Set – IV

(1)

INTERPOLATION

→ The interpolation has been defined as the art of reading between the lines of a table, and in elementary mathematics the term usually denotes the process of computing intermediate values of a function from a set of given values of that function.

For example:

Consider the table that lists the population of Delhi. The population census is taken every ten years and the table gives poulation for the years 1901, 1911, 1961, 1971, 1981 and 1991 in Delhi

we would like to know whether this table could be used to estimate the poupulation of Delhi in 1936 say or even in 1996. Such estimates of population can be made using a function that fits the given data.

→ Let $y=f(x)$ be a real valued function defined on the interval $[a,b]$ and we denote $f(x_k)$ by f_k or y_k.

Suppose that the values of the function $f(x)$ are given to be $f_0, f_1, f_2, \dots f_n$ when $x = x_0, x_1, \dots x_n$ respectively, where $x_0 < x_1 < \dots\dots < x_{n-1} < x_n$ lying in the interval $[a,b]$.

The function $f(x)$ may not be known to us. This technique of determining the value $f(x)$

for a non-tabular value of x which lies in the interval $[a,b]$ is called 'interpolation'.

The process of determining the value of $f(x)$ for a value of 'x' lying outside the interval $[a,b]$ is called extrapolation.

It may be noted that if the function $f(x)$ is known, the value of $y=f(x)$ corresponding to any x can be readily computed to the desired accuracy. But, in practice, it may be difficult or sometimes impossible to know the function $y=f(x)$ in its exact form. In such cases the function $f(x)$ is replaced by a polynomial $\phi(x)$ of degree $\leq n$ which agrees with the values of $f(x)$ at the given $(n+1)$ distinct points, called nodes or abscissas. In other words, we can find a polynomial $\phi(x)$ such that $\phi(x_j)=f_j$, $j=0,1,2,\ldots n$. Such a polynomial $\phi(x)$ is called the interpolating polynomial of $f(x)$.

In general, for interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

Interpolation with Equal Intervals:—

Finite Difference Operators:-

Assume that we have a table of values (x_k, y_k), $k=0,1,2\ldots n$ of any function $y=f(x)$, the values of x being equally spaced i.e, $x_k = x_0+kh$, $k=0,1,2\ldots n$.

Forward Differences:

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If $y_0, y_1, y_2 \ldots\ldots y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots y_n - y_{n-1}$ are called the differences of y.

Denoting these differences by $\Delta y_0, \Delta y_1, \ldots\ldots \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}.$$

The symbol Δ is called forward difference operator and $\Delta y_0, \Delta y_1 \ldots\ldots$ are called first forward differences.

— The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots\ldots \Delta^2 y_{n-1}$.

we have
$$\Delta^2 y_0 = \Delta[\Delta y_0]$$
$$= \Delta[y_1 - y_0]$$
$$= \Delta y_1 - \Delta y_0.$$
$$= (y_2 - y_1) - (y_1 - y_0).$$

$$\boxed{\Delta^2 y_0 = y_2 - 2y_1 + y_0}$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$= (y_3 - y_2) - (y_2 - y_1)$$

$$\boxed{\Delta^2 y_1 = y_3 - 2y_2 + y_1} \text{ etc.}$$

Thus, in general

$$\boxed{\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}}$$

The symbol Δ^2 is called second forward difference operator.

Similarly $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$$= y_3 - 2y_2 + y_1 - [y_2 - 2y_1 + y_0].$$

$$\therefore \boxed{\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0}$$

Thus, continuing, we can define, rth difference of y_n as

$$\boxed{\Delta^r y_{n-1} = \Delta^{r-1} y_n - \Delta^{r-1} y_{n-1}.}$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a differce table for values (x_k, y_k), $k = 0, 1, \ldots 6$ as shown below:

Forward Differce Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

This difference table is called forward difference table or diagonal difference table. Here, each difference is located in its appropriate column, midway between the elements of the previous column. It can be noted that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the leading term, while the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots\ldots$ are called leading differences. (3)

→ construct a forward difference table for the following values of x and y.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	0.000
		0.100		0.003		
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

* Backward Differences :-

Let $y=f(x)$ be a function given by the values $y_0, y_1, y_2, \ldots\ldots y_n$ which it takes for the equally spaced abscissas $x_0, x_1, x_2 \ldots\ldots x_n$ of the independent variable x. Then $y_1-y_0, y_2-y_1, \ldots\ldots y_n-y_{n-1}$ are called the first backward differences of $y=f(x)$

Denoting these differences by $\nabla y_1, \nabla y_2, \ldots \nabla y_n$ respectively. we have

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\vdots$$

$$\nabla y_n = y_n - y_{n-1}.$$

The differences of these differences are called second differences and they are denoted by $\nabla^2 y_2, \nabla^2 y_3 \ldots\ldots \nabla^2 y_n$. That is,

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$

$$\vdots$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Thus, in general, the second backward difference are

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}.$$

while the k-th backward differences are given as $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$.

These backward differences can be systematically arranged for a table of values of (x_k, y_k), $k = 0, 1, 2, \ldots 6$ as indicated below:

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Central Differences:-

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts of the two members of the difference.

Thus, we write

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1 \quad \text{, etc.}$$

In general

$$\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

Higher differences are defined as follows:

$$\delta^2 y_n = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$$

$$\delta^k y_n = \delta^{k-1} y_{n+\frac{1}{2}} - \delta^{k-1} y_{n-\frac{1}{2}}$$

These central differences can be systematically arranged as shown below.

Central Difference Table.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y=f(x)$ be a functional relation between x and y, which is also denoted by y_x.

Suppose, we are given consecutive values of x differing by h say $x, x+h, x+2h, x+3h$, etc. The corresponding values of y are y_x, y_{x+h}, y_{x+2h}, y_{x+3h} etc.

As before, we can form the differences of these values.

Thus, $\Delta y_x = y_{x+h} - y_x$
$$= f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly, $\nabla y_x = y_x - y_{x-h}$
$$= f(x) - f(x-h)$$

and $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$
$$= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Note: we should note that it is only the notation which changes and not the differences.

i.e, it is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward, (or) central differences.

Thus we obtain

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2};$$

$$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2} \text{ etc.}$$

* **Symbolic Relations and seperation of symbols:**

— **shift operator (E):—**

Let $y = f(x)$ be a function of x and $x, x+h, x+2h, x+3h, \ldots\ldots$ etc be the consecutive values of x, then the operator E is defined as

$$E f(x) = f(x+h)$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on $f(x)$,

we get

$$E^2 f(x) = E[E f(x)]$$
$$= E[f(x+h)]$$
$$= f(x+2h)$$

Thus, in general, if we apply the operator E n times on $f(x)$, we arrive at

$$E^n f(x) = f(x+nh)$$

In terms of new notation, we can write

$$E^n y_x = y_{x+nh}.$$

(or)

$$E^n f(x) = f(x+nh).$$

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for all real values of n. Also, if $y_0, y_1, y_2 \ldots$ are the consecutive values of the function y_x, then we can also write

$$Ey_0 = y_1, \quad E^2 y_0 = y_2, \quad E^4 y_0 = y_4, \ldots \quad E^2 y_2 = y_4$$

and so on.

The inverse operator E^{-1} is defined as

$$\boxed{E^{-1} f(x) = f(x-h)}$$

and similarly

$$\boxed{E^{-n} f(x) = f(x-nh)}$$

Average operator (μ):-

The average operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[y_{x+\left(\frac{h}{2}\right)} + y_{x-\left(\frac{h}{2}\right)}\right].$$

or

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

Differential operator (D):-

The differential operator is defined as

$$D f(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x) \quad \text{etc.}$$

*** Relation b/w operators $\Delta, \nabla, \delta, E$ and μ:-**

$\boxed{I}$. We know that $\Delta y_x = y_{x+h} - y_x$

$$= E y_x - y_x$$

$$\Delta y_x = (E-1)y_x$$

$$\therefore \boxed{\Delta = E-1}$$

$$\boxed{E = 1+\Delta}$$

[2]. we know that

$$\nabla y_x = y_x - y_{x-h}$$

$$= y_x - E^{-1}y_x.$$

$$= (1-E^{-1})y_x$$

$$\therefore \nabla y_x = (1-E^{-1})y_x$$

$$\therefore \boxed{\nabla = 1-\frac{1}{E}}$$

$$\boxed{E = (1-\nabla)^{-1}}.$$

[3]. The definition of operators δ and E gives $\delta y_x = y_{x+(h/2)} - y_{x-(h/2)}$

$$= E^{1/2}y_x - E^{-1/2}y_x$$

$$= (E^{1/2} - E^{-1/2})y_x$$

Hence $\boxed{\delta = E^{1/2} - E^{-1/2}}$

[4]. The definition of operations μ and E gives $\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-h/2}\right]$

$$= \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right]y_x$$

$$\therefore \boxed{\mu = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)}$$

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15. We know that $Ey_x = y_{x+h} = f(x+h)$

Using Taylor's series expansion, we have

$$Ey_x = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \cdots\cdots$$

$$= \left[1 + \frac{hD}{1!} + \frac{h^2}{2!}D^2 + \frac{h^3D^3}{3!} + \cdots\cdots\right]f(x)$$

$$\therefore Ey_x = e^{hD}y_x$$

$$\boxed{E = e^{hD}}$$

$$\boxed{hD = \log E}.$$

Hence, all the operators can be expressed in terms of operators E.

* <u>The properties of operator</u> Δ:

(1) → If c is a constant then $\Delta c = 0$

<u>sol</u>: Let $f(x) = c$

then $f(x+h) = c$.

where h is the interval of differencing.

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= c - c = 0$$

$$\Rightarrow \boxed{\Delta c = 0}.$$

(2) → $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$

(3) → If c is a constant then $\Delta[c\,f(x)] = c\,\Delta f(x)$.

(4) If m and n are +ve integers then

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x).$$

(5) $\Delta[f_1(x) + f_2(x) + \cdots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \cdots + \Delta f_n(x).$

(6) $\Delta[f(x)\, g(x)] = f(x)\,\Delta g(x) + g(x)\,\Delta f(x). + (\Delta f(x))(\Delta g(x))$

(7) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x)\,g(x+h)}.$

Note:

1. from the properties (2) & (3) it is clear that Δ is a linear operator.

2. If n is a +ve integer $\Delta^n[\Delta^{-n} f(x)] = f(x)$ and in particular when $n=1$,

$$\Delta[\Delta^{-1} f(x)] = f(x).$$

Example: Find (a) Δe^{ax} (b) $\Delta^2 e^{ax}$ (c) $\Delta \sin x$ (d) $\Delta \log x$ (e) $\Delta \tan^{-1} x$.

