IAS MATHEMATICS (OPT.)-2014

PAPER - II : SOLUTIONS

1(a) let 'G' be the set of all 2x2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ where, $xz \neq 0$, show that 'G' is a group under matrix multiplication.

let N denote the subset $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} ; a \in R \right\}$

Is 'N' a normal subgroup of 'G'? Justify your answer.

Sol^n:-

Let $(G, *)$ be an algebric structure. where '$*$' implies multiplication between its element in 'G' is the set of all 2x2 matrices of type $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$; $xz \neq 0$, $x, y, z \in IR$

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / xz \neq 0, x, y, z \in R \right\}.$$

i) closure property :-

$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G$, $x, y, z, a, b \in IR$ — (i)

$xz \neq 0$, $ac \neq 0$ — (ii)

$$\Rightarrow \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ax & xb+yc \\ 0 & zc \end{bmatrix} = A.$$

$ax \in IR$.

$(ax).(cz) = (ac)(xz) \neq 0$ — from (ii)

$xb + yc \in IR$,

$zc \in IR$ —— from (i).

$\therefore A \in G \implies (G, *)$ is closed.

ii) Associative pro :-

$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \in G,$ $x,y,z,a,b,c,p,q,r \in \mathbb{R}$

$xz \neq 0,\ ac \neq 0,\ pr \neq 0.$

$$\Rightarrow \left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} ax & bx+yc \\ 0 & zc \end{bmatrix} * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$$

$$= \begin{bmatrix} apx & aqx+bxr+ycr \\ 0 & zcr \end{bmatrix} \quad \text{— (iii)}$$

$$\Rightarrow \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \right) = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} ap & aq+br \\ 0 & cr \end{bmatrix}$$

$$= \begin{bmatrix} apx & aqx+bxr+ycr \\ 0 & zcr \end{bmatrix} \quad \text{— (iv)}$$

Here, (iii) = (iv)

i.e $\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \right).$

$\therefore$ $(G, *)$ satisfy associative property.

iii) Existence of left Identity :-

let $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in G$, where $xz \neq 0$, $x,y,z \in \mathbb{R}$. — (v)

let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G$, where $ac \neq 0$, $a,b,c \in \mathbb{R}$. — (vi)

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\begin{bmatrix} ax & ay+bz \\ 0 & cz \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

equating its element,

$$cz = z \Rightarrow z(c-1) = 0$$

From (vi) $ac \neq 0 \Rightarrow a \neq 0,\ c \neq 0$.

From (v) $xz \neq 0 \Rightarrow x \neq 0,\ z \neq 0$.

$$\therefore c-1 = 0 \Rightarrow \boxed{c=1}$$

$$ax = x \Rightarrow x(1-a) = 0$$

From(v) $x \neq 0$ $\quad 1-a = 0$

$$\boxed{a=1}$$

$$ay + bz = y$$

$$y + bz = y \Rightarrow bz = 0.$$

From(v) $z \neq 0$

$$\therefore \boxed{b=0}$$

$\therefore \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ left identity.

iv) <u>Existence of left Inverse</u> $\Rightarrow$

let $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \in G$ where, $xz \neq 0$, $pr \neq 0$
$x, y, z, p, q, r \in \mathbb{R}$.
$\neq 0$.

$$\therefore \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} px & py+qz \\ 0 & rz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equating the elements,

$px = 1$ | $rz = 1$ | $py + qz = 0$

$p = \frac{1}{x}$ | $r = \frac{1}{z}$ | $\frac{y}{x} + qz = 0$

$$q = -\left(\frac{y}{xz}\right)$$

$\therefore \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}$ — left inverse of $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ —(vii).

$\therefore (G, *)$ is a group.

$N = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} ; a \in \mathbb{R} \right\} \subset G.$

$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N \quad ; \quad N \neq \phi$

let $A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N \quad x \in \mathbb{R}$

$A^{-1} = \begin{bmatrix} 1/1 & -x/1 \\ 0 & 1/1 \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$ — from (vii)

i) $A^{-1} \in N.$

ii) let $B = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \in N \quad ; \quad y \in \mathbb{R}$

$A * B = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y+x \\ 0 & 1 \end{bmatrix} \in N.$

$\therefore N < G$ ('N' is subgroup of G).

For 'N' to be a normal subgroup.

$\forall \; X \in G \Rightarrow X * n * X^{-1} \in N.$
$n \in N.$

let $X = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, $X^{-1} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}$

$$xnx^{-1} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/x & -y/xz \\ 0 & 1/z \end{bmatrix}$$

$$= \begin{bmatrix} x & xx+y \\ 0 & z \end{bmatrix} * \begin{bmatrix} 1/x & -y/xz \\ 0 & 1/z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{-xy}{xz} + \frac{xx}{z} + \frac{y}{z} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{xx}{z} \\ 0 & 1 \end{bmatrix} \in N.$$

$\therefore$ $N \trianglelefteq G$. (Normal subgroup of G).

1(b) Test the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$.

Sol^n: Given that $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$.

Let $f(x) = \frac{1}{x^2(1+e^{-x})}$

let $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1 \neq 0$

Since $\int_1^\infty g(x) = \int_1^\infty \frac{1}{x^2}dx$ is convergent

Here $n=2>1$ $\left(\because \int_a^\infty \frac{dx}{x^n} \text{ is convergent } (a>0) \text{ iff } n>1\right)$

$\therefore$ By comparison test,

$\int_1^\infty f(x)\,dx$ is convergent.

$\int_1^\infty \frac{1}{x^2(1+e^{-x})}dx$ is convergent

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1(c) Prove that the function $f(z) = u+iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \ z \neq 0 \ ; \ f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z=0$ doesnot exist.

Sol^n: Here $u = \frac{x^3 - y^3}{x^2+y^2}$, $v = \frac{x^3+y^3}{x^2+y^2}$ where $z \neq 0$.

Here we see the both u and v are rational and finite for all values of $z \neq 0$.

so u and v are continuous at all those points for which $z \neq 0$.

Hence $f(z)$ is continuous where $\neq 0$.

At the origin $u=0, v=0$ [since $f(0) = 0$]

Hence u and v are both continuous at the origin. consequently $f(z)$ is continuous at the origin.

$\therefore f(z)$ is continuous every-where.

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \left(\frac{-y}{y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \left(\frac{y}{y}\right) = 1.$$

$\therefore$ we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Hence Cauchy-Riemann conditions are satisfied at $z=0$.

Again $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$

$= \lim_{z\to 0} \left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy}\right]$

Now let $z\to 0$ along $y=x$, then

$f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2} \cdot \frac{1}{x+ix}$

$= \lim_{x\to 0} \frac{2i}{2(1+i)}$

$= \frac{1}{2}(1+i)$

Again let $z\to 0$ along $y=0$, then

$f'(0) = \lim_{x\to 0} \frac{x^3(1+i)}{x^3} = 1+i$

$\therefore$ we see that $f'(0)$ is not unique.

i.e., the values of $f'(0)$ are not the same as $z\to 0$ along different curves.

$\therefore$ $f'(z)$ doesnot exist at the origin.

1(d) Expand in Laurent series the function

$f(z) = \frac{1}{z^2(z-1)}$ about $z=0$ and $z=1$.

Sol^n: Given that $f(z) = \frac{1}{z^2(z-1)}$.

About $z=0$, the Laurent series is given by

$$\frac{1}{z^2(z-1)} = \frac{1}{-z^2(1-z)} = -\frac{1}{z^2}(1-z)^{-1}$$

$$= -\frac{1}{z^2}(1+z+z^2+z^3+\cdots)$$

$$= -\left(\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\cdots\right)$$

Let $z-1=u \Rightarrow z=u+1$ and

$$\frac{1}{z^2(z-1)} = \frac{1}{(u+1)^2 u} = \frac{1}{u}(1+u)^{-2}$$

$$= \frac{1}{u}\left[1-(2)(u)+\frac{(-2)(-3)}{2!}(u)^2+\frac{(-2)(-3)(-4)}{3!}(u)^3+\cdots\right]$$

$$= \frac{1}{u}\left[1-2u+3u^2-4u^3+\cdots\right]$$

1(e) Solve graphically:

Maximise $Z = 6x_1 + 5x_2$

subject to

$$2x_1 + x_2 \le 16$$
$$x_1 + x_2 \le 11$$
$$x_1 + 2x_2 \ge 6$$
$$5x_1 + 6x_2 \le 90$$
$$x_1, x_2 \ge 0$$

Sol:-

Since every point which satisfies the condition $x_1 \ge 0$, $x_2 \ge 0$ lies in the first quadrant only.

