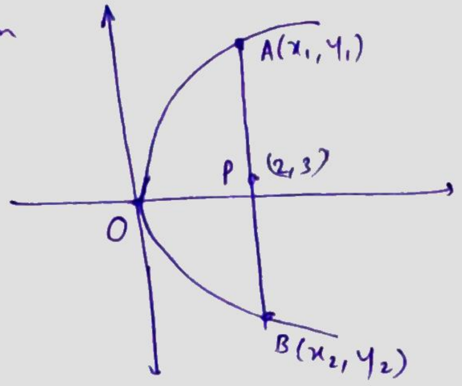
ANALYTIC GEOMETRY

:IFoS-2016:

①(d) If the point (3,2) is the mid point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord.

→ If P is the midpoint of AB, then



$$\frac{x_1 + x_2}{2} = 2 \quad \& \quad \frac{y_1 + y_2}{2} = 3$$

$$x_1 + x_2 = 4 \quad \text{—①} \qquad \& \quad y_1 + y_2 = 6 \quad \text{—②}$$

A & B lie on the parabola.

$\therefore y_1^2 = 4x_1$ & $y_2^2 = 4x_2$.

$$y_1^2 - y_2^2 = 4(x_1 - x_2)$$

$$\Rightarrow (y_1 + y_2)(y_2 - y_1) = 4(x_2 - x_1)$$

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{4}{y_1 + y_2} = \frac{4}{6} = \frac{2}{3}.$$

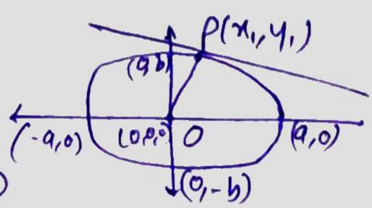
$\therefore$ Slope of the chord is $\frac{2}{3}$.

$\therefore$ Eqⁿ of chord is $y - 3 = \frac{2}{3}(x - 2)$

$$\Rightarrow 3y - 9 = 2x - 4 \Rightarrow \boxed{2x - 3y + 5 = 0}$$

②(b). A perpendicular is drawn from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of foot of perpendicular is given by $(x^2+y^2)^2 = a^2x^2 + b^2y^2$.

→ The tangent to the ellipse is given by the equation



$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad \text{—①}$$

for some value of m.

Slope of OP is $\frac{y_1-0}{x_1-0} = \frac{y_1}{x_1}$.

Slope of tangent is m $\Rightarrow$ $m \cdot \frac{y_1}{x_1} = -1$ [OP ⊥ tangent line]

$\Rightarrow m = -\frac{x_1}{y_1}$

Eqn of tangent line: $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$\Rightarrow$ $y = -\frac{x_1}{y_1}x + \frac{x_1^2}{y_1} + y_1$

$\Rightarrow$ $y = \frac{-x_1}{y_1}x + \frac{x_1^2+y_1^2}{y_1}$ — (2)

comparing the two tangent eqns (1) & (2), we have

$\pm\sqrt{a^2m^2+b^2} = \frac{x_1^2+y_1^2}{y_1}$ $\Rightarrow \sqrt{a^2\left(\frac{x_1^2}{y_1^2}\right)+b^2} = \frac{x_1^2+y_1^2}{y_1}$

$\Rightarrow$ $a^2x_1^2 + b^2y_1^2 = (x_1^2+y_1^2)^2$

$\therefore$ Reqd locus of (x_1, y_1) is: $\boxed{(x^2+y^2)^2 = a^2x^2+b^2y^2}$

(3) (d) Obtain the equation of the sphere on which the intersection of the plane $5x-2y+4z+7=0$ with the sphere which has (0,1,0) and (3,-5,2) as the end points of its diameters is a great circle.

$\longrightarrow$ Eqn of sphere whose diameters end points are (0,1,0), (3,-5,2) is

$(x-0)(x-3) + (y-1)(y+5) + (z-0)(z-2) = 0$

$\Rightarrow x^2+y^2+z^2-3x+4y-2z-5=0$ — (1)

Sphere through the intersection of plane $5x-2y+4z+7=0$ with the sphere (1) is given by

$S \equiv x^2+y^2+z^2-3x+4y-2z-5+\lambda(5x-2y+4z+7)=0$ — (2)

$\Rightarrow$ $x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z+(-5+7\lambda)=0$

Since the plane $5x-2y+4z+7=0$ cuts this sphere into a great circle, the centre of this sphere lies on this plane.

Centre of the sphere S is $\left(\frac{1}{2}(3-5\lambda),\ \lambda-2,\ 1-2\lambda\right)$

$\therefore\quad 5\left(\frac{1}{2}\right)(3-5\lambda) - 2(\lambda-2) + 4(1-2\lambda) + 7 = 0$

$\Rightarrow\quad \frac{45\lambda}{2} = \frac{45}{2} \quad \Rightarrow \lambda = 1$

$\therefore$ The required sphere is :

(2) $\equiv x^2+y^2+z^2-3x+4y-2z-5+1(5x-2y+4z+7)=0$

$\Rightarrow\quad x^2+y^2+z^2+2x+2y+2z+2=0$

4 (d) A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate planes at A, B, C. Find the equation of the cone with vertex at the origin and guiding curve as a circle passing through A, B, C.

$\rightarrow$ The plane intercepts are a, b, c. Hence, it passes through $(a,0,0), (0,b,0), (0,0,c)$.

WKT the equation of a sphere through the origin and passing through $A(a,0,0)$, $B(0,b,0)$ & $C(0,0,c)$ is given by

$x^2+y^2+z^2-ax-by-cz=0$

Then, the circle through A,B,C is given by

$\underbrace{x^2+y^2+z^2-ax-by-cz=0}_{(1)},\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ——— (2)

~~Let the equation of any generator of the cone be~~

The required cone is obtained by making the eqn (1) homogenous with the help of eqn (2)

$\therefore\ x^2+y^2+z^2-(ax+by+cz)=0$

$\Rightarrow x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$

$\Rightarrow \boxed{yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0}$