6 Vector Analysis

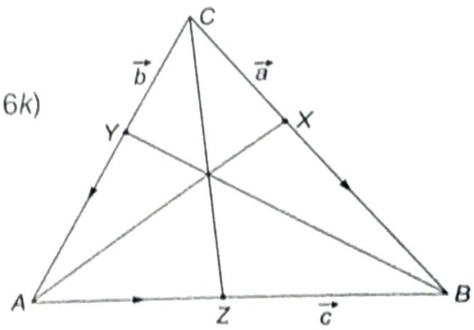
1. Scalar and Vector Fields

1.1 Prove that the vectors $\vec{a} = 3i + j - 2k$, $\vec{b} = -i + 3j + 4k$, $\vec{c} = 4i - 2j - 6k$ can form the sides of a triangle. Find the lengths of the medians of the triangle.

(2016 : 10 Marks)

Solution:

Here, we find that



$$\begin{aligned}\vec{b}+\vec{c} &= (-i + 3j + 4k) + (4i - 2j - 6k)\\ &= 3i + j - 2k\\ &= \vec{a}\end{aligned}$$

i.e.,

$$\vec{b}+\vec{c} = \vec{a}$$

And also we notice that these three vectors are not collinear (components are not proportional). Hence, these form the sides of a triangle. Let AX, BY and CZ be medians.

By triangle law of vector addition :

$$\overrightarrow{AX} = \overrightarrow{AB}+\overrightarrow{BX} = \vec{C}-\frac{\vec{a}}{2} = \frac{1}{2}(5i-5j-14k)$$

$$\left|\overrightarrow{AX}\right| = \sqrt{\frac{1}{4}(25+25+196)} = \sqrt{\frac{246}{4}} = \frac{\sqrt{246}}{2}$$

$$\overrightarrow{BY} = \overrightarrow{BA}+\overrightarrow{AY} = -\left(\vec{C}+\frac{\vec{b}}{2}\right) = -\frac{1}{2}(7\hat{i}-\hat{j}-8\hat{k})$$

$$\left|\overrightarrow{BY}\right| = \frac{1}{2}\sqrt{49+1+64} = \frac{\sqrt{114}}{2}$$

$$\overrightarrow{CZ} = \overrightarrow{CA}+\overrightarrow{AZ} = \vec{b}+\frac{\vec{c}}{2} = \frac{1}{2}(2i+4j+2k)$$

$$\left|\overrightarrow{CZ}\right| = |i+2j+k| = \sqrt{1+4+1} = \sqrt{6}$$

2. Differentiation of a Vector Field of a Scalar Variable

2.1 For two vectors $\vec{a}$ and $\vec{b}$ given respectively by

$$\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$$

and

$$\vec{b} = \sin t\hat{i} - \cos t\hat{j}$$

Determine : (i) $\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$ and $\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)$

(2009 : 10 Marks)

Solution :

$$\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$$
$$\vec{b} = \sin 5t\,\hat{i} - \cos t\,\hat{j}$$
$$\vec{a}\cdot\vec{b} = 5t^2 \sin 5t - t\cos t \qquad (\because \hat{i}\cdot\hat{i} = 1 \text{ etc., } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0)$$

$$\therefore \quad \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}(5t^2 \sin 5t - t\cos t)$$
$$= 5(2t\sin 5t + t^2\cdot 5\cos 5t) - (1\cdot\cos t - t\sin t)$$
$$= 10t\sin 5t + 25t^2\cos 5t - \cos t + t\sin t$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin 5t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}(0 - t^3\cos t) + \hat{j}(-t^3\sin 5t - 0) + \hat{k}(-5t^2\cos t - t\sin 5t)$$
$$= -t^3\cos t\,\hat{i} - t^3\sin 5t\,\hat{j} - (5t^2\cos t + t\sin 5t)\hat{k}$$

$$\therefore \quad \frac{d}{dt}(\vec{a}\times\vec{b}) = (-3t^2\cos t + t^3\sin t)\hat{i} - (3t^2\sin 5t + 5t^3\cos t)\hat{j} -$$
$$(10t\cos t - 5t^2\sin t + t\cos 5t + 1\cdot\sin 5t)\hat{k}$$

2.2 If

$$\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$$
$$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$$

find the value of $\dfrac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B})$ at (1, 0, –2).

(2012 : 12 Marks)

Solution:

Given :

$$\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$$
$$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$$

$$\therefore \quad \vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= \vec{i}(2x^3z^3 - xyz^2) + \vec{j}(2xz^3 + x^4yz) + \vec{k}(x^2y^2z + 4xz^4)$$

$$\frac{\partial}{\partial y}(\vec{A}\times\vec{B}) = \vec{i}(-xz^2) + \vec{j}(x^4z) + \vec{k}(2x^2yz)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B}) = \vec{i}(-z^2) + \vec{j}(4x^3z) + \vec{k}(4xyz)$$

$\therefore$ At (1, 0, –2)

$$\frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B}) = -4\vec{i} - 8\vec{j}$$

2.3 The position vector of a moving point at time t is, $\vec{r} = (\sin t)i + (\cos 2t)j + (t^2 + 2t)k$. Find the components of acceleration $\vec{a}$ in the directions parallel to the velocity vector $\vec{v}$ and perpendicular to the plane of $\vec{r}$ and $\vec{v}$ at time $t = 0$.

(2017 : 10 Marks)

Solution:

$$\vec{r} = (\sin t)i + (\cos 2t)i + (t^2 + 2t)k \quad \ldots(i)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (\cos t)i - 2\sin 2t\, j + (2t + 2)k \quad \ldots(ii)$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (-\sin t)i - 4\cos 2t\, j + 2k \quad \ldots(iii)$$

At $t = 0$,

$$\vec{r} = j,\ \vec{v} = i + 2k,\ \vec{a} = -4j + 2k$$

Component of $\vec{a}$ in direction of $\vec{v}$

$$\vec{a}_v = \frac{\vec{a}\cdot\vec{v}}{|v|}\cdot\frac{\vec{v}}{|v|} = \frac{+4}{(1+4)}\vec{v} = \frac{4}{5}(i+2k)$$

Component of $\vec{a}$ in the direction of vector perpendicular to the plane of $\vec{r}$ and $\vec{v}$.

Let $\vec{p}$ be the vector perpendicular to $\vec{r}$ and $\vec{v}$

$$\vec{p} = \vec{r}\times\vec{v} = 2\hat{i} - \hat{k},\ |\vec{p}|^2 = 5$$

$$\vec{A}_p = \frac{\vec{P}\cdot\vec{a}}{|\vec{P}|}\cdot\frac{\vec{P}}{|\vec{P}|} = \frac{-2}{5}(2i - k)$$

3. Gradient, Divergence and Curl in Cartesian and Cylindrical Coordinate and Directional Derivative

3.1 Find the directional derivative of :

(i) $4xz^3 - 3x^2y^2z^2$ at (2, –1, 2) along z-axis

(ii) $x^2yz + 4xz^2$ at (1, –2, 1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$

(2009 : 6 + 6 = 12 Marks)

Solution:

Approach : The directional derivative in any direction is the dot product of the gradient witht hat direction.

(i)

$$f(x, y, z) = 4xz^3 - 3x^2y^2z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= (4z^3 - 6xy^2z^2)\hat{i} - 6x^2yz^2\hat{j} + (12xz^2 - 6x^2y^2z)\hat{k}$$

Directional derivative along z-axis

$$= \nabla f\cdot\hat{k} = 12xz^2 - 6x^2y^2z$$

Directional derivative along z-axis at (2, –1, 2)

$$= (\nabla f\cdot\hat{k})\Big|_{(2,-1,2)} = 12\cdot2\cdot2^2 - 6\cdot2^2\cdot(-1)^2\cdot2 = 48$$

(ii)
$$f(x, y, z) = x^2yz + 4xz^2$$
$$\nabla f = (2xyz+4z^2)\hat{i}+(x^2z)\hat{j}+(x^2y+8xz)\hat{k}$$
$$\nabla f|_{(1,-2,1)} = \hat{j}+6\hat{k}$$

∴ Directional derivative along $(2\hat{i}-\hat{j}-2\hat{k})$

$$= (\hat{j}+6\hat{k})\cdot\left(\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}\right)$$
$$= \left(\frac{-1-12}{3}\right)=\frac{-13}{3}$$

3.2 Examine whether the vectors ∇u, ∇v and ∇w are coplanar, where u, v and w are the scalar functions defined by :

$$u = x + y + z$$
$$v = x^2 + y^2 + z^2$$
and $$w = yz + zx + xy$$

(2011 : 15 Marks)

Solution:

$$\nabla u = \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(x+y+z)$$
$$= \vec{i}+\vec{j}+\vec{k}$$
$$\nabla v = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

Similarly, $$\nabla w = (y+z)\vec{i}+(x+z)\vec{j}+(x+y)\vec{k}$$

Now, ∇u, ∇v, ∇w would be co-planar if their scalar triple product is zero.

$$\nabla v\times\nabla w = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2x & 2y & 2z\\ y+z & x+z & x+y\end{vmatrix}$$
$$= \vec{i}(2xy+2y^2-2xz-2z^2) + \vec{j}(2yz+2z^2-2x^2-2xy) + \vec{k}(2x^2+2xz-2y^2-2yz)$$

∴ $$\nabla u(\nabla v\times\nabla w) = (\vec{i}+\vec{j}+\vec{k})\cdot\left[(2xy+2y^2-2xz-2z^2)\vec{i} + (2yz+2z^2-2x^2-2xy)\vec{j} + (2x^2+2xz-2y^2-2yz)\vec{k}\right]$$
$$= 2xy+2y^2-2x^2-2z^2+2yz+2z^2-2x^2-2xy+2x^2+2xz-2y^2-2yz$$
$$= 0$$

∴ The vector ∇u, ∇v, ∇w are co-planar.

3.3 If $\vec{r}$ be the position vector of a point, find the value(s) of n for which the vector $r^n\vec{r}$ is (i) irrotational, (ii) solenoidal.

(2011 : 15 Marks)

Solution:

A vector $\vec{V}$ is said to be solenoidal if divergence of $\vec{V} = 0$.

i.e.,
$$\nabla\cdot\vec{V} = 0$$

Also,
$$\text{div}(\phi\vec{V}) = (\text{grad }\phi)\cdot\vec{V}+\phi\text{ div }\vec{V}$$

$\therefore$ $r^n\vec{r}$ will be solenoidal if

$$\text{div}(r^n\vec{r}) = 0$$

$$\Rightarrow \quad (\text{grad } r^n)\cdot\vec{r}+r^n\text{div }(\vec{r}) = 0$$

$$\Rightarrow \quad (nr^{n-1}\text{grad } r)\cdot\vec{r}+r^n\cdot 3 = 0$$

$$\because \quad \text{div }\vec{r} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$
$$= 1+1+1 = 3$$

and
$$\text{grad } f(4) = f'(4)\text{ grad } u$$

$$\Rightarrow \quad \left(nr^{n-1}\cdot\frac{\vec{r}}{r}\right)\cdot\vec{r}+3r^n = 0 \qquad \left[\because \text{grad } r = \frac{\vec{r}}{r}\right]$$

$$\Rightarrow \quad nr^{n-2}(\vec{r}\cdot\vec{r})+3r^n = 0$$

$$\Rightarrow \quad nr^{n-2}\cdot r^2+3r^n = 0$$

$$\Rightarrow \quad r^n(n+3) = 0 \Rightarrow n = -3$$

A vector $\vec{V}$ is said to be irrotational if

$$\nabla\times\vec{V} = 0$$

Also,
$$\nabla\times(\phi\vec{V}) = (\text{grad }\phi)\times\vec{V}+\phi(\nabla\times\vec{V})$$

$\therefore$ $r^n\vec{r}$ will be irrotational if

$$\nabla\times(r^n\vec{r}) = 0$$

$$\Rightarrow \quad (\text{grad } r^n)\times\vec{r}+r^n(\nabla\times\vec{r}) = 0$$

$$\Rightarrow \quad \left(nr^{n-1}\frac{\vec{r}}{r}\right)\times\vec{r}+r^n\cdot 0 = 0 \qquad (\because \nabla\times\vec{r} = 0)$$

Hence, $r^n\vec{r}$ is irrotational for all the real values of n.

