Motion in a Resisting Medium

(In a straight line only)

§ 1. Introduction.

It is a well known fact that a body moving in a medium (like air) feels a resistance to its motion which increases with the increase in the velocity of the body. Thus the resistance on a body moving in a medium may be assumed to be equal to some function of the velocity of the body. The resistance of the medium always acts opposite to the direction of motion of the body.

Experimentally it has been found out that when a particle is projected in air, the force of resistance varies as the square of the velocity upto a velocity of 800 ft./sec. and as cube of the velocity between 800 ft./sec and 1350 ft./sec. Beyond this velocity the resistance again varies as the square of the velocity.

Therefore in this chapter we shall mostly discuss the motion of a particle (or body) in a resisting medium where the resistance varies as the square of the velocity.

§ 2. Terminal Velocity.

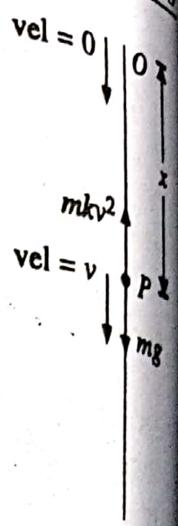
If a particle falls under gravity in a resisting medium the force of resistance acts vertically upwards on the particle while the force of gravity acts vertically downwards. As the velocity of the particle goes on increasing the force of resistance also goes on increasing. Suppose the force of resistance becomes equal to the force of gravity when the particle has attained the velocity V. Then the resultant downward acceleration of the particle becomes zero and so during its subsequent motion the particle falls with constant velocity V, called the *terminal velocity* or the *limiting velocity*. The terminal velocity is maximum for the downward motion.

Definition. *If a particle is falling under gravity in a resisting medium, then the velocity V when the downward acceleration is zero is called the* **terminal (or limiting) velocity.** **[Meerut 95]**

§ 3. Motion of a Particle Falling Under Gravity.

A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion. **[Lucknow 80; Meerut 78, 81S, 83P, 84S, 86, 88S, 90S, 97]**

Let a particle of mass m fall from rest under gravity from the fixed point O.



Let P be the position of the particle after time t, where $OP = x$. If v is the velocity of the particle at P, then mkv^2 is the resistance of the medium on the particle acting in the upwards direction *i.e.*, in the direction of x decreasing. Here kv^2 is the resistance per unit mass so that the resistance on the particle of mass m is mkv^2.

The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv^2$$

or
$$\frac{d^2x}{dt^2} = g\left(1 - \frac{k}{g}v^2\right) \qquad \text{...(1)}$$

If V is the terminal velocity, then when $v = V, d^2x/dt^2 = 0$.

$\therefore$ from (1), we have, $0 = g\left(1 - \frac{k}{g}V^2\right)$ or $\frac{k}{g} = \frac{1}{V^2}$.

$$\therefore \quad \frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{V^2}\right) \quad \text{or} \quad \frac{d^2x}{dt^2} = \frac{g}{V^2}(V^2 - v^2) \qquad \text{...(2)}$$

To find the relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx} = \frac{g}{V^2}(V^2 - v^2) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or
$$\frac{-2g}{V^2}dx = \frac{-2v\,dv}{V^2 - v^2}.$$

Integrating, $\frac{-2g}{V^2}x = \log(V^2 - v^2) + A$, where A is a constant.

But initially at O, when $x = 0, v = 0$.

$$\therefore \quad 0 = \log V^2 + A \quad \text{or} \quad A = -\log V^2.$$

$$\therefore \quad \frac{-2gx}{V^2} = \log(V^2 - v^2) - \log V^2 = \log\left(\frac{V^2 - v^2}{V^2}\right)$$

or
$$\frac{V^2 - v^2}{V^2} = e^{-2gx/V^2}$$

or
$$V^2 - v^2 = V^2e^{-2gx/V^2} \quad \text{or} \quad v^2 = V^2(1 - e^{-2gx/V^2}), \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t. Again the equation (2) can be written

$$\frac{dv}{dt}=\frac{g}{V^2}(V^2-v^2) \qquad \left[\because \frac{d^2x}{dt^2}=\frac{dv}{dt}\right]$$

or

$$\frac{g}{V^2}dt=\frac{dv}{V^2-v^2}.$$

Integrating, $\dfrac{g}{V^2}t=\dfrac{1}{2V}\log\dfrac{V+v}{V-v}+B$, where B is a constant.

Initially at O, when $t=0, v=0$.

$$\therefore\ 0=\frac{1}{2V}\log 1+B, \text{ or } B=0.$$

$$\therefore\ \frac{gt}{V^2}=\frac{1}{2V}\log\frac{V+v}{V-v}$$

or

$$t=\frac{V}{g}\cdot\frac{1}{2}\log\frac{1+(v/V)}{1-(v/V)}=\frac{V}{g}\tanh^{-1}\frac{v}{V}$$

$$\left[\because \tanh^{-1}z=\frac{1}{2}\log\frac{1+z}{1-z}\right]$$

or

$$\frac{gt}{V}=\tanh^{-1}\frac{v}{V} \text{ or } v=V\tanh(gt/V), \qquad \text{...(4)}$$

which gives the velocity of the particle at any time.

Relation between x and t. Eliminating v between (3) and (4), we have

$$V^2\tanh^2(gt/V)=V^2(1-e^{-2gx/V^2})$$

or

$$e^{-2gx/V^2}=1-\tanh^2(gt/V)=\text{sech}^2(gt/V)$$

or

$$e^{2gx/V^2}=\cosh^2(gt/V)$$

or

$$\frac{2gx}{V^2}=2\log\cosh(gt/V) \text{ or } x=\frac{V^2}{g}\log\cosh(gt/V), \qquad \text{...(5)}$$

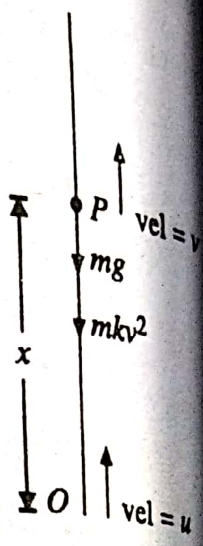
which gives the position of the particle at any time.

Remark. To evaluate $\displaystyle\int\frac{dv}{V^2-v^2}$, we can directly apply the formula $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a}$. **Remember this formula.**

§ 4. Motion of a particle Projected Vertically Upwards.

A particle is projected vertically upwards under gravity, supposed constant, in a resisting medium whose resistance veries as the square of the velocity; to discuss the motion. **[Lucknow 79; Meerut 78, 90S, 91, 92]**

Let a particle of mass m be projected, vertically upwards from the point O, with velocity u. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are



(i) The force mkv^2 due to resistance acting against the direction of motion *i.e.*, acting vertically downwards.

(ii) The weight mg of the particle also acting vertically downwards.

Both these forces act in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or
$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{k}{g}v^2\right).$$

Let V be the terminal velocity of the particle during its downwards motion *i.e.*, the velocity when the resultant acceleration of the particle during its downwards motion is zero. Then

$$0 = mg - mkV^2, \text{ or } k = g/V^2.$$

Putting this value of k in the above equation of motion of the particle, we get

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right) \text{ or } \frac{d^2x}{dt^2} = \frac{-g}{V^2}(V^2 + v^2). \quad ...(1)$$

Relation between v and x.

Equation (1) can be written as

$$v\frac{dv}{dx} = \frac{-g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or
$$\frac{-2g}{V^2}dx = \frac{2v\,dv}{V^2 + v^2}, \text{ separating the variables.}$$

Integrating, $\dfrac{-2gx}{V^2} = \log(V^2 + v^2) + A$, where A is a constant.

Initially at O, $x = 0$ and $v = u$.

$$\therefore 0 = \log(V^2 + u^2) + A$$

or
$$A = -\log(V^2 + u^2).$$

$$\therefore \frac{-2gx}{V^2} = \log(V^2 + v^2) - \log(V^2 + u^2)$$

$$x = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2 + v^2}, \qquad \ldots(2)$$

which gives the velocity of the particle in any position.

Relation between v and t.

Equation (1) can be written as

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or $$dt = \frac{-V^2}{g} \cdot \frac{dv}{V^2 + v^2}, \text{ separating the variables.}$$

Integrating, $t = \dfrac{-V^2}{g} \cdot \dfrac{1}{V} \tan^{-1} \dfrac{v}{V} + B$, where B is a constant

or $$t = \frac{-V}{g} \tan^{-1} \frac{v}{V} + B.$$

Initially at O, when $t = 0, v = u$.

$$\therefore\ 0 = -\frac{V}{g} \tan^{-1} \frac{u}{V} + B \text{ or } B = \frac{V}{g} \tan^{-1} \frac{u}{V}.$$

$$\therefore\ t = \frac{V}{g}\left(\tan^{-1} \frac{u}{V} - \tan^{-1} \frac{v}{V}\right), \qquad \ldots(3)$$

which gives the velocity of the particle at any time t.

Relation between x and t.

A relation between x and t can be obtained by eliminating v between (2) and (3).

Illustrative Examples

Ex. 1 (a). *A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $(1/\mu V)\,[\sqrt{(1 + 2\mu V^2 t)} - 1]$ and that its velocity then is $V/\sqrt{(1 + 2\mu V^2 t)}$.*

[Meerut 73, 76]

Solution. Take the point of projection O as origin. Let v be the velocity of the particle at time t at a point distant x from the fixed point O. Then the resistance at this point will be $m\mu v^3$, acting in the direction of x decreasing. Here the resistance is the only force acting on the particle during its motion.

$\therefore$ the equation of motion of the particle is

$$m \frac{dv}{dt} = -m\mu v^3$$

or $$\frac{dv}{v^3} = -\mu dt.$$

Integrating, $-\dfrac{1}{2v^2} = -\mu t + A$, where A is a constant.

But initially, when $t=0, v=V$; $\therefore A=-\dfrac{1}{2V^2}$.

$$\therefore -\frac{1}{2v^2}=-\mu t-\frac{1}{2V^2}$$

or $\quad 1/v^2=(2\mu V^2 t+1)/V^2$ or $v=V/\sqrt{(1+2\mu V^2 t)}$, ...(1)

which gives the velocity of the particle at time t.

Since the particle is moving in the direction of x increasing, therefore from the equation (1), we have

$$\frac{dx}{dt}=v=V/\sqrt{(1+2\mu V^2 t)}$$

or $$dx=V(1+2\mu V^2 t)^{-1/2}\,dt.$$

Integrating, $x=\dfrac{1}{\mu V}(1+2\mu V^2 t)^{1/2}+B$, where B is a constant.

But initially when $t=0, x=0$; $\therefore B=-\dfrac{1}{\mu V}$.

$$\therefore x=\frac{1}{\mu V}(1+2\mu V^2 t)^{1/2}-\frac{1}{\mu V}$$

or $$x=\frac{1}{\mu V}[\sqrt{(1+2\mu V^2 t)}-1],$$

which gives the distance described in time t.

(b) *A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is k (velocity), show that the velocity after a time t and the distane s in that time are given by*

$$v=ue^{-kt} \text{ and } s=u(1-e^{-kt})/k. \quad \text{(Meerut 1989, 91)}$$

Sol. Proceed as in Ex. 1. (a).

