Similarly

$$\frac{Z_5[x]}{\langle x+3\rangle} \cong Z_5$$

$$\therefore \frac{Z_5[x]}{\langle x+1\rangle} \cong Z_5 \times Z_5$$

$a = (1,0) \in Z_5 \times Z_5$

$b = (0,1) \in Z_5 \times Z_5$

$a.b = (1,0).(0,1)$

$= (0,0)$

$\therefore Z_5 \times Z_5$ is not an I.D.

Year-2010

57. Let $C = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. Show C is a commutative ring with 1 under point wise addition and multiplication. Determine whether C is an integral domain. Explain. (15)

Sol.: Let f be the set of all real valued continuous function defined on [0, 1]. '+' and '.' are defined as:

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$; $f, g \in f$.

We have to show C is a commutative ring. First, we will show $(C,+)$ form an abelian group.

We will show property holds w.r.t. '+'.

Let $f, g \in C$.

$\therefore f(x), g(x)$ is its function on [0, 1]. And we know, sum of two continuous function is continuous.

$\therefore f(x)+g(x)$ is continuous on [0, 1].

Hence, $f+g \in C$.

Similarly we can show associative property holds.

We will show the existence of identity.

Let $f \in C$.

$\therefore f(x)$ is continuous on [0, 1].

Define $O : [0,1] \to \mathbb{R}$.

$O(x) = 0 \; \forall x \in [0,1]$.

Here,

$f(x) + O(x) = f(x) = O(x) + f(x) \; \forall f \in C.$

Hence, zero mapping is the identity map.

We will show the existence of inverse.

Let $f \in C$

$\therefore f(x)$ is continuous on [0, 1]

$\Rightarrow f(x)$ is also continuous on [0, 1]

Further,

$f(x) + (-f(x)) = O(x) = (-f(x))$

$+ f(x) \; \forall f \in C.$

$\therefore -f(x)$ is the inverse function of f.

We will show commutative properly holds.

Let $f, g \in C$

$\therefore f(x), g(x)$ is continuous ~~on [0, 1]~~.

Consider

$f(x) + g(x) = g(x) + f(x)$.

($\because f, g$ are real valued continuous function and $(\mathbb{R},+)$ commutative holds)

Hence, commutative holds.

We will show closure properly holds w.r.t.

Let $f, g \in C$

$\therefore f(x), g(x)$ is continuous function on [0, 1].

Hence, $f \cdot g \in C$.

Similarly we can show associative properly holds.

We will show existence of identity w.r.t. multiplication

Let $f \in C$

$\therefore f(x)$ is continuous function on [0,1].

Define $I : [0,1] \to R$

$I(x) = 1 \; \forall x \in [0,1]$.

$\because \quad I(x)$ is constant

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$\therefore f(x)$ is continuous.

Hence, $I(x) \in C$.

Now,

$f(x) \cdot I(x) = f(x) = I(x) \cdot f(x) \ \forall f \in C$

Hence $I(x) = 1$ is the identity element.

We will show commutative property holds w.r.t. multiplication.

Let $f, g \in C$.

$\therefore f(x), g(x)$ are the continuous function on $[0, 1]$.

Consider,

$f(x) \cdot g(x) = g(x) \cdot f(x)$

($\because f, g$ are real valued continuous function and $(\mathbb{R}, \cdot)$ commutatively holds.

Easily we can show left distributive and right distributive holds.

Hence C is a commutative ring with identity.

$C = \{f : [0,1] \to \mathbb{R} / f \text{ is continuous}\}$

Let $f(x) = \begin{cases} x - \frac{1}{2} & ; \ 0 \le x \le \frac{1}{2} \\ 0 & ; \ \frac{1}{2} < x \le 1 \end{cases}$

$g(x) = \begin{cases} 0 & ; \ 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} & ; \ \frac{1}{2} < x \le 1 \end{cases}$

As $f(x), g(x) \neq 0$.

$f(x), g(x)$ are continuous function.

But $f(x) \cdot g(x) = 0 \ \ \forall x \in [0,1]$.

Hence, C is not an I.D.

58. Consider the polynomial ring. Show $p(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$. Let I be the ideal in $\mathbb{Q}[x]$ generated by $p(x)$. Then show that $\mathbb{Q}[x]/I$ is a field and that each element of it is of the form $a_0 + a_1 t + a_2 t^2$ with a_0, a_1, a_2 in Q and $t = x + I$. (15)

Sol.: Let $p(x) = x^3 - 2$.

