New College

# ADVANCED CALCULUS

**B.A. / B.Sc. II**
**Semester - III**

**JEEVANSONS PUBLICATIONS**

# New College
# ADVANCED CALCULUS

FOR

**B.A. / B.Sc. II (Third Semester)**

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# PREFACE

The present book is designed to meet the requirements of students of B.A./B.Sc. II (Third Semester) of Kurukshetra University, Kurukshetra, Maharishi Dayanand University, Rohtak, C.D.L.U., Sirsa, I.G.U., Meerpur, C.B.L.U., Bhiwani, C.R.S.U., Jind, G.J.U., Hissar and various other universities of India.

In this book an humble attempt has been made to present the subject matter in a comprehensive, lucid and easy to understand style. At the same time the matter is rigorous and exhaustive and nothing of importance has been omitted.

The special feature of the book are the notes and remarks given wherever it is felt that the students need an explanation of any particular point. Thus the students will find that they can understand the topics after reading them in the similar manner as they do when discussed and explained in the class room. Keeping in the mind that this subject is generally considered tough by the student fraternity, quite a number of illustrative examples are given at the end of each topic. In fact, almost all types of questions have been tackled in these solved examples. The questions given in the unsolved exercises are based of these solved examples so that the students can attempt these questions on their own. The articles and results, which constitute a fair amount of the subject matter, have been discussed in a very simple and easy language so that the students can understand and then reproduce the same instead of cramming them up. And the last but not the least, almost all the questions appearing in the latest University papers of previous years have been incorporated, giving an idea of the standard demand from the students in the examination.

We would like to extend our heartiest gratitude and thanks to the modern writers whose works we have freely consulted in preparing the manuscript of the book. We would also take this opportunity to thank the publishers and the printers for their whole hearted co-operation in bringing out the present volume.

Although, every care has been taken to keep the book free of errors but still some misprints might have crept in. We would be grateful to our colleagues if they can point these out. Also, any suggestions or shortcomings in the matter, may be sent to us so that the same can be incorporated in the subsequent editions of the book.

**- Authors**

# SYLLABUS

## B.A./B.Sc. 2nd Year

### THIRD SEMESTER

### ADVANCED CALCULUS : (BM - 231)

**K.U. Kurukshetra, M.D.U. Rohtak, C.D.L.U. Sirsa, C.R.S.U. Jind, C.B.L.U. Bhiwani, G.J.U. Hissar, I.G.U. Meerpur**

Time Allowed : 3 hours

Maximum Marks : B.Sc. : 40, B.A. : 27

### Section - I

Continuity, Sequential continuity, properties of continuous functions, Uniform continuity, Chain rule of differentiability. Mean value theorems; Rolle's theorem and Lagrange's mean value theorem and their geometrical interpretations. Taylor's theorem with various form of remainders, Darboux intermediate value theorem for derivatives, Indeterminate forms.

### Section - II

Limit and continuity of real valued functions of two variables. Partial differentiation. Total differentials; Composite functions and implicit functions. Change of variables. Homogeneous functions and Euler's theorem on homogeneous functions. Taylor's theorem for functions of two variables.

### Section - III

Differentiability of real valued functions of two variables. Schwarz and Young's theorem. Implicit function theorem. Maxima, Minima and saddle points of two variables. Lagrange's method of multipliers.

### Section - IV

Curves : Tangents, Principal normals, Binormals, Serret-Frenet formulae. Locus of the centre of curvature, Spherical curvature, Locus of centre of spherical curvature, Involutes, Evolutes, Bertrand curves. Surfaces : Tangent planes, one parameter family of surfaces, Envelopes.

# CONTENTS

## AN IMPORTANT NOTE FOR THE STUDENTS

Dear Students,

At many places it is observed that when you go to a bookseller to buy text books, they tell you to buy photocopies of these books. We request you to kindly refuse them and do not buy photocopies as these are not at all beneficial for you. Only the bookseller/photocopier gains from this.

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**– Publishers**

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# PRELIMINARIES

We shall begin our study by learning some definitions related to sequences which will be useful in our present syllabus. The sequences will be studied in detail in fourth semester.

## SEQUENCE

**Definition.** *A sequence is a function whose domain is the set of natural numbers and range can be any set*. Thus, a sequence is an ordered set of numbers $a_1, a_2, a_3, ....., a_n, .....$ . It is denoted by $< a_n >$, where $a_n$ denotes the $n$th term of the sequence and is called **general term** of the sequence $< a_n >$.

## REAL SEQUENCE

A sequence whose range is the subset of R, the set of real numbers, is called a **real sequence**. In our present syllabus, we shall deal with real sequences only. Therefore, we shall use the term sequence instead of "real sequence".

*Note.*

1. The terms of a sequence occurring at different positions are treated as distinct terms even if they have the same value.
2. The number of terms of a sequence is always infinite. However, its range may be finite.

   If $a_n = (-1)^n$, then $< a_n >$ is a sequence whose range is a finite set $\{1, -1\}$ but the sequence is described as $< -1, 1, -1, 1, -1, 1, ..... >$.

## METHODS TO DESCRIBE A SEQUENCE

A sequence can be described in following different ways :

(*i*) We list first few terms of the sequence in order, till the rule for writing down different terms becomes evident. For example, $< 1, 4, 9, 16, 25, ..... >$ is a sequence whose $n$th term is $n^2$.

(*ii*) Another method of writing a sequence is to give a formula for its $n$th term. For example, if sequence $< a_n > = < n^2 >$, then the sequence $< a_n >$ is defined as $< 1, 4, 9, 16, 25, ..... >$.

(*iii*) Another convenient method to describe a sequence is to write first few terms and then give a rule for obtaining its other terms using the preceeding terms.

# 8 CURVES IN SPACE

## 8.1. MEANING OF DIFFERENTIAL GEOMETRY

Differential geometry is that part of geometry which is studied with the help of *differential calculus.*

There are two branches of differential geometry :

(*i*) **Local differential geometry.** In this branch we study the properties of curves and surfaces in the neighbourhood of a point.

(*ii*) **Global differential geometry.** In this branch we study the properties of these curves and surfaces as a whole.

In this book we shall confine ourself to the study of Local differential geometry.

## 8.2. DESCRIPTION OF CURVES IN SPACE

(*i*) A curve in space is the locus of a point whose position vector $\vec{r}$ relative to a fixed origin O is a function of single parameter $t$, say.

Cartesian coordinates $(x, y, z)$ of a point P are the components of $\vec{r}$ and are the functions of parameter $t$

Thus,
$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k}\\ &= x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}\\ &= \vec{r}(t)\end{aligned}$$

Thus we may define curves in space as follows :

*A curve in space is the locus of a point whose cartesian coordinates are the functions of a single variable 't' called parameter.*

(*ii*) Parametric equations of the curve can be written as

$$x = x(t),\ y = y(t),\ z = z(t)$$

where $x, y, z$ are real valued functions of a single variable $t$ over some interval $a \le t \le b$.

(*iii*) A curve in space can be represented as the intersection of two surfaces

$$f_1(x, y, z) = 0 \quad \text{and} \quad f_2(x, y, z) = 0$$

Both equations together represents the curve of intersection of the surfaces.

Another form of curve in space is obtained by eliminating the parameter from the parametric equation of the curve.

For example, if the parametric equations of the curve are

$$x = t,\quad y = t^3,\quad z = t^6 \qquad \text{...(1)}$$

Then eliminating $t$, we get

$$y = x^3 \quad \text{and} \quad z = y^2$$

which can be written as $f_1(x, y) = 0$ and $f_2(y, z) = 0$

Thus, the intersection of two surfaces $f_1(x, y) = 0$ and $f_2(y, z) = 0$ is the curve in space given by (1).

### 8.2.1. Regular Curve

*The vector valued curve $\vec{r}$, defined on I is said to be* ***regular*** *if $\frac{d\vec{r}}{dt}$ i.e., derivative w.r.t parameter t never vanishes.*

In other words, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are never zero simultaneously.

## 8.3. TANGENT TO A CURVE

Let C be a curve and P be any point on C. The tangent at P to the curve C is the limiting position of a straight line through points P and Q, where Q is another point on the curve such that Q approaches P along the curve [Ref. fig. 8.1].

## 8.4. UNIT VECTOR ALONG THE TANGENT TO A GIVEN CURVE

Let P and Q be two consecutive points on the curve C whose position vectors are $\vec{r}$ and $\vec{r} + \delta\vec{r}$ with respect to origin O.

Thus, $\quad \vec{OP} = \vec{r}$

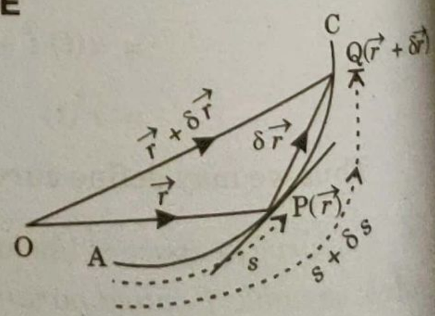


*Fig. 8.1*

$$\vec{OQ} = \vec{r} + \delta\vec{r}$$

and

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$\therefore \quad = \delta\vec{r}$$

Suppose, A is some fixed point on the curve such that Arc $AP = s$ and Arc $AQ = s + \delta s$.

Now unit vector along the chord PQ $= \dfrac{\delta\vec{r}}{|\delta\vec{r}|}$

$$= \frac{\delta\vec{r}}{\delta s}\frac{\delta s}{|\delta\vec{r}|}$$

Let $Q \to P$, then the chord PQ becomes the tangent at P.

$\therefore$ Unit vector along the tangent at P $= \lim\limits_{Q\to P} \dfrac{\delta\vec{r}}{\delta s}\dfrac{\delta s}{|\delta\vec{r}|}$

$$= \lim_{Q\to P} \frac{\delta\vec{r}}{\delta s}\left(\frac{\text{arc PQ}}{\text{chord PQ}}\right)$$

$$= \frac{d\vec{r}}{ds} \qquad \left[\because \textit{As } Q \to P, \lim\left(\frac{\textit{arc PQ}}{\textit{chord PQ}}\right) \to 1\right]$$

Denoting unit tangent vector by $\hat{t}$, we have

$$\boxed{\hat{t} = \frac{d\vec{r}}{ds} = \vec{r}\,'} \qquad ...(1)$$

Also,

$$\hat{t} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} \qquad ...(2)$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \hat{t}\,\frac{ds}{dt}$$

or

$$\left|\frac{d\vec{r}}{dt}\right| = |\hat{t}|\left|\frac{ds}{dt}\right| = \left|\frac{ds}{dt}\right| \qquad [\because |\hat{t}| = 1]$$

$$\therefore \qquad \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt} \qquad \ldots(3)$$

From (2) and (3), we get

$$\hat{t} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|} \qquad \ldots(4)$$

Form (4) of unit tangent vector is important for numerical purpose.

## 8.5. FORMULA FOR LENGTH OF ARC OF A CURVE

We shall use the symbol $\hat{t}$ for unit vector along the tangent at point P and it is considered positive in the direction of $s$ increasing.

Thus, $$\hat{t} = \frac{d\vec{r}}{ds} = \vec{r}\,' \qquad \ldots(1)$$

If $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \text{ then}$$

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

*i.e.,* $$\hat{t} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

We have seen that if $x, y, z$ are components of $\vec{r}$, then components of unit vector $\hat{t}$ along the tangent at P are $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}$.

Now, $$|\hat{t}| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} \qquad \text{[Formula for length of vector]}$$

$$\therefore \qquad 1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 \qquad [\because |\hat{t}| = 1]$$

or $$1 = \left(\frac{dx}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\cdot\frac{dt}{ds}\right)^2$$

where $t$ is any other parameter

$$\therefore \quad 1 = \left(\frac{dt}{ds}\right)^2\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]$$

or

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 \quad \text{...(2)}$$

Now,

$$\left|\frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{ds}\cdot\frac{ds}{dt}\right| = \left|\frac{d\vec{r}}{ds}\right|\left|\frac{ds}{dt}\right|$$

or

$$\left|\frac{d\vec{r}}{dt}\right| = |\hat{t}|\left|\frac{ds}{dt}\right| \qquad \left[\because \frac{d\vec{r}}{ds} = \hat{t} \textit{ is unit vector along the tangent}\right]$$

$$\therefore \quad \left|\frac{d\vec{r}}{dt}\right| = \left|\frac{ds}{dt}\right| \quad \text{...(3)} \quad [\because |\hat{t}| = 1]$$

Using (2), we get

$$\left|\frac{d\vec{r}}{dt}\right| = \left|\frac{ds}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} \quad \text{...(4)}$$

or

$$|\dot{\vec{r}}| = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

where ***dots denotes the derivatives w.r.t parameter t.***

If $s$ is the length of the arc between the points $t_0$ and $t$, then

$$s = \int_{t_0}^{t} ds = \int_{t_0}^{t} |\dot{\vec{r}}|\, dt = \int_{t_0}^{t} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, dt$$

**Normal Form of a curve :**

The equation of the curve is said to be in normal form if the parameter used is $s$, where $s$ is the arc length of the curve.

## SOLVED EXAMPLES

**Example 1.** ***Find the length of the circular helix***

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k},\ -\infty < t < \infty, \textit{ from } (a, 0, 0) \textit{ to } (a, 0, 2\pi c).$$

***Also obtain its equation in terms of parameter 's'.*** [M.D.U. 2014]

**Solution.** Here equation of circular helix is

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k} \quad \text{...(1)}$$

Differentiating w.r.t. $t$, we have

$$\dot{\vec{r}}(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + c\,\hat{k}$$

$$\therefore \quad |\dot{\vec{r}}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2+c^2} \qquad \left[\because |\dot{\vec{r}}| = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right]$$

We have to find the length of circular helix from $(a, 0, 0)$ to $(a, 0, 2\pi c)$.

As $ct$ varies from 0 to $2\pi c$ therefore $0 \le t \le 2\pi$.

$$\therefore \text{ Length of circular helix from } (a, 0, 0) \text{ to } (a, 0, 2\pi c) = \int_0^{2\pi} |\dot{\vec{r}}|\, dt = \int_0^{2\pi} \sqrt{a^2+c^2}\, dt$$

$$= 2\pi\sqrt{a^2+c^2}$$

**To obtain the equation in terms of $s$ :**

Suppose $s$ is the length of the arc from $t = 0$ to any point $t$. Then

$$s = \int_0^t |\dot{\vec{r}}(t)|\, dt = \int_0^t \sqrt{a^2+c^2}\, dt = \sqrt{a^2+c^2}\; t$$

$$\therefore \quad t = \frac{s}{\sqrt{a^2+c^2}}$$

Substituting the value of $t$ in (1), the equation of helix in terms of parameter $s$ is

$$\vec{r}(s) = \left[a\cos\frac{s}{\sqrt{a^2+c^2}}\right]\hat{i} + \left[a\sin\frac{s}{\sqrt{a^2+c^2}}\right]\hat{j} + \frac{cs}{\sqrt{a^2+c^2}}\hat{k}$$

which is the normal form of the curve.

**Example 2.** ***Find the length of the curve given as the intersection of the surfaces***

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } x = a\cosh\left(\frac{z}{a}\right)$$

***from the point (a, 0, 0) to the point (x, y, z).***

**Solution.** The given curve is the intersection of the following two surfaces

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and
$$x = a\cosh\left(\frac{z}{a}\right) \quad \ldots(2)$$

Here $x = a\cosh t$ and $y = b\sinh t$ satisfy equation (1).

Now, putting $z = at$ in (2), we get

$$x = a\cosh t$$

$\therefore$ Parametric equations of the curve are

$$x = a\cosh t, \quad y = b\sinh t \quad \text{and} \quad z = at$$

Position vector of any point on the curve is

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= a\cosh t\,\hat{i} + b\sinh t\,\hat{j} + at\,\hat{k}$$

$\therefore$
$$\dot{\vec{r}}(t) = a\sinh t\,\hat{i} + b\cosh t\,\hat{j} + a\,\hat{k} \qquad \left[\because \dot{\vec{r}} = \frac{d\vec{r}}{dt}\right]$$

and
$$|\dot{\vec{r}}(t)| = \sqrt{a^2\sinh^2 t + b^2\cosh^2 t + a^2}$$
$$= \sqrt{a^2(1+\sinh^2 t) + b^2\cosh^2 t}$$
$$= \sqrt{a^2\cosh^2 t + b^2\cosh^2 t} \qquad [\because \cosh^2 t - \sinh^2 t = 1]$$
$$= \sqrt{(a^2+b^2)\cosh^2 t}$$
$$= \sqrt{(a^2+b^2)}\cosh t$$

The length of curve is required from point $(a, 0, 0)$ to $(x, y, z)$, so the limits of $t$ are from $t = 0$ to any point $t$.

$\therefore$ Length of the curve is given by

$$s = \int_0^t |\dot{\vec{r}}(t)|\,dt$$
$$= \int_0^t \sqrt{a^2+b^2}\cosh t\,dt$$
$$= \left[\sqrt{a^2+b^2}\sinh t\right]_0^t$$
$$= \sqrt{a^2+b^2}\sinh t = \frac{\sqrt{a^2+b^2}\,y}{b}$$

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**Example 3.** ***Show that the length of the curve***

$$x = 2a\,(\sin^{-1} t + t\sqrt{1-t^2}),\quad y = 2at^2,\quad z = 4at$$

***between the points $t = t_1$ and $t = t_2$ is $4a\sqrt{2}\,(t_2 - t_1)$.*** [M.D.U. 2015]

**Solution.** The position vector $\vec{r}$ of any point on the curve is given by

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= 2a\,[\sin^{-1} t + t\sqrt{1-t^2}]\,\hat{i} + 2at^2\hat{j} + 4at\,\hat{k}$$

$$\therefore \quad \dot{\vec{r}}(t) = 2a\left[\frac{1}{\sqrt{1-t^2}} + \frac{t}{2}\left(\frac{-2t}{\sqrt{1-t^2}}\right) + \sqrt{1-t^2}\right]\hat{i} + 4at\,\hat{j} + 4a\,\hat{k}$$

$$= 2a\left[\frac{1-t^2+1-t^2}{\sqrt{1-t^2}}\right]\hat{i} + 4at\,\hat{j} + 4a\,\hat{k}$$

$$= 4a\sqrt{1-t^2}\,\hat{i} + 4at\,\hat{j} + 4a\,\hat{k}$$

$$\therefore \quad |\dot{\vec{r}}(t)| = [16a^2(1-t^2) + 16a^2t^2 + 16a^2]^{1/2}$$

$$= 4a[1 - t^2 + t^2 + 1]^{1/2} = 4a\sqrt{2}$$

Thus, the length of the curve is given by

$$s = \int_{t_1}^{t_2} |\dot{\vec{r}}(t)|\,dt = \int_{t_1}^{t_2} 4a\sqrt{2}\,dt = 4a\sqrt{2}\,(t_2 - t_1).$$

**Example 4.** ***Find the normal form of the curve***

$$2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 6t\,\hat{k},\quad -\infty < t < \infty$$

[M.D.U. 2018; K.U. 2015]

**Solution.** The equation of the given curve is

$$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 6t\,\hat{k} \qquad ...(1)$$

Differentiating w.r.t. $t$, we have

$$\frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 6\,\hat{k}$$

.U. 2015]

$$\therefore \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t + 36}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 36} = \sqrt{4 + 36} = \sqrt{40}$$

$$= 2\sqrt{10}$$

Let $s$ be the length of arc from $t = 0$ to any point $t$ on the curve, then

$$s = \int_0^t \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^t 2\sqrt{10}\, dt = 2\sqrt{10}\, t$$

$$\therefore \quad s = 2\sqrt{10}\, t \quad \Rightarrow \quad t = \frac{s}{2\sqrt{10}}$$

Putting the value of $t$ in (1), we get

$$\vec{r} = 2\cos\frac{s}{2\sqrt{10}}\,\hat{i} + 2\sin\frac{s}{2\sqrt{10}}\,\hat{j} + \frac{3s}{2\sqrt{10}}\,\hat{k}$$

which is the normal form of the given curve.

**Example 5.** ***Express the curve*** $\vec{r} = e^{2t}\cos t\,\hat{i} + e^{2t}\sin t\,\hat{j} + e^{2t}\,\hat{k}$, $-\infty < t < \infty$ **in the normal form.**

*K.U. 2017, 11; M.D.U. 2017]*

**Solution.** The equation of the given curve is

$$\vec{r} = e^{2t}\cos t\,\hat{i} + e^{2t}\sin t\,\hat{j} + e^{2t}\,\hat{k} \qquad \text{...(1)}$$

Differentiating w.r.t. $t$, we have

$$\frac{d\vec{r}}{dt} = [2e^{2t}\cos t - e^{2t}\sin t]\,\hat{i} + [2e^{2t}\sin t + e^{2t}\cos t]\,\hat{j} + 2e^{2t}\,\hat{k}$$

$$\therefore \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{[e^{2t}(2\cos t - \sin t)]^2 + [e^{2t}(2\sin t + \cos t)]^2 + [2e^{2t}]^2}$$

$$= e^{2t}\sqrt{4\cos^2 t + \sin^2 t - 4\sin t\cos t + 4\sin^2 t + \cos^2 t + 4\sin t\cos t + 4}$$

$$= e^{2t}\sqrt{5\cos^2 t + 5\sin^2 t + 4}$$

$$= e^{2t}\sqrt{5(\cos^2 t + \sin^2 t) + 4}$$

$$= e^{2t}\sqrt{5 + 4} = 3e^{2t}$$

Let $s$ be the length of arc from $t = 0$ to any point $t$ on the curve, then

$$s = \int_0^t \left| \frac{d\vec{r}}{dt} \right| dt = \int_0^t 3e^{2t}\, dt = \frac{3}{2}\left[e^{2t}\right]_0^t$$

or $$s = \frac{3}{2}[e^{2t} - 1]$$

$$\Rightarrow \quad \frac{2s}{3} = e^{2t} - 1 \quad \Rightarrow \quad e^{2t} = \frac{2s}{3} + 1$$

$$\Rightarrow \quad 2t = \log\left(\frac{2s}{3} + 1\right) \quad \Rightarrow \quad t = \frac{1}{2}\log\left(\frac{2s}{3} + 1\right)$$

Putting the value of $t$ in (1), we get

$$\vec{r} = \left[\left(\frac{2s}{3}+1\right)\cos\frac{1}{2}\log\left(\frac{2s}{3}+1\right)\right]\hat{i} + \left[\left(\frac{2s}{3}+1\right)\sin\frac{1}{2}\log\left(\frac{2s}{3}+1\right)\right]\hat{j} + \left(\frac{2s}{3}+1\right)\hat{k}$$

which is the normal form of the given curve.

**Example 6.** ***Find the unit tangent vector t and the direction cosines of the tangent at any point to the circular helix***

$$x = a\cos t, \quad y = a\sin t, \quad z = bt$$ [M.D.U. 2017; K.U. 2013]

**Solution.** The position vector $\vec{r}$ of any point on the curve is given as

$$\vec{r} = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + bt\,\hat{k}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = (-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + b\,\hat{k} \qquad ...(1)$$

$$\text{Unit tangent vector at any point} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$$

$$= \frac{1}{\sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}}\,[-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}]$$

$$= \frac{1}{\sqrt{a^2 + b^2}}\,[-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}]$$

$$= \frac{-a \sin t}{\sqrt{a^2+b^2}}\hat{i} + \frac{a \cos t}{\sqrt{a^2+b^2}}\hat{j} + \frac{b}{\sqrt{a^2+b^2}}\hat{k}$$

$\therefore$ Direction cosines of the tangent line at any point to the circular helix are

$$\frac{-a \sin t}{\sqrt{a^2+b^2}}, \frac{a \cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}.$$

**Remember:**

Unit vector along the tangent is $\dfrac{d\vec{r}}{dt} \Big/ \left|\dfrac{d\vec{r}}{dt}\right|$.

## EXERCISE 8.1

1. Find the length of one complete turn of the circular helix

$$\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k},\ -\infty < t < \infty$$

[**Hint :** Limits for one complete turn of the helix are from any point $t_0$ to $t_0 + 2\pi$]

2. Obtain the equation of the circular helix

$$\vec{r} = (a\cos t, a\sin t, bt), -\infty < t < \infty, \text{ where } a > 0,$$

referred to $s$ as parameter and show that length of one complete turn of the helix is $2\pi c$, where $c = \sqrt{a^2+b^2}$.