Sol^n: (a) $\Delta e^{ax} = e^{a(x+h)} - e^{ax}$
$= e^{ax}[e^{ah} - 1]$

(b) $\Delta^2 e^{ax} = \Delta(\Delta e^{ax})$
$= \Delta[e^{ax}(e^{ah}-1)]$
$= (e^{ah}-1)\,\Delta e^{ax}$
$= (e^{ah}-1)[e^{ah}-1]e^{ax}$
$= (e^{ah}-1)^2 e^{ax}.$

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(c) $\Delta \sin x = \sin(x+h) - \sin x$

$= 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$

$= 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

(d) $\Delta \log x = \log(x+h) - \log x$

$= \log\left(\frac{x+h}{x}\right)$

$= \log\left(1+\frac{h}{x}\right)$

(e) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$= \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right]$ $\left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$

$= \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$

→ Prove that $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

Soln: we know that

$e^{hD} = E = 1+\Delta$

$\Rightarrow hD = \log(1+\Delta)$

$= -\log E^{-1}$

$= -\log(1-\nabla)$ $\quad (\because E^{-1} = 1-\nabla)$

$\therefore hD = \log(1+\Delta) = -\log(1-\nabla)$ —— (1)

we have $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

and $\delta = \left(E^{1/2} - E^{-1/2}\right)$

$\therefore \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$

$$= \frac{1}{2}(E - E^{-1})$$

$$= \frac{1}{2}(e^{hD} - e^{-hD})$$

$$\mu\delta = \sinh D.$$

$$\Rightarrow hD = \sinh^{-1}(\mu\delta) \text{ --- } (2)$$

$\therefore$ from (1) & (2)

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

→ Show that

(i) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \frac{\delta}{2}$

(iii) $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+(\delta^2/4)}$ (iv) $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$

(v) $\mu\delta = \frac{\Delta+\nabla}{2}$

Sol^n:

(i) we have

$$\mu\delta = \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$$

$$= \frac{1}{2}(E-E^{-1})$$

$$\Rightarrow 1+\mu^2\delta^2 = 1+\frac{1}{4}(E-E^{-1})^2$$

$$= 1+\frac{1}{4}(E^2-2+E^{-2})$$

$$= \frac{1}{4}(E^2+2+E^{-2})$$

$$= \frac{1}{4}(E+E^{-1})^2 \text{ --- } (1)$$

Also $1+\frac{\delta^2}{2} = 1+\frac{1}{2}(E^{1/2}-E^{-1/2})^2$

$$= 1+\frac{1}{2}(E-2+E^{-1})$$

$$= \frac{1}{2}(E+E^{-1})$$

$$\Rightarrow \left(1+\frac{\delta^2}{2}\right)^2 = \frac{1}{4}(E+E^{-1})^2 \text{ --- } (2)$$

$\therefore$ from (1) & (2) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$

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(ii) $\mu + \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2}) + \frac{1}{2}(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}(2E^{1/2})$

$= E^{1/2}$.

(iii) $\frac{1}{2}\delta^2 + \delta\sqrt{1+(\delta^2/4)} = \frac{(E^{1/2} - E^{-1/2})^2}{2} + (E^{1/2} - E^{-1/2})\sqrt{1 + \frac{1}{4}(E^{1/2} - E^{-1/2})^2}$

$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})\sqrt{(4 + E - 2 + E^{-1})}$

$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})\sqrt{(E^{1/2} + E^{-1/2})^2}$

$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$

$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E - E^{-1})$

$= \frac{1}{2}(E - 2 + E^{-1} + E - E^{-1})$

$= \frac{1}{2}(2E - 2)$

$= E - 1$

$= \Delta$ $\quad (\because \Delta = E - 1)$.

(iv) $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}(E - E^{-1})$ $\quad (\because E = 1 + \Delta)$

$= \frac{1}{2}(1 + \Delta - E^{-1})$

$= \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1})$

$= \frac{\Delta}{2} + \frac{1}{2}\left(1 - \frac{1}{E}\right)$ $\quad (\because E - 1 = \Delta)$

$= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right)$

$= \frac{\Delta}{2} + \frac{1}{2}\frac{\Delta}{E} = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$.

(v) $\mu\delta = \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$

$= \frac{1}{2}(E-E^{-1})$

$= \frac{1}{2}[1+\Delta-(1-\nabla)]$

$= \frac{1}{2}(\Delta+\nabla)$. $\quad (\because E-1=\Delta$ and $1-E^{-1}=\nabla)$

→ Construct the forward difference table for the following data.

x	0	10	20	30
y	0	0.174	0.347	0.518

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0	0.174	−0.001	
10	0.174	0.173		−0.001
20	0.347	0.171	−0.002	
30	0.518			

→ Construct a difference table for $y=f(x)=x^3+2x+1$ for $x=1,2,3,4,5$.

Sol:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4	9	12	6
2	13	21	18	6
3	34	39	24	
4	73	63		
5	136			

Differences of a polynomial:

(10)

The n^{th} differences of a polynomial of the n^{th} degree are constant and all higher order differences are zero when the values of independent variable are at equal intervals.

proof: Let the polynomial of the n^{th} degree in x, be

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots\cdots + a_{n-1}x + a_n$$

where $a_0, a_1, a_2, \ldots\ldots a_n$ are constants and $a_0 \neq 0$.

$$\therefore f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots \cdots + a_{n-1}(x+h) + a_n$$

where h is the interval of differencing.

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots\cdots + a_{n-1}(x+h) + a_n - a_0x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n$$

$$= a_0\left[(x+h)^n - x^n\right] + a_1\left[(x+h)^{n-1} - x^{n-1}\right] + \cdots\cdots + a_{n-1}\left[(x+h) - x\right]$$

$$= a_0\left[\cancel{x^n} + nC_1x^{n-1}h + nC_2x^{n-2}h^2 + \cdots + nC_{n-1}h^{n-1} + h^n - \cancel{x^n}\right] +$$

$$a_1\left[\cancel{x^{n-1}} + {}^{n-1}C_1x^{n-2}h + {}^{n-1}C_2x^{n-3}h^2 + \cdots + h^{n-1} - \cancel{x^{n-1}}\right] + \cdots + a_{n-1}h$$

$$= a_0nhx^{n-1} + \left[a_0h^2nC_2 + a_1h(n-1)C_1\right]x^{n-2} + \cdots + a_{n-1}h.$$

$$= a_0 nh x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots\cdots + b_{n-1} x + b_n \quad \text{(1)}$$

where $b_2, b_3, \ldots\ldots b_n$ are constants.

from (1), it is Clear that the first difference of $f(x)$ is a polynomial of $(n-1)^{th}$ degree.

Similarly. $\Delta^2 f(x) = \Delta(\Delta f(x))$

$$= \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= a_0 nh[(x+h)^{n-1} - x^{n-1}] + b_2[(x+h)^{n-2} - x^{n-2}] + \cdots$$

$$\cdots\cdots + b_{n-1}[x+h-x]$$

$$= a_0 n(n-1) h^2 x^{n-2} + c_3 x^{n-3} + c_4 x^{n-4} + \cdots + c_{n-1} x + c_n$$

where $c_3, c_4, \ldots\ldots c_{n-1}$ are constants.

Therefore the second differences of $f(x)$ reduces to a polynomial of $(n-2)^{th}$ degree. proceeding as above and differencing for n times we get

$$\Delta^n f(x) = a_0 n(n-1)\cdots\cdots 3\times2\times1\, h^n x^{n-n}$$

$$= a_0 n!\, h^n.$$

which is a constant

and $\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$

$$= a_n n!\, h^n - a_0 n!\, h^n = 0$$

i.e, the $(n+1)^{th}$ and higher order differences of a polynomial of n^{th} degree will be zero.

Note: The converse of the above theorem is true. i.e, if the n^{th} differences of a tabulated function and the values of the independent variable are equally spaced then the function is a polynomial of degree n.

Effect of an Error on a difference table :- (11)

Difference tables can be used to check errors in tabular values. Suppose that there is an error ϵ in the entry y_5 of a table. As higher differences are formed, this error spreads out and is considerably magnified.

Let us see, how it effects the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1+\epsilon$
		Δy_3		$\Delta^3 y_2+\epsilon$	
x_4	y_4		$\Delta^2 y_3+\epsilon$		$\Delta^4 y_2-4\epsilon$
		$\Delta y_4+\epsilon$		$\Delta^3 y_3-3\epsilon$	
x_5	$y_5+\epsilon$		$\Delta^2 y_4-2\epsilon$		$\Delta^4 y_3+6\epsilon$
		$\Delta y_5-\epsilon$		$\Delta^3 y_4+3\epsilon$	
x_6	y_6		$\Delta^2 y_5+\epsilon$		$\Delta^4 y_4-4\epsilon$
		Δy_6		$\Delta^3 y_5-\epsilon$	
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_4+\epsilon$
		Δy_7		$\Delta^3 y_6$	
x_8	y_8		$\Delta^2 y_7$		
		Δy_8			
x_9	y_9				

The above table shows that:

(i) The error increases with the order of differences.

(ii) The coefficients of ϵ's in any column are binomial coefficients of $(1-\epsilon)^n$. Thus the errors in the fourth difference column are $\epsilon, -4\epsilon, 6\epsilon, -4\epsilon, \epsilon$.

(iii) The algebraic sum of the errors in any difference column is zero.

(iv) The maximum error in each column, occurs opposite to the entry containing the error i.e, y_5.

The above facts enable us to detect errors in a difference table.

→ One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		-4		
1	21		1	
		-3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

y being a polynomial of the 3rd degree, $\Delta^3 y$ must be constant. i.e, the same.

The sum of the third differences being 15, (12)
each entry under $\Delta^3 y$ must be $15/5$ i.e, 3.
Thus the four entries under $\Delta^3 y$ are in error which can be written as

$2 = 3 + (-1)$, $6 = 3 - 3(-1)$, $0 = 3 + 3(-1)$,
$4 = 3 - (-1)$.

Taking $\epsilon = -1$, we find that the entry corresponding to $x=3$ is in error.

$\therefore y + \epsilon = 18$
$\Rightarrow y = 18 - \epsilon$
$\Rightarrow y = 18 - (-1) = 19$.

Thus the true value of $y = 19$ at $x = 3$.

→ The following is a table of values of a polynomial of degree 5. It is given that $f(3)$ is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1054	3126	7777

Sol^n: It is given that

$y = f(x)$ is a polynomial of degree 5.

$\therefore \Delta^5 y$ must be constant.

Also given $f(3)$ is in error.

Let $254 + \epsilon$ be the true value,

now we form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
				$160+\epsilon$	$200-4\epsilon$	
		31				
2	33		$190+\epsilon$			$220+10\epsilon$
				$360-3\epsilon$		
		$221+\epsilon$			$420+6\epsilon$	
3	$254+\epsilon$		$550-2\epsilon$			$20-10\epsilon$
				$780+3\epsilon$		
		$771-\epsilon$			$440-4\epsilon$	
4	1025		$1330+\epsilon$			
				$1220-\epsilon$		
		2101				
5	3126		2550			
		4651				
6	7777					

Since the fifth differences of y are constant.

$$220+10\epsilon = 20-10\epsilon$$

$$\Rightarrow 20\epsilon = -200 \Rightarrow \boxed{\epsilon = -10}$$

$$\therefore f(3) = 254+\epsilon = 254-10 = 244.$$

Missing values: Let a function $y=f(x)$ be given for equally spaced values $x_0, x_1, x_2, \dots, x_n$ of the argument and $y_0, y_1, y_2, \dots, y_n$ denote the corresponding values of the function. If one or more values of $y=f(x)$ are missing we can find the missing values by using the construction of difference table or using the relation between the operators E and Δ

→ By constructing a difference table and taking the second order differences as a constant, find the sixth term of the series

8, 12, 19, 29, 42, ?.

Sol: Let k be the sixth term of the series

The difference table is.

x	y	Δy	$\Delta^2 y$
1	8		
		4	
2	12		3
		7	
3	19		3
		10	
4	29		3
		13	
5	42		$k-55$
		$k-42$	
6	k		

Since the second order differences are constant

$\therefore k-55=3$

$\Rightarrow \boxed{k=58}$

$\therefore$ The sixth term of the series is 58.

→ Assuming that the following values of y belong to a polynomial degree 4, compute the next three values.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	1	—	—	—

Sol: we construct the difference table from the given data.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0=1$				
		-2			
1	$y_1=-1$		4		
		2		-8	
2	$y_2=1$		-4		$16=\Delta^4 y_0$
		-2		8	
3	$y_3=-1$		4		$16=\Delta^4 y_1$
		2		$\Delta^3 y_2$	
4	$y_4=1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		16.
		Δy_5		$\Delta^3 y_4$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y_0 = 16$.

$\therefore$ The other fourth order differences must also 16.

