IFoS MATHS PAPER-2
(2009 to 2020)

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G-20 MATHS

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CHAPTER-01 : IFoS P-2 2020 G-20(MATHS)

1(a) Let p be a prime number. Then show that
$$(p-1)!+1 \equiv 0 \bmod(p)$$
Also, find the remainder when
$$6^{44}.(22)!+3 \text{ is divided by } 23. \quad (8)$$

Sol: This statement is true both ways and it is called 'Wilson's Theorem'.

The statement is clear when $p=2$, so we assume that $p>2$.
We first assume that p is prime and prove that $(p-1)!+1 \equiv 0 \bmod(p)$.

If $a \in \{1,2,\dots,p-1\}$, then the equation
$$ax \equiv 1 \ (\bmod p)$$
has a unique solution $a' \in \{1,2,\dots,p-1\}$.
If $a=a'$, then $a^2 \equiv 1 \bmod(p)$,
so $p|(a^2-1) = (a-1)(a+1)$,
so $p|(a-1)$ or $p|(a+1)$, so $a \in \{1, p-1\}$.

We can thus pair off the elements of $\{2,3,\dots,p-2\}$, each with their inverse.
Thus
$$2.3.\dots.(p-2) \equiv 1 \bmod(p)$$
multiplying both sides by $(p-1)$ proves that
$$(p-1)! \equiv -1 \bmod(p)$$
$$\Rightarrow \quad (p-1)!+1 \equiv 0 \bmod(p).$$

Now, we will prove the converse part.
(Not asked in Question though ☺)

Assume that $(p-1)!+1 \equiv 0 \mod (p)$

We have to prove that p must be prime.

Suppose not,

Let $p \geq 4$ is a composite number.

Let m be a prime divisor of p.

Then $m < p$, so that $m \mid (p-1)!$

Also, by our above assumption

$$\cancel{m}\ p \mid (p-1)!+1$$

$$\Rightarrow m \text{ divides } (p-1)!+1$$

This is a contradiction, because a prime cannot divide a number a and also divide $a+1$, since it would then have to divide $(a+1)-a=1$.

It completes our proof.

Now, we find the remainder when

$x = 6^{44}\cdot(22)!+3$ is divided by 23.

By above result, $(23-1)!+1 \equiv 0 \mod (23)$

i.e. $22! \equiv -1 \mod (23)$

$$\therefore\ 6^{44}\cdot(22)!+3 \equiv 6^{44}\cdot(-1)+3 \mod (23)$$

Using Euler's Theorem, i.e. $a^{\phi(n)} \equiv 1 \mod n$

$\cancel{for}$ $(n \in \mathbb{N},\ a \in \mathbb{Z},\ \gcd(a,n)=1)$

$$6^{\phi(23)} \equiv 1 \mod 23$$

$$\Rightarrow 6^{22} \equiv 1 \mod 23$$

$$\therefore\ 6^{44} = (6^{22})^2 \equiv 1 \mod 23$$

Hence, $6^{44}\cdot(22)!+3 \equiv 1\cdot(-1)+3 \mod 23$

$$\equiv 2 \mod (23)$$

1(b) (i) If $u = u(y-z, z-x, x-y)$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

(ii) If $u(x,y,z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

(8)

Sol: (i) $u = u(y-z, z-x, x-y)$

$u = u(\alpha, \beta, \gamma)$ (say)

where,

$\alpha = y-z, \quad \beta = z-x, \quad \gamma = x-y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} + \frac{\partial u}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial x}$$

$$= \frac{\partial u}{\partial \alpha}\cdot(0) + \frac{\partial u}{\partial \beta}\cdot(-1) + \frac{\partial u}{\partial \gamma}(1)$$

$$= -\frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} \quad \text{—— (1)}$$

similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial y} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial y}$$

$$= \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(0) + \frac{\partial u}{\partial \gamma}(-1)$$

$$= \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \gamma} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial z} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial z} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial z}$$

$$= \frac{\partial u}{\partial \alpha}(-1) + \frac{\partial u}{\partial \beta}(1) + \frac{\partial u}{\partial \gamma}(0)$$

$$= -\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \quad \text{—— (3)}$$

Adding (1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

classmate
Date
Page

(ii) $u(x,y,z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$

Consider,

$$u(\lambda x, \lambda y, \lambda z) = \frac{\lambda x}{\lambda y+\lambda z} + \frac{\lambda y}{\lambda z+\lambda x} + \frac{\lambda z}{\lambda x+\lambda y}$$

$$= \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

$$= u(x,y,z)$$

i.e. $u(\lambda x, \lambda y, \lambda z) = \lambda^0 . u(x,y,z)$

$\therefore$ $u(x,y,z)$ is a homogeneous function of degree zero.