[**Hint :** $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$ ]

3. Find the length of the arc of the curve $x = 3\cosh 2t$, $y = 3\sinh 2t$, $z = 6t$ from $t = 0$ to $t = \pi$.

4. Express the curve $\vec{r} = e^t\cos t\,\hat{i} + e^t\sin t\,\hat{j} + e^t\,\hat{k}$ in the normal form. [K.U. 2013, 12]

5. For the helix $x = a\cos t, y = a\sin t, z = at\tan\alpha$, show that $\dfrac{ds}{dt} = a\sec\alpha$ and that the length of the curve measured from the point $t = 0$ is $at\sec\alpha$.

6. Find the arc length as a function of $\theta$ along the epicycloid

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

$$y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right),\quad z = 0$$

7. Find the length of the arc of the curve $x = 2t$, $y = t^2$, $z = (\sin^{-1} t + t\sqrt{1-t^2})$ from $t = 0$ to $t = \pi$. [M.D.U. 2014, 08]

EXERCISE 8.1

8. Find the length of the curve given by the intersection of the surfaces $x^2 - y^2 = 1$, $x = \cosh z$ from the point (1, 0, 0) to the point $(x, y, z)$.

[**Hint.** Parametric equations of curve are $x = \cosh t$, $y = \sinh t$, $z = t$]

9. Find the length of the curve

$$\vec{r} = 4\cosh 2t\,\hat{i} + 4\sinh 2t\,\hat{j} + 8t\,\hat{k},\ 0 \le t \le \pi.$$

10. Find $\hat{t}$ for the curve

$$\vec{r} = \log\cos t\,\hat{i} + \log\sin t\,\hat{j} + t\sqrt{2}\,\hat{k}.$$

**ANSWERS**

1. $2\pi\sqrt{a^2+c^2}$

2. Equation of the helix is $\vec{r} = a\cos\left(\frac{s}{c}\right)\hat{i} + a\sin\left(\frac{s}{c}\right)\hat{j} + \left(\frac{bs}{c}\right)\hat{k}$

3. $3\sqrt{2}\sinh 2\pi$

4. $\vec{r} = \left[\left(\frac{s}{\sqrt{3}}+1\right)\cos\log\left(\frac{s}{\sqrt{3}}+1\right)\right]\hat{i} + \left[\left(\frac{s}{\sqrt{3}}+1\right)\sin\log\left(\frac{s}{\sqrt{3}}+1\right)\right]\hat{j} + \left[\frac{s}{\sqrt{3}}+1\right]\hat{k}$

6. $s = \frac{4(a+b)\,b}{a}\left[1-\cos\frac{a}{2b}\theta\right]$

7. $2\sqrt{2}\,\pi$ 8. $\sqrt{2}\,y$ 9. $4\sqrt{2}\sinh 2\pi$

10. $\hat{t} = -\sin^2 t\,\hat{i} + \cos^2 t\,\hat{j} + \sqrt{2}\sin t\cos t\,\hat{k}$

## 8.6. EQUATION OF A TANGENT LINE AT A POINT ON A SPACE CURVE

Let $\vec{r}$ be the position vector of point P on the curve. Then unit vector along the tangent at P is $\frac{d\vec{r}}{ds}$ and denoted by $\vec{r}\,'$ or $\hat{t}$.

Let $\vec{R}$ be the position vector of any point Q on the tangent line at P.

Obviously, tangent line is a line through P and parallel to vector $\vec{r}\,'$

$\therefore$ $\vec{PQ} = |\vec{PQ}|\,\vec{r}\,' = \lambda\,\vec{r}\,'$, where $\lambda$ is a scalar parameter.

Using triangle law, we have

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

$$\vec{R} = \vec{r} + \lambda \vec{r}' \quad \text{...(1)}$$

which is the required equation of tangent line at P.

If any other parameter is used in place of parameter $\lambda$, we modify the above equation as

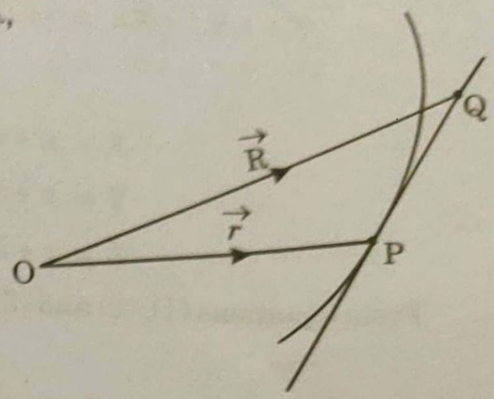


Fig. 8.2

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} = \vec{r}'\frac{ds}{dt}$$

or

$$\vec{r}' = \frac{\dot{\vec{r}}}{\frac{ds}{dt}}$$

Substituting the value of $\vec{r}'$ in (1), we have

$$\vec{R} = \vec{r} + \lambda\frac{\dot{\vec{r}}}{\frac{ds}{dt}}$$

i.e.,

$$\vec{R} = \vec{r} + \mu\dot{\vec{r}}, \quad \text{where } \frac{\lambda}{\frac{ds}{dt}} = \mu \quad \text{...(2)}$$

which is the required equation of tangent line at point P.

**Remember :**

If equation of curve is given in terms of parameter $s$, then equation (1) is the equation of tangent and if equation of curve is given in terms of parameter $t$, then equation (2) is the equation of tangent.

## 8.7. CARTESIAN FORM OF EQUATION OF TANGENT LINE

Let the coordinates of point P be $(x, y, z)$ and that of neighbouring point Q be (X, Y, Z).

$\therefore$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$$

where $x', y', z'$ denotes the derivatives w.r.t. $s$

and $$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$\therefore$ Equation of tangent line is

$$\vec{R} = \vec{r} + \lambda \vec{r}'$$

which can be written as

$$X\hat{i} + Y\hat{j} + Z\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k}) + \lambda(x'\hat{i} + y'\hat{j} + z'\hat{k})$$

[*Putting the values of* $\vec{R}, \vec{r}$ *and* $\vec{r}'$]

$\therefore$
$$X = x + \lambda x' \Rightarrow X - x = \lambda x' \quad ...(1)$$
$$Y = y + \lambda y' \Rightarrow Y - y = \lambda y' \quad ...(2)$$
$$Z = z + \lambda z' \Rightarrow Z - z = \lambda z' \quad ...(3)$$

From equations (1), (2) and (3), we have

$$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'} (= \lambda) \quad ...(4)$$

which is the equation of tangent line in cartesian form.

Also we know that $x'^2 + y'^2 + z'^2 = 1$

$\therefore$ $x', y', z'$ are direction cosines of the tangent line.

If we take the equation of tangent line at P$(x, y, z)$ to the curve as $\vec{R} = \vec{r} + \mu\dot{\vec{r}}$, then as above the cartesian form is

$$\frac{X-x}{\dot{x}} = \frac{Y-y}{\dot{y}} = \frac{Z-z}{\dot{z}} \quad ...(5)$$

where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}, \dot{z} = \frac{dz}{dt}$

Here $\dot{x}, \dot{y}, \dot{z}$ are not the direction cosines of the tangent line but these are the direction ratios.

### 8.7.1. To find the equation of the tangent line to the curve of intersection of $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$.

The equation of curve of intersection of given surfaces is given as $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$.

Differentiating w.r.t. parameter $s$, we have

$$\frac{\partial f_1}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f_1}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial f_1}{\partial z}\cdot\frac{dz}{ds} = 0$$

$$\frac{\partial f_2}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial f_2}{\partial y}\cdot\frac{dy}{ds}+\frac{\partial f_2}{\partial z}\cdot\frac{dz}{ds}=0,$$

Solving these equations, we have

$$\frac{\frac{dx}{ds}}{\frac{\partial f_1}{\partial y}\cdot\frac{\partial f_2}{\partial z}-\frac{\partial f_1}{\partial z}\cdot\frac{\partial f_2}{\partial y}}=\frac{\frac{dy}{ds}}{\frac{\partial f_1}{\partial z}\cdot\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial z}}=\frac{\frac{dz}{ds}}{\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\cdot\frac{\partial f_2}{\partial x}} \quad ...(1)$$

But equation of tangent line is

$$\frac{X-x}{\frac{dx}{ds}}=\frac{Y-y}{\frac{dy}{ds}}=\frac{Z-z}{\frac{dz}{ds}} \quad ...(2)$$

$\therefore$ From (1) and (2), the equation of tangent line is

$$\frac{X-x}{\frac{\partial f_1}{\partial y}\cdot\frac{\partial f_2}{\partial z}-\frac{\partial f_1}{\partial z}\cdot\frac{\partial f_2}{\partial y}}=\frac{Y-y}{\frac{\partial f_1}{\partial z}\cdot\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial z}}=\frac{Z-z}{\frac{\partial f_1}{\partial x}\cdot\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\cdot\frac{\partial f_2}{\partial x}}$$

## SOLVED EXAMPLES

**Example 1.** ***Find the equation of tangent line at the point $t=1$ to the curve***

$$x=1+t,\quad y=-t^2,\quad z=1+t^2$$ [K.U. 2014]

**Solution.** The given equation of curve is

$$x=1+t,\quad y=-t^2,\quad z=1+t^2$$

$$\therefore \quad \frac{dx}{dt}=1 \quad \frac{dy}{dt}=-2t,\quad \frac{dz}{dt}=2t$$

At $t=1$, we have

$$x=2,\quad y=-1,\quad z=2$$

and

$$\frac{dx}{dt}=1,\quad \frac{dy}{dt}=-2,\quad \frac{dz}{dt}=2$$

$\therefore$ Equation of the tangent line at the point $t=1$ is

$$\frac{X-x}{\frac{dx}{dt}}=\frac{Y-y}{\frac{dy}{dt}}=\frac{Z-z}{\frac{dz}{dt}}$$

i.e.,

$$\frac{X-2}{1}=\frac{Y+1}{-2}=\frac{Z-2}{2}$$

**Example 2.** ***Show that the tangent at any point of the curve whose equation referred to rectangular axes are $x = 3t$, $y = 3t^2$, $z = 2t^3$ makes a constant angle with the line $y = z - x = 0$.*** [K.U. 2016]

**Solution.** Here $\vec{r} = 3t\,\hat{i} + 3t^2\,\hat{j} + 2t^3\,\hat{k}$

$$\therefore \quad \frac{d\vec{r}}{dt} = 3\,\hat{i} + 6t\,\hat{j} + 6t^2\,\hat{k}$$

and
$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9 + 36t^2 + 36t^4}$$

Unit vector along the tangent,
$$\hat{t} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$$

$$= \frac{3\hat{i} + 6t\hat{j} + 6t^2\hat{k}}{\sqrt{9 + 36t^2 + 36t^4}} = \frac{3\hat{i} + 6t\hat{j} + 6t^2\hat{k}}{3(1 + 2t^2)}$$

Direction cosines of tangent to the curve at any point are $\dfrac{1}{1+2t^2}, \dfrac{2t}{1+2t^2}, \dfrac{2t^2}{1+2t^2}$.

Now, equation of given line $y = z - x = 0$ can be written as $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{1}$.

Its direction cosines are $< \dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} >$.

Let the tangent makes an angle $\theta$ with the given line

$$\therefore \quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

*i.e.,*
$$\cos\theta = \frac{1}{\sqrt{2}}\left(\frac{1}{1+2t^2}\right) + 0.\left(\frac{2t}{1+2t^2}\right) + \frac{1}{\sqrt{2}}\left(\frac{2t^2}{1+2t^2}\right)$$

*i.e.,* $\cos\theta = \dfrac{1}{\sqrt{2}}$. Hence, $\theta = \dfrac{\pi}{4}$, which is a constant angle.

**Example 3.** ***Show that the tangent at a point of the curve of intersection of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and the confocal $\dfrac{x^2}{a^2 - \lambda} + \dfrac{y^2}{b^2 - \lambda} + \dfrac{z^2}{c^2 - \lambda}$ is given by***

$$\frac{x(X - x)}{a^2(b^2 - c^2)(a^2 - \lambda)} = \frac{y(Y - y)}{b^2(c^2 - a^2)(b^2 - \lambda)} = \frac{z(Z - z)}{c^2(a^2 - b^2)(c^2 - \lambda)}$$

**Solution.** Here equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad \ldots(1)$$

and equation of confocal to (1) is

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}-1=0 \qquad \ldots(2)$$

Let $\vec{r}=\vec{r}(t)$ be the curve of intersection of surfaces (1) and (2).

Differentiating (1) w.r.t. $t$, we have

$$\frac{2x}{a^2}\frac{dx}{dt}+\frac{2y}{b^2}\frac{dy}{dt}+\frac{2z}{c^2}\frac{dz}{dt}=0$$

or

$$\frac{x\dot{x}}{a^2}+\frac{y\dot{y}}{b^2}+\frac{z\dot{z}}{c^2}=0 \qquad \ldots(3)$$

Differentiating (2) w.r.t. $t$, we have

$$\frac{2x}{a^2-\lambda}\cdot\frac{dx}{dt}+\frac{2y}{b^2-\lambda}\cdot\frac{dy}{dt}+\frac{2z}{c^2-\lambda}\cdot\frac{dz}{dt}=0$$

or

$$\frac{x\dot{x}}{a^2-\lambda}+\frac{y\dot{y}}{b^2-\lambda}+\frac{z\dot{z}}{c^2-\lambda}=0 \qquad \ldots(4)$$

Solving (3) and (4) for $\dot{x}$, $\dot{y}$ and $\dot{z}$, we have

$$\frac{\dot{x}}{\dfrac{yz}{b^2(c^2-\lambda)}-\dfrac{yz}{c^2(b^2-\lambda)}}=\frac{\dot{y}}{\dfrac{zx}{c^2(a^2-\lambda)}-\dfrac{zx}{a^2(c^2-\lambda)}}=\frac{\dot{z}}{\dfrac{xy}{a^2(b^2-\lambda)}-\dfrac{xy}{b^2(a^2-\lambda)}}$$

or

$$\frac{\dot{x}}{\dfrac{yz\lambda\,(b^2-c^2)}{b^2c^2\,(c^2-\lambda)\,(b^2-\lambda)}}=\frac{\dot{y}}{\dfrac{zx\lambda\,(c^2-a^2)}{c^2a^2\,(a^2-\lambda)\,(c^2-\lambda)}}=\frac{\dot{z}}{\dfrac{xy\lambda\,(a^2-b^2)}{a^2b^2\,(b^2-\lambda)\,(a^2-\lambda)}}$$

or

$$\frac{x\dot{x}}{a^2\,(a^2-\lambda)\,(b^2-c^2)}=\frac{y\dot{y}}{b^2\,(b^2-\lambda)\,(c^2-a^2)}=\frac{z\dot{z}}{c^2\,(c^2-\lambda)\,(a^2-b^2)}$$

or

$$\frac{\dot{x}}{\dfrac{a^2\,(a^2-\lambda)\,(b^2-c^2)}{x}}=\frac{\dot{y}}{\dfrac{b^2\,(b^2-\lambda)\,(c^2-a^2)}{y}}=\frac{\dot{z}}{\dfrac{c^2\,(c^2-\lambda)\,(a^2-b^2)}{z}}$$

Equation of the tangent line at $(x, y, z)$ is

$$\frac{X-x}{\dot{x}}=\frac{Y-y}{\dot{y}}=\frac{Z-z}{\dot{z}}$$

*i.e.,*
$$\frac{x(X-x)}{a^2(a^2-\lambda)(b^2-c^2)}=\frac{y(Y-y)}{b^2(b^2-\lambda)(c^2-a^2)}=\frac{z(Z-z)}{c^2(c^2-\lambda)(a^2-b^2)}$$

Hence the result.

## EXERCISE 8.2

1. Find the unit tangent to the circular helix $\vec{r}=(a\cos t, a\sin t, bt)$, $-\infty<t<\infty$, where $a>0$. Write down the equation of tangent at any point. Also find the length of one complete turn of the helix.

2. Find the equation of tangent to the curve $\vec{r}=\cos t\,\hat{i}+2\sin t\,\hat{j}$ at the point $\left(\frac{1}{2}, \sqrt{3}, 0\right)$.

3. Find the intersection of $xy$ plane and tangent line to the curve $\vec{r}=(1+t)\hat{i}-t^2\hat{j}+(1+t^3)\hat{k}$ at $t=1$.

4. Show that tangent at any point on the curve $\vec{r}=at\,\hat{i}+bt^2\hat{j}+t^3\hat{k}$, $2b^2=3a$ makes a constant angle with the line $x-z=0, y=0$.

## ANSWERS

1. $\dfrac{X-a\cos t}{-a\sin t}=\dfrac{Y-a\sin t}{a\cos t}=\dfrac{Z-bt}{b}$; $2\pi\sqrt{a^2+b^2}$

2. $\vec{R}=\frac{1}{2}\hat{i}+\sqrt{3}\hat{j}+\lambda\left(-\frac{\sqrt{3}}{2}\hat{i}+\hat{j}\right)$

3. $\left(\frac{4}{3}, \frac{1}{3}, 0\right)$

## 8.8. FUNCTIONS OF CLASS m

**Definition :** *A real valued function defined on an interval I is said to be of class $m(\geq 1)$, if it has continuous derivatives of mth order at every point of I.*

We denote it as **$C^m$ function.**

If $f$ is differentiable infinite number of times then we say it is of *infinite class* or it is a $C^\infty$ function.

## 8.3 OSCULATING PLANE (or PLANE OF CURVATURE)

**Definition :** *The osculating plane at a point P of a curve of class $\geq 2$, is the limiting position of the plane which contains the tangent line at P and a neighbouring point Q on the curve as $Q \to P$.*

### 8.3.1 Derivation of Equation of the Osculating Plane at a Point P of the Curve

Here we shall derive the equation of osculating plane is terms of parameter 's'.

Let $\vec{r} = \vec{r}(s)$ be the equation of the given curve C of class $\geq 2$. Suppose arc length 's' is measured from some fixed point A on the curve.

Suppose P and Q are two neighbouring points on the curve C such that

Arc AP $= s$

and

Arc AQ $= s + \delta s$

$\therefore$ Position vector of P *i.e.,* $\overrightarrow{OP} = \vec{r}(s)$

and position vector of Q *i.e.,* $\overrightarrow{OQ} = \vec{r}(s + \delta s)$

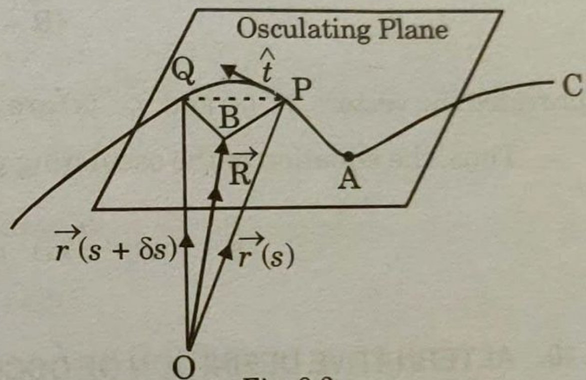

*Fig. 8.3*

Let $\vec{R}$ be the position vector of current point B on the plane containing the tangent line at P and the point Q.

$\therefore$ The vector $\overrightarrow{PB}$, $\hat{t}$ (unit vector along the tangent) and $\overrightarrow{PQ}$ lies in the plane BPQ. Hence these vectors are coplanar. Therefore, their scalar triple product is zero

*i.e.,* $\overrightarrow{PB} \cdot \hat{t} \times \overrightarrow{PQ} = 0$ ...(1)

Now, $\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \vec{R} - \vec{r}(s)$

Also $\hat{t} = \vec{r}\,'(s)$

and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}(s + \delta s) - \vec{r}(s)$

$\therefore$ From (1), $\{\vec{R} - \vec{r}(s)\} . \vec{r}\,'(s) \times \{\vec{r}(s + \delta s) - \vec{r}(s)\} = 0$ ...(2)

Now, $\vec{r}(s + \delta s) = \vec{r}(s) + \delta s\, \vec{r}\,'(s) + \dfrac{(\delta s)^2}{2!} \vec{r}\,''(s) + O\{(\delta s)^3\}$

[*By Taylor's theorem $O(\delta s)^3$ implies sum of terms having power of $\delta s$ three or more*]

$$\therefore \quad \vec{r}(s+\delta s) - \vec{r}(s) = \delta s\, \vec{r}'(s) + \frac{(\delta s)^2}{2!} \vec{r}''(s) + O\{(\delta s)^3\}$$

Using this in (2), we have

$$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}'(s) \times \left[\delta s\, \vec{r}'(s) + \frac{(\delta s)^2}{2!} \vec{r}''(s) + O\{(\delta s)^3\}\right] = 0$$

or $$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}'(s) \times \left[\frac{(\delta s)^2}{2!} \vec{r}''(s) + O\{(\delta s)^3\}\right] = 0 \qquad [\because \vec{r}'(s) \times \vec{r}'(s) = \vec{0}]$$

or $$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}'(s) \times [\vec{r}''(s) + O(\delta s)] = 0$$

Proceeding to limits as $Q \to P$ *i.e.*, $\delta s \to 0$, the limiting position of plane is

$$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}'(s) \times \vec{r}''(s) = 0 \qquad ...(3)$$

provided the vectors $\vec{r}'(s)$ and $\vec{r}''(s)$ are not linearly dependent.

Thus, the equation of the osculating plane is

$$[\vec{R} - \vec{r}(s) \;\; \vec{r}'(s) \;\; \vec{r}''(s)] = 0.$$

## 8.10. ALTERNATIVE DEFINITION OF OSCULATING PLANE

*The limiting position of the plane passing through the tangent at P on a curve and parallel to the tangent at the neighbouring point Q on the curve when $Q \to P$ is defined as the osculating plane or plane of curvature at the point P.*

### 8.10.1. Equation of the Osculating Plane from above definition

Here we shall derive the equation of osculating plane in terms of parameter '$t$'.

Let the equation of curve C be $\vec{r} = \vec{r}(t)$.

Let $P(t)$ and $Q(t + \delta t)$ be two neighbouring points on the curve C.

$\therefore$ $\vec{r}(t)$ and $\vec{r}(t + \delta t)$ are the position vectors of the points P and Q referred to some origin O. Then tangents to the curve at P and Q are parallel to the vectors $\dot{\vec{r}}(t)$ and $\dot{\vec{r}}(t + \delta t)$ respectively.

∴ Plane through tangent at P and parallel to tangent at Q is perpendicular to the vector

$$\dot{\vec{r}}(t) \times \dot{\vec{r}}(t+\delta t)$$

i.e., perpendicular to the vector $\dot{\vec{r}}(t) \times [\dot{\vec{r}}(t+\delta t) - \dot{\vec{r}}(t)]$ $\qquad [\because \dot{\vec{r}}(t) \times \dot{\vec{r}}(t) = \vec{0}]$

i.e., plane is perpendicular to $\dot{\vec{r}}(t) \times \left[\dfrac{\dot{\vec{r}}(t+\delta t) - \dot{\vec{r}}(t)}{\delta t}\right]$ $\qquad [\because \delta t \text{ is a scalar}]$

As $Q \to P$, $\delta t \to 0$.

In the limiting position, the plane becomes the osculating plane.