Here the equation of motion of the particle is

$$m\frac{dv}{dt}=-mkv \quad \text{or} \quad \frac{dv}{dt}=-kv.$$

$$\therefore \frac{dv}{v}=-k\,dt.$$

Integrating, $\log v=-kt+C_1$.

But initially, when $t=0, v=u$; $\therefore C_1=\log u$.

$\therefore \log v=-kt+\log u$, or $\log(v/u)=-kt$

or $\quad v/u=e^{-kt}$ or $v=ue^{-kt}$.

But $v=ds/dt$.

$\therefore ds/dt=ue^{-kt}$ or $ds=ue^{-kt}\,dt$.

Integrating, $s=\dfrac{ue^{-kt}}{-k}+C_2$.

But initially, when $t=0, s=0$; $\therefore C_2=u/k$.

$$\therefore s=\frac{u}{k}-\frac{ue^{-kt}}{k}=\frac{u(1-e^{-kt})}{k}.$$

Ex. 2. *A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v be the velocity actually acquired by it, v_0 the velocity it would have acquired, had there been no resisting medium and V the terminal velocity, show that*

$$\frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots\ldots$$

(Meerut 1985 P)

Sol. If v is the velocity of the particle acquired in falling through a distance x in the given resisting medium, then proceeding as in § 3, it is given by

$$v^2=V^2\left(1-e^{-2gx/V^2}\right). \qquad \ldots(1)$$

If v_0 is the velocity of the particle acquired in falling freely through a distance x, if there is no resisting medium, then

$$v_0^2=0+2gx=2gx. \qquad \ldots(2)$$

Subtituting $2gx=v_0^2$ in (1), we have

$$v^2=V^2\left(1-e^{-v_0^2/V^2}\right)$$

$$=V^2\left[1-\left\{1-\frac{v_0^2/V^2}{1!}+\frac{v_0^4/V^4}{2!}-\frac{v_0^6/V^6}{3!}+\ldots\ldots\ldots\right\}\right]$$

$$=V^2\left[\frac{v_0^2/V^2}{1!}-\frac{v_0^4/V^4}{2!}+\frac{v_0^6/V^6}{3!}-\ldots\ldots\right]$$

$$=v_0^2\left[1-\frac{1}{2!}\cdot\frac{v_0^2}{V^2}+\frac{1}{3!}\frac{v_0^4}{V^4}-\frac{1}{4!}\cdot\frac{v_0^6}{V^6}+\ldots\right]$$

or
$$\frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2\cdot3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots\ldots$$

Ex. 3. *A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda-\log(1+\lambda)]$, where V is the terminal velocity of the particle and λV is the initial velocity.*

(Lucknow 1981; Meerut 71, 76, 84P, 86, 87P, 88, 92S, 93)

Sol. Suppose a particle of mass m is projected vertically upwards from O with velocity λV in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP=x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are,

(i) The force mkv due to the resistance acting vertically downwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = -mg - mkv$$

or
$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{k}{g}v\right). \quad \text{...(1)}$$

Now V is given to be the terminal velocity of the particle during its downward motion. Then V is the velocity of the particle when during the downward motion its acceleration is zero. If the particle falls vertically downwards, the resistance acts vertically upwards. Therefore the equation of motion of the particle in downward motion is

$$m\frac{d^2x}{dt^2} = mg - mkv. \quad \text{...(2)}$$

Putting $v = V$ and $d^2x/dt^2 = 0$ in (2), we get

$$0 = mg - mkV \quad \text{or} \quad k = g/V.$$

Substituting this value of k in (1), the equation of motion of the particle in the upward motion is

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v}{V}\right)$$

or
$$v\frac{dv}{dx} = -\frac{g}{V}(V + v), \quad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or
$$dx = -\frac{V}{g}\frac{v\,dv}{V + v}, \text{ separating the variables}$$

or
$$dx = -\frac{V}{g}\left\{\frac{(v + V) - V}{v + V}\right\}dv = -\frac{V}{g}\left(1 - \frac{V}{V + v}\right)dv.$$

Integrating, we have

$$x = -\frac{V}{g}\{v - V\log(V + v)\} + A, \text{ where } A \text{ is a constant.}$$

But initially when $x = 0, v = \lambda V$ (given).

$$\therefore \quad 0 = -\frac{V}{g}\{\lambda V - V\log(V + \lambda V)\} + A$$

or
$$A = \frac{V}{g}[\lambda V - V\log\{V(1 + \lambda)\}].$$

$$\therefore \quad x = \frac{V}{g}\left[\lambda V - v - V\log\frac{V(1 + \lambda)}{(V + v)}\right], \text{ giving the velocity of}$$

the particle at any position.

If h is the greatest height attained by the particle, we have $v = 0$ when $x = h$.

$$\therefore \quad h = \frac{V}{g}\left[\lambda V - V\log\frac{V(1 + \lambda)}{V}\right] = \frac{V^2}{g}[\lambda - \log(1 + \lambda)].$$

Ex. 4. *A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Find the greatest height attained by the particle.* **(Meerut 96, 97)**

Sol. Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resitance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. Then proceeding as in Ex. 3. the equation of motion of the particle at time t is

$$m \frac{d^2x}{dt^2} = -mg - mkv$$

or $$\frac{d^2x}{dt^2} = -(g + kv) \quad \text{or} \quad v\frac{dv}{dx} = -(g + kv).$$

$$\therefore \quad dx = -\frac{v}{g + kv} dv = -\frac{1}{k} \frac{kv}{g + kv} dv$$

$$= -\frac{1}{k} \frac{(g + kv) - g}{g + kv} dv = -\frac{1}{k}\left[1 - \frac{g}{g + kv}\right] dv.$$

Integrating, we get

$$x = -\frac{1}{k}\left[v - \frac{g}{k}\log(g + kv)\right] + A, \text{ where } A \text{ is a constant.}$$

But initially when $x = 0$, we have $v = u$.

$$\therefore \quad 0 = -\frac{1}{k}\left[u - \frac{g}{k}\log(g + ku)\right] + A$$

or $$A = \frac{1}{k}\left[u - \frac{g}{k}\log(g + ku)\right].$$

$$\therefore \quad x = -\frac{1}{k}\left[v - \frac{g}{k}\log(g + kv)\right] + \frac{1}{k}\left[u - \frac{g}{k}\log(g + ku)\right]$$

$$= \frac{1}{k}(u - v) - \frac{g}{k^2}\log\frac{g + ku}{g + kv},$$ giving the velocity of the particle at any position.

If h is the greatest height attained by the particle, we have $v = 0$ when $x = h$.

$$\therefore \quad h = \frac{u}{k} - \frac{g}{k^2}\log\left(\frac{g + ku}{g}\right) = \frac{u}{k} - \frac{g}{k^2}\log\left(1 + \frac{ku}{g}\right).$$

Ex. 5. *A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is* $\dfrac{gm^2}{\mu^2}\left[e^{-(\mu/m)t} - 1 + \dfrac{\mu t}{m}\right]$. **(Meerut 1975, 79, 83, 87S 88S, 90S)**

Sol. Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards *i.e.*, in the direction of x decreasing. The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

or $$\frac{dv}{dt} = g - \frac{\mu}{m}v, \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or $$dt = \frac{dv}{g - (\mu/m)v}.$$

Integrating, we have

$$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A, \text{ where } A \text{ is a constant.}$$

But initially when $t = 0, v = 0$; $\therefore$ $A = (m/\mu)\log g$.

$\therefore$ $$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + \frac{m}{\mu}\log g$$

or $$t = -\frac{m}{\mu}\log\left\{\frac{g - (\mu/m)v}{g}\right\}$$

or $$-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm}v\right) \quad \text{or} \quad 1 - \frac{\mu}{gm}v = e^{-\mu t/m}$$

or $$v = \frac{dx}{dt} = \frac{gm}{\mu}(1 - e^{-\mu t/m}) \quad \text{or} \quad dx = \frac{gm}{\mu}(1 - e^{-\mu t/m})\,dt.$$

Integrating, we have

$$x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B, \qquad \text{...(1)}$$

where B is a constant.

But initially when $t = 0, x = 0$.

$\therefore$ $$0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B. \qquad \text{...(2)}$$

Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right\} = \frac{gm^2}{\mu^2}\left\{e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right\}.$$

Ex. 6. *Discuss the motion of a particle projected upwards with a velocity u in a medium whose resistance varies as the velocity.* **(Meerut 89)**

Sol. Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are

(i) The force mkv due to the resistance acting vertically downwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle in upwards motion at time t is

$$m\frac{d^2x}{dt^2} = -mg - mkv$$

or
$$\frac{d^2x}{dt^2} = -(g + kv). \qquad \text{...(1)}$$

If the particle moves downwards in the same resisting medium and its velocity is v at time t at distance x from the starting point, then its equation of motion in downwards motion will be

$$m\frac{d^2x}{dt^2} = mg - mkv \quad \text{or} \quad \frac{d^2x}{dt^2} = g - kv.$$

If V is the terminal velocity of the particle during its downward motion, then

$$0 = g - kV \quad \text{or} \quad k = g/V.$$

$\therefore$ the equation of motion (1) in upwards motion becomes

$$\frac{d^2x}{dt^2} = -\left(g + \frac{g}{V}v\right) = \frac{-g(V+v)}{V}. \qquad \text{...(2)}$$

Relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx} = -\frac{g}{V}(V + v)$$

or
$$dx = -\frac{V}{g}\frac{v}{v+V}dv = -\frac{V}{g}\frac{(v+V) - V}{v+V}dv$$
$$= -\frac{V}{g}\left[1 - \frac{V}{v+V}\right]dv.$$

Integrating, $x = -\dfrac{V}{g}[v - V\log(v+V)] + A$, where A is a constant.

But initially at O, $x = 0$ and $v = u$.

$$\therefore \quad A = \frac{V}{g}[u - V\log(u+V)].$$

$$\therefore \quad x = -\frac{V}{g}[v - V\log(v+V)] + \frac{V}{g}[u - V\log(u+V)]$$

or
$$x = \frac{V}{g}\left[(u - v) + V\log\left(\frac{v+V}{u+V}\right)\right], \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t.

The equation (2) can also be written as

$$\frac{dv}{dt} = -\frac{g}{V}(v + V).$$

$$\therefore \quad dt = -\frac{V}{g}\frac{dv}{v+V}.$$

Integrating, $t = -\frac{V}{g}\log(v + V) + B$, where B is a constant.

But initially at O, $t = 0$ and $v = u$.

$$\therefore \quad B = \frac{V}{g}\log(u + V).$$

$$\therefore \quad t = -\frac{V}{g}\log(v + V) + \frac{V}{g}\log(u + V)$$

or
$$t = \frac{V}{g}\log\frac{u+V}{v+V},$$

which gives the velocity of the particle at any time t. ...(4)

Relation between x and t.

