VIRTUAL WORK

CHAPTER 8

8.1. WORK

Definition. *A force acting on a body is said to do work, when its point of application is displaced from one position to the other along the line of action of force.*

In order to distinguish it from other kinds of work, the work done by a force is called *mechanical work*.

8.1.1 Measurement of Work

(*i*) Let $\vec{F}$ be the force acting at a point P whose position vector is $\vec{r}$.

Let the point P be displaced by distance $\vec{dr}$ by the application of force $\vec{F}$.

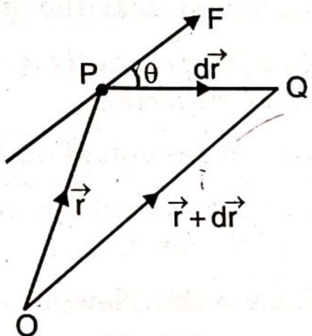

Fig. 8.1

Then, Work done $= \vec{F}.\vec{dr}$

$= F\,(dr \cos\theta)$, where θ is the angle between $\vec{F}$ and $\vec{dr}$.

$= F$ (resolved part of dr along the direction of $\vec{F}$) ...(1)

Also, work done $= dr\,(F\cos\theta)$

$= dr$ (resolved part of F along $\vec{dr}$) ...(2)

Thus amount of work done is the product of F with the resolved part of dr along the direction of $\vec{F}$ or it is product of the displacement dr with the resolved part of F along the direction of $\vec{dr}$. Work done is positive or negative according as θ is acute or obtuse.

(*ii*) In case the point P moves in the direction of $\vec{F}$, then $\theta = 0$

$\therefore$ Amount of work done $= \vec{F}.\vec{dr} = F\,(dr\cos 0°) = F.dr$

Here the work done is positive.

(*iii*) In case the point P moves in the direction opposite to $\vec{F}$, then $\theta = \pi$

$\therefore$ Amount of work done $= \vec{F} \cdot \vec{dr} = F \cdot (dr \cos \pi) = - F \cdot dr$

Here the work done is against the force and is termed as negative.

(*iv*) In case the work done is zero, then $F\, dr \cos \theta = 0$

$$\Rightarrow \quad \text{either } dr = 0 \quad \text{or} \quad \cos \theta = 0 \quad i.e., \quad \theta = \frac{\pi}{2}$$

Thus the work done by a force is zero, if the point of application does not have any displacement or has a displacement at right angle to the direction of $\vec{F}$.

8.2. UNITS OF WORK

(*a*) (*i*) In F.P.S. (English) system, the unit of work is **foot-poundal.**

It is the work done by a force of one poundal in displacing its point of application through one foot in the direction of the force.

(*ii*) In the C.G.S. (French) system, the unit of work is **Erg.**

It is the work done by a force of one dyne in displacing its point of application through one centimetre in the direction of the force.

(*iii*) In the M.K.S. system the unit of work is **Joule.**

It is the work done by a force of one Newton in displacing its point of application through one metre in the direction of the force.

$$\therefore \quad 1 \text{ Joule} = \text{One Newton} \times \text{One metre}$$

$$= 100{,}000 \text{ dynes} \times 100 \text{ cm} = 10^7 \text{ ergs.}$$

[*One Newton of force is that force which produces an acceleration of one metre/sec² in a mass of one kg.*]

(*b*) **The gravitational or practical unit of work is the work done in lifting the weight of a unit mass through a height of one unit.**

(*i*) In the F.P.S.system, the practical unit of work is **Foot Pound.** It is the work done in raising a mass of one pound vertically through a height of one foot.

(*ii*) In the C.G.S. system, the practical unit of work is **gm.cm.** It is the work done in raising a mass of one gramme vertically through a height of one centimetre.

(*iii*) In the M.K.S. system, the practical unit of work is **kg.m.** It is the work done in raising a mass of one kg vertically through a height of 1 metre.

The relation between the practical units and the absolute units are as follows:

One foot pound $= g$ foot poundals $= 32$ ft. poundals

One gm cm $= g$ ergs $= 981$ ergs

One kg metre $= g$ joules $= 9{\cdot}8$ joules.

8.3. WORK DONE BY A NUMBER OF FORCES

If a system of forces acting at a point produces in it a small displacement, the algebraic sum of the work done by the separate forces is equal to the work done by their resultant.

[*M.D.U. 2007*]

Let P_1, P_2, P_3, ...be a system of coplanar forces acting at a point O which displaces it from O to A. Let R be their resultant, then

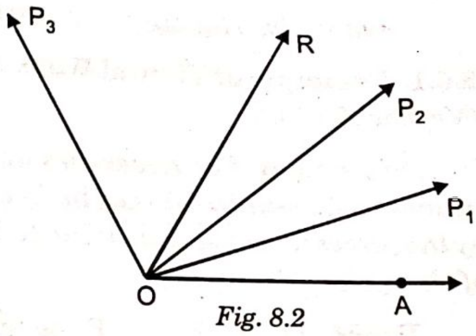

Fig. 8.2

Work done by P_1 = OA × (the resolved part of P_1 along OA)

Work done by P_2 = OA × (the resolved part of P_2 along OA)

.... and so on.

∴ Algebraic sum of the work done by all forces

= OA × (algebraic sum of resolved parts of the forces along OA)

= OA × (resolved part of their resultant R along OA)

= Work done by the resultant force R.

Cor. If the system of forces is in equilibrium, the resultant R = 0 and therefore, the algebraic sum of work done = 0.

8.4. VIRTUAL WORK

If a rigid body undergoes some displacement under the action of a system of forces, then some amount of actual work is done. If the body is in equilibrium under the action of the system of forces, then there is no displacement and hence no actual work is done.

Now suppose that a system of forces acting on a body is in equilibrium and the body be imagined to undergo a slight displacement with the geometrical conditions under which the system exists. If a point A of the body which is acted upon by system of forces be imagined to be displaced to B, then AB is called the *virtual displacement* of the point A.

The word virtual is used to imply that the displacement is only imagined and not an actual displacement.

The theorems on work which are true when there is actual displacement must also be true when there is virtual displacement.

Hence *virtual work done by a force = force × projection of virtual displacement in the direction of the force.*

8.5. PRINCIPLE OF VIRTUAL WORK

Statement. *If a system of forces acting on a body or a system of bodies be in equilibrium and if the system is supposed to undergo a small displacement, consistent with its geometrical conditions, the algebraic sum of the virtual works done by the system of forces is zero.*

Conversely, if the algebraic sum of the virtual works be zero, then the forces are in equilibrium.

8.5.1. Principle of Virtual Work for a System of Coplanar Forces Acting on a Particle (Vector Method)

[M.D.U. 2009, 08]

Statement. ***The necessary and sufficient condition that a particle acted upon by a number of coplanar forces be in equilibrium, is that the sum of the virtual works done by the forces in any small virtual displacement, consistent with the geometrical conditions of the system, is zero.***

[M.D.U. 2013, 11]

Proof. Let $F_1, F_2, \ldots.. F_n$ be number of coplanar forces acting on a particle at O. Due to these forces, let OO′ be a small virtual displacement where O′ is such that R is not perpendicular to OO′.

$\therefore$ Sum of virtual works done

$$= \vec{F_1}\cdot\vec{OO'} + \vec{F_2}\cdot\vec{OO'} + \ldots.. + \vec{F_n}\cdot\vec{OO'}$$

$$= (\vec{F_1} + \vec{F_2} + \ldots. + \vec{F_n})\cdot\vec{OO'}$$

$$= \vec{R}\cdot\vec{OO'} \qquad \ldots(1)$$

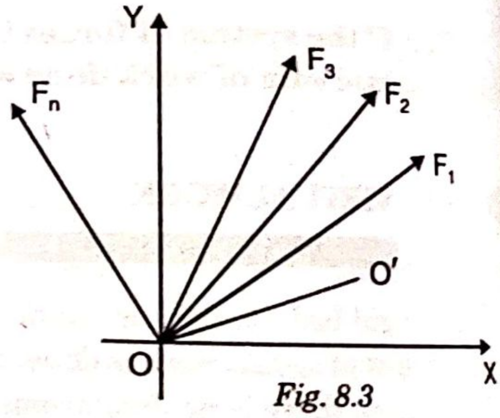

Fig. 8.3

The condition is necessary :

It is given that the particle acted upon by the above forces is in equilibrium

$$\therefore \qquad \vec{R} = \vec{0}$$

From (1), we have

Sum of virtual works done = $\vec{R}\cdot\vec{OO'} = 0$

The condition is sufficient :

It is given that the algebraic sum of the virtual works done by the system of forces acting on a particle is zero.

$\therefore$ Sum of virtual works done = 0

$$\Rightarrow \qquad \vec{R}\cdot\vec{OO'} = 0 \qquad \text{[Using (1)]}$$

$$\Rightarrow \qquad \text{either } \vec{R} = \vec{0} \quad \text{or} \quad \vec{OO'} = \vec{0}$$

But $\vec{OO'} \neq \vec{0}$, therefore $\vec{R} = \vec{0}$ [$\because$ *R is not perpendicular to* OO′]

Hence, the particle is in equilibrium.

Cartesian Method :

Let $F_1, F_2, \ldots, F_n$ be number of coplanar forces acting on a particle at O. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the resolved parts of these forces along two mutually perpendicular directions OX and OY respectively.

