Set-I

Co-ordinate System : (1)

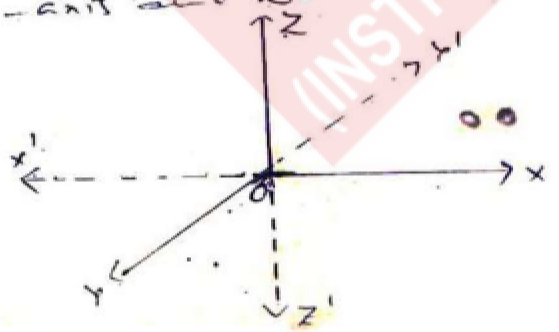
Introduction:

In analytical geometry of two dimensions the position of a point is determined with respect to two axes of reference but in the space it is not sufficient to determine the point w.r.t two axes.

Thus to locate the position of a point in space, another (third) axis is required in addition to the two axes. That is why, the co-ordinate system in space is called a three dimensional system.

Origin: Let XOX', YOY' and ZOZ' be three mutually perpendicular straight lines in space, intersecting at 'O'. Then the point 'O' is called the origin.

Axes: The three fixed straight lines XOX', YOY' and ZOZ' respectively called x-axis, y-axis and z-axis.



The three lines taken together are called rectangular co-ordinate axes.

* Co-ordinate planes:

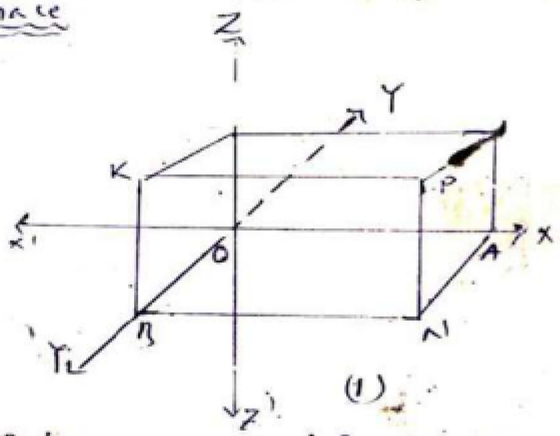
— The plane containing the axes of Y and Z is called the YZ-plane.
Thus YOZ is the YZ-plane.

— The plane containing the axes of Z and X is called the ZX-plane.
Thus ZOX is the ZX-plane.

— The plane containing the axes of X and Y is called the XY-plane.
Thus XOY is the XY-plane.

The above three planes are together called the rectangular co-ordinate planes or simply co-ordinate planes.

* Co-ordinates of a point in space



(1)

Let P be any point in space.

Draw through 'p' three planes parallel to the three co-ordinate planes and cutting x, y and z axes in A, B and C respectively as in the figure.

These planes, together with the co-ordinate planes form a rectangular parallelopiped.

The position of P relative to the co-ordinate system is given by its perpendicular distances from the co-ordinate planes and these distances are given by lengths OA, OB and OC.

Let OA = a, OB = b and OC = c.

Then a, b, c are called x-co-ordinate, y-co-ordinate and z-co-ordinate respectively of the point P. The point P referred as (a, b, c) or P(a, b, c).

Any one of these a, b, c will be +ve or −ve according as it is measured from 'O' along the corresponding axis in the positive or negative direction.

— Let $P(x, y, z)$ be a point in the space.

Then (i) P lies in the xy-plane $\Rightarrow z = 0$

(ii) P lies in the zx-plane $\Rightarrow y = 0$;

(iii) P lies in the yz-plane $\Rightarrow x = 0$

(iv) P lies in the x-axis $\Rightarrow y = 0, z = 0$

(v) P lies in the y-axis $\Rightarrow x = 0;\ z = 0$

(vi) P lies in the z-axis $\Rightarrow x = 0, y = 0$

(vii) $P = O \Rightarrow x = 0, y = 0, z = 0$

* Octants:

The three co-ordinate planes divide the whole space into 8 parts and these parts are called octants.

The sign of a point determine the octant in which it lies.

The signs for the eight octants are given in the tabular form below:

octants	$OXYZ$	$OX'YZ$	$OX'Y'Z$	$OXY'Z$	$OXYZ'$	$OX'YZ'$	$OX'Y'Z'$	$OXY'Z'$
x	+	−	−	+	+	−	−	+
y	+	+	−	−	−	+	−	−
z	+	+	+	+	−	−	−	−

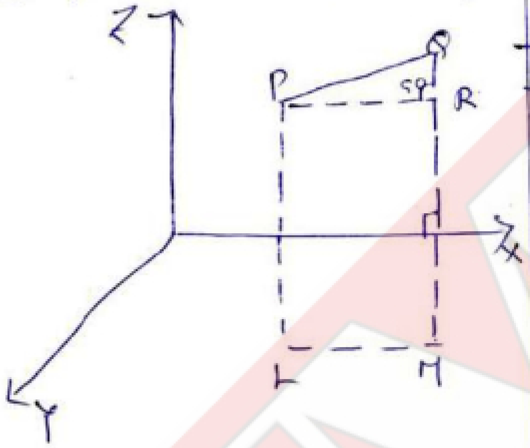
Note:— The co-ordinates of the origin 'O' are (0, 0, 0) and those of A, B, C, N, K and M in fig (1) are (a, 0, 0); (0, b, 0); (0, 0, c); (a, b, 0); (0, b, c) and (a, 0, c) respectively.

* Distance b/w two points:-

→ To find the distance b/w two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.

Through P and Q draw PL and QM $\perp^s$ to the xy-plane meeting it in the points L and M respectively.



Then in the xy-plane L is the point (x_1, y_1) and M is (x_2, y_2)

so that

$$LM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad (1)$$

Now through P, draw PR and $\perp$ lar to QM

Then clearly $PR = LM$

and $QR = QM - RM$

$= QM - PL$

$= z_2 - z_1$

$\therefore$ in the rt. angled triangle PQR,

$PQ^2 = PR^2 + QR^2$ (by Pythagoras theorem)

$= LM^2 + QR^2 \quad (2)$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$$\therefore \boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

→ The distance of the point $P(x, y, z)$ from the origin $(0, 0, 0)$ is $\boxed{OP = \sqrt{x^2 + y^2 + z^2}}$

→ Method to prove by distance that the three points A, B, C are collinear.

(1) find the three distances AB, BC and CA

(2) Then if the sum of any two distances is equal to the third, the three given points are collinear.

$AB + BC = AC \qquad AC + CB = AB.$

→ Triangles*

If three points are not collinear then they form a triangle

(1) A triangle is said to be an equilateral triangle if three sides of the triangle are equal

(2) A triangle is said to be an isosceles triangle if any two sides of the triangle are equal.

(3). A triangle is said to be a right angled triangle if one angle of the triangle is a right angle

(4). A triangle is said to be obtuse angled triangle if one angle of the triangle is an obtuse angle

5. A triangle is said to be acute angled triangle if the three angles are acute.

Quadrilaterals :-

1. A quadrilateral is said to be a parallelogram if opposite sides are parallel and equal. If the opposite sides are equal, Then clearly they are parallel.

2. A quadrilateral is said to be a rectangle if opposite sides are equal and diagonals are equal.

3. A quadrilateral is said to be a rhombus if the four sides are equal and the two diagonals are not equal.

4. A quadrilateral is said to be a square if the four sides are equal and the two diagonals are equal.

Note :- In a parallelogram (or) rectangle (or) rhombus (or) square, the diagonals bisect each other.

Note :- In a rhombus (or) a square, the diagonals are perpendicular to each other.

Problems :-

→ Find the distance between the points $(-1,0,6)$ and $(5,3,0)$

→ Show that the points $(3,-2,4)$, $(1,1,1)$, $(-1,4,-2)$ are collinear.

→ If $A=(-1,3,5)$ and $B(4,-12,-20)$ find whether O, A, B are collinear.

→ Show that the following points are collinear

(i) $(-1,0,7)$, $(3,2,1)$, $(5,3,-2)$ $\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}=0$

(ii) $(1,2,3)$, $(7,0,1)$, $(-2,3,4)$

→ Show that the three points $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ form an equilateral triangle.

→ $(1,1,1)$, $(-2,4,1)$, $(-1,5,5)$ form a right angles isosceles triangle.

→ Show that the points $(-1,-2,-1)$, $(2,3,2)$, $(4,7,6)$ and $(1,2,3)$ form a parallelogram.

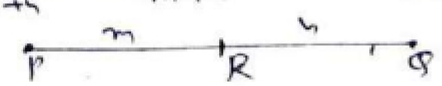
→ Show that the points $(1,3,4)$, $(-1,6,10)$, $(-7,4,7)$, $(-5,1,1)$ are the vertices of a rhombus.

→ Prove that the four points A, B, C, D whose coordinates are $(1,1,1)$, $(-2,4,1)$, $(-1,5,5)$ and $(2,2,5)$ are the vertices of a square.

* Section formulae for External division:

→ The co-ordinates of a point R which divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m:n$ are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$



③

* **Section formulae for external division:-**

The co-ordinates of a point R which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio m:n are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

* **Midpoint formula:**

The co-ordinates of the mid-point R of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

→ If $P(x, y, z)$ lies on the line joining $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ then $\frac{x_1 - x}{x - x_2} = \frac{y_1 - y}{y - y_2} = \frac{z_1 - z}{z - z_2}$.

and P divides $\overline{AB}$ in the ratio $x_1 - x : x - x_2$ or

$y_1 - y : y - y_2$ or

$z_1 - z : z - z_2$.

→ xy-plane divides the line segment joining (x_1, y_1, z_1) (x_2, y_2, z_2) in the ratio

$-z_1 : z_2$.

similarly others.

→ If $D(\alpha_1, \beta_1, \gamma_1)$, $E(\alpha_2, \beta_2, \gamma_2)$, $F(\alpha_3, \beta_3, \gamma_3)$ are midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the $\triangle ABC$ then

$A(\alpha_2 + \alpha_3 - \alpha_1, \beta_2 + \beta_3 - \beta_1, \gamma_2 + \gamma_3 - \gamma_1)$

$B(\alpha_1 + \alpha_3 - \alpha_2, \beta_1 + \beta_3 - \beta_2, \gamma_1 + \gamma_3 - \gamma_2)$

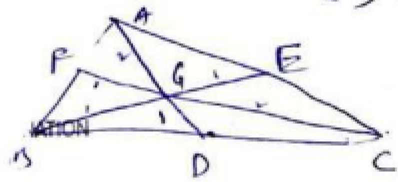
and

$C(\alpha_1 + \alpha_2 - \alpha_3, \beta_1 + \beta_2 - \beta_3, \gamma_1 + \gamma_2 - \gamma_3)$

→ **Centroid of a triangle**

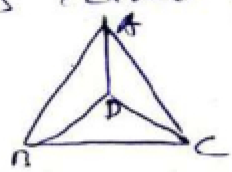
The centroid of a triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$



* **Tetrahedron:-**

Let ABC be a triangle and D is a point in the space which is not in the plane of the triangle ABC. Then ABCD is called a tetrahedron.



