Generating lines of conicoids

(1)

Ruled Surfaces:

The surfaces which are generated by a moving straightline are called ruled surfaces.

For example, cones, cylinders, the hyperboloids of one sheet and hyperbolic paraboloids are ruled surfaces.

— A ruled surface can also be defined as one through every point of which a straightline can be drawn so as to lie completely on it. The lines which lie on the surfaces are called its generating lines.

* Generating Lines of a hyperboloid of one sheet:

we write the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. ——(1)

of a hyperboloid of one sheet in the form $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$

$$\Rightarrow \left(\frac{x}{a}\right)^2-\left(\frac{z}{c}\right)^2=1-\left(\frac{y}{b}\right)^2$$

$$\Rightarrow \left[\left(\frac{x}{a}\right)-\frac{z}{c}\right]\left[\frac{x}{a}+\frac{z}{c}\right]=\left(1-\frac{y}{b}\right)\left(1+\frac{y}{b}\right)$$

$$\Rightarrow \frac{\frac{x}{a}-z/c}{1-\frac{y}{b}}=\frac{1+\frac{y}{b}}{\frac{x}{a}+z/c}=\lambda \text{ (say)} \quad \text{——(2)}$$

and $$\frac{x/a-z/c}{1+y/b}=\frac{1-\frac{y}{b}}{\frac{x}{a}+z/c}=\mu \text{ say.} \quad \text{——(3)}$$

$$\Rightarrow \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right);\ \left(1+\frac{y}{b}\right)=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$$

& —— (A)

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right);\ 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right)$$

—— (B)

are two families of lines

where λ, μ are parameters.

Clearly, ~~ever~~ it will be shown that every point of each of the lines (A) & (B) lies on the hyperboloid (1)

because, if (x_0, y_0, z_0) be a point on a family (A) obtained some value λ_0 of λ, we have $\frac{x_0}{a}-\frac{z_0}{c}=\lambda_0\left(1-\frac{y_0}{b}\right);\ 1+\frac{y_0}{b}=\lambda_0\left(\frac{x_0}{a}+\frac{z_0}{c}\right)$

On eliminating λ_0 from these, we obtain

$$\frac{x_0^2}{a^2}-\frac{z_0^2}{c^2}=1-\frac{y_0^2}{b^2} \Leftrightarrow \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1.$$

Clearly, it shows that (x_0, y_0, z_0) is a point of the hyperboloid (1).

A similar proof holds for the family of lines (B).

Thus, as λ and μ vary, we get two families of lines (A) and (B) each member of which lies wholly on the hyperboloid. These two families of lines are called two systems of generating lines (or generators) of the hyperboloid.

Generally, the ruled surfaces may be divided into two categories : (i) developable surfaces
(ii) skew surfaces

A developable surface is one on which the consecutive generators intersect while on a skew surface, ~~at all~~ the consecutive generating lines donot intersect. The cone is a developable surface as all the generators pass through a common vertex and the cylinder is also a developable surface as consecutive generators touch all along their length. The hyperboloid of one sheet and the hyperbolic paraboloid are skew surfaces.

* Properties of generating lines of hyperboloid of one sheet:

Let the equation of hyperboloid of one sheet be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ —— (1)

we know that the system of generators of (1) are given by the equations

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \;;\; \left(1+\frac{y}{b}\right)=\lambda\left(\frac{x}{a}+\frac{z}{c}\right) \quad \text{—— (2)}$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right) \;;\; \left(1-\frac{y}{b}\right)=\mu\left(\frac{x}{a}+\frac{z}{c}\right) \quad \text{—— (3)}$$

Property (I) One generator of each system passes through every point of the

hyperboloid. &
i.e, through every point of the hyperboloid there passes one generator of each system.

Sol. let $P(x_0, y_0, z_0)$ be a point of the hyperboloid (1)
so that, we have

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1. \quad \text{--- (4)}$$

Now the generator

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$$

will pass through the point (x_0, y_0, z_0)
$\Leftrightarrow$ λ has a value equal to each of the two fractions

$$\frac{\frac{x_0}{a}-\frac{z_0}{c}}{1-\frac{y_0}{b}}=\frac{1+\frac{y_0}{b}}{\frac{x_0}{a}+\frac{z_0}{c}}$$

$$\Leftrightarrow \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1 \quad \text{--- (5)}$$

which is same as (4)

Thus, the member of the system (2) corresponding to either of the equal values (5) of λ will pass through the given point (x_0, y_0, z_0). Similarly it can be shown that the member of the system (3) corresponding to either of the equal values

$$\frac{\left(\frac{x_0}{a}-\frac{z_0}{c}\right)}{\left(1+\frac{y_0}{b}\right)},\ \frac{\left(1-\frac{y_0}{b}\right)}{\left(\frac{x_0}{a}+\frac{z_0}{c}\right)}$$

of μ passes through the given point (x_0, y_0, z_0)

Property (ii):

No two generators of the same system intersect.

Let
1. (i) $\frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right)$, (ii) $1+\frac{y}{b}=\lambda_1\left(\frac{x}{a}+\frac{z}{c}\right)$

2. (iii) $\frac{x}{a}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right)$, (iv) $1+\frac{y}{b}=\lambda_2\left(\frac{x}{a}+\frac{z}{c}\right)$

be any two different generators of the λ system.

It will be shown that these four equations in x, y, z are not consistent.

Subtracting (iii) from (i), we obtain

$(\lambda_1-\lambda_2)\left(1-\frac{y}{b}\right)=0 \Rightarrow y=b$ for $\lambda_1\neq\lambda_2$

Again, from (ii) and (iv), we obtain

$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\left(1+\frac{y}{b}\right)=0 \Rightarrow y=-b$ for $\lambda_1\neq\lambda_2$

Thus, we see that these four equations are inconsistent and accordingly the two lines donot intersect.

Property (iii):

Any two generators belonging to different systems intersect.

Let
1. (i) $\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right)$, (ii) $1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$

2. (i) $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$, (iv) $1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right)$

be two generators, one of each system.