Let X, Y be the resolved parts of their resultant R along OX and OY respectively.

$$\therefore \quad X = X_1 + X_2 + X_3 + \ldots$$

$$Y = Y_1 + Y_2 + Y_3 + \ldots$$

and $$R^2 = X^2 + Y^2$$

Let the forces displace particle from O to O′ so that OO′ is the virtual displacement.

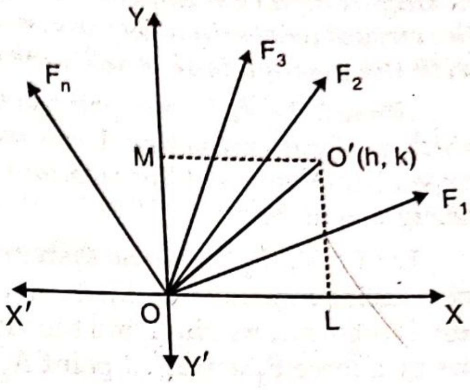

Fig. 8.4

Let (h, k) be the co-ordinates of point O′ so that $OL = h$ and $OM = k$

Virtual work done by the force F_1

= the algebraic sum of the virtual works done by its resolved parts X_1 and Y_1

$= X_1 h + Y_1 k$

Similarly, virtual works done by the force F_2

$= X_2 h + Y_2 k$

....

....

Virtual works done by the force $F_n = X_n h + Y_n k$

$\therefore$ Algebraic sum of the virtual works done by the system of forces

$$= (X_1 h + Y_1 k) + (X_2 h + Y_2 k) + (X_3 h + Y_3 k) + \ldots$$

$$= h\,(X_1 + X_2 + X_3 + \ldots) + k\,(Y_1 + Y_2 + Y_3 + \ldots)$$

$$= h\,.\,X + k\,.\,Y \qquad \ldots(1)$$

The condition is necessary :

If a particle acted upon by a number of coplanar forces be in equilibrium, then the algebraic sum of the virtual works done by the forces in any small virtual displacement, consistent with the geometrical conditions of the system, is zero.

Since the particle is in equilibrium, therefore R = 0

$$\therefore \quad X = 0 \quad \text{and} \quad Y = 0$$

From (1), we have that the algebraic sum of the work done by the forces = 0.

The condition is sufficient : (Converse of the principle of virtual work)

If the algebraic sum of the virtual works done by the system of forces acting on a particle for any small virtual displacement be zero, then the particle is in equilibrium.

It is given that the algebraic sum of the works done by the system of forces = 0

$$\therefore \quad hX + kY = 0$$

Since h, k are independent of one another, therefore

$$X = 0 \quad \text{and} \quad Y = 0$$

$$R = \sqrt{X^2 + Y^2} = 0$$

$\Rightarrow$ Thus the system is in equilibrium and hence the result.

8.5.2. Principle of Virtual Work for a System of Coplanar Forces Acting at Different Points of a Rigid Body

Statement. ***The necessary and sufficient condition that a system of coplanar forces acting at different points of a rigid body be in equilibrium is that the algebraic sum of the virtual works done by the system of forces in any small virtual displacement, consistent with the geometrical conditions of the system, is zero.*** [M.D.U. 2012, 10]

Proof. Let O be any point in the plane in which the forces are acting. Draw two mutually perpendicular straight lines OX and OY passing through point O.

Let $F_1, F_2, F_3, \ldots$ be the system of coplanar forces acting at points $A_1, A_2, A_3, \ldots$ of the rigid body. First of all, we shall find the virtual works done by a force F_1 acting at point A_1.

Let (x_1, y_1) and (r, θ) be the cartesian and polar co-ordinates of the point A_1, then $OA_1 = r$ and $\angle XOA_1 = \theta$

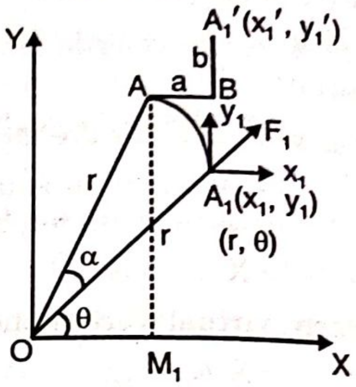

Fig. 8.5

Also,

$$x_1 = r \cos \theta$$

$$y_1 = r \sin \theta$$

Suppose the body undergoes a small virtual displacement, consistent with the geometrical conditions of the system. A body in a plane can be moved in three and only three independent ways *viz.*,

1. *Motion of rotation about point O*
2. *Motion of translation parallel to OX*
3. *Motion of translation parallel to OY.*

Thus first of all, the body is rotated through an angle α about O, the origin and then given a motion of translation $AB = a$, parallel to OX and $BA_1' = b$, parallel to OY.

But every point of the body shares its motion, therefore the points of application of all forces rotate about point O through the same angle α and move the same distances a and b parallel to OX and OY respectively.

Let (x_1', y_1') be the co-ordinates of point A_1', then

$$x_1' = OM_1 + AB$$
$$= OA \cos(\theta + \alpha) + a = r\cos(\theta + \alpha) + a$$
$$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha + a$$
$$= x_1 \cos\alpha - y_1 \sin\alpha + a \qquad [\because x_1 = r\cos\theta,\ y_1 = r\sin\theta]$$
$$= x_1 - y_1\alpha + a \qquad [\because \text{For small } \alpha, \cos\alpha = 1 \text{ and } \sin\alpha = \alpha]$$

Also $\quad y_1' = M_1A + BA_1'$

$= OA \sin(\theta + \alpha) + b = r \sin(\theta + \alpha) + b$

$= r \sin\theta \cos\alpha + r \cos\theta \sin\alpha + b$

$= y_1 \cos\alpha + x_1 \sin\alpha + b$

$= y_1 + x_1\alpha + b$

[$\because$ *For small* α, $\cos\alpha = 1$ *and* $\sin\alpha = \alpha$]

Virtual displacement of A_1 parallel to x-axis

$= x_1' - x_1 = x_1 - y_1\alpha + a - x_1 = a - \alpha y_1$

Virtual displacement of A_1 parallel to y-axis

$= y_1' - y_1 = b + \alpha x_1$

Let X_1, Y_1 be the resolved parts of the force F_1 parallel to the axes respectively.

$\therefore$ Virtual works done by the force F_1

= the algebraic sum of the virtual works done by X_1 and Y_1

$= X_1$ (virtual displacement parallel to x-axis)

$+ Y_1$ (virtual displacement parallel to y-axis)

$= X_1(a - \alpha y_1) + Y_1(b + \alpha x_1)$

$= aX_1 + bY_1 + \alpha(x_1Y_1 - y_1X_1)$.

Similarly the virtual works done by the force F_2 acting at the point (x_2, y_2) is $aX_2 + bY_2 + \alpha(x_2Y_2 - y_2X_2)$ and we get similar expressions for the other forces.

Hence the algebraic sum of the virtual works done by the system of forces acting upon the rigid body

$$= a\,\Sigma X_1 + b\,\Sigma Y_1 + \alpha\,\Sigma(x_1Y_1 - y_1X_1) \quad ...(1)$$

The condition is necessary :

If a system of coplanar forces acting at different points of a rigid body, keep the body in equilibrium and if the body be given a small arbitrary virtual displacement, consistent with the geometrical conditions of the system, then the algebraic sum of the virtual works done by the forces is zero.

It is given that the body is in equilibrium.

$\therefore \quad \Sigma X_1 = 0, \quad \Sigma Y_1 = 0 \quad \text{and} \quad G = \Sigma(x_1Y_1 - y_1X_1) = 0$

$\therefore$ From (1), the algebraic sum of the virtual works done by the system of forces = 0.

The condition is sufficient :

It is given that the algebraic sum of the virtual works done by the forces is zero.

i.e., $\quad a\,\Sigma X_1 + b\,\Sigma Y_1 + \alpha\,\Sigma(x_1Y_1 - y_1X_1) = 0$ for all displacements.

We shall show that the body will be in equilibrium.

As the displacement is perfectly arbitrary; hence a, b, α are all independent of one another. The above equation is true for any positive or negative independent values of a, b and α. Hence we must have

$$\Sigma X_1 = 0, \quad \Sigma Y_1 = 0 \quad \text{and} \quad \Sigma(x_1Y_1 - y_1X_1) = 0$$

$\therefore$ The conditions of equilibrium are satisfied and hence the system of forces is in equilibrium.

8.6. TWO IMPORTANT RESULTS

8.6.1. The virtual works done by the tension in a virtual extension of a string from length l to $l + \delta l$ is $-T\,\delta l$, where T is the tension in the string

[K.U. 2015, 14; M.D.U. 2013, 09, 07]

Let a string AB be displaced to the position A′B′ such that A′B′ makes an angle θ with AB.

Tension T in the string AB is equivalent to a force T at A in the direction of AB and a force T at B in the direction of BA.

Draw A′L and B′M perpendiculars to AB. Then LM = A′B′ cos θ, where AB = l and A′B′ = $l + \delta l$.

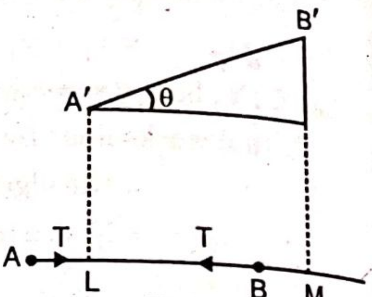

Fig. 8.6

The virtual work done by the tension when string is displaced from the position AB to A′B′

$$= \text{sum of the virtual works done by T at A and T at B}$$
$$= T.AL - T.BM$$
$$= T(AB - LB) - T(LM - LB)$$
$$= T(AB - LM) = T(AB - A'B'\cos\theta)$$
$$= T\left[l - (l + \delta l)\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\right)\right]$$
$$= T[l - (l + \delta l)(1)] = -T\delta l. \qquad [\because \theta \text{ is small}]$$

8.6.2. The virtual works done by the thrust in a virtual extension of a light rod from length l to $l + \delta l$ is $T.\,\delta l$, where T is the thrust in the rod.