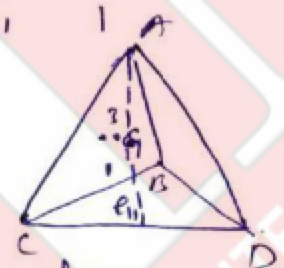
→ The tetrahedron ABCD has four faces namely $\Delta ABC, \Delta ACD, \Delta ABD$ & ΔBCD.

→ It has four vertices, namely A, B, C, D and it has six edges, namely AB, AC, AD, BC, BD, CD

→ The centroid G of the tetrahedron ABCD divides the line joining any vertex to centroid of its opposite face in the ratio 3:1. (from statics)

Thus if G_1 is the centroid of ΔBCD, then G the centroid of tetrahedron ABCD divides AG_1 in the ratio 3:1

i.e $\frac{AG}{GG_1} = \frac{3}{1}$.



→ If all the edges are of equal length, then it is called a regular tetrahedron.

→ Centroid of tetrahedron:

Let ABCD be a tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$.

Then ~~the~~ co-ordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

→ Find the points dividing the line segment joining (1,−1,2) and (2,3,7) in the ratio.

(i) 2:3 (ii) −2:3.

→ Find the middle point of the line segment with end points (1,2,−3) and (−1,6,7).

→ Find the ratio in which the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) is ~~divided~~ bisects divided by xy-plane.

Sol: Let the ratio be $\lambda:1$ and let R be the point of intersection of plane and line segment

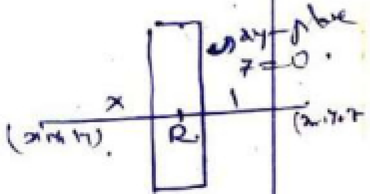
∴ the coordinates of R are

$$\left[\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right]$$

Since the point R lies on XY-plane

∴ z-coordinate must be zero.

$$\therefore \frac{\lambda z_2 + z_1}{\lambda+1} = 0$$



$$\Rightarrow \lambda z_2 + z_1 = 0$$

$$\Rightarrow \lambda z_2 = -z_1$$

$$\Rightarrow \lambda = \frac{-z_1}{z_2}$$

$$\Rightarrow \frac{\lambda}{1} = \frac{-z_1}{z_2}$$

$$\Rightarrow \boxed{\lambda : 1 = -z_1 : z_2}$$

Note: → on XY-plane is $-z_1 : z_2$

→ on YZ-plane is $-x_1 : x_2$

→ on ZX-plane is $-y_1 : y_2$.

Problem :—

→ Find the ratio in which the line joining the points (2,4,5), (3,5,−4) is divided by the XY-plane

→ Find the ratio in which the line joining the points (2,4,5), (3,5,−4) divided by the coordinate planes

Sol^n: (1) XY-plane $-5:-4$
$= 5:4$ (internally)
and $-5:4$ (externally)

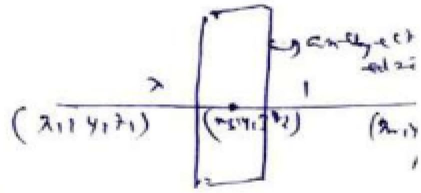
(2) YZ-plane $-2:3$ (internally)
and $2:3$ (externally)

(3) ZX-plane $-4:5$ (internally)
and $4:5$ (externally).

→ Find the ratio in which the coordinate planes divide the line joining the points (−2,4,7), (3,−5,8).

→ Show that the plane $ax+by+cz+d=0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $\dfrac{-(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$

Sol Let the given plane $ax+by+cz+d=0$ meet the line joining the given points (x_1, y_1, z_1), (x_2, y_2, z_2) in (x_3, y_3, z_3).



Then $ax_3+by_3+cz_3+d=0$. ——(1)

Also let the point (x_3, y_3, z_3) divide the line joining the given points in the ratio $\lambda:1$.

Then $(x_3, y_3, z_3) = \left(\dfrac{\lambda x_2+x_1}{\lambda+1}, \dfrac{\lambda y_2+y_1}{\lambda+1}, \dfrac{\lambda z_2+z_1}{\lambda+1}\right)$

$\Rightarrow x_3 = \dfrac{\lambda x_2+x_1}{\lambda+1}$, $y_3 = \dfrac{\lambda y_2+y_1}{\lambda+1}$,
$z_3 = \dfrac{\lambda z_2+z_1}{\lambda+1}$

Substituting these values in (1) we get

$a\left(\dfrac{\lambda x_2+x_1}{\lambda+1}\right)+b\left(\dfrac{\lambda y_2+y_1}{\lambda+1}\right)+c\left(\dfrac{\lambda z_2+z_1}{\lambda+1}\right)+d=0$

$$\Rightarrow \boxed{\lambda = \dfrac{-(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}}.$$

→ Find the ratio in which the join of (2,1,5) and (3,4,3) is divided by the plane $x+y-z=\frac{1}{2}$.

Sol $\lambda = \dfrac{-(2+1-5-\frac{1}{2})}{(3+4-3-\frac{1}{2})}$

$= \dfrac{-(-2-\frac{1}{2})}{(4-\frac{1}{2})}$

$= \left(\frac{5}{2} \times \frac{2}{7}\right)$

$= \frac{5}{7}$.

$\boxed{\lambda = 5:7}$

→ Find the co-ordinates of the point at which the line joining the points $(4,3,1)$ and $(1,-2,6)$ meets the plane $3x-2y-z+3=0$.

Sol

The plane $3x-2y-z+3=0$ divides the line joining the points $(4,3,1)$ and $(1,-2,6)$ in the ratio

$$-\frac{(3(4)-2(3)-1+3)}{3(1)-2(-2)-6+3}$$

$$=-\frac{2}{1}.$$

Let R be the point of intersection of the plane & the line segment

$(4,3,1)\ \xrightarrow{-2}\ \boxed{\ 3x-2y-z+3\ }\ \xrightarrow{1}\ (1,-2,6)$

∴ The co-ordinates of R are

$$\left(\frac{-2+4}{-2+1},\frac{4+3}{-2+1},\frac{-12+1}{-2+1}\right)$$

$$=(-2,-7,11).$$

→ P.T the points $P(3,2,-4)$, $Q(5,4,-6)$, $R(9,8,-10)$ are collinear. Also find the ratio in which the point Q divides PR.

Sol Half solution

Let the ratio $\lambda:1$

∴ The co-ordinates of Q are $\left(\frac{9\lambda+3}{\lambda+1},\frac{8\lambda+2}{\lambda+1},\frac{-10\lambda-4}{\lambda+1}\right)$

P $\xrightarrow{\lambda}$ Q $\xrightarrow{1}$ R

$$\therefore \frac{9\lambda+3}{\lambda+1}=5,\quad \frac{8\lambda+2}{\lambda+1}=4$$

$$\frac{-10\lambda-4}{\lambda+1}=-6$$

$$\Rightarrow 9\lambda+3=5\lambda+5,$$

$$8\lambda+2=4\lambda+4$$

$$-10\lambda-4=-6\lambda-6$$

$$\Rightarrow 4\lambda=2;\quad 4\lambda=2,\quad \lambda=\tfrac{1}{2}$$

$$\Rightarrow \lambda=\tfrac{1}{2}\quad \lambda=\tfrac{1}{2},\quad \lambda=\tfrac{1}{2}.$$

∴ reqd ratio $= 2:1$, $1:2$

→ Find the ratio in which C divides AB where $A(2,3,4)$, $B(3,7,2)$, $C(-6,-17,-4)$.

→ Find the ratio in which the line segment join the points $A(0,-1,2)$ and $B(5,4,-3)$ is divided by the xy-plane. Also find the co-ordinates of the point of intersection.

→ Find the point of intersection of the line through (−2, 3, 4), (1, 2, 3) with the xz-plane.

→ Find the ratio in which the yz-plane divides the line joining the points (3, 5, −7) and (−2, 1, 8). Find also the point of division.

→ Find the distance of the point (1, 2, 0) from the point where the line joining (2, −3, 1) and (3, −4, −5) cuts the plane $2x+y+z=7$.

ans:- The point where the line joining (2, −3, 1) and (3, −4, −5) meets the plane $2x+y+z=7$ is $\left(\frac{7}{3}, \frac{-10}{3}, -1\right)$

∴ The reqd distance

= distance b/w (1, 2, 0) and $\left(7/3, \frac{-10}{3}, -1\right)$

$= \frac{1}{3}\sqrt{281}$

→ (i) $A = (1, 2, 3)$ and $B = (2, 0, 1)$. If the points A, B, Q are collinear and if the x-co-ordinate of Q be −1. Find the y-co-ordinate z-coordinate of Q.

(5)

Sol Given that

$A = (1, 2, 3)$, $B = (2, 10, 1)$

Let Q divide AB in the ratio $\lambda : 1$

∴ The co-ordinates of Q are $\left(\frac{2\lambda+1}{\lambda+1}, \frac{10\lambda+2}{\lambda+1}, \frac{\lambda+3}{\lambda+1}\right)$ —— (1).

A ——λ—— Q (x, y, z) ——1—— B

Since x-co-ordinate of Q is −1

∴ $\frac{2\lambda+1}{\lambda+1} = -1$

$\Rightarrow \boxed{\lambda = -2/3}$

∴ The ratio is $\lambda : 1 = \frac{2}{3} : 1$

$\Rightarrow -2 : 3$.

(1) ≡ y-coordinate is

$\frac{10\lambda+2}{\lambda+1} = \frac{10(-2/3)+2}{-2/3+1}$

$= \frac{-14}{1} = -14$.

(1) ≡ z-co-ordinate is

$\frac{\lambda+3}{\lambda+1} = \frac{-2/3+3}{-2/3+1} = \frac{7}{1}$

$= 7$.

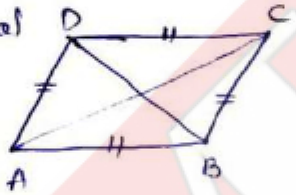
(ii) → If $A = (-1, 0, 7)$, $B = (3, 2, t)$, $C = (5, 3, -2)$ are collinear then show that $t = 1$

→ Three vertices of a parallelogram ABCD are $A(3,-1,2)$ $B(1,2,-4)$ and $C(-1,1,2)$. Find the coordinates of the fourth vertex D.

