Paper : 2011 – I

Ques: 1 (c) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3+y^3}$ if it exists.

Solution:- Given $f(x,y) = \dfrac{x^2y}{x^3+y^3}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

Firstly; $y=0$; and x approaching to 0, from x-axis

$$\lim_{x\to 0} \frac{x^2.0}{x^3+0} = 0$$

$\Rightarrow x=0$, $y\to 0$, from y axis.

$$\lim_{y\to 0} \frac{y.0}{y^3+0} = 0$$

Let, $y=mx$, and $(x,y)\to(0,0)$, through this line

$$\lim_{x\to 0} \frac{x^2.mx}{x^3+m^3x^3} = \lim_{x\to 0} \frac{\cancel{x^3}.m}{\cancel{x^3}(1+m^3)} = \lim_{x\to 0} \frac{m}{1+m^3}$$

Since; the value of limit depends on 'm' (i.e path) thus it is independent of x,y and hence limit does not exist

Ques: 1) d) 'f' be a function defined on $\mathbb{R}$ such that $f(0) = -3$ and $f'(x) \leq 5$ for all the values of 'x' in $\mathbb{R}$. How large can $f(2)$ possibly be?

Solution:-

Since; $f'(x) \leq 5 \quad \forall\, x \in \mathbb{R}$

thus $f(x)$ will be a linear polynomial

let; $f(x) = ax + b$ ——— (1)

$f(0) = a \times 0 + b = -3$ $\quad [\because f(0) = -3]$

$\Rightarrow \boxed{b = -3}$

$\therefore f(x) = ax - 3$

Differentiate w.r.t x.

$f'(x) = a \leq 5 \quad \forall\, x \in \mathbb{R}$

$\therefore$ In eqn (1)

$\boxed{f(2) = 2a - 3}$ ——— (2)

and value of $a \leq 5$

The largest possible value of $f(2)$ is attained only when 'a' is maximum, thus put $a = 5$ in eqn (2), we get

$f(2) = 2 \times 5 - 3 = 7$

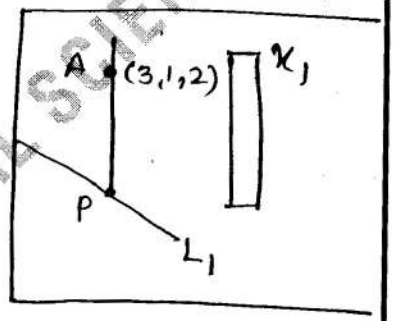
$\boxed{\therefore \text{ largest value of } f(2) \text{ be } 7,}$ required Solution.

Ques: 1) e) Find the equations of the straight line through the point $(3,1,2)$ to intersect the straight line $x+4 = y+1 = 2(z-2)$ and parallel to the plane $4x+y+5z=0$.

Solution:-

Given, point $(3,1,2)$ to intersect the Straight line $x+4 = y+1 = 2(z-2)$.

$\therefore L_1 \Rightarrow \frac{x+4}{2} = \frac{y+1}{2} = \frac{z-2}{1}$



and the general point on L_1,

let P be the point

$$\frac{x+4}{2} = \frac{y+1}{2} = \frac{z-2}{1} = r$$

$$P\left(2r-4,\ 2r-1,\ r+2\right)$$

Let, straight line through $(3,1,2)$ meet the given line 'p', therefore Direction ratio's of line AP to be determined by.

$AP\,(2r-4-3,\ 2r-1-1,\ r+2-2] \Rightarrow AP(2r-7, 2r-2, r)$

The line is parallel to the plane $4x+y+5z=0$

$\therefore$ Dr's of normal to the plane is $\perp^r$ to Dr's of determined line thus

$$(2r-7)4 + 2r-2+5r = 0 \Rightarrow 15r = 30 \Rightarrow \boxed{r=2}$$

$\therefore$ Dr's of $AP = (-3, 2, 2)$

And Point $P = (0,3,4)$

Therefore; straight line AP, which intersect $(3,1,2)$.

