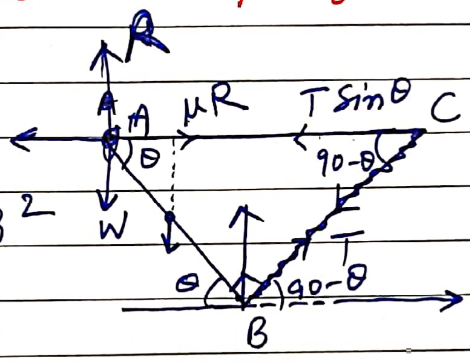
CSE 2019 to 2010

STATICS.

2019/5(c)/10m

1. One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. When the rod is on the point of sliding, then $AC^2 - AB^2 = BC^2$.
If θ is the angle between AB and the horizontal line, then prove that the coefficient of friction is $\dfrac{\cot\theta}{2+\cot^2\theta}$.



As $AC^2 = BC^2 + AB^2$

$\therefore \angle B = 90^\circ$

Since system is in equilibrium, we compare the horizontal and vertical components of forces at A.
Let weight of rod AB be W.

Horizontally,
$$T \cdot \sin\theta = \mu R$$

Vertically,
$$R + T\cos\theta = W$$

eliminating R,
$$\frac{T \cdot \sin\theta}{\mu} + T \cdot \cos\theta = W$$

$$T(\sin\theta + \mu \cdot \cos\theta) = \mu \cdot W \quad \text{(1)}$$

Also, the moment at A is zero.

$$T\cdot(AB) - W\cdot\left(\frac{AB}{2}\right)\cdot\cos\theta = 0$$

clockwise

$$T = \frac{W\cos\theta}{2}.$$

using it in ①

$$\frac{\cancel{W}\cos\theta}{2}\left(\sin\theta + \mu\cos\theta\right) = \mu\cdot\cancel{W}$$

$$\sin\theta\cdot\cos\theta + \mu\cdot\cos^2\theta = 2\mu$$

$$\mu\left(2-\cos^2\theta\right) = \sin\theta\cos\theta$$

$$\mu = \frac{\sin\theta\cos\theta}{2-\cos^2\theta}$$

dividing by $\sin^2\theta$

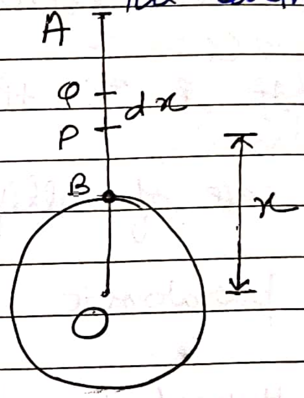
$$= \frac{\cot\theta}{2\cdot\operatorname{cosec}^2\theta - \cot^2\theta}$$

$$= \frac{\cot\theta}{2+2\cot^2\theta - \cot^2\theta}$$

$$\boxed{\mu = \frac{\cot\theta}{2+\cot^2\theta}}$$

CSE 2019 (S&D)

5(d) The force of attraction of a particle by the earth is inversely proportional to the square of its distance from the earth's centre. A particle, whose weight on the surface of the earth is W, falls to the surface of the earth from a height 3h above it. Show that the magnitude of work done by the earth's attraction force is $\frac{3}{4}hW$, where h is the radius of the Earth.



Let P be the position of the particle at a height x above the centre of the earth.

Then attraction $= \frac{\lambda}{x^2}$

At the surface of the earth, when $x = h$, attraction is W, the weight of the particle.

$$\therefore W = \frac{\lambda}{h^2} \Rightarrow \lambda = h^2 W$$

$\therefore$ Attraction at height x above centre $= \frac{h^2 W}{x^2}$

$\therefore$ Work done by gravity in moving the particle through a distance δx

$$= -\frac{h^2 W}{x^2}\,\delta x$$

Since displacement is in a direction opposite to the direction of the force of gravity, therefore -ve sign has been taken.