3.4 A vector field is given by

$$\vec{F} = (x^2+xy^2)i+(y^2+x^2y)j$$

Verify that the field $\vec{F}$ is irrotational or not. Find the scalar potential.

(2015 : 12 Marks)

Solution:

A vector field $\vec{F}$ is said to be irrotational if curl $\vec{F} = 0$, i.e.,

$$\nabla\times\vec{F} = 0$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$= \hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(2xy-2xy)$$

$$= \vec{0}$$

$\Rightarrow \vec{F}$ is irrotational.

Now, it can be written as grad of a scalar field, i.e., to find ϕ so that

$$\nabla\phi = \vec{F}$$

i.e., $$\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y} = (x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{k}$$

$\therefore$ $$\frac{\partial\phi}{\partial x} = x^2+xy^2;\quad \frac{\partial\phi}{\partial y} = y^2+x^2y \qquad \ldots(*)$$

$\Rightarrow$ $$\phi = \frac{x^3}{3}+\frac{x^2y^2}{2}+f(y)$$

Differentiating w.r.t. y and comparing with $(*)$

$$\frac{\partial\phi}{\partial y} = x^2y+f'(y)$$

$\Rightarrow$ $$f'(y) = y^2$$

$$f(y) = \frac{y^3}{3}+C$$

$\therefore$ $$\phi(x,y) = \frac{x^3}{3}+\frac{y^3}{3}+\frac{x^2y^2}{2}+C$$

3.5 For what values of the constants a, b and c the vector

$$\vec{V} = (x+y+az)\hat{i}+(bx+2y-z)\hat{j}+(-x+cy+2z)\hat{k}$$

is irrational. Find the divergence in cylindrical coordinates of this vector with these values.

(2017 : 10 Marks)

Solution:

Irrational $\Rightarrow$ Curl $\bar{V}=0$

$$\text{Curl }\bar{V} = \nabla\times\vec{V}\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{bmatrix} \text{ (if } V = fi+gi+hk)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix}$$

$$= i(c-(-1)-j(-1-a)+k(b-1)$$

$$= (c+1)i+(a+1)j+(b-1)k$$

$$= 0$$

$\therefore$ $$a = -1,\ b = 1,\ c = -1$$

$\therefore$ $$\vec{V} = (x+y-z)i+(x+2y-z)j+(-x-y+2z)k$$

We find div $\vec{V}$ and express it in cylinderical coordinates.

$$\vec{V} = (x+y-z)i + (x+2y-z)j + (-x-y+2z)k$$

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= 1 + 2 + 2 = 5 \text{ (constant)}$$

Hence, divergence in cylinderical co-ordinates = 5.

3.6 Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$. Show that curl (curl $\vec{v}$) = grad(div $\vec{v}$) $-\nabla^2\vec{v}$.

(2018 : 12 Marks)

Solution:

$$\text{Curl (Curl } \vec{v}) = \nabla \times (\nabla \times \vec{v})$$

Now, we know that

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Given : $v_1\hat{i} + v_2\hat{j} + v_3\hat{k} = \vec{v}$

$$\because \quad \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\therefore \quad \nabla \times (\nabla \times \vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 v_2}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial v_3}{\partial z \partial x}\right) - \hat{j}\left(\frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_1}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z^2}\right) + \hat{k}\left(\frac{\partial^2 v_1}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_2}{\partial y \partial z}\right)$$

$$= \hat{i}\left(\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x}\right) + \hat{j}\left(\frac{\partial^2 y}{\partial x \partial y} + \frac{\partial^2 v_3}{\partial y \partial z}\right) + \hat{k}\left(\frac{\partial^2 v_1}{\partial x \partial z} + \frac{\partial^2 v_2}{\partial y \partial z}\right)$$

$$= -\hat{i}\left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) - \hat{j}\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2}\right) - \hat{k}\left(\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial y^2}\right)$$

$$= \hat{i}\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial x \partial y} + \frac{\partial^2 v_3}{\partial x \partial z}\right) + \hat{j}\left(\frac{\partial^2 v_1}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_3}{\partial y \partial z}\right) + \hat{k}\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2}\right) - \hat{i}\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) -$$

$$\hat{j}\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2}\right) - \hat{k}\left(\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_3}{\partial z^2}\right)$$

$$= \nabla(\nabla \cdot \vec{v}) - \nabla^2\vec{v} = \text{grad(div } \vec{v}) - \nabla^2\vec{v}$$

3.7 Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point (1, 1, 1).

(2019 : 10 Marks)

Solution:

Let $$\phi(x, y, z) = xy^2 + yz^2 + zx^2$$

Then, $$\text{grad}\,\phi = \left(\frac{\partial\phi}{\partial x}\right)i + \left(\frac{\partial\phi}{\partial y}\right)j + \left(\frac{\partial\phi}{\partial z}\right)k$$

$$\text{grad}\,\phi = (y^2 + 2zx)i + (z^2 + 2xy)j + (x^2 + 2yz)k$$

$$\text{grad}\,\phi_{\text{at}(1,1,1)} = 3i + 3j + 3k$$

Also, for the curve $x = t$, $y = t^2$ and $z = t^3$

We have, $$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t,\ \frac{dz}{dt} = 3t^2$$

At point (1, 1, 1) on the curve $x = t$, $y = t^2$, $z = t^3$

we have, $t = 1$

Now, vector along the tangent to the above curve at the point (x, y, z)

$$= \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} + \left(\frac{dz}{dt}\right)\hat{k}$$

$$= \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

Putting t = 1, a vector along the tangent to the curve at the point (1, 1, 1), we have

$$= \hat{i} + 2\hat{j} + 3\hat{k}$$

If $\hat{a}$ be the unit vector in the direction of this tangent then

$$\hat{a} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{|\hat{i} + 2\hat{j} + 3\hat{k}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{1+4+9}}$$

$$\hat{a} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

∴ The required directional derivative

$$= \hat{a}\ \text{grad}\,\phi \text{ at } (1, 1, 1)$$

$$= \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}(3\hat{i} + 3\hat{j} + 3\hat{k})$$

$$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

∴ $$\hat{a}\ \text{grad}\,\phi = \frac{18}{\sqrt{14}}.$$

4. Vector Identity and Vector Equations

4.1 Show that div(grad r^n) = $n(n + 1)r^{n-2}$

(2009 : 12 Marks)

Solution:

$$\nabla\cdot(\nabla r^n) = n(n+1)r^{n-2}$$

$$r^2 = x^2 + y^2 + z^2$$

Differentiating partially with respect to x on both sides

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } r^n = \sum \frac{\partial}{\partial x}\hat{i}(r^n) = \sum \frac{\partial r^n}{\partial x}\hat{i}$$

$$= \sum nr^{n-1}\frac{\partial r}{\partial x}\hat{i} = nr^{n-1}\sum \frac{x}{r}\hat{i}$$

$$= nr^{n-2}\vec{r}$$

where,
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div(grad } r^n) = \sum \frac{\partial}{\partial x}\hat{i}\cdot nr^{n-2}\sum x\hat{i} = \sum \frac{\partial}{\partial x}(nr^{n-2}x)$$

$$= \sum\left(nr^{n-2} + n(n-2)r^{n-3}\cdot x\frac{\partial r}{\partial x}\right)$$

$$= 3nr^{n-2} + \sum n(n-2)r^{n-3}\frac{x^2}{r}$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}\sum x^2$$

$$= n(n-2)r^{n-2} + 3nr^{n-2} = n(n+1)r^{n-2}$$

4.2 Find the directional derivative of

$$f(x, y) = x^2y^3 + xy$$

at the point (2, 1) in the direction of a unit vector which makes an angle of $\frac{\pi}{3}$ with the x-axis.

(2010 : 12 Marks)

Solution:

Given :
$$f(x, y) = x^2y^3 + xy$$

$\because$
$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \frac{\partial(x^2y^3+xy)}{\partial x}\hat{i} + \frac{\partial(x^2y^3+xy)}{\partial y}\hat{j}$$

$$= (2xy^3 + y)\hat{i} + (3x^2y^2 + x)\hat{j}$$

Now,
$$(\nabla f)_{(2,1)} = (2\times 2\times 1^3 + 1)\hat{i} + (3\times 2^2\times 1^2 + 2)\hat{j}$$

$$= 5i + 14j$$

$(\nabla f)_{(2,1)}$ is the direction of unit vector at angle $\frac{\pi}{3}$ with x-axis is

$$(\nabla f)_{(2,1)}\cdot\left(\cos\frac{\pi}{3}\hat{i} + \sin\frac{\pi}{3}\hat{j}\right) = (5\hat{i} + 14\hat{j})\cdot\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = \frac{5}{2} + 7\sqrt{3}$$

4.3 Show that the vector field defined by the vector function

$$\vec{v} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$$

is conservative.

(2010 : 12 Marks)

Solution:

Given :
$$\vec{v} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$$
$$= xy^2z^2\hat{i} + x^2yz^2\hat{j} + x^2y^2z\hat{k}$$
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$
$$= \hat{i}\left(\frac{\partial}{\partial y}x^2y^2z - \frac{\partial}{\partial z}x^2yz^2\right) - \hat{j}\left(\frac{\partial}{\partial x}x^2y^2z - \frac{\partial}{\partial z}xy^2z^2\right) + \hat{k}\left(\frac{\partial}{\partial x}x^2yz^2 - \frac{\partial}{\partial y}xy^2z^2\right)$$
$$= \hat{i}(2x^2y^2 - 2x^2y^2) - \hat{j}(2xy^2z - 2xy^2z) + \hat{k}(2xyz^2 - 2xyz^2) = 0$$

as $\nabla \times \vec{v} = 0$. $\therefore$ $\vec{v}$ is conservative.

4.4 Prove that :
$$div(f\vec{v}) = f(div\,\vec{v}) + (\text{grad}\,f)\cdot\vec{v}$$

where f is a scalar function.

(2011 : 20 Marks)

Solution:
$$\text{LHS} = div(f\vec{v})$$
$$\text{RHS} = f\,div\,\vec{v} + (grad\,f)\cdot\vec{v}$$

Take LHS :
$$div(f\vec{v}) = \Sigma i\frac{\partial}{\partial x}(f\vec{v})$$
$$= \left(\Sigma i\frac{\partial f}{\partial x}\right)\cdot\vec{v} + f\Sigma i\frac{\partial}{\partial x}(\vec{v})$$
$$= \text{grad}\,f\cdot\vec{v} + f(div\,\vec{v})$$
$$= f(\text{div}\,\vec{v}) + (\text{grad}\,f)\cdot\vec{v}$$
$$= \text{RHS}$$

So, LHS = RHS. Hence Proved.

4.5 If u and v are two scalar fields and $\vec{f}$ is a vector field, such that
$$u\vec{f} = \text{grad}\,v,$$

find the value of $\vec{f}\cdot\text{curl}\,\vec{f}$.

(2011 : 10 Marks)

Solution:

Given :
$$u\vec{f} = \text{grad}\,v$$
$$\Rightarrow \quad \vec{f} = \frac{1}{u}\cdot\text{grad}\,v$$
$$\therefore \quad \vec{f}\cdot\text{curl}\,\vec{f} = \left(\frac{1}{u}\,\text{grad}\,v\right)\cdot\text{curl}\left(\frac{1}{u}\,\text{grad}\,v\right)$$

$$= \left(\frac{1}{u}\text{ grad } v\right)\cdot\left[\left(\text{grad }\frac{1}{u}\right)\times(\text{grad } v)+\frac{1}{u}(\text{curl grad } v)\right]$$

$$\left[\because \text{curl }(\phi\vec{A}) = (\text{grad }\phi)\times\vec{A}+\phi \text{ Curl }\vec{A}\right]$$

$$= \left(\frac{1}{u}\text{ grad } v\right)\cdot\left[\left(\text{grad }\frac{1}{u}\right)\times(\text{grad } v)+0\right] \quad (\because \text{Curl grad }\phi = 0)$$

$$= \frac{1}{u}\left[\text{grad } v \quad \text{grad }\frac{1}{u} \quad \text{grad } u\right] = 0$$

[∵ If a vector repeats in a scalar triple product, then its value is zero].

4.6 Calculate $\nabla^2(r^n)$ and find its expression in terms of r and n, r being the distance of any point (x, y, z) from the origin, n being a constant and ∇^2 being the Laplace operator.

