CSE - 2012

Q1

c) Let $f(z) = u(x,y) + iV(x,y)$

$u(x,0) = 0 \qquad u(0,0) = 0$

$\therefore u_x(0,0) = \lim_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = \dfrac{0-0}{x} = 0$

$V(x,0) = 0 \qquad V(0,0) = 0$

$V_x(0,0) = \lim_{x \to 0} \dfrac{V(x,0) - V(0,0)}{x} = \dfrac{0-0}{x} = 0$

$u(0,y) = 0 \qquad u(0,0) = 0$

$\therefore u_y(0,0) = \lim_{y \to 0} \dfrac{u(0,y) - u(0,0)}{y} = \dfrac{0-0}{y} = 0$

$V(0,y) = 0 \qquad V(0,0) = 0$

$\therefore V_y(0,0) = \lim_{y \to 0} \dfrac{V(0,y) - V(0,0)}{y} = \dfrac{0-0}{y} = 0$

$\therefore V_x = V_y = u_x = u_y = 0$

Hence CR conditions

$u_x = V_y = 0$ True

$u_y = -V_x = 0$ True

are satisfied.

for differentiability at $z = 0$

$$f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3y^5(x+iy)}{x^6+y^{10}} \Big/ (x+iy)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3y^5}{x^6+y^{10}}$$

Along ~~x-axis, $y = 0$~~

$$\cancel{\lim_{\substack{(x,y)\to(0,0)\\ x\to 0}} \frac{x^3(0)^5}{x^6+(0)^{10}} \neq 0}$$

Along path $x^3 = my^5$

$$\lim_{\substack{\cancel{(x,y)\to(0,0)}\\ y\to 0}} \frac{(my^5)y^5}{(my^5)^2+y^{10}} = \lim_{y\to 0} \frac{my^{10}}{y^{10}(m^2+1)} = \frac{m}{m^2+1}$$

As limit is dependent upon m

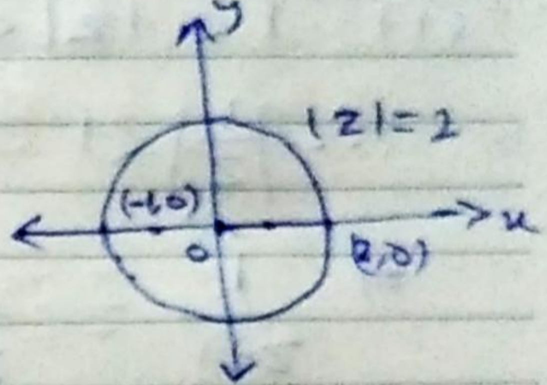
$\therefore$ $f'(z)$ is not differentiable at $z = 0$

Q2 Acc. to Cauchy's Integral formula

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}$$

Given C is circle $|z|=2$

If $f(z) = e^{3z}$ then $f(z)$ is analytic inside $|z|=2$.



$$\int_C \frac{e^{3z}}{(z+1)^4}dz = \int_C \frac{e^{3z}}{(z-(-1))^4}dz$$

then using Cauchy's integral formula

$$\int_C \frac{e^{3z}}{(z-(-1))^4}dz = \frac{2\pi i}{3!} f^3(-1) = \frac{2\pi i}{6}\left|\frac{d^3}{dz^3}(e^{3z})\right|_{z=-1}$$

$$= \frac{i\pi}{3}\left|27e^{3z}\right|_{z=-1} = \frac{i\pi}{3}\cdot 27e^{-3}$$

$$= \frac{9\pi i}{e^3}$$

$$\boxed{\therefore \int_C \frac{e^{3z}}{(z+1)^4}dz = \frac{9\pi i}{e^3}}$$

Q3 $f(z) = \dfrac{1}{(z+1)(z+3)}$

$\Rightarrow f(z) = \dfrac{1}{2}\left[\dfrac{1}{z+1} - \dfrac{1}{z+3}\right]$

(i) $1 < |z| < 3$

$\therefore \left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$

$\Rightarrow \dfrac{1}{z+1} = \dfrac{1}{z(1+\frac{1}{2})} = \dfrac{1}{z}(1+1/2)^{-1}$

$= \dfrac{1}{z}\left(1 - \dfrac{1}{2} + \dfrac{1}{z^2} - \dfrac{1}{z^3} \dots\right)$

$\therefore \dfrac{1}{z+1} = \dfrac{1}{z} - \dfrac{1}{z^2} + \dfrac{1}{z^3} - \dots$

$\dfrac{1}{z+3} = \dfrac{1}{3(1+\frac{z}{3})} = \dfrac{1}{3}\left(1+\dfrac{z}{3}\right)^{-1}$

$= \dfrac{1}{3}\left(1 - \dfrac{z}{3} + \dfrac{z^2}{9} - \dfrac{z^3}{27} + \dots\right)$

$\therefore \dfrac{1}{z+3} = \dfrac{1}{3} - \dfrac{z}{9} + \dfrac{z^2}{27} - \dots$

THINGS TO DO

Hence laurent series of $f(z)$ is

$$\frac{1}{2}\left[\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}\cdots\right)-\left(\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}\cdots\right)\right]$$

$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}\cdots\right)+\left(-\frac{1}{3}+\frac{z}{9}-\frac{z^2}{27}\cdots\right)$$

