IFoS (M) MATHS PAPER-II
2017
SEC-A

1.a

Prove that every group of order four is abelian. (8)

Sol: Let $G = \{e, a, b, c\}$ be a group of order 4.

We show that $ab = ba$, $bc = cb$, $ac = ca$.

Case-I: $x^2 = e$, for all $x \in G \Rightarrow x = x^{-1} \; \forall x \in G$

$\Rightarrow (xy)^{-1} = xy$ ie $y^{-1}x^{-1} = xy \Rightarrow yx = xy$

$\Rightarrow G$ is abelian.

Case-II: There is an element $x \in G$ s.t. $x^2 \neq e$.

Let $a \in G$ be such element s.t. $a^2 \neq e$ ie $a \neq a^{-1}$.

also $a^{-1} \neq e$, because $a^{-1} = e \Rightarrow a = e$, which is not true.

$\therefore a^{-1} = b$ or c.

for the sake of definiteness, let $a^{-1} = b$.

$\therefore ab = ba = e$ (by defn of inverse). —①

Consider, $ac \in G$, $ac = e, a, b$ or c.

But $ac \neq e$ because $ac = e \Rightarrow a^{-1} = c \Rightarrow b = c$, not true.

$ac \neq a$ because $ac = a \Rightarrow c = e$, not true

$ac \neq c$ because $ac = c \Rightarrow a = e$, not true

so, we must have $ac = b$

By similar arguement $ca = b$. $\Rightarrow ac = ca$. —②

$\therefore b(ac)b = b(ca)b \Rightarrow (ba)(cb) = (bc)(ab)$

$\Rightarrow cb = bc$ as $ab = e = ba$ $\therefore G$ is abelian.

—③

b). $\boxed{1.b}$ A function, $f: \mathbb{R} \to \mathbb{R}$ is defined as below:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. (10).

Sol: First, let $a \neq \frac{1}{2}$ be any rational number, so that $f(a) = a$

Since in every interval there lies an infinite number of rational and irrational numbers, therfore for each positive integer n, we can choose an irrational number a_n s.t.

$$|a_n - a| < \frac{1}{n}$$

Thus sequence $\langle a_n \rangle$ converges to 'a'.

But, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (1-a_n) = 1-a, \quad a \neq \frac{1}{2}$

$$\therefore \lim_{n\to\infty} f(a_n) \neq f(a), \quad a \neq \frac{1}{2}.$$

Hence, the function is discontinuous at any rational number, other than zero.

In a similary manner, the function $f(x)$ may be shown to be discontinuous at every irrational point.

[* Limit of a function (sequential approach) —
A number l is called the limit of a fn 'f' as x tends to c if ~~the~~ sequence $\langle f(x_n) \rangle \to$ ~~[illegible]~~ l for any sequence, $\langle x_n \rangle \to c$.]

~~Let us show~~

It may be seen from above, that the function is continuous at $x = \frac{1}{2}$ ie $a = \frac{1}{2}$.
However, it can be shown to be continuous at $x = \frac{1}{2}$ as follows –

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$, then

Consider, $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}| < \varepsilon$

whenever $|x - \frac{1}{2}| < \delta$ & x is rational.

and

$$|f(x) - f(\tfrac{1}{2})| = |(1-x) - \tfrac{1}{2}| = |x - \tfrac{1}{2}| < \varepsilon$$

whenever $|x - \frac{1}{2}| < \delta$ & x is irrational.

$$\therefore |x - \tfrac{1}{2}| < \delta \Rightarrow |f(x) - f(\tfrac{1}{2})| < \varepsilon$$

or $$\lim_{x \to \frac{1}{2}} f(x) = f(\tfrac{1}{2})$$

Hence, the function, f is continuous at $x = \frac{1}{2}$.

$\boxed{1.c}$ If $f(z) = u(x,y) + iv(x,y)$ is an analytic function of $z = x+iy$ and
$$u + 2v = x^3 - 2y^3 + 3xy(2x-y),$$
then find $f(z)$ in terms of z. (8)

Sol: $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$ — ①

$2i\,f(z) = 2i\,u(x,y) - 2v(x,y)$ — ②

(1) − (2) ⇒ $f(z) - 2i\,f(z) = (u+2v) + i(v - 2u)$

ie. $F(z) = (u+2v) + i(-2u+v)$

$f(z)$ is analytic ⇒ $F(z) = (1-2i)f(z)$ is analytic.

Let $F(z) = U + iV$

$U = x^3 - 2y^3 + 3xy(2x-y)$

We use Milne-Thomson Method to find V

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = 3x^2 + 12xy - 3y^2$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = 6x^2 - 6xy - 6y^2$$

$$F'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$
$$= 3z^2 - i\,6z^2 = 3(1-2i)z^2$$

Integrating w.r.t z

$$F(z) = \cancel{3}\cdot(1-2i)\frac{z^3}{\cancel{3}} + C$$

$$\Rightarrow (1-2i)f(z) = (1-2i)z^3 + C$$

$$\Rightarrow f(z) = z^3 + C_1.$$

When $V(x,y)$ is given
$F'(z) = \psi_1(z,0) + i\,\psi_2(z,0)$
$\psi_1(x,y) = \frac{\partial V}{\partial y}$
$\psi_2(x,y) = \frac{\partial V}{\partial x}$

1.d Solve by simplex method the following LPP: minimize $z = x_1 - 3x_2 + 2x_3$

Subject to the constraints

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 0$$

and $x_1, x_2, x_3 \ge 0$. (14)

Sol: Adding the slack variables $S_1, S_2, S_3 \ge 0$ the given LPP reduces to standard from

minimize, $z = x_1 - 3x_2 + 2x_3 + 0S_1 + 0S_2 + 0S_3$

Sub to, $3x_1 - x_2 + 2x_3 + S_1 = 7,$

$-2x_1 + 4x_2 + S_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + S_3 = 0$

$x_i \ge 0, \; s_i \ge 0.$

The first simplex table is —

		C_j	1	-3	2	0	0	0	min Ratio
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_2
S_1	0	7	3	-1	2	1	0	0	—
S_2	0	12	-2	4	0	0	1	0	$12/4 = 3$
S_3	0	0	-4	3	8	0	0	1	$0/3 = 0$ →
		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	1	-3 ↑	2	0	0	0	

We select the most negative value of $C_j - Z_j$, that will be our entering variable and corresponding minimum positive ratio of X_B/x_2 will be new entering variable.

Iteration - 2.

B	C_B	c_j / X_B	1 / x_1	-3 / x_2	2 / x_3	0 / S_1	0 / S_2	0 / S_3	X_B/x_4
s_1	0	7	5/3	0	14/3	1	0	1/3	21/5
s_2	0	12	10/3	0	-32/3	0	1	-4/3	18/5 →
x_2	-3	0	-4/3	1	8/3	0	0	1/3	—
		z_j	4	-3	-8	0	0	-1	
		$c_j - z_j$	-3 ↑	0	10	0	0	1	

Iteration - 3.

s_1	0	1	0	0	10	1	-1/2	1	1 →
x_1	1	18/5	1	0	-16/5	0	3/10	-2/5	—
x_2	-3	24/5	0	1	-8/5	0	2/5	-1/5	—
		z_j	1	-3	8/5	0	-9/10	1/5	
		$c_j - z_j$	0	0	2/5	0	9/10	-1/5 ↑	

Iteration-4

S_3	0	1	0	0	10	1	-1/2	1
x_1	1	4	1	0	4/5	2/5	1/10	0
x_2	-3	5	0	1	2/5	1/5	3/10	0
$z=-11$		z_j	1	-3	-2/5	-1/5	-4/5	0
		$c_j - z_j$	0	0	12/5	1/5	4/5	0

All values of $c_j - z_j$ are non-negative.
Hence, optimal solution is,

$z = -11$ at $x_1 = 4$, $x_2 = 5$, $x_3 = 0$
which is a degenerate solution.