(2013 : 10 Marks)

Solution:

$$\nabla^2(r^n) = \nabla\cdot\nabla(r^n)$$

$$\nabla(r^n) = \sum\frac{\partial}{\partial x}(r^n)\hat{i} = nr^{n-1}\sum\frac{\partial r}{\partial x}\hat{i} = nr^{n-1}\sum\frac{\partial r}{\partial x}\hat{i}$$

Now $\quad r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore \quad \nabla(r^n) = nr^{n-1}\sum\frac{x}{r}\hat{i} = nr^{n-2}\vec{r}$

where $\quad \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\nabla\cdot\nabla(r^n) = \left(\sum\frac{\partial}{\partial x}\hat{i}\right)\cdot\left(nr^{n-2}\sum x\hat{i}\right)$$

$$= \sum\frac{\partial}{\partial x}(nr^{n-2}x)$$

$$= \sum\left(nr^{n-2}+n(n-2)r^{n-3}\frac{\partial r}{\partial x}\cdot x\right)$$

$$= \left(3nr^{n-2}+\sum n(n-2)r^{n-3}\frac{x^2}{r}\right)$$

$$= 3nr^{n-2}+n(n-2)r^{n-4}\Sigma x^2$$

$$= 3nr^{n-2}+n(n-2)r^{n-2}$$

$$= n(n+1)r^{n-2}$$

4.7 Find f(r) such that $\nabla f = \frac{\vec{r}}{r^5}$ and $f(1) = 0$.

(2016 : 10 Marks)

Solution:

We know that

$$\nabla f = f'(r)\nabla r = f'(r)\frac{\vec{r}}{r} \qquad \left(\because \nabla r = \frac{\vec{r}}{r}\right)$$

We have, $\quad \nabla f = \frac{\vec{r}}{r^5}$

$$\therefore \quad f'(r)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^5}$$

$$\Rightarrow \quad \vec{r}\left[\frac{f'(r)}{r} - \frac{1}{r^5}\right] = 0$$

Since, $\vec{r} \neq 0$, $\therefore$

$$f'(r) = \frac{1}{r^4}$$

Integrating, we get

$$f(r) = \frac{-1}{3r^3} + c$$

$$f(1) = 0 \Rightarrow 0 = \frac{-1}{3\cdot 1} + c \Rightarrow c = \frac{1}{3}$$

$$\therefore \quad f(r) = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$$

$$\left[\begin{array}{l} \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \\ \vec{r} = xi + yj + zk \\ r = \sqrt{x^2 + y^2 + z^2} \end{array}\right]$$

5. Application to Geometry

5.1 A curve in space is defined by the vector equation $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Determine the angle between the tangents to this curve at the $t = +1$ and $t = -1$.

(2013 : 10 Marks)

Solution:

$$\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2 + 4 + 9t^4}$$

$$\therefore \quad \hat{t} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{2 + \hat{i} + 2\hat{j} - 3t^2\hat{k}}{\sqrt{4t^2 + 4 + 9t^4}}$$

where $\hat{t}$ is the direction of the tangent

$$\hat{t}\Big|_{t=1} = \frac{2\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{17}}$$

$$\hat{t}\Big|_{t=-1} = \frac{-2\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{17}}$$

Let θ be the angle

Then

$$\cos\theta = \hat{t}_1 \cdot \hat{t}_2 = \frac{1}{17}(2\hat{i} + 2\hat{j} - 3\hat{k})\cdot(-2\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{-4 + 4 + 9}{17} = \frac{9}{17}$$

$$\therefore \quad \theta = \cos^{-1}\frac{9}{17}$$

5.2 Find the angle between the surfaces $x^2 + y^2 + z^2 - 9 = 0$ and $z = x^2 + y^2 - 3$ at (2, –1, 2).

(2015 : 10 Marks)

Solution:

Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $f_1 = x^2 + y^2 + z^2$

and $f_2 = x^2 + y^2 - z$

Then $\text{grad}(f_1) = 2xi + 2yj + 2zk$

$\text{grad}(f_2) = 2xi + 2yj - k$

Let $n_1 = \text{grad } f_1$ at point $(2, -1, 2) = 4i - 2j + 4k$

$n_2 = \text{grad } f_2$ at point $(2, -1, 2) = 4i - 2j - 4k$

The vectors n_1 and n_2 are along normals to the two surfaces at the point (2, –1, 2). If θ is the angle between these two vectors then,

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

$$= \frac{16+4-4}{\sqrt{16+4+16}\ \sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

5.3 Find the values of λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at (1, –1, 2).

(2015 : 12 Marks)

Solution:

Let $f_1 = \lambda x^2 - (\lambda + 2)x - \mu yz$

$f_2 = 4x^2y + z^3$

$\text{grad } f_1 = \nabla f_1 = (2\lambda x - \lambda - 2)i - \mu zj - \mu yk$

$\nabla f_2 = 8xyi + 4x^2j + 3z^2k$

At point (1, –1, 2)

$n_1 = (\lambda - 2)i - 2\mu j + \mu k$

$n_2 = -8i + 4j + 12k$

$n_1 \cdot n_2 = 0$ (intersect orthogonally)

$-8(\lambda - 2) - 8\mu + 12\mu = 0$

$-8\lambda + 4\mu = -16 \Rightarrow 2\lambda - \mu = 4$ ...(i)

Point (1, –1, 2) lies on surface $\lambda x^2 - \mu yz = (\lambda + 2)x$

$\therefore \quad \lambda + 2\mu = (\lambda + 2) \Rightarrow \mu = 1$

$\therefore \quad 2\lambda - 1 = 4 \Rightarrow \lambda = \frac{5}{2}$

5.4 Find the curvature vector and its magnitude at any point $\vec{r} = (\theta)$ of the curve $\vec{r} = (a\cos\theta, a\sin\theta, a\theta)$. Show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2 + y^2 - z^2 = a^2$.

(2017 : 16 Marks)

Solution:

$$\vec{r} = a\cos\theta i + a\sin\theta j + a\theta k$$

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta i + a\cos\theta j + ak$$

$$\frac{ds}{d\theta} = \left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{2}a$$

$$\vec{\tau} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{d\theta}}{\frac{ds}{d\theta}}$$

$$= \frac{1}{\sqrt{2}}(-\sin\theta i + \cos\theta j + k)$$

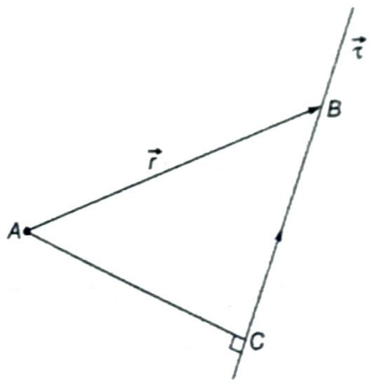
$$\frac{d\vec{\tau}}{d\theta} = \frac{1}{\sqrt{2}}\left(-\cos\theta\hat{i} - \sin\theta\hat{j}\right)$$

$\therefore$ Curvature vector,

$$\vec{k} = \frac{d\vec{\tau}}{ds} = \frac{d\vec{\tau}/d\theta}{ds/d\theta}$$

$$= \frac{1}{\sqrt{2}a}\cdot\frac{1}{\sqrt{2}}(-\cos\theta i - \sin\theta j)$$

$$= \frac{-\cos\theta i}{2a} - \frac{\sin\theta j}{2a}$$



Magnitude,

$$|K| = \frac{1}{2a}$$

Locus of feet of perpendicular from origin to tangent

$$\overrightarrow{AC} = \vec{r} - (\vec{r}\cdot\vec{\tau})\vec{\tau}$$

$$C = \text{Foot of perpendicular}$$

$$\vec{r}\cdot\vec{\tau} = (a\cos\theta i + a\sin\theta j + a\theta k)\cdot(-\sin\theta i + \cos\theta + k)\frac{1}{\sqrt{2}}$$

$$= \frac{a\theta}{\sqrt{2}}$$

$$\vec{r} - (\vec{r}\cdot\vec{\tau})\vec{\tau} = (a\cos\theta i + a\sin\theta j + a\theta k) - \frac{a\theta}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(-\sin\theta i + \cos\theta + k)$$

$$= \left(a\cos\theta + \frac{a\theta\sin\theta}{2}\right)i + \left(a\sin\theta - \frac{a\theta\cos\theta}{2}\right)j + \frac{a\theta}{2}k$$

Let foot of perpendicular be $C(x, y, z)$

$\therefore$

$$x = a\cos\theta + \frac{a\theta\sin\theta}{2}$$

$$y = a\sin\theta - \frac{a\theta\cos\theta}{2}$$

$$z = \frac{a\theta}{2}$$

$\therefore$

$$x^2 + y^2 - z^2 = a^2 + \frac{a^2\theta^2}{4} - \frac{a^2\theta^2}{4} = a^2$$

$x^2 + y^2 - z^2 = a^2$ is the required locus.

5.5 Find the angle between tangent at general point of curve whose equations are $x = 3t$, $y = 3t^2$, $z = 3t^3$ and the line $z - x = y = 0$.

(2018 : 10 Marks)

Solution:

The radius vector of given curve, $\vec{r}$ can be written as

$$\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 3t^3\hat{k}$$

$$\frac{d\vec{r}}{dr} = 3\hat{i} + 6t\hat{j} + 9t^2\hat{k}$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{9 + 36t^2 + 81t^4} = 3\sqrt{1 + 4t^2 + 9t^4}$$

Tangent vector,

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{1}{3\sqrt{1+4t^2+9t^4}} \times (3\hat{i} + 6t\hat{j})$$

$$\Rightarrow \quad \vec{T} = \frac{\hat{i} + 2t\hat{j} + 3t^2\hat{k}}{\sqrt{1+4t^2+9t^4}}$$

The given line can be written as

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$\therefore$ angle between line and tangent vector, θ

$$\cos\theta = \frac{1\times 1 + 2t\times 0 + 3t^2 \times 1}{\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}} = \frac{1+3t^2}{\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}}$$

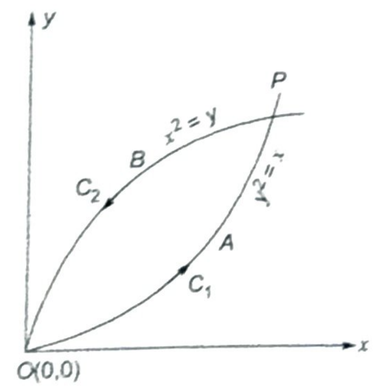
$$\theta = \cos^{-1}\left(\frac{1+3t^2}{\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}}\right)$$

5.6 Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = (2x + y^2)\hat{i} + (3y - 4x)\hat{j}$ and C is the curve $y = x^2$ from (0, 0) to (1, 1) and the curve $y^2 = x$ from (1, 1) to (0, 0).