(ii) $|z|>3$

$$\therefore \frac{3}{|z|}<1 \Rightarrow \frac{1}{|z|}<1$$

$$f(z)=\frac{1}{2}\left[\frac{1}{z+1}-\frac{1}{z+3}\right]$$

$$\frac{1}{z+1}=\frac{1}{z(1+1/z)}=\frac{1}{z}(1+1/z)^{-1}$$

$$=\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$=\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots$$

$$\frac{1}{z+3}=\frac{1}{z(1+3/z)}=\frac{1}{z}(1+3/z)^{-1}$$

$$=\frac{1}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right)$$

$$= \frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \frac{27}{z^4} + \cdots$$

$$f(z) = \frac{1}{2}\left[\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} \cdots\right) - \left(\frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} \cdots\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{z^2} - \frac{8}{z^3} + \frac{26}{z^4} \cdots + (-1)^n \frac{(3^{n-1} - 1)}{z^n} + \cdots\right]$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots + (-1)^n \left(\frac{3^{n-1}-1}{2}\right)\frac{1}{z^n} + \cdots]$$

(iii) $0 < |z+1| < 2$

Let $z+1 = u$

$0 < |u| < 2$

$$\therefore f(z) = \frac{1}{(z+1)(z+3)} \Rightarrow f(u) = \frac{1}{u(u+2)}$$

As $\frac{|u|}{2} < 1$

$$\therefore \frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right]$$

$$f(u) = \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \cdots$$

$$\therefore f(z) = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{(z+1)}{8} - \frac{(z+1)^2}{16} + \dots$$

(iv) $|z| < 1$ $\quad \therefore \frac{|z|}{3} < \frac{1}{3} < 1$

$$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$\frac{1}{z+1} = (z+1)^{-1} = 1 - z + z^2 - z^3 + \dots$$

$$\frac{1}{z+3} = \frac{1}{3(1+z/3)} = \frac{1}{3}(1+z/3)^{-1} = \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots\right]$$

$$= \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \dots$$

$$\therefore f(z) = \frac{1}{2}\left[\left(1 - z + z^2 - z^3 + \dots\right) - \left(\frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \dots\right)\right]$$

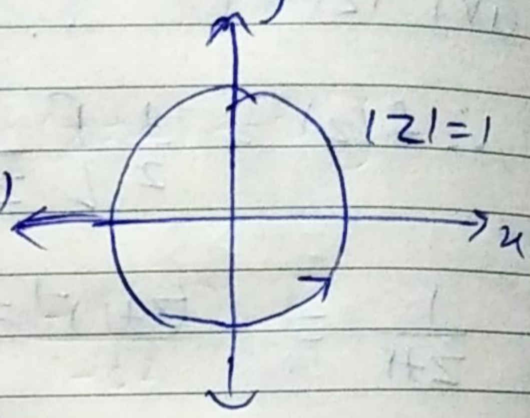
$$= \frac{1}{2}\left[\frac{2}{3} - \frac{8}{9}z + \frac{26}{27}z^2 \dots + (-1)^n\left(\frac{3^{n+1}-1}{3^{n+1}}\right)z^n + \dots\right]$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 \dots + (-1)^n\frac{(3^{n+1}-1)}{2\cdot 3^{n+1}}z^n + \dots$$

d4

$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}, \quad a^2 < 1$$

Consider a contour $C: |z| = 1$



$\therefore \; z = e^{i\theta} = \cos\theta + i\sin\theta$

$$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$z + \frac{1}{z} = 2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

Also $dz = i e^{i\theta} d\theta$

$$\frac{dz}{iz} = d\theta$$

$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \oint_C \frac{1}{1-a2\left(\frac{z^2+1}{2z}\right)+a^2} \frac{dz}{iz}$$

$$= -i \oint_C \frac{dz}{z\left[\frac{(a^2+1)z - a(z^2+1)}{z}\right]} = -i \oint_C \frac{dz}{(a^2+1)z - az^2 - a}$$

$$= i \oint \frac{dz}{az^2 - (a^2+1)z + a}$$

$$az^2 - (a^2+1)z + a = 0$$

$$z = \frac{(a^2+1) \pm \sqrt{(a^2+1)^2 - 4a^2}}{2a}$$

THINGS TO DO

$$z = \frac{(a^2+1) \pm \sqrt{a^4+2a^2+1-4a^2}}{2a}$$

$$z = \frac{(a^2+1) \pm \sqrt{(a^2-1)^2}}{2a}$$

$$z = \frac{(a^2+1) \pm (a^2-1)}{2a}$$

$$z = \frac{2a^2}{2a}, \frac{2}{2a}$$

$$z = a, \frac{1}{a}$$

$$\therefore I = i \oint \frac{dz}{a(z-a)(z-1/a)}$$

As $a^2 < 1$ $\therefore \frac{1}{a} > 1$ & lies outside $|z| = 1$

Hence, $z = a$ is only singularity inside $|z| = 1$ which is a pole of order 1.

<u>Residue at $z = a$</u>

$$\lim_{z \to a} (z-a) \frac{1}{a(z-a)(z-1/a)} = \lim_{z \to a} \frac{1}{a(z-1/a)} = \frac{1}{a(a-1/a)}$$

$$= \frac{1}{(a^2-1)} = \frac{-1}{1-a^2}$$

THINGS TO DO

Using Cauchy's residue theorem

$$\oint_C f(z)\,dz = 2\pi i\,(\text{Sum of residues})$$

$$\therefore \oint \frac{dz}{a(z-a)(z-1/a)} = 2\pi i\,[\text{Re}(z=a) + 0]$$

$$= 2\pi i\left(\frac{-1}{1-a^2}\right)$$

$$= \frac{-2\pi i}{1-a^2}$$

Now, $I = i\oint \frac{dz}{a(z-a)(z-1/a)}$

$$= i\left(\frac{-2\pi i}{1-a^2}\right) = \frac{-2\pi i^2}{1-a^2} = \frac{2\pi}{1-a^2}$$

$$\boxed{\therefore\ I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} = \frac{2\pi}{1-a^2}}$$