From (4), we have

$$\log\frac{u+V}{v+V} = \frac{gt}{V} \quad \text{or} \quad \frac{u+V}{v+V} = e^{gt/V}$$

or
$$v + V = (u + V)\,e^{-gt/V}$$

or
$$v = \frac{dx}{dt} = -V + (u + V)\,e^{-gt/V}$$

or
$$dx = [-V + (u + V)\,e^{-gt/V}]\,dt.$$

Integrating, we get

$$x = -Vt - \frac{V}{g}(u + V)\,e^{-gt/V} + C,$$ where C is a constant.

Initially at O, $x = 0$ and $t = 0$.

$$\therefore \quad C = \frac{V}{g}(u + V).$$

$$\therefore \quad x = -Vt - \frac{V}{g}(u + V)\,e^{-gt/V} + \frac{V}{g}(u + V)$$

or
$$x = -Vt + \frac{V}{g}(u + V)\,[1 - e^{-gt/V}],$$

which gives the distance covered by the particle at any time t. ...(5)

Ex. 7. *Discuss the motion of a particle falling under gravity in a medium whose resistance varies as the velocity.* **(Meerut 1992S, 93S)**

Sol. Suppose a particle of mass m starts at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P.

The forces acting on the particle at P are

(i) The force mkv due to the resistance acting vertically upwards *i.e.,* against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

By Newton's second law of motion the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2}=mg-mkv$$

or $$\frac{d^2x}{dt^2}=g-kv. \qquad ...(1)$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0=g-kV \quad \text{or} \quad k=g/V.$$

Putting $k=g/V$ in (1), we get

$$\frac{d^2x}{dt^2}=g-\frac{g}{V}v=\frac{g}{V}(V-v). \qquad ...(2)$$

Relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx}=\frac{g}{V}(V-v)$$

or $$dx=\frac{V}{g}\frac{v}{V-v}dv=-\frac{V}{g}\frac{-v}{V-v}dv$$

$$=-\frac{V}{g}\frac{(V-v)-V}{V-v}dv=-\frac{V}{g}\left[1-\frac{V}{V-v}\right]dv.$$

Integrating, $x=-\frac{V}{g}[v+V\log(V-v)]+A$, where A is a constant.

But initially at $O, x=0$ and $v=0$.

$$\therefore \quad A=\frac{V^2}{g}\log V.$$

$$\therefore \quad x=-\frac{V}{g}v-\frac{V^2}{g}\log(V-v)+\frac{V^2}{g}\log V$$

or $$x=-\frac{V}{g}v+\frac{V^2}{g}\log\frac{V}{V-v}, \qquad ...(3)$$

which gives the velocity of the particle at any position.

Relation between v and t.

The equation (2) can also be written as

$$\frac{dv}{dt}=\frac{g}{V}(V-v).$$

$$\therefore \quad dt=\frac{V}{g}\frac{dv}{V-v}.$$

Integrating, we have

$$t = -\frac{V}{g}\log(V - v) + B, \text{ where } B \text{ is a constant.}$$

Initially at $O, t = 0$ and $v = 0$.

$$\therefore \quad B = \frac{V}{g}\log V.$$

$$\therefore \quad t = -\frac{V}{g}\log(V - v) + \frac{V}{g}\log V$$

or $$t = \frac{V}{g}\log\frac{V}{V - v},$$

which gives the velocity of the particle at any time t. ...(4)

Relation between x and t.

From (4), we have

$$\log\frac{V}{V - v} = \frac{gt}{V} \quad \text{or} \quad \frac{V}{V - v} = e^{gt/V}$$

or $$V - v = Ve^{-gt/V}$$

or $$v = V[1 - e^{-gt/V}]$$

or $$\frac{dx}{dt} = V[1 - e^{-gt/V}]$$

or $$dx = V[1 - e^{-gt/V}]\,dt.$$

Integrating, we get

$$x = Vt + \frac{V^2}{g}e^{-gt/V} + C, \text{ where } C \text{ is a constant.}$$

Initially at $O, x = 0$ and $t = 0$.

$$\therefore \quad C = -\frac{V^2}{g}.$$

$$\therefore \quad x = Vt + \frac{V^2}{g}e^{-gt/V} - \frac{V^2}{g}$$

or $$x = Vt + \frac{V^2}{g}(e^{-gt/V} - 1),$$

which gives the distance fallen through in time t. ...(5)

Ex. 8. *A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is $\frac{1}{2k}\log\frac{g + ku^2}{g}$.*

(Meerut 1979)

Sol. Let a particle of mass m be projected vertically upwards from a point O with velocity u. If v is the velocity of the particle at time t at a distance x from the starting point O, then the resistance on the particle is mkv^2 in the downward direction *i.e.*, in the direction of x decreasing. The weight mg of the particle also acts vertically downwards. So the equation of motion of the particle during its upward motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$v\frac{dv}{dx} = -(g + kv^2), \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

or $$\frac{2kv\,dv}{g + kv^2} = -2k\,dx, \text{ separating the variables.}$$

Integrating, $\log(g + kv^2) = -2kx + A$, where A is a constant.

But initially $x = 0, v = u$; $\therefore \quad A = \log(g + ku^2)$.

$\therefore \quad \log(g + kv^2) = -2kx + \log(g + ku^2)$

or $$2kx = \log(g + ku^2) - \log(g + kv^2)$$

or $$x = \frac{1}{2k}\log\frac{g + ku^2}{g + kv^2} \qquad ...(1)$$

which gives the velocity of the particle at a distance x.

If h is the greatest height attained by the particle then at $x = h$, $v = 0$. Therefore from (1), we have

$$h = \frac{1}{2k}\log\frac{g + ku^2}{g}.$$

Ex. 9. *A particle is projected vertically upwards with a velocity V and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity V' with which the particle will return to the point of projection is given by*

$$\frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}.$$

(Allahabad 1979; Meerut 73, 80, 82S, 84, 86P, 88, 93)

Sol. Let a particle of mass m be projected vertically upwards with a velocity V.

If v is the velocity of the particle at time t, at a distance x from the starting point, the resistance there is mkv^2 in the downward direction (*i.e.*, in the direction of x decreasing). The weight mg of the particle also acts vertically downwards.

$\therefore$ the equation of motion of the particle in the upward motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$v\frac{dv}{dx} = -(g + kv^2)$$

or $$\frac{2kv\,dv}{g + kv^2} = -2k\,dx.$$

Integrating, $\log(g + kv^2) = -2kx + A$, where A is a constant.
Initially when $x = 0, v = V$; $\therefore$ $A = \log(g + kV^2)$.

$\therefore$ $\log(g + kv^2) = -2kx + \log(g + kV^2)$

or
$$x = \frac{1}{2k}\log\frac{g + kV^2}{g + kv^2}.$$

If h is the maximum height attained by the particle, then $v = 0$, when $x = h$.

$$\therefore \quad h = \frac{1}{2k}\log\frac{g + kV^2}{g}. \quad \text{...(1)}$$

Now from the highest point O' the particle falls vertically downwards.

Let y be the depth of the particle below the highest point O' after time t and v be the velocity there. Then the resistance at this point is mkv^2 acting in the vertically upwards direction.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2y}{dt^2} = mg - mkv^2$$

or
$$v\frac{dv}{dy} = g - kv^2 \quad \text{or} \quad \frac{-2kv\,dv}{g - kv^2} = -2k\,dy.$$

Integrating, $\log(g - kv^2) = -2ky + B$, where B is a constant.
At the highest point O', $y = 0, v = 0$; $\therefore$ $B = \log g$.

$\therefore$ $\log(g - kv^2) = -2ky + \log g$

or
$$y = \frac{1}{2k}\log\frac{g}{g - kv^2}.$$

If the particle returns to the point of projection O with velocity V', then $v = V'$ when $y = h$.

$$\therefore \quad h = \frac{1}{2k}\log\frac{g}{g - kV'^2} \quad \text{...(2)}$$

From (1) and (2), equating the values of h, we have

$$\frac{1}{2k}\log\frac{g + kV^2}{g} = \frac{1}{2k}\log\frac{g}{g - kV'^2}$$

or
$$\frac{g + kV^2}{g} = \frac{g}{g - kV'^2}$$

or
$$(g + kV^2)(g - kV'^2) = g^2$$

or
$$-gkV'^2 + gkV^2 - k^2V^2V'^2 = 0.$$

Dividing by $kgV^2V'^2$, we have

$$-\frac{1}{V^2} + \frac{1}{V'^2} - \frac{k}{g} = 0 \quad \text{or} \quad \frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}.$$

Ex. 10. *A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is $(1/k) \log \cosh \{t\sqrt{(gk)}\}$.*

If the particle were ascending, show that at any instant its distance from the highest point of its path is $(1/k) \log \sec \{t\sqrt{(gk)}\}$, where t now denotes the time it will take to reach its highest point.

Sol. When the particle is falling vertically downwards, let x be its distance from the starting point after time t. If v is its velocity at this point, then the resistance on the particle is mkv^2 in the vertically upwards direction. The weight mg of the particle acts vertically downwards.

$\therefore$ the equation of motion of the particle during the downward motion is

$$m\frac{d^2x}{dt^2} = mg - mkv^2 \quad \text{or} \quad \frac{d^2x}{dt^2} = g - kv^2$$

or $$\frac{dv}{dt} = g - kv^2 \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

or $$\frac{dv}{g - kv^2} = dt \quad \text{or} \quad \frac{dv}{k\,[(g/k) - v^2]} = dt.$$

Integrating, we get

$$\frac{1}{k} \cdot \frac{1}{\sqrt{(g/k)}} \tanh^{-1} \frac{v}{\sqrt{(g/k)}} = t + C_1.$$

But initially when $t = 0, v = 0$; $\qquad \therefore \quad C_1 = 0.$

$\therefore$ $$\frac{1}{\sqrt{(gk)}} \tanh^{-1} \frac{v}{\sqrt{(g/k)}} = t$$

or $$\tanh^{-1} \frac{v}{\sqrt{(g/k)}} = t\sqrt{(gk)} \quad \text{or} \quad \frac{v}{\sqrt{(g/k)}} = \tanh \{t\sqrt{(gk)}\}$$

or $$v = \sqrt{\left(\frac{g}{k}\right)} \cdot \frac{\sinh \{t\sqrt{(gk)}\}}{\cosh \{t\sqrt{(gk)}\}}$$

or $$\frac{dx}{dt} = \sqrt{\left(\frac{g}{k}\right)} \cdot \frac{1}{\sqrt{(gk)}} \cdot \frac{\sqrt{(gk)} \,.\, \sinh \{t\sqrt{(gk)}\}}{\cosh \{t\sqrt{(gk)}\}}$$

or $$dx = (1/k) \cdot \frac{\sqrt{(gk)} \,.\, \sinh \{t\sqrt{(gk)}\}}{\cosh \{t\sqrt{(gk)}\}}\, dt.$$

Integrating, we get

$$x = (1/k) \,.\, \log \cosh \{t\sqrt{(gk)}\} + C_2.$$

But initially when $t = 0, x = 0$.

$\therefore \quad 0 = (1/k) \,.\, \log \cosh 0 + C_2 = (1/k) \,.\, \log 1 + C_2 = 0 + C_2.$

$\therefore \quad C_2 = 0.$

$\therefore \quad x = (1/k) \log \cosh \{t\sqrt{(gk)}\}$, which proves the first part of the question.