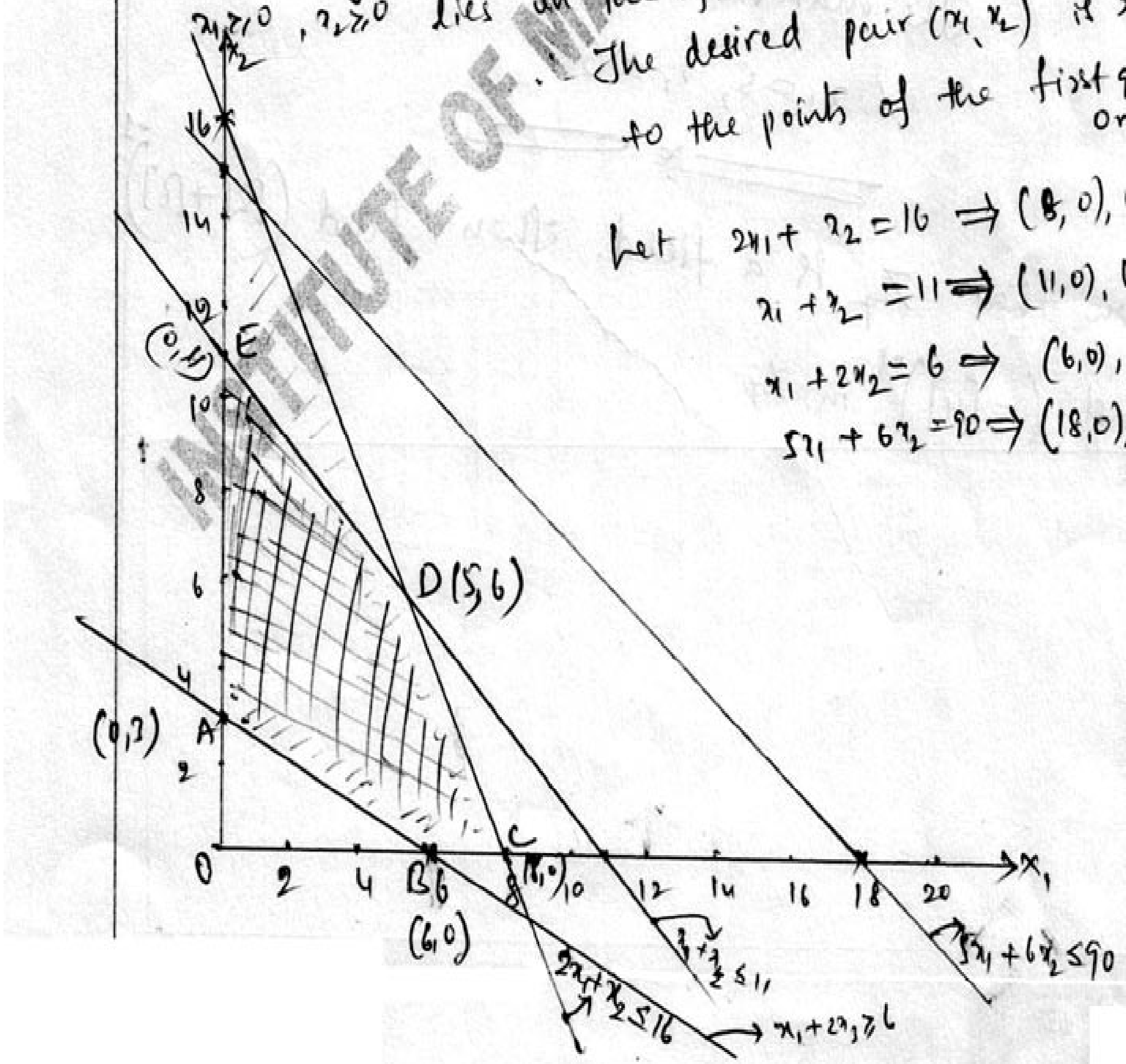
$\therefore$ The desired pair (x_1, x_2) is restricted to the points of the first quadrant only.

Let $2x_1 + x_2 = 16 \Rightarrow (8, 0), (0, 16)$

$x_1 + x_2 = 11 \Rightarrow (11, 0), (0, 11)$

$x_1 + 2x_2 = 6 \Rightarrow (6, 0), (0, 3)$

$5x_1 + 6x_2 = 90 \Rightarrow (18, 0), (0, 15)$



The shaded region ABCDE is the feasible region corresponding to the given constraints, with $A(0,3)$, $B(6,0)$, $C(8,0)$, $D(5,6)$, $E(0,11)$ as the extreme points.

The values of the objective function $z = 6x_1 + 5x_2$ at these extreme points are

$z(0,3) = 0 + 15 = 15$

$z(6,0) = 36 + 0 = 36$

$z(8,0) = 48 + 0 = 48$

$z(5,6) = 30 + 30 = 60 \longrightarrow$

$z(0,11) = 0 + 55 = 55$

The value of z is maximum at $D(5,6)$.

$\therefore$ The max value value is $z = 60$

$\therefore x_1 = 5, x_2 = 6$

Q2(a) 2014 Show that $\mathbb{Z}_7$ is a field. Then find $([5]+[6])^{-1}$ and $(-[4])^{-1}$ in $\mathbb{Z}_7$

Solⁿ:-

$$\mathbb{Z}_7 = \{[0],[1],[2],[3],[4],[5],[6]\}.$$

For Z_7 to be a field, it should satisfy

i) $(Z_7,+)$ is an abelian.

ii) $(Z_7,\times)$ is an abelian group, where $Z_7^* = Z_7 - \{0\}$.

iii) Distributive law.

i) $(\mathbb{Z}_7,+)$

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

- All elements belong to $\mathbb{Z}_7$ - closure pro.
- First Row coincide with the top row, then [0] is identity element.
- [0] is in every row & coloumn
  ∴ Inverse property satisfied.

– $([a]+[b])+[c] = [a+b]+[c] = [a+b+c]$

$= [a]+[b+c] = [a]+([b]+[c])$

$\therefore$ Associative pro. satisfied.

– Transpose of matrix equal to original matrix.

$\therefore$ commutative pro. satisfy.

ii) $(\mathbb{Z}_7^*, \times)$.

$\times$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

– every element belong to $\mathbb{Z}_7^*$ – closure pro.

– Top row co-incide with First Row, Hence [1] is an Identity element.

– [1] is in every row & column.

$\therefore$ Inverse pro. satisfied.

– $([a]\times[b])\times[c] = [(a\times b)]\times[c] = [a\times b\times c]$

$= [a]\times[b\times c] = [a]\times([b]\times[c])$.

$\therefore$ Associative pro. satisfied.

- Transpose of matrix is equal to original matrix.

$\therefore$ commutative pro. satisfy.

iii) Distributivity :-

$[a]\times([b]+[c]) = [a]\times([b+c]) = [a(b+c)]$

$= [ab+ac] = [ab]+[ac]$

$= [a][b]+[a][c]$.

$([b]+[c])\times[a] = [b][a]+[c][a]$ — similarly.

$\Rightarrow ([5]+[6])^{-1} = [4]^{-1} = [2]$

$\Rightarrow (-[4])^{-1} = ([3])^{-1} = [5]$

Ques) 2(b) $\int_0^1 f(x)\,dx$ (Integrate), where.

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Sol. The function

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}; & x \in (0,1) \\ 0; & x = 0 \end{cases}$$

is not continuous on $[0,1]$; (it is discontinuous at $x=0$), but it is bounded and continuous on $(0,1]$ and thus Riemann-integral on $[0,1]$.

The function:

$$g(x) = \begin{cases} x^2\sin\frac{1}{x}; & x \in (0,1] \\ 0; & x = 0 \end{cases}$$

is differentiable on $[0,1]$ and satisfies

$$g'(x) = f(x); \quad \forall x \in [0,1]$$

$$\therefore \int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx = g(1) - g(0)$$

$$= (1)^2 \cdot \sin\frac{1}{1} - 0$$

$$= 1\cdot\sin 1 - 0 = \sin 1.$$

$$\therefore \int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx = \sin 1$$

2(c) Find the initial basic feasible solution to the following transportation problem by Vogel's approximation method. Also, find its optimal solution and the minimum transportation cost:

Destinations

		D_1	D_2	D_3	D_4	Supply
Origins	O_1	6	4	1	5	14
	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
Demand →		6	10	15	4	

Sol^n: Since the total supply and total demand being equal.

$\therefore$ The transportation problem is balanced.

Find the initial basic feasible solution:

Using Vogel's Approximation method, the initial basic feasible solution is:

The differences between the smallest and next to the smallest costs in each row and each column are first computed and displayed inside parenthesis against the respective rows and columns

6	4	1	5	14	(3)
8	9	(15) 2	7	16	(5) ⇐
4	3	6	2	5	(1)
6	10	15	4		
(2)	(1)	(1)	(3)		

The largest of these differences is (5) which is associated with the 2^{nd} row.
Since $c_{23} = 2$, is the minimum cost, we allocate $x_{23} = \min(15, 16) = 15$ in the cell $(2,3)$.
This exhausts the demand of the 3^{rd} column and therefore we cross it.

Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are as shown below.

6	4	5	14	(1)
8	9	7	1	(1)
4	3	(4) 2	5	(1)
6	10	4		
(2)	(1)	(3) ↑		

6	4	14	(2)
8	9	1	(1)
(1) 4	3	1	(1)
6	10		
↑ (2)	(1)		

(4) 6	(10) 4	14	(2)
(1) 8	9	1	(1)
5	10		
(2)	(5)		

∴ The initial basic feasible solution is

(4) 6	(10) 4	1	5
(1) 8	9	(15) 2	7
(1) 4	3	6	(4) 2

The no. of allocations $= m+n-1$
$= 3+4-1 = 6$
(basic variables)

Now, find the values of u_i and v_j:

As the maximum no. of basic cells exist in the first row.

$\therefore$ Let $v_1 = 0$

				$u_i \downarrow$
(4) 6	(10) 4	(–) 1	(–) 5	6
(1) 8	(–) 9	(15) 2	(–) 7	8
(1) 4	(–) 3	(–) 6	(4) 2	4
$v_j \rightarrow$ 0	–2	–6	–2	

The net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unoccupied cells are less than or equal to zero

i.e., $\Delta_{ij} \le 0$.

$\therefore$ The current basic feasible solution is optimal.

Hence the optimal allocation is given by

$x_{11} = 4,\ x_{12} = 10,\ x_{21} = 1,\ x_{23} = 15,\ x_{31} = 1,\ x_{34} = 4.$

$\therefore$ The optimal (minimum) transportation

$$\text{cost} = 6\times4 + 10\times4 + 1\times8 + 15\times2 + 1\times4 + 4\times2$$
$$= 24 + 40 + 8 + 30 + 4 + 8$$
$$= \underline{114}.$$

3(a) 2014. Show that the set $\{a+b\omega : \omega^3=1\}$ where, 'a' & 'b' are real no. is a field w.r.t usual addition and multiplication.

Sol^n:-

let $G = \{(a+b\omega) ; \omega^3=1 ; a,b \in \mathbb{R}\}$.

i) let $(G,+)$ be an algebric structure.

$\forall (a+b\omega), (c+d\omega) \in G$.

