Mains Test Series - 2019

TEST No. 18 [Paper-II Full Syllabus)

Section-A

Ques:1(a) Is the ideal $M = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ a maximal ideal of $Z/\langle 12 \rangle$, the ring of integers modulo 12? Justify your answer?

Solution:- Let $U = \langle 12 \rangle = 12Z = \{\ldots\ldots -36, -24, -12, 0, 12 \ldots\}$

The only ideals of Z which contain U are Z and $12Z$, and;

$2Z = \{\ldots\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$

$3Z = \{\ldots\ldots -6, -3, 0, 3, 6, \ldots\ldots\}$

$4Z = \{\ldots\ldots -8, -4, 0, 4, 8 \ldots\ldots\}$

$6Z = \{\ldots\ldots -12, -6, 0, 6, 12, \ldots\ldots\}$

Hence, all the ideals of Z/U are

$$Z/U = \{\bar{0}, \bar{1}, \bar{2}, \ldots\ldots \overline{11}\}$$

$U/U = \bar{0} = U$

$2Z/U = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \ldots\ldots\}$

$3Z/U = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \ldots\ldots\}$

$4Z/U = \{\bar{0}, \bar{4}, \bar{8}\}$

$6Z/U = \{\bar{0}, \bar{6}\}$. (Here $\bar{n} = n+U$)

Here $\bar{M} = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ is a maximal ideal of

$Z/\langle 12\rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots, \bar{11}\}$

Since, there doesnot exist any proper ideal between $M = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ and

$$Z/\langle 12\rangle = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots \bar{11}\}.$$

$\Rightarrow$ [Notice that $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}$ is also a maximal ideal of $Z/\langle 12\rangle$.

However $\{\bar{0}\}$, $\{\bar{0}, \bar{4}, \bar{8}\}$, $\{\bar{0}, \bar{6}\}$ are not maximal ideals of $Z/\langle 12\rangle$, since for example,

$$\{\bar{0}, \bar{6}\} \subset \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\} \subset Z/\langle 12\rangle \;]$$

Ques:1) b) Let M be the set of all 3X3 matrices of the following form: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$

where $a, b, c \in Z_2$. Show that with standard matrix addition and multiplication (over Z_2), M is a commutative ring. Find all the idempotent elements of M.

Solution:- Given Matrix $M = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$; $a, b, c \in Z_2$

Let $A_1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix}$ $a_2, b_2, c_2 \in Z_2$

$a_1, b_1, c_1 \in Z_2$

For Matrix Addition

$$A_1 + A_2 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & a_1+a_2 & 0 \\ b_1+b_2 & c_1+c_2 & a_1+a_2 \end{bmatrix}$$

$$A_2 + A_1 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & a_1+a_2 & 0 \\ b_1+b_2 & c_1+c_2 & a_1+a_2 \end{bmatrix}$$

Since $a_1 + a_2 = a_2 + a_1$, $b_1 + b_2 = b_2 + b_1$, $c_1 + c_2 = c_2 + c_1$

as $a_1, a_2, b_1, b_2, c_1, c_2 \in Z_2$

and addition of two is always commutative in Z_2.

Hence, M is commutative in standard matrix addition.

For Matrix Multiplication

$$A_1 \cdot A_2 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & a_1a_2 & 0 \\ b_1a_2+a_1b_2 & c_1a_2+a_1c_2 & a_1a_2 \end{bmatrix}$$

M is closed under matrix Multiplication.

$$A_2 A_1 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2a_1 & 0 & 0 \\ 0 & a_2a_1 & 0 \\ b_2a_1+a_2b_1 & c_2a_1+a_2c_1 & a_2a_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & a_1a_2 & 0 \\ a_1b_2+b_1a_2 & a_1c_2+c_1a_2 & a_1a_2 \end{bmatrix}$$

as multiplication is always commutative in Z_2.

therefore ; $\boxed{A_1A_2 = A_2A_1}$

Hence, M is commutative

Idempotent means $\Rightarrow \boxed{M^2 = I}$

$$M^2 = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 2ab & 2ac & a^2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e $a^2 = 1$, $2ab = 0 = 2ac$ i.e $b = c = 0$

$a = \pm 1$.

Hence, Idempotent elements of M are.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \& \ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Ques:1(c)) Determine whether
$f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ is Riemann-integrable on $[0,1]$ and justify your answer.

Solution:-

Given; $f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ on $[0,1]$

$$f(x) = \begin{cases} 2x\sin 1/x - \cos 1/x & , x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

f is not continuous on $[0,1]$, 0 being the point of discontinuity.

Let; $\phi : [0,1] \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Thus $\phi'(x) = f(x)$ for all $x \in [0,1]$

Thus ϕ is an antiderivative of 'f' on $[0,1]$, although 'f' is not continuous on $[0,1]$

Hence; $f(x)$ is not Reimann integrable on $[0,1]$.

Justification:

$\phi'(x) = f(x) = 2x\sin 1/x - \cos 1/x$; when $x \neq 0$

and $f(0) = \lim_{x\to 0} \frac{x^2\sin 1/x - 0}{x} = 0$

So, f is differentiable.

The derivative is not continuous at '0' because $\lim_{x\to 0} f(x)$ does not exist.

The existence of $\lim_{x\to 0} f(x)$ implies the existence of $\lim_{x\to 0} \cos 1/x$ as $\lim_{x\to 0} x \sin 1/x$ exists.

However $\lim_{x\to 0} \cos 1/x$ does not exists.

Proof:- Let $\lim_{x\to 0} \cos 1/x = m$, then gives $0 < \varepsilon < 1$ there exist $\delta > 0$, such that $0 < |x| < \delta$

$\Rightarrow |\cos 1/x - m| < \varepsilon/2$.

If $0 < |x_1| < \delta$; $0 < |x_2| < \delta$

then $|\cos 1/x_1 - m| < \varepsilon/2$; $|\cos 1/x_2 - m| < \varepsilon/2$

$\Rightarrow |\cos 1/x_1 - \cos 1/x_2| = |\cos 1/x_1 - m + m - \cos 1/x_2| < \varepsilon$

But, for any $\delta > 0$, we find $x_1 = \frac{2}{(2n+1)\pi}$; $x_2 = \frac{1}{2n\pi}$

such that $0 < |x_1| < \delta$; $0 < |x_2| < \delta$

However; $|\cos 1/x_1 - \cos 1/x_2| = 1 \not< \varepsilon$

Thus, $\lim_{x\to 0} \cos 1/x$ does not exist.

Hence; $\lim_{x\to 0} f(x)$ does not exist.

So, $f(x)$ is not continuous at $x = 0$

$\therefore f(x)$ is not Reimann Integral

Ques: 1(d)) Show that $u(x,y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x,y)$. Hence find the analytic function 'f' for which $u(x,y)$ is the real part.

Solution:- Given; $u(x,y) = 2x - x^3 + 3xy^2$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6x$$

$$\frac{\partial u}{\partial y} = 6xy$$

$$\frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$ Hence; $u(x,y)$ is harmonic.

Now, to find the conjugate harmonic.

$$\frac{\partial u}{\partial y} = 6xy = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = -6xy$$

Integrate $\Rightarrow$ $v = -3x^2 y + f(y)$.

Now; differentiate w.r.t y.

$$\frac{\partial v}{\partial y} = -3x^2 + f'(y)$$

and $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$$

Comparing both $\frac{\partial v}{\partial y}$; we have.

$$-\cancel{3x^2}+f'(y) = 2-\cancel{3x^2}+3y^2$$

$$f'(y) = 2+3y^2$$

Integrate both side

$$\boxed{f(y) = 2y+y^3+c}$$

$$\therefore \; v = -3x^2y+2y+y^3+c$$

$$\boxed{v = y^3-3x^2y+2y+c}$$

which is the harmonic conjugate of $u(x,y)$]

$$F = u(x,y)+i\,v(x,y)$$

$$\therefore \boxed{F(u,v) = 2x-x^3+3xy^2+i(y^3-3x^2y+2y+c)}$$

required solution.

Q.1 (e) Obtain the dual of the LP problem:

$\text{Min } z = x_1 + x_2 + x_3$

subject to the constraints:

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, and x_3 is unrestricted.

Sol^n: Since the problem is of minimization type, all constraints should be of ($\geq$) type.

multiply the second constraint throughout by -1, so that $-x_1 + 2x_2 \geq -3$.

and we write the first equality constraint in the form of two inequalities of $\geq$ type.

$\therefore$ The given problem can be written as

Minimize $z = x_1 + x_2 + x_3$

subject to

$$\left.\begin{aligned} x_1 - 3x_2 + 4x_3 &\geq 5 \\ -x_1 + 3x_2 - 4x_3 &\geq -5 \\ -x_1 + 2x_2 &\geq -3 \\ 2x_2 - x_3 &\geq 4 \end{aligned}\right\} \longrightarrow (1)$$

$x_1, x_2 \geq 0$, x_3 is unrestricted

Since x_3 is unrestricted.

put $x_3 = x_3' - x_3''$

The equation (1) can be written as

$\text{Min } z = x_1 + x_2 + x_3' - x_3''$

subject to

$x_1 - 3x_2 + 4x_3' - 4x_3'' \geq 5$

$-x_1 + 3x_2 - 4x_3' + 4x_3'' \geq -5$

$-x_1 + 2x_2 \qquad \geq -3$

$$2x_2 - x_3' + x_3'' \geq 4.$$

$$x_1, x_2, x_3', x_3'' \geq 0.$$

Let y_1, y_2, y_3 and y_4 be the dual variables associated with the above 4 constraints.

Then the dual is given by

Maximize $w = 5y_1 - 5y_2 - 3y_3 + 4y_4$

subject to

$$y_1 - y_2 - y_3 + 0y_4 \leq 1$$

$$-3y_1 + 3y_2 + 2y_3 + 2y_4 \leq 1$$

$$4y_1 - 4y_2 \quad - y_4 \leq 1$$

$$-4y_1 + 4y_2 \quad + y_4 \leq -1$$

$$y_i \geq 0.$$

This dual can be written in more compact form as:

Max $w = 5y' - 3y_3 + 4y_4$

Subject to

$$y' - y_3 \leq 1$$

$$-3y' + 2y_3 + 2y_4 \leq 1$$

$$4y' - \quad - y_4 \leq 1$$

$$-4y' \quad + y_4 \leq -1$$

$y', y_3, y_4 \geq 0$ and $y' (= y_1 - y_2)$ unrestricted

(or)

Max $w = 5y'_1 - 3y_3 + 4y_4$

subject to

$$y' - y_3 \leq 1$$

$$-3y' + 2y_3 + 2y_4 \leq 1$$

$$4y' - y_4 = 1$$

$y_3, y_4 \geq 0$ and y' is unrestricted.

2(a) (i) If A, B to be two ideals of a ring R, then $AB \subseteq A\cap B$. Give an example to show that there exist ideals A and B such that $AB \neq A\cap B$.

(ii) Let R be a ring with unity. If R has no right ideals except R and {0}, then prove that R is a division ring.

