SuccessClap

Online Coaching for UPSC MATHEMATICS

QUESTION BANK SERIES

PAPER 2 : 07 ALGEBRA

Content:

SuccessClap : Question Bank for Practice

01 GROUPS

(1) Let (G,*) be a group, then
(i) The identity element is unique.
(ii) Every element of G has unique inverse in G.

(2) If (G,*) is a group, then
(i) $(a^{-1})^{-1} = a; \forall\, a \in G$.
(ii) $(a*b)^{-1} = b^{-1}*a^{-1}; \forall a, b \in G$ (Reverse rule)

(3) In a group G, the equation a*x = b and y*a = b where $a,b \in G$ have unique solution in G.

(4) The left identity is also the right identity.

(5) The left inverse of an element is also its right inverse.

(6) A finite set G, with a binary operation* which is associative, is a group iff the cancellation laws hold.

(7) Show that the set {1,-1,i,-i} is an abelian finite group of order 4 under multiplication.

(8) Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a * b = \frac{(ab)}{2}$.

(9) Show that the set Z of all integers form a group with respect to binary operation * defined by $a * b = a+b+1; \forall a, b \in Z$ is an abelian group.

(10) Prove that the set of all n nth roots of unity forms an abelian group w.r.t multiplication.

(11) The set M_2 of all 2×2 matrices
$M_2 = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$: $a, b, c, d \in R$ is an abelian group under the addition of two matrices.

(12) Show that the set of matrices G = $\left\{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \alpha\epsilon R\right\}$ forms a group under matrix multiplication.

(13) Show that G = $\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \neq 0 \in R\right\}$ is an abelian group under matrix multiplication.

(14) Prove that the following matrices:
I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, A = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, B = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and C = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ form a group under the multiplication of two matrices.

(15) Show that the set S of all 2×2 non – singular matrices over R is a group under matrix multiplication.

(16) Show that the set S = {1,5,7,11} is a group w.r.t multiplication modulo 12.

(17) Prove that {S,$\odot_{14}$} is a group, S = {2,4,8}.

(18) Show that the set G = {f_1,f_2,f_3,f_4}, where $f_1(x) = x_1 f_2(x) = -x_1 f_3(x) = 1/x$, $f_4(x) = -1/x$ $\forall$ $x \in R - \{0\}$ is a group w.r.t the product of two mappings.

(19) Show that the set G = {f_1,f_2,f_3,f_4,f_5,f_6}, where $f_1(x) = x$, $f_2(x) = 1-x$, $f_3(x) = 1/x$, $f_4(x) = 1/(1-x)$, $f_5(x) = (x-1)/x$, $f_6(x) = x/(x-1)$ $\forall$ $R - \{0,1\}$ is a group w.r.t 'composite of functions'.

(20) Show that the set G = {x+y$\sqrt{3}$: $x, y \in Q$} is a group w.r.t addition.

(21) Show that the set 1 of all integers with binary operation, defined as a.b = a+b+I $\forall$ $a, b \in I$ is an abelian group.

(22) Show that the set Q of all rational numbers other than -1 is an abelian group w.r.t the binary composition a*b = a+b+ab.

(23) Let G = {(a,b):a$\neq 0, b \in R$} and * be a binary composition defined by (a,b) * (c,d) = (ac,bc+d). Show that (G,*) is a non – abelian group.

(24) Let G = {(a,b): a,b ∈ R and not both zero} and * be a binary composition defined by (a,b) * (c,d) = (ac-bd, a +bc). Show that (G, *) is a commutative group.

(25) Let G = {(a,b): a,b ∈ R}, and $*$ be a binary compositive defined by $(a, b) * (c, d) = (a + c, b + d)\ \forall\ a, b, c, d \in R$. Show that $(G,*)$ is a commutative group.

(26) Prove that if G is an abelian group, then $(a,b)^n = a^n . b^n$ for all a,b ∈ G and all positive integers n.

(27) If G is a group and if a,b ∈ G, show that a.b = b.a ⇒ $(a.b)^n = a^n . b^n$, n being any positive integer.

(28) Show that a group G satisfying $a^2 = e\ \forall\ a \in G$ must be abelian.

(29) Prove that a group G is abelian if and only if $(a.b)^2 = a^2.b^2\ \forall\ a, b \in G$.

(30) Show that if every element of the group G is its own inverse, then G is abelian.

(31) Prove that a group G is abelian if and only if $(a.b)^{-1} = a^{-1}.b^{-1}$ for all a,b ∈ G.

(32) Show that the equation x.a. x=b is solvable for x in a group G if and only if a.b is the square of some element in G.

(33) Show the equation $x^2. a . x = a^{-1}$ is solvable for x in a group G if and only if a is the cube of some element in G.

(34) If G is a group such that $(a.b)^n = a^n.b^n$ for three consecutive integers n and for all a,b ∈ G, show that G is abelian.

