MECHANICS

1. MOMENT OF INERTIA
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1. MOMENT OF INERTIA

1. 5e 2020

Prove that the moment of inertia of a triangular lamina ABC about any axis through A in its plane is

$$\frac{M}{6}\left(\beta^2+\beta\gamma+\gamma^2\right)$$

where M is the mass of the lamina and β, γ are respectively the length of perpendiculars from B and C on the axis. 10

2. 5e 2017

Show that the moment of inertia of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length r is $\frac{1}{4}M\frac{a^2b^2}{r^2}$. Further, prove that the moment of inertia about a tangent is $\frac{5M}{4}p^2$, where p is the perpendicular distance from the centre of the ellipse to the tangent. *10*

3. 5c 2017 IFoS

5.(c) A uniform rectangular parallelopiped of mass M has edges of lengths $2a$, $2b$, $2c$. Find the moment of inertia of this rectangular parallelopiped about the line through its centre parallel to the edge of length $2a$. 10

4. 5e 2016 IFoS

5.(e) Calculate the moment of inertia of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) relative to the x-axis

(ii) relative to the y-axis and

(iii) relative to the origin

5. 8a 2016 IFoS

8.(a) Find the moment of inertia of a right solid cone of mass M, height h and radius of whose base is a, about its axis. 14

6. 5e 2015

Calculate the moment of inertia of a solid uniform hemisphere $x^2+y^2+z^2=a^2$, $z \geq 0$ with mass m about the OZ-axis. 10

7. 7b 2015 IFoS

(b) Find the moment of inertia of a uniform mass M of a square shape with each side a about its one of the diagonals. *12*

8. 6b 2014 IFoS

(b) Show that the moment of inertia of a uniform rectangular mass M and sides $2a$ and $2b$ about a diagonal is $\dfrac{2Ma^2b^2}{3(a^2+b^2)}$. 10

9. 5e 2013

(e) Four solid spheres A, B, C and D, each of mass m and radius a, are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about a diagonal of the square. *10*

10. 8a 2012

8. (a) A pendulum consists of a rod of length $2a$ and mass m; to one end of which a spherical bob of radius $a/3$ and mass 15 m is attached. Find the moment of inertia of the pendulum :

(i) about an axis through the other end of the rod and at right angles to the rod. 15

(ii) about a parallel axis through the centre of mass of the pendulum.

[Given : The centre of mass of the pendulum is $a/12$ above the centre of the sphere.] 15

11. 5e 2011

12

(e) Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis.

12. 8a 2011 IFoS

8. *(a)* From a uniform sphere of radius a, a spherical sector of vertical angle 2α is removed. Find the moment of inertia of the remainder mass M about the axis of symmetry. 14

13. 5e 2010

(e) A uniform lamina is bounded by a parabolic arc of latus rectum $4a$ and a double ordinate at a distance b from the vertex.

If $b = \frac{a}{3}\left(7+4\sqrt{7}\right)$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there. 12

14. 5d 2010 IFoS

(d) Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same. 10

2. EQUATION OF MOTION IN 2D/ D'ALEMBERT PRINCIPLE

1. 5c 2019

A uniform rod OA, of length $2a$, free to turn about its end O, revolves with angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ; find the value of α. 10

2. 6c 2019

A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$. 20

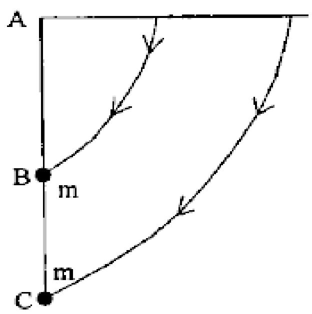
3. 7c 2014 IFoS

(c) A uniform rod OA of length $2a$ is free to turn about its end O, revolves with uniform angular velocity ω about a vertical axis OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or

$$\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$$ 15

4. 8b 2012 IFoS

(b) A weightless rod ABC of length 2a is movable about the end A which is fixed and carries two particles of mass m each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$. 13



5. 8a 2011

8. (a) The ends of a heavy rod of length 2a are rigidly attached to two light rings which can respectively slide on the thin smooth fixed horizontal and vertical wires O_x and O_y. The rod starts at an angle α to the horizon with an angular velocity $\sqrt{[3g(1-\sin\alpha)/2a]}$ and moves downwards. Show that it will strike the horizontal wire at the end of time

$$-2\sqrt{a/(3g)}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right].$$ 30

3. LAGRANGE'S EQUATION OF MOTION

1. 8a 2020 IFoS

A particle is attracted to a center by a force which varies inversely as the cube of its distance from the center. Identify the generalized coordinates and write down the Lagrangian of the system. Derive then the equations of motion and solve them for the orbits. Discuss how the nature of orbits depends on the parameters of the system. 20

2. 6c 2019 IFoS

(c) For a dynamical system

$$T = \frac{1}{2}\{(1+2k)\,\dot{\theta}^2 + 2\dot{\theta}\dot{\varphi} + \dot{\varphi}^2\},$$

$$V = \frac{n^2}{2}\{(1+k)\,\theta^2 + \varphi^2\},$$

where θ, φ are coordinates and n, k are positive constants, write down the Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\varphi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \varphi) = 0.$$

Further show that if $\theta = \varphi$, $\dot{\theta} = \dot{\varphi}$ at t = 0, then $\theta = \varphi$ for all t. *15*

3. 6c 2018

Suppose the Lagrangian of a mechanical system is given by

$$L = \tfrac{1}{2}m(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \tfrac{1}{2}k(ax^2 + 2bxy + cy^2),$$

where a, b, c, $m(>0)$, $k(>0)$ are constants and $b^2 \neq ac$. Write down the Lagrangian equations of motion and identify the system. 20