It will be shown that these four equations in x, y, z are not consistent

firstly, we solve simultaneously the equations (i), (ii) and (iii). Now, (i) and (ii) give

$$\lambda\left(1-\frac{y}{b}\right)=\mu\left(1+\frac{y}{b}\right) \implies y=b\,\frac{\lambda-\mu}{\lambda+\mu}$$

substituting this value of y in (i) and (ii), we obtain $\frac{x}{a}-\frac{z}{c}=\frac{2\lambda\mu}{\lambda+\mu}$, $\frac{x}{a}+\frac{z}{c}=\frac{2}{\lambda+\mu}$

These give, on adding and subtracting,

$$x=a\,\frac{1+\lambda\mu}{\lambda+\mu}, \quad z=c\,\frac{1-\lambda\mu}{\lambda+\mu}$$

Now, as may easily be seen, these values of x, y, z satisfy (iv) also.

Thus, the two lines intersect and the point of intersection is

$$\left(a\,\frac{1+\lambda\mu}{\lambda+\mu},\ b\,\frac{\lambda-\mu}{\lambda+\mu},\ c\,\frac{1-\lambda\mu}{\lambda+\mu}\right) \longleftarrow 6$$

Another Method:-

The planes

$$\frac{x}{a}-\frac{z}{c}-\lambda\left(1-\frac{y}{b}\right)-k\left[1+\frac{y}{b}-\lambda\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0$$

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)-k'\left[1-\frac{y}{b}-\mu\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0$$

pass through the two lines respectively for all values of k and k'.

Now, obviously these equations becomes identical for $k=\mu$ and $k'=\lambda$.

Thus, the two lines are coplanar and as such they intersect. Also the plane through the two lines, obtained by putting $k=\mu$ or $k'=\lambda$ is

$$\frac{1+\lambda\mu}{\lambda+\mu}\cdot\frac{x}{a}+\frac{\lambda-\mu}{\lambda+\mu}\frac{y}{b}-\frac{1-\lambda\mu}{\lambda+\mu}\cdot\frac{z}{c}=1.$$

* Parametric equations of the hyperboloid:-

The co-ordinates (6) show that

$$x=a\frac{1+\lambda\mu}{\lambda+\mu},\quad y=b\frac{\lambda-\mu}{\lambda+\mu}\quad z=c\frac{1-\lambda\mu}{\lambda+\mu}$$

are the parametric equations of the hyperboloid, λ,μ being the parameters. These co-ordinates satisfy the equation of the hyperboloid ~~for all~~ values of the parameters λ and μ

Generating Lines

§ 13.01. Ruled Surfaces.

We are already aware that cylinder and cone are the surfaces which are generated by the motion of a straight line. Similarly hyperboloid of one sheet and hyperbolic paraboloid are also generated by the motion of a straight line. This type of surfaces which are generated by the motion of a straight line is known as a **ruled surface.** Thus through every point on a ruled surface a straight line can be drawn which lies wholly (entirely) on the ruled surface. These straight lines are called the **generating lines or generators.**

**§ 13.02. Generating Lines of a hyperboloid of one sheet.

(*Gorakhpur* 97)

We know equation of the hyperboloid of one sheet is

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \qquad \text{...(i)}$$

Consider any straight line whose equations are

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right). \qquad \text{...(ii)}$$

where λ is constant. (arbitrary constant)

If we multiply above two equations given by (ii), we find that λ is eliminated and we get $\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=1-\dfrac{y^2}{b^2}$ or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1.$

which is the equation (i) of the hyperboloid of one sheet.

Thus we conclude that all those points which lie on the straight line (ii) *i.e.* whose coordinates satisfy the equation given by (ii), must also satisfy the equation (i) of the hyperboloid of one sheet.

Hence the straight line whose equations are given by (ii) lies wholly on the surface given by (i).

Assigning different values to the constant λ, we find that equations (ii) represents an infinite number of straight lines all of which lie wholly on the hyperboloid of one sheet given by the equation (i) so that these lines cover the whole surface. These lines are the generators or the generating lines of the surface given by (i).

In a manner similar to above we can show that the system of lines given by the equations

$$\boxed{\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)} \qquad \text{...(iii)}$$

lie wholly on the hyperboloid of one sheet given by (i) and as such are its generators or generating lines.

Hence as λ and μ vary, we obtain, two families of straight lines such that every member of each family lies wholly on the hyperboloid of one sheet given by (i). These two families of straight lines given by (ii) and (iii) are known as two systems of generating lines of (i).

181/XIII/1

$$\Rightarrow \left(\frac{y}{b}-\frac{z}{c}\right)=\mu\left(1+\frac{x}{a}\right),\ \left(\frac{y}{b}-\frac{z}{c}\right)=\frac{1}{\mu}\left(1-\frac{x}{a}\right)$$

(or

§ 13.03. Properties of generating lines (or generators) of hyperboloid of one sheet.

Let the equation of hyperboloid of one sheet be

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{...(i)}$$

~~From § 13.02 above~~ we know that the systems of generators of (i) are given by the equations

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right) \quad \text{...(ii)}$$

and

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad \text{...(iii)}$$

Prop I. *One generator of each system passes through every point of the hyperboloid.*

Let $P(\alpha, \beta, \gamma)$ be any point on the hyperboloid (i), then

$$(\alpha^2/a^2)+(\beta^2/b^2)-(\gamma^2/c^2)=1 \quad \text{...(iv)}$$

Now, the generators of first system of (i) given by (ii) will pass through the point $P(\alpha, \beta, \gamma)$ if and only if λ has a value equal to each of the fractions

$$\lambda = \frac{(\alpha/a)-(\gamma/c)}{1-(\beta/b)}, \frac{1+(\beta/b)}{(\alpha/a)+(\gamma/c)} = \lambda \text{ [obtained from (ii)]} \quad \text{...(v)}$$

or $[(\alpha/a)-(\gamma/c)]\,[(\alpha/a)+(\gamma/c)]=[1+(\beta/b)]\,[1-(\beta/b)]$, equating the two values of γ from (v)

or $(\alpha^2/a^2)-(\gamma^2/c^2)=1-(\beta^2/b^2)$ or $(\alpha^2/a^2)+(\beta^2/b^2)-(\gamma^2/c^2)=1$, which is true by virtue of (iv).

Thus if λ is chosen equal to the values given by either of the fractions (v), the corresponding line (generator) of the system of generators (ii) will pass through the point $P(\alpha, \beta, \gamma)$.

Similarly we can show that if μ is equal to either of the fraction $\dfrac{(\alpha/a)-(\gamma/c)}{1+(\beta/b)}$ or $\dfrac{1-(\beta/b)}{(\alpha/a)+(\gamma/c)}$ [obtained by evaluating μ from the equations given by (iii)], then a member of the system of generators (iii) corresponding to either of equal values of μ will pass through the point $P(\alpha, \beta, \gamma)$.