Vertically Upwards Motion. When the particle is ascending vertically upwards, let y be its distance from the starting point after time T. If v is its velocity at this point, then the resitance is mkv^2 in the downward direction. The weight mg of the particle also acts vertically downwards.

$\therefore$ the equation of motion of the particle during the upward motion is

$$m\frac{d^2y}{dT^2} = -mg - mkv^2 \quad \text{or} \quad \frac{d^2y}{dT^2} = -(g + kv^2)$$

or $$\frac{dv}{dT} = -(g + kv^2) \qquad \left[\because \frac{d^2y}{dT^2} = \frac{dv}{dT}\right]$$

or $$\frac{dv}{g + kv^2} = -dT \quad \text{or} \quad \frac{dv}{k\,[(g/k) + v^2]} = -dT.$$

Integrating, we get

$$\frac{1}{k}\cdot\frac{1}{\sqrt{(g/k)}}\tan^{-1}\frac{v}{\sqrt{(g/k)}} = -T + C_1.$$

Let t_1 be the time from the point of projection to reach the highest point. Then $T = t_1, v = 0$.

$$\therefore \qquad 0 = -t_1 + C_1 \quad \text{or} \quad C_1 = t_1.$$

$$\therefore \qquad \frac{1}{\sqrt{(gk)}}\tan^{-1}\frac{v}{\sqrt{(g/k)}} = t_1 - T$$

or $$\tan^{-1}\frac{v}{\sqrt{(g/k)}} = (t_1 - T)\sqrt{(gk)}$$

or $$\frac{v}{\sqrt{(g/k)}} = \tan\{(t_1 - T)\sqrt{(gk)}\}$$

or $$v = \frac{dy}{dT} = \sqrt{\left(\frac{g}{k}\right)}\cdot\tan\{(t_1 - T)\sqrt{(gk)}\}. \qquad ...(1)$$

If h is the greatest height attained by the particle and x be the depth below the highest point of the point distant y from the point of projection, then

$$x = h - y. \qquad ...(2)$$

Also if t denotes the time from the distance y from the point of projection to reach the highest point, then

$$t = t_1 - T. \qquad ...(3)$$

From (2), we have $dx = -dy$

and from (3), we have $dt = -dT$.

$$\therefore \qquad \frac{dx}{dt} = \frac{dy}{dT}.$$

$\therefore$ from (1), we have

$$\frac{dx}{dt} = \sqrt{\left(\frac{g}{k}\right)} \cdot \tan \{t\sqrt{(gk)}\}.$$

Integrating, we get

$$x = \sqrt{\left(\frac{g}{k}\right)} \cdot \frac{\log \sec \{t\sqrt{(gk)}\}}{\sqrt{(gk)}} + C_2$$

$$[\because \int \tan x \, dx = \log \sec x]$$

or $$x = (1/k) \log \sec \{t\sqrt{(gk)}\} + C_2.$$

But from (2) and (3), it is obvious that $x = 0$, when $t = 0$.

$\therefore \quad 0 = (1/k) \log \sec 0 + C_2 \quad$ or $\quad C_2 = 0.$

$\therefore \quad x = (1/k) \log \sec \{t\sqrt{(gk)}\}$, which gives the required ditance of the particle from the highest point.

Ex. 11. *A particle of unit mass is projected vertically upwards with velocity V in a medium for which the resistance is kv when the speed of the particle is v. Prove that the particle returns to the point of projection with speed V_1 such that*

$$V + V_1 = \frac{g}{k} \log \left(\frac{g + kV}{g - kV_1}\right).$$

(Meerut 1974)

Sol. Let x be the distance of the particle of unit mass from the starting point O at time t in its upward motion. If v is its velocity at this point, then the resistance is kv. The weight $1.g$ of the particle acts vertically downwards.

$\therefore$ the equation of motion of the particle during its upward motion is

$$1 . \frac{d^2x}{dt^2} = -g - kv$$

or $$v \frac{dv}{dx} = -(g + kv)$$

or $$dx = -\frac{v \, dv}{kv + g}$$

or $$k \, dx = -\frac{kv \, dv}{kv + g}$$

or $$k \, dx = -\frac{(kv + g) - g}{kv + g} dv = -\left(1 - \frac{g}{kv + g}\right) dv.$$

Integrating, $kx = -v + \frac{g}{k} \log (kv + g) + A$, where A is a constant.

But initially when $x = 0, v = V$; $\quad \therefore A = V - \frac{g}{k} \log (kV + g).$

$$\therefore \quad kx = -v + \frac{g}{k} \log (kv + g) + V - \frac{g}{k} \log (kV + g)$$

or $$x = \frac{V - v}{k} + \frac{g}{k^2} \log \left(\frac{kv + g}{kV + g} \right).$$

Let h be the maximum height attained by the particle. Then at the heighest point O', $x = h$ and $v = 0$.

$$\therefore \quad h = \frac{V}{k} + \frac{g}{k^2} \log \left(\frac{g}{kV + g} \right). \quad \text{...(1)}$$

Now after coming to instantaneous rest at O', the particle begins to fall vertically downwards. If v be its velocity at the point distant y from O' after time t (measured from the instant it started from O'), then the resistance there is kv in the upward direction and the weight $1.g$ acts vertically downwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$1 . \frac{d^2y}{dt^2} = g - kv \quad \text{or} \quad v \frac{dv}{dy} = g - kv \quad \text{or} \quad dy = \frac{v\,dv}{g - kv}$$

or $$k\,dy = \frac{kv\,dv}{g - kv} = \frac{g - (g - kv)}{g - kv} dv = \left(\frac{g}{g - kv} - 1 \right) dv.$$

Integrating, $ky = -(g/k) \log (g - kv) - v + B$, where B is a constant.

But at O', $y = 0$ and $v = 0$; $\therefore$ $B = (g/k) \log g$.

$$\therefore \quad ky = \frac{g}{k} \log g - \frac{g}{k} \log (g - kv) - v$$

or $$y = -\frac{v}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kv} \right).$$

If the particle returns to the point O with velocity V_1, then at O, $v = V_1$ and $y = h$.

$$\therefore \quad h = -\frac{V_1}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kV_1} \right). \quad \text{...(2)}$$

From (1) and (2), we have

$$\frac{V}{k} + \frac{g}{k^2} \log \left(\frac{g}{g + kV} \right) = -\frac{V_1}{k} + \frac{g}{k^2} \log \left(\frac{g}{g - kV_1} \right)$$

or $$V + V_1 = \frac{g}{k} \left[\log \left(\frac{g}{g - kV_1} \right) - \log \left(\frac{g}{g + kV} \right) \right]$$

or $$V + V_1 = \frac{g}{k} \log \left(\frac{g + kV}{g - kV_1} \right).$$

Ex. 12. *A particle of unit mass is projected vertically upwards with velocity v_0 in a medium for which the resitance is kv when the speed of the particle is v, show that the distance covered when the velocity is v is given by*

$$x = \frac{v_0 - v}{k} + \frac{g}{k^2} \log \left(\frac{kv + g}{kv_0 + g} \right).$$

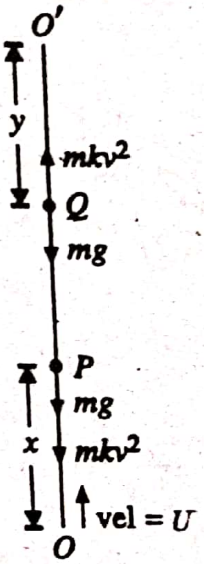
(Meerut 94)

Sol. For complete solution of this problem proceed as in the first part of Ex. 11. Simply replace V by v_0.

Ex. 13. *A particle projected upwards with a velocity U, in a medium whose resistance varies as the square of the velocity, will return to the point of projection with velocity* $v_1 = \dfrac{UV}{\sqrt{(U^2 + V^2)}}$ *after a time* $\dfrac{V}{g}\left(\tan^{-1}\dfrac{U}{V} + \tanh^{-1}\dfrac{v_1}{V}\right)$, *where V is the terminal velocity.*

[Meerut 86S, 96; Kanpur 88]

Solution. Upward motion. Let a particle of mass m be projected vertically upwards from the point O with velocity U. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is mkv^2 acting vertically downwards. Since the weight mg of the particle also acts vertically downwards, therefore the equation of motion of the particle is



$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or
$$\frac{d^2x}{dt^2} = -(g + kv^2) \quad \text{...(1)}$$

or
$$v\frac{dv}{dx} = -(g + kv^2)$$

or
$$\frac{2kv\,dv}{g + kv^2} = -2k\,dx.$$

Integrating, $\log(g + kv^2) = -2kx + A$, where A is a constant.

But initially at O, $x = 0$, $v = U$;

$$\therefore A = \log(g + kU^2).$$

$$\therefore \log(g + kv^2) = -2kx + \log(g + kU^2)$$

or
$$x = \frac{1}{2k}\log\left(\frac{g + kU^2}{g + kv^2}\right).$$

If $OO' = h$ is the maximum height attained by the particle, then at O', $x = h$ and $v = 0$.

$$\therefore h = \frac{1}{2k}\log\left(\frac{g + kU^2}{g}\right). \quad \text{...(2)}$$

Now to find the time from O to O', we write the equation (1) as

$$\frac{dv}{dt} = -(g + kv^2) = -k\left(\frac{g}{k} + v^2\right)$$

or $$dt = -\frac{1}{k}\frac{dv}{(g/k)+v^2}.$$

Integrating, the time t_1 from O to O' is given by

$$\int_{t=0}^{t_1} dt = -\frac{1}{k}\int_{v=U}^{0}\frac{dv}{(g/k)+v^2}$$

or $$t_1 = -\frac{1}{k}\cdot\frac{1}{\sqrt{(g/k)}}\left[\tan^{-1}\frac{v}{\sqrt{(g/k)}}\right]_U^0$$

or $$t_1 = \frac{1}{\sqrt{(kg)}}\tan^{-1}\{U\sqrt{(k/g)}\}. \quad ...(3)$$

Downward motion. Now the particle comes to instantaneous rest at the highest point O' and then falls downwards. If v is its velocity at the point Q at the time t (measured from the instant the particle starts falling downwards from O') such that $O'Q = y$, then resistance at Q is mkv^2 acting vertically upwards. Since the weight mg of the particle acts vertically downwards, therefore the equation of motion during this downward motion is

$$m\frac{d^2y}{dt^2} = mg - mkv^2 \quad \text{or} \quad \frac{d^2y}{dt^2} = g - kv^2.$$

If V is the terminal velocity of the particle during its downward motion, then $d^2y/dt^2 = 0$, when $v = V$.

$$\therefore\ 0 = g - kV^2 \quad \text{or} \quad k/g = 1/V^2 \quad ...(4)$$

$$\therefore\ \frac{d^2y}{dt^2} = g\left(1 - \frac{k}{g}v^2\right) = g\left(1 - \frac{v^2}{V^2}\right)$$

or $$\frac{d^2y}{dt^2} = \frac{g}{V^2}(V^2 - v^2) \quad ...(5)$$

or $$v\frac{dv}{dy} = \frac{g}{V^2}(V^2 - v^2) \quad \text{or} \quad \frac{-2v\,dv}{V^2 - v^2} = \frac{-2g}{V^2}dy.$$

Integrating, $\log(V^2 - v^2) = \dfrac{-2g}{V^2}y + B$, where B is a constant.