4. 8c 2018 IFoS

(c) A particle of mass m is constrained to move on the inner surface of a cone of semi-angle α under the action of gravity. Write the equation of constraint and mention the generalized coordinates. Write down the equation of motion. 10

5. 6c 2017

Two uniform rods AB, AC, each of mass m and length 2a, are smoothly hinged together at A and move on a horizontal plane. At time t, the mass centre of the rods is at the point (ξ, η) referred to fixed perpendicular axes Ox, Oy in the plane, and the rods make angles $\theta \pm \phi$ with Ox. Prove that the kinetic energy of the system is

$$m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\ \dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\ \dot{\phi}^2\right].$$

Also derive Lagrange's equations of motion for the system if an external force with components [X, Y] along the axes acts at A. *20*

6. 8a 2017 IFoS

8. (a) Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume that the mass can only move up and down in the vertical direction. Show that

$$L = \tfrac{1}{2}my'^2 - \tfrac{1}{2}k(y-l)^2 + mgy$$

Also determine and solve the corresponding Euler-Lagrange equations of motion. 12

7. 8b 2016

A hoop with radius r is rolling, without slipping, down an inclined plane of length l and with angle of inclination ϕ. Assign appropriate generalized coordinates to the system. Determine the constraints, if any. Write down the Lagrangian equations for the system. Hence or otherwise determine the velocity of the hoop at the bottom of the inclined plane. *15*

8. 8b 2016 IFoS

8.(b) A bead slides on a wire in the shape of a cycloid described by the equations

$$x = a(\theta - \sin\theta)$$
$$y = a(1 + \cos\theta)$$

where $0 \leqslant \theta \leqslant 2\pi$ and the friction between the bead and the wire is negligible. Deduce Lagrange's equation of motion. 10

9. 8a 2013

(a) Two equal rods AB and BC, each of length l, smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$. 15

10. 5d 2011 IFoS

(d) Find the Lagrangian for a simple pendulum and obtain the equation describing its motion. 10

4. HAMILTON'S EQUATION OF MOTION

1. 6c 2020

By writing down the Hamiltonian, find the equations of motion of a particle of mass m constrained to move on the surface of a cylinder defined by $x^2+y^2=R^2$, R is a constant. The particle is subject to a force directed towards the origin and proportional to the distance r of the particle from the origin given by $\vec{F}=-k\vec{r}$, k is a constant. 20

2. 5d 2020 IFoS

Find the condition on a, b, c (real numbers) such that the dynamical system with equations $\dot{p}=aq-q^2$, $\dot{q}=bp+cq$ is Hamiltonian. Compute also the Hamiltonian of the system. 8

3. 7a 2019

Using Hamilton's equation, find the acceleration for a sphere rolling down a rough inclined plane, if x be the distance of the point of contact of the sphere from a fixed point on the plane. 15

4. 7b 2019 IFoS

(b) Consider a mass-spring system consisting of a mass m and a linear spring of stiffness k hanging from a fixed point. Find the equation of motion using the Hamiltonian method, assuming that the displacement x is measured from the unstretched position of the string. *10*

5. 7c 2018

The Hamiltonian of a mechanical system is given by,

$H=p_1q_1-aq_1^2+bq_2^2-p_2q_2$, where a, b are the constants. Solve the Hamiltonian equations and show that $\dfrac{p_2-bq_2}{q_1}=$ constant. 20

6. 6c 2018 IFoS

(c) For a particle having charge q and moving in an electromagnetic field, the potential energy is $U = q(\phi - \vec{v} \cdot \vec{A})$, where ϕ and $\vec{A}$ are, respectively, known as the scalar and vector potentials. Derive expression for Hamiltonian for the particle in the electromagnetic field. 8

7. 5c 2016

Consider a single free particle of mass m, moving in space under no forces. If the particle starts from the origin at t = 0 and reaches the position (x, y, z) at time τ, find the Hamilton's characteristic function S as a function of x, y, z, τ. 10

8. 6b 2015

Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion. 15

9. 7c 2015

A Hamiltonian of a system with one degree of freedom has the form

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{k}{2} q^2$$

where α, b, k are constants, q is the generalized coordinate and p is the corresponding generalized momentum.

(i) इस हैमिल्टोनियन के संगत एक लग्रांजी ज्ञात कीजिए।

Find a Lagrangian corresponding to this Hamiltonian.

(ii) एक तुल्य लग्रांजी ज्ञात कीजिए, जो कि समय पर स्पष्ट रूप से आश्रित नहीं है।

Find an equivalent Lagrangian that is not explicitly dependent on time.

10+10=20

10. 5c 2015 IFoS

(c) Derive the Hamiltonian and equation of motion for a simple pendulum. 10

11. 5e 2014

Find the equation of motion of a compound pendulum using Hamilton's equations. 10

12. 5c 2013 IFoS

(c) Derive the Hamiltonian and equation of motion for a simple pendulum. *14*

13. 5d 2012

(d) Obtain the equations governing the motion of a spherical pendulum. 12

14. 6b 2012 IFoS

(b) Derive the differential equation of motion for a spherical pendulum. 13

15. 8a 2010

8. (a) A sphere of radius a and mass m rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations. 30

Miscellaneous

Work & Energy (Equilibrium/Centre of Mass)

1. 8b 2014 IFoS

(b) A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

where a is the length of the plank. 15

Work & Energy (Statics)

42 (8a) 2010 IFoS

8. *(a)* A mass m_1, hanging at the end of a string, draws a mass m_2 along the surface of a smooth table. If the mass on the table be doubled, the tension of the string is increased by one-half. Show that $m_1 : m_2 = 2 : 1$. 13