∴ Osculating plane is perpendicular to the vector $\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)$

$$\left[\because \lim_{\delta t \to 0} \frac{\dot{\vec{r}}(t+\delta t) - \dot{\vec{r}}(t)}{\delta t} = \ddot{\vec{r}}(t)\right]$$

If $\vec{R}$ is the position vector of any current point B on osculating plane, then vector

$\vec{PB} = \vec{R} - \vec{r}(t)$ lies on osculating plane to the curve at P

∴ $\vec{PB}$ is perpendicular to $\dot{\vec{r}}(t) \times \ddot{\vec{r}}(t)$

Hence their dot product is zero *i.e.*, $\{\vec{R} - \vec{r}(t)\} \cdot \dot{\vec{r}}(t) \times \ddot{\vec{r}}(t) = 0$

∴ Equation of osculating plane is

$$[\vec{R} - \vec{r}(t) \quad \dot{\vec{r}}(t) \quad \ddot{\vec{r}}(t)] = 0 \cdot \qquad \text{...(4)}$$

### 8.10.2. Cartesian Form of Osculating Plane

Let (X, Y, Z) be the coordinates of current point on the osculating plane and coordinates of P be $(x(s), y(s), z(s))$. Then

$$\vec{R} - \vec{r}(s) = \{X - x(s)\}\hat{i} + \{Y - y(s)\}\hat{j} + \{Z - z(s)\}\hat{k}$$

and

$$\vec{r}\,'(s) = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

i.e.,

$$\vec{r}\,'(s) = x'(s)\hat{i} + y'(s)\hat{j} + z'(s)\hat{k}$$

and

$$\vec{r}\,''(s) = x''(s)\hat{i} + y''(s)\hat{j} + z''(s)\hat{k}$$

The equation of osculating plane is

$$[\vec{R} - \vec{r}(s)\ \vec{r}\,'(s)\ \vec{r}\,''(s)] = 0$$

or
$$\begin{vmatrix} X - x(s) & Y - y(s) & Z - z(s) \\ x'(s) & y'(s) & z'(s) \\ x''(s) & y''(s) & z''(s) \end{vmatrix} = 0$$

In case, coordinates $(x, y, z)$ of point P are functions of parameter $t$, then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\dot{\vec{r}} = \dot{x}(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + \dot{z}(t)\,\hat{k}$$

and
$$\ddot{\vec{r}} = \ddot{x}(t)\,\hat{i} + \ddot{y}(t)\,\hat{j} + \ddot{z}(t)\,\hat{k}$$

Then, the equation of osculating plane is

$$[\vec{R} - \vec{r}(t)\ \dot{\vec{r}}(t)\ \ddot{\vec{r}}(t)] = 0$$

or
$$\begin{vmatrix} X - x(t) & Y - y(t) & Z - z(t) \\ \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \\ \ddot{x}(t) & \ddot{y}(t) & \ddot{z}(t) \end{vmatrix} = 0$$

where single dot stands for derivative with respect to parameter $t$ and two dots stands for the second derivative with respect to parameter $t$.

## 8.11. ANALYTIC FUNCTION

*A function f is said to be* ***analytic*** *over I if it is single valued and possesses continuous derivatives of all orders at every point of I.*

### 8.11.1 To show that if a curve is analytic, there exists a definite osculating plane at a point of inflexion unless the curve is a straight line.

**Proof :** Let $\vec{r}$ be the position vector of any point P on the curve. If $\hat{t}$ denotes the unit tangent vector at P, then we know that

$$\hat{t} = \frac{d\vec{r}}{ds} = \vec{r}\,'$$

$$|\hat{t}| = |\vec{r}\,'| = 1$$

and

$$\vec{r}\,' \cdot \vec{r}\,' = 1$$

∴ Differentiating both sides w.r.t. 's', we get

$$\vec{r}\,' \cdot \vec{r}\,'' + \vec{r}\,'' \cdot \vec{r}\,' = 0$$

or

$$2(\vec{r}\,' \cdot \vec{r}\,'') = 0$$

or

$$\vec{r}\,' \cdot \vec{r}\,'' = 0$$

Differentiating again w.r.t. 's', we get

$$\vec{r}\,' \cdot \vec{r}\,''' + \vec{r}\,'' \cdot \vec{r}\,'' = 0 \qquad ...(1)$$

As P is point of inflexion, therefore $\vec{r}\,'' = 0$

∴ From (1), $\vec{r}\,' \cdot \vec{r}\,''' = 0$ ...(2)

If $\vec{r}\,''' \neq 0$, then $\vec{r}\,'$ is linearly independent of $\vec{r}\,'''$.

Differentiating (2), w.r.t. 's', we get

$$\vec{r}\,' \cdot \vec{r}\,'''' + \vec{r}\,'' \cdot \vec{r}\,''' = 0$$

∴ $\vec{r}\,' \cdot \vec{r}\,'''' = 0$ [∵ $\vec{r}\,'' = 0$ *at point of inflexion*]

If $\vec{r}\,'''' \neq 0$, then $\vec{r}\,'$ is linearly independent of $\vec{r}\,''''$.

Repeating the above argument again and again, we have the result

$$\vec{r}\,' . \vec{r}\,^{m} = 0, m > 2$$

where $\vec{r}\,^{m}$ is the first non-zero derivative of $\vec{r}$ at P and $\vec{r}\,'' = \vec{r}\,''' = \vec{r}\,'''' = .... = \vec{r}\,^{m-1} = 0$

Then from Art. 8.9.1, we obtain

$$\vec{r}(s + \delta s) - \vec{r}(s) = \frac{(\delta s)^m}{m!}\,\vec{r}\,^{m}(s) + O(\delta s)^{m+1}$$

Using this in the equation (2) of the article 8.9.1, we have

$$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}\,'(s) \times \left[\frac{(\delta s)^m}{m!}\,\vec{r}\,^{m}(s) + O(\delta s)^{m+1}\right] = 0$$

$$\vec{R} - \vec{r}(s) \cdot \vec{r}'(s) \times \{\vec{r}^m(s) + O(\delta s)\} = 0$$

or

Proceeding to limits as $Q \to P$ *i.e.*, $\delta s \to 0$, the equation of the osculating plane is,

$$\{\vec{R} - \vec{r}(s)\} \cdot \vec{r}'(s) \times \vec{r}^m(s) = 0$$

If $\vec{r}^m = 0$ for all $m \geq 2$, then the curve being analytic we conclude that $\vec{r}'(=\hat{t})$ is constant *i.e.*, the tangent vector is same at each point of the curve and hence the curve is a straight line.

## 8.12. ORDER OF CONTACT BETWEEN CURVE AND SURFACE

**Definition.** *If P, $P_1$, $P_2$, ......, $P_n$ points on a given curve lie on a given surface and $P_1, P_2, \ldots\ldots P_n$ coincide with P, then the curve and the surface are said to have the contact of nth order at the point P.*

Let S be the surface whose equation is $f(x, y, z) = 0$

and C be the curve whose equation is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where $x, y, z$ are functions of parameter $t$.

The common points of curve and surface are the roots of equation

$$f[x(t),\ y(t),\ z(t)] = 0$$

which can be written as $\quad f(t) = 0$

Let $t_0$ be one root of $f(t) = 0$

$$\therefore \quad f(t_0) = 0 \qquad \ldots(1)$$

Now,
$$f(t) = f[t_0 + (t - t_0)]$$
$$= f(t_0 + h) \text{ where } h = t - t_0$$

Expanding $f(t)$ by Taylor's theorem, we have

$$f(t) = f(t_0) + h f'(t_0) + \frac{h^2}{2!} f''(t_0) + \ldots\ldots + \frac{h^n}{n!} f^n(t_0) + \ldots..$$

or

$$f(t) = h f'(t_0) + \frac{h^2}{2!} f''(t_0) + \ldots\ldots + \frac{h^n}{n!} f^n(t_0) + \ldots.. \qquad [\because f(t_0) = 0, \text{ by } (1)]$$

**Case I.** $f'(t_0) \neq 0$ :

In this case $t_0$ is said to be simple zero of $f(t)$. The curve and the surface have contact of **zero order**.

*Case* **II.** $f'(t_0) = 0$ *and* $f''(t_0) \neq 0$ :

In this case $t_0$ is said to be double zero of $f(t)$. The curve and the surface have double point contact and this contact is of **first order**.

*Case* **III.** $f'(t_0) = f''(t_0) = 0$ *and* $f'''(t_0) \neq 0$ :

In this case $t_0$ is said to be triple zero of $f(t)$. The curve and the surface have three point contact and this contact is of **second order**.

In general, if $f'(t_0) = f''(t_0) = \ldots\ldots = f^n(t_0) = 0$ and $f^{n+1}(t_0) \neq 0$, then the given curve and surface have contact at $(n + 1)$ points and this contact is of ***n*th order**.

**Inflexional Tangent.** *A straight line is called inflexional tangent to the surface S if it has a second order point of contact with the surface at that point.*

## SOLVED EXAMPLES

**Example 1.** ***Find the equation of the plane that has three point contact at the origin with the curve***

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1.$$

**Solution.** Any plane through the origin is

$$ax + by + cz = 0 \qquad \ldots(1)$$

Equations of given curve are

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1 \qquad \ldots(2)$$

At origin, we have $t = 1$

Points which are common to (1) and (2) are obtained by solving (1) and (2). Substituting the values of $x, y, z$ from (2) in (1), we have

$$F(t) = a(t^4 - 1) + b(t^3 - 1) + c(t^2 - 1) = 0 \qquad \ldots(3)$$

Since the plane has three point contact with the curve at the origin

$$\therefore \quad F'(1) = 0 \text{ and } F''(1) = 0 \qquad \ldots(4)$$

$\therefore$ From (3),

$$\left.\begin{aligned} F'(t) &= 4at^3 + 3bt^2 + 2ct \\ F''(t) &= 12at^2 + 6bt + 2c \end{aligned}\right\} \qquad \ldots(5)$$

$\therefore$ From (4) and (5), we have

$$F'(1) = 4a + 3b + 2c = 0$$

$$F''(1) = 12a + 6b + 2c = 0$$

Solving these equations, we have

$$\frac{a}{6-12} = \frac{b}{24-8} = \frac{c}{24-36}$$

or

$$\frac{a}{3} = \frac{b}{-8} = \frac{c}{6}$$

Hence, the equation of the required plane is $3x - 8y + 6z = 0$.

**Example 2.** *Show that the condition that four consecutive points of a curve should be coplanar is*

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

**Solution.** Let us suppose that parametric equations of the curve are

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad \text{...(1)}$$

Equation of the plane through a point $(x_0, y_0, z_0)$ on the curve is given by

$$(x - x_0)\, l + (y - y_0)\, m + (z - z_0)\, n = 0 \qquad \text{...(2)}$$

Let $\quad F(t) = [x(t) - x(t_0)]\, l + [y(t) - y(t_0)]\, m + [z(t) - z(t_0)]\, n$

The plane will pass through four consecutive points if it has four point contact with the curve

*i.e.*, if $\quad F(t_0) = F'(t_0) = F''(t_0) = F'''(t_0) = 0$

which gives

$$x'l + y'm + z'n = 0$$

$$x''l + y''m + z''n = 0$$

$$x'''l + y'''m + z'''n = 0$$

where dashes denote the differentiation w.r.t. $t$.

Eliminating $l, m, n$ from above equations, we get

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

**Example 3.** *Find the equation of osculating plane of the curve*

$$x = 2 \log t, \quad y = 4t, \quad z = 2t^2 + 1$$

[K.U. 2014, 13; M.D.U. 2014, 12; C.D.L.U. 2013]

**Solution.** The position vector $\vec{r}$ of any point on the curve is given by

$$\vec{r} = (2 \log t)\,\hat{i} + 4t\,\hat{j} + (2t^2 + 1)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = \frac{2}{t}\hat{i} + 4\hat{j} + 4t\hat{k}$$

$\therefore$

$$\frac{d^2\vec{r}}{dt^2} = \left(-\frac{2}{t^2}\right)\hat{i} + 0\hat{j} + 4\hat{k}$$

and

If (X, Y, Z) denotes the current coordinates of a point on osculating plane, then

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$\therefore$

$$\vec{R} - \vec{r} = (X - 2\log t)\hat{i} + (Y - 4t)\hat{j} + (Z - 2t^2 - 1)\hat{k}$$

The equation of the osculating plane is

$$\left[\vec{R} - \vec{r} \quad \frac{d\vec{r}}{dt} \quad \frac{d^2\vec{r}}{dt^2}\right] = 0$$

or

$$\begin{vmatrix} X - 2\log t & Y - 4t & Z - (2t^2 + 1) \\ \frac{2}{t} & 4 & 4t \\ -\frac{2}{t^2} & 0 & 4 \end{vmatrix} = 0$$

or

$$(X - 2\log t)16 - (Y - 4t)\left(\frac{8}{t} + \frac{8t}{t^2}\right) + (Z - (2t^2 + 1))\left(\frac{8}{t^2}\right) = 0$$

or

$$16X - Y\left(\frac{8}{t} + \frac{8t}{t^2}\right) + \frac{8}{t^2}Z + \left[-32\log t + 4t\left(\frac{8}{t} + \frac{8t}{t^2}\right) - \frac{8}{t^2}(2t^2 + 1)\right] = 0$$

Multiplying by $t^2$ and dividing by 8, we have

$$2t^2X - Y\,2t + Z - 4t^2\log t + 6t^2 - 1 = 0$$

or

$$2t^2X - 2t\,Y + Z = 4t^2\log t - 6t^2 + 1.$$

**Example 4.** ***For the curve $x = 3t, y = 3t^2, z = 2t^3$, show that any plane meets it in three points and find the equation to the osculating plane at $t = t_1$.***

*[K.U. 2018, 15, 14; M.D.U. 2015, 12, 11; C.D.L.U. 2016; 13]*

**Solution.** Parametric equations of the curve are

$$x = 3t, y = 3t^2, z = 2t^3, \quad ...(1)$$

The equation of the plane is of the form

$$Ax + By + Cz + D = 0 \quad ...(2)$$

Points which are common to (1) and (2) are obtained by solving them. Substituting the values of $x, y, z$ from (1) in (2), we have

$$F(t) = 3At + 3Bt^2 + 2Ct^3 + D = 0$$

or

$$F(t) = 2Ct^3 + 3Bt^2 + 3At + D = 0 \qquad ...(3)$$

which is a cubic in $t$ and gives three values of $t$.

From (1), we get three points of intersection. Thus, any plane meets the curve (1) in three points.

Now, $\quad x = 3t, \quad y = 3t^2, \quad z = 2t^3$

$$\therefore \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 6t, \quad \frac{dz}{dt} = 6t^2$$

and

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 6, \quad \frac{d^2z}{dt^2} = 12t$$

$\therefore$ Equation of osculating plane at $t = t_1$ is

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 2t_1^3 \\ 3 & 6t_1 & 6t_1^2 \\ 0 & 6 & 12t_1 \end{vmatrix} = 0$$

i.e.,

$$(x - 3t_1)(72t_1^2 - 36t_1^2) - (y - 3t_1^2)\, 36t_1 + (z - 2t_1^3)\, 18 = 0$$

or

$$(x - 3t_1)\, 2t_1^2 - 2t_1(y - 3t_1^2) + (z - 2t_1^3) = 0$$

or

$$2t_1^2 x - 2t_1 y + z - 2t_1^3 = 0.$$

**Example 5.** ***Find the equation of the osculating plane at a general point on the cubic curve given by $\vec{r} = (t, t^2, t^3)$ and show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these three points.***

[M.D.U. 2007]

**Solution.** Parametric equations of given curve are

$$x = t, \quad y = t^2, \quad z = t^3$$

$$\therefore \quad \frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2$$

and

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 2, \quad \frac{d^2z}{dt^2} = 6t$$

Equation of osculating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

$$\begin{vmatrix} X-t & Y-t^2 & Z-t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 0$$

$$(X-t)(12t^2-6t^2)-(Y-t^2)(6t-0)+(Z-t^3)(2-0)=0$$

$$(X-t)(12t^2-6t^2)-6t(Y-t^2)+2(Z-t^3)=0$$

$$(X-t)6t^2-6t(Y-t^2)+2(Z-t^3)=0$$

$$6Xt^2-6tY+2Z+(-6t^3+6t^3-2t^3)=0$$

$$3t^2X-3tY+Z-t^3=0 \quad \text{...(1)}$$

Let $t_1, t_2, t_3$ be any three points on the given curve.

$\therefore$ Equations of osculating plane at these three points are

$$3t_1^2X-3t_1Y+Z-t_1^3=0 \quad \text{...(2)}$$

$$3t_2^2X-3t_2Y+Z-t_2^3=0 \quad \text{...(3)}$$

$$3t_3^2X-3t_3Y+Z-t_3^3=0 \quad \text{...(4)}$$

Since these three equations are independent, therefore point $(X_0, Y_0, Z_0)$ satisfy equations (2), (3), (4)

$\therefore$ We suppose osculating plane meet at $(X_0, Y_0, Z_0)$

Therefore,
$$3t_1^2X_0-3t_1Y_0+Z_0-t_1^3=0 \quad \text{...(5)}$$

$$3t_2^2X_0-3t_2Y_0+Z_0-t_2^3=0 \quad \text{...(6)}$$

$$3t_3^2X_0-3t_3Y_0+Z_0-t_3^3=0 \quad \text{...(7)}$$

Equations (5), (6) and (7), suggest that $t_1, t_2, t_3$, are roots of the equation

$$3t^2X_0-3tY_0+Z_0-t^3=0$$

or
$$t^3-3t^2X_0+3tY_0-Z_0=0 \quad \text{...(8)}$$

Now, let the equation of plane through the points $t_1, t_2, t_3$ be

$$AX+BY+CZ+D=0 \quad \text{...(9)}$$

$\therefore t_1, t_2, t_3$ satisfy equation (9), the condition for which is

$$At+Bt^2+Ct^3+D=0 \quad [\because X=t, Y=t^2, Z=t^3]$$

or
$$Ct^3+Bt^2+At+D=0 \quad \text{...(10)}$$

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$\therefore$ $t_1, t_2, t_3$, are roots of both (8) and (10).

Comparing the coefficients of two equations, we have

$$\frac{C}{1} = \frac{B}{-3X_0} = \frac{A}{3Y_0} = \frac{D}{-Z_0}$$

which gives $A = 3CY_0$, $B = -3CX_0$, $D = -CZ_0$

$\therefore$ Equation (1) of the plane becomes

$$3CY_0X - 3CX_0Y + CZ - CZ_0 = 0$$

or

$$3Y_0X - 3X_0Y + Z - Z_0 = 0 \qquad ...(11)$$

Clearly $(X_0, Y_0, Z_0)$ satisfy this plane as

L.H.S. $= 3Y_0X_0 - 3X_0Y_0 + Z_0 - Z_0 = 0 =$ R.H.S., which is true

Hence the result.

**Example 6.** ***Show that the osculating plane at a point P has in general three points of contact (contact of second order) with the curve at P.***

**Solution.** Let the equation of curve be $\vec{r} = \vec{r}(s)$ ...(1)

Suppose the arc is measured from P, therefore $s = 0$ at P.

Equation of the osculating plane at P is given by

$$[\vec{R} - \vec{r}(0) \;\; \vec{r}\,'(0) \;\; \vec{r}\,''(0)] = 0 \qquad ...(2)$$

*i.e.*,

$$\{\vec{R} - \vec{r}(0)\} \cdot (\vec{r}\,'(0) \times \vec{r}\,''(0)) = 0$$

Points common to (1) and (2) are given by

$$(\vec{r}(s) - \vec{r}(0)) \cdot (\vec{r}\,'(0) \;\; \vec{r}\,''(0)) = 0$$

Let $\quad F(s) = (\vec{r}(s) - \vec{r}(0)) \cdot (\vec{r}\,'(0) \;\; \vec{r}\,''(0))$ ...(3)

Now by Maclaurin's theorem, we have

$$\vec{r}(s) = \vec{r}(0) + s\,\vec{r}\,'(0) + \frac{s^2}{2!}\vec{r}\,''(0) + \frac{s^3}{3!}\vec{r}\,'''(0) + O(s^4)$$

$$\therefore \quad \vec{r}(s) - \vec{r}(0) = s\,\vec{r}\,'(0) + \frac{s^2}{2!}\vec{r}\,''(0) + \frac{s^3}{3!}\vec{r}\,'''(0) + O(s^4) \qquad ...(4)$$

Using (4) in (3), we have

$$F(s) = \left(s\,\vec{r}\,'(0) + \frac{s^2}{2!}\vec{r}\,''(0) + \frac{s^3}{3!}\vec{r}\,'''(0) + O(s^4)\right) \cdot \left\{\vec{r}\,'(0) \times \vec{r}\,''(0)\right\}$$

$$= s\vec{r}'(0)\,.\,\vec{r}'(0)\times\vec{r}''(0)+\frac{s^2}{2!}\vec{r}''(0)\,.\,\vec{r}'(0)\times\vec{r}''(0)$$

$$+\frac{s^3}{3!}\vec{r}'''(0)\,.\,\vec{r}'(0)\times\vec{r}''(0)+O(s^4)$$

Now, we know that scalar triple product in which two of the vectors are same is zero.

$$\therefore\quad F(s)=\frac{s^3}{3!}[\vec{r}'''(0)\quad \vec{r}'(0)\quad \vec{r}''(0)]+O(s^4) \qquad ...(5)$$

From (5), we conclude that $F(s) = 0$ as $s = 0$ at P. Differentiating (5), we get

$F'(s) = 0$ at P

and $F''(s) = 0$ at P $\qquad [\because s = 0]$

Clearly $F'''(s) \neq 0$ at $s = 0$ *i.e.*, at P provided $[\vec{r}'''(0)\quad \vec{r}'(0)\quad \vec{r}''(0)] \neq 0$

$[\because$ *$F'''(s)$ is independent of s*$]$

This shows that the osculating plane has in general three points of contact with the curve.

Further, in case we find $[\vec{r}'''(0)\quad \vec{r}'(0)\quad \vec{r}''(0)]=0$, then we observe that $F'''(s)=0$.

In this case, the osculating plane has at least four points of contact with the curve at P.

**Example 7.** ***Normals are drawn from the point*** $(\alpha, \beta, \gamma)$ ***to the ellipsoid***

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

***Find the equation to the osculating plane at*** $(\alpha, \beta, \gamma)$ ***of the cubic curve through the feet of the normals.***

**Solution.** Equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Equation of tangent plane at $(x, y, z)$ to (1) is

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}+\frac{Zz}{c^2}=1$$

$\therefore$ Direction ratios of normal are $\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right)$

Equation of the normal to the ellipsoid at $(x, y, z)$ is

$$\frac{X-x}{\left(\frac{x}{a^2}\right)}=\frac{Y-y}{\left(\frac{y}{b^2}\right)}=\frac{Z-z}{\left(\frac{z}{c^2}\right)}=t \text{ (say), where } t \text{ is a parameter.}$$

As it passes through $(\alpha, \beta, \gamma)$,

$$\therefore \quad \frac{\alpha - x}{\left(\frac{x}{a^2}\right)} = \frac{\beta - y}{\left(\frac{y}{b^2}\right)} = \frac{\gamma - z}{\frac{z}{c^2}} = t$$

$\therefore$ Foot of the normal through $(\alpha, \beta, \gamma)$ is given by

$$x = \frac{a^2\alpha}{a^2 + t}, \quad y = \frac{b^2\beta}{b^2 + t}, \quad z = \frac{c^2\gamma}{c^2 + t} \qquad \text{...(1)}$$

Equation (1) gives the parametric equation of the cubic curve through the feet of normals through $(\alpha, \beta, \gamma)$.

Now, we have to find the osculating plane at $(\alpha, \beta, \gamma)$ to the space curve given by (1)

From (1), we have

$$\frac{dx}{dt} = -\frac{a^2\alpha}{(a^2+t)^2}, \quad \frac{dy}{dt} = -\frac{b^2\beta}{(b^2+t)^2}, \quad \frac{dz}{dt} = \frac{-c^2\gamma}{(c^2+t)^2}$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{2a^2\alpha}{(a^2+t)^3}, \quad \frac{d^2y}{dt^2} = \frac{2b^2\beta}{(b^2+t)^3}, \quad \frac{d^2z}{dt^2} = \frac{2c^2\gamma}{(c^2+t)^3}$$

At $(\alpha, \beta, \gamma)$, we see from eq. (1), that parameter $t = 0$.