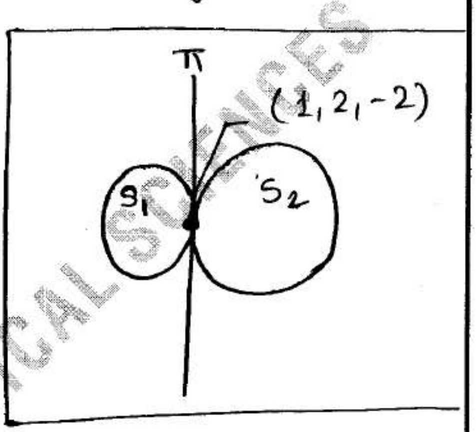
$$\left[\ \frac{x-3}{-3} = \frac{y-1}{2} = \frac{z-2}{2}\ \right]$$

Ques:1(f) Show that the equation of the sphere which touches the sphere

$$4(x^2+y^2+z^2)+10x-25y-2z=0$$

at point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is $x^2+y^2+z^2+2x-6y+1=0$?

Solution:-



Given sphere S_1:

$$4(x^2+y^2+z^2)+10x-25y-2z=0$$

$$\Rightarrow x^2+y^2+z^2+\frac{5}{2}x-\frac{25}{4}y-\frac{z}{2}=0$$

Centre $C_1=\left(-\frac{5}{4},+\frac{25}{8},\frac{1}{4}\right)$

To evaluate sphere(s) that passes through $(-1,0,0)$ and touches S_1 at $(1,2,-2)$, this means that line through C_1 and $(1,2,-2)$ passes through centre of sphere 'S', line formed by joining C_1 and $P(1,2,-2)$ is

$$\frac{x-1}{9/4}=\frac{y-2}{-9/8}=\frac{z+2}{-9/4}$$

$$\Rightarrow \frac{x-1}{2}=\frac{y-2}{-1}=\frac{z+2}{-2}=\lambda$$

$$\Rightarrow x=2\lambda+1,\quad y=-\lambda+2,\quad z=-2\lambda-2$$

So any point on this line i.e.

$(2\lambda+1,-\lambda+2,-(2\lambda+2)]$ be the centre of sphere S.

So the equation of sphere be -

$$x^2+y^2+z^2+2(-2\lambda-1)x+2(\lambda-2)y+2(2\lambda+2)z+d=0 \quad \text{--- (1)}$$

as $(1,0,0)$ passes through it

i.e $1+0+0+2(-2\lambda-1)\cdot 1+0+0+d=0$

$\Rightarrow 1+2(2\lambda+1)+d=0$

$\Rightarrow \boxed{d=-4\lambda-3}$

Now put $(3,2,-2)$ in eqn (1)

$9+2(-2\lambda-1)+4(\lambda-2)-8(\lambda+1)-4\lambda-3=0$

$9-\cancel{4\lambda}-2+\cancel{4\lambda}-8-8\lambda-8-4\lambda-3=0$

$12\lambda=9-2-8-8-3$

$12\lambda=-12$

$\boxed{\lambda=-1}$

Put $\lambda=-1$ in eq (1), we will get.

$$x^2+y^2+z^2+2(-2\times-1-1)x+2(-1-2)y+2(2\times-1+2)z+(-4\times-1)-3=0$$

$$x^2+y^2+z^2+2(1)x-6y+2(2-2)z+4-3=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2+2x-6y+1=0}$$

Which is required solution

Ques: 3(a)) Evaluate

(i) $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & , x \neq 2 \\ \pi & , x = 2 \end{cases}$

Solution:-

given; $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & ; x \neq 2 \\ \pi & ; x = 2 \end{cases}$

To check whether limit exist or not

$\boxed{f(2+h) = f(2-h) = f(2)}$ - if exist. not

$\therefore$ R.H.L $= \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \dfrac{(2+h)^2-4}{(2+h)-2}$

$= \lim_{h\to 0} \dfrac{\cancel{4}+4h+h^2-\cancel{4}}{\cancel{2}+h-\cancel{2}}$

$= \lim_{h\to 0} \dfrac{\cancel{h}(4+h)}{\cancel{h}} = 4.$

L.H.L $= \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \dfrac{(2-h)^2-4}{(2-h)-2}$

$= \lim_{h\to 0} \dfrac{\cancel{4}-4h+h^2-\cancel{4}}{\cancel{2}-h-\cancel{2}}$

$= \lim_{h\to 0} \dfrac{\cancel{h}(-4+h)}{-\cancel{h}} = +4.$

since; $f(2) = \pi$.