We have to show $p(x)$ is irreducible over $\mathbb{Q}$.

By Eisenstein's criteria;

If $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a non-constant polynomial in $\mathbb{Z}[x]$. Suppose $\exists$ a prime p such that

(i) $p / a_i \quad i = 0 \text{ to } n-1$

(ii) $p \nmid a_n$

(iii) $p \nmid a_n$

Then $f(x)$ is irreducible over $\mathbb{Q}$.

Here $p(x) = x^3 - 2 = x^3 + 0x^2 + 0x - 2$.

$\exists \ p = 2$.

So that,

$p = 2 / a_0, p / a_1, p / a_2$

but $p \nmid a_3$

and $p^2 = 4 \nmid (-2)$

Hence, $p(x) = x^3 - 2$ is irreducible over Q.

And, we know,

If F be a field and let $p(x) \in F[x]$. Then $\langle p(x) \rangle$ is a maximal ideal in $F[x] \Leftrightarrow p(x)$ is irreducible over F.

And $\dfrac{F[x]}{\langle p(x) \rangle}$ is a field.

As $\mathbb{Q}$ is a field

And $p(x) = x^3 - 2$ is irreducible over $\mathbb{Q}$

$\therefore \langle x^3 - 2 \rangle$ is maximal idea of $\mathbb{Q}[x]$

Hence, $\dfrac{\mathbb{Q}[x]}{\langle x^3 - 2 \rangle}$ is a field.

$\therefore \dfrac{\mathbb{Q}[x]}{\langle x^3 - 2 \rangle} = \{f(x) + \langle x^3 - 2 \rangle : f(x) \in \mathbb{Q}[x]\}$

By division algorithm on $f(x)$ and $x^3-2=p(x)\ \exists\ q(x), r(x)$ so that

$f(x)=p(x)q(x)+r(x)$; where $r(x)=0$ or $\deg. r(x)<\deg p(x)$

$$f(x)+\langle x^3-2\rangle=(x^3-2)q(x)+r(x)+\langle x^3-2\rangle$$

$$=(x^3-2)q(x)+\langle x^3-2\rangle+r(x)+\langle x^3-2\rangle$$

$$=\langle x^3-2\rangle+r(x)+\langle x^3-2\rangle$$

$$=r(x)+\langle x^3-2\rangle.$$

$$\frac{\mathbb{Q}[x]}{\langle x^3-2\rangle}=\{r(x)+\langle x^3-2\rangle : \deg r(x)<3; r(x)\in\mathbb{Q}[x]\}$$

$$=\{a_0+a_1x+a_2x^2+\langle x^3-2\rangle : a_i\in\mathbb{Q}\}$$

59. Show that the quotient ring $\mathbb{Z}[i]/(1+3i)$ is isomorphic to the ring $\mathbb{Z}/10\mathbb{Z}$ where $\mathbb{Z}[i]$ denotes the ring of Gaussian integers.

Sol.: We have to show

$$\frac{\mathbb{Z}[i]}{\langle 1+3i\rangle}\cong\frac{\mathbb{Z}}{10\mathbb{Z}}=\mathbb{Z}_{10}.$$

Let $R=\dfrac{\mathbb{Z}[i]}{\langle 1+3i\rangle}=\{a+ib+\langle 1+3i\rangle : a,b\in\mathbb{Z}\}$.

As $1+3i+\langle 1+3i\rangle=0+\langle 1+3i\rangle$

It is equivalent to $1+3i=0$.

We will show ϕ is surjective and ker $\phi=10\mathbb{Z}$.

As $1+3i=0$.

i.e. $i=3$ or $10=0$.

Define $\phi:\mathbb{Z}\to\dfrac{\mathbb{Z}[i]}{I}$; where $I=\langle 1+3i\rangle$

$\phi(n)=n+I$.

We will show ϕ is a ring homo.

Let $n_1,n_2\in\mathbb{Z}$.

Consider,

$$\phi(n_1+n_2)=n_1+n_2+I$$
$$=n_1+I+n_2+I$$
$$=\phi(n_1)+\phi(n_2)$$

$\therefore\ \phi(n_1+n_2)=\phi(n_1)+\phi(n_2)$ for all $n_1,n_2\in\mathbb{Z}$.

And

$$\phi(n_1n_2)=n_1n_2+I$$
$$=(n_1+I)(n_2+I)$$
$$=\phi(n_1)\phi(n_2)$$

$\therefore\ \phi$ is a ring homomorphism.