Therefore at $(\alpha, \beta, \gamma)$ *i.e.*, at $t = 0$, we have

$$\frac{dx}{dt} = -\frac{\alpha}{a^2}, \quad \frac{dy}{dt} = -\frac{\beta}{b^2}, \quad \frac{dz}{dt} = -\frac{\gamma}{c^2}$$

and

$$\frac{d^2x}{dt^2} = \frac{2\alpha}{a^4}, \quad \frac{d^2y}{dt^2} = \frac{2\beta}{b^4}, \quad \frac{d^2z}{dt^2} = \frac{2\gamma}{c^4}$$

Equation of the osculating plane at $(\alpha, \beta, \gamma)$ is

$$\begin{vmatrix} X - \alpha & Y - \beta & Z - \gamma \\ -\frac{\alpha}{a^2} & -\frac{\beta}{b^2} & -\frac{\gamma}{c^2} \\ \frac{2\alpha}{a^4} & \frac{2\beta}{b^4} & \frac{2\gamma}{c^4} \end{vmatrix} = 0$$

or $$(X - \alpha)\left[-\frac{2\beta\gamma}{b^2c^4} + \frac{2\beta\gamma}{b^4c^2}\right] - (Y - \beta)\left[-\frac{2\gamma\alpha}{a^2c^4} + \frac{2\alpha\gamma}{a^4c^2}\right] + (Z - \gamma)\left[-\frac{2\alpha\beta}{a^2b^4} + \frac{2\beta\alpha}{a^4b^2}\right] = 0$$

which can be written as [*Expanding by first row*]

$$(X-\alpha)\frac{2\beta\gamma}{b^4c^4}[c^2-b^2]+(Y-\beta)\frac{2\gamma\alpha}{a^4c^4}[a^2-c^2]+(Z-\gamma)\frac{2\alpha\beta}{a^4b^4}[b^2-a^2]=0$$

Dividing by $2\alpha\beta\gamma$ and multiplying by $a^4b^4c^4$, we have

$$\frac{(X-\alpha)(c^2-b^2)a^4}{\alpha}+\frac{(Y-\beta)(a^2-c^2)b^4}{\beta}+\frac{(Z-\gamma)(b^2-a^2)c^4}{\gamma}=0$$

$$\frac{(X-\alpha)a^4}{\alpha(a^2-c^2)(b^2-a^2)}+\frac{(Y-\beta)b^4}{(c^2-b^2)(b^2-a^2)\beta}+\frac{(Z-\gamma)c^4}{(c^2-b^2)(a^2-c^2)\gamma}=0$$

$$\frac{Xa^4}{\alpha(a^2-c^2)(b^2-a^2)}+\frac{Yb^4}{(c^2-b^2)(b^2-a^2)\beta}+\frac{Zc^4}{(c^2-b^2)(a^2-c^2)\gamma}$$
$$-\left[\frac{a^4}{(a^2-c^2)(b^2-a^2)}+\frac{b^4}{(c^2-b^2)(b^2-a^2)}+\frac{c^4}{(c^2-b^2)(a^2-c^2)}\right]=0$$

$$\frac{Xa^4}{\alpha(a^2-c^2)(b^2-a^2)}+\frac{Yb^4}{(c^2-b^2)(b^2-a^2)\beta}+\frac{Zc^4}{(c^2-b^2)(a^2-c^2)\gamma}$$
$$-\left[\frac{a^4(c^2-b^2)+b^4(a^2-c^2)+c^4(b^2-a^2)}{(c^2-b^2)(b^2-a^2)(a^2-c^2)}\right]=0$$

$$\frac{Xa^4}{\alpha(a^2-c^2)(b^2-a^2)}+\frac{Yb^4}{(c^2-b^2)(b^2-a^2)\beta}+\frac{Zc^4}{(c^2-b^2)(a^2-c^2)\gamma}+1=0$$

[*Factorizing the numerator in last term*]

**Example 8.** ***Prove that there are three points on the cubic***

$$x=at^3+b,$$
$$y=3ct^2+3dt,$$
$$z=3et+f$$

***such that the osculating planes pass through the origin and the points lie in the plane***

$$3ce\,x+af\,y=0.$$

**Solution.** Position vector of any point $(x, y, z)$ on the curve is given by

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

i.e.,

$$\vec{r}=(at^3+b)\hat{i}+(3ct^2+3dt)\hat{j}+(3et+f)\hat{k}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = 3at^2\,\hat{i} + (6ct + 3d)\,\hat{j} + 3e\,\hat{k}$$

and
$$\frac{d^2\vec{r}}{dt^2} = 6at\,\hat{i} + 6c\,\hat{j} + 0\,\hat{k}$$

Let (X, Y, Z) be the coordinates of any point on the osculating plane, then

$$\vec{R} = X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}$$

Then the equation of osculating plane is $[\vec{R} - \vec{r} \quad \dot{\vec{r}} \quad \ddot{\vec{r}}\,] = 0$

i.e.,
$$(\vec{R} - \vec{r}) \cdot \dot{\vec{r}} \times \ddot{\vec{r}} = 0$$

i.e.,
$$\{(X - at^3 - b)\,\hat{i} + (Y - 3ct^2 - 3dt)\,\hat{j} + (Z - 3et - f)\,\hat{k}\} \cdot \dot{\vec{r}} \times \ddot{\vec{r}} = 0 \qquad \text{...(1)}$$

Now,
$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3at^2 & 6ct + 3d & 3e \\ 6at & 6c & 0 \end{vmatrix}$$

$$= -18ec\,\hat{i} + 18aet\,\hat{j} - 18(act^2 + atd)\,\hat{k}$$

$\therefore$ From (1), equation of the osculating plane is

$$[(X - at^3 - b)\,\hat{i} + (Y - 3ct^2 - 3dt)\,\hat{j} + (Z - 3et - f)\,\hat{k}\,] \cdot (-18)\,[ec\,\hat{i} - aet\,\hat{j} + (act^2 + adt)\,\hat{k}\,] = 0$$

Since osculating plane passes through (0, 0, 0)

$$\therefore \quad [-(at^3 + b)\,\hat{i} + (-3ct^2 - 3dt)\,\hat{j} + (-3et - f)\,\hat{k}\,] \cdot (-18)\,[ec\,\hat{i} - aet\,\hat{j} + (act^2 + adt)\,\hat{k}\,] = 0$$

or
$$(at^3 + b)\,ec - (3ct^2 + 3dt)\,(aet) + (3et + f)\,(act^2 + adt) = 0$$

or
$$acet^3 + bec - 3aect^3 - 3aedt^2 + 3acet^3 + 3aedt^2 + acft^2 + afdt = 0$$

or
$$acet^3 + acft^2 + afdt + bce = 0 \qquad \text{...(2)}$$

which is a cubic in $t$ and gives three values of $t$. Hence there are three points on the cubic such that osculating plane passes through (0, 0, 0).

Corresponding to three values of $t$, we get from

$$\left.\begin{aligned} x &= at^3 + b \\ y &= 3ct^2 + 3dt \\ z &= 3et + f \end{aligned}\right\} \qquad \text{...(3)}$$

the coordinates of the points.

Now, we find the locus of these points satisfying condition (2). For this, we eliminate $t$ from (1) and (3).

As
$$x = at^3 + b \Rightarrow t^3 = \frac{x-b}{a}$$
$$z = 3et + f \Rightarrow t = \frac{z-f}{3e}$$

Then
$$y = 3ct^2 + 3dt \Rightarrow 3ct^2 = y - 3dt$$
$$\Rightarrow t^2 = \frac{y - 3d\left(\frac{z-f}{3e}\right)}{3c}$$
$$\Rightarrow t^2 = \frac{ey - dz + df}{3ce}$$

Substituting the value of $t, t^2, t^3$ in (2), we have

$$ace\left(\frac{x-b}{a}\right) + acf\left(\frac{ey-dz+df}{3ce}\right) + afd\left(\frac{z-f}{3e}\right) + bce = 0$$

Multiplying by $3ce$, we get

$$3\,c^2e^2(x-b) + afc\,(ey - dz + df) + afdc\,(z - f) + 3bc^2e^2 = 0$$

or
$$3\,xc^2e^2 - 3c^2e^2b + afcey - afcdz + acdf^2 + afcdz - af^2dc + 3bc^2e^2 = 0$$

or
$$3xc^2e^2 + afcey = 0$$

or
$$3cex + afy = 0$$

which is the equation of plane. Thus three points lie on a plane.

Hence the result.

**Example 9.** ***Show that the curve*** $\vec{r} = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$ ***lies in a plane.*** [M.D.U. 2014]

**Solution.** Here $\vec{r} = t\,\hat{i} + \frac{1+t}{t}\hat{j} + \frac{1-t^2}{t}\hat{k}$

Parametric equations of the given curve are

$$x = t, \qquad y = \frac{1+t}{t}, \qquad z = \frac{1-t^2}{t}$$

$$\therefore \quad \frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = -\frac{1}{t^2}, \qquad \frac{dz}{dt} = -\frac{1}{t^2} - 1$$

$$\Rightarrow \qquad \frac{d^2x}{dt^2}=0, \qquad \frac{d^2y}{dt^2}=\frac{2}{t^3}, \qquad \frac{d^2z}{dt^2}=\frac{2}{t^3}$$

Equation of osculating plane to the given curve at point $t$ is

$$\begin{vmatrix} X-t & Y-\frac{1+t}{t} & Z-\frac{1-t^2}{t} \\ 1 & -\frac{1}{t^2} & -\frac{1}{t^2}-1 \\ 0 & \frac{2}{t^3} & \frac{2}{t^3} \end{vmatrix}=0$$

Operating $C_2 \to C_2 - C_3$, we have

$$\begin{vmatrix} X-t & Y-Z-1-t & Z-\frac{1-t^2}{t} \\ 1 & 1 & -\frac{1}{t^2}-1 \\ 0 & 0 & \frac{2}{t^3} \end{vmatrix}=0$$

Expanding along third row, we get

$$\frac{2}{t^3}[(X-t).1-(Y-Z-1-t)]=0$$

Thus the equation of osculating plane is

$$X-Y+Z+1=0 \qquad ...(1)$$

If the given curve is a plane curve *i.e.*, curve lying in a plane then it must lie on its osculating plane.

In this case, the given equation of curve must satisfy osculating plane (1).

Putting $x=t,\ y=\frac{1+t}{t}, z=\frac{1-t^2}{t}$ in (1) *i.e.*, in equation of osculating plane, we have

$$t-\frac{1+t}{t}+\frac{1-t^2}{t}+1=0$$

*i.e.*,

$$0=0, \text{ which is true for all } t$$

Hence the given curve is a plane curve.

EXERCISE 8.3

1. Prove that the osculating plane at the point $t = 1$ of the curve given by

$$\vec{r} = (3at, 3bt^2, ct^3) \text{ is } \frac{x}{a} - \frac{y}{b} + \frac{z}{c} = 1.$$

2. Find the equation of osculating plane at the point $t$ on the curve helix

$$\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$$

[M.D.U. 2011]

3. Find the equation of osculating plane at the point $t$ of the curve

$$x = a\cosh t, \quad y = a\sinh t, \quad z = bt$$

4. Find the equation of osculating plane at any point of the curve

$$\vec{r} = 4a\cos^3 t\,\hat{i} + 4a\sin^3 t\,\hat{j} + 2a\cos 2t\,\hat{k}$$

5. Show that the curve for which

$$e^{ax} = \frac{b-t}{c-t}, \quad e^{by} = \frac{c-t}{a-t}, \quad e^{cz} = \frac{a-t}{b-t}$$

is a plane curve which lies in the plane $ax + by + cz = 0$.

6. Find the osculating plane at any point $t$ on the curve given by

$$x = a\cos 2t, \quad y = a\sin 2t, \quad z = 2a\sin t.$$

[K.U. 2014; M.D.U. 2009]

7. Find the plane that has three point contact at the point (3, 5, 7) of the curve

$$x = 2t + 1, \quad y = 3t^2 + 2, \quad z = 4t^3 + 3$$

8. Find the equation of osculating plane at any point $t$ on the curve $\vec{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$.

9. Show that there are three points on the cubic $\vec{r} = \vec{a}t^3 + 3\vec{b}t^2 + 3\vec{c}t + \vec{d}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ etc., the osculating plane at which passes through the origin and they lie in the plane $3[\vec{r}\ \vec{b}\ \vec{c}] = [\vec{r}\ \vec{a}\ \vec{d}]$.

10. Determine $a, h, b$ so that the paraboloid $2z = ax^2 + 2hxy + by^2$ may have the closed contact at the origin with the curve $x = t^3 - 2t^2 + 1$, $y = t^3 - 1$, $z = t^2 - 2t + 1$. Find also the order of contact.

11. Find the inflexional tangents at $(x, y, z)$ on the surface $y^2z = 4ax$.

**[Hint :** Inflexional tangents are lines which have three point contact $(x_1, y_1, z_1)$ on the surface]

12. Prove that if the circle $lx + my + nz = 0$, $x^2 + y^2 + z^2 = 2cz$ has three point contact at the origin within the paraboloid $ax^2 + by^2 = 2z$, then $c = \dfrac{(l^2 + m^2)}{bl^2 + am^2}$.

13. Show that the curve $x = t, y = t^2, z = t^3$ has six point contact with the paraboloid $x^2 + z^2 = y$ at the origin.

**ANSWERS**

2. $bx \sin t - by \cos t + az = abt$
3. $bx \sinh t - by \cosh t + az = abt$
4. $2x \cos t - 2y \sin t - 3z - 2a \cos 2t = 0$
6. $x(\sin 3t + 3 \sin t) - y(\cos 3t + 3 \cos t) + 4z - 6a \sin t = 0$
7. $6x - 4y + z = 5$
8. $3t^2x - 3ty + z = t^3$
10. Contact of 4th order
11. $\dfrac{X-x}{y^2} = \dfrac{Y-y}{0} = \dfrac{Z-z}{4a}$ and also $\dfrac{X-x}{3x} = \dfrac{Y-y}{2y} = \dfrac{Z-z}{-z}$.

## 8.13. EQUATION OF TANGENT PLANE AT ANY POINT OF THE SURFACE $f(x, y, z) = 0$

Equation of surface is $f(x, y, z) = 0$ ...(1)

Let P $(x, y, z)$ be any point on the surface. Let $s$ be the arc length of any curve through P measured from a fixed point to the point P$(x, y, z)$.

From (1),
$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = 0$$

which can be written as,

$$\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = 0$$

or
$$\vec{r}\,' \cdot \nabla f = 0 \qquad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

This shows that all the tangent lines at P are perpendicular to the vector $\nabla f$ (*i.e.*, gradient $f$).

Thus, all the tangent lines at P lie in a plane which is perpendicular to the vector $\nabla f$.

*This plane is called the tangent plane to the surface at P.*

Let $\vec{R}$ be the position vector of a current point on the tangent plane. Then, vector $\vec{R} - \vec{r}$ is in the tangent plane and perpendicular to vector $\nabla f$.

$$\therefore \qquad (\vec{R} - \vec{r}) \cdot \nabla f = 0$$

is the equation of tangent plane at P.

**Deductions :**

**(*i*) Normal Plane :** ***Plane through P and perpendicular to the tangent line at P is called normal plane at P to the curve.***

If $\vec{r}$ is the position vector of P, then $\vec{r}\,'$ is the unit vector along the tangent.

Let $\vec{R}$ be the position vector of current point on the normal plane, then vector $\vec{R}-\vec{r}$ is perpendicular to $\vec{r}'$,

$$(\vec{R}-\vec{r})\cdot\vec{r}'=0 \quad i.e., \quad (\vec{R}-\vec{r})\cdot\hat{t}=0$$

which is the equation of normal plane at P.

Equation of normal plane can also be written as

$$(\vec{R}-\vec{r})\cdot\frac{d\vec{r}}{dt}=0 \quad i.e., \quad (\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0$$

### (ii) Equation of normal plane in cartesian form

Let $\quad \vec{R}=X\hat{i}+Y\hat{j}+Z\hat{k}$

where (X, Y, Z) denotes the current point on plane

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\therefore \quad \vec{R}-\vec{r}=(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}$$

Also, $\quad \vec{r}'=x'\hat{i}+y'\hat{j}+z'\hat{k}$

$\therefore$ Equation of normal plane is

$$(\vec{R}-\vec{r})\cdot\vec{r}'=0$$

i.e., $\quad (X-x)x'+(Y-y)y'+(Z-z)z'=0$

If $x, y, z$ are functions of parameter $t$, then equation of normal plane can be written as

$$(X-x)\dot{x}+(Y-y)\dot{y}+(Z-z)\dot{z}=0.$$

## 8.14. TO SHOW THAT NORMAL PLANE IS PERPENDICULAR TO THE OSCULATING PLANE

**Proof :** Equation of normal plane at a point P whose position vector is $\vec{r}$ is given by

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0 \qquad ...(1)$$

Equation of osculating plane at P is

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}\times\ddot{\vec{r}}=0 \qquad ...(2)$$

(1) shows that normal plane is perpendicular to the vector $\dot{\vec{r}}$.

(2) shows that osculating plane is perpendicular to the vector $\dot{\vec{r}} \times \ddot{\vec{r}}$.

Thus, perpendiculars to normal plane and osculating planes are $\dot{\vec{r}}$ and $\dot{\vec{r}} \times \ddot{\vec{r}}$ respectively.

Their scalar triple product is $\dot{\vec{r}} \cdot (\dot{\vec{r}} \times \ddot{\vec{r}}) = 0$.

[*Scalar triple product is zero when two of the three vectors are same*]

$\therefore$ Perpendiculars to two planes are at right angles.

Hence, normal plane and osculating planes are perpendicular.

## 8.15. EQUATION OF OSCULATING PLANE AT A POINT ON THE CURVE OF INTERSECTION OF TWO SURFACES $f(\vec{r}) = 0$ and $\psi(\vec{r}) = 0$

Let $P(\vec{r})$ be any point on the curve of intersection of surfaces

$$f(\vec{r}) = 0$$

and

$$\psi(\vec{r}) = 0$$

Then by Art. 8.13, tangent line at P is perpendicular to the vector $\nabla f$ (*i.e.*, normal to surface at P)

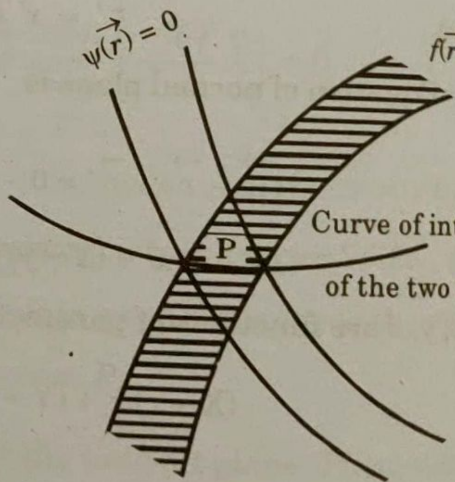


*Fig. 8.4*

$$\therefore \quad \dot{\vec{r}} \cdot \nabla f = 0 \quad ...(1)$$

Similarly, $\dot{\vec{r}} \cdot \nabla \psi = 0$ ...(2)

From (1) and (2), we have

$$\lambda \dot{\vec{r}} = \nabla f \times \nabla \psi \quad ...(3)$$

Differentiating (3) w.r.t. $t$, we have

$$\lambda \ddot{\vec{r}} + \dot{\lambda} \dot{\vec{r}} = (\nabla f) \times (\nabla \psi)^{\cdot} + (\nabla f)^{\cdot} \times \nabla \psi$$

Taking cross product on both sides with $\dot{\vec{r}}$, we have

$$\dot{\vec{r}} \times (\lambda \ddot{\vec{r}} + \dot{\lambda} \dot{\vec{r}}) = \dot{\vec{r}} \times [(\nabla f) \times (\nabla \psi)^{\cdot} + (\nabla f)^{\cdot} \times (\nabla \psi)]$$

or

$$\dot{\vec{r}} \times \lambda \ddot{\vec{r}} = \dot{\vec{r}} \times [(\nabla f) \times (\nabla \psi)^{\cdot}] + \dot{\vec{r}} \times [(\nabla f)^{\cdot} \times (\nabla \psi)] \quad [\because \dot{\vec{r}} \times \dot{\vec{r}} = 0]$$

$$\lambda \dot{\vec{r}} \times \ddot{\vec{r}} = [\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}]\, \nabla f - [\dot{\vec{r}} \cdot (\nabla f)]\, (\nabla\psi)^{\cdot} + [\dot{\vec{r}} \cdot (\nabla\psi)]\, (\nabla f)^{\cdot} - [\dot{\vec{r}} \cdot (\nabla f)^{\cdot}]\, \nabla\psi$$

$$[\because\ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}]$$

$$\lambda \dot{\vec{r}} \times \ddot{\vec{r}} = [\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}]\, \nabla f - 0 + 0 - [\dot{\vec{r}} \cdot (\nabla f)^{\cdot}]\, \nabla\psi \qquad \text{[Using (1) and (2)]}$$

$$\lambda \dot{\vec{r}} \times \ddot{\vec{r}} = [\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}]\, \nabla f - [\dot{\vec{r}} \cdot (\nabla f)^{\cdot}]\, \nabla\psi \qquad \text{...(4)}$$

We know that equation of osculating plane at P is

$$(\vec{R} - \vec{r}) \cdot \dot{\vec{r}} \times \ddot{\vec{r}} = 0$$

$\therefore$ Equation of osculating plane is

$$\frac{1}{\lambda} (\vec{R} - \vec{r}) \cdot \{[\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}]\, \nabla f - [\dot{\vec{r}} \cdot (\nabla f)^{\cdot}] \nabla\psi\} = 0 \qquad \text{[Using (4)]}$$

$$(\vec{R} - \vec{r}) \cdot [\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}]\, \nabla f = (\vec{R} - \vec{r}) \cdot [\dot{\vec{r}} \cdot (\nabla f)^{\cdot}]\, \nabla\psi$$

where $\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}$ and $\dot{\vec{r}} \cdot (\nabla f)^{\cdot}$ are scalars.

