Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

DIFFERENTIAL EQUATIONS

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Diff. Eq. :- Equation involving derivative of one dependant variable w.r.t. one or more independent variable

Ordinary Diff. Eq :- Involves derivative of dependent variable w.r.t. single independent variable

Partial Diff. Eq. :- Involves derivative of dependent variable with more than 1 independent variable

Order :- Order of highest order deriv. is called order of the differential equation.

Degree :- Degree of highest order derivative when all derivatives are made free from radicals & fractions is called degree.

Note : $y = \sin\left(\frac{dy}{dx}\right) = y' - \frac{y'^3}{3!} + \frac{y'^5}{5!} - \dots$

$\therefore$ Degree not defined.

$y = \sin\left(\frac{dy}{dx}\right) + y''' \rightarrow$ order 3, degree 1.

Linear Differential Equations :- A diff eq is linear if
(i) Depending variable & derivatives occur in first degree
(ii) No product of derivatives with each other
(iii) No product of dependent variable & derivative

i.e. $\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n(x)y = Q(x)$

Non-linear :- Diff-eq. which is not linear

★ 3 Types of solutions :-

(i) General Sol^n : Sol^n where number of arbitrary constants is equal to the order

(ii) Particular Sol^n :- Sol^n obtained by assigning specific value to arbitrary constants of gen. sol.

(iii) Singular Sol^n :- A solution that does not contain arbitrary constants & cannot be obtained by assigning specific values to parameters of gen. sol^n. Generally this will be envelope of gen. solution.

★ Formulating a differential equation :-

Diff. given eq. as many times as there are parameters. Eliminate the parameters to get diff eq.

You can also put it in matrix form & equate it to 0.

e.g. $y = Ae^{x} + Be^{-2x}$

$y' = Ae^{x} - 2Be^{-2x}$

$y'' = Ae^{x} + 4Be^{-2x}$

Since, these are concurrent eq. w.r.t. A & B;

$$\begin{vmatrix} y & -e^{x} & -e^{-2x} \\ y' & -e^{x} & 2e^{-2x} \\ y'' & -e^{x} & -4e^{-2x} \end{vmatrix} = 0$$ is the diff. eq.

IMP :- Differentiate only as many times as the no. of arbitrary constants !!

★ SOLVING FIRST ORDER - FIRST DEGREE EQ.

(a) First order - first degree will be of 2 forms

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \quad \text{or} \quad M\,dx + N\,dy = 0$$

There are 4 methods to solve them.

Note 1st order - 1st degree does not mean linear as we don't know if y is 1st degree or a function like ...

(a) VARIABLE SEPARABLE METHOD

We try to separate function of x & y s.t.

$\frac{dy}{dx} = XY$ where X is function of x
& Y is function of y.

& then $\int \frac{dy}{Y} = \int X\,dx + c$

Adding constant is important to ensure it is gen. sol^n!!

(a1) Sometimes eq. will be in the form of

$\frac{dy}{dx} = f(ax+by+c)$ then we put $ax+by+c=t$

i.e. $a + b\frac{dy}{dx} = \frac{dt}{dx}$

Or we could have

$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c_2}$ then put $ax+by+c=t$

this simplifies the matter rathan than putting $ax+by=t$

(b) Homogeneous Differential Equation :-

A function is homo. of degree 'n' if

$f(kx, ky) = k^n f(x,y)$

A diff. eq is said to be homogeneous if it is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f & g are homo of same degree.

Working rule :- Put $y = vx$

Otherwise also if diff. equation has terms as functions over y/x, then also we put $y = vx$

i.e. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-x/y\right)dy = 0$ (Here we put $x = yv$)

(b) Non-homo reducible to homogeneous

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

Put $x_1 = x+h$ & $y_1 = y+k$ & find (h,k)
such that $ah+bk+c=0$ & $a_1h+b_1k+c_1=0$
this converts above into homogeneous equation.

☆ EXACT DIFFERENTIAL EQUATION

$Mdx + Ndy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\because \exists u$ s.t. $\frac{\partial u(x,y)}{\partial x} = M$ & $\frac{\partial u(x,y)}{\partial y} = N$

$\therefore$ above condition holds i.e. $\frac{\partial M}{\partial y} = \frac{\partial u(x,y)}{\partial x \partial y} = \frac{\partial N}{\partial x}$

Then General sol^n is

$$\underset{\substack{y \text{ considered} \\ \text{as constant}}}{\int M\,dx} + \int \begin{pmatrix}\text{terms of } N \text{ not} \\ \text{containing } x\end{pmatrix} dy = C$$

Try to remember some imp. exact eq which can be detected by inspection.

① $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ ② $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

③ $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2ydy - 2xy^2dx}{x^4}$ ④ $d\left(\frac{e^y}{x}\right) = \frac{xe^ydx - e^ydy}{x^2}$

⑤ $d(\log|xy|) = \frac{xdy + ydx}{xy}$ ⑥ $d\left(\log\left|\frac{y}{x}\right|\right) = \frac{xdy - ydx}{xy}$

⑦ $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy - ydx}{x^2+y^2}$ ☆
very useful.

⑧ $d\left[\log\sqrt{x^2+y^2}\right] = \frac{xdx + ydy}{x^2+y^2}$
this also very useful.