$\boxed{\therefore f(2+h) = f(2-h) = 4 \neq f(2)}$ Thus, limit does exist, but it is not continuous.

Ques: 3(a) Evaluate:

(ii) $\int_0^1 \ln x \, dx$

Solution: $I = \int_0^1 \ln x \, dx$

Integration by parts.

$$I = \int_0^1 \ln x \, dx = \left[\ln x \cdot \int 1 \, dx\right]_0^1 - \int_0^1 \frac{d}{dx} \ln x \cdot \int 1 \, dx$$

$$= [x \cdot \ln x]_0^1 - \int_0^1 \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$$

$$= [x \ln x]_0^1 - \int_0^1 dx$$

$$= [\ln 1 - 0] - [x]_0^1$$

$$= 0 - [1 - 0] = -1.$$

$$\boxed{\therefore I = \int_0^1 \ln x \, dx = -1}$$

which is required solution.

Ques: 3(b)) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.

Solution:-

given sphere $\Rightarrow g(x,y) = x^2+y^2+z^2-4=0$ —— (1)

We need to find the closest and farthest from the point $(3,1,-1)$.

Let any point on sphere (x,y,z) to determine using Lagrangian multiplier

$$d^2 = (x-3)^2+(y-1)^2+(z+2)^2 \quad \text{—— (2)}$$

Let; $f(x,y,z) = d^2 + \lambda g(x,y)$

$$f(x,y,z) = (x-3)^2+(y-1)^2+(z+2)^2+\lambda(x^2+y^2+z^2-4) \quad \text{—— (3)}$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2(x-3)+\lambda(2x)=0 \Rightarrow x = \frac{3}{1+\lambda} \\ \frac{\partial F}{\partial y} &= 2(y-1)+\lambda(2y)=0 \Rightarrow y = \frac{1}{1+\lambda} \\ \frac{\partial F}{\partial z} &= 2(z+1)+\lambda(2z)=0 \Rightarrow z = \frac{-1}{1+\lambda} \end{aligned}\right\} \text{—— (4)}$$

Put the values of x, y, z from (4) in eq (1)

$$\left(\frac{3}{1+\lambda}\right)^2+\left(\frac{1}{1+\lambda}\right)^2+\left(\frac{-1}{1+\lambda}\right)^2 = 4$$

$$3^2+1^2+1^2 = 4(1+\lambda)^2$$

$$(1+\lambda)^2 = \frac{11}{4}$$

$$\Rightarrow \boxed{1+\lambda = \pm\frac{\sqrt{11}}{2}}.$$

$\therefore$ The stationary points are

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right) \text{ and } \left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

from 4

When $\lambda+1 = \sqrt{11}/2$

$$x = \frac{6}{\sqrt{11}}, \quad y = \frac{2}{\sqrt{11}}, \quad z = \frac{-2}{\sqrt{11}}$$