$\therefore$ Total work done $= \displaystyle\int_{4h}^{h} \frac{-h^2 W}{x^2}\, dx$

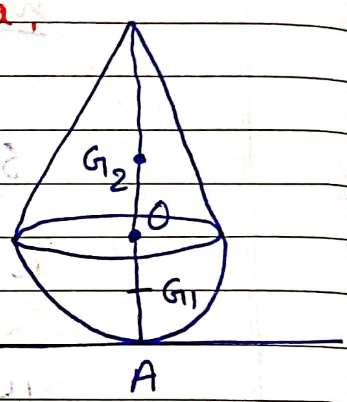
$$= -h^2 W \left[\frac{-1}{x} \right]_{4h}^{h}$$

$$= -h^2 W \left[\frac{-1}{h} + \frac{1}{4h} \right]$$

$$= \frac{3}{4} hW.$$

2019 / 6(a) / 15m

2. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius 'a'. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.



Let us first try to find out the C.G. of the whole body:

As we know C.G. of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its flat base, where 'a' is radius of sphere.

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

w_1 = weight of hemisphere

$$= \frac{2}{3}\pi a^3 \rho g$$

x_2 = distance of centre of gravity of cone from table

$$= AO + OG_2 = a + \frac{H}{4}$$

w_2 = weight of cone

$$= \frac{1}{3}\pi a^2 H \rho g,$$ H – Height of cone

h = distance of C.G. of combined body from horizontal plane

$$= \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$$

$$= \frac{\frac{2}{3}\pi a^3 \rho g \cdot \frac{5a}{8} + \frac{1}{3}\pi a^2 H \rho g\left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H \rho g}$$

$$= \frac{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)}{2a + H}$$

$$h = \frac{5a^2 + H(4a+H)}{4(2a+H)}$$

Let R = radius of lower surface $= \infty$
r = radius of upper surface $= a$

For stable equilibrium, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$\frac{4(2a+H)}{5a^2 + H(4a+H)} > \frac{1}{a} + \frac{1}{\infty}$$

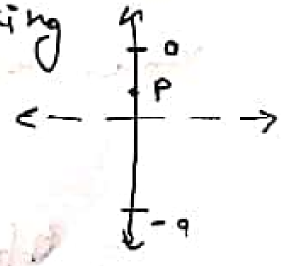
$$a(8a + \cancel{4H}) > 5a^2 + \cancel{4aH} + H^2$$

$$3a^2 > H^2$$

or $\boxed{H < \sqrt{3}\,a}$

(Q) A particle moving along the y-axis has an acceleration Fy towards the origin, where F is a positive and even function of y. The periodic time, when the particle vibrates between $y=-a$ and $y=a$ is T. S.T $\frac{2\pi}{\sqrt{F_1}} < T < \frac{2\pi}{\sqrt{F_2}}$ where F_1 and F_2 are the greatest and least value of F within the range $[-a, a]$. Further S.T when a simple pendulum of length l oscillates through 30° on either side of the vertical line, T lies between $2\pi\sqrt{l/g}$ and $2\pi\sqrt{l/g}\sqrt{\pi/3}$

Solⁿ let P be the particle vibrating between $-a$ to a in y-axis



Thus, $V_a = V_{-a} = 0$ —(1)

Motion of particle :-

$$\frac{d^2y}{dt^2} = -Fy$$

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} = -Fy\frac{2dy}{dt}$$

Integrating w.r.t t, we get

$$\left(\frac{dy}{dt}\right)^2 = -2\int_0^y Fy\,dy + c \qquad y \in [0,a]$$

At $y=a$, $\frac{dy}{dt}=0$ From 1 $\Rightarrow c = \int_0^a 2Fy\,dy$

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$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = \int_0^a 2Fy\,dy - \int_0^y 2Fy\,dy$$

$$= \int_y^a 2Fy\,dy$$

$$\Rightarrow \int_0^a \frac{dy}{\sqrt{\int_y^a 2Fy\,dy}} = \int_0^{T/4} dt$$

$$\Rightarrow \int_0^a \frac{dy}{\sqrt{\int_y^a 2Fy\,dy}} = T/4 \quad — (2)$$

Given,

$$F_2 \leq F \leq F_1$$

$$\Rightarrow \int_y^a 2F_2 y\,dy \leq \int_y^a 2Fy\,dy \leq \int_y^a 2F_1 y\,dy$$

$$\Rightarrow F_2(a^2-y^2) \leq \int_y^a 2Fy\,dy \leq F_1(a^2-y^2)$$

$$\Rightarrow \int_0^a \frac{dy}{\sqrt{F_2(a^2-y^2)}} \leq \int_0^a \frac{1}{\sqrt{\int_y^a 2Fy\,dy}} \leq \int_0^a \frac{dy}{\sqrt{F_1(a^2-y^2)}}$$

$$\Rightarrow \frac{1}{\sqrt{F_2}}\left[\sin^{-1}\frac{x}{a}\right]_0^a \leq \frac{T}{4} \leq \frac{1}{\sqrt{F_1}}\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