Rather than directly trying to find the integrating factor, check by inspection if diff. eq. can be rearranged into regular known differentials

example :-

$$y(2x^2y+e^x)\,dx - (e^x+y^3)\,dy = 0$$

i.e. $ye^x dx - e^x dy + 2x^2y^2 dx - y^3 dy = 0$

$$\therefore \frac{ye^x dx - e^x dy}{y^2} + 2x^2 dx - y\,dy = 0$$

$$\therefore d\left(\frac{e^x}{y}\right) + 2x^2 dx - y\,dy = 0$$

$$\therefore \frac{e^x}{y} + \frac{2}{3}x^3 - \frac{y^2}{2} = c$$

This is more efficient than finding integrating factor

Even cooler example :-

$$x\,dy = \left[y + x\cos^2(y/x)\right] dx$$

$$\therefore x\,dy - y\,dx = x\cos^2(y/x)\,dx$$

Seeing (y/x) should give you the inspiration

$$\therefore \frac{x\,dy - y\,dx}{x^2} = \cos^2(y/x)\,\frac{dx}{x}$$

$$\therefore \sec^2(y/x)\,d(y/x) = dx/x$$

$$\therefore \tan(y/x) = \log x + c$$

But these can be solved by finding integrating factor also

☆ Solve $(x^3+xy^2-y)\,dx + (y^3+x^2y+x)\,dy = 0$

$\rightarrow (x^2+y^2)(x\,dx+y\,dy) + (x\,dy - y\,dx) = 0 \quad \therefore d\left(\frac{x^2+y^2}{2}\right) + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

$$\therefore d\left(\frac{x^2+y^2}{2}\right) + d\left(\tan^{-1}\frac{y}{x}\right) = 0 \text{ \& so on}$$

$\therefore$ useful to know $d\left(\tan^{-1}\frac{y}{x}\right)$ & $d\left(\log\sqrt{x^2+y^2}\right)$

INTEGRATING FACTOR FOR CONVERSION TO EXACT & $Mx+Ny \neq 0$;

(a) If $Mdx + Ndy = 0$ is homogeneous then $\frac{1}{Mx+Ny}$ is I.F.

(b) If $Mdx + Ndy = 0$ is s.t.
$M = y f_1(x;y)$ & $N = x f_2(x;y)$ then
$\frac{1}{Mx-Ny}$ is I.F. given $Mx - Ny \neq 0$

(c) If $Mdx + Ndy = 0$ s.t.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ or constant then}$$

$$I.F. = e^{\int f(x)dx} \text{ or } e^{\int k dx}$$

(If M starts numerator, N in denominator & function of x needed)

(d) If $Mdx + Ndy = 0$ is s.t.

(If N starts numerator, function of y needed, & M in denominator)

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \text{ or constant then}$$

$$I.F. = e^{\int f(y)dy} \text{ or } e^{\int k dy}$$

(e) If $Mdx + Ndy = 0$ can be put in the form

$$x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1} y^{\beta_1}(m_1 y\,dx + n_1 x\,dy) = 0$$

Where $\alpha_i, \beta_i, m_i, n_i$ are constants then an I.F. $x^h y^k$ exists & h & k can be found by applying exact diff. eq. condition.

★ FIRST ORDER LINEAR DIFFERENTIAL EQUATION

Type ① $\frac{dy}{dx} + P(x)y = Q(x)$

$$I.F. = e^{\int P(x)dx}$$

Sol^n ⇒ $y.(I.F) = \int Q.(I.F)dx + c$

Type ② $\frac{dx}{dy} + P(y)x = Q(y)$

$$I.F. = e^{\int P(y)dy}$$

Sol^n ⇒ $x.(I.F.) = \int Q.(I.F.)dy + c$

Tip:- Don't try to force eq. in dy/dx.
Keep an open mind for dx/dy also.

example:- $(x+2y^3)\frac{dy}{dx} = y$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2 \qquad I.F. = e^{\int \frac{1}{y}dy} = 1/y$$

$\therefore \quad \frac{x}{y} = \int \frac{2y^2}{y} + c = y^2 + c$ is G.S.

★ EQUATIONS REDUCIBLE TO LINEAR

$$f'(y)\frac{dy}{dx} + P f(y) = Q$$

Putting $f(y) = t \Rightarrow \frac{dt}{dx} + Pt = Q$

Bernouli's Theorem

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \Rightarrow y^{-n}\frac{dy}{dx} + y^{1-n}p(x) = Q(x)$$

Put $y^{1-n} = Z$ & it converts to linear.

* LINEAR DIFF. EQ. WITH CONSTANT COEFFICIENTS

$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n] y = Q(x)$ — (1)

Where a_i are constants is Lin. eq. with constant coeff.

It is called homogeneous when $Q = 0$

$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n] y = 0$ ——→ (2)

The solution of (2) is known as complimentary function y_c & solution of (1) not containing arbitrary constants is called particular integral, y_p.

Then gen. sol^n of (1) is $y_c + y_p$.