Prop II. *No two generators of the same system intersect.*

Consider two generators of the λ-system given by (ii) corresponding to two distinct values λ_1, λ_2 of λ.

$$\frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right), \quad \text{...(vi)}$$

$$\frac{x}{b}+\frac{z}{c}=\frac{1}{\lambda_1}\left(1+\frac{y}{b}\right), \quad \text{...(vii)}$$

and
$$\frac{x}{c}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right), \quad \text{...(viii)}$$

$$\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda_2}\left(1+\frac{y}{b}\right), \quad \text{...(ix)}$$

Substracting (viii) from (vi) we have $(\lambda_1-\lambda_2)\left(1-\frac{y}{b}\right)=0$

or $1-(y/b)=0, \quad \because \quad \lambda_1 \neq \lambda_2$

or $y=b$

Similarly subtracting (ix) from (vii), we get $y=-b$

Thus we find that the four equations giving two generators of the same system are inconsistent and so we conclude that the two generators of the same system do not intersect.

Prop. III. *Any two generators of the different systems intersect.*

Let us consider two generator one of each system given by (ii) and (iii) *i.e.*

$$(x/a)-(z/c)=\lambda\,[1-(y/b)], \quad \text{...(x)}$$

$$(x/a)+(z/c)=(1/\lambda)\,[1+(y/b)], \quad \text{...(xi)}$$

and
$$(x/a)-(z/c)=\mu\,[1+(y/b)], \quad \text{...(xii)}$$

$$(x/a)+(z/c)=(1/\mu)\,[1-(y/b)] \quad \text{...(xiii)}$$

Solving (x) and (xii), we get $\lambda\,[1-(y/b)]=\mu\,[1+(y/b)]$

or
$$y/b=(\lambda-\mu)/(\lambda+\mu) \quad \text{...(xiv)}$$

$\Rightarrow$
$$1-(y/b)=1-[(\lambda-\mu)/(\lambda+\mu)]=2\mu/(\lambda+\mu) \quad \text{...(xv)}$$

$\therefore$ From (x) and (xi) with the help of (xv) we get

$(x/a)-(z/c)=2\lambda\,\mu/(\lambda+\mu)$ and $(x/a)+(z/c)=2/(\lambda+\mu)$

Solving these, we get $\frac{x}{a}=\frac{1+\lambda\mu}{\lambda+\mu}, \frac{z}{c}=\frac{1-\lambda\mu}{\lambda+\mu}$ ...(xvi)

The values of x, y and z given by (xvi) and (xiv) also satisfy the equation (xiii) which show that the two generators of λ, μ systems interrect at the point

$$\left(\frac{a(1+\lambda\mu)}{\lambda+\mu}, \frac{b(\lambda-\mu)}{\lambda+\mu}, \frac{c(1-\lambda\mu)}{\lambda+\mu}\right) \quad \text{...(xvii)}$$

For all values of λ and μ the coordinates of this point satisfy the equation (i) of the hyperboloid of one sheet and hence the parametric equations of the hyperboloid of one sheet given by (i) can be taken as

$$x=\frac{a(1+\lambda\mu)}{\lambda+\mu}, y=\frac{b(\lambda-\mu)}{\lambda+\mu}, z=\frac{c(1-\lambda\mu)}{\lambda+\mu} \quad \text{...(xviii)}$$

§ 13.04. An Imporatnt Theorem.

Statement. *If three points of any straight line* $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ *lie on the conicoid* $F(x,y,z)=0$, *then the line wholly lies on the coni. oid.*

Proof. Any point on the given line is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$

If this point lies on the given conicoid $F(x, y, z) = 0$, which is a second degree equation in x, y and z, then we will get a second degree equation in r, say

$$Lr^2 + Mr + N = 0 \quad \text{(identity } (0=0)) \quad ...(i)$$

As three points of the line lie on the given conicoid, we will have three values of r which will satisfy (i) and as such it should be an identity which leads to $L = 0$, $M = 0$ and $N = 0$.

In this case, (i) is satisfied by all values of r which shows that every point on the line lies on the conicoid *i.e.* the line lies entirely (*i.e.* wholly) on the conicoid.

****§ 13.05. Condition for a given line to be a generator of a given conicoid.** (*Kumaun* 94, 93)

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$ ...(i)

and those of the given straight line be $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = r$ (say) ...(ii)

Any point on the line (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$

If this point lies on the conicoid (i), then we have

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

or $$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad ...(iii)$$

If the line (ii) is a generator of the conicoid (i), then it lies wholly on the conicoid and the conditions for which are $al^2 + bm^2 + cn^2 = 0$ ...(iv)

$$al\alpha + bm\beta + cn\gamma = 0 \quad ...(v)$$

and $$a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0 \quad ...(vi)$$

[obtained with the help of § 13.04 wherein the conditions are given as $L = 0 = M = N$]

The avove three conditions are analysed as follows :

The condition (iv) shows that the lines parallel to the generating lines (ii) and passing through the centre (0, 0, 0) of the conicoid (i) *i.e.* the lines $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ are generators of the cone $ax^2 + by^2 + cz^2 = 0$, which is known as the **asymptotic cone.**

The condition (v) shows that the generating lines whose direction cosines are l, m, n should lie on the plane $a\alpha x + b\beta y + c\gamma z = 1$, which is the equation of the tangent plane to the conicoid (i) at the point (α, β, γ).

The condition (vi) shows that the point (α, β, γ) lies on the conicoid (i).

Also the equations (iv) and (v) give the **direction ratios** l, m, n of the generating lines.

Solved Examples on § 13.01—§ 13.05.

Ex. 1. Find the equations of the generting lines of the hyperboloid $yz + 2zx + 3xy + 6 = 0$ which pass through the point $(-1, 0, 3)$.

