CSE - 2011 ODE

1. $\frac{dy}{dx} = (4x+y+1)^2$ if $y(0)=1$. — (1)

Put $4x+y+1 = z$ — (2) $\Rightarrow 4 + \frac{dy}{dx} = \frac{dz}{dx}$ (Diff wrt x)

ie. $\frac{dy}{dx} = \frac{dz}{dx} - 4$

$\Rightarrow \frac{dz}{dx} - 4 = z^2$ (using (1))

$\Rightarrow \frac{dz}{z^2+4} = dx \Rightarrow$ Integrating both sides,

$\frac{1}{2}\tan^{-1}\frac{z}{2} = x + c$ $\left[\text{using } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$ [using (2)].

or $\boxed{2\tan(2x+c') = 4x+y+1}$ is the sol.

$y(0)=1 \Rightarrow c' = \frac{\pi}{4} \Rightarrow 4x+y+1 = 2\tan\left(2x+\frac{\pi}{4}\right)$.

2. $r = a(1-\cos\theta)$; orthogonal trajectory

Taking logarithm on both sides,

$\log r = \log a + \log(1-\cos\theta)$

Differentiating w.r.t θ, we get

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta} \times \sin\theta$

Now since constant a is eliminated,
for finding orthogonal trajectory,
replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we get

$\frac{1}{r} \times -r^2\frac{d\theta}{dr} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

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$$-r\frac{d\theta}{dr} = \cot\frac{\theta}{2}$$

$$\Rightarrow \frac{dr}{r} = -\tan\frac{\theta}{2}\,d\theta$$

Integrating both sides we get

$$\log r = 2\log\cos\frac{\theta}{2} + \log c \quad \left[\text{using } \int \tan\theta = -\log\ldots\right.$$

i.e. $\log r = \log c\cos^2\frac{\theta}{2}$ (or)

Taking exponential of both sides we get,

$$e^{\log r} = e^{\log c\cos^2\frac{\theta}{2}}$$

$$\Rightarrow r = c\cos^2\frac{\theta}{2} = c\left(\frac{1+\cos\theta}{2}\right) = c'(1+\cos\theta)$$

$\boxed{r = c(1+\cos\theta)}$ is the required og traje…

3. obtain Clairaut's form and find general sol.

Doubt?

$$\left(x\frac{dy}{dx} - y\right)\left(y\frac{dy}{dx} + y\right) = a^2\frac{dy}{dx}$$

$$(xp-y)(yp+y) = a^2 p \quad \text{--- (1)}$$

Put $x^2 = X$, $y^2 = Y$ --- (2)

$2x\,dx = dX$, $2y\,dy = dY$

$$P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p$$

$$\Rightarrow \boxed{p = \frac{x}{y}P} \quad \text{Putting --- (3) into (1), we get}$$

$$\left(x\times\frac{x}{y}P - y\right)\left(y\times\frac{x}{y}P + y\right) = a^2\frac{x}{y}P$$

$$\left(\frac{x^2}{y}P - y\right)(xP+y) = a^2\frac{x}{y}P$$

$$(x^2P - y^2)(xP+y) = a^2xP$$

$$x^3P^2 + x^2yP - xy^2P - y^3 = a^2xP$$

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$y'' - 2y' + 2y = x + e^x \cos x$

Auxillary equation corresponding to homogeneous equation $y'' - 2y' + 2y = 0$ is

$$m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm 2i}{2} = 1 \pm i$$

ie. complementary function, $y_c = e^x (c_1 \cos x + c_2 \sin x)$

for finding particular integral,

$$f(D)\, y = x + e^x \cos x$$

$$y_p = \frac{1}{f(D)} (x + e^x \cos x) = \frac{1}{D^2 - 2D + 2} (x + e^x \cos x)$$

$$y_p = \frac{1}{D^2 - 2D + 2} (x) + \frac{1}{D^2 - 2D + 2} e^x \cos x$$

$$= \frac{1}{2\left(1 - D + \frac{D^2}{2}\right)} x + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \cos x$$

$\left[\text{using } \frac{1}{f(D)} e^{ax} f(x) = e^{ax} \frac{1}{f(D+a)} f(x)\right]$

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$$= \frac{1}{2}\left(1-\left(D-\frac{D^2}{2}\right)\right)^{-1}(x) + e^x \times \frac{1}{D^2+1}\cos x$$

$$= \frac{1}{2}\left[\left(1 + D - \frac{D^2}{2} + \cdots\right)(x)\right] + e^x \times \frac{x}{2D}\cos x$$