Sol'n :- Let the required point be $D(x,y,z)$.

Mid point of diagonal BD is $\left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$

mid point of diagonal AC is



$\left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right)$

i.e. $(1,0,2)$

But, the mid points of the diagonals of a parallelogram coincide.

$\therefore \frac{x+1}{2} = 1;\quad \frac{y+2}{2} = 0;\quad \frac{z-4}{2} = 2$

$x = 1;\quad y = -2;\quad z = 8.$

→ Find the Centroid of the triangle with vertices $(7,-4,7)$, $(1,-6,10)$, $(5,-1,1)$

Sol'n - The Centroid of a triangle ABC is $\left(\frac{13}{3}, \frac{-11}{3}, \frac{18}{3}\right)$.

→ A, B, C are the vertices of a triangle $A(1,1,1)$, $B(-2,4,1)$ If the centroid of the ΔABC is the origin then find 'C'.

Sol'n :- Given $A(1,1,1)$, $B(-2,4,1)$ & let $C(x,y,z)$

Since the Centroid of the ΔABC is the origin.

$\therefore \left(\frac{-1+x}{3}, \frac{5+y}{3}, \frac{2+z}{3}\right) = (0,0,0)$

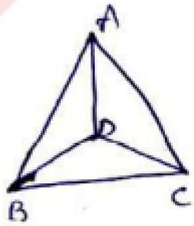
$\Rightarrow \frac{-1+x}{3} = 0;\ \frac{5+y}{3} = 0,\ \frac{2+z}{3} = 0$

$\Rightarrow x = 1,\ y = -5\quad z = -2$

$\therefore$ The point $C = (1,-5,-2)$.

→ Show that $(5,-1,-1)$, $(-1,5,-1)$, $(-1,-1,5)$, $(-3,-3,-3)$ are the vertices of a regular tetrahedron.

Sol'n :- Given $A(5,-1,-1)$, $B(-1,5,-1)$, $C(-1,-1,5)$, $D(-3,-3,-3)$.



$AB = \sqrt{36+36}$

$= \sqrt{72} = 6\sqrt{2}$

$BC = \sqrt{36+36} = 6\sqrt{2}$

Similarly $AC = 6\sqrt{2}$, $AD = 6\sqrt{2}$,

$BD = 6\sqrt{2}$, $CD = 6\sqrt{2}$

$\therefore AB = BC = AC = BD = CD = AD$

$\therefore$ All the edges are of equal length.

$\therefore$ ABCD is a regular tetrahedron.

* Locus *

Def'n (1):-

The set of all points in the space satisfying given Condition (or) a given property is called a locus.

Def'n (2):- An equation $f(x,y,z)=0$ is said to be an equation of a locus S', if every point in S' satisfies $f(x,y,z)=0$ and every point that satisfies $f(x,y,z)=0$ belongs to S'.

Note (1):- If every point $P(x_1,y_1,z_1)$ in a locus S' satisfies the condition $f(x_1,y_1,z_1)=0$ then the equation of locus S' is $f(x,y,z)=0$

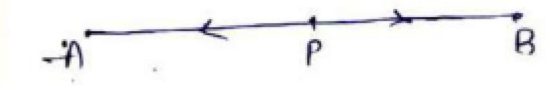
(2) Generally a locus can be described by its equation.

Problems:-

→ (1) Find the equation to the locus of points which are equidistant from the points $(-2,2,3)$, $(3,4,5)$

Sol'n:- Let $A(-2,2,3)$, $B(3,4,5)$ & let $P(x,y,z)$ be a point in the locus.

Given Condition is $PA=PB$



$\Rightarrow \sqrt{(x+2)^2+(y-2)^2+(z-3)^2} = \sqrt{(x-3)^2+(y-4)^2+(z-5)^2}$

$\Rightarrow x^2+4+4x+y^2-2y+4+z^2-6z+9 = x^2-6x+9+$
$-8y+16+z^2-10z+2$

$\Rightarrow 10x+4y+4z-33=0$

$\therefore$ The equation to the locus of P is $10x+4y+4z-33=0$

→ Find the locus of the point which is at a distance of 5 units from $(2,1,-3)$.

Sol'n:- Let $P(x,y,z)$ be a point in locus. and $A(2,1,-3)$ be the given point.

$\therefore$ Given Condition is $PA=5$

$\Rightarrow \sqrt{(x-2)^2+(y-1)^2+(z+3)^2}=5$

$\Rightarrow x^2-4x+4+y^2-2y+1+z^2+6z+9=25$

$\Rightarrow x^2+y^2-4x-2y+6z-11=0$

$\therefore$ The equation to the locus of a point P is $x^2+y^2-4x-2y+6z-11=0$

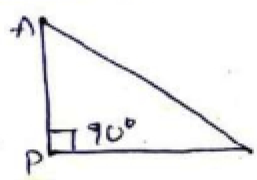
→ Find the locus of the point if the join of the points $(-3,2,4)$, $(2,1,-2)$ subtends a right angle at P.

Sol'n:- Let $A(-3,2,4)$, $B(2,1,-2)$

Let $P(x,y,z)$ be a locus of the point

Given Condition is $\angle APB=90^\circ$

$\Rightarrow PA^2+PB^2=AB^2$



$\Rightarrow (x+3)^2+(y-2)^2+(z-4)^2+(x-2)^2+(y-1)^2+(z+2)^2=(-3-2)^2+(2-1)^2+(4+2)^2$

$\Rightarrow x^2+9+6x+y^2+4-4y+z^2+16-8z+x^2+4-4x+y^2+1-2y+z^2+4+4z = 25+1+36$

$\Rightarrow 2x^2+2y^2+2z^2+2x-6y-4z-24=0$

$\Rightarrow x^2+y^2+z^2+x-3y-2z-12=0$

$\therefore$ The equation to the locus of P is

$x^2+y^2+z^2+x-3y-2z-12=0$

→ Find the locus of the point P s.t PA + PB = 8 where A (1, 2, 3), B (1, -2, 2)

Sol'n: Let P(x, y, z) be the locus of the point

Given PA + PB = 8

$\Rightarrow \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}+\sqrt{(x-1)^2+(y+3)^2+(z-2)^2}=8$

$\Rightarrow (x-1)^2+(y-2)^2+(z-3)^2 = 64+(x-1)^2+(y+2)^2+(z-2)^2-16\sqrt{(x-1)^2+(y+2)^2+(z-2)^2}$

$\Rightarrow y^2+4-4y+z^2+9-6z = 64+y^2+4+4y+z^2+4-4z-16\sqrt{x^2+1-2x+y^2+4+4y+z^2+4-4z}$

$\Rightarrow -8y-2z-59 = -16\sqrt{x^2+y^2+z^2-2x+4y-4z+9}$

$\Rightarrow 64y^2+4z^2+3481+32yz+236z+944y = 256(x^2+y^2+z^2-2x+4y-4z+9)$

$\Rightarrow 256x^2+192y^2+252z^2-512x+8y-1260z-1177=0$

$\therefore$ The eqn to the locus of P is

$256x^2+192y^2+252z^2-512x+8y-1260z-1177=0$

→ If A(-1, 0, 0), B(0, 1, 0) Find the locus of P $3(PA)^2=2(PB)^2$.

→ If A and B are the points (3, 4, 5) and (-1, 3, -7) then find the locus of P which moves so that $PA^2-PB^2=3$.

→ Find the locus of a point P which is at a distance r from the point (a, b, c).

→ Find the locus of the point the difference of whose distances from (2, 0, 0) and (-2, 0, 0) is 1.

→ Find the locus of point P(x, y, z), the difference of whose distances from (0, 0, -4) and (0, 0, 4) is 4.

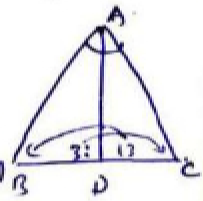
→ Find the locus of the point the sum of whose distances from (4, 0, 0) and (-4, 0, 0) is equal to 10.

→ Find the locus of the point P(x, y, z) if $PA^2+PB^2=2k^2$, where A and B are the points (3, 4, 5) and (-1, 3, -7).

→ $A(3,2,0)$, $B(5,3,2)$, $C(-9,6,-3)$ are three points forming a triangle. AD, the bisector of the angle BAC, meets BC in D. Find the co-ordinates of the point D.

Sol

Since AD is the internal bisector of $\angle BAC$.



$\therefore \frac{BD}{DC} = \frac{AB}{AC}$ (elementary geometry) —(1)

Now $AB = \sqrt{(5-3)^2+(3-2)^2+(2-0)^2}$
$= \sqrt{4+1+4} = 3.$

$AC = \sqrt{(-9-3)^2+(6-2)^2+(-3-0)^2}$
$= \sqrt{144+16+9} = 13.$

$\therefore$ from (1), $\frac{BD}{DC} = \frac{AB}{AC} = \frac{3}{13}$.

$\therefore$ the co-ordinates of the point D are

$\left(\frac{3(-9)+13(5)}{3+13}, \frac{3(6)+13(3)}{3+13}, \frac{3(-3)+13(2)}{3+13}\right)$

$= \left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right).$

→ Find the ratios in which the join of the points $(3,2,1)$ $(1,3,2)$ is divided by the locus of $3x^2-72y^2+128z^2=3$.

Sol Let P and Q denote the given points $(3,2,1)$ and $(1,3,2)$. Let the given locus meet PQ in R, and let R divide PQ in the ratio $k:1$.

Then the co-ordinates of R are $\left(\frac{k+3}{k+1}, \frac{3k+2}{k+1}, \frac{2k+1}{k+1}\right)$.

Since this point 'R' lies on $3x^2-72y^2+128z^2=3$. ⑦

$\therefore 3\left(\frac{k+3}{k+1}\right)^2 - 72\left(\frac{3k+2}{k+1}\right)^2 + 128\left(\frac{2k+1}{k+1}\right)^2 = 3.$

$\Rightarrow 3(k+3)^2 - 72(3k+2)^2 + 128(2k+1)^2 = 3(k+1)^2$

$\Rightarrow 3(k^2+6k+9) - 72(9k^2+12k+4) + 128(4k^2+4k+1) = 3(k^2+2k+1).$

$\Rightarrow 176k^2 + 340k + 136 = 0.$

$\Rightarrow 2k^2+5k+2=0.$

$\Rightarrow (2k+1)(k+2)=0$

$\therefore$ either $2k+1=0 \Rightarrow k=-\frac{1}{2}$
or $k+2=0 \Rightarrow k=-2.$

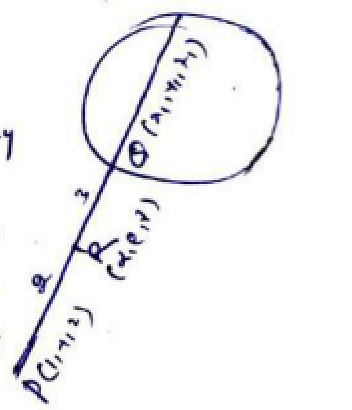
$\therefore$ The line PQ is divided by the given surface in the ratio $-1:2$ and $-2:1$.