Sol. Any line through $(-1, 0, 3)$ is $\dfrac{x+1}{l} = \dfrac{y-0}{m} = \dfrac{z-3}{n} = r$ ...(i)

$\therefore$ any point on this line is $(lr-1, mr, nr+3)$ and it lies on the given hyperboloid if $mr(nr+3) + 2(nr+3)(lr-1) + 3(lr-1)(mr) + 6 = 0$

or $$r^2(mn + 2nl + 3lm) + r(3m - 2n + 6l - 3m) = 0$$

or $$r^2(mn + 2nl + 3lm) + r(6l - 2n) = 0 \qquad ...(ii)$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the conditions for which from (ii) are

$$mn + 2nl + 3lm = 0 \quad \text{and} \quad 6l - 2n = 0 \qquad [L = 0 = M = N]$$

$\therefore$ $n = 3l$ and hence $m(3l) + 2(3l)l + 3lm = 0$, on eliminating n.

or $$6l(l+m) = 0 \Rightarrow l = 0 \quad \text{or} \quad l = -m$$

and when $l = -m$, $n = 3l \Rightarrow \dfrac{l}{1} = \dfrac{m}{-1} = \dfrac{n}{3}$

Hence from (i), the generators are

$$\frac{x+1}{0} = \frac{y}{m} = \frac{z-3}{0} \quad \text{and} \quad \frac{x+1}{1} = \frac{y}{-1} = \frac{z-3}{3}$$

i.e. $$x+1 = 0,\ z-3 = 0 \quad \text{and} \quad \frac{x+1}{1} = \frac{y}{-1} = \frac{z-3}{3}$$ Ans.

***Ex. 2. Find the equations to the generating lines of the hyperbloid $(x^2/4) + (y^2/9) - (z^2/16) = 1$ which pass through the points $(2, 3, -4)$ and $(2, -1, 4/3)$.** (*Garhwal* 95)

Sol. Any line through $(2, 3, -4)$ is $\dfrac{x-2}{l} = \dfrac{y-3}{m} = \dfrac{z+4}{n} = r$ (say) ...(i)

$\therefore$ Any point on this line is $(lr+2, mr+3, nr-4)$ and it lies on the given hyperboloid if $[(lr+2)^2/4] + [(mr+3)^2/9] - [(nr-4)^2/16] = 1$

or $$r^2\left[\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16}\right] + 2r\left[\frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16}\right] = 0 \qquad ...(ii)$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the conditions for which from (ii) are

$$\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0 \quad \text{and} \quad \frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16} = 0$$

i.e. $$(l^2/4) + (m^2/9) - (n^2/16) = 0 \quad \text{and} \quad (l/2) + (m/3) = -(n/4) \qquad ...(iii)$$

Eliminating n, we get $$\frac{l^2}{4} + \frac{m^2}{9} - \left(\frac{l}{2} + \frac{m}{3}\right)^2 = 0$$

or $-(1/3)\,lm = 0 \Rightarrow$ either $l = 0$ or $m = 0$

When $l = 0$, from (iii) we get $m/3 = -n/4$

When $m = 0$, from (iii) we get $l/2 = -n/4$ i.e. $l/1 = -n/2$

Hence from (i), equations of the required generator through $(2, 3, -4)$ are

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \quad \text{and} \quad \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}$$ Ans.

In a similar manner we can find that the generators through the point $(2, -1, 4/3)$ are

$$\frac{x-2}{0}=\frac{y+1}{3}=\frac{z-(4/3)}{-4} \quad \text{and} \quad \frac{x-2}{3}=\frac{y+1}{6}=\frac{z-(4/3)}{10}$$ Ans.

****Ex. 3. Find the equations to the generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ through any point of the principal elliptic section $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1,\ z=0$ by the plane $z=0$.**

Or

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Find the equations of the generators of the hyperboloid $(x^2/a^2)+y^2/b^2)-(z^2/c^2)=1$ which pass through the point $(a\cos\theta, b\sin\theta, 0)$

(*Garhwal* 94, *Gorakhpur* 96)

Sol. Any point on the elliptic section of the hyperboloid is $(a\cos\theta, b\sin\theta, 0)$.

$\therefore$ Equations of any line through this point is

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}=r,\text{ say} \quad \text{...(i)}$$

Any point on this line is $(lr+a\cos\theta, mr+b\sin\theta, nr)$ and it lies on the given hyperboloid if

$$\frac{(lr+a\cos\theta)^2}{a^2}+\frac{(mr+b\sin\theta)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

or $$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2+2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r=0 \quad \text{...(ii)}$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the condition for which from (ii) are

$$(l^2/a^2)+(m^2/b^2)-(n^2/c^2)=0 \quad \text{...(iii)}$$

and $$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0 \quad \text{...(iv)}$$

From (iv) we get $\dfrac{l}{a\sin\theta}=\dfrac{m}{-b\cos\theta}$ or $\dfrac{(l/a)}{\sin\theta}=\dfrac{(m/-b)}{\cos\theta}$

$$\Rightarrow \frac{(l/a)}{\sin\theta}=\frac{(m/-b)}{\cos\theta}=\frac{\sqrt{[(l^2/a^2)+(m^2/b^2)]}}{\sqrt{(\sin^2\theta+\cos^2\theta)}}=\frac{\sqrt{(n^2/c^2)}}{1},\text{ from (iii)}$$

$$\Rightarrow \frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta}=\frac{n}{\pm c} \quad \text{(Note)}$$

$\therefore$ The equation to the required generators from (i) are

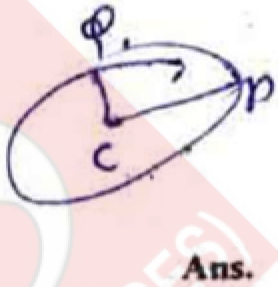
$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}$$ Ans.

***Ex. 4. A point 'm' on the parabola $y = 0, cx^2 = 2a^2z$ is $(2am, 0, 2cm^2)$ and the point 'n' on the parabola $x = 0, cy^2 = -2b^2z$ is $(0, 2bn, -2cn^2)$. Obtain the locus of the lines joining the points for which (i) $m = n$ and (ii) $m = -n$.**

Sol. The equations of the line joining "m and 'n' points are

$$\frac{x-2am}{2am} = \frac{y-0}{-2bm} = \frac{z-2cn^2}{2cm^2+2cn^2} = r \text{ (say)}$$

If $m = \pm n$, then $\dfrac{x \mp 2an}{\pm 2an} = \dfrac{y}{-2bn} = \dfrac{z-2cn^2}{2c(2n^2)} = r$