(2019 : 15 Marks)

Solution:

Here the closed curve 'C' consists of arcs OAP and PBO. Let C_1 denote the arc OAP and C_2 denote arc PBO. Along C_1, we have $y = x^2$ so that $dy = 2x dx$ and x varies from $0 \to 1$ along C_2, we have $x = y^2$ so that $dx = 2y dy$ and y varies from 1 to also.



$$F.dr = [(2x+y^2)\hat{i}+(3y-4x)\hat{j}][dx\hat{i}+dy\hat{j}]$$

$$F.dr = (2x+y^2)dx+(3y-4x)dy$$

Now circulation of F round C.

$$\therefore \quad \oint_C F.dr = \oint_{C_1} \bar{F}.\bar{d}x + \oint_{C_2} \bar{F}.\bar{d}r$$

$$= \oint_{C_1}(2x+y^2)dx+(3y-4x)dy + \oint_{C_2}(2x+y^2)dx+(3y-4x)dy$$

$$= \int_{x=0}^{1}(2x+x^4)dx+(3x^2-4x)2xdx +$$

$$\int_{y=1}^{0}(2y^2+y^2)2ydy+(3y-4y^2)dy$$

$$= \int_0^1(2x-8x^2+6x^3+x^4)dx + \int_0^1(3y-4y^2+6y^3)dy$$

$$= \left[x^2-\frac{8}{3}x^3+\frac{3}{2}x^4+\frac{x^5}{5}\right]_0^1 + \left[\frac{3}{2}y^2-\frac{4y^3}{3}+\frac{8}{2}y^4\right]_1^0$$

$$= \left[1-\frac{8}{3}+\frac{3}{2}+\frac{1}{5}\right]-\left[\frac{3}{2}-\frac{4}{3}+\frac{3}{2}\right]$$

$$= \left[\frac{30-80+45+6}{30}\right]-\left[\frac{9-8+9}{6}\right]$$

$$= \frac{1}{30}-\frac{5}{3}=\frac{1-50}{30}=\frac{-49}{30}$$

$$\therefore \quad \oint_C \bar{F}\cdot\bar{d}r = \frac{-49}{30}$$

6. Curves in Space

6.1 Find k/z for the curve

$$\vec{r}(t) = a\cos t\hat{i}+a\sin t\hat{j}+bt\hat{k}$$

(2010 : 12 Marks)

Solution:

Given :

$$\vec{r}(t) = a\cos t\hat{i}+a\sin t\hat{j}+bt\hat{k}$$

$$\therefore \quad \frac{d\vec{r}(t)}{dt} = -a\sin t\hat{i}+a\cos t\hat{j}+b\hat{k}$$

$$\frac{d^2\vec{r}(t)}{dt^2} = -a\cos t\hat{i}-a\sin t\hat{j}$$

$$\frac{d^3\vec{r}(t)}{dt^3} = a\sin t\hat{i}-a\cos t\hat{j}$$

$$\therefore \quad \frac{d\vec{r}(t)}{dt}\times\frac{d^2\vec{r}(t)}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}(0 + ab\sin t) - \hat{j}(0 + ab\cos t) + \hat{k}(a^2 \sin^2 t + a^2 \cos^2 t)$$

$$= ab\sin t\hat{i} - ab\cos\hat{j} + a^2\hat{k}$$

$$\therefore \quad k = \frac{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{\left|ab\sin t\hat{i} - ab\cos t\hat{j} + a^2\hat{k}\right|}{\left|-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}\right|^3}$$

$$= \frac{(a^2b^2\sin^2 t - a^2b^2\cos^2 t + a^4)^{1/2}}{(a^2\sin^2 t + a^2\cos^2 t + b^2)^{3/2}} = \frac{(a^2b^2 + a^4)^{1/2}}{(a^2 + b^2)^{3/2}}$$

$$= \frac{a(a^2 + b^2)^{1/2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}$$

$$z = \frac{\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2} = \frac{\left(\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right) \cdot \frac{d^3r}{dt^3}}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

$$= \frac{(ab\sin t\hat{i} - ab\cos t\hat{j} + a^2\hat{k}) \cdot (a\sin t i - a\cos t\hat{j})}{\left|ab\sin t\hat{i} - ab\cos t\hat{j} + a^2\hat{k}\right|^2}$$

$$= \frac{a^2b\sin^2 t + a^2b\cos^2 t}{(a^2b^2 + a^4)} = \frac{a^2b}{a^2(a^2 + b^2)} = \frac{b}{a^2 + b^2}$$

$$\therefore \quad \frac{k}{z} = \frac{\frac{a}{a^2 + b^2}}{\frac{b}{a^2 + b^2}} = \frac{a}{b}$$

6.2 **Derive the Frenet-Serret formulae. Define the curvature and torsion for a space curve. Compute them for the space curve**

$$x = t,\ y = t^2,\ z = \frac{2}{3}t^3$$

Show that the curvature and torsion are equal for this curve.

(2012 : 20 Marks)

Solution:

The following three relations are known as Serret-Frenet formulae :

$$t' = kn \qquad \text{...(i)}$$

$$n' = \tau b - kt \qquad \text{...(ii)}$$

$$b' = -\tau n \qquad \text{...(iii)}$$

where, k is the magnitude of curvature, τ is the magnitude of torsion, t, n, b are the unit tangent vector, the unit principal normal vector and the unit bi-normal vector respectively and '1' denotes the differentiation w.r.t. the arc length S.

Derivation : We know

$$t^2 = 1$$

Differentiating it w.r.t. the arc length S,

$$t \cdot t' = 0 \Rightarrow t' \text{ is perpendicular to } t.$$

The equation of the oscillating plane at a point $P(r)$ of the curve is

$$[R - r, t, t'] = 0$$

This equation shows that t' lies in the oscillating plane and hence t' is perpendicular to the binormal b (since oscillating plane is perpendicular to b).

$\therefore$ t' is parallel to $b \times t$.

$\Rightarrow$ t' is parallel to n. $\quad (\because n = b \times t)$

$\Rightarrow$ $$t' = \pm kn$$

But we choose the direction n so that the curvature k is always positive, i.e., we take $t' = kn$ which proves (i).

Again, we know that $b^2 = 1$.

Differentiating w.r.t. the arc length S,

$$b \cdot b' = 0 \Rightarrow b' \text{ is perpendicular to } b.$$

Also, $$b \cdot t = 0 \qquad \text{...(iv)}$$

Differentiating w.r.t. S,

$$b' \cdot t + b \cdot t' = 0 \Rightarrow b' \cdot t + b \cdot kn = 0$$

i.e., $$b' \cdot t = 0 \qquad (\because b \cdot n = 0)$$

$\Rightarrow$ $$b' \perp t \qquad \text{...(v)}$$

From (iv) and (v), b' is perpendicular to both t and b.

$\Rightarrow$ b' is parallel to $b \times t$.

$\Rightarrow$ b' is parallel to n.

We may write, $\quad b' = \tau n$

By convention, we have $\quad b' = -\tau n$, which proves (iii).

We know, $\quad n = b \times t$

Differentiating w.r.t. S,

$$\begin{aligned} n' &= b' \times t + b \times t' \\ &= -\tau n \times t + b \times kn \qquad \text{(from (i) and (iii))} \\ &= \tau b - kt, \text{ which proves (ii).} \end{aligned}$$

Curvature : The arc rate at which the tangent changes direction as the point $P(r)$ moves along the curve is called the curvature vector of the curve and its magnitude is denoted by k.

Torsion : The arc rate at which the bi-normal changes direction as the point $P(r)$ moves along the curve is called the torsion vector of the curve and its magnitude is denoted by τ.

Given : $$x = t,\ y = t^2,\ z = \frac{2}{3}t^3$$

Let $$\vec{r} = \left(t, t^2, \frac{2}{3}t^3\right)$$

$$\dot{\vec{r}} = (1, 2t, 2t^2),\ \ddot{\vec{r}} = (0, 2, 4t)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}$$

$$= \vec{i}(8t^2 - 4t^2) + \vec{j}(0 - 4t) + \vec{k}(2)$$

$$= (4t_1^2 - 4t_1 + 2)$$

$\therefore$ $$\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right| = \sqrt{16t^4 + 4t^2 + 4} = 2(2t^2 + 1)$$

$$\left[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}\right] = \dot{\vec{r}} \times \ddot{\vec{r}} \times \dddot{\vec{r}}$$

$$= (4t_1^2 - 4t_1, 2)(0, 0, 4)$$
$$= 8$$

$\therefore$ Torsion, $K = \dfrac{\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right|}{\left|\dot{\vec{r}}\right|} = \dfrac{2(2t^2+1)}{(1+4t^2+4t^4)^{3/2}}$

$$= \frac{2}{(2t^2+1)^3}$$

Curvature, $\tau = \dfrac{\left[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}\right]}{\left|\dot{\vec{r}} \times \ddot{\vec{r}}\right|^2}$

$$= \frac{8}{4(2t^2+1)^2} = \frac{2}{(2t^2+1)^2}$$

6.3 Show that the curve $\vec{x}(t) = t\hat{i} + \left(\dfrac{1+t}{t}\right)\hat{j} + \dfrac{(1-t^2)}{t}\hat{k}$ lies in a plane.

(2013 : 10 Marks)

Solution:

For the curve to lie in a plane the binormal B must point in a constant direction as one moves along the curve.

i.e., $\dfrac{dB}{ds} = 0$

$\Rightarrow$ $-\tau N = 0 \Rightarrow \tau = 0$

i.e., torsion must be zero.

$$\tau = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{(\dot{r} \times \ddot{r}) \cdot (\dot{r} \times \ddot{r})}$$

Now, $\dot{r} = \dfrac{d\vec{x}}{dt} = \hat{i} + (-t^{-2})\hat{j} - (t^{-2}+1)\hat{k}$

$$\ddot{r} = \frac{d^2\vec{x}}{dt^2} = 2t^{-3}\hat{j} + 2t^{-3}\hat{k}$$

$$\dddot{r} = \frac{d^3\vec{x}}{dt^3} = -6t^{-4}\hat{j} - 6t^{-4}\hat{k}$$

$$\ddot{r} \times \dddot{r} = \begin{vmatrix} i & j & k \\ 0 & 2t^{-3} & 2t^{-3} \\ 0 & -6t^{-4} & -6t^{-4} \end{vmatrix} = 0$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = 0$$

$\therefore$ $\tau = 0$

So, the curve lies in a plane.

6.4 Find the curvature vector at any point of the curve $\vec{r}(t) = t\cos t\hat{i} + t\sin t\hat{j}$, $0 \le t \le 2\pi$. Give its magnitude also.

(2014 : 10 Marks)

Solution:

The position vector $\vec{r}$ of any point on the given curve is $\vec{r}$.

$$\vec{r} = t\cos t\hat{i} + t\sin t\hat{j}$$

i.e.,
$$\vec{r} = (t\cos t, t\sin t)$$

$\therefore$
$$\frac{d\vec{r}}{dt} = (\cos t - t\sin t + \sin t + t\cos t)$$

Now,
$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$$
$$= \sqrt{\cos^2 t + t^2\sin^2 t - 2t\cos t\sin t + \sin^2 t + t^2\cos^2 t + 2t\cos t\sin t}$$
$$= \sqrt{1+t^2}$$

Hence, T
$$= \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{1}{\sqrt{1+t^2}}(\cos t - t\sin t, \sin t + t\cos t)$$

Differentiating this w.r.t. S, we get

$$\frac{dT}{dS} = kN = \frac{dT/dt}{dS/dt}$$
$$= \frac{1}{\sqrt{1+t^2}} \cdot \frac{dT}{dt}$$
$$= \frac{1}{\sqrt{1+t^2}}\left[(-\sin t - \sin t - t\cos t)\sqrt{1+t^2} - \frac{1}{2\sqrt{1+t^2}}(2t)(\cos t - t\sin t),\right.$$
$$\left.(\cos t + \cos t - t\sin t)\sqrt{1+t^2} - \frac{1}{2\sqrt{1+t^2}}(2t)(\sin t + t\cos t) \cdot \frac{1}{(1+t^2)}\right]$$
$$= \frac{1}{\sqrt{1+t^2}} \cdot \frac{1}{(1+t^2)}\left[\frac{(-2\sin t - t\cos t)(1+t^2) - t(\cos t + \sin t)}{\sqrt{1+t^2}},\right.$$
$$\left.\frac{(2\cos t - t\sin t)(1+t^2) - t(t\sin t + t\cos t)}{\sqrt{1-t^2}}\right]$$

$$kN = \frac{1}{(1+t^2)^2}(-(2+t^2)(\sin t + \cos t), (2+t^2)(\cos t - \sin t))$$

$$kN = \frac{(2+t^2)}{(1+t^2)^2}(\sin t + \cos t)\hat{i} + \left(\frac{2+t}{(1+t)^2} \cdot \cos t - \sin t\right)\hat{j}$$

which is the required curvature vector.