[K.U. 2016, 10, 09; M.D.U. 2008]

Let a rod AB (= l) be displaced to the position A′B′ such that A′B′ makes an angle θ with AB. Let T, T be the thrusts in the rod AB at the ends A and B along BA and AB respectively.

Draw A′L and B′M perpendiculars to AB.

Then LM = A′B′ cos θ where A′B′ = $l + \delta l$

Now, virtual works done by the thrusts

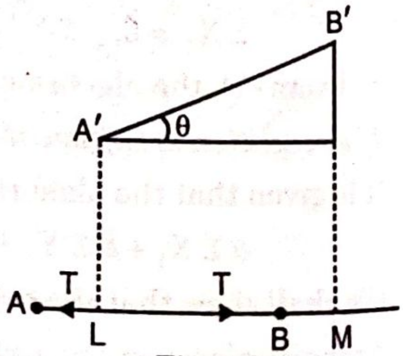

Fig. 8.7

$$= -T.AL + T.BM$$
$$= -T(AB - LB) + T(LM - LB)$$
$$= T(LM - AB)$$
$$= T(A'B'\cos\theta - AB)$$
$$= T[(l + \delta l)(1) - l]$$
$$= T.\,\delta l \qquad [\because \theta \text{ is small}]$$

8.7. FORCES WHICH MAY BE OMITTED IN FORMING THE EQUATION OF VIRTUAL WORK

There are a number of cases in which either a pair of forces *do equal and opposite works* or the forces *do no work*. Certain number of forces can be omitted in writing the equations of equilibrium. For virtual displacements, which do not violate the geometrical conditions of the system, the following forces which are not required in the final result may be omitted as they contribute nothing in forming the equation of virtual work of the system.

(i) ***The tension of an inextensible string (Pull).***

Let T be the tension in the string AB *i.e.*, a force T acts at A along AB and another force T acts at B along BA.

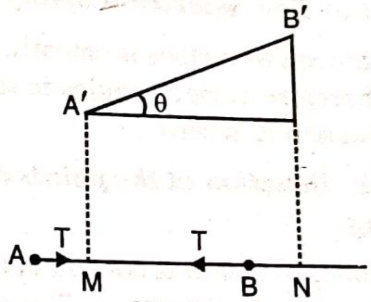

Fig. 8.8

On account of the slight displacement, let A′B′ be the position of the string which makes an angle θ with the direction of AB and A′B′ = AB.

Let M and N be the projections of A′ and B′ on AB respectively.

The sum of the virtual works done by T at A and T at B

$$= T\,.\,AM - T\,.\,BN$$

$$= T\,(AM - BN) = T\,[(AB - MB) - (MN - MB)]$$

$$= T\,(AB - MN) = T\,(AB - A'B'\cos\theta) = T\,(AB - AB\cos\theta)$$

$$= T\,(AB)\,(1 - \cos\theta) = T\,(AB)\left[1 - \left(1 - \frac{\theta^2}{2!} + ...\right)\right]$$

$$= T\,(AB)\,[1 - 1] = 0$$ [*When θ is small, its higher powers can be neglected*]

(ii) ***The thrust of an inextensible rod (Push).***

Let T be the thrust in the rod AB *i.e.*, a force T acts at A along BA and another force T at B along AB.

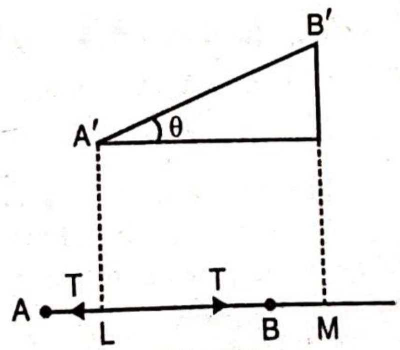

Fig. 8.9

On account of the slight displacement, let A′B′ be the position of the rod which makes an angle θ with the direction of AB and A′B′ = AB.

Let L and M be the projections of A′ and B′ respectively on AB.

The sum of the virtual works done by T at A and T at B

$$= -T\,.\,AL + T\,.\,BM$$

$$= -T\,(AB - LB) + T\,(LM - LB)$$

$= T\,(LM - AB)$

$= T\,(A'B' \cos\theta - AB) = T\,(l \cos\theta - l)$

$$= T\left[l - \left(1 - \frac{\theta^2}{2} + \ldots\right) - l\right] = T\,[l - l] \qquad [\because \theta \text{ is small}]$$

$= 0.$

(*iii*) ***The mutual action and reaction between two bodies whose equilibrium is being considered together.***

In this case the action and reaction are equal and opposite, therefore their virtual works are equal and opposite and hence the sum of their virtual works is zero.

(*iv*) ***The reaction R of any smooth surface with which the body is in contact.***

Since the surface is smooth, the reaction R is normal to the surface at the point of contact and therefore at right angles to the direction of the displacement. Hence virtual work done by the reaction R is zero.

(*v*) ***Reaction at the points of contact of a fixed surface on which a body rolls without sliding.***

Since the point of contact of the body with the surface is instantaneously at rest, it does not undergo any displacement. Therefore no work is done by the normal reaction or the force of friction at the point of contact.

(*vi*) ***The mutual reaction between two bodies which roll upon one another.***

Since both the bodies are taken together, the action and reaction are equal and opposite, thus the sum of their virtual works is zero.

Remark

The students must remember the following while solving the problem on virtual work.

(*i*) Draw the diagram showing the directions of the given forces.

(*ii*) Replace the rod by two forces T and T acting outwards.

$\therefore$ Virtual works done by the thrust is positive and equal to $T\,.\,\delta l$

$= T \times$ [increase in length of the rod by a small virtual displacement]

(*iii*) Replace the string by two forces T and T acting inwards.

$\therefore$ Virtual work done by the tension is negative and equal to

$-T\,.\,\delta l = -T \times$ [increase in length of the string by a small virtual displacement]

(*iv*) If the distances of various points from a fixed point are measured in the same direction in which the force is acting, then the work done is positive and if they are measured in the direction opposite to the direction of force, the work done is negative.

SOLVED EXAMPLES

Solved Examples Based on Simple Frameworks :

Example 1.

Two equal uniform rods AB and AC, each of length 2b are freely joined at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then $b \sin^3 \theta = a \cos \theta$. [K.U. 2016, 15, 09, 08; M.D.U. 2013, 08, 06; C.D.L.U. 2012]

Solution. Let O be the centre of the given circle and hence O is a fixed point. Let W be the weight of each rod. Then 2W (total weight of two rods) is the only force which will appear in the equation of virtual work. This will act at the common centre of gravity.

Let G_1 and G_2 be the mid-points of AB and AC.

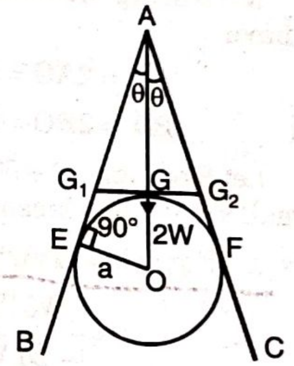

Fig. 8.10

Then
$$AG_1 = \frac{1}{2} AB = b$$
$$AG_2 = \frac{1}{2} AC = b$$
$$\therefore \quad \angle BAO = \angle CAO = \theta$$

Let G be the mid point of G_1G_2

$\therefore$ The weight 2W of the rods acts at G. Let the distances be measured from fixed point O

$$\therefore \quad AG = AG_1 \cos\theta = b\cos\theta$$
and
$$OG = OA - AG = OE \operatorname{cosec}\theta - b\cos\theta$$
$$= a\operatorname{cosec}\theta - b\cos\theta$$

Let the system be given a vertical small displacement such that θ becomes θ + δθ.

$\therefore$ Equation of virtual work is

$$-2W.\delta(OG) = 0 \qquad \textit{[Since G lies above O]}$$
$$\therefore \quad \delta(OG) = 0 \qquad [\because W \neq 0]$$
or
$$\delta[a\operatorname{cosec}\theta - b\cos\theta] = 0$$
or
$$[-a\operatorname{cosec}\theta\cot\theta + b\sin\theta]\,\delta\theta = 0$$
or
$$b\sin\theta - a\operatorname{cosec}\theta\cot\theta = 0 \qquad [\because \delta\theta \neq 0]$$
or
$$b\sin\theta = a\frac{\cos\theta}{\sin^2\theta}$$
i.e.,
$$b\sin^3\theta = a\cos\theta.$$

Example 2.

Five weightless rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. If the weight W be attached to C and the system be suspended from A; show that there is a thrust in BD equal to $\frac{W}{\sqrt{3}}$.

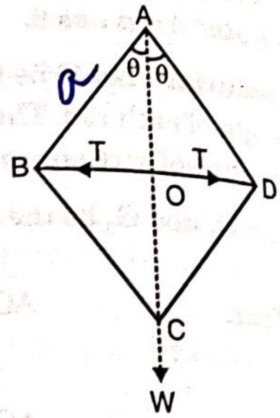

Fig. 8.11

Solution. Let the diagonals AC and BD of the rhombus ABCD intersect at point O. Let 2θ be the angle between AB and AD. The system is suspended from A with a weight W at C.