(35) If G is a group of even order, prove that it has an element $a \neq e$ satisfying $a^2 = e$.

(36) If G is a finite group, show that there exists a positive integer N such that $a^N = e$ for all a ∈ G.

(37) Show that if G is a finite semi – group with cross – cancellation laws i.e., $x.y = y.z \Rightarrow x = z$ then G is an abelian group.

(38) If number of elements in a group G is less than or equal to four, then group must be abelian.

(39) If G is a group of even order, then show that there exists an element a, other than the identity e such that $a^2 = e$.

(40) In a group G if $xy^2 = y^3x$ and $yx^2 = x^3y$, show that $x = y = e$, where e is the identity of G.

(41) Let G be a finite group whose order is not divisible by 3. Suppose $(ab)^3 = a^3b^3$ for all $a,b \in G$, then show that G is abelian.

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02 SUBGROUPS

(1) A non – empty subset H of a group G is a subspace of G if and only if
(i) a,b $\in H \Rightarrow ab \in H$.
(ii) a $\in H \Rightarrow a^{-1} \in H$, where a^{-1} is the inverse of a $\in G$.

(2) Let H be a non – empty subset of a group G. Then H is a subgroup of G iff a,b $\in H \Rightarrow ab^{-1} \in H$, where b^{-1} is the inverse of b in G.

(3) The necessary and sufficient condition of a non – empty subset H of a group G to be a subgroup is $HH^{-1} \subset H$.

(4) A necessary and sufficient condition of a non – empty subset H of a group G to be a subgroup is that is $HH^{-1} = H$.

(5) Let H be any complex of group G, then $(HK)^{-1} = K^{-1}H^{-1}$.

(6) If H is any subgroup of G, then $H^{-1} = H$. Also, show that converse is not true.

(7) If H,K are subgroups of a group G, then HK is a subgroup of G iff HK = KH.

(8) If H and K are subgroups of an abelian group G, then HK is a subgroup of G.

(9) The necessary and sufficient condition for a non – empty finite subset H of a group G, with respect to multiplication to be a subgroup is that H must be closed with respect to multiplication, i.e., a $\in H$, $b \in H \Rightarrow ab \in H$.

(10) The intersection of any two subgroups of a group G is a subgroup of G.

(11) The union of two subgroups of a group G is a subgroup of G iff one is contained in the other.

(12) Let G be the additive group of integers and H = {nl :n is a fixed integer and $l \in Z$}. Show that H is a subgroup of G.

(13) If G is a group, then show that the set Z, defined by Z = {xz = zx: $x \in G, z \in Z$}. (it is called centre of the group) is a subgroup of G.

(14) If a is any element of a group G, then show that {a^n: $n \in Z$} is a subgroup of G.

(15) If a is a fixed element of a group G, then prove that the set N(a) = {$x \in G: xa = ax$} is a subgroup of G.

(16) Show that H = {(1,b) $b \in R$} is a subgroup of the group $G = \{(a, b): a \neq 0, b \in R\}$ is a subgroup of the group $G = \{(a, b): a \neq 0, b \in R\}$ under the composition $*$ given by $(a, b) * (c, d) = (ac, bc + d)$.

(17) Show that H = $\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}: a \neq 0; a, b \in R\right\}$ is a subgroup of the multiplicative group of 2×2 non − singular matrices over R.

(18) Show that $aHa^{-1} = \{aha^{-1}: h \in H\}$ is a subgroup of G, where H is a subgroup of G and $a \in G$.

(19) If a be a fixed element of group G and if $H = \{x \in G: xa^2 = a^2x\}$, $K = \{x \in G: xa = ax\}$, then show that $H < G$ and $K < H$.

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03 ORDERS

(1) Consider the multiplicative group G = {1,-1,-i,-i} of cube roots of unity. Find the order of each element of G.

(2) The order of every element of a finite group is finite.

(3) If the element a of a group G is of order n, then $a^m = e$, iff n is a divisor of m.

(4) The order of an element of a group is the same as that of its inverse.

(5) The order of any integral power of an element a cannot exceed that order of a.

(6) If a and b are any two elements of a group G, then $o(a) = o(b^{-1}ab)$.

(7) For any two elements a,b of a group G, o(ab) = o(ba).

(8) The order of any integral power of an element of a group is a divisor of the order of that element.

(9) If a is an element of order n and p is prime to n, then a^p is also of order n. Let r be the order of a^p.

(10) In a group, if $ba = a^m b^n$, prove that the elements $a^m b^{n-2}$, $a^{m-2} b^n$, ab^{-1} have the same order.

(11) In any group G if $a^5 = e$, $aba^{-1} = b^2$ for $a,b \in G$. Find $o(b)$.

(12) If a,b are two elements of a group G such that ab = ba and (o(a),o(b)) =1, then o(ab) = o (a) o(b).