$\therefore$
$$k = |kN|$$
$$= \sqrt{\left[\frac{2+t^2}{(1+t^2)^2}\right]^2((\sin t + \cos t)^2 + (\cos t - \sin t)^2}$$
$$= \frac{2+t^2}{(1+t^2)^2}\sqrt{2\sin^2 t + 2\cos^2 t}$$
$$= \frac{\sqrt{2}(2+t^2)}{(1+t^2)^2}$$

6.5 Find the curvature and torision of the curve :

$$\vec{r} = a(u - \sin u)\hat{i} + a(1 - \cos u)\hat{j} + bu\hat{k}$$

(2018 : 12 Marks)

Solution:

Given,
$$\vec{r} = a(u - \sin u)\hat{i} + a(1-\cos u)\hat{j} + bu\hat{k}$$
$$\frac{d\vec{r}}{du} = a(1-\cos u)\hat{i} + a\sin u\hat{j} + b\hat{k}$$
$$\frac{d^2\vec{r}}{du^2} = a\sin u\hat{i} + a\cos u\hat{j}$$
$$\frac{d^3\vec{r}}{du^3} = a\cos u\hat{i} - a\sin u\hat{j}$$
$$\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(1-\cos u) & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix}$$
$$= \hat{i}(-ab\cos u) - \hat{j}(-ab\sin u) + \hat{k}(a^2(\cos u - \cos^2 u) - a^2\sin^2 u)$$
$$= (-ab\cos u)\hat{i} + ab\sin u\hat{j} + \hat{k}(a^2(\cos u - 1))$$

Curvature,
$$K = \frac{\left|\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2}\right|}{\left|\frac{d\vec{r}}{du}\right|^3}$$
$$= \frac{\sqrt{a^2b^2\cos^2 u + a^2b^2\sin^2 u + a^2(\cos u - 1)^2}}{\left(\sqrt{a^2(1-\cos u)^2 + a^2\sin^2 u + b^2}\right)^3}$$
$$= \frac{\sqrt{a^2b^2 + a^4(\cos u - 1)^2}}{\left(\sqrt{2a^2 - 2a^2\cos u + b^2}\right)^3} = \frac{a\sqrt{b^2 + a^2(\cos u - 1)^2}}{\left(\sqrt{2a^2(1-\cos u) + b^2}\right)^3}$$

Torsion,
$$\tau = \frac{\left[\frac{d\vec{r}}{du}\ \frac{d^2\vec{r}}{du^2}\ \frac{d^3\vec{r}}{du^3}\right]}{\left|\frac{d\vec{r}}{du}\times\frac{d^2r}{du^2}\right|^2}$$
$$= \frac{-a^2b\cos^2 u - a^2b\sin^2 u}{a^2b^2 + a^4(\cos u - 1)^2} = \frac{-a^2b}{a^2b^2 + a^4(\cos u - 1)^2}$$

Thus,
$$K = \frac{a\sqrt{b^2 + a^2(\cos u - 1)^2}}{(2a^2(1-\cos u) + b^2)^{3/2}}$$
$$\tau = \frac{-a^2b}{a^2b^2 + a^4(\cos u - 1)^2}$$

6.6 Find the radius of curvature and radius of torsion of the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$.

(2019 : 15 Marks)

Solution:

Given :
$$x = a\cos u;\ y = a\sin u;\ z = au\tan\alpha$$

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = a\cos u\hat{i} + a\sin u\hat{j} + au\tan\alpha\hat{k}$$

$$k = \frac{\left|\frac{dr}{du} \times \frac{d_2 r}{du^2}\right|}{\left|\frac{dr}{du}\right|^3} \quad \text{...(2)}$$

$$F = \frac{\left|\frac{dr}{du} \times \frac{d^2 r}{du^2} \times \frac{d^3 r}{du^3}\right|}{\left|\frac{dr}{du} \times \frac{d^2 r}{du^2}\right|^2} \quad \text{...(3)}$$

From (1), differentiate w.r.t. u

$$\frac{dr}{du} = -a\sin u\hat{i} + a\cos u\hat{j} + a\tan\alpha\hat{k}$$

$$\frac{d^2 r}{du^2} = -a\cos u\hat{i} - a\sin u\hat{j} + 0\hat{k}$$

$$\frac{d^3 r}{du^3} = a\sin u\hat{i} - a\cos u\hat{j}$$

$$\left|\frac{dr}{dt}\right| = \sqrt{(-a\sin u)^2 + (a\cos u)^2 + (a\tan\alpha)^2}$$

$$= \sqrt{a^2(\sin^2 u + \cos^2 u + \tan^2\alpha}$$

$$= a\sqrt{1 + \tan^2\alpha} = a\sqrt{\sec^2\alpha}$$

$$\left|\frac{dr}{dt}\right| = a\sec\alpha$$

$$\frac{dr}{du} \times \frac{d^2 r}{du^2} = \begin{vmatrix} i & j & k \\ -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

$$= a^2\begin{vmatrix} i & j & k \\ -\sin u & \cos u & \tan\alpha \\ -\cos u & -\sin u & 0 \end{vmatrix}$$

$$= a^2\{\hat{i}(-\sin u\tan\alpha) - \cos u\tan\alpha\hat{j} + \hat{k}]$$

$$\frac{dr}{du} \times \frac{d^2 r}{du^2} = \sqrt{a^2(\sin^2 u\tan^2\alpha + \cos^2 u\tan^2\alpha) + a^4}$$

$$= a^2\sqrt{\tan^2\alpha(\sin^2 u + \cos^2 u) + 1}$$

$$= a^2\sqrt{\tan^2\alpha + 1} = a^2\sec\alpha$$

$$\text{Curvature, } K = \frac{a^2\sec\alpha}{(a\sec\alpha)^3}$$

$$K = \frac{1}{a}\cdot\frac{1}{\sec^2\alpha} = \frac{1}{a}\cos^2\alpha$$

$$\text{Radius of curvature} = \frac{1}{K}\cdot a\sec^2\alpha$$

$$\left[\frac{dr}{du}\times\frac{d^2r}{du^2}\times\frac{d^3r}{du^3}\right] = \begin{vmatrix} a\sin u & -a\cos u & 0 \\ -a\sin u & a\cos u & a\tan\alpha \\ a\cos u & a\sin u & 0 \end{vmatrix}$$

$$= a^3\begin{vmatrix} \sin u & -\cos u & 0 \\ -\sin u & \cos u & \tan\alpha \\ \cos u & \sin u & 0 \end{vmatrix} \quad R_2 \to R_2 + R_1$$

$$= a^3\begin{vmatrix} \sin u & -\cos u & 0 \\ 0 & 0 & \tan\alpha \\ \cos u & \sin u & 0 \end{vmatrix}$$

$$= a^3[\sin u(-\sin u\tan\alpha) + \cos u(-\cos u\tan\alpha) + 0]$$
$$= a^3[-(\sin^2 u\tan\alpha + \cos^2 u\tan\alpha)]$$
$$= -a^3[\tan\alpha(\sin^2 u + \cos^2 u)]$$
$$= -a^3\tan\alpha \qquad [\because \sin^2 u + \cos^2 u = 1]$$

$$T = \frac{|-a^3\tan\alpha|}{(a^2\sec\alpha)^2} = \frac{a^3\tan\alpha}{a^4\sec^2\alpha}$$

$$T = \frac{1}{a}\cdot\frac{1}{2}\sin^2\alpha = \frac{1}{2a}\sin 2\alpha$$

$$\text{Radius of tension} = \frac{1}{T} = \frac{a\sec^2\alpha}{\tan\alpha}$$

$$\text{Radius tension} = \frac{2(a)}{\sin 2\alpha} \text{ (Required Solution)}$$

7. Gauss-Divergence Theorem, Stokes Theorem and Green's Identity

7.1 Find the work done in moving the particle one round the ellipse $\frac{x^2}{25}+\frac{y^2}{16} = 1$, $z = 0$ under the field of force given by

$$\vec{F} = (2x-y+z)\hat{i} + (x+y-z)\hat{j} + (3x-2y+4z)\hat{k}$$

(2009 : 20 Marks)

Solution:

Let $C: \frac{x^2}{25}+\frac{y^2}{16} = 1$, $z = 0$ denote the ellipse.

$$\text{Work done} = \oint_C \vec{F}\cdot d\vec{r}$$

By Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}dS$$

where S is the surface of the ellipse

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y+z & x+y-z & 3x-2y+4z \end{vmatrix}$$

$$= -\hat{i}-2\hat{j}+2\hat{k}$$

For S normal is along positive z-axis

$$\therefore \qquad \hat{n} = \hat{k}$$

$$(\nabla\times\vec{F})\cdot\hat{n} = 2$$

$$\therefore \qquad \iint_S (\nabla\times\vec{F})\cdot\hat{n}dS = \iint_S 2dS = 2\times\text{Area of ellipse}$$

$$= 2\pi ab = 2\pi\cdot 5\cdot 4$$

$$= 40\pi$$

7.2 **Using divergent theorem evaluate $\iint_S \vec{A}\cdot d\vec{S}$ where $\vec{A} = x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$.**

(2009 : 20 Marks)

Solution:

By gauss divergence theorem

$$\iint_S \vec{A}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{A})dV$$

$$\nabla\cdot\vec{A} = \sum\frac{\partial}{\partial x}\hat{i}\cdot\vec{A} = 3x^2+3y^2+3z^2$$

$$= 3(x^2+y^2+z^2)$$

S is surface of sphere $x^2+y^2+z^2=a^2$.

$\therefore$ V is the volume compressing the whole sphere.

$$\iint_S \vec{A}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{A})dV$$

$$= \iiint_V 3(x^2+y^2+z^2)dV$$

Converting to spherical polar coordinates.

$$x^2+y^2+z^2 = r^2$$

$$dV = r^2\sin\theta\, dr d\theta d\phi$$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} 3r^2\cdot r^2\sin\theta dr d\theta d\phi$$

$$= \int_0^{2\pi}\left(\int_0^{2\pi}\int_0^{a}(3r^4dr)\sin\theta d\theta\right)d\phi$$

$$= \int_0^{\pi}\left[\frac{3a^5}{5}\int_0^{\pi}\sin\theta\, d\theta\right]d\phi$$

$$= \frac{3a^5}{5}[-\cos\theta]_0^{\pi}\pi = \frac{6\pi a^5}{5}$$

7.3 Find the value of $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the boundaring curve lies on the plane $z = 0$ when

$$\vec{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$$

(2009 : 20 Marks)

Solution:

The given surface S_1

$$x^2 + y^2 - 2ax + az = 0$$

$$\Rightarrow \quad (x-a)^2 + y^2 = a^2 - az$$

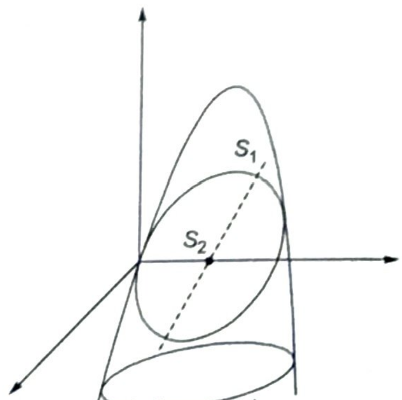
is of an inverted paraboloid as there is maximum value of zmax = a. Assumption a = +ve. Had 'a' been –ve, it would not be inverted paraboloid.

Consider the part of the paraboloid enclosed by S_1 and S_2 i.e., the circle of intersection in the XY-plane.

$$S_2 (x-a)^2 + y^2 = a^2,\ z = 0$$

These surfaces enclosed a volume.

Using gauss divergence theorem



$$\iint_{S=S_1+S_2} (\nabla\times\vec{F})\cdot d\vec{S} = \iiint_V \nabla\cdot(\nabla\times\vec{F})dV$$

$$= 0 \text{ as } \nabla\cdot(\nabla\times\vec{F}) = 0$$

$$\therefore \quad \iint_{S_1}(\nabla\times\vec{F})\cdot dS = -\iint_{S_2}(\nabla\times\vec{F})\cdot dS$$

Now for S_2, $\hat{n} = -\hat{k}$ (outward normal is towards downward direction).

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$$= (2y-2z)\hat{i} + 2(z-x)\hat{j} + 2(x-y)\hat{k}$$

$$\therefore \quad (\nabla\times\vec{F})\cdot\hat{n} = 2(y-x)$$

$$\therefore \quad I = -\iint_{S_2} 2(y-x)dS = \iint 2(x-y)dS$$

Converting to polar coordinates.

$$x = a + r\cos\theta$$

$$y = r\sin\theta$$

$$dS = r\,dr\,d\theta$$

$$I = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{a}(a + r\cos\theta - r\sin\theta)r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi}\left[\frac{ar^2}{2}+\frac{r^3}{3}(\cos\theta-\sin\theta)\right]_0^a d\theta$$

$$= 2\int_{\theta=0}^{2\pi}\frac{a^3}{2}+\frac{a^3}{3}(\cos\theta-\sin\theta)d\theta$$

$$= 2\cdot\frac{a^3}{2}\cdot 2\pi \text{ (as integral of } \sin\theta \text{ and } \cos\theta \text{ over 0 to } 2\pi \text{ is zero).}$$

$$= 2\pi a^3$$

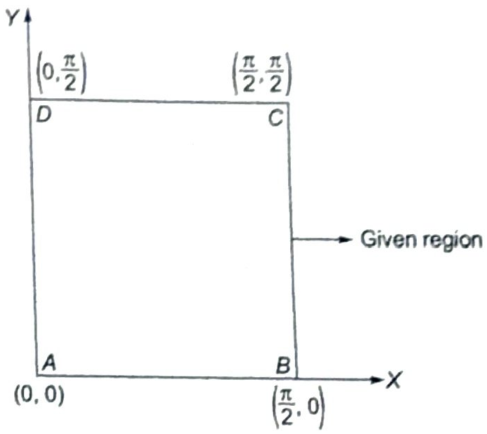
7.4 Verify Green's Theorem for

$$e^{-x}\sin y dx + e^{-x}\cos y dy$$

the path of integration being the boundary of the square whose vertices are (0, 0), $\left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\left(0,\frac{\pi}{2}\right)$.