$\therefore$ By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$$= 0.u \quad (\because n = 0 \text{ here})$$

$$= 0$$

Method -2 : Without Euler's Theorem

$$\frac{\partial u}{\partial x} = \frac{1}{y+z} - \frac{y}{(z+x)^2} - \frac{z}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{(y+z)^2} + \frac{1}{z+x} - \frac{z}{(x+y)^2}$$

$$\frac{\partial u}{\partial z} = \frac{-x}{(y+z)^2} + \frac{(-y)}{(z+x)^2} + \frac{1}{x+y}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(\frac{x}{y+z} - \frac{xy}{(z+x)^2} - \frac{zx}{(x+y)^2}\right)$$

$$+\left(\frac{-xy}{(y+z)^2} + \frac{y}{z+x} - \frac{zy}{(x+y)^2}\right) + \left(\frac{-xz}{(y+z)^2} - \frac{yz}{(z+x)^2} + \frac{z}{x+y}\right)$$

$$= \sum \left\{\frac{x}{y+z} - x\left(\frac{y}{(y+z)^2} + \frac{z}{(y+z)^2}\right)\right\} = \sum \frac{x}{y+z} - \frac{x}{y+z} = 0$$

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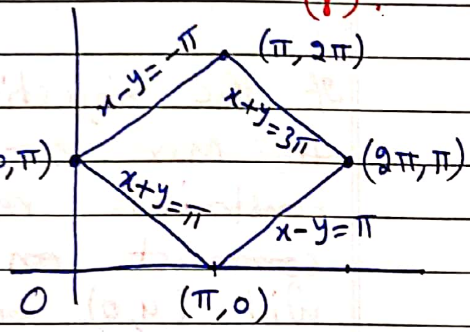
2(c) Evaluate the integral

$$\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy,$$

where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$. (8).

Sol: Changing the coordinates



Let $u = x+y$

$v = x - y$

$\therefore \; x = \frac{1}{2}(u+v),$

$y = \frac{1}{2}(u-v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= \left| -\frac{1}{4} - \frac{1}{4} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

$\therefore$ Limits changes to: $u = \pi$ to 3π

$v = -\pi$ to π

$$I = \iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy$$

$$= \int_{u=\pi}^{3\pi} \int_{v=-\pi}^{\pi} v^2 \cos^2 u \left(+\frac{1}{2}\right) du\, dv$$

$$= +\frac{1}{2} \int_{-\pi}^{\pi} v^2\, dv \cdot \int_{\pi}^{3\pi} \cos^2 u\, du$$

$$= +\frac{1}{2} \left[\frac{v^3}{3}\right]_{-\pi}^{\pi} \times \left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_{\pi}^{3\pi}$$

$$= +\frac{1}{6} \times 2\pi^3 \times \frac{\cancel{2}\pi}{\cancel{2}} = \frac{\pi^4}{3}$$

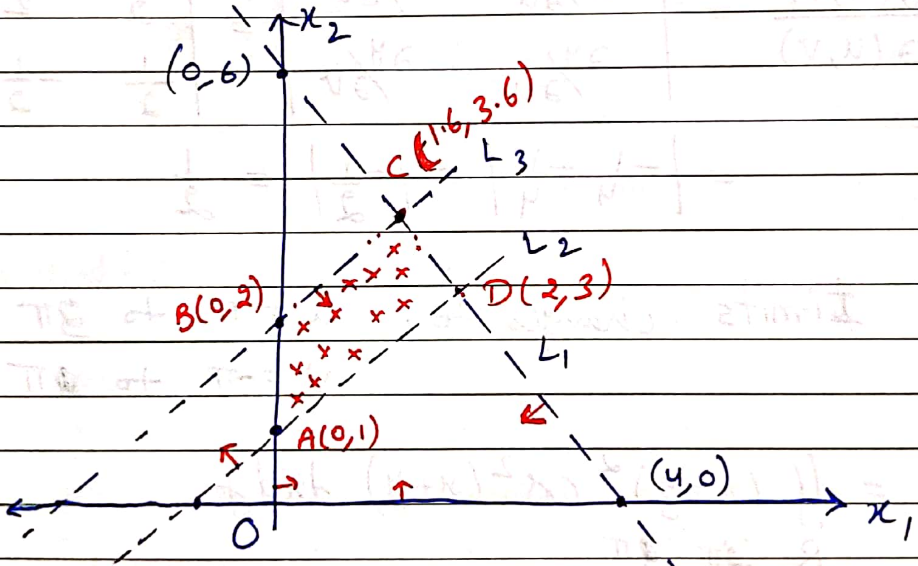
1(d) Solve the following LPP graphically,