Auxillary Equation:-

$m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$ is called A.E. of (1)

* Finding the complimentary function

case i) When roots of A.E. are real & distinct

$m = m_1, m_2 \dots m_n$

then $C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

case ii) When 2 roots are equal & other distinct

then $C.F. = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_2 x} + \dots + c_n e^{m_{n-1} x}$

case iii) When all roots are equal

then $C.F. = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}) e^{m_1 x}$

case iv) When $\alpha \pm i\beta$ pair is in roots

then $C.F. = e^{\alpha x}[A \cos \beta x + B \sin \beta x]$

& if imaginary roots are repeated

$C.F. = e^{\alpha x}[(A + Bx) \cos \beta x + (C + Dx) \sin \beta x]$

Case v) Not a special case thought but still
If surds are there $(\alpha \pm \sqrt{\beta})$ then
$$C.F. = e^{\alpha x}\left[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x\right]$$

☆ Finding the particular integral
Let the original eq. be $f(D).y = Q$

So for particular integral we find $y_p = \frac{1}{f(D)} Q$

Case i) Finding P.I. when $Q = e^{ax}$
then
$$P.I. = \frac{1}{f(a)} e^{ax} \quad \text{provided } f(a) \neq 0$$

Case ii) When $Q = \sin ax$ or $\cos ax$ & $f(-a^2) \neq 0$
then we rewrite $f(D)$ in terms of $f(D^2)$ & if any D term is there, we keep it in denominator & put $D^2 = -a^2$ for rest.
Afterwards if we have $D+\alpha$ in denominator, we multiply & divide by $D-\alpha$ & again put $D^2 = -a^2$

So, $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ similar for $\cos ax$.

example $(D^2 - 2D + 5)\, y = \sin 3x$
$$\therefore y_p = \frac{1}{(D^2 - 2D + 5)} \sin 3x = \frac{1}{-9 - 2D + 5} \sin 3x = \frac{-1}{2D+4} \sin 3x$$
$$= \frac{-(2D-4)}{4D^2 - 16} \sin 3x = \frac{4-2D}{-36-16} \sin 3x = \frac{4-2D}{-52} \sin 3x$$
$$\therefore = -\frac{\sin x}{13} + \frac{3\cos 3x}{26}$$

Case iii) When $Q = x^m$ or polynomial of degree m. So
We takeout common the lowest degree term so
that denominator will be in the form
$[1+F(D)]$ or $[1-F(D)]$ & then expand it in the
numerator using below formulae (binomial)

$$(1-x)^{-1} = 1+x+x^2+\dots$$
$$(1+x)^{-1} = 1-x+x^2-\dots$$
$$(1-x)^{-2} = 1+2x+3x^2+4x^3\dots \quad \text{(Obtained by diff. } (1-x)^{-1})$$
$$(1+x)^{-2} = 1-2x+3x^2-4x^3\dots$$

Case iv) When $Q(x) = e^{ax} \cdot f(x)$ where
$f(x) = \sin ax, \cos ax$ or x^m

$$P.I. = \frac{1}{f(D)} e^{ax} \cdot f(x)$$
$$= e^{ax} \frac{1}{f(D+a)} f(x)$$

[Simple logic being derivative of $e^{ax} \cdot f(x)$ will always have a term $a e^{ax} f(x)$ with other term.]

Case v) Finding P.I. when $Q(x) = e^{ax}$ & $f(a) = 0$

$$P.I. = \frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(D+a)} \cdot 1$$

OR

Since $f(a)=0 \Rightarrow f(D) = g(D) \cdot (D-a)^r$
so first put $D=a$ for $g(D)$ & then use
$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$$

OR

Keep on differentiating $f(D)$ till it has a as sol^n;
& then for remaining term put $D=a$; every
time you differentiate $f(D)$, put x in numerator
& appropriate factorial in denominator.

example: $[D^3+D^2-D-1]\, y = e^x$

$\therefore y_p = \dfrac{1}{D^3+D^2-D-1} e^x = \dfrac{x}{3D^2+2D-1} e^x$ (diff. once)

Putting $D=1 \Rightarrow \quad = \dfrac{x}{4} e^x$

OR $y_p = \dfrac{1}{(D+1)^2(D-1)} e^x = \dfrac{1}{(D-1)}\left[\dfrac{1}{(D+1)^2} e^x\right]$

$= \dfrac{1}{(D-1)}\left[\dfrac{e^x}{4}\right]$ putting $D=1$

$= e^x \dfrac{1}{(D-1+1)} \cdot \dfrac{1}{4} = \dfrac{xe^x}{4}$

case vi) We use above method again when
$Q(x) = \sin ax$ or $\cos ax$ & $f(-a^2) = 0$

then simply put $P.I. = \dfrac{1}{f(D)} \sin ax$

$= \text{imagi. part. of} \left(\dfrac{1}{f(D)} e^{iax}\right)$

& proceed with same steps as case v)

Only for $\sin ax$ or $\cos ax$ with D^2+a^2 as $f(D)$.
Otherwise, check the note below.