→ From the point $(1,-1,2)$ lines are drawn to meet the sphere $x^2+y^2+z^2=1$ and they are divided in the ratio $2:3$. Prove that the points of section lie on the surface

$5x^2+5y^2+5z^2-6x+6y-12z+10=0.$

Sol

Let PRQ be any line through P which meets the sphere $x^2+y^2+z^2=1$ in $Q(x_1,y_1,z_1)$ say



$\therefore x_1^2+y_1^2+z_1^2=1$ —(1)

Let $R(\alpha,\beta,\gamma)$ divide PQ in the ratio $2:3$

Then $\alpha = \frac{2x_1+3}{2+3} \Rightarrow x_1 = \frac{5\alpha-3}{2}$

$\beta = \frac{2y_1 - 3}{5} \Rightarrow y_1 = \frac{5\beta + 3}{2}$

$\gamma = \frac{2z_1 + 6}{5} \Rightarrow z_1 = \frac{5\gamma - 6}{2}$.

$\therefore$ putting these values of x_1, y_1, z_1 in ①, we have

$\left(\frac{5\alpha - 3}{2}\right)^2 + \left(\frac{5\beta + 3}{2}\right)^2 + \left(\frac{5\gamma - 6}{2}\right)^2 = 1$

$\Rightarrow 25\alpha^2 + 25\beta^2 + 25\gamma^2 - 30\alpha + 30\beta - 60\gamma + 50 = 0$

$\Rightarrow 5\alpha^2 + 5\beta^2 + 5\gamma^2 - 6\alpha + 6\beta - 12\gamma + 10 = 0$

Hence the locus of $R(\alpha, \beta, \gamma)$ is

$5x^2 + 5y^2 + 5z^2 - 6x + 6y - 12z + 10 = 0$

(By changing α, β, γ to x, y, z respectively)

which is the reqd result.

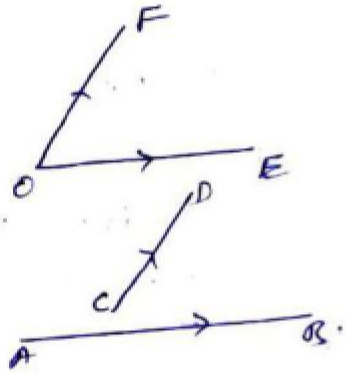
→ From the point $(1, -2, 3)$, lines are drawn to meet the sphere $x^2 + y^2 + z^2 = 4$ and they are divided in the ratio $2:3$. Prove that the points of section lie on the surface.

$5x^2 + 5y^2 + 5z^2 - 6x + 12y - 18z + 22 = 0$

* DIRECTION COSINES AND DIRECTION RATIOS OF A LINE *

* We, now define the angle b/w two non-coplanar or skew lines as below:

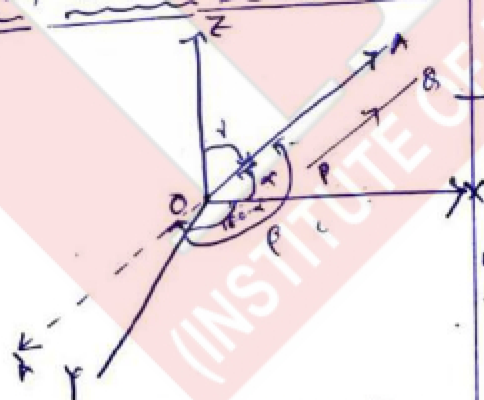
1) Defn



Let AB and CD be two non-coplanar and non-intersecting lines. Take a point 'O' and through 'O' draw two lines OE and OF parallel to AB and CD respectively.

Then the angle b/w AB and CD is equal to the angle EOF.

* Direction cosines of a line:



If α, β, γ are the angles that a given line $\overrightarrow{PQ}$ makes with the positive directions x, y and z axes then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines (or d.c's) of the line PQ.

Generally the directions cosines are represented by l, m, n.

i.e $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Note:-

① The angles b/w a given line $\overline{PQ}$ and the co-ordinate axes are angles which the line drawn through the origin parallel to the the given line makes with the axes.

② The angles α, β, γ are called the direction angles of the line $\overline{PQ}$.

③ The direction cosines of the $\overrightarrow{QP}$ are $\cos(\pi-\alpha)$, $\cos(\pi-\beta)$ and $\cos(\pi-\gamma)$

i.e $-\cos\alpha, -\cos\beta, -\cos\gamma$.

i.e $-l, -m, -n$.

Since the direction angles of the line $\overrightarrow{OA'}$ through 'O' parallel to $\overrightarrow{QP}$ are $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$.

④ Here the angles α, β, γ are not coplanar.

→ Direction cosine's of the co-ordinate axes:

Since the line x-axis makes with x-axis, y-axis and z-axis the angles $0°, 90°$ and $90°$ respectively.

∴ Its direction cosines are $\cos 0°, \cos 90°, \cos 90°$

$\Rightarrow 1, 0, 0.$

Thus the d.c's of x-axis are 1,0,0

similarly d.c's of y-axis are 0,1,0

and d.c's of z-axis are 0,0,1

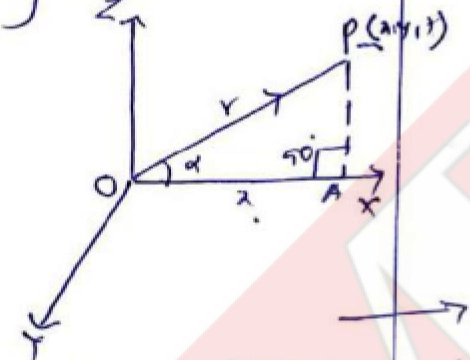
A useful result :-

If l, m, n be the d.c's of line OP and $OP = r$, then the co-ordinates of P are (lr, mr, nr)

Sol Let (x, y, z) be the co-ordinates of P and A, the foot of $\perp$ from P on x-axis.

then by definition

$x = OA$ —①



Now in the rt. $\angle$d ΔOAP

$$\frac{OA}{OP} = \cos\alpha$$

$\therefore OA = OP \cos\alpha$

$OA = r\cos\alpha$

$\therefore x = r\cos\alpha$ (from ①)

$\boxed{x = rl}$ —②

Similarly $\boxed{y = mr}$ —③, $\boxed{z = nr}$ —④

Thus the co-ordinates of P are (lr, mr, nr).

→ It follows from above that if (x, y, z) be the co-ordinates of a point P, such that $OP = r$ then the d.c's of OP are l, m, n

i.e $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

(from ②, ③ & ④)

* Relation b/w direction cosines :-

If l, m, n are the d.c's of a line, then $l^2+m^2+n^2=1$

Sol Let $OP = r$ & $P(x, y, z)$

then $x = lr$; $y = mr$; $z = nr$

squaring and adding, we get

$x^2+y^2+z^2 = r^2(l^2+m^2+n^2)$

$\Rightarrow r^2 = r^2(l^2+m^2+n^2)$.

$\Rightarrow l^2+m^2+n^2 = 1$.

→ If α, β, γ be angles which a line makes with the axes, then $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Since $l^2+m^2+n^2 = 1$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1-\sin^2\alpha) + (1-\sin^2\beta) + (1-\sin^2\gamma) = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3-1$

$$\boxed{\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2}$$

→ A line make angles of 45° and 60° with the positive axes of x and y respectively. What angle does it make with the positive axis of z?

Sol Let the line makes an angle γ with the +ve z-axis.

then since the line makes angles 45°, 60° and γ with the axes,

$\therefore$ d.c's are $\cos 45^\circ, \cos 60^\circ, \cos\gamma$

$\Rightarrow \frac{1}{\sqrt{2}}, \frac{1}{2}, \cos\gamma$.

But we know that

$l^2+m^2+n^2=1$

$\Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1$

$\Rightarrow \cos^2\gamma = 1-3/4$
$= 1/4$

$\therefore \boxed{\cos\gamma = \pm\frac{1}{2}}$

$\Rightarrow \gamma = 60^\circ$ or 120°.

$\rightarrow$ What are the d.c's of the line equally inclined to the axes?

(Or)

Find the direction cosines of a line that makes equal angles with the axes.

Sol Let l, m, n be the d.c's of a line.

given that line makes equal angles with the axes.

So we have

$l=m=n$.

(i.e $\cos\alpha = \cos\beta = \cos\gamma$.
i.e $\alpha=\beta=\gamma$)

But w.k.T $l^2+m^2+n^2=1$

$\Rightarrow l^2+l^2+l^2=1$

$\Rightarrow 3l^2=1$

$\Rightarrow l^2=\frac{1}{3}$

$\Rightarrow l=\pm\frac{1}{\sqrt{3}}$

9

$\therefore$ The reqd d.c's are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

(OR)

If a line makes angles α, β, γ with the axes, we have here

$\alpha=\beta=\gamma$

$\therefore \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow l=m=n$.

Since $l^2+m^2+n^2=1$

$\therefore l^2+l^2+l^2=1$

$\Rightarrow 3l^2=1$

$\Rightarrow l^2=1/3$

$\Rightarrow \boxed{l=\pm\frac{1}{\sqrt{3}}}$

Hence the d.c's of a line are

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Note:- Here four such lines are possible and their d.c's are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; + + +

$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; − + +

$\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ + − +

and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ + + −

$\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}$ + − −

$-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}$ − − +

$-\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, -\frac{1}{\sqrt3}$ − + −

$-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}$ − − −

or

$-\frac{1}{\sqrt3}, +\frac{1}{\sqrt3}, +\frac{1}{\sqrt3}$ − + +

$+\frac{1}{\sqrt3}, +\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}$ + + −

$\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}$ + − +

$\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}$ + + +

as l, m, n and $-l, -m, -n$ are the d.c's of the same line.

→ If a straight line makes angle of $\frac{\pi}{4}$ with each of x-axis and y-axis, then what angle does it make with the z-axis?