Thus. $\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$

i.e, $\Delta^3 y_2 = \Delta^4 y_1 + \Delta^3 y_1$

$\Delta^3 y_2 = 16 + 8 = 24$

Now $\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$

$24 = \Delta^2 y_3 - 4$

$\Rightarrow \boxed{\Delta^2 y_3 = 28}$

$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$

$\Rightarrow 28 = \Delta y_4 - 2$

$\Rightarrow \boxed{\Delta y_4 = 30}$

$\Delta y_4 = y_5 - y_4$

$30 = y_5 - 1 \Rightarrow \boxed{y_5 = 31}$

Similarly starting with $\Delta^4 y_2 = 16$,

we get $\Delta^3 y_3 = 40$, $\Delta^2 y_4 = 68$,

$\Delta y_5 = 98$, $y_6 = 129$

Starting with $\Delta^4 y_3 = 16$;

we obtain $\Delta^3 y_4 = 56$, $\Delta^2 y_5 = 124$

$\Delta y_6 = 222$, $y_7 = 351$

(14)

4.3 Newton's binomial expansion formula

Let $y_0, y_1, y_2, \ldots, y_n$ denote the values of the function $y = f(x)$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ of x and let one of the values of y be missing since n values of the functions are known. We have

$$\Delta^n y_0 = 0$$

$$\Rightarrow (E-1)^n y_0 = 0$$

$$\Rightarrow [E^n - {}^nc_1 E^{n-1} + {}^nc_2 E^{n-2} \mp \ldots + (-1)^n] y_0 = 0$$

$$\Rightarrow E^n y_0 - nE^{n-1} y_0 + \frac{n(n-1)}{1\times 2} E^{n-2} y_0 + \ldots + (-1)^n y_0 = 0$$

$$\Rightarrow y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \ldots + (-1)^n y_0 = 0$$

The above formula is called **Newton's binomial expansion formula** and is useful in finding the missing values without constructing the difference table.

Example 1 : Find the missing entry in the following table

x	0	1	2	3	4
y_x	1	3	9	–	81

Solution : Given $y_0 = 1, y_1 = 3, y_2 = 9, \ldots, y_3 = ?, y_4 = 81$ four values of y are given. Let y be polynomial of degree 3

$$\therefore \Delta^4 y_0 = 0$$

$$(E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\therefore 81 - 4y_3 + 6\times 9 - 4\times 3 + 1 = 0$$

$$y_3 = 31.$$

Example 2 : Following are the population of a district

Year (x)	1881	1891	1901	1911	1921	1931
Population (y)	363	391	421	?	467	501

Find the population of the year 1911.

Solution : We have

$$y_0 = 363$$
$$y_1 = 391$$
$$y_2 = 421$$
$$y_3 = ?$$

$$y_4 = 467$$
$$y_5 = 501$$

Five values of y are given. Let us assume that y is a polynomial in x of degree 4.

$$\Delta 5 y_0 = 0$$
$$\Rightarrow (E-1)^5 y_0 = 0$$
$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0 = 0$$
$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$
$$501 - 5 \times 461 + 10y_3 - 10 \times 421 + 5 \times 391 - 363 = 0$$
$$10\, y_3 - 4452 = 0$$
$$y_3 = 445.2$$

The population of the district in 1911 is 445.2 lakhs.

Example 3 : Interpolate the missing entries

x	0	1	2	3	4	5
$y = f(x)$	0	–	8	15	–	35

Solution : Given $y_0 = 0$, $y_1 = ?$, $y_2 = 8$, $y_3 = 15$, $y_4 = ?$, $y_5 = 35$. Three values are known. Let us assume that $y = f(x)$ is a polynomial of degree 3.

$$\Delta^4 y_0 = 0$$
$$\Rightarrow (E-1)^4 y_0 = 0$$
$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_0 = 0$$
$$\therefore\ y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$
$$\therefore\ y_4 - 4 \times 15 + 6 \times 8 - 4y_1 - 0 = 0$$
$$\therefore\ y_4 - 4y_1 = 12 \qquad (1)$$

and

$$\Delta^5 y_0 = 0$$
$$\Rightarrow (E-1)^5 y_0 = 0$$
$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0 = 0$$
$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$
$$35 - 5y_4 + 10 \times 15 - 10 \times 8 + 5y_1 - 0 = 0$$
$$y_4 - y_1 = 21 \qquad (2)$$

solving (1) and (2) we get $y_1 = 3$, $y_4 = 24$.

INTERPOLATING POLYNOMIALS USING FINITE DIFFERENCES:

4.4 Newton's forword interpolation formula (Gregory).

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the $(n+1)$ values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values x be equally spaced i.e.,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

where h is the interval of differencing. Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$ i.e., $\phi(x)$ represents the continuous function $y = f(x)$ such that $f(x_r) = \phi(x_r)$, $r = 0, 1, 2, \ldots, n$ and at all other points $f(x) = \phi(x) + R(x)$ where $R(x)$ is called the **error term** (Remainder term) of the interpolation formula. Ignoring the error term let us assume

$$f(x) \approx \phi(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad (3)$$

the constants $a_0, a_1, a_2, \ldots a_n$ can be determine as follows.

Putting $x = x_0$ in (3) we get

$$f(x_0) \approx \phi(x_0) = a_0$$
$$\Rightarrow y_0 = a_0$$

putting $x = x_1$ in (3) we get

$$f(x_1) \approx \phi(x_1) = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$
$$\therefore y_1 = y_0 + a_1 h$$
$$\Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}.$$

Putting $x = x_2$ in (3) we get

$$f(x_2) \approx \phi(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\therefore y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$
$$\Rightarrow y_2 = y_0 + 2(y_1 - y_0) + a_2(2h^2)$$
$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly by putting $x = x_3, x = x_4, \ldots, x = x_n$ in (3) we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3},\ a_4 = \frac{\Delta^4 y_0}{4!h^4},\ \ldots,\ a_n = \frac{\Delta^n y_0}{n!h^n}$$

putting the values of $a_0, a_1, \ldots a_n$ in (3) we get

$$f(x) \approx \phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad (4)$$

writing $u=\frac{x-x_0}{h}$, we get

$$x - x_0 = uh$$

$$x-x_1 = x-x_0+x_0-x_1$$

$$=(x-x_0)-(x_1-x_0)$$

$$=uh-h=(u-1)h$$

similarly

$$x-x_2=(u-2)h$$

$$x-x_3=(u-3)h$$

$$\ldots$$

$$x-x_{n-1}=(u-n+1)h$$

Equation (4) can be written as

$$\phi(x)=y_0+u\frac{\Delta y_0}{1!}+\frac{u(u-1)}{2!}\Delta^2 y_0+\ldots+\frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_o.$$

The above formula is called **Newton's forward interpolation formula**.

Note :

1. Newton forward interpolation formula is used to interpolate the values of y near the beginning of a set of tabular values.

2. y_0 may be taken as any point of the table, but the formula contains only those values of y which come after the value chosen as y_0.

Example 4 : Given that

$$\sqrt{12500}=111.8034,\ \sqrt{12510}=111.8481$$

$$\sqrt{12520}=111.8928,\ \sqrt{12530}=111.9375$$

find the value of $\sqrt{12516}$.

Solution : The difference table is

x	$y=\sqrt{x}$	Δy	$\Delta^2 y$
12500 x_0	111.8034 y_0		
		0.0447 Δy_0	
12510	111.8481		0 $\Delta^2 y_0$
		0.0447	
12520	111.8928		0
		0.0447	
12530	111.9375		

We have $x_0 = 12500$, $h = 10$ and $x = 12516$

$$u = \frac{x - x_0}{h} = \frac{12516 - 12500}{10} = 1.6$$

from Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow f(12516) = 111.8034 + 1.6 \times 0.0447 + 0 + \ldots$$

$$= 111.8034 + 0.07152$$

$$= 111.87492$$

$$\therefore \sqrt{12516} = 111.87492.$$

Example 5 : Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.255

Solution : The difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.000	1.0000				
0.05		0.2214			
0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have $x_0 = 0.00$, $x = 0.05$, $h = 0.1$.

$$\therefore u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} 0.5$$

Using Newton's forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$f(0.05) = 1.0000 + 0.5 \times\ 0.2214 + \frac{0.5(0.5-1)}{2}(0.0490)$$

$$+\frac{0.5(0.5-1)(0.5-2)}{6}(0.0109)$$

$$+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(0.0023)$$

$$= 1.000 - 0.006125 + 0.000681 - 0.000090$$

$$= 1.105166$$

$\therefore f\ (0.05) \approx 1.052.$

Example 6 : The values of sin$x$ are given below for different values of x. find the value fo sin 32°

x	30°	35°	40°	45°	50°
y = sin$x$	0.5000	0.5736	0.6428	0.7071	0.7660

Solution : $x = 32°$ is very near to the starting value $x_o = 30°$. By using Newton's forward interpolation formula the difference table is

x	y=sin$x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30°	0.5000				
		0.0736			
35°	0.5736		–0.0044		
		0.0692		–0.005	
40°	0.6428		–0.0049		0
		0.0643		–0.005	
45°	0.7071		–0.0054		
		0.0589			
50°	0.7660				

$$u = \frac{x - x_0}{h} = \frac{32^0 - 30^0}{5} = 0.4$$

We have $y_0 = 0.5000$, $\Delta y_0 = 0.0736$, $\Delta^2 y_0 = -0.0044$, $\Delta^3 y_0 = -0.005$

putting these values in Newton's forward interpolation formula we get

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\Rightarrow f(32^0) = 0.5000 + 0.4 \times 0.0736 + \frac{(0.4)(0.4-1)}{2}(-0.0044)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.0005)$$

$f(32°) = 0.5000 + 0.02944 + 0.000528 - 0.00032 = 0.529936 = \cancel{0.299}$.

0.529936

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(17)

Example 7 : In an examination the number of candidates who obtained marks between certain limits were as follows

Marks	30–40	40–50	50–60	60–70	70–80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution : First of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x marks	y cumulative frequencies	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

we have $x_0 = 40$, $x = 45$, $h = 10$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

and $y_0 = 31$, $\Delta y_0 = 42$ $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = -25$, $\Delta^4 y_0 = 37$.

From Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$\therefore f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2} \times 9$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} \times (37)$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4452$

$= 47.8673$

$= 48$ (approximately)

$\therefore$ The number of students who obtained mark less than 45 = 48, and the number of students who scored marks between 45 and 50 = 73 − 48 = 25.

Example 8 : A second degree polynomial passes through the points (1, −1), (2, −1), (3, 1), (4, 5). Find the polynomial.

Solution : We construct difference table with the given values of x and y

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	−1			
		0		
2	−1		2	
		2		0
3	1		2	
		4		
4	5			

We have $x_0 = 1, h = 1, y_0 = -1, \Delta y_0 = 0, \Delta^2 y_0 = 2,$

$$u = \frac{x - x_0}{h} = (x - 1).$$

From Newton's forward interpolation we get

$$y = f(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots$$

$$\Rightarrow f(x) = -1 + (x-1)\cdot 0 + \frac{(x-1)(x-1-1)}{2}\cdot 2$$

$$\therefore f(x) = x^2 - 3x + 1.$$

Note : There may be polynomials of higher degree which also fit the data, but Newton's formula gives us the polynomial of least degree which fits the data.

4.5 Newton - Gregory backward interpolation formula

Newton's forward interpolation formula cannot be used for interpolating a value of y near the end of a table of values. For this purpose, we use another formula known as Newton - Gregory backward interpolation formula. It can be derived as follows.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values $x_0, x_1, x_2, \dots x_n$ of the independent variable x. Let the values of x be equally spaced with h as the interval of differencing, i.e.,

(18)

Let

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots x_n$ i.e., $\phi(x)$ represents $y = f(x)$ such that $f(x_r) = \phi(x_r)$, $r = 0, 1, 2, \ldots, n$ we may write $\phi(x)$ as

$$f(x) \approx \phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1) \qquad (5)$$

putting $x = x_n$ in (5) we get

$$f(x_n) \approx \phi(x_n) = a_0$$
$$\Rightarrow y_n = a_0.$$

Putting $x = x_{n-1}$ in (5) we get

$$f(x_{n-1}) \approx \phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$
$$\Rightarrow y_{n-1} = y_n + a_1(-h)$$
$$\Rightarrow a_1 h = y_n - y_{n-1} = \nabla y_n$$
$$\Rightarrow a_1 = \frac{\nabla y_n}{1!h}$$

Putting $x = x_{n-2}$, we get

$$f(x_{n-2}) \approx \phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$
$$\Rightarrow y_{n-2} = y_n + \left(\frac{y_n - y_{n-1}}{h}\right)(-2h) + a_2(-2h)(-h)$$
$$\Rightarrow y_{n-2} = y_n - 2y_n + 2y_{n-1} + (2h^2)a_2$$
$$\Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!h^2}$$

similarly putting $x = x_{n-3}, x = x_{n-4}, \ldots, x = x_{n-5}, \ldots$ we get

$$a_3 = \frac{\nabla^3 y_n}{3!h^3}, a_4 = \frac{\nabla^4 y_n}{4!h^4}, \ldots, a_n = \frac{\nabla^n y_n}{n!h^n}$$

substituting these values in (5)

$$f(x) \approx \phi(n) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1})$$
$$+ \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2})$$
$$+ \ldots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\ldots(x - x_1) \qquad (6)$$

writing $u = \dfrac{x - x_n}{h}$ we get

$$x - x_n = uh$$

$$\therefore x - x_{n-1} = x - x_n + x_n - x_{n-1} = (uh) + h = (u+1)h$$

$$\Rightarrow x - x_{n-2} = (u+2)h, \ldots, (x - x_1) = (u + n - 1)h$$

$\therefore$ The equation (6) may be written as

$$f(x) \approx \phi(x) = y_n + \frac{u \nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+ \ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n y_n.$$

The above formula is known as Newton's backward interpolation formula.

