CSE - 2014

Q) prove b/w two real roots of $e^x \cos x + 1 = 0$ a real root of $e^x \sin x + 1 = 0$ lies

Sol Let a, b be roots of

$e^x \cos x + 1 = 0$

$\therefore e^a \cos a = -1, \; e^b \cos b = -1$

$\cos a = -e^{-a}, \; \cos b = -e^{-b}$

Let $f(x) = -\cos x - e^{-x}$

$\forall x \in [a, b]$

→ $\cos x$ & e^{-x} are continuous in $[a, b]$

$\therefore f$ is continuous on $[a, b]$

→ $f'(x) = \sin x + e^{-x}$

↳ exists $\forall x \in (a, b)$

$\therefore f$ is differentiable at (a, b)

$\Rightarrow \exists$ atleast one point in (a, b) i.e. c s.t $f'(c) = 0$

[∵ Rolles theorem]

$f'(c) = 0 \Rightarrow \sin c + e^{-c} = 0$

$\Rightarrow e^c \sin c + 1 = 0$

[∵ multiplying e^c]

$x = c \in (a, b)$ is a root of

$e^x \sin x + 1 = 0$

$\therefore e^x \sin x + 1 = 0$ has a real root b/w any two roots of eqn.

Q) $$\int_0^1 \frac{\log_e(1+x)}{1+x^2}\, dx$$

Sol Given $I = \int_0^1 \frac{\log_e(1+x)}{1+x^2} dx$

put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)\cancel{\sec^2\theta}}{\cancel{1+\tan^2\theta}} \quad \text{—} (1)$$

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right)$$

$$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\because \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow I = \int_0^{\pi/4} \log \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)$$

$$\Rightarrow I = \int_0^{\pi/4} [\log 2] - \int_0^{\pi/4} (1+\tan\theta)\, d\theta$$

$$I = \int_0^{\pi/4} \log 2 \qquad [\because ①]$$

$$\Rightarrow I = \frac{1}{4} \log 2 \cdot \pi/4$$

$$\Rightarrow \therefore I = \frac{\pi}{8} \log 2 \;,,$$

) By using $x+y=u$, $y=uv$

evaluate $\iint (xy)(1-x-y)^{1/2}$

lines $x=0, y=0, x+y=1$.

Given $x+y=u$ ①, $y=uv$

$x = u-y$, $x = u(1-v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u - \cancel{uv} + uv$

$= u$

also $dx\,dy = \frac{\partial(x,y)}{\partial(u,v)} du\,dv$

$\therefore dx.dy = u\,du\,dv$

& $\sqrt{xy(1-x-y)} =$

$\sqrt{u(1-v)\,uv\,(1-u)}$

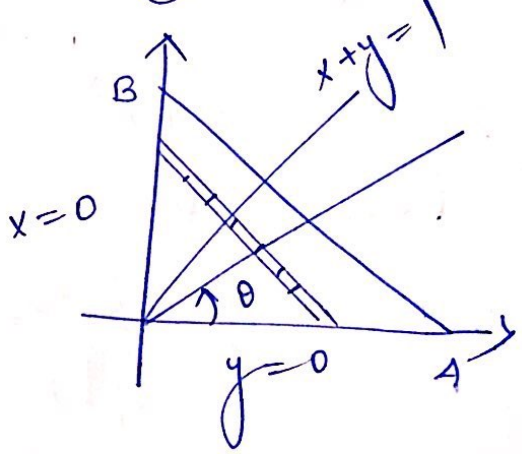
$= u v^{1/2} \sqrt{(1-u)(1-v)}$

Given

$y = uv \quad [\because y = (x+y)v$

$y = \frac{v}{1-v} x \qquad \because ①]$

②



Here $y = \frac{v}{1-v} x$

$\Rightarrow y = mx$

$m = \tan\theta$

Here θ varies from

$0 \to \pi/2$

as v varies from $0 \to 1$ since

$u = x + y$ varies from $0 \to 1$

$$\therefore I = \int_0^1 \int_0^1 u v^{1/2} \sqrt{(1-u)(1-v)}\, u\, du\, dv$$

$$= \int_0^1 u^2 (1-u)^{1/2} du \int_0^1 v^{1/2} (1-v)^{1/2} dv$$

$$= \int_0^1 u^{3-1} (1-u)^{3/2-1} du \int_0^1 v^{3/2-1} (1-v)^{3/2-1} dv$$

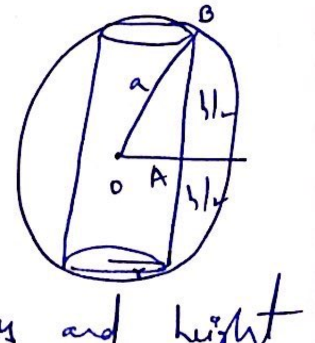
$$= \beta(3, 3/2) \cdot \beta(3/2, 3/2)$$