$$(2) \equiv d^2 = 1.7325 \Rightarrow \boxed{d_{min} = 1.316}$$

When $\lambda+1 = -\sqrt{11}/2$

$$(2) \equiv d^2 = 28.266 \Rightarrow \boxed{d_{max} = 5.316}$$

again partially differentiating (4) , w.r.t x, y, z respectively

$$\boxed{F_{xx} = 1+\lambda = f_{yy} = F_{zz}}$$

Similarly :- $f_{yx} = 0 = f_{zx}$

$$f_{xz} = f_{xy} = 0$$

$$f_{yz} = f_{zy} = 0$$

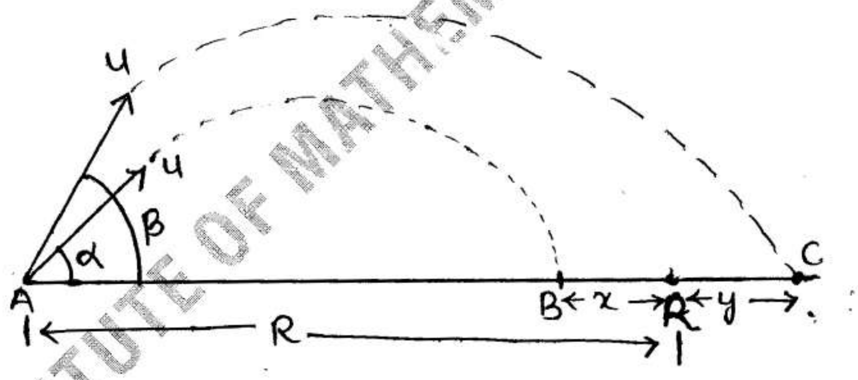
$$|F_{xx}|, \begin{vmatrix} F_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix}, \begin{vmatrix} f_{xx} & F_{xy} & F_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & F_{zz} \end{vmatrix}$$ retain same +ve sign; thus it is a closest point.

$$\boxed{\begin{aligned} &\therefore \text{Closest point} = d = 1.316 = \left[\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right] \\ &\text{farthest point} = d = 5.316 = \left[\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right] \end{aligned}}$$

which is required solution.

Ques:-5(d)> A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls 'x' meter short of it when the angle of projection is α and goes y meter beyond when the angle of projection is 'β'. If the velocity of projection is assumed same in all cases, find the correct angle of projection.

Solution:-



Let, correct angle of projection be θ

such that, $\boxed{\alpha < \theta < \beta}$

Correct Range $= R$ at point R.

When projected at angle α,

$$\text{Range} = AB = R - x.$$

When projected at angle β.

$$\text{Range} = AC = R + y$$

and 'u' is the initial velocity of projection which is same for all angles.

$\therefore$ Range at α projected at angle α;

$$R - x = \frac{u^2 \sin 2\alpha}{g} \quad \text{--- (1)}$$

Range when projected at angle β:

$$R + y = \frac{\mu^2 \sin 2\beta}{g} \quad \text{--- (2)}$$

Correct Range when projected at angle θ

$$R = \frac{\mu^2 \sin 2\theta}{g}$$

Put value of R in (1) & (2), we get

$$\frac{\mu^2 \sin 2\theta}{g} - x = \frac{\mu^2 \sin 2\alpha}{g}$$

$$\boxed{x = \frac{\mu^2}{g}(\sin 2\theta - \sin 2\alpha)} \quad \text{--- (A)}$$

$$\frac{\mu^2 \sin 2\theta}{g} + y = \frac{\mu^2 \sin 2\beta}{g}.$$

$$\boxed{y = \frac{\mu^2}{g}(\sin 2\beta - \sin 2\theta)} \quad \text{--- (B)}$$

$\Rightarrow$ A $\div$ B

$$\frac{x}{y} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$$

$$\Rightarrow x \sin 2\beta - x \sin 2\theta = y \sin 2\theta - y \sin 2\alpha$$

$$\Rightarrow x \sin 2\beta + y \sin 2\alpha = (y + x) \sin 2\theta$$

$$\sin 2\theta = \frac{x\sin 2\beta + y\sin 2\alpha}{x+y}$$

$$2\theta = \sin^{-1}\left[\frac{x\sin 2\beta + y\sin 2\alpha}{x+y}\right]$$

$$\boxed{\theta = \frac{1}{2}\sin^{-1}\left[\frac{x\sin 2\beta + y\sin 2\alpha}{x+y}\right]}$$

is required angle of projection.

