ORDINARY DIFFERENTIAL EQUATIONS ANALYSIS
CLASS TEST (23 June-2020)
ANSWER KEY

Q.1

(i) solve $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0$ — (1)

(ii) solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

Sol (i) Let $x^2 = X$, $y^2 = Y$ so that $2x\,dx = dX$, $2y\,dy = dY$

$\therefore$ from (1) $\dfrac{dY}{dX} = \dfrac{2X+3Y-7}{3X+2Y-8}$

Now putting $X = u+h$, $Y = v+k$

$dX = du$, $dY = dv$

so that $\dfrac{dv}{du} = \dfrac{2u+3v+(2h+3k-7)}{3u+2v+(3h+2k-8)}$ — (2)

Choose h, k so that $2h+3k-7=0$ & $3h+2k-8=0$ — (3)

solving (3), we get $h=2$, $k=1$

$X = u+2$, $Y = v+1$

$\Rightarrow u = X-2$, $v = Y-1$

Equation (2) becomes

$\dfrac{dv}{du} = \dfrac{2u+3v}{3u+2v} = \dfrac{2+3(v/u)}{3+2(v/u)}$ — (4)

Taking $\dfrac{v}{u} = w \Rightarrow v = uw$

$\Rightarrow \dfrac{dv}{du} = w + u\dfrac{dw}{du}$ — (5)

from (4) & (5)

$w + u\dfrac{dw}{du} = \dfrac{2+3w}{3+2w}$

$\Rightarrow u\dfrac{dw}{du} = \dfrac{2+3w}{3+2w} - w$

$\Rightarrow u\dfrac{dw}{du} = \dfrac{2-2w^2}{3+2w}$

$\Rightarrow \dfrac{3+2w}{2(1-w^2)}\,dw = \dfrac{du}{u}$

$\Rightarrow \dfrac{du}{u} = \dfrac{1}{2}\left[\dfrac{1}{2}\dfrac{dw}{1+w} + \dfrac{5}{2}\dfrac{dw}{1-w}\right]$

$\Rightarrow \log u = \dfrac{1}{4}\log(1+w) - \dfrac{5}{4}\log(1-w)$

$\Rightarrow 4\log u + \log C = \log(1+w)(1-w)^{-5}$

$\Rightarrow Cu^4 = (1+w)(1-w)^{-5}$

$$\Rightarrow C = \frac{1+w}{(1-w)^5 u^4}$$

$$\Rightarrow C = \frac{1+v/u}{(1-v/u)^5 u^4} = \frac{u+v}{(u-v)^5}$$

$$\Rightarrow C = \frac{x-2+y-1}{(x-2-y+1)^5} = \frac{x^2+y^2-3}{(x^2-y^2-1)^5}$$

$$\Rightarrow (x^2+y^2-3) = C(x^2-y^2-1)^5$$

which is the required solution

(ii) Here $M = 2xy^4e^y + 2xy^3 + y$; $N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 1 \; ; \; \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$\therefore$ the given eqn is not exact.

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4$

and $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-4}{y}$

which is a function of y alone.

$$I.F = e^{-\int \frac{4}{y} dy} = e^{-4\log y} = \frac{1}{y^4}.$$

Multiplying (1) by $\frac{1}{y^4}$ we have

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left[x^2e^y - \frac{x^2}{y^2} - 3\left(\frac{x}{y^4}\right)\right]dy = 0$$

which must be exact and its solution is given by

$$\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$$

If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find

$$L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}.$$

Solution:- Given that; $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t = F(t)$ (say)

Since; $f(ap) = \frac{1}{a}F(t/a)$.

$$\therefore L^{-1}\left\{\frac{ap}{[(ap)^2+1]^2}\right\} = \frac{1}{2}\cdot\left(\frac{1}{a}\right)\frac{t}{a}\sin\frac{t}{a}$$

putting $a = 4$

$$\Rightarrow L^{-1}\left\{\frac{32p}{((ap)^2+1)^2}\right\} = L^{-1}\left\{\frac{8\cdot 4p}{[(4p)^2+1]^2}\right\}.$$

$$= 8L^{-1}\left\{\frac{4p}{((4p)^2+1)^2}\right\}.$$

$$= 8\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{4}}\cdot\frac{t}{4}\cdot\sin\frac{t}{4}$$

$$= \frac{t}{4}\cdot\sin\frac{t}{4}.$$

$$\therefore \boxed{L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}}$$

required solution

2. Use the method of variation of parameters to solve $y'' + y = 1/(1+\sin x)$.

Solution:- Given; $y'' + y = \frac{1}{1+\sin x}$ —— (1)

Comparing (1), with $y'' + Py' + Qy = R$

hence; Here; $P = 0$, $Q = 1$, $R = \frac{1}{1+\sin x}$ —— (2)

Consider: $y'' + y = 0$ or $(D^2+1)y = 0$

$D^2 + 1 = 0 \Rightarrow D = \pm i$

$\boxed{\therefore C.F = c_1 \cos x + c_2 \sin x}$ where; c_1 & c_2 being arbitrary constants.

