CSE-2013

Q Show that the curve $\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$ lies in a plane.

Sol. A curve lies in plane if bi-normal vector (B) is constant $\Rightarrow \frac{dB}{dt} = 0 \quad \therefore \tau N = \frac{dB}{dt} = 0$

$\Rightarrow$ Torsion (τ) is zero for a planar curve.

$$\tau = \frac{[x'(t)\ x''(t)\ x'''(t)]}{\|x' \times x''\|^2} = 0$$

$$x(t) = t\hat{i} + \left(1+\frac{1}{t}\right)\hat{j} + \left(\frac{1}{t} - t\right)\hat{k}$$

$$x'(t) = \hat{i} + \left(-\frac{1}{t^2}\right)\hat{j} + \left(-\frac{1}{t^2} - 1\right)\hat{k}$$

$$x''(t) = 0\hat{i} + \frac{2}{t^3}\hat{j} + \frac{2}{t^3}\hat{k}$$

$$x'''(t) = 0\hat{i} - \frac{6}{t^4}\hat{j} - \frac{6}{t^4}\hat{k}$$

$$x'(t) \times x''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1/t^2 & -\frac{1}{t^2} - 1 \\ 0 & -6/t^4 & -6/t^4 \end{vmatrix}$$

$$= \left(\frac{6}{t^6} - \frac{6}{t^6} + \frac{6}{t^4}\right)\hat{i} - \left(-\frac{6}{t^4} - 0\right)\hat{j} + \left(\frac{-6}{t^4}\right)\hat{k}$$

$$= \frac{6}{t^4}\left(\hat{i} + \hat{j} - \hat{k}\right)$$

$$(x'(t) \times x''(t)) \cdot x'''(t) = \left(\frac{6}{t^4}(\hat{i} + \hat{j} - k)\right) \cdot \left(0\hat{i} - \frac{6}{t^4}\hat{j} - \frac{6}{t^4}\hat{k}\right)$$

$$= -\frac{36}{t^8} + \frac{36}{t^8} = 0$$

$$\therefore \tau = \frac{[x'(t)\ x''(t)\ x'''(t)]}{\|x'(t) \times x''(t)\|^2} = 0$$

Hence the given curve is planar.

Q Calculate $\nabla^2(r^n)$ in terms of r and n where $r = \sqrt{x^2+y^2+z^2}$

sol. $r^2 = x^2+y^2+z^2 \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \therefore \frac{\partial \vec{r}}{\partial x} = \hat{i}, \quad \frac{\partial \vec{r}}{\partial y} = \hat{j}, \quad \frac{\partial \vec{r}}{\partial z} = \hat{k}$

Now $\nabla^2(r^n) = \nabla\cdot(\nabla r^n) = \nabla\cdot\left(\sum i \frac{\partial}{\partial x}(r^n)\right) = \nabla\cdot\left(\sum i\, n r^{n-1} \frac{\partial r}{\partial x}\right)$

$= \nabla\cdot\left(\sum i\, n r^{n-1}\left(\frac{x}{r}\right)\right) = \nabla\cdot\left(\sum i\, n r^{n-2} x\right) = \nabla\cdot\left(n r^{n-2} \sum x i\right)$

$= \nabla\cdot\left(n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})\right) = \nabla\cdot\left(n r^{n-2}\vec{r}\right)$

Now, As $\nabla\cdot(\phi\vec{V}) = \nabla\phi\cdot\vec{V} + \phi\nabla\vec{V}$

$\therefore \nabla\cdot(n r^{n-2}\vec{r}) = \nabla(n r^{n-2})\cdot\vec{r} + n r^{n-2}\nabla\vec{r}$

$= \left(n\sum i \frac{\partial}{\partial x} r^{n-2}\right)\cdot\vec{r} + n r^{n-2}\sum i\cdot\frac{\partial \vec{r}}{\partial x}$

$= \left(n\sum i (n-2) r^{n-3}\frac{\partial r}{\partial x}\right)\cdot\vec{r} + n r^{n-2}\sum \hat{i}\cdot\hat{i}$

$= \left(n(n-2)\sum i\, r^{n-3}\cdot\frac{x}{r}\right)\cdot\vec{r} + n r^{n-2}\sum 1$

$= \left(n(n-2) r^{n-4}\sum x\hat{i}\right)\cdot\vec{r} + 3n r^{n-2}$

$= \left(n(n-2) r^{n-4}\vec{r}\right)\cdot\vec{r} + 3n r^{n-2}$

$= n(n-2) r^{n-4}(\vec{r}\cdot\vec{r}) + 3n r^{n-2}$

$= n(n-2) r^{n-4} r^2 + 3n r^{n-2}$

$= (n^2-2n) r^{n-2} + 3n r^{n-2} = (n^2-2n+3n) r^{n-2}$

$= (n^2+n) r^{n-2} = \boxed{n(n+1) r^{n-2}}$ Ans.

