Q-4 use divergence thm $\iint_S \vec{V}\cdot\hat{n}\,dA$

$\vec{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$; S is boundary of region bounded by paraboloid $z = x^2+y^2$ and the plane $z = 4y$

$$\iint \vec{V}\cdot\hat{n}\,dA = \iiint_V (\nabla\cdot\vec{V})\,dV$$

V = volume bounded by region S

ROUGH

intersection of plane $z = 4y$ with paraboloid $z = x^2 + y^2$ is

$x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4$

$x^2 + (y-2)^2 = (2)^2$

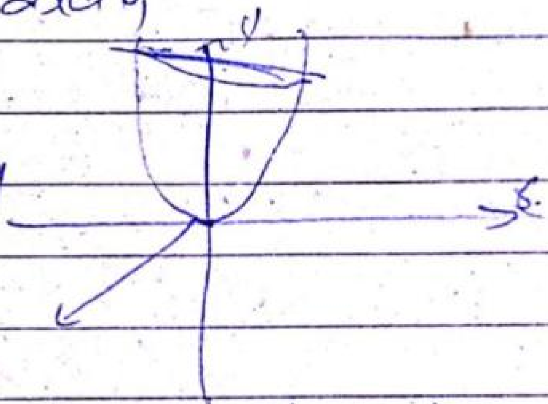
which is circle with centre $(0,2)$ and radius 2

$\nabla \cdot \vec{V} = 2xz + 1 - 2xz = 1$

so $I = \iiint 1 \, dV = \iiint dx\,dy\,dz$

$= \iint \left(\int_{z = x^2+y^2}^{4y} 1 \, dz \right) dx\,dy$

$= \iint (4y - x^2 - y^2) \, dx\,dy$

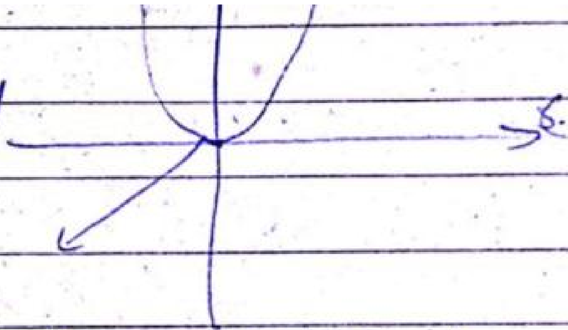


Put $x = r\cos\theta$,

$y = 2 + r\sin\theta$

$0 \le \theta \le 2\pi$ → converting into polar co-ordinate

$$= \iint (4y - x^2 - y^2)\, dx\, dy$$



Put $x = r\cos\theta$,

$y = 2 + r\sin\theta$

~~XXF~~. $0 \le \theta \le 2\pi$ $\longrightarrow$ converting into polar co-ordinates

Then

$$I = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} \left(4(2+r\sin\theta) - r^2\cos^2\theta - (2+r\sin\theta)^2\right) r\,dr\,d\theta$$

$$= \iint \left(8 + 4r\sin\theta - r^2\cos^2\theta - 4 - r^2\sin^2\theta - 4r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{0}^{2} (4r - r^3)\,dr\,d\theta = \left[2r^2 - \frac{r^4}{4}\right]_0^2 \times 2\pi$$

$$= (8-4)2\pi = \boxed{8\pi} \text{ . Ans.}$$

4. $\vec{u} = 4y\hat{i} + x\hat{j} - 2z\hat{k}$

Find $\iint (\nabla \times \vec{u}) \cdot \vec{ds}$ over the hemisphere given by $x^2+y^2+z^2 = a^2$, $z \geq 0$

Using stokes Thm.