$\therefore$ AC must be vertical and $\angle BAO = \angle OAD = \theta$.

Let T be the thrust in the rod BD. Let the length of each side of the rhombus be a.

Measuring the distances from the fixed point A, we have

$$AC = 2AO = 2AB\cos\theta = 2a\cos\theta$$

and
$$BD = 2BO = 2AB\sin\theta = 2a\sin\theta = y \text{ (say)}$$

Let θ becomes $\theta + \delta\theta$ and y becomes $y + \delta y$ due to a small virtual displacement in the system.

$\therefore$ Equation of virtual work is $\quad W.\delta(AC) + T.\delta(BD) = 0$

or
$$W\delta(2a\cos\theta) + T\delta(2a\sin\theta) = 0$$

or
$$(-2aW\sin\theta + 2aT\cos\theta)\,\delta\theta = 0$$

or
$$T\cos\theta = W\sin\theta \qquad [\delta\theta \neq 0]$$

$\therefore$
$$T = W\tan\theta$$

In the position of equilibrium, $\quad AB = AD = BD$ — No given.

$\therefore$ Δ ABD is an equilateral triangle

$$\angle BAD = 2\theta = 60° \quad \Rightarrow \quad \theta = 30°$$

Hence, $\quad T = W\tan 30° = \frac{W}{\sqrt{3}}$.

Example 3.

Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely joined so as to form a hexagon. The rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Prove that its tension is 3W.

[M.D.U. 2014, 12; K.U. 2006]

Solution. Let G_1, G_2, G_3, G_4, G_5 and G_6 be the middle points of the rods.

Let T be the tension in the string G_1G_4 and O be the centre of the hexagon ABCDEF.

Since AB is fixed in a horizontal position, therefore we measure the distances from AB.

Let $\quad OG_1 = x$

$\therefore \quad G_1G_4 = 2x = l$ (say)

The total weight 6W of the rods acts at O.

Let the system undergo a displacement in the vertical plane such that x becomes $x + \delta x$ and l becomes $l + \delta l$.

$\therefore$ By the principle of virtual work, we have

$$6W.\delta(OG_1) - T.\delta(G_1G_4) = 0$$

?

or $$6W.\delta(x) - T.\delta(2x) = 0$$

or $$6W.1.\delta x - T.2\delta x = 0$$

or $$6W - 2T = 0$$

$$\therefore \quad T = 3W. \qquad [\because \delta x \neq 0]$$

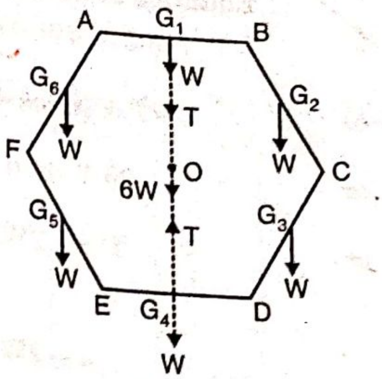

Fig. 8.12

Example 4.

A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rod is supported in a horizontal position, prove that the tension of the string is

$$T = \frac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$$

[M.D.U. 2014, 09, 06; K.U. 2013, 07]

Solution. Let ABCD be a rhombus formed by four uniform rods, each of length b and let side AD be fixed in a horizontal position. Let AC be the smaller diagonal.

$$\therefore \quad AC = a$$

Let T be the tension in the string AC. Let G be the point of intersection of the diagonals.

$\therefore$ G is C.G. of the rhombus and weight 4W of the four rods can be taken to act vertically downwards through this point.

$$\therefore \quad AG = AD \cos\theta = b\cos\theta$$

$$\therefore \quad AC = 2AG = 2b\cos\theta$$

From G draw GL $\perp$ AD

Depth of G below $AD = GL = AG\sin\theta = b\cos\theta\sin\theta = \frac{b}{2}\sin 2\theta$.

Now suppose that the system undergoes a slight displacement in such a way that the angle θ is changed to $\theta + \delta\theta$ and the length of the string AC is altered.

$\therefore$ Equation of virtual work is given by

$$-T\,.\,\delta(AC) + 4W.\delta(\text{depth of G below AD}) = 0$$

or
$$-T.\delta(2b\cos\theta) + 4W.\delta\left(\frac{1}{2}\,b\sin 2\theta\right) = 0$$

or
$$2b\,T\sin\theta\,.\,\delta\theta + 4Wb\cos 2\theta\,.\,\delta\theta = 0$$

or
$$T = -\,2W\,\frac{\cos 2\theta}{\sin\theta} \qquad [\because \delta\theta \neq 0]$$

$$= -\,2W\,\frac{(2\cos^2\theta - 1)}{\sin\theta} \qquad \text{...(1)}$$

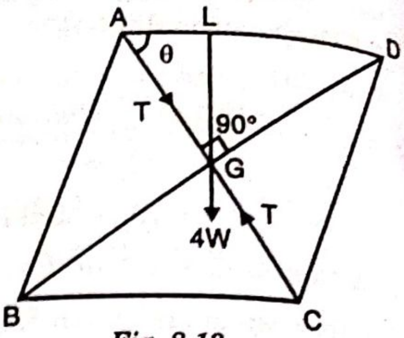

Fig. 8.13

In the position of equilibrium,

$$AG = \frac{a}{2}$$

But
$$AG = b\cos\theta$$

$\therefore$
$$\frac{a}{2} = b\cos\theta \;\Rightarrow\; \cos\theta = \frac{a}{2b}$$

$\therefore$
$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{a^2}{4b^2}} = \sqrt{\frac{4b^2-a^2}{4b^2}} = \frac{\sqrt{4b^2-a^2}}{2b}$$

Putting the values of cos θ and sin θ in (1), we get

$$T = \frac{-\,2W\left[\dfrac{2a^2}{4b^2}-1\right]}{\dfrac{\sqrt{4b^2-a^2}}{2b}} = \frac{-\,2W(2a^2-4b^2)}{2b\sqrt{4b^2-a^2}}$$

$$= \frac{2W(2b^2-a^2)}{b\sqrt{4b^2-a^2}}.$$

Example 5.

Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which hung up by the corner A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts in these rods are $\frac{1}{2}5\sqrt{3}\,W$ and $\frac{1}{2}\sqrt{3}\,W$, where W is the weight of each rod.

[K.U. 2011; M.D.U. 2004]

Solution. Let G be the C.G. of the hexagon where the total weight 6W acts.

Suppose that each slanting rod makes an angle θ with the horizontal in the position of equilibrium. Let T_1 and T_2 be the thrusts in the rods CE and BF. Replace the rod CE by forces T_1 at C and E and the rod BF by forces T_2 at B and F in the directions as shown in Fig. 8.14.

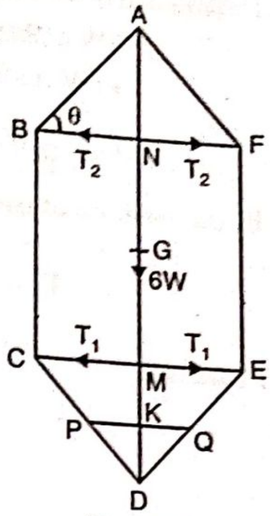

Fig. 8.14

Let $2a$ be the length of each rod. Since A is fixed, therefore we measure all distances from A downwards.

Now $AN = AB \sin\theta = 2a \sin\theta$

$BN = AB \cos\theta = 2a \cos\theta$

$AG = AN + NG = 2a \sin\theta + a$

$BF = 2(BN) = 2(2a\cos\theta) = 4a\cos\theta = CE$

The total weight 6W of the rods is acting at G.

Let the system be given a small virtual displacement such that θ becomes θ + δθ for upper as well as lower slant rods.

The equation of the virtual work is

$$6W.\delta(AG) + T_1.\delta(CE) + T_2.\delta(BF) = 0$$

$$\Rightarrow \quad 6W\delta(2a\sin\theta + a) + T_1.\delta(4a\cos\theta) + T_2.\delta(4a\cos\theta) = 0$$

$$\Rightarrow \quad 6W(2a\cos\theta)\,\delta\theta + T_1(-4a\sin\theta)\,\delta\theta + T_2(-4a\sin\theta)\,\delta\theta = 0$$

$$\Rightarrow \quad 12aW\cos\theta - 4a(T_1 + T_2)\sin\theta = 0 \qquad [\because \delta\theta \neq 0]$$

$$\Rightarrow \quad T_1 + T_2 = 3W\cot\theta$$

In the position of equilibrium, $\theta = 30°$

$$\therefore \quad T_1 + T_2 = 3W\cot 30° = 3\sqrt{3}\,W \qquad ...(1)$$

Let the system be given another small virtual displacement such that only the lower slant rods are displaced and the length CE alters. The upper slant rods and length of BF remain unchanged.

In this case θ becomes θ + δθ only for the lower slant rods.

The weights of two rods AB and AF do not work due to this displacement

The equation of the virtual work is

$$2W\,\delta(AG) + 2W\,\delta(AK) + T_1\,\delta(CE) = 0 \qquad ...(2)$$

where K is the middle point of PQ, the line joining the mid-points of the two lower slant rods.

