2018

Date:
Page:

1) a) let R be an integral domain with unit element. show that any unit in $R[x]$ is a unit in R. (10)

proof:- Given R is an integral domain.
let $p(x) \in R[x]$ is arbitrary element.
let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is unit in $R[x]$.
we have to show that $p(x)$ is unit in R.
$\because$ R is I.D. with unit element
$\Rightarrow$ if $p \in R$ is unit
$\therefore \exists\ p^{-1} \in R \ni p \cdot p^{-1} = 1 \in R$
$\therefore$ R is I.D. with unity 1
we know that
If R has unity 1 then $R[x]$ is also has unity 1.
$\therefore$ if $p(x)$ is unit $\exists\ q(x) \in R[x]$
$\ni p(x) \cdot q(x) = 1$ ——— (1)
{ because $1 = 1 + 0x + 0x^2 + \ldots$ }

let $q(x) = b_0 + b_1 x + b_2 x^2 + \ldots$
$\therefore p(x) \cdot q(x) = a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + \ldots$
but two polynomials are equal iff there coefficient are equal.
hence by (1) we have
$a_0 \cdot b_0 = 1$
$a_1 = a_2 = \ldots 0 = b_1 = b_2 = \ldots$
$\therefore$ only $a_0 \neq 0,\ b_0 \neq 0$
$\therefore$ a_0 is unit in $R[x]$

$\because\ a_1 = a_2 \cdots = 0$

$p(x) = a_0$

$\therefore$ unit of $R[x]$ is also unit of R.

2] a] Show that the quotient group of $(\mathbb{R}, +)$ modulo $\mathbb{Z}$ is isomorphic to multiplicative group of complex no. on the unit circle in the complex plane. there $\mathbb{R}$ is set of Real no. and $\mathbb{Z}$ is the set of integers. [15]

Proof:- $f : \mathbb{R}/\mathbb{Z} \to S^1$

where $S^1 = \{z \mid z \in \mathbb{C}, |z| = 1\}$

$$f(x + \mathbb{Z}) = e^{2\pi i x}$$

to show that f is well defined and 1-1, we have.

$x + \mathbb{Z} = y + \mathbb{Z}$

$\Leftrightarrow x - y = n, n \in \mathbb{Z}$

$\Leftrightarrow e^{i2\pi(x-y)} = e^{i2\pi n} = 1$

$\Leftrightarrow e^{i2\pi x} \cdot e^{i2\pi(-y)} = 1$

$\Leftrightarrow e^{i2\pi x} = e^{i2\pi y}$

To show f is onto

$\forall\ e^{i\theta} \in S^1, \exists\ x = \frac{\theta}{2\pi} \in \mathbb{R}$

$\ni f(x+\mathbb{Z}) = f\left(\frac{\theta}{2\pi}+\mathbb{Z}\right)$

$= e^{i2\pi\frac{\theta}{2\pi}}$

$= e^{i\theta}$

For homomorphism,

$f(x+\mathbb{Z} + y+\mathbb{Z})$

$= f(x+y+\mathbb{Z})$

$= e^{i2\pi(x+y)}$

$= e^{2\pi i x} \cdot e^{2\pi i y}$

$= f(x+\mathbb{Z}) \cdot \cancel{f(x+y)}\, f(y+\mathbb{Z})$

we prove f is one-one, onto and homomorph
then f is isomorphism.

2nd way:-

$f : \mathbb{R} \to S^1$

$f(x) = e^{2\pi i x}$

$\forall\ e^{i\theta} \in S^1\ \exists\ x = \frac{\theta}{2\pi} \in \mathbb{R}$

$\therefore f(x) = e^{2\pi i \frac{\theta}{2\pi}} = e^{i\theta}$

$\therefore$ f is onto.

$f(x+y) = e^{2\pi i(x+y)}$

$= e^{2\pi i x} \cdot e^{2\pi i y}$

$= f(x) \cdot f(y)$

$\therefore$ f is homomorphism.

by I^{st} theorem on Isomorphism.

Scanned with CamScanner

$$\frac{\mathbb{R}}{\ker f} \cong S^1$$

what is $\ker f$?

$\ker f = \{ x \in \mathbb{R} \ni f(x) = 1 \}$

$\therefore e^{2\pi i x} = 1$

iff $x \in \mathbb{Z}$

$\therefore \ker f = \mathbb{Z}$

$$\therefore \frac{\mathbb{R}}{\mathbb{Z}} \cong S^1$$

2] c] Find all proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 respectively.

(~~1~~5 marks)

Soln:- we know that $\mathbb{Z}_p$ is a cyclic Group iff p is prime.

$\therefore \mathbb{Z}_{13}^{\times} = \{1, 2, \text{———} \, 12\}$ is a cyclic group.

now $\because$ 2 is a generator of $\mathbb{Z}_{13}$

because $(2, 13) = 1$

$2^1 = 2$	$2^4 = 3$	$2^7 = 11$	$2^{10} = 10$
$2^2 = 4$	$2^5 = 6$	$2^8 = 9$	$2^{11} = 7$
$2^3 = 8$	$2^6 = 12$	$2^9 = 5$	$2^{12} = 1$

$\therefore \mathbb{Z}_{13}^{\times} = \langle 2 \rangle$ is a cyclic group of order 12, with generator 2.

by lagrange's theorem we know that If H is a subgroup of order m of a group G of order n then m/n.

in case of cyclic group converse of lagrange's theorem is applicable.

that means for each divisior of n $\exists$ a unique subgroup (cyclic) of order $\left(\frac{n}{m}\right)$.

$\because$ division of 12 are, 1, 2, 3, 4, 6, 12

$\therefore$ subgroups of $\mathbb{Z}_{13}$ are

$\langle 1 \rangle = \langle 2^{12} \rangle$, $\langle 2^6 \rangle = \langle 12 \rangle$

$\langle 2^4 \rangle = \langle 3 \rangle$,

$\langle 2^3 \rangle = \langle 8 \rangle$

$\langle 2^2 \rangle = \langle 4 \rangle$

$\langle 2^1 \rangle = \langle 2 \rangle = \mathbb{Z}_{13}$

$\therefore$ proper subgroup are $\langle 12 \rangle, \langle 3 \rangle, \langle 8 \rangle, \langle 4 \rangle$.