$\boxed{2a}$ Let G be the set of all real nos. except -1 and define $a*b = a+b+ab$ $\forall\, a,b \in G$. Examine if G is an Abelian group under $*$. (10)

Sol: Let $G = \mathbb{R} - \{-1\}$, we examine G for properties of a Group.

i) Closure: Let $a, b \in G$

Then $a, b \in \mathbb{R}$ and $a \neq -1$, $b \neq -1$,

$\therefore$ $a+b+ab \in \mathbb{R}$. We prove, $a+b+ab \neq -1$

As $a \neq -1$, $b \neq -1$ $\Rightarrow (a+1) \neq 0$; $(b+1) \neq 0$

$\Rightarrow (a+1)(b+1) \neq 0$

$\Rightarrow a+b+ab+1 \neq 0$

$\Rightarrow a+b+ab \neq -1 \Rightarrow a*b \in G$.

ii) Associative: Let $a, b, c \in G$.

$$a*(b*c) = a*(b+c+bc)$$
$$= a+(b+c+bc) + a(b+c+bc)$$
$$= (a+b+c) + (bc+ab+ac) + abc$$

$$(a*b)*c = (a+b+ab)*c$$
$$= (a+b+ab) + c + (a+b+ab)c$$
$$= (a+b+c) + (ab+bc+ca) + abc$$

$\therefore (a*b)*c = a*(b*c)$

$\therefore$ G is associative.

iii) Identity: $0 \in G$ such that for all $a \in G$

$a * 0 = a + 0 + a\cdot 0 = a = 0 * a$

Thus 0 is identity element of G.

iv) Inverse: For any $a \in G$, $a \neq -1$,

$\therefore$ $a+1 \neq 0$ and $\frac{-a}{a+1} \neq -1$,

Also, $\frac{-a}{a+1} \in \mathbb{R}$ as $a \in \mathbb{R}$ and $a \neq -1$.

$\therefore b = \frac{-a}{a+1} \in G$

$$\left[\begin{array}{c} a+b+ab=0 \\ a+b(1+a)=0 \\ b=\frac{-a}{1+a} \end{array}\right]$$

and $a * b = a+b+ab$

$= a - \frac{a}{a+1} - \frac{a^2}{a+1} = 0$

Similarly, $b * a = 0 \Rightarrow b$ is inverse of a.

$\therefore$ $(G, *)$ is a group.

v) Abelian (Commutativity):

for all $a, b \in G$.

$a * b = a+b+ab$

$= b+a+ba$

$= b * a$

$\therefore$ $*$ is commutative in G

$\Rightarrow$ $(G, *)$ is an abelian group.

2b Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh \ \forall h \in H, k \in K$. (10)

Sol: Let $o(H) = m$; $o(K) = n$ s.t. $(m, n) = 1$

Now, Let $a \in H \cap K \Rightarrow a \in H$ & $a \in K$

$\therefore a^m = e$ & $a^n = e$

$\therefore o(a) \mid m$ & $o(a) \mid n$, let $o(a) = k$

$\therefore k \mid m$; $k \mid n$ ~~$\Rightarrow$ Let $\frac{m}{k} = r$ & $\frac{n}{k} = q$~~

$\Rightarrow k \mid \gcd(m, n) \Rightarrow k \mid 1$ ie $k = 1$

$\therefore o(a) = 1 \Rightarrow a = e \Rightarrow H \cap K = \{e\}$ — ①

Now, To prove $hk = kh \ \forall h \in H, k \in K$.

As H is normal in $G \Rightarrow ghg^{-1} \in H \ \forall g \in G$

ie. $khk^{-1} \in H$ [taking $g = k \in K \subseteq G$]

or $(khk^{-1})h^{-1} \in H$ ($h^{-1} \in H$, closure in H)

$\Rightarrow \underline{khk^{-1}h^{-1} \in H}$

Again, K is normal in G and by similar argument, $gk^{-1}g^{-1} \in K \ \forall g \in G$

$\Rightarrow hk^{-1}h^{-1} \in K$ (taking $g = h$)

$\Rightarrow \underline{k(hk^{-1}h^{-1}) \in K}$ (closure in K)

Hence, $khk^{-1}h^{-1} \in H \cap K = \{e\}$

$\therefore khk^{-1}h^{-1} = e \Rightarrow kh = hk$

$\forall h \in H, k \in K$

Hence, proved.

2C Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A$, for some positive integer $n\}$

Is B an ideal of R? Justify your answer. (10)

Sol: Let $a, b \in B$.

Then $a^m \in A$ and $b^n \in A$, for some positive integers m and n,

Since R is commutative,

$$(a-b)^{m+n} = a^{m+n} - {}^{m+n}C_1 \, a^{m+n-1} b + \ldots + (-1)^{m+n} b^{m+n}$$

$$= a^m \cdot a^n - ({}^{m+n}C_1)\, a^m \cdot a^{n-1} b + \ldots + (-1)^{m+n} b^m b^n$$

$$\in A$$

Since $a^m \in A$, $b^n \in A$ and A is an ideal of R.

Thus $(a-b) \in B$.

For any $r \in A$, $a \in B$, we have

$(ra)^m = r^m a^m$, since R is commutative.

again, $r^m a^m \in A$, ~~B~~

[since $a^m \in A$, $r^m \in R$ & A is ideal of R]

$\therefore$ $ra \in B$.

Similarly, $ar \in B$

Hence B is an ideal of R.

[Ideal: A non-empty subset S of a ring R is called an ideal of R, if

i) $(S, +)$ is a subgroup of $(R, +)$

ie. $a, b \in S \Rightarrow a-b \in S$.

ii) $a \in S$ and $r \in R \Rightarrow ar \in S$ and $ra \in S$.]

2(d) Prove that the ring

$$Z[i] = \{a + ib : a, b \in \mathbb{Z},\ i = \sqrt{-1}\}$$

of Gaussian integers is a Euclidean Domain. (10)

Sol: Euclidean Domain (ED): An integral domain R is called a ED, if for each ~~$a \neq 0$~~ $a \in R, a \neq 0$, there is associated a non-negative integer, denoted by $d(a)$, such that

i) $d(a) \leq d(ab)$, $\forall$ ~~$a \neq 0, b$~~ $a, b \in R,\ a, b \neq 0$

ii) For each pair, $a, b \in R$, $a, b \neq 0$, there exist $t, r \in R$ such that

$a = tb + r$, where either $r = 0$ or $d(r) < d(b)$.

Here: It is easy to verify that $Z[i]$ is an integral domain with unity $1 = 1 + i0$.

Let $x = m_1 + in_1 \neq 0$, $y = m_2 + in_2 \neq 0$
$x, y \in \mathbb{Z}[i]$.

