IAS MATHEMATICS (OPT.)-2015

PAPER - 1 : SOLUTIONS

Q 1(a) The Vectors $V_1(1,1,2,4)$, $V_2=(2,-1,-5,2)$, $V_3(1,-1,-4,0)$ and $V_4(2,1,1,6)$ are linearly Independent. Is it true? Justify your Answer. (10)

Solution In order to determine if given four Vectors are L.I, We form a matrix with these given vectors as Rows and Row-Reduce it to Row-echelon form to Investigate its Rank $\rho(A)$

$$A=\begin{bmatrix}1&1&2&4\\2&-1&-5&2\\1&-1&-4&0\\2&1&1&6\end{bmatrix}$$

$$=\begin{bmatrix}1&1&2&4\\0&-3&-9&-6\\0&-2&-6&-4\\0&-1&-3&-2\end{bmatrix}\quad \begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-R_1\\R_4\to R_4-2R_1\end{array}$$

$$=\begin{bmatrix}1&1&2&4\\0&-3&-9&-6\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad \begin{array}{l}R_3\to R_3-2/3\,R_2\\R_4\to R_4-R_2/3\end{array}$$

clearly it is in the echelon form. and the number of non-zero rows equal to 2.

$\therefore$ The given set of vectors are linearly dependent.

1(b) Reduce the following Matrix to Row Echelon form and hence find its Rank.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

Solution. Let us denote given Matrix by A, then Reducing A to Row Echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to R_4 + 5R_3 \end{matrix}$$

This is the Row-Echelon form of A

Rank of Matrix A, $\rho(A) = 2$

Q1(c) Evaluate the following limit :-

$$\underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

Solution Let,

$$L = \underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

Taking Logarithm on both sides :-

$$\log L = \log \underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$\log L = \underset{x\to a}{Lt} \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \underset{x\to a}{Lt}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left[0/0 \text{ form}\right]$$

$$= \underset{x\to a}{Lt}\frac{(-1/a)}{\left(2-x/a\right)} \times \frac{1}{\left(-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\right)\cdot\left(\frac{\pi}{2a}\right)} \quad \left[\text{L-Hospital Rule}\right]$$

$$= \frac{(-1/a)}{1\times(-1)\times\left(\frac{\pi}{2a}\right)} \quad \left[\because \operatorname{cosec}\left(\frac{\pi x}{2a}\right)\to 1 \text{ as } x\to a\right]$$

$$\log L = \frac{2}{\pi}$$

$$\therefore \boxed{L = e^{2/\pi}}$$

Q 1(d) Evaluate the following Integral :-

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$$

Sol^n

$$I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}\,dx}{(\sin x)^{1/3}+(\cos x)^{1/3}} \quad \text{--- (1)}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\left[\sin\left(\pi/6+\pi/3-x\right)\right]^{1/3}}{\left[\sin\left[\pi/6+\pi/3-x\right)\right]^{1/3}+\left[\cos\left(\pi/6+\pi/3-x\right)\right]^{1/3}}\,dx$$

$$\left[\text{as } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

So, $$I = \int_{\pi/6}^{\pi/3} \frac{(\cos x)^{1/3}}{(\cos x)^{1/3}+(\sin x)^{1/3}}\,dx \quad \text{--- (2)}$$

Adding Equation (1) and (2)

$$2I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}+(\cos x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\,dx.$$

$$\Rightarrow \int_{\pi/6}^{\pi/3} dx = \left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6}$$

$\therefore\ 2I = \pi/6$

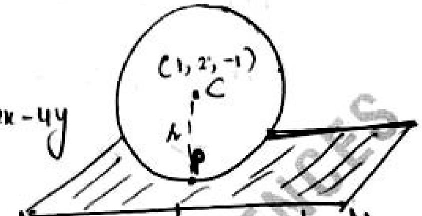
$\therefore\ \boxed{I = \pi/12}$

Q 1 (e) For what positive value of a, the plane $ax-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and Hence find the point of Contact.

Sol^n :- If the given plane :

$ax-2y+z+12=0$ —— (1)

touches the sphere $S \equiv x^2+y^2+z^2-2x-4y+2z-3=0$ —— (2)



then the length of perpendicular from Centre on to the plane = Radius of the sphere.

Centre C : $(1, 2, -1)$ and

Radius $r = \sqrt{1+(2)^2+(-1)^2-(-3)}$

$r = \sqrt{9} = 3$

Since plane is a Tangent plane

$CP \Rightarrow r = \left|\dfrac{a\cdot 1 - 2\cdot 2 + (-1) + 12}{\sqrt{a^2+4+1}}\right| = 3$

$\Rightarrow (a+7)^2 = 9(a^2+5)$

$\Rightarrow a^2+49+14a = 9a^2+45$

$\Rightarrow 4a^2-7a-2=0$

$\Rightarrow (a-2)(4a+1)=0$

$a=2$ or $a=-\frac{1}{4}$

Rejecting Negative value :-

Equation of st. line CP is :-

$\dfrac{x-1}{2} = \dfrac{y-2}{-2} = \dfrac{z+1}{1} = \lambda$ (perpendicular to given plane and taking $a=2$)

Any point on this line $(2\lambda+1, -2\lambda+2, \lambda-1)$ — (A)

It satisfy the Eqⁿ of plane for $\lambda = t$ (say)

$$\therefore\ 2(2t+1) - 2(-2t+2) + (t-1) + 12 = 0$$

$$(4t+4t+t) + 2-4-1+12 = 0$$

$$\Rightarrow\ t = -1$$

$\therefore$ point of Contact :- $(-1, 4, -2)$ — from (A)

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Q 2 (a) If Matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find A^{30}? (12)

Soln

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now, $A + A^2 - I \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^3 = A + A^2 - I$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = 3$

Let say $A^n = A^{n-2} + A^2 - I$ is true for $n = K$

$\therefore A^K = A^{K-2} + A^2 - I$

Let $n = k+1$

$A^{k+1} = A \cdot A^{k}$

$= A \cdot (A^{k-2} + A^2 - I)$

$A^{k+1} = A^{k-1} + A^2 - I$

$\therefore$ $A^n = A^{n-2} + A - I$ is true for $n = k+1$

$\therefore$ By Mathematical Induction Rule, It is true for Every Integer $n \geq 3$.

$\therefore$ $A^3 = A + A^2 - I$

$A^4 = 2A^2 - I$

$A^6 = 3A^2 - 2I$

$A^8 = 4A^2 - 3I$

$\vdots$

$A^{2n} = (n)A^2 - (n-1)I$

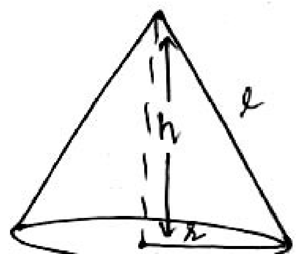
$\therefore$ $A^{30} = A^{2\times 15} = 15A^2 - 14I$

$$A^{30} = 15\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{30} = \begin{bmatrix} 15 & 0 & 0 \\ 15 & 15 & 0 \\ 15 & 0 & 15 \end{bmatrix} - \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$$

(b) A Conical tent is of given Capacity. For the Least Amount of Canvas Required, for it, find the Ratio of its height to the Radius of its base.