2011 = Paper - II

Ques:- 2(d)) Find the shortest distance from the origin (0,0) to hyperbola.

$$x^2+8xy+7y^2=225$$

Solution)

Let (x,y) be the point on hyperbola which is at shortest distance from the origin (0,0)

So, evaluating minimum value of x^2+y^2 using Lagrange's Multiplier.

$$f(x,y) = x^2+y^2+\lambda(x^2+8xy+7y^2-225)$$

$$\frac{\partial f}{\partial x} = 2x+2x\lambda+8y\lambda = 0$$

$$\Rightarrow x+\lambda(x+4y) = 0 \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial y} = 2y+14y\lambda+8x\lambda = 0$$

$$y+\lambda(7y+4x) = 0 \quad \text{—— (2)}$$

Multiplying (1) with x and (2) with y and adding both equation.

$$x^2+\lambda(x^2+4xy)+y^2+\lambda(7y^2+4xy)=0$$

$$x^2+y^2+\lambda(x^2+7y^2+8xy)=0$$

Let $x^2+y^2 = u$

$\therefore \lambda = \dfrac{-u}{225}$ $\quad [\because x^2+7y^2+8xy=225]$

from ① and ②

$$\frac{-x}{x+4y} = \frac{-y}{4x+7y} = \lambda = \frac{-u}{225}$$

$$\frac{x+4y}{x} = \frac{225}{u} \Rightarrow \frac{225}{u} - 1 = \frac{4y}{x}$$

$$\frac{4x+7y}{y} = \frac{225}{u} \Rightarrow \frac{225}{u} - 7 = \frac{4x}{y}$$

Multiplying both

$$\frac{(225-u)}{u} \times \frac{(225-7u)}{u} = \frac{4\cancel{y}}{\cancel{x}} \times \frac{4\cancel{x}}{\cancel{y}}$$

$$\Rightarrow (225-u)(225-7u) = 16u^2$$

$$\Rightarrow 9u^2 + 8\times 225u - (225)^2 = 0$$

$$(9u-225)(u+225) = 0$$

$$\boxed{u = \frac{225}{9}, -225}$$

u cannot be negative, as it is sum of squares i.e $u = x^2+y^2$.

$\therefore u = -225$ neglected.

so $\boxed{u = 25}$ i.e $x^2+y^2 = 25$

and hence $d = \sqrt{x^2+y^2} = \sqrt{25} = 5$

$\boxed{\text{Minimum Distance} = 5 \text{ units}}$ required solution

Ques: 3(b) Show that the series for which the sum of first n terms

$$f_n(x) = \frac{nx}{1+n^2x^2} \; ; \; 0 \leq x \leq 1$$

cannot be differentiated term by term at $x=0$. What happens at $x \neq 0$.

Solution:-

given ; $f_n(x) = \frac{nx}{1+n^2x^2}$; $0 \leq x \leq 1$

for $f_n(x)$ to be term by term differentiable $\lim\limits_{n\to\infty} f_n'(x)$ must be equal to $f'(x)$, where $f(x)$ is a limit function.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2}$$

$$= \lim_{n\to\infty} \frac{x}{2nx^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$f(x) = 0 \quad \Rightarrow \quad f'(x) = 0 \quad \forall x \in \mathbb{R}$$

Differentiating f_n with respect to x:-

$$f_n'(x) = \frac{(1+n^2x^2)n - nx(2n^2x)}{(1+n^2x^2)^2}$$

$$= \frac{n + n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2}$$

$$\boxed{f_n'(x) = \frac{n - n^3x^2}{(1+n^2x^2)^2}}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & ; x \neq 0 \\ \infty & ; x = 0 \end{cases}$$

Thus, $f_n(x)$ is not differentiable term by term at $x=0$ and differentiable term by term at $\forall\ x \neq 0$.

