Q.1 Prove that the funct^n $f(z) = u+iv$, where

$f(z) = \frac{x^3(1+i) - y^3(1+i)}{x^2+y^2}$, $z \neq 0$, $f(0)=0$ satisfies C-R Eq^{ns}

at the origin, but the derivative of f at origin

does not exist.

Sol^n... $f(z) = \frac{x^3(1+i) - y^3(1+i)}{x^2+y^2}$ ... (Given)

$$= \frac{(x^3-y^3)}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

$\therefore$ $u = \frac{x^3-y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$.

$\Rightarrow$ When $z \neq 0$: u, v are rational functions of x & y, therefore they are continuous and at origin, their values becomes zero, so continuous there as well.

Now, at $(0,0)$: (1) $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$

(2) $\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1$

(3) $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = 1$

(4) $\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{y}{y} = 1$.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ C-R Eqn's satisfied.

Derivative at (0,0): $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{z\to 0} \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy}$$

Now, let $z\to 0$ along $y=x$, then

$$f'(x) = \lim_{x\to 0} \frac{2i}{2(1+i)} = \frac{1}{2}(1+i)$$

Again let $z\to 0$ along x-axis, then, $f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^3}$

$\because$ Two limits are different $\therefore$ Not Differentiable.

Q.2 Expand Laurent series the function $f(z) = \frac{1}{z^2(z-1)}$

about $z=0$ & $z=1$.

Soln... After taking partial fractions: $f(z) = \frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2}$

① About $z=0$: $f(z) = \frac{-1}{(1-z)} - \frac{1}{z} - \frac{1}{z^2}$

$$= -\left(\frac{1}{z^2} + \frac{1}{z} + (1-z)^{-1}\right)$$

$$= -\left(\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \dots\right)$$

About $z=1$: let $z-1=u \Rightarrow z=u+1$

$$f(z)=\frac{1}{z^2(z-1)}=\frac{1}{(u+1)^2 u}=\frac{1}{u}\cdot(1+u)^{-2}$$

$$=\frac{1}{u}\left[1-2u+\frac{(-2)(-3)}{2!}u^2+\dots\right]=\frac{1}{u}\left[1-2u+3u^2-4u^3+\dots\right]$$

Q. Evaluate the integral $\int_0^{\pi}\frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}$ using residues.

Sol^n: let $I=\int_0^{\pi}\frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}=\frac{1}{2}\int_0^{2\pi}\frac{4\,d\theta}{(2+\cos\theta)^2}$

$$=\int_0^{2\pi}\frac{2\,d\theta}{(2+\cos\theta)^2}$$

let the contour 'C' be unit circle $|z|=1$ with centre at the origin.

let $z=e^{i\theta}$ then $\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$

Also $dz=ie^{i\theta}d\theta$

$\Rightarrow d\theta=\frac{dz}{iz}$

Substituting: $I=\int_0^{2\pi}\frac{2\,d\theta}{(2+\cos\theta)^2}=\int_C\frac{2\,dz}{iz\left(\frac{2+z^2+1}{2z}\right)^2}$

$$=\frac{1}{i}\int_C\frac{8z}{(z^2+4z+1)^2}dz=\frac{8}{i}\int_C\frac{z}{(z^2+4z+1)^2}dz$$

$$=\frac{8}{i}\int_C f(z)\,dz \quad \text{where } f(z)=\frac{z}{(z^2+4z+1)^2}$$

Now the poles of $f(z)$ are given by:

$(z^2+4z+1)=0 \Rightarrow z=\frac{-4\pm\sqrt{16-4}}{2}=-2\pm\sqrt{3}$ (twice)

$\therefore$ $f(z)$ has poles of order 2. at $z = -2 \pm \sqrt{3}$ (twice)

let $\alpha = -2+\sqrt{3}$ \$ $\beta = -2-\sqrt{3}$

$\because$ $|\beta| \not< 1$ \$ $|\alpha\beta| = 1$ $\therefore$ $|\alpha| < 1$.

Hence the pole inside C is at $z=\alpha$ and it is of order 2.

$$\therefore \int_C f(z)\,dz = \int \frac{2\,dz}{(z^2+4z+1)^2}$$

$$= 2\pi i\,(\text{Residue at } z=\alpha)$$

Residue at $z=\alpha$: $\lim_{z\to\alpha} \frac{d}{dz} (z-\alpha)^2 \frac{z}{(z^2+4z+1)^2} = \lim_{z\to\alpha} \frac{d}{dz} \cancel{(z-\alpha)^2} \times \frac{z}{\cancel{(z-\alpha)^2}(z-\beta)^2}$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left(\frac{z}{(z-\beta)^2}\right) = \lim_{z\to\alpha}\left[\frac{(z-\beta)^2 - z\cdot 2(z-\beta)}{(z-\beta)^4}\right]$$

$$= \lim_{z\to\alpha}\left[\frac{(z-\beta)-2z}{(z-\beta)^3}\right] = \lim_{z\to\alpha}\left[\frac{-(\beta+z)}{(z-\beta)^3}\right]$$

$$= \frac{-(\alpha+\beta)}{(\alpha-\beta)^3} = \frac{+4}{(2\sqrt{3})^3} = \frac{1}{6\sqrt{3}}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \times \frac{1}{6\sqrt{3}} = \frac{\pi i}{3\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{2\,d\theta}{(2+\cos\theta)^2} = \frac{8}{i}\left[\frac{\pi i}{3\sqrt{3}}\right] = \frac{8\pi}{3\sqrt{3}} \quad \underline{\text{Ans}}.$$