Example 9 : The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees c and P is the percentage of lead in the alloy.

P	40	50	60	70	80	90
t	180	204	226	250	276	304

Find the melting point of the alloy containing 84 percent lead.

Solution : The value of 84 is near the end of the table therefore we use the Newton's backward interpolation formula. The difference table is

P	t	∇	∇^2	∇^3	∇^4	∇^5
40	180					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have $x_n = 90,\ x = 84,\ h = 10,\ t_n = y_n = 304,\ \nabla t_n = \nabla y_n = 28,\ \nabla^2 y_n = 2,$ and

$$\nabla^3 y_n = \nabla^4 y_n = \nabla^5 y_n = 0,$$

$$u = \frac{x - x_n}{h} = \frac{84-90}{10} = -0.6.$$

From Newton backward formula

$$f(84) = t_n + u\nabla t_n + \frac{u(u+1)}{2}\nabla^2 t_n + \ldots$$

$$f(84) = 304 - 0.6 \times 28 + \frac{(-0.6)(-0.6+1)}{2}2$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96.$$

Example 10 : Calculate the value of $f(7.5)$ for the table

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Solution : 7.5 is near to the end of the table, we use numbers backward formula to find $f(7.5)$.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		0
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

We have

$x_n = 8,\ x = 7.5, h = 1, y_n = 512,\ \nabla y_n = 169,\ \nabla^2 y_n = 42,\ \nabla^3 y_n = 6,\ \nabla^4 y_n = \nabla^5 y_n = \ldots = 0$

$$u = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5.$$

$\therefore$ We get

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2}(42)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.87.$$

4.6 Error in the interpolation formula

Let the function $f(x)$ be continuous and possess continuous derivatives of all orders with in the interval $[x_0, x_n]$ and let $\phi(x)$ denote the interpolating polynomial. Define the auxiliary function $F(t)$ as given below.

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\}\frac{(t - x_0)(t - x_1)\ldots(t - x_n)}{(x - x_0)(x - x_1)\ldots(x - x_n)}$$

The function $F(t)$ is continuous in $[x_0, x_n]$. $F(t)$ possesses continuous derivatives of all orders in $[x_0, x_n]$ and variables for the values $t = x, x_0, \ldots, x_n$. Therefore $F(t)$ satisfies all the conditions of Rollers Theorem in each of the subintervals $(x_0, x_1), (x_1, x_2) \ldots (x_{n-1}, x_n)$. Hence $F^{I}(t)$ vanishes at least once in each of the sub intervals. Therefore $F'(t)$ vanishes at least $(n + 1)$ times in (x_0, x_n), $F''(t)$ vanishes at least n times in the interval (x_0, x_n), ..., $F^{n+1}(t)$ vanishes at least once in (x_0, x_n) say at ζ, where $x_0 < \zeta < x_n$.

The expression $(t - x_0)(t - x_1) \ldots (t - x_n)$ is a polynomial of degree $(n + 1)$ in t and the coefficient of $t = 1$.

$\therefore$ The $(n + 1)$th derivative of the polynomial is $(n + 1)!$

$$\therefore F^{n+1}(\xi) = f^{n+1}(\xi) - \{f(x) - \phi(x)\}\frac{(n+1)!}{(x - x_0)(x - x_1)\ldots(x - x_n)} = 0$$

$\phi(t)$ is a poly. of nth degree $\phi^{n+1}(t) = 0$.

$$\Rightarrow f(x) - \phi(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

If $R(x)$ denotes the error in the formula then $R(x) = f(x) - \phi(x)$

$$\therefore R(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

But $x - x_0 = uh \Rightarrow x - x_1 = (u - 1)h, \ldots (x - x_n) = (u - n)h$ where h is the interval of differencing therefore we can write

$$\text{Error } R(x) = \frac{h^{n+1} f^{n+1}(\xi)}{(n+1)!} u(u-1)(u-2)\ldots(u-n).$$

Using the relation $D = \frac{1}{h}\Delta$

we get

$$D^{n+1} \approx \frac{1}{h^{n+1}}\Delta^{n+1}$$

20

$$\Rightarrow f^{n+1}(\xi) \approx \frac{\Delta^{n+1} f(x_0)}{n+1}$$

The error in the forward interpolation formula is

$$\boxed{R(x) = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2)\ldots(u-n)}$$

Similarly by taking the auxiliary function $F(t)$ in the form

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\} \frac{(t-x_n)(t-x_{n-1})\ldots(t-x_0)}{(x-x_n)(x-x_{n-1})\ldots(x-x_0)},$$

and proceeding as above we get the error in the Newton backward interpolation formula as

$$\boxed{R(x) = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1)\ldots(u+n) \quad \text{where } uh = x - x_n .}$$

Example 11 : Use Newton's forward interpolation formula and find the value of sin 52° from the following data. Estimate the error.

x	45°	50°	55°	60°
$y = \sin x$	0.7071	0.7660	0.8192	0.8660

Solution : The difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		–0.0057	
		0.0532		–0.0007
55°	0.8192		–0.0064	
		0.0468		
60°	0.8660			

$\therefore$ We have $x_0 = 45°$, $x_1 = 52°$, $y_0 = 0.7071$, $\Delta y_0 = 0.0589$, $\Delta^2 y_0 = -0.0057$ and $\Delta^3 y_0 = -0.0007$,

$$u = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4 .$$

From Newton's formula

$$y = u_o + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\therefore\ f(52) = 0.7071 + 1.4 \times 0.0589 + \frac{(1.4)(1.4-1)}{2} \times (-0.0057)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007)$$

$$= 0.7071 + 0.8246 - 0.001596 + 0.0000392$$

$$= 0.7880032$$

$$\therefore\ \sin 52^\circ = 0.7880032$$

$$\text{Error } = \frac{u(u-1)(u-2)\ldots(u-n)}{(n+1)!}\Delta^{n+1}y_0$$

taking $n = 2$ we get

$$\text{Error } = \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007) = 0.0000392 .$$

Exercise 4.1

1. Find the missing term in the following table

x	1	2	3	4	5	6	7
$f(x)$	2	4	8	–	32	64	128

2. Estimate the production of cotton in the year 1985 from the data given below

year (x)	1981	1982	1983	1984	1985	1986	1987
production (y)	17.1	13.0	14.0	9.6	–	12.4	18.2

3. Find the missing figure in the frequency table

x	15–19	20–24	25–29	30–34	35–39	40–44
f	7	21	35	?	57	58

4. Find the missing figures in the following table

x	0	5	10	15	20	25
y	7	11	–	18	–	32

5. Complete the table

x	2	3	4	5	6	7	8
$f(x)$	0.135	–	0.111	0.100	–	0.082	0.074

6. Find $f(1.1)$ from the table

x	1	2	3	4	5
$f(x)$	7	12	29	64	123

7. The following are data from the steam table

Temperature °C	140	150	160	17	180
Pressure kgt/cm2	3.685	4.84	6.302	8.076	10.225

Using the Newton's formula, find the pressure of the steam for a temperature of 142°C.

8. The area A of circle of diameter d is given for the following values

A	80	85	90	95	100
d	5026	5674	6362	708	7854

Find approximate values for the areas of circles of diameter 82 and 91 respectively.

9. Compute (1) $f(1.38)$ from the table

x	1.1	1.2	1.3	1.4
$f(x)$	7.831	8.728	9.627	10.744

10. Find the value of y when $x = 0.37$, using the given values

x	0.000	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.2255

11. Find the value of $\log_{10} 2.91$, using table given below

x	2.0	2.2	2.4	2.6	2.8	3.0
$y = \log_{10} x$	0.30103	0.34242	0.38021	0.41497	0.44716	0.47721

12. Find $f(2.8)$ from the following table

x	0	1	2	3
$f(x)$	1	2	11	34

13. Find the polynomial which takes on the following values

x	0	1	2	3	4	5
$f(x)$	41	43	47	53	61	71

14. Find a polynomial y which satisfies the following table

x	0	1	2	3	4	5
y	0	5	34	111	260	505

15. Given the following table find $f(x)$ and hence find $f(4.5)$

x	0	2	4	6	8
$f(x)$	-1	13	43	89	151

16. A second degree polynomial passes through (0,1) (1,3 (2, 7) (3, 13), find the polynomial.

17. Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	0	1	10

18. $u_0 = 560$, $u_1 = 556$, $u_2 = 520$, $u_4 = 385$ show that $u_3 = 465$.

19. In an examination the number of candidates who secured marks between certain limit were as follows :

Marks	0 – 19	20 – 39	40 – 59	60 – 79	80 – 99
No. of candidates	41	62	65	50	17

Estimate the number of candidates whose marks are less than 70.

20. Given the following score distribution of statistics

Marks	30 – 40	40 – 50	50 – 60	60 – 70
No of students	52	36	21	14

Find

(i) the number of students who secured below 35

(ii) the number of students who secured above 65

(iii) the number of students who secured between 35–45

Answers

1. 17 **2.** 6.60 **3.** 48 **4.** 23.5, 14.25 **5.** $f(3) = 0.123$, $f(6) = 0.090$ **6.** 7.13 **7.** 3.899 **8.** 5281, 6504 **9.** 10.963 **10.** 2.0959 **11.** 0.46389 **12.** 27.992 **13.** $x^2 + x + 41$ **14.** $4x^3 + x$ **15.** $2x^3 + 3x - 1$ **16.** $x^2 + x + 1$ **17.** $x^3 - 2x^2 + 1$ **19.** 197 **20.** (i) 26 (ii) 7 (iii) 46

→ The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x = height	100	150	200	250	300	350	400
y = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

find the values of y when

(i) $x = 218$ ft

(ii) $x = 410$ ft.

Sol: The difference table is:—

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		−0.39		
		2.01		0.15	
200	15.04		−0.24		−0.07
		1.77		0.08	
250	16.81		−0.16		−0.05
		1.61		0.03	
300	18.42		−0.13		−0.01
		1.48		0.02	
350	19.90		−0.11		
		1.37			
400	21.27				

(i) If we take $x_0 = 200$, then $y_0 = 15.04$,

$$\Delta y_0 = 1.77, \quad \Delta^2 y_0 = -0.16, \quad \Delta^3 = 0.03 \text{ etc.}$$

Since $x = 218$ and $h = 50$.

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{18}{50} = 0.36.$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{(0.36)(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

$$= 15.697$$

$$= 15.7 \text{ nautical miles.}$$

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,$$
$$\nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$f(410) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 21.27 + (0.2)(1.37) + \frac{(0.2)(1.2)}{2!}(-0.11) + \frac{(0.2)(1.2)(2.2)}{3!}(0.02) + \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$= 21.53$ nautical miles.

*Interpolation With unequal Intervals:

The various interpolation formulae derived so-far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae:

(i) Lagrange's Interpolation formula.