(13) If a is any element of a group G, show that $o(a^n) = \frac{o(a)}{(n,o(a))}$, where n is a positive integer and (n,o(a)) means the g.c.d of n and o(a).

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04 COSETS LAGRANGE

(1) Let $H < G$ and $a,b \in G$. Prove that

(i) $Ha = H$ iff $a \in H$

(ii) $Ha = Hb$ iff $ab^{-1} \in H$

(iii) $aH = bH$ iff $a^{-1}b \in H$.

(iv) $(Ha)^{-1} = a^{-1}H$.

(2) Let H be a subgroup of a group G and $a,b \in G$. Show that either $Ha \cap Hb = \varphi$ or $Ha = Hb$.

Or

Prove that any two right cosets of H in G are either identical or disjoint, H being a subgroup of G.

(3) Prove that there is a one-to-one correspondence between any two right cosets of H in G.

(4) **(Lagrange's Theorem)**

The order of a subgroup of a finite group divides the order of the group.

Or

If G is a finite group and H is a subgroup of G, then o(H) is a divisor of o(G).

(5) The index of a subgroup H of a finite group G divides the order of the group and $i_G(H) = \frac{o(G)}{o(H)}$

(6) If G is a finite group and $a \in G$, then order of a divides o(G).

(7) Every group of prime order is cyclic.

(8) Let H be a subgroup of G and $a,b \in G$.

Show that $Ha \neq Hb \Rightarrow a^{-1}H \neq b^{-1}H$.

(9) If $H \dot{\subseteq} K$ be two subgroups of a finite group G, then show that $[G:H] = [G:K][K:H]$.

(10) Show that there exist a one – to – one correspondence between the right and left cosets of H in G, where H is any subgroup of a group G.

(11) If $H < G$, prove that
(i) $Hh = H \Rightarrow h \in H$,
(ii) $b \in Ha \Rightarrow Ha = Hb$.

(12) Show that if H and K are subgroups of a group G and $a \in G$, then $Ha \cap Ka = (H \cap K)a$.

(13) If G is a group and H, K are two subgroups of finite index in G, prove that $H \cap K$ is of finite index.

(14) If H and K be two subgroups of a group G, then HK is a subgroup of G if and only if HK = KH.

(15) If H and K are finite subgroups of a group G, then $o(HK) = \frac{o(H)o(K)}{o(H \cap K)}$.

(16) If H and K are subgroups of a group G and $o(H) > \sqrt{o(G)}$, $o(K) > \sqrt{o(G)}$; then $o(H \cap K) > 1$ i.e., $H \cap K \neq \{e\}$.

(17) If G is a group of order 35, show that it cannot have two subgroups of order 7.

(18) Suppose G is a finite group of order pq, where p and q are primes ($p > q$). Show that G has at most one subgroup of order p.

(19) Show that a group G of order 2p, where p is prime and $p > 2$, has exactly one subgroup of order p.

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05 CYCLIC GROUP

(1) Every subgroup of a cyclic group is cyclic.

(2) Every group of prime order is cyclic.

(3) If cyclic group G is generated by an element a of order n, then a^m is a generator of G iff (m,n) = 1, i.e., the GCD of m and n is 1.

(4) A finite group of order n continuing an element of order n must be cyclic.

(5) Every isomorphic image of a cyclic group is cyclic.

(6) A cyclic group G with a generator of finite order n, is isomorphic to the multiplicative group of n, n^{th} roots of unity.

(7) The order of a cyclic group is equal to the order of any generator of the group.

(8) If the generator of a cyclic group G is of infinite order (or of zero order), then G is isomorphic to the additive group of integers.

(9) Every cyclic group is necessarily abelian but the converse is not necessarily true.

(10) How many generators are there of the cyclic group of order 8?

(11) Show that the group G = [{1,-1,i,-i},.] is cyclic.

(12) Show that number of generators of an infinite cyclic group is two.

(13) If a is a generator of cyclic group G, then a^{-1} is also a generator of G.

(14) Show that the Klein's 4 – group is not cyclic.

(15) Converse of Lagrange's theorem holds in finite cyclic groups.

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06 NORMAL SUBGROUP

(1) A subgroup N of a group G is a normal subgroup of G if and only if $gNg^{-1} = N$ for each $g \in G$.

(2) A subgroup N of a group G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G.

(3) If N is a normal subgroup of a group G, then
(i) NaNb = Nab
(ii) aNbN = abN; $a,b \in G$.

(4) Prove that H is not normal subgroup of a group G iff the product of any two right cosets of H in G is a right coset of H in G.

(5) If N and M are normal subgroups of a group G, then $N \cap M$ is a normal subgroup of G.

(6) Show that $Z = \{a \in G ; ax = xa \forall\, x \in G\}$ is a normal subgroup of G.