Sol'n: By corollary theorem

If A and B are two ideals of a ring R, then
$$AB \subseteq A\cap B.$$

Let $A = \left\{\begin{bmatrix} a & 0 \\ b & 0\end{bmatrix} : a,b\in \mathbf{Z}\right\}$, $B = \left\{\begin{bmatrix} c & d \\ 0 & 0\end{bmatrix} : c,d\in Z\right\}$

Then A is left ideal of M_2 and B is a right ideal of M_2 (M_2 being the ring of all 2×2 matrices over the integers). It follows that AB is an ideal of M_2.

we have
$$\begin{bmatrix} a & 0 \\ b & 0\end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0\end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd\end{bmatrix}$$

Consequently, $AB = \left\{\begin{bmatrix} p & q \\ r & s\end{bmatrix} : p,q,r,s\in Z\right\}$

However, $A\cap B = \left\{\begin{bmatrix} x & 0 \\ 0 & 0\end{bmatrix} : x\in Z\right\}$

Hence $AB \neq A\cap B$.

For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix} \in AB$, but $\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix} \subseteq A\cap B$.

(ii) Sol'n: we are given that $1 \in R$ and so R becomes a division ring if we show that each non-zero element in R has its multiplicative inverse in R. Let $a \neq 0 \in R$ be arbitrary.

Let $aR = \{ax : x \in R\}$

Since $0 = a.0 \in aR$, aR is non-empty.

Let $\alpha, \beta \in aR$ and $\gamma \in R$. Then

$\alpha = ax,\ \beta = ay,$ for some $x, y \in R$, by ①

$\Rightarrow \alpha - \beta = ax - ay = a(x-y) \in aR$, as $x-y \in R$,

and $\alpha\gamma = (ax)\gamma = a(x\gamma) \in aR$, as $x\gamma \in R$.

thus aR is a right ideal of R and so by the given hypothesis.

$$aR = \{0\} \text{ or } aR = R$$

Since $1 \in R$ and $a = a.1 \in aR$ and $a \neq 0$, so $aR \neq \{0\}$

Hence $aR = R$. Since $1 \in R$, so $1 \in aR$. Consequently,

$1 = ab$, for some $b \in R$

from ②, it follows that each non-zero element of R has a right inverse. Since $b \neq 0 \in R$ (for otherwise, $1 = ab = 0$, a contradiction), there exists some $c \in R$ such that $bc = 1$.

we have $ba = ba.1$, since 1 is the unity of R.

$= (ba)(bc)$, using ③

$= b(ab)c$, by ②

$= bc$

$= 1$, by ③

$\therefore ab = ba = 1$

$\Rightarrow a^{-1} = b \in R$

Hence R is a division ring.

Ques:- 2(b)(i) Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$, is uniformly convergent but not absolutely for all real values of x.

Solution:-

To show uniform convergence

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n+x^2}$$

$$f(x) = (-1)^{n-1}$$

$$g(x) = \frac{1}{n+x^2}$$

Using dirichlet theorem

(1) Sum function; $S_n = \sum_{n=1}^{r} (-1)^{n-1} = \begin{cases} 0 & ; r \text{ is even.} \\ 1 & ; r \text{ is odd.} \end{cases}$

Thus; S_n is bounded.

(2) To check sequence $\langle g(x) \rangle$ is positive, monotonic decreasing sequence converging to zero.

$$u_n - u_{n+1} = \frac{1}{n+x^2} - \frac{1}{n+1+x^2}$$

$$= \frac{1}{(n+x^2)(n+1+x^2)} > 0$$

$u_n > u_{n+1}$; thus $g(x)$ is positive monotonic decreasing sequence converging to zero.

$\therefore$ By Dirichlet theorem;

$$\boxed{\sum (-1)^{n-1} \frac{1}{n+x^2} \text{ is uniformly convergent.}}$$

Now, to check whether 'f' is absolutely convergent or not.

$\sum f(x)$ is absolutely convergent if $\sum |f(x)|$ is convergent.

$$\sum \left|\frac{(-1)^{n-1}}{n+x^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n+x^2}$$

Let $u_n = \dfrac{1}{n+x^2}$; $u_{n+1} = \dfrac{1}{n+1+x^2}$

$$\frac{u_n}{u_{n+1}} = \frac{n+1+x^2}{n+x^2} = \left(1+\frac{1}{n+x^2}\right)$$

$$= \left(1+\frac{1+x^2}{n}\right)\left(1+\frac{x^2}{n}\right)^{-1}$$

<u>Doing binomial Expansion</u>

$$= \left(1+\frac{1+x^2}{n}\right)\left(1-\frac{x^2}{n}+\frac{x^4}{2n^2}+\cdots\right)$$

$$= \left[1+\frac{1+x^2}{n}-\frac{x^2}{n}-\frac{x^2(1+x^2)}{n^2}+\frac{x^4}{2n^2}+\cdots\right]$$

$$= \left[1+\frac{1}{n}-\frac{x^2(1+x^2)}{n^2}+\frac{x^4}{2n^2}\cdots\right]$$

$\lambda = 1$; i.e coefficient of $1/n$ is 1, then by Gauss's test given series is divergent, thus.

$$\boxed{\sum \frac{(-1)^{n-1}}{n+x^2} \text{ is not absolutely convergent}}$$

which is required result

Ques: 2(b)(II) Define $\{x_n\}$ by $x_1 = 5$ and
$x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$.
Show that the sequence converges to $\frac{(1+\sqrt{17})}{2}$.

Solution:-

Given; $x_{n+1} = \sqrt{4 + x_n} \quad \forall\, n > 1$ —— (1)

and $x_1 = 5$

when; $n = 1$. $x_2 = \sqrt{4 + x_1} = \sqrt{4+5}$

$x_2 = \sqrt{9} = 3 < x_1$

$\therefore\ x_2 < x_1$

Let $x_{k+1} < x_k$

$\Rightarrow\ 4 + x_{k+1} < 4 + x_k$

$\sqrt{4 + x_{k+1}} < \sqrt{4 + x_k}$

$x_{k+2} < x_{k+1} \quad \forall\, k > 2$

$\therefore\ x_{k+1} < x_k \quad \forall\, k > 1$

$\therefore\ x_{n+1} < x_n \quad \forall\, n > 1$.

$\therefore\ \{x_n\}$ is monotonically decreasing.

Now; $\because\ x_1 = 5 < 12$.

$x_2 = \sqrt{x_1 + 4} = 3 < 12$

let $x_k < 12$

$4 + x_k < 16$

$\sqrt{4+x_k} < \sqrt{16}$

$x_{k+1} < 4 < 12$

i.e $x_{k+1} < 12$

$\Rightarrow x_{n+1} < 12 \quad \forall\, n > 1$ [by mathematical Induction]

$\Rightarrow x_n < 12 \quad \forall\, n \geq 1$

$\therefore \{x_n\}$ is bounded above by 12.

Hence, $\{x_n\}$ is convergent.

Let, $\lim\limits_{n\to\infty} x_n = l$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = l$

$\because x_{n+1} = \sqrt{4+x_n}$

$\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \sqrt{4+x_n}$

$l = \sqrt{4+l}$

Squaring both side.

$l^2 = 4+l$

$l^2 - l - 4 = 0$

$\therefore l = \dfrac{+1 \pm \sqrt{1+4\times 4}}{2\times 1} = \dfrac{1 \pm \sqrt{1+16}}{2}$

$l = \dfrac{1 \pm \sqrt{17}}{2}$; But $l \neq \dfrac{1-\sqrt{17}}{2}$

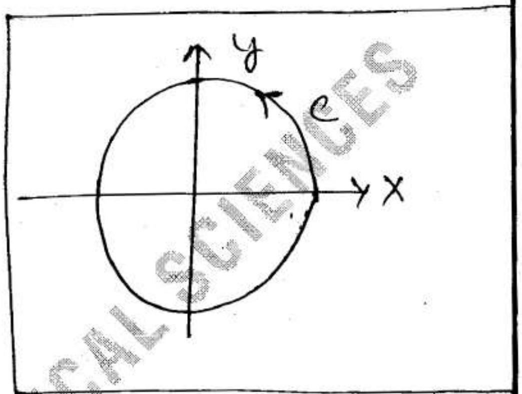
because $x_1 = 5 > 0$

$\therefore \boxed{l = \dfrac{1+\sqrt{17}}{2}}$ Ans hence proved.

Ques: 2(c) If α, β, γ are real numbers such that $\alpha^2 > \beta^2+\gamma^2$. Show that :

$$\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}$$

Solution:-



Let $z = e^{i\theta}$, then

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$dz = i\cdot e^{i\theta}\,d\theta$$

$$\boxed{dz = iz\,d\theta}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \oint_C \frac{dz}{iz\left[\alpha+\beta\left(\frac{z+z^{-1}}{2}\right)+\gamma\left[\frac{z-z^{-1}}{2i}\right]\right]}$$

$$= \oint_C \frac{2i\,dz}{iz\left[2\alpha i+\frac{2i\beta}{2}(z+z^{-1})+\gamma(z-z^{-1})\right]}$$

$$= \oint_C \frac{2\,dz}{2\alpha zi+z\left[\frac{z^2\beta i+\beta i}{z}+\frac{\gamma(z^2-1)}{z}\right]}$$

$$= \oint_C \frac{2\,dz}{2\alpha zi+z^2\beta i+\beta i+\gamma z^2-\gamma}$$

$$= \oint_C \frac{2\,dz}{z^2(\beta i+\gamma)+2\alpha zi+(\beta i-\gamma)}$$

$$= \oint_C \frac{2\,dz}{(\gamma+i\beta)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left[\frac{\beta i-\gamma}{\gamma+\beta i}\right]\right]}$$

$$= \oint_c \frac{2\,dz}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z + \frac{-\gamma+\beta i}{\gamma+\beta i}\right]} \quad \text{--- (1)}$$

where 'c' is the circle of unit radius with centre at the origin.

The poles of $\dfrac{2}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$

are the single poles —

$$z = \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \sqrt{\frac{-4(\alpha^2)}{(\gamma+\beta i)^2} - 4(1)\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)}}{2}$$

$$z = \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \frac{2}{\gamma+\beta i}\sqrt{-\alpha^2 - (-\gamma+\beta i)(\gamma+\beta i)}}{2}$$

$$= \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \frac{2i}{\gamma+\beta i}\sqrt{\alpha^2+(-)(\gamma^2+\beta^2)}}{2}$$

$$\Rightarrow \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-\gamma^2-\beta^2}$$

or

$$z = \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)} \quad \text{--- (2)}$$

Given, $\alpha^2 > \beta^2+\gamma^2$.

$\therefore z = \dfrac{-\alpha i + i\sqrt{\alpha^2-(\gamma^2+\beta^2)}}{\gamma+\beta i} = z_1$ (say) lies inside 'c' because $|z|<1$.