(ii) Newton's general Interpolation formula with divided differences.

* Lagrange's Interpolation Formula :-

Let $x_0, x_1, x_2, ---- x_n$ be $n+1$ distinct points on the real line and let $f(x)$ be a real valued function defined on some interval $I=[a,b]$ containing these points. Then, there exists exactly one polynomial $P_n(x)$ of degree $\le n$, which interpolates $f(x)$ at $x_0, x_1, ---- x_n$ that is $P_n(x_i) = f(x_i) = f_i$ ——— (A)

$i = 0, 1, 2, 3 ------ n.$

Proof:

First of all, we discuss the uniqueness of the interpolating polynomial.

Let $P_n(x)$ and $Q_n(x)$ be two distinct interpolating polynomials of degree $\le n$, which interpolate $f(x)$ at $(n+1)$ distinct points $x_0, x_1, x_2, --- x_n$.

and also satisfy $P_n(x_i) = f(x_i)$ and $Q_n(x_i) = f(x_i)$; $i = 0, 1, 2, --- n.$

Now let us consider the polynomial

$$h(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ are both polynomials of degree $\le n$. then $h(x)$ is also a polynomial of degree $\le n$ and satisfying the condition.

$$h(x_i) = P_n(x_i) - Q_n(x_i); \quad i = 1, 2 \dots n.$$

$$= f(x_i) - f(x_i) = 0.$$

$$\therefore h(x_i) = 0; \quad i = 0, 1, 2, \dots n$$

i.e. $h(x)$ has $(n+1)$ distinct zero's. But $h(x)$ is of degree $\le n$.

This implies that $h(x) = 0$.

because a polynomial $h(x)$ of degree 'n' has exactly 'n' roots real or complex.

$$\therefore P_n(x) = Q_n(x).$$

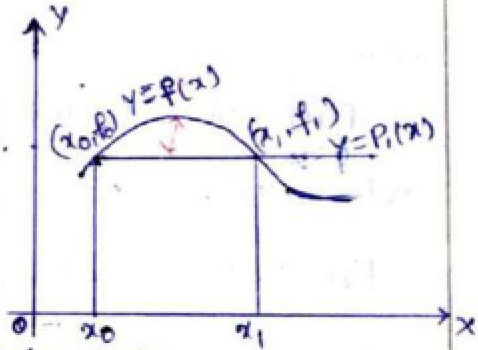
this shows that the uniqueness of the polynomial.

Let the data be given at the points

$(x_0, f_0), (x_1, f_1), (x_2, f_2), \dots (x_n, f_n)$.

* **Linear Interpolation:**

Before deriving the general formula, we first consider a simpler case namely, the equation of the straight line (a linear polynomial) passing through two points (x_0, f_0) and (x_1, f_1).



Such a polynomial say $P_1(x)$.

$$P_1(x) = \frac{x - x_1}{(x_0 - x_1)} f_0 + \frac{x - x_0}{x_1 - x_0} f_1.$$

$$\left[\because \underset{x_1}{(x_0}, \underset{y_1}{f_0)}, \underset{x_2}{(x_1}, \underset{y_2}{f_1)} \right.$$

$$\Rightarrow (y - f_0)(x_1 - x_0) = (f_1 - f_0)(x - x_0)$$

$$\Rightarrow y = \frac{x - x_1}{(x_0 - x_1)} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$\left. \Rightarrow P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \right].$$

$$= l_0(x) f_0 + l_1(x) f_1$$

$$\boxed{P_1(x) = \sum_{i=0}^{1} l_i(x) f_i} \quad \text{——— (1)}$$

where $$\left.\begin{aligned} l_0(x) &= \frac{x-x_1}{x_0-x_1} \\ l_1(x) &= \frac{x-x_0}{x_1-x_0} \end{aligned}\right\} \quad \text{—— } (2)$$

Putting $x = x_0$ in (2), we get

$$l_0(x_0) = 1$$
$$l_1(x_0) = 0$$

Putting $x = x_1$ in (2), we get

$$l_0(x_1) = 0$$
$$l_1(x_1) = 1$$

∴ the relations can be expressed in a more convenient form as

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{—— } (3)\ (A)$$

from (2), we have

$$l_0(x) + l_1(x) = \frac{x-x_1}{x_0-x_1} + \frac{x-x_0}{x_1-x_0}$$

$$= 1$$

$$\boxed{\sum_{i=0}^{1} l_i(x) = l_0(x) + l_1(x) = 1} \quad \text{—— } (4)\ (B)$$

Equation (1) is the Lagrange Polynomial of degree one passing through two points $(x_0, f_0), (x_1, f_1)$.

Similarly, the Lagrange polynomial of degree two passing through three points $(x_0, f_0), (x_1, f_1)$ & (x_2, f_2) is written as

$$P_2(x) = \sum_{i=0}^{2} l_i(x) f_i$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

—— (5)

where the $l_i(x)$ satisfy the conditions given in (3) & (4).

(or)

Linear Interpolation:

Let us consider first degree poly.

$P_1(x) = a_1 x + a_0$ —— ①

where a_0 & a_1 are arbitrary constants, which satisfies the interpolating conditions $f(x_0) = P_1(x_0)$ and $f(x_1) = P_1(x_1)$

$\therefore$ we have $\left.\begin{aligned} f(x_0) &= P_1(x_0) = a_1 x_0 + a_0 \\ f(x_1) &= P_1(x_1) = a_1 x_1 + a_0 \end{aligned}\right\}$ —— ②

Now eliminating a_0 & a_1 determinantly, b/w ② & ③ we obtain the required linear interpolating poly. as

$$\begin{vmatrix} P_1(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow P_1(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$\Rightarrow P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1)$$

where $l_0(x) = \frac{x - x_1}{x_0 - x_1}$, $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

The functions $l_0(x)$, and $l_1(x)$ are called the lagrange fundamental polynomials and it can be verified that they satisfy the conditions

$l_0(x) + l_1(x) = 1.$

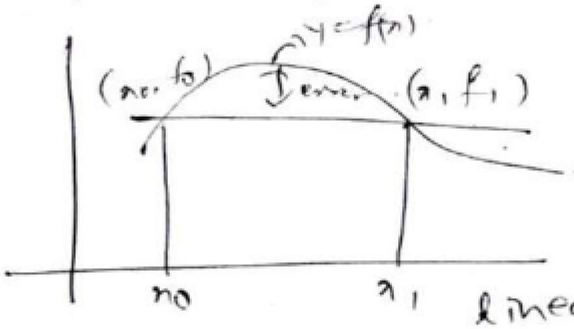
where $l_0(x_0) = 1$; $l_1(x_0) = 0$ if $x = x_0$.

$l_0(x_1) = 0$ $l_1(x_1) = 1$ if $x = x_1$.

(or) $l_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$

The degree of a polynomials $l_0(x)$ & $l_1(x)$ is one.

which can also see in graphically,



linear interpolation

24(i)

Quadratic interpolation:-

Let us consider 2nd degree poly.

$$P_2(x) = a_0 + a_1 x + a_2 x^2. \quad \text{---} (7)$$

where a_0, a_1 and a_2 are arbitrary constants, which satisfies the interpolating conditions

$P_2(x_0) = f(x_0)$, $P_2(x_1) = f(x_1)$, $P_2(x_2) = f(x_2)$.

$\therefore$ we have from (7),

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 \end{aligned}\right\} \text{---} (ii).$$

Eliminating a_0, a_1 & a_2 ~~from~~ determinantally from (7) & (ii), we obtain the quadratic interpolating polynomial as:

$$\begin{vmatrix} P_2(x) & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{vmatrix} = 0.$$

$$\Rightarrow P_2(x) D_0 - f(x_0) D_1 + f(x_1) D_2 - f(x_2) D_3 = 0.$$

where $D_0 = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_0 - x_1)(x_1 - x_2)(x_2 - x_0)$

$$D_1 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x - x_1)(x_1 - x_2)(x_2 - x)$$

$$D_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x - x_0)(x_0 - x_2)(x_2 - x)$$

$$D_3 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \end{vmatrix} = (x - x_0)(x_0 - x_1)(x_1 - x)$$

$$\therefore \quad P_2(x) = \frac{D_1}{D_0} f(x_0) + \frac{D_2}{D_0} f(x_1) + \frac{D_3}{D_0} f(x_2).$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2). \qquad \text{—— (5)}$$

* **To derive the general formula:**

Let $P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ —— (6).

be the required polynomial of the n^{th} degree such that conditions (A) (called the interpolatory conditions) are satisfied.

Substituting these Conditions in ⑥ 24(ii)

we get the equations

$$\left.\begin{aligned} f_0 &= a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n \\ f_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n \\ f_2 &= a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_n x_2^n \\ &\cdots \\ f_n &= a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n \end{aligned}\right\} \quad \text{---} \quad (7)$$

The set of equations ⑦ will have a solution if

$$\begin{vmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \cdots & & & & & \\ \cdots & & & & & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{vmatrix} \neq 0$$

The value of this determinant, called Vandermonde's determinant, is $(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)(x_1 - x_2) \cdots (x_1 - x_n)$
$(x_2 - x_3)(x_2 - x_4) \cdots (x_2 - x_n) \cdots (x_{n-1} - x_n)$.

Eliminating $a_0, a_1, a_2 \cdots a_n$ from equations ⑥ and ⑦, we obtain

$$\begin{vmatrix} P_n(x) & 1 & x & x^2 & \cdots & x^n \\ f_0 & 1 & x_0 & x_0^2 & \cdots & x_0^n \\ f_1 & 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdots & & & & & \\ \cdots & & & & & \\ f_n & 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}_{(n+2)\times(n+2)} = 0 \quad \text{---} \quad (8)$$

which shows that $P_n(x)$ is a linear combination of $f_0, f_1, f_2 \cdots f_n$.

Hence write $P_n(x) = \sum_{i=0}^{n} l_i(x) f_i$ ——— ⑨

where $l_i(x)$ are polynomials in x of degree 'n'.

since $P_n(x_j) = f_j$ for $j = 0, 1, 2 \ldots n$.

the equation (4)(7) gives

$$l_i(x_j) = 0 \text{ if } i \neq j$$
$$l_j(x_j) = 1 \text{ for all } i = j.$$

which are the same as (3)(A) Hence $l_i(x)$ may be written as

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \quad \text{---} (10)$$

which obviously satisfies the condition (3)(A)

If we now set

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_i)(x-x_{i+1})\cdots(x-x_n) \quad \text{---} (11)$$

then $\pi'_{n+1}(x_i) = \frac{d}{dx}\left[\pi_{n+1}(x)\right]_{x=x_i}$

$$= (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n) \quad \text{---} (12)$$

so that (10) becomes

$$l_i(x) = \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} \quad \text{---} (13)$$

Hence (9) gives

$$P_n(x) = \sum_{i=0}^{n} \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} f_i \quad \text{---} (14)$$

which is called Lagrange's interpolation formula.

— The coefficients $l_i(x)$, defined in (10) are called Lagranges Interpolation Coefficients.

*Inverse Interpolation Formula :

In inverse interpolation, in a table of values of x and $y = f(x)$ one is given a number $\bar{y}$ and wishes to find the point $\bar{x}$ so that $f(\bar{x}) = \bar{y}$., where $f(x)$ is the tabulated function This problem can always be solved if $f(x)$ is continuous and strictly increasing (or) decreasing (i.e. the inverse of 'f' exists). This is done by considering the table of values $x_i, f(x_i)$; $i=0,1,2,3,\ldots n$ to be a table of values $y_i, g(y_i)$, $i=0,1,2\ldots n$ for the inverse function $g(y)=f^{-1}(y)=x$ by taking $y_i=f(x_i), g(y_i)=x_i$, $i=0,1,2\ldots n$. Then we can interpolate for the unknown value $g(\bar{y})$ in this table.

$$P_n(y)=\sum_{i=1}^{n}\frac{\pi_{n+1}(y)}{(y-y_i)\pi'_{n+1}(y_i)}x_i \quad \text{—— (15)}$$

and $\bar{x}\approx P_n(\bar{y})$.