(7) Show that $H = \{(1\text{-}b) : b \in R\}$ is a normal subgroup of $G = \{(a, b); a \neq 0, b \in R\}$ under the composition $*$ defined by $(a, b) * (c, d) = (ac, bc + d)$.

(8) If H is a subgroup of G and N is a normal subgroup of G, then $H \cap N$ is a normal subgroup of H.

(9) If N and M are normal subgroups of a group G and if $N \cap M = (e)$, then $nm = mn$ for each $n \in N$ and $m \in M$.

(10) If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G.

(11) If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, prove that H is a normal subgroup of G.

(12) If H is the only subgroup of finite order m in the group G, then show that H is a normal subgroup of G.

(13) Let $H < G$ and $N(H) = (g \in G : gHg^{-1} = H)$. Prove that
(i) N(H) is a subgroup of G.
(ii) H is normal in N (H).
(iii) If H is a normal subgroup of the subgroup K of G, then $K \subset N(H)$.
(iv) H is normal in $G \Rightarrow N(H) = G$.

(14) If N is a normal sub group of G and H is a subgroup of G, then show that HN is a subgroup of G.

(15) If H and K are normal subgroups of G, then $HK = (hk : h \in H, k \in K)$ is a normal subgroup of G.

(16) Show that a subgroup H of a group G is normal iff $Ha \neq Hb \Rightarrow aH \neq bH$.

(17) Let H be a non-empty subset of a group G. Show that H is a normal subgroup of G iff $(gx)(gy)^{-1} \in H \forall g \in G$ and $x, y \in H$.

(18) Show that a subgroup N of a group G is normal if and only if $xy \in N \Rightarrow yx \in N$.

(19) If a cyclic subgroup T of G is normal in G, then show that every subgroup of T is normal in G.

(20) For any two real number $a,b \in R$; define a maping $f_{ab}: R \to R$ as $f_{ab}(x) = ax+b \ \forall x \in R$.
Let $G = \{f_{ab} : a \neq 0\}$. Prove that G is a group under the composition of mappings. Further show that $N = \{f_{lb} \in G\}$ is a normal subgroup of G.

(21) Show that a normal subgroup is commutative with every complex.

(22) If N is a normal subgroup of G and H is any subgroup of G, show that NH is a subgroup of G.

(23) If N and M are normal subgroups of G, then NM is also a normal subgroup of G.

(24) Let G be a group of order 2p, where p is prime. Show that G has a normal subgroup of order p.

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07 COSETS HOMOMORPHISM

(1) Suppose N is a normal subgroup of a group G. Let $\frac{G}{N}$ denote the set of all right cosets of N in G i.e.,$\frac{G}{N} = \{Na: a \in G\}$. Show that $\frac{G}{N}$ is a group under the composition: NaNb = Nab for all a,b ∈ G. The group $\frac{G}{N}$ is called the quotient group or factor group of G by N.

(2) If G is a finite group and N is a normal subgroup of G, then o$\left(\frac{G}{N}\right) = \frac{o(G)}{o(N)}$.

(3) If G is an abelian group and N is a normal subgroup of G, then G/N is abelian. Show by an example that the converse need not be true.

(4) If G is a cyclic group and N a subgroup of G, then G/N is cyclic. However, the converse need not be true.

Or

Show that every quotient group of a cyclic group is cyclic. However, the converse need not be true.

(5) If N is a normal subgroup of a group G and a ∈ G is of order o(a), prove that o(Na) divides o(a). Also show that $a^m \in N$ if and only if o(Na) divides m.

(6) Let N be a normal subgroup of G. Show that G/N is abelian iff $xyx^{-1}y^{-1} \in N \text{ for all } x, y \in G$.

(7) If H is a subgroup of a group G such that $x^2 \in$ H for all x $\in G$. Prove that G/H is abelian.

(8) Let G be the set of all real 2 × 2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where ad $\neq 0$, under matrix multiplication. Let $N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$. Prove that

(a) N is a normal subgroup of G.

(b) G/N is abelian.

(9) Let H and K be normal subgroups of a group G such that H ⊂ K, show that K/H is a normal subgroup of G/H.

(10) If Z is the centre of a group G such that $\frac{G}{Z}$ is cyclic, then show that G is abelian.

(11) If f : G→ G' *be a homomorphism of group and* e, e' *be the identities in* G and G' respectively. Then
(i) f(e) = e'
(ii) $f(a^{-1}) = [f(a)]^{-1}$ where a ∈ G
(iii) If the order of an element x ∈ G is finite, then the order of f(x) is a divisor of the order of x.

(12) If G : G' is an isomormhism of groups, then the order of an element a ∈ G is equal to order of the f – image of a, i.e., o(a) = o[f(a)].

(13) Show that every quotient group of a group is a homomorphic image of the group.

Or

If N is a normal subgroup of a group G, show that there is a homomorphism f of G onto G/N with Ker f = N.