Now residue of z_1

$$\Rightarrow \lim_{z \to z_1} (z - z_1) \frac{2}{(\gamma+\beta i)\left[z^2 + \left(\frac{2\alpha i}{\gamma+\beta i}\right) z + \left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$$

$$\Rightarrow \lim_{z \to z_1} \frac{2}{\cancel{(\gamma+\beta i)}\left[\frac{z + \alpha i + i\sqrt{\alpha^2 - (\beta^2+\gamma^2)}}{\cancel{\gamma+\beta i}}\right]}$$

$$\Rightarrow \frac{1}{i\sqrt{\alpha^2 - (\beta^2+\gamma^2)}}.$$

$$\therefore (1) \equiv \oint_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = 2\pi i \times \left(\begin{array}{c}\text{residue at} \\ z_1\end{array}\right)$$

$$\therefore \oint_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = 2\pi \cancel{i} \times \frac{1}{\cancel{i}\sqrt{\alpha^2 - (\beta^2+\gamma^2)}}$$

$$\therefore \boxed{\oint_c^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - (\beta^2+\gamma^2)}}}$$

Hence proved

Ques: 3(a)(i) Let R be a ring and $M_2(R)$ be the ring of 2×2 matrices with entries from R. Explain why these two rings have the same Charateristic.

(ii) Is $\{Z_6, \oplus_6, \otimes_6\}$ a field? Justify.

Sol'n: $M_2(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ a,b,c,d \in R \right\}$ be the ring of 2×2 matrices over R w.r.t $+^n$ & $\times^n$

let $\text{Char}(R) = n$

then $na = 0 \;\forall\; a \in R$ —— (1).

where n is the least +ve integer.

let $A \in M_2$: $m = \text{Char}(M_2)$

then we have

$$mA = m\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ma & mb \\ mc & md \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow mA = 0$ —— (2) $\quad$ ($\because ma = 0 = mb = mc = md = 0 \;\forall\; a,b,c,d \in R$)

$\Rightarrow ma = 0 \;\forall\; a \in R$

$\Rightarrow n \le m$ —— (3)

we have

$$nA = \begin{bmatrix} na & nb \\ nc & nd \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow nA = 0 \;\forall\; A \in M_2$

$\Rightarrow m \le n$ —— (4)

from (3) & (4) $\underline{\underline{m = n}}$

Ques: 3(a)(ii) Is $\{Z_6, \oplus_6, \otimes_6\}$ a field? Justify?

Solution:

Given; $\{Z_6, \oplus_6 \otimes_6\}$

$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

Z_6 is not a field.

because; Z_6 has zero divisor.

i.e $\bar{2} \cdot \bar{3} = 0$

$\Rightarrow$ $\bar{2}$ and $\bar{3}$ are zero divisors.

Hence, Z_6 cannot be a field.

Ques: 3(b) i) Consider the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

Find the values of x for which it is convergent and also the sum function.
Is the convergence uniform? Justify your answer.

Solution:- Given series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

$$S_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n}$$

$$S_n(x) = x^2 \left\{ 1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n} \right\}$$

$$S_n(x) = x^2 \cdot \frac{1\left(1 - \frac{1}{(1+x^2)^{n+1}}\right)}{1 - \frac{1}{1+x^2}}$$

$$S_n(x) = \frac{(1+x^2)^{n+1} - 1}{(1+x^2)^n}$$

$$\boxed{S_n(x) = (1+x^2) - \frac{1}{(1+x^2)^n}}$$

Sum function; $S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} (1+x^2) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

$$\left| S_n(x) - S(x) \right| = \left| \frac{1}{(1+x^2)^n} \right| < \varepsilon$$

$$\Rightarrow (1+x^2)^n > 1/\varepsilon$$

$$\Rightarrow n \log(1+x^2) > \log 1/\varepsilon$$

$$\boxed{n > \frac{\log 1/\varepsilon}{\log(1+x^2)}}$$

At $x=0$; there exist no integer 'm' (large +ve number) such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall \ n \geq m$$

thus, the given series is not uniformly convergent at $x=0$, but it is convergent uniformly for all other 'x'.

Ques: 3(b) ii) Test the convergence of $\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}dx$?

Solution: Given, $I = \int_0^1 \frac{\sin(1/x)}{\sqrt{x}}dx$

Let $g(x) = \frac{1}{\sqrt{x}}$ and $f(x) = \frac{\sin 1/x}{\sqrt{x}}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin 1/x}{\sqrt{x}\cdot 1/\sqrt{x}} = L$$

$L \in [-1, 1]$; which is a finite limit

$\therefore \int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converges and diverges together.

Now, $\int_0^1 \frac{1}{\sqrt{x}}dx = \int_0^1 \frac{dx}{(x)^{1/2}}$

is convergent since $n = 1/2 < 1$. [By p-test]

$\Rightarrow g(x)$ is convergent.

Hence, $f(x)$ is convergent

$\therefore \int_0^1 \frac{\sin(1/x)}{\sqrt{x}}dx$ is convergent.

Ques: 3(c)) Find the optimum solution of the following Transportation table.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

Solution:-

From the given table

Total Demand $= \sum b_j = 20+40+30+10+50+25 = 175$

Total Availability $= \sum a_i = 30+50+75+20 = 175$.

Hence, $\boxed{\sum a_i = \sum b_j = 175}$

$\therefore$ Given transportation problem is balanced. Hence, for initial solution using Low cost Entry method:

	D_1	D_2	D_3	D_4	D_5	D_6	
O_1	20 $_{1}$	X $_{2}$	10 $_{1}$	X $_{4}$	X $_{5}$	X $_{2}$	30
O_2	X $_{3}$	X $_{3}$	20 $_{2}$	10 $_{1}$	20 $_{4}$	X $_{3}$	~~50~~ ~~40~~ ~~20~~ 0
O_3	X $_{4}$	40 $_{2}$	X $_{5}$	X $_{9}$	10 $_{6}$	25 $_{2}$	~~75~~ ~~35~~ 10
O_4	X $_{3}$	X $_{1}$	X $_{7}$	X $_{3}$	20 $_{4}$	X $_{6}$	20
	~~20~~ 0	~~40~~ 0	~~30~~ ~~20~~ 0	~~10~~ 0	~~50~~ ~~30~~ ~~20~~ 0	~~25~~ 0	

Here ; the number of positive allocations = 9

and $m+n-1 = 6+4-1 = 9$

$\therefore$ Initial feasible solution $= 20\times1 + 10\times1 + 20\times2 + 10\times1 + 20\times4 + 40\times2 + 10\times6 + 25\times2 + 20\times4$

$\boxed{\text{IBFS} = 430}$

Lets check the optimality

Using MODI method:

For Basic Cells ; $\Delta_{ij} = u_i + v_j - c_{ij} = 0$

$= u_i + v_j = c_{ij}$

$u_1 + v_1 = 1$

$u_1 + v_3 = 1$

$u_2 + v_3 = 2$

$u_2 + v_4 = 1$

$u_2 + v_5 = 4$

$u_3 + v_2 = 2$

$u_3 + v_5 = 6$

$u_3 + v_6 = 2$

$u_4 + v_5 = 4$

let $u_2 = 0$, $u_1 = -1$, $u_3 = 2$, $u_4 = 0$

$v_1 = 2$, $v_2 = 0$, $v_3 = 2$, $v_4 = 1$, $v_5 = 4$, $v_6 = 0$

Non- check for non-basic cells (Δ_{ij})

$\Delta_{12} = u_1 + v_2 - 2 = -1 + 0 - 2 = -3$

$\Delta_{14} = u_1 + v_4 - 4 = -1 + 1 - 4 = -4$

$\Delta_{15} = u_1 + v_5 - 5 = -1 + 4 - 5 = -2$

$\Delta_{16} = u_1 + v_6 - 2 = -1 + 0 - 2 = -3$

$\Delta_{21} = u_2 + v_1 - 3 = 0 + 2 - 3 = -1$

$\Delta_{22} = u_2 + v_2 - 3 = 0 + 0 - 3 = -3$

$\Delta_{26} = u_2 + v_6 - 3 = 0 + 0 - 3 = -3$

$\Delta_{31} = u_3 + v_1 - 4 = 2 + 2 - 4 = 0$

$\Delta_{33} = u_3 + v_3 - 5 = 2 + 2 - 5 = -1$

$\Delta_{34} = u_3 + v_4 - 9 = 2 + 1 - 9 = -6$

$\Delta_{41} = u_4 + v_1 - 3 = 0 + 2 - 3 = -1$

$\Delta_{42} = u_4 + v_2 - 1 = 0 + 0 - 1 = -1$

$\Delta_{43} = u_4 + v_3 - 7 = 0 + 2 - 7 = -5$

$\Delta_{44} = u_4 + v_4 - 3 = 0 + 1 - 3 = -2$

$\Delta_{46} = u_4 + v_6 - 6 = 0 + 0 - 6 = -6$

As we observe, there all $\Delta_{ij} \leq 0$ in non-basic cells; Hence optimality obtained.

$\therefore$ optimal Transportation cost at

$x_{11} = 20$, $x_{13} = 10$

$x_{23} = 20$, $x_{24} = 10$, $x_{25} = 20$

$x_{32} = 40$, $x_{35} = 10$, $x_{36} = 25$.

$x_{45} = 20$

$\therefore$ Minimum cost of Transportation (i.e optimal value)

$= 1\times20 + 1\times10 + 2\times20 + 1\times10 + 4\times20 + 2\times40 + 6\times10 + 2\times25 + 4\times20$

$\Rightarrow 430.$

$\boxed{\therefore \text{ Min. Cost of Transportation} = 430}$

which was our initial solution.

Ques: 4(a)) Show that $4x^2+6x+2$ is not a primitive polynomial in $Z[x]$, where Z is the ring of integers. Will $4x^2+6x+2$ be a primitive polynomial over $Q[x]$? Justify your answer?

Solution:-

(i) Let $Z = \{ \ldots\ldots -3,-2,-1,0,1,2,\ldots\ldots \}$ be a U.F.D

then $Z[x]$ is a ring of all polynomials

Let $P(x) = 4x^2+6x+2 \in Z[x]$

Such that the content of $P(x) = \gcd[4,6,2]$
$= 2$

Clearly, content of $P(x)$ is not a unit in Z.

$\therefore$ By Definition of primitive polynomial,

$P(x) = 4x^2+6x+2$ is not a primitive in $Z[x]$

Hence proved

(ii) Let $Q = \{ p/q \mid (p,q)=1 \ \& \ q \neq 0 \}$, $p, q \in Z$

be a U.F.D.

then $Q[x]$ is a ring of all polynomials

Let $q(x) = 4x^2+6x+2 \in Q[x]$

Such that the content of $q(x) = \gcd(4,6,2)$
$= 2$

Clearly, the content of $q(x)$ is a unit in Q.

$\therefore$ By definition of primitive polynomial,
$q(x) = 4x^2+6x+2$ is primitive in $\mathbb{Q}[x]$

Hence, the result.

Ques: 4(b)> Show that the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{4}{6}\right)^2+\cdots+\left(\frac{1\cdot4\cdot7\ldots(3n-2)}{3\cdot6\cdot9\ldots3n}\right)^2+\cdots$$

converges.