Upon solving; you would realise

$$\frac{1}{D^2+a^2} \sin ax = \frac{x}{2}\int \sin ax \, dx$$

$$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2}\int \cos ax \, dx$$

Note that you can't directly differentiate $f(D)$ here
First you have to convert it into e^{iax}
Now, while solving problems, generally the factor of $f(D)$ that makes $f(-a^2)=0$ would be (D^2+a^2). Keep it outside & solve for remaining by putting $D^2=-a^2$

& then for $\frac{1}{(D^2+a^2)}$; remember $\frac{x}{2}\begin{cases}\sin ax \text{ or } \cos ax\end{cases}$

example:-

$$\frac{1}{(D^2-2D+7/4)(D^2+3/4)}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{1}{(D^2+3/4)}\left[\frac{1}{(D^2-2D+7/4)}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

put $D^2 = -3/4$ here & so on.

In case of repeated (D^2+a^2), handle one at a time

$$\frac{1}{D^2+a^2}\left[\frac{1}{D^2+a^2}\cos ax\right] \quad \text{\& so on.}$$

Case vi) When $Q = x\,V(x)$ ← This is very useful. Mug it up. Quite intuitive

$$P.I. = \frac{1}{f(D)}x\,V(x) \quad \text{as } f(D) = D \text{ means integration by parts here.}$$

$$= x\frac{1}{f(D)}V(x) - \frac{f'(D)}{[f(D)]^2}V(x)$$

Case vii) When $Q = x^m \sin ax$ or $x^m \cos ax$

Put

$$P.I. = \frac{1}{f(D)}x^m \,\text{Real}\left(e^{iax}\right)$$

$$= \text{Real}\left(\frac{1}{f(D)}x^m e^{iax}\right) = \text{Real}\left(e^{iax}\frac{1}{f(D+ia)}x^m\right)$$

& then expand $(f(D+ia))^{-1}$ using binomial thm to handle x^m.

Case viii) If Q is a function of x then

$$\frac{1}{D-\alpha} Q(x) = e^{\alpha x}\int e^{-\alpha x} Q(x)\,dx$$

$$\frac{1}{D+\alpha} Q(x) = e^{-\alpha x}\int e^{\alpha x} Q(x)\,dx$$

This simply follows from $\frac{1}{f(D)} e^{x}\cdot v(x) = e^{x}\frac{1}{f(D+1)} v(x)$

Good example

Tip :- Inspiration in non-usual trigo. etc. functions

example :- $(D^2+a^2)y = \sec ax$

$$y_p = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ai}\right]\sec ax$$

now, $\frac{1}{D-ia}\sec ax = e^{iax}\int e^{-iax}\sec ax\,dx$

& so on.

METHOD OF UNDETERMINED COEFFICIENTS

From special form of $Q(x)$; assume a particular integral solution & try to find its coefficients

$Q(x)$	Trial solution
x^n or $a_0+a_1x+\ldots+a_nx^n$	$A_0+A_1x+\ldots+A_mx^m$
e^{ax}	Ae^{ax}
$x^n e^{ax}$	$(A_0+A_1x+\ldots+A_nx^n)e^{ax}$
$p\sin ax + q\cos ax$	$A\sin ax + B\cos ax$
$x^n \sin ax$	$(A_0+A_1x+\ldots+A_nx^n)\sin ax + (B_0+B_1x+\ldots+B_nx^n)\cos ax$

Try to find undetermined coefficients by putting them in original diff. equation.

PAGE FOR TIPS/TRICKS LEARNT OVER PRACTICE

(1) When finding $y_p = \frac{1}{f(D)} F(x)$; be careful in applying algebra in denominator $f(D)$ as shown below

$$\frac{1}{D^2+D}\, x = \frac{1}{D(1+D)}\, x = \frac{1}{D}\left[1-D+D^2\dots\right] x$$

now here don't divide numerator function of D by denominator D as shown in case (2)

Case 1 $\quad \frac{1}{D}\left[1-D+D^2\dots\right]x = \frac{1}{D}\left[x-1\right] = \frac{x^2}{2} - x$

Case 2 $\quad \frac{1}{D}\left[1-D+D^2\dots\right]x = \left[\frac{1}{D} - 1 + D\right]x = \frac{x^2}{2} - x + 1$

So, case 2 adds an unnecessary constant which shouldn't be there in particular sol^n bcoz in this case complimentary function will already have C_1.

$(D^2+D) f(x) = 0 \Rightarrow C.F. = C_1 + C_2 e^{-x}$

$\therefore$ Keep denominator factors of D separate & deal with them 1 by 1.

CAUCHY-EULER EQUATION

An equation of following form is Cauchy-Euler Equation.

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

Solved by putting $x = e^z$ which transforms above equation into

$$\left[[D_1(D_1-1)\dots(D_1-(n-1))] + a_1 [D_1(D_1-1)(D_1-2)\dots(D_1-(n-2))] + \dots + a_{n-1} D_1 + a_n \right] y = Q(e^z)$$

& above is linear with constant coefficients.