$= \iint \text{curl}\vec{F} \cdot \vec{ds}$ $\iint \text{curl}\vec{F} \cdot \vec{n}\, ds = \oint_C \vec{F} \cdot dr$

where C is the circle $x^2+y^2=a^2$ on xy-plane

$\oint_C \vec{F} \cdot dr = \int (4y\hat{i} + x\hat{j} - 2z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

on $C: z=0, dz=0$

$I \Rightarrow \oint_C 4y\,dx + x\,dy$

let us parametrize C by $x = a\cos\theta$, $y = a\sin\theta$

$dx = -a\sin\theta \quad dy = a\cos\theta$

$I \int_{\theta=0}^{2\pi} -4a^2\sin^2\theta\, d\theta + \int_{\theta=0}^{2\pi} a^2\cos^2\theta\, d\theta$

$= \int \left(-4a^2\left(\frac{1-\cos 2\theta}{2}\right) + a^2\left(\frac{1+\cos 2\theta}{2}\right) \right) d\theta$

$= -\frac{3}{2}a^2 \times 2\pi = -3a^2\pi$

$= -\frac{3}{2} a^2 \times 2\pi = -3a^2\pi$

OR By using divergence theorem,

$$\iint_S (\nabla \times \vec{u}) \cdot d\vec{s} + \iint_{S_1} (\nabla \times \vec{u}) \cdot d\vec{s} = \iiint \nabla \cdot (\nabla \times \vec{u})\, dV$$

where S_1 is the enclosing plane $z=0$, $x^2+y^2=a^2$

so $\iint_S (\nabla \times \vec{u}) \cdot d\vec{s} = -\iint_{S_1} (\nabla \times \vec{u}) \cdot d\vec{s}$

$$= -\iint (-3\hat{k}) \cdot (-\hat{k})\, dx\,dy = -3 \iint_S dx\,dy$$

$$= \boxed{-3\pi a^2}$$

ROUGH

Q1⇒ If $\vec{A} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$; $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$

Find the value of $\frac{\delta^2}{\delta x\,\delta y}(\vec{A}\times\vec{B})$ at $(1,0,-2)$.

Sol^n

$$\vec{A} = x^2yz\,\hat{i} - \cancel{2xz}\;2xz^3\,\hat{j} + xz^2\,\hat{k}\ ,\quad \vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\hat{i} - \hat{j}\,(-x^4yz - 2xz^3) + \hat{k}\,(x^2y^2z + 2xz^4)$$

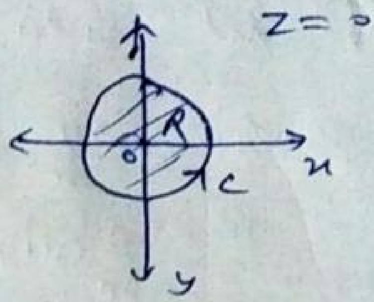
$$\vec{A}\times\vec{B} = (2x^3z^3 - xyz^2)\,\hat{i} + (x^4yz + 2xz^3)\,\hat{j} + (x^2y^2z + 2xz^4)\,\hat{k}$$

$$\cancel{\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y}} = \frac{\delta(\vec{A}\times\vec{B})}{\delta y} = -xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}$$

$$\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y} = -z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}$$

$$\left.\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y}\right|_{at\,(1,0,-2)} = -4\,\hat{i} + 4\times1\times-2\,\hat{j} = \boxed{-4\,\hat{i} - 8\,\hat{j}}$$

Q3 Find value of line integral over circular path $x^2+y^2=a^2, z=0$ where $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$



Ans. $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

As $z=0 \quad dz = 0$

$\therefore\ d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (\sin y\,\hat{i} + x(1+\cos y)\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_C \sin y\,dx + x(1+\cos y)\,dy$$

As the path C is closed & contains region R, therefore Applying Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$P = \sin y \quad \therefore\ \dfrac{\partial P}{\partial y} = \cos y$

$Q = x(1+\cos y) \quad \therefore\ \dfrac{\partial Q}{\partial x} = 1+\cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1+\cos y - \cos y = 1$$

$$\therefore\ \oint_C \sin y\,dx + x(1+\cos y)\,dy = \iint_R 1.\,dx\,dy = \iint_R dx\,dy$$

$= \text{Area of circle} = \pi a^2$

$P = \sin y \quad \therefore \frac{\partial P}{\partial y} = \cos y$

$Q = x(1+\cos y) \quad \therefore \frac{\partial Q}{\partial x} = 1+\cos y$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1+\cos y - \cos y = 1$

$\therefore \oint_C \sin y\, dx + x(1+\cos y)\, dy = \iint_R 1 \cdot dx\, dy = \iint_R dx\, dy$