Solution:-

given series; $\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{4}{6}\right)^2+\cdots+\left(\frac{1\cdot4\cdot7\cdots(3n-2)}{3\cdot6\cdot9\cdots3n}\right)^2+\cdots$

Consider;

$$u_n = \left[\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}\cdots\cdots\frac{3n-2}{3n}\right]^2$$

$$u_{n+1} = \left[\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}\cdots\cdots\frac{3n-2}{3n}\cdot\frac{3n+1}{3n+3}\right]^2$$

Using ratio test

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left[\frac{3n+3}{3n+1}\right]^2$$

$$= \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{3+3/n}{3+1/n}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \left[\frac{3}{3}\right]^2 = 1.$$

Thus ratio test fails

Modifying; $\frac{u_n}{u_{n+1}} = \frac{3n+3}{3n+1}$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{3n}\right)^{-1}$$

By Gauss's test

Expand $\frac{u_n}{u_{n+1}}$ using binomials expansions

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1-\frac{1}{3n}+\frac{1}{(3n)^2}-\cdots\cdots\right)$$

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{n}-\frac{1}{3n}+\frac{1}{(3n)^2}-\frac{1}{3n^2}+\frac{1}{9n^3}+\cdots\right]$$

$$\boxed{\frac{u_n}{u_{n+1}} = \left[1+\frac{2}{3n}-\frac{2}{9n^2}+\cdots\cdots\right]}$$

Since; $\lambda = \frac{2}{3} < 1$, that is coefficient of $1/n$ is less than 1, then by gauss's test given series is convergent.

Ques: 4(c)> Prove that the function 'f' is defined by-

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0 & ; z=0 \end{cases}$$ is not differentiable at $z=0$?

Solution:-

Given function is -

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$$

Consider, $\lim\limits_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim\limits_{z\to 0} \frac{z^{\cancel{5}4}/|z|^4}{z}$

$$= \lim_{z\to 0} z^4/|z|^4 = \lim_{(x,y)\to(0,0)} \frac{(x+iy)^4}{(x^2+y^2)^2}$$

put $z=x$, $y=0$; i.e along x-axis, we get

$$\lim_{z\to 0} \frac{(x+iy)^4}{(x^2+y^2)^2} = \frac{x}{x} = 1.$$

Along st. line i.e $y=x$:

$\lim\limits_{x,y\to(0,0)} \frac{(x+iy)^4}{(x^2+y^2)^2}$ becomes

$$= \lim_{y\to 0} \frac{(y+iy)^4}{(y^2+y^2)^4} = \frac{\cancel{y^4}(1+i)^4}{\cancel{y^4}(4)} = \frac{(1+i)^4}{4}$$

$$= \frac{1}{4}[4i^2] = i^2 = -1.$$

Since, $1 \neq -1$

$\therefore$ limit does not exist.

Hence; function; $f(z) = \begin{cases} z^5/|z|^4 & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

is not differentiable at $z=0$

i.e $f'(z)$ does not exist at $z=0$

Ques: 4(d)) Solve the L.P.P. by simplex method.

Maximize $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 2$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

and $x_1, x_2 \geq 0$

Solution:- The objective function of the given LPP is of maximize type.

Now we write the given LPP in standard form

Max. $Z = 5x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3$

$x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 2$

$5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 10$

$3x_1 + 8x_2 + 0S_1 + 0S_2 + S_3 = 12$

$x_1, x_2, S_1, S_2, S_3 \geq 0$

where S_1, S_2, S_3 are slack variables.

Now, the IBFS is $x_1 = x_2 = 0$ (non-basic)

$S_1 = 2,\ S_2 = 10,\ S_3 = 12$ (basic)

for which $\boxed{Z = 0}$

Now put the above information in the simplex tableau:

	C_j	5	3	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	1	1	1	0	0	2	$2/1 \to 2 \to$
0	S_2	5	2	0	1	0	10	$10/5 = 2$
0	S_3	3	8	0	0	1	12	$12/3 = 4$
$Z_j = \sum C_B a_{ij}$		0	0	0	0	0	0	
$C_j = C_j - Z_j$		5	3	0	0	0		

$\uparrow$

From the above table ;
x_1 is the entering variable , S_1 is the outgoing variable and (1) is the key element and other elements in this column equal to zero.
Then the revised simplex Tableau is :

	C_j	5	3	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
5	x_1	1	1	1	0	0	2	2
0	S_2	0	-3	-5	1	0	0	0
0	S_3	0	5	-3	0	1	6	6/5
$Z_j = \sum C_B a_{ij}$		5	5	5	0	0	10	
$C_j = C_j - Z_j$		0	-2	-5	0	0		

As C_j is either zero or negative (i.e. $C_j \leq 0$) under all columns, the above table gives the optimal basic feasible solution.

$\therefore$ The optimal solution is $x_1 = 2$, $x_2 = 0$

and Maximize $Z = 5x_1 + 3x_2$
$= 5\times2 + 3\times0$
$= 10$

$\boxed{\therefore \; Z_{max} = 10}$ Required solution.

Section - B

Ques: 5(a) Obtain the general solution of

$$(D-3D'-2)^2 z = 2e^{2x}\sin(y+3x)$$

where $D=\frac{\partial}{\partial x}$ and $D'=\frac{\partial}{\partial y}$.

Solution:- Given; $(D-3D'-2)^2 z = 2e^{2x}\sin(y+3x)$ —— (1)

The auxillary equation of above eqn

$$(D-3D'-2)^2 = 0$$

$$\therefore C.F = e^{2x}\left[\phi_1(y+3x) + x\phi_2(y+3x)\right]$$

and Now for P.I

$$P.I = \frac{1}{(D-3D'-2)^2}\cdot 2e^{2x}\cdot\sin(y+3x)$$

$$= \frac{1}{(D-3D'-2)^2}\cdot 2\cdot e^{2x+0y}\cdot\sin(y+3x).$$

$$= 2\cdot e^{2x}\cdot\frac{1}{((D+2)-3(D'+0)-2)^2}\sin(y+3x)$$

$$= 2e^{2x}\cdot\frac{1}{(D-3D')^2}\sin(y+3x)$$

$$= 2e^{2x}\frac{x^2}{1^2\cdot 2!}\sin(y+3x) = x^2e^{2x}\cdot\sin(y+3x)$$

$\therefore$ General solution $=$ Cf + P I

$$\boxed{z = e^{2x}\left[\phi_1(y+3x) + x\phi_2(y+3x) + x^2\sin(y+3x)\right]}$$

Required solution.

Ques: 5(b) Reduce the equation

$$y\frac{\partial^2 z}{\partial x^2} + (x+y)\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial^2 z}{\partial y^2} = 0$$

to its canonical form when $x \neq y$.

Solution:- Given,

$$y\frac{\partial^2 z}{\partial x^2} + (x+y)\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial^2 z}{\partial y^2} = 0$$

which can be re-written as.

$$yr + (x+y)s + xt = 0 \quad \text{——} (1)$$

comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

Here; $R = y$, $S = x+y$, $T = x$ & $f(x,y,z,p,q) = 0$

So that $S^2 - 4RT = (x+y)^2 - 4 \times y \times x$

$$S^2 - 4RT = x^2 + y^2 + 2xy - 4xy$$

$$\boxed{S^2 - 4RT = (x-y)^2} > \text{ for } x \neq y$$

and so (1) is hyperbolic. It's λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$y\lambda^2 + (x+y)\lambda + x = 0$$

$$(y\lambda + x)(\lambda + 1) = 0$$

So ; $\lambda = -1, -x/y$

Then, the corresponding characteristic equations are given by -

$$\frac{dy}{dx} - 1 = 0 \quad \text{and} \quad \frac{dy}{dx} - (x/y) = 0$$

Integrating both equations

$$y-x=c_1 \quad \text{and} \quad \frac{y^2}{2}-\frac{x^2}{2}=c_2$$

In order to reduce one (1) to its canonical form—
we choose —

$$u=y-x \quad \text{and} \quad v=y^2/2-x^2/2 \quad \text{—— (2)}$$

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}.$$

$$p=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right), \text{ using (2)} \quad \text{—— (3)}$$

$$q=\frac{\partial z}{\partial y}=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}\right)+\left(\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)=\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right) \text{—— (4)}$$

(using 2)

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right) \text{ (using 3)}$$

$$r=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]$$

$$r=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

By solving it,

$$\boxed{r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}} \quad \text{—— (5)}$$

$$\text{Now, } t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)$$

using (4),

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}$$

By solving it

$$\boxed{t = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}} \quad \text{—— (6)}$$

Also ; $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$ [using (4)]

By solving –

$$\boxed{s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}} \quad \text{—— (7)}$$

using (5), (6) and (7) in (1), we get.

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right) + (x+y)\left[-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v}\right.$$

$$\left. - xy\frac{\partial^2 z}{\partial v^2}\right] + x\left[\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right) = 0$$

$$\Rightarrow (\text{or}) \Rightarrow \{4xy - (x+y)^2\}\frac{\partial^2 z}{\partial u \partial v} - y\frac{\partial z}{\partial v} + x\frac{\partial z}{\partial v} = 0$$

$$\Rightarrow (y-x)^2\frac{\partial^2 z}{\partial u \partial v} + (y-x)\frac{\partial z}{\partial v} = 0$$

$$\Rightarrow \text{or} \quad u^2\frac{\partial^2 z}{\partial u \partial v} + u\frac{\partial z}{\partial v} = 0 \Rightarrow u\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0 \quad \text{—— (8)}$$

[$\because u \neq 0$ and $y - x = u$, by (2)]

(8) is the required canonical form of (1).

i.e $\boxed{u\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0}$

Ques:- 5(c) Apply Lagrange's interpolation formula to find $f(5)$ and $f(6)$; given that $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(7)=128$.

Solution:- From above, we can draw a table.

x	1^{x_0}	2^{x_1}	3^{x_2}	$7\ x_3$
$f(x)$	2 $f(x_0)$	4 $f(x_1)$	8 $f(x_2)$	128 $f(x_3)$

To find $f(5)$ and $f(6)$.

Lagranges's formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\cdot f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\cdot f(x_3)$$

$$\therefore f(5) = \frac{(5-2)(5-3)(5-7)}{(1-2)(1-3)(1-7)}\times 2 + \frac{(5-1)(5-3)(5-7)}{(2-1)(2-3)(2-7)}\times 4$$

$$+ \frac{(5-1)(5-2)(5-7)}{(3-1)(3-2)(3-7)}\times 8 + \frac{(5-1)(5-2)(5-3)}{(7-1)(7-2)(7-3)}\times 128$$

$$f(5) = \frac{\cancel{3}\times\cancel{2}\times-\cancel{2}}{-\cancel{1}\times-\cancel{2}\times-\cancel{6}}\times 2 + \frac{4\times 2\times +2}{1\times +1\times -5}\times 4$$

$$+ \frac{\cancel{4}\times 3\times\cancel{+2}}{\cancel{2}\times 1\times\cancel{+4}}\times 8 + \frac{\cancel{4}\times\cancel{3}\times\cancel{2}}{\cancel{6}\times 5\times\cancel{4}}\times 128$$

$$f(5) = 2 - \frac{64}{5} + 24 + \frac{128}{5} = 26 + \frac{64}{5} = \frac{130}{5} + \frac{64}{5}$$

$$\boxed{f(5) = \frac{194}{5}}$$ required value.