(2010 : 20 Marks)

Solution:



By Green's Theorem,

$$\int(Mdx+Ndy) = \iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

Given equation is $e^{-x}\sin y dx + e^{-x}\cos y dy$

Here,

$$M = e^{-x}\sin y$$
$$N = e^{-x}\cos y$$

$$\therefore \quad \int(e^{-x}\sin y dx+e^{-x}\cos y dx) = \iint\left(\frac{\partial e^{-x}\cos y}{\partial x}-\frac{\partial e^{-x}\sin y}{\partial y}\right)dxdy \quad \text{(By Green's Theorem)}$$

R.H.S. : $\iint\left(\frac{\partial e^{-x}\cos y}{\partial x}-\frac{\partial e^{-x}\sin y}{\partial y}\right)dxdy$

$$= \iint(-e^{-x}\cos y-e^{-x}\cos y)dxdy$$

$$= -2\int_{y=0}^{\pi/2}\int_{x=0}^{\pi/2}e^{-x}\cos y dxdy$$

$$= -2\left[\frac{e^{-x}}{-1}\right]_0^{\pi/2}[\sin y]_0^{\pi/2} = \frac{+2}{+1}\left(e^{-\pi/2}-1\right)(1-0) = 2\left(e^{-\pi/2}-1\right)$$

L.H.S. : $\int e^{-x}\sin y dx + e^{-x}\cos y dy$

In figure, from A to B :

$$\int(e^{-x}\sin y dx + e^{-x}\cos y dy) = \int 0dx + 0 = 0$$

as $\sin y = 0$

and $dy = 0$

from B to C :

$$\int(e^{-x}\sin y dx + e^{-x}\cos y dy) = 0 + \int_{y=0}^{\pi/2} e^{-\pi/2}\cos y dy \qquad \left(dx = 0; x = \frac{\pi}{2}\right)$$

$$= e^{-\pi/2}[\sin y]_0^{\pi/2} = e^{-\pi/2}$$

from C to D :

$$\int(e^{-x}\sin y dx + e^{-x}\cos y dy) = \int_{x=\frac{\pi}{2}}^{0} e^{-x}dx \qquad \left(dy = 0, y = \frac{\pi}{2}\right)$$

$$= -\left[e^{-x}\right]_{\pi/2}^{0} = -\left[1 - e^{-\pi/2}\right] = e^{-\pi/2} - 1$$

from D to A :

$$\int(e^{-x}\sin y dx + e^{-x}\cos y dy) = 0 + \int e^{-x}\cos y dy \qquad (dx = 0, x = 0)$$

$$= \int_{y=\frac{\pi}{2}}^{0} \cos y dy = [\sin y]_{\pi/2}^{0} = 0 - 1 = -1$$

$\therefore$ Line integral along $A \to B \to C \to D \to A$ is $0 + e^{-\pi/2} + e^{-\pi/2} - 1 - 1$

$$= 2e^{-\pi/2} - 2 = 2(e^{-\pi/2} - 1)$$

$$= \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

Green's theorem is verified.

7.5 If $\vec{u} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$, calculate the double integral $\iint(\nabla \times \vec{u})\cdot d\vec{s}$ over the hemisphere given by

$$x^2 + y^2 + z^2 = a^2, z \geq 0.$$

(2011 : 15 Marks)

Solution:

Let C be the boundary of the sphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. Then C is a circle in the xy-plane of radius a and centre origin.

The equations of the curve C are

$$x^2 + y^2 = a^2, z = 0$$

Suppose $x = a\cos t$, $y = a\sin t$, $z = 0$; $0 \leq t < 2\pi$ are parametric equations of C.

Then using Stoke's theorem,

$$\iint_S \text{Curl } u \cdot d\vec{s} = \oint_C \vec{u}\cdot d\vec{r}$$

$$= \oint_C (4y\hat{i} + x\hat{j} + 2z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C 4ydx + xdy + 2zdz$$

$$= \oint_C (4ydx + xdy)\text{, as on } C,\ z = 0 \text{ and } dz = 0.$$

$$= \int_{t=0}^{2\pi} 4a\sin t(-a\sin t)dt + \int_{t=0}^{2\pi} a\cos t(a\cos t)dt$$

$$= -2a^2\int_0^{2\pi} 2\sin^2 t\,dt + \frac{a^2}{2}\int_0^{2\pi} 2\cos^2 t\,dt$$

$$= -2a^2\int_0^{2\pi} (1-\cos 2t)dt + \frac{a^2}{2}\int_0^{2\pi} (1+\cos 2t)dt$$

$$= -2a^2\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{a^2}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -2a^2[2\pi + 0] + \frac{a^2}{2}[2\pi - 0]$$

$$= -4\pi a^2 + \pi a^2 = -3\pi a^2$$

7.6 Verify gauss' Divergence Theorem for the vector

$$\vec{v} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

taken over the cube

$$0 \le x, y, z \le 1$$

(2011 : 15 Marks)

Solution:

We need to prove that

$$\iiint_V \nabla\cdot\vec{V}dV = \iint_S \vec{V}\cdot\vec{n}dS$$

Given : $\vec{V} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the cube $0 \le x, y, z \le 1$.

$$\nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= 2x + 2y + 2z = 2(x + y + z)$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{V}dV = \iiint_V 2(x+y+z)dV$$

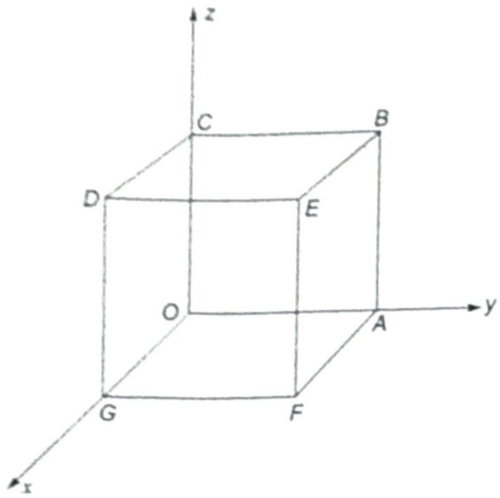
$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (x+y+z)dz\,dy\,dx$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\left(xz + yz + \frac{z^2}{2}\right)_0^1 dy\,dx$$

$$= 2\int_{x=0}^{1}\left(xy + \frac{y^2}{2} + \frac{y}{2}\right)_0^1 dx$$

$$= 2\int_{x=0}^{1}\left(x + \frac{1}{2} + \frac{1}{2}\right)dx = 2\int_{x=0}^{1}(x+1)dx$$

$$= 2\left(\frac{x^2}{2} + x\right)_0^1 = 2\left(\frac{1}{2} + 1\right) = 2\left(\frac{3}{2}\right) = 3$$



Now we will calculate $\iint_S \vec{V}\cdot\hat{n}dS$ over the six faces of the cube.

Over the face DEFG, $\hat{n} = \hat{i}$, $x = 1$

$$\therefore \quad \iint_{DEFG} \vec{V}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{y=0}^{1}(x^2\hat{i}+y^2\hat{j}+z^2\hat{k})\cdot\hat{i}\,dydz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} 1\cdot dy\,dz = \int_{z=0}^{1}(y)_0^1 dz$$

$$= \int_{z=0}^{1} dz = (z)_0^1 = 1$$

Over the face $ABCO$, $\hat{n} = -\hat{i}$, $x = 0$

$$\therefore \quad \iint_{ABCO} \vec{V}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{y=0}^{1} 0dydz = 0$$

Over the face $ABEF$, $\hat{n} = \hat{j}$, $y = 1$

$$\therefore \quad \iint_{ABEF} \vec{V}\cdot\hat{n}dS = \int_{z=0}^{1}\int_{x=0}^{1}(x^2\hat{i}+1^2\hat{j}+z^2\hat{k})\cdot\hat{j}dxdz$$

$$= \int_{z=0}^{1}\int_{x=0}^{1} dxdz = 1$$

Over the face $OGDC$, $\hat{n} = -\hat{j}$, $y = 0$

$$\therefore \quad \iint_{OGDC} \vec{V}\cdot\hat{n}dS = 0$$

Over the face $BCDE$, $\hat{n} = \hat{k}$, $z = 1$

$$\therefore \quad \iint_{BCDE} \vec{V}\cdot\hat{n}dS = \int_{x=0}^{1}\int_{y=0}^{1} dxdy = 1$$

Over the face AFGO, $\hat{n} = -\hat{k}$, $z = 0$

$$\therefore \quad \iint_{AFGO} \vec{V}\cdot\hat{n}dS = 0$$

Adding the six surface integrals, we get,

$$\iint_S \vec{F}\cdot\hat{n}dS = 1+0+1+0+1+0=3$$

Hence, the theorem is verified.

7.7 Verify Green's theorem in the plane for

$$\oint_C [(xy+y^2)dx + x^2dy]$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

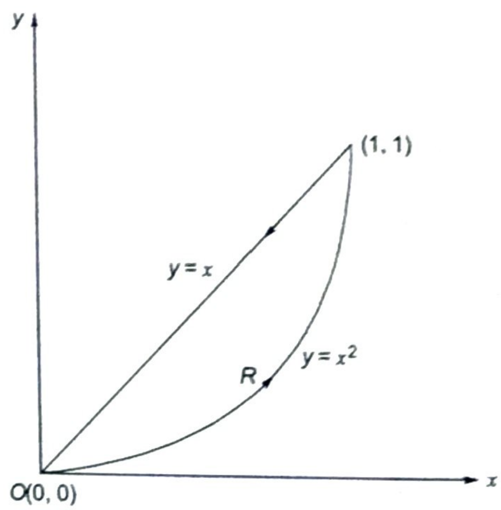
(2012 : 20 Marks)

Solution:

By Green's theorem, if R is the plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \oint_C (Mdx + Ndy)$$

Here, $M = xy + y^2,\ N = x^2$



We have, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

$$= \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right)dxdy$$

$$= \iint_R (\partial x - x - \partial y)dxdy = \iint_R (x - \partial y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x}(x-\partial y)dydx = \int_{x=0}^{1}[xy - y^2]_{y=x^2}^{x}\,dx$$

$$= \int_0^1 (x^4 - x^3)dx = \frac{-1}{20}$$

Again, along $y = x^2$, $dy = 2xdx$

$\therefore$ along $y = x$, the line integral becomes

$$\int_1^0 (x\cdot x + x^2)dx + x^2dx = \int_1^0 3x^2dx = -1$$

$\therefore$ The required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$

Hence, the theorem is verified.

7.8 If $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$, evaluate

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} d\vec{s}$$

where S is the surface of the sphere x2 + y2 + z2 = a2 above the xy-plane.

(2012 : 20 Marks)

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \vec{i}(-x+2x) + \vec{j}(0+y) + \vec{k}(1-2z-1)$$

$$= x\vec{i} + y\vec{j} - 2z\vec{k} \quad \text{...(i)}$$

A normal to the surface $x^2 + y^2 + z^2 = a^2$ is

$$\nabla(x^2+y^2+z^2) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$\therefore$ $\vec{n}$ = a unit normal to the surface $x^2 + y^2 + z^2 = a^2$ is given by

$$\vec{n} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2+4y^2+2z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{n}}{a} \quad [\because x^2 + y^2 + z^2 = a^2]$$

The projection of S on the xy-plane is the region R bounded by the circle $x^2 + y^2 = a^2$, $z = 0$.