$= \text{Area of circle} = \pi a^2$

$\boxed{\therefore \text{ Value of line integral} = \pi a^2}$

Q Calculate $\nabla^2(r^n)$ in terms of r and n where $r=\sqrt{x^2+y^2+z^2}$

sol. $r^2 = x^2+y^2+z^2 \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k} \quad \therefore \frac{\partial \vec{r}}{\partial x} = \hat{i}, \quad \frac{\partial \vec{r}}{\partial y} = \hat{j}, \quad \frac{\partial \vec{r}}{\partial z} = \hat{k}$

Now
$$\nabla^2(r^n) = \nabla\cdot(\nabla r^n) = \nabla\cdot\left(\Sigma i \frac{\partial}{\partial x}(r^n)\right) = \nabla\cdot\left(\Sigma i\, n r^{n-1}\frac{\partial r}{\partial x}\right)$$
$$= \nabla\cdot\left(\Sigma i\, n r^{n-1}\left(\frac{x}{r}\right)\right) = \nabla\cdot\left(\Sigma i\, n r^{n-2} x\right) = \nabla\cdot\left(n r^{n-2}\Sigma x i\right)$$
$$= \nabla\cdot\left(n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})\right) = \nabla\cdot\left(n r^{n-2}\vec{r}\right)$$

Now, As $\nabla\cdot(\phi\vec{V}) = \nabla\phi\cdot\vec{V}+\phi\,\nabla\vec{V}$

$$\therefore \nabla\cdot(n r^{n-2}\vec{r}) = \nabla(n r^{n-2})\cdot\vec{r} + n r^{n-2}\nabla\vec{r}$$
$$= \left(n\Sigma i \frac{\partial}{\partial x} r^{n-2}\right)\cdot\vec{r} + n r^{n-2}\Sigma i\cdot\frac{\partial \vec{r}}{\partial x}$$
$$= \left(n\,\Sigma i (n-2) r^{n-3}\frac{\partial r}{\partial x}\right)\cdot\vec{r} + n r^{n-2}\Sigma \hat{i}\cdot\hat{i}$$
$$= \left(n(n-2)\,\Sigma i\, r^{n-3}\cdot\frac{x}{r}\right)\cdot\vec{r} + n r^{n-2}\Sigma 1$$
$$= \left(n(n-2) r^{n-4}\Sigma x\hat{i}\right)\cdot\vec{r} + 3n r^{n-2}$$
$$= \left(n(n-2) r^{n-4}\vec{r}\right)\cdot\vec{r} + 3n r^{n-2}$$
$$= n(n-2) r^{n-4}(\vec{r}\cdot\vec{r}) + 3n r^{n-2}$$
$$= n(n-2) r^{n-4} r^2 + 3n r^{n-2}$$
$$= (n^2-2n) r^{n-2} + 3n r^{n-2} = (n^2-2n+3n) r^{n-2}$$
$$= (n^2+n) r^{n-2} = \boxed{n(n+1) r^{n-2}} \text{ Ans.}$$

Q.2 Evaluate by Stokes theorem $\int_\Gamma y\,dx + z\,dy + x\,dz$ where Γ is the curve given by $x^2+y^2+z^2-2ax-2ay=0$, $x+y=2a$, starting from $(2a,0,0)$ and then going below the z-plane.

Soln.. The given sphere $x^2+y^2+z^2-2ax-2ay=0$ has the centre $(a,a,0)$. The plane passes through the centre and therefore, their intersection is a circle C of radius $\sqrt{2}a$. ...[C is great circle of sphere]

Now acc. to Stokes theorem:

$$\int F \cdot dR = \iint (\nabla \times F) \cdot \hat{n}\, ds \quad \text{———} \quad (1)$$

$$\text{Now, } \int_C y\,dx + z\,dy + x\,dz = \int_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot dR$$

$$\therefore F = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\text{Now, } \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} = -(\hat{i} + \hat{j} + \hat{k})$$

Now acc. to Stokes theorem:

$$\int F \cdot dR = \iint (\nabla \times F) \cdot \hat{n}\, ds \quad \text{———} \quad (1)$$

Now, $\int_c y\,dx + z\,dy + x\,dz = \int_c (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_c (y\hat{i} + z\hat{j} + x\hat{k}) \cdot dR$$

$\therefore\ F = y\hat{i} + z\hat{j} + x\hat{k}$

Now, $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} = -(\hat{i} + \hat{j} + \hat{k})$

$\because$ S is surface of plane $x+y = 2a$, then a vector normal to S is grad $[x+y-2a] = \hat{i} + \hat{j}$.