Now; for f(6).

$$\therefore f(6) = \frac{(6-2)(6-3)(6-7)}{(1-2)(1-3)(1-7)} \times 2 + \frac{(6-1)(6-3)(6-7)}{(2-1)(2-3)(2-7)} \times 4$$

$$\frac{(6-1)(6-2)(6-7)}{(3-1)(3-2)(3-7)} \times 8 + \frac{(6-1)(6-2)(6-3)}{(7-1)(7-2)(7-3)} \times 128$$

$$\therefore f(6) = \frac{\cancel{4} \times \cancel{3} \times \cancel{-1}}{\cancel{-1} \times \cancel{-2} \times \cancel{-6}} \times 2 + \frac{\cancel{5} \times 3 \times -1}{1 \times \cancel{-1} \times \cancel{-5}} \times 4$$

$$+ \frac{5 \times \cancel{4} \times \cancel{-1}}{\cancel{2} \times 1 \times \cancel{-4}} \times \cancel{8}\,4 + \frac{\cancel{5} \times \cancel{4} \times \cancel{3}}{\cancel{6}_2 \times \cancel{5} \times \cancel{4}} \times \cancel{128}\,64$$

$$f(6) = 2 - 12 + 20 + 64$$

$$f(6) = 86 - 12$$

$$\boxed{f(6) = 74}$$

$$\therefore \boxed{f(5) = \frac{194}{5} \text{ and } f(6) = 74}$$

Required solution.

Ques:- 5(d)) (i) Realize the following expression by using NAND gates only:

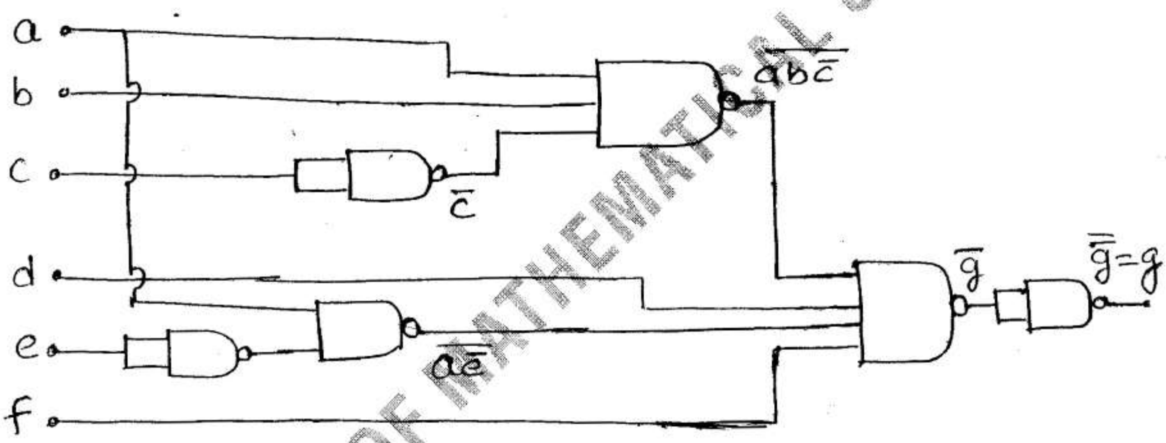
$$g = (\bar{a}+\bar{b}+c)\bar{d}\,(\bar{a}+e)f$$

where $\bar{x}$ denotes the complement of x.

Solution:-

given; $g = (\bar{a}+\bar{b}+c)\bar{d}\,(\bar{a}+e)f$

$$g = \overline{a \cdot b\bar{c}} \cdot d \cdot (\overline{a \cdot \bar{e}})f$$



(ii) Find the decimal equivalent of $(357.32)_8$?

Solution:- $(357.32)_8 \longleftrightarrow (?)_{10}$

357
$7 \times 8^0 = 7$
$5 \times 8^1 = 40$
$3 \times 8^2 = 192$
$(232)_{10}$

0.32
$2 \times 8^{-2} = 0.03125$
$3 \times 8^{-1} = 0.375$
$(0.40625)_{10}$

$$\boxed{\therefore\ (357.32)_8 \longleftrightarrow (232.40625)_{10}}$$

required solution.

Ques:-5(d)

(iii) Compute $(3205)_{10}$ to the base 8.

Solution:- To find $(3205)_{10} \longleftrightarrow (\_\_\_)_8$

8	3205	
8	400	5
8	50	0
	6	2

$\Rightarrow (6205)_8$

$\therefore \boxed{(3205)_{10} \longleftrightarrow (6205)_8}$

Ques: 5(e) The velocity components in a 3-Dimensional flow field for an incompressible fluid are $(2x, -y, -z)$. Is it a possible field? Determine the equations of the streamline passing through the point $(1, 1, 1)$?

Solution:- Given; $u = 2x$, $v = -y$ and $w = -z$

Streamlines are given by -

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{i.e} \quad \frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z} \quad \text{——(1)}$$

Taking first two of (1), we have

$$\frac{dx}{2x} = \frac{dy}{-y} \Rightarrow (\text{Integrate}) \Rightarrow \frac{1}{2}\log x = -\log y + \log c_1$$

$$\Rightarrow \log x + 2\log y = \log c_1 \Rightarrow \boxed{xy^2 = c_1} \quad \text{—(2)}$$

Again taking first and third of (1), we have.

$$\frac{dx}{2x} = \frac{dy}{-y} \Rightarrow \boxed{xz^2 = c_2} \quad \text{——— (3)}$$

Here c_1 and c_2 are arbitrary constants.

The streamlines are given by the curves of intersection of (2) and (3).

The required streamline passes through $(1, 1, 1)$ so that $c_1 = 1$ and $c_2 = 1$.

Thus, the desired streamline is given by the intersection of $xy^2 = 1$ and $xz^2 = 1$.

we also have.

$$\frac{\partial u}{\partial x} = 2 \quad ; \quad \frac{\partial v}{\partial y} = -1 \quad \text{and} \quad \frac{\partial w}{\partial z} = -1.$$

So that;

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2-1-1 = 0}$$

Showing that the equation of continuity is satisfied for the given flow for an incompressible fluid.

Hence, the given velocity components corresponds to a possible field.

Ques: 6(a)

(i) Form the partial differential equation by eliminating the arbitrary function 'f' given by → $f(x^2+y^2, z-xy) = 0$

Solution:- Given; $f(x^2+y^2, z-xy) = 0$ ——(1)

Let $x^2+y^2 = u$ and $z - xy = v$

Differentiating (1) w.r.t x,

$$\frac{\partial f}{\partial x}(u,v) = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) \quad (2)$$

Differentiating (1) w.r.t y

$$\frac{\partial f}{\partial y}(u,v) = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) \quad (3)$$

(2) ≈ $\frac{\partial f}{\partial u}(2x + 0\cdot p) + \frac{\partial f}{\partial v}(-y+p) = 0$ ——(4)

(3) ≈ $\frac{\partial f}{\partial u}(2y + 0\cdot p) + \frac{\partial f}{\partial v}(-x+q) = 0$ ——(5)

Divide (4) by (5), we get.

$$\frac{2x}{2y} = \frac{p-y}{p-x} \Rightarrow \frac{x}{y} = \frac{p-y}{q-x}$$

$$\Rightarrow \quad yp - y^2 = xq - x^2$$

$$\Rightarrow \quad \boxed{yp - xq = y^2 - x^2}$$

Which is required PDE.

Ques: 5(a) ii) Find the integral surface of:

$x^2 p + y^2 q + z^2 = 0$; which passes through the curve : $xy = x+y$; $z = 1$.

Solution:- Given — $x^2 p + y^2 q + z^2 = 0$

Lagrange's auxillary equation is given by-

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{——} (1)$$

Taking first two parts:

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \frac{-1}{x} = \frac{-1}{y} + c_1 \quad \text{——} (2)$$

Taking first and last part:

$$\frac{dx}{x^2} = \frac{dz}{z^2} \Rightarrow -\frac{1}{x} = \frac{-1}{z} + c_2 \quad \text{——} (3)$$

Also given that $xy = x+y \Rightarrow \frac{1}{x} + \frac{1}{y} = 1$

and $z = 1$. $\Rightarrow \frac{1}{x} = 1 - \frac{1}{y}$

$\therefore$ from (2) and (3)

$$-\left(1 - \frac{1}{y}\right) + \frac{1}{y} = c_1$$

$$\Rightarrow 1 - \frac{2}{y} + c_1 = 0$$

$$\Rightarrow \boxed{\frac{1}{y} = \frac{1+c_1}{2}}$$

$$-\left(1 - \frac{1}{y}\right) = -\frac{1}{z} + c_2$$

$$\Rightarrow \cancel{-1} + \frac{1}{y} + \cancel{1} = c_2$$

$$\Rightarrow c_2 = \frac{1}{y}$$

$$\Rightarrow \boxed{-\frac{1}{2} + c_2 - \frac{c_1}{2} = 0}$$

$$= -\frac{1}{2} + \left[\frac{1}{z} - \frac{1}{x}\right] - \frac{1}{2}\left[\frac{1}{y} - \frac{1}{x}\right] = 0$$

$$\Rightarrow -\frac{1}{2} + \frac{1}{z} - \frac{1}{x} - \frac{1}{2y} + \frac{1}{2x} = 0$$

By solving

$$\frac{-xyz - yz + (-xz) + 2xy}{2xyz} = 0$$

$$\Rightarrow -xyz - yz - xz + 2xy = 0$$

$$\Rightarrow \boxed{xyz + yz + xz = 2xy}$$

is the required integral surface.

Ques: 6(b)) Find the characteristic strips of the equation $xp + yq - pq = 0$ and then find the equation of the integral surface through the curve $z = x/2$, $y = 0$.

Solution:- Here $f(x, y, z, p, q) = xp + yq - pq = 0$ —— ①

The integral surface passes through the curve $z = x/2$, $y = 0$, whose parametric equation can be written as →

$$x = f_1(\lambda) = \lambda \,;\ y = f_2(\lambda) = 0 \,;\ z = f_3(\lambda) = \lambda/2$$

λ being a parameter

∴ Initial values for x, y, z are $x = x_0 = \lambda$, $y = y_0 = 0$, $z = z_0 = \lambda/2$, when $t = 0$

And corresponding initial values p_0 and q_0 of p and q are determined by the relations.

$$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \Rightarrow \frac{1}{2} = p_0 \cdot 1 + q_0 \cdot 0 \Rightarrow p_0 = \frac{1}{2}$$

and $f\{f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

$$\Rightarrow x_0 p_0 + y_0 q_0 - p_0 q_0 = 0 \qquad \Rightarrow q_0 = \lambda.$$

The characteristic equations of the given partial Differential equation ① are —

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = x - q \quad \text{—— ②}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = y - p \quad \text{—— ③}$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = p(x-q) + q(y-p) = px + qy - 2pq = -pq$$

using ① —④

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -p \quad \text{—— ⑤}$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -q \quad \text{—— ⑥}$$

from (5) and (6), we get $p = Ae^{-t}$ and $q = Be^{-t}$

But initially, when $t=0$, $p = p_0 = \frac{1}{2}$ and $q = q_0 = \lambda$

$\therefore$ $A = p_0 = \frac{1}{2}$ and $B = q_0 = \lambda$

$\Rightarrow$ $\boxed{p = \frac{1}{2}e^{-t} \quad \text{and} \quad q = \lambda e^{-t}}$ —— (7)

Using (7), from (2), we have $\frac{dx}{dt} - x = -\lambda e^{-t}$

which is a L.D.E, with $I.F = e^{\int -dt} = e^{-t}$.