But at O', $y = 0$ and $v = 0$; $\therefore\ B = \log V^2$.

$$\therefore\ \log(V^2 - v^2) = \frac{-2g}{V^2}y + \log V^2$$

or $$y = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v^2}\right).$$

If the particle returns to the point O with velocity v_1, then at O, $v = v_1$ and $y = h$.

$$\therefore\ h = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v_1^2}\right). \quad ...(6)$$

Substituting $\frac{k}{g} = \frac{1}{V^2}$ in (2), we have

$$h = \frac{V^2}{2g} \log \left(1 + \frac{U^2}{V^2}\right). \quad ...(7)$$

From (6) and (7), we have

$$\frac{V^2}{2g} \log \left(\frac{V^2 + U^2}{V^2}\right) = \frac{V^2}{2g} \log \left(\frac{V^2}{V^2 - v_1^2}\right)$$

or
$$\frac{V^2 + U^2}{V^2} = \frac{V^2}{V^2 - v_1^2}.$$

or
$$V^2 (V^2 + U^2) - (V^2 + U^2) v_1^2 = V^4$$

or
$$(V^2 + U^2) v_1^2 = U^2 V^2.$$

$\therefore$
$$v_1 = UV/\sqrt{(U^2 + V^2)},$$

which proves the first part of the question. To find the time from O' to O, we write the equation (5) as

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2) \quad \text{or} \quad dt = \frac{V^2}{g} \frac{dv}{(V^2 - v^2)}.$$

Integrating, the time t_2 from O' to O is given by

$$\int_{t=0}^{t_2} dt = \frac{V^2}{g} \int_{v=0}^{v_1} \frac{dv}{V^2 - v^2}$$

or
$$t_2 = \frac{V^2}{g} \cdot \frac{1}{V} \left[\tanh^{-1} \frac{v}{V}\right]_0^{v_1} = \frac{V}{g} \tanh^{-1} \frac{v_1}{V}.$$

$\therefore$ the particle returns to the point of projection O in time

$$= t_1 + t_2 = \frac{1}{\sqrt{(kg)}} \tan^{-1} \{U \sqrt{(k/g)}\} + \frac{V}{g} \tanh^{-1} \frac{v_1}{V}.$$

Substituting $k/g = 1/V^2$ from (4), we get

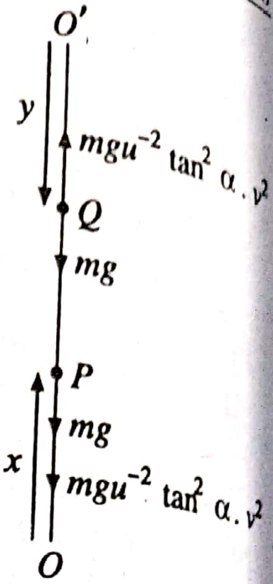
$$t_1 + t_2 = \frac{V}{g} \tan^{-1} \frac{U}{V} + \frac{V}{g} \tanh^{-1} \frac{v_1}{V}$$

$$= \frac{V}{g} \left(\tan^{-1} \frac{U}{V} + \tanh^{-1} \frac{v_1}{V}\right).$$

Ex. 14. *A particle is projected upwards with velocity u in a medium, the resistance of which is $gu^{-2} \tan^2 \alpha$ times the square of the velocity, α being a constant. Show that the particle will return to the point of projection with velocity $u \cos \alpha$ after a time*

$$ug^{-1} \cot \alpha \left[\alpha + \log \frac{\cos \alpha}{1 - \sin \alpha}\right].$$

(Meerut 82P, 90P)



Sol. **Upward Motion.** Let a particle of mass m be projected vertically upwards from the point O with velocity u. If v is the velocity of the particle at time t at the point P such that $OP = x$, then resistance at P is $mgu^{-2} \tan^2 \alpha \, . \, v^2$ acting in vertically downward direction. Since the weight mg of the particle also acts vertically downwards, therefore the equation of motion of the particle during its upward motion is

$$m \frac{d^2x}{dt^2} = - mg - mgu^{-2} \tan^2 \alpha \, . \, v^2$$

or $$\frac{d^2x}{dt^2} = - gu^{-2} \tan^2 \alpha \, (u^2 \cot^2 \alpha + v^2). \qquad ...(1)$$

From (1), we have

$$v \frac{dv}{dx} = - gu^{-2} \tan^2 \alpha \, (u^2 \cot^2 \alpha + v^2)$$

or $$dx = - \frac{u^2 \cot^2 \alpha}{2g} \cdot \frac{2v \, dv}{v^2 + u^2 \cot^2 \alpha}.$$

Integrating, we have

$$x = - \frac{u^2 \cot^2 \alpha}{2g} \log (v^2 + u^2 \cot^2 \alpha) + A,$$ where A is a constant.

But at $O, x = 0$ and $v = u$.

$$\therefore \quad 0 = - \frac{u^2 \cot^2 \alpha}{2g} \log (u^2 \operatorname{cosec}^2 \alpha) + A$$

or $$A = \frac{u^2 \cot^2 \alpha}{2g} \log (u^2 \operatorname{cosec}^2 \alpha).$$

$$\therefore \quad x = - \frac{u^2 \cot^2 \alpha}{2g} \log (v^2 + u^2 \cot^2 \alpha) + \frac{u^2 \cot^2 \alpha}{2g} \log (u^2 \operatorname{cosec}^2 \alpha)$$

or $$x = \frac{u^2 \cot^2 \alpha}{2g} \log \frac{u^2 \operatorname{cosec}^2 \alpha}{v^2 + u^2 \cot^2 \alpha}.$$

If the particle rises to the point O' such that $OO' = h$, then at $O', x = h, v = 0.$

$$\therefore \quad h = \frac{u^2 \cot^2 \alpha}{2g} \log \frac{u^2 \operatorname{cosec}^2 \alpha}{u^2 \cot^2 \alpha} = \frac{u^2 \cot^2 \alpha}{2g} \log \sec^2 \alpha. \qquad ...(2)$$

To find the time from O to O', the equation (1) can be written as

$$\frac{dv}{dt} = - gu^{-2} \tan^2 \alpha \, (u^2 \cot^2 \alpha + v^2)$$

or $$dt = - \frac{u^2 \cot^2 \alpha}{g} \frac{dv}{u^2 \cot^2 \alpha + v^2}.$$

Let t_1 be the time from O to O'. Then from O to O', t varies from 0 to t_1 and v varies from u to 0. So integrating from O to O' we get

$$\int_0^{t_1} dt = -\frac{u^2 \cot^2 \alpha}{g \,.\, u \cot \alpha}\left[\tan^{-1}\frac{v}{u \cot \alpha}\right]_u^0$$

or
$$t_1 = \frac{u}{g} \cot \alpha \,.\, \tan^{-1} \tan \alpha = \frac{u\alpha}{g} \cot \alpha. \qquad ...(3)$$

Downward Motion. Now from the highest point O', the particle falls downwards. If y is its distance after time t from O' and if v is the velocity there, then

the total resistance at this point is $mgu^{-2} \tan^2 \alpha \,.\, v^2$, acting vertically upwards.

The weight mg of the particle acts vertically downwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2y}{dt^2} = mg - mgu^{-2} \tan^2 \alpha \,.\, v^2$$

or
$$\frac{d^2y}{dt^2} = gu^{-2} \tan^2 \alpha \,(u^2 \cot^2 \alpha - v^2), \qquad ...(4)$$

or
$$v\frac{dv}{dy} = gu^{-2} \tan^2 \alpha \,(u^2 \cot^2 \alpha - v^2)$$

or
$$dy = -\frac{u^2 \cot^2 \alpha}{2g} \cdot \frac{-2v\, dv}{u^2 \cot^2 \alpha - v^2}.$$

Integrating, we have

$$y = -\frac{u^2 \cot^2 \alpha}{2g} \log (u^2 \cot^2 \alpha - v^2) + B.$$

But at O', $y = 0$ and $v = 0$.

$$\therefore \quad 0 = -\frac{u^2 \cot^2 \alpha}{2g} \log (u^2 \cot^2 \alpha) + B.$$

Subtracting, we get $y = \dfrac{u^2 \cot^2 \alpha}{2g} \cdot \log \dfrac{u^2 \cot^2 \alpha}{u^2 \cot^2 \alpha - v^2}$.

If v_1 is the velocity at the lowest point O, then at O, $y = h$, $v = v_1$.

$$\therefore \quad h = \frac{u^2 \cot^2 \alpha}{2g} \cdot \log \frac{u^2 \cot^2 \alpha}{u^2 \cot^2 \alpha - v_1^2}. \qquad ...(5)$$

From (2) and (5), we have

$$\frac{u^2 \cot^2 \alpha}{2g} \cdot \log \sec^2 \alpha = \frac{u^2 \cot^2 \alpha}{2g} \cdot \log \frac{u^2 \cot^2 \alpha}{u^2 \cot^2 \alpha - v_1^2}$$

or
$$\sec^2 \alpha = \frac{u^2 \cot^2 \alpha}{u^2 \cot^2 \alpha - v_1^2}$$

or $\quad u^2 \cot^2 \alpha - v_1^2 = u^2 \cot^2 \alpha \,.\, \cos^2 \alpha$

or $\quad v_1^2 = u^2 \cot^2 \alpha \,.\, (1 - \cos^2 \alpha) = u^2 \cot^2 \alpha \sin^2 \alpha = u^2 \cos^2 \alpha$

or $\quad v_1 = u \cos \alpha,$

i.e., the particle returns to the point of projection with velocity $v_1 = u \cos \alpha$. This proves the first part of the question. Again to find the time from O' to O, the equation (4) can be written as

$$\frac{dv}{dt} = gu^{-2} \tan^2 \alpha \,(u^2 \cot^2 \alpha - v^2)$$

or
$$dt = \frac{u^2}{g} \cot^2 \alpha \,.\, \frac{dv}{u^2 \cot^2 \alpha - v^2}.$$