Let; $u = \cos x$ and $v = \sin x$

Here; $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$\boxed{W = \cos^2 x + \sin^2 x = 1 \neq 0}$

Then;

P.I. of (1) $= u f(x) + v g(x)$, where —— (3)

$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin x}{1+\sin x} dx$

$f(x) = -\int \frac{\sin x(1-\sin x)}{1^2 - \sin^2 x} dx = -\int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$

$f(x) = -\int (\sec x \tan x - \tan^2 x) dx$

$f(x) = -\int \{\sec x \tan x - (\sec^2 x - 1)\} dx$

$f(x) = -(\sec x - \tan x + x)$

$\boxed{f(x) = \tan x - \sec x - x}$ —— (A)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{\cos x}{1+\sin x} dx$

$\boxed{g(x) = \log(1+\sin x)}$ —— (B)

Using (A) & (B), in (3), we get.

$P.I = -\cos x(\sec x - \tan x + x) + \sin x \log(1+\sin x)$

$\boxed{\therefore P.I = -1 + \sin x - x\cos x + \sin x \log(1+\sin x)}$

$\therefore$ Required solution $y = C.F + P.I$

$$\boxed{y = c_1 \cos x + c_2 \sin x - 1 + \sin x - x\cos x + \sin x \log(1+\sin x)}$$

which is required solution.

3.

Find the general and singular solution of

$y^2(y - xp) = x^4 p^2$.

Solution:

The given equation is $y^2(y-xp) = x^4p^2$ —— (1)

Putting $x = 1/u$, $y = 1/v$ so that

$dx = -(1/u^2)\,du$, $dy = -(1/v^2)\,dv$, we get,

$$\frac{dy}{dx} = \frac{u^2}{v^2}\frac{dv}{du}$$

$\Rightarrow p = \frac{u^2}{v^2} P$, where $P = \frac{dv}{du}$ and $p = \frac{dy}{dx}$.

$\therefore$ Putting $x = 1/u$, $y = 1/v$, $p = (u^2 P)/v^2$ in (1),

we have,

$$\left(\frac{1}{v^2}\right)\left\{\frac{1}{v} - \frac{1}{u}\frac{u^2P}{v^2}\right\} = \left(\frac{1}{u^4}\right)\left(\frac{u^4P^2}{v^4}\right)$$

$\Rightarrow$ $\underline{v = uP + P^2}$, which is in Clairaut's form.

$\therefore$ The required general solution is

$$v = uC + C^2$$

$\Rightarrow$ $1/y = C/x + C^2$

$\Rightarrow$ $x = cy + c^2xy$.

i.e. $\boxed{xyc^2 + yc - x = 0}$ —— (2)

which is a quadratic equation in c and so it's c-discriminant relation is

$$y^2 - 4(xy)(-x) = 0$$

$$\Rightarrow \quad y(y + 4x^2) = 0.$$

Now, $y = 0$ gives $p = \frac{dy}{dx} = 0$.

These values satisfy (1). So, $\boxed{y = 0 \text{ is a singular solution}}$. Again, $y = -4x^2$ gives $p = \frac{dy}{dx} = -8x$.

These values satisfy (1). Hence, $\boxed{y + 4x^2 = 0 \text{ is also singular solution}}$.

4. Reduce the equation $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$, $(x>0)$ into the normal form and hence solve it.

<u>Solution :</u>

The given equation is $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$.

Rewriting it, we get,

$$y'' - \frac{2}{x}(1+x)y' + \frac{2(1+x)}{x^2}y = x \quad \text{—— (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2(1+x)}{x^2}, \quad R = x. \quad \text{—— (2)}$$

We choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x}(1+x)dx} = e^{\int\left(\frac{1}{x}+1\right)dx}$

$$= e^{\ln x + x} = x \cdot e^{x}. \quad \text{—— (3)}$$

Let the required general solution be $y = uv$ —(4)

Then v is given by the normal form,

$$\frac{d^2v}{dx^2} + Iv = S, \quad \text{—— (5)}$$

where,

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{2(1+x)}{x^2} - \frac{1}{4}\times\frac{4}{x^2}(1+x)^2 - \frac{1}{2}\times\frac{2}{x^2}$$

$$= \cancel{\frac{2}{x^2}} + \cancel{\frac{2}{x}} - 1 - \cancel{\frac{1}{x^2}} - \cancel{\frac{2}{x}} - \cancel{\frac{1}{x^2}}$$

$$= -1$$

$$S = \frac{R}{u} = \frac{x}{x \cdot e^{x}} = e^{-x}$$

Then (5) becomes, $\frac{d^2V}{dx^2} - V = e^{-x}$

$$\Rightarrow (D^2 - 1)v = e^{-x} \text{ where } D \equiv d/dx \quad \text{——— (6)}$$

The A.E. of (6) is given by $D^2 - 1 = 0$

$$\Rightarrow D = \pm 1$$

$$\therefore \quad C.F. = c_1 e^{x} + c_2 e^{-x} \quad \text{——— (7).}$$

$$P.I. = \frac{1}{(D+1)(D-1)} \cdot e^{-x} = \frac{x}{1!} \times -\frac{1}{2} e^{-x} = \frac{-x \cdot e^{-x}}{2}$$

$$\therefore \quad v = c_1 e^{x} + c_2 e^{-x} - \frac{x \cdot e^{-x}}{2} \quad \text{——— (8)}$$

from (4), (3) and (8), we have

$$\boxed{y = c_1 x e^{2x} + c_2 x - \frac{x^2}{2}}$$

which is the required solution.

Hence, the result

5.

Solve $(x^2+y^2)(1+p^2) - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$.