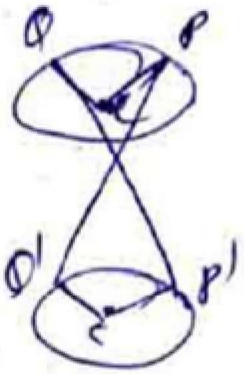


$\Rightarrow x/(2a) = \pm n(r+1), y/(2b) = -nr, \ z = 2cn^2(2r+1)$

$\therefore \dfrac{x^2}{4a^2} - \dfrac{y^2}{4b^2} = n^2(r+1)^2 - n^2r^2 = n^2(2r+1) = \dfrac{z}{2c}$

$\therefore$ The required locus is $(x^2/a^2) - (y^2/b^2) = 2z/c$ **Ans.**

****Ex. 5. CP, CQ are any two conjugate semi-diametrs of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = c$, CP′, CQ′ are the conjugate diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = -c$ drawn in the same directions as CP and CQ. Prove that the hyperboloid $(2x^2/a^2) + (2y^2/b^2) - (z^2/c^2) = 1$ is generated by either PQ′ or P′Q.** $P'Q$



Sol. The coordinates of P, Q, P' and Q' are given by

$$P(a\cos\theta, b\sin\theta, c), Q(-a\sin\theta, b\cos\theta, c),$$
$$P'(a\cos\theta, b\sin\theta, -c) \text{ and } Q'(-a\sin\theta, b\cos\theta, -c)$$

$\therefore$ Equations to PQ' are

$$\frac{x-a\cos\theta}{-a\sin\theta-a\cos\theta} = \frac{y-b\sin\theta}{b\cos\theta-b\sin\theta} = \frac{z-c}{-c-c} = r \text{ (say)}$$

$\therefore x - a\cos\theta = r[-a(\sin\theta+\cos\theta)]$,

$y - b\sin\theta = r[b(\cos\theta-\sin\theta)]$ and $z - c = -2cr$

$\Rightarrow x/a = \cos\theta - r(\sin\theta+\cos\theta), y/b = \sin\theta + r(\cos\theta-\sin\theta)$

and $z = c(1-2r)$ ...(i)

Eliminating r from these we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = [\cos\theta - r(\sin\theta+\cos\theta)]^2 + [\sin\theta + r(\cos\theta-\sin\theta)]^2$$

$$= 1 + r^2\{(\sin\theta+\cos\theta)^2 + (\cos\theta-\sin\theta)^2\} - 2r[\cos\theta(\sin\theta+\cos\theta) - \sin\theta(\cos\theta-\sin\theta)]$$

$$= 1 + r^2(2) - 2r(1) = 2r^2 - 2r + 1$$

or $\quad 2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = 4r^2 - 4r + 2 = (1-2r)^2 + 1 = \left(\dfrac{z}{c}\right)^2 + 1$, from (i)

or $\quad 2(x^2/a^2) + 2(y^2/b^2) - (z^2/c^2) = 1$, which is a hyperboloid. Proved.

Similarly we can show that the surface generated by $P'Q$ is also the above hyperboloid.

****Ex. 6. Show that if two generators of the surface $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ through the points P $(a\cos\alpha, b\sin\alpha, 0)$ and Q $(a\cos\beta, b\sin\beta, 0)$ intersect at right angles, their projection on the plane $z=0$ intersect at an angle θ, where $\tan\theta=[ab\sin(\alpha-\beta)]/c^2$**

Sol. As in Ex. 3. Page 6 we can prove that the generator of one system through $P(a\cos\alpha, b\sin\alpha, 0)$ is

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z-0}{c} \quad \text{...(i)}$$

Generator of other system through $Q(a\cos\beta, b\sin\beta, 0)$ is

$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z-0}{-c} \quad \text{...(ii)}$$

These will be perpendicular if

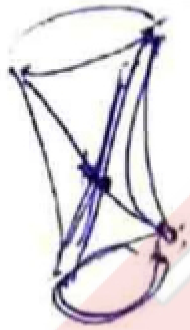
$$(a\sin\alpha)(a\sin\beta)+(-b\cos\alpha)(-b\cos\beta)+(c)(-c)=0$$

or
$$a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta-c^2=0 \quad \text{...(iii)}$$

Now the projecton of above generators on the plane $z=0$ are tangents to the ellipse $(x^2/a^2)+(y^2/b^2)=1$ at $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ and their equations are

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \quad \text{and} \quad \frac{x\cos\beta}{a}+\frac{y\sin\beta}{b}=1$$

[See Ex. 7. below]



Their slopes are $-(b\cos\alpha)/(a\sin\alpha)$ and $-(b\cos\beta)/(a\sin\beta)$

∴ If θ be the angle between them, then

$$\tan\theta=\frac{\left(\frac{-b\cos\alpha}{a\sin\alpha}\right)-\left(\frac{-b\cos\beta}{a\sin\beta}\right)}{1+\left(\frac{-b\cos\alpha}{a\sin\alpha}\right)\left(\frac{-b\cos\beta}{a\sin\beta}\right)}=\frac{-\frac{b}{a}\left[\frac{\cos\alpha}{\sin\alpha}-\frac{\cos\beta}{\sin\beta}\right]}{1+\frac{b^2\cos\alpha\cos\beta}{a^2\sin\alpha\sin\beta}}$$

$\tan\theta = \frac{m_1 \sim m_2}{1+m_1m_2}$

$$=\frac{-ab[\sin\beta\cos\alpha-\sin\alpha\cos\beta]}{a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta}=\frac{ab\sin(\alpha-\beta)}{c^2}, \text{ from (iii)}$$

****Ex. 7. Prove that the projections of the generators of a hyperboloid on coordinate plane are tangents to the section of the hyperboloid by that plane.**

Sol. Let the equation of the hyperboloid be

$$(x^2/c^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{...(i)}$$

From Ex. 3 Page 6 we know that a generator of the hyperboloid (i) is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}. \quad \text{...(ii)}$$

Now consider the coordinate plane $z=0$. The section of the hyperboloid (i) by this plane $z=0$ is given by $\quad (x^2/a^2)+(y^2/b^2)=1,\ z=0 \quad$ ...(iii)

The projection of the generator (ii) on the plane $z=0$ is given by

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta},\ z = 0$$

which is a plane through the generator perpendicular to the plane $z = 0$.