This process is called Inverse Interpolation.

Note (1): The Lagrange form [equation (9)] of interpolating polynomial makes it easy to show the existence of an interpolating polynomial. But its evaluation at a point x_i involves a lot of computation.

Note (2): Moreover the lagrange form of interpolating polynomial can be determined for equally spaced or un-equally spaced nodes.

Note (3): The lagrangian coefficients in (13) can conventially be computed in pratice by the following scheme. We first compute the differences, row wise, as given below.

$$\begin{array}{ccccc}
x-x_0 & x_0-x_1 & x_0-x_2 & \cdots\cdots & x_0-x_n\\
x_1-x_0 & x-x_1 & x_1-x_2 & \cdots\cdots & x_1-x_n\\
x_2-x_0 & x_2-x_1 & x-x_2 & \cdots\cdots & x_2-x_n\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
x_n-x_0 & x_n-x_1 & x_n-x_2 & \cdots\cdots & x-x_n.
\end{array}$$

we note that the product of the elts along the diagonal line is $\Pi_{n+1}(x)$.

Similarly, the product of the elements of the first row $(x-x_0)\Pi'_{n+1}(x_0)$, of the second row is $(x-x_1)\Pi'_{n+1}(x_1)$ and of the $(n+1)^{th}$ row is $(x-x_n)\Pi'_{n+1}(x_n)$.

The Lagrangian coefficients can then be computed by using formula (13).

— The Lagrange's interpolation formula (14) can be written as

$$P_n(x) = \sum_{i=0}^{n} \frac{\Pi_{n+1}(x)}{(x-x_i)\,\Pi'_{n+1}(x_i)} f_i$$

$$= \frac{\Pi_{n+1}(x)}{(x-x_0)\,\Pi'_{n+1}(x_0)} f_0 + \frac{\Pi_{n+1}(x)}{(x-x_1)\,\Pi'_{n+1}(x_1)} + \cdots\cdots + \frac{\Pi_{n+1}(x)}{(x-x_n)\,\Pi'_{n+1}(x_n)} f_n.$$

$$\Rightarrow P_n(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} f_0 + \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} f_1 + \cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})} f_n.$$

$\longrightarrow$ (16).

— The Lagrange's inverse ~~inverse~~ interpolation formula (15) can be written as

$$x = P_n(f) = \frac{(f-f_1)(f-f_2)\cdots\cdots(f-f_n)}{(f_0-f_1)(f_0-f_2)\cdots\cdots(f_0-f_n)} x_0 + \frac{(f-f_0)(f-f_2)\cdots(f-f_n)}{(f_1-f_0)(f_1-f_2)\cdots(f_1-f_n)} x_1 + \cdots\cdots + \frac{(f-f_0)(f-f_1)\cdots\cdots(f-f_{n-1})}{(f_n-f_0)(f_n-f_1)\cdots\cdots(f_n-f_{n-1})} x_n.$$

$\longrightarrow$ (17)

problems

→ Certain corresponding values of x and $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ and $(307, 2.4871)$ find $\log_{10} 301$

Sol We have
$x_0 = 300,\ x_1 = 304,\ x_2 = 305,\ x_3 = 307$
and $f_0 = 2.4771,\ f_1 = 2.4829,\ f_2 = 2.4843.$
$f_3 = 2.4871.$

Now the Lagrange's interpolation formula with 4 points is

$$f(x) = P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_4$$

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}(2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}(2.4829)$$
$$+ \frac{(1)(-3)(-6)}{(5)(1)(-2)}(2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)}(2.4871)$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106$$

$$= \underline{2.4786}.$$

→ ② If $f_1 = 4$. $f_3 = 12$, $f_4 = 19$ and $f_x = 7$ find x. by using Lagrange's inverse interpolation formula.

Sol We have $f_1 = 4$, $f_3 = 12$, $f_4 = 19$
$f_x = 7$ as

i.e the given points are

$(1, 4)$, $(3, 12)$, $(4, 19)$ and $(x, 7)$
$\;\;x_0\; y_0 \quad x_1\; y_1 \quad x_2\; y_2 \quad x\; y$

now we find x by using the Lagrange's inverse interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$\Rightarrow x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3)$$

$$+ \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

$$= \frac{(-5)(-12)}{(-8)(-15)}(1) + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7}$$

$$= \underline{\underline{1.86}}$$

→ The function $y = \sin x$ is obtained below

x	0	$\pi/4$	$\pi/2$
$y = \sin x$	0	0.70711	1.0

Using Lagrange's interpolation formula, find the value of $\sin(\pi/6)$.

Sol we have

$$\sin \pi/6 = \frac{\left(\frac{\pi}{6}-0\right)\left(\pi/6-\pi/2\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\pi/2\right)}(0.70711) + \frac{\left(\frac{\pi}{6}-0\right)\left(\pi/6-\pi/4\right)}{\left(\pi/2-0\right)\left(\pi/2-\pi/4\right)}(1)$$

$$= \frac{8}{9}(0.70711) - \frac{1}{9}$$

$$= \underline{\underline{0.51743.}}$$

→ Using Lagrange's interpolation formula, find the form of the function $y(x)$ from the following table

x	0	1	3	4
y	-12	0	12	24

Sol since $y=0$ when $x=1$

$\therefore (x-1)$ is a factor

Let $y(x) = (x-1)R(x)$ —— ①

then $R(x) = \frac{y(x)}{x-1}$

we now tabulate the values of x, $R(x)$

x	0	3	4
$R(x)$	12	6	8

Applying Lagrange's formula to the above table, we find

$$R(x) = \frac{(x-3)(x-4)}{(-3)(-4)}(12) + \frac{(x-0)(x-4)}{(3-0)(3-4)}(6)$$

$$+ \frac{(x-0)(x-3)}{(4-0)(4-3)}(8)$$

$$= (x-3)(x-4) - 2x(x-4) + 2x(x-3)$$

$$= x^2 - 5x + 12.$$

Hence the reqd polynomial approximation to $y(x)$ is given by

$$\boxed{\therefore y(x) = (x-1)(x^2-5x+12).}$$

2002 10M → Find the cubic polynomial which takes the following values: $y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$. Hence, or otherwise, obtain $y(4)$.

2005 15M → Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1) = 1$, $P(3) = 27$, $P(4) = 64$. Using the Lagrange's interpolation formula.

→ If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$ and $y(6) = 132$, find the four-point Lagrange interpolation polynomial that takes the same values as the function y at the given points.

→ Given table of values

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

Evaluate $\sqrt{155}$ using Lagrange's interpolation formula.

→ Show that $l_0(x) + l_1(x) + l_2(x) + l_3(x) = 1$ for all x.

→ Applying Lagrange's formula, find a cubic polynomial which approximates the following data

x	-2	-1	2	3
$y(x)$	-12	-8	3	5

Note: Lagrange's interpolation formula can be used to express a rational function as a sum of partial fractions in the following way.

Let the given rational function be

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}.$$

We consider the numerator $f(x) = 3x^2+x+1$ and tabulate its values for $x = 1, 2, 3$. We get

x	1	2	3
$f(x) = 3x^2+x+1$	5	15	31

Using Lagrange's interpolation formula, we get

$$f(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

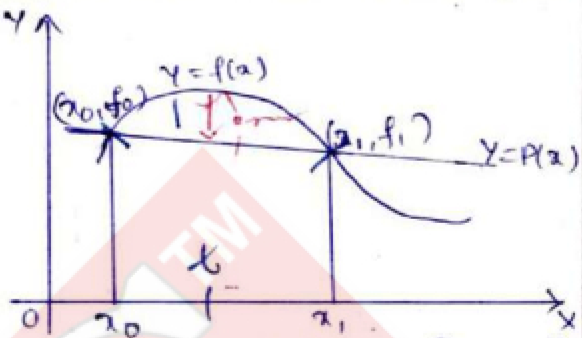
Hence $$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}$$

→ Using Lagrange's interpolation formula, express the function $\frac{x^2+x-3}{x^3-2x^2-x+2}$ as sums of partial fractions.

→ Express the functions $\frac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sums of partial fractions.

* Truncation Error Bounds:

The polynomial $P(x)$ coincides with the $f(x)$ at x_0 and x_1, and it deviates at all other points, in the interval (x_0, x_1) as shown in the figure.



This deviation is called the truncation error and may be written as $E_1(f; x) = f(x) - P(x)$ ——— ①

we will now derive an expression for $E_1(f; x)$ for $x \in [x_0, x_1]$

we use the result ROLLE Theorem.

If $g(x)$ is continuous function $[a, b]$ and differentiable on (a, b) and if $g(a) = g(b) = 0$.

then there exist at least one point $\xi \in (a, b)$ such that $g'(\xi) = 0$.

— If $x = x_0$ (or) $x = x_1$ then $E_1(f; x) = 0$

— If $x \in [a, b]$, $x \neq x_0, x_1$ be fixed then for this x, we define a function $g(t)$ as

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t - x_0)(t - x_1)}{(x - x_0)(x - x_1)} \quad \text{——— ②}$$

clearly $g(t) = 0$ at the three distinct points $t = x_0$, $t = x_1$ and $t = x$.

Differentiating ② twice with respect to 't', we get

$$g''(t) = f''(t) - \frac{2(f(x) - P(x))}{(x - x_0)(x - x_1)} \quad \text{——— ③}$$

Now by Rolle's Theorem,

$$g''(\xi) = 0$$

solving ③ for $f(x)$, we get

$$f(x) = P(x) + \tfrac{1}{2}(x - x_0)(x - x_1) f''(\xi) \quad \text{——— ④}$$

where $\text{Min}(x_0, x_1, x) < \xi < \text{Max}(x_0, x_1, x)$.

$\Rightarrow f(x) - P(x) = \frac{1}{2}(x-x_0)(x-x_1) f''(\xi)$

$$\boxed{E_1(f;x) = \tfrac{1}{2}(x-x_0)(x-x_1) f''(\xi_1)} \quad \text{—— (5)}$$

which is known as the truncation error in linear Interpolation.

↳ first degree poly

If we determine a bound for $f''(x)$ in $[x_0, x_1]$

i.e. $|f''(x)| < M_2 \quad x \in [x_0, x_1]$

then $|f(x) - P(x)| = \left|\frac{1}{2}(x-x_0)(x-x_1) f''(\xi)\right|$

Let $w(x) = (x-x_0)(x-x_1)$
setting $w'(x) = 0$.
we obtain the critical point of $w(x)$ at $x = \frac{x_0+x_1}{2}$

$$\le \frac{1}{2} \max_{x_0 \le \xi \le x_1} |(x-x_0)(x-x_1) f''(\xi_1)|$$

$$\le \frac{1}{2} \max_{x_0 \le \xi \le x_1} |(x-x_0)(x-x_1)| M_2 \quad \text{—— (6)}$$

Hence The maximum value of $|(x-x_0)(x-x_1)|$ occures at $x = \frac{1}{2}(x_0+x_1)$

and (6) becomes

$$\boxed{|f(x) - P(x)| \le \tfrac{1}{8}(x_1 - x_0)^2 M_2} \quad \text{—— (7)}$$

Further the equation (7) may also be used to construct a table of values for a function $f(x)$ ~~for a function f(x)~~ for equally spaced nodal points $x_i = a + ih$, $i = 0, 1, 2 \ldots n$

$h = \frac{b-a}{n}$;

so that the maximum truncation error using the linear interpolating polynomial $P(x)$ is less than given $\epsilon > 0$, we have

$$\frac{h^2}{8} \max_{a \le x \le b} |f''(x)| \le \epsilon.$$

→ Using $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.19687$, find an approximate value of $\sin(0.15)$ by Lagrange interpolation. Obtain a bound on the truncation error.