[using $(1-x)^{-1} = 1+x+x^2+\cdots$ and
$\frac{1}{f(D)}\cos ax = \frac{x}{f'(D)}\cos ax$ if $f(D^2) = f(-a^2) = 0$]

$$= \frac{1}{2}(x+1) + e^x \times \frac{x}{2} \times -\sin x + C$$

[$\because \frac{1}{D}\cos x = \int \cos x = \sin x$]

$$y_p = \frac{1}{2}(x+1) - \frac{x e^x \sin x}{2} + C$$

so general solution is $y = y_c + y_p$

$$\boxed{y = e^x(c_1 \cos x + c_2 \sin x) + \left(\frac{x+1}{2}\right) - \frac{x e^x \sin x}{2} + C}$$

5. $y'' + 4y = \tan 2x$ using Method of variation of parameters

Auxillary equation is $m^2+4=0$ i.e. $m = \pm 2i$ so, complementary function CF. $y_c = c_1 \cos 2x + c_2 \sin 2x$

Let $u = \cos 2x$, $v = \sin 2x$ Then let $y = A\cos 2x + B\sin 2x$ be the particular solution to given equation.

Then wronskian $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2 \neq 0.$

$\boxed{R = \tan 2x}$ = R.H.S of given eq.

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Then using $A = \int -\frac{vR}{W}\,dx$

$B = \int \frac{uR}{W}\,dx$ we get

$$A = -\int \frac{\sin 2x \times \tan 2x}{2}\,dx = -\int \frac{\sin^2 2x}{2\cos 2x}\,dx$$

$$= -\int \frac{1-\cos^2 2x}{2\cos 2x}\,dx = \int \frac{\cos 2x}{2}\,dx - \int \frac{\sec 2x}{2}\,dx$$

$$= \frac{\sin 2x}{4} - \frac{1}{4}\log(\sec 2x + \tan 2x) + c$$

$$B = \int \frac{\cos 2x \times \tan 2x}{2}\,dx = -\frac{\cos 2x}{4} + c'$$

i.e. y_p = particular solution is given by

$$y_p = Au + Bv = \cancel{\frac{\sin 2x \times \cos 2x}{4}} - \frac{1}{4}\cos 2x \times \log(\sec 2x + \tan 2x)$$

$$\cancel{-\frac{\sin 2x \cos 2x}{4}} = -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

gen. sol. is $y = y_c + y_p = c_1\cos 2x + c_2\sin 2x - \frac{\cos 2x}{4} \times \log(\sec 2x + \tan 2x)$.

6. Use laplace transform → $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, $x(0) = 2$, $x'(0) = -1$.

Apply laplace transform on both sides,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^t]$$

$$p^2 L[x(t)] - p\,x(0) - x'(0) - 2 \times [p\,L[x(t)] - x(0)]$$

$$+ L[x(t)] = \frac{1}{p-1}$$

$$\left[\text{using } L(F^{(n)}(t)) = p^n f(p) - p^{n-1}F(0) - \cdots - F^{(n-1)}(0) \text{ and } L(e^{at}) = \frac{1}{p-a}\right]$$

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$$(p^2-2p+1)\,L[x(t)] = \frac{1}{p-1} + 2p-1-4$$

$$L[x(t)] = \frac{1}{(p-1)(p-1)^3} + \frac{2(p-1)}{(p-1)^2} - \frac{3}{(p-1)^2}$$

$$= \frac{1}{(p-1)^3} + \frac{2}{p-1} - \frac{3}{(p-1)^2} \quad \text{--- } (*)$$

Using $L(t^n F(t)) = (-1)^n \frac{d^n}{dp^n}(f(p))$ — (1)

where $f(p) = L[F(t)]$

let $f(p) = \frac{1}{p-1}$ i.e. $L(F(t)=e^t) = \frac{1}{p-1} = f(p)$

$f'(p) = \frac{-1}{(p-1)^2}$ $\quad f''(p) = \frac{2}{(p-1)^3}$

using $n=1 \Rightarrow L(te^t) = \frac{1}{(p-1)^2}$

$\Rightarrow$ using $n=2$ in (1),

$$L(t^2e^t) = \frac{2}{(p-1)^3} \Rightarrow \frac{1}{(p-1)^3} = \frac{1}{2}L(t^2e^t).$$

$\Rightarrow$ (*) becomes

$$L[x(t)] = \frac{1}{2}L(t^2e^t) + 2L[e^t] - 3L[te^t]$$

using laplace inverse on both sides we get

$$\boxed{x(t) = \frac{t^2e^t}{2} - 2e^t - 3te^t}$$

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