Thus the equation of the osculating plane is

$$\frac{(\vec{R} - \vec{r}) \cdot \nabla f}{\dot{\vec{r}} \cdot (\nabla f)^{\cdot}} = \frac{(\vec{R} - \vec{r}) \cdot \nabla\psi}{\dot{\vec{r}} \cdot (\nabla\psi)^{\cdot}} \qquad \text{...(5)}$$

**8.15.1. Equation in Cartesian Form**

Let
$$\vec{R} = (X, Y, Z)$$

$$\vec{r} = (x, y, z)$$

$$f(\vec{r}) = f(x, y, z) \quad \text{and} \quad \psi(\vec{r}) = \psi(x, y, z)$$

$\therefore$
$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

and
$$\nabla \psi = \frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j} + \frac{\partial \psi}{\partial z} \hat{k}$$

$$(\nabla f)^{\cdot} = \left(\frac{\partial f}{\partial x}\right)^{\cdot} \hat{i} + \left(\frac{\partial f}{\partial y}\right)^{\cdot} \hat{j} + \left(\frac{\partial f}{\partial z}\right)^{\cdot} \hat{k} \qquad \text{...(6)}$$

and
$$(\nabla \psi)^{\cdot} = \left(\frac{\partial \psi}{\partial x}\right)^{\cdot} \hat{i} + \left(\frac{\partial \psi}{\partial y}\right)^{\cdot} \hat{j} + \left(\frac{\partial \psi}{\partial z}\right)^{\cdot} \hat{k} \qquad \text{...(7)}$$

Now $$\left(\frac{\partial f}{\partial x}\right)^{\cdot} = \frac{d}{dt}(\mathrm{P}), \text{ where } \mathrm{P} = \frac{\partial f}{\partial x}$$

$$= \frac{\partial \mathrm{P}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathrm{P}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathrm{P}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial^2 f}{\partial x^2}\dot{x} + \frac{\partial^2 f}{\partial x \partial y}\dot{y} + \frac{\partial^2 f}{\partial x \partial z}\dot{z} = \mathrm{L} \text{ (say)}$$

Similarly, $$\left(\frac{\partial f}{\partial y}\right)^{\cdot} = \frac{\partial^2 f}{\partial x\, \partial y}\dot{x} + \frac{\partial^2 f}{\partial y^2}\dot{y} + \frac{\partial^2 f}{\partial y \partial z}\dot{z} = \mathrm{M} \text{ (say)}$$

and $$\left(\frac{\partial f}{\partial z}\right)^{\cdot} = \frac{\partial^2 f}{\partial x\, \partial z}\dot{x} + \frac{\partial^2 f}{\partial y \partial z}\dot{y} + \frac{\partial^2 f}{\partial z^2}\dot{z} = \mathrm{N} \text{ (say)}$$

From (6), $$(\nabla f)^{\cdot} = \mathrm{L}\,\hat{i} + \mathrm{M}\,\hat{j} + \mathrm{N}\,\hat{k}$$

Also, $$\vec{\mathrm{R}} - \vec{r} = (\mathrm{X} - x)\,\hat{i} + (\mathrm{Y} - y)\,\hat{j} + (\mathrm{Z} - z)\,\hat{k}$$

As $$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$\therefore$ $$\dot{\vec{r}} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}$$

and $$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$\therefore$ L.H.S. of (5) becomes,

$$\frac{(\vec{\mathrm{R}} - \vec{r})\cdot \nabla f}{\dot{\vec{r}}\cdot(\nabla f)^{\cdot}} = \frac{(\mathrm{X} - x)\frac{\partial f}{\partial x} + (\mathrm{Y} - y)\frac{\partial f}{\partial y} + (\mathrm{Z} - z)\frac{\partial f}{\partial z}}{\dot{x}\mathrm{L} + \dot{y}\mathrm{M} + \dot{z}\mathrm{N}}$$

$$= \frac{(\mathrm{X} - x)\frac{\partial f}{\partial x} + (\mathrm{Y} - y)\frac{\partial f}{\partial y} + (\mathrm{Z} - z)\frac{\partial f}{\partial z}}{\dot{x}^2\frac{\partial^2 f}{\partial x^2} + \dot{y}^2\frac{\partial^2 f}{\partial y^2} + \dot{z}^2\frac{\partial^2 f}{\partial z^2} + 2\dot{x}\dot{y}\frac{\partial^2 f}{\partial x\, \partial y} + 2\dot{x}\dot{z}\frac{\partial^2 f}{\partial x\, \partial z} + 2\dot{y}\dot{z}\frac{\partial^2 f}{\partial y\, \partial z}} \quad \text{...(8)}$$

Similarly replacing $f$ by $\psi$, we have

$$\frac{(\vec{\mathrm{R}} - \vec{r})\cdot \nabla \psi}{\dot{\vec{r}}\cdot(\nabla \psi)^{\cdot}} = \frac{(\mathrm{X} - x)\frac{\partial \psi}{\partial x} + (\mathrm{Y} - y)\frac{\partial \psi}{\partial y} + (\mathrm{Z} - z)\frac{\partial \psi}{\partial z}}{\dot{x}^2\frac{\partial^2 f}{\partial x^2} + \dot{y}^2\frac{\partial^2 f}{\partial y^2} + \dot{z}^2\frac{\partial^2 f}{\partial z^2} + 2\dot{x}\dot{y}\frac{\partial^2 f}{\partial x\, \partial y} + 2\dot{x}\dot{z}\frac{\partial^2 f}{\partial x\, \partial z} + 2\dot{y}\dot{z}\frac{\partial^2 f}{\partial y\, \partial z}} \quad \text{...(9)}$$

Using (8) and (9) in (5), we obtain the equation of osculating plane as

$$\frac{(X-x)\dfrac{\partial f}{\partial x}+(Y-y)\dfrac{\partial f}{\partial y}+(Z-z)\dfrac{\partial f}{\partial z}}{\left(\dot{x}^2\dfrac{\partial^2 f}{\partial x^2}+\ldots\ldots\right)+\left(2\dot{y}\dot{z}\dfrac{\partial^2 f}{\partial y\partial z}+\ldots\ldots\right)}=\frac{(X-x)\dfrac{\partial \psi}{\partial x}+(Y-y)\dfrac{\partial \psi}{\partial y}+(Z-z)\dfrac{\partial \psi}{\partial z}}{\left(\dot{x}^2\dfrac{\partial^2 \psi}{\partial x^2}+\ldots\ldots\right)+\left(2\dot{y}\dot{z}\dfrac{\partial^2 \psi}{\partial y\partial z}+\ldots\ldots\right)}$$

## SOLVED EXAMPLES

**Example 1.** ***Find the equation of normal plane to the curve***

$$\vec{r}=e^{2t}\cos t\,\hat{i}+e^{2t}\sin t\,\hat{j}+e^{2t}\hat{k},\text{where }-\infty<t<\infty.$$ [K.U. 2011]

**Solution.** The equation of the curve is

$$\vec{r}=e^{2t}\cos t\,\hat{i}+e^{2t}\sin t\,\hat{j}+e^{2t}\hat{k}$$

$$\therefore \quad \frac{d\vec{r}}{dt}=[(-e^{2t}\sin t+2e^{2t}\cos t)\,\hat{i}+(e^{2t}\cos t+2e^{2t}\sin t)\,\hat{j}+2e^{2t}\,\hat{k}]$$

$$=e^{2t}\,[(2\cos t-\sin t)\,\hat{i}+(2\sin t+\cos t)\,\hat{j}+2\hat{k}]$$

Squaring both sides, we have

$$\left(\frac{d\vec{r}}{dt}\right)^2=e^{4t}\,[(2\cos t-\sin t)^2+(2\sin t+\cos t)^2+(2)^2]$$

or
$$\left(\frac{d\vec{r}}{dt}\right)^2=e^{4t}\,[4(\cos^2 t+\sin^2 t)+(\sin^2 t+\cos^2 t)+4]$$

i.e.,
$$\left(\frac{d\vec{r}}{dt}\right)^2=e^{4t}\,[4+1+4]=9e^{4t}$$

$$\therefore \quad \left|\frac{d\vec{r}}{dt}\right|=3e^{2t}$$

Now,
$$\hat{t}=\frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$$

$$\therefore \qquad \hat{t} = \frac{1}{3e^{2t}} \cdot e^{2t} \, [(2\cos t - \sin t)\,\hat{i} + (2\sin t + \cos t)\,\hat{j} + 2\hat{k}]$$

$$= \frac{1}{3} [(2\cos t - \sin t)\,\hat{i} + (2\sin t + \cos t)\,\hat{j} + 2\hat{k}]$$

Thus the equation of the normal plane is $(\vec{R} - \vec{r}) \cdot \hat{t} = 0$, where $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

*i.e.*,
$$[(X\hat{i} + Y\hat{j} + Z\hat{k}) - (e^{2t}\cos t\,\hat{i} + e^{2t}\sin t\,\hat{j} + e^{2t}\hat{k})] \cdot \left\{\frac{1}{3}[(2\cos t - \sin t)\,\hat{i} + (2\sin t + \cos t)\,\hat{j} + 2\hat{k}]\right\} = 0$$

$$\Rightarrow \quad \left[(X - e^{2t}\cos t)\,\hat{i} + (Y - e^{2t}\sin t)\,\hat{j} + (Z - e^{2t})\,\hat{k}\right] \cdot \left\{\frac{1}{3}[(2\cos t - \sin t)\,\hat{i} + (2\sin t + \cos t)\,\hat{j} + 2\,\hat{k}]\right\} = 0$$

or $\quad (X - e^{2t}\cos t)(2\cos t - \sin t) + (Y - e^{2t}\sin t)(2\sin t + \cos t) + (Z - e^{2t})(2) = 0$

or $\quad X(2\cos t - \sin t) + Y(2\sin t + \cos t) + 2Z$
$$- e^{2t}\{2\cos^2 t - \cos t \sin t + 2\sin^2 t + \sin t \cos t + 2\} = 0$$

or $\quad X(2\cos t - \sin t) + Y(2\sin t + \cos t) + 2Z - e^{2t}\{2(\cos^2 t + \sin^2 t) + 2\} = 0$

or $\quad X(2\cos t - \sin t) + Y(2\sin t + \cos t) + 2Z - 4e^{2t} = 0$

or $\quad X(2\cos t - \sin t) + Y(2\sin t + \cos t) + 2Z = 4e^{2t}$.

**Example 2.** ***Find the osculating plane at $(x_1, y_1, z_1)$ on the curve of intersection of cylinders $x^2 + z^2 = a^2$, $y^2 + z^2 = b^2$.***

[M.D.U. 2016, 13]

**Solution.** The given equations of surfaces are

$$f(x, y, z) = x^2 + z^2 - a^2 = 0$$

and
$$\psi(x, y, z) = y^2 + z^2 - b^2 = 0$$

Differentiating both equations w.r.t. '$t$', we have

$$x\dot{x} + z\dot{z} = 0 \quad \text{and} \quad y\dot{y} + z\dot{z} = 0$$

which gives
$$\frac{\dot{x}}{\frac{1}{x}} = \frac{\dot{y}}{\frac{1}{y}} = -\frac{\dot{z}}{\frac{1}{z}}$$

Also, $\dfrac{\partial f}{\partial x} = 2x,\quad \dfrac{\partial f}{\partial y} = 0,\quad \dfrac{\partial f}{\partial z} = 2z,$

$$\frac{\partial^2 f}{\partial x^2} = 2,\quad \frac{\partial^2 f}{\partial y^2} = 0,\quad \frac{\partial^2 f}{\partial z^2} = 2$$

and $\dfrac{\partial^2 f}{\partial x \partial y} = 0,\quad \dfrac{\partial^2 f}{\partial x \partial z} = 0,\quad \dfrac{\partial^2 f}{\partial y \partial z} = 0$

$$\frac{\partial \psi}{\partial x} = 0,\quad \frac{\partial \psi}{\partial y} = 2y,\quad \frac{\partial \psi}{\partial z} = 2z$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0,\quad \frac{\partial^2 \psi}{\partial y^2} = 2,\quad \frac{\partial^2 \psi}{\partial z^2} = 2$$

and $\dfrac{\partial^2 \psi}{\partial x \partial y} = 0,\quad \dfrac{\partial^2 \psi}{\partial y \partial z} = 0,\quad \dfrac{\partial^2 \psi}{\partial z \partial x} = 0$

We can find their values at $(x_1, y_1, z_1)$. Using the formula obtained in Art. 8.15.1, the equation of osculating plane at $(x_1, y_1, z_1)$ is

$$\frac{(X - x_1)\, 2x_1 + (Y - y_1) . 0 + (Z - z_1)\, 2z_1}{\dfrac{2}{x_1^2} + \dfrac{2}{z_1^2}} = \frac{2y_1 (Y - y_1) + 2z_1 (Z - z_1)}{\dfrac{2}{y_1^2} + \dfrac{2}{z_1^2}}$$

or
$$\frac{(X x_1 + Z z_1 - x_1^2 - z_1^2)\, x_1^2 z_1^2}{x_1^2 + z_1^2} = \frac{(Y y_1 + Z z_1 - y_1^2 - z_1^2)\, y_1^2 z_1^2}{y_1^2 + z_1^2}$$

or
$$\frac{(X x_1 + Z z_1 - a^2) x_1^2}{a^2} = \frac{(Y y_1 + Z z_1 - b^2)\, y_1^2}{b^2}$$

$$[\because\ x_1^2 + z_1^2 = a^2,\ y_1^2 + z_1^2 = b^2]$$

or
$$\frac{X x_1^3 + (Z z_1 - a^2)(a^2 - z_1^2)}{a^2} = \frac{Y y_1^3 + (Z z_1 - b^2)(b^2 - z_1^2)}{b^2}$$

or
$$\frac{X x_1^3 - Z z_1^3 - a^4 + a^2 (z_1^2 + Z z_1)}{a^2} = \frac{Y y_1^3 - Z z_1^3 - b^4 + b^2 (z_1^2 + Z z_1)}{b^2}$$

$$\Rightarrow \quad \frac{X x_1^3 - Z z_1^3 - a^4}{a^2} = \frac{Y y_1^3 - Z z_1^3 - b^4}{b^2}.$$

**EXERCISE**

1. Find the osculating plane at the point of the curve of intersection of conicoids

$$ax^2 + by^2 + cz^2 = 1,$$
$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1$$

2. Show that osculating plane at $(x, y, z)$ on the curve

$$x^2 + 2ax = y^2 + 2by = z^2 + 2cz \text{ is}$$

$$(b^2 - c^2)(X - x)(x + a)^3 + (c^2 - a^2)(Y - y)(y + b)^3 + (a^2 - b^2)(Z - z)(z + c)^3 = 0$$

**ANSWERS**

1. $\dfrac{BCx^3X}{(\beta-b)(\gamma-c)}+\dfrac{CAy^3Y}{(\gamma-c)(\alpha-a)}+\dfrac{ABz^3Z}{(\alpha-a)(\beta-b)}+1=0$

where $A=b\gamma-c\beta$, $B=c\alpha-a\gamma$, $C=a\beta-b\alpha$

## 8.16. NORMAL LINE AT A POINT

**Definition :** *Any line through a point P on the curve, perpendicular to the tangent to the curve at P is defined as the normal line to the curve at P.*

From the definition, we observe that :

(*i*) There will be an infinite number of such normal lines drawn through P.

(*ii*) All these normals lie on the normal plane.

In other words, *locus of the normal lines at the point P of the curve is called the normal plane at P.*

**Note.**

We have already deduced the equation of normal plane as

$$(\vec{R}-\vec{r})\cdot\vec{r}\,'=0 \qquad \text{[Deduction of Article 8.13]}$$

or

$$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}=0$$

## 8.17. TWO SPECIAL NORMALS

[M.D.U. 2017; K.U. 2011]

(*a*) **Principal Normal :** The normal which lies in the osculating plane at a point on the curve is called the *principal normal* at P.

The unit vector along the principal normal is denoted by $\hat{n}$.

(*b*) **Bi-normal :** The normal which is perpendicular to the osculating plane at a point is called the *bi-normal*.

The unit vector along the bi-normal is denoted by $\hat{b}$.

**Important Observations :**

(i) Bi-normal is perpendicular to the principal normal.

(ii) Since principal normal lies in the osculating plane and also lies in the normal plane, therefore principal normal is the line of intersection of osculating plane and the normal plane at a point on the curve.

(iii) Osculating plane at a point of the curve contains the tangent to the curve at the point and also the principal normal.

(iv) Bi-normal to a curve at a point lies on the normal plane to the curve at that point and is perpendicular to the osculating plane of the curve at that point.

## 8.18. FUNDAMENTAL UNIT VECTORS AND FUNDAMENTAL PLANES

### 8.18.1. Fundamental Unit Vectors

The unit vectors along the tangent, principal normal and bi-normal are denoted by $\hat{t}, \hat{n}$ and $\hat{b}$ respectively and they form an orthogonal right handed triad of vectors. Thus

$$\hat{t} \cdot \hat{n} = 0, \quad \hat{n} \cdot \hat{b} = 0 \text{ and } \hat{b} \cdot \hat{t} = 0$$

and

$$\hat{t} \times \hat{n} = \hat{b}, \quad \hat{n} \times \hat{b} = \hat{t} \text{ and } \hat{b} \times \hat{t} = \hat{n}$$

The vectors $\hat{t}, \hat{n}, \hat{b}$ at any point on a curve are called the fundamental unit vectors of a given curve.

### 8.18.2. Fundamental Planes

The three planes - *osculating plane*, *normal plane* and *rectifying plane* associated with each point of a curve are known as **Fundamental planes.** These planes are perpendicular to each other.

The plane through the tangent and the principal normal to the curve at P is called **osculating plane** at P.

In adjoining fig, plane TPN is the osculating plane.

Plane NPB in adjoining fig. *i.e.*, plane through principal normal and binormal to curve at P is the **normal plane** to the curve at P.

Plane BPT *i.e.*, plane through binormal and tangent to the curve at P is known as **rectifying** plane to the curve at P.

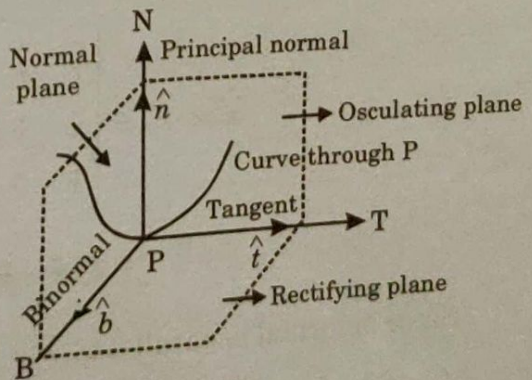


*Fig. 8.5*

**Equation of osculating plane :** This plane is normal to $\hat{b}$ and contains vectors $\hat{t}$ and $\hat{n}$. If $\vec{R}$ is the position vector of a point on the plane and $\vec{r}$ is the position vector of point $P(x, y, z)$ on the curve, then equation of osculating plane is $(\vec{R} - \vec{r}) \cdot \hat{b} = 0$.

**Equation of normal plane :** This plane is normal to $\hat{t}$ and contains vectors $\hat{b}$ and $\hat{n}$. If $\vec{R}$ is the position vector of a point on the plane and $\vec{r}$ is the position vector of point $P(x, y, z)$ on the curve, then equation of normal plane is $(\vec{R} - \vec{r}) \cdot \hat{t} = 0$.

**Equation of rectifying plane :** This plane is normal to $\hat{n}$ and contains vectors $\hat{b}$ and $\hat{t}$. If $\vec{R}$ is the position vector of a point on the plane and $\vec{r}$ is the position vector of point $P(x, y, z)$ on the curve, then equation of rectifying plane is $(\vec{R} - \vec{r}) \cdot \hat{n} = 0$.

## 8.19. DIRECTIONS OF PRINCIPAL NORMAL AND BINORMAL

[M.D.U. 2007]

**Direction ratios of binormal :**

Since binormal is perpendicular to the osculating plane and osculating plane is perpendicular to the vector $\dot{\vec{r}} \times \ddot{\vec{r}}$.

Thus, **binormal is parallel to the vector $\dot{\vec{r}} \times \ddot{\vec{r}}$.**

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $x, y, z$ are functions of parameter $t$, then

$$\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

and

$$\ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\therefore \quad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}$$

$$= (\dot{y}\ddot{z} - \ddot{y}\dot{z})\hat{i} + (\dot{z}\ddot{x} - \ddot{z}\dot{x})\hat{j} + (\dot{x}\ddot{y} - \ddot{x}\dot{y})\hat{k} \qquad \ldots(1)$$

Since binormal is parallel to $\dot{\vec{r}} \times \ddot{\vec{r}}$

$\therefore \quad \dot{y}\ddot{z} - \ddot{y}\dot{z}, \; \dot{z}\ddot{x} - \ddot{z}\dot{x}, \; \dot{x}\ddot{y} - \ddot{x}\dot{y}$ are direction ratios of the binormal.

If parameter are of arc length $s$, then direction ratios of binormal are given by

$$y'z'' - z'y'',\ z'x'' - z''x',\ x'y'' - x''y'$$

**Direction ratios of principal normal :**

Since principal normal is perpendicular to the tangent and the binormal, therefore principal normal is parallel to the cross product $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$.

Again, if $t$ is the parameter, then

$$\dot{\vec{r}} \times \ddot{\vec{r}} = (\dot{y}\ddot{z} - \ddot{y}\dot{z})\hat{i} + (\dot{z}\ddot{x} - \ddot{z}\dot{x})\hat{j} + (\dot{x}\ddot{y} - \ddot{x}\dot{y})\hat{k} \qquad \text{[Using (1)]}$$

$$\therefore \quad \dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ \dot{y}\ddot{z} - \ddot{y}\dot{z} & \dot{z}\ddot{x} - \ddot{z}\dot{x} & \dot{x}\ddot{y} - \ddot{x}\dot{y} \end{vmatrix}$$

$$= \left[\dot{y}(\dot{x}\ddot{y} - \ddot{x}\dot{y}) - \dot{z}(\dot{z}\ddot{x} - \ddot{z}\dot{x})\right]\hat{i} + \left[\dot{z}(\dot{y}\ddot{z} - \ddot{y}\dot{z}) - \dot{x}(\dot{x}\ddot{y} - \ddot{x}\dot{y})\right]\hat{j}$$

$$+ \left[\dot{x}(\dot{z}\ddot{x} - \ddot{z}\dot{x}) - \dot{y}(\dot{y}\ddot{z} - \ddot{y}\dot{z})\right]\hat{k}$$

As principal normal is parallel to vector $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$.

$\therefore$ Direction ratios of principal normal are given by

$$\dot{y}(\dot{x}\ddot{y} - \ddot{x}\dot{y}) - \dot{z}(\dot{z}\ddot{x} - \ddot{z}\dot{x});\ \dot{z}(\dot{y}\ddot{z} - \ddot{y}\dot{z}) - \dot{x}(\dot{x}\ddot{y} - \ddot{x}\dot{y});\ \dot{x}(\dot{z}\ddot{x} - \ddot{z}\dot{x}) - \dot{y}(\dot{y}\ddot{z} - \ddot{y}\dot{z})$$

If parameter is of arc length $s$, then $\dfrac{d\vec{r}}{ds}$ *i.e.*, $\vec{r}\,'$ is a unit vector along the tangent

$$\therefore \quad \vec{r}\,' \cdot \vec{r}\,' = 1$$

Differentiating w.r.t. $s$, we have

$$2\vec{r}\,' \cdot \vec{r}\,'' = 0$$

or

$$\vec{r}\,' \cdot \vec{r}\,'' = 0$$

which shows that vector $\vec{r}\,''$ is perpendicular to $\vec{r}\,'$. Hence the principal normal will be parallel to $\vec{r}\,''$.

$\therefore$ Direction ratios of principal normal are $x'', y'', z''$.

## 8.20. CURVATURE

**Definition :** *The curvature of a curve at a point is defined as the arc-rate of rotation of the tangent.*

Let A be a fixed point and P(s) and Q (s + δs) be two neighbouring points on the curve so that AP = s and AQ = s + δs. Suppose δθ is the angle between the tangents to the curve at P and Q.

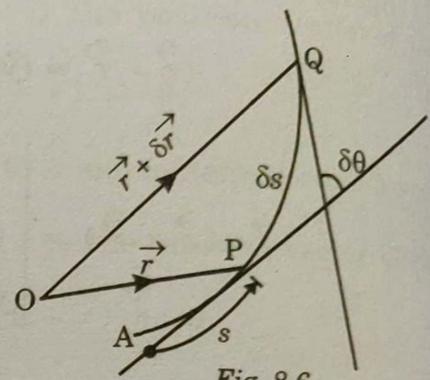

*Fig. 8.6*

Average curvature (*i.e.*, rotation) of arc (PQ)

$$= \frac{\delta\theta}{\delta s} \quad \left[ i.e. \ \frac{\textit{Total rotation}}{\textit{Arc } PQ} \right]$$

Proceeding to limits as Q → P, we have

$$\text{Curvature at } P = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$$

The magnitude of curvature is denoted by κ (**kappa**).

Thus curvature at a point P is given by

$$\boxed{\kappa = \frac{d\theta}{ds}}$$

Reciprocal of curvature is called the ***radius of curvature*** and is denoted by ρ

$$\therefore \quad \boxed{\rho = \frac{1}{\kappa}}$$

**8.21. To find an expression for the curvature (κ) at a given point to the curve**

Let P and Q be two given points on a curve such that length of arc measured from a fixed point F to P and Q are $s$ and $s + \delta s$ respectively [Ref. fig. 8.7]

∴ Length of arc PQ = δs

Let $\hat{t}$ and $\hat{t} + \delta\hat{t}$ denote the unit vectors along the tangents at P and Q and δθ be the angle between the tangent vectors at P and Q.

Let $\vec{QA}$ and $\vec{QB}$ represents the unit vectors along the tangents at P and Q.

$$|\vec{QA}| = 1 \text{ and } |\vec{QB}| = 1$$

$\therefore$

$$|\vec{QA} \times \vec{QB}| = 1.1.\sin \delta\theta$$

Now,

$$|\hat{t} \times (\hat{t} + \delta \hat{t})| = \sin \delta\theta$$

or

$$|\hat{t} \times \hat{t} + \hat{t} \times \delta \hat{t}| = \sin \delta\theta$$

or

$$|\hat{t} \times \delta \hat{t}| = \sin \delta\theta \qquad [\because \hat{t} \times \hat{t} = 0]$$

i.e.,

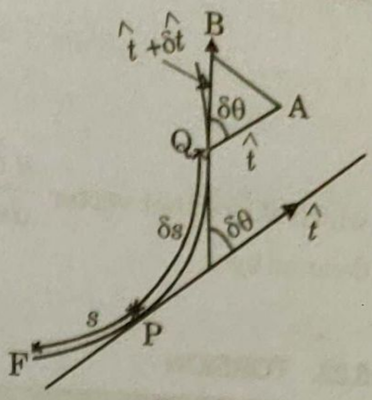


*Fig. 8.7*

or

$$\left| \hat{t} \times \frac{\delta \hat{t}}{\delta s} \right| = \frac{\sin \delta\theta}{\delta s} \qquad [\because \delta s \text{ is a scalar}]$$

$$= \frac{\sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta s}$$

Proceeding to limits as $Q \to P$, we get

$$\left| \hat{t} \times \frac{d\hat{t}}{ds} \right| = 1.\frac{d\theta}{ds}$$

$\therefore$ Curvature at a point P *i.e.,*

$$\kappa = \left| \hat{t} \times \frac{d\hat{t}}{ds} \right| \qquad \ldots(1)$$

Since $\hat{t}$ is a unit vector

$\therefore$

$$\hat{t}^2 = |\hat{t}|^2 = 1$$

or

$$\hat{t} \cdot \hat{t} = 1$$

Differentiating w.r.t. $s$, we get

$$2\hat{t} \cdot \frac{d\hat{t}}{ds} = 0$$

i.e.,

$$\hat{t} \cdot \frac{d\hat{t}}{ds} = 0$$

It shows that $\hat{t}$ is perpendicular to $\dfrac{d\hat{t}}{ds}$

$$\therefore \text{ From (1), (curvature at P)} = |\hat{t}| \left| \frac{d\hat{t}}{ds} \right| \sin \frac{\pi}{2} \qquad \left[\because \hat{t} \perp \frac{d\hat{t}}{ds}\right]$$

$$= \left|\frac{d\hat{t}}{ds}\right| = \left|\frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)\right| = |\vec{r}\,''| \qquad [\because |\hat{t}| = 1]$$

which shows that vector $\frac{d\hat{t}}{ds}$ *i.e.*, $\vec{r}\,''$ is the curvature vector of the curve and its magnitude is denoted by $\kappa$.

