Calculus, IAS

$\boxed{2013}$

Q. Evaluate $\int_0^1 \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) dx$

$$f(x) = \begin{cases} 2x \sin 1/x - \cos 1/x & x \in (0,1) \\ 0 & x = 0 \end{cases}$$

is not continuous on $[0,1]$ but it is bounded & continuous on $(0,1]$ & thus RI on $[0,1]$

The function

$$g(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

is differentiable on $[0,1]$ & satisfies

$$g'(x) = f(x), \quad \forall\ x \in [0,1]$$

$$\therefore \int_0^1 \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) dx = g(1) - g(0) = \sin 1.$$

Q. Using Lagrange's Multiplier Method, find the shortest distance b/w the line $y = 10 - 2x$ & the ellipse.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Let $p(x,y)$ be any point on the given ellipse.

Then, length of $\perp r$ from $P(x,y)$ to line $y = 10-2x$ is

$$F(x,y) = \frac{1}{\sqrt{5}}(2x+y-10) + \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right)$$

$$F_x = \frac{2}{\sqrt{5}} + \frac{2\lambda x}{y}$$

$$F_y = \frac{1}{\sqrt{5}} + 2\lambda\frac{y}{9}$$

$$\Rightarrow x = \frac{-4}{\sqrt{5}}\cdot\frac{1}{\lambda}, \quad y = \frac{-9}{2\sqrt{5}}\frac{1}{\lambda}$$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{9} = 1$$

$$= \frac{1}{4}\left(\frac{16}{5}\cdot\frac{1}{\lambda^2}\right) + \frac{1}{\cancel{9}}\left(\frac{\overset{9}{\cancel{81}}}{20}\cdot\frac{1}{\lambda^2}\right) = 1$$

$$\frac{25}{20}\cdot\frac{1}{\lambda^2} = 1$$

$$\Rightarrow \lambda^2 = 5/4 \quad \Rightarrow \quad \lambda = \pm\sqrt{5}/2$$

$$x = \frac{-4}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{8}{5}, \quad y = \frac{-9}{2\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{9}{5}$$

Sattionary Point is $(8/5, 9/5)$

Hence, shortest distance $= \frac{1}{\sqrt{5}}\left(2\times\frac{8}{5}+\frac{9}{5}-10\right) = \frac{1}{\sqrt{5}}\left(\frac{\overset{5}{\cancel{25}}}{\cancel{5}}-10\right) = \sqrt{5}$

Q. Compute $f_{xy}(0,0)$ & $f_{yx}(0,0)$ for the func^n

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Also discuss the continuity of f_{xy} & f_{yx} at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2} & , x,y \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

(i) $$f_{xy}(0,0) = \lim_{x \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{\cancel{k}} \left(\frac{hk^{\cancel{3}}}{h+k^2} \right) = 0$$

$$f_y(0,0) = \lim_{x \to 0} \frac{0-0}{k} = 0$$

(ii) $$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \left(\frac{\cancel{h}k^3}{h+k^2} \right) = k$$

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{k-0}{k} = 1$$

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clearly, f_{xy} & f_{yx} are not equal at $(0,0)$

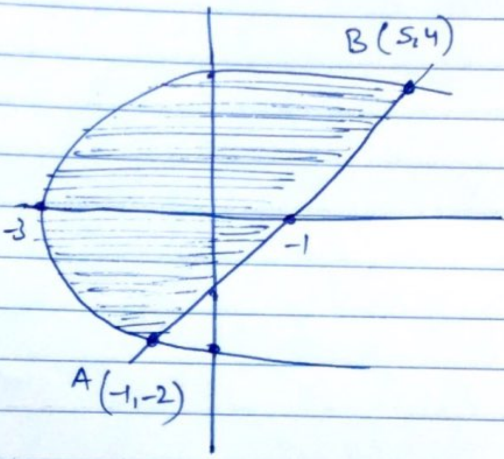
$\therefore$ f_{xy} & f_{yx} are not continuous at $(0,0)$.