On simplifying it reduces to $\dfrac{x}{a\sin\theta} - \dfrac{\cos\theta}{\sin\theta} = \dfrac{y}{-b\cos\theta} + \dfrac{\sin\theta}{\cos\theta},\ z = 0$

i.e. $\dfrac{x}{a\sin\theta} + \dfrac{y}{b\cos\theta} = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sin\theta\cos\theta},\ z = 0$

i.e. $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1,\ z = 0$

which is evidently a tangent to the section (iii) of the hyperboloid (i) by the plane $z = 0$ at the point $(a\cos\theta, b\sin\theta, 0)$

Again consider the coordinate plane $x = 0$. The section of the hyperboloid (i) by this plane $x = 0$ is given by $(y^2/b^2) - (z^2/c^2) = 1,\ x = 0$ ...(iv)

The projection of the generator (ii) on the plane $x = 0$ is given by $\dfrac{y - b\sin\theta}{-b\cos\theta} = \dfrac{z}{c},\ x = 0$ which is a plane through the generator perpendicular to the plane $x = 0$.

On simplifying it reduces to $\dfrac{y}{-b\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{z}{c},\ x = 0$

or $\dfrac{y}{b} + \dfrac{z\cos\theta}{c} = \sin\theta, x = 0$ or $\dfrac{y}{b}\operatorname{cosec}\theta + \dfrac{z}{c}\cot\theta = 1,\ x = 0$

which is evidently a tangent to the section (iv) of the hyperboloid (i) by the plane $x = 0$ at the point $(0, b\operatorname{cosec}\theta, -c\cot\theta)$.

Similarly we can prove the result by considering the plane $y = 0$.

****Ex. 8. Prove that in general two generators of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z = 0$ in P and Q, PQ touches the ellipse $(x^2/a^6) + (y^2/b^6) = c^4/(a^4b^4)$.**

Sol. We know for the given hyperboloid, the generator belonging to λ-system is given by $\dfrac{x}{a} - \dfrac{z}{c} = \lambda\left(1 - \dfrac{y}{b}\right)$ and $\dfrac{x}{a} + \dfrac{z}{c} = \dfrac{1}{\lambda}\left(1 + \dfrac{y}{b}\right)$ ...(i)

or $\dfrac{x}{a} + \dfrac{\lambda y}{b} - \dfrac{z}{c} = \lambda$ and $\lambda\dfrac{x}{a} - \dfrac{y}{b} + \lambda\dfrac{z}{c} = 1$

$\therefore$ If l_1, m_1, n_1 be the d.r.'s of the generator (i), then

$$\frac{l_1}{a} + \frac{\lambda m_1}{b} - \frac{n_1}{c} = 0 \quad \text{and} \quad \frac{\lambda l_1}{a} - \frac{m_1}{b} + \frac{\lambda n_1}{c} = 0$$

Solving these simultaneously, we get

$$\frac{l_1/a}{\lambda^2 - 1} = \frac{m_1/b}{-\lambda - \lambda} = \frac{n_1/c}{-1 - \lambda^2}$$

or $$\frac{l_1}{-a(\lambda^2 - 1)} = \frac{m_1}{2\lambda b} = \frac{n_1}{c(1 + \lambda^2)} \quad \text{...(ii)}$$

Similarly the direction ratios l_2, m_2, n_2 of the generator belonging to μ-system viz. $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$ and $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$ ...(iii)

are given by $$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)} \quad \text{...(iv)}$$

If these two generators given by (i) and (iii) are perpendicular then
$$-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(\mu^2+1)=0 \quad \text{...(v)}$$

Now if λ-generator is given, then λ is constant and (v) will be a quadratic equation in μ which gives two values of μ and this shows that there will be two generators of μ-system which will be perpendicular to a generator of λ-system.

Now let the generators of μ-system meet the plane $z=0$ in the points $P(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$

$\therefore$ The generator of the μ-system through these points are given by
$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c} \quad \text{...(vi)}$$
and
$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z}{c} \quad \text{...(vii)}$$

[See Ex. 3. Page 6 of this chapter]

These two generators intersect at right angles a generator of λ-system through any point $(a\cos\theta, b\sin\theta, 0)$ say whose equations are
$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c} \quad \text{...(viii)}$$

As (vi) and (vii) are both perpendicular to (viii), so
$$a^2\sin\alpha\sin\theta+b^2\cos\alpha\cos\theta-c^2=0$$
and
$$a^2\sin\beta\sin\theta+b^2\cos\beta\cos\theta-c^2=0$$

Solving these simultaneouly for $a^2\sin\theta$, $b^2\cos\theta$ and $-c^2$, we get
$$\frac{a^2\sin\theta}{\cos\alpha-\cos\beta}=\frac{b^2\cos\theta}{\sin\beta-\sin\alpha}=\frac{-c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$
or
$$\frac{a\sin\theta}{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}=\frac{b^2\cos\theta}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}$$
$$=\frac{-c^2}{\sin(\alpha-\beta)}=\frac{-c^2}{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \quad \frac{a^2\sin\theta}{c^2}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \qquad \frac{b^2\cos\theta}{c^2}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad \text{...(ix)}$$

Also equation of the line joining P and Q is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2},\ z=0$$

or
$$\frac{x}{a}\left(\frac{b^2\cos\theta}{c^2}\right)+\frac{y}{b}\left(\frac{a^2\sin\theta}{c^2}\right)=1,\ z=0 \qquad \text{...(x)}$$

using the results of (ix).

Now in order to find its envelope, we should differentiate (x) with respect to θ and then eliminate θ.

Differentiating (x) w. r. to θ, we get

$$\frac{-xb^2}{ac^2}\sin\theta+\frac{ya^2}{bc^2}\cos\theta=0,\ z=0 \qquad \text{...(xi)}$$

Squaring and adding (x) and (xi), θ is eliminated and we get the required envelope of PQ as $\dfrac{x^2b^4}{a^2c^4}+\dfrac{y^2a^4}{b^2c^4}=1,\ z=0$ or $\dfrac{x^2}{a^6}+\dfrac{y^2}{b^6}=\dfrac{c^4}{a^4b^4},\ z=0$

which represents an ellipse on the plane $z=0$. Hence proved.