Sol Let the line make angle γ with the z-axis.

Then since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$ $\quad (\because \alpha = \beta = \frac{\pi}{4})$

$\Rightarrow \frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 0.$

$\Rightarrow \boxed{\gamma = 90^\circ}$

∴ the line makes an angle of $\pi/2$ with the z-axis.

* **Direction ratios of a line:-**

A set of three numbers a, b, c, which are proportional to the direction cosines l, m, n respectively of a line are called the direction ratio's of a line.

Note:- Sum of squares of the direction cosines of a line is equal to one whereas the sum of squares of the direction ratios of a line is not equal to one.

* **Relation b/w direction cosines and direction ratios:-**

We have $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say)

Then $l = ka, m = kb, n = kc$.

But $l^2 + m^2 + n^2 = 1$

$\Rightarrow k^2(a^2 + b^2 + c^2) = 1$

$\Rightarrow k^2 = \frac{1}{a^2+b^2+c^2}$

$\Rightarrow K = \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$.

$\therefore l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}$

$m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}$

$n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$.

where the same sign either +ve or −ve is to be taken throughout.

i.e. If AB be the line whose d.r's are a, b, c then the d.c's of AB are given by +ve sign and those of the line BA by − sign.

Rule:- If a, b, c are d.r's of a given line, then to find actual d.c's of this line, divide each of a, b, c by $\sqrt{(a^2+b^2+c^2)}$.

Note:- ① d.c's of a line are unique, but d.r's of a line are not unique.

If ⟨a, b, c⟩ are d.r's of a line then ⟨[redacted], kb, kc⟩ are also d.r's of the same line (if $K \neq 0$)

② If a, b, c are d.r's of a line then $a^2+b^2+c^2 \neq 1$

→ The d.r's of a line are 6, 2, 3. find the d.c's..

Sol. The d.r's of a line are 6, 2, 3.

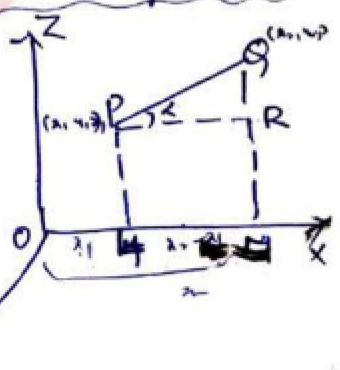
$\therefore$ D.c's are

$\frac{6}{\sqrt{6^2+2^2+3^2}}, \frac{2}{\sqrt{6^2+2^2+3^2}}, \frac{3}{\sqrt{6^2+2^2+3^2}}$

$\Rightarrow \frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

H.W. 1, −2, −2 are d.r's of a line. what are its d.c's.?

* Direction ratios of a line joining two points:-



Let $P(x_1, y_1, z_1)$

$Q(x_2, y_2, z_2)$ be two points.

Join P, Q.

and let l, m, n be the d.c's of the line PQ.

Then if α, β, γ be the angles which the line PQ makes with the axes.

$\therefore l = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

Draw PL, QM ⊥s on x-axis

Draw PR ⊥ QM s.t

$\angle RPQ = \alpha$

$\therefore PR = LM = OM - OL$

$= x_2 - x_1$.

Now from the rt. ∠d ΔPQR,

$\cos\alpha = l = \frac{PR}{PQ}$

$= \frac{x_2-x_1}{PQ}$

similarly $m = \frac{y_2-y_1}{PQ}$

$n = \frac{z_2-z_1}{PQ}$.

Hence the d.c's of PQ are

$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$

where $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

the d.c's are proportional to $x_2-x_1, y_2-y_1, z_2-z_1$.

i.e d.r's of the line are $x_2-x_1, y_2-y_1, z_2-z_1$.

Note:- To be remember:

The d.c's of the line joining two points are proportional to (i.e d.r's are) $x_2-x_1, y_2-y_1, z_2-z_1$.

→ Find the d.c's of the line joining the points $P(4,3,-5)$ and $Q(-2,1,-8)$

Sol Now the d.c's PQ are proportional to

$-2-4, 1-3, -8-(-5)$ ($\because x_2-x_1, y_2-y_1, z_2-z_1$)

$\Rightarrow -6, -2, -3$

or : $6, 2, 3$.

∴ The actual d.c's are (dividing the dr's by $\sqrt{6^2+2^2+3^2} = \sqrt{36+4+9} = 7$)

$\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

HW → The d.r's of a line are 2, 3, 4. What are its direction cosines?

Ans: $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$

* **Lagrange's Identity:**

$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1l_2+m_1m_2+n_1n_2)^2 = (l_1m_2-m_1l_2)^2+(m_1n_2-n_1m_2)^2 + (n_1l_2-l_1n_2)^2$.

proof LHS $=$

$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1l_2+m_1m_2+n_1n_2)^2$

$= l_1^2l_2^2+l_1^2m_2^2+l_1^2n_2^2 + m_1^2l_2^2+m_1^2m_2^2+m_1^2n_2^2 + n_1^2l_2^2+n_1^2m_2^2+n_1^2n_2^2 - (l_1^2l_2^2+m_1^2m_2^2+n_1^2n_2^2 + 2l_1l_2m_1m_2 + 2l_1l_2n_1n_2 + 2m_1m_2n_1n_2)$.

$= (l_1^2m_2^2+m_1^2l_2^2-2l_1l_2m_1m_2) + (m_1^2n_2^2+m_2^2n_1^2-2m_1m_2n_1n_2) + (n_1^2l_2^2+l_1^2n_2^2-2l_1l_2n_1n_2)$

$= (l_1m_2-m_1l_2)^2 + (m_1n_2-n_1m_2)^2 + (n_1l_2-l_1n_2)^2$

$=$ RHS.

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* condition for $\perp$ and $\parallel$:

→ If the lines are $\perp$ then $\theta = 90^\circ$.

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$

→ If the lines are $\parallel$ then $l_1 = l_2$; $m_1 = m_2$, $n_1 = n_2$

$$\Rightarrow \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \boxed{\frac{a_1}{a_2} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}}.$$

Similarly $\dfrac{b_1}{b_2} = \dfrac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}$

and $\dfrac{c_1}{c_2} = \dfrac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Note:-

$$\sin\theta = \frac{\sqrt{\sum (a_1b_2-b_1a_2)^2}}{\sqrt{\sum a_1^2}\cdot\sqrt{\sum a_2^2}}.$$

$$\tan\theta = \frac{\sqrt{\sum (a_1b_2-a_2b_1)^2}}{\sum a_1a_2}$$

→ Find the angle b/w the lines whose direction ratio's are (2, 3, 4) and (1, −2, 1).

Sol:- Let the required angle be θ, then

$$\cos\theta = \frac{(2)(1)+(3)(-2)+(4)(1)}{\sqrt{2^2+3^2+4^2}\sqrt{1^2+(-2)^2+1^2}}$$

$$= \frac{2-6+4}{\sqrt{29}\,\sqrt{6}} = 0$$

$$= \cos\pi/2$$

$$\therefore \theta = \pi/2$$

Hence the given lines are at right angles.

→ If points P, Q are (2, 3, −6) and (3, −4, 5), find the angle that OP makes with OQ.

Sol^n:- direction ratio's of OP are

$2-0,\ 3-0,\ -6-0$

i.e. $2, 3, -6$

direction ratio's of OQ are

$3-0,\ -4-0,\ 5-0$

i.e., $3, -4, 5$

Let the angle b/w OP and OQ be θ. Then

$$\cos\theta = \frac{(2)(3)+(3)(-4)+(-6)(5)}{\sqrt{2^2+3^2+(-6)^2}\sqrt{3^2+(-4)^2+5^2}}$$

$$= \frac{6-12-30}{\sqrt{49}\,\sqrt{50}} = \frac{-36}{35\sqrt{2}}$$

$$= \frac{-18\sqrt{2}}{35}$$

$\therefore$ Angle b/w the lines OP and OQ is $\theta = \cos^{-1}\left(\dfrac{-18\sqrt{2}}{35}\right)$

→ If A, B, C, D are the points (3, 4, 5), (4, 6, 3), (−1, 2, 4) and (1, 0, 5), find the angle between CD and AB.

Sol^n: direction ratio's of CD are $1+1;\ 0-2,\ 5-4$

i.e. $2, -2, 1$

direction ratio's of AB are

$4-3,\ 6-4,\ 3-5$

i.e. $1, 2, -2$

Let the angle between CD and AB be θ. Then

$$\cos\theta = \frac{(2)(1)+(-2)(2)+(1)(-2)}{\sqrt{2^2+(-2)^2+3^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$= \frac{2-4-2}{3.3} = \frac{-4}{9}$$

Hence the acute angle between AB and CD is given by

$\cos^{-1}\left(\frac{4}{9}\right)$

→ If A, B, C are (−1, 3, 2), (2, 3, 5) and (1, 2, −7), find the angles of the triangle ABC.

Sol: The dr's of the line AB are $2-(-1),\ 3-3,\ 5-2$

i.e $3, 0, 3$.

The d.c's of line AB are

$$\frac{3}{\sqrt{3^2+0^2+3^2}}, \frac{0}{\sqrt{3^2+0^2+3^2}}, \frac{3}{\sqrt{3^2+0^2+3^2}}$$

$$\Rightarrow \frac{3}{3\sqrt{2}}, \frac{0}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}$$

i.e $\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

Similarly d.c's of the BC and AC are

$\frac{1}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}$ and

$\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$ respectively

∴ $\cos A$ = cosine of the angle b/w AB and AC

$= l_1l_2+m_1m_2+n_1n_2$

$= \frac{1}{\sqrt{2}}\cdot\frac{2}{3} + 0\cdot\frac{1}{3} + \frac{1}{\sqrt{2}}\left(\frac{-2}{3}\right)$

$= 0$.

∴ $\angle A = 90^\circ$

Now $\cos B$ = cosine of the angle b/w BC and BA

$= \frac{1}{3\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{2}{3\sqrt{6}}(0) - \frac{7}{3\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right)$

$= \frac{6}{3\sqrt{6}\cdot\sqrt{2}} = \frac{1}{\sqrt{3}}$.

∴ $\angle B = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Similarly we find that $\angle C = \cos^{-1}\left(\frac{1}{3}\right)$.

→ Find the angles of triangle ABC whose vertices A, B and C are the points (1, 3, 5), (−1, 2, 3) and (3, 4, −2)

Note:-

W.K.T angle b/w two lines

$\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$

If on putting the values of l_1, m_1, n_1 and l_2, m_2, n_2 we get the value of $\cos\theta$ as negative, then θ gives the obtuse angle between the two lines. The angle required is generally acute angle which in this case will be $180^\circ - \theta$. But this can also be obtained directly by taking the numerical values of $\cos\theta$ (if it is $-ve$).