Sol^n: we have

$$P_1(0.15) = \frac{(0.15-0.2)}{(0.1-0.2)}(0.09983) + \frac{(0.15-0.1)}{(0.2-0.1)}(0.19867)$$

$$= (0.5)(0.09983) + (0.5)(0.19867)$$

$$= 0.14925$$

The truncation error is

$$E_1(f;x) = \frac{(x-0.1)(x-0.2)}{2}(-\sin\xi) \qquad \left[\begin{array}{l} \text{Let } f(x) = \sin x \\ f''(x) = -\sin x \end{array}\right]$$

where $0.1 < \xi < 0.2$

The maximum value of $|-\sin\xi|$, $\xi \in [0.1, 0.2]$ is $\sin(0.2) = 0.19867$

$$\therefore |E_1(f;x)| \le \left|\frac{(0.15-0.1)(0.15-0.2)}{2}\right| (0.19867)$$

$$= (0.19867)(0.00125) \qquad f''(\xi) = |-\sin\xi| = 0.19867,\ \xi \in [0.1, 0.2]$$

$$\approx 0.00025$$

→ Determine the stepsize h to be used in tabulation of $f(x) = \sin x$ in the interval $[1,3]$ so that the linear interpolation will be correct to four decimal places.

<u>sol</u>^{n}: $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow f''(x) = -\sin x$

θ: $0°$ $30°$ $45°$ $60°$ $90°$ $120°$ etc

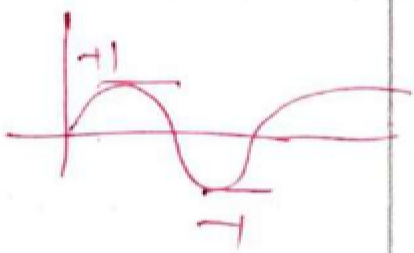
$f(\theta)$: 0 $\frac{1}{2}$ $\frac{1}{\sqrt{2}}$ $\frac{\sqrt{3}}{2}$ 1 $\frac{\sqrt{3}}{2}$ etc

Also $\max\limits_{1 \le x \le 3} |-\sin x| = 1$

$\pi/2 = \frac{3.14}{2} = 1.57$

Hence we obtain

$$\frac{h^2}{8}(1) \le 5 \times 10^{-5}$$



This gives

$$h \le 0.02$$

* **Truncation error for Lagrange interpolating polynomial of degree n:-**

we know that the Lagrange interpolating polynomial of degree n is

$$P(x) = \sum_{i=0}^{n} \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} \quad \text{———} (1)$$

where $\Pi_{n+1}(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$

the truncation error in the lagrange interpolation is given by

$E_n(f;x) = f(x) - P(x)$

since $E_n(f;x) = 0$ at $x = x_i$, $i = 0, 1, 2 \ldots n$, then for $x \in [a,b]$ and $x \neq x_i$,

we define a function $g(t)$ as

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1) \cdots (t-x_n)}{(x-x_0)(x-x_1) \cdots (x-x_n)} \quad \text{--- (2)}$$

clearly $g(t) = 0$ at $t = x$ and $t = x_i$, $i = 0, 1, 2 \ldots n$

Differentiating (2) $n+1$ times with respect to t, we get

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{(n+1)!\,[f(x) - P(x)]}{(x-x_0)(x-x_1) \cdots (x-x_n)} \quad \text{--- (3)}$$

By using the Rolle's theorem, $g^{(n+1)}(\xi) = 0$

solving (3) for $f(x)$, we get

$$f(x) = P(x) + \frac{\Pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\xi)$$

where 'ξ' is some point in $[x_0, x_1, x_2 \ldots x_n, x]$.

Hence the truncation error in Lagrange's interpolation is

$$E_n(f;x) = \frac{\Pi_{n+1}(x)}{(n+1)!} f^{n+1}(\xi) \quad \text{--- (4)}$$

Note: the error formula (equation (4)) is an important theoretical result because lagrange interpolating polynomials are extensively used in deriving important formulae for numerical differentiation and numerical integration.

→ the following table gives the values of $f(x) = e^x$. If we fit an interpolating polynomial of degree four to the data, find the magnitude of the maximum possible error in the Computed value of $f(x)$ when $x = 1.25$

x	1.2	1.3	1.4	1.5	1.6
$f(x)$	3.3201	3.6692	4.0552	4.4817	4.9530

Sol'n: The magnitude of the error associated with the 4th degree polynomial approximation is given by

$$E_4(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\frac{f^5(\xi)}{5!}$$

$$|E_4(x)| = \left|(x-1.2)(x-1.3)(x-1.4)(x-1.5)(x-1.6)\frac{f^5(\xi)}{5!}\right| \quad \text{—— (1)}$$

Since $f(x) = e^x \Rightarrow f^{(5)}(x) = e^x$

$$\max_{1.2\le x\le 1.6} |f^{(5)}(x)| = e^{1.6} = 4.9530 \quad \text{—— (2)}$$

Substituting (2) in (1) and putting $x = 1.25$, the upperbound on the magnitude of the error

$$= \left|(0.05)(-0.05)(-0.15)(-0.25)(-0.35)\right|\left(\frac{4.9530}{120}\right)$$

$$= 0.00000135.$$

→ Find the Lagrange interpolating polynomial of degree 2 approximating the function $y = \log_e x$ defined by the following table of values. Hence determine the value of 2.7 and also obtain a bound on the error.

x	2	2.5	3.0
$y = \log_e x$	0.69315	0.91629	1.09861

Sol'n: Using the Lagrange's interpolation formulae we have

$$P_2(x) = \frac{(x-2.5)(x-3)}{(-0.5)(-1.0)}(0.69315) + \frac{(x-2)(x-3)}{(0.5)(-0.5)}(0.91629) + \frac{(x-2)(x-2.5)}{(1)(0.5)}$$

$$= (2x^2-11x+15)(0.69315) - (4x^2-20x+14)(0.91629) + (2x^2-9x+10)(1.09861)$$

$$= -0.08164x^2 + 0.81366x - 0.60761 \quad \text{——} \; (1)$$

which is the required quadratic polynomial

putting $x = 2.7$ in the equation (1)

$$\log_e(2.7) = 0.9941164.$$

Since $y = \log_e x$

$$\Rightarrow y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2} \text{ and } y''' = \frac{2}{x^3}$$

The magnitude of the error associated with 2nd degree polynomial approximation is given by

$$|E_2(x)| = \left| (x-x_0)(x-x_1)(x-x_2) \frac{f^{(3)}(\xi)}{3!} \right|$$

$$|E_2(x)| = \left| (2.7-2)(2.7-2.5)(2.7-3.0) \frac{f^{(3)}(\xi)}{3!} \right| \quad \text{——} \; (2)$$

$$\max_{2 \le x \le 3} |f^{(3)}(x)| = \max_{2 \le x \le 3} \left| \frac{2}{x^3} \right|$$

$$= \frac{1}{4} \quad \text{——} \; (3)$$

Substituting (3) in (2),

the magnitude of the error

$$|E_2(x)| \le \left| (0.7)(0.2)(0.3) \frac{1}{4} \times \frac{1}{6} \right|$$

$$\approx 0.00175$$

<u>Note</u>: Actual value $\log_e 2.7 = 0.9932518$

sothat $|\text{Error}| = |0.9941164 - 0.9932518|$

$= 0.0008646$ (Actual Error)

and we have $|E_2(x)| \le 0.00175$ (truncation error)

which agrees with the actual Error.

Example : Using Lagrange's interpolation formula find a polynomial which passes the points (0, −12), (1, 0), (3, 6), (4, 12).

Solution : We have

$x_0 = 0$ $x_1 = 1$, $x_2 = 3$, $x_3 = 4$, $y_0 = f(x_0) = -12$, $y_1 = f(x_1) = 0$, $y_2 = f(x_2) = 6$, $y_3 = f(x_3) = 12$.

Using Lagrange's interpolation formula we can write

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)\ldots(x_3-x_2)} f(x_3).$$

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \times (-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \times 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times (6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times (12)$$

$$= \frac{(x^3-8x^2+19x-12)}{12} \times 12 + \frac{(x^3-5x^2+4x)}{(-6)} \times (6) + \frac{(x^3-4x^2+3x)}{(12)} \times 12$$

$\therefore f(x) = x^3 - 7x^2 + 18x - 12$

is the required polynomial.

Example : Using Lagrange's interpolation formula, find the value of y corresponding to $x = 10$ from the following table

x	5	6	9	11
$y = f(x)$	12	13	14	16

Solution : We have

$x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $y_0 = 5$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

Using Lagrange's Interpolation formula we can write

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3.$$

Substituting we get

$$f(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times(13)$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times(16)$$

$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=\frac{42}{3}.$$

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Exercise

1. Find by Lagrange's formula the interpolation polynomial which corresponds to the following data

x	-1	0	2	5
$f(x)$	9	5	3	15

2. Find the polynomial degree three relevant to the following data

x	-1	0	1	2
$f(x)$	1	1	1	-3

3. Find the polynomial of the least degree which attains the prescribed values at the given points

x	-2	1	2	4
$f(x)$	25	-8	-15	-25

4. Compute $f(0.4)$ using the table

x	0.3	0.5	0.6
$f(x)$	0.61	0.69	0.72

5. The following table gives the sales of a concern for the five years. Estimate the sales for the years (a) 1986 (b) 1992

Year	1985	1987	1989	1991	1993
Sales	40	43	48	52	57

(in thousands)

6. Compute sin39° form the table

x°	0	10	20	30	40
$\sin x^\circ$	0	1.1736	0.3420	0.5000	0.6428

7. Find log 5.15 from the table

x	5.1	5.2	5.3	5.4	5.5
$\log_{10}x$	0.7076	0.7160	0.7243	0.7324	0.7404

8. Use Lagrange's interpolation formula and find $f(0.35)$

x	0.0	0.1	0.2	0.3	0.4
$f(x)$	1.0000	1.1052	1.2214	1.3499	1.4918

9. Use Lagrange's formula and compute

9. Use Lagrange's formula and compute

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

10. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x = 0$ given the following table

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

Answers

1. $x^2 - 3x + 5$ **2.** $-x^3 + x + 1$ **3.** $x^2 - 10x + 1$ **4.** 0.65 **5.** (a) 41.02 (b) 54.46
6. 0.6293 **7.** 0.7118 **8.** 1.4191 **9.** 1.6751 **10.** 1

Example : The following table gives the value of the elliptical integral

$$F(\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}$$

for certain values of ϕ. Find the values of ϕ if $F(\phi) = 0.3887$

ϕ	21°	23°	25°
$F(f)$	0.3706	0.4068	0.4433

Solution : We have

$\phi = 21°$, $\phi_1 = 23°$, ϕ_2, $25°$, $F = 0.3887$, $F_0 = 0.3706$, $F_1 = 0.4068$ and $F_2 = 0.4433$.

Using the inverse interpolation formula we can write

$$\phi=\frac{(F-F_1)(F-F_2)}{(F_0-F_1)(F_0-F_2)}\phi_0+\frac{(F-F_0)(F-F_2)}{(F_1-F_0)(F_1-F_2)}\phi_1+\frac{(F-F_0)(F-F_1)}{(F_2-F_0)(F_2-F_2)}\phi_2,$$

$$\Rightarrow \phi=\frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}\times 21$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}\times 23$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)}\times 25$$

$$= 7.884 + 17.20 - 3.087$$

$$= 21.999\ 22$$

$$\therefore \phi = 22°.$$

Example : Find the value of x when $y = 0.3$ by applying Lagrange's formula inversely

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897

Solution : From Lagrange's inverse interpolation formula we get

$$x=\frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0+\frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2.$$

Substituting $x_0 = 0.4$, $x_1 = 0.6$, $x_2 = 0.8$, $y_0 = 0.3683$, $y_1 = 0.3332$, $y_2 = 0.2899$ in the above formula, we get

$$x = \frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)} \times (0.4)$$
$$+ \frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)} \times (0.6)$$
$$+ \frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)} \times (0.8)$$
$$= 0.757358.$$

Example : The following table gives the values of the probability integral $y = \frac{2}{\sqrt{\pi}}\int_0^n e^{-x^2}dx$ corresponding to certain values of x. For what value of x is this integral equal to $\frac{1}{2}$?

x	0.46	0.47	0.48	0.49
$y = \frac{2}{\sqrt{\pi}}\int_0^n e^{-x^2}dx$	0.484655	0.4937452	0.5027498	0.5116683

Solution : Here $x_0 = 0.46$, $x_1 = 0.47$, $x_2 = 0.48$, $x_3 = 0.48$, $y_0 = 0.484655$, $y_1 = 0.4937452$, $y_2 = 0.5027498$, $y_3 = 0.5116683$ and $y = \frac{1}{2}$.