We define,

$$d(x) = d(m_1 + in_1) = m_1^2 + n_1^2 \quad \text{— (1)}$$

$\therefore d(x)$ is a positive integer for each non-zero $x \in \mathbb{Z}[i]$.

and $d(x) \geq 1 \quad \forall$ non-zero $x \in \mathbb{Z}[i]$ — (2)

and, we have

$$xy = (m_1 + in_1)(m_2 + in_2) = (m_1m_2 - n_1n_2) + i(m_1n_2 + m_2n_1)$$

$$\therefore d(xy) = (m_1m_2 - n_1n_2)^2 + (m_1n_2 + m_2n_1)^2$$

$$= m_1^2m_2^2 + n_1^2n_2^2 + m_1^2n_2^2 + m_2^2n_1^2$$

$$= (m_1^2 + n_1^2)(m_2^2 + n_2^2) = d(x)\,d(y).$$

Thus, $d(xy) = d(x)\,d(y)$, $\forall$ non-zero $x, y \in \mathbb{Z}[i]$ — (3)

Notice that,

$$d(x) = d(x)\cdot 1$$
$$\leq d(x)\cdot d(y) \qquad [\because d(y) \geq 1 \text{ by } (2)]$$
$$= d(xy) \qquad \text{by } (3)$$

$\therefore$ $d(x) \leq d(xy)$, $\forall$ non-zero $x, y \in \mathbb{Z}[i]$

Now, we verify second Property of ED.

$$\frac{x}{y} = \frac{m_1+in_1}{m_2+in_2} = \frac{(m_1+in_1)(m_2-in_2)}{(m_2+in_2)(m_2-in_2)}$$

$$= \frac{(m_1m_2+n_1n_2)}{(m_2^2+n_2^2)} + i\left(\frac{m_2n_1-m_1n_2}{m_2^2+n_2^2}\right)$$

$$= p+iq \text{ (say).} \quad \text{where } p, q \in \mathbb{Q}$$

Corresponding to rational numbers p and q, we can find suitable integers p' and q'

s.t. $|p'-p| \leq \frac{1}{2}$ and $|q'-q| \leq \frac{1}{2}$ — (4)

Let $\underline{t = p'+iq'}$ $\therefore$ $t \in \mathbb{Z}[i]$

We have, $\frac{x}{y} = \lambda$ where $\underline{\lambda = p+iq}$

or $x = \lambda y = (\lambda - t)y + ty = ty + r,$

where $r = (\lambda - t)y$

As, $x, y, t \in \mathbb{Z}[i] \Rightarrow r = x - ty \in \mathbb{Z}[i]$

Thus, there exist $t, r \in \mathbb{Z}[i]$ such that

$x = ty + r$, where $r = 0$ or

$$d(r) = d[(\lambda - t)y] = d[(p-p') + i(q-q')]\, d(y)$$
$$= [(p-p')^2 + (q-q')^2]\, d(y)$$
$$\leq \left[\frac{1}{4} + \frac{1}{4}\right] d(y) \qquad \text{by } (4)$$
$$= \frac{1}{2} d(y) < d(y)$$

$\Rightarrow$ $\underline{\mathbb{Z}[i] \text{ is an ED.}}$

3a Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$ given

$$f(x,y) = \begin{cases} x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}, & \text{if } xy \neq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Sol:

$$f_{xy}(a,b) = \underset{h\to 0}{Lt} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \underset{k\to 0}{Lt} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \underset{k\to 0}{Lt} \frac{f(h,k) - f(h,0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{1}{k}\left[\left(h^2 \tan^{-1}\frac{k}{h} - k^2 \tan^{-1}\frac{h}{k}\right) - h^2 \tan^{-1} 0/h\right]$$

$$= \underset{k\to 0}{Lt} \frac{h^2}{k} \tan^{-1}\frac{k}{h} - \underset{k\to 0}{Lt} k \cdot \tan^{-1}\frac{h}{k}$$

$$= h - 0 = h$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1 \qquad (1)$$

~~$f_x(0,k) = \lim$~~

Now, $$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{0-0}{h} = 0$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left(h^2 \tan^{-1}\frac{k}{h} - k^2 \tan^{-1}\frac{h}{k}\right) - 0\right]$$

$$= \lim_{h\to 0} h \tan^{-1}\frac{k}{h} - \lim_{h\to 0} \frac{k^2}{h} \cdot \tan^{-1}\frac{h}{k}$$

$$= 0 \cdot \left(\frac{\pi}{2}\right) - \lim_{h\to 0} k \cdot \left(\frac{\tan^{-1} h/k}{h/k}\right)$$

$$= 0 - k(1) = -k \qquad \left(\because \frac{\tan^{-1} x}{x} = 1 \text{ as } x\to 0\right)$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$$

$\_\_\_\_\_$ (2)

from (1) and (2)

$$f_{xy} \neq f_{yx} \text{ at } (0,0).$$

3b Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the condition

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \qquad (10)$$

Sol: Let $u(x,y,z) = x^2+y^2+z^2$

$$\phi(x,y,z) = \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1$$

Consider, $F = u+\lambda\phi$

$$dF = \left(2x+\lambda\frac{2x}{4}\right)dx + \left(2y+\lambda\frac{2y}{5}\right)dy + \left(2z+\lambda\frac{2z}{25}\right)dz$$

for stationary points, $dF=0$

$dx=0 \Rightarrow 2x+\lambda\frac{2x}{4}=0$ —(1)

$dy=0 \Rightarrow 2y+\lambda\frac{2y}{5}=0$ —(2)

$dz=0 \Rightarrow 2z+\lambda 2z/25=0$ —(3)

multiplying (1), (2) and (3) by x, y, z respectively and adding, we get

$$2(x^2+y^2+z^2)+2\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}\right)=0$$

ie $u+\lambda(1)=0 \Rightarrow u=-\lambda$ —(*)

From (1), $2x\left(1+\frac{\lambda}{4}\right) \Rightarrow x=0$ or $\lambda=-4$ —(4)

from (2), $2y\left(1+\frac{\lambda}{5}\right) \Rightarrow y=0$ or $\lambda=-5$ —(5)

from (3), $2z\left(1+\frac{\lambda}{25}\right) \Rightarrow z=0$ or $\lambda=-25$ —(6)

From (*) and (4), (5), (6) we get that

$u=4$ or $u=5$ or $u=25$ or $u=0$ at $(0,0,0)$

$\therefore$ Maximum value of $u=25$

minimum value of $u=4$.

3.c Prove that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent but not absolutely convergent. (12)

Sol: Here, point 0 is not a point of infinite discontinuity because $\frac{\sin x}{x} \to 1$ as $x \to 0$.

So, we take $\int_0^\infty \frac{\sin x}{x} = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$

Now, $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral.

$\Rightarrow$ Convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$ at ∞

$$\left|\int_1^X \sin x\,dx\right| = |\cos 1 - \cos X| \le |\cos 1| + |\cos X| < 2$$

So that $\left|\int_1^X \sin x\,dx\right|$ is bounded above for all $X \ge 1$

also, $\frac{1}{x}$ is a monotone decreasing function tending to 0 as $x \to \infty$.

Hence, by Dirichlet's test, $\int_1^\infty \frac{\sin x}{x}\,dx$ is cgt.

Hence, $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

[Dirichlet's test: If ϕ is bounded and monotonic in $[a, \infty)$ and tends to 0 as $x \to \infty$, and $\int_a^X f\,dx$ is bounded for $X \ge a$, then $\int_a^\infty f \cdot \phi\,dx$ is convergent at ∞].

To show $\int_0^\infty \frac{\sin x}{x}\,dx$ is not absolutely cgt.

Consider for $n \geq 1$, the proper integral

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx$$

Now $\forall\, x \in [(r-1)\pi, r\pi]$

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx \geq \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{r\pi}\,dx$$

Putting, $x = (r-1)\pi + y$

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{r\pi}\,dx = \int_0^{\pi} \frac{|\sin((r-1)\pi + y)|\,dy}{r\pi}$$

$$= \frac{1}{r\pi}\int_0^{\pi} \sin y\,dy = \frac{2}{r\pi}$$

Hence, $$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{r=1}^{n} \frac{2}{r\pi}$$

But $\sum_{r=1}^{n} \frac{2}{r\pi}$ is a divergent series.