$(a+b\omega)+(c+d\omega) = (a+c)+(b+d)\omega$

$= p+q\omega$ — $(a+c) \in \mathbb{R}$

$\in G$ $\quad (b+d) \in \mathbb{R}$.

$\therefore$ closure pro. satisfy.

ii) let, $(a+b\omega), (c+d\omega), (e+f\omega) \in G$.

$[(a+b\omega)+(c+d\omega)]+(e+f\omega)$

$=((a+c)+(b+d)\omega)+(e+f\omega)$

$=(a+c+e)+(b+d+f)\omega$ — ⓐ $\quad (a+c+e) \in \mathbb{R}$. $(b+d+f) \in \mathbb{R}$.

$(a+b\omega)+[(c+d\omega)+(e+f\omega)]$

$=(a+b\omega)+[(c+e)+(d+f)\omega]$

$= (a+c+e)+(b+d+f)\omega$ — ⓑ

From (a) & (b) $\Rightarrow$ Asso. pro. satisfy.

iii) let $(a+b\omega) \in G$, $0 \in G$.

$(a+b\omega)+0 = (a+b\omega)$.

∴ '0' is the identity element of $(G,+)$.

iv) let $(a+b\omega) \in G$, $(-a-b\omega) \in G$.

$$(a+b\omega)+(-a-b\omega) = (a-a)+(b-b)\omega$$
$$= 0$$

∴ Inverse pro. satisfied.

v) let $(a+b\omega), (c+d\omega) \in G$.

$$\therefore\ a+b\omega + c+d\omega = (a+c)+(b+d)\omega$$
$$= (c+a)+(d+b)\omega$$
$$= c+d\omega+a+b\omega$$

∴ commutative pro. satisfied.

∴ $(G,+)$ is an abelian group.

II) $(G^*, \times)$ $\qquad G^*$ denote $G-\{0\}$.

i) let $(a+b\omega), (c+d\omega) \in G$.

$$(a+b\omega)\times(c+d\omega) = ac+bd\omega^2+(ad+bc)\omega$$
$$= ac+bd(-1-\omega)+(ad+bc)\omega$$
$$= (ac-bd)+\omega(ad+bc-bd) \in G.$$

∴ closure pro. satisfied.

ii) Associative property is satisfies over complex numbers.

iii)
$$(a+b\omega)(c+d\omega) = (a+b\omega)$$

let $c=1$; $d=0$

$(a+b\omega)(1) = (a+b\omega) \quad 1 \in G^*$

$\therefore$ Identity pro. satisfied.

iv) $(a+b\omega)(c+d\omega) = 1$

$= (ac-bd) + \omega(bc+ad-bd) = 0$

$$c+d\omega = \frac{1}{a+b\omega} \times \frac{a+b\omega^2}{a+b\omega^2}$$

$$= \frac{a+b\omega^2}{a^2+b^2\omega^3+ab(\omega+\omega^2)}$$

$$= \frac{a+b(-1-\omega)}{a^2+b^2+ab(-1)} = \frac{a-b-b\omega}{a^2+b^2+(-ab)}.$$

$$= \frac{(a-b)}{a^2+b^2-ab} + \frac{(-b\omega)}{a^2+b^2-ab}$$

$\therefore \quad c+d\omega = \alpha + \beta\omega$

where,

$$\alpha = \frac{a-b}{a^2+b^2-ab}, \quad \beta = \frac{-b}{a^2+b^2-ab}$$

$(a-b)^2 \geq 0$

$a^2+b^2 \geq 2ab > ab$

as $a, b \neq 0$

$\therefore$ Inverse exists.

v) $(a+b\omega).(c+d\omega) = ac + bc\omega + ad\omega + bd\omega^2$

$= c(a+b\omega) + d(a\omega + b\omega^2).$

$= (c+d\omega).(a+b\omega)$

$\therefore$ commutative satisfy.

$\therefore (G^*, \times)$ is an abelian group.

III) <u>Distributive</u> <u>pro</u>:-

$(a+b\omega)\times[(c+d\omega)+(e+f\omega)]$

$= (a+b\omega)\times[(c+e)+(d+f)\omega]$

$= ac+ae+bc\omega+be\omega+(ad+af)\omega+(bd+bf)\omega^2$

$= (a+b\omega)\times(c+d\omega)+(a+b\omega)\times(e+f\omega).$

Q.3(b) Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x,y)=\begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

Also; discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$?

Sol: Given; $f(x,y)=\begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & ;(x,y)\neq(0,0) \\ 0 & ;(x,y)=0 \end{cases}$

Now;
$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{0-0}{k}=\frac{0}{k}=0$$

$$f_y(h,0)=\lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k}$$

$$=\lim_{k\to 0}\frac{\frac{hk(3h^2-2k^2)}{h^2+k^2}-0}{k}$$

$$=\lim_{k\to 0}\frac{h\cancel{k}(3h^2-2k^2)}{\cancel{k}(h^2+k^2)}$$

$$=\frac{h(3\cancel{h^2})}{\cancel{h^2}}=3h.$$

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\frac{3h-0}{h}$$

$$\boxed{f_{xy}(0,0)=3}$$

Again;

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

But $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}k(3h^2-2k^2)-0}{\cancel{h}(h^2+k^2)}$$

$$= \lim_{h\to 0} \frac{k\cdot(3h^2-2k^2)}{h^2+k^2}$$

$$= \frac{-k\cdot 2\cancel{k^2}}{\cancel{k^2}} = -2k.$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \frac{-2k}{k} = -2$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$$

$\therefore f(x,y)$ is not a continuous function.

3(c) Evaluate the integral $\int_0^{\pi}\frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}$ using residues.

Sol^n: Let $I=\int_0^{\pi}\frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}$

$$=\frac{1}{2}\int_0^{2\pi}\frac{4\,d\theta}{(2+\cos\theta)^2}$$

$$=\int_0^{2\pi}\frac{2\,d\theta}{(2+\cos\theta)^2}$$

Let the contour 'C' be the unit circle $|z|=1$ with centre at the origin.

Let $z=e^{i\theta}$ then $\cos\theta=\frac{1}{2}(z+\frac{1}{z})$

$\Rightarrow dz=ie^{i\theta}d\theta$

$\Rightarrow d\theta=\frac{dz}{iz}$

$$\therefore I=\int_0^{2\pi}\frac{2\,d\theta}{(2+\cos\theta)^2}=\int_0^{2\pi}\frac{2\,dz}{iz\left(2+\frac{z^2+1}{2z}\right)^2}$$

$$=\frac{1}{i}\int_0^{2\pi}\frac{8z}{(z^2+4z+1)^2}\,dz$$

$$=\frac{8}{i}\int_C\frac{z}{(z^2+4z+1)^2}\,dz$$

$$=\frac{8}{i}\int_C f(z)\,dz.\quad\text{where } f(z)=\frac{z}{(z^2+4z+1)^2}\qquad\text{---(1)}$$

$$=\frac{8}{i}\int_C f(z)\,dz.\quad\text{where } f(z)=\frac{z}{(z^2+4z+1)^2}\qquad\text{---(1)}$$

Now the poles of $f(z)$ are given by

$$(z^2+4z+1)^2=0 \Rightarrow z=\frac{-4\pm\sqrt{16-4}}{2}=\frac{-4\pm\sqrt{12}}{2}$$
$$=-2\pm\sqrt{3} \text{ (twice)}$$

$\therefore f(z)$ has poles of order 2 at $z=-2\pm\sqrt{3}$ (twice)

Let $\alpha=-2+\sqrt{3}$, $\beta=-2-\sqrt{3}$

Clearly $|\beta|>1$

Since $|\alpha\beta|=1 \Rightarrow |\alpha|<1$ $(\because |\beta|>1)$

Hence the only pole inside C is at $z=\alpha$ of order 2.

$$\therefore \int_C f(z)\,dz = \int \frac{z\,dz}{(z^2+4z+1)^2}$$
$$= 2\pi i\,(\text{Residue at } z=\alpha)$$

Now the residue at $z=\alpha$ is

$$\lim_{z\to\alpha}\frac{d}{dz}(z-\alpha)^2\frac{z}{(z^2+4z+1)^2} = \lim_{z\to\alpha}\frac{d}{dz}\frac{z}{(z-\beta)^2}$$
$$= \lim_{z\to\alpha}\frac{-(\beta+z)}{(z-\beta)^3}$$
$$= \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$
$$= \frac{-(-4)}{(2\sqrt{3})^3}$$
$$= \frac{4}{8(3\sqrt{3})} = \frac{1}{6\sqrt{3}}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{1}{6\sqrt{3}}\right) = \frac{2\pi i}{6\sqrt{3}} = \frac{\pi i}{3\sqrt{3}}$$

$\therefore$ from ① $\displaystyle\int_0^{2\pi}\frac{2\,d\theta}{(2+\cos\theta)^2} = \frac{8}{i}\left(\frac{\pi i}{3\sqrt{3}}\right) = \frac{16}{9}\cdot\frac{8\pi}{3\sqrt{3}}$

$$\therefore I = \int_0^{\pi}\frac{d\theta}{\left(1+\frac{\cos\theta}{2}\right)^2} = \frac{16}{9}\cdot\frac{8\pi}{3\sqrt{3}}$$

4(a) 2014. Prove that, the set $Q(\sqrt{5}) = \{a+b\sqrt{5}\ ;\ a,b \in Q\}$ is a commutative ring with identity.