$AG = AN + NG = 2a\sin\theta + a$

$AK = AN + NM + MK$

$= 2a\sin\theta + 2a + a\sin\theta = 3a\sin\theta + 2a$

$CE = 4a\cos\theta$

Putting these values in (2), we get

$$2W\,\delta(2a\sin\theta + a) + 2\,W\,\delta(3a\sin\theta + 2a) + T_1\,\delta(4a\cos\theta) = 0$$

or $$4a\,W\cos\theta\,\delta\theta + 6aW\cos\theta\,\delta\theta - 4a\,T_1\sin\theta\,\delta\theta = 0$$

or $$T_1 = \frac{5}{2}W\cot\theta \qquad [\because \delta\theta \neq 0]$$

In the position of equilibrium, $\theta = 30°$

$\therefore$ $$T_1 = \frac{5}{2}W\cot 30° = \frac{5\sqrt{3}}{2}W$$

From (1), $$T_1 + T_2 = 3\sqrt{3}\,W$$

$\therefore$ $$T_2 = 3\sqrt{3}\,W - T_1$$

$$= 3\sqrt{3}\,W - \frac{5\sqrt{3}}{2}W$$

$$= \frac{6\sqrt{3}\,W - 5\sqrt{3}\,W}{2} = \frac{\sqrt{3}\,W}{2}$$

Hence thrusts in rods are $\frac{1}{2}(5\sqrt{3})W$ and $\frac{1}{2}\sqrt{3}\,W$ respectively.

Solved Examples Based on Bodies Resting on Inclined Planes or Pegs:

Example 6.

A heavy uniform rod of length 2a rests with its ends in contact with two smooth inclined planes of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove by the principle of virtual work that

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta).$$ [C.D.L.U. 2013; K.U. 2012, 05; M.D.U. 2007]

Solution. Let AB be the uniform rod of length $2a$ which is resting on the inclined planes OA and OB which are inclined at angles α and β to the horizon. The weight W of the rod acts at G, the mid-point of AB.

Draw AE, GH and BF perpendiculars from A, G and B on the horizontal plane through O.

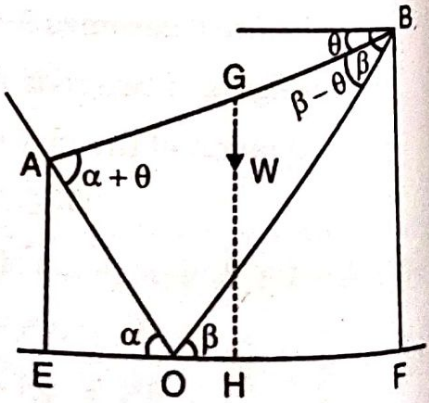

Fig. 8.15

$\therefore$ $$\angle AOB = \pi - \overline{\alpha + \beta}$$

$$\angle OBA = \beta - \theta$$

$$\angle OAB = \alpha + \theta$$

Let the distance be measured from the horizontal plane through O.

Let $$GH = Z$$

Let the rod be given a small displacement such that θ changes to $\theta + \delta\theta$, whereas α, β and a remain unchanged.

$\therefore$ By the principle of virtual work, we have

$$W . \delta Z = 0 \quad \text{or} \quad \delta Z = 0 \qquad \text{...(1)} \quad [\because W \neq 0]$$

Now, Z = Height of G above the horizontal plane

$$= \frac{AE + BF}{2} = \frac{1}{2}[OA \sin\alpha + OB \sin\beta] \qquad \text{...(2)}$$

In Δ OAB, by sine formula, we have

$$\frac{OA}{\sin \angle OBA} = \frac{OB}{\sin \angle OAB} = \frac{AB}{\sin \angle AOB}$$

or
$$\frac{OA}{\sin(\beta - \theta)} = \frac{OB}{\sin(\alpha + \theta)} = \frac{AB}{\sin[180^\circ - (\alpha + \beta)]}$$

or
$$OA = AB.\frac{\sin(\beta - \theta)}{\sin(\alpha + \beta)} = \frac{2a \sin(\beta - \theta)}{\sin(\alpha + \beta)} \qquad [\because AB = 2a]$$

Also
$$OB = AB\frac{\sin(\alpha + \theta)}{\sin(\alpha + \beta)} = \frac{2a \sin(\alpha + \theta)}{\sin(\alpha + \beta)}$$

Substituting the values of OA and OB in (2), we get

$$Z = \frac{1}{2} \cdot \frac{2a}{\sin(\alpha + \beta)}[\sin(\beta - \theta)\sin\alpha + \sin(\alpha + \theta)\sin\beta]$$

$$= \frac{a}{\sin(\alpha + \beta)}[\sin(\beta - \theta)\sin\alpha + \sin(\alpha + \theta)\sin\beta]$$

From (1),
$$\delta Z = 0$$

or
$$\delta\left[\frac{a}{\sin(\alpha + \beta)}\sin(\beta - \theta)\sin\alpha + \sin(\alpha + \theta)\sin\beta\right] = 0$$

or
$$\frac{a}{\sin(\alpha + \beta)}[-\cos(\beta - \theta)\sin\alpha + \cos(\alpha + \theta)\sin\beta]\delta\theta = 0$$

or
$$\sin\beta(\cos\alpha\cos\theta - \sin\alpha\sin\theta) = \sin\alpha(\cos\beta\cos\theta + \sin\beta\sin\theta)$$

or
$$\cos\theta(\sin\beta\cos\alpha - \sin\alpha\cos\beta) = 2\sin\alpha\sin\beta\sin\theta$$

or
$$\frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = 2\tan\theta$$

or
$$2\tan\theta = \cot\alpha - \cot\beta$$

Hence,
$$\tan\theta = \frac{1}{2}[\cot\alpha - \cot\beta].$$

Example 7.

Four equal rods, each of length 2a and weight W, are freely joined to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB and AD in contact with two fixed smooth pegs which are at distance 2b apart on the same level. Find the stress in the rod BD.

Solution. Let ABCD be a square formed by four equal uniform rods each of length $2a$.

Let W be the weight of each rod and G be the centre of gravity of square ABCD. Therefore weight 4W of four rods acts at G.

Let P and Q be the two given smooth pegs, such that the reactions at these pegs do not appear in the equation of virtual work.

Let T be the thrust in light rod BD. Replace the rod BD by a force T at B and a force T and D in the directions DB and BD respectively.

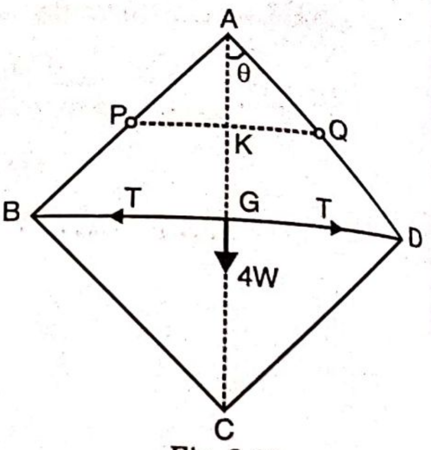

Fig. 8.16

Let $\angle DAG = \theta$

The distances are measured downwards from the line PQ *i.e.,* the line joining the pegs P and Q.

Let K be the mid-point of PQ

$$\therefore \quad KQ = \frac{1}{2} PQ = b \qquad [\because PQ = 2b]$$

In Δ AKQ, $$\frac{AK}{KQ} = \cot\theta$$

or $$AK = b\cot\theta$$

$\therefore$ In Δ AGD, $$AG = AD\cos\theta = 2a\cos\theta$$

$\Rightarrow$ $$KG = AG - AK = 2a\cos\theta - b\cot\theta$$

and $$BD = 2GD = 2AD\sin\theta = 4a\sin\theta$$

Let the system be given a small displacement such that θ changes to $\theta + \delta\theta$, then by the principle of virtual work, we have

$$4W\,\delta(KG) + T\,\delta(BD) = 0$$

or $$4W\,\delta(2a\cos\theta - b\cot\theta) + T\delta(4a\sin\theta) = 0$$

or $$[W(-2a\sin\theta + b\,\text{cosec}^2\,\theta) + T(a\cos\theta)\,\delta\theta] = 0$$

or $$W(-2a\sin\theta + b\,\text{cosec}^2\theta) + T(a\cos\theta) = 0 \qquad [\because \delta\theta \neq 0]$$

$$\therefore \quad T = \frac{W(2a\sin\theta - b\,\text{cosec}^2\,\theta)}{a\cos\theta}$$

In the position of equilibrium, $\theta = 45°$

$$\therefore \quad T = \frac{W(2a \sin 45° - b \operatorname{cosec}^2 45°)}{a \cos 45°}$$

$$= \frac{W(a\sqrt{2} - 2b)}{\frac{a}{\sqrt{2}}} = \frac{2W}{a}(a - b\sqrt{2}).$$

Example 8.

A rectangular lamina ABCD rests with the sides AB, AD on two smooth pegs in a horizontal line. Prove that if the distance between the pegs is half a diagonal of the rectangle, then AB, AD bisect the angle between AC and the horizon. [K.U. 2016]

Solution. Let P_1 and P_2 be the two pegs so that P_1P_2 is horizontal and $P_1P_2 = AG$ where G is the middle point of diagonal AC.

Let AX be the horizontal through A,

$\angle BAX = \theta$ and $\angle BAG = \phi$

We have to show that $\theta = \phi$.

Let the distances be measured from horizontal line P_1P_2.

From G and P_2 draw $GM \perp AX$ and $P_2N \perp AX$.