Sol^n Let r be the Radius of the Cone, h its height and l its slant height. Then,



$\frac{1}{3}\pi r^2 h = \text{volume}, V$

and S = lateral Surface Area $= \pi r l = \pi r\sqrt{r^2+h^2}$

Now S^2 Minimum $\Rightarrow$ S is minimum.

Hence,

$$S^2 = \pi^2 r^2 (r^2+h^2)$$

$$= \pi^2\left(\frac{3V}{\pi h}\right)\left[\frac{3V}{\pi h}+h^2\right]$$

$$= 3\pi V\left[\left(\frac{3V}{\pi}\right)\frac{1}{h^2}+h\right]$$

Differentiating w.r.t. h :-

$$\frac{d(S^2)}{dh} = \frac{d}{dh}\left[3\pi V\left[\frac{3V}{\pi}\frac{1}{h^2}+h\right]\right]$$

$$= 3\pi V\left[-\frac{6V}{\pi}\frac{1}{h^3}+1\right]$$

$$\frac{d^2(S^2)}{dh^2} = 3\pi V\left[\frac{18V}{\pi}\frac{1}{h^4}\right] > 0$$

for Critical points, $\frac{d(s^2)}{dh} = 0$

$\therefore 3\pi v\left[\frac{-6v}{\pi h^3}+1\right] = 0$

$\Rightarrow \frac{6v}{\pi h^3} = 1$

$\Rightarrow 6v = \pi h^3$

$\Rightarrow \cancel{6}^2 \cdot \frac{1}{\cancel{3}}\cancel{\pi} r^2 h = \cancel{\pi} h^3$

$\Rightarrow 2r^2 = h^2$

$\Rightarrow \sqrt{2}r = h$

$\therefore \frac{h}{r} = \sqrt{2}$ and $\frac{d^2(s^2)}{dh^2} > 0$ $\therefore$ Minima.

Q 2 (c) Find the Eigen Values and Eigen Vectors of the Matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

(12)

Sol^n Let $A =$ ~~given~~ $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, then its characteristic polynomial is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \begin{array}{l} \lambda^3 - 7\lambda^2 + 36 = 0 \\ (\lambda+2)(\lambda-3)(\lambda-6) = 0 \\ \lambda = -2, 3, 6. \end{array}$$

Here, Scalars λ is called Eigen Value of A. If there exists a non-Zero (Column) vector v such that $Av = \lambda v$

the v is called an Eigen Vector of A belonging to the Eigen Value λ.

$(A-2I)\, v = 0$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} 3x+y+3z = 0 \\ x+7y+z = 0 \\ \Rightarrow y = 0 \\ x+z = 0, \quad z = -x. \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = -x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \leftarrow \text{Eigen Vector for } \lambda = -2.$$

for $\lambda = 3$, $(A-3I)v = 0$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{aligned} -2x + y + 3z &= 0 \\ x + 2y + z &= 0 \\ 3x + y - 2z &= 0 \end{aligned}$$

$\therefore 5y + 5z = 0 \Rightarrow y = -z$ and $x = -y$

$\therefore V^T = (1, -1, 1)$ for $\lambda = 3$

for $\lambda = 6$; $(A - 6I)V = 0$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{aligned} -5x + y + 3z &= 0 \\ x - y + z &= 0 \\ 3x + y - 5z &= 0 \end{aligned}$$

$x = z$; $y = 2x$

$V^T = (1, 2, 1)$ for $\lambda = 6$

Q 2 (d) If $6x = 3y = 2z$ Represents one of the three mutually perpendicular Generators of the Cone $5yz - 8zx - 3xy = 0$ then obtain the Equations of the other two Generators.

Sol^n If $x/1 = y/2 = z/3$ is one of the three mutually perpendicular generators, then it is normal to the plane through the vertex cutting the cone in two perpendicular generators and therefore the Eq^n of the plane is

$$x + 2y + 3z = 0 \quad \text{—— (1)}$$

Now, we are to find the lines of Intersection of this plane and the given Cone.

Let one of the lines of Intersection be, $\frac{x}{l} = y/m = z/n$

$\therefore$ $l + 2m + 3n = 0$ and $5mn - 8nl - 3lm = 0$ —— (2)

Eliminating l between these,

$5mn - (8n+3n)\left[-(2m+3n)\right] = 0$

$\Rightarrow 24n^2 + 30mn + 6m^2 = 0$

$\Rightarrow m^2 + 5mn + 4n^2 = 0$ or $(m+n)(m+4n) = 0$

If $m = -n$, from (2), $l = -n$ $\therefore$ $l/1 = m/1 = n/-1$

If $m = -4n$, from (2), $l = 5n$ $\therefore$ $l/5 = m/-4 = n/1$

Hence, other two Generators are :-

$x/1 = y/1 = z/-1$ and $x/5 = y/-4 = z/1$

and Evidently these three form a set of mutually $\perp$ generators.

Q 3 (a) Let $V = \mathbb{R}^3$ and $T \in A(V)$, for all $a_i \in A(V)$, be defined by $T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_1 + 5a_2 + a_3 \\ -3a_1 + a_2 - a_3 \\ -a_1 + 2a_2 + 3a_3 \end{bmatrix}$

what is the matrix T Relative to the basis.

$v_1 = (1, 0, 1)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, -1, 1)$?

Sol^n first we find the Co-ordinates of (a, b, c) in the given basis.

$$(a, b, c) = x(1, 0, 1) + y(-1, 2, 1) + z(3, -1, 1)$$

$\therefore$ $x - y + 3z = a$
$2y - z = b$
$x + y + z = c$

Solving,
$x = \frac{1}{2}[-3a - 4b + 5c]$
$y = \frac{1}{2}[a + 2b - c]$
$z = [a + b - c]$

$T(1, 0, 1) = (3, -4, 2)$
$= \frac{17}{2}(1, 0, 1) - \frac{7}{2}(-1, 2, 1) - 3(3, -1, 1)$

$T(-1, 2, 1) = (9, 4, 8)' = -\frac{3}{2}(1, 0, 1) + \frac{9}{2}(-1, 2, 1) + 5(3, -1, 1)$

$T(3, -1, 1) = (2, -11, -2)$
$= 14(1, 0, 1) + (-9)(-1, 2, 1) - 7(3, -1, 1)$

$\therefore$ Matrix of Linear Transformation w.r.t. given Basis

$$M = \begin{bmatrix} 17/2 & -7/2 & -3 \\ -3/2 & 9/2 & 5 \\ 14 & -9 & -7 \end{bmatrix}^T = \begin{bmatrix} 17/2 & -3/2 & 14 \\ -7/2 & 9/2 & -9 \\ -3 & 5 & -7 \end{bmatrix}$$

Q(3)(b) Which point of the Sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$?

Sol^n Calculus Approach :- Let (x,y,z) be such point

Then maximize, $f=(x-2)^2+(y-1)^2+(z-3)^2 \rightarrow$ (1)

Such that, $x^2+y^2+z^2=1 \longrightarrow$ (2)

Let $g = x^2+y^2+z^2-1$

Then, applying lagrange'. Multipliers method,

Consider, $F=f+\lambda g$.

