GROUP THEORY

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1. GROUPS AND SUBGROUPS

1. 1a 2020 IFoS

Let p be a prime number. Then show that

$$(p-1)!+1 \equiv 0 \bmod (p)$$

Also, find the remainder when $6^{44} \cdot (22)! + 3$ is divided by 23. 8

2. 3a 2019 IFoS

If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some a, b $\in$ G, find the order of b. 10

3. 1a 2017 IFoS

1. (a) Prove that every group of order four is Abelian.

4. 2a 2017 IFoS

2. (a) Let G be the set of all real numbers except -1 and define $a*b = a+b+ab$ $\forall$ $a, b \in G$. Examine if G is an Abelian group under $*$. 10

5. 1a 2016 IFoS

1. (a) Prove that the set of all bijective functions from a non-empty set X onto itself is a group with respect to usual composition of functions. 8

6. 1a 2014 IFoS

(a) If G is a group in which $(a \cdot b)^4 = a^4 \cdot b^4$, $(a \cdot b)^5 = a^5 \cdot b^5$ and $(a \cdot b)^6 = a^6 \cdot b^6$, for all a, $b \in G$, then prove that G is Abelian. 8

7. 1b 2013

Give an example of an infinite group in which every element has finite order. 10

8. 1b 2013 IFoS

(b) Prove that if every element of a group (G, 0) be its own inverse, then it is an abelian group.

9. 1a 2012

1. (a) How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$. 12

10. 1a 2011

1. (a) Show that the set

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

of six transformations on the set of Complex numbers defined by

$$f_1(z) = z, \ f_2(z) = 1 - z$$

$$f_3(z) = \frac{z}{(z-1)}, \ f_4(z) = \frac{1}{z}$$

$$f_5(z) = \frac{1}{(1-z)} \text{ and } f_6(z) = \frac{(z-1)}{z}$$

is a non-abelian group of order 6 w.r.t. composition of mappings. 12

11. 4a 2011

4. (a) Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4a$.

Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$. 20

12. 1a 2011 IFoS

(a) Let G be a group, and x and y be any two elements of G. If $y^5 = e$ and $yxy^{-1} = x^2$, then show that $O(x) = 31$, where e is the identity element of G and $x \neq e$. 10

13. 1a 2010

(a) Let $G = \mathbb{R} - \{-1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a * b = a + b + ab$. Show $(G, *)$ is a group and it is abelian 12

14. 1a 2010 IFoS

(a) Let

$$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \, a \in \mathbb{R}, \, a \neq 0 \right\}$$

Show that G is a group under matrix multiplication. 10

15. 3a 2010 IFoS

(a) Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. 13

2. CYCLIC GROUPS

1. 2a 2020

Let G be a finite cyclic group of order n. Then prove that G has $\phi(n)$ generators (where ϕ is Euler's ϕ-function). 15

2. 3b 2020 IFoS

Let G be a finite group and let p be a prime. If p^m divides order of G, then show that G has a subgroup of order p^m, where m is a positive integer. 15

3. 2b 2016

Let p be a prime number and $\mathbf{Z}_p$ denote the additive group of integers modulo p. Show that every non-zero element of $\mathbf{Z}_p$ generates $\mathbf{Z}_p$. *15*

4. 2b 2016 IFoS

2.(b) Let G be a group of order pq, where p and q are prime numbers such that $p > q$ and $q \nmid (p-1)$. Then prove that G is cyclic. 15

5. 1a 2015

How many generators are there of the cyclic group G of order 8? Explain.

Taking a group $\{e, a, b, c\}$ of order 4, where e is the identity, construct composition tables showing that one is cyclic while the other is not.

5+5=10

6. 1a 2015 IFoS

Q1. (a) If in a group G there is an element a of order 360, what is the order of a^{220} ? Show that if G is a cyclic group of order n and m divides n, then G has a subgroup of order m. *10*

7. 1e 2011

(e) (i) Prove that a group of Prime order is abelian. 6

(ii) How many generators are there of the cyclic group $(G\,,\,\cdot)$ of order 8 ? 6

8. 2a 2011

2. (a) Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic. 15

9. 3b 2011 IFoS

(b) Let G be a group of order $2p$, p prime. Show that either G is cyclic or G is generated by $\{a, b\}$ with relations $a^p = e = b^2$ and $bab = a^{-1}$. 13

