SHANKAR
IAS ACADEMY ™

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MAINSTORMING – 2019

Mathematics Mains Test -7

Paper 1 (Full)

Time : 3 hours | Maximum marks: 250

Instructions

1. Question paper contains **8** questions out of this candidate need to answer **5** questions in the following pattern
2. Candidate should attempt question No's **1** and **5** compulsorily and any **three** of the remaining questions selecting **atleast one** question from each section.
3. The number of marks carried by each question is indicated at end of each question.
4. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/Notations carry their usual meanings, unless otherwise indicated.

Section- A

1.

a) Let A be a square matrix of order 3 such that each of its diagonal elements is a and off--diagonal elements is 1. If $B = bA$ is orthogonal, determine values of a and b. (10 marks)

b) Let V be a vector space of all 2×2 matrices over the field R. Show that W Is not a subspace of V, where

I. W contains all 2×2 matrices with zero determinant

II. W contains all 2×2 matrices A $such\ that$ $A^2 = A$ (15 marks)

c) Using mean value theorem show that

I. $f(x)$ is constant in [a b], $f'(x) = 0$ in $[a\ b]$ (5 marks)

II. $f(x)$ is decreasing function in $(a\ b)$ if $f'(x)$ exists and is less than zero everywhere in $(a\ b)$. (10 marks)

d) Let $u = ax^2 + 2hxy + by^2$ and $v = Ax^2 + 2Hxy + By^2$, find the Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)}$ and hence show that u and v are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$

(10 marks)

2.

a) State Cayley Hamilton Theorem. Verify theorem for the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Hence find A^{-1}. (10 marks)

b) Show that $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta = \frac{1}{2} \frac{\Gamma\frac{(p+1)}{2}\Gamma\frac{(q+1)}{2}}{\Gamma\frac{(p+q+2)}{2}}, p, q > -1.$

Hence evaluate

I. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx$

II. $\int_0^1 x^3 (1-x^2)^{5/2} \, dx$

III. $\int_0^1 x^4 (1-x)^3 \, dx$ (20 marks)

c) Find maxima and minima for the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. Also find saddle point of the function if any. (10 marks)

d) A function $f(x,y)$ is defined as $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, if \ (x,y) \neq (0,0) \\ 0, if \ (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$ (10 marks)

3.

a. Show that angel between the planes given by the equation $2x^2 - y^2 + 3z^2 - xy + 7xz + 2yz = 0 \ is \ \tan^{-1}\frac{\sqrt{50}}{4}$. (10 marks)

b. Given the set $\{u, v, w\}$ is linearly independent. Examine the sets

I. $\{u+v, v+w, w+u\}$

II. $\{u+v, u-v, u-2v+2w\}$ for linear independence. (10 marks)

c. Find the angel between the lines whose direction cosines are given by the relations $l+m+n = 0, 2lm + 2ln - mn = 0$. (10 marks)

d. Find eigen values and eigen vectors for matrix $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Examine whether A is diagonalizable. Obtain matrix D such that $D = P^{-1}AP$ (10 marks)

e. Find the equation of right circular cone with vertex at origin and whose axis makes equal angels with coordinate axes and the generator is the line passing through the origin with direction ratios $(1, -2, 2)$ (10 marks)

4.

a) Show that the equation $2y^2 + 4zx + 2x - 4y + 6z + 5 = 0$ represents a right circular cone. Show that the semi-vertical angel of this cone is $\frac{\pi}{4}$ and that axis is given by $x + z + 2 = 0, y = 1$. (15 marks)

b) Prove that $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$ represents a cylinder whose cross section is an ellipse of eccentricity $1/\sqrt{2}$ and find the equations to its axis. (15 marks)

c) Planes are drawn through the origin O and the generators through any point P of the paraboloid given by $x^2 - y^2 = az$. Prove that the angel between them is $\tan^{-1} 2r/a$ where r is length of OP. (10 marks)

d) Show that the equations $y - \lambda z + \lambda + 1 = 0, (\lambda + 1)x + y + \lambda = 0$ represent for different values of λ generators of one system of the hyperboloid $yz + zx + xy + 1 = 0$ and find the equations to the generators of the other system. (10 marks)

Section- B

5.

a) Solve, $(2D^3 - 7D^2 + 7D - 2)y = e^{-8x}$ *where* $D = \frac{dy}{dx}$ (10 marks)

b) Solve the differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ (10 marks)

c) A particle is undergoing SHM, of period T about a center O and it passes through the position P ($OP = b$) with velocity V in the direction of OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}(\frac{VT}{2\pi b})$ (10 marks)

d) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if side of the cube be $\frac{\pi r}{2}$, prove that the total angel through which the cube can swing without falling is 90^o. (10 marks)

e) Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$ and that $r^n\vec{r}$ is irrotational. (10 marks)

6.

a) Solve the differential equation $(\frac{dy}{dx})^2 + 2\frac{dy}{dx}y\cot x = y^2$ (15 marks)

b) A string of length a forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together.if one of the rods be supported in a horizontal position, prove that the tension of the string is $\frac{2W(2b^2-a^2)}{b\sqrt{4b^2-a^2}}$. (15 marks)

c) Using Stokes theorem, evaluate $\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz$
Where C is the boundary of triangle with vertices at $(2,0,0), (0,3,0), (0,0,6)$.
(10 marks)

d) Solve the differential equation $e^{3x}\left(\frac{dy}{dx}-1\right)+\left(\frac{dy}{dx}\right)^3 e^{2y}=0.$ (10 marks)

7.

a) A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angel to the major axis of the path and that velocity then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, *where* $2a$ *is the major axis*, e *is the eccentricity and* T *is the periodic time.* (15 marks)

b) A semi ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as depth. If the minor axis lies on the surface, the find the eccentricity in order that the focus may be center of the pressure.
(10 marks)

c) Evaluate $\iint_S (\nabla \times \vec{f}).\hat{n}ds$, *where S is is the surface of the cone,* $z = 2-\sqrt{x^2+y^2}$ above XY *plane* and $\vec{f} = (x-z)\hat{\imath} + (x^3+yz)\hat{\jmath} - 3x^2y\hat{k}$
(15 marks)

d) Solve $\frac{d^2y}{dx^2} + 4y = tan2x$ by using method of variation of parameters.(10 marks)

8.

a) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(a^5/x^2\right)^{1/3}$ *when its at a distance x from O*. if it starts from rest at a distance a *from O*, show that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$. (15marks)

b) Find the curvature and Torsion of the circular Helix, $\vec{r} = a(cos\theta, sin\theta, \theta cot\beta)$, β is the constant angel at which it cuts its generators. (15 marks)

c) Write if the tangent to the curve makes a constant angel α , with a fixed line, then prove that $Kcos\alpha \pm \tau sin\alpha = 0$. Conversely, if $\frac{K}{\tau}$ is a constant, then show that the tangent makes a constant angel with a fixed direction. (20 marks)