$F=(x-2)^2+(y-1)^2+(z-3)^2+\lambda(x^2+y^2+z^2-1)$

for Critical points, $df=0$

$2[(x-2)+\lambda x]dx+2[(y-1)+\lambda y]dy+2[(z-3)+\lambda z]dz=0$

$(\lambda+1)x=2$

$(\lambda+1)y=1 \quad \Rightarrow \quad x=2/\lambda+1, \; y=1/\lambda+1, \; z=3/\lambda+1$

$(\lambda+1)z=3$

from (2), $\dfrac{4+1+9}{(\lambda+1)^2}=1 \Rightarrow \lambda+1=\pm\sqrt{14}$

Taking $\lambda+1=\sqrt{14}$, $(x,y,z)=\left(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14}\right)$

$\therefore f=\left(2/\sqrt{14}-2\right)^2+\left(1/\sqrt{14}-1\right)^2+\left(3/\sqrt{14}-3\right)^2=(\sqrt{14}-1)^2$

Taking, $\lambda+1=-\sqrt{14}$, $(x,y,z)=\left(\dfrac{-2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$

$$\therefore\ f = \left(\frac{-2}{\sqrt{14}}-2\right)^2+\left(\frac{-1}{\sqrt{14}}-1\right)^2+\left(\frac{-3}{\sqrt{14}}-3\right)^2$$

$$= \left(\sqrt{14}+1\right)^2$$

Hence, the point $\left(\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$ of the sphere is at the maximum distance from point $(2, 1, 3)$

Max. distance $= \sqrt{f} = \left(\sqrt{14}+1\right)$

Geometrical Approach :- The Eq^n of st. line through Centre $(0,0,0)$ and point $(2, 1, 3)$ is $x-2/_2 = y-1/_1 = z-3/_3$.

This line will cut the sphere in two points (one max., one min)

then $\dfrac{x-2}{2/\sqrt{14}} = \dfrac{y-1}{1/\sqrt{14}} = \dfrac{z-3}{3/\sqrt{14}} = \lambda$

$$\Rightarrow (x, y, z) \equiv \left(2+\frac{2\lambda}{\sqrt{14}}, 1+\frac{\lambda}{\sqrt{14}}, 3+\frac{3\lambda}{\sqrt{14}}\right)$$

Q(3)(c)

(i) Obtain the Equation of the plane passing through the points $(2,3,1)$ and $(4,-5,3)$ parallel to x-Axis.

Sol^n

The Eq^n of any plane passing through $(2,3,1)$ is

$$a(x-2)+b(y-3)+c(z-1)=0 \quad \text{— (1)}$$

It passes through $(4,-5,3)$

$$\therefore a(4-2)+b(-5-3)+c(3-1)=0$$

i.e. $a-4b+c=0$ — (2)

If the plane (1) is parallel to x-Axis, then it is perpendicular to x plane.

i.e. $x=0$ i.e. $1\cdot x+0\cdot y+0\cdot z=0$

$$\therefore 1\cdot a+0\cdot b+0\cdot c=0 \Rightarrow a=0$$

$\therefore$ from (2) ; $-4b+c=0$ i.e. $c=4b$

$$\therefore \frac{a}{0}=\frac{b}{1}=\frac{c}{4}$$

Hence, (1) becomes :-

$$0+1(y-3)+4(z-1)=0$$

$$\boxed{y+4z-7=0}$$

Q(3)(c)(iii) Verify if the lines

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta} \text{ and } \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar. If yes, then find the equation of the plane in which they lie.

Sol^n Two straight lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

and Eq^n of plane containing them, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

Here, in this case,

$$\begin{vmatrix} (b-c)-(a-d) & b-a & b+c-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} \qquad C_1 \to C_1-C_2, \quad C_3 \to C_3-C_2$$

$$\Rightarrow \begin{vmatrix} d-c & b-a & c-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix}=0 \quad \text{as } C_1=-C_3.$$

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Hence, the given Lines are Coplanar.
The Equation of the plane Containing them, is.

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$
$C_3 \to C_3 - C_2$

$$\begin{vmatrix} x-y+d & y-a & z-y-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2y+z & y-a & z-y-d \\ 0 & \alpha & \delta \\ 0 & \beta & \gamma \end{vmatrix} = 0$$

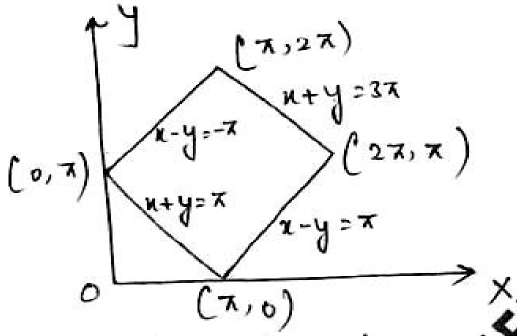
$\Rightarrow x-2y+z = 0$

Q(3) (d) Evaluate the Integral

$$\iint_R (x-y)^2 \cos^2 (x+y)\, dx\, dy$$

where R is the Rhombus with successive Vertices as $(\pi, 0)$, $(2\pi, \pi)$; $(\pi, 2\pi)$, $(0, \pi)$

Sol^n



Using the Transformation

$x-y = u$ and $x+y = v$

i.e. $x = \dfrac{u+v}{2}$, $y = \dfrac{-u+v}{2}$

Our Region of Integration gets transformed to a Square

Jacobian

$$\frac{J(x,y)}{J(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/4 + 1/4 = 1/2$$

$$\iint_R f(x,y)\, dx\, dy = \iint_{R_1} f_1(u,v)\, J(u,v)\, du\, dv$$

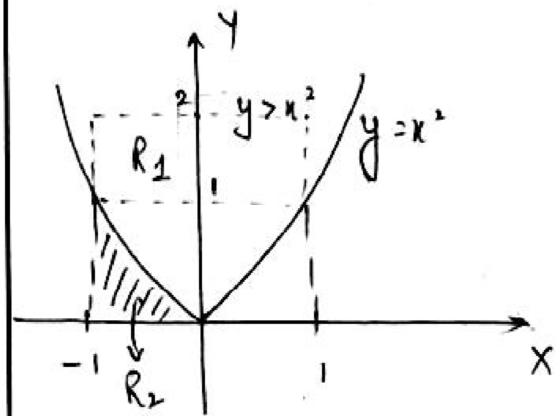
$$\Rightarrow \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2 \cos^2 v \left(\frac{1}{2}\right) du\, dv$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} u^2\, du \cdot \int_{\pi}^{3\pi} \left(\frac{1}{2}\right)(1+\cos 2v)\, dv$$

$$= \boxed{\pi^4/3}$$

Q(4)(a) Evaluate $\iint_R \sqrt{|y-x^2|}\, dx\,dy$ where $R = [-1, 1; 0, 2]$

Sol^n



We divide the domain R into two parts R_1 & R_2.