$\therefore \lim_{n\to 0} \int_0^{n\pi} \frac{|\sin x|}{x}\,dx \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{2}{r\pi} \Rightarrow \lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x}\,dx$ is infinite.

Now, let t be a real numb. $\exists\, n \in \mathbb{N}$ s.t. $n\pi \leq t < (n+1)\pi$

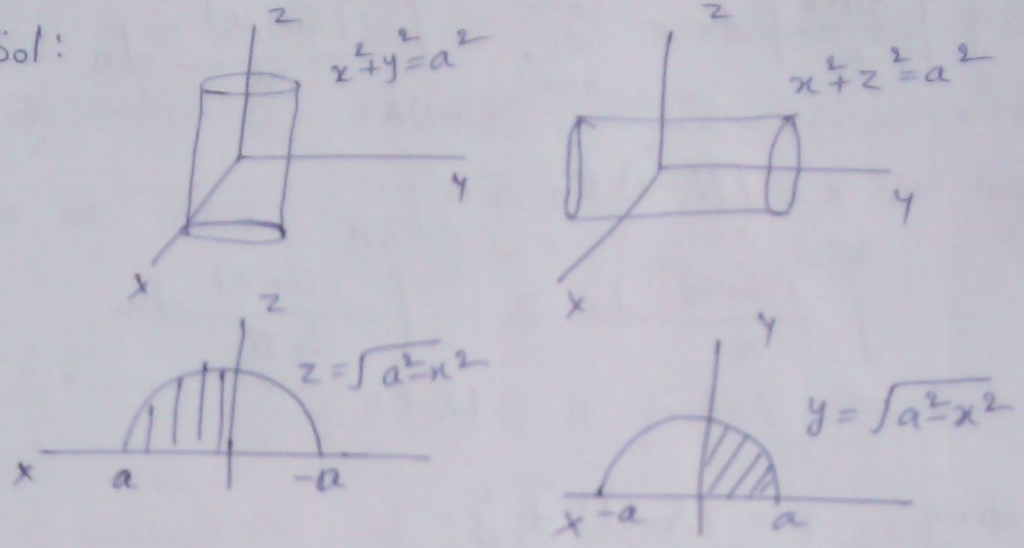
$$\therefore \int_0^{t} \frac{|\sin x|}{x}\,dx \geq \int_0^{n\pi} \frac{|\sin x|}{x}\,dx.$$

Let $t \to \infty$, so that $n \to \infty$, $\therefore \int_0^{t} \frac{|\sin x|}{x}\,dx \to \infty$

$\Rightarrow \int_0^{\infty} \frac{|\sin x|}{x}\,dx$ does not converge.

$\boxed{3.d}$ Find the volume of the region common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$. (8)

Sol:



$$V = 8 \text{ (Volume bounded in Ist Octant)}$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy\, dx = 8 \int_{0}^{a} (a^2-x^2)\, dx$$

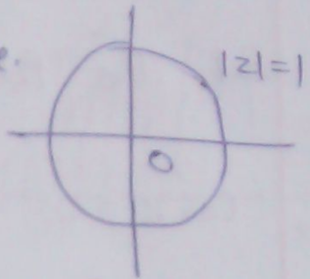
$$= 8 \left[a^2 x - \frac{x^3}{3}\right]_0^a = 8\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{16a^3}{3}.$$

4a) Prove by the method of contour integration that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0. \qquad (12)$$

Sol: Let contour be a unit circle.



$\therefore \; z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta, \quad z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$$I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$$

$$= \frac{1}{2i}\int_C \frac{1+z+\frac{1}{z}}{5+2\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{z} = \frac{1}{2i}\int_C \frac{(z^2+z+1)\,dz}{z(2z^2+5z+2)}$$

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z(z+\frac{1}{2})(z+2)}\, dz = \frac{1}{4i}\int_C f(z)\, dz$$

$z = 0, -\frac{1}{2}, -2$ are the simple poles of $f(z)$.

Out of these, only $z = 0, -\frac{1}{2}$ lie inside C.

Residue at $z=0$ is

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z(z^2+z+1)}{z(z+\frac{1}{2})(z+2)} = 1$$

Residue at $z = -\frac{1}{2}$ is $= \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) \cdot \frac{z^2+z+1}{z(z+\frac{1}{2})(z+2)} = -1$

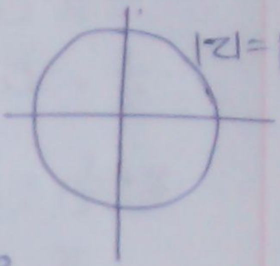
By Cauchy-Residue Theorem, $\quad I$ = Sum of residues

$= 1 + (-1) = 0.$

4.b Find the sum of the residues of $f(z) = \frac{\sin z}{\cos z}$ at its poles inside the circle $|z| = 2$. (8)

Sol: The Poles of $f(z) = \frac{\sin z}{\cos z}$ are given by

$\cos z = 0$ ie $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$



Of these poles, $z = \frac{\pi}{2}$ and $-\frac{\pi}{2}$ only are within the given circle.

$$\therefore \text{Res}\, f\left(\frac{\pi}{2}\right) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \frac{\sin z}{-\sin z} = -1$$

$$\text{Res}\, f\left(-\frac{\pi}{2}\right) = \lim_{z \to -\frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \frac{\sin z}{-\sin z} = -1$$

$\left[\because \text{If } f(z) = \frac{\phi_1(z)}{\phi_2(z)}, \text{ then } \text{Res}\, f(a) = \frac{\phi_1(a)}{\phi_2'(a)}\right]$

Hence, Sum of residues —

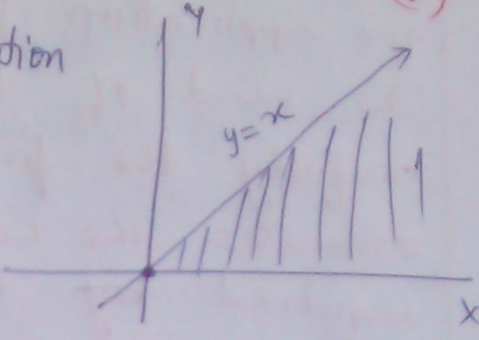
$-1 + (-1) = -2.$

4.c Evaluate $\int_{x=0}^{\infty}\int_{y=0}^{x} x\cdot e^{-x^2/y}\,dy\,dx$ (8)

Sol: Given region of integration

$x=0$ to $x\to\infty$

$y=0$ to $y=x$



Changing order of integration, we get limits as

$y=0$ to $y\to\infty$ and $x=y$ to $x\to\infty$

$$I=\int_{y=0}^{\infty}\int_{x=y}^{\infty} x\cdot e^{-x^2/y}\,dx\,dy$$

$$=\left(\frac{-1}{2}\right)\int_{y=0}^{\infty}\int_{x=y}^{\infty} -\frac{2x}{y}\cdot e^{-x^2/y}\cdot y\,dx\,dy$$

$$=\frac{-1}{2}\int_{y=0}^{\infty} y\cdot\left[e^{-x^2/y}\right]_{x=y}^{\infty} dy$$

$$=-\frac{1}{2}\int_{0}^{\infty} y\left[0-e^{-y^2/y}\right]dy=+\frac{1}{2}\int_{0}^{\infty} y\cdot e^{-y}\,dy$$

$$=\frac{1}{2}\left[y\cdot(-e^{-y})\Big]_0^{\infty}+\int_0^{\infty}1\cdot e^{-y}\,dy\right]$$

$$=+\frac{1}{2}\left(\left[\frac{-y}{e^{y}}\right]_0^{\infty}+\left[-e^{-y}\right]_0^{\infty}\right)=\frac{+1}{2}\left([0-0]+[0-(-1)]\right)$$

$$=\frac{1}{2}.$$

4.d A computer centre has four expert programmers. The centre needs for application programs to be developed. The head of the centre after studying carefully the programs to be developed, estimates the computer times in hours required by the experts to the application programs as follows :

Programs

Programmer	A	B	C	D
P_1	5	3	2	8
P_2	7	9	2	6
P_3	6	4	5	7
P_4	5	7	7	8

Assign the programs to the programmers in such a way that total computer time is least. (12)

Sol: Subtracting the minimum element of a row from the corresponding row and applying the same in columns in next step.