Now, if $Q(x)$ is x^m or $\log x$ etc. then putting $x = e^z$ is fine for solution.
If $Q(x)$ is $\sin x$ etc. then we can keep $Q(x)$ in terms of x only instead of converting to z & then after splitting $f(D)$ into factors; use following rule

$$\frac{1}{D_1 - \alpha} Q(x) = x^{\alpha} \int x^{-\alpha} \frac{Q(x)}{x} dx$$

$$\frac{1}{D_1 + \alpha} Q(x) = x^{-\alpha} \int x^{\alpha} \frac{Q(x)}{x} dx$$

This is based on earlier formula

$$\frac{1}{D - \alpha} Q(x) = e^{\alpha x} \int e^{-\alpha x} Q(x) dx$$

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☆ LEGENDRE'S LINEAR EQUATION

$$(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} A_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots + A_{n-1} (a+bx) \frac{dy}{dx}$$

$$+ A_n y = \phi(x)$$

Solution obtained by putting $a+bx = e^z$ which then converts it to

$$\left[b^n D_1 (D_1 - 1) \dots (D_1 - (n-1)) + b^{n-1} A_1 D_1 (D_1 - 1) \dots\dots + A_{n-1} b D_1 + A_n \right] y = \phi\left(\frac{e^z - a}{b}\right)$$

Tip:- Remember that powers of coefficient of x (i.e. b) come in new eq.

SOLVING 2^{nd} ORDER LIN. DIFF. EQ. WITH VARIABLE COEFFICIENTS

There are 4 methods for this

(1) When a part of complimentary function is known:-

Working Rule:-

(a) Rewrite given equation in the standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$$

(b) Either a sol^n of homo. eq. would be given or try to check if it is x^m or e^{ax}.

$a^2 + aP + Q = 0$ if it is e^{ax}

& $m(m-1) + Pmx + Qx^2 = 0$ if it is x^m.

(c) Assume the required general sol^n is $y = uv$ & v is obtained by solving

$$\frac{d^2v}{dx^2} + \left(P + \frac{2u'}{u}\right)\frac{dv}{dx} = \frac{R}{u}$$

Here, putting $\frac{dv}{dx} = t$ gives a 1^{st} order linear eq.

In case you can't remember above steps, just check if given problem has x^m or e^{ax} as a solution of homo. & if yes; put $y = uv$ as gen. sol^n. This should give diff eq. to find v.

Note this method is also known as 'reduction of order.'

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(2) By changing ~~first order derivative~~ dependent variable & removing first order derivative

Working steps:-

(1) Rewrite eq. in standard form

$$y'' + P(x)y' + Q(x)y = R(x)$$

(2) To remove first derivative; use choose

$$u = e^{-\frac{1}{2}\int P\,dx}$$

(3) Assume gen. sol^n is $y = uv$ where v is given by normal form

$$\frac{d^2v}{dx^2} + Iv = S$$

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \qquad S = \frac{R}{u}$$

We must get I constant for finding v.

How to remember :- Put $t = +\frac{P}{2}$

then $u = e^{-\int t}$ $\quad I = -t^2 - \frac{dt}{dx} + Q$ & $S = \frac{R}{u}$ in 1^{st} & 2^{nd} method.

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③ By changing the independent variable & making $P_1 = 0$

Working steps:-

① Consider new independent variable $z = f(x)$
Now the diff. eq. becomes

$$y'' + P_1 y' + Q_1 y = R_1$$

for remembering $t = \frac{dz}{dx}$

where
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{Pt + \frac{dt}{dx}}{t^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{Q}{t^2} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{R}{t^2}$$

② We try to find z such that $Q_1 = \pm a^2$
This z should give P_1 constant. This rule is useful only when P_1 & Q_1 both come as constant.
Also say $Q = -k f(x)$, then we consider $Q_1 = -k$
so that $\left(\frac{dz}{dx}\right)^2 = f(x)$ (not dealing with complex no. here)

So, typically if you see Q is $\pm a^2 (f(x))^2$, you can try this method (say $Q = -4\sin^2 x$)

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(4) By variation of parameters when C.F. is known

Working Steps

(1) Find solution of $y'' + Py' + Qy = 0$
Let it be $C_1 y_1(x) + C_2 y_2(x)$

(2) Let the particular integral be given by
$A(x)\, y_1(x) + B(x)\, y_2(x)$

(3) Find Wronskien $(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

(4) Then $A = \int \frac{-y_2 R}{W(y_1, y_2)}$ & $B = \int \frac{y_1 R}{W(y_1, y_2)}$

(5) And G.S. is $y = \left(C_1 y_1(x) + C_2 y_2(x) + A(x)\, y_1(x) + B(x)\, y_2(x)\right)$

Since we need to know full C.F. here, this method is inferior to 1st method (reduction of order) where we only need to know part of C.F.

★ Choosing Between these 4 methods while solving second order diff. equation

① Check if e^{ax} or x^m is part of C.F. (or if C.F. is provided)
i.e. $a^2 + pa + q = 0$ or $m(m-1) + pxm + qx^2 = 0$
If yes. Try 1st method by quickly finding out v

from $v'' + \left(P + \frac{2u'}{u}\right) v' = R/u$

② You might also get full C.F. here but don't go for variation of parameters directly since generally it is lengthy compared to first method.
Choose variation of parameters only when specifically asked.

③ If e^{ax} or x^m is not a solⁿ. See if Q has a form $-a^2 f(x)^2$
Then we can try putting $\frac{-a^2 f(x)^2}{\left(\frac{dz}{dx}\right)^2} = -a^2$

Try to quickly find out if above $\frac{dz}{dx}$ eliminate the term

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

If P_1 is eliminated; go ahead & solve otherwise next method.