$\therefore$ $x \cdot e^{-t} = C_1 + \int(-\lambda e^{-t})(e^{-t})\,dt = C_1 - \lambda\int e^{-2t}dt$

$= C_1 + \frac{1}{2}\lambda e^{-2t}$

But, when $t=0$, $x = x_0 = \lambda$

$\therefore$ $x_0 = \lambda = C_1 + \frac{1}{2}\lambda \Rightarrow \boxed{C_1 = \lambda/2}$

$\therefore$ $xe^{-t} = \frac{\lambda}{2} + \frac{\lambda}{2}e^{-2t}$

$\Rightarrow$ $\boxed{x = \frac{\lambda}{2}(1+e^{-2t})e^{t}}$ —— (8)

Again using (7) from (3), we have $\Rightarrow \frac{dy}{dt} - y = -\frac{1}{2}e^{-t}$

which is L.D.E with $I.F = e^{\int -dt} = e^{-t}$

$\therefore$ $ye^{-t} = C_2 + \int\left(\frac{1}{2}e^{-t}\right)e^{-t}dt = C_2 - \frac{1}{2}\int e^{-2t}dt = C_2 + \frac{1}{4}e^{-2t}$

But when $t=0$, $y = y_0 = 0$, $\therefore$ $y_0 = 0 = C_2 + \frac{1}{4} \Rightarrow C_2 = -\frac{1}{4}$

$\therefore$ $ye^{-t} = -\frac{1}{4} + \frac{1}{4}e^{-2t} \Rightarrow \boxed{y = \frac{1}{4}(e^{-2t} - 1)e^{t}}$ —— (9)

Now using (7), from (4), we have

$\frac{dz}{dt} = -\frac{\lambda}{2}e^{-2t}$

Integrating, we get

$\boxed{z = \frac{\lambda}{4}e^{-2t} + C_3}$

But, when $t = 0$

$z = z_0 = \lambda/2$

$\therefore z_0 = \frac{\lambda}{2} = \frac{\lambda}{4} + c_3 \Rightarrow \boxed{c_3 = \frac{\lambda}{4}}$

$\therefore z = \frac{\lambda}{4} e^{-2t} + \frac{\lambda}{4} \Rightarrow \boxed{z = \frac{\lambda}{4}(1 + e^{-2t})}$ —— (10)

Thus, the characteristic strips of the given equation are given by-

$x = \frac{\lambda}{2}(1 + e^{-2t})e^{t}$

$y = \frac{1}{4}(e^{-2t} - 1)e^{t}$ and

$z = \frac{\lambda}{4}(1 + e^{-2t})$

where, λ and t are two parameter.

The required integral surface is obtained by eliminating λ and t between x, y and z.

we have,

$\frac{x}{z} = 2e^{t} \Rightarrow e^{t} = \frac{x}{2z}$

$\therefore y = \frac{1}{4}(e^{-t} - e^{t}) = \frac{1}{4}\left(\frac{2z}{x} - \frac{x}{2z}\right)$

$$\boxed{y = \frac{4z^2 - x^2}{8xz}}$$

$$\Rightarrow \boxed{4z^2 = x^2 + 8xyz}$$

which is the required Integral Surface.

6(c). A tightly stretched string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x)=\mu x(l-x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t > 0$.

SOLUTION

Weve equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots(1)$$

string of length 'l'.

Boundary conditions given by

$$u(0,t) = u(l, t) = 0 \quad \ldots(2)$$

Initial conditions given by

$$u(x,0) = \mu x(l-x) \quad \ldots(3)$$

$$u_t(x,0) = 0 \quad \ldots(4)$$

Let $u(x,t) = X(x)T(t)$ be the trail solution from(2) $X(0)T(t) = X(l)T(t) = 0$ for some $t > 0$, there exist $T(t)\neq 0$.

$\therefore \quad X(0)=X(l) = 0 \quad \ldots(5)$

from(1) $\quad XT'' = c^2X''T$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \mu \text{ (say)} \quad \ldots(6)$$

from(6) $X''-\mu X = 0$; $X(0) = X(l) = 0$

Case-(i)

$\mu = 0$, $X = Ax + B$

from(5) A=0; B=0 we reject $\mu=0$

Case-(ii)

$\mu = \lambda^2$, $X''-\lambda^2X=0 \Rightarrow X = Ae^{\lambda x}+Be^{-\lambda x}$

from (5) A=0; B=0 we riject $\mu = \lambda^2$

Case-(iii)

$\mu = -\lambda^2$, $X''+\lambda^2X=0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x$

from (5) A=0; B=0 $0=A\cos\lambda l+B\sin(\lambda l)$

$\therefore \quad A = 0, \lambda l = n\pi$

$$\lambda = \left(\frac{n\pi}{l}\right)$$

from(6) $T''-\mu c^2T = 0$

putting $\mu = -\lambda^2 = -\left(\frac{n\pi}{l}\right)^2$

$$T_n'' + \left(\frac{n\pi c}{l}\right)^2 T_n = 0$$

$$\therefore \quad T_n = C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} X_nT_n = \sum_{n=1}^{\infty}\left(E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

from(4) $\quad u_t(x,0) = 0$

$$\left(\frac{n\pi c}{l}\right)F_n.\sin\left(\frac{n\pi x}{l}\right) = 0$$

$\therefore \quad$ Fn = 0

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) = \mu x(l-x)$$

$$\mathrm{E_n} = \frac{2}{l}\int_0^l \mu x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{\mu x(l-x)\cos\left(\frac{n\pi x}{l}\right)}{\frac{-n\pi}{l}}\right]_0^1 - \frac{2}{l}\int_0^l \frac{\mu(l-2x)\cos\left(\frac{n\pi x}{l}\right)}{-\left(\frac{n\pi}{l}\right)}$$

$$= \frac{2}{l}\left[\frac{\mu(l-2x)\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2} - \frac{(-2\mu)\left(-\cos\frac{n\pi x}{l}\right)}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n]$$

$$\therefore \quad \boxed{u(x,t) = \frac{4\mu l^2}{\pi^3}\sum \frac{(1-(-1)^n)}{n^3}\cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)}$$

Required Solution

Ques:-7(a) Solve the following system of linear equations correct to two decimal places by Gauss-Seidel method for four iterations

$$x+4y+z=-1\ ;\ 3x-y+z=6\ ;\ x+y+2z=4.$$

Solution:-

Given Equations are

$$x+4y+z=-1$$
$$3x-y+z=6$$
$$x+y+2z=4$$

Re-arrang the equation, such that ~~the~~ elements in the coefficient matrix are diagonally dominant.

i.e
$$3x-y+z=6$$
$$x+4y+z=-1$$
$$x+y+2z=4$$

from above equation:-

$$x=\frac{1}{3}(6+y-z)$$

$$y=\frac{1}{4}(-1-x-z)=-\frac{1}{4}(1+x+z)$$

$$z=\frac{1}{2}(4-x-y)$$

Initial values of (x, y, z) are $(0, 0, 0)$

Ist Iteration

$$x_1=\frac{1}{3}[6+0-0]=2$$

$$y_1=\frac{-1}{4}[1+2+0]=-\frac{3}{4}=-0.75$$

$$z_1=\frac{1}{2}[4-2+0.75]=\frac{2.75}{2}=1.375\simeq 1.38$$

$\underline{\text{II}^{nd}\text{ iteration}}$:- $(x_1, y_1, z_1) = (2, -0.75, 1.38)$

$$x_2 = \frac{1}{3}[6 + y_1 - z_1] = \frac{1}{3}[6 - 0.75 - 1.38] = 1.29$$

$$y_2 = -\frac{1}{4}[1 + 1.29 + 1.38] = -\frac{1}{4} \times 3.67 = -0.92.$$

$$z_2 = \frac{1}{2}[4 - (1.29) + 0.92] = \frac{1}{2} \times (+3.63) = 1.81$$

$\underline{\text{III}^{rd}\text{ Iteration}}$: $(x_2, y_2, z_2) = (1.29, -0.92, 1.81)$

$$x_3 = \frac{1}{3}[6 + y_2 - z_2] = \frac{1}{3}[6 - 0.92 - 1.81] = 1.09.$$

$$y_3 = -\frac{1}{4}[1 + 1.09 + 1.81] = -\frac{1}{4} \times 3.9 = -0.98$$

$$z_3 = \frac{1}{2}[4 - 1.09 + 0.98] = \frac{1}{2} \times 3.89 = 1.95$$

$\underline{\text{IV}^{th}\text{ Iteration}}$: $(x_3, y_3, z_3) = (1.09, -0.98, 1.95).$

$$x_4 = \frac{1}{3}[6 + y_3 - z_3] = \frac{1}{3}[6 - 0.98 - 1.95] = 1.02$$

$$y_4 = -\frac{1}{4}[1 + 1.02 + 1.95] = -\frac{1}{4}[3.97] = -0.99 \approx -1$$

$$z_4 = \frac{1}{2}[4 - 1.02 + 1] = \frac{3.98}{2} = 1.99$$

$$\therefore (x_4, y_4, z_4) = (1.02, -0.99, 1.99)$$

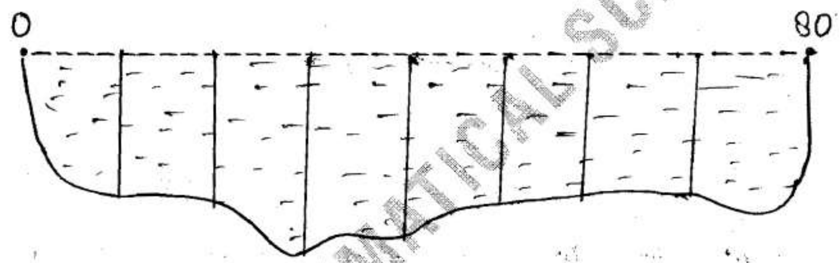
Hence ; equivalent values of x, y, z after 4 iteration are $\boxed{\begin{matrix} x = 1 \\ y = -1 \\ z = 2 \end{matrix}}$ Hence the result

Ques: 7(b) A river is 80 meter wide, the depth y, in metre, of the river at a distance 'x' from one bank is given by the following table :-

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the area of cross-section of the river using Simpson's $\frac{1}{3}$rd rule.