Sol^n.

$$Q(\sqrt{5}) = \{a+b(\sqrt{5})\ ;\ a,b \in Q\}.$$

I) let $(Q\sqrt{5}, +)$ be an ~~abelian~~ algebraic structure.

i) $\Rightarrow \forall\ (a+b\sqrt{5}), (c+d\sqrt{5}) \in Q(\sqrt{5})$, $a,b,c,d \in Q$.

$$(a+b\sqrt{5})+(c+d\sqrt{5}) = (a+c)+(b+d)\sqrt{5}$$
$$= (l+m\sqrt{5}) \in Q\sqrt{5}$$
$l, m \in Q$.

closure pro. satisfy.

ii) $[(a+b\sqrt{5})+(c+d\sqrt{5})]+(e+f\sqrt{5})$

$= [(a+c)+(b+d)\sqrt{5}]+(e+f\sqrt{5})$

$= (a+c+e)+(b+d+f)\sqrt{5}$

$= a+b\sqrt{5}+[(c+e)+(d+f)\sqrt{5}]$

$= a+b\sqrt{5}+[(c+d\sqrt{5})+(e+f\sqrt{5})]$

$\therefore$ Associative pro. satisfy.

iii) let $c+d\sqrt{5} \in Q\sqrt{5}$

if $c=d=0 \quad \Rightarrow 0 \in Q$.

$\therefore (a+b\sqrt{5})+0 = a+b\sqrt{5}$

$\therefore$ '0' is additive ~~inverse~~ Identity.

iv) let $c+d\sqrt{5} \in Q\sqrt{5}$

$(a+b\sqrt{5})+(c+d\sqrt{5})=0$

$(a+c)+(b+d)\sqrt{5}=0.$

$\Rightarrow a+c=0 \qquad b+d=0$

$\underline{c=-a} \qquad \underline{d=-b}$

$\therefore c+d\sqrt{5} = -a-b\sqrt{5} \in Q$

$\therefore$ Additive inverse exists.

v) $(a+b\sqrt{5})+(c+d\sqrt{5}) = a+d\sqrt{5}+c+b\sqrt{5}$

$= c+d\sqrt{5}+a+b\sqrt{5}$

$\therefore$ commutative satisfy.

* II) let $(Q^*\sqrt{5}, \times)$ be an algebric structure.

i) $\forall (a+b\sqrt{5}),(c+d\sqrt{5}) \in Q^*\sqrt{5} \quad a,b,c,d \in Q.$

$Q^*\sqrt{5} \in \mathbb{R}.$

$\therefore (a+b\sqrt{5})\times(c+d\sqrt{5})$

$= ac+5bd+\sqrt{5}(bc+ad)$

$= l+\sqrt{5}(m) \qquad (l = ac+5bd \in Q$

$m = bc+ad \in Q).$

$\therefore$ closure pro. satisfied.

ii) $\forall (e+f\sqrt{5}) \in Q^*\sqrt{5} \qquad e,f \in Q.$

$[(a+b\sqrt{5}).(c+d\sqrt{5})].(e+f\sqrt{5})$

$= [ac+5bd+\sqrt{5}(bc+ad)](e+f\sqrt{5}).$

$= ace+5(bcf+aef)+5bde+\sqrt{5}(bce+ade)+\sqrt{5}(acf+5bdf).$

$= p + \sqrt{5}(q).$ $\quad p = ace + 5(bcf + aef) + 5de$
$q = bce + ade + ae + 5bd \in Q.$

$= a(ce + 5df + \sqrt{5}de + \sqrt{5}cf) + b\sqrt{5}(bcf + \sqrt{5}de + ce + 5df)$

$= (a + b\sqrt{5})(ce + 5af + \sqrt{5}de + \sqrt{5}cf)$

$= (a + b\sqrt{5})(c(e + \sqrt{5}f) + d\sqrt{5}(\sqrt{5}f + e)).$

$= (a + b\sqrt{5}).[(c + d\sqrt{5}) * (e + f\sqrt{5})]$

$\therefore$ Asso. pro. satisfied.

iii) $Q^*\sqrt{5} = Q\sqrt{5} - \{0\}.$ $\quad$ let $c + d\sqrt{5}$ = Identity.

$(a + b\sqrt{5}) \times (c + d\sqrt{5}) = (a + b\sqrt{5})$

$ac + 5bd + \sqrt{5}(bc + ad) = a + b\sqrt{5}$

$\Rightarrow ac + 5bd = a$ $\quad$ $bc + ad = b$

$\Rightarrow a(c-1) + 5bd = 0$ $\quad$ $b(c-1) + ad = 0$

$c - 1 = \frac{-5bd}{a} = \frac{-ad}{b} \Rightarrow 5b^2d = a^2d$

$d(5b^2 - a^2) = 0$

$5b^2 \neq a^2.$

$\therefore \boxed{d = 0}$ — (i)

using $\underline{d = 0}$

$a(c-1) + 5b(0) = 0$

$a(c-1) = 0$

$a \neq 0 \quad \therefore c = 1$

$\therefore$ Identity $= 1$ exists.

iv) let $c + d\sqrt{5}$ be inverse of $a + b\sqrt{5}$

$\therefore (a + b\sqrt{5}).(c + d\sqrt{5}) = I = 1$

$ac + 5bd = 1$ $\qquad$ $bc + ad = 0$

$bc = -ad.$

$\frac{c}{d} = \frac{-a}{b} = k$

$-bd[k^2+5] = 1$

$c = dk.$

$d = \frac{-1}{(5+k^2)b} = \frac{-b}{(5b^2+a^2)}$ $\qquad$ $a = -bk.$

$c = \frac{a}{5b^2+a^2}$

$\therefore c + d\sqrt{5} = \frac{a}{5b^2+a^2} - \frac{b\sqrt{5}}{5b^2+a^2} \Rightarrow$ Inverse of $a+b\sqrt{5}$.

v)
$(a+b\sqrt{5}) \times (c+d\sqrt{5})$

$= ac + 5bd + \sqrt{5}(bc+ad)$

$= c[a+\sqrt{5}.b] + \sqrt{5}.d[\sqrt{5}b+a]$

$= (c+d\sqrt{5}) \times (a+b\sqrt{5})$

commutative pro. satisfied.

(III) Distributivity $\Rightarrow$

$(a+b\sqrt{5}) \times [(c+d\sqrt{5}) + (e+f\sqrt{5})]$

$= (a+b\sqrt{5}) \times [c+e+(d+f)\sqrt{5}]$

$= ac + ae + 5(bd+bf) + \sqrt{5}(bc+be+ad+af)$

$= a(c+\sqrt{5}.d) + \sqrt{5}.b(c+\sqrt{5}.d) + a(e+\sqrt{5}.f)$
$+ b\sqrt{5}(e+\sqrt{5}f).$

$= (a+\sqrt{5}b).(c+\sqrt{5}d)+(a+b\sqrt{5}).(e+f\sqrt{5}).$

$[(a+b\sqrt{5})+(c+d\sqrt{5})]\times(e+f\sqrt{5}).$

$= (a+b\sqrt{5})\times(e+f\sqrt{5})+(c+d\sqrt{5})\times(e+f\sqrt{5})$

similarly,

$\therefore (Q(\sqrt{5}),+,\cdot)$ is a field.

Q.4(b) Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multipliers.

Sol: Let $f_1 = x^2+y^2+z^2$; which is subject to $f_2 = xyz = a^3$ or $f_2 = xyz - a^3 = 0$

Now, with the lagrange multipliers form.

$F = f_1 + \lambda f_2 = 0$

$F = x^2+y^2+z^2 + \lambda(xyz - a^3)$.

Now; we need to find partial derivative of F w.r.t x, y, z respectively ;

Now put

$F_x = 2x + \lambda(yz) \Rightarrow F_x = 0$

$F_y = 2y + \lambda(zx) \Rightarrow F_y = 0$

$F_z = 2z + \lambda(xy) \Rightarrow F_z = 0$

we get $\lambda = \dfrac{-2x}{yz}$ for F_x

$\lambda = \dfrac{-2y}{zx}$ for F_y & $\lambda = \dfrac{-2z}{xy}$ for F_z

Equating λ values of F_x & F_y, we get

$-\dfrac{\cancel{2}x}{\cancel{y}\cancel{z}} = \dfrac{-\cancel{2}y}{\cancel{z}x} \Rightarrow x^2 = y^2 \Rightarrow x = y$ [Since; we are ignoring the negative values]

Similarly $y = z$ for F_y & F_z

$z = x$ for F_z & F_x.

Thus, $x = y = z$

then from given condition $xyz = a^3$; we get $x = y = z = a$

So; minimum value of $x^2+y^2+z^2 = a^2+a^2+a^2 = \underline{\underline{3a^2}}$

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4(c) Find all optimal solutions of the following linear programming problem by the simplex method:

Maximize $Z = 30x_1 + 24x_2$

Subject to

$5x_1 + 4x_2 \le 200$

$x_1 \le 32$

$x_2 \le 40$.

$x_1, x_2 \geq 0$.

Sol^n: The objective function of the given LPP is of maximization type and RHS of all constraints are ≥ 0.

Now we write the given LPP in the standard form:

$$\text{Max } Z = 30x_1 + 24x_2 + 0s_1 + 0s_2 + 0s_3.$$

Subject to

$$5x_1 + 4x_2 + s_1 \qquad = 200$$

$$x_1 \qquad + s_2 \quad = 32$$

$$x_2 \qquad + s_3 = 40$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Where s_1, s_2, s_3 are slack variables.

Now the initial basic feasible solution is given by setting $x_1 = x_2 = 0$ (Non-basic)

$s_1 = 200$, $s_2 = 32$, $s_3 = 40$.

$\therefore$ The Initial Basic feasible solution is

$(0, 0, 200, 32, 40)$ for which $Z = 0$.

Now we move from the current basic feasible solution to the next better basic feasible solution.

put the above information in the tableau form.

	C_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	5	4	1	0	0	200	$\frac{200}{5} = 40$
0	s_2	(1)	0	0	1	0	32	$\frac{32}{1} = 32 \rightarrow$
0	s_3	0	1	0	0	1	40	—
$Z_j = \sum a_{ij} c_B$		0	0	0	0	0	0	
$C_j = c_j - Z_j$		30 $\uparrow$	24	0	0	0		

from the above table,

x_1 is the incoming variable as $C_j = 30$ is maximum and the corresponding column is known as key column.

The minimum +ve ratio θ occurs in the second row.

$\therefore$ s_2 is the outgoing variable and the column common intersection element (1) is the key element. and convert all other elements in its column to zero.

