}{h^2+k^2}-0}{h}=k\left(\frac{-2\cancel{k^2}}{\cancel{k^2}}\right)=-2k$$

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \lim_{h\to 0} \frac{-2h - 0}{h} = \boxed{-2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \quad (\text{at } (0,0))$$

$$\lim_{x\to 0, y\to 0} \frac{\partial^2 f}{\partial x \partial y} = -2 \neq \frac{\partial^2 f}{\partial x \partial y} \; ($$

$\therefore$ $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are discontinous at $(0,0)$

7. (d) Show that $\iint_D \frac{(x-y)}{(x+y)^3}\, dx\, dy$ does not exist, where

$D = \{(x, y) \in R^2 / 0 \le x \le 1, 0 < y < 1\}$ **[10]**

Sol $\iint_D \frac{(x-y)}{(x+y)^3}\, dx dy$

let $U = x-y$ and $\vartheta = (x+y)$

$x = \frac{U+\vartheta}{2}$, $y = \frac{\vartheta - U}{2}$

$$J\frac{(x,y)}{(U,\vartheta)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \end{vmatrix} = \frac{1}{2} \quad \left(\frac{J(x,y)}{(U,\vartheta)} : \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial \vartheta} \end{vmatrix}\right.$$

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$$\iint \frac{x-y}{(x+y)^3} = \frac{1}{2}\iint \frac{U}{\vartheta^3}\, dU\, d\vartheta$$

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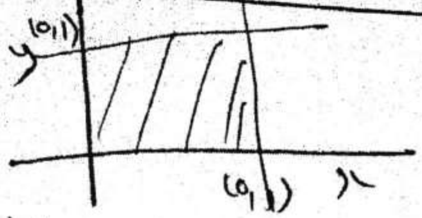

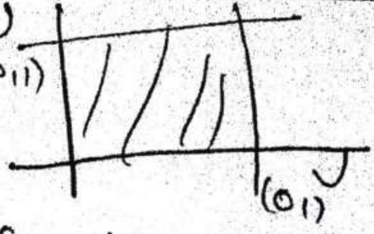

$= \int_0^1\int_0^1 \frac{u}{v^3}\,du\,dv = \frac{1}{4}\int_0^1 \frac{1}{v^3}\,dv \int_0^1 u\,du$

$= \frac{1}{4}\int_0^1 \frac{1}{v^3}\,dv \times \frac{1}{2}(1-0) = \frac{1}{8}\int_0^1 \frac{1}{v^3}\,dv$

$\iint = \frac{1}{8}\int_0^1 \frac{1}{v^3}\,dv$ (improper integral)

$f(x) = \frac{1}{v^3}$, let $g(x) = \frac{1}{v^3}$ $\ni \lim_{x\to 0}\frac{f(x)}{g(x)} = 1$

$\therefore \int f(x)$ and $g(x)$ converge together but by p-test $g(x)$ diverges

$\therefore \int f(x)$ diverges $\Rightarrow$ $\boxed{\iint \frac{(x-y)}{(x+y)^3} \text{ diverges}}$ $\Rightarrow$ does not exist

8. (a) Using Cauchy's theorem and / or Cauchy's integral formula, calculate the following integrals :

(i) $\int_C \frac{\cosh(\pi z)\,dz}{z(z^2+1)}$, where C is circle $|z| = 2$

(ii) $\int_C \frac{e^{az}\,dz}{(z-\pi i)}$, where C is the ellipse $|z-2| + |z+2| = 6$.

(iii) $\int_C \frac{(\sin z)^2\,dz}{\left(z-\frac{\pi}{6}\right)^3}$, where C is circle $|z| = 1$. **[12]**