Since $a+ib$ and $a+3b$ belong to the same cosets in $\dfrac{\mathbb{Z}[i]}{I}$, where $I=\langle 1+3i\rangle$

$\therefore\quad \phi(a+3b)=a+ib.$

Thus ϕ is onto.

Note if $n\in10\mathbb{Z}$.

Then $n=10m$, for some $m\in\mathbb{Z}$

Hence $\phi(n)=\phi(10m)=10m+I=I$.

$\therefore\ 10\mathbb{Z}\subseteq\ker\phi$. (1)

Also if $n\in\ker\phi$

$\phi(n)=0$

Then $n\in I$.

i.e., $n=(a+b)(1+3i)=(a-3b)+(3a+b)i$

for some $a,b\in\mathbb{Z}$.

Hence $3a+b=0$

$\Rightarrow\quad b=-3a.$

$\therefore\ n=a-3b$

$=a+9a=10a\in10\mathbb{Z}.$

As $(1+3i)=0$

$\Rightarrow\quad(1+3i)(1-3i)=0$

$\Rightarrow\quad 10=0.$

By first homomorphism theorem of rings;

$$\frac{\mathbb{Z}}{\ker\phi}\cong\phi(\mathbb{Z}) \qquad (*)$$

$\therefore\quad \ker\phi\subseteq10\mathbb{Z}$ (2)

From (1) and (2)

$Ker\phi=10\mathbb{Z}$

And (*) reduces to

$$\frac{\mathbb{Z}}{10\mathbb{Z}}\cong\frac{\mathbb{Z}[i]}{I}$$

$$\therefore\quad\frac{\mathbb{Z}}{10\mathbb{Z}}\cong\frac{\mathbb{Z}[i]}{\langle 1+3i\rangle}$$

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Year-2011

60. Let F be the set of all real valued, continuous functions defined on the closed interval $[0, 1]$. Prove that $(F, +, \cdot)$ is a Commutative Ring with unity with respect to addition and multiplication of functions defined pointwise as below:

$$\left.\begin{array}{l}(f+g)(x)=f(x)+g(x) \\ \text{and } (f\cdot g)(x)=f(x)\cdot g(x)\end{array}\right\} x\in[0,1]$$

where $f, g \in F$. **(15)**

Sol. Let F be the set of all real valued continuous function defined on $[0, 1]$. '+' and '.' are defined as:

$(f+g)(x)=f(x)+g(x)$

$(f\cdot g)(x)=f(x)\cdot g(x)$; $f, g\in f$.

We have to show $(F,+)$ is a commutative ring first, we will show $(F,+)$ form an abelian group.

We will show property holds w.r.t. '+'.

Let $f, g\in F$.

$\therefore f(x), g(x)$ is continuous function on $[0,1]$

And we know, sum of two continuous function is continuous.

$\therefore$ $f(x)+g(x)$ is its on $[0, 1]$.

Hence, $f+g\in F$.

Similarly we can show, associative property holds. We will show the existence of identity.

Let $f\in F$.

$\therefore$ $f(x)$ is continuous on $[0,1]$.

Define $O:[0,1]\to R$.

$O(x)=0$; $\forall x\in[0,1]$.

Here,

$f(x)+O(x)=f(x)=O(x)+f(x)\forall f\in F$.

Hence, zero mapping is the identity map.

We will show the existence of inverse.

Let $f\in F$

$\therefore$ $f(x)$ is continuous on $[0, 1]$

$\Rightarrow$ $-f(x)$ is also continuous on $[0, 1]$.

Further,

$f(x)+(-f(x))=O(x)=(-f(x))$
$+f(x)\forall f\in F$.

$\therefore$ $-f(x)$ is the inverse function on f.

We will show commutative properly holds.

Let $f, g\in f$

$\therefore$ $f(x), g(x)$ is continuous on $[0, 1]$.

Consider

$f(x)+g(x)=g(x)+f(x)$

($\because f, g$ are real valued its function and $(R,+)$ commutative holds)

Hence, commutative holds.

We will show closure properly holds w.r.t. '.'.

Let $f, g\in F$

$\therefore f(x), g(x)$ is continuous function on $[0,1]$

Hence, $f\cdot g\in F$.

We can show associative properly holds. We will show existence of identity w.r.t. multiplication

Let $f\in F$

$\therefore$ $f(x)$ is continuous function on $[0,1]$.