Ques: 4(b)) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$,

then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \text{; for all } x.$$

Solution:-

given function; $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$

∴ By Wiesstress M-test ΣS_n is uniformly convergent, if there exist ΣM_n convergent series of positive sequence M_n such that $|S_n| \le M_n$.

$$S_n(x) = \frac{1}{n^3+n^4x^2}$$

thus; $|S_n| = \left|\frac{1}{n^3+n^4x^2}\right|$

$$|S_n(x)| = \left|\frac{1}{n^3+n^4x^2}\right| \le \frac{1}{n^3} = M_n$$

$[\because n^4x^2 \geq 0] \ \forall \ x$

$\Sigma \frac{1}{n^3}$ is convergent series by p-test

$\left[\because \frac{1}{n^p} \ ; \ p > 1 - \text{convergent}\right]$

$\therefore \ \Sigma \frac{1}{n^3+n^4x^2}$ is uniformly convergent and can be differentiated term by term for all x, thus.

$$S'(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n^3+n^4x^2}\right)' = \sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left[\frac{1}{1+nx^2}\right]'$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left[\frac{(1+nx^2)(0.) - 2xn(1)}{(1+nx^2)^2}\right]$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{\cancel{3}2}}\cdot\frac{-2x\cancel{n}}{(1+nx^2)^2}$$

$$\boxed{S'(x) = -2x\sum_{n=1}^{\infty}\frac{1}{(1+nx^2)^2}}$$

Which is required solution

Ques:- 5 (d)

(i) Compute $(3205)_{10}$ to the base 8.

Solution:- To find $(3205)_{10} \longleftrightarrow (\_\_\_)_8$

8	3205	
8	400	5
8	50	0
	6	2

$\Rightarrow (6205)_8$

$\therefore \boxed{(3205)_{10} \longleftrightarrow (6205)_8}$

(ii) Let A be an arbitrary but fixed Boolean algebra with operations $\wedge, \vee$ and $'$ and the zero and the unit element denoted by 0 and 1 respectively. Let $x, y, z \ldots$ be elements of A.
If $x, y \in A$ be such that $x \wedge y = 0$ and $x \vee y = 1$ then prove that $y = x'$.

Solution:-

given that $x, y \in A$

and $x \wedge y = 0$ & $x \vee y = 1$ —— (1)

$(x \wedge y)' = 1$ $\quad \left[\because \begin{array}{l} x \neq x' \\ y \neq y' \end{array}\right]$

$x' \vee y' = 1$ → [By De Morgan law]

$x' \vee y' = x \vee y$ —— from (1)

Since; $x' \neq x$ & $y' \neq y$.

$\therefore \boxed{x' = y \text{ and } y' = x}$

Hence proved

Ques:- 7(b) Find the logic circuit the represents the following Boolean function. Find also an equivalent simpler circuit.

x	y	z	$f(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

Solution:-

x	y	z	$f(x,y,z)=m_i$		Minterm
1	1	1	1	m_1	xyz
1	1	0	0	m_2	$xy\bar{z}$
1	0	1	0	m_3	$x\bar{y}z$
1	0	0	0	m_4	$x\bar{y}\bar{z}$
0	1	1	1	m_5	$\bar{x}yz$
0	1	0	0	m_6	$\bar{x}y\bar{z}$
0	0	1	0	m_7	$\bar{x}\bar{y}z$
0	0	0	0	m_8	$\bar{x}\bar{y}\bar{z}$

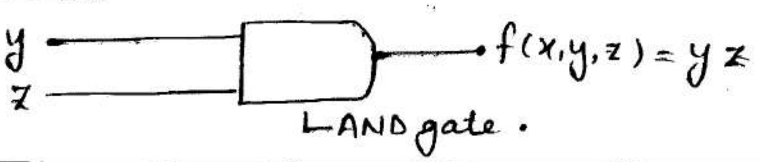
To get the Boolean function, let us add minterms corresponding to output '1'.

$$\therefore \ f(x,y,z) = m_1 + m_5 = xyz + \bar{x}yz$$

$$f(x,y,z) = (x+\bar{x})(yz) = 1\cdot yz = yz \quad [\because A+\bar{A}=1]$$

$$\therefore \ \boxed{f(x,y,z) = yz}$$

Simpler circuit is:-



Ques:- 7 (c)) Draw a flow chart for Lagrange's interpolation formula.

Solution:-