④ If these steps don't work, put $u = e^{-\int P/2}$
& check if $I = -\left(\frac{P}{2}\right)^2 - \frac{d\left(P/2\right)}{dx} + Q$ turns out to be a const

If I is a constant, go ahead with this method.

One of these steps should guide you to appropriate met

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DIFF. EQ. OF FIRST ORDER BUT NOT OF FIRST DEGREE

$p^n + A_1 p^{n-1} + \dots + A_{n-1} p + A_n = 0$
where $p = \frac{dy}{dx}$ & A_i are not constants but functions of x.

4 methods exist for solving them

(i) SOLVABLE FOR P

Let diff. eq. be solvable as
$[p - f_1(x,y)][p - f_2(x,y)] \dots [p - f_n(x,y)] = 0$
i.e. $p = f_1(x,y)$, $p = f_2(x,y) \dots$
On integration; let the solution be given by
$F_1(x,y,c_1) = 0$; $F_2(x,y,c_2) = 0 \dots F_n(x,y,c_n) = 0$
but since it is of first order, there can be only one C.

$\therefore$ G.S. $= F_1(x,y,c) F_2(x,y,c) \dots F_n(x,y,c) = 0$

(ii) SOLVABLE FOR x

Let us assume we can rewrite given eqn in the form
$x = f(y,p)$ — (1)
then differentiating both sides, we get
$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$ — (2)

Let the solution of (2) be
$\phi(y,p,c) = 0$ — (3)

Then by eliminating P from (1) & (3); we get the sol^n.

In case it is difficult to eliminate P, we say (1) & (3) together give solution.
Note that while solving (2); we might get a singular sol^n which we have to ignore. example $(p - y^2)\left(\frac{dp}{dy} - yp\right) = 0$
we simply ignore $p - y^2 = 0$ & solve for $\frac{dp}{dy} - yp = 0$

iii)

iv

iii) SOLVABLE FOR Y

Similar to method 2; we put $y = f(x, p)$ — ①
& then $p = F\left(x, p, \frac{dp}{dx}\right)$ — ②

Let solⁿ of ② be $\phi(x, p, c) = 0$ — ③
By eliminating p from ① & ③; we get the solution.

If elimination not possible; ① & ③ together give solution.

iv) Clairaut's Equation

If possible rearrange given equation in terms of

$$y = xp + f(p)$$

then G. solⁿ is given by

$$y = c\cancel{p}^{x} + f(c)$$

For any diff. eq. question involving powers of P; try to first put it in clairnaut form. Makes life easy!!

v) Equations reducible to Clairaut form

There is no method here. Just by observation if you can find; it would be good.

(a) If the eq. involves $(y - px)$ factor; then $X = x^2$ & $Y = y^2$ would mostly work.

(b) If $e^{by}(a - bp)$ involved; try putting $X = e^{ax}$ & $Y = e$

(c) If y/x is involved, try ~~$Y = vx$~~ $Y = vx$

Generally bit difficult to find these substitutions.

SINGULAR SOLUTIONS / ENVELOPES

A circle is an envelope of its family of tangents.

★ P-Discriminant

Let $f(x,y,P)$ be a differential equation.
Then P-discriminant is obtained by eliminating P between $f(x,y,P)=0$ & $\frac{df}{dP}=0$.

Now, if $f(x,y,P)$ is a quadratic in P; then this obviously means $b^2-4ac=0$.

★ C-discriminant

Let $\phi(x,y,c)$ be solution of $f(x,y,P)$.
Then C-discriminant is obtained by eliminating C between $\phi(x,y,c)=0$ & $\frac{d\phi}{dC}=0$

Again, if it is a quadratic in C, then $\Delta=0$ is the C-discriminant.

★ SINGULAR SOLUTION

If $E(x,y)$ is an envelope of general sol^n of $f(x,y,P)=0$ then

(a) $E(x,y)$ must satisfy $f(x,y,P)=0$

(b) $E(x,y)$ must be a factor of both P & C discriminants

★ Extraneous Loci

Now, each discriminant could have factors other than envelope. These would be extraneous loci who do not satisfy the differential equation.

3 Types (1) Tac-locus (T) (2) Node-locus (N) (3) Cusp-locus (C)

So, a p-discriminant can have ① envelope once
② Tac-locus twice ③ cusp-locus once

$\therefore$ P- Discriminant = ET^2C

& a C-discriminant can have ① envelope once
② Node locus twice ③ Cusp locus thrice

$\therefore$ C- discriminant = EN^2C^3

So, once you find P & C discriminant; the common factor that is present once would be E.
& Rest T, C, N can be identified similarly.

They might ask geometric significance of rest of the factors. That time you mention the loci ~~point~~ names.

★ Note :- When equation is in Clairaut's form, both C & P discriminants are same & are envelope eq.

ORTHOGONAL TRAJECTORIES

Two families $f(x,y,c)$ & $g(x,y,c)$ are orthogonal trajectories if they intersect at right angle everywhere.