Solution :

Let $x^2+y^2 = v$ and $x+y = u$ —— (1)

Differentiating (1), we have,

$2(x\,dx + y\,dy) = dv$ and $dx + dy = du$

$$\therefore \frac{dv}{du} = \frac{2(x\,dx + y\,dy)}{dx+dy}$$

$$= \frac{2\left\{x + y\left(\frac{dy}{dx}\right)\right\}}{1+\left(\frac{dy}{dx}\right)}$$

$$P = \frac{2(x+yp)}{1+p} \quad \text{—— (2)}$$

where $P = dv/du$ and $p = dy/dx$

Re-writing the given equation, we get,

$$(x^2+y^2) - 2(x+y)\left(\frac{x+yp}{1+p}\right) + \left(\frac{x+yp}{1+p}\right)^2 = 0$$

$$\Rightarrow v - 2u \times \frac{P}{2} + \left(\frac{P}{2}\right)^2 = 0 \quad \text{— using (1) \& (2)}$$

$$\Rightarrow v = uP - \frac{P^2}{4}$$ which is in Clairaut's form.

form).

Hence, it's general solution is given as

$$v = uC - \frac{C^2}{4}$$

$$\Rightarrow \boxed{(x^2+y^2) = c(x+y) - \frac{c^2}{4}} \quad \text{— by (1)}$$

is the required solution.

Hence, the result.

6. (i) If $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, find $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$

where $a>0$.

(ii) Find $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$.

Solution:

(i) Since $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$

$$\therefore L^{-1}\left\{\frac{e^{-1/pk}}{(pk)^{1/2}}\right\} = \frac{1}{k}\cdot\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)/k}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/pk}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)}}$$

Taking $k = 1/a$, we have,

$$\boxed{L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{(\pi t)}}} \quad \text{—— (1)}$$

(ii) Let $f(p) = \log\left(1+\frac{1}{p^2}\right) = -\log\frac{p^2}{p^2+1}$.

i.e. $f(p) = -2\log p + \log(p^2+1)$

$$\therefore \ f'(p) = -2\left(\frac{1}{p} - \frac{p}{p^2+1}\right)$$

$$\therefore \ L^{-1}\{f'(p)\} = -2(1-\cos t)$$

$$\Rightarrow -t\,L^{-1}\{f(p)\} = -2(1-\cos t)$$

$$\Rightarrow \boxed{L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2}{t}(1-\cos t)} \quad \text{--- (ii)}$$

Hence, the result.

7.(i)

Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4 \sin x$

Solution:

The given equation is $x^2y'' - 4xy' + 6y = -x^4 \sin x$ —— (1)

Re-writing the given equation, we have

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = -x^2 \sin x \quad \text{—— (2)}$$

Comparing (2) with $y'' + Py' + Qy = R$, we have

$P = -4/x$, $Q = 6/x^2$, $R = -x^2 \sin x$

Consider, $y'' - (4/x)y' + (6/x^2)y = 0$

$\Rightarrow (x^2D^2 - 4xD + 6)y = 0$ where $D \equiv d/dx$ —— (3)

In order to apply the method of variation of parameters, we shall reduce (3) into linear differential equation with constant co-efficients.

Let $x = e^z$ i.e. $\log x = z$ and let $D_1 \equiv d/dz$ —— (4)

Then, $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$ and so on,

Eq.^{n} (3) reduces to

$$\{D_1(D_1 - 1) - 4D_1 + 6\}y = 0$$

$\Rightarrow (D_1^2 - 5D_1 + 6)y = 0$, whose auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$ giving $D_1 = 2, 3$.

$\therefore$ C.F. of (1) $= c_1 e^{2z} + c_2 e^{3z}$

$= c_1 (e^z)^2 + c_2 (e^z)^3$

$= c_1 x^2 + c_2 x^3$ —— (5)

Let $u = x^2$ and $v = x^3$. Also, here $R = -x^2 \sin x$. —— (6)

Here, $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4$

$= x^4 \neq 0$

Hence, P.I. of (1) $= u f(x) + v g(x)$, —— (7)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^3 \times (-x^2 \sin x)}{x^4} dx$

$= \int x \sin x \, dx$

$= x(-\cos x) - \int \{(-\cos x) \times 1\} dx$

$= -x \cos x + \sin x$ —— (8)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{x^2 \times (-x^2 \sin x)}{x^4} dx$

$= -\int \sin x \, dx = \cos x$ —— (9)

Using (6), (8) and (9), (7) reduces to

P.I. of (1) $= x^2(-x\cos x + \sin x) + x^3 \cos x$

$= x^2 \sin x$ —— (10)

Hence, the required general solution is

$y =$ C.f + P.I.

$\therefore \boxed{y = c_1 x^2 + c_2 x^3 + x^2 \sin x}$ where c_1, c_2 are arbitrary constants.

Hence, the result.