(14) If f : G → G' is a homomorphism, then kernel of f is a subgroup of G.

(15) If f : G → G' *is a homomorphism, then* $Ker\ f = \{e\} \Rightarrow f$ *is one − to − one*.

(16) If f : G→ G' is a homomorphism, then Im f is a subgroup of G'.

Or

Show that a homomorphic image of a group is a group.

(17) **(Fundamental Theorem of Homomorphism)**

If f is a homomorphism of G onto G' with kernel K, then $\frac{G}{Ker f}$ = G' or $\frac{G}{K} = G'$

Or

Show that every homomorphic image of a group G is isomorphic to a quotient group.

(18) If H and K are subgroups of a group G and H is normal in G, then $\frac{HK}{H} \sim \frac{K}{H \cap K}$.

(19) If H and K are subgroups of a group G and K is normal in G, then $\frac{HK}{K} = \frac{H}{H \cap K}$.

(20) If H and K are two normal subgroups of a group G such that H⊆K, then show that $\frac{G}{K} = \frac{G/H}{K/H}$.

(21) Let $\bar{G}$ be the group of non – zero real numbers under multiplication and G = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \text{ and } ad - bc \neq 0\right\}$ be a group under matrix multiplication. Exhibit a homomorphism of G onto G.

(22) Let G be any group, g a fixed element in G. Define $\varphi: G \to G$ by $\varphi(x)$ =gxg^{-1}. Prove that φ is an isomorphism of G onto G.

(23) Prove that a group G is abelian if and only if the mapping f : G → G, given by f(x) = x^2, is a homomorphism.

(24) Prove that a group G is abelian if and only if the mapping f : G→ G, given by f(x) = x^{-1}, is a homomorphism.

(25) Show that:
(i) Every homomorphic image of an abelian group is abelian.
(ii) Every homomorphic image of a cyclic group is cyclic.
(iii) Show, by means of an example, that the converse of each of the above results is not true.

(26) Let f : G → G' be a homomorphism and H a subgroup G. show that f(H) is a subgroup of group G'.

(27) If N and M are normal subgroups of G, prove that $\frac{NM}{M} = \frac{N}{N \cap M}$.

(28) For any group G, show that $\frac{G}{(e)} = G \text{ and } \frac{G}{G} = (e)$.

(29) Let R be the set of real numbers. For a,b ∈ $R\ (a \neq 0)$; *let* f_{ab} : R → R be defined as $f_{ab}(x)$ = ax+b.
Let G = {f_{ab}: a,b ∈ $R \text{ and } a \neq 0$} and N = {f_{lb} ∈ G}. Prove that N is a normal subgroup of G and that G/N is isomorphic to the group of non – zero real numbers under multiplication.

(30) Let G be the group of non – zero complex numbers under multiplication and N the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication.

(31) Let G be the group of all non – zero complex numbers under multiplication and let $\bar{G}$ be the group of all real 2×2 *matrices of the form* $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are not both zero , under matrix multiplication. Show that G and $\bar{G}$ are isomorphic by exhibiting an isomorphism of G onto $\bar{G}$.

(32) Let G be the group of real numbers under addition and let N be the subgroup of G consisting of all the integers. Prove that G/N is isomorphic to the group of all complex numbers of absolute value I under multiplication.

(33) Show that any infinite cyclic group is isomorphic to (Z, +).

(34) Show that a finite cyclic group of order n is isomorphic to Z_n, the group of integers modulo n.

(35) Show that a finite cyclic group of order n is isomorphic to the multiplicative group of n nth roots of unity.

(36) Show that any two cyclic groups of the same order are isomorphic.

(37) Show that a finite cyclic group of order n is isomorphic to the quotient group Z/N , where N = {nx: x ∈ $Z\} = (n)$.

(38) Show that the relation = of isomorphism in groups is am equivalence relation.

(39) Every group G is isomorphic to a permutation group.

(40) Prove that a group of order 36 is not simple.

(41) Prove that a group of order 99 is not simple.

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08 RINGS

(1) Show that the set R = {a+b$\sqrt{3}$: $a, b \in Q$} is a ring under the usual addition and multiplication as binary compositions.

(2) Show that the set I of integers with two binary compositions * and o defined by a*b = a+b-1, aob = a+b-ab for all integers a and b is a commutative ring with unity.

(3) If {R,+,*} be a ring with unit element, show that {R,$\oplus$,$\otimes$} is also a ring with unit element, where a$\oplus$b = a+b+1 and a $\otimes$ b = a.b +a+b $\forall\ a, b \in R$.

(4) If E denotes the set of all even integers, then prove that {E,+,*} is a commutative ring, where a*b = ab/2 and + is the usual addition.

(5) Prove that the set S of all ordered pairs (a,b) of real numbers is a commutative ring under the addition and multiplication compositions defined as (a,b)+(c,d)=(a+c,b+d) and (a,b) (c,d) = (ac,bd).