★ Working steps for finding ortho. trajectories of $f(x,y,c)=0 \Rightarrow$

① Eliminate c by differentiating once & form equation $\phi(x,y,P)=0$ $\quad P=\frac{dy}{dx}$

② Replace P by $\frac{-1}{P}$ in ϕ.

③ Solve for $\phi\left(x,y,\frac{-1}{P}\right)=0$ & it gives orthogonal trajectory equation.

★ Self orthogonal family
When replacing P by $-\frac{1}{P}$ gives same differential equation; we say the family is self-orthogonal.

★ Orthogonal trajectories in polar co-ordinates
Working steps:-
① Form diff. eq. $F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ by eliminating c from $f(r,\theta,c)=0$

② Replace $\frac{dr}{d\theta}$ by $\frac{-r^2}{dr/d\theta}$ in F.

③ Solve $F\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$ to get orthogonal trajectories classmate

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★ Working steps for finding Oblique trajectories at angle α.

Here, replace P by $\frac{P+\tan\alpha}{1-P\tan\alpha}$ or $\frac{1+P\cot\alpha}{\cot\alpha-P}$

Solve for $F\left(x, y, \frac{P+\tan\alpha}{1-P\tan\alpha}\right)=0$ to get oblique trajectories.

LAPLACE TRANSFORM FORMULAE

① Laplace transform of $f(t)$

$$F(P) = \int_0^\infty e^{-Pt} f(t)\,dt$$

It follows linearity

$$L(a_1 f_1(t) + a_2 f_2(t)) = a_1 F_1(P) + a_2 F_2(P)$$

Laplace exists if

ⓐ $f(t)$ is piecewise continuous on $t \geq 0$

ⓑ It is of exponential order i.e. $\exists\, M, \alpha > 0$ s.t. $|f(t)| \leq Me^{\alpha t}$ $t \geq 0$. or $|e^{-\alpha t} F(t)| < M \;\forall t \geq t_0$

Also called $f(t)$ is function of class A.

② Laplace of standard functions

ⓐ $L(1) = \frac{1}{P}$ $P > 0$

ⓑ $L(t^n) = \frac{n!}{P^{n+1}}$ $n \in N$

ⓒ $L(t^n) = \frac{\Gamma(n+1)}{P^{(n+1)}}$ $-1 < n < 0$ or $n > 0$

Gamma function $\Gamma(n+1) = n\Gamma(n)$

& remember $\Gamma\frac{1}{2} = \sqrt{\pi}$

$\therefore \Gamma\frac{5}{2} = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{4}\sqrt{\pi}$

ⓓ $L(e^{at}) = \frac{1}{P-a}$ $\underline{P > a}$

ⓔ $L(\sin at) = \frac{a}{P^2+a^2}$

ⓕ $L(\cos at) = \frac{P}{P^2+a^2}$

ⓖ $L(\sinh at) = \frac{a}{P^2-a^2}$

ⓗ $L(\cosh at) = \frac{P}{P^2-a^2}$

(3) First Transition Theorem

$$L(e^{at}F(t)) = f(p-a)$$

Second Transition Theorem

If $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$

the $L(G(t)) = e^{-ap} f(p)$

Scaling Theorem

$$L(aF(t)) = \frac{1}{a} f\left(\frac{p}{a}\right)$$

(4) $L(F'(t)) = pf(p) - F(0)$ (obv. from integration by parts)

If $F(t)$ has different limits at $t = a$ then

$$L(F'(t)) = pf(p) - F(0) - e^{-ap}\left(F(a_+) - F(a_-)\right)$$

(5) Laplace of n^{th} derivative

$$L\{F^n(t)\} = p^n f(p) - p^{n-1}F(0) - p^{n-2}F'(0) - \dots - F^{n-1}(0)$$

(6) Initial & Final Value Thm.

$$\lim_{t \to 0} F(t) = \lim_{p \to \infty} pf(p)$$

$$\lim_{t \to \infty} F(t) = \lim_{p \to 0} pf(p)$$

(7) Laplace of integral

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{f(p)}{p}$$

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(8) Multiplication by t^n

$$L(t\,F(t)) = -f'(p)$$

$$L(t^n F(t)) = (-1)^n \frac{d^n f(p)}{dp^n}$$

$$L\left(\frac{F(t)}{t}\right) = \int_p^\infty f(x)\,dx$$

(9) Evaluation of integral using integral

$$\int_0^\infty F(t)\,dt = \lim_{p \to 0} f(p)$$

(10) Periodic function of period T

$$L(F(t)) = \frac{\int_0^T e^{-pt} F(t)\,dt}{1 - e^{-pT}}$$

(11) Shifting Theorem of Laplace Inverse

If $L^{-1}(f(p)) = F(t)$

(a) $L^{-1}(f(p-a)) = e^{at} F(t)$

(b) $L^{-1}(e^{-ap} f(p)) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$

(12) Scaling Property

$$L^{-1}(a\,f(p)) = \frac{1}{a} F\left(\frac{t}{a}\right)$$

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(13) $L^{-1}\{f^{n}(p)\} = (-1)^n t^n F(t)$

(14) $L^{-1}\left\{\int_p^\infty f(x)\,dx\right\} = \dfrac{F(t)}{t}$

$L^{-1}\{p\,f(p)\} = F'(t)$ if $F(0) = 0$

(15) $L^{-1}\left\{\dfrac{f(p)}{p}\right\} = \int_0^t F(x)\,dx$

$$L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \underbrace{\int_0^t\int_0^t \cdots \int_0^t}_{\leftarrow n \text{ times} \rightarrow} F(x)\,dx^n$$

Obv. F(t) should be well behaved etc.