Q Curve in space is defined by $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Find the angle between the tangents at point $t = 1$ & $t = -1$

Sol: Given $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$

equation of tangent is given by $\frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right|_{t=1} = 2(1)\hat{i} + 2\hat{j} - 3(1)^2\hat{k} = 2\hat{i} + 2\hat{j} - 3\hat{k} = \vec{A}$$

& $$\left|\frac{d\vec{r}}{dt}\right|_{t=-1} = 2(-1)\hat{i} + 2\hat{j} - 3(-1)^2\hat{k} = -2\hat{i} + 2\hat{j} - 3\hat{k} = \vec{B}$$

$\therefore$ Angle between the tangents $= \frac{\vec{A}\cdot\vec{B}}{|A|\,|B|}$
($\cos\theta$)

$$\therefore \cos\theta = \frac{(2\hat{i} + 2\hat{j} - 3\hat{k})\cdot(-2\hat{i} + 2\hat{j} - 3\hat{k})}{\sqrt{2^2 + 2^2 + (-3)^2}\,\sqrt{(-2)^2 + 2^2 + (-3)^2}} = \frac{-4+4+9}{\sqrt{17}\,\sqrt{17}}$$

$$\cos\theta = \frac{9}{17}$$

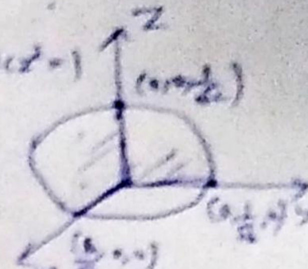
$$\therefore \theta = \cos^{-1}\left(\frac{9}{17}\right)$$

Hence angle between the tangents at $t = 1$ & $t = -1$ is $\cos^{-1}\left(\frac{9}{17}\right)$

Q

Q By using Divergence theorem of Gauss, evaluate the surface integral $\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,dS$, where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$, $a,b,c>0$

Sol: Surface S is surface of ellipsoid $ax^2+by^2+cz^2=1$ enclosing volume V.



unit normal to surface $(\hat{n}) = \dfrac{\nabla S}{|\nabla S|}$

$$\hat{n} = \frac{\nabla(ax^2+by^2+cz^2-1)}{|\nabla(ax^2+by^2+cz^2)|} = \frac{2ax\hat{i}+2by\hat{j}+2cz\hat{k}}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}}$$

$$= \frac{2(ax\hat{i}+by\hat{j}+cz\hat{k})}{2\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

Let vector field $\vec{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ then

$$\iint \vec{F}\cdot\hat{n}\,dS = \iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,dS$$

$$\therefore \vec{F}\cdot\hat{n} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$$

$$\Rightarrow \frac{F_1ax+F_2by+F_3cz}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$$\Rightarrow F_1ax+F_2by+F_3cz = 1$$

$$\therefore F_1 = x \quad F_2 = y \quad F_3 = z \quad (\because ax^2+by^2+cz^2=1)$$

Hence, $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,dS$$

<u>Using Gauss divergence theorem</u>

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V (\operatorname{div}\vec{F})\,dV$$ where $\operatorname{div}\vec{F}$ = Divergence of F, V = Volume enclosed by surface S.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V 3\,dV = 3\iiint_V dV = 3\iiint_V dx\,dy\,dz$$

~~Converting to spherical co-ordinates~~

Let $u = \sqrt{a}\,x \Rightarrow du = \sqrt{a}\,dx$
$v = \sqrt{b}\,y \Rightarrow dv = \sqrt{b}\,dy$
$w = \sqrt{c}\,z \Rightarrow dw = \sqrt{c}\,dz$

$ax^2 + by^2 + cz^2 = 1$
$\Rightarrow u^2 + v^2 + w^2 = 1$

$$\therefore dx\,dy\,dz = \frac{du\,dv\,dw}{\sqrt{abc}}$$

$$\therefore 3\iiint_V dx\,dy\,dz = 3\iiint_V \frac{du\,dv\,dw}{\sqrt{abc}} = \frac{3}{\sqrt{abc}}\iiint_V du\,dv\,dw$$