Then
$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = \iint_R (\nabla \times \vec{F}) \cdot \vec{n} \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

$$= \iint_R (x\vec{i} + y\vec{j} - 2z\vec{k}) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right) \frac{dx dy}{z/a}$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2) - 2a^2}{\sqrt{a^2-x^2-y^2}} dy dx$$

Changing to polar co-ordinates, by substituting

$$x = r\cos\theta,\ y = r\sin x$$

and
$$dx dy = r dr d\theta$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} \cdot r dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3(r^2 - a^2) + a^2}{\sqrt{a^2 - r^2}} \cdot r dr d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\left(-3r\sqrt{a^2-r^2}+\frac{a^2r}{\sqrt{a^2-r^2}}\right)dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left((a^2-r^2)^{3/2}-a^2\sqrt{(a^2-r^2)}\right)_0^a d\theta$$

$$= \int_{\theta=0}^{2\pi}(a^3-a^3)d\theta = 0$$

7.9 By using Divergence Theorem of Gauss, evaluate the surface integral $\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2}dS$ where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$. a, b, c being all positive constants.
(2013 : 15 Marks)

Solution:

We have to write the integral in the form $\iint \vec{F}\cdot\hat{n}dS$

So we first find suitable $\vec{F}$

Now $\phi(x, y, z) = ax^2+by^2+cz^2 = 1$ is the given surface.

$$\text{Normal to surface} = \nabla\phi = 2ax\hat{i}+2by\hat{j}+2cz\hat{k}$$

$$\therefore \quad \hat{n} = \text{unit normal} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}}$$

Taking $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{F}\cdot\hat{n} = \frac{ax^2+by^2+cz^2}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}} = \frac{1}{(a^2x^2+b^2y^2+c^2z^2)^{1/2}} \text{ on } S$$

By Gauss divergence theorem

$$\iint \vec{F}\cdot\hat{n}dS = \iiint_V(\nabla\cdot\vec{F})dV = \iiint_V \nabla\cdot(x\hat{i}+y\hat{j}+z\hat{k})dV$$

$$= 3\iiint_V dV = 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{abc}}$$

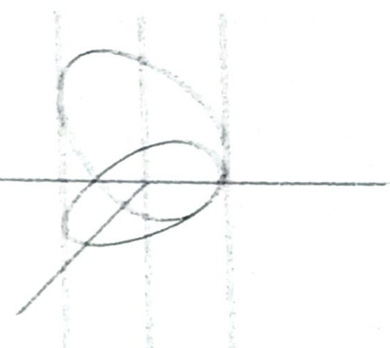
$$= \frac{4\pi}{\sqrt{abc}}$$

7.10 Use Stoke's theorem to evaluate the line integral $\int_C(-y^3dx+x^3dy-z^3dz)$ where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$.
(2013 : 15 Marks)

Solution:

By Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S(\nabla\times\vec{F})\cdot\hat{n}dS$$



$$\vec{F}\cdot d\vec{r} = F\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$
$$= -y^3dx+x^3dy-z^3dz$$

$$\therefore \quad \vec{F} = -y^3\hat{i}+x^3\hat{j}-z^3\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 3(x^2+y^2)\hat{k}$$

δ is part of the plane $x + y + z = 1$.

$\therefore$ Normal to $S = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ [(1, 1, 1) are direction ratio of normal]

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S 3(x^2+y^2)\hat{k}\cdot\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}dS$$

$$= \iint_S \sqrt{3}(x^2+y^2)dS$$

$$= \iint_S \sqrt{3}(x^2+y^2)dA \quad \text{(where } dA \text{ is projection on } XY \text{ plane)}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{3}r^2 r dr d\theta = \frac{\sqrt{3}}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi$$

7.11 Evaluate the Stoke's theorem $\int_\Gamma (ydx + zdy + xdz)$ where Γ is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0$, $x + y = 2a$, starting from $(2a, 0, 0)$ and then going below the z-plane.

(2014 : 20 Marks)

Solution:

The centre of the sphere $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ is the point $(a, a, 0)$. Since the plane $x + y = 2a$ passes through the point $(a, a, 0)$. Therefore the circle C is great circle of this sphere

$\therefore$ Radius of the circle C = Radius of the sphere $= \sqrt{(a^2+a^2)} = a\sqrt{2}$

Now
$$\int_C (ydx + zdy + xdz) = \int_C (yi + zj + xk)\cdot dr$$

$$= \iint_S [\text{curl}(yi + zj + xk)]\cdot nds$$

where S is any surface of which circle C is boundary [Stoke's theorem]

Now
$$\text{curl } (yi + zj + zk) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{dx} & \frac{\delta}{dy} & \frac{\delta}{dz} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} = -(\hat{i}+\hat{j}+\hat{k})$$

Let us take S as the surface of the plane $x + y = 2a$ bounded by the circle C. Then a vector normal to S is grad $(x+y) = \hat{i}+\hat{j}$.

$$\therefore \quad n = \text{Unit normal to } \delta = \frac{1}{2}(\hat{i}+\hat{j})$$

$$\therefore \int_C (ydx + zdy + xdz) = \iint_C (\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)ds$$

$$= \frac{-2}{\sqrt{2}}\iint_S ds = \frac{-2}{\sqrt{2}} \text{ (area of the circle of radius } a\sqrt{2})$$

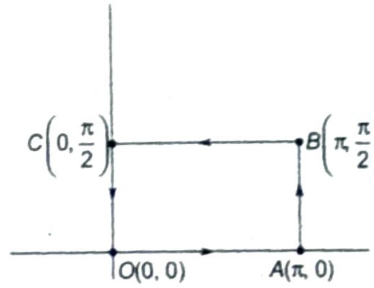
$$= -\sqrt{2}(2\pi a^2)$$

7.12 Evaluate : $\int_C e^{-x}(\sin y\, dx + \cos y\, dy)$, where C is the rectangle with vertices (0, 0), (π, 0), $\left(\pi, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$.

(2015 : 12 Marks)

Solution:

By Green's theorem in plane,



$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here,

$$M = e^{-x} \sin y,$$
$$N = e^{-x} \cos y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x} \cos y - e^{-x} \cos y = -2e^{-x} \cos y$$

$\therefore$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R -2e^{-x} \cos y dxdy$$

$$= -2 \int_{x=0}^{\pi} e^{-x} dx \int_{y=0}^{\pi/2} \cos y dy$$

$$= -2[-e^{-x}]_0^{\pi} [\sin y]_0^{\pi/2}$$

$$= 2(e^{-\pi} - 1) \cdot \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(e^{-\pi} - 1)$$

7.13 Prove that $\oint_C f d\vec{r} = \iint_S d\vec{S} + \nabla f$.

(2016 : 15 Marks)

Solution:

Let $\vec{p}$ be any arbitrary constant vector and let $\vec{F} = f\vec{p}$.

Using Stoke's theorem for $\vec{F}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = \iint_S (\nabla \times f\vec{p}) \cdot \hat{n} dS$$

$$= \iint_S [\nabla f \times \vec{p} + f(\nabla \times \vec{p})] \cdot d\vec{S}$$

$$= \iint_S (\nabla f \times \vec{p}) \cdot d\vec{S} \qquad (\because \text{Curl } \vec{p} = 0)$$

$\therefore$

$$\oint_C f\vec{p} \cdot d\vec{r} = \iint_S (\nabla f \times \vec{p}) \cdot d\vec{S}$$

$\Rightarrow$

$$\vec{p} \cdot \left[\oint_C (f) d\vec{r} - \iint_S d\vec{S} \times \nabla f\right] = 0 \qquad \left[\because (\nabla f \times \vec{p}) \cdot d\vec{S} = [\nabla f\, \vec{p}\, d\vec{S}] = [d\vec{S}\, \nabla f\, \vec{p}] = (d\vec{S} \times \nabla f) \cdot \vec{p}\right]$$

Since, $\vec{p}$ was arbitrary, hence

$$\oint_C f d\vec{r} = \iint_S d\vec{S} \times \nabla f$$

7.14 (i) Evaluate the integral : $\iint_S \vec{F}\cdot\vec{x}\,ds$ where $\vec{F} = 3xy^2i + (yx^2 - y^3)j + 3zx^2k$ and s is a surface of the cylinder $y^2 + z^2 \le 4$, $-3 \le x \le 3$, using divergence theorem.

(2017 : 9 Marks)

(ii) Using Green's theorem, evaluate $\int_C F(\vec{r})\cdot d\vec{r}$ counterclockwise, where

$$F(\vec{r}) = (x^2 + y^2)i + (x^2 - y^2)j \text{ and } d\vec{r} = dxi + dyj$$

and curve C is the boundary of the region, $R = \{(x, y): 1 \le y \le 2 - x^2\}$.

(2017 : 8 Marks)

Solution:

(i) Let V be the volume enclosed by surface, S. By divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}dS = \iiint_V \text{div}\,\vec{F}\cdot dV \qquad ...(i)$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(yx^2 - y^3) + \frac{\partial}{\partial z}(3zx^2)$$

$$= 4x^2$$

$\therefore$
$$\iint_S \vec{F}\cdot\hat{n}dS = \iiint_V 4x^2 dV,\ V \text{ bounded by } y^2 + z^2 = 4,\ x = \pm 3$$

In cylinderical co-ordinates

$$y = r\cos\theta,\ z = r\sin\theta,\ x = x$$

r : 0 to 2, θ : 0 to 2π, x : -3 to 3

$$\iint_S \vec{F}\cdot\hat{n}dS = 4\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{x=-3}^{3} x^2 r\,dr\,d\theta\,dx$$

$$= 4\int_0^2 rdr\int_0^{2\pi} d\theta\int_{-3}^{3} x^2dx$$

$$= 4\left[\frac{r^2}{2}\right]_0^2 [\theta]_0^{2\pi}\left[\frac{x^3}{3}\right]_{-3}^{3}$$

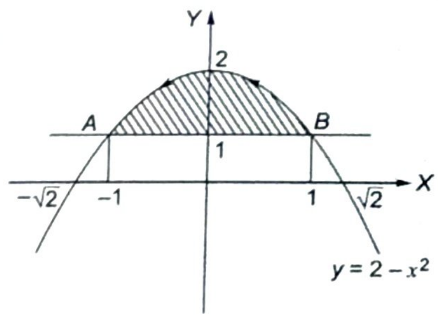
$$= 4\times\frac{4}{2}\times 2\pi\times 2\times\frac{27}{3} = 288\pi$$

(ii) We write

$$\int_C F(\vec{r})\cdot d\vec{r} = \int_C (x^2 + y^2)dx + (x^2 - y^2)dy$$

$$= \int_C Mdx + Ndy$$

$$M = x^2 + y^2$$
$$N = x^2 - y^2$$



C : Curve $ABCA$

S : Area shaded bounded by $y \ge 1$ and $y \le 2 - x^2$

Using Green's Theorem

$$\int_C Mdx + Ndy = \iint_S\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dS$$

$$= 2\iint_S (x - y)dxdy$$

$$= 2\int_{x=-\sqrt{2-y}}^{\sqrt{2-y}}\int_{y=1}^{2}(x-y)dxdy$$

$$= 2\int_{1}^{2}\left[\frac{x^2}{2}-yx\right]_{x=-\sqrt{2-y}}^{\sqrt{2-y}}dy$$

$$= -4\int_{1}^{2}y\sqrt{2-y}\,dy \qquad \text{...(i)}$$

$$I = -4\int_{1}^{2}y\sqrt{2-y}\,dy$$

Put $y = t + 1$, $dy = dt$

$$I = -4\int_{0}^{1}(1+t)\sqrt{1-t}\,dt$$

Let $t = \sin^2\theta$, $\quad dt = 2\sin\theta\cos\theta\, d\theta,\ \theta : 0 \to \frac{\pi}{2}$

$$= -4\int_{0}^{\pi/2}(1+\sin^2\theta)\cos\theta\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= -8\left[\int_{0}^{\pi/2}\sin\theta\cos^2\theta\, d\theta t+\int_{0}^{\pi/2}\sin^3\theta\cos^2\theta\, d\theta\right]$$

$$= -8\left[\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{1+2+2}{2}\right)}+\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{3+2+2}{2}\right)}\right]$$

$$= -8\left[\frac{1}{3}+\frac{2}{15}\right] = -\frac{56}{15}$$

$$\therefore \qquad \int_{0}^{\pi/2}\sin^P\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{P+q+2}{2}\right)}$$

7.15 If S is the surface of sphere $x^2 + y^2 + z^2 = a^2$, then evaluate

$$\iint (x+2)dydz + (y+2)dzdx + (x+y)dxdy$$

using Gauss's Divergence Theorem.