## 8.22. TORSION

**Definition:** *The arc-rate of rotation of the binormal at P on the curve is defined as torsion vector of the curve.*

If P($s$) and Q($s + \delta s$) are two neighbouring points on the curve and $\delta\phi$ is the angle between the binormals at P and Q, then

Average rotation of binormal from P to Q is $\frac{\delta\phi}{\delta s}$

Proceeding to limits as Q $\to$ P, we have

$$\text{Torsion at P} = \lim_{\delta s \to 0} \frac{\delta\phi}{\delta s} = \frac{d\phi}{ds}.$$

Torsion at P is denoted by $\tau$

$$\therefore \qquad \tau = \frac{d\phi}{ds}$$

Reciprocal of torsion at P *i.e.*, $\frac{1}{\tau}$ is known as **radius of torsion** at P and is denoted by $\sigma$.

## 8.23. EXPRESSION FOR TORSION TO THE CURVE AT THE GIVEN POINT

Let P and Q be two given points on a curve such that length of arc from a fixed point F to P and Q are $s$ and $s + \delta s$ respectively.

$\therefore$ Length of the arc PQ = $\delta s$.

Let $\hat{b}$ and $\hat{b} + \delta\hat{b}$ denote the unit vectors along the binormal at P and Q respectively and $\delta\phi$ is the angle between binormals at P and Q.

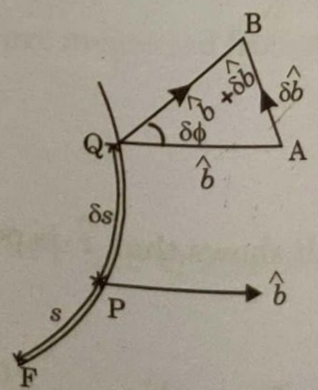


*Fig. 8.8*

Let $\overrightarrow{QA}$ and $\overrightarrow{QB}$ represents the vectors along the binormals at P and Q

$$|\overrightarrow{QA}| = 1 \text{ and } |\overrightarrow{QB}| = 1$$

$\therefore$

$$\overrightarrow{QA} = \hat{b},$$

$$\overrightarrow{QB} = \hat{b} + \delta\hat{b}$$

We have,

$$|\overrightarrow{QA} \times \overrightarrow{QB}| = |\overrightarrow{QA}|\,|\overrightarrow{QB}| \sin\delta\phi$$

$$\Rightarrow \quad |\hat{b} \times (\hat{b} + \delta\hat{b})| = \sin\delta\phi$$

$$\Rightarrow \quad |\hat{b} \times \delta\hat{b}| = \sin\delta\phi \qquad [\because \hat{b}\times\hat{b} = 0]$$

$$\Rightarrow \quad \left|\hat{b} \times \frac{\delta\hat{b}}{\delta s}\right| = \frac{\sin\delta\phi}{\delta\phi}\,\frac{\delta\phi}{\delta s} \qquad [\delta s \textit{ is a scalar}]$$

Proceeding to limits as $Q \to P$, we have

$$\left|\hat{b} \times \frac{d\hat{b}}{ds}\right| = \frac{d\phi}{ds}$$

$\therefore$ Torsion at P *i.e.*, $\tau = \left|\hat{b} \times \dfrac{d\hat{b}}{ds}\right|$ ...(1)

Since $\hat{b}$ is a unit vector

$$\therefore \quad \hat{b}^2 = |\hat{b}|^2 = 1$$

*i.e.*,

$$\hat{b}\cdot\hat{b} = 1$$

Differentiating w.r.t. $s$, we have

$$2\hat{b}\cdot\frac{d\hat{b}}{ds} = 0$$

$\therefore$ $\hat{b}$ and $\dfrac{d\hat{b}}{ds}$ are perpendicular to each other.

$\therefore$ From (1), torsion at P $= |\hat{b}|\left|\dfrac{d\hat{b}}{ds}\right| \sin\dfrac{\pi}{2} = \left|\dfrac{d\hat{b}}{ds}\right|$

which shows that vector $\frac{d\hat{b}}{ds}$ *i.e.*, $\hat{b}'$ is the torsion vector at P and its magnitude is the torsion

$$\tau = \left|\frac{d\hat{b}}{ds}\right|$$

## 8.24. SCREW CURVATURE

**Definition :** *The arc-rate at which the principal normal changes the direction i.e.,* $\frac{d\hat{n}}{ds}$ *as* $P(\vec{r})$ *moves along the curve is called the screw curvature vector.*

*Its magnitude is denoted by* $\sqrt{\kappa^2+\tau^2}$ .

## 8.25. SERRET-FRENET FORMULAE

[M.D.U. 2017, 14, 07]

**To show that :**

(1) $\frac{d\hat{t}}{ds} = \kappa\hat{n}$ (2) $\frac{d\hat{b}}{ds} = -\tau\hat{n}$ (3) $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}$

**Proof (1) :** We denote unit vector along tangent by $\frac{d\vec{r}}{ds}$ or $\hat{t}$

$\therefore$
$$|\hat{t}| = 1$$

*i.e.,*
$$\hat{t}\,.\,\hat{t} = 1$$

Differentiating, w.r.t. 's', we have

$$2\hat{t}\,.\,\hat{t}' = 0$$

$\therefore$ $\hat{t}'$ is perpendicular to $\hat{t}$ ...(1)

Equation of osculating plane is

$$[\vec{R} - \vec{r}\ \ \vec{r}'\ \ \vec{r}''] = 0$$

*i.e.,*
$$[\vec{R} - \vec{r}\ \ \hat{t}\ \ \hat{t}'] = 0$$

It shows that $\hat{t}'$ lies in the osculating plane, which is perpendicular to binormal $\hat{b}$ .

$\therefore\ \hat{t}'$ is perpendicular to both $\hat{t}$ and $\hat{b}$.

Hence $\hat{t}'$ is parallel to $\hat{t} \times \hat{b}$ i.e., $\hat{t}'$ is collinear with $\hat{n}$.

As
$$|\hat{t}'| = |\vec{r}\,''| = \kappa$$ [Refer Art. 8.21]

$$\hat{t}' = \pm \kappa \hat{n}$$

We choose the direction of $\hat{n}$ such that curvature $\kappa$ is always positive

$$\therefore\ \hat{t}' = \kappa \hat{n}$$

$$\frac{d\hat{t}}{ds} = \kappa\,\hat{n} \qquad \text{...(A)}$$

**Proof (2) :** Since $\hat{t}$ and $\hat{b}$ are perpendicular vectors

$$\therefore\ \hat{t}\,.\,\hat{b} = 0$$

Differentiating w.r.t. 's', we have

$$\hat{t}\,.\,\hat{b}' + \hat{t}'\,.\,\hat{b} = 0$$

$$\Rightarrow\ \hat{t}\,.\,\hat{b}' + \kappa\hat{n}\,.\,\hat{b} = 0 \qquad \text{[By (A)]}$$

$$\Rightarrow\ \hat{t}\,.\,\hat{b}' = 0 \qquad [\because\ \hat{n}\,.\,\hat{b} = 0]$$

$\therefore\ \hat{b}'$ is perpendicular to $\hat{t}$ ...(2)

We have $\hat{b}\,.\,\hat{b} = 1$

Differentiating w.r.t. $s$, $2\hat{b}\,.\,\hat{b}' = 0$

$\therefore\ \hat{b}'$ is perpendicular to $\hat{b}$ ...(3)

From (2) and (3), $\hat{b}'$ is normal to the plane containing $\hat{t}$ and $\hat{b}$

$\therefore\ \hat{b}'$ is parallel to $\hat{b} \times \hat{t}$ i.e., $\hat{b}'$ is parallel to $\hat{n}$

Since
$$|\hat{b}'| = \left|\frac{d\hat{b}}{ds}\right| = \tau$$

$$\therefore\ \hat{b}' = \pm \tau \hat{n}$$

or
$$\frac{d\hat{b}}{ds} = \pm \tau \hat{n} \qquad \text{...(B)}$$

By convention, direction of $\hat{b}'$ is taken opposite to $\hat{n}$

$$\therefore \qquad \frac{d\hat{b}}{ds} = -\tau\,\hat{n}$$

**Proof (3) :**

We know that $\qquad \hat{n} = \hat{b} \times \hat{t}$

Differentiating w. r. t. $s$, we have

$$\frac{d\hat{n}}{ds} = \hat{b} \times \frac{d\hat{t}}{ds} + \frac{d\hat{b}}{ds} \times \hat{t}$$

$$= \hat{b} \times \kappa\hat{n} + (-\tau\hat{n}) \times \hat{t} \qquad \text{[Using (A) and (B)]}$$

$$= \kappa\,(\hat{b} \times \hat{n}) - \tau\,(\hat{n} \times \hat{t})$$

$$= \kappa\,(-\hat{t}) - \tau\,(-\hat{b})$$

$$= \tau\,\hat{b} - \kappa\,\hat{t}$$

$$\therefore \qquad \frac{d\hat{n}}{ds} = \tau\,\hat{b} - \kappa\,\hat{t} \qquad \text{...(C)}$$

**Note.**

(*i*) The above relations (A), (B), and (C) are known as **Serret-Frenet formulae.**

(*ii*) Remember Serret Frenet formula as matrix equation

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}$$

## 8.26. To find the curvature and torsion at a point on the curve r = r(t) in terms of the derivative of the position vector r of that point with respect to parameter t.

### (a) To find the formula for curvature

If $\vec{r}$ is the position vector of any point P on the curve, then

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \times \frac{ds}{dt}$$

$$= (\vec{r}\,')(\dot{s})$$

[Dash denotes derivative w.r.t. s and dot denotes derivative w.r.t t]

$$\dot{\vec{r}} = \hat{t}\,\dot{s} \qquad \text{...(1)} \qquad [\because \hat{t} = \vec{r}\,']$$

$\therefore$

$$|\dot{\vec{r}}| = |\hat{t}\,\dot{s}| = \dot{s} \qquad \text{...(2)} \qquad [\because |\hat{t}| = 1]$$

$\therefore$ Differentiating (1) w.r.t $t$, we have

$$\ddot{\vec{r}} = \hat{t}\,'(\dot{s})^2 + \hat{t}\,\ddot{s} \qquad \left[\text{Derivative of } \hat{t} = \left(\frac{d\hat{t}}{ds}\right)\left(\frac{ds}{dt}\right)\right]$$

or

$$\ddot{\vec{r}} = (\kappa\hat{n})\,\dot{s}^2 + \hat{t}\,\ddot{s} \qquad \text{...(3)}$$

[$\because$ By Serret-Frenet formula, $\hat{t}\,' = \kappa\hat{n}$]

Taking the cross product of (1) and (3), we have

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \hat{t}\,\dot{s} \times [(\kappa\hat{n})\,\dot{s}^2 + \hat{t}\,\ddot{s}]$$

or

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \kappa\dot{s}^3\,(\hat{t} \times \hat{n}) \qquad [\because \hat{t} \times \hat{t} = 0]$$

or

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \kappa\dot{s}^3\,\hat{b} \qquad \text{...(4)}$$

$\therefore$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \kappa\dot{s}^3 \qquad \text{...(5)} \qquad [\because |\hat{b}| = 1]$$

or

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \kappa|\dot{\vec{r}}|^3 \qquad [\text{Using (2)}]$$

$\therefore$

$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}, \text{ which gives the curvature.}$$

**To find the formula for torsion :**

Differentiating (4) w.r.t. $t$, we have

$$(\dot{\vec{r}} \times \dddot{\vec{r}} + \ddot{\vec{r}} \times \ddot{\vec{r}}) = \dot{s}^3\kappa\,\hat{b}\,'\dot{s} + \hat{b}\,\frac{d}{dt}(\dot{s}^3\kappa) \qquad \left[\because \frac{d\hat{b}}{dt} = \left(\frac{d\hat{b}}{ds}\right)\left(\frac{ds}{dt}\right) = \hat{b}\,'\dot{s}\right]$$

$$= \dot{s}^4\kappa\,\hat{b}\,' + \hat{b}\,\frac{d}{dt}(\dot{s}^3\kappa) \qquad \text{...(6)}$$

Taking the dot product of (3) and (6), we have

$$\ddot{\vec{r}}\cdot(\dot{\vec{r}} \times \dddot{\vec{r}} + \ddot{\vec{r}} \times \ddot{\vec{r}}) = [(\kappa\hat{n})\,\dot{s}^2 + \hat{t}\,\ddot{s}]\cdot\left[\dot{s}^4\kappa(-\tau\hat{n}) + \hat{b}\,\frac{d}{dt}(\dot{s}^3\kappa)\right] \qquad [\because \hat{b}\,' = -\tau\hat{n}]$$

or $$[\dddot{\vec{r}}\ \dot{\vec{r}}\ \dddot{\vec{r}}] = -\dot{s}^6 \kappa^2 \tau$$

$[\because \hat{n}\cdot\hat{b} = 0,\ \hat{t}\cdot\hat{n} = 0,\ \hat{t}\cdot\hat{b} = 0$ and $\ddot{\vec{r}} \times \ddot{\vec{r}} = 0]$

or $$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \dot{s}^6 \kappa^2 \tau \quad ...(7)$$

or $$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = |\dot{\vec{r}} \times \ddot{\vec{r}}|^2 \tau \quad \text{[Using (5)]}$$

$\therefore$ $$\tau = \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

which gives the torsion.

## 8.27. TO SHOW THAT :

(i) $\kappa = |\vec{r}\,' \times \vec{r}\,''|$. [K.U. 2011]

(ii) $\tau = \dfrac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2}$ [K.U. 2012]

**Proof:** (i) We know that $\vec{r}\,' = \hat{t}$ and $\vec{r}\,'' = \kappa\hat{n}$

$\therefore$ $$\vec{r}\,' \times \vec{r}\,'' = \hat{t} \times \kappa\hat{n}$$

$$= \kappa\hat{b} \quad [\because \hat{t} \times \hat{n} = \hat{b}]$$

$\therefore$ $$|\vec{r}\,' \times \vec{r}\,''| = \kappa|\hat{b}| = \kappa \quad [\because |\hat{b}| = 1]$$

Thus, $$\kappa = |\vec{r}\,' \times \vec{r}\,''|$$
which proves the first part.

(ii) We have $$\vec{r}\,' = \hat{t} \quad ...(1)$$

and $$\vec{r}\,'' = \kappa\hat{n} \quad ...(2)$$

Differentiating (2), w.r.t. $s$, we have

$$\vec{r}\,''' = \kappa\frac{d\hat{n}}{ds} + \frac{d\kappa}{ds}\hat{n}$$

$$= \kappa(\tau\hat{b} - \kappa\hat{t}) + \kappa'\hat{n} \quad \left[\because \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}\right]$$

$$= -\kappa^2\hat{t} + \kappa'\hat{n} + \kappa\tau\hat{b} \quad ...(3)$$

Equations (1), (2) and (3) can be written as

$$\vec{r}\,' = 1\hat{t} + 0\hat{n} + 0\hat{b}$$

$$\vec{r}\,'' = 0\hat{t} + \kappa\hat{n} + 0\hat{b}$$

$$\vec{r}\,''' = -\kappa^2\hat{t} + \kappa'\hat{n} + \kappa\tau\hat{b}$$

Writing the triple product in determinant form, we have

$$[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,'''] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \end{vmatrix} = \kappa^2\tau$$

$$\therefore \qquad \tau = \frac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{\kappa^2}$$

*i.e.,*

$$\tau = \frac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2} \qquad \text{[Using part (i)]}$$

## 8.28. SOME IMPORTANT THEOREMS

**8.28.1. The necessary and sufficient condition for the curve to be a straight line is that the curvature $\kappa = 0$ at all points of the curve.**

[*M.D.U. 2009*]

**Proof.** *Condition is necessary* :

Let the curve be a straight line. Its vector equation can be written as

$$\vec{r} = s\,\vec{a} + \vec{b}$$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Now,

$$\frac{d\vec{r}}{ds} = \vec{a} \qquad [\because \vec{b} \text{ is a constant vector}]$$

$$\therefore \qquad \frac{d^2\vec{r}}{ds^2} = 0 \qquad [\because \vec{a} \text{ is a constant vector}]$$

*i.e.,*

$$\vec{r}\,'' = 0$$

As $\kappa = |\vec{r}''| = |0| = 0$

$\therefore$ Curvature $\kappa = 0$ at all points of the straight line.

*Condition is sufficient :*

Let $\kappa = 0$ for all points on the curve

$$\therefore \quad \kappa = |\vec{r}''| = 0$$

$$\therefore \quad \vec{r}'' = 0$$

Integrating, $$\vec{r}' = \vec{a}$$

Integrating again, $$\vec{r} = \vec{a}s + \vec{b} \quad ...(1)$$

where $\vec{a}$ and $\vec{b}$ are arbitrary constant vectors. Equation (1) represents straight line for all values of $\vec{a}$ and $\vec{b}$.

**8.28.2. The necessary and sufficient condition for a given curve to be a plane curve is that $\tau = 0$ at all points of the curve.**

**Proof :** *Condition is necessary :*

By a plane curve we mean the tangent and normal at all points of the curve lie in the plane of the curve.

We conclude that osculating plane at all points of the curve is the plane of the curve.

$\therefore$ Unit vector $\hat{b}$ along the binormal is constant

$$\therefore \quad \frac{d\hat{b}}{ds} = 0$$

*i.e.,* $$-\tau \hat{n} = 0 \quad \text{[By Serret-Frenet formula]}$$

or $$\tau = 0$$

Hence the condition is necessary.

*Condition is sufficient : .*

Let $\tau = 0$ at all points of the curve

$$\therefore \quad \frac{d\hat{b}}{ds} = -\tau\hat{n} = 0 \quad ...(1)$$

*i.e.,* $\hat{b}$ is a constant vector

Now, $$\frac{d}{ds}(\vec{r} \cdot \hat{b}) = \frac{d\vec{r}}{ds} \cdot \hat{b} + \vec{r} \cdot \frac{d\hat{b}}{ds}$$

$$= \hat{t} \cdot \hat{b} + 0$$

...(2) [*Using (1)*]

Since $\hat{t}$ and $\hat{b}$ are perpendicular, therefore $\hat{t} \cdot \hat{b} = 0$. Also from (1), $\hat{b}' = 0$.

$\therefore$ From (2), $\dfrac{d}{ds}(\vec{r} \cdot \hat{b}) = 0$

$\Rightarrow$ $\vec{r} \cdot \hat{b}$ is constant.

As we know $\vec{r} \cdot \hat{b}$ is the projection of position vector $\vec{r}$ on $\hat{b}$ (which is a constant vector) constant.

Thus, the projection of position vector $\vec{r}$ on $\hat{b}$ is the same at all points of the curve.

It shows that the curve must lie in a plane.

**8.28.3. The necessary and sufficient condition for the curve to be a plane curve is**

$$[\vec{r}', \vec{r}'', \vec{r}'''] = 0$$

**Proof.** *Condition is necessary* : [*M.D.U. 2015*]

Let the curve be a plane curve. Then $\tau = 0$ at all points on the curve

Since $[\vec{r}', \vec{r}'', \vec{r}'''] = \kappa^2\tau$ [*Refer Art. 8.27*]

$\therefore$ $[\vec{r}', \vec{r}'', \vec{r}'''] = 0$ [$\because \tau = 0$]

which is the necessary condition.

*Condition is sufficient* :

Here it is given that

$$[\vec{r}', \vec{r}'', \vec{r}'''] = 0 \qquad \text{...(1)}$$

at all points on the curve

Since $[\vec{r}', \vec{r}'', \vec{r}'''] = \kappa^2\tau$

$\therefore$ From (1), $\kappa^2\tau = 0$

$\Rightarrow$ Either $\kappa = 0$ or $\tau = 0$

If possible, let $\tau \neq 0$, at some point of the curve. Then in the neighbourhood of this point $\tau \neq 0$. This implies that $\kappa = 0$ in the neighbourhood of this point.

[*Refer Art. 8.28.1*]

Hence the curve is a straight line

$\therefore \tau = 0$ on this line,

which is a contradiction to our assumption *i.e.*, $\tau \neq 0$.

$\therefore \quad \tau = 0$, at all points. Thus, curve is a plane curve.

## SOLVED EXAMPLES

**Example 1.** ***For the curve $x = 4a\cos^3 t$, $y = 4a\sin^3 t$, $z = 3c\cos 2t$; find***

***(i) the equation of the principal normal***

***(ii) the equation of the osculating plane***

***(iii) curvature***

***at any point of the curve.*** [M.D.U. 2009]

**Solution.** Here $\quad \vec{r} = (4a\cos^3 t, 4a\sin^3 t, 3c\cos 2t)$

$$\therefore \quad \vec{r}\,' = \hat{t} = (-12a\cos^2 t\sin t, 12a\sin^2 t\cos t, -6c\sin 2t)\frac{dt}{ds}$$

$$= 6(-2a\cos^2 t\sin t, 2a\sin^2 t\cos t, -c\sin 2t)\frac{dt}{ds}$$

$$= 12\sin t\cos t(-a\cos t, a\sin t, -c)\frac{dt}{ds} \quad \text{...(1)}$$

and
$$|\hat{t}| = \sqrt{144\sin^2 t\cos^2 t(a^2\cos^2 t + a^2\sin^2 t + c^2)\left(\frac{dt}{ds}\right)^2}$$

or
$$1 = 12\sin t\cos t\sqrt{a^2+c^2}\,\frac{dt}{ds} \qquad [\because |\hat{t}| = 1]$$

or
$$\frac{ds}{dt} = 12\sin t\cos t\sqrt{a^2+c^2} \quad \text{...(2)}$$

$\therefore$ From (1), we have

$$\vec{r}\,' = \hat{t} = \frac{1}{\sqrt{a^2+c^2}}(-a\cos t, a\sin t, -c)$$

$$\vec{r}\,'' = \frac{1}{\sqrt{a^2+c^2}}(a\sin t, a\cos t, 0)\frac{dt}{ds}$$

or
$$\vec{r}\,'' = \frac{1}{\sqrt{a^2+c^2}}(a\sin t, a\cos t, 0)\left(\frac{1}{12\sin t\cos t\sqrt{a^2+c^2}}\right)$$

[Using (2)]

$$= \left(\frac{1}{12(a^2+c^2)}\right)\left(\frac{1}{\sin t \cos t}\right)(a\sin t, a\cos t, 0)$$

$$= \frac{a}{12(a^2+c^2)}(\sec t, \operatorname{cosec} t, 0)$$

Since principal normal is parallel to vector $\vec{r}''$

$\therefore$ $< \frac{a \sec t}{12(a^2+c^2)}, \frac{a \operatorname{cosec} t}{12(a^2+c^2)}, 0 >$ are direction ratios of the principal normal.