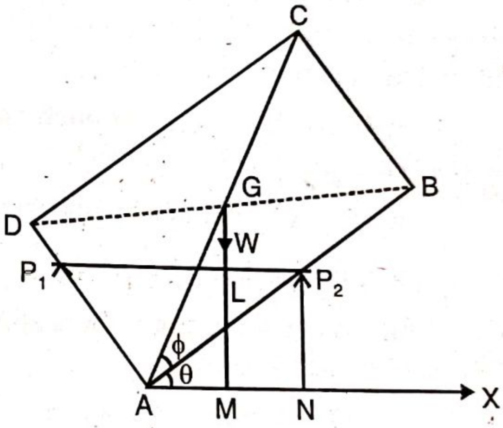

Fig. 8.17

Now, $\quad GL = GM - LM = GM - P_2N$

$$= AG \sin(\theta + \phi) - AP_2 \sin\theta$$

$$= AG \sin(\theta + \phi) - P_1P_2 \cos \angle AP_2P_1 \sin\theta$$

$$= AG \sin(\theta + \phi) - AG \cos\theta \sin\theta$$

$$= AG\left[\sin(\theta + \phi) - \frac{\sin 2\theta}{2}\right] \quad \text{...(1)}$$

Let the system be given a small virtual displacement such that θ becomes $\theta + \delta\theta$ and ϕ does not alter.

Let W be the weight of system acting at G. The reactions at P and Q do not appear in the equation of the virtual work.

The equation of the virtual work is $- W.\delta(GL) = 0$

or $$\delta(GL) = 0$$

or $$\delta\left[AG\left\{\sin(\theta + \phi) - \frac{\sin 2\theta}{2}\right\}\right] = 0 \quad \text{[Using (1)]}$$

or $$AG[\cos(\theta + \phi) - \cos 2\theta] = 0$$

or $$\cos(\theta + \phi) - \cos 2\theta = 0 \quad \Rightarrow \quad \cos 2\theta = \cos(\theta + \phi)$$

$\Rightarrow$ $$2\theta = \theta + \phi \quad \Rightarrow \quad \theta = \phi$$

Thus AB bisects the angle between diagonal AC and the horizontal AX. But AD is perpendicular to AB. Therefore, AD also bisects the angle between AC and the horizon. Hence the result.

Solved Examples Based on Elastic Strings and Endless Strings :

Example 9.

Four equal jointed rods each of length a, are hung from an angular point which is connected by an elastic string with the opposite point. If the rods hang in the form of a square and if the modulus of elasticity of the string be equal to the weight of a rod, show that the natural length of the string is $\frac{a\sqrt{2}}{3}$. *[C.D.L.U. 2013; M.D.U. 2008]*

Solution. Let ABCD be the square hung from the angular point A whose diagonals AC and BD intersect at O, the centre of the square.

Let W be the weight of each rod. The weight 4W of the four equal rods acts at O.

Let T be the tension in the string AC.

Let $\angle BAD = 2\theta$.

Since A is a fixed point, therefore all the distances will be measured from A.

$\therefore$ $AO = AD\cos\theta = a\cos\theta$ and $AC = 2\,AO = 2a\cos\theta$

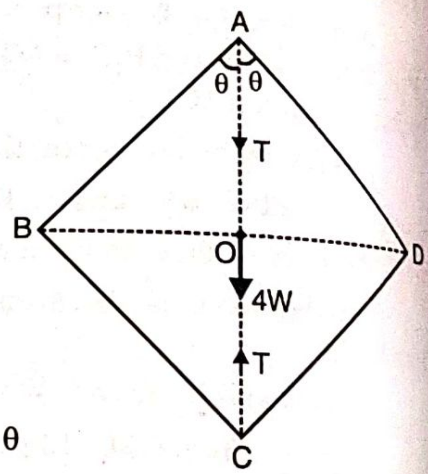

Fig. 8.18

Let the system be given a small vertical displacement so that θ changes to $\theta + \delta\theta$.

$\therefore$ Equation of virtual work is

$$4W\,\delta(AO) - T\,\delta(AC) = 0$$

or $$4W.\delta(a\cos\theta) - T.\delta(2a\cos\theta) = 0$$

or $$4W(-a\sin\theta)\,\delta\theta - T(-2a\sin\theta)\,\delta\theta = 0$$

or $$2aT\sin\theta = 4aW\sin\theta \qquad [\because \delta\theta \neq 0]$$

$\therefore$ $$T = 2W \qquad ...(1)$$

Let l be the natural length of the string

Extended length of the string $= AC = 2a\cos\theta$

$\therefore$ By Hooke's law, $$T = \lambda\frac{\text{extension}}{\text{natural length}}$$

i.e., $$T = \lambda\frac{AC - l}{l}$$

$$\therefore \quad T = W.\frac{2a\cos\theta - l}{l}$$

$$2W = W.\frac{2a\cos\theta - l}{l} \qquad [\because \lambda = W\,(given)]$$

$$2l = 2a\cos\theta - l \qquad [Using\ (1)]$$

$$\therefore \quad l = \frac{2a}{3}\cos\theta = \frac{2a}{3}\cos 45^\circ$$

$$= \frac{2a}{3}\cdot\frac{1}{\sqrt{2}} \qquad [\because In\ equilibrium\ position,\ \theta = 45^\circ]$$

Hence, natural length $= \dfrac{a\sqrt{2}}{3}$.

Example 10.

A heavy elastic string, whose natural length $2\pi a$ is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is $a\left[1 + \dfrac{W\cot\alpha}{2\pi\lambda}\right]$. [K.U. 2015; M.D.U. 2014, 05]

Solution. Here, V is the vertex, C is the centre of the base of the cone and r, the radius of the endless string AB.

Let the ring be in equilibrium at a depth x below the vertex V of the cone.

Then the radius of the circle formed by the string

$$= AG = x\tan\alpha.$$

Length of the string $= 2\pi r = 2\pi x\tan\alpha$.

The weight W of the string acts at G.

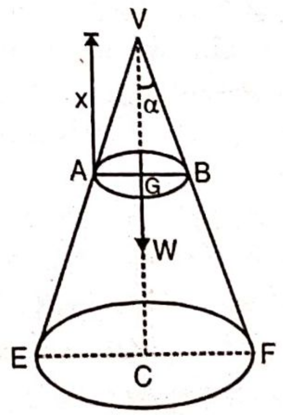

Fig. 8.19

Let the string be given a small virtual displacement downwards by a distance δx.

Then, equation of virtual work is

$$W.\delta(x) - T.\delta(2\pi x\tan\alpha) = 0$$

or

$$W.\delta x - 2\pi T\tan\alpha.\delta x = 0$$

or

$$T = \frac{W}{2\pi}\cot\alpha \qquad ...(1)\ [\because \delta x \neq 0]$$

Also, by Hooke's law, $T = \lambda\dfrac{\text{extension}}{\text{natural length}} = \lambda\dfrac{2\pi x\tan\alpha - 2\pi a}{2\pi a} = \lambda\left(\dfrac{x\tan\alpha - a}{a}\right)$ ...(2)

From (1) and (2), $\lambda\left(\dfrac{x\tan\alpha - a}{a}\right) = \dfrac{W}{2\pi}\cot\alpha$

or $$x\tan\alpha = \frac{aW}{2\pi\lambda}\cot\alpha + a = a\left[1 + \frac{W}{2\pi\lambda}\cot\alpha\right]$$

$\therefore$ Radius of string in position of equilibrium

$$r = a\left[1 + \frac{W}{2\pi\lambda}\cot\alpha\right].$$

Example 11.

One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall and the other end B is smoothly joined to an equal rod BC. The middle points of the rods are jointed by an elastic string of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A and the angle between the rods is $2\sin^{-1}\left(\dfrac{3}{4}\right)$.

Solution. Let G_1, G_2 be the mid-points of AB and BC. Therefore $AG_1 = G_1B = a$.

Let T be the tension in the string G_1G_2. The total weight 2W of two equal rods acts at G, the mid-point of G_1G_2.

Let each rod be inclined to the horizontal at an angle θ.

Since the point A is fixed, therefore the distances are measured from A downwards

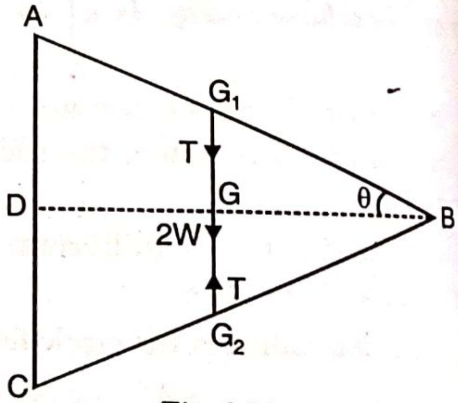

Fig. 8.20

Now, $G_1G_2 = 2G_1G = 2a\sin\theta$

$AD = AB\sin\theta = 2a\sin\theta$

Let the system be given a small virtual displacement such that θ changes to $\theta + \delta\theta$.