From Lagrange's inverse interpolation formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3.$$

$$\therefore x = \frac{(0.5-0.4937452)(0.5-0.5027498)(0.5-0.5116683)}{(0.4846555-0.4937452)(0.484655-0.5027498)(0.4846555-0.5116683)} \times 0.46$$
$$+ \frac{(0.5-0.4846555)(0.5-0.5027498)(0.5-0.5116683)}{(0.4937452-0.4846555)(0.4937452-0.5027498)(0.4937452-0.5116683)} \times 0.47$$
$$+ \frac{(0.5-0.4846555)(0.5-0.4937452)(0.5-0.5116683)}{(0.5027498-0.484655)(0.5027498-0.4937452)(0.5027498-0.5116683)} \times 0.48$$
$$+ \frac{(0.5-0.484655)(0.5-0.4937452)(0.5-0.5027498)}{(0.5116683-0.484655)(0.5116683-0.4937452)(0.5-0.5027498)} \times 0.49$$
$$= -0.0207787 + 0.157737 + 0.369928 - 0.0299495 = 0.476937.$$

Example : Show that Lagrange's interpolation formula can be written in the form

$$f(x)=\sum_{r=0}^{r=n}\frac{\phi(x_r)}{(x-x_r)\phi^1(x_r)}$$

where $\phi(x)=(x-x_0)(x-x_1)\dots(x-x_n)$

and $\phi^1(x_r)=\frac{d}{dx}[\phi(x)]$ at $x=x_r$

Solution : we have

$$\phi(x)=(x-x_0)(x-x_1)\dots(x-x_n)$$

$$\phi'(x)=(x-x_1)(x-x_2)\dots(x-x_n)$$
$$+(x-x_0)(x-x_2)\dots(x-x_n)$$
$$+\dots+(x-x_0)(x-x_1)\dots(x-x_{r-1})(x-x_{r+1})\dots(x-x_n)$$
$$+\dots+(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$\therefore\ \phi'(x_r)=(x_r-x_0)(x_r-x_1)\dots(x_r-x_{r-1})(x_r-x_{r+1})\dots(x_r-x_n)$$

$$\therefore\ f(x)=\sum_{r=0}^{r=n}\frac{\phi(x)}{(x-x_r)\phi^1(x_r)}f(x_r)$$

Example : By means of lagrange's formula prove that

(i) $y_1=y_3-0.3[y_5-y_{-3}]+0.2[y_{-3}-y_{-5}]$

(ii) $y_o=\frac{1}{2}(y_1+y_{-1})-\frac{1}{8}\left[\frac{1}{2}(y_3-y_1)-\frac{1}{2}(y_{-1}-y_{-3})\right]$

Solution : (i) y_{-5}, y_{-3}, y_3 and y_5 are given, therefore the values of the arguments are -5, -3, 3, and 5, y_1 is to be obtained. By lagrange's formula

$$y_x=\frac{[x-(-3)](x-3)(x-5)}{[-5-(-3)](-5-3)(-5-5)}y_{-5}+\frac{[x-(-5)](x-3)(x-5)}{[-3-(-5)](-3-3)(-3-5)}y_{-3}$$
$$+\frac{[x-(-5)][x-(-3)](x-5)}{[3-(-5)][3-(-3)](3-5)}y_3+\frac{[x-(-5)][x-(-3)](x-3)}{[5-(-5)][5-(-3)](5-3)}y_5$$

Taking $x=1$, we get

$$y_1=\frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)}y_{-5}+\frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)}y_{-3}$$
$$+\frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)}y_3+\frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)}y_5$$

$$\Rightarrow y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_5 + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= \frac{y_{-5}}{5} + \frac{y_{-3}}{2} + y_3 - \frac{3}{10} y_5$$

$$= y_3 + 0.2y_{-5} + 0.5y_{-3} - 0.3y_5$$

$$= y_3 + 0.2y_{-5} + 0.2y_{-3} + 0.3y_{-3} - 0.3y_5$$

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

(ii) y_{-3}, y_{-1}, y_1, and y_3 are given, y_0 is to be obtained. By Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_3 + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(0+3)(0+1)(0-1)}{(3+3)(3+1)(3-1)} y_3$$

$$= -\frac{1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}\left[(y_3 - y_1) - (y_{-1} - y_{-3})\right]$$

$$\therefore y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Example : The values of $f(x)$ are given at a, b, and c. Show that the maximum is obtained by

$$\frac{f(a)\cdot(b^2 - c^2) + f(b)\cdot(c^2 - a^2) + f(c)\cdot(a^2 - b^2)}{2[f(a)\cdot(b-c) + f(b)\cdot(c-a) + f(c)\cdot(a-b)]}$$

Solution : By lagrange's formula $f(x)$ for the arguments a, b, and c is given by

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b)$$

$$+ \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

$$= \frac{x^2 - (b+c)x + bc}{(a-b)(a-c)} f(a) + \frac{x^2 - (c+a)x + ca}{(b-c)(b-a)} f(b) + \frac{x^2 - (a+b)x + ab}{(c-a)(c-b)} f(c)$$

for maximum or minimum we have $f'(x)=0$

$$\therefore -\frac{2x-(b+c)}{(a-b)(c-a)}f(a)-\frac{2x-(a+c)}{(a-b)(b-c)}f(b)-\frac{2x-(a+b)}{(b-c)(c-a)}f(c)=0$$

$$\Rightarrow 2x\left[(b-c)f(a)+(c-a)f(b)+(a-b)f(c)\right]$$

$$-\left[\left(b^2-c^2\right)f(a)+\left(c^2-a^2\right)f(b)+\left(a^2-b^2\right)f(c)\right]=0$$

$$\therefore x=\frac{\left(b^2-c^2\right)f(a)+\left(c^2-a^2\right)f(b)+\left(a^2-b^2\right)f(c)}{2\left[(b-c)f(a)+(c-a)f(b)+(a-b)f(c)\right]}$$

Example : Given $\log_{10}654 = 2.8156$, $\log_{10}658 = 2.8182$, $\log_{10}659 = 2.8189$, $\log_{10}661 = 2.8202$, find $\log_{10}656$.

Solution : Here $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$ and $f(x) = \log_{10}656$.

By Lagrange's formula we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\therefore \log_{10}656=\frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}\times(2.8156)$$

$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}\times(2.8182)$$

$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}\times(2.8189)$$

$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}\times(\;\;8202)$$

$$=\frac{3}{14}(2.8156)+\frac{5}{2}(2.8182)-2(2.8189)+\frac{2}{7}(2.8202)$$

$$=0.6033+7.045-5.6378+0.8057$$

$$=2.8170\,.$$

Example : Write down the Lagrange's polynomial passing through (x_0, f_0), (x_1, f_1) and (x_2, f_2). Hence express $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.

Solution : The Lagrange's polynomial through the points (x_0, f_0), (x_1, f_1) and (x_2, f_2) is given by

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\times f_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\times f_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\times f_2$$

Consider the numerator $3x^2+x+1$.

Let

$$f(x) = 3x^2+x+1,$$

tabulating the values of $f(x)$ for $x = 1, 2, 3$ we get

x	1	2	3
$f(x)$	5	15	31

Using Lagrange's formula, we get

$$f(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}\times 5+\frac{(x-1)(x-3)}{(2-1)(2-3)}\times 15+\frac{(x-1)(x-2)}{(3-1)(3-2)}\times 31$$

$$=\frac{5}{2}(x-2)(x-3)-15(x-1)(x-3)+\frac{31}{2}(x-1)(x-2)$$

$$\therefore\ \frac{3x^2+x+1}{(x-1)(x-2)(x-3)}=\frac{5}{2}(x-2)(x-3)-15(x-1)(x-3)+\frac{31}{2}(x-1)(x-2).$$

Exe[redacted]

1. Show that the sum of coefficients of y_i s' in the Lagrange's interpolation formula is unity.

2. Given u_{-1}, y_0, u_1, and u_2. Using Lagrange's formula show that

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0 \quad \text{where } x + y = 1.$$

3. If $y_0, y_1, y_2, \dots, y_6$ are the consecutive terms of a series then prove that

$$y_3 = 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(y_2 + y_4).$$

4. If all terms except y_5 of the sequence $y_1, y_2, y_3, \ldots, y_9$ be given, show that the value of y_5 is

$$\left[\frac{156(y_4+y_6)-28(y_3+y_7)+8(y_2+y_8)-(y_1+y_9)}{70}\right]$$

5. The following table is given

x	0	1	2	5
$f(x)$	2	3	12	147

show that the form of $f(x)$ is $x^3 + x^2 - x + 2$.

6. Using Lagrange's interpolation formula, express the function

$$\frac{x^2+x-3}{x^3-2x^2-x+2}$$ as sum of partial fractions.

7. Express the function $\dfrac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sum of partial functions.

8. Using Lagrange's formula show that

a. $$\frac{x^3-10x+13}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)}+\frac{3}{(x-2)}-\frac{4}{(x-3)}$$

b. $$\frac{x^2+6x+1}{(x^2-1)(x-4)(x+6)}=\frac{-2}{25(x+1)}-\frac{4}{21(x-1)}+\frac{41}{150(x-4)}-\frac{1}{350(x+6)}$$

9. The following values of the function $f(x)$ for values of x are given: $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the values of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

Answers

6. $$\frac{1}{2(x-1)}-\frac{1}{(x-2)}-\frac{1}{2(x+1)}$$

7. $$\frac{2}{35(x+1)}+\frac{4}{15(x-1)}-\frac{41}{30(x-4)}+\frac{73}{70(x-6)}$$

9. $f(6) = 5.66$, maximum at $x = 4.5$

7.13. DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots$ be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$.

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

Divided difference (d.d) Table

		first d.d	second d.d	third d.d
x_0	y_0			
		$[x_0, x_1]$		
x_1	y_1		$[x_0, x_1, x_2]$	
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$
x_2	y_2		$[x_1, x_2, x_3]$	
		$[x_2, x_3]$		
x_3	y_3			

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.* For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$, $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$= \dfrac{1}{2!\,h^2}\,\Delta^2 y_0$ and in general, $[x_0, x_1, x_2, \ldots\ldots, x_n] = \dfrac{1}{n!\,h^n}\,\Delta^n y_0$.

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

7.14. NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that
$$y = y_0 + (x - x_0)[x, x_0] \qquad \ldots(1)$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives
$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \ldots(2)$$

Also
$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0\, x_1, x_2]}{x - x_2}$$

which gives
$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$\begin{aligned} y = f(x) = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots \\ &+ (x - x_0)(x - x_1)\ldots\ldots(x - x_n)[x, x_0, x_1, \ldots\ldots x_n] \qquad \ldots(3) \end{aligned}$$

which is called *Newton's general interpolation formula with divided differences.*

Example 7.12. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate f(9), using (i) Lagrange's formula

(ii) Newton's divided difference formula. (P.T.U., B. Tech., 2005)

Sol. (*i*) Here $x_0 = 5, \quad x_1 = 7, \quad x_2 = 11, \quad x_3 = 13, \quad x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

(*ii*) The divided differences table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$

$$= 150 + 484 + 192 - 16 = 810.$$

Example 7.13. *Using Newton's divided differences formula, evaluate f(8) and f(15) given :*

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(V.T.U., B. Tech., 200[illegible])

Sol. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)(8-5)\,15 + (8-4)(8-5)(8-7)\,1$$
$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

Example 7.14. *Determine f(x) as a polynomial in x for the following data :*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335.

(V.T.U., B. Tech., 2007)

Sol. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$
$$+ (x+4)(x+1)(x-0)(-14) + (x+4)(x+1)x(x-2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$