Q. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x-1$ & the parabola $y^2 = 2x+6$

Sol. Given,

$y = x-1$ — ①

$y^2 = 2x+6$ — ②



By solving ① & ②, we get

A $(-1,-2)$, B $(5,4)$

$\therefore$ $\iint_D xy\, dA$ where D is the shaded region in the fig.

$$D = \left\{ (x,y) \in \mathbb{R}^2 : y \in [-2,4] ,\ \frac{y^2}{2} - 3 \le x \le y+1 \right\}$$

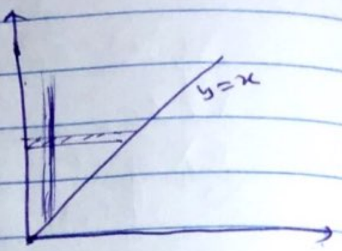
$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4} \left[\frac{x^2}{2} \cdot y \right]_{\frac{y^2}{2}-3}^{y+1} dy$$

$$= \frac{1}{2} \int_{-2}^{4} y \left[(y+1)^2 - \left(\frac{y^2}{2}-3\right)^2 \right] dy = \frac{1}{2} \left[\frac{-y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2 \right]_{-2}^{4}$$

$$= 36$$

Q. Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx$ by changing the order of integration.

Given, $\int_{x=0}^{\infty} \int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$



$y=0$
$y=x$

By changing the order of integration.

$$= \int_0^\infty \int_y^\infty \underline{x}\, \underline{e}^{-x^2/y}\, dx\, dy = \frac{1}{2} \int_0^\infty \int_{y^2}^\infty e^{-t/y}\, dt\, dy$$

$x=y$
Put $x^2 = t$
$2x\,dx = dt$
$x\,dx = \frac{dt}{2}$

$$= \frac{1}{2} \int_0^\infty \left[\frac{e^{-t/y}}{-1/y} \right]_{y^2}^\infty dy = \frac{1}{2} \int_0^\infty \left(\frac{-e^{-y}}{1/y} \right) dy$$

$$= -\frac{1}{2} \int_0^\infty y^1 e^{-y}\, dy = -\frac{1}{2} \Gamma 2 = -\frac{1}{2} \times 2 \times 1 = -1$$

$\int_0^\infty e^{-x} \cdot x^{n-1}\, dx = \Gamma n$

$n-1 = 1$
$\underline{\underline{n=2}}$

Q. Find c of the MVT, if $f(x) = x(x-1)(x-2)$, $a=0$, $b=1/2$ & c has usual meaning.

Given $f(x) = x(x-1)(x-2)$

By MVT $\therefore \exists$ a $c \in (a,b)$ such that
$a < c < b$

here $a=0$, $b=1/2$

$f(x) = (x^2-x)(x-2) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$\Rightarrow f'(c) = 3c^2 - 6c + 2$

By MVT, $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$3c^2 - 6c + 2 = \frac{b(b-1)(b-2) - a(a-1)(a-2)}{b-a}$$

$$= \frac{\cancel{1/2}(1/2 - 1)(1/2 - 2) - 0(0-1)(0-2)}{\cancel{1/2} - 0}$$

$$= (-1/2)(-3/2) = \frac{3}{4}$$

$$3c^2 - 6c + 2 = \frac{3}{4}$$

$$3c^2 - 6c + 2 - \frac{3}{4} = 0$$

$$3c^2 - 6c + \frac{5}{4} = 0$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{6 \pm \sqrt{21}}{6}$$

Q. locate stationary points of the func^n $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ & determine their nature.

Let $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 + \ \ -4x \ \ + 4y = 0$

$f_y = 4y^3 + 4x - 4y = 0$

$4x^3 - 4x + 4y = 0$ $\qquad$ $4y^3 + 4x - 4y = 0$

$x^3 - x + y = 0$ $\qquad$ $y^3 + x - y = 0$

$y = x - x^3$

$(x - x^3)^3 + \cancel{x} - \cancel{x} + x^3 = 0$

$x^3 - x^9 + 3x \cdot x^6 - 3x^2 \cdot x^3 + x^3 = 0$

$x = 0$

$x^3 \left(1 - x^6 + 3x^4 - 3x^2 + 1\right) = 0$

Q

$$4x^3 + 4y^3 = 0$$

$\cancel{4+4=4}$

$$x^3 + y^3 = 0$$

$(1+1)^2$

4

16 $\quad 3\times2\times2$

So

$$a^3+b^3 = (a+b)(a^2+b^2 - ab)$$

$$(x+y)(x^2+y^2-xy) = 0$$

hat

$\boxed{x=-y}$ or $x^2+y^2-xy = 0$

$$x^2+y^2 \underline{-xy+3xy} - 3xy = 0$$

$$(x+y)^2 - 3xy = 0$$

$$(x+y)^2 = \underline{3xy}$$

$$2x^2+2y^2-2xy = 0$$

$$x^2+y^2+(x-y)^2 = 0$$

This can only be possible when $x=0$, $\underline{y=0}$

At $(0,0)$

Other stationary points are :- $\underline{(x,-x),\ (0,0)}$

Q. Prove that if $a_0, a_1, a_2, \dots, a_n$, are the real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$

then there exists at least one real number x between 0 & 1 such that $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$

$$(1+x)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} \dots {}^nC_n$$

(from Binomial expansion)