Let t_2 be time from O' to O. Then from O' to O, t varies from 0 to t_2 and v varies from 0 to $u \cos \alpha$. Therefore integrating from O' to O, we have

$$\int_0^{t_2} dt = \frac{u^2}{g} \cot^2 \alpha \,.\, \int_{v=0}^{v = u\cos\alpha} \frac{dv}{u^2 \cot^2 \alpha - v^2}.$$

$$\therefore \quad t_2 = \frac{u^2 \cot^2 \alpha}{2gu \cot \alpha} \cdot \left[\log \frac{u \cot \alpha + v}{u \cot \alpha - v}\right]_0^{u \cos \alpha}$$

$$= \frac{u}{2g} \cot \alpha \cdot \left[\log \frac{u \cot \alpha + u \cos \alpha}{u \cot \alpha - u \cos \alpha} - \log 1\right]$$

$$= \frac{u}{2g} \cot \alpha \,.\, \log \frac{1 + \sin \alpha}{1 - \sin \alpha} = \frac{u}{2g} \cot \alpha \,.\, \log \frac{(1 + \sin \alpha)\,.\,(1 - \sin \alpha)}{(1 - \sin \alpha)\,.\,(1 - \sin \alpha)}$$

$$= \frac{u}{2g} \cot \alpha \,.\, \log \frac{(1 - \sin^2 \alpha)}{(1 - \sin \alpha)^2} = \frac{u}{2g} \cot \alpha \,.\, \log \left(\frac{\cos \alpha}{1 - \sin \alpha}\right)^2$$

$$= \frac{u}{g} \cot \alpha \,.\, \log \frac{\cos \alpha}{1 - \sin \alpha}.$$

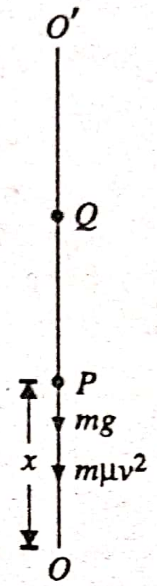
$$\therefore \quad \text{the required time} = t_1 + t_2 = \frac{u}{g} \cot \alpha \left[\alpha + \log \frac{\cos \alpha}{1 - \sin \alpha}\right].$$

Ex. 15. *A heavy particle is projected vertically upwards in a medium the resistance of which varies as the square of velocity. If it has a kinetic energy K in its upwards path at a given point, when it passes the same point on the way down, show that its loss of energy is* $\dfrac{K^2}{K + K'}$, *where K' is the limit to which the energy approaches in its downwards course.*

[Meerut 85S; Lucknow 79]

Sol. Let a particle of mass m be projected vertically upwards with a velocity u from the point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is $m\mu v^2$

acting vertically downwards. The weight mg of the particle also acts vertically downwards.



$\therefore$ The equation of motion of the particle during its upwards motion is

$$m\frac{d^2x}{dt^2} = -mg - m\mu v^2$$

or
$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{\mu}{g}v^2\right) \qquad ...(1)$$

If H is the maximum height attained by the particle, then at the highest point O' the particle comes to rest and starts falling vertically downwards. If y is the distance fallen in time t from O' and v is the velocity of the particle at this point, then the resistance is $m\mu v^2$ acting vertically upwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2y}{dt^2} = mg - m\mu v \quad \text{or} \quad \frac{d^2y}{dt^2} = g - \mu v^2. \qquad ...(2)$$

If V is the terminal velocity of the particle during its downward motion, then $d^2y/dt^2 = 0$ when $v = V$. Therefore $0 = g - \mu V^2$

or
$$\frac{\mu}{g} = \frac{1}{V^2}. \qquad ...(3)$$

$\therefore$ From (2), the equation of motion of the particle in downward motion is

$$\frac{d^2y}{dt^2} = g\left(1 - \frac{1}{V^2}v^2\right) \quad \text{or} \quad v\frac{dv}{dy} = \frac{g}{V^2}(V^2 - v^2)$$

or
$$\frac{-2v\,dv}{V^2 - v^2} = -\frac{2g}{V^2}dy.$$

Integrating, $\log(V^2 - v^2) = -\dfrac{2g}{V^2}y + A$, where A is a constant.

But at O', $y = 0$ and $v = 0$; $\therefore$ $A = \log V^2$.

$$\therefore \quad \log(V^2 - v^2) = -\frac{2g}{V^2}y + \log V^2$$

or
$$\frac{2gy}{V^2} = \log V^2 - \log(V^2 - v^2)$$

or
$$y = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v^2}\right). \qquad ...(4)$$

If v_1 is the velocity of the particle at the point Q at distance h from O', when falling downwards, then from (4),

$$h = \frac{V^2}{2g} \log \left(\frac{V^2}{V^2 - v_1^2} \right). \quad ...(5)$$

Upward Motion. When the particle is moving upwards from O, then from (1) with the help of (3), the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right) \quad \text{or} \quad v\frac{dv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$

or

$$\frac{2v\,dv}{V^2 + v^2} = -\frac{2g}{V^2}\,dx.$$

Integrating, $\log(V^2 + v^2) = -\frac{2g}{V^2}x + B$, where B is a constant.

But at $O, x = 0, v = u$; $\therefore$ $B = \log(V^2 + u^2)$.

$$\therefore \quad \log(V^2 + v^2) = -\frac{2g}{V^2}x + \log(V^2 + u^2)$$

or

$$x = \frac{V^2}{2g} \log \left(\frac{V^2 + u^2}{V^2 + v^2} \right). \quad ...(6)$$

If v_2 is the velocity of the particle at the point Q in its upward motion then at $Q, x = OQ = H - h, v = v_2$.

$$\therefore \; H - h = \left(\frac{V^2}{2g}\right) \log \left(\frac{V^2 + u^2}{V^2 + v_2^2} \right). \quad ...(7)$$

Since H is the maximum height attained by the particle therefore putting $x = H$ and $v = 0$ in (6), we get

$$H = \frac{V^2}{2g} \log \left(\frac{V^2 + u^2}{V^2} \right). \quad ...(8)$$

Substituting the values of h and H from (5) and (8) in (7), we get

$$\frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2} - \frac{V^2}{2g} \log \frac{V^2}{V^2 - v_1^2} = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2 + v_2^2}$$

or

$$\log \frac{(V^2 + u^2)}{V^2} - \log \left(\frac{V^2 + u^2}{V^2 + v_2^2} \right) = \log \left(\frac{V^2}{V^2 - v_1^2} \right)$$

or

$$\log \left\{ \left(\frac{V^2 + u^2}{V^2} \right) \cdot \left(\frac{V^2 + v_2^2}{V^2 + u^2} \right) \right\} = \log \frac{V^2}{V^2 - v_1^2}$$

$$\frac{V^2 + v_2^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$$

or

$$(V^2 + v_2^2)(V^2 - v_1^2) = V^4$$

or

$$(V^2 + v_2^2)\, V^2 - (V^2 + v_2^2)\, v_1^2 = V^4$$

or

$$v_1^2 = \frac{v_2^2 V^2}{V^2 + v_2^2}. \qquad ...(9)$$

or

Now the kinetic energy K of the particle at the point Q at depth h below O' during its upward motion $= \frac{1}{2} m v_2^2$ and the K.E. at Q during downward motion $= \frac{1}{2} m v_1^2$.

Also the terminal K.E. $= \frac{1}{2} m V^2$.

The requried loss of K.E. $= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$

$$= \frac{1}{2} m \left[v_2^2 - \frac{v_2^2 V^2}{V^2 + v_2^2} \right], \text{ substituting for } v_1^2 \text{ from (9)}$$

$$= \frac{m}{2} \cdot \frac{v_2^4}{V^2 + v_2^2} = \frac{(\frac{1}{2} m v_2^2)^2}{\frac{1}{2} m V^2 + \frac{1}{2} m v_2^2} = \frac{K^2}{K' + K},$$

where $K' = \frac{1}{2} m V^2$ = limiting K.E. in the medium.

Ex. 16. *If the resistance vary as the 4th power of the velocity, the energy of m lbs. at a depth x below the highest point when moving in a vertical line under gravity will be E tan (mgx/E) when rising, and E tanh (mgx/E) when falling, where E is the terminal energy in the medium.* **(Allahabad 1977; Meerut 77, 94)**

Sol. Let a particle of mass m be projected vertically upwards with a velocity u in the given resisting medium. If v is the velocity of the particle at time t at the point whose distance is y from the starting point O, then the resistance on the particle is $m\mu v^4$ acting vertically downwards.

$\therefore$ the equation of motion of the particle during its upward motion is

$$m \frac{d^2y}{dt^2} = -mg - m\mu v^4$$

or

$$\frac{d^2y}{dt^2} = -(g + \mu v^4). \qquad ...(1)$$

If h is the maximum height attained by the particle, then at the highest point, say O', the particle will come to rest and will start falling downwards. If x is the distance fallen in time t from O' and v is the velocity of the particle at this point, then resistance is $m\mu v^4$ acting vertically upwards.

$\therefore$ the equation of motion of the particle during its downward motion is

$$m\frac{d^2x}{dt^2} = mg - m\mu v^4$$

or $$\frac{d^2x}{dt^2} = g - \mu v^4. \quad \text{...(2)}$$

If V is the terminal velocity, then $0 = g - \mu V^4$

or $$\frac{\mu}{g} = \frac{1}{V^4}. \quad \text{...(3)}$$

$\therefore$ from (2), the equation of motion of the particle when moving vertically downwards is

$$\frac{d^2x}{dt^2} = g\left(1 - \frac{1}{V^4}v^4\right) = \frac{g}{V^4}(V^4 - v^4) \quad \text{...(4)}$$

or $$v\frac{dv}{dx} = \frac{g}{V^4}(V^4 - v^4)$$

or $$\frac{2v\,dv}{V^4 - v^4} = \frac{2g}{V^4}dx$$

or $$\frac{dz}{V^4 - z^2} = \frac{2g}{V^4}dx, \text{ putting } v^2 = z \text{ so that } 2v\,dv = dz.$$

Integrating, $\frac{1}{V^2} \cdot \tanh^{-1}\frac{z}{V^2} = \frac{2g}{V^4}x + A$, where A is a constant

or $$\frac{1}{V^2}\tanh^{-1}\frac{v^2}{V^2} = \frac{2gx}{V^4} + A.$$

But at O', $x = 0$ and $v = 0$; $\therefore$ $A = 0$.

$$\therefore \quad \frac{1}{V^2}\tanh^{-1}\frac{v^2}{V^2} = \frac{2gx}{V^4} \quad \text{or} \quad \tanh^{-1}\frac{v^2}{V^2} = \frac{2gx}{V^2}$$

or $$v^2 = V^2 \tanh(2gx/V^2).$$

$\therefore$ the K.E. at a depth x below the highest point when moving downwards is $= \frac{1}{2}mv^2 = \frac{1}{2}mV^2 \tanh(2gx/V^2)$

$$= \tfrac{1}{2}mV^2 \tanh(mgx/\tfrac{1}{2}mV^2) = E \tanh(mgx/E),$$

where $E = \frac{1}{2}mV^2 =$ the terminal energy in the medium.