$\therefore$ Equation of virtual work is $2W.\delta(AD) - T.\delta(G_1G_2) = 0$

or $$2W\delta(2a\sin\theta) - T\,\delta(2a\sin\theta) = 0$$

or $$4aW\cos\theta - 2aT\cos\theta = 0$$

or $$2aT\cos\theta = 4aW\cos\theta \qquad [\because \delta\theta \neq 0]$$

or $$T = 2W \qquad ...(1)$$

By Hooke's law, $$T = \lambda\frac{\text{change in length}}{\text{original length}}$$

i.e., $$T = 4W\left(\frac{G_1G_2 - a}{a}\right) \qquad [\because \lambda = 4W]$$

or $$T = 4W\left(\frac{2a\sin\theta - a}{a}\right)$$

or $$T = 4W(2\sin\theta - 1)$$

From (1) and (2), we have $2W = 4W(2\sin\theta - 1)$ ...(2)

or $$W = 2W(2\sin\theta - 1)$$

or $$4\sin\theta = 3 \Rightarrow \theta = \sin^{-1}\left(\frac{3}{4}\right)$$

Hence, the required angle between AB and BC $= 2\theta = 2\sin^{-1}\left(\frac{3}{4}\right)$.

Some Miscellaneous Solved Examples :

Example 12.

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ and φ are the inclinations of the string and the plane base of the hemisphere to the vertical, show that $\tan\phi = \frac{3}{8} + \tan\theta$.   [K.U. 2018, 17, 16, 15, 08, 05; M.D.U. 2011, 08]

Solution. Let O be the point of suspension in the wall, AB the base of the hemisphere, C its centre, G its centre of gravity, M the point of contact of the hemisphere and the wall and OA the string. Let l be the length of the string OA and let a be the radius of the hemisphere.

$$\therefore \quad CA = a \quad \text{and} \quad CG = \frac{3a}{8}$$

Since O is a fixed point, so all the distances will be measured from this point O.

Let d be the depth of G below O

$$\therefore \quad d = OM + FG = OL + LM + CG\sin\phi$$

$$= l\cos\theta + AC\cos\phi + \frac{3a}{8}\sin\phi$$

$$\Rightarrow \quad d = l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi \quad ...(1)$$

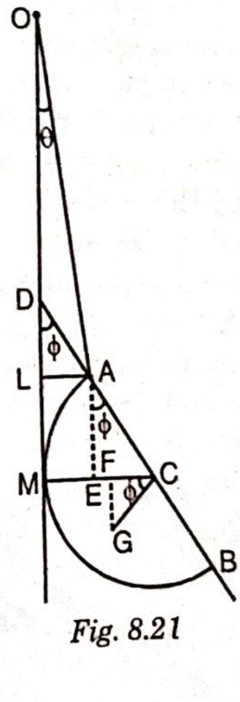

Fig. 8.21

The normal reaction at M is perpendicular to the wall.

∴ MC is horizontal

Let the system be given a small virtual displacement such that θ becomes θ + δθ and φ becomes φ + δφ.

W, the weight of the hemisphere will be the only force doing work. The reaction at M does not appear in the equation of virtual work

∴ Equation of virtual work is $W.\delta(d) = 0$

or $$\delta(d) = 0 \quad [\because W \neq 0]$$

or $\delta\left[l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right] = 0$

or $-l\sin\theta\,.\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3a}{8}\cos\phi\,\delta\phi = 0$

or $l\sin\theta.\delta\theta = \left(\frac{3}{8}\cos\phi - \sin\phi\right).\,a\delta\phi$ ...(2)

Again, $a = CM = CE + EM = CE + AL$ $[\because EM = AL]$

$= CA\sin\phi + OA\sin\theta$

$= a\sin\phi + l\sin\theta$

or $l\sin\theta = a - a\sin\phi$

Differentiating, $l\cos\theta\,.\,\delta\theta = -a\cos\phi\,\delta\phi$ ...(3)

Dividing (2) by (3), we get $\tan\theta = -\frac{3}{8} + \tan\phi$

Hence $\tan\phi = \frac{3}{8} + \tan\theta$.

Example 13.

Five equal uniform rods, freely jointed at their ends form a regular pentagon ABCDE and BE is jointed by a light rod. The system is suspended from A in a vertical plane. Prove that the thrust in BE is W cot $\left(\frac{\pi}{10}\right)$, where W is the weight of each rod.

[K.U. 2014, 04; M.D.U. 2007]

Solution. Let T be the thrust in the rod BE. Replace this rod by two forces T at B and T at E acting along EB and BE respectively.

The weights W at S_1, the mid-point of AB and W at S_4, the mid-point of AE are equivalent to weight 2W at K, the mid-point of S_1S_4.

The weights W, W at S_2 and S_3, the mid-points of BC and ED, are equivalent to the weight 2W at P, the mid-point of $S_2\,S_3$.

Let θ be the angle which the upper slant rods make with the vertical AF and ϕ, the angle which lower slant rods make with the vertical. The framework is suspended from A, therefore, we measure all the distances from A downwards.

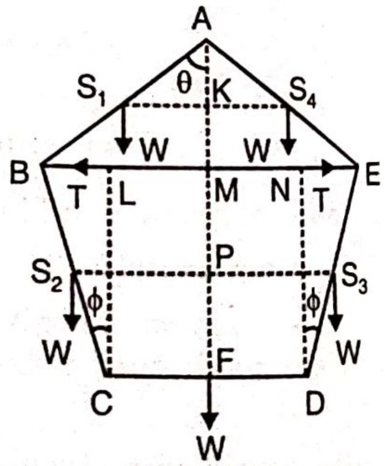

Fig. 8.22

Let 2*a* be the length of each rod.

Now, $AK = AS_1\cos\theta = a\cos\theta$

$AP = AM + MP = AB\cos\theta + \frac{1}{2}CL = 2a\cos\theta + a\cos\phi$

$$AF = AM + MF = 2a\cos\theta + 2a\cos\phi$$
$$BE = 2BM = 2(2a\sin\theta) = 4a\sin\theta.$$

Let the system be given a small virtual displacement such that θ becomes $\theta + \delta\theta$ and ϕ becomes $\phi + \delta\phi$.

The equation of virtual work is

$$2W\delta(AK) + 2W\delta(AP) + W\delta(AF) + T\delta(BE) = 0$$
$$2W\delta(a\cos\theta) + 2W\delta(2a\cos\theta + a\cos\phi) + W\delta(2a\cos\theta + 2a\cos\phi) + T\delta(4a\sin\theta) = 0$$
$$2W(-a\sin\theta)\,\delta\theta + 2W(-2a\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi) + W(-2a\sin\theta\,\delta\theta - 2a\sin\phi\,\delta\phi) + T(4a\cos\theta)\,\delta\theta = 0$$
$$(4T\cos\theta - 8W\sin\theta)\,\delta\theta - 4W\sin\phi\,\delta\phi = 0$$
$$(T\cos\theta - 2W\sin\theta)\,\delta\theta = W\sin\phi\,\delta\phi \quad ...(1)$$

But ϕ and θ are not independent of each other.

Let us find the relation between them.

In Δ ABM, $BM = AB\sin\theta = 2a\sin\theta$

In quadrilateral BCFM, $BM = BL + LM = 2a\sin\phi + a$

$$\therefore \quad 2a\sin\phi + a = 2a\sin\theta$$
$$2\sin\theta = 2\sin\phi + 1$$

Differentiating, we get $2\cos\theta\,\delta\theta = 2\cos\phi\,\delta\phi$

$$\cos\theta\,\delta\theta = \cos\phi\,\delta\phi \quad ...(2)$$

Dividing (1) by (2), we have

$$(T - 2W\tan\theta) = W\tan\phi$$
$$T = 2W\tan\theta + W\tan\phi \quad ...(3)$$

In the position of equilibrium, $2\theta = 108°$ $\left[\because \text{ Each external angle} = \dfrac{360°}{5} = 72°\right]$

$$\therefore \quad \theta = 54° = \frac{3\pi}{10}$$

$$\phi = 90° - 72° = 18° = \frac{\pi}{10}$$

$\therefore$ From (3),
$$T = W\left[2\tan\frac{3}{10}\pi + \tan\frac{\pi}{10}\right]$$
$$= W\left[2\cot\frac{2}{10}\pi + \tan\frac{\pi}{10}\right] \quad \left[\because \frac{3\pi}{10} = \frac{\pi}{2} - \frac{2\pi}{10}\right]$$
$$= W\left[\frac{2}{\left(\tan\frac{2\pi}{10}\right)} + \tan\frac{\pi}{10}\right] = W\left[\frac{2\left(1 - \tan^2\frac{\pi}{10}\right)}{2\tan\frac{\pi}{10}} + \tan\frac{\pi}{10}\right]$$

$$= W\left[\frac{1-\tan^2\frac{\pi}{10}+\tan^2\frac{\pi}{10}}{\tan\frac{\pi}{10}}\right] = W\cot\frac{\pi}{10}.$$

Hence, thrust along rod BE= $W\cot\left(\frac{\pi}{10}\right)$.

Example 14.

A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find the position of equilibrium and show that its length must be at least $\frac{3}{4}a+\frac{1}{4}a\sqrt{1+8e^2}$, where 2a is length of major axis and e is the eccentricity.

Solution. Let AB = 2l be the length of the rod of weight W which passes through the ring at S. Let weight W of the rod act at G.

$\therefore$ AG = GB = l

Let $\angle CSA = \theta$

The equation of the ellipse referred to S as pole is

$$\frac{l'}{r} = 1+e\cos(\pi+\theta) = 1-e\cos\theta \qquad ...(1)$$

where l' is length of semi latus-rectum.

Let (r, θ) be the polar co-ordinates of point A.