Define $I:[0,1]\to R$

$I(x)=1\,\forall x\in[0,1]$.

$\because$ $I(x)$ is constant

$\therefore$ $I\in f(x)$ is continuous.

Hence, $I(x)\in F$.

Now,

$f(x)\cdot I(x)=f(x)=I(x)\cdot f(x)\,\forall f\in F$

Hence $I(x)=1$ is the identity element.

We will show commutative property holds w.r.t. multiplication.

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Let $f, g \in F$.

$\therefore$ $f(x), g(x)$ are the continuous function on $[0, 1]$.

Consider,

$f(x)\cdot g(x) = g(x)\cdot f(x)$

($\because$ f, g are real valued continuous function and $(R, \cdot)$ commutatively holds.

Easily we can show right distributive as well as left distributive holds.

Hence $(F, +, \cdot)$ is a commutative ring with identity.

Year-2012

61. Is the ideal generated by 2 and X in the polynomial ring $\mathbb{Z}[X]$ of polynomials in a single variable X with coefficients in the ring of integer $\mathbb{Z}$, a principal ideal? Justify your answer. (15)

Sol.: $A = \{xf(x) + 2g(x) | f(x), g(x) \in \mathbb{Z}[x]\}$

Consider

$A = \{xf(x) + 2g(x) | f(x), g(x) \in \mathbb{Z}[x]\}$

Then A is an ideal of $\mathbb{Z}[x]$ as

$[xf(x) + 2g(x)] - [xf'(x) + 2g'(x)]$

$= x[f(x) - f'(x)] + 2[g(x) - g'(x)] \in A$

and for $h(x) \in \mathbb{Z}[x]$,

$[xf(x) + 2g(x)]h(x) = xf(x)h(x)$

$+2g(x)h(x) \in A$

Again, $A \neq \mathbb{Z}[x]$, because if $A = \mathbb{Z}[x]$ then as $1 \in \mathbb{Z}[x]$

$1 \in A \Rightarrow 1 = xf(x) + 2g(x)$

$\Rightarrow 1 = x(a_0 + a_1x + ... + a_mx^m)$

$+2(b_0 + b_1x + ... + b_nx^n)$

$\Rightarrow 1 = 2b_0, b_0 \in \mathbb{Z}$

a contradiction

Hence $A \neq \mathbb{Z}[x]$

62. Describe the maximal ideals in the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$. (20)

Sol.: We define $\delta(a + ib) = |a + ib|^2 = a^2 + b^2$ for all $a + ib \in \mathbb{Z}[i] \backslash 0$.

Clearly, $\delta(u) > 0$ for all $u \neq 0$ in $\mathbb{Z}[i]$

Further for any $u, v \in \mathbb{Z}[i] \backslash \{0\}$,

$\delta(uv) = |u|^2|v|^2 \geq |u|^2 = \delta(u)$

Next let $u, v \in \mathbb{Z}[i]$ with $v \neq 0$. Then $u = a + ib$ and $v = c + id$ for some $a, b, c, d \in \mathbb{Z}$ such that $(c, d) \neq (0, 0)$.

Now, $\frac{u}{v} = \frac{(a + ib)(c + id)}{c^2 + d^2} = \alpha + i\beta$ (say),

where α, β are rational numbers. Then there exist integers m and n such that $|m - \alpha| \leq \frac{1}{2}$ and $|n - \beta| \leq \frac{1}{2}$.

So $u = (\alpha + i\beta)v = (m + in)v$

$+[(\alpha - m) + i(\beta - n)]v$

Now

$[(\alpha - m) + i(\beta - n)v = u - (m + in)v \in \mathbb{Z}[i]$,

as is a ring and $u, v, m + in \in \mathbb{Z}[i]$. Let $r = [(\alpha - m) + i(\beta - n)]v$. Then

$\delta(r) = |r|^2 = [(\alpha - m)^2 + (\beta - n)^2]|v|^2$

$\leq \left(\frac{1}{4} + \frac{1}{4}\right)|v|^2$

Thus taking $q = m + in$, we have $u = vq + r$ where either $r = 0$ or $\delta(r) < \delta(v)$. Hence $\mathbb{Z}[i]$ is a Euclidean domain.

As every Euclidean domain is a principal ideal domain.

$\therefore$ $\mathbb{Z}[i]$ is P.I.D.