7.(ii) Solve $x^2\left(\frac{d^3y}{dx^3}\right) + 2x\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{y}{x}\right) = 10\left(1+\frac{1}{x^2}\right)$

Solution :

Multiplying both sides by x, the given equation becomes,

$$x^3\left(\frac{d^3y}{dx^3}\right) + 2x^2\left(\frac{d^2y}{dx^2}\right) + 2y = 10\left(x+\frac{1}{x}\right)$$

i.e. $(x^3D^3 + 2x^2D^2 + 2)y = 10(x + x^{-1})$ —— (1)

where $D \equiv d/dx$

Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$

Then (1) becomes,

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2]y = 10[e^z + e^{-z}]$$

$\Rightarrow (D_1^3 - D_1^2 + 2)y = 10e^z + 10e^{-z}$ —— (2)

A.E. of (2) is $D_1^3 - D_1^2 + 2 = 0$

$\Rightarrow (D_1+1)(D_1^2 - 2D_1 + 2) = 0$ giving

$D_1 = -1,\ 1 \pm i$.

$\therefore$ C.F. $= c_1e^{-z} + e^z(c_2\cos z + c_3\sin z)$

i.e. $= c_1x^{-1} + x(c_2\cos\log x + c_3\sin\log x)$

Now,

P.I. corresponding to $10e^z = 10.\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z$

$= 10.\frac{1}{2(1-2+2)}.e^z$

$= 5x$

P.I. corresponding to $10e^{-z} = 10.\frac{1}{(D_1+1)(D_1^2-2D_1+2)}.e^{-z}$

$= 10.\frac{1}{D_1+1}.\frac{1}{1+2+2}e^{-z} = 2.\frac{1}{D_1+1}e^{-z}.1$

$= 2e^{-z}.\frac{1}{D_1-1+1}.1 = 2.e^{-z}.\frac{1}{D_1}.1 = 2e^{-z}z$

$= 2x^{-1}\log x$.

$\therefore$ Required solution is $y = C.F + P.I.$

$$\therefore \boxed{y = c_1x^{-1} + x(c_2\cos\log x + c_3\sin\log x) + 5x + 2x^{-1}\log x}$$

8. Solve $(px^2+y^2)(px+y)=(p+1)^2$ by reducing it to Clairaut's form and find its singular solution.

Solution:

given: $(px^2+y^2)(px+y)=(p+1)^2$ —— (A)

Let ; $x+y=u$ and $xy=v$ —— (1)

$\therefore \dfrac{dv}{du}=\dfrac{xdy+ydx}{dx+dy}=\dfrac{x(dy/dx)+y}{1+(dy/dx)}$

or $P=\dfrac{xp+y}{1+p}$ —— (2)

where $P=\dfrac{dv}{du}$ & $p=\dfrac{dy}{dx}$

Now, (px^2+y^2) can be written as

$$px^2+y^2=(px+y)(x+y)-xy(p+1)$$

Hence, the equation can be written as

$$\{(px+y)(x+y)-xy(p+1)\}(px+y)=(p+1)^2$$

or $\left(\dfrac{px+y}{p+1}\right)^2(x+y)-xy\left(\dfrac{px+y}{p+1}\right)=1$

or $P^2u-vP=1$ — by (1) and (2)

or $\boxed{v=uP-1/P}$, which is in Clairaut's form.

$\therefore$ Its general solution is

$$v=uc-(1/c)$$

$$\Rightarrow xy=(x+y)c-(1/c)$$

$$\Rightarrow \boxed{c^2(x+y)-cxy-1=0}$$

c being an arbitrary constant.

Its c-discriminant relation is

$B^2-4AC=0$ i.e.

$$(xy)^2-4(x+y)\times(-1)=0$$

$$\boxed{x^2y^2+4(x+y)=0}$$

This relation satisfies (A), and hence it is the singular solution.

9. (i) solve $x(1-x^2)dy + (2x^2y - y)dx = ax^3dx$

(ii) solve $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$.

Sol^n: (i) the given equation can be written as.

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)} \cdot y = \frac{ax^2}{1-x^2}, \text{ which is linear.}$$

Here $P = \dfrac{2x^2-1}{x(1-x^2)} = \dfrac{2x^2-1}{x(1-x).(1+x)}$

$$= -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}, \text{ by partial fractions}$$

$Q = \dfrac{ax^2}{1-x^2}$

we have

$$\int P dx = -\int \left[\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right] dx$$

$$= -\left[\log x + \tfrac{1}{2}\log(x-1) + \tfrac{1}{2}\log(x+1)\right]$$

$$= -\left[\log x + \tfrac{1}{2}\log(x^2-1)\right]$$

$$= -\log\{x\sqrt{(x^2-1)}\} = \log\left[\frac{1}{x\sqrt{(x^2-1)}}\right]$$

$\therefore$ I.F. $= e^{\int P dx} = \dfrac{1}{x\sqrt{x^2-1}}$

$\therefore$ the solution is

$y(\text{IF}) = \int [Q.(\text{I.F})] dx + c$

i.e. $y \cdot \dfrac{1}{x\sqrt{(x^2-1)}} = \displaystyle\int \frac{ax^2}{(1-x^2)} \cdot \frac{1}{x\sqrt{(x^2-1)}} dx + c$