{using (2)

$$\Rightarrow \frac{\pi}{2\sqrt{F_2}} \leq \frac{T}{4} \leq \frac{\pi}{2\sqrt{F_1}}$$

$$\Rightarrow \boxed{\frac{2\pi}{\sqrt{F_2}} \leq T \leq \frac{2\pi}{\sqrt{F_1}}} \quad — (3)$$

Equation of motion for simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

$$= -\left(\frac{g}{l}\frac{\sin\theta}{\theta}\right)\theta$$

let, $F = \frac{g}{l}\frac{\sin\theta}{\theta}$, where $\theta \neq 0$

For $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ F is positive and even.

Max F at $\theta \to 0$ $F_1 = \frac{g}{l}$

Min F at $\theta = \pi/6$ $F_2 = \frac{3g}{\pi l}$

From equation (3).

$$\boxed{2\pi\sqrt{\frac{l}{g}} \leq T \leq 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{\pi}{3}}}$$

(S&D CSE 2019)

classmate

Date

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8(b) Prove that the path of a planet, which is moving so that its acceleration is always directed to a fixed point (star) and is equal to $\mu/(\text{distance})^2$, is a conic section.

Find the conditions under which the path becomes i) ellipse ii) parabola and iii) hyperbola.

Since the force is directed to a fixed point, so that path of the particle is a central orbit.

$\therefore$ Central acceleration $= \dfrac{\mu}{r^2}$

The D.E. of central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = F = \frac{\mu}{r^2}$$

$$\therefore \quad 2h^2p^{-3}\,dp = -2\mu r^{-2}\,dr$$

Integrating,

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + c_1 \qquad \text{— (1)}$$

But in central orbit, $v = \dfrac{h}{p} \Rightarrow v^2 = \dfrac{h^2}{p^2}$

from (1)

$$\therefore \quad v^2 = \frac{2\mu}{r} + c_1 \qquad \text{— (2)}$$

Case-I: Elliptic Path

If a, b are lengths of semi-major and minor axes respectively, then reffered to focus as pole, the pedal equation of ellipse is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \qquad \text{— (3)}$$

Comparing ① and ③, we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c_1}{-1}$$

$$\therefore h^2 = \frac{\mu b^2}{a} \text{ and } c_1 = \frac{-\mu}{a}$$

$\therefore$ from ②,

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

~~In~~

Case-II: Parabolic Path

If $4a$ is the length of latus-rectum then ~~reffered~~ referred to focus as pole, the pedal eqn of parabola is

$$p^2 = ar$$

or $$\frac{1}{p^2} = \frac{1}{ar}$$

Comparing ① and ④

$$\frac{h^2}{1} = \frac{2\mu}{1/a} = \frac{c_1}{0}$$

$$\Rightarrow h^2 = 2\mu a \text{ and } c_1 = 0$$

$\therefore$ ② gives,
$$v^2 = \frac{2\mu}{r}$$

Case-III: Hyperbolic path

If a and b are the lengths of semi-transverse and conjugate axes respectively, the referred to focus as pole, the pedal eqn of hyperbola is

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1$$

Comparing ① and ⑤

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c_1}{1}$$

$$\therefore \quad h^2 = \frac{\mu b^2}{a} \quad \text{and} \quad c_1 = \frac{\mu}{a}$$

② gives, $$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

Thus eqn ① represents a conic section whose focus is at the centre of force.

If l is the length of semi-latus rectum.

In case of elliptic path, $h^2 = \frac{\mu b^2}{a} = \mu l$

parabolic path $\Rightarrow$ $h^2 = 2\mu a = \mu l$

hyperbolic path $\Rightarrow$ $h^2 = \frac{\mu b^2}{a} = \mu l$.