(6) Prove that a ring R is commutative if and only if $(a+b)^2 = a^2+2ab+b^2$ for all a,b $\in R$.

(7) If R is a system satisfying all the conditions for a ring with unit element with the possible exception of a+b = b+a, prove that the axiom a+b= b+a must hold in R and that R is thus a ring.

(8) Let R be a ring such that $a^2 = a$ for all a $\in$ R. Prove that R is commutative.

(9) If R is a ring with unity satisfying $(xy)^2=x^2y^2$ for all x,y $\in R$, prove that R is commutative.

(10) Show that a ring R is commutative if and only if $a^2-b^2=(a+b)(a-b)$ for all a,b $\in R$.

(11) Let R be a ring such that for $x \in R$, *there exists* a unique a $\in$ R satisfying xa =x. Show that ax =x. Hence deduce that if R has a unique right unity e, then e is the unity of R.

(12) Let R be a ring with unity 1 ∈ R. Suppose for $x \neq 0 \in R$, *there exists a unique* $y \in R$ such that xyx = x. Prove that xy=yx=1 i.e., x is invertible in R.

(13) Let R be a ring with unity e. If some $x \in R$, *there* exists unique $y \in R$ such that $xy = e$, prove that x is invertible.
(iii) If H is a normal subgroup of the subgroup K of G, then K ⊂ N (H)
(iv) H is normal in G⇔ N(H) =G.

(14) The set C = {a+bi: a,b∈ R} of complex numbers is a field under usual addition and multiplication of complex numbers.

(15) The set S =$\left\{\begin{pmatrix} x & y \\ -x & y \end{pmatrix} : x, y \in C\right\}$ is a division ring which is not a field.

(16) The set Q ={$a_0+a_1i+a_2j+a_3k$: a_0,a_1,a_2,a_3 are real numbers} where $i^2=j^2=k^2=ijk=-1$,ij=-ji=k,jk=-kj=I,ki=-ik=j is a division ring , which is not a field.

(17) Let C be the set of all ordered pairs (a,b) where a, b are real numbers. Let the compositions of addition and multiplication in C be defined as (a,b)+(c,d)=(a+c,b+d), (a,b).(c,d)=(ac-bd,bc+ad). Then C is a field.

(18) Let R be a commutative ring. Then R is an integral domain if and only if ab = ac ⇒b=c, where a,b,c ∈ R *and* $a \neq 0$.

(19) Prove that every field is an integral domain.

(20) Prove that a finite integral domain is a field.

(21) Show that the ring Z_P of integers modulo p is a field if and only if p is prime.

(22) Let R be a ring such that the equation ax =b has a solution for all $a \neq 0 \in R$ and for all $b \in R$. Show that R is a division ring.

(23) A non – empty subset S of a ring R is a subring of R if and only if (i) a=b $\in S$ and $(ii) ab \in S$ for all $a, b \in S$.

(24) Show that the centre of a ring R is a subring of R.

(25) Show that the set S = $\left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in Z\right\}$ is a subring of the ring M_2 of 2×2 matrices over integers.

(26) If a is a fixed element of a ring R, show that $I_a = \{x \in R : ax = 0\}$ is a $subring$ of R.

(27) Prove or disprove that subring of a non – commutative ring is non – commutative.

(28) Let e be idempotent in a ring R. Show that eRe= {eae: a $\in R$} is a subring of R with unity e.

(29) Let R be a ring such that $x^3 = x \ \forall x \in R$. Show that R is commutative.

(30) Let R be a ring such that for each a $\in R$ there exists x $\in R$ such that $a^2x = a$. Prove the following:

(i) R has no non – zero nilpotent elements.

(ii) axa – a is nilpotent and so axa = a

(iii) ax and xa are idempotents.

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09 IDEAL RING HOMOMORPHISM

(1) If Z be the ring of integers and n be any integer, then (n) =[nx: x $\in \mathbb{Z}$] is an ideal of Z.

(2) Every ideal of a ring R is a subring of R, but the converse need not be true

(3) Show that the set S = $\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$: a, b are integers$\}$ is a left ideal in the ring M_2 of 2×2 matrices over integers. Further show that S is not a right ideal in M_2.

(4) Show that the set S = $\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$: a, b are integers$\}$ is a right ideal of M_2, the ring of 2×2 matrices over integers, which is not a left ideal of M_2

(5) If S be an ideal of a ring R and $1 \in S$, prove that S =R.

(6) If F is a field, prove its only ideals are (0) and F itself.

(7) Let R be a ring and $a \in R$. Show that the set $S = \{r \in$ R:ra=0$\}$ is a left ideal of R.

(8) Let R be the ring of all real valued, continuous functions on [0,1]. Show that the set S = $\{f \in R: f\left(\frac{1}{2}\right)=0\}$ is an ideal of R.