$$= -\tfrac{1}{2}a\int (x^2-1)^{-3/2}(2x)\,dx + c$$

$$= -\tfrac{1}{2}a\,\frac{(x^2-1)^{-1/2}}{-1/2} + c, \text{ by power formula}$$

$$= \frac{a}{\sqrt{(x^2-1)}} + c$$

Hence the required solution is

$$y = ax + cx\sqrt{(x^2-1)}$$

(ii) The given equation can be written as

$$(xy+2x^2y^2)y\,dx+(xy-x^2y^2)x\,dy.$$

Here $Mx-Ny=(xy+2x^2y^2)xy-(xy-x^2y^2)xy$

$$=x^2y^2+2x^3y^3-x^2y^2+x^3y^3=3x^3y^3\neq 0.$$

$\therefore$ the Integrating factor $=\dfrac{1}{Mx-Ny}=\dfrac{1}{3x^3y^3}$

Multiplying the given equation by the I.F $\frac{1}{3x^3y^3}$, we get

$$\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0 \quad \text{—— (1)}$$

Now the eqn (1) is of the form $M\,dx+N\,dy=0$

$\therefore M=\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)$ and $N=\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)$

we have

$\dfrac{\partial M}{\partial y}=\dfrac{1}{3x^2}\left(-\dfrac{1}{y^2}\right)$ and $\dfrac{\partial N}{\partial x}=\dfrac{1}{3y^2}\left(-\dfrac{1}{x^2}\right)$, Thus $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

and hence equation (1) is exact.

Now $\int M\,dx=\int\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx=\frac{1}{3}\left(-\frac{1}{xy}+2\log x\right)$ —— (2)

and $\int N\,dy=\int -\frac{1}{3}\cdot\frac{1}{y}dy=-\frac{1}{3}\log y.$ —— (3)

the solution is given by

(2) + (3) = c (an arbitrary constant).

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=\frac{1}{3}\log c$$

$$\Rightarrow -\frac{1}{xy}+\log\frac{x^2}{y}=\log c$$

$$\Rightarrow \log\frac{x^2}{cy}=\frac{1}{xy}$$

$$\Rightarrow \frac{x^2}{cy}=e^{1/xy}$$

$$\Rightarrow x^2=cye^{1/xy}$$

10. By using Laplace transformation solve $(D^2+m^2)x = a\sin nt$, $t>0$. When x, Dx equal to x_0 and x_1, when $t=0$, $n\neq m$.

Ans: $\boxed{x_0\cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}\left(\sin nt - \frac{n}{m}\sin mt\right)}$

11. Solve the initial value problem.

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t \; ; \; y(0)=0, \; y'(0)=0$$

by using Laplace transform.

<u>Solution</u>: Given that; $\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t$

and $y(0)=0$; $y'(0)=0$

$$\frac{d^2f}{dt^2} = L f''(t) = p^2 f(p).$$

$$\therefore \; L f''(t) + L f(t) = 8 L e^{-2t}\sin t$$

$$p^2 f(p) + f(p) = 8 \cdot \frac{1}{(p+2)^2+1}$$

$$f(p)[p^2+1] = \frac{8}{(p+2)^2+1} = \frac{8}{p^2+4p+5}$$

$$f(p) = \frac{8}{(p^2+1)(p^2+4p+5)} = 8 \cdot I.$$

Where; $I = \frac{1}{(p^2+1)(p^2+4p+5)} = \frac{Ap+B}{p^2+4p+5} + \frac{Cp+D}{p^2+1}$

$$1 = (Ap+B)(p^2+1) + (Cp+D)(p^2+4p+5)$$

$$1 = (A+C)p^3 + (B+4C+D)p^2 + (A+5C+4D)p + (B+5D).$$

$\therefore \; A+C=0 \Rightarrow \boxed{A=-C}$

$B+4C+D=0 \Rightarrow \boxed{B=3D}$

$A+5C+4D=0 \Rightarrow 4C=-4D \Rightarrow \boxed{C=-D}$

$B+5D=1$
$3D+5D=1$ $\Rightarrow$ $\boxed{D=1/8, \; A=1/8, \; C=-1/8, \; B=3/8}$

Put the values of A, B, C, D in I.

$$I = \frac{1}{8}\left[\frac{p+3}{p^2+48+5} + \frac{1-p}{p^2+1}\right]$$

$$f(p) = 8 \times \frac{1}{8}\left[\frac{p+2}{(p+2)^2+1} + \frac{1}{(p+2)^2+1} + \frac{1}{p^2+1} - \frac{p}{p^2+1}\right].$$

$$L^{-1}[f(p)] = L^{-1}\left[\frac{p+2}{(p+2)^2+1} + \frac{1}{(p+2)^2+1} + \frac{1}{p^2+1} - \frac{p}{p^2+1}\right].$$

$$\therefore \; \boxed{f(t) = e^{-2t}\cos t + e^{-2t}\sin t + \sin t - \cos t}$$

required solution.

12.(i)

(i) Find $L^{-1}\left\{\log\frac{p+3}{p+2}\right\}$, (ii) Find $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$

Sol'n: (i) Let $f(p)=\log\frac{p+3}{p+2}$

$$=\log(p+3)-\log(p+2)$$

$$\therefore f'(p)=\frac{1}{p+3}-\frac{1}{p+2}$$

$$\therefore L^{-1}\{f^{-1}(p)\}=e^{-3t}-e^{-2t}$$

$$\Rightarrow (-1)t\,L^{-1}\{f(p)\}=e^{-3t}-e^{-2t}$$

$$\Rightarrow L^{-1}\log\left\{\frac{p+3}{p+2}\right\}=\frac{1}{t}\left(e^{-2t}-e^{-3t}\right)$$

12.(ii) Prove that $L\left\{\frac{\cos at-\cos bt}{t}\right\}=\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