Now, SG = SA − AG = $r - l$

$\therefore$ GM = SG sin θ = $(r-l)\sin\theta$ ...(2)

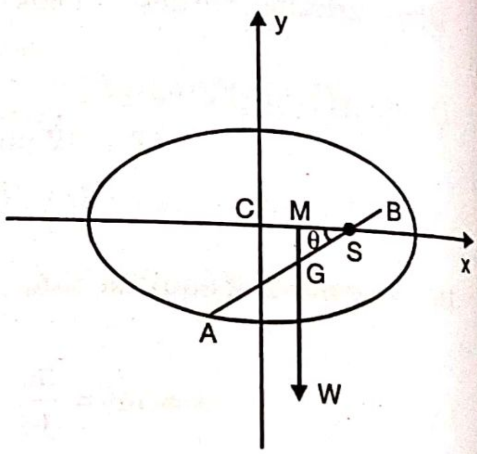

Fig. 8.23

Let the system be given a small virtual displacement *i.e.*, θ is changed to θ + δθ

$\therefore$ Equation of virtual work is $W\delta(GM) = 0$

i.e. $W\delta(r-l)\sin\theta = 0$ [Using (2)]

$$\therefore \quad W\delta\left[\frac{l'}{1-e\cos\theta}-l\right]\sin\theta = 0 \qquad \text{[Using (1)]}$$

$$\therefore \quad \left[\frac{-l'e\sin\theta}{(1-e\cos\theta)^2}\sin\theta+\left(\frac{l'}{1-e\cos\theta}-l\right)\cos\theta\right]\delta\theta = 0$$

$$\Rightarrow -l'e\sin^2\theta + l'\cos\theta(1-e\cos\theta) - l\cos\theta(1-e\cos\theta)^2 = 0$$

or $$l'\cos\theta - l'e(\sin^2\theta+\cos^2\theta) = l\cos\theta(1-e\cos\theta)^2 \qquad [\because \delta\theta\neq 0]$$

or $$l'\cos\theta - l'e = l\cos\theta(1-e\cos\theta)^2 \qquad ...(3)$$

Now for the least length of the rod, B must coincide with S

$$AB = 2l = r = \frac{l'}{1 - e\cos\theta} \qquad ...(4)$$

Eliminating l from (3) and (4), $l'\cos\theta - l'e = \dfrac{l'}{2(1-e\cos\theta)}\cos\theta\,(1-e\cos\theta)^2$

$$2l'\cos\theta - 2l'e = l'\cos\theta - l'e\cos^2\theta$$

$$e\cos^2\theta + \cos\theta - 2e = 0$$

$$\cos\theta = \frac{-1 \pm \sqrt{1+8e^2}}{2e} \qquad ...(5)$$

Now eliminating θ from equations (4) and (5), we have

$$2l = \frac{l'}{1 - e\left(\dfrac{(-1+\sqrt{1+8e^2})}{2e}\right)} = \frac{2l'}{2+1-\sqrt{1+8e^2}}$$

$$\therefore \quad AB = 2l = \frac{2l'}{3-\sqrt{1+8e^2}} \qquad ...(6)$$

Also we know that $l' = \dfrac{b^2}{a} = \dfrac{a^2(1-e^2)}{a} = a(1-e^2)$ ...(7)

Eliminating l' from (6) and (7), we have

$$AB = \frac{2a(1-e^2)}{3-\sqrt{1+8e^2}} = \frac{2a(1-e^2)(3+\sqrt{1+8e^2})}{(9-1-8e^2)}$$

$$= \frac{2a(1-e^2)(3+\sqrt{1+8e^2})}{8(1-e^2)} = \frac{3a}{4} + \frac{1}{4}a\sqrt{1+8e^2}.$$

EXERCISE 8.1

1. Four rods of equal weight w form a rhombus, with smooth hinges at the joints. The frame is suspended by the point A and a weight W is attached to point C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping them inclined at an angle α to AC. Show that the thrust in the stiffening rod is $(2W + 4w)\tan\alpha$.
2. A square framework formed of uniform heavy rods of equal weight W joined together, is hung up by one corner A. Weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.
3. Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A and is kept in shape by a string AC. Prove that the tension of the string is equal to half the whole weight. *[M.D.U. 2007]*

4. Four equal heavy uniform rods are hinged together to form a rhombus ABCD and the frame is suspended from A. The corners A and C are connected by an inelastic string to prevent them from collapsing. Prove that the tension in the string is T = 2W. [M.D.U. 2011]

5. Four equal uniform rods each of weight w are freely joined to form a rhombus ABCD. The frame work is suspended freely from A and a weight W is attached to each of the joints B, C and D. If two horizontal forces each of magnitude P acting at B and D keep the angle BAD equal to 120°, prove that $P = (W + w)\,2\sqrt{3}$.

6. Six equal heavy beams are freely jointed at their ends to form a hexagon and are placed in a vertical plane with one beam resting on the horizontal plane, the middle points of the two upper slant beams, which are inclined at an angle θ to the horizontal are connected by a light string. Find the tension in terms of W and θ, where W is the weight of each beam.

7. A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.

8. Six equal bars are freely joined at the extremities forming a regular hexagon ABCDEF, which is kept in shape by vertical strings joining the middle points of BC, CD and AF, EF respectively; the side AB being horizontal and uppermost. Prove that the tension of each string is three times the weight of each bar. [M.D.U. 2015]

9. Six equal light rods are jointed to form a regular hexagon ABCDEF which is suspended at A and F so that AF is horizontal. A light rod BE also keeps the frame from collapsing and is of such a length that the rods ending in the points B, E are inclined at angles of 45° to the vertical. Equal weights are suspended from B, C, D and E. Find the stress in BE. [M.D.U. 2009]

10. Four equal uniform rods, each of weight W are jointed so as to form a square ABCD, the side AB is fixed in vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC. Find the tension of the string. [K.U. 2015]

11. Four rods are jointed to form a parallelogram, the opposite points are joined by strings forming diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths. [K.U. 2008; M.D.U. 2006]

12. A regular hexagon is composed of six heavy uniform rods freely joined together and two opposite edges are connected by the string which is horizontal, one rod being in contact with a horizontal plane, at the middle point of the opposite rod is placed a weight W_1. If W be the weight of each rod, show that the tension of the string is $\dfrac{3W + W_1}{\sqrt{3}}$. [M.D.U. 2007]

13. Two rods each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension at the string and if the string be slack, show that ϕ satisfies the equation $\tan^3\phi + \tan\phi = \dfrac{l}{2r}$. [M.D.U. 2011, 08; K.U. 2004]

VIRTUAL

14. A rh fron Pro

T =

15. A If th

ar

16. A a

17.

18

14. A rhombus ABCD is formed of four equal uniform rods freely jointed together and suspended from the point A. It is kept in position by a light rod joining the mid-points of BC and CD. Prove that if T be the thrust in this rod and W the weight of the rhombus, then $T = W \tan \frac{A}{2}$.

[M.D.U. 2014, 12]

15. A frame consists of five bars forming the sides of a rhombus ABCD with diagonal AC. If four equal forces P act inwards at the middle points of the sides and at right angle to the respective sides, prove that the tension in AC is $\frac{P \cos 2\theta}{\sin \theta}$, where θ denotes the angle BAC.

16. A uniform beam of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium, the rod is inclined to the wall at an angle $\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$.

[C.D.L.U. 2012; M.D.U. 2008, 07; K.U. 2007]

17. A rhombus is formed of rods each of weight W and length l with smooth joints. It rests symmetrically with its upper sides in contact with two smooth pegs at the same level and at a distance $2a$ apart. A weight W′ is hung at the lowest point. If the sides of rhombus make an angle θ with the vertical, prove that $\sin^3 \theta = \frac{a}{l} \frac{(4W + W')}{(4W + 2W')}$. [M.D.U. 2005]

18. ABCD is a rhombus formed with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD. B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is 2α, show that the tension in the string is $W \tan \alpha \left(\frac{c}{2l} \operatorname{cosec}^3 \alpha - 1\right)$.

19. An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be 2α. [K.U. 2014; M.D.U. 2013, 04]

20. A smooth cone of weight W stands inverted in a circular hole with its axis vertical. A string is wrapped twice round the cone just above the hole and pulled tight. What must be the tension in the string so that it will just raise the cone ? [K.U. 2006]

21. Two uniform rods AB and AC, smoothly joined at A are in equilibrium in a vertical plane. B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is $\frac{W}{\tan B + \tan C}$, where W is the total weight of the rods AB and AC.

22. A regular pentagon ABCDE formed of equal uniform rods each of weight W is suspended from the point A and is maintained in shape by a light rod joining the middle points of BC and DE. Prove that the stress in the light rod is $2W \cot\left(\frac{\pi}{10}\right)$.

[M.D.U. 2016; C.D.L.U. 2016]

23. A uniform square lamina rests in a equilibrium in a vertical plane under gravity with two of its sides in contact with smooth pegs in the same horizontal line at a distance C apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by $C(\sin\theta + \cos\theta) = a$.

24. Four equal rods, each of length a are joined to form a rhombus ABCD and the angles B and D are joined by a string of length l. The system is placed in a vertical plane with A resting on a horizontal plane and AC is vertical. Prove that the tension of the string is $2W.\frac{l}{\sqrt{4a^2 - l^2}}$, where W is the weight of each rod. [M.D.U. 2012, 08]

ANSWERS

2. $T = 4W$ **6.** $T = 6W \cot\theta$ **9.** $T = 3W$ **10.** $T = 4\sqrt{2}\,W$

13. $T = w\left(r \sec^2\phi - \frac{1}{2} l \cot\phi\right)$ **19.** $T = \frac{W \cot\alpha}{2\pi}$ **20.** $T = \frac{W \cot\alpha}{4\pi}$.