3	1	0	6
5	7	0	4
2	0	1	3
0	2	2	3

→

3	1	0	3
5	7	0	1
2	0	1	0
0	2	2	0

Now, ~~we cross~~ there is at least one zero in each row and each column. We cross all the 0's using minimum number of lines. We get number of lines as 3, but the number of rows is 4, so the solution is not optimal.

Now, we select the minimum of all the uncovered elements (which is 1) and subtract it from all the uncovered elements.

2	0	0	2
4	6	0	0
2	0	2	0
0	2	3	0

Now, here minimum number of lines required to cover all the zero's is 4. Hence this is an optimal solution.

2	$\boxed{0}$	~~0~~	2
4	6	$\boxed{0}$	0
2	0	2	$\boxed{0}$
$\boxed{0}$	2	3	0

By Row-scanning (encircling a zero in a row which is single zero in that row and putting a line across corresponding column) and column scanning we see that one of the optimal solution is

$P_1 \to B$, $P_2 \to C$, $P_3 \to D$, $P_4 \to A$, Total cost $= 3+2+7+5$
$= \underline{\underline{17}}$

SEC-B

5.a From the PDE by eliminating arbitrary functions ϕ and ψ from the relation
$$z = \phi(x^2-y) + \psi(x^2+y). \quad (8)$$

Sol: — ①

Differentiating ① partially w.r.t x and y

$$\frac{\partial z}{\partial x} = \phi'(x^2-y)\cdot 2x + \psi'(x^2+y)\cdot 2x \quad \text{— ②}$$

$$\frac{\partial z}{\partial y} = \phi'(x^2-y)(-1) + \psi'(x^2+y)(1) \quad \text{— ③}$$

Differentiating again w.r.t x and y

$$\frac{\partial^2 z}{\partial x^2} = 2\phi'(x^2-y) + 2x\,\phi''(x^2-y)\cdot 2x + 2\psi'(x^2+y) + 2x\cdot\psi''(x^2+y)\cdot 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2\left[\phi'(x^2-y)+\psi'(x^2+y)\right] + 4x^2\left[\phi''(x^2-y)+\psi''(x^2+y)\right] \quad \text{— ④}$$

$$\frac{\partial^2 z}{\partial y^2} = \phi''(x^2-y) + \psi''(x^2+y) \quad \text{— ⑤}$$

From ④, ④ and ②

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\cdot\frac{\partial z}{\partial x} + 4x^2\cdot\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \boxed{x\cdot\frac{\partial^2 z}{\partial x^2} - 4x^3\cdot\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} = 0} \Leftarrow \text{Required PDE.}$$

5.b Write a BASIC program to compute the multiplicative inverse of a non-singular square matrix. (12)

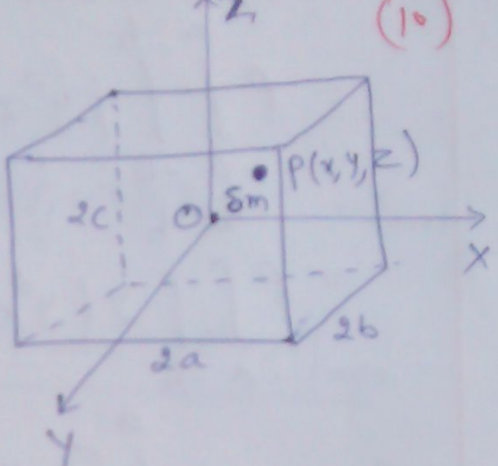
Sol: BASIC Program to compute the multiplicative inverse of a non-singular square matrix, A.

```
10 Print "Enter Dimension of matrix"
20 Input n
30 Dim A(n,n)
40 MAT INPUT A
50 MAT B = INV(A)
60 PRINT "The inverse is"
70 PRINT B
80 END.
```

5.C A uniform rectangular parallelopiped of mass M has edges of lengths $2a, 2b, 2c$. Find the moment of inertia of this rectangular parallelopiped about the line through its centre parallel to the edge of length $2a$. (10)

Sol: If M is the mass, then mass per unit volume

$$\rho = \frac{M}{V} = \frac{M}{8abc}$$



Let OX, OY, OZ be the axes through the centre and parallel to the edges of the rectangular parallelopiped. (RP)

Consider an elementary volume $\delta x \delta y \delta z$ of the parallelopiped at the point $P(x,y,z)$

Its mass, $\delta m = \rho(\delta x \delta y \delta z)$

Distance of the point $P(x, y, z)$ from OX is
$$= \sqrt{y^2+z^2}$$

$\therefore$ MI of the elementary volume of mass δm at P about OX

$$= \rho(y^2+z^2)\,\delta x \delta y \delta z$$

Hence, MI of the R.P. about OX (OX is parallel to the edge 2a)

$$= \int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} \rho(y^2+z^2)\,dx\,dy\,dz$$

$$= \rho\int_{-a}^{a}\int_{-b}^{b}\left[y^2z+\frac{z^3}{3}\right]_{z=-c}^{c} dy\,dx$$

$$= \rho\int_{-a}^{a}\int_{-b}^{b} 2\left(y^2c+\frac{c^3}{3}\right) dy\,dx$$

$$= 2\rho\int_{-a}^{a}\left[c\cdot\frac{y^3}{3}+\frac{c^3}{3}\cdot y\right]_{-b}^{b} dx$$

$$= \frac{2}{3}\rho\int_{-a}^{a} 2(b^3c+c^3b)\,dx$$

$$= \frac{4\rho}{3} bc(b^2+c^2)\left[x\right]_{-a}^{a}$$

$$= \frac{4}{3}\cdot\frac{M}{8abc}\cdot bc(b^2+c^2)\,2a \quad \left[\because \rho=\frac{M}{8abc}\right]$$

$$= \frac{M}{3}(b^2+c^2).$$

5.d Evaluate $\int_0^1 e^{-x^2}dx$ using the composite trapezoidal rule with four decimal precision, ie with the absolute value of the error not exceeding 5×10^{-5}. (10)

Sol: Trapezoidal Rule:

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right]$$

Error bound, $|E| = \dfrac{h^2(b-a)\,|f''(x)|_{max}}{12}$

Here, $f(x)=e^{-x^2}$, $f'(x) = -2x.e^{-x^2}$

$$f''(x) = -2e^{-x^2}+4x^2e^{-x^2}$$

Considering, $|E| = 10^{-3}$ (i.e. correct upto three decimal places

* $E=5\times10^{-5}$ is not feasible)

we get

$$h = 0.077$$

$$n \geq \frac{1}{h} \qquad \therefore \boxed{n \geq 13}$$

x	0	.077	.1544	.2314	.3084	.3854	.4624
$y=f(x)$	1	.9941	.9764	.9479	.9093	.8619	.8075

x	.5394	.6164	.6934	.7704	.8474	.9244	1
$y=f(x)$	.7476	.6839	.6183	.5524	.4877	.4255	.3679

$$\therefore \int_0^1 e^{-x^2}dx = \frac{h}{2}\left[(y_0+y_{13})+2(y_1+y_2+\dots+y_{12})\right]$$

$$= \frac{0.077}{2}\left[1.3679+2(9.0125)\right] = \underline{0.7466}$$

6a Solve the PDE :

$$(x-y)\frac{\partial z}{\partial x} + (x+y)\frac{\partial z}{\partial y} = 2xz \qquad (8)$$