And in P.I.D. every non-zero non unit element (say a) is prime element iff $\langle a \rangle$ is a maximal ideal.

$\therefore$ maximal ideals in $\mathbb{Z}[i]$ is the ideal generated by prime element.

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Year-2013

63. Show that the set of matrices $S=\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a,b\in\mathbb{R}\right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f:\mathbb{C}\to S$ defined by $f(a+ib)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.)

Sol. Let $S=\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a,b\in\mathbb{R}\right\}$

We will show S is a field under the ordinary addition and matrix multiplication.

First we will show $(S,+)$ forms on abelian group.

We will show closure property holds w.r.t. +

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B=\begin{bmatrix} c & -d \\ b & c \end{bmatrix}\in S$.

$$A+B=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ b & c \end{bmatrix}$$

$$=\begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}\in S.$$

∴ Closure property holds.

Similarly we can show associative property holds w.r.t. +.

We will show existence of identity w.r.t. +.

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\in S$.

And $B=\begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}\in S$

$\therefore A+B=E=B+A$

Hence $\begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$ is the inverse of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

we can easily show commutative property holds in S w.r.t. +.

We will show closure property holds w.r.t. multiplication.

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix}\in S$.

Consider

$$A.B=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

$$=\begin{bmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{bmatrix}$$

$$=\begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}\in S$$

Hence closure property holds.

Similarly associative property can be verified easily.

We will show existence of identity element w.r.t. multiplication.

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\in S$

As $E=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\in S$

And $A.E=A=E.A\ \forall A\in S$.

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element.

We will show the existence of inverse w.r.t. multiplication.

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$$|A|=a^2+b^2$$

$$\therefore A^{-1}=\frac{1}{|A|}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

And $A.A^{-1}=E=A^{-1}.A$

As A^{-1} exists if $|A|\neq 0$.

∴ Every non-zero element has multiplication inverse.

We will show the commutative property also holds w.r.t. multiplication.

Let $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

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Consider

$$A.B=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

$$=\begin{bmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{bmatrix}$$

$$=\begin{bmatrix} ca-db & -da-cb \\ cb+da & -bd+ca \end{bmatrix}=\begin{bmatrix} c & -d \\ d & c \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$=B.A.$$

$\therefore\ A.B=B.A\ \forall B,A\in S.$

Further left distributive and right distributive can be verified easily.

Hence, S forms field w.r.t. matrix addition and matrix multiplication.

Let $P=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$|P|=1+1=2\neq 0$

$$\therefore P^{-1}=\frac{1}{|P|}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\therefore PP^{-1}=I.$

Consider the map;

$f:\mathbb{C}\to S$

$$f(a+ib)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

We will show f is an homomorphism

Let $a+ib,c+id\in\mathbb{C}$

Consider

$$f(a+ib,c+id)=f((a+c)+i(b+d))$$

$$=\begin{pmatrix} a+c & -(b+a) \\ b+d & a+c \end{pmatrix}$$

$$=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}+\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$=f(a+ib)+f(c+id)$$

$$\therefore f((a+ib)+(c+id))=f(a+ib)+f(c+id)$$

$\forall\ a+ib,c+id\in\mathbb{C}$

Hence f is homomorphism.

$$\ker f=\{z\in\mathbb{C}/f(z)=0\}$$

$$=\left\{a+ib/f(a+ib)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

$$=\left\{a+ib/\begin{bmatrix} a & -b \\ b & a \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

$$=\{a+ib/a=0,b=0\}$$

$$=\{0\}.$$

Hence, f is one-one

For any $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\in S$

We have $a+ib\in\mathbb{C}$ so that

$$f(a+ib)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$\therefore f$ is onto .

$\therefore f$ is a homomorphism, and onto

Hence, f is an homomorphism.

64. Let $J=\{a+bi\,|\,a,b\in\mathbb{Z}\}$ be the ring of Gaussian integers (subring of $\mathbb{C}$). Which of the following is J : Euclidean domain, principal ideal domain, unique factorization domain ? Justify your answer.

Sol. Define $\delta(a+ib)=|a+ib|^2=a^2+b^2$ for all $a+ib\in\mathbb{Z}[i]\setminus 0$.