→ Find the acute angle between the lines whose direction cosines are proportional to $(2,3,-6)$ and $(3,-4,5)$.

Sol^n: If θ be the angle between the given lines, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{(2)(3)+3(-4)+(-6)(5)}{\sqrt{2^2+3^2+(-6)^2}\sqrt{3^2+(-4)^2+5^2}}$$

$$= \frac{-36}{35\sqrt{2}}$$

= negative Hence θ is obtuse angle

∴ Required acute angle $= \pi - \theta$, where $\cos\theta = (-36/35\sqrt{2})$

H.W → Find the acute angle between lines CD and AB where A, B, C, D are the points $(3,4,5)$, $(4,6,3)$, $(-1,2,4)$ and $(1,0,5)$.

→ Find the d.c's l, m, n of two lines which are connected by the relation $l-5m+3n=0$ and $7l^2+5m^2-3n^2=0$

Sol Given that

$l-5m+3n=0$ —— (1)

$7l^2+5m^2-3n^2=0$ —— (2)

(1) $\Rightarrow l = 5m - 3n$ —— (3)

substituting this in (2), we get

$7(5m-3n)^2+5m^2-3n^2=0$

$\Rightarrow 180m^2-210mn+60n^2=0$

$\Rightarrow 6m^2-7mn+2n^2=0$

$\Rightarrow (3m-2n)(2m-n)=0$

$\Rightarrow 3m-2n=0 \quad | \quad 2m-n=0$

$\Rightarrow \frac{m}{n}=\frac{2}{3} \qquad \frac{m}{n}=\frac{1}{2}$

If $\frac{m}{n}=\frac{2}{3}$ then $\frac{m}{2}=\frac{n}{3}$

$\frac{5}{5}\left(\frac{m}{2}\right)=\frac{3}{3}\left(\frac{n}{3}\right)$

$\Rightarrow \frac{5m-3n}{5\cdot2-3\cdot3}=\frac{l}{1}$ (from (3))

$\therefore \frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+2^2+3^2}}=\frac{1}{\sqrt{14}}$

∴ The d.c's of one line are

$l=\frac{1}{\sqrt{14}}, m=\frac{2}{\sqrt{14}}, n=\frac{3}{\sqrt{14}}$

If $\frac{m}{n} = \frac{1}{2}$ then we have

$$\frac{m}{1} = \frac{n}{2} = \frac{5m-3n}{5\cdot 1 - 3\cdot 2} = \frac{l}{-1}$$
(from (3))

$$\Rightarrow \frac{m}{1} = \frac{n}{2} = \frac{l}{-1}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+2^2+(-1)^2}}$$

$$= \frac{1}{\sqrt{6}}$$

∴ The d.c's of the other line are $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

→ Show that the straight lines whose direction cosines are given by the eqns: $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$ are (i) perpendicular if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$.

(ii) parallel if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0$

Sol: The d.c's of the lines are given by

$ul+vm+wn=0$ and — (1)

$al^2+bm^2+cn^2=0$. — (2)

Now eliminating n b/w (1) & (2) we get

$$al^2+bm^2+c\left[\frac{-(ul+vm)}{w}\right]^2=0$$

$$\Rightarrow (aw^2+cu^2)l^2+(bw^2+cv^2)m^2+2cuvlm=0$$

$$\Rightarrow (aw^2+cu^2)\left(\frac{l}{m}\right)^2+2cuv\left(\frac{l}{m}\right)+(bw^2+cv^2)=0$$
— (3)

(i) Its roots are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

If the d.c's of the two lines be taken as (l_1,m_1,n_1) and (l_2,m_2,n_2)

∴ from (3), we have

$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}$ = product of the roots

$= \frac{c}{a}$ (formula)

$$= \frac{bw^2+cv^2}{aw^2+cu^2}.$$

$$\therefore \frac{l_1l_2}{m_1m_2} = \frac{bw^2+cv^2}{aw^2+cu^2}$$

$$\Rightarrow \frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{cu^2+aw^2} = \frac{n_1n_2}{av^2+bu^2}$$
(by symmetry)

∴ If the two lines are perpendicular, then we have

$l_1l_2+m_1m_2+n_1n_2=0$

$$\Rightarrow (bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0.$$

$$\Rightarrow u^2(b+c)+v^2(c+a)+w^2(a+b)=0.$$

(ii) If the two lines are parallel, then their d.c's are equal ~~and~~ consequently the roots of (3) are equal, the condition for the same being "$b^2=4ac$" i.e.

$(2cuw)^2 = 4(aw^2+cu^2)(bw^2+cv^2)$

(or) $c^2u^2v^2 = abw^4 + acw^2v^2 + bcu^2w^2 + c^2u^2v^2$

(or) $abw^4 + acw^2v^2 + bcu^2w^2 = 0$ (or)

$abw^2 + acv^2 + bcu^2 = 0$

(or) $\frac{w^2}{c} + \frac{v^2}{b} + \frac{u^2}{a} = 0$, dividing each term by abc.

→ Find the angle between the lines whose d.c's (l,m,n) satisfy the equations $l+m+n=0$ and $2lm+2nl-mn=0$

Sol'n: Eliminating n between the given relations, we get

$2lm + 2(-l-m)l - m(-l-m) = 0$

(or) $2l^2 - lm - m^2 = 0$ (or)

$(2l+m)(l-m) = 0$ —— (i)

If $2l+m=0$, then from $l+m+n=0$, we get $n=l$

$\therefore$ we have $2l = -m = 2n$

$\Rightarrow \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$.

$\therefore$ The direction cosines of one line are

$\frac{1}{\sqrt{[1^2+(-2)^2+1^2]}}, \frac{-2}{\sqrt{[1^2+(-2)^2+1^2]}}, \frac{1}{\sqrt{[1^2+(-2)^2+1^2]}}$

$\Rightarrow \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}, \frac{1}{\sqrt6}$

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If $l-m=0$, then from $l+m+n=0$, we get $n=-2m$

$\therefore$ we have $l=m=\frac{n}{-2}$

$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$ —— (ii)

$\therefore$ The direction cosines of the second line are

$\frac{1}{\sqrt{[1^2+1^2+(-2)^2]}}, \frac{1}{\sqrt{[1^2+1^2+(-2)^2]}}, \frac{-2}{\sqrt{[1^2+1^2+(-2)^2]}}$

$\Rightarrow \frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}$

$\therefore$ If θ be the required angle then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$= \frac{1}{\sqrt6}\cdot\frac{1}{\sqrt6} + \left(\frac{-2}{\sqrt6}\right)\frac{1}{\sqrt6} + \frac{1}{\sqrt6}\left(\frac{-2}{\sqrt6}\right)$

$= \frac{-3}{6}$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = 120°$

→ Prove that the acute angle between the lines whose direction cosines are given by the relations $l+m+n=0$ and $l^2+m^2-n^2=0$ is $\pi/3$.

Sol'n: Eliminating 'n' between the given relations, we get

$l^2+m^2-(-l-m)^2=0$ (or) $lm=0$

$\therefore$ Either $l=0$ (or) $m=0$

If $l=0$, then from $l+m+n=0$

we get $m=-n$

$\therefore$ we have, $m=-n$ (or) $\frac{l}{0}=\frac{m}{-1}=\frac{n}{1}$

$\therefore$ Direction cosines of one line are

$$\frac{0}{[0^2+(-1)^2+1^2]}=\frac{(-1)}{\sqrt{[0^2+(-1)^2+1^2]}}=\frac{1}{\sqrt{[0^2+(-1)^2+1^2]}}$$

$\Rightarrow \; 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

If $m=0$, then from $l+m+n=0$

we get $l=-n$

$\therefore$ we have $\frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$

$\therefore$ Direction cosines of the second line as above are $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$

$\therefore$ If θ be the angle between these two lines, then

$$\cos\theta = l_1l_2+m_1m_2+n_1n_2$$

$$= 0.\frac{1}{\sqrt{2}}+\left(-\frac{1}{\sqrt{2}}\right).0+\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{2}$$

$\therefore \theta = 120^\circ$

$\therefore$ Required acute angle between the lines $= 180^\circ-120^\circ$

$= 60^\circ$

i.e. $\pi/3$ proved.

→ Find the angle between the two lines whose d.c's (l, m, n) satisfy the equations $l+m+n=0$ and $l^2+m^2+n^2=0$

Sol'n :- Eliminating n between the given relations, we get

$l^2+m^2+(-l-m)^2=0$ (or)

$$l^2+m^2+lm=0$$

$\Rightarrow (l/m)^2+(l/m)+1=0$, dividing each term by m^2

Let its roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$, then

$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}$ = Product of the roots $=\frac{1}{1}$

$\Rightarrow \frac{l_1l_2}{1}=\frac{m_1m_2}{-1}$ —— (i)

Again eliminating m between the given relations we get as above

$$l^2+ln+n^2=0$$

$\Rightarrow (l/n)^2+\left(\frac{l}{n}\right)+1=0$ dividing each term by n^2

If its roots be $\frac{l_1}{n_1}$ and $\frac{l_2}{n_2}$

Then $\frac{l_1}{n_1}\cdot\frac{l_2}{n_2}$ = Product of the roots $=-\frac{1}{1}$

$\Rightarrow \frac{l_1l_2}{1}=\frac{n_1n_2}{-1}$ —— (ii)

$\therefore$ from (i) and (ii) we get

$$\frac{l_1l_2}{1}=\frac{m_1m_2}{-1}=\frac{n_1n_2}{-1}$$

$\therefore$ If θ be the required angle, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$\Rightarrow \cos\theta = 1-1-1$

$\Rightarrow \quad = -1$

$\Rightarrow \quad \theta = \pi$

→ Prove that the straight lines whose direction cosines are given by relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are

(i) perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and

(ii) parallel if $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$

Sol'n: (i) Let the d.c's of the two lines be (l_1, m_1, n_1) and (l_2, m_2, n_2)

Eliminating 'n' between the given relations we get

$$-fm\left[-(al+bm)/c\right] + gl\left[-(al+bm)/c\right] + hlm = 0$$

$\Rightarrow -afml - bfm^2 - agl^2 - bglm + chlm = 0$

$\Rightarrow ag(l/m)^2 + (af+bg-ch)\left(\frac{l}{m}\right) + bf = 0$ —(1)

dividing each term by m^2

Its roots are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

$\therefore \frac{l_1}{m_1}\cdot\frac{l_2}{m_2}$ = Product of roots $\frac{bf}{ag}$

$\Rightarrow \frac{l_1l_2}{bf} = \frac{m_1m_2}{ag}$

$\Rightarrow \frac{l_1l_2}{(f/a)} = \frac{m_1m_2}{(g/b)} = \frac{n_1n_2}{(h/c)}$, by symmetry

If the lines are perpendicular, then

$l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow -f/a + (g/b) + (h/c) = 0$

Hence proved.