In R_1, $y > x^2$

$\therefore |y-x^2| = (y-x^2)$

In R_2, $y < x^2$

$\therefore |y-x^2| = -(y-x^2) = x^2-y$

$$\iint_R \sqrt{|y-x^2|}\,dx\,dy = \iint_{R_1} \sqrt{|y-x^2|}\,dx\,dy + \iint_{R_2} \sqrt{|y-x^2|}\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=x^2}^{2} \sqrt{y-x^2}\,dx\,dy + \int_{x=-1}^{1}\int_{y=0}^{x^2} \sqrt{x^2-y}\,dx\,dy$$

$$= \int_{-1}^{1} \frac{2}{3}(y-x^2)^{3/2}\Big|_{y=x^2}^{y=2} dx + \int_{-1}^{1} \frac{-2}{3}(x^2-y)^{3/2}\Big|_{y=0}^{y=x^2} dx$$

$$= \left(\frac{4}{3}\right)\int_0^1 (2-x^2)^{3/2}\,dx + \frac{2}{3}\left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$\left(\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\right)$$

$$= \left(\frac{16}{3}\right)\int_0^{\pi/4} \cos^4\theta\, d\theta \quad \text{taking } x = \sqrt{2}\sin\theta$$

$$= \frac{4}{3}\int_0^{\pi/4} (1+\cos 2\theta)^2\, d\theta = \frac{4}{3} + \frac{\pi}{2}$$

Q(4)(b) find the dimension of the subspace of $\mathbb{R}^4$, spanned by the set.

$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$

Hence find its basis.

Sol^n We find the Echelon form of the matrix formed by given vectors taking as Rows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim$ $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3.$$

Hence, Rank of matrix $= 3$

$\therefore$ Dimension of Subspace $= 3$

for the basis, we take the vectors from the original Matrix which correspond the non-zero Rows in the Echl-eon form

$\therefore$ Basis $= \{(1,0,0,0), (0,1,0,0), (1,2,0,1)\}$

Q4(c) Two $\perp$ tangent planes to the parabloid $x^2+y^2=2z$ intersect in a straight line in plane $x=0$. Obtain the curve to which this straight line touches.

Soln Let the line of intersection of the two planes be:-

$my+nz=\lambda,\ x=0$ — (1)

Since, this lies on the plane $x=0$ (Given)

$\therefore$ Eqn of the plane through the line (1) is

$(my+nz-\lambda)+kx=0$

or $kx+my+nz=\lambda$ — (2)

If the plane (2) touches the parabloid, then

$l^2/a^2+m^2/b^2+2\lambda n/c=0$ [Condition]

i.e. $k^2+m^2+2\lambda n=0$ — (3)

This being quadratic in k, gives two values of k, say k_1 and k_2 such that

$k_1 \cdot k_2 = m^2+2\lambda n$ — (4)

Also, from (2), the dr's of the normal to the two tangent planes whose line of intersection is (2) are k_1, m, n and k_2, m, n.

Also, as these two tangent planes are $\perp$

$\therefore k_1 \cdot k_2 + m \cdot m + n \cdot n = 0$

$\Rightarrow (m^2 + 2\lambda n) + m^2 + n^2 = 0$ - from (4)

$\Rightarrow (2m^2) + n^2 + 2\lambda n = 0$ — (5)

Now, we are to prove that the line (1) touches a parabola (to be ~~ground~~ found).

So, we are to find the Envelope of (1) which satisfies the Condn (5)

Eliminating λ between (1) and (5), the Equations of the line of Intersection of two tangent planes is :-

$2m^2 + n^2 + 2(my + nz) = 0 \; ; \; x = 0$

$\Rightarrow 2\left(\frac{m}{n}\right)^2 + 2y\left(\frac{m}{n}\right) + (1 + 2z) = 0 \; ; \; x = 0$

It is quadratic in m/n, so its Envelope is given by

$B^2 - 4AC = 0 \; , \; x = 0$

$\Rightarrow (2y)^2 - 4.2(1 + 2z) = 0 \; ; \; x = 0$

$\Rightarrow y^2 = 2(2z + 1) \; , \; x = 0$

This is the Required Curve.

5.(a) Solve the differential equation :

$$x\cos x\frac{dy}{dx}+y\,(x\sin x+\cos x)=1.$$

SOLUTION

Re-arranging the terms

$$\frac{dy}{dx}+\left(\tan x+\frac{1}{x}\right)y = \frac{1}{x\cos x}$$

$$\text{I.F} = e^{\int\left(\tan x+\frac{1}{x}\right)dx}=e^{\ell n\sec x+\ell n x}=x\sec x$$

Multiplying by I.F.

$$x.\sec x\frac{dy}{dx}+y\left(x\sec x\tan x+\sec x\right)=\sec^2 x$$

$$\frac{d}{dx}\left[x.\sec x.y\right] = \sec^2 x$$

Integrating both sides :

$$(x\ \sec\ x)\ y = \int\sec^2 x\,dx$$

$$= \tan x + c$$

$$\therefore \quad \boxed{y=\frac{\tan x}{x\sec x}+\frac{c}{x\sec x}}$$

5.(b) Sove the differential equation :

$(2xy^4e^y + 2xy^3 + y)\ dx + (x^2y^4e^y - x^2y^2-3x)dy = 0$

SOLUTION

Given differential equation is of the form, Mdx + Ndy = 0

where $M = 2xy^4 e^y + 2xy^3 + y$

$N = x^2y^4 e^y - x^2 y^2 - 3x$

$$\therefore \quad \frac{\partial M}{\partial y} = 8xy^3 e^y + 2xy^4 e^y + 6xy^2 + 1$$

$$\text{and,} \quad \frac{\partial N}{\partial x} = 2xy^4 e^y - 2xy^2 - 3$$

$$\because \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

Given differential equation is NOT exact. To convert into an exact equation we find the I.F. (Integrating factor) as follows

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3 e^y - 8xy^2 - 4$$

$$\frac{I}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-4}{y}$$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{4}{y}dy} = e^{-4\ell n\, y} = \frac{1}{y^4}$$

Multiplying the differential equation by I.F. :

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$$\underset{\left[\substack{\text{treating y} \\ \text{as constant}}\right]}{\int M\,dx} + \int(\text{terms in N not contaning x})dy = C$$

$$\Rightarrow \quad \int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \int 0\,dy = C$$

$$\Rightarrow \quad \boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C}$$

5(c) → A body moving under SHM has an amplitude 'a' and time period 'T'. If the velocity is trebled, when the distance from mean position is '$\frac{2}{3}a$' the period being unaltered, find the new amplitude.

Sol^n → The velocity of a particle executing SHM at any instant is defined as the time rate of change of its displacement at that instant.

$$v = \omega\sqrt{A^2 - x^2}$$

where ω is angular frequency, A is amplitude and x are displacements of a particle.

Suppose that the new amplitude of the modules be A'.

The initial velocity of a particle performs SHM.

$$v^2 = \omega^2\left[A^2 - \left(\frac{2A}{3}\right)^2\right] \quad \text{——} (1)$$

where A, is initial amplitude and ω is angular frequency.