$\therefore\ \hat{n}$ = unit normal vector to S $= \dfrac{\text{grad}(x+y)}{|\text{grad}(x+y)|} = \dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$.

$\therefore$ from (1):

$$\int_C y\,dx + z\,dy + x\,dz = \iint_S (\nabla \times F) \cdot \hat{n}\, ds$$

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k}) \cdot \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) ds$$

$$= -\sqrt{2} \iint_S ds$$

$= -\sqrt{2}$ (area of circle with $r = a\sqrt{2}$)

$$= -\sqrt{2} \times \pi \times (a\sqrt{2})^2$$

$$= -2\sqrt{2}\pi a^2 .$$

Q.1 Find the angle betn the surfaces $x^2+y^2+z^2-9=0$ and $z=x^2+y^2-3$ at $(2,-1,2)$

Soln Let $f_1 = x^2+y^2+z^2-9$ and $f_2 = x^2+y^2-z-3$

$\therefore$ Grad $f_1 = \nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ ... $\left[\because \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right]$

Grad $f_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$.

Let n_1 and n_2 be grad f_1 and grad f_2 at point $(2,-1,2)$ respectively.

$\therefore n_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ & $n_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

Here n_1 and n_2 are the normals to the surfaces and θ is the angle between them.

$\therefore \cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k})}{\left(\sqrt{4^2+2^2+4^2}\right)\left(\sqrt{4^2+2^2+1^2}\right)}$ .... $(A\cdot B = |A||B|\cos\theta)$

Here n_1 and n_2 are the normals to the surfaces and θ is the angle between them.

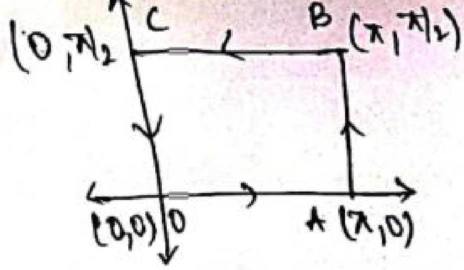
$$\therefore \cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k})}{(\sqrt{4^2 + 2^2 + 4^2})(\sqrt{4^2 + 2^2 + 1^2})} \quad \ldots (A \cdot B = |A||B| \cos\theta)$$

$$\cos\theta = \frac{16 + 4 - 4}{\sqrt{36} \cdot \sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$\therefore$ Angle betn the surfaces $= \theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$.

Q.4 Evaluate $\int_C e^{-x}(\sin y\, dx + \cos y\, dy)$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

Sol^n: The path C can be broken into paths OA, AB, BC and CO.



$\therefore \int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$.

Along OA: $y=0$; $dy=0$ and x varies from 0 to π

$\therefore \int_{OA} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_0^{\pi} e^{-x}(\sin 0\, dx + (\cos 0)\cdot 0) = 0$.

Along AB: $dx=0$, $x=\pi$ & y varies from 0 to $\pi/2$.

$\int_{AB} = \int_0^{\pi/2} e^{-\pi}(\cos y\, dy) = e^{-\pi}\left[\sin y\right]_0^{\pi/2} = e^{-\pi}$.

Along BC: $dy=0$, $y=\pi/2$ and x varies from π to 0.