(ii) If the lines are parallel, then their d.c's must be the same i.e. the roots of (1) must be equal the condition for the same is

$b^2 = 4ac$

i.e. $(af+bg-ch)^2 = 4ag.bf$ —(2)

$\Rightarrow af+bg-ch = \pm 2\sqrt{(af)}\sqrt{(bg)}$

$\Rightarrow af+bg \pm 2\sqrt{(af)}\sqrt{(bg)} = ch$

$\Rightarrow \left[\sqrt{(af)} \pm \sqrt{(bg)}\right]^2 = ch = \left[\sqrt{(ch)}\right]^2$

$\Rightarrow \sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$ is the required condition.

Also from (2), we get

$a^2f^2+b^2g^2+c^2h^2+2abfg-2acfh-2bcgh = 4abfg$

$\Rightarrow a^2f^2+b^2g^2+c^2h^2-2bcgh-2cafh-2abfg=0$

→ Show that the lines whose direction cosines are given by $l+m+n=0$, $2mn+3ln-5lm=0$ are perpendicular to one another.

Sol'n: Given $l+m+n=0$ —(i)

$2mn+3ln-5lm=0$ —(ii)

Eliminating n between (i) and (ii), we have

$2m(-l-m)+3l(-l-m)-5lm=0$

$\Rightarrow 3l^2+10lm+2m^2=0$ (or)

$3(l/m)^2+10(l/m)+2=0$

$\Rightarrow \frac{l}{m}=\frac{-10\pm\sqrt{100-24}}{6}$

$\Rightarrow \frac{l}{m}=\frac{-5\pm\sqrt{19}}{3}$ ——(iii)

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of two lines, Then from (iii) we have

$\frac{l_1}{m_1}=\frac{-5+\sqrt{19}}{3}$ and $\frac{l_2}{m_2}=\frac{-5-\sqrt{19}}{3}$

If $\frac{l_1}{m_1}=\frac{\sqrt{19}-5}{3}$; then $\frac{l_1}{\sqrt{19}-5}=\frac{m_1}{3}=K_1$ (say)

$\therefore l_1=K_1[\sqrt{19}-5]$; $m_1=3K_1$

Also from (i) $n_1=-l_1-m_1=K_1(5-\sqrt{19})-3K_1$

$=K_1(2-\sqrt{19})$

$\therefore \frac{l_1}{\sqrt{19}-5}=\frac{m_1}{3}=\frac{n_1}{2-\sqrt{19}}$ ——(iv)

similarly taking

$\frac{l_2}{m_2}=\frac{-5-\sqrt{19}}{3}$ we can find

that $\frac{l_2}{-\sqrt{19}-5}=\frac{m_2}{3}=\frac{n_2}{2+\sqrt{19}}$ ——(v)

From (iv) and (v), we have the direction ratios of two lines and if they are at right angles then we must have

$a_1a_2+b_1b_2+c_1c_2=0$

Here $a_1a_2+b_1b_2+c_1c_2=[\sqrt{19}-5][-\sqrt{19}-5]+(3\times3)+[2-\sqrt{19}][2+\sqrt{19}]$

$=(-5)^2-19+9+(2)^2-19$

$=38-38=0$

Hence the two lines are at right angles. Hence proved.

→ Show that the lines whose direction cosines are given by the equations $2l+2m-n$ and $mn+nl+lm=0$ are at right angles.

Sol^n:- Eliminating n between given equations we get

$m(2l+2m)+(2l+2m)l+lm=0$ (or)

$2l^2+2m^2+5lm=0$

$\Rightarrow 2(l/m)^2+5(l/m)+2=0$, dividing each term by m^2

$\Rightarrow \frac{l}{m}=\frac{-5\pm\sqrt{(5)^2-4(2)(2)}}{2(2)}$

$=\frac{-5\pm3}{4}=-\frac{1}{2},-2$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c's of the two lines, then we have $l_1/m_1=-\frac{1}{2}$, $l_2/m_2=-2$ ——(ii)

If $\frac{l_1}{m_1}=\frac{-1}{2}$, then $\frac{l_1}{-1}=\frac{m_1}{2}=K_1$ (say)

$\Rightarrow l_1=-K_1$, $m_1=2K_1$

Also from $n=2l+2m$ we have $n_1=2l_1+2m_1$

$\Rightarrow n_1=2(-K_1)+2(2K_1)=2K_1$

$\therefore$ we have

$l_1=-K_1, m_1=2K_1, n_1=2K_1$ ——(iii)

Again from (i)

$$\frac{l_2}{m_2} = \frac{-2}{1} \quad (\text{or}) \quad \frac{l_2}{-2} = \frac{m_2}{1} = k_2 \text{ (say)}$$

$\Rightarrow \quad l_2 = -2k_2, \; m_2 = k_2$

And from $n = 2l + 2m$ we have

$$n_2 = 2l_2 + 2m_2$$

$\Rightarrow \quad n_2 = 2(-2k_2) + 2k_2 = -2k_2$

$\therefore$ we have $l_2 = -2k_2$,

$m_2 = k_2, \; n_2 = -2k_2$

$\therefore$ If θ be the angle between the given lines, then

$$\cos\theta = a_1a_2 + b_1b_2 + c_1c_2$$

$$= (-k_1)(-2k_2) + (2k_1)(k_2) + (2k_1)(-2k_2)$$

$$= 2k_1k_2 + 2k_1k_2 - 4k_1k_2$$

$\cos\theta = 0$

Hence the given lines are at right angles.

1987 → The direction cosines (l, m, n) of a line are connected by relations $2l + 2m + n = 0$, $3l^2 + 5m^2 = 5n^2$. Show that two such lines are possible and that they are perpendicular to each other.

→ If l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, then show that the line whose direction cosines are

$$\frac{l_1+l_2+l_3}{3}, \quad \frac{m_1+m_2+m_3}{3} \text{ and } \frac{n_1+n_2+n_3}{3}$$

makes equal angles with them.

Sol'n :-

Since l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, we have

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{array}\right\}$$

$$\left.\begin{array}{l} l_1l_2+m_1m_2+n_1n_2=0 \\ l_2l_3+m_2m_3+n_2n_3=0 \\ l_3l_1+m_3m_1+n_3n_1=0 \end{array}\right\}$$

Let the angle between the lines whose direction cosines are l_1, m_1, n_1 and

$$\frac{l_1+l_2+l_3}{3}, \frac{m_1+m_2+m_3}{3}, \frac{n_1+n_2+n_3}{3}$$

be θ. Then

$$\cos\theta = l_1\left(\frac{l_1+l_2+l_3}{3}\right)+m_1\left(\frac{m_1+m_2+m_3}{3}\right)+n_1\left(\frac{n_1+n_2+n_3}{3}\right)$$

$$= \frac{(l_1^2+m_1^2+n_1^2)+(l_1l_2+m_1m_2+n_1n_2)+(l_1l_3+m_1m_3+n_1n_3)}{3}$$

$$= \frac{1+0+0}{3} = \frac{1}{3}$$

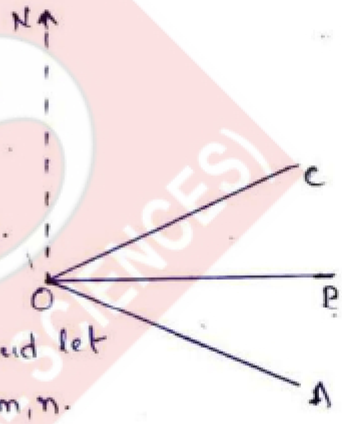
$\Rightarrow \theta = \cos^{-1}(1/3)$

similarly, we can prove that the other angles are also equal to $\cos^{-1}(1/3)$ each.

→ show that three concurrent lines with direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 are coplanar if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

Sol'n :- Let OA, OB, OC be the given three concurrent lines with d.c's l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 respectively.



Now if these lines are coplanar, then these lines have a common normal.

Let ON be the common normal and let its d.c's be l, m, n.

Since $ON \perp OA, OB, OC$.

$$\therefore \quad ll_1+mm_1+nn_1=0$$
$$ll_2+mm_2+nn_2=0$$
$$ll_3+mm_3+nn_3=0$$

Eliminating l, m, n determinatically from the above equations,

we have $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$ [∵ writing coefficients of l in one column, and coefficient of m and n in second and third columns.]

which is the required condition.

→ If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c's of two lines, then the direction ratios of another which is perpendicular to both the given lines are

$$(m_1n_2-m_2n_1), (n_1l_2-n_2l_1), (l_1m_2-l_2m_1)$$

prove further if the given lines are at right angles to each other then these direction ratios are the actual direction cosines.

Sol'n:- Let l, m, n be the required d.c's of the line

then as this line is perpendicular to both the given lines, so we have

$ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

solving these, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1},$$

which give the required direction ratios.

$\therefore$ The d.c's of this line are

$$\frac{m_1n_2 - m_2n_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}, \frac{n_1l_2 - n_2l_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}, \frac{l_1m_2 - l_2m_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}} \quad \text{—— (i)}$$

If θ be the angle between the two given lines whose d.c's are (l_1, m_1, n_1) and (l_2, m_2, n_2) then we know that

$$\sin\theta = \sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}$$

If these lines are at right angles, then $\theta = 90^\circ$ or $\sin\theta = 1$

and therefore $\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]} = 1$

$\therefore$ From (i) the actual d.c's are

$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1)$ and $(l_1m_2 - l_2m_1)$.

Hence Proved

→ Lines OA, OB are drawn from O with d.c's proportional to $(1,-2,-1)$, $(3,-2,3)$. Find the d.c's of the normal to the plane AOB.

Sol'n:- Let a, b, c be the direction ratios of the required normal to the plane AOB.