$\therefore$ Equation of principal normal is

$$\frac{x - 4a\cos^3 t}{\sec t} = \frac{y - 4a\sin^3 t}{\operatorname{cosec} t} = \frac{z - 3c\cos 2t}{0}.$$

(ii) The equation to the osculating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

i.e.,
$$\begin{vmatrix} X-4a\cos^3 t & Y-4a\sin^3 t & Z-3c\cos 2t \\ -a\cos t & a\sin t & -c \\ \sec t & \operatorname{cosec} t & 0 \end{vmatrix} = 0$$

(iii) From part (i),
$$\hat{t} = \frac{1}{\sqrt{a^2+c^2}}(-a\cos t, a\sin t, -c)$$

$\therefore$
$$\hat{t}' = \frac{1}{\sqrt{a^2+c^2}}(a\sin t, a\cos t, 0)\frac{dt}{ds}$$

$$\hat{t}' = \frac{1}{\sqrt{a^2+c^2}} \cdot \frac{1}{12\sin t\cos t\sqrt{a^2+c^2}}(a\sin t, a\cos t, 0)$$

$\therefore$
$$\kappa\hat{n} = \left(\frac{1}{12(a^2+c^2)}\right)(a\sec t, a\operatorname{cosec} t, 0) \qquad [\because \hat{t}' = \kappa\hat{n}]$$

$\therefore$
$$|\kappa\hat{n}| = \left(\frac{a^2\sec^2 t}{144(a^2+c^2)^2} + \frac{a^2\operatorname{cosec}^2 t}{144(a^2+c^2)^2}\right)^{1/2}$$

or $$|\kappa| = \frac{1}{12(a^2+c^2)}\left[\frac{a^2}{\cos^2 t}+\frac{a^2}{\sin^2 t}\right]^{1/2} \qquad [\because |\hat{n}|=1]$$

Hence the required curvature is $\dfrac{a}{12(a^2+c^2)\sin t\cos t}$.

**Example 2.** ***Find the radius of curvature for the curve***

$$\vec{r} = 3a\cos 2\theta\,\hat{i} + 4c\sin^3\theta\,\hat{j} + 4c\cos^3\theta\,\hat{k} \text{ at } \theta = \frac{\pi}{4}.$$

[M.D.U. 2014, 08]

**Solution.** We have $\vec{r} = 3a\cos 2\theta\,\hat{i} + 4c\sin^3\theta\,\hat{j} + 4c\cos^3\theta\,\hat{k}$

$$\therefore \quad \vec{r}\,' = \hat{t} = (-6a\sin 2\theta\,\hat{i} + 12c\sin^2\theta\cos\theta\,\hat{j} - 12c\cos^2\theta\sin\theta\,\hat{k})\frac{d\theta}{ds}$$

$$= 6(-a\sin 2\theta\,\hat{i} + 2c\sin^2\theta\cos\theta\,\hat{j} - 2c\cos^2\theta\sin\theta\,\hat{k})\frac{d\theta}{ds}$$

$$= 12\sin\theta\cos\theta(-a\,\hat{i} + c\sin\theta\,\hat{j} - c\cos\theta\,\hat{k})\frac{d\theta}{ds} \qquad ...(1)$$

Squaring (1), we have

$$1 = 144\sin^2\theta\cos^2\theta\,(a^2+c^2\sin^2\theta+c^2\cos^2\theta)\left(\frac{d\theta}{ds}\right)^2 \qquad [\because \hat{t}^2 = \hat{t}\cdot\hat{t} = 1]$$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2 = 144\sin^2\theta\cos^2\theta\,(a^2+c^2)$$

$$\Rightarrow \quad \frac{ds}{d\theta} = 12\sin\theta\cos\theta\sqrt{a^2+c^2} \qquad ...(2)$$

Substituting the value of $\dfrac{ds}{d\theta}$ in (1), we have

$$\hat{t} = \frac{1}{\sqrt{a^2+c^2}}(-a\,\hat{i} + c\sin\theta\,\hat{j} - c\cos\theta\,\hat{k})$$

Differentiating w.r.t. $s$, we have

$$\hat{t}\,' = \frac{1}{\sqrt{a^2+c^2}}(0\,\hat{i} + c\cos\theta\,\hat{j} + c\sin\theta\,\hat{k})\frac{d\theta}{ds}$$

or $$\hat{t}\,' = \frac{1}{\sqrt{a^2+c^2}}(c\cos\theta\,\hat{j} + c\sin\theta\,\hat{k})\frac{1}{12\sin\theta\cos\theta\sqrt{a^2+c^2}}$$

$$\kappa \hat{n} = \frac{1}{(a^2+c^2)\,12 \sin\theta \cos\theta}(c \cos\theta\, \hat{j} + c \sin\theta\, \hat{k})$$

Squaring, $\kappa^2 = \dfrac{c^2}{36\,(a^2+c^2)^2 \sin^2 2\theta} \Rightarrow \kappa = \dfrac{c}{6\,(a^2+c^2) \sin 2\theta}$

At $\theta = \dfrac{\pi}{4}$, $\kappa = \dfrac{c}{6\,(a^2+c^2)\sin\dfrac{\pi}{2}} = \dfrac{c}{6\,(a^2+c^2)}$.

**Example 3.** ***Use Serret-Frenets formulae to find the direction cosines of the principal normal and binormal.***

**Solution.** By Serret-Frenets formula,

$$\hat{t}\,' = \kappa \hat{n}$$

$$\hat{n} = \frac{\hat{t}\,'}{\kappa}$$

$$\hat{n} = \frac{\vec{r}\,''}{\kappa} \qquad \left[\because \hat{t} = \frac{d\vec{r}}{ds}\right]$$

$$\hat{n} = \frac{1}{\kappa}(x''\hat{i} + y''\hat{j} + z''\hat{k}) \qquad ...(1)$$

Since $\hat{n}$ is a unit vector

$\therefore$
$$|\hat{n}| = 1$$

i.e.,
$$\frac{x''^2}{\kappa^2} + \frac{y''}{\kappa^2} + \frac{z''}{\kappa^2} = 1$$

i.e.,
$$\kappa^2 = x''^2 + y''^2 + z''^2$$

$\therefore$
$$\kappa = \sqrt{x''^2 + y''^2 + z''^2} \qquad ...(2)$$

$\therefore$ Direction cosines of $\hat{n}$ from (1) are

$$\frac{x''}{\sqrt{x''^2+y''^2+z''^2}}, \frac{y''}{\sqrt{x''^2+y''^2+z''^2}}, \frac{z''}{\sqrt{x''^2+y''^2+z''^2}}$$

Now, $\hat{b} = \hat{t} \times \hat{n}$

i.e., $$\hat{b} = \vec{r}\,' \times \frac{\vec{r}\,''}{\kappa} \qquad \left[\because \ \hat{n} = \frac{\hat{t}'}{\kappa} = \frac{\vec{r}\,''}{\kappa}\right]$$

i.e., $$\hat{b} = \frac{1}{\kappa}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

or $$\hat{b} = \frac{1}{\kappa}\left[(y'z''-y''z')\,\hat{i} + (z'x''-x'z'')\,\hat{j} + (x'y''-x''y')\,\hat{k}\right]$$

As $\hat{b}$ is a unit vector, therefore direction cosines of the binormal are

$$\frac{y'z''-y''z'}{\kappa}, \frac{z'x''-x'z''}{\kappa}, \frac{x'y''-x''y'}{\kappa}$$

**Note.**

We have already determined the direction ratios of principal normal and binormal in article 8.19.

**Example 4.** ***Find the curvature and torsion of the helix***

$$x = a\cos t, \quad y = a\sin t, \quad z = at\tan\alpha$$

[K.U. 2018, 15, 14, 13, 10; M.D.U. 2016, 15, 14, 13]

**Solution.** Here $\vec{r} = (a\cos t, a\sin t, at\tan\alpha)$.

$\therefore$ $$\dot{\vec{r}} = (-a\sin t, a\cos t, a\tan\alpha)$$

and $$\ddot{\vec{r}} = (-a\cos t, -a\sin t, 0)$$

$\therefore$ $$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= a^2\tan\alpha\sin t\,\hat{i} + a^2\tan\alpha\cos t\,\hat{j} + a^2\,\hat{k}$$

$\therefore$ $$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4} = a^2\sec\alpha$$

and $$|\dot{\vec{r}}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2\tan^2\alpha}$$

$$= \sqrt{a^2\sec^2\alpha}$$

$$= a \sec \alpha$$

$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{a^2 \sec \alpha}{a^3 \sec^3 \alpha} = \frac{1}{a} \cos^2 \alpha$$

$\therefore$ Radius of curvature $= \frac{1}{\kappa} = a \sec^2 \alpha$

We have deduced that $\ddot{\vec{r}} = (-a \cos t, -a \sin t, 0)$

$$\dddot{\vec{r}} = (a \sin t, -a \cos t, 0)$$

$$\therefore \quad [\dot{\vec{r}} \; \ddot{\vec{r}} \; \dddot{\vec{r}}] = \begin{vmatrix} -a \sin t & a \cos t & a \tan \alpha \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}$$

$$= a^3 \tan \alpha$$

$$\therefore \quad \tau = \frac{[\dot{\vec{r}} \; \ddot{\vec{r}} \; \dddot{\vec{r}}]}{[\dot{\vec{r}} \times \ddot{\vec{r}}]^2} = \frac{a^3 \tan \alpha}{a^4 \sec^2 \alpha} = \frac{1}{a} \frac{\sin \alpha}{\sec \alpha}$$

$\therefore$ Radius of torsion $= a \sec \alpha \operatorname{cosec} \alpha.$

**Example 5.** ***Find the curvature and torsion of the curve given by***

$$\vec{r} = (at - a \sin t, a - a \cos t, bt)$$ [K.U. 2015; M.D.U. 2015, 14, 12; C.D.L.U. 2013]

**Solution.** We have $\vec{r} = (at - a \sin t, a - a \cos t, bt)$

$$\therefore \quad \dot{\vec{r}} = (a - a \cos t, a \sin t, b)$$

$$\ddot{\vec{r}} = (a \sin t, a \cos t, 0)$$

$$\dddot{\vec{r}} = (a \cos t, -a \sin t, 0)$$

Now,
$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a - a \cos t & a \sin t & b \\ a \sin t & a \cos t & 0 \end{vmatrix}$$

$$= (-ab \cos t)\,\hat{i} + ab \sin t\,\hat{j} + (a^2 \cos t - a^2)\,\hat{k}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{a^2 b^2 \cos^2 t + a^2 b^2 \sin^2 t + (a^2 \cos t - a^2)^2}$$

$$= a\sqrt{b^2(\cos^2 t + \sin^2 t) + a^2(\cos t - 1)^2}$$

$$= a\sqrt{b^2 + 4a^2 \sin^4 \frac{t}{2}}$$

$$|\dot{\vec{r}}| = \sqrt{(a - a\cos t)^2 + a^2 \sin^2 t + b^2}$$

$$= \sqrt{a^2 + a^2(\cos^2 t + \sin^2 t) - 2a^2 \cos t + b^2}$$

$$= \sqrt{2a^2(1 - \cos t) + b^2}$$

$$= \sqrt{4a^2 \sin^2 \frac{t}{2} + b^2}$$

$$\therefore \quad \kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{a\sqrt{b^2 + 4a^2 \sin^4 \frac{t}{2}}}{\left(b^2 + 4a^2 \sin^2 \frac{t}{2}\right)^{3/2}}$$

Also, $$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \begin{vmatrix} a - a\cos t & a\sin t & b \\ a\sin t & a\cos t & 0 \\ a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= b(-a^2 \sin^2 t - a^2 \cos^2 t) \qquad \text{[Expanding by 3rd column]}$$

$$= -a^2 b$$

$$\therefore \quad \tau = \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$= -\frac{a^2 b}{a^2\left(b^2 + 4a^2 \sin^4 \frac{t}{2}\right)}$$

$$= \frac{-b}{b^2 + 4a^2 \sin^4 \frac{t}{2}}.$$

**Example 6.** ***For a point on the curve of intersection of the surfaces $x^2 - y^2 = c^2$, $y = x \tanh \frac{z}{c}$, prove that***

$$\rho = -\sigma = \frac{2x^2}{c}.$$

**Solution.** Let $x = c\cosh t$.

Now
$$y = x\tanh\frac{z}{c} = x\frac{\sinh\frac{z}{c}}{\cosh\frac{z}{c}} = c\cosh t\,\frac{\sinh t}{\cosh t}\left(\frac{d\vec{r}}{dt}\right), \text{ where } \frac{z}{c} = t$$

$\therefore$
$$y = c\sinh t \text{ and } z = ct$$

Clearly, these equations satisfy the equations of given surfaces.

$\therefore$ These equations can be taken as parametric equations of the curve of intersection of the surfaces.

If $\vec{r}$ is the position vector of any point on the curve, then

$$\vec{r} = c\cosh t\,\hat{i} + c\sinh t\,\hat{j} + ct\,\hat{k}$$

i.e.,
$$\vec{r} = (c\cosh t,\ c\sinh t,\ ct)$$

$\therefore$
$$\dot{\vec{r}} = (c\sinh t,\ c\cosh t,\ c)$$

$$\ddot{\vec{r}} = (c\cosh t,\ c\sinh t,\ 0)$$

and
$$\dddot{\vec{r}} = (c\sinh t,\ c\cosh t,\ 0)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c\sinh t & c\cosh t & c \\ c\cosh t & c\sinh t & 0 \end{vmatrix}$$

$$= -c^2\sinh t\,\hat{i} + c^2\cosh t\,\hat{j} - c^2\,\hat{k}$$

$\therefore$
$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = (c^4\sinh^2 t + c^4\cosh^2 t + c^4)^{1/2}$$
$$= c^2(\sinh^2 t + \cosh^2 t + 1)^{1/2}$$
$$= c^2(2\cosh^2 t)^{1/2} \qquad [\because \cosh^2 t - \sinh^2 t = 1]$$
$$= \sqrt{2}c^2\cosh t$$

Also,
$$|\dot{\vec{r}}| = (c^2\sinh^2 t + c^2\cosh^2 t + c^2)^{1/2}$$
$$= [c^2(1+\sinh^2 t) + c^2\cosh^2 t]^{1/2}$$
$$= [2c^2\cosh^2 t]^{1/2} = \sqrt{2}c\cosh t$$

$\therefore$
$$\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{\sqrt{2}\,c^2\cosh t}{2\sqrt{2}\,c^3\cosh^3 t}$$

$$= \frac{1}{2c\cosh^2 t}$$

$$= \frac{c}{2c^2 \cosh^2 t} = \frac{c}{2x^2}$$

$$\therefore \qquad \rho = \frac{1}{\kappa} = \frac{2x^2}{c} \qquad \ldots(1)$$

Also,
$$[\dot{\vec{r}} \ \ddot{\vec{r}} \ \dddot{\vec{r}}] = \begin{vmatrix} c \sinh t & c \cosh t & c \\ c \cosh t & c \sinh t & 0 \\ c \sinh t & c \cosh t & 0 \end{vmatrix}$$

Expanding the determinant by third column, we have

$$[\dot{\vec{r}} \ \ddot{\vec{r}} \ \dddot{\vec{r}}] = c\,[c^2 \cosh^2 t - c^2 \sinh^2 t]$$

$$= c^3$$

$$\therefore \qquad \tau = \frac{[\dot{\vec{r}} \ \ddot{\vec{r}} \ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$= \frac{c^3}{2c^4 \cosh^2 t} = \frac{c}{2x^2}$$

$$\therefore \qquad \sigma = \frac{1}{\tau} = \frac{2x^2}{c} \qquad \ldots(2)$$

From (1) and (2), we have $\rho = \sigma = \dfrac{2x^2}{c}$

But for a left hand system, $\sigma = -\dfrac{2x^2}{c}$

$$\therefore \qquad \rho = -\sigma = \frac{2x^2}{c}$$

Hence the result.

**Example 7.** ***Show that the Serret-Frenet formulae can be written as***

$$\frac{d\hat{t}}{ds} = \omega \times \hat{t}, \ \frac{d\hat{n}}{ds} = \omega \times \hat{n} \quad \textit{and} \quad \frac{d\hat{b}}{ds} = \omega \times \hat{b}$$

***and determine the vector $\omega$ (Darboux vector).***

*Solution.* We know that

$$\frac{d\hat{t}}{ds} = \kappa\hat{n} \qquad \textit{[Serret-Frenet formula]}$$

We can write it as

$$\frac{d\hat{t}}{ds} = 0 + \kappa\hat{n}$$

$$= \tau\hat{t}\times\hat{t} + \kappa(\hat{b}\times\hat{t}) \qquad [\because\ \hat{t}\times\hat{t} = 0]$$

$$\therefore \quad \frac{d\hat{t}}{ds} = (\tau\hat{t} + \kappa\hat{b})\times\hat{t} \qquad ...(1)$$

Again,

$$\frac{d\hat{n}}{ds} = (\tau\hat{b} - \kappa\hat{t}) \qquad \textit{[Serret-Frenet formula]}$$

$$= \tau(\hat{t}\times\hat{n}) + \kappa(\hat{b}\times\hat{n}) \qquad [\because\ \hat{b} = \hat{t}\times\hat{n} \textit{ and } \hat{t} = \hat{n}\times\hat{b}]$$

$$\therefore \quad \frac{d\hat{n}}{ds} = (\tau\hat{t} + \kappa\hat{b})\times\hat{n} \qquad ...(2)$$

Also,

$$\frac{d\hat{b}}{ds} = -\tau\hat{n} \qquad \textit{[Serret-Frenet formula]}$$

$$= \tau(\hat{t}\times\hat{b}) + \kappa\hat{b}\times\hat{b} \qquad [\because\ \hat{n} = \hat{b}\times\hat{t} \textit{ and } \hat{b}\times\hat{b} = 0]$$

$$\therefore \quad \frac{d\hat{b}}{ds} = (\tau\hat{t} + \kappa\hat{b})\times\hat{b} \qquad ...(3)$$

Let $(\tau\hat{t} + \kappa\hat{b}) = \omega$ in each of (1), (2) and (3). Then

$$\frac{d\hat{t}}{ds} = \omega\times\hat{t}, \quad \frac{d\hat{n}}{ds} = \omega\times\hat{n}, \quad \frac{d\hat{b}}{ds} = \omega\times\hat{b},$$

where $\omega = \tau\hat{t} + \kappa\hat{b}$.

**Remark**

$\omega = \tau\hat{t} + \kappa\hat{b}$ is known as Darboux vector.

**Example 8.** *If the position vector $\vec{r}$ of a current point on a curve is function of any parameter t and dot denotes the differentiation w.r.t t, then prove that*

(i) $\dot{\vec{r}} = \dot{s}\hat{t}$ (ii) $\ddot{\vec{r}} = \ddot{s}\hat{t} + \kappa\dot{s}^2\hat{n}$

(iii) $\dddot{\vec{r}} = (\dddot{s} - \kappa^2\dot{s}^3)\hat{t} + \dot{s}(3\kappa\ddot{s} + \dot{\kappa}\dot{s})\hat{n} + \kappa\tau\dot{s}^3\hat{b}$

*Hence deduce that :*

(iv) $\hat{b} = \dfrac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\kappa\dot{s}^3}$ (v) $\hat{n} = \dfrac{\dot{s}\,\ddot{\vec{r}} - \ddot{s}\,\dot{\vec{r}}}{\kappa\dot{s}^3}$

(vi) $\tau = \dfrac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{\kappa^2\dot{s}^6}$ (vii) $\kappa^2 = \dfrac{(\ddot{\vec{r}})^2 - (\ddot{s})^2}{\dot{s}^4}$

**Solution.** (*i*) We know that $\dot{\vec{r}} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}$

*i.e.,*
$$\dot{\vec{r}} = \vec{r}\,'\dot{s}$$

or
$$\dot{\vec{r}} = \dot{s}\hat{t} \quad \ldots(1) \quad [\because \vec{r}\,' = \hat{t}]$$

(*ii*) Differentiating (1) w.r.t $t$, we get

$$\ddot{\vec{r}} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{ds}\left(\frac{ds}{dt}\right)$$

or
$$\ddot{\vec{r}} = \ddot{s}\hat{t} + \dot{s}^2\kappa\hat{n}$$
[*By Serret-Frenet formula*]

$\therefore$
$$\ddot{\vec{r}} = \ddot{s}\hat{t} + \kappa\dot{s}^2\hat{n} \quad \ldots(2)$$

(*iii*) Differentiating (2) w.r.t. parameter $t$, we get

$$\dddot{\vec{r}} = \left(\dddot{s}\hat{t} + \ddot{s}\left(\frac{d\hat{t}}{ds}\right)\left(\frac{ds}{dt}\right)\right) + \kappa\left(\dot{s}^2\frac{d\hat{n}}{dt} + 2\dot{s}\ddot{s}\hat{n}\right) + \dot{\kappa}\dot{s}^2\hat{n}$$

[*Please do not confuse vector $\hat{t}$ with parameter t*]

$$= \left(\dddot{s}\hat{t} + \ddot{s}\dot{s}\kappa\hat{n}\right) + \kappa\left(\dot{s}^2\frac{d\hat{n}}{dt}\dot{s} + 2\dot{s}\ddot{s}\hat{n}\right) + \dot{\kappa}\dot{s}^2\hat{n}$$

$$= \left(\dddot{s}\,\hat{t} + \ddot{s}\,\dot{s}\,\kappa\hat{n}\right) + \kappa\dot{s}^3(\tau\hat{b} - \kappa\hat{t}) + 2\dot{s}\ddot{s}\kappa\hat{n} + \dot{\kappa}\dot{s}^2\hat{n}$$

$$= \left(\dddot{s} - \kappa^2\dot{s}^3\right)\hat{t} + \dot{s}\,(3\ddot{s}\,\kappa + \dot{\kappa}\,\dot{s})\,\hat{n} + \kappa\dot{s}^3\tau\hat{b}$$

(iv) $$\dot{\vec{r}} \times \ddot{\vec{r}} = \dot{s}\hat{t} \times (\ddot{s}\hat{t} + \kappa\dot{s}^2\hat{n})$$

[Using part (i) and (ii)]

$$= \dot{s}\ddot{s}\,\hat{t} \times \hat{t} + \kappa\,\dot{s}^2\dot{s}\,\hat{t} \times \hat{n}$$

...(3)

$$= \kappa\dot{s}^3\hat{b}$$

$[\because\ \hat{t} \times \hat{t} = 0 \text{ and } \hat{t} \times \hat{n} = \hat{b}]$

$$\therefore \quad \hat{b} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3\kappa}$$

(v) Multiplying (2) by $\dot{s}$ and (1) by $\ddot{s}$ and subtracting, we have

$$\dot{s}\,\ddot{\vec{r}} - \ddot{s}\,\dot{\vec{r}} = \dot{s}\,(\ddot{s}\hat{t} + \kappa\dot{s}^2\hat{n}) - \ddot{s}\dot{s}\hat{t} = \kappa\dot{s}^3\hat{n}$$

$$\therefore \quad \hat{n} = \frac{\dot{s}\,\ddot{\vec{r}} - \ddot{s}\,\dot{\vec{r}}}{\kappa\dot{s}^3}.$$

(vi) $$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \dot{\vec{r}} \times \ddot{\vec{r}} \cdot \dddot{\vec{r}}$$

$$\therefore \quad [\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \kappa\dot{s}^3\hat{b}\cdot[(\dddot{s} - \kappa^2\dot{s}^3)\,\hat{t} + \dot{s}\,(3\ddot{s}\kappa + \dot{\kappa}\,\dot{s})\,\hat{n} + \kappa\dot{s}^3\tau\hat{b}]$$

[Using part (iv) and part (iii)]

i.e., $$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \kappa^2\dot{s}^6\tau$$

$[\because\ \hat{b}\cdot\hat{t} = 0,\ \hat{b}\cdot\hat{n} = 0 \text{ and } \hat{b}\cdot\hat{b} = 1]$

$$\therefore \quad \tau = \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{\dot{s}^6\kappa^2}$$

(vii) Squaring (2), we have

$$(\ddot{\vec{r}})^2 = (\ddot{s}\hat{t} + \kappa\dot{s}^2\hat{n})^2$$

$$= \ddot{s}^2\hat{t}^2 + 2\kappa\ddot{s}\dot{s}^2\,\hat{t}\cdot\hat{n} + \kappa^2\dot{s}^4\hat{n}\cdot\hat{n}$$