Solution :-



From the table

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$h=10$	0	4	7	9	12	15	14	8	3

Simpson's $\frac{1}{3}$rd Rule ;

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\Big]$$

Here:

$$\text{Area} = \int_0^{80} (\text{depth function})\,dx$$

$$= \frac{10}{3}\left[(0+3)+4(4+9+15+8)+2(7+12+14)\right]$$

$$= \frac{10}{3}\left[3+144+66\right] = 710\,m^2.$$

$\boxed{\therefore \text{ Area of crosssection of the river } = 710\,m^2.}$

Ques:- 7(c) Solve $\frac{dy}{dx} = xy$ for $x = 1.4$ by fourth order classical Runge kutta method, initially $x=1$, $y=2$ (Take $h = 0.2$) ?

Solution:- Given; $\frac{dy}{dx} = f(x) = xy$.

given at $x = 1$; $y(1) = 2$.

$h = 0.2$

To find $y(1.4) = ?$

Using Runge kutta method (4th order classical).

$y(1.2) = y(1) + k$

$y(1.4) = y(1.2) + k$.

Hence; first to find y at $x = 1.2$.

i.e $y(1.2) = y(1) + k$

where $k = \frac{1}{6}(k_1 + k_4 + 2(k_2 + k_3))$.

$k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$k_4 = h f(x_1 + h, y_1 + k_3)$

$\therefore$ $k_1 = 0.2 f(1, 2) = 0.2 \times 2 = 0.4$

$k_2 = 0.2 f(1.1, 2.2) = 0.2 \times 2.42 = 0.484$

$k_3 = 0.2 f(1.1, 2.242) = 0.2 \times 1.1 \times 2.242 = 0.493$

$k_4 = 0.2 f(1.2, 2.493) = 0.2 \times 1.2 \times 2.493$

$= 0.598$

$\therefore\ y(1.2) = y(1) + \frac{1}{6}\left(0.4 + 0.598 + 2(0.484 + 0.493)\right]$

$y(1.2) = 2 + \frac{1}{6}\left[0.998 + 2(0.977)\right]$

$y(1.2) = 2 + 0.492 = \underline{\underline{2.492}}.$

Now, for $y(1.4) = ?$ $\quad y(1.2) = 2.492,\ x = 1.2$
$h = 0.2$

$K_1 = 0.2 \times f(1.2, 2.492) = 0.2 \times 1.2 \times 2.492 = 0.5981$

$K_2 = 0.2 f(1.3, 2.791) = 0.2 \times 1.3 \times 2.791 = 0.72566$

$K_3 = 0.2 f(1.3, 2.8548) = 0.2 \times 1.3 \times 2.8548 = 0.74225$

$K_4 = 0.2 f(1.4, 3.234) = (0.2 \times 1.4 \times 3.234) = 0.9055.$

$\therefore\ y(1.4) = y(1.2) + \frac{1}{6}\left(0.5981 + 0.9055 + 2(0.72566 + 0.74225)\right]$

$y(1.4) = 2.492 + \frac{1}{6}\left[1.5036 + 2[1.46791]\right]$

$y(1.4) = 2.492 + \frac{1}{6}[4.43942]$

$y(1.4) = 2.492 + 0.7399$

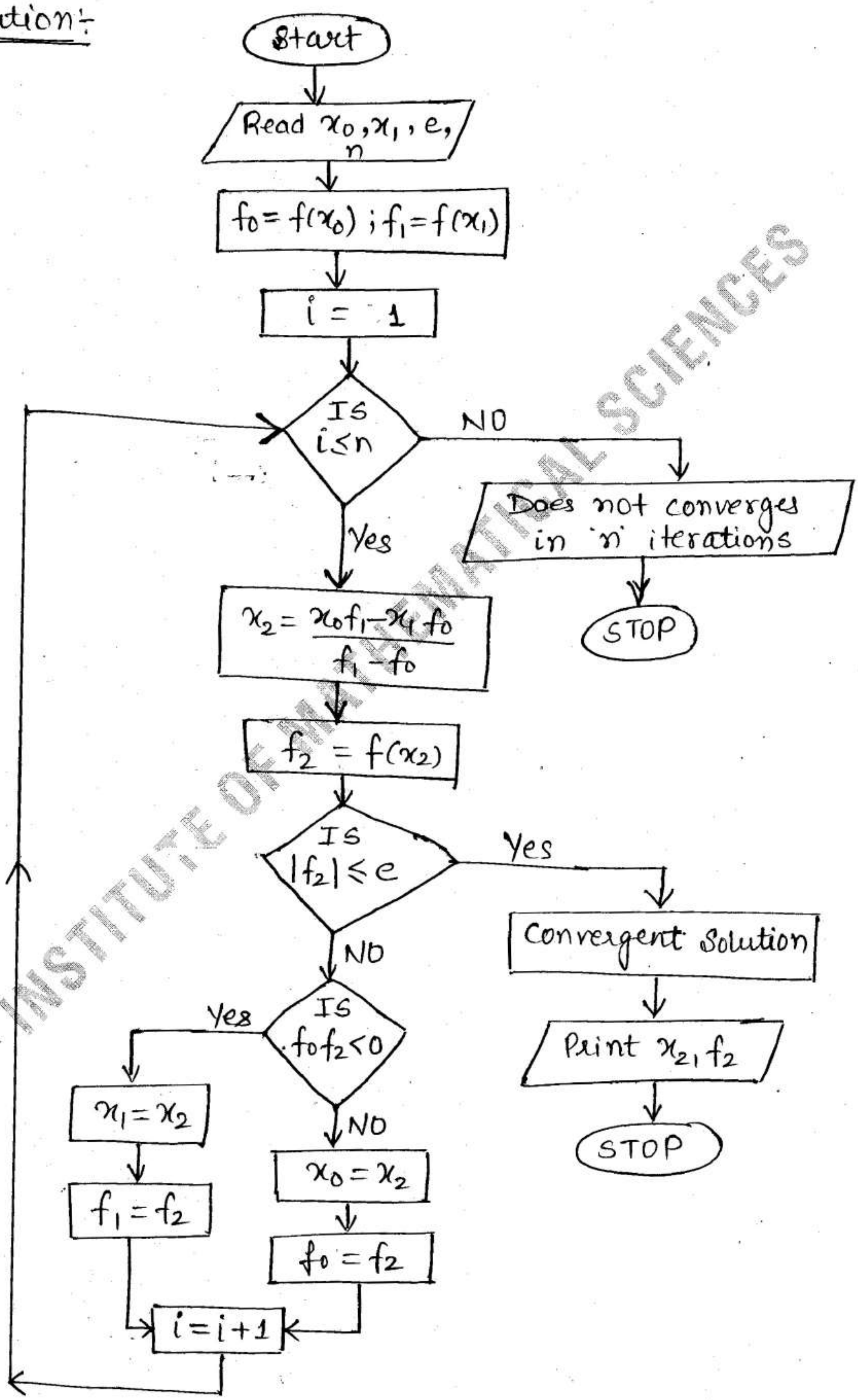
$\boxed{y(1.4) = 3.2319}$

$\boxed{\text{Hence; value of } y \text{ at } x = 1.4 \text{ is } y(1.4) = 3.2319}$

Required Result

Ques: 7(d)(i) Draw a flow chart for Regula falsi method?

Solution:-



Ques:- 7(d) ii)) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved?

Solution: For the above problem, we have to make a logic circuit with three input and provide output as 1 (when the light ON) when there is majority approval (i.e 1). Hence, the truth table and the circuit is shown below:

x	y	z	Light
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Light On $(T) = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$

Light (on) $T = \bar{x}yz + x\bar{y}z + xy(z+\bar{z})$

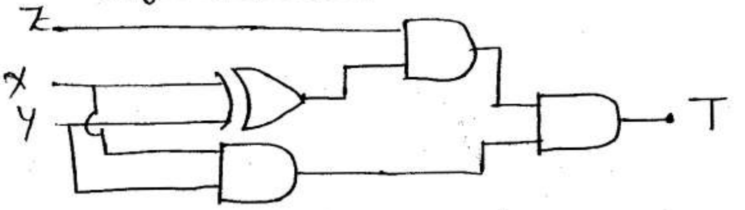
$T = \bar{x}yz + x\bar{y}z + xy$ $[\because z+\bar{z}=1]$

$\therefore \boxed{T = \bar{x}yz + x\bar{y}z + xy}$

or $T = z(x\bar{y} + \bar{x}y) + xy$

$\boxed{T = z(x \oplus y) + xy}$

Logic circuit.



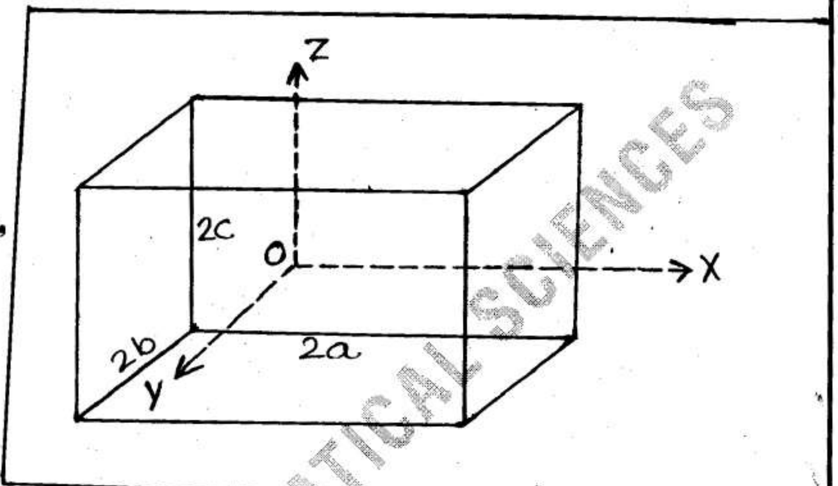
$(x \oplus y) = $ Exor

EXOR gate

Ques: 8(a)(i) A uniform rectangular parallelopiped of mass M has edges of length 2a, 2b, 2c. Find the moment of inertia of this rectangular parallelopiped about the line through its centre parallel to the edge of length 2a.

Solution:-

Let O be the centre and 2a, 2b, 2c the lengths of the edges of a rectangular parallelopiped.



If M is the mass of the parallelopiped, the mass per unit volume

$$= \rho = \frac{M}{2a\cdot 2b\cdot 2c} = \frac{M}{8abc}$$

Let OX, OY, OZ be the axes through the centre and parallel to the edge of the rectangular parallelopiped.

Consider an elementary volume $\delta x \delta y \delta z$ of the parallelopiped at the point $P(x, y, z)$, then its mass

$$= \rho\, \delta x\, \delta y\, \delta z = \delta m$$

Distance of the point $P(x, y, z)$ from OX is $\sqrt{(y^2+z^2)}$

$\therefore$ M.I. of the elementary volume of mass δm at P about OX $= P(x^2+y^2)\delta x \delta y \delta z$.