10. 1b 2010

(b) Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this ? Justify. 12

11. 1b 2009

(b) Determine the number of homomorphisms from the additive group $\mathbb{Z}_{15}$ to the additive group $\mathbb{Z}_{10}$. ($\mathbb{Z}_n$ is the cyclic group of order n). 12

3. COSETS, NORMAL SUBGROUPS & QUOTIENT GROUPS

1. 1a 2019

Let G be a finite group, H and K subgroups of G such that $K \subset H$. Show that $(G:K) = (G:H)(H:K)$. 10

2. 2b 2019

Write down all quotient groups of the group Z_{12}. 10

3. 1a 2018 IFoS

1. (a) Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. 8

4. 4c 2018 IFoS

(c) Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. 10

5. 2b 2017 IFoS

2. (b) Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh \ \forall h \in H$ and $k \in K$. 10

6. 1a 2014

Let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz \neq 0$. Show that G is a group under matrix multiplication. Let N denote the subset $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}$. Is N a normal subgroup of G? Justify your answer. 10

7. 1a 2009 IFoS

(a) Prove that a non-empty subset H of a group G is normal subgroup of $G \Leftrightarrow$ for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$. 10

8. 1d 2009 IFoS

(d) If G is a finite Abelian group, then show that $O(a, b)$ is a divisor of l.c.m. of $O(a)$, $O(b)$. 10

4. HOMOMORPHISM AND AUTOMORPHISMS

1. 2a 2019

If G and H are finite groups whose orders are relatively prime, then prove that there is only one homomorphism from G to H, the trivial one. 10

2. 2a 2018

Show that the quotient group of $(\mathbb{R}, +)$ modulo $\mathbb{Z}$ is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. 15

3. 2a 2018 IFoS

2. *(a)* Find all the homomorphisms from the group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$.

4. 3a 2017

Show that the groups $\mathbb{Z}_5 \times \mathbb{Z}_7$ and $\mathbb{Z}_{35}$ are isomorphic. *15*

5. 2a 2011 IFoS

(a) Let G be the group of non-zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G / N is isomorphic to the group of all positive real numbers under multiplication. 13

6. 2a 2010

2. (a) Let $(\mathbb{R}^*, \cdot)$ be the multiplicative group of non-zero reals and $(GL(n, \mathbb{R}), X)$ be the multiplicative group of $n \times n$ non-singular real matrices. Show that the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ and $(\mathbb{R}^*, \cdot)$ are isomorphic where

$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / \det A = 1\}$.

What is the centre of $GL(n, \mathbb{R})$? 15

7. 2b 2010 IFoS

(b) Prove or disprove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. 13

8. 1a 2009

(a) If $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers, show that $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and $\mathbb{R}_+$ under multiplication $(\mathbb{R}_+, \cdot)$ are isomorphic. Similarly if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers, are $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, \cdot)$ isomorphic? Justify your answer. 4+8=12

5. PERMUTATION GROUPS

1. 1a 2020

Let S_3 and Z_3 be permutation group on 3 symbols and group of residue classes module 3 respectively. Show that there is no homomorphism of S_3 in Z_3 except the trivial homomorphism. 10

2. 4a 2019 IFoS

Show that the smallest subgroup V of A_4 containing $(1, 2)(3, 4)$, $(1, 3)(2, 4)$ and $(1, 4)(2, 3)$ is isomorphic to the Klein 4-group. *10*

3. 1b 2017

Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n. *10*

4. 2a 2016 IFoS

2.(a) Show that any non-abelian group of order 6 is isomorphic to the symmetric group S_3. 15

5. 3a 2015 IFoS

3. (a) What is the maximum possible order of a permutation in S_8, the group of permutations on the eight numbers $\{1, 2, 3, \ldots, 8\}$? Justify your answer. (Majority of marks will be given for the justification). *13*

6. 2a 2013

What are the orders of the following permutations in S_{10} ?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$$ and $(1\,2\,3\,4\,5)(6\,7)$. *10*

7. 2b 2013

What is the maximal possible order of an element in S_{10} ? Why ? Give an example of such an element. How many elements will there be in S_{10} of that order ? *13*

8. 2a 2012

2. (a) How many conjugacy classes does the permutation group S_5 of permutations 5 numbers have ? Write down one element in each class (preferably in terms of cycles). 15

9. 2a 2012 IFoS

(a) Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. 13

10. 2b

(b) Show that the alternating group on four letters A_4 has no subgroup of order 6. 15