LINEAR ALGEBRA

: IFoS 2012 :

① Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2)$, $\alpha_2 = (0,1,3)$, $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)$. be the elements of V. Find the basis for the intersection of subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

→ Let $W_1 = \text{Span}(\alpha_1, \alpha_2) = a(1,1,2) + b(0,1,3) = (a, a+b, 2a+3b)$

Let $W_2 = \text{Span}(\alpha_3, \alpha_4) = c(2,4,5) + d(-1,0,-1) = (2c-d, 4c, 5c-d)$

Let (x,y,z) be ~~the~~ elt of intersection of W_1 & W_2 i.e. $(x,y,z) \in W_1 \cap W_2$.

Then, $(x,y,z) = (a, a+b, 2a+3b) = (2c-d, 4c, 5c-d)$

$\Rightarrow \quad = (a, a+b, 2a+3b) - (2c-d, 4c, 5c-d) = (0,0,0)$

$= (a-2c+d,\ a+b-4c,\ 2a+3b-5c+d) = (0,0,0)$

Let $A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & -4 & 0 \\ 2 & 3 & -5 & 1 \end{bmatrix} \underset{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -1 & -1 \end{bmatrix} \underset{R_3 \to R_3 - 3R_2}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 5 & 2 \end{bmatrix}$

$R_1 \to 5R_1 + 2R_3,\ R_2 \to 5R_2 + 2R_3 \qquad R_1 \to R_1/5,\ R_2 \to R_2/5,\ R_3 \to R_3/5$

$\sim \begin{bmatrix} 5 & 0 & 0 & 9 \\ 0 & 5 & 0 & -1 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 9/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$

$\therefore \quad a + \frac{9}{5}d = 0, \quad b - \frac{1}{5}d = 0, \quad c + \frac{2}{5}d = 0$

$a = -\frac{9}{5}d, \quad b = \frac{1}{5}d, \quad c = -\frac{2}{5}d.$

$(x,y,z) = (a, a+b, 2a+3b) = \left(-\frac{9}{5}d,\ -\frac{9}{5}d + \frac{1}{5}d,\ 2\cdot\left(-\frac{9}{5}d\right) + 3\left(\frac{1}{5}d\right)\right)$

$= d\left(-\frac{9}{5}, -\frac{8}{5}, -3\right)$

$= k(-9, -8, -15)$

$= k_1(9, 8, 15)$

$\therefore$ Basis of $W_1 \cap W_2$ is $\{(9, 8, 15)\}$.

② Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $f(a,b,c) = (a, a+b, 0)$. Find the matrices A and B respectively of the linear transformation f wrt the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1,1,0)$, $e_2' = (0,1,1)$ and $e_3' = (1,1,1)$. Also show that there exist an invertible matrix P such that $B = P^{-1}AP$.

→ $S_1 = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 (0,0,1)$ is the standard basis of $\mathbb{R}^3$.

$T(e_1) = (1,1,0) = e_1 + e_2 + 0e_3$
$T(e_2) = (0,1,0) = 0e_1 + e_2 + 0e_3$
$T(e_3) = (0,0,0) = 0e_1 + 0e_2 + 0e_3$

∴ Matrix of T wrt standard basis is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

<u>Now</u>: $S_2 = \{e_1', e_2', e_3'\}$ where $e_1' = (1,1,0)$, $e_2' = (0,1,1)$ & $e_3' = (1,1,1)$

Let $(x,y,z) = ae_1' + be_2' + ce_3'$
$= (a+c, a+b+c, b+c)$

On comparing, $a+c = x$, $b+c = z$, $a+b+c = y$

$a+b+c = y$
$\underline{a+0b+c = x}$
$b = y-x$.

$c = z-b$
$c = z-y+x$
$c = x-y+z$.

$a = x-c$
$a = x-x+y-z$
$a = y-z$

$\therefore (x,y,z) = (y-z)(1,1,0) + (-x+y)(0,1,1) + (x-y+z)(1,1,1)$
$= (y-z)e_1' + (-x+y)e_2' + (x-y+z)e_3'$

$\therefore T(e_1') = T(1,1,0) = (1,2,0) = 2e_1' + 1 \cdot e_2' + (-1)e_3'$
$T(e_2') = T(0,1,1) = (0,1,0) = 1 \cdot e_1' + 1 \cdot e_2' + (-1)e_3'$
$T(e_3') = T(1,1,1) = (1,2,0) = 2 \cdot e_1' + 1 \cdot e_2' + (-1)e_3'$

∴ Matrix of T wrt basis S_2 is $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

To prove that $B = P^{-1}AP$ for some non-singular matrix P, we need to show that A & B are similar i.e. the char. eqn and the roots of A & B are the same

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(-\lambda) = 0$$

$$\lambda = 1, 1, 0.$$

$$B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow |B - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 1-\lambda & 1 \\ -1 & -1 & -(1+\lambda) \end{vmatrix} = (2-\lambda)[\lambda^2] + [\lambda] + 2[-\lambda] = 0$$

$$\Rightarrow \lambda[\lambda - 1]^2 = 0$$

$$\Rightarrow \lambda = 1, 1, 0.$$

$\therefore$ A & B are similar. Hence, $\exists$ a nonsingular matrix P such that $B = P^{-1}AP$.

③ Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ & find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as linear polynomial in A.

→ Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

The characteristic equation of A is given by $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \quad \text{—— ①}$$

Putting A in the LHS of ①

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\therefore$ A satisfies its characteristic equation.

Hence, Cayley Hamilton theorem is verified

Now: $A^2 = 4A + 5I$ —— (2)

Premultiplying with A on both sides:

$A^3 = 4A^2 + 5A$

$A^4 = 4A^3 + 5A^2$

$A^5 = 4A^4 + 5A^3 \Rightarrow A^5 - 4A^4 - 5A^3 = 0.$

Now: given equation is $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$\Rightarrow (A^5 - 4A^4 - 5A^3) - 2A^3 + 11A^2 - A - 10I$

$\Rightarrow 0 - 2\cdot[4A^2 + 5A] + 11A^2 - A - 10I$

$\Rightarrow \cancel{-19}\, 3 A^2 - \cancel{10A - A - 10I}$

$\Rightarrow \cancel{-19(4A+5I) - 11A - 10I}$

$\Rightarrow \cancel{-87A - 105I.}$

$\Rightarrow 3A^2 - 11A - 10I$

$\Rightarrow 3(4A + 5I) - 11A - 10I$

$\Rightarrow \underline{A + 5I}$

Premultiplying (1) with $A^{-1} \Rightarrow A = 4I + 5A^{-1}$

$\Rightarrow 5A^{-1} = A - 4I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$\rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

(4) Show that there are 3 real values of λ for which the equations $(a-\lambda)x + by + cz = 0$, $bx + (c-\lambda)y + az = 0$ and $cx + ay + (b-\lambda)z = 0$ are simultaneously true and that the product of these values of λ is $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

$\rightarrow$
$$\begin{aligned} (a-\lambda)x + by + cz &= 0 \\ bx + (c-\lambda)y + az &= 0 \\ cx + ay + (b-\lambda)z &= 0 \end{aligned}$$

These 3 are simultaneously true iff $\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$

$(a-\lambda)[(c-\lambda)(b-\lambda) - a^2] - b[b(b-\lambda) - ac] + c[ab - c[c-\lambda]] = 0$

$(a-\lambda)[(c-\lambda)(b-\lambda) - a^2] - b[b^2 - b\lambda - ac] + c[ab - c^2 + c\lambda] = 0$

$\Rightarrow (a-\lambda)[\lambda^2 - (b+c)\lambda + bc - a^2] - b[b^2 - b\lambda - ac] + c[ab - c^2 + c\lambda]$

$\Rightarrow$ $a\lambda^2 - \lambda^3 - a(b+c)\lambda + (b+c)\lambda^2 + abc - bc\lambda - a^3 + a^2\lambda - b^3 + b^2\lambda$

$+ abc + abc - c^3 + c^2\lambda = 0$

$\Rightarrow$ $\lambda^3 - \lambda^2(a+b+c) - \lambda(a^2+b^2+c^2 - ab - bc - ac) \cancel{+3abc=0} + (a^3+b^3+c^3 - 3abc) = 0$

which is a cubic equation. Hence, there are 3 such values of λ.

Now: If α, β, γ are roots of this equation, then

$\cancel{\alpha+\beta+\gamma=}$ $\alpha \cdot \beta \cdot \gamma = -\frac{d}{a} = -\frac{(a^3+b^3+c^3-3abc)}{1}$ — ①

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc-a^2) + b(ac-b^2) + c(ab-c^2)$$
$$= -(a^3+b^3+c^3-3abc). \text{ — ②}$$

① = ②. Hence, verified

5. Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a,b,c) = (2b+c, a-4b, 3a)$ corresponding to the basis $B = \{(1,1,1), (1,1,0), (1,0,0)\}$

$\rightarrow$ Let $(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$
$= (a+b+c, a+b, a)$

On comparison, $x = a+b+c$, $y = a+b$, $z = a$.
$\Rightarrow a = z$, $b = y-z$, $c = x-z-y+z = x-y$

$\therefore (x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$T(1,1,1) = (3,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 6(1,0,0)$

$T(1,1,0) = (2,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 5(1,0,0)$

$T(1,0,0) = (0,1,3) = 3(1,1,1) + (-2)(1,1,0) + (-1)(1,0,0)$

$\therefore$ Reqd. matrix is: $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$