(9) If A and B are two ideals of a ring R such that B$\dot{\subseteq}$ A, then $\frac{R}{A} = \frac{R/B}{A/B}$.

(10) Prove that a division ring is a simple ring.

(11) Let R be a commutative simple ring with unity. Prove that R is a field.

Or

If R be a commutative ring with unity whose only ideals are {0} are R, then show that R is a field.

(12) Show that $M_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Q\}$ is a simple ring.

(13) Let R be a ring with unity. If R has no right ideals except R and {0}, then prove that R is a division ring.

(14) Let R be a ring having more than one element such that aR = R $\forall\ a \neq 0 \in R$, then R is a division ring.

(15) Let R be a ring such that the only right ideals of R are {0} and R. Prove that either R is a division ring or that R is a ring with prime number of elements in which ab =0 for a,b $\in$ R.

(16) If R is a ring, then the mapping f:R $\rightarrow$ R defined as $f(x) = x \forall x \in R$ is a homomorphism.

(17) If R is a ring, the maping f:R $\rightarrow$ R defined as $f(x) = 0 \forall x \in R$ is a homomorphism.

(18) Let $Z(\sqrt{2}) = \{m + \ n\sqrt{2}: m, n \text{ are integers}\}$. The mapping $f: Z[\sqrt{2}] \rightarrow Z[\sqrt{2}]$ defined as $f(m + n\sqrt{2}) = m - n\sqrt{2}$ is a homomorphism.

(19) Let R = Z and R' = set of all even integers. Then (R',+,*) is a ring, where a*b = $\frac{1}{2}ab \forall a, b \in R'$. The mapping $f: R \rightarrow R'$ defined as $f(a) = 2a \forall a \in R$ is a homomorphism.

(20) Let R be the ring of all complex numbers an R' the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are real numbers. Then the mapping f:R$\rightarrow$ R' defined as $f(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a homomorphism.

(21) Let R be a commutative ring such that 2x =0 $\forall\ x \in R$. Then the mapping $f: R \rightarrow R$ defined as $f(x) = x^2$ is a homomorphism.

(22) Show that f: $Z_2 \rightarrow Z_2$ defined by f(n)=n^2-n is a ring homomorphism.

(23) If $R \rightarrow R'$ is a homomorphism, then

1. f(0)=0'
2. f(-a)=-f(a), a $\in$ R.

(24) Show that:
(a) The homomorphic image of a commutative ring is a commutative ring. The converse need not be true.
(b) The homomorgraphic image of a ring with unity is a ring with unity. The converse need not be true.

(25) If R $\rightarrow$ R' is a homomorphism, then Ker f is a two- sided ideal of R.

(26) If f: R$\rightarrow$ R' is a homomorphism, then Ker f = {0} if and only if f is to one – to – one.

(27) Let f be an isomorphism of a ring onto a ring R'. Show that
(a) If R is an integral domain, then R' is also an integral domain.
(b)If R is a field, then R' is also a field.

(28) Let R be a ring with unity. Using its elements, we define a ring R' by defining a⊕b = a+b+1 and a ⊕b = ab+a+b $\forall$ $a, b \in \mathrm{R}$. Prove that R is isomorphic to R'.

(29) If U is an ideal of a ring R, then R/U is a ring and is a homomorphic image of R with kernel U.

(30) **(Fundamental Theorem of Homomorphism)**
Let f : R $\rightarrow$ R' be a homomorphism of a ring R onto a ring R'. Then $\frac{\mathrm{R}}{\mathrm{Kerf}} = \mathrm{R}'$.

(31) If A and B are two ideals of a ring R, then $\frac{\mathrm{A+B}}{\mathrm{B}} = \frac{\mathrm{A}}{\mathrm{A \cap B}}$.

(32) If A and B are two ideals of a ring R, then $\frac{\mathrm{A+B}}{\mathrm{A}} = \frac{\mathrm{B}}{\mathrm{A \cap B}}$.

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10 EMBEDDING MAX PRIME IDEALS

(1) Every ring can be imbedded in a ring with unity.

(2) Every ring R with unity can be imbedded in a ring of endomorphisms of some additive abelian group.

(3) Every ring R can be imbedded in a ring of endomorphisms of some additive abelian group.

(4) Every integral domain can be imbedded in a field.

(5) Let R be the ring of all the real – valued continuous functions on the closed unit interval. Show that
M = $\{f \in R : f\left(\frac{1}{3}\right) = 0\}$ is a maximum ideal of R.

(6) If R is a commutative ring with unity, then an ideal M of R is maximal if and only if R/M is a field.

(7) Let R be a ring with unity. Prove that an ideal M of R is maximal if and only if M +(a) = R $\forall\, a \notin M$.

(8) Show that in the ring R = $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Q\right\}$,
The set M = $\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in Q\right\}$, is a maximal ideal of R.

(9) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal if and only if R/P is an integral domain.