$$= \frac{\Gamma 3\, \Gamma 3/2}{\Gamma 3+3/2} \cdot \frac{\Gamma 3/2\, \Gamma 3/2}{\Gamma 3}$$

$$= 2\pi/105$$

Q) find the height of cylinder of max volume than can be inscribed in a sphere of radius 'a'

Sol) Let 'O' be centre of sphere of radius 'a'



'r', 'h' be radius and height of cylinder respectively

$$OA = r = \sqrt{a^2 - h^2/4}$$

$$V(\text{cylinder}) = \pi r^2 h$$

$$= \pi \left(a^2 - \frac{h^2}{4}\right) h$$

for Max / min $\frac{dV}{dh} = 0$

$$\pi a^2 - \frac{3h^2 \pi}{4} = 0$$

$$\Rightarrow a = \frac{\sqrt{3}}{2} h \Rightarrow h = \frac{2a}{\sqrt{3}}$$

$\therefore$ V will be max if $h = \frac{2a}{\sqrt{3}}$

$$V_{max} = \pi r^2 \cdot \frac{2a}{h}$$

$$= \pi \left(a^2 - \frac{1}{4} \cdot \frac{4a^2}{3}\right) \frac{2a}{\sqrt{3}}$$

$$= \frac{4\pi a^3}{3\sqrt{3}}$$

Q) find max & min values of $x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$, interpret them geometrically

Sol) Given $f(x, y, z)$

$$= x^2 + y^2 + z^2 \quad \text{—(1)}$$

Conditions,

$$ax^2 + by^2 + cz^2 - 1 = 0 \quad \text{—(2)}$$

$$lx + my + nz = 0 \quad \text{—(3)}$$

Let F be $F = f_1 + \lambda_1 ② + \lambda_2$ ③

also $dF = 0$

$\Rightarrow (2x + 2ax\lambda_1 + l\lambda_2)\,dx$

$+ (2y + 2by\lambda_1 + m\lambda_2)\,dy$

$+ (2z + 2\lambda_1 cz + n\lambda_2)\,dz$

$= dF$

$dF = 0$ @ stationary points.

$$\left.\begin{aligned} F_x = 0;\ & 2x + 2a\lambda_1 + l\lambda_2 = 0 \\ F_y = 0,\ & 2y + 2b\lambda_1 + m\lambda_2 = 0 \\ F_z = 0,\ & 2z + 2c\lambda_1 + n\lambda_2 = 0 \end{aligned}\right\} ④$$

multiplying ⑥ with x, y & z and adding we get

$2(x^2+y^2+z^2) + 2(ax^2+by+cz^2)$

$+ (lx + my + nz))\lambda_2 = 0$

Let $x^2+y^2+z^2 = t$

$2t + 2(1)\lambda_1 + 0(\lambda_2) = 0$

$\Rightarrow \lambda_2 = \frac{-t}{\ }$ ⑤

put ⑤ in ④ we get

$x = \frac{-l\lambda_2}{2(1-au)}$ $\left[\because 2x + l\lambda_2 + 2ax(-u)\right]$

similarly $y = \frac{-m\lambda_2}{2(1-bu)}$

$z = \frac{-n\lambda_2}{2(1-cu)}$

put these values in ③

$\Sigma l\left(\frac{-l\lambda_2}{2(1-au)}\right) = 0$

$\Rightarrow \lambda_2\left[\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu}\right] = 0$

If $\lambda_2 = 0$ then we get $x = y = z = 0$

this won't satisfy ②

$\therefore \lambda_2 \neq 0$

$$\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu} = 0$$

gives max & min of 'u'

ie $u = x^2+y^2+z^2$