Sol: Comparing with, $Pp + Qq = R$

$P = (x-y)$, $Q = (x+y)$, $R = 2xz$

Langrange's Auxiallary equations:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \quad \text{--- (1)}$$

Taking first two equations,

$$\frac{dx}{x-y} = \frac{dy}{x+y} \Rightarrow (x+y)dx - (x-y)dy = 0$$

$$\Rightarrow (x\,dx + y\,dy) + (y\,dx - x\,dy) = 0$$

$$\Rightarrow \frac{x\,dx + y\,dy}{x^2+y^2} + \frac{y\,dx - x\,dy}{x^2+y^2} = 0$$

$$\therefore \frac{1}{2} d(\log(x^2+y^2)) + d\left(\tan^{-1}\frac{y}{x}\right) = 0$$

$$\therefore \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = C_1 \quad \text{--- (2)}$$

Using multipliers, $1, 1, -1/z$ in (1)

$$\text{Each fraction} = \frac{dx + dy - \frac{1}{z}dz}{(x-y)+(x+y) - \frac{1}{z}2xz} = \frac{dx+dy-\frac{dz}{z}}{0}$$

$$\therefore dx + dy - \frac{1}{z}dz = 0 \Rightarrow x + y - \log z = C_2 \quad \text{--- (3)}$$

Hence, General solution from (2) and (3)

$$f\left(\frac{1}{2}\log(x^2+y^2) + \tan^{-1}y/x,\ x+y-\log z\right) = 0.$$

6.b Find the surface which is orthogonal to the family of surfaces, $z(x+y) = c(3z+1)$ and which passes through the circle, $x^2+y^2=1$, $z=1$. (8)

Sol: Orthogonal to $\frac{z(x+y)}{3z+1} = c$

Let $f = \frac{z(x+y)}{3z+1}$, $P = \frac{z}{3z+1}$, $Q = \frac{z}{3z+1}$, $R = \frac{x+y}{(3z+1)^2}$

$\left[P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}, R = \frac{\partial f}{\partial z}\right]$

Orthogonal surfaces are generated by integral curves of, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

ie $$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \quad \text{— (1)}$$

Taking first two fractions,

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} \Rightarrow dx = dy$$
$$\Rightarrow \boxed{x-y=c_1} \quad \text{— (2)}$$

Now, $$\frac{x\,dx+y\,dy}{z\cancel{(x+y)}(3z+1)} = \frac{dz}{\cancel{(x+y)}}$$

ie $$x\,dx+y\,dy = z(3z+1)\,dz$$

Integrating, $$x^2+y^2 = 2z^3+z^2+c_2$$

$$\therefore \boxed{x^2+y^2-2z^3-z^2=c_2} \quad \text{— (3)}$$

Hence, any surface which is orthogonal to given surface is given by

$$f(c_1, c_2) = f(x-y,\ x^2+y^2-2z^3-z^2)$$

For any f of one variable,

$$x^2+y^2-2z^3-z^2 = f(x-y) \text{ is solution.}$$

In particular, for surface passing through $x^2+y^2=1$, $z=1$, taking $f=-2$

$\cancel{x^2+}$ $\therefore$ Required surface is

$$x^2+y^2 = 2z^3+z^2-2.$$

$\boxed{6.c}$ Find complete integral of

$$xp - yq = xqf(z - px - qy). \qquad (12)$$

Sol: Let

$$F(x, y, z, p, q) \equiv xp - yq - xqf(z - px - qy) = 0 \quad \text{— ①}$$

Charpit's auxilary eqns are —

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-(pF_p + qF_q)} = \frac{dx}{-F_p} = \frac{dy}{-F_q} \quad \text{— ②}$$

$$\therefore \frac{dp}{p - qf + xqpf' - pqxf'} = \frac{dq}{-q + xq^2f' - xq^2f'} \quad \text{— ③}$$

$$\text{Each fraction} = \frac{x\,dp + y\,dq}{xp - yq - qxf} = \frac{x\,dp + y\,dq}{0} \quad \text{by ②}$$

$$\Rightarrow \quad x\,dp + y\,dq = 0$$

$$\therefore x\,dp + y\,dp + p\,dx + q\,dy = p\,dx + q\,dy$$

(adding $p\,dx + q\,dy$ to both sides)

$$\therefore dz - d(xp) - d(yp) = 0 \quad \text{as } dz = p\,dx + q\,dy$$

Integrating,

$$z - xp - yp = a \quad \text{— ④} \quad (a \text{ is constant})$$

$$\therefore xp + yq = z - a \quad \text{— ⑤}$$

Using ④ in ①

$$xp - yq = xq\,f(a) \quad \text{— ⑥}$$

Subtracting 6 from 5

$$2yq = z - a - xqf(a)$$

$$\Rightarrow \quad q = \frac{z-a}{2y + xf(a)} \quad — 7$$

using 7, we get p from 5 as

$$p = \frac{(z-a)(y+xf(a))}{x(2y+xf(a))} \quad — 8$$

using 7 and 8,

$$dz = p\,dx + q\,dy$$

$$= (z-a)\left[\frac{(y+xf(a))\,dx}{x(2y+xf(a))} + \frac{dy}{2y+xf(a)}\right]$$

$$\therefore \frac{2dz}{(z-a)} = \frac{2y\,dx + 2xf(a)\,dx + 2x\,dy}{x[2y+xf(a)]}$$

Integrating, we get

$$2\log(z-a) = \log[2xy + x^2 f(a)] + \log b$$

$$\therefore \boxed{(z-a)^2 = b(2xy + x^2 f(a))}$$

Required Complete Integral.

6.d A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by, $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement $y(x,t)$. (12)

Sol: Given, $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

Boundary Conditions: $y(0,t) = y(\pi,t) = 0 \quad \forall t \geq 0$

Initial Condition:

$$\text{Initial Velocity} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{for } 0 \leq x \leq \pi \qquad \text{— (1)}$$

$$\text{Initial displacement} = y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) \qquad \text{— (2)}$$

We know that, solutions of one dimensional wave, satisfying given boundary & initial conditions can be given as

$$y(x,t) = \sum_{n=1}^{\infty}\left[E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \qquad \text{— (3)}$$

where, $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

For initial velocity, differentiate (3) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{l} E_n \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l} F_n \cos\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad \text{—④}$$

Put, $t=0$ in ③ & ④ and using initial Conditions

$$③ \equiv y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{—⑤}$$

$$④ \equiv \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l} \quad \text{—⑥}$$

where $F_n = \frac{2}{n\pi c}\int_0^l (0).\sin\frac{n\pi l}{l}\,dx = 0.$

$$\therefore y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

$$\left(\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)\right)$$

$$\frac{1}{4}\left[y_0\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)\right] = E_1 \sin\frac{\pi l}{l} + E_2 \sin\frac{2\pi l}{l} + E_3 \sin\frac{3}{} + \dots$$

Comparing coefficients of similar terms,

$$E_1 = \frac{3}{4}y_0, \quad E_2 = 0, \quad E_3 = -\frac{y_0}{4}, \quad E_4 = E_5 = E_6 = \dots = E_n =$$

Putting these values in ③, the required displacement is given by

$$\boxed{y(x,t) = \frac{3}{4}y_0 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}}$$

7.(a) Find the real root of the equation $x^3+x^2+3x+4=0$ correct up to five places of decimal using Newton-Raphson method. (10)

Sol: $f(x) = x^3+x^2+3x+4$

$f(-1.2) = 0.112$ (+ve), $f(-1.3) = -0.406$ (-ve)

Root lies between -1.2 and -1.3

As $|f(-1.2)| < |f(-1.3)|$

So, we take, $x_0 = -1.2$.