Then the new iterated simplex table is.

	c_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	4	1	-5	0	40	$\frac{40}{4}=10$ →
30	x_1	1	0	0	1	0	32	$\frac{32}{30}$
0	s_3	0	1	0	0	1	40	$\frac{40}{1}$
$z_j=\Sigma a_{ij}c_B$		30	0	0	30	0	960	
$C_j=c_j-z_j$		0	24 ↑	0	-30	0		

from the above table, x_2 is the incoming variable, s_1 is the outgoing variable and (4) is the key element. Now convert the key element to unity and all other elements in its column to zero.

Then we get the new iterated simplex table as

	c_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
24	x_2	0	1	$1/4$	$-5/4$	0	10	—
30	x_1	1	0	0	1	0	32	32
0	s_3	0	0	$-1/4$	$5/4$	1	30	$\frac{30\times4}{5}=30$ →
$z_j=\Sigma a_{ij}c_B$		30	24	6	0	0	1200	
$C_j=c_j-z_j$		0	0	-6	0 ↑	0		

Since all $C_j\le 0$, an optimal solution has been reached.

∴ The optimum basic feasible solution is $x_2=10,\ x_1=32$ and $Z_{Max}=1200$

Existence of alternative optima:

from the above table, net evaluation for the non-basic variable s_2 is zero. Clearly this is an indication that the current solution is not unique. We can bring s_2 into the basis in place of s_3, which satisfies the exist criterian.

	c_j	30	24	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
24	x_2	0	1	0	0	1	40	
30	x_1	1	0	1/5	0	-4/5	8	
0	s_2	0	0	-1/5	1	4/5	24	
$z_j = \sum a_{ij} c_B$		30	24	6	0	0	1200	
$C_j = c_j - z_j$		0	0	-6	0	0		

Since all $C_j \leq 0$.

$\therefore$ An alternative optimum solution is

$$x_1 = 8, \; x_2 = 24, \; Z_{max} = 1200$$

5(a) Solve the partial differential equation
$(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$.

Sol^n.
The auxiliary of the given equation is

$2m^2 - 5m + 2 = 0$

$\Rightarrow m = \frac{1}{2}, 2$

$\therefore$ C.F $= \phi_1(y+2x) + \phi_2(2y+x)$

ϕ_1, ϕ_2 being arbitrary functions.

Now, $P.I = \dfrac{1}{2D^2 - 5DD' + 2D'^2} 24(y-x)$

$= 24 \dfrac{1}{2D^2 - 5DD' + 2D'^2}(y-x)$

$= \dfrac{24}{2(-1)^2 - 5(-1)(2) + 2\times(2)^2} \iint v\,dv\,dv$ where $v = y - x$

$= \dfrac{24}{20} \int \dfrac{v^2}{2} dv$

$= \dfrac{6}{5}\left(\dfrac{v^3}{6}\right)$

$= \dfrac{1}{5} v^3 = \dfrac{1}{5}(y-x)^3$.

Hence the required general solution is
$z = \phi_1(y+2x) + \phi_2(2y+x) + \frac{1}{5}(y-x)^3$

5(b) Apply Newton-Raphson method to determine a root of the Equation $\cos x - xe^x = 0$ correct up to four decimal places.

Sol^n: $f(x) = \cos x - xe^x$.

So that $f(0) = 1$ and $f(1) = \cos 1 - e$
$= -2.17798$

$\therefore f(0)\, f(1) < 0$

Hence the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$.

$\therefore f(x_0) = 1$ and $f(x_1) = -2.17798$

By the method of false position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{——} (1)$$

$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$

$$= 0.31467$$

$\therefore$ The first approximation to the root is

$x_2 = 0.31467$

Now $f(x_2) = 0.51987 > 0$.

$\therefore f(x_2)\, f(x_1) < 0$.

$\therefore$ The root lies b/w 0.31467 and 1

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (1),

$$x_3 = \frac{(0.3146)(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987}$$

$x_3 = 0.44673$

The 2nd approximation to the root is

$x_3 = 0.44673$

Now repeating this process, the successive approximations are

$x_4 = 0.49402$, $x_5 = 0.50995$,

$x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$.

$x_9 = 0.51767$, $x_{10} = 0.51775$, etc.

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places

5(c) Use five subintervals to integrate $\int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rule.

Solution : Here

$$f(x) = \frac{1}{1+x^2},$$

$a = 0$, $b = 1$, and $n = 5$.

$$\therefore h = \frac{1-0}{5} = \frac{1}{5} = 0.2.$$

x	0.0	0.2	0.4	0.6	0.8	1
$y=f(x)$	1.000000	0.961538	0.832069	0.735294	0.609756	0.500000
	y_0	y_1	y_2	y_3	y_4	y_5

Using trapezoidal rule we get

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(1.000000+.500000)+2(0.961538+0.862069+0.735294+0.609756)\right]$$

$$= 0.7837314,$$

$\therefore I = 0.78373$, correct to five significant figures.

The exact value

$$= \int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} = 0.7853981$$

$$\int_0^1 \frac{1}{1+x^2}dx = 0.78540,$$

correct to five significant figures.

$\therefore$ The error is $= 0.78540 - 0.78373$

$= 0.00167$

$\therefore$ Absolute error $= 0.00167$.

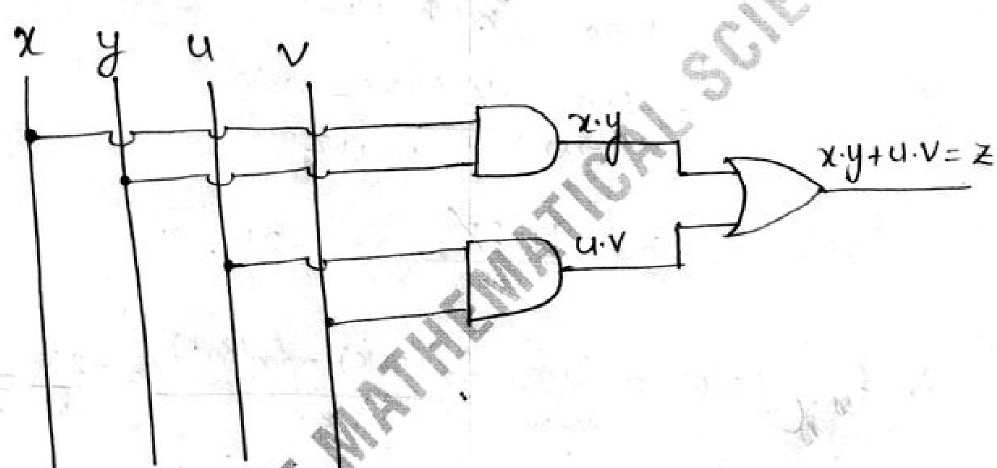
Q.5(d) Use only AND and OR logic gates to construct a logic circuit for Boolean expression

$$z = xy + uv.$$

Sol: Here ; $x \cdot y$ uses one AND gate

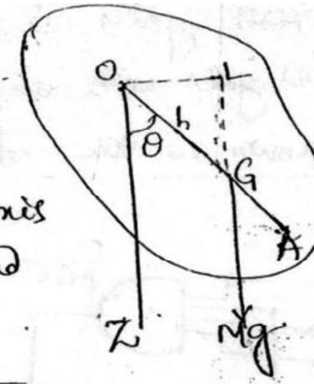
$u \cdot v$ uses another AND gate

and one OR gate will be used to provide $z = xy + uv$.



5(c) Find the equation of motion of a Compound pendulum using Hamilton's equations.

Sol^n:



At time t, let θ be the angle between the vertical plane through the fixed axis (plane fixed in space) and the plane through the centre of Gravity 'G' and the fixed axis (plane fixed in the body).

Let $OG = h$.

If T and V are the kinetic & potential energies of the pendulum then

$T = \frac{1}{2}MK^2\dot{\theta}^2$ and $V = -Mgh\cos\theta$.

(-ve sign is taken because G is below the fixed axis).

$\therefore L = T - V$

$= \frac{1}{2}MK^2\dot{\theta}^2 + Mgh\cos\theta$.

Here θ is the only generalised coordinate.

$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = MK^2\dot{\theta}$ ——— (1)

$H = T + V = \frac{1}{2}MK^2\dot{\theta}^2 - Mgh\cos\theta$

$= \frac{1}{2MK^2}p_\theta^2 - Mgh\cos\theta$. (from (1))

Hence the two Hamilton's equations are

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mgh\sin\theta$ ——— (2) and $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{MK^2}p_\theta$ ——— (3)

Differentiating (3) and substituting from (2), we get

$\ddot{\theta} = \frac{1}{MK^2}\dot{p}_\theta = \frac{1}{MK^2}(-Mgh\sin\theta) \Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta$.

which is the equation of motion of Compound pendulum.

6(a) Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to Canonical form?

Sol: let the given equation

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

Re-writing the given eqn becomes $r - x^2 t = 0$ —(1)

comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

we have $R = 1,\ S = 0,\ T = -x^2$

Now, the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$

$$\lambda^2 - x^2 = 0 \Rightarrow \lambda = \pm x$$

Here; $\lambda_1 = x$ and $\lambda_2 = -x$ (Real & distinct roots)

Hence, the characteristic equations.

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad dy/dx + \lambda_2 = 0$$

$$\frac{dy}{dx} + x = 0 \quad \text{and} \quad \frac{dy}{dx} - x = 0$$

Integrating these two equations, we get

$$y + (x^2/2) = c_1 \qquad y - (x^2/2) = c_2$$

Hence, in order to reduce (1) to canonical form, we change x, y, to u, v by taking.

$$u = y + x^2/2 \quad \text{and} \quad v = y - x^2/2 \quad \text{—(2)}$$

Now; $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}$ — from (2)

$$p = x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \text{—(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \text{ (from (2))} \text{—(4)}$$

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$r = x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \text{ — from (3)}$$

$$x^2\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = r$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ from (4)}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \text{ — from (4)}$$

Put, the values of r & t in eqn (1)

$$x^2\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left[\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] = 0$$

or. $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

which is the required canonical form of the given equation.