(2018 : 12 Marks)

Solution:

Given : $\iint (x+z)dydz+(y+z)dzdx+(x+y)dxdy$

$$= \iint \underbrace{((x+z)\hat{i}+(y+z)\hat{j}+(x+y)\hat{k})}_{\vec{F}}\cdot\hat{n}dS$$

Here, $$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

By Gauss's Divergence theroem,

$$\iint \vec{F}\cdot\hat{n}dS = \iiint \nabla\vec{F}dV$$

$$\nabla\vec{F} = 1+1+0=2$$

$$\therefore \quad \iiint \nabla\cdot\vec{F}dV = \iiint 2dV = 2\iiint dV = 2\times\frac{4}{3}\pi a^2$$

as volume of sphere is $\frac{4\pi}{3}a^3$.

$$\therefore \quad \iiint \nabla\cdot\vec{F}dV = \frac{8\pi}{3}a^3$$

7.16 Evaluate the line integral $\int_C -y^3dx + x^3dy + z^3dz$ using Stokes' theorem. Here, C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$. The orientation on C corresponds to counter-clockwise motion in xy-plane.

(2018 : 13 Marks)

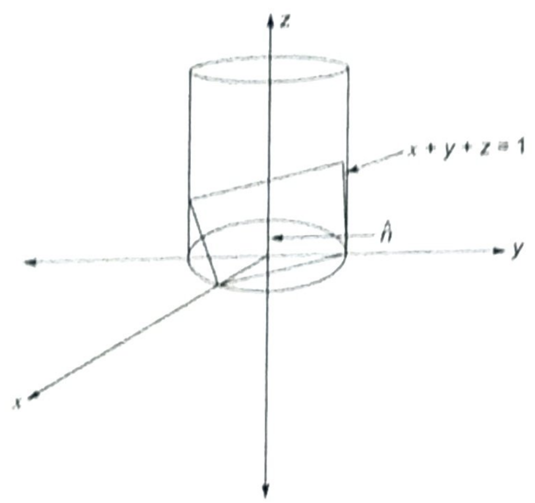
Solution:

Given :

$$\int_C -y^3dx + x^3dy + z^3dz = \int_C(-y^3\hat{i}+x^3\hat{j}+z^3\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_C \vec{F}\cdot dr$$

where, $$\vec{F} = -y^3\hat{i}+x^3\hat{j}+z^3\hat{k}$$



By Stoke's theorem, $$\int_C \vec{F}\cdot dr = \iint(\nabla\times\vec{F})\cdot dS$$

$$\nabla\times\vec{F} = 3(x^2+y^2)\hat{k}$$

$$\hat{n} = \frac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\therefore \quad \iint \nabla\times\vec{F}\cdot dS = \iint 3(x^2+y^2)\hat{k}\cdot\hat{n}\,dS$$

$$= 3\iint (x^2+y^2)\hat{k}\cdot\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}dS$$

$$= 3\iint (x^2+y^2)dS$$

Taking projection of $\hat{n}$ in xy-plane, it forms a circle with equation $x^2+y^2=1$, as plane of intersection is ellipse.

$$\therefore \quad \iint \nabla\times\vec{F}\cdot dS = \sqrt{3}\iint (x^2+y^2)\frac{dxdy}{\hat{n}\cdot\hat{k}}$$

$$= \sqrt{3}\iint (x^2+y^2)\frac{dxdy}{1/\sqrt{3}} =$$

$$= 3\iint (x^2+y^2)dxdy$$

In the region $x^2+y^2 \le 1$, let $x = r\cos\theta$, $y = r\sin\theta$, the integral becomes

$$\iint \nabla\times\vec{F}\cdot dS = 3\iint r^2\times r dr d\theta = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3 dr d\theta$$

$$= 3\times\frac{1}{4}\times 2\pi = \frac{3\pi}{2}$$

$$\therefore \quad \int \vec{F}\cdot dr = \iint(\nabla\times\vec{F})dS = \frac{3\pi}{2}$$

7.17 Let $\vec{F} = xy^2\hat{i} + (y+x)\hat{j}$. Integrate $(\nabla\times\vec{F})\hat{k}$ over the region in first quadrant bounded by the curves $y = x^2$ and $y = x$ using Green's theorem.

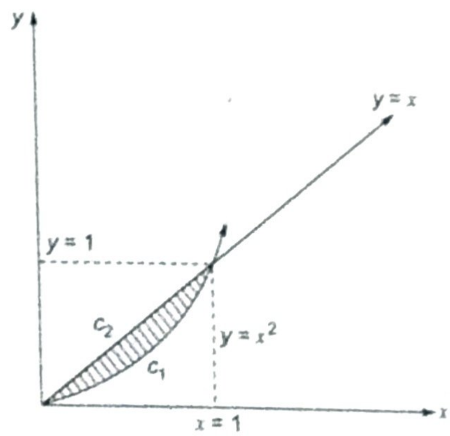
(2018 : 13 Marks)

Solution:

Given :
$$\vec{F} = xy^2\hat{i} + (y+x)\hat{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y+x & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(1-2xy) = (1-2xy)\hat{k}$$

$$\therefore \quad (\nabla\times\vec{F})\cdot\hat{k} = 1-2xy$$



Now, $$\iint(\nabla\times\vec{F})\cdot\hat{k}\,dx\,dy = \int P\,dx+Q\,dy$$ (By Green's theorem)

$$\Rightarrow \iint\left(\frac{\partial(y+x)}{\partial x}-\frac{\partial(xy^2)}{\partial y}\right)dx\,dy = \iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial x}\right)dx\,dy$$

Comparing LHS and RHS, we get

$$Q = y-x,\ P = xy^2$$

Therefore, we have to find $\int(xy^2)dx+(y+x)dy$ over C_1 and C_2.

Over C_1 :
$$\int_{C_1}(Pdx+Qdy) = \int_{C_1}(xy^2)dx+(y+x)dy = \int_{C_1}(xy^2)dx+\int_{C_1}(y+x)dy$$

$$= \int_{C_1}x^5dx+\int(x^2+x)\times 2nd\,x$$ (as $y = x^2$)

$$= \int_{x=0}^{1}x^5dx+2\int_{x=0}^{1}(x^3+x^2)dx = \frac{1}{6}+2\times\frac{1}{4}+2\times\frac{1}{3}$$

$$= \frac{1}{6}+\frac{1}{2}+\frac{2}{3} = \frac{8}{6} = \frac{4}{3}$$

Over C_2 :
$$\int_{C_2}Pdx+Qdy = \int_{C_2}(xy^2)dx+\int_{C_2}(y+x)dy$$

$$= \int_{C_2}x\cdot x^2dx+\int_{C_2}2xdx$$ (as $y = x$)

$$= \int_{x=1}^{0}x^3dx+\int_{x=1}^{0}2xdx = \left[\frac{x^4}{4}\right]_1^0+[x^2]_1^0$$

$$= \frac{-1}{4}-1 = \frac{-5}{4}$$

$$\therefore \int_C Pdx+Qdy = \int_{C_1}Pdx+Qdy+\int_{C_2}Pdx+Qdy$$

$$= \frac{4}{3}-\frac{5}{4} = \frac{16-15}{12} = \frac{1}{12}$$

7.18 (i) State Gauss divergence theorem. Verify this theorem for $\vec{F} = 4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$, taken over the region bounded by $x^2+y^2 = 4$, $z = 0$ and $z = 3$

(2019 : 15 Marks)

(ii) Evaluate by Stokes' theorem $\oint_C e^x dx+2ydy-dz$, where C is the curve $x^2+y^2 = 4$, $z = 2$.

(2019 : 5 Marks)

Solution:

(i) **Gauss Divergence Theorem** states that if v is the volume bounded by a closed surface S and 'A' is a vector function of position with continuous derivatives, then

$$\iiint_V \Delta\cdot\vec{A}dV = \iint_S \vec{A}\cdot\hat{n}dS$$

where 'n' is the positive outward drawn normal to S.

Now, $$\vec{F} = 4x\hat{i}+2y^2\hat{j}+z^2\hat{k}$$

taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

$\therefore$ By Gauss divergence theorem

$$\iiint_V \Delta \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} dS$$

$$\text{Volume integral} = \iiint_V \Delta \cdot \vec{F} du$$

$$= \iiint \left[\frac{\partial}{\partial x}(ux) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \right] dV$$

$$= \iiint_V (4 - 4y - 2z) dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{-4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z) dz dy dx$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{-4-x^2}}^{\sqrt{4-x^2}} [4z - 4yz + z^2]_0^3 dy dx$$

$$\left[\because \text{ Radius of circle} : x^2 + y^2 = 4, r = 2, y = \pm\sqrt{4 - x^2} \right]$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12 - 12y + 9] dy dx$$

Using polar coordinates

$$x = l \cos\theta;\ y = r \sin\theta$$

$$dx = r(-\sin\theta) d\theta;\ dy = r\cos\theta\, d\theta$$

$\Rightarrow$

$$dx dy = r d\theta dr$$

$$= \int_{0=0}^{2\pi} \int_{0}^{2} (21 - 12 r \sin\theta) r d \cdot d\theta$$

$$= \int_0^{2\pi} \left[\frac{21 l^2}{2} - \frac{r^3}{3} \sin\theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[42 - \frac{8}{3}\sin\theta \right] d\theta = \left[42\theta + \frac{8}{3}\cos\theta \right]_0^{2\pi}$$

$$= \left[84\pi + \left[\frac{8}{3}(1) - \frac{8}{3}(1) \right] \right] = 84\pi$$

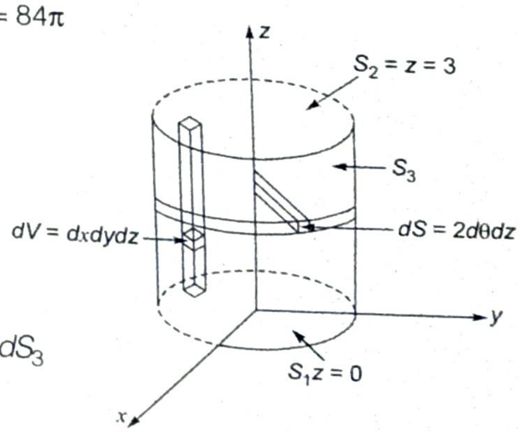
$\Rightarrow$

$$\iiint_V \Delta \cdot \vec{F} dV = 84\pi$$

The surface S of the cylinder consists of a base $S_1 (z = 0)$ the top $S_2 (z = 3)$ and the convex portion S_3 $(x^2 + y^2 = 4)$ then

$$\text{Surface Integral} = \iint_S \vec{F} \cdot \hat{n} dS$$

$$= \iint_{S_1} \vec{F} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{F} \cdot \hat{n} dS_2 + \iint_{S_3} \vec{F} \cdot \hat{n} dS_3$$



On S2 (z = 3): $\hat{n} = k$,

$$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$$

$$\vec{F}\cdot\hat{n} = 9, \text{ so that}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,dS_2 = \iint_{S_2} 9\,dS_2 = 9\cdot\pi r^2 = 9\pi(2)^2 \quad [\because r = 2]$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,dS_2 = 36\pi$$

On $S_3 (x^2 + y^2 = u)$

A perpendicular to $x^2 + y^2 = u$ has the direction

$$\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

Then a unit normal is

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}}$$

$$\hat{n} = \frac{2(x\hat{i} + y\hat{j})}{2\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{\alpha} \qquad [\because x^2 + y^2 = 4]$$

$$\vec{F}\hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\left(\frac{x\hat{i} + y\hat{j}}{2}\right)$$

$$\vec{F}\hat{n} = 2x^2 - y^3$$

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\hat{n}\,dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

$\therefore$ Then the surface integral

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$$\therefore \quad \iint_{2} \vec{F}\cdot\hat{n}\,dS = \iiint_V \vec{F}\,dV = 84\pi. \text{ Hence verified.}$$

(ii)

$$\vec{F} = e^x dx + 2y\,dy - dz$$

C is the curve

$$x^2 + y^2 = 4;\ z = 2$$

Using Strices theorem

$$\oint_C \vec{F}\,dr = \iint (\nabla \times \vec{F})\hat{n}\,dS$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$\nabla \times \vec{F} = \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$\nabla \times F = 0$$

$$\therefore \quad \oint_C e^x dx + 2y\,dy - dz = (\nabla \times \vec{F})\hat{n}\,dS = 0 \text{ which is the required result.}$$

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