****Ex. 9. Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.** (*Gorakhpur* 97, 95)

Sol. As in Ex. 8 above we can find that if one generator of λ-system and one generator of μ-system intersect at right angles, then

$$-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(1+\mu^2)=0$$

...See result (v) Page 10 of this chapter

or $\quad a^2(\lambda^2\mu^2-\lambda^2-\mu^2+1)-4b^2\lambda\mu+c^2(\lambda^2\mu^2+\lambda^2+\mu^2+1)=0$

or $\quad a^2[(1+\lambda\mu)^2-(\lambda+\mu)^2]+b^2[(\lambda-\mu)^2-(\lambda+\mu)^2]$
$$+c^2[(1-\lambda\mu)^2+(\lambda+\mu)^2]=0 \quad \textbf{(Note)}$$

or $\quad a^2(1+\lambda\mu)^2+b^2(\lambda-\mu)^2+c^2(1-\lambda\mu)^2=(\lambda+\mu)^2(a^2+b^2-c^2)$

This relation shows that the point of intersection of the above two generators *i.e.* $\left[\dfrac{a(1+\lambda\mu)}{(\lambda+\mu)},\ \dfrac{b(\lambda-\mu)}{\lambda+\mu},\ \dfrac{c(1-\lambda\mu)}{\lambda+\mu}\right]$

...See § 13.03 Prop. III Page 3 of this chapter

lies on the sphere $x^2+y^2+z^2=a^2+b^2-c^2$, which is known as the director sphere.

Hence the required locus is the curve of intersection of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ and the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2. \qquad \textbf{Ans.}$$

[Alternative method is given in Ex. 20 Page 22 of this chapter]

****Ex. 10. Prove that the tangent planes to the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ which are parallel to tangent planes to the cone $\dfrac{b^2c^2x^2}{c^2-b^2}+\dfrac{c^2a^2y^2}{c^2-a^2}+\dfrac{a^2b^2z^2}{a^2+b^2}=0$ cut the surface in perpendicular generators.**

Sol. We know that the equation of the cone reciprocal to the cone $ax^2+by^2+cz^2=0$ is $(x^2/a)+(y^2/b)+(z^2/c)=0$.

$\therefore$ The equation of the cone reciprocal to the given cone is

$$\frac{c^2-b^2}{b^2c^2}x^2+\frac{c^2-a^2}{c^2a^2}y^2+\frac{a^2+b^2}{a^2b^2}z^2=0. \quad \text{...(i)}$$

Let $lx+my+nz=0$ be a tangent plane to the given cone so that by definition its normal with d.ratios l, m, n is a generator of its reciprocal cone (i).

$$\therefore \text{ We know } \frac{c^2-b^2}{b^2c^2}l^2+\frac{c^2-a^2}{c^2a^2}m^2+\frac{a^2+b^2}{a^2b^2}n^2=0 \quad \text{...(ii)}$$

Let any plane parallel to the tangent plane to the given cone be

$$lx+my+nz=p \quad \text{...(iii)}$$

If it is a tangent plane to the given hyperboloid, then

$$p^2=a^2l^2+b^2m^2-c^2n^2 \qquad \text{(See Ch. IX)} \quad \text{...(iv)}$$

Again if it is a tanegnt plane at the point (x_1, y_1, z_1) then its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1 \quad \text{...(v)}$$

Comparing (iii) and (v), we get $\dfrac{x_1/a^2}{l}=\dfrac{y_1/b^2}{m}=\dfrac{z_1/c^2}{-n}=\dfrac{1}{p}$

or
$$\frac{x_1}{la^2}=\frac{y_1}{mb^2}=\frac{z_1}{-nc^2}=\frac{1}{p} \quad \text{...(vi)}$$

Also the plane (iii) cuts the given hyperboloid in perpendicular generators if (x_1, y_1, z_1) lies on the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2 \qquad \text{...See Ex. 9 Page 11}$$

$$\therefore x_1^2+y_1^2+z_1^2=a^2+b^2-c^2$$

or
$$\left(\frac{a^2l}{p}\right)^2+\left(\frac{b^2m}{p}\right)^2+\left(\frac{-c^2n}{p}\right)^2=a^2+b^2-c^2, \text{ from (vi)}$$

or
$$a^4l^2+b^4m^2+c^4n^2=(a^2+b^2-c^2)p^2$$
$$=(a^2+b^2-c^2)(a^2l^2+b^2m^2-c^2n^2), \text{ from (iv)}$$

or
$$a^2l^2(b^2-c^2)+b^2m^2(a^2-c^2)-c^2n^2(a^2+b^2)=0$$

or
$$\frac{l^2(c^2-b^2)}{b^2c^2}+\frac{m^2(c^2-a^2)}{c^2a^2}+\frac{n^2(a^2+b^2)}{a^2b^2}=0, \text{ dividing each term by } -a^2b^2c^2$$

which is true by virtue of (ii). Hence proved.

Ex. 11. Find the point of intersection P, Q of the generators of opposite system drawn through the points A $(a\cos\alpha, b\sin\alpha, 0)$ and B $(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1.$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$(x^2/a^2)+(y^2/b^2)=2, \ z=\pm c.$$

Sol. Let the coordinates of the point P be (x_1, y_1, z_1).

The equation of the tangent plane to the given hyperboloid at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$ and it meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0 \qquad \text{...(i)}$$

which is the same as the line joining the points A and B

i.e. $$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right),\ z=0 \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get

$$\frac{x_1/a^2}{(1/a)\cos\frac{\alpha+\beta}{2}}=\frac{y_1/b^2}{(1/b)\sin\frac{\alpha+\beta}{2}}=\frac{1}{\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}},\ \frac{y_1}{b}=\frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \qquad \text{...(iii)}$$

Again $(x_1^2/a^2)+(y_1^2/b^2)-(z_1^2/c^2)=1$

or $$\left[1/\cos^2\left(\frac{\alpha-\beta}{2}\right)\right]-\frac{z_1^2}{c^2}=1, \text{ substituting values from (iii)}$$

or $$\frac{z_1^2}{c^2}=\sec^2\left(\frac{\alpha-\beta}{2}\right)-1=\tan^2\left(\frac{\alpha-\beta}{2}\right)$$

or $$\frac{z_1}{c}=\pm\frac{\sin\{(\alpha-\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \qquad \text{...(iv)}$$

From (iii) and (iv) we get the coordinates of $P\,(x_1, y_1, z_1)$ as

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{\pm c\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

Again as A and B are extremities of two semi-conjugate diameters, we have

$$\alpha-\beta=\pi/2 \qquad \text{...(v)}$$

$\therefore$ From (iii) we get

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\frac{1}{\cos^2\{(\alpha-\beta)/2\}}=\frac{1}{\cos^2(\pi/4)}, \text{ from (v)}$$

or $$(x_1^2/a^2)+(y_1^2/b^2)=2$$

And from (iv), $z_1=\pm c\tan\left(\frac{\alpha-\beta}{2}\right)=\pm c\tan\left(\frac{\pi}{4}\right)$, from (v)

or $$z_1=\pm c$$

$\therefore$ The locus of P and Q are $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2,\ z=\pm c$ Proved.