Then as OA lies in this plane so it is perpendicular to the normal to this plane and consequently we have

$a(1) + b(-2) + c(-1) = 0$ —— (i)

Similarly OB is also perpendicular to this normal and so we have

$a(3) + b(-2) + c(3) = 0$ —— (ii)

solving (i) and (ii) simultaneously, we have

$$\frac{a}{(-2)(3)-(-1)(-2)} = \frac{b}{(-1)(3)-(1)(3)} = \frac{c}{(1)(-2)-(-2)(3)}$$

(or) $\frac{a}{-8} = \frac{b}{-6} = \frac{c}{4}$ (or)

$a : b : c = 4 : 3 : -2$

Also $4^2 + 3^2 + (-2)^2 = 29$

$\therefore$ Required d.c's are $\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}$

→ The direction ratios of two lines are $1, -2, -2$ and $0, 2, 1$. Find the direction cosines of the line perpendicular to the above lines.

Sol'n:- Let a, b, c be the direction ratio's of the line whose direction cosines are required.

Then as this line is perpendicular to the given lines so we have $a(1)+b(-2)+c(-2)=0$

and $a(0)+b(2)+c(1)=0$

solving these simultaneously, we get

$$\frac{a}{(-2)(1)-(-2)(2)}=\frac{b}{(-2)(0)-(1)(1)}=\frac{c}{(1)(2)-(0)(-2)}$$

or $\frac{a}{2}=\frac{b}{-1}=\frac{c}{2}$

i.e. $a:b:c=2:-1:2$

$\therefore$ The required direction cosines are

$$\frac{2}{\sqrt{(2^2+1^2+2^2)}},\ \frac{-1}{\sqrt{(2^2+1^2+2^2)}},\ \frac{2}{\sqrt{2^2+1^2+2^2}}.$$

i.e. $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$

4.40. Find the direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to $3,-1,1$ and $-3,2,4$.

→ Prove that the three lines drawn from a point with direction cosines proportional to $(1,-1,1)$, $(2,-3,0)$ and $(1,0,3)$ are coplanar.

Sol'n :- Let PA, PB and PC be three given lines with d.c's proportional to $(1,-1,1)$ $(2,-3,0)$ and $(1,0,3)$

Let l,m,n be the direction ratios of the normal to the plane APB. Then as PA and PB are perpendicular to this normal so we have

$$l.1+m(-1)+n.1=0 \quad \text{—(i)}$$

and $$l.2+m(-3)+n.0=0 \quad \text{—(ii)}$$

solving (i) and (ii) we have

$$\frac{l}{(-1)(0)-(1)(-3)}=\frac{m}{(1)(2)-(1)(0)}=\frac{n}{(1)(-3)-(-1)(2)}$$

i.e. $\frac{l}{3}=\frac{m}{2}=\frac{n}{-1}$ ——(iii)

If PC also lies in this plane PAB, then PC must be at right angles to this normal whose direction ratios are given by (iii), therefore we get

$$a_1a_2+b_1b_2+c_1c_2=0$$

Here $a_1a_2+b_1b_2+c_1c_2$

$= l.1+m.0+n.3$

$= 3.1+2.0+(-1).3$ from (iii)

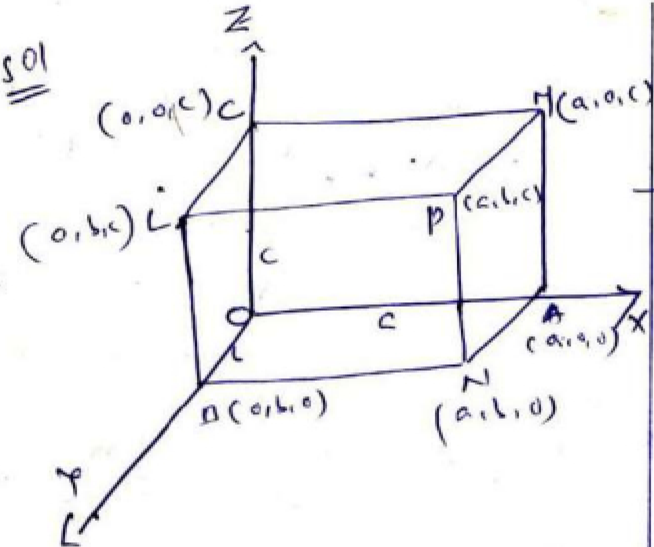
$= 0$

Hence proved.

→ The edges of a rectangular parallelopiped are a,b,c. Show that the angles b/w the four diagonals are given by

$$\cos^{-1}\left(\frac{\pm a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$$

Sol



Take 'O', a corner of the rectangular parallelopiped as the origin and three edges OA, OB, OC through it (i.e 'O') as the axes.

Then the co-ordinates of the various corners are $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$, $L(0,b,c)$, $M(a,0,c)$, $N(a,b,0)$, $P(a,b,c)$.

The four diagonals are AL, BM, CN and OP.

The d.c's of AL are proportional to
$0-a, b-0, c-0$
$-a, b, c$ | Using $x_2-x_1, y_2-y_1, z_2-z_1$)

similarly,
d.c's of BM are proportional to $a, -b, c$.
d.c's of CN " " " $a, b, -c$
d.c's of OP " " " a, b, c.

If α is the angle b/w the diagonals OP and AL then

$$\cos\alpha = \frac{a(-a)+b(b)+c(c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}}$$

$$= \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right) \quad (1)$$

similarly

angle b/w OP & BM $= \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$ (2)

" " OP & CN $= \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$ (3)

" " AL & BM $= \cos^{-1}\left(\frac{-a^2-b^2+c^2}{a^2+b^2+c^2}\right)$

$= \cos^{-1}\left|\frac{-(a^2+b^2-c^2)}{a^2+b^2+c^2}\right|$

$= \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$

(when acute angle is taken)

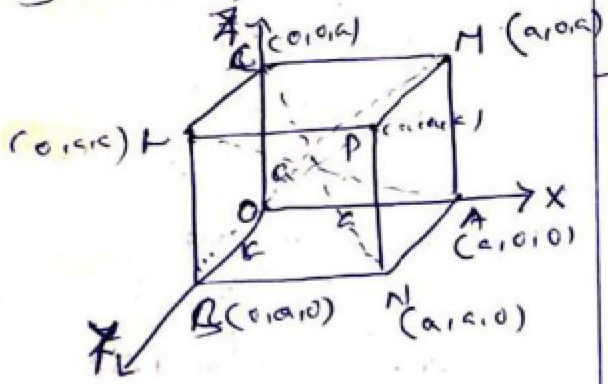
This angle is the same as the angle b/w OP and CN.

similarly we can show that all other angles b/w other two diagonals are repeated and we get only three different angles as given by (1), (2) and (3).

Hence the angles b/w four diagonals are given by $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$

Note: The sign in the numerators cannot be all +ve, since if they are all +ve, then the angle $= \cos^{-1}\left(\frac{a^2+b^2+c^2}{a^2+b^2+c^2}\right) = \cos^{-1}(1) = 0$: i.e the two diagonals will be ||, which is impossible as it is clear from the figure

Find the angle b/w the diagonals of a cube.



Take O, a corner of the cube as origin and the three edges OA, OB, OC through it (i.e 'O') as the axes.

Let $OA = OB = OC = a$

Then the co-ordinates of the various points are

$O(0,0,0)$, $A(a,0,0)$, $B(0,a,0)$, $C(0,0,a)$, $L(0,a,a)$, $M(a,0,a)$, $N(a,a,0)$, $P(a,a,a)$

∴ The four diagonals are AL, BM, CN and OP.

∴ The dr's of the diagonal AL are $0-a, a-0, a-0$ | using $x_2-x_1, y_2-y_1, z_2-z_1$.

i.e $-a, a, a$

or $-1, 1, 1$ (cancelling a).

Similarly the dr's of the diagonals BM are

$a-0, 0-a, a-0$

or $a, -a, a$

or $1, -1, 1$ (cancelling a).

If θ be the angle b/w the diagonals AL and BM

then $\cos\theta = \dfrac{1(-1)+(-1)(1)+1(1)}{\sqrt{1+1+1}\sqrt{1+1+1}}$

$= \dfrac{-1-1+1}{\sqrt{3}\cdot\sqrt{3}}$

$= \dfrac{-1}{3}$

$= \dfrac{1}{3}$ (numerically)

$\therefore \boxed{\theta = \cos^{-1}(1/3)}$

Similarly the angle b/w other two diagonals is also the same.

(1993) A line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube.

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$.

Sol

From the above problem figure cube,

The four diagonals are AL, BM, CN and OP.

The d.r's of the diagonal OP are proportional to $a-0, a-0, a-0$. | using $x_2-x_1, y_2-y_1, z_2-z_1$

a, a, a

or $1, 1, 1$.

∴ Actual d.c's of OP are

$\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Similarly the d.c's of diagonal AL are
$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

the d.c's of diagonal BM are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

and the d.c's of diagonal CN are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

Let l, m, n be the d.c's of the given line which makes the angles $\alpha, \beta, \gamma, \delta$ with the four diagonals OP, AL, BM and CN respectively (say)

$$\therefore \cos\alpha = \frac{1}{\sqrt{3}}.l + \frac{1}{\sqrt{3}}.m + \frac{1}{\sqrt{3}}.n$$

$$= \frac{1}{\sqrt{3}}(l+m+n).$$

Similarly $\cos\beta = \frac{1}{\sqrt{3}}(-l+m+n)$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n).$$

and $\cos\delta = \frac{1}{\sqrt{3}}(l+m-n)$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}(l+m+n)^2 + \frac{1}{3}(-l+m+n)^2 + \frac{1}{3}(l-m+n)^2 + \frac{1}{3}(l+m-n)^2$$

$$= \frac{1}{3}\left[4(l^2+m^2+n^2)\right]$$

$$= \frac{4}{3}. \quad (\because l^2+m^2+n^2=1).$$

→ If a variable line in two adjacent positions had direction cosines l, m, n; $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Sol'n :- Since l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$ are dc's,

hence $l^2+m^2+n^2=1$ —— ①

and $(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$

$$\Rightarrow \delta l^2+\delta m^2+\delta n^2 = -2(l\delta l+m\delta m+n\delta n) \quad \text{—— ②}$$

Now, $\cos\delta\theta = l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$

$$= l^2+m^2+n^2+l\delta l+m\delta m+n\delta n$$

$$= 1-\frac{1}{2}\left\{\delta l^2+\delta m^2+\delta n^2\right\}, \text{ from ① and ②}$$

$$\Rightarrow \delta l^2+\delta m^2+\delta n^2 = 2(1-\cos\delta\theta)$$

$$= 2.2\sin^2\frac{\delta\theta}{2}$$

$$= 4\left(\tfrac{1}{2}\delta\theta\right)^2$$

$$= \delta\theta^2$$

$l+\delta l, m+\delta m, n+\delta n$

$\delta\theta$

l, m, n