3(a) If $f_n(x) = \frac{3}{n+x}$, $0 \le x \le 2$, state with reasons whether $\langle f_n \rangle$ converges uniformly on $[0, 2]$ or not. (10)

M_n- Test : Let $\langle f_n \rangle$ be a sequence of functions defined on an interval I such that $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in [a, b]$

and let $M_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\}$

then $\langle f_n \rangle$ converges uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

Consider,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{3}{n+x}$$

$$= 0 \quad , \quad x \in [0, 2]$$

Now, $$|f_n(x) - f(x)| = \left| \frac{3}{n+x} - 0 \right|$$

$$= \frac{3}{n+x} \le \frac{3}{n} \quad , x \in [0, 2]$$

Take, $M_n = \frac{3}{n}$

As $\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{3}{n} = 0$

Hence, $\langle f_n \rangle$ converges uniformly on $[0, 2]$.

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3(b) Examine the continuity of

$$f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ \dfrac{1}{2}, & (x,y) = (0,0) \end{cases}$$

at the point (0,0). (8)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{2(x+2y)} \times \frac{2(x+2y)}{\tan^{-1}(2x+4y)}$$

$$= \lim_{\theta\to 0} \frac{\sin^{-1}\theta}{2\theta} \cdot \lim_{\theta\to 0} \frac{2\theta}{\tan^{-1}2\theta}$$

$$= \frac{1}{2}\times 1 = \frac{1}{2} \qquad (\text{taking } x+2y=\theta)$$

Also, $f(0,0) = \frac{1}{2}$

As $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = \frac{1}{2}$

Hence, $f(x,y)$ is continuous at $(0,0)$.

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3(c) If $u(x,y) = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0<x<1$, $0<y<1$

then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. (10)

~~Let~~ $$u(x,y) = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

Let $$z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+y/x\right)}{\sqrt{x}\left(1+\sqrt{y/x}\right)}$$

$$= x^{1/2} f\left(\frac{y}{x}\right)$$

$\therefore$ z is homogeneous function of order $\frac{1}{2}$.

Using Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\left(-\sin u \cdot \frac{\partial u}{\partial x}\right) + y\left(-\sin u \cdot \frac{\partial u}{\partial y}\right) = \frac{1}{2}\cdot \cos u$$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

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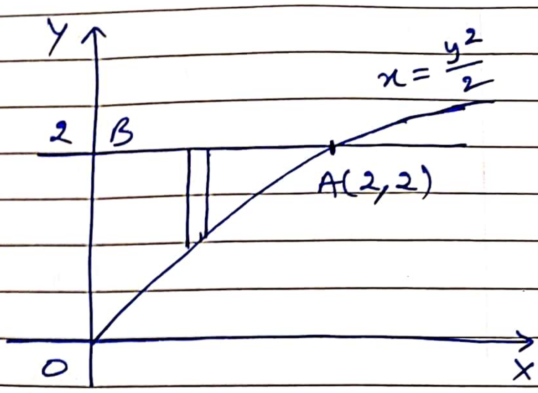
3(d) Evaluate $\displaystyle\int_0^2\int_0^{y^2/2} \frac{y}{(x^2+y^2+1)^{1/2}}\,dx\,dy$ (12)

Region of integration:

$y: 0$ to 2

$x=0$ to $x=\frac{y^2}{2}$.



On change of order of integration.

$x=0$ to $x=2$

$y=\sqrt{2x}$ to $y=2$

$$I = \int_{x=0}^{2}\int_{\sqrt{2x}}^{2} \frac{y}{(x^2+y^2+1)^{1/2}}\,dy\,dx$$

$$= \int_0^2 (x^2+y^2+1)^{1/2}\Big|_{\sqrt{2x}}^{2}\,dx$$

$$= \int_0^2 \sqrt{x^2+4+1} - \sqrt{x^2+2x+1}\,dx$$

$$= \int_0^2 \sqrt{x^2+5} - \int_0^2 (x+1)\,dx$$

$$= \left[\frac{x}{2}\sqrt{x^2+5}\right]_0^2 + \left[\frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|\right]_0^2 - \left(\frac{x^2}{2}+x\right)_0^2$$

$$= \left(\frac{2}{2}\sqrt{4+5}-0\right) + \left(\frac{5}{2}\log\left|2+\sqrt{4+5}\right| - \frac{5}{2}\log\sqrt{5}\right) - \left(\frac{4}{2}+2\right)$$

$$= (3-4) + \frac{5}{2}\log 5 - \frac{5}{4}\log 5$$

$$\boxed{I = \frac{5}{4}\log 5 - 1}$$

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