LINEAR ALGEBRA

: CSE-2014 :

1(a) Find one vector in $\mathbb{R}^3$ which generates the intersection of V & W where V is the xy plane and W is the space generated by the vectors (1,2,3) & (1,-1,1).

$\longrightarrow$ Basis of V = {(1,0,0), (0,1,0)}

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \sim^{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \end{bmatrix} \to$ echelon form

$\rho(A) = 2 \Rightarrow (1,2,3)$ & $(1,-1,1)$ are L.I. vectors.

$\therefore$ Basis of W = {(1,2,3), (1,-1,1)}.

$V = \{ a(1,0,0) + b(0,1,0) / a,b \in \mathbb{R} \} = \{(a,b,0) / a,b \in \mathbb{R}\}$

$W = \{ x(1,2,3) + y(1,-1,1) / x,y \in \mathbb{R} \} = \{(x+y, 2x-y, 3x+y) / x,y \in \mathbb{R}\}$.

For $V \cap W$: $\quad a = x+y, \quad b = 2x - y, \quad 0 = 3x + y$

$\cancel{b=5} \qquad y = -3x$

$\therefore a = x + y = x - 3x = -2x$

$b = 2x - y = 2x + 3x = 5x$

$\therefore (a, b, 0) = (-2x, 5x, 0) = x(-2, 5, 0)$.

$\therefore$ The vector which generates the intersection of V & W is given by (-2,5,0)

1.(b) Using elementary row or column operations, find the rank of the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$\longrightarrow$ Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} R_4 \leftrightarrow R_1$

$R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ $\cancel{R_3 \to R_3 + 2R_4}$ $R_3 \to R_3 + 2R_4$ $\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ $R_3 \leftrightarrow R_4$ $\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$ Echelon form. Since it has three non-zero rows, $\rho(A) = 3$

2(a) Let V & W be the following subspaces of $\mathbb{R}^4$.

$V = \{(a,b,c,d) \mid b-2c+d=0\}$ and $W = \{(a,b,c,d) \mid a=d, b=2c\}$.

Find a basis and dimension of (i) V (ii) W (iii) $V \cap W$.

$\longrightarrow$ (i) $V = \{(a,b,c,d) \mid b-2c+d=0\}$.

$b-2c+d=0 \Rightarrow b = 2c-d$.

$\therefore V = \{(a, 2c-d, c, d)\}$. Therefore, we have

~~(1,0,0,0), (0,2,1,0) and (0,-1,0,1)~~

$(a, 2c-d, c, d) = a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$\therefore$ Basis of $V = \{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$.

Dim $V = 3$.

(ii) $W = \{(a,b,c,d) \mid a=d, b=2c\}$.

$\Rightarrow W = \{(a, 2c, c, a) \mid a, c \in \mathbb{R}\}$.

$\therefore (a, 2c, c, a) = 1(1,0,0,1) + c(0,2,1,0)$.

$\therefore$ Basis of $W = \{(1,0,0,1), (0,2,1,0)\}$.

Dim $W = 2$.

(iii) $V \cap W = \{(a,b,c,d) \mid b-2c+d=0, a=d, b=2c\}$

$a=d, b=2c$, $\quad b-2c+d=0$

$\longrightarrow 2c-2c+d=0 \Rightarrow d=0$.

Since $a=d \Rightarrow a=d=0$.

$\therefore V \cap W = \{(0, 2c, c, 0)\}$ where $c \in \mathbb{R}$.

$\therefore (0, 2c, c, 0) = c(0,2,1,0)$

$\therefore$ Basis of $V \cap W = \{(0,2,1,0)\}$

Dim $V \cap W = 1$

2(b)(i) Investigate values of λ & μ so that the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have
(i) No solution (ii) A unique solution (iii) Infinitely many solutions.

→ The given system of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}.$$ Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$.

Aug. matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

The given system has:
(i) No solution: if rank of $A \neq$ rank of $[A|B]$
(ii) Unique solution: if rank of A = rank of $[A|B]$ = No. of unknowns $(=3)$
(iii) Infinitely many solutions: if rank of A = rank of $[A|B]$ $<$ No. of unknowns (3).

Now: Reducing $[A|B]$ to echelon form:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Now:
(i) No solution: If $\lambda = 3$ and $\mu \neq 10$, we have rank of $A = 2 \neq 3$ = rank of $[A|B]$. Hence, for ~~$\neq$~~ $\lambda = 3$ and $\mu \neq 10$, we have no solution.

(ii) A unique solution: if $\lambda \neq 3$, we have rank of $A = 3$ = rank of $[A|B]$ = No of unknowns.
Hence, for $\lambda \neq 3$, we have a unique solution.
μ can take any value here.

(iii) Infinitely Many Solutions: If $\lambda = 3$, $\mu = 10$, then,
Rank of A = Rank of $A|B = 2 < 3$ = No. of unknowns.
Hence, for ~~$\lambda 3$~~ $\lambda = 3$, $\mu = 10$, we have infinitely many solutions.

2(b)(ii) Verify Cayley - Hamilton Theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence, find its inverse. Also, find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

$\rightarrow$ Cayley - Hamilton's Theorem states that each and every square matrix satisfies its characteristic equation.