Newton's iterative formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{x_n^3+x_n^2+3x_n+4}{3x_n^2+2x_n+3}\right]$$

$$x_{n+1} = \frac{2x_n^3+x_n^2-4}{3x_n^2+2x_n+3}$$

$n=0$, $x_1 = x(-1.2) = -1.222764$

$x_2 = x(-1.222764) = -1.222494$

$x_3 = x(-1.222494) = -1.222494$

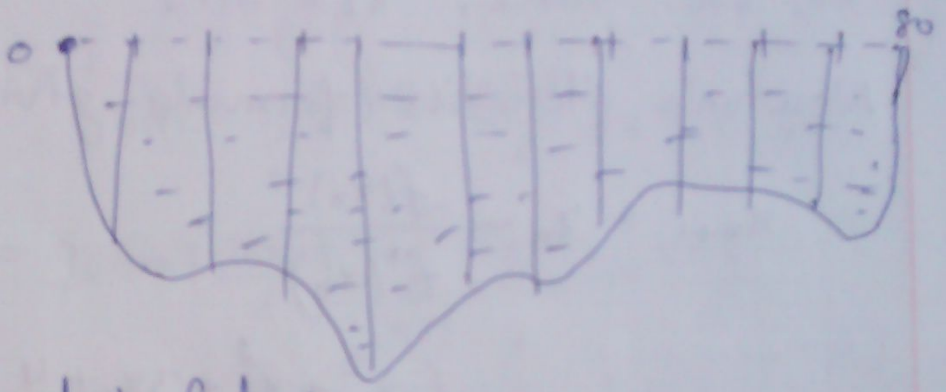
Hence, the required root correct upto five decimal places is -1.222494.

7.b A river is 80 metre wide, the depth y, in metre, of the river at a distance x from one bank is given by the table:

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

find the area of cross section of the river using simpson's $\frac{1}{3}$rd rule. (10)

Sol:



Simpson's $\frac{1}{3}$rd Rule:

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\dots+y_{n-1}) + 2(y_2+y_4+y_6+\dots+y_{n-2})\Big]$$

Here,

$$\text{Area} = \int_0^{80} (\text{depth function})\,dx$$

$$= \frac{10}{3}\Big[(0+3)+4(4+9+15+8)+2(7+12+14)\Big]$$

$$= \frac{10}{3}\left[3+144+66\right] = 710\ m^2.$$

7.c Find y for $x = 0.2$ taking $h = 0.1$ by modified Euler's method and compute the error, given that:

$$\frac{dy}{dx} = x + y, \quad y(0) = 1. \qquad (10)$$

Sol: Modified Euler's method:

Let $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, then

$$\cancel{y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2) \text{ where}}$$

$$\cancel{k_1 = hf(x_n, y_n) \; ; \; k_2 = h\, f(x_n + h, y_n + k_1)}.$$

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad \text{[Euler's Method]} \quad (1)$$

$$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right] \quad (2)$$

where, $x_n = x_0 + nh$

Here, $f(x, y) = x + y$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$

To find $y(0.2) = ?$

$x_1 = x_0 + 1 \cdot h = 0.1$, $x_2 = x_0 + 2 \cdot h = 0.2$

Step-1: $n = 0$ in (1)

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1)(0 + 1) = 1.1$$

Put $n = 0$, $m = 1$ in (2) (first modification in y_1)

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{0})\right]$$

$$= 1 + \frac{0.1}{2}\left[(0 + 1) + (0.1 + 1.1)\right]$$

$$= 1 + \frac{0.1 \times 2.2}{2} = 1 + 0.11 = 1.11$$

STEP-2: Put $n=1$ in ①

$$y_2 = y(0.2) = y_1 + h f(x_1, y_1)$$

$$= 1.11 + 0.1\,(0.1 + 1.11) = 1.231$$

To modify it put $n=1,\ m=1$ in ②

(first modification in y_2)

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{0})\right]$$

$$= \cancel{1.231}\ 1.11 + \frac{0.1}{2}\left[(0.1+1.11) + (0.2+1.231)\right]$$

$$= 1.11 + \frac{0.1}{2}\left[2.641\right] = 1.24205$$

Error in Modified Euler's Method

$$|E| = \frac{h^3 \cdot y_2'''}{12} + \frac{h^4 \cdot y_2''''}{24} = \frac{h^3}{24}\left[2y_2''' + h \cdot y_2''''\right]$$

here, $\quad y' = x + y$

$$\therefore\ y'' = 1 + y' = 1 + x + y \qquad (\text{as } y' = x+y)$$

$$y''' = 1 + y' = 1 + x + y$$

$$y'''' = 1 + y' = 1 + x + y$$

$$h = 0.1$$

$$|E| = \frac{(0.1)^3}{24}\left[2(1 + x_2 + y_2) + 0.1(1 + x_2 + y_2)\right]$$

$$= \frac{1}{24}(0.1)^3(2.1)(1 + 0.2 + 1.24205)$$

$$= \cancel{0.00}\ 2.1368 \times 10^{-4}$$

Error (Method - 2)

$$\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - 1\,y = x$$

Solving, $y \cdot e^{-x} = \int x e^{-x} dx = -e^{-x}(x+1) + c$

$y(0) = 1 \Rightarrow c = 2$

$\therefore \; y = -(x+1) + 2e^{x}$

$x = 0.2 \Rightarrow y = -1.2 + 2e^{0.2} = 1.24281$

$|Error| = 1.24281 - 1.24205$
$= 7.6 \times 10^{-4}.$

7.d Assuming a 32 bit computer representation of signed integers using 2's complement representation, add the two numbers -1 and -1024 and give the answer in 2's complement representation. (10).

Sol: To convert a signed number into binary representation, first we find the binary of the number without sign. To negate the number, we invert the bits (0 to 1 and 1 to 0) and add 1 to it.

$(1024)_{10} = (00000000\ 00000000\ 00000100\ 00000000)_2$

$(-1024)_{10} = (11111111\ 11111111\ 11111100\ 00000000)_2$

$(-1)_{10} = (11111111\ 11111111\ 11111111\ 11111111)_2$

in 2's complement. rep

2 | 1024
2 | 512 – 0
2 | 256 – 0
2 | 128 – 0
2 | 64 – 0
2 | 32 – 0
2 | 16 – 0
2 | 8 – 0
2 | 4 – 0
2 | 2 – 0
1 – 0

$(-1)_{10} + (-1024)_{10}$

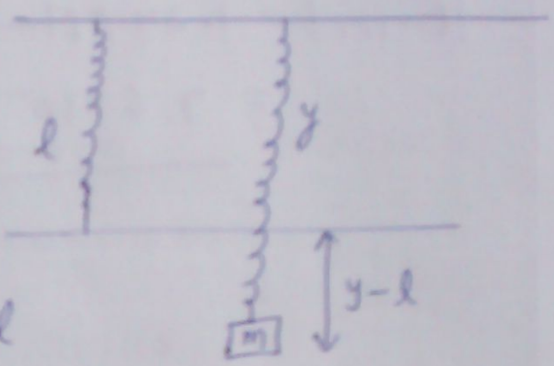
$$
\begin{array}{lllll}
 & 11111111 & 11111111 & 11111110 & 00000000 \\
+ & 11111111 & 11111111 & 11111111 & 11111111 \\
\hline
1] & 11111111 & 11111111 & 11111101 & 11111111
\end{array}
$$

8a Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assum that the mass can only move up and down in the vertical direction. Show that

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-l)^2 + mgy$$

Also determine and solve the corresponding Euler-Lagrange equation of motion. (12)

Sol: Spring Constant k acts in upward direction. and gravity, g acts in downwards.