6(b) Solve the system of equations $2x_1 - x_2 = 7$, $-x_1 + 2x_2 - x_3 = 1$, $-x_2 + 2x_3 = 1$ using Gauss-Seidel iteration method.

Sol^n: The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(7 + x_2) \\ x_2 &= \frac{1}{2}(1 + x_1 + x_3) \\ x_3 &= \frac{1}{2}(1 + x_2) \end{aligned}\right\} \longrightarrow (1)$$

By the Gauss-Seidel method, system (1) can be written as

$$x_1^{k+1} = \frac{1}{2}\left(7 + x_2^{(k)}\right)$$

$$x_2^{k+1} = \frac{1}{2}\left(1 + x_1^{(k+1)} + x_3^{k}\right)$$

$$x_3^{k+1} = \frac{1}{2}\left(1 + x_2^{(k+1)}\right) \quad \text{where } k = 0, 1, 2, 3, \ldots$$

Now taking $x^{(0)} = 0$, we obtain the following iterations

k=0:

$$x_1^{(1)} = \frac{1}{2}(7 + 0) = \frac{7}{2} = 3.5$$

$$x_2^{(1)} = \frac{1}{2}(1 + 3.5 + 0) = \frac{4.5}{2} = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1 + 2.25) = \frac{1}{2}(3.25) = 1.625.$$

k=1:

$$x_1^{(2)} = \frac{1}{2}\left(7 + x_2^{(1)}\right) = \frac{1}{2}(7 + 2.25) = \frac{9.25}{2} = 4.625$$

$$x_2^{(2)} = \frac{1}{2}\left(1 + x_1^{(2)} + x_3^{(1)}\right) = \frac{1}{2}(1 + 4.625 + 1.625) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1 + x_2^{(2)}\right) = \frac{1}{2}(1 + 3.625) = 2.3125$$

k=2:

$$x_1^{(3)} = \frac{1}{2}\left(7 + x_2^{(2)}\right) = \frac{1}{2}(7 + 3.625) = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}\left(1 + x_1^{(3)} + x_3^{(2)}\right) = \frac{1}{2}(1 + 5.3125 + 2.3125) = 4.3125$$

$x_3^{(3)} = \frac{1}{2}(1 + x_2^{(3)}) = \frac{1}{2}(1 + 4.3125) = 2.6563$

$\underline{K=3}$:

$x_1^{(4)} = \frac{1}{2}(7 + x_2^{(3)}) = \frac{1}{2}(7 + 4.3125) = 5.6563$

$x_2^{(4)} = \frac{1}{2}(1 + x_1^{(4)} + x_3^{(3)}) = \frac{1}{2}(1 + 5.6563 + 2.6563) = 4.6563$

$x_3^{(4)} = \frac{1}{2}(1 + x_2^{(4)}) = \frac{1}{2}(1 + 4.6563) = 2.8282$

$\underline{k=4}$;

$x_1^{(5)} = \frac{1}{2}(7 + x_2^{(4)}) = \frac{1}{2}(7 + 4.6563) = 5.8282$

$x_2^{(5)} = \frac{1}{2}(1 + x_1^{(5)} + x_3^{(4)}) = \frac{1}{2}(1 + 5.8282 + 2.8282) = 4.8282$

$x_3^{(5)} = \frac{1}{2}(1 + x_2^{(5)}) = \frac{1}{2}(1 + 4.8282) = 2.9141$.

which is the good approximation to the exact solution $X = (6\ 5\ 3)^T$.

6(c) Use Runge-Kutta formula of fourth order to find the value of y at $x=0.8$, where $\frac{dy}{dx} = \sqrt{x+y}$, $y(0.4)=0.41$. Take the step length $h=0.2$.

Sol: Given that $\frac{dy}{dx} = \sqrt{x+y} = f(x,y)$

To find $y(0.6)$:

Here $x_0=0.4$, $y_0=0.41$, $h=0.2$

$$f(x_0,y_0) = \sqrt{0.81}$$

$$\therefore k_1 = hf(x_0,y_0) = (0.2)\sqrt{0.81} = 0.18$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.2)f(0.5, 0.5) = 0.2$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0.2)f(0.5, 0.51) = 0.20099$$

$$k_4 = hf(x_0+h, y_0+k_3) = (0.2)f(0.6, 0.61099) = 0.220089$$

$$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}[0.18+2(0.2)+2(0.20099)+0.220089]$$

$$= \frac{1}{6}(1.202069982)$$

$$= 0.2003449$$

$$y_1 = y(0.6) = y_0+K = 0.41+0.2003449 = 0.6103449$$

To find $y(0.8)$:

Here $x_1=0.6$, $y_1=0.6103449$, $h=0.2$

$$k_1 = hf(x_1,y_1) = (0.2)f(0.6, 0.6103449) = 0.220031$$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = (0.2)f(0.7, 0.72036) = 0.23836$$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = (0.2)f(0.7, 0.72952) = 0.23911$$

$$k_4 = hf(x_1+h, y_1+k_3) = (0.2)f(0.8, 0.852571) = 0.257105$$

$$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.238686$$

$$y_2 = y(0.8) = y_1+K = 0.6103449+0.238686$$

$$= 0.8490309$$

7(a) Find the deflection of a vibrating string (length $= \pi$, ends fixed, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$) corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$

Sol: The vibration of the string is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{---(1)}$$

At the end points of the string are fixed for all time

B.C: $u(0,t) = 0$ and $u(\pi,t) = 0$ ---(2)

I.C: initial velocity $= \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ for $0 \le x \le \pi$ ---(3)

and initial displacement $= u(x,0) = f(x) = k(\sin x - \sin 2x)$ ---(4)

Suppose that (1) has the solution of the form

$u(x,t) = X(x)\,T(t)$ ---(5)

substituting this value of u in (1), we have

$XT'' = X''T \Rightarrow \frac{X''}{X} = \frac{T''}{T} = \mu$ (say)

$\Rightarrow X'' - \mu X = 0$ ---(6) and $T'' - \mu T = 0$ ---(7)

Using (2), (5) gives

$X(0)\,T(t) = 0$ and $X(\pi)\,T(t) = 0$ ---(8)

Since $T(t) = 0$ leads to $u \equiv 0$ ~~$\forall t$~~

so suppose that $T(t) \neq 0$.

Then (8) gives $X(0) = 0$ and $X(\pi) = 0$ ---(9)

we now solve (6) under boundary conditions (9).

Three cases arise.

Case(i): Let $\mu = 0$

Then solution of (6) is $X(x) = Ax + B$ ---(10)

Using B.C. (9), (10) gives $B = 0$, $A = 0$

$\Rightarrow X(x) = 0$

This leads to $u \equiv 0$, which doesnot satisfy I.C (3) & (4)

so we reject $\mu = 0$.

Case (ii): Let $\mu = \lambda^2$, $\lambda \neq 0$, then the solution of (6) is

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ —— (11)

Using B.C. (9), (11) gives $A = 0, B = 0$

$\Rightarrow X(x) = 0$

This leads to $u \equiv 0$ which doesnot satisfy (3) & (4).

So reject $\mu = \lambda^2$.

Case (iii): Let $\mu = -\lambda^2$, $\lambda \neq 0$

The solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ —— (12)

Using B.C. (9), (12) gives

$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$

and $X(\pi) = 0 = 0 + B\sin\lambda\pi \Rightarrow B\sin\lambda\pi = 0$

$\Rightarrow \sin\lambda\pi = 0$ ($\because B \neq 0$)

$\Rightarrow$ ~~$\lambda\pi = n\pi$~~

$\Rightarrow \lambda = n$, $n = 1, 2, \ldots$

from (12), we have

$X(x) = B\sin nx$ $n = 1, 2, \ldots$

Hence non-zero solutions $X_n(x)$ of (6) are

given by $X_n(x) = B_n \sin nx$ —— (13)

from (7), $T'' - \mu T = 0 \Rightarrow T'' + \lambda^2 T = 0$ ($\because \mu = -\lambda^2$)

$\Rightarrow T'' + n^2 T = 0$ ($\because \lambda = n$)

whose general solution is

$T_n(t) = C_n\cos(nt) + D_n\sin(nt)$

$\therefore u_n(x,t) = X_n(x)\,T_n(t)$

$= B_n\sin(nx)\left[C_n\cos(nt) + D_n\sin(nt)\right]$

$= \left(E_n\cos nt + F_n\sin nt\right)\cdot\sin(nx)$ (14)

are solutions of (1) satisfying (2).

Here $E_n = B_nC_n$ and $F_n = B_nD_n$.

In order to obtain a solution also satisfying (3) and (4). we consider more general solution

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

ie, $u(x,t) = \sum_{n=1}^{\infty} (E_n \cos nt + F_n \sin nt) \sin nx$ ——(15)

Differentiating (15) partially w.r.t t, we get

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-nE_n \sin nt + nF_n \cos nt) \sin nx$ ——(16)

putting $t=0$ in (15) and (16) and using initial conditions (3) and (4), we get

(15) ≡ $u(x,0) = k(\sin x - \sin 2x) = \sum_{n=1}^{\infty} E_n \sin nx$ ——(17)

(16) ≡ $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = \sum_{n=1}^{\infty} nF_n \sin nx.$

where $F_n = \frac{2}{n}\int_0^{\pi} (0)\sin nx\,dx = 0$

from (17), we have

$k(\sin x - \sin 2x) = \sum_{n=1}^{\infty} E_n \sin nx$

comparing the coefficients of like terms on both sides, we have

$E_1 = 1, \quad E_2 = -k \text{ and } E_3 = E_4 = \cdots = 0$

∴ from (15), we have

$u(x,t) = E_1 \sin x \cos t + E_2 \sin 2x \cos 2t$

$= k \cos t \sin x - k \cos 2t \sin 2x$

$u(x,t) = k(\cos t \sin x - \cos 2t \sin 2x)$

which is the required solution.