—— final velocity,

$$(3v^2) = \omega^2\left[A'^2 - \left(\frac{2A}{3}\right)^2\right] \quad \text{——} (2)$$

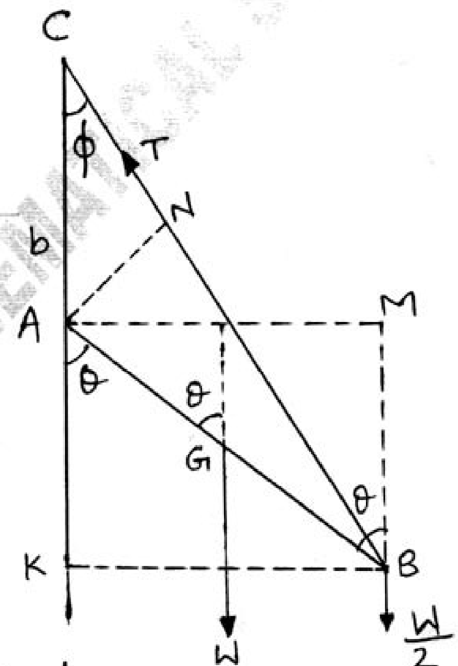
from equation (1) and (2) we get

$$\frac{1}{9} = \frac{A^2 - \frac{4A^2}{9}}{A'^2 - \frac{4A^2}{9}}$$

$$A' = \frac{7A}{3}$$

5(d) → A rod of 8 kg is movable in a verticle plane about a hinge at one end, another end is fastened a weight equal to half of the rod, this end is fastened by a string of length l to a point at a height b above the hinge vertically. Obtain the tension in the string.

Soln → Let a rod AB of length $2a$ (say) be movable in a vertical plane about a smooth hinge at the end A. A weight $W/2$ is attached at the other end B of the rod and this end is fastened by a string BC of length l to a point C at a height $AC = b$ vertically over the hinge at A.



The rod is in equilibrium under the action of the following forces

(i) W, weight of the rod at its mid-point G, acting vertically down-wards.

(ii) $W/2$, weight attached at the end B, acting vertically down-wards,

(iii) T, tension in the string along BC, and

(iv) the reaction at the hinge at A.

Let θ and ϕ be the angles of inclination of the rod and the string respectively to the vertical.

To avoid reaction at A, taking moments about the point A, we have,

$$T \cdot AN = W \cdot AL + \frac{1}{2} W \cdot AM$$

or $$T \cdot AC\sin\phi = W \cdot AG\sin\theta + \frac{1}{2} W \cdot AB\sin\theta$$

or $$T \cdot b\sin\phi = W \cdot a\sin\theta + \frac{1}{2} W \cdot 2a\sin\theta$$

$[\because AB = 2a]$

or, $$T = W\frac{2a\sin\theta}{b\sin\phi} \quad \text{—— (1)}$$

Now from the ΔCBK, $BK = BC\sin\phi = l\sin\phi$

from the ΔABK, $BK = AB\sin\theta = 2a\sin\theta$

$$\therefore\ l\sin\phi = 2a\sin\theta \quad \text{—— (2)}$$

$\therefore$ from (1) and (2), we get

$$T = \frac{Wl}{b} = \frac{8l}{b} \quad [\because \text{given } W = 8\,kg]$$

5.(e) Find the angle between the surfaces $x^2+y^2+z^2-9=0$ and $z = x^2+y^2-3$ at (2, –1, 2).

SOLUTION

Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $f_1 = x^2 + y^2 + z^2$ and $f_2 = x^2 + y^2 - z$

Then $\quad \text{grad}(f_1) = 2xi + 2yj + 2zk$

$\qquad \text{grad}(f_2) = 2xi + 2yj - k$

Let

n_1 = grad f_1 at point (2,–1,2) = $4i-2j+4k$

n_2 = grad f_2 at point (2,–1,2) = $4i-2j-k$

The vector n_1 and n_2 are along normals to the two surfaces at the point (2,–1, 2) If θ is the angle between these two vectors then,

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

$$= \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \quad \boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

6.(a) Find the constant a so that $(x + y)^a$ is the Integrating factor of $(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0$ and hence solve the differential eauation.

SOLUTION

As $(x+y)^a$ is an I.F. so,

$$\frac{\partial}{\partial y}\left[(x+y)^a\left(4x^2+2xy+6y\right)\right]=\frac{\partial}{\partial x}\left[(x+y)^a\left(2x^2+9y+3x\right)\right]$$

$a(x+y)^{a-1}(4x^2+2xy+6y) + (x+y)^a(2x+6) = a(x+y)^{a-1}(2x^2+9y+3x)+(x+y)^a(4x+3)$

Dividing both sides by $(x+y)^{a-1}$

$a(4x^2+2xy+6y)+(x+y)(2x+6) = a(2x^2+9y+3x) + (x+y)(4x+3)$

$x^2(4a+2-2a-4)+xy(2a+2-4)+y(6a+6-9a+3)+x(6-3a-3) = 0$

$x^2(2a-2)+xy(2a-2)+y(3-3a)+x(3-3a) = 0$

This equation is satisfied universally for a = 1.

$\therefore$ I.F. = $(x+y)$

Multiplying differential equation by I.F.

$(4x^3 + 2x^2y + 6xy+4yx^2+2xy^2+6y^2)\,dx+ (2x^3 + 9xy +3x^2+2x^2y+9y^2+3xy)dy = 0$

$(4x^3+6x^2y+2xy^2+6xy+6y^2)\,dx + (2x^3 + 2x^2y +12xy+3x^2+9y^2)dy = 0$

$$\int\left(4x^3+6x^2y+2xy^2+6xy+6y^2\right)dx+\int 9y^2dy=0$$

$$\boxed{x^4+2x^3y+x^2y^2+3x^2y+6xy^2+3y^3=0}$$

6.(c) Find the value of λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally at (1, –1, 2).

SOLUTION

Let $f_1 = \lambda x^2 - (\lambda+2)x - \mu yz$

$f_2 = 4x^2y + z^3 - 4$

grad $f_1 = \nabla f_1 = (2\lambda x - \lambda - 2)i - \mu zj - \mu yk$

$\nabla f_2 = 8xy\,i + 4x^2\,j + 3z^2\,k$

At point (1, –1, 2)

$n_1 = (\lambda-2)i - 2\mu\,j + \mu\,k$

$n_2 = -8i + 4j + 12k$

$n_1.n_2 = 0$ [Intersect orthogonally]

$-8(\lambda-2) - 8\mu + 12\mu = 0$

$-8\lambda + 4\mu = -16$

$\Rightarrow 2\lambda - \mu = 4$ .....(1)

point (1, –1, 2) lies on surface

$\lambda x^2 - \mu yz = (\lambda+2)x$

$\therefore \quad \lambda + 2\mu = (\lambda+2) \Rightarrow \mu = 1$

$\therefore \quad 2\lambda - 1 = 4 \Rightarrow \lambda = \dfrac{5}{2}$

6(d) A mass starts from rest at a distance 'a' from the centre of force which attracts inversely as the distance. find the time of arriving at the centre.

Sol^n If x is the distance of the particle from the centre of force at time t, then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x}.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.to t we have

$(dx/dt)^2 = -2\mu \log x + A$, where A is constant.

But initially at $x=a$, $\frac{dx}{dt}=0$

$\therefore\ 0 = -2\mu \log a + A$ or $A = 2\mu \log a$.