Upward motion. When the particle is moving upwards from O then from (1) with the help of (3), the equation of motion of the particle is

$$\frac{d^2y}{dt^2} = -g\left(1 + \frac{v^4}{V^4}\right)$$

$$v\frac{dv}{dy} = -\frac{g}{V^4}(V^4 + v^4)$$

or

$$\frac{2v\,dv}{V^4 + v^4} = -\frac{2g}{V^4}dy$$

or

$$\frac{dz}{V^4 + z^2} = -\frac{2g}{V^4}dy, \text{ putting } v^2 = z \text{ so that } 2v\,dv = dz.$$

or

Integrating, $\dfrac{1}{V^2}\tan^{-1}\dfrac{z}{V^2} = -\dfrac{2gy}{V^4} + B$, where B is a constant

or

$$\frac{1}{V^2}\tan^{-1}\frac{v^2}{V^2} = -\frac{2gy}{V^4} + B.$$

But at $O, y = 0, v = u$; $\therefore$ $B = \dfrac{1}{V^2}\tan^{-1}\dfrac{u^2}{V^2}$.

$$\therefore \quad \frac{1}{V^2}\tan^{-1}\frac{v^2}{V^2} = -\frac{2gy}{V^4} + \frac{1}{V^2}\tan^{-1}\frac{u^2}{V^2}$$

or

$$\frac{2gy}{V^2} = \tan^{-1}\frac{u^2}{V^2} - \tan^{-1}\frac{v^2}{V^2}. \qquad \text{...(5)}$$

At the highest point $O', y = h$ and $v = 0$.

$$\therefore \quad \frac{2gh}{V^2} = \tan^{-1}\frac{u^2}{V^2}. \qquad \text{...(6)}$$

If v_1 is the velocity during the upward motion at a depth x below the highest point O', *i.e.*, at a height $y = h - x$ from the starting point O, then from (5), we have

$$\frac{2g}{V^2}(h - x) = \tan^{-1}\frac{u^2}{V^2} - \tan^{-1}\frac{v_1^2}{V^2}. \qquad \text{...(7)}$$

Subtracting (7) from (6), we have

$$\frac{2gx}{V^2} = \tan^{-1}\left(\frac{v_1^2}{V^2}\right) \quad \text{or} \quad v_1^2 = V^2\tan\left(\frac{2gx}{V^2}\right).$$

$\therefore$ the K.E. at a depth x below the highest point when rising is

$$= \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}mV^2\tan\left(\frac{2gx}{V^2}\right)$$

$$= \tfrac{1}{2}mV^2\tan\{mgx/(\tfrac{1}{2}mV^2)\} = E\tan(mgx/E).$$

Ex. 17. *A particle moving in a straight line is subjected to a resistance kv^3, where v is the velocity. Show that if v is the velocity at time t when the distance is s, $v = u/(1 + kus)$ and $t = (s/u) + \frac{1}{2}ks^2$, where u is the initial velocity.* **(Meerut 1983, 93S, 95)**

Sol. Suppose a particle of mass m starts with velocity u from a point O. Let v be the velocity of the particle at a distance s from the point O at time t. Then the resitance on the particle is mkv^3 acting against the direction of motion of the particle.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2s}{dt^2} = -mkv^3$$

or $$v\frac{dv}{ds} = -kv^3, \quad \left[\because \frac{d^2s}{dt^2} = v\frac{dv}{ds}\right]$$

or $$\frac{dv}{v^2} = -k\,ds.$$

Integrating, $-\dfrac{1}{v} = -ks + A$, where A is a constant.

But initially when $s = 0, v = u$; $\quad\therefore\quad A = -1/u$.

$$\therefore \quad -\frac{1}{v} = -ks - \frac{1}{u} = -\frac{kus+1}{u}$$

or $\quad v = u/(1 + kus)$,

which proves the first part of the question.

Now $$v = \frac{ds}{dt} = \frac{u}{1+kus}.$$

$$\therefore \quad dt = \left(\frac{1+kus}{u}\right)ds = \left(\frac{1}{u} + ks\right)ds.$$

Integrating, $t = \left(\dfrac{s}{u} + \dfrac{1}{2}ks^2\right) + B$, where B is a constant.

But $t = 0, s = 0$; $\quad\therefore\quad B = 0$.

$$\therefore \quad t = \frac{s}{u} + \frac{1}{2}ks^2.$$

Ex. 18. *A heavy particle is projected vertically upwards with velocity U in a medium, the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.*

(Meerut 1980)

Sol. Let a particle of mass m be projected vertically upwards with velocity U. If v be the velocity of the particle at time t at the point whose distance is x from the starting point O, then the resistance on the particle is $m\mu v^3$ acting vertically downwards. Also the weight mg of the particle acts vertically downwards.

$\therefore$ The equation of motion of the particle during its upwards motion is

$$m\frac{d^2x}{dt^2} = -mg - m\mu v^3$$

or $$v\frac{dv}{dx} = -g - \mu v^3. \qquad ...(1)$$

If the particle is moving downwards in the given resisting medium then the resistance will act vertically upwards and the equation of motion will be

$$m\,(d^2x/dt^2) = mg - m\mu v^3.$$

If V is the terminal velocity, then $d^2x/dt^2 = 0$, when $v = V$.

$\therefore$ $0 = g - \mu V^3$ or $\mu/g = 1/V^3$.

$\therefore$ from (1), we have

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^3}{V^3}\right) = -\frac{g}{V^3}(V^3 + v^3)$$

or
$$dx = -\left(\frac{V^3}{g}\right)\frac{v\,dv}{v^3 + V^3}$$

or
$$dx = -\frac{V^3}{g}\frac{v\,dv}{(v+V)(v^2 - vV + V^2)}. \qquad \text{...(2)}$$

Now let
$$\frac{v}{(v+V)(v^2 - vV + V^2)} = \frac{A}{v+V} + \frac{Bv + C}{v^2 - Vv + V^2} \qquad \text{...(3)}$$

or
$$v = A(v^2 - vV + V^2) + (Bv + C)(v + V).$$

Equating the coefficients of like powers of v from the two sides, we get

$$0 = A + B,\ 1 = -AV + BV + C \text{ and } 0 = AV^2 + CV.$$

Solving, we have, $A = -1/3V$, $B = 1/3V$ and $C = 1/3$.

Substituting in (3), we have

$$\frac{v}{(v+V)(v^2 - vV + V^2)} = -\frac{1}{3V(v+V)} + \frac{v+V}{3V(v^2 - Vv + V^2)}.$$

$\therefore$ from (2), we have

$$dx = -\frac{V^3}{g}\cdot\left[-\frac{1}{3V(v+V)} + \frac{v+V}{3V(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{\frac{1}{2}(2v - V) + \frac{3}{2}V}{(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{(2v - V)}{2(v^2 - Vv + V^2)} + \frac{3V}{2(v^2 - Vv + V^2)}\right]dv$$

$$= -\frac{V^2}{3g}\left[-\frac{1}{v+V} + \frac{(2v - V)}{2(v^2 - Vv + V^2)} + \frac{3V}{2}\frac{1}{(v - \frac{1}{2}V)^2 + (\sqrt{3}V/2)^2}\right]dv.$$

Integrating, we have

$$x = -\frac{V^2}{3g}\left[-\log(v+V) + \tfrac{1}{2}\log(v^2 - Vv + V^2) + \frac{3V}{2}\cdot\frac{2}{\sqrt{3}V}\tan^{-1}\frac{v - \frac{1}{2}V}{(\sqrt{3}\,V/2)}\right] + D,$$

where D is a constant

or
$$x = -\frac{V^2}{3g}\left[-\log(v+V) + \tfrac{1}{2}\log(v^2 - Vv + V^2) + \sqrt{3}\tan^{-1}\frac{2v - V}{\sqrt{3}V}\right] + D. \qquad \text{...(4)}$$

But initially when $x = 0, v = U$.

$$\therefore \quad 0 = -\frac{V^2}{3g}\left[-\log(U+V) + \tfrac{1}{2}\log(U^2 - UV + V^2) + \sqrt{3}\tan^{-1}\frac{2U-V}{\sqrt{3}V}\right] + D. \quad ...(5)$$

Subtracting (5) from (4), we have

$$x = \frac{V^2}{3g}\left[-\log(U+V) + \log(v+V) + \tfrac{1}{2}\log(U^2 - UV + V^2) - \tfrac{1}{2}\log(v^2 - Vv + V^2) + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\frac{2v-V}{\sqrt{3}V}\right)\right]$$

or $$x = \frac{V^2}{3g}\left[\log\frac{v+V}{U+V} + \tfrac{1}{2}\log\frac{U^2 - UV + V^2}{v^2 - Vv + V^2} + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\frac{2v-V}{\sqrt{3}V}\right)\right].$$

If h is the height to which the particle will ascend, then $v = 0$, when $x = h$.

$$\therefore \quad h = \frac{V^2}{3g}\left[\log\left(\frac{V}{U+V}\right) + \tfrac{1}{2}\log\frac{U^2 - UV + V^2}{V^2} + \sqrt{3}\left\{\tan^{-1}\frac{2U-V}{\sqrt{3}V} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right\}\right]$$

or $$h = \frac{V^2}{3g}\left[\log\frac{(U^2 - UV + V^2)^{1/2}}{(U+V)} + \sqrt{3}\left(\tan^{-1}\frac{2U-V}{\sqrt{3}V} + \frac{\pi}{6}\right)\right].$$

Ex. 19. *A particle of mass m falls from rest at a distance a from the centre of the earth, the motion meeting with a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity; show that the kinetic energy at a distance x from the centre is*

$$mgr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu\log\left(\frac{a}{x}\right)\right],$$

the square of μ being neglected and r is the radius of the earth.

[Meerut 90, 95BP]

Sol. Let a particle of mass m fall from rest at a distance a from the centre O of the earth. If v is the velocity of the particle at time t at the point P whose distance from the centre of the earth is x *i.e.*, $OP = x$, then the two accelerations *i.e.*, the forces acting on the unit mass of the particle at P are:

(i) The attraction of the earth towards its centre $= \lambda/x^2$. But on the surface of the earth, the attraction (acceleration) is g and $x = r =$ the radius of the earth.

$\therefore\ \lambda/r^2 = g$ or $\lambda = r^2 g$.

$\therefore$ the attraction of the earth towards the centre (*i.e.*, in the direction of x decreasing) is $r^2 g/x^2$.

(ii) The resistance of the medium on the particle $= kv^2$, acting against the direction of motion. But for $v = 1$, the retardation due to the resistance is μ.

$\therefore\ \mu = k \cdot 1^2$ or $k = \mu$.