$[\because\ \hat{t}\,.\,\hat{n} = 0]$

$$= \ddot{s}^2 + \kappa^2\dot{s}^4$$

$$\kappa^2 = \frac{(\ddot{\vec{r}})^2 - (\ddot{s})^2}{\dot{s}^4}.$$

**Example 9.** ***Prove the following :*** [K.U. 2015]

(*i*) $\vec{r}''' = \kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b}$

(*ii*) $\vec{r}'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)\hat{n} - 3\kappa\kappa'\hat{t} + (2\kappa'\tau + \tau'\kappa)\hat{b}$

(*iii*) $[\vec{r}'\ \vec{r}''\ \vec{r}'''] = \kappa^2\tau$

(*iv*) $\vec{r}' \cdot \vec{r}'' = 0$

(*v*) $\vec{r}' \cdot \vec{r}''' = -\kappa^2$

(*vi*) $\vec{r}' \cdot \vec{r}'''' = -3\kappa\kappa'$

(*vii*) $\vec{r}'' \cdot \vec{r}''' = \kappa\kappa'$

(*viii*) $\vec{r}'' \cdot \vec{r}'''' = \kappa(\kappa'' - \kappa^3 - \kappa\tau^2)$ [K.U. 2016]

(*ix*) $\vec{r}''' \cdot \vec{r}'''' = \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa^2\tau\tau' + \kappa\kappa'\tau^2$

(*x*) $[\hat{t}'\ \hat{t}''\ \hat{t}'''] = [\vec{r}''\ \vec{r}'''\ \vec{r}''''] = \kappa^3(\kappa\tau' - \kappa'\tau) = \kappa^5 \dfrac{d}{ds}\left(\dfrac{\tau}{\kappa}\right)$

(*xi*) $[\hat{b}'\ \hat{b}''\ \hat{b}'''] = \tau^3(\kappa'\tau - \kappa\tau') = \tau^5 \dfrac{d}{ds}\left(\dfrac{\kappa}{\tau}\right)$

***where dash denotes the derivatives w.r.t. s.***

**Solution.** By Serret-Frenet formula

$$\frac{d\hat{t}}{ds} = \kappa\hat{n}$$

or
$$\frac{d\left(\dfrac{d\vec{r}}{ds}\right)}{ds} = \kappa\hat{n} \qquad \left[\because\ \hat{t} = \frac{d\vec{r}}{ds}\right]$$

*i.e.,*
$$\vec{r}'' = \kappa\hat{n}$$

Differentiating w.r.t. $s$, we have

$$\vec{r}''' = \kappa\hat{n}' + \kappa'\hat{n}$$

$$= \kappa(\tau\hat{b} - \kappa\hat{t}) + \kappa'\hat{n} \qquad \left[\because \hat{n}' = \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}\right]$$

$$= \kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b} \qquad ...(1)$$

(ii) Differentiating (1) w.r.t. $s$, we have

$$\vec{r}'''' = \kappa''\hat{n} + \kappa'\hat{n}' - 2\kappa\kappa'\hat{t} - \kappa^2\hat{t}' + (\kappa\tau' + \kappa'\tau)\hat{b} + \kappa\tau\hat{b}'$$

We know that $\hat{n}' = \dfrac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}$

$$\hat{t}' = \frac{d\hat{t}}{ds} = \kappa\hat{n} \qquad \text{[Serret-Frenet formulae]}$$

and $\hat{b}' = \dfrac{d\hat{b}}{ds} = -\tau\hat{n}$

Using these, we have

$$\vec{r}'''' = \kappa''\hat{n} + \kappa'(\tau\hat{b} - \kappa\hat{t}) - 2\kappa\kappa'\hat{t} - \kappa^2\kappa\hat{n} + (\kappa\tau' + \kappa'\tau)\hat{b} + \kappa\tau(-\tau\hat{n})$$

$$= (\kappa'' - \kappa^3 - \kappa\tau^2)\hat{n} - 3\kappa\kappa'\hat{t} + (2\kappa'\tau + \tau'\kappa)\hat{b}$$

(iii) We know that $\dfrac{d\vec{r}}{ds} = \hat{t}$ i.e., $\vec{r}' = \hat{t}$

$$\therefore \quad \vec{r}'' = \frac{d\hat{t}}{ds} = \kappa\hat{n} \qquad \text{[Differentiating w.r.t. } s\text{]}$$

and $\vec{r}''' = \kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b}$ [By part (i)]

$$\therefore \quad [\vec{r}'\ \vec{r}''\ \vec{r}'''] = \vec{r}' \cdot \vec{r}'' \times \vec{r}'''$$

$$= \hat{t} \cdot \kappa\hat{n} \times (\kappa'\hat{n} - \kappa^2\hat{t} + \kappa\tau\hat{b})$$

$$= \hat{t} \cdot (-\kappa^3(-\hat{b}) + \kappa^2\tau\hat{t}) \qquad [\because \hat{n} \times \hat{t} = -\hat{b} \text{ and } \hat{n} \times \hat{b} = \hat{t}]$$

$$[\because \hat{t} \cdot \hat{b} = 0]$$

$$= \kappa^2\tau \qquad [\because \vec{r}' = \hat{t}]$$

(iv) $\vec{r}' \cdot \vec{r}'' = \hat{t} \cdot \hat{t}'$ $\left[\because \hat{t}' = \dfrac{d\hat{t}}{ds} = \kappa\hat{n}\right]$

$= \hat{t} \cdot \kappa\hat{n}$

$= \kappa \hat{t} \cdot \hat{n}$

$= 0$ $\quad [\because \hat{t} \cdot \hat{n} = 0]$

(v) $\vec{r}' \cdot \vec{r}''' = \hat{t} \cdot (\kappa' \hat{n} - \kappa^2 \hat{t} + \kappa\tau \hat{b})$ $\quad$ [By part (i)]

$= -\kappa^2$ $\quad [\because \hat{t} \cdot \hat{n}$ and $\hat{t} \cdot \hat{b} = 0]$

(vi) $\vec{r}' \cdot \vec{r}'''' = \hat{t} \cdot [(\kappa'' - \kappa^3 - \kappa\tau^2)\, \hat{n} - 3\kappa\kappa' \hat{t} + (2\kappa'\tau + \tau'\kappa)\, \hat{b}]$ $\quad$ [By part (ii)]

$= -3\kappa\kappa' \hat{t} \cdot \hat{t}$ $\quad [\because \hat{t} \cdot \hat{n} = 0$ and $\hat{t} \cdot \hat{b} = 0]$

$= -3\kappa\kappa'$ $\quad [\because \hat{t} \cdot \hat{t} = 0]$

(vii) $\vec{r}'' = \kappa \hat{n}$

$\vec{r}''' = \kappa' \hat{n} - \kappa^2 \hat{t} + \kappa\tau \hat{b}$ $\quad$ [By part (i)]

$\therefore$ $\vec{r}'' \cdot \vec{r}''' = \kappa \hat{n} \cdot (\kappa' \hat{n} - \kappa^2 \hat{t} + \kappa\tau \hat{b})$

$= \kappa\kappa' \hat{n} \cdot \hat{n}$ $\quad [\because \hat{n} \cdot \hat{t} = 0$ and $\hat{n} \cdot \hat{b} = 0]$

$= \kappa\kappa'$ $\quad [\because \hat{n} \cdot \hat{n} = 1]$

(viii) $\vec{r}'' = \kappa \hat{n}$

$\vec{r}'''' = (\kappa'' - \kappa^3 - \kappa\tau^2)\, \hat{n} - 3\kappa\kappa' \hat{t} + (2\kappa'\tau + \tau'\kappa)\, \hat{b}$

$\therefore$ $\vec{r}'' \cdot \vec{r}'''' = \kappa \hat{n} \cdot [(\kappa'' - \kappa^3 - \kappa\tau^2)\, \hat{n} - 3\kappa\kappa' \hat{t} + (2\kappa'\tau + \tau'\kappa)\, \hat{b}]$

$= \kappa(\kappa'' - \kappa^3 - \kappa\tau^2)$ $\quad [\because \hat{n} \cdot \hat{t} = 0$ and $\hat{n} \cdot \hat{b} = 0]$

(ix) $\vec{r}''' \cdot \vec{r}'''' = (\kappa' \hat{n} - \kappa^2 \hat{t} + \kappa\tau \hat{b}) \cdot [(\kappa'' - \kappa^3 - \kappa\tau^2)\, \hat{n} - 3\kappa\kappa' \hat{t} + (2\kappa'\tau + \tau'\kappa)\, \hat{b}]$

[Using parts (i) and (ii)]

$= \kappa'(\kappa'' - \kappa^3 - \kappa\tau^2)(\hat{n} \cdot \hat{n}) + \kappa^2 3\kappa\kappa' (\hat{t} \cdot \hat{t}) + \kappa\tau(2\kappa'\tau + \tau'\kappa)(\hat{b} \cdot \hat{b})$

$= \kappa'\kappa'' - \kappa'\kappa^3 - \kappa\kappa'\tau^2 + 3\kappa^3\kappa' + 2\kappa\kappa'\tau^2 + \kappa^2\tau\tau'$

$= \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa\kappa'\tau^2 + \kappa^2\tau\tau'$

(x) We know that $\hat{t} = \vec{r}\,'$

$\therefore$ $\hat{t}' = \vec{r}\,''$, $\hat{t}'' = \vec{r}\,'''$ and $\hat{t}''' = \vec{r}\,''''$

Now $[\hat{t}'\ \hat{t}''\ \hat{t}'''] = [\vec{r}\,''\ \vec{r}\,'''\ \vec{r}\,'''']$ ...(1)

Also, we have $\vec{r}\,'' = 0\hat{t} + \kappa\hat{n} + 0\hat{b}$

$$\vec{r}\,''' = -\kappa^2\hat{t} + \kappa'\hat{n} + \kappa\tau\hat{b}$$

$$\vec{r}\,'''' = -3\kappa\kappa'\hat{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\hat{n} + (2\kappa'\tau + \tau'\kappa)\hat{b}$$ [By part (i) and (ii)]

Using these in (1), we have

$$[\hat{t}'\ \hat{t}''\ \hat{t}'''] = [\vec{r}\,''\ \vec{r}\,'''\ \vec{r}\,''''] = \begin{vmatrix} 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \\ -3\kappa\kappa' & \kappa'' - \kappa^3 - \kappa\tau^2 & 2\kappa'\tau + \tau'\kappa \end{vmatrix}$$

$$= -\kappa[-\kappa^2(2\kappa'\tau + \tau'\kappa) - \kappa\tau(-3\kappa\kappa')]$$

$$= -\kappa[-2\kappa^2\kappa'\tau - \kappa^2\tau'\kappa + 3\kappa^2\kappa'\tau]$$

$$= -\kappa[\kappa^2\kappa'\tau - \kappa^2\tau'\kappa]$$

$$= \kappa^3[\kappa\tau' - \kappa'\tau]$$

$$= \kappa^3\left[\kappa\frac{d\tau}{ds} - \tau\frac{d\kappa}{ds}\right]$$

$$= \kappa^5\left[\frac{\kappa\dfrac{d\tau}{ds} - \tau\dfrac{d\kappa}{ds}}{\kappa^2}\right]$$

$$= \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

(xi) We have $\hat{b}' = -\tau\hat{n} = 0\hat{t} - \tau\hat{n} + 0\hat{b}$ ...(1)

and $\hat{b}'' = -\tau\hat{n}' - \tau'\hat{n}$

$$= -\tau(\tau\hat{b} - \kappa\hat{t}) - \tau'\hat{n} \qquad \left[\because \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}\right]$$

$$= -\tau^2\hat{b} + \kappa\tau\hat{t} - \tau'\hat{n}$$

$$= \tau\kappa\hat{t} - \tau'\hat{n} - \tau^2\hat{b} \qquad \ldots(2)$$

Differentiating w.r.t. $s$, we have

$$\hat{b}''' = -(\tau^2\hat{b}' + 2\tau\tau'\hat{b}) + \tau\kappa\hat{t}' + (\tau\kappa' + \tau'\kappa)\hat{t} - (\tau'\hat{n}' + \tau''\hat{n})$$

or $$\hat{b}''' = -\tau^2(-\tau\hat{n}) - 2\tau\tau'\hat{b} + \tau\kappa\kappa\hat{n} + (\tau\kappa' + \tau'\kappa)\hat{t} - \tau'(\tau\hat{b} - \kappa\hat{t}) - \tau''\hat{n}$$

$$\therefore \quad \hat{b}''' = (\tau\kappa' + 2\tau'\kappa)\hat{t} + (\tau^3 + \tau\kappa^2 - \tau'')\hat{n} - 3\tau\tau'\hat{b} \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$[\hat{b}'\ \hat{b}''\ \hat{b}'''] = \begin{vmatrix} 0 & -\tau & 0 \\ \tau\kappa & -\tau' & -\tau^2 \\ \tau\kappa' + 2\tau'\kappa & \tau^3 + \tau\kappa^2 - \tau'' & -3\tau\tau' \end{vmatrix}$$

$$\therefore \quad [\hat{b}'\ \hat{b}''\ \hat{b}'''] = \tau[\tau\kappa(-3\tau\tau') + \tau^2(\tau\kappa' + 2\tau'\kappa)]$$

$$= \tau[-3\tau^2\tau'\kappa + \tau^3\kappa' + 2\tau^2\tau'\kappa]$$

$$= \tau^3[\tau\kappa' - \tau'\kappa]$$

$$= \tau^5\left[\frac{\tau\dfrac{d\kappa}{ds} - \kappa\dfrac{d\tau}{ds}}{\tau^2}\right]$$

$$= \tau^5\frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

**Example 10.** ***Prove that the position vector of the current point on a curve satisfies the differential equation***

$$\frac{d}{ds}\left[\sigma\frac{d}{ds}\rho\left(\frac{d^2\vec{r}}{ds^2}\right)\right] + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{d\vec{r}}{ds}\right] + \frac{\rho}{\sigma}\frac{d^2\vec{r}}{ds} = 0.$$

**Solution.** We know that

$$\tau = \frac{1}{\sigma} \quad \text{and} \quad \kappa = \frac{1}{\rho}$$

Also, $$\frac{d\vec{r}}{ds} = \vec{r}\,' = \hat{t}, \quad \frac{d^2\vec{r}}{ds^2} = \vec{r}\,'' = \hat{t}' = \kappa\hat{n}$$

Putting the above values in the given differential equation, we have

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\kappa\hat{n}\right)\right]+\frac{d}{ds}\left[\frac{\kappa}{\tau}\hat{t}\right]+\frac{\tau}{\kappa}\kappa\hat{n}=0$$

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\hat{n}\right]+\frac{d}{ds}\left[\frac{\kappa}{\tau}\hat{t}\right]+\tau\hat{n}=0$$

$$\frac{d}{ds}\left[\frac{1}{\tau}\hat{n}'\right]+\frac{d}{ds}\left[\frac{\kappa}{\tau}\hat{t}\right]+\tau\hat{n}=0$$

$$\frac{d}{ds}\left[\frac{1}{\tau}(\tau\hat{b}-\kappa\hat{t})\right]+\frac{d}{ds}\left[\frac{\kappa}{\tau}\hat{t}\right]+\tau\hat{n}=0$$

$$\frac{d}{ds}\left[\hat{b}-\frac{\kappa}{\tau}\hat{t}\right]+\frac{d}{ds}\left[\frac{\kappa}{\tau}\hat{t}\right]+\tau\hat{n}=0$$

or $$\frac{d\hat{b}}{ds}-\frac{d}{ds}\left(\frac{\kappa}{\tau}\hat{t}\right)+\frac{d}{ds}\left(\frac{\kappa}{\tau}\hat{t}\right)+\tau\hat{n}=0$$

or $$\hat{b}'+\kappa\hat{n}=0$$

or $$-\tau\hat{n}+\tau\hat{n}=0 \qquad [\because \hat{b}'=-\tau\hat{n}]$$

i.e., $0 = 0$, which is true.

Hence the result.

**Example 11.** ***If the tangent and binormal at a point of a curve makes angle $\theta$ and $\phi$ with a fixed direction, show that***

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}=-\frac{\kappa}{\tau}.$$ [M.D.U. 2013, 08, 07]

**Solution.** Let $\theta$ and $\phi$ be the angles made by tangent and normal respectively with a fixed direction of vector $\vec{a}$ in space

Let $|\vec{a}| = a$

$\therefore$ $\hat{t}\cdot\vec{a} = a\cos\theta$ ...(1) $[\because |\hat{t}| = 1]$

$\hat{b}\cdot\vec{a} = a\cos\phi$ ...(2) $[\because |\hat{b}| = 1]$

Differentiating (1) w.r.t. $s$, we have

$$\hat{t}' \cdot \vec{a} = -a \sin\theta \frac{d\theta}{ds} \qquad [\because \vec{a} \text{ is a fixed i.e., constant vector}]$$

or
$$\kappa \hat{n} \cdot \vec{a} = -a \sin\theta \frac{d\theta}{ds} \qquad ...(3)$$

Differentiating (2) w.r.t. $s$, we have

$$\hat{b}' \cdot \vec{a} = -a \sin\phi \frac{d\phi}{ds} \qquad [\because \vec{a} \text{ is a constant vector}]$$

or
$$-\tau \hat{n} \cdot \vec{a} = -a \sin\phi \frac{d\phi}{ds} \qquad ...(4)$$

Dividing (3) by (4), we get $-\dfrac{\kappa}{\tau} = \dfrac{\sin\theta}{\sin\phi} \dfrac{d\theta}{d\phi}$.

## EXERCISE 8.5

1. Calculate the curvature and torsion of the cubic curve given by $\vec{r} = (t, t^2, t^3)$.

2. For the curve $x = a(3t - t^3), y = 3at^2, z = a(3t + t^3)$, show that $\kappa = \tau = \dfrac{1}{3a(1+t^2)^2}$. [M.D.U. 2013, 07]

3. For the curve $\vec{r} = (2abt, a^2 \log t, b^2 t^2)$, show that $\kappa = -\tau = \dfrac{2abt}{(a^2 + 2b^2t^2)^2}$.

4. For the curve $x = 3t, y = 3t^2, z = 2t^3$, show that $\rho = -\sigma = \dfrac{3}{2}(1 + 2t^2)^2$.

5. Find the curvature and torsion of the curve given by
$$\vec{r} = \left(a + a\cos t,\ a \sin t,\ 2a \sin\frac{t}{2}\right).$$

6. Find the radius of curvature and radius of torsion at the point $\theta = \dfrac{\pi}{4}$ of the curve,
$$\vec{r} = (a\cos\theta,\ a\sin\theta,\ a\cos 2\theta).$$

7. For the curve $x = 4a\cos^3 t, y = 4a \sin^3 t, z = 3c \cos 2t$, prove that
$$\kappa = \frac{a}{6(a^2 + c^2)\sin 2t}. \qquad \text{[M.D.U. 2017]}$$

8. Find the curvature and torsion of the curves
$$x = a\cos t, y = a \sin t, z = a t \cot\alpha \qquad \text{[M.D.U. 2017]}$$

9. For the curve $x = a\tan\theta, y = a\cot\theta, z = \sqrt{2}\, a \log \tan\theta$, prove that
$$\rho = -\sigma = \frac{2\sqrt{2}\, a}{\sin^2 2\theta}$$

# EXERCISE 8.5

10. Find the radii of curvature and torsion at a point of the curve

$$x^2 + y^2 = a^2, \; x^2 - y^2 = az$$

[K.U. 2011]

[**Hint :** Parametric equation of the curve are $x = a\cos\theta$, $y = a\sin\theta$, $z = a\cos 2\theta$]

11. Find the osculating plane, curvature and torsion at any point of the curve

$$x = a\cos 2t, \; y = a\sin 2t, \; z = 2a\sin t.$$

12. If $m_1, m_2, m_3$ are the moments about the origin of unit vector $\hat{t}, \hat{n}, \hat{b}$ localised in the tangent, principal normal and binormal, show that

$$m_1' = \kappa m_2, \quad m_2' = \hat{b} - \kappa m_1 + \tau m_3, \quad m_3' = -\hat{n} - \tau m_2$$

13. Given the curve $\vec{r} = \left(e^{-u}\sin u, e^{-u}\cos u, e^{-u}\right)$. At any point $u$ of this curve, find

(*i*) the unit tangent vector $\hat{t}$.

(*ii*) the equations of the tangent.

(*iii*) the equation of the normal plane.

(*iv*) the curvature.

(*v*) the unit principal normal vector $\hat{n}$.

(*vi*) the equation of the principal normal.

(*vii*) the unit binormal vector $\hat{b}$.

(*viii*) the equation of the binormal.

14. Let C be a curve given by the equation $\vec{r} = (t, t^2, t^3)$. Find the curvature and torsion of C at the point (0, 0, 0). Also find the equation of normal line and normal plane at the point(1, 1, 1).

15. Determine the function $f(u)$, so that the curve given by

$\vec{r} = \{a\cos u, a\sin u, f(u)\}$ should be a plane.

16. Prove that at the point of intersection of the surfaces

$$x^2 + y^2 = z^2, \; z = a\tan^{-1}\frac{y}{x}, \text{ where } y = x\tan\theta,$$

$$\rho = \frac{a(2+\theta^2)^{3/2}}{(8+5\theta^2+\theta^4)^{1/2}} \text{ and } \sigma = \frac{a(8+5\theta^2+\theta^4)}{6+\theta^2}.$$

17. Show that the principal normal at consecutive points do not intersect unless $\tau = 0$.

## ANSWERS

**1.** $\kappa = \dfrac{2(9t^4+9t^2+1)^{1/2}}{(1+4t^2+9t^4)^{3/2}}, \quad \tau = \dfrac{3}{(9t^4+9t^2+1)}$

**5.** $\kappa = \dfrac{(13+3\cos t)^{1/2}}{a\,(3+\cos t)^{3/2}}, \quad \tau = \dfrac{6\cos\dfrac{t}{2}}{a(13+3\cos t)}.$

**6.** $\rho = 5a, \ \sigma = \dfrac{5a}{6}$

**8.** $\kappa = \dfrac{1}{a}\sin^2\alpha, \quad \tau = \dfrac{1}{a}\sin\alpha\cos\alpha$

**10.** $\rho^2 = \dfrac{(5a^2-4z^2)^3}{a^2\,(5a^2+12z^2)}, \quad \sigma = \dfrac{5a^2+12z^2}{6\sqrt{a^2-z^2}}$, where $z = a\cos 2\theta$

**11.** Equation of osculating plane is $X(\sin t + \sin 2t\cos t) - 2Y\cos^3 t + 2Z = 3a\sin t$

$$\kappa = \frac{(5+3\cos^2 t)^{1/2}}{2a\,(1+\cos^2 t)^{3/2}}; \quad \tau = \frac{3}{a\,(5\sec t + 3\cos t)}$$

**13.** (*i*) $\hat{t} = \dfrac{1}{\sqrt{3}}(\cos u - \sin u, -\sin u - \cos u, -1)$

(*ii*) $\dfrac{X - e^{-u}\sin u}{\cos u - \sin u} = \dfrac{Y - e^{-u}\cos u}{-\sin u - \cos u} = \dfrac{Z - e^{-u}}{-1}$

(*iii*) $X(\sin u - \cos u) + Y(\sin u + \cos u) + Z = 2e^{-u}$

(*iv*) $\kappa = \dfrac{\sqrt{2}}{3}e^{u}$

(*v*) $\hat{n} = -\dfrac{1}{\sqrt{2}}[(\sin u + \cos u)\,\hat{i} + (\cos u - \sin u)\,\hat{j} + 0\hat{k}]$

(*vi*) $\dfrac{X - e^{-u}\sin u}{\sin u + \cos u} = \dfrac{Y - e^{-u}\cos u}{\cos u - \sin u} = \dfrac{Z - e^{-u}}{0}$

(*vii*) $\hat{b} = \dfrac{1}{\sqrt{6}}[(\sin u - \cos u)\,\hat{i} + (\sin u + \cos u)\,\hat{j} - 2\hat{k}]$

(*viii*) $\dfrac{X - e^{-u}\sin u}{\sin u - \cos u} = \dfrac{Y - e^{-u}\cos u}{\sin u + \cos u} = \dfrac{Z - e^{-u}}{-2}$

**14.** $\kappa = 2$ and $\tau = 3$

Equation of binormal is $\dfrac{X-1}{3} = \dfrac{Y-1}{-3} = \dfrac{Z-1}{1}$

Equation of normal plane is $X + 2Y + 3Z = 6.$

**15.** $f(u) = A - B\sin(C - u).$

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