$\int_{BC} = \int^{0} e^{-x}$ ... 0

Along AB: $dx=0$, $x=\pi$ & y varies from 0 to $\pi/2$.

$$\int_{AB} = \int_0^{\pi/2} e^{-\pi}(\cos y\, dy) = e^{-\pi}\left[\sin y\right]_0^{\pi/2} = e^{-\pi}.$$

Along BC: $dy=0$, $y=\pi/2$ and x varies from π to 0.

$$\int_{BC} = \int_\pi^0 e^{-x}(\sin \pi/2)\, dx = \int_\pi^0 e^{-x}\, dx = \left[-e^{-x}\right]_\pi^0$$

$$= e^{-\pi} - 1$$

Along CO: $y=0$ $x=0$, $dx=0$ and y varies from $\pi/2$ to 0.

$$\int_{CO} = \int_{\pi/2}^0 \cos y\, dy = \left(\sin y\right]_{\pi/2}^0 = -1.$$

$$\therefore \int_C e^{-x}(\sin y\, dx + \cos y\, dy) = e^{-\pi} + e^{-\pi} - 1 - 1.$$

$$= 2(e^{-\pi} - 1)$$

Ans.

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Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$ for $\bar{f} = (2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$

where S is upper half of sphere $x^2+y^2+z^2=1$ bounded by its projection on xy plane.

The sphere meets $z=0$ in circle C given by

$c: \quad x^2+y^2=1 \;, \; z=0$

let s_1 be plane region bounded by circle C.

Let S be the surface above xy plane and S' be the whole surface, i.e., $S' = S + S_1$

Let V be the volume bounded by S'.

on S_1, $\hat{n} = -\hat{k}$

Now,

$$\iint_{S'} \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iiint_V (\text{div curl}\,\vec{F})\,dv = 0$$

$[\text{div curl}\,\bar{F} = 0]$

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds + \iint_{S_1} \text{curl}\,\vec{F} \cdot \hat{n}\, ds = 0$$

$$\Rightarrow \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{curl}\,\vec{F} \cdot \hat{k}\, dS$$

Now,

$$\iint_{S'} \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \iiint_V (\text{div curl}\, \vec{F})\, dv = 0$$

$$[\text{div curl}\, \vec{F} = 0]$$

$$\iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds + \iint_{S_1} \text{curl}\, \vec{F} \cdot \hat{n}\, ds = 0$$

$$\Rightarrow \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{curl}\, \vec{F} \cdot \hat{k}\, dS$$

$$\text{curl}\, \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$$

$$\therefore \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \hat{k} \cdot \hat{k}\, ds$$

$$= \iint_{S_1} dS$$

$$= \pi \cdot (1)^2$$

[S_1 = area bounded by circle C]

$$\boxed{\iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \pi}$$

Ans

5(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2 dy$

along the path $x^4 - 6xy^3 = 4y^2$.

The integral is of the form

$$\int_C M dx + N dy$$

where $M = 10x^4 - 2xy^3$

$N = -3x^2y^2$

$$\frac{\partial M}{\partial y} = -6xy^2 \quad , \quad \frac{\partial N}{\partial x} = -6xy^2$$

Method-1

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the given integral is path-independent.

It means we can use any path,

Let the path consists of straight line L_1 : from $(0,0)$ to $(2,0)$ and then L_2 : from $(2,0)$ to $(2,1)$.

Along L_1 : $y = 0 \Rightarrow dy = 0$

Along L_2 : $x = 2 \Rightarrow dx = 0$

Meth

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the given integral is path - independent.

It means we can use any path,

Let the path consists of straight line L_1 : from $(0,0)$ to $(2,0)$ and then L_2 : from $(2,0)$ to $(2,1)$.

Along L_1 : $y=0 \Rightarrow dy = 0$

Along L_2 : $x = 2 \Rightarrow dx = 0$

$\therefore$ Value of integral $\int_{x=0}^{2} 10x^4 dx + \int_{y=0}^{1} -3(2)^2 y^2 dy$

$$= 2x^5 \Big|_0^2 - 4y^3 \Big|_0^1 = 64 - 4 = 60.$$

Method-2 :

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ $(10x^4 - 2xy^3)\,dx - (3x^2y^2)\,dy$ is an exact differential of $(2x^5 - x^2y^3)$.

$$\therefore \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$$

$$= \int_{(0,0)}^{(2,1)} d(2x^5 - x^2y^3),$$

$$= (2x^5 - x^2y^3)\Big|_{(0,0)}^{(2,1)}$$

$$= 64 - 4 = 60.$$