Characteristic equation of A is given by $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$ —— ①

Now $A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow A^2 - 4A - 5I = O$ —— ②

$\therefore$ A satisfies the characteristic eqn. ①

Hence, Cayley-Hamilton's theorem is verified.

Now: Since $|A| \neq 0$, A^{-1} exists.

Premultiplying with A^{-1} on both sides of ②

$A^{-1}.A^2 - 4.A^{-1}A - 5.A^{-1}I = A^{-1}O$

$\Rightarrow A - 4I - 5A^{-1} = O \quad \Rightarrow 5A^{-1} = A - 4I$

$\Rightarrow 5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$,

From ② $\quad A^2 - 4A - 5I = 0$

$\Rightarrow A^2 = 4A + 5I$

Premultiplying A on both sides, $\quad A^3 = 4A^2 + 5A$

$= 4(4A + 5I) + 5A$

$= 21A + 20I$

Sly $\quad A^4 = 21A^2 + 20A$

$= 21(4A + 5I) + 20A \quad = 104A + 105I$

$A^5 = 104A^2 + 105A = 104(4A + 5I) + 105A = 521A + 520I$

$\therefore A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$\Rightarrow 521A + 520I - 4[104A + 105I] - 7[21A + 20I] + 11[4A + 5I] - A - 10I$

$\Rightarrow A - 5I$

$\Rightarrow \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-4 & 4\\2 & -2\end{bmatrix}$

4(c) Let $A = \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$. Find the eigen values of A and corresponding eigen vectors.

$\rightarrow$ Characteristic equation of $A \equiv |A - \lambda I| = 0 \Rightarrow \begin{vmatrix}-2-\lambda & 2 & -3\\2 & 1-\lambda & -6\\-1 & -2 & -\lambda\end{vmatrix} = 0$

$(-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + (1-\lambda)] = 0$

$(-2-\lambda)[\lambda^2 - \lambda - 12] - 2[-2\lambda - 6] - 3[-3 - \lambda] = 0$

$-2\lambda^2 - \lambda^3 + 2\lambda + \lambda^2 + 24 + 12\lambda + 4\lambda + 12 + 9 + 3\lambda = 0$

$\rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow \lambda = 5, -3, -3.$

$\Rightarrow (\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0$

$\rightarrow (\lambda - 5)(\lambda + 3)^2 = 0$

Eigen values of A are $-3, -3, 5$

Eigen Vectors corresponding to eigen values:

(i) $\lambda = -3$ $(A - (-3)I)X = 0$

$\rightarrow (A + 3I)X = 0$

$\begin{bmatrix}1 & 2 & -3\\2 & 4 & -6\\-1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$\begin{bmatrix}1 & 2 & -3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$x + 2y - 3z = 0 \Rightarrow x = -2y + 3z$

$X = \begin{bmatrix}-2y + 3z\\y\\z\end{bmatrix} = y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}3\\0\\1\end{bmatrix}$

$\therefore X_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$, $X_2 = \begin{bmatrix}3\\0\\1\end{bmatrix}$

$\therefore$ Eigen Vectors corr. to $\lambda = 3$ are X_1 & X_2 & $\lambda = 5$ is X_3

(ii) $\lambda = 5$ $(A - 5I)X = 0$

$\begin{bmatrix}-7 & 2 & -3\\2 & -4 & -6\\-1 & -2 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_1 \leftrightarrow R_3$ $\begin{bmatrix}-1 & -2 & -5\\2 & -4 & -6\\-7 & 2 & -3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\rightarrow R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 7R_1$

$\begin{bmatrix}-1 & -2 & -5\\0 & -8 & -16\\0 & 16 & 32\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_3 \to R_3 + 2R_2$

$\begin{bmatrix}-1 & -2 & -5\\0 & -8 & -16\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$-8y - 16z = 0 \Rightarrow y = -2z$

$-x - 2y - 5z = 0$

$x = -2y - 5z$

$= 4z - 5z$

$x = -z$

$\therefore X = \begin{bmatrix}-z\\-2z\\z\end{bmatrix} = z\begin{bmatrix}-1\\-2\\1\end{bmatrix}$

$\therefore X_3 = \begin{bmatrix}-1\\-2\\1\end{bmatrix}$

4 (c) (ii) Prove that eigen values of a unitary matrix have absolute value 1.

→ Let A be a unitary matrix. Then $A^{\theta} A = I$

Let X be an eigen vector of A corr. to eigen value λ.

Then, $X \neq 0$ and $AX = \lambda X$ ...... (1)

Taking Tranjugate both sides, we have

$(AX)^{\theta} = (\lambda X)^{\theta} \Rightarrow X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta}$ — (2)

$(2) \cdot (1) \equiv X^{\theta} A^{\theta} \cdot A X = \bar{\lambda} X^{\theta} \lambda X$

$\Rightarrow X^{\theta} I X = \bar{\lambda} \lambda X^{\theta} X$

$\Rightarrow X^{\theta} X = |\lambda|^2 X^{\theta} X$

$\Rightarrow X^{\theta} X \left[|\lambda|^2 - 1 \right] = 0$

Since $X \neq 0 \Rightarrow X^{\theta} X \neq 0$. Then

$|\lambda|^2 = 1 \Rightarrow \lambda = 1$

∴ The absolute value of any eigen values of a unitary matrix ~~are~~ is 1.