$\therefore\ (dx/dt)^2 = 2\mu(\log a - \log x) = 2\mu \log(a/x)$

or. $\frac{dx}{dt} = -\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}}$

where the -ve sign has been taken since the particle is moving in the direction of x decreasing.

seperating the varriables, we have.

$$dt = -\frac{1}{\sqrt{(2\mu)}}\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Integrating from $x=a$ to $x=0$, the required time t_1 to reach the centre is given by

$$t_1 = \frac{1}{\sqrt{(2\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

put $\log\left(\frac{a}{x}\right) = u^2$ ie. $x = ae^{-u^2}$

so that $dx = -2ae^{-u^2}\,u\,du$.

when $x=a$, $u=0$ and when $x\to 0, u\to\infty$

$$\therefore\ t_1 = \frac{2}{\sqrt{(2\mu)}}\int_0^\infty e^{-u^2}du. \text{ But,}$$

$$\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ t_1 = \frac{2a}{\sqrt{(2\mu)}}\frac{\sqrt{\pi}}{2} = a\sqrt{\left(\frac{\pi}{2\mu}\right)}.$$

7(a) (i) Obtain Laplace Inverse transform of

$$\left\{\ell n\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+25}e^{-\pi s}\right\}.$$

(ii) Using Laplace transform, solve
$y'' + y = t$, $y(0) = 1$, $y'(0 = -2$.

SOLUTION

(i) $F(s) = \log\left(s^2+1\right)-2\log s+\frac{s}{s^2+25}e^{-\pi s}$

Now, $L^{-1}\left[\frac{s}{s^2+25}e^{-\pi s}\right] = u(t-\pi)\cos 5(t-\pi)$

$= -u(t-\pi)\cos 5t$

$L^{-1}\left[\log(s^2+1)-2\log s\right] = f(t)$

$\Rightarrow \quad L(f(t)) = \log(1+s^2) - 2\log s.$

$$L[tf(t)] = \frac{-d}{ds}\left[\log\left(1+s^2\right)-2\log s\right]$$

$$= \frac{-2s}{1+s^2}+\frac{2}{s}$$

$$t.f(t) = L^{-1}\left[\frac{-2s}{s^2+1}+\frac{2}{s}\right]$$

$= -2\cos t + 2$

$$\Rightarrow \quad f(t) = \frac{-2\cos t}{t}+\frac{2}{t}$$

$$\therefore \quad L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+25}e^{-\pi s}\right] = \frac{2}{t}-\frac{2\cos t}{t}-u(t-\pi)\cos 5t$$

(ii) Taking Laplace

$$s^2L - sy(0) - y(0) + L = \frac{1}{s^2}$$

$$s^2L - s + 2 + L = \frac{1}{s^2}$$

i.e. $$L\left(1+s^2\right) = \frac{1}{s^2}+s-2$$

$$L = \frac{1}{s^2\left(1+s^2\right)}+\frac{s}{1+s^2}-\frac{2}{1+s^2}$$

$$= \frac{1}{s^2}-\frac{1}{1+s^2}+\frac{s}{1+s^2}-\frac{2}{1+s^2}$$

$$= \frac{1}{s^2}-\frac{3}{1+s^2}+\frac{s}{1+s^2}$$

$$y = L^{-1}\left[\frac{1}{s^2}-\frac{3}{1+s^2}+\frac{s}{1+s^2}\right]$$

$= t - 3\sin t + \cos t$

Q.7(b) A particle is projected from the base of a hill whose slope is that of a right circular cone, whose axis is verticle. The projectile graves the vertex and strikes the hill again at a point on the base. If the semiverticle angle of the cone is 30°, h is the height, determine the initial velocity 'u' of the projection and its angle of projection?

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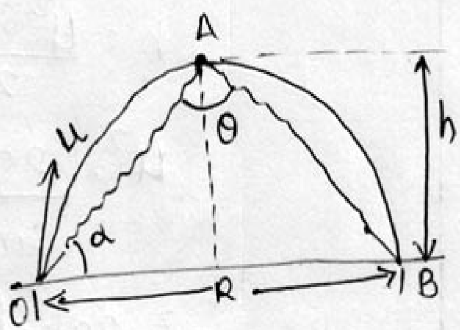
Sol. Given; Semiverticle angle = θ

initial velocity of the projectile = u

and angle of projection with the horizontal $= \alpha$.

(i) we know that maximum height through which the particle will rise



$$h = \frac{u^2 \sin^2\alpha}{2g}$$

and horizontal range $OB \Rightarrow R = \dfrac{u^2 \sin 2\alpha}{g}$

- from the geometry of the figure we find that

$$\cot\theta = \frac{h}{R/2} = \frac{2h}{R} = \frac{\cancel{2} \times \dfrac{\cancel{u^2}\sin^2\alpha}{\cancel{2g}}}{\dfrac{\cancel{u^2}\sin 2\alpha}{\cancel{g}}} = \frac{\sin^2\alpha}{\sin 2\alpha}$$

$$= \frac{\sin^2\alpha}{2\sin\alpha\cdot\cos\alpha} = \frac{\sin\alpha}{2\cos\alpha} = \frac{1}{2}\tan\alpha$$

$$\tan\alpha = 2\cot\theta \quad \text{——} \quad (1)$$

put $\theta = 30°$

$$\tan\alpha = 2\times\cot 30°$$

$$\boxed{\tan\alpha = 2\sqrt{3}}\ \text{A.}$$

(ii) The above equation (1) can be written as

$$\frac{1}{\tan\alpha} = \frac{1}{2\cot\theta} \quad \text{or} \quad \cot\alpha = \frac{1}{2}\tan\theta$$

$$\cot^2\alpha = \frac{1}{4}\tan^2\theta \quad [\because \text{squaring both side})$$

we know that maximum height through which the particle will rise:

$$h = \frac{u^2\sin^2\alpha}{2g}$$

$$u^2 = \frac{2gh}{\sin^2\alpha} = 2gh\cdot\operatorname{cosec}^2\alpha$$

$$u^2 = 2gh\cdot[1+\cot^2\alpha]$$

$$u^2 = 2gh\left[1+\frac{1}{4}\tan^2\theta\right]$$

$$u^2 = 2gh\left[1+\frac{1}{4}\cdot\tan^2 30°\right]$$

$$u^2 = 2gh\left[1+\frac{1}{2}\cdot\left(\frac{1}{\sqrt{3}}\right)^2\right]$$

$$u^2 = 2gh\left[1+\frac{1}{2}\times\frac{1}{3}\right] = \cancel{2}gh\left[\frac{6+1}{\cancel{6}}\right]$$

$$\boxed{u^2 = gh\,(7/3)}\ \text{A.}$$

7.(c) A vector field is given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}$

Verify that the field $\vec{F}$ is irrotational or not. Find the scalar potential.