$\therefore$ the retardation on the particle due to the resistance of the medium is μv^2 acting in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\frac{r^2 g}{x^2} + \mu v^2$$

or $$v\frac{dv}{dx} = -\frac{r^2 g}{x^2} + \mu v^2 \quad \text{or} \quad \frac{1}{2}\frac{d(v^2)}{dx} = -\frac{r^2 g}{x^2} + \mu v^2$$

or $$\frac{d(v^2)}{dx} - (2\mu)v^2 = -\frac{2r^2 g}{x^2}, \qquad \text{...(1)}$$

which is a linear differential equation in v^2.

$$\text{I.F.} = e^{\int -2\mu\, dx} = e^{-2\mu x}.$$

$\therefore$ the solution of (1) is

$$v^2 e^{-2\mu x} = C - \int \frac{2r^2 g}{x^2} e^{-2\mu x}\, dx, \text{ where } C \text{ is a constant}$$

or $$v^2(1 - 2\mu x) = C - 2r^2 g \cdot \int \frac{1}{x^2}(1 - 2\mu x)\, dx,$$

[expanding $e^{-2\mu x}$ and neglecting the squares and higher powers of μ]

or $$v^2(1 - 2\mu x) = C - 2r^2 g \cdot \int \left(\frac{1}{x^2} - \frac{2\mu}{x}\right) dx$$

or $$v^2(1 - 2\mu x) = C + 2r^2 g\left(\frac{1}{x} + 2\mu \log x\right). \qquad \text{...(2)}$$

But initially at $x = a$, $v = 0$.

$$\therefore\ 0 = C + 2r^2 g\left(\frac{1}{a} + 2\mu \log a\right). \qquad \text{...(3)}$$

Subtracting (3) from (2), we have

$$v^2(1 - 2\mu x) = 2r^2 g\left(\frac{1}{x} - \frac{1}{a} + 2\mu \log x - 2\mu \log a\right)$$

or $$v^2 = 2r^2 g\left[\frac{1}{x} - \frac{1}{a} - 2\mu \log\left(\frac{a}{x}\right)\right] \cdot (1 - 2\mu x)^{-1}$$

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} - 2\mu \log\left(\frac{a}{x}\right)\right] \cdot (1 + 2\mu x)$$

[Expanding by binomial theorem and neglecting the squares and higher powers of μ]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu x\left(\frac{1}{x} - \frac{1}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right]$$

[Neglecting μ^2]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

$\therefore$ the kinetic energy of the particle at a distance x from the centre

$$= \frac{1}{2} mv^2$$

$$= mgr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

Ex. 20. *A particle moves from rest at a distance a from a fixed point O under the action of a force to O equal to μ times the distance per unit of mass. If the resistance of the medium in which it moves be k times the square of the velocity per unit mass, then show that the square of its velocity when it is at a distance x from O, is $\frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2}[1 - e^{2k(x-a)}]$. Show also that when it first comes to rest it will be at a distance b given by*

$$(1 - 2bk)\, e^{2bk} = (1 + 2ak)\, e^{-2ak}.$$

Sol. Let a particle of mass m start from rest at a distance a from the fixed point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the two forces acting on the particle are :

(i) the force $m\mu x$ towards O (*i.e.*, in the direction of x decreasing) :

and (ii) the resistance of the medium $= mkv^2$ acting opposite to the direction of motion *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -m\mu x + mkv^2$$

or $$v\frac{dv}{dx} = -\mu x + kv^2 \quad \text{or} \quad \frac{1}{2}\frac{d}{dx}(v^2) = -\mu x + kv^2$$ ...(1)

or $$\frac{d}{dx}(v^2) - 2kv^2 = -2\mu x,$$

which is a linear differential equation in v^2.

$\therefore$ I.F. $= e^{\int -2k\,dx} = e^{-2kx}$.

$\therefore$ the solution of the equation (1) is

$$v^2 . e^{-2kx} = C - \int 2\mu x\, e^{-2kx}\, dx, \text{ where } C \text{ is a constant}$$

$$= C - 2\mu \left[x \frac{e^{-2kx}}{-2k} - \int 1 \cdot \frac{e^{-2kx}}{-2k}\, dx \right], \quad \text{[integrating by parts]}$$

$$= C - 2\mu \left[-\frac{x}{2k} e^{-2kx} - \frac{e^{-2kx}}{(-2k)^2} \right]$$

or $$v^2 e^{-2kx} = C + \frac{\mu}{k} \left[x e^{-2kx} + \frac{e^{-2kx}}{2k} \right]. \quad ...(2)$$

But initially when $x = a, v = 0$.

$$\therefore \quad 0 = C + \frac{\mu}{k} \left[a e^{-2ka} + \frac{e^{-2ka}}{2k} \right]. \quad ...(3)$$

Subtracting (3) from (2), we have

$$v^2 e^{-2kx} = \frac{\mu}{k} \left[x\, e^{-2kx} + \frac{e^{-2kx}}{2k} - a e^{-2ka} - \frac{e^{-2ka}}{2k} \right].$$

or $$v^2 = \frac{\mu x}{k} + \frac{\mu}{2k^2} - \frac{\mu a}{k} e^{2k(x-a)} - \frac{\mu}{2k^2} e^{2k(x-a)}$$

or $$v^2 = \frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2} \left[1 - e^{2k(x-a)} \right], \quad ...(4)$$

which proves the first part of the question.

Let V be the velocity of the particle at the centre of force O so that $v = V$ when $x = 0$. Then from (4), we have

$$V^2 = -\frac{\mu a}{k} e^{-2ka} + \frac{\mu}{2k^2} [1 - e^{-2ka}] \neq 0$$

i.e., the particle does not come to rest at the centre of force O. Therefore the particle moves to the left of O with velocity V. As the particle moves to the left of O, the force of attraction and the resistance of the medium will act towards O, and therefore the velocity of the particle will go on decreasing. If the particle comes to instantaneous rest at a distance b from O on its left, then $v = 0$, when $x = -b$.

$\therefore$ from (4), we have

$$0 = -(\mu b/k) - (\mu a/k)\, e^{2k(-b-a)} + (\mu/2k^2)\, [1 - e^{2k(-b-a)}]$$

or $$\frac{\mu}{2k^2} - \frac{\mu b}{k} = \left(\frac{\mu a}{k} + \frac{\mu}{2k^2} \right) e^{-2kb - 2ka}$$

or $$(1 - 2bk) = (2ak + 1)\, e^{-2ak} . e^{-2bk}$$

or $$(1 - 2bk)\, e^{2bk} = (1 + 2ak)\, e^{-2ak}.$$

Ex. 21. *What do you understand by 'terminal velocity' ? Give reasons that the terminal velocity obtained from vertically downward motion is also used for the motion vertically upwards. Why is it so ?*

(Meerut 95)

Sol. Suppose a particle falls under gravity in a resisting medium. The force of resistance acts vertically upwards on the particle while the force of gravity acts vertically downwards. As the velocity of the particle goes on increasing the force of resistance also goes on increasing. Suppose the force of resistance becomes equal to the weight of the particle when it has attained the velocity V. Then the resultant downward acceleration of the particle becomes zero and so during the subsequent motion the particle falls with constant velocity V, called the **terminal velocity.**

Thus if a particle is falling under gravity in a resisting medium, then the velocity V when the force of resistance on the particle becomes equal to the weight of the particle so that the downward acceleration of the particle is zero is called the **terminal velocity.**

If a particle is projected vertically upwards in a resisting medium, then both the force of resistance and the weight of the particle act vertically downwards *i.e.*, act against the direction of motion. Consequently the velocity of the particle goes on decreasing and becomes zero when the particle reaches the point of maximum height. Thus in the upwards motion the question of terminal velocity does not arise. The terminal velocity in a resisting medium arises only in the downward motion.

If a particle is projected vertically upwards in a resisting medium and we have to change the constant of proportionality of the force of resistance in terms of the terminal velocity in that resisting medium, then first we write the equation of motion of the particle during its downward motion and using this equation the value of constant of proportionality of the force of resistance is expressed in terms of terminal velocity and is then used in the equation of upward motion.

Ex. 22. *A heavy particle is projected vertically downwards with an initial velocity U, in a resisting medium. Discuss the behaviour of its velocity when U is less than, equal to or greater than the terminal velocity V in the medium.*

(Allahabad 1987)

Sol. If the initial velocity U is less than the terminal velocity V, then the velocity of the particle goes on increasing till it becomes equal to the terminal velocity V. After attaining this terminal velocity V, the particle will move with constant velocity V.

If the initial velocity U is equal to the terminal velocity V, then the particle will continue moving with this constant velocity U.

If the initial velocity of projection U is greater than the terminal velocity V, then at first the velocity of the particle goes on decreasing till it becomes equal to the terminal velocity V. After attaining this **terminal velocity** V, **the particle** will move with constant velocity V.

Ex. 23. *A particle is projected vertically upwards. Prove that if the resistance of air were constant and equal to* $(1/n)$*th of its weight, the time of ascent and descent would be as*

$$(n-1)^{1/2} : (n+1)^{1/2}.$$

(Allahabad 1987)

Sol. Suppose a particle of mass m is projected vertically upwards from O with velocity u. Let P be its position at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are,

(i) The weight mg of the particle acting vertically downwards, and
(ii) the force of resistance $(1/n)\, mg$ acting vertically upwards.

The equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = -mg - \frac{1}{n}mg$$

or
$$\frac{d^2x}{dt^2} = -\left(\frac{n+1}{n}\right)g. \qquad ...(1)$$

The equation (1) can be written as

$$\frac{dv}{dt} = -\left(\frac{n+1}{n}\right)g.$$

$\therefore$
$$dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}\,dv. \qquad ...(2)$$

Let O' be the point of maximum height *i.e.*, the velocity of the particle becomes zero at O'. Let $OO' = h$ and let t_1 be the time of ascent from O to O'. Then from (2), we have

$$\int_0^{t_1} dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 dv = \left(\frac{n}{n+1}\right)\frac{1}{g}\int_0^u dv$$

or
$$t_1 = \left(\frac{n}{n+1}\right)\frac{u}{g}. \qquad ...(3)$$

Again the equation (1) can also be written as

$$v\frac{dv}{dx} = -\left(\frac{n+1}{n}\right)g.$$

$\therefore$
$$dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}\,v\,dv.$$

Integrating from O to O', we get

$$\int_0^h dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 v\,dv$$

or
$$h = \left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2}.$$

During the downwards motion from O' to O, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = g - \frac{g}{n} = \left(\frac{n-1}{n}\right)g. \qquad ...(4)$$

The equation (4) can be written as

$$v\frac{dv}{dx}=\left(\frac{n-1}{n}\right)g.$$

$$\therefore \qquad dx=\left(\frac{n}{n-1}\right)\frac{1}{g}\,v\,dv. \qquad \ldots(5)$$

Suppose the particle reaches back O with velocity u_1. Then integrating (5) from O' to O, we get

$$\int_0^h dx=\left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} v\,dv$$

or
$$h=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}$$

or
$$\left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2}=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}, \qquad \text{substituting for } h$$

or
$$u_1^2=\left(\frac{n-1}{n+1}\right)u^2 \qquad \text{or} \qquad u_1=\sqrt{\left(\frac{n-1}{n+1}\right)}\,u.$$

Now the equation (4) can also be written as

$$\frac{dv}{dt}=\left(\frac{n-1}{n}\right)g.$$

$$\therefore \qquad dt=\left(\frac{n}{n-1}\right)\frac{1}{g}\,dv. \qquad \ldots(6)$$

Let t_2 be the time of descent from O' to O. Then integrating (6) from O' to O, we get

$$\int_0^{t_2} dt=\left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} dv$$

or
$$t_2=\left(\frac{n}{n-1}\right)\frac{1}{g}u_1=\left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\sqrt{\left(\frac{n-1}{n+1}\right)}\,u$$

$$=\frac{u}{g}\cdot\frac{n}{\sqrt{(n-1)}\sqrt{(n+1)}}.$$

$$\therefore \qquad \frac{t_1}{t_2}=\left(\frac{n}{n+1}\right)\frac{u}{g}\cdot\frac{g\sqrt{(n-1)}\sqrt{(n+1)}}{un}=\frac{\sqrt{(n-1)}}{\sqrt{(n+1)}}.$$

Hence $\quad t_1 : t_2=(n-1)^{1/2} : (n+1)^{1/2}.$

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