Changing to spherical co-ordinates

$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\Rightarrow \frac{3}{\sqrt{abc}} \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho = \frac{3}{\sqrt{abc}} \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left| \frac{\rho^3}{3}\sin\phi \right|_0^1 d\theta \, d\phi$$

$$= \frac{3}{\sqrt{abc}} \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left(\frac{1}{3} - 0\right)\sin\phi \, d\theta \, d\phi = \frac{3}{\sqrt{abc}} \cdot \frac{1}{3} \int_{\theta=0}^{2\pi} \left| -\cos\phi \right|_0^{\pi} d\theta$$

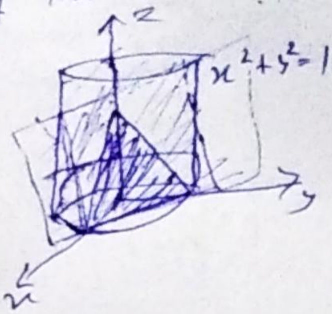
$$= \frac{1}{\sqrt{abc}} \int_{\theta=0}^{2\pi} (-\cos\pi + \cos 0)\, d\theta = \frac{2}{\sqrt{abc}} \int_{\theta=0}^{2\pi} d\theta = \frac{2}{\sqrt{abc}} \left|\theta\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{abc}} (2\pi - 0) = \frac{4\pi}{\sqrt{abc}}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V (\operatorname{div}\vec{F})\, dV = \frac{4}{\sqrt{abc}}$$

$$\Rightarrow \boxed{\iint (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS = \frac{4}{\sqrt{abc}}}$$

Q. Use stoke's theorem to evaluate $\int_C -y^3dx + x^3dy - z^3dz$ where C is intersection of the cylinder $x^2+y^2=1$ & the plane $x+y+z=1$.



Sol. According to stokes theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds \text{ or } \iint_S \text{curl}(\vec{F})\cdot\hat{n}\,ds$$

where C is the boundary & S is the region enclosed by C. $\hat{n}$ is unit normal of surface S.

$$\therefore \int_C -y^3dx + x^3dy - z^3dz = \int_C (-y^3\hat{i} + x^3\hat{j} - z^3\hat{k})\cdot d\vec{r}$$

$$\Rightarrow \vec{F} = -y^3\hat{i} + x^3\hat{j} - z^3\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & +x^3 & -z^3 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (3x^2+3y^2)\hat{k}$$

$$= 3(x^2+y^2)\hat{k}$$

Using stokes theorem

$$\int_C (-y^3\hat{i} + x^3\hat{j} - z^3\hat{k})\cdot d\vec{r} = \iint_S 3(x^2+y^2)\hat{k}\cdot\hat{n}\,ds$$

Surface S is on the plane $x+y+z=1$

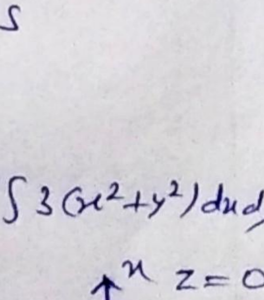
$$\therefore \hat{n} = \frac{\nabla s}{|\nabla s|} = \frac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1^2+1^2+1^2}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\therefore \iint_S 3(x^2+y^2)\hat{k}\cdot\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)ds = \iint_S \sqrt{3}(x^2+y^2)\,ds$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{1/\sqrt{3}} = \sqrt{3}\,dx\,dy$$

$$\therefore \iint_S \sqrt{3}(x^2+y^2)\,ds = \iint_S \sqrt{3}(x^2+y^2)\sqrt{3}\,dx\,dy = \iint 3(x^2+y^2)\,dx\,dy$$

Changing to cylindrical co-ordinates



$$\Rightarrow \int_0^{2\pi}\int_0^1 3r^2(r\,dr\,d\theta) = \int_0^{2\pi} \left[\frac{3r^4}{4}\right]_0^1 = \frac{3}{4}\int_0^{2\pi} d\theta$$

$$= \frac{3}{4}[\theta]_0^{2\pi} = \frac{3}{4}\times 2\pi = \boxed{\frac{3\pi}{2}} \text{ Ans.}$$