SOLUTION

A vector field $\vec{F}$ is said to be irrotational if $\text{curl}\vec{F} = 0$ i.e. $\nabla \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(2xy - 2xy)$$

$$= \vec{0} \Rightarrow \vec{F} \text{ is irrotational.}$$

Now, it can be written as grad of a scalar field.

i.e. To find ϕ s.t. $\nabla\phi = \vec{F}$

i.e. $$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} = (x^2 + xy^2)i + (y^2 + x^2 y)k$$

$\therefore$ $$\frac{\partial\phi}{\partial x} = x^2 + xy^2;$$

$$\frac{\partial\phi}{\partial y} = y^2 + x^2 y$$

$\Rightarrow$ $$\phi = \frac{x^3}{3} + \frac{x^2 y^2}{2} + F(y)$$

Differentiating w.r.t. y and comparing with (*)

$$\frac{\partial\phi}{\partial y} = x^2 y + f'(y)$$

$\Rightarrow$ $$f'(y) = y^2$$

$$f(y) = \frac{y^3}{3} + c$$

$\therefore$ $$\boxed{\phi(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2 y^2}{2} + c}$$

7.(d) Solve the differential equation

$x = py - p^2$ where $p = \dfrac{dy}{dx}$

SOLUTION

Differentiate w.r.t. y

$$\frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$$

$$\Rightarrow \quad 1-p^2 = \left(py - 2p^2\right)\frac{dp}{dy}$$

$$\Rightarrow \quad (py - 2p^2)\, dp + (p^2 - 1)dy = 0 \qquad .....(1)$$

$(Mdx + Ndy = 0)$

Then $$\frac{\partial M}{\partial y} = p,$$

$$\frac{\partial N}{\partial p} = 2p$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial p}\right) = \frac{-p}{p^2-1}$$

$$\therefore \quad IF = e^{\int \frac{-p}{p^2-1}dp} = e^{-\frac{1}{2}\log\left(p^2-1\right)} = \frac{1}{\sqrt{p^2-1}}$$

Multiplying (1) throughout by I.F.

$$\Rightarrow \quad \left(\frac{py}{\sqrt{p^2-1}} - \frac{2p^2}{\sqrt{p^2-1}}\right)dp + \left(\sqrt{p^2-1}\right)dy = 0$$

$$\Rightarrow \quad \int\frac{py}{\sqrt{p^2-1}} - \int\frac{2p^2}{\sqrt{p^2-1}} = \text{constant}$$

$$\Rightarrow \quad y\sqrt{p^2-1} - 2\int p\frac{p}{\sqrt{p^2-1}} = c_1$$

$$\Rightarrow \quad y\sqrt{p^2-1} - 2p\int\frac{p}{\sqrt{p^2-1}} + 2\iint\frac{p}{\sqrt{p^2-1}} = c_1$$

$$\Rightarrow \quad y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + 2\int\sqrt{p^2-1} = c_1$$

$$\Rightarrow \quad y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + p\sqrt{p^2-1} - \log\left|\left(p+\sqrt{p^2-1}\right)\right| = c_1$$

$$\Rightarrow \quad y = p + \frac{\log\left|p+\sqrt{p^2-1}\right| + c_1}{\sqrt{p^2-1}}$$

$$\& \quad x = py - p^2 = p\left[\log\left|p+\sqrt{p^2-1}\right| + c_1\right]/\sqrt{p^2-1}$$

7(d). Solve the differential equation

$x = py - p^2$ where $p = \dfrac{dy}{dx}$

SOLUTION

Differentiate w.r.t. y

$$\frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$$

$$\Rightarrow \quad 1-p^2 = \left(py - 2p^2\right)\frac{dp}{dy}$$

$\Rightarrow$ $(py - 2p^2)\,dp + (p^2 - 1)dy = 0$ .....(1)

$(Mdx + Ndy = 0)$

Then $$\frac{\partial M}{\partial y} = p,$$

$$\frac{\partial N}{\partial p} = 2p$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial p}\right) = \frac{-p}{p^2-1}$$

$$\therefore \quad IF = e^{\int \frac{-p}{p^2-1}dp} = e^{-\frac{1}{2}\log\left(p^2-1\right)} = \frac{1}{\sqrt{p^2-1}}$$

Multiplying (1) throughout by I.F.

$$\Rightarrow \left(\frac{py}{\sqrt{p^2-1}} - \frac{2p^2}{\sqrt{p^2-1}}\right)dp + \left(\sqrt{p^2-1}\right)dy = 0$$

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$$\Rightarrow y\sqrt{p^2-1} - 2\int p\frac{p}{\sqrt{p^2-1}} = c_1$$

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$$\Rightarrow y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + 2\int\sqrt{p^2-1} = c_1$$

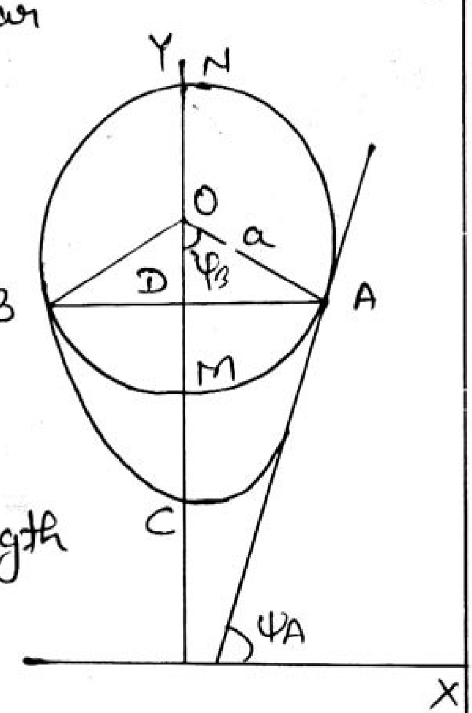
$$\Rightarrow y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + p\sqrt{p^2-1} - \log\left|\left(p+\sqrt{p^2-1}\right)\right| = c_1$$

$$\Rightarrow y = p + \frac{\log\left|p+\sqrt{p^2-1}\right| + c_1}{\sqrt{p^2-1}}$$

$$\& \quad x = py - p^2 = p\left[\log\left|p+\sqrt{p^2-1}\right| + c_1\right]/\sqrt{p^2-1}$$

8(a) Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two thirds of circumference of the pulley.

Soln Let ANBMA be the circular pulley of radius a and ANBCA the endless chain hanging over it.



Since the chain is in contact with the two third of circumference of pulley, hence the length of this portion ANB of the chain.

$$= \frac{2}{3}(\text{Circumference of the pulley})$$

$$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a$$

Let the catenary portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e. the vertex CO'N the axis and OX the directrix of this catenary.

Let $OC = c =$ the parameter of the catenary.

The tangent at A will be perpendicular to the radius O'A.

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal, then.

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left(\frac{1}{3}\cdot 2\pi\right)$$
$$= \frac{1}{3}\pi.$$

from the triangle AO'D, we have.

$$DA = O'A \sin\frac{1}{3}\pi = a\sqrt{3}/2.$$

$\therefore$ from $x = c\log(\tan\psi + \sec\psi)$, for the point A, we have,

$$x = DA = c\log(\tan\psi_A + \sec\psi_A)$$

or $$\frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$
$$= c\log(\sqrt{3}+2)$$
$$\therefore\quad c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $s = c\tan\psi$ applied for the point A, we have.

$$\text{arc } CA = c\tan\psi_A = c\tan\frac{1}{3}\pi$$
$$= c\sqrt{3} = \frac{3a}{2\log(2+\sqrt{3})}$$

Hence the total length of the chain.