Clearly, $\delta(u)>0$ for all $u\neq 0$ in $\mathbb{Z}[i]$

Further for any $u,v\in\mathbb{Z}[i]\setminus\{0\}$,

$$\delta(u\ v)=|u|^2|v|^2\geq|u|^2=\delta(u)$$

Next let $u,v\in\mathbb{Z}[i]$ with $v\neq 0$. Then $u=a+ib$ and $v=c+id$ for some $a,b,c,d\in\mathbb{Z}$ such that $(c,d)\neq(0,0)$.

Now, $\dfrac{u}{v}=\dfrac{(a+ib)(c+id)}{c^2+d^2}=\alpha+i\beta$ (say), where α, β are national numbers. Then there exist integers m and n such that $|m-\alpha| \le \frac{1}{2}$ and $|n-\beta| \le \frac{1}{2}$.

So,

$$u = (\alpha + i\beta)v = (m+in)v + [(\alpha - m) + i(\beta - n)]v$$

Now

$$[(\alpha - m) + i(\beta - n)v = u - (m+in)v \in \mathbb{Z}[i],$$

as is a ring and $u, v, m+in \in \mathbb{Z}[i]$. Let $r = [(\alpha - m) + i(\beta - n)]v$. Then

$$\delta(r) = |r|^2 = \left[(\alpha - m)^2 + (\beta - n)^2\right]|v|^2$$

$$= \frac{1}{2}|v|^2 < |v|^2 = \delta(v)$$

Thus taking $q = m + in$, we have $u = vq + r$ where either $r = 0$ or $\delta(r) < \delta(v)$. Hence $\mathbb{Z}[i]$ is a Euclidean domain.

And, we know every E.D implies P.I.D and every P.I.D implies U.F.D. Hence, $\mathbb{Z}[i]$ is E.D, P.I.D as well as U.F.D.

65. Let R^C = ring of all real valued continuous functions on $[0,1]$, under the operations

$$(f+g)x = f(x) + g(x)$$

$$(fg)x = f(x)g(x).$$

Let $M = \left\{ f \in R^C \middle| f\left(\frac{1}{2}\right) = 0 \right\}$.

Is M a maximal ideal of R? Justify your answer.

Sol. Let R^C = ring all real valued continuous functions on $[0,1]$.

$= \{ f / f : [0,1] \to \mathbb{R};$ where f is a continuous function$\}$

R^C forms ring under the operations

$$(f+g)(x) = f(x) + g(f)$$

$$(fg)(x) = f(x)g(x)$$

We have to show $N = \left\{ f \in R^C / f\left(\frac{1}{2}\right) = 0 \right\}$ is a maximal ideal of R^C Define $\phi : R^C \to \mathbb{R}$

$$\phi(f) = f\left(\frac{1}{2}\right)$$

We will show ϕ is a ring homomorphism.

Let $f, g \in R^C$

Consider

$$\phi(f+g) = (f+g)\frac{1}{2}$$

$$= f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right)$$

$$= \phi(f) + \phi(y)$$

$$\therefore \phi(f+g) = \phi(f) + \phi(g). \ \forall f, y \in R^C$$

And $\phi(f.g) = (fg)\frac{1}{2}$

$$= f\left(\frac{1}{2}\right) g\left(\frac{1}{2}\right) = \phi(f).\phi(g)$$

$$\therefore \phi(f.g) = \phi(f).\phi(g) \quad \forall f, g \in R^C$$

ϕ is a ring homomorphism.

Clearly ϕ is a onto homomorphism.

By first isomorphism theorem on rings;

$$\frac{R^C}{\ker \phi} \cong \phi(R^C) = \mathbb{R}$$

$$\ker \phi = \{ f \in R^C / \phi(f) = 0 \}$$

$$= \left\{ f \in R^C / f\left(\frac{1}{2}\right) = 0 \right\}$$

$$\therefore \frac{R^C}{\left\{ f \in R^C / f\left(\frac{1}{2}\right) = 0 \right\}} \cong \mathbb{R}$$

As we know,

If R be a commutative ring with unity.

Let N be any ideal of R.

Then $\frac{R}{N}$ is a field $\Leftrightarrow$ N is maximal ideal.

$\therefore$ We have

$$\frac{R^C}{\left\{f : f\left(\frac{1}{2}\right)=0\right\}} \cong \mathbb{R}.$$

As $\mathbb{R}$ is a field

$$\therefore \frac{R^C}{\left\{f : f\left(\frac{1}{2}\right)=0\right\}} \text{ is a field}$$

By above theorem;

$\left\{f : f\left(\frac{1}{2}\right)=0\right\}$ is a maximal ideal of R^C