Sol'n: Here $L\{\cos at-\cos bt\}=L\{\cos at\}-L\{\cos bt\}$

$$\therefore L\{\cos at-\cos bt\}=\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}=f(s),\text{ say}$$

$$\therefore L\left\{\frac{\cos at-\cos bt}{t}\right\}=\int_s^\infty f(s)\,ds$$

$$=\int_s^\infty\left\{\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right\}ds$$

$$=\left[\frac{1}{2}\log(s^2+a^2)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$=\frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2}-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$=\frac{1}{2}\lim_{s\to\infty}\log\frac{1+a^2/s^2}{1+b^2/s^2}+\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$=0+\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$=\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

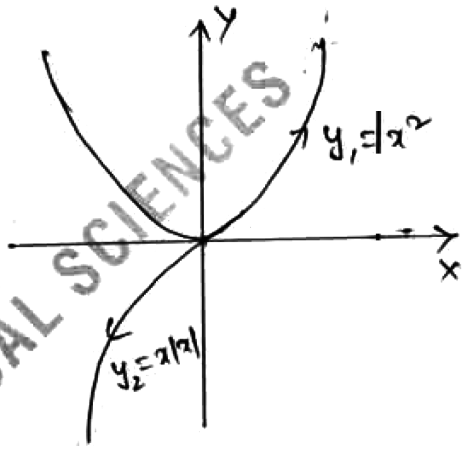
13. Show graphically that $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent on $-\infty < x < \infty$, however wronskian vanishes for every real value of x.

Sol'n: Given that $y_1(x) = x^2$ and $y_2(x) = x|x|$, $-\infty < x < \infty$

Here $y_2(x) = x|x|$

$$= \begin{cases} x(-x) \; ; & x < 0 \\ 0 \; ; & x = 0 \\ x(x) & x > 0 \end{cases}$$

$$= \begin{cases} -x^2 : & x < 0 \\ 0 \; ; & x = 0 \\ x^2 : & x > 0 \end{cases}$$



According to graph

clearly for $x<0$:

$y_1(x)$ & $y_2(x)$ are linearly independent but wronskian vanishes for every real value of x

i.e. for $x<0$: $W(y_1, y_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$

for $x>0$: $W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$

14. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$, λ being the parameter.

Sol'n: The given family of curves is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1, \text{ with } \lambda \text{ as parameter} \quad \text{—— (1)}$$

Differentiating (1) w.r.t x, we get

$$\frac{2x}{a^2}+\frac{2y}{a^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a^2+\lambda}=\frac{-x}{a^2y}\frac{dx}{dy} \quad \text{—— (2)}$$

Eliminating λ from (1) & (2), we get

$$\frac{x^2}{a^2}+y^2\left(\frac{-x}{a^2y}\cdot\frac{dx}{dy}\right)=1 \Rightarrow \frac{x^2}{a^2}-\frac{xy}{a^2}\frac{1}{dy/dx}=1 \quad \text{—— (3)}$$

which is the differential equation of the given family of curves (1).

Replacing dy/dx by $-dx/dy$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{x^2}{a^2}-\frac{xy}{a^2}\left(-\frac{1}{dx/dy}\right)=1$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{xy}{a^2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{xy}{a^2}\frac{dy}{dx}=\frac{a^2-x^2}{a^2}$$

$$\Rightarrow y\,dy=\left(\frac{a^2}{x}-x\right)dx$$

Integrating, we get

$$y^2/2=a^2\log x-\tfrac{1}{2}x^2+\tfrac{1}{2}c$$

$$x^2+y^2=2a^2\log x+c$$

$$\Rightarrow x^2+y^2-2a^2\log x=c$$

which is the required equation of the orthogonal trajectories.

15. Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin \log x + 1}{x}$

Sol: Given that $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin \log x + 1}{x}$ ——(1)

Equation (1) can be written as

$$(x^2D^2 - 3xD + 1)y = x^{-1}[1 + \log x \sin \log x] \quad \text{——(2)}$$

Let $x = e^z$ so that $z = \log x$ and let $D_1 = \frac{d}{dz}$ ——(3)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ ——(4)

using (3) & (4), (2) reduces to

$$[D_1(D_1-1) - 3D_1 + 1]y = e^{-z}[1 + z\sin z]$$

$$\Rightarrow (D_1^2 - 4D_1 + 1)y = e^{-z} + e^{-z} z\sin z \quad \text{——(5)}$$

A.E of (5) is $D_1^2 - 4D_1 + 1 = 0$

$$\Rightarrow D_1 = 2 \pm \sqrt{3}$$

$$\therefore C.F. = e^{2z}[c_1\cosh(\sqrt{3}z) + c_2\sinh\sqrt{3}z]$$

$$y_c = x^2[c_1\cosh(\sqrt{3}\log x) + c_2\sinh(\sqrt{3}\log x)]$$

P.I. corresponding to e^{-z}

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z} = \frac{1}{1+4+1}e^{-z} = \frac{1}{6}x^{-1} = \frac{1}{6x}$$

and P.I corresponding to $e^{-z} z\sin z$

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z}(z\sin z)$$

$$= e^{-z}\frac{1}{(D_1-1)^2 - 4(D_1-1) + 1} z\sin z$$

$$= e^{-z}\frac{1}{D_1^2 - 6D_1 + 6} z\sin z$$

$$= e^{-z}\left[\frac{1}{D_1^2 - 6D_1 + 6}\sin z - \frac{2D_1 - 6}{(D_1^2 - 6D_1 + 6)^2}\sin z\right]$$