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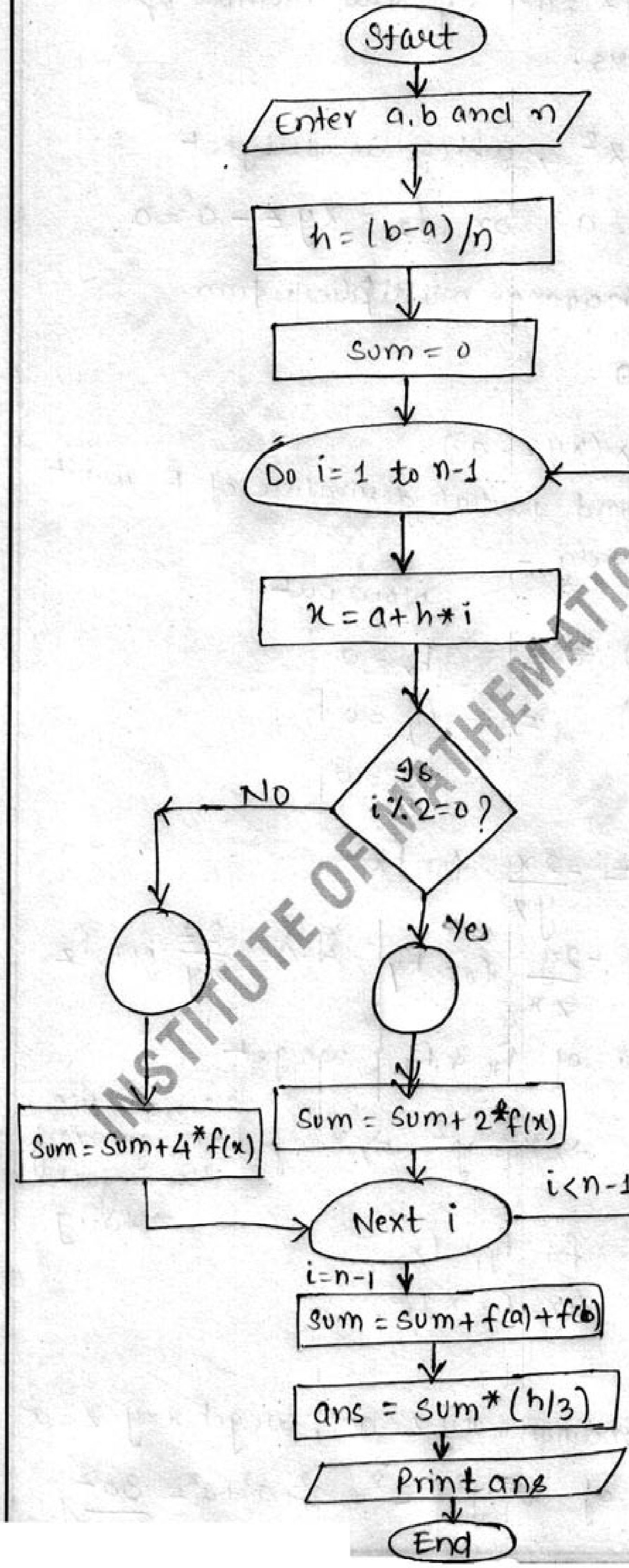
7(b). Draw a flowchart for Simpson's one-third rule?



7(c) Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$; determine the streamlines.

Sol:- velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$

To determine stream lines

$$\frac{-\partial\phi}{\partial x} = u = -\frac{\partial\psi}{\partial y} \; ; \quad -\frac{\partial\phi}{\partial y} = v = \frac{\partial\psi}{\partial x}$$

Hence $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$

Now, $\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$

Integrating w.r.t y

$$\psi = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) + f(x). \quad \text{--- (1)}$$

where $f(x)$ is constant of Integration. To determine $f(x)$

$$\frac{\partial\psi}{\partial x} = \frac{-\partial\phi}{\partial y} = \frac{-y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \quad \text{--- (2)}$$

by (1) $\Rightarrow$ $\frac{\partial\psi}{\partial x} = \frac{-y}{(x+a)^2+y^2} + \frac{y}{(x+a)^2+y^2} + F'(x)$ --- (3)

Equating (2) & (3) $\Rightarrow$ $F'(x) = 0$; Integrating this $F(x)$ = absolute constand hence neglected. Since, it has no effect on the fluid motion.

Now (1) becomes

$$\boxed{\psi = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) = \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}} \quad \text{--- (4)}$$

Stream lines are given by ψ = constant. i.e.

$\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right]$ = Constant or $\frac{y}{x^2-a^2+y^2}$ = constant

If we take
constant $= 0$; then
weget $y = 0$; i.e x-axis.

If we take
constant $= \infty$; then
we get circle
$$x^2 - a^2 + y^2 = 0$$
i.e $x^2 + y^2 = a^2$

Thus; stream lines include x-axis and circle.

8(a) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0<x<1$, $t>0$, given that

(i) $u(x,0)=0$, $0\le x\le 1$

(ii) $\frac{\partial u}{\partial t}(x,0)=x^2$, $0\le x\le 1$

(iii) $u(0,t)=u(1,t)=0$ for all t.

Solⁿ: The vibration of the string is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{---(1)}$$

As the ends points of the string are fixed for all time.

B.C: $u(0,t)=u(1,t)=0 \quad \forall t$ ---(2)

I.C. $u(x,0)=0$, $0\le x\le 1$ ---(3)

and $\frac{\partial u}{\partial t}(x,0)=x^2$, $0\le x\le 1$ ---(4)

suppose that (1) has the solution of the form

$$u(x,t)=X(x)\,T(t) \quad \text{---(5)}$$

substituting this value of u in (1), we have

$XT''=X''T$

$\Rightarrow \frac{X''}{X}=\frac{T''}{T}=\mu$ (say)

$\Rightarrow X''-\mu X=0$ ---(6) and $T''-\mu T=0$ ---(7)

using (2), (5) gives

$X(0)T(t)=0$ and $X(1)T(t)=0$ ---(8)

since $T(t)=0$ leads to $y\equiv 0 \ \forall t$

so suppose that $T(t)\neq 0$

Then (8) gives $X(0)=0$ and $X(1)=0$ ---(9)

which are boundary conditions.

we now solve (6) under boundary conditions (9).

Three cases arise.

Case (i): Let $\mu = 0$

Then solution of (6) is $X(x) = Ax + B$ —(10)

Using B.C. (9), (10) gives $B = 0, A = 0$

$\Rightarrow X(x) = 0$

This leads to $u \equiv 0$ which doesnot satisfy I.C (3) & (4)

so we reject $\mu = 0$

Case (ii): Let $\mu = \lambda^2$, $\lambda \neq 0$, Then the solution of (6).

is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ —(11)

Using B.C. (9), (11) gives $A = 0, B = 0$

$\Rightarrow X(x) = 0$

This leads to $u \equiv 0$ which doesnot satisfy (3) & (4)

so reject $\mu = \lambda^2$.

Case (iii): Let $\mu = -\lambda^2$, $\lambda \neq 0$

The solution of (6) is

$X(x) = A\cos\lambda x + B\sin\lambda x$ —(12)

Using B.C (9), (12) gives

$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$

and $X(1) = 0 = 0 + B\sin\lambda(1) \Rightarrow B\sin\lambda = 0$

$\Rightarrow \sin\lambda = 0 \quad (\because B \neq 0)$

$\Rightarrow \lambda = n\pi, \ n = 1, 2, 3, \ldots$

From (12), we have

$X(x) = B\sin n\pi x, \ n = 1, 2, \ldots$

Hence non-zero solutions $X_n(x)$ of (6)

are given by $X_n(x) = B_n \sin(n\pi x)$ —(13)

From (7), $T'' - \mu T = 0 \Rightarrow T'' + \lambda^2 T = 0$

$\Rightarrow T'' + n^2\pi^2 T = 0$

whose general solution is

$T_n(t) = C_n \cos n\pi t + D_n \sin n\pi t$.

$\therefore u_n(x,t) = X_n(x)\, T_n(t)$

$= B_n \sin(n\pi x)\left[C_n \cos n\pi t + D_n \sin n\pi t\right]$

$= \left[E_n \cos n\pi t + F_n \sin n\pi t\right] \sin n\pi x$ — (14)

are solutions of (1) satisfying (2)

Here $E_n = B_n C_n$ and $F_n = B_n D_n$

In order to obtain a solution also satisfying (3) and (4), we consider more general solution

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$u(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos n\pi t + F_n \sin n\pi t\right) \sin n\pi x$ — (15)

Diff. (15) partially w.r.t. t, we get

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-E_n n\pi \sin n\pi t + F_n n\pi \cos n\pi t\right) \sin n\pi x$ — (16)

putting $t=0$ in (15) and (16) and using I.C (3) and (4), we get

(15) $\Rightarrow u(x,0) = 0 = \sum_{n=1}^{\infty} E_n \sin n\pi x$

where $E_n = \frac{2}{1}\int_0^1 (0) \sin(n\pi x)\,dx = 0$

(16) $\Rightarrow \left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} n\pi F_n \sin n\pi x = x^2$

$n\pi F_n = \frac{2}{1}\int_0^1 x^2 \sin n\pi x\, dx$

$\Rightarrow F_n = \frac{2}{n\pi}\int_0^1 x^2 \sin(n\pi x)\,dx$

$= \frac{2}{n\pi}\left[x^2\left(-\frac{\cos n\pi x}{n\pi}\right) + \int 2x \frac{\cos n\pi x}{n\pi}\,dx\right]_0^1$

$= \frac{2}{n\pi}\left[-\frac{1}{n\pi}\cos n\pi + \frac{2}{n\pi}\left(x\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^1$

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$$= \frac{2}{n^2\pi^2}\left[-\cos n\pi + 2\left(0 + \frac{\cos n\pi}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^{n+1} + 2\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]\right]$$

$$= \begin{cases} \frac{2}{n^2\pi^2}\left(1 - \frac{4}{n^2\pi^2}\right), & \text{if } n \text{ is odd} \\ -\frac{2}{n^2\pi^2}, & \text{if } n \text{ is even.} \end{cases}$$

$$f_n = \begin{cases} \frac{2}{(2m-1)^2\pi^2}\left(1 - \frac{4}{(2m-1)^2\pi^2}\right) & \text{if } n = 2m-1, \; m = 1,2,\ldots \\ -\frac{2}{(2m)^2\pi^2}, & \text{if } n = 2m, \; m = 1,2,\ldots \end{cases} \quad (17)$$

$\therefore$ The required displacement is given by

$$u(x,t) = \sum_{n=1}^{\infty} f_n \sin n\pi t \sin n\pi x$$

where f_n is given by (17)

8(b) For any Boolean variables x and y, show that $x + xy = x$.