Now, Integrating both sides, we get

$$\int (1+x)^n\, dx = \frac{{}^nC_0\, x^{n+1}}{n+1} + \frac{{}^nC_1\, x^n}{n} + \frac{{}^nC_2\, x^{n-1}}{n-1} + \dots \frac{{}^nC_n\, x}{1}$$

For $x = 1$

$$R.H.S = \frac{{}^nC_0}{n+1} + \frac{{}^nC_1}{n} + \frac{{}^nC_2}{n-1} + \dots {}^nC_n$$

here ${}^nC_0 = a_0$, ${}^nC_1 = a_1$, ${}^nC_2 = a_2 \dots {}^nC_n = a_n$ (say)

Acc. to the ques,

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots a_n = 0$$

Thus the given series,

$$(1+x)^n = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_n = 0$$

converging for $x \in (0,1)$

Q. $$\int_0^{\pi/2} \frac{x \sin x \cos x \, dx}{\sin^4 x + \cos^4 x}$$

$$= \int_0^{\pi/2} \cancel{\frac{x \sin x \cos x}{\cos^4 x (1+\tan^4 x)} dx} = \int_0^{\pi/2} \cancel{\frac{x \tan x \sec^2 x \, dx}{1+\tan^4 x}}$$

$$= \cancel{\int_0^{\infty}}$$

~~Put $\tan^4 x = t$~~

~~$4\tan^3 x \cdot \sec^2 x \, dx = dt$~~

~~$\tan x \cdot \sec^2 x \, dx = \frac{1}{4} \frac{dt}{\tan^2}$~~

Put $x = \pi/2 - y \Rightarrow dx = -dy$

$\sin x = \cos y \qquad x = 0 \Rightarrow y = \pi/2$

$\cos x = \sin y \qquad x = \pi/2 \Rightarrow y = 0$

$$I = \int_{\pi/2}^{0} \frac{\left(\frac{\pi}{2} - y\right) \cos y \sin y}{(\cos^4 y + \sin^4 y)} (-dy) = \int_0^{\pi/2} \frac{(\pi/2 - y) \cos y \sin y}{\cos^4 y + \sin^4 y} dy$$

$$I = \int_0^{\pi/2} \frac{\pi}{2} \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} dx - \int_0^{\pi/2} \frac{x \cos x \sin x}{\sin^4 x + \cos^4 x} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} dx$$

$$I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\sin^4 x + \cos^4 x}$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x \, dx}{\cos^2 x\,(1+\tan^4 x)} = \frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x \, dx}{1+\tan^4 x}$$

Put $\tan^4 x = t$

$4\tan^3 x \cdot \sec^2 x \cdot dx = dt$

$\tan x \cdot \sec^2 x \cdot dx = \frac{1}{4}\frac{dt}{\tan^2 x}$

$= \frac{1}{4}\frac{dt}{\sqrt{t}}$

$x = 0 \Rightarrow t = 0$

$x = \pi/2 \Rightarrow t = \infty$

$$= \frac{\pi}{4}\int_0^{\infty} \frac{1}{4} \cdot \frac{dt}{(1+t)\,(t)^{1/2}}$$

$$= \frac{\pi}{16}\int_0^{\infty} \frac{t^{-1/2}}{(1+t)}\,dt$$

$$\boxed{\int_0^{\infty} \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = B(m,n)}$$

$$= \frac{\pi}{16}\, B(1/2, 1/2)$$

$$= \frac{\pi}{16} \cdot \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \frac{\pi}{16} \cdot \sqrt{\pi} \cdot \sqrt{\pi} = \frac{\pi^2}{16}$$

Q. Find all asymptotes of the curve

$$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$$

let eq^n of Asymptotes : $y = mx + c$

$\phi_4 = x^4 - y^4$

Putting $x = 1,\ y = m$

$\phi_4(m) = 1 - m^4$

$\phi_4(m) = 0 \Rightarrow m = 1, -1$

$\phi_3 = 3x^2y + 3xy^2$

$\phi_3(m) = 3m + 3m^2$ $(x=1$

Also, $c = \frac{-\phi_3(m)}{\phi_4'(m)} = \frac{-3\cancel{m}(1+m)}{-4m^{\cancel{3}}} = \frac{3(1+m)}{4m^2}$

For $m = 1 \Rightarrow c = 3/2$

$m = -1 \Rightarrow c = 0$

hence, eq^n of asymptotes are : $\boxed{\begin{array}{l} y = x + 3/2 \\ y = -x \end{array}}$