$$= e^{-z}\left[z\,\frac{1}{-1-6D_1+6}\sin z-(2D_1-6)\frac{1}{(-1-6D_1+6)^2}\sin z\right]$$

$$= e^{-z}\left[z\,\frac{1}{5-6D_1}\sin z-\frac{2D_1-6}{(5-6D_1)^2}\sin z\right]$$

$$= e^{-z}\left[z\,\frac{5+6D_1}{25-36D_1^2}\sin z-(2D_1-6)\frac{1}{25-60D_1+36D_1^2}\sin z\right]$$

$$= e^{-z}\left[z\,(5+6D_1)\frac{1}{25+36}\sin z-(2D_1-6)\frac{1}{25-60D_1-36}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5+6D_1)\sin z+(2D_1-6)\frac{1}{11+60D_1}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{(2D_1-6)(60D_1-11)}{3600D_1^2-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120D_1^2-382D_1+66}{-3600-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120(-\sin z)-382\cos z+66\sin z}{-3721}\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{54\sin z+382\cos z}{3721}\right]$$

$$= \frac{1}{x}\left[\frac{\log x}{61}(5\sin\log x+6\cos\log x)+\frac{54\sin(\log x)+382\cos(\log x)}{3721}\right]$$

$$\therefore\ y = x^2\left[c_1\cosh(\sqrt{3}\log x)+c_2\sinh(\sqrt{3}\log x)\right]+\frac{1}{6x}$$

$$+\frac{1}{x}\left[\frac{\log x}{61}\{5\sin(\log x)+6\cos(\log x)\}+\frac{54\sin\log x+382\cos\log x}{3721}\right]$$

which is the required solution.

16.(i) Find the orthogonal trajectories of $r = a(1+\cos n\theta)$.

Sol'n: Given family is $r = a(1+\cos n\theta)$, where a is parameter. —— ①

Taking logarithm of bothsides,

$$\log r = \log a + \log(1+\cos n\theta) \quad \text{—— ②}$$

Differentiating ② w.r.t θ

$$\left(\frac{1}{r}\right)\left(\frac{dr}{d\theta}\right) = -(n\sin n\theta)/(1+\cos n\theta) \quad \text{—— ③}$$

which is differential equation of the family of curves ①.

Replacing $dr/d\theta$ by $-r^2\left(\frac{d\theta}{dr}\right)$ in ③, the differential equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta}$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}d\theta$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{2\cos^2(n\theta/2)\,d\theta}{2\sin(n\theta/2)\cos(n\theta/2)}$$

$$\Rightarrow n\frac{dr}{r} = \cot(n\theta/2)\,d\theta$$

Integrating $n\log r = \frac{2}{n}\times\log\sin(n\theta/2) + \left(\frac{1}{n}\right)\log c$, c being parameter

$$\Rightarrow n^2\log r = \log\sin^2(n\theta/2) + \log c$$

$$\Rightarrow r^{n^2} = c\sin^2(n\theta/2)$$

$$\Rightarrow r^{n^2} = \left(\frac{c}{2}\right)(1-\cos n\theta)$$

$$\Rightarrow r^{n^2} = b(1-\cos n\theta) \quad \text{taking } b = c/2$$

which is the equation of required orthogonal trajectories with b as parameter.

16.(ii) Solve $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

Sol^n: Comparing the given equation with

$\dfrac{dy}{dx} + Py = Q$, here $P = \dfrac{1}{(1-x^2)^{3/2}}$ & $Q = \dfrac{x+\sqrt{1-x^2}}{(1-x^2)^2}$

Hence $\displaystyle\int P\,dx = \int \frac{1}{(1-x^2)^{3/2}}\,dx$

$= \displaystyle\int \frac{\cos\theta\, d\theta}{\cos^3\theta\, d\theta}$, putting $x = \sin\theta$

$= \displaystyle\int \sec^2\theta\, d\theta = \tan\theta$

$= \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}}$

$\therefore$ I.F $= e^{\int P dx} = e^{\frac{x}{\sqrt{1-x^2}}}$

Solution of the given differential equation is

$y(\text{I.F}) = \int Q(\text{I.F}) + c$

Now $\displaystyle\int Q(\text{I.F}) = \int \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}\, e^{\frac{x}{(1-x^2)^{1/2}}}\,dx$ —— (1)

put $\dfrac{x}{\sqrt{1-x^2}} = t$

$\Rightarrow \dfrac{\sqrt{1-x^2}\cdot 1 - x\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}{(1-x^2)}\,dx = dt$

$\Rightarrow \dfrac{\sqrt{(1-x^2)} + [x^2/\sqrt{1-x^2}]}{1-x^2}\,dx = dt$

$\Rightarrow \dfrac{1}{(1-x^2)^{3/2}}\,dx = dt$

from (1),

$$\int Q(I.F)\,dx = \int \frac{\left[x/\sqrt{1-x^2}+1\right]}{(1-x^2)^{3/2}} e^{\frac{x}{(1-x^2)^{1/2}}}\,dx$$

$$= \int (t+1)e^t\,dt$$

$$= (t+1)e^t - \int e^t\,dt$$

$$= te^t.$$

$$\therefore \int Q\ldots = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}}$$

$$\therefore y e^{x/(1-x^2)^{1/2}} = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}} + c$$

$$\Rightarrow \boxed{y = \frac{x}{\sqrt{1-x^2}} + c e^{-\frac{x}{\sqrt{1-x^2}}}}$$

which is the required solution

17. solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$.