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CONVOLUTION & LAPLACE

Definition :- Convolution of 2 functions $F(t), G(t)$ is given by

$$F * G = \int_0^t F(x) \cdot G(t-x)\, dx$$

(1) Laplace inverse of $f(p) \cdot g(p)$

$$L^{-1}\left(f(p) \cdot g(p)\right) = F(t) * G(t)$$

(I) e.g. $L^{-1}\left\{\frac{p}{(p^2+4)^3}\right\} = L^{-1}\left\{\frac{1}{(p^2+4)} \cdot \frac{p}{(p^2+4)^2}\right\}$

$$\frac{1}{(p^2+4)} \downarrow L\left(\frac{\sin 2t}{2}\right) \qquad \frac{p}{(p^2+4)^2} \downarrow \frac{d}{dp}\frac{1}{(p^2+4)} \cdot \frac{-1}{2} \downarrow L\left(\frac{-t \sin 2t}{2}\right)$$

(II) Prove that $\int_0^\infty e^{-tx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$

$$\rightarrow L\left\{\int_0^\infty e^{-tx^2} dx\right\} = \int_0^\infty e^{-pt} \int_0^\infty e^{-tx^2} dx\, dt = \int_0^\infty\int_0^\infty e^{-pt} \cdot e^{-tx^2} dx\, dt$$

$$= \int_0^\infty L\left(e^{-tx^2}\right) dx = \int_0^\infty \frac{1}{p+x^2} dx$$

$$= \frac{1}{p}\int_0^\infty \frac{1}{1+\left(x/\sqrt{p}\right)^2} dx = \frac{1}{\sqrt{p}} \Big| \tan^{-1}\frac{x}{\sqrt{p}} \Big|_0^\infty = \frac{1}{\sqrt{p}} \cdot \frac{\pi}{2}$$

$$\therefore \int_0^\infty e^{-tx^2} dx = \frac{\pi}{2} L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = \frac{\sqrt{\pi}}{2} L^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{p}}\right\} = \frac{\sqrt{\pi}}{2} L^{-1}\left\{\frac{\Gamma\frac{1}{2}}{\sqrt{p}}\right\}$$

$$= \frac{\sqrt{\pi}}{2} t^{-1/2}$$

Remember these type of examples / not easily intuitive

Putting $t=1$ gives $\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$

Remember this comes in error function

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$$

Complimentary error function is defined as

$$erfc(t) = 1 - erf(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx$$

III Find $L^{-1}\left\{e^{-\sqrt{p}}/p\right\}$

→ Expand $e^{-\sqrt{p}}$ by exponential series & then solve.

(2) Heaviside's Expansion Formula

Let $F(P)$ & $G(P)$ be 2 polynomials with deg $F(P) <$ deg $G($
Let $G(P)$ has n distinct roots α_1 to α_n.

Then $$L^{-1}\left\{\frac{F(P)}{G(P)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

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WRONSKIAN & INDEPENDENT SOL^n

1

Let $y_1, y_2 \dots y_n$ be solutions of a differential equation

$$\text{Wronskian} = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Now,

Wronskian $\neq 0 \Rightarrow$ Solutions are linearly independent.

But

Wronskian $= 0 \not\Rightarrow$ Sol^n are lin dependent.

e.g. $y_1 = x^2$ $y_2 = x|x|$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0$$

but x^2 & $x|x|$ are linearly independent.

Remember this result & fact that converse is not true.

DIFF EQ. TIPS

Partial fractions

(1) In laplace inverse problems, you need to find partial fractions to make life simple.

Some basic rules

Factor in denominator	Partial fraction
$(2x+3)$	$\frac{A}{2x+3}$
$(2x+3)^3$	$\frac{A_1}{(2x+3)} + \frac{A_2}{(2x+3)^2} + \frac{A_3}{(2x+3)^3}$
(x^2+2x+3)	$\frac{Ax+B}{x^2+2x+3}$
$(x^2+2x+3)^2$	$\frac{A_1x+B_1}{(x^2+2x+3)} + \frac{A_2x+B_2}{(x^2+2x+3)^2}$

In a question, it is much better to write all steps rather than trying to find factors with a trick.

e.g. $\dfrac{4x^2}{(x-1)(x-2)^2}$

$$\rightarrow \frac{4x^2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\therefore 4x^2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1) \quad \text{--- (1)}$$

$$= A(x^2-4x+4) + B(x^2-3x+2) + C(x-1)$$

$$= (A+B)x^2 + ((-3B-4A)x + (4A+2B-C)$$

$\therefore A+B=4$

$C = 3B+4A$ & $C = 4A+2B$

$\therefore B=0 \quad \therefore A=4 \quad \therefore C=16$

It doesn't take much time really.

Be careful about factors like here B needs to be multiplied by $(x-1)(x-2)$ in (1) & not just $(x-1)$ or $(x-2)$.