Max $z = 5x_1 - 3x_2$

subject to
$$3x_1 + 2x_2 \le 12$$
$$-x_1 + x_2 \ge 1$$
$$-x_1 + x_2 \le 2$$
$$x_1, x_2 \ge 0.$$

If the objective function z is changed to Max $z = 6x_1 + 4x_2$, while the constraints remain the same, comment on the number of solutions. Will $(4,0)$ be also a solution? (8)

Sol: We plot the feasible region on graph.



L_1: $3x_1 + 2x_2 = 12 \Rightarrow$

x_1	0	4
x_2	6	0

L_2: $-x_1 + x_2 = 1 \Rightarrow$

x_1	0	-1
x_2	1	0

L_3: $-x_1 + x_2 = 2 \Rightarrow$

x_1	0	-2
x_2	2	0

To get point C : $-x_1 + x_2 = 2$ i.e. $x_2 = 2 + x_1$

$3x_1 + 2x_2 = 12 \Rightarrow 3x_1 + 2(x_1+2) = 12$

i.e. $5x_1 = 8 \Rightarrow x_1 = 1.6, x_2 = 3.6$

Point D : $-x_1 + x_2 = 1 \Rightarrow x_2 = x_1 + 1$

$3x_1 + 2(x_1+1) = 12 \Rightarrow 5x_1 = 10 \Rightarrow x_1 = 2$

$x_2 = 3$

Points	$z_1 = 5x_1 - 3x_2$	$z_2 = 6x_1 + 4x_2$
A (0,1)	−3	4
B (0,2)	−6	8
C (1.6, 3.6)	−2.8	24
D (2, 3)	1	24

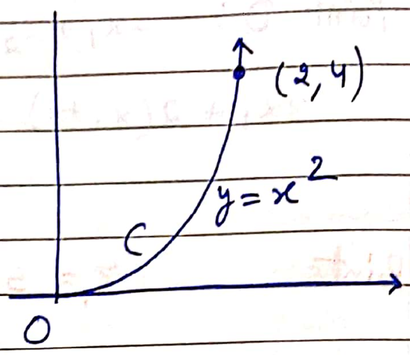
As the feasible region is closed.
Max value of $z_1 = 5x_1 - 3x_2$ is obtained at point D(2, 3) and Max $z_1 = 1$.

In case, objective function is $z_2 = 6x_1 + 4x_2$ we notice that maximum value i.e. Max $z_2 = 24$ occurs at two points C(1.6, 3.6) and D(2, 3).
Hence, z_2 will have same value at all points on line segment CD.
Hence there will be infinite many solutions.

finally, Point (4,0) will not be a solution as it lies outside feasible region and it does not satisfy the given constraints.

1(e) Evaluate the integral
$$\int_C Re(z^2)\,dz \text{ from } 0 \text{ to } 2+4i$$
along the curve $C: y = x^2$. (8)

Sol: $I = \int_C Re(z^2)\,dz$



$$= \int_C Re(x+iy)^2\,dz$$

$$= \int_C (x^2-y^2)(dx+idy)$$

$$= \int_C (x^2-y^2)\,dx + i\int_C (x^2-y^2)\,dy$$

Parameterizing C, $x = t,\ y = t^2$
with $t = 0$ to $t = 2$

$$\therefore I = \int_0^2 (t^2-t^4)\,dt + i\int_0^2 (t^2-t^4)\,2t\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^5}{5}\right)_0^2 + 2i\left(\frac{t^4}{4} - \frac{t^6}{6}\right)_0^2$$

$$= \left(\frac{8}{3} - \frac{32}{5}\right) + i\left(\frac{16}{2} - \frac{64}{3}\right)$$

$$= -\frac{56}{15} - i\,\frac{40}{3}$$

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