Hence; M.I. of the rectangular parallelopiped about OX (which is parallel to 2a).

$$= \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} \rho(y^2+z^2)dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-b}^{b}\rho\left[y^2z+\frac{z^3}{3}\right]_{-c}^{c}dy\,dx$$

$$= \rho\int_{-a}^{a}\int_{-b}^{b}2\left(y^2c+\frac{1}{3}c^3\right)dx\,dy$$

$$= 2\rho\int_{-a}^{a}\left[\frac{1}{3}y^3c+\frac{1}{3}c^3y\right]_{-b}^{b}dx$$

$$= \frac{2}{3}\rho\int_{-a}^{a}2\left[b^3c+c^3b\right]dx$$

$$= \frac{4}{3}\rho\left[b^3cx+c^3bx\right]_{-a}^{a} = \frac{4}{3}\rho bc\left[b^2+c^2\right]\left[x\right]_{-a}^{a}$$

$$= \frac{4}{3}\rho\, bc(b^2+c^2)\cdot 2a = \frac{\cancel{4}}{3}\cdot\frac{M}{\cancel{8abc}}\cdot \cancel{bc}(b^2+c^2)\cdot\cancel{2a}$$

$$\left[\because \rho = \frac{M}{8abc}\right]$$

$$= \frac{1}{3}M(b^2+c^2).$$

Hence; M.I. of rectangular parallelopiped about the lines OX i.e $2a$ $= \frac{1}{3}M(b^2+c^2)$.

Similarly, MI of rectangular parallelopiped about the lines OY and OZ, through centre 'O' and parallel to $2b$ and $2c$ are $\frac{1}{3}M(a^2+c^2)$ and $\frac{1}{3}M(a^2+b^2)$ respectively.

Note; If it is a cube than $MI = \frac{2}{3}Ma^2$

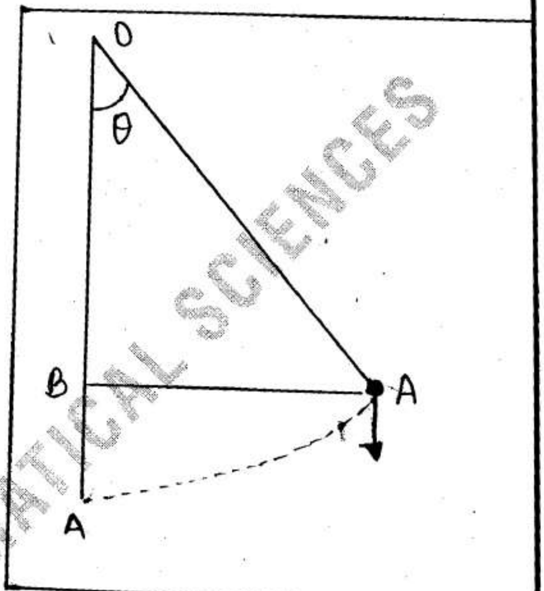
$\because 2a=2b=2c$

Ques:- 8(a) ii)> For a simple pendulum

(i) find the Lagrangian function and

(ii) Obtain an equation describing its motion.

Solution:-



Let 'l' be the length of the simple pendulum and θ the angle made by the string with the velocity at time 't'. Thus θ is the only generalised co-ordinate. Then the velocity of mass M at A will be $v = l\dot{\theta}$

$\therefore$ Total Kinetic Energy,

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$$

And the potential function.

$$V = Mg\,(A'B) = Mg\,(l - l\cos\theta) = Mgl\,(1-\cos\theta).$$

(i) The Lagrangian function

$$\boxed{L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)}$$

(ii) Lagrange's θ- equation is -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{i.e} \quad \frac{d}{dt}(Ml^2\dot{\theta}) + Mgl\sin\theta = 0$$

or $\quad Ml^2\ddot{\theta} + Mgl\sin\theta = 0$

$$\ddot{\theta} = -\frac{gl\sin\theta}{l^2} \Rightarrow \ddot{\theta} = -\frac{g}{l}\cdot\sin\theta$$

$\left[\because \theta << \text{(very small)} \quad \sin\theta \approx \theta\right]$

$$\boxed{\ddot{\theta} = -(g/l)\cdot\theta}$$

which is required equation of motion.

Ques: 8(b) Write Hamilton's equations in polar co-ordinates for a particle of mass 'm' moving in three dimensions in a force field of potential V?

Solution :- At time t, let (r, θ, ϕ) be the polar co-ordinates of the particle 'm' at 'P'. If (x, y, z) are the cartesian co-ordinates of 'P', then

$$x = r\sin\theta\cos\phi \quad ; \quad y = r\sin\theta . \sin\phi$$
$$z = r\cos\theta$$

$\therefore$ K.E. ; $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$T = \frac{1}{2}m\left[(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\cdot\sin\phi)^2 + (\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cdot\cos\phi)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2\right]$$

$$T = \frac{1}{2}m\left[(\dot{r}^2[\sin^2\theta\cos^2\phi + \sin^2\theta\cdot\sin^2\phi + \cos^2\theta) + r^2\dot{\theta}^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta) + r^2\dot{\phi}^2\sin^2\theta(\sin^2\phi + \cos^2\phi)\right]$$

$$T = \frac{1}{2}m\left[(\dot{r}^2[\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)] + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right].$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V$$

Here, r, θ and ϕ are the generalised co-ordinates.

$$\left.\begin{aligned} \therefore p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad ; \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta \end{aligned}\right] \text{——} (1)$$

Since, L does not contain, t explicitly,

$$\therefore H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta + V$$

Eliminating $\dot{r}, \dot{\theta}, \dot{\phi}$ with the help of relations (1).

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right] + V$$

Hence; the six Hamilton's equations are —

[Note: V is the function of r, θ, ϕ and t).

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{-1}{2m}\left[-\frac{2p_\theta^2}{r^3} - \frac{2p_\phi^2}{r^3\sin^2\theta}\right] - \frac{\partial V}{\partial r}$$

$$\therefore \boxed{\dot{p}_r = \frac{1}{mr^3}\left[p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right] - \frac{\partial V}{\partial r}} \quad \text{——} \quad (H_1)$$

$$\boxed{\dot{r} = \frac{\partial H}{\partial p_r} = p_r/m} \quad \text{——} \quad (H_2)$$

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = \frac{-1}{2m}\left[\frac{-2\cos\theta}{r^2\sin^3\theta}p_\phi^2\right] - \frac{\partial V}{\partial \theta}$$

$$\boxed{\dot{p}_\theta = \frac{\cos\theta}{mr^2\sin^3\theta}p_\phi^2 - \frac{\partial V}{\partial \phi}} \quad \text{——} \quad (H_3)$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}} \quad \text{——} \quad (H_4)$$

$$\boxed{\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}} \quad \text{——} \quad (H_5)$$

$$\boxed{\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}} \quad \text{——} \quad (H_6)$$

$\therefore$ $H_1 - H_6$ are the required Hamilton's equations in polar Co-ordinate of a mass 'm' moving in 3-D in a force field of potential V.

Ques: 8(c)) Prove that $\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1.$ is a possible form for the bounding surface of a liquid and find the velocity components.

Solution:- For the present two dimensional motion $\left(\partial F/\partial z = 0 \text{ and } \partial w/\partial z = 0\right)$, the surface

$$F(x,y,t) = \left(\frac{x^2}{a^2}\right)\tan^2 t + \left(y^2/b^2\right)\cot^2 t - 1 = 0 \quad \text{———} (1)$$

can be possible boundary surface of a liquid, if it satisfies the boundary condition

$$\frac{\partial F}{\partial t} + u\left(\frac{\partial F}{\partial x}\right) + v\left(\frac{\partial F}{\partial y}\right) = 0 \quad \text{———} (2)$$

and the same values of u and v satisfy the equation of continuity.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{———} (3)$$

from (1); $\frac{\partial F}{\partial t} = \frac{x^2}{a^2}\cdot 2\tan t\cdot \sec^2 t - \frac{y^2}{b^2}\cdot 2\cot t\cdot \operatorname{cosec}^2 t$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2}\tan^2 t.$$

$$\frac{\partial F}{\partial y} = \frac{2y}{b^2}\cot^2 t$$

with these above values (2), reduces to -

$$2\frac{x^2}{a^2}\cdot\tan t\sec^2 t - \frac{2y^2}{b^2}\cot t\cdot\operatorname{cosec}^2 t + u\frac{2x}{a^2}\tan^2 t + v\frac{2y}{b^2}\cot^2 t = 0$$

By solving, we get

$$\frac{x\tan t}{a^2}(x\sec^2 t + u\tan t) + \frac{y\cot t}{b^2}(-y\operatorname{cosec}^2 t + v\cot t) = 0$$

which is identically satisfied if we take

$$x\sec^2 t + u\tan t = 0$$
$$-y\operatorname{cosec}^2 t + v\cot t = 0$$

$$\Rightarrow \quad u = \frac{-x\sec^2 t}{\tan t} = \frac{-x\,\cancel{\cos t}\cdot\frac{1}{\cos^2 t}}{\sin t}$$

$$\therefore \boxed{u = \frac{-x}{\sin t\cos t}} \quad \text{———} \quad 4(a).$$

$$\Rightarrow \boxed{v = \frac{y\operatorname{cosec}^2 t}{\cot t} = \frac{y}{\sin t\cos t}} \quad \text{———} \quad 4(b)$$

from 4(a) from 4(b)

$$\frac{\partial u}{\partial x} = \frac{-1}{\sin t\cos t} \qquad \frac{\partial v}{\partial y} = \frac{1}{\sin t\cdot\cos t} \quad \text{———} \quad (5)$$

Using (5), we find that (3) is also satisfied by above values of u and v.

Hence; (1) is a possible bounding surace with velocity components given by (4).

The velocity component is given by

$$= \frac{u(\partial F/\partial x) + v(\partial F/\partial y)}{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2}}$$

$$= \frac{\dfrac{-x}{\sin t\cos t}\cdot\dfrac{2x\tan^2 t}{a^2} + \dfrac{y}{\sin t\cos t}\cdot\dfrac{2y\cot^2 t}{b^2}}{\sqrt{\left(\dfrac{2x\tan^2 t}{a^2}\right)^2 + \left(\dfrac{2y\cot^2 t}{b^2}\right)^2}}$$

$$= \frac{\dfrac{1}{\sin t\cos t}\left[\dfrac{2y^2}{b^2}\cot^2 t - \dfrac{2x^2}{a^2}\tan^2 t\right]}{\sqrt{\dfrac{4x^2\tan^4 t}{a^4} + \dfrac{4y^2\cot^4 t}{b^4}}}$$

$$\Rightarrow \frac{\cancel{2}(a^2y^2\cot t\operatorname{cosec}^2 t - b^2x^2\tan t\sec^2 t)}{\dfrac{\cancel{2a^2b^2}}{\cancel{a^2b^2}}\sqrt{x^2b^4\tan^4 t + y^2a^4\cot^4 t}}$$

$$\boxed{\text{Velocity component} = \frac{a^2y^2\cot t\operatorname{cosec}^2 t - b^2x^2\tan t\sec^2 t}{\sqrt{x^2b^4\tan^4 t + y^2a^4\cot^4 t}}}$$

required solution.