Displacement of mass $= y - l$

Now,
$$v^2 = \dot{x}^2 + \dot{y}^2 = 0^2 + \dot{y}^2$$
$$v^2 = \dot{y}^2$$

$$\therefore \text{P.E.} = \frac{1}{2}k(y-l)^2 - mgy$$
$$\text{K.E.} = \frac{1}{2}m\dot{y}^2$$
$$\therefore L = \text{K.E} - \text{PE}$$
$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-l)^2 + mgy$$

Lagrange's Equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial}{\partial t}(m\dot{y}) - \left(-\frac{1}{2}k\cdot 2(y-l) + mg\right) = 0$$

$$m\ddot{y} + k(y-l) - mg = 0$$

$$\ddot{y} = +g - \frac{k}{m}(y-l). \quad \text{---} *$$

Solving this eqn :

Putting $y - l = r \Rightarrow \frac{dy}{dt} = \frac{dr}{dt}$

Hence, — $(*)$ becomes

$$\frac{d^2 r}{dt^2} + \left(\frac{k}{m}\right) r = g$$

Homogeneous, $\left(D^2 + \frac{k}{m}\right) r = 0$

$$m = \pm i\sqrt{\frac{k}{m}}$$

Hence, general solution is,

$$r = a\sin\left(\sqrt{\frac{k}{m}}\right)t + b\cos\left(\sqrt{\frac{k}{m}}\right)t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{k}{m}}(g) = \frac{m}{k}g$$

$$\therefore\ r = a\sin\sqrt{\frac{k}{m}}\,t + b\cos\sqrt{\frac{k}{m}}\,t$$

$$\therefore\ y = a\sin\sqrt{\frac{k}{m}}\,t + b\cos\sqrt{\frac{k}{m}}\,t + \frac{m}{k}g + l$$

8.b Find the streamlines and pathlines of the two dimensional velocity field:

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0. \qquad (8)$$

Sol: Streamlines: are the solutions of PDE

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

ie
$$\underset{(i)}{\frac{dx}{x/(1+t)}} = \underset{(ii)}{\frac{dy}{y}} = \underset{(iii)}{\frac{dz}{0}} \quad \text{--- (1)}$$

from (i) & (ii) ⇒ $\frac{(1+t)}{x}dx = \frac{dy}{y}$

⇒ $(1+t)\log x = \log y + \log c_1$ ⇒ $x^{(1+t)} = c_1 y$ --- (2)

from (ii) & (iii) ⇒ $\frac{dy}{y} = \frac{dz}{0}$ ⇒ $z = c_2$ --- (3)

These two equations (2) & (3) represents streamlines.

PATHLINES: are solutions of ODEs given by

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0$$

∴ $\frac{dx}{x} = \frac{dt}{1+t}$ ⇒ $\log x = \log(1+t) + \log k_1$

⇒ $x = k_1(1+t)$

$\frac{dy}{dy} = dt$ ⇒ $\log y = t + \log k_2$ ⇒ $y = k_2 e^{t}$

⇒ $\boxed{y = k_2 e^{(x/k_1 - 1)}}$ --- (4)

$\frac{dz}{dt} = 0$ ⇒ $\boxed{z = k_3}$ --- (5)

These two equations (4) & (5) represent pathlines where k_1, k_2, k_3 are arbitrary constants.

8.C The velocity vector in the flow field is given by

$$\vec{q} = (az - by)i + (bx - cz)j + (cy - ax)k$$

where a, b, c are non-zero constants. Determine the equations of vortex lines. (8).

Sol: Let $\vec{q} = ui + vj + wk$, then we have

$$u = az - by, \quad v = bx - cz, \quad w = cy - ax \quad \text{—} (1)$$

Let $\Omega = \Omega_x i + \Omega_y j + \Omega_z k$ be the vorticity vector.

Then $\Omega = \text{curl}\,\vec{q}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ az-by & bx-cz & cy-ax \end{vmatrix} = i(c+c) - j(-a-a) + k(b+b)$$

$$= 2ci + 2aj + 2bk$$

$\therefore \Omega_x = 2c, \quad \Omega_y = 2a, \quad \Omega_z = 2b$

The equation of the vortex lines are solutions of

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \quad \text{ie.} \quad \frac{dx}{2c} = \frac{dy}{2a} = \frac{dz}{2b} \quad \text{—} (2)$$

$$\frac{dx}{2c} = \frac{dy}{2a} \Rightarrow \boxed{ax - cy = c_1} \quad \text{—} (3)$$

$$\frac{dy}{2a} = \frac{dz}{2b} \Rightarrow \boxed{by - az = c_2} \quad \text{—} (4)$$

where c_1 and c_2 are arbitrary constants.

The required vortex lines are the straight lines of the intersection of (3) and (4).

8.d Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the conditions

$u(0,y)=u(l,y)=u(x,0)=0$ and $u(x,a)=\sin\frac{n\pi x}{l}$. (12)

Sol: Given $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ — ①

Three possible solutions of ① are

$u=(c_1e^{px}+c_2e^{-px})(c_3\cos py+c_4\sin py)$ — ②

$u=(c_5\cos px+c_6\sin px)(c_7e^{py}+c_8e^{-py})$ — ③

$u=(c_9x+c_{10})(c_{11}y+c_{12})$ — ④

Solving ① to satisfy boundary conditions,

$u(0,y)=0,\quad u(l,y)=0$ — ⑤

$u(x,0)=0,\quad u(x,a)=\sin\frac{n\pi x}{l}$ — ⑥

using ⑤ in ②, we get

$c_1+c_2=0$ & $c_1e^{pl}+c_2e^{-pl}=0$

$\therefore c_1=c_2=0$, which is trivial solution.

Similarly using ⑤ in ④ we get trivial solⁿ.

Hence, only possible solution is ③

using ⑤ in ③

$c_5(c_7e^{py}+c_8e^{-py})=0$ ie. $c_5=0$

$\therefore$ ③ becomes, $u=c_6\sin px(c_7e^{py}+c_8e^{-py})$ — ⑦

$\therefore$ Either $c_6=0$ or $\sin pl=0$

If we take, $c_6=0$, we get trivial solⁿ.

$\therefore \sin pl = 0 \Rightarrow pl = n\pi$ or $p = \frac{n\pi}{l}$, $n = 0, 1, 2 \ldots$

from ⑦,

$$u = c_6 \sin \frac{n\pi x}{l} \left(c_7 e^{\frac{n\pi y}{l}} + c_8 e^{-\frac{n\pi y}{l}} \right)$$

using ⑥, we get

$$0 = c_6 \sin \frac{n\pi y}{l} (c_7 + c_8) \quad \text{ie. } c_7 = -c_8$$

$\therefore$ Solution for problem is,

$$u(x, y) = b_n \sin \frac{n\pi x}{l} \left(e^{n\pi y/l} - e^{-n\pi y/l} \right)$$

where $b_n = c_6 c_7$

from ⑤,

$$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l} \left(e^{n\pi a/l} - e^{-n\pi a/l} \right)$$

$$\therefore b_n = \frac{1}{e^{n\pi a/l} - e^{-n\pi a/l}}$$

Hence, required equation is

$$u(x, y) = \left(\frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \right) \sin \frac{n\pi x}{l}$$

$$= \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin\left(\frac{n\pi x}{l}\right).$$