Soln:

$$\begin{aligned} LHS = x + xy &= x\cdot 1 + xy \\ &= x(1+y) \\ &= x\cdot 1 \quad (\because 1+y=1) \\ &= x \end{aligned}$$

$\therefore x + xy = x.$

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8(c) Find Navier-stokes equation for a steady laminar flow of a viscous incompressible fluid b/w two infinite parallel plates?

Sol: By Laminar flow, we mean that fluid moves in layers parallel to the plates.

We suppose that ~~the~~ an incompressible fluid with constant viscosity is confined between two parallel plates $y = a/2$ and $y = -a/2$.

Let, the fluid be moving with velocity u parallel to x-axis with Laminar flow. In order to maintain such a motion, the difference of pressure in x-direction must be balanced by shearing stresses.

Here; $\quad q = q(u, 0, 0)$

Equation of continuity

$$\frac{\partial u}{\partial x} = 0 \; ; \text{ so that } \; u = u(y, t)$$

Navier-Stokes equation in absence of external force is

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + (q \cdot \nabla) q = -\frac{1}{\rho} \nabla p + \nu \nabla^2 q .$$

or $\quad i\frac{\partial u}{\partial t} + iu\frac{\partial u}{\partial x} = -\frac{1}{\rho} \nabla p + \nu i \nabla^2 u .$

or $\quad i\frac{\partial u}{\partial t} = -\frac{1}{\rho} \nabla p + \nu i \nabla^2 u \quad$ as $\quad \frac{\partial u}{\partial x} = 0$

This $\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u} \quad$ ——— (1)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{and} \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

The last two $\Rightarrow p = p(x, t)$ ——— (2)

Now (1) $\Rightarrow \quad \boxed{\frac{\partial p}{\partial x} = \rho\frac{\partial u}{\partial t} + \mu\frac{\partial^2 u}{\partial y^2}} \quad$ ——— (3)

Also: $\boxed{p = p(x, t), \; u = u(y, t)}$

R.H.S of (3) is constant or function of y, t.

Consequently (3) declares that either $\frac{\partial p}{\partial x}$ is constant or function of 't'. Now consider the case of steady motion so that (3) becomes.

$$\mu\frac{\partial^2 u}{\partial y^2}=\frac{\partial p}{\partial x}=\frac{dp}{dx} \quad \text{or} \quad \frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{dp}{dx}$$

Integrating

$$\frac{du}{dy}=\frac{y}{\mu}\frac{dp}{dx}+A \quad \text{or} \quad u=\frac{y^2}{2\mu}\frac{dp}{dx}+Ay+B. \quad \text{——(4)}$$

Case I. Plane Couette flow:

In this case $\frac{dp}{dx}=0$; the lower plate is stationary while the upper is moving with uniform velocity 'U' parallel to x-axis. The boundary conditions are

(i) $u=0$; $y=-h/2$ (ii) $u=U=$ constant ; $y=h/2$

Substituting (4) to (i) & (ii) we get.

$$0=\frac{h^2}{8\mu}\cdot 0+A\left(-\frac{h}{2}\right)+B \quad \text{and} \quad U=\frac{h^2}{8\mu}\cdot 0+A\cdot\frac{h}{2}+B$$

This $\Rightarrow -Ah+2B=0$; $Ah+2B=2U$

$\Rightarrow 2B=U$; $-Ah+U=0$

Now (4) becomes

$$\boxed{u=\frac{U}{h}y+\frac{U}{2}} \quad \text{——(5)}$$

Evidently; velocity distribution is linear.

Case II. Plane Poiseuille flow:

In this case $\frac{dp}{dx}=$ const. $=a\neq 0$ and both the walls are at rest.

The boundary conditions are.

(i) $u=0$; $y=-h/2$

(ii) $u=0$; $y=h/2$.

Subjecting (4) to condition (i) & (ii),

$$\frac{ah^2}{8\mu} + A\left(-\frac{h}{2}\right) + B = 0 \text{ and } \frac{ah^2}{8\mu} + A\left(\frac{h}{2}\right) + B = 0$$

subtracting we get.

$A = 0$ and $B = -ah^2/8\mu$.

Now, (4) becomes

$$u = \frac{ay^2}{2\mu} - \frac{ah^2}{8\mu} = \frac{-h^2}{8\mu}\left(1 - \frac{4y^2}{h^2}\right)\frac{dp}{dx} \quad \text{——— (6)}$$

$$u = u_m\left(1 - \frac{4y^2}{h^2}\right) \quad \text{where } u_m = -\frac{h^2}{8\mu}\frac{dp}{dx} \text{——— (7)}$$

is the maximum velocity in the flow occuring at $y=0$; Evidently, velocity distribution is parabolic.

Drag (shear stress) at lower plate

$$= \left(\mu\frac{du}{dy}\right)_{y=-h/2} = \mu\left(-\frac{8y}{h^2}\cdot u_m\right)_{y=-h/2} = \underline{\underline{\frac{4\mu u_m}{h}}}.$$

∴ The average velocity distribution for the present flow is given by

$$u_a = \frac{1}{h}\int_{-h/2}^{h/2} u\cdot dy \quad .(\text{using}(6)); \text{ we get}$$

$$u_a = \frac{1}{h}u_m\int_{-h/2}^{h/2}\left(1 - \frac{4y^2}{h^2}\right)dy = \frac{2}{h}u_m\int_0^{h/2}\left(1 - \frac{4y^2}{h^2}\right)dy$$

$$u_a = \frac{2}{h}\left(-\frac{h^2}{8\mu}\cdot a\right)\left[\frac{h}{2}\left(\frac{-4}{h^2}\right)\cdot\frac{1}{3}\left(\frac{h}{2}\right)^3\right] = \left(-\frac{ha}{4\mu}\right)\left(\frac{h}{3}\right)$$

$$u_a = \frac{2}{3}\left(-\frac{h^2 a}{8\mu}\right) = \frac{2}{3}u_m$$

or $\boxed{u_a = \frac{2}{3}u_m}$ ——— (8)

where; u_a = average velocity ; $a = \frac{dp}{dx}$ = constant

u_m = maximum velocity.

Case II Generalised Plane Couette Flow:

In this case $\frac{dp}{dx}$ = const. $= a \neq 0$; the lower plate is at rest while the upper plate is in motion with velocity U. The boundary conditions are.

(i) $u=0$; $y=-h/2$ (ii) $u=U$; $y=h/2$

Substituting in (4) from (i) & (ii)

$$\frac{ah^2}{8\mu} + A\left(-\frac{h}{2}\right) + B = 0 \quad ; \quad \frac{ah^2}{8\mu} + A\left(\frac{h}{2}\right) + B = U$$

This $\Rightarrow$ $B = \frac{U}{2} - \frac{ah^2}{8\mu}$; $A = \frac{U}{h}$

Now (4) becomes.

$$u = \frac{ay^2}{2\mu} + \frac{U}{h}y + \frac{U}{2} - \frac{ah^2}{8\mu}$$

or $$u = \frac{a}{8\mu}(4y^2 - h^2) + \frac{U}{2}\left(1 + \frac{2y}{h}\right)$$

Evidently ∴ velocity distribution is parabolic

$$\mu\frac{du}{dy} = \frac{a}{8\mu}(8y - 0) + \mu\cdot\frac{U}{2}\left(0 + \frac{2}{h}\right) = ay + \frac{\mu}{h}\cdot U \quad \text{—(9)}$$

Drag per unit area on boundaries.

$$= \mu\frac{du}{dy} \quad \text{at } y = \pm h/2$$

$$= \mu\frac{U}{h} \pm \frac{h}{2}\frac{dp}{dx}.$$

Total flux (flow) per unit breadth across a plane perpendicular to x-axis is Q

$$= \int_{-h/2}^{h/2} u\,dy = \left[\frac{a}{8\mu}\left(\frac{4}{3}y^3 - h^2 y\right) + \frac{U}{2}\left(y + \frac{y^3}{h}\right)\right]_{y=-h/2}^{y=h/2}$$

or $$Q = U\cdot\frac{h}{2} - \frac{h^3}{12}\cdot\frac{a}{\mu}$$

Vortricity $W(\xi, \eta, \zeta)$ at any point is given by

$$\xi = 0 \,;\ \eta = 0 \,;\ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{-1}{2}\frac{\partial u}{\partial y} = \frac{-1}{2}\frac{du}{dy}.$$

$$\zeta = \frac{-1}{2}\left(\frac{ay}{\mu} + \frac{U}{h}\right) \text{ by } (9)$$

Rate ω of dissipation of energy per unity area is given by

$$D = 4\mu \int_{-h/2}^{h/2} \zeta^2 dy = \mu \int_{-h/2}^{h/2} \left[\frac{ay}{\mu} + \frac{U}{h}\right]^2 dy$$

$$D = \mu \int_{-h/2}^{h/2} \left(\frac{a^2 y^2}{\mu^2} + \frac{U^2}{h^2} + \frac{2ayU}{\mu h}\right) dy$$

$$\boxed{D = \frac{a^2 h^3}{12\mu} + \frac{U^2 a}{h}}$$