= arc ABC + length of the chain in contact with the pulley.

$$= 2\,(\text{arc}\cdot CA) + \frac{4}{3}\pi a$$

$$= 2\,\frac{3a}{2\log\left(2+\sqrt{3}\right)} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log\left(2+\sqrt{3}\right)} + \frac{4\pi}{3}\right\}.$$

8(b) A particle moves in a plane under a force, towards a fixed centre, proportional to the distance. If the path of the particle has two apsidal distance a, b (a>b), then find the equation of the path.

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Sol^n Here, the central accelaration P

$= -\lambda$ (distance [$\because$ Central force $\propto$ distance])

$= -\lambda r = -\dfrac{\lambda}{u}$

(we choose negative sign as force in direction towards the fixed centre).

The differential equation of the path is

$$P = u^2h^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$\Rightarrow \frac{-\lambda}{u} = u^2h^2\left(u + \frac{d^2u}{d\theta^2}\right) \Rightarrow \frac{\lambda}{u^3} = h^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow \frac{-\lambda}{u^3}\cdot 2\frac{du}{d\theta} = h^2\left[u\cdot\frac{2du}{d\theta} + \frac{d^2u}{d\theta^2}\cdot\frac{2du}{d\theta}\right]$$

$$\Rightarrow A + \frac{\lambda}{u^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

integrating.

or $u^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\lambda}{u^2} + A$

At $u = \frac{1}{a}$, $\frac{du}{d\theta} = 0 \Rightarrow A + \lambda a^2 = h\left[\frac{1}{a^2}\right]$

$\Rightarrow A + \lambda a^2 = \frac{h^2}{a^2}$

Similary $A + \lambda b^2 = h^2/b^2$

$\therefore \lambda(a^2 - b^2) = h^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \Rightarrow \frac{\lambda}{h^2} = -\frac{1}{a^2b^2}$

$\therefore A + \lambda a^2 = -\frac{\lambda a^2 b^2}{a^2} \Rightarrow A = -\lambda(a^2+b^2)$

Thus $-\lambda a^2b^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\lambda(a^2+b^2) + \frac{\lambda}{u^2}$

$u^2 + \left(\frac{du}{d\theta}\right)^2 = \left(a^2+b^2 - \frac{1}{u^2}\right)\Big/a^2b^2$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{a^2+b^2}{a^2b^2} - \frac{1}{u^2a^2b^2} - u^2$

$= \frac{1}{u^2}\left[\frac{a^2+b^2}{a^2\ b^2}u^2 - \frac{1}{a^2b^2} - u^4\right]$

$= \frac{1}{u^2}\left[-\left(\frac{1}{2}\ \frac{a^2+b^2}{a^2b^2} - u^2\right)^2 + \frac{1}{u}\left(\frac{a^2+b^2}{a^2b^2}\right)^2 - \frac{1}{a^2b^2}\right]$

Let $k_1^2 = \frac{1}{4}\left(\frac{a^2+b^2}{a^2b^2}\right)^2 - \frac{1}{a^2b^2} = \frac{1}{4(a^2b^2)^2}\left[(a^2+b^2)^2 - 4a^2b^2\right]$

$= \frac{(a^2-b^2)^2}{4(a^2b^2)^2} = \frac{1}{4}\left[\frac{1}{b^2}-\frac{1}{a^2}\right]^2$

and $k_2^2 = \frac{1}{2}\frac{a^2+b^2}{a^2b^2} = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^2}\left(k_1^2 - (k_2^2-u^2)^2\right)$

$\Rightarrow \frac{u\,du}{\sqrt{k_1^2-(k_2^2-u^2)^2}} = d\theta$ | Let $k_2^2 - u^2 = v$, $-2u\,du = dv$

$\therefore -\frac{1}{2}\frac{dv}{\sqrt{k_1^2-v^2}} = d\theta \Rightarrow \theta + c = -\frac{1}{2}\sin^{-1}\frac{v}{k_1}$

$\Rightarrow \frac{v}{k_1} = \sin\left[-2(\theta+c)\right] \Rightarrow \frac{k_2^2-u^2}{k_1} = -\sin 2(\theta+c)$

$\Rightarrow \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - u^2 = -\frac{1}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin 2(\theta+c)$

$u^2 = \frac{1}{2}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{b^2}\sin 2(\theta+c) - \frac{1}{a}\sin 2(\theta+c)\right]$

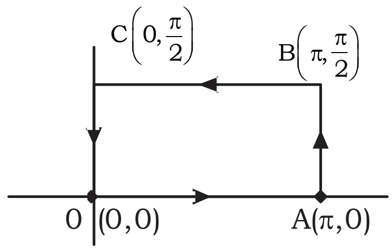
Let $c_1 = c - \pi/4 \Rightarrow u^2 = \frac{\sin^2(\theta+c_1)}{a^2} + \frac{\cos^2(\theta+c_1)}{b^2}$

8.(c) Evaluate $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$, where C is the rectangle with vertices (0, 0), (π, 0), $\left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$.

SOLUTION

By Green's Theorem in plane,

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



Here,
$$M = e^{-x} \sin y$$
$$N = e^{-x} \cos y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$$

$$\therefore \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R -2e^{-x}\cos y\,dx\,dy$$

$$= -2\int_{x=0}^{\pi} e^{-x}dx \int_{y=0}^{\pi/2} \cos y\,dy$$

$$= -2\left[-e^{-x}\right]_0^{\pi} \left[\sin y\right]_0^{\pi/2}$$

$$= 2\left(e^{-\pi} - 1\right)\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \boxed{2\left(e^{-\pi} - 1\right)}$$

8(d) Solve :

$$x^4\frac{d^4y}{dx^4}+6x^3\frac{d^3y}{dx^3}+4x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\ \cos(\log_e x).$$

SOLUTION

Putting, $x = e^z$ equation becomes

$[D(D-1)(D-2)(D-3)+6D(D-1)(D-2)+4D(D-1) - 2D-4] = e^{2z} + 2\cos z$

$[D(D-1)(D-2)(D-3)+6D(D-1)(D-2)+2(D-2)(D+1)] = e^{2z} + 2\cos z$

$\Rightarrow \quad (D^2+1)(D^2-4)\ y= e^z + 2\cos z$

For homogeneous part,

Complementary fuction,

$$y_c = c_1e^{2z} + c_2e^{-2z} + c_3 \sin z + c_4 \cos z$$

Particular integral,

$$y_P = \frac{1}{(D^2+1)(D^2-4)}e^{2z}+\frac{1}{(D^2+1)(D^2-4)}2\cos z$$

$$= \frac{1}{5}\cdot\frac{1}{D^2-4}\cdot e^{2z}-\frac{2}{5}\frac{1}{(D^2+1)}\cos z$$

$$= \frac{ze^{2z}}{20}-\frac{z}{5}\sin z$$

$\therefore$ Complete solution :

$$y = y_H + y_P$$

$$y = c_1e^{2z}+c_2e^{-2z}+c_3\sin z+c_4\cos z+\frac{ze^{2z}}{20}-\frac{z\sin z}{5}$$

$$\left[y(x)=c_1x^2+\frac{c_2}{x^2}+c_3\sin(\ell n\,x)+c_4\cos(\ell n\,x)+\frac{x^2\ell nx}{20}-\frac{\ell nx}{5}\sin(\ell nx)\right]$$