Sol'n: Given $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

The auxiliary equation is $D^4+D^2+1=0$

$$\Rightarrow (D^2+1)^2 - D^2 = 0$$

$$\Rightarrow (D^2+D+1)(D^2-D+1)=0$$

$$\Rightarrow D^2+D+1=0 \text{ (or) } D^2-D+1=0$$

$$\Rightarrow D = \frac{-1\pm i\sqrt{3}}{2},\ \frac{1\pm i\sqrt{3}}{2}$$

$$\therefore \text{C.F} = e^{-x/2}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right)+c_4\sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

$$\text{P.I} = \frac{1}{D^4+D^2+1}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})(D^2-2D+\frac{7}{4})}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{-\frac{3}{4}-2D+\frac{7}{4}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{(1-2D)}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{(1+2D)}{1-4D^2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+3/4)}\,\frac{1}{4}\,(1+2D)\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{e^{-x/2}}{4}\,\frac{1}{(D^2+3/4)}\left(\cos\frac{\sqrt{3}}{2}x-\sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{1}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2}\cos\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2}\sin\frac{\sqrt{3}}{2}x\right]$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{x}{2\left(\sqrt{3}/2\right)}\sin\left(\frac{x\sqrt{3}}{2}\right)-\frac{\sqrt{3}x}{2\left(\sqrt{3}/2\right)}\left(-\cos\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{x}{4\sqrt{3}}e^{-x/2}\left[\sin\left(\frac{\sqrt{3}}{2}x\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$\therefore\ y = C.F + PI$

$$= e^{-x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$+e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right)+c_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$+\frac{x}{4\sqrt{3}}e^{-x/2}\left[\sin\left(\frac{\sqrt{3}}{2}x\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

18.(i)

(i) The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, find the number of bacteria which will be present at the end of 5 hours.

(ii) Solve $(D^2+1)y = x^2 \sin 2x$.

Sol'n: (i) Suppose that the number of bacteria is x_0 when $t=0$, and it is x at time t (in hrs).

Then given that

$$\frac{dx}{dt} \propto x$$

$$\Rightarrow \frac{dx}{dt} = kx \text{ ——— (1)}, \text{ where } k = \text{constant of proportionality}$$

$$(1) \Rightarrow \frac{dx}{x} = k\,dt = \int \frac{dx}{x} = k\int dt$$

$$\Rightarrow \log x - \log c = kt$$

$$\therefore \log\left(\frac{x}{c}\right) = kt \text{ so that } x = ce^{kt} \text{ ——— (2)}$$

By our assumption, when $t=0$, $x = x_0$ so that

$$(2) \Rightarrow x_0 = c \text{ and so } (2) \Rightarrow x = x_0 e^{kt} \text{ ——— (3)}$$

Given $x = 3x_0$ when $t=1$, so (3) yields

$$3x_0 = x_0 e^k$$

$$\Rightarrow e^k = 3 \text{ ——— (4)}$$

let $x = x'$ when $t=5$, then (3) yields

$$x' = x_0 e^{5k} = x_0 (e^k)^5 = x_0 \cdot 3^5 \text{ by (4)}$$

Hence, the bateria is expected to grow 3^5 times at the end of 5hrs.

(ii) the auxiliary equation is $m^2+1=0$

$$\therefore m = \pm i = 0 \pm i$$

$\therefore$ C.F $= e^{0x}(c_1\cos x + c_2 \sin x) = c_1\cos x + c_2\sin x$

And P.I $= \dfrac{1}{(D^2+1)} x^2 \sin 2x$

$= \text{Imaginary part of } \dfrac{1}{(D^2+1)} x^2 e^{2ix}$

$= \text{I.P of } e^{2ix} \dfrac{1}{\{(D+2i)^2+1\}} x^2$

$= \text{I.P. of } e^{2ix} \dfrac{1}{(D^2+4iD-3)} x^2 \quad [\because i^2=-1]$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1-\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)\right]^{-1} x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1+\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)+\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)^2+\cdots\right]x^2$

$= \text{I.P. of } \dfrac{e^{2ix}}{-3}\left[1+\dfrac{4iD}{3}+\dfrac{D^2}{3}-\dfrac{16D^2}{9}+\cdots\right]x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1+\dfrac{4Di}{3}-\dfrac{13D^2}{9}+\cdots\right]x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[x^2+\dfrac{4i}{3}.2x-\dfrac{13}{9}.2\right]$

$= \text{I.P of } -\frac{1}{3}(\cos 2x + i\sin 2x)\{x^2+(8/3)ix-(26/9)\}$

$= -\frac{1}{3}\cdot\left(\frac{8}{3}x\cos 2x\right)-\frac{1}{3}\{x^2-(26/9)\}\sin 2x.$

$= -\left(\frac{1}{27}\right)[24x\cos 2x-(9x^2-26)\sin 2x]$

Hence the Complete solution is $y =$ C.F + P.I

$\Rightarrow y = c_1\cos x + c_2\sin x - \frac{1}{27}[24x\cos 2x-(9x^2-26)\sin 2x]$