(10) Let A and B be two primal ideals of a commutative ring R. Show that $x^2 \in A \cap B \Rightarrow A \cap B, for\ all$ x $\in R$.

(11) If R be a commutative ring with unity, then every maximal ideal of R is a prime ideal of R.

(12) If R is a finite commutative ring with unity, then every prime ideal of R is a maximal ideal of R.

(13) Show that a commutative ring R is an integral domain if f(0) is a prime ideal.

(14) Let R be commutative ring with unit element in which every ideal is a prime ideal. Prove that R is a field.

(15) Let R be a commutative ring with unity and let M be a maximal ideal of R such that $M^2 = (0)$. Show that if N is any other maximal ideal of R, then N = M.

(16) Let R be a P.I.D. Show that every ascending chain of ideals $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \ldots \subseteq (a_n) \subseteq \ldots$ is finite

(17) Prove that if an ideal U of a ring R contains a unit of R, then U = R.

(18) Let R be a principal ideal domain. Show that any non – zero ideal $P \neq R$ is prime if and only if it is maximal.

(19) Prove that the units in a commutative ring R with a unit element form an abelian group.

(20) Let R be a P.I.D ., which is not a field. Prove that an ideal A = (a) is a maximal ideal if and only if a is an irreducible element of R.

(21) Let R be an integral domain with unity and a,b be any two non – zero elements of R. Show that a and b are associates iff a/b and b/a.

(22) I the ring $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, show that $\bar{2}$ is a prime element but not irreducible,

(23) Show that 1+i is an irreducible element in Z [i].

(24) Show that 3 is not a prime element of Z[$\sqrt{-5}$].

(25) Show that $\sqrt{-5}$ $is\ a\ prime\ element\ of\ Z\{\sqrt{-5}\}$.

(26) Show that 3 is an irreducible element of Z{$\sqrt{-5}$ }.

(27) If R be a commutative ring and a $\in R, then\ (a) = \{ar + na: r \in R. n \in Z\}$.

(28) If R is a commutative ring with unity and a $\in R, then$ (a) = {ar: r $\in R\} = aR$.

(29) Show that Z (all integers) is a P.I.D

(30) Prove that every field is a P.I.D. Is the converse true? Justify your answer.

(31) Find all the units of Z ($\sqrt{-5}$).

(32) Show the ring of polynomials over a field of reals is a Euclidean ring.

(33) In a P.I.D. an element is prime if and only if it is irreducible.

(34) Show that Z {$\sqrt{-5}$ } is not a P.I.D.

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11 ED PID

(1) The ring Z of integers is a Euclidean domain.

(2) Every field F is a Euclidean domain.

(3) Show that Z[i] = {m+ni: m, n∈ $Z, i = \sqrt{-1}$ }is a Euclidean domain. {Z[i] is called the Ring of Gaussian integers}.

(4) Show that Z[$\sqrt{-2}$] = {m+n$\sqrt{2}$: $m, n \in Z$}, is a Euclidean domain.

(5) Every Euclidean domain is a principal ideal domain i.e., E.D ⇒ P.I.D

(6) Show Z{$\sqrt{-5} = \{a + b\sqrt{-5} : a, b \in Z\}$ is not a Euclidean domain.

(7) If possible, find g.c.d and i.c.m of 10+11i and 8+i in Z[i].

(8) Show that 3+4i and 4-3i are associates in Z [i].

(9) Find the g.c.d in Z[i] of 2 and 3+5i.

(10) Find the g.c.d of 11+7i and 18-I in Z[i].

(11) Let f(x) and g(x) be two non – zero polynomials in R[x], R being any ring.
(i) If f(x) +g(x) $\neq 0$, then $\deg(f(x) + g(x)) \leq \max(\deg f(x), \deg g(x))$.
(ii) If f(x) g(x) $\neq 0$, then $\deg\big(f(x)g(x)\big) \leq \deg f(x) + \deg g(x)$.
(iii) If R is an integral domain, then deg(f(x) g(x)) = deg f(x) +deg g(x).

(12) If R is an integral domain, then R [x] is an integral domain.

(13) Show that every ring R can be imbedded in the polynomial ring R[x].

Or

Show that every ring R is isomorphic to a subring of R[x].

(14) Show that a ring R is an integral domain if and only if R[x] is an integral domain.

(15) **(Division Algorithm)**
If f(x) and g(x) are two non – zero polynomials in F[x] (F being a field), then there exist two polynomials t(x) and r(x) in F(x) such that f(x) = t(x) g(x) +r(x), where r(x) =0 or deg r(x) < deg g(x).

(16) If F is a field, then F[x] is a Euclidean domain.

(17) Show that (x+2) is a maximal ideal of Q[x] and hence Q[x]/(x+2) is a field.

(18) If R is a ring, prove that $\frac{R[x]}{x} = R$, (x)is the ideal generated by x.