****Ex. 12. The generators through points on the principal elliptic section of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$, such that the eccentric angle of the one is double that of the other, intersect on the curve given by $x=\dfrac{a(1-3t^2)}{1+t^2},\ y=\dfrac{bt(3-t^2)}{1+t^2},\ z=\pm ct$.**

Sol. Let $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$ be the two points on the priencipal elliptic section of the given hyperbolid by the plane $z=0$.

Then the points of intersection P and Q of the generators of opposite system through them are given by

$$\frac{x}{a}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{y}{b}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{z}{c}=\pm\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad ...(i)$$

...See Ex. 11 Pages 12-13 of this chapter.

Now here we are given $\alpha=2\beta$

Putting $\alpha=2\beta$ in (i), we get $\dfrac{z}{c}=\pm\tan\dfrac{\beta}{2}$ or $z=\pm c\tan\dfrac{\beta}{2}=\pm ct$, ...(ii)

where $t=\tan(\beta/2)$.

$$\frac{x}{a}=\frac{\cos(3\beta/2)}{\cos(\beta/2)}=\frac{4\cos^3(\beta/2)-3\cos(\beta/2)}{\cos(\beta/2)}=4\cos^2\frac{\beta}{2}-3$$

$$=\frac{4-3\sec^2(\beta/2)}{\sec^2(\beta/2)}=\frac{4-3[1+\tan^2(\beta/2)]}{1+\tan^2(\beta/2)}$$

or $\dfrac{x}{a}=\dfrac{1-3t^2}{1+t^2}$, where $t=\tan\dfrac{\beta}{2}$

or $x=a(1-3t^2)/(1+t^2)$ ...(iii)

And $\dfrac{y}{b}=\dfrac{\sin(3\beta/2)}{\cos(\beta/2)}=\dfrac{3\sin(\beta/2)-4\sin^3(\beta/2)}{\cos(\beta/2)}$

$$=3\tan\frac{\beta}{2}-4\tan\frac{\beta}{2}\sin^2\frac{\beta}{2}=\tan\frac{\beta}{2}\left[3-4\sin^2\frac{\beta}{2}\right]$$

$$=\tan\frac{\beta}{2}\left[\frac{3\sec^2(\beta/2)-4\tan^2(\beta/2)}{\sec^2(\beta/2)}\right]$$

multiplying num and denom. by $\sec^2(\beta/2)$

$$=\tan\frac{\beta}{2}\left[\frac{3\{1+\tan^2(\beta/2)\}-4\tan^2(\beta/2)}{1+\tan^2(\beta/2)}\right]$$

$$=t\left[\frac{3(1+t^2)-4t^2}{1+t^2}\right]=\frac{t(3-t^2)}{1+t^2}$$

or $$y = bt(3 - t^2)/(1 + t^2) \qquad \text{...(iv)}$$

Hence from (ii), (iii) and (iv) we get the required result.

****Ex. 13. The generators through P of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ meets the principal elliptic section of A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x + \beta y + \gamma z = 0$, show that P lies on the surface**

$$z(\alpha x + \beta y) + \gamma(c^2 + z^2) = 0.$$

Sol. Let the coordinates of P, A and B be (x_1, y_1, z_1), $(a\cos\theta, b\sin\theta, 0)$ and $(a\cos\phi, b\sin\phi, 0)$ respectively.

The values of x_1, y_1, z_1 can be found as given in (iii) and (iv) of Ex. 11 Page 13 of this chapter.

Also the coordinates of F, the mid-point of AB are

$$[\tfrac{1}{2}a(\cos\theta + \cos\phi), \tfrac{1}{2}b(\sin\theta + \sin\phi), 0]$$

or $$\left[a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ 0\right]$$

$\therefore$ Direction ratios of the median PF through P are

$$x_1 - a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ y_1 - b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ z_1 - 0$$

or $$\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} - a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} - b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\qquad \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}$$

from (iii) and (iv) of Ex. 11 Page 13

or $$\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),\ \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),\ \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}$$

or $$a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\ c\tan\frac{\theta-\phi}{2}\operatorname{cosec}^2\frac{\theta-\phi}{2}$$

or $$x_1, y_1, z_1\ \operatorname{cosec}^2\frac{\theta-\phi}{2} \qquad \text{...See (iii), (iv) of Ex. 11 P. 13}$$

or $$x_1, y_1, z_1\left(1 + \cot^2\frac{\theta-\phi}{2}\right), \text{ where } \frac{z_1}{c} = \tan\frac{\theta-\phi}{2}$$

or $$x_1, y_1, z_1\,[1 + (c^2/z_1^2)]$$

As PF is parallel to the plane $\alpha x + \beta y + \gamma z = 0$

$\therefore\ \alpha x_1 + \beta y_1 + \gamma z_1 [1 + (c^2/z_1^2)] = 0$ or $(\alpha x_1 + \beta y_1) z_1 + \gamma (z_1^2 + c^2) = 0$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is $z(\alpha x + \beta y) + \gamma (z^2 + c^2) = 0$

Hence proved.

***Ex. 14. If the generators through a point P on the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ whose centre is at O, meet the plane $z = 0$ in A and B and the volume of the tetrahedron OAPB is constant and equal to $abc/6$, prove that P has on one of the planes $z = \pm c$.**

Sol. Let the coordinates of P, A and B be (x_1, y_1, z_1), $(a \cos \alpha, b \sin \alpha, 0)$ and $(a \cos \beta, b \sin \beta, 0)$ respectively. The values of x_1, y_1, z_1 can be found as given in (iii) and (iv) of Ex. 11 Page 13. Also O is the origin.

$\therefore$ The volume of the tetrahedraon $OAPB$

$$= \text{“}\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\text{”}, \text{ as } O \text{ is origin}$$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ a\cos\alpha & b\sin\alpha & 0 \\ a\cos\beta & b\sin\beta & 0 \end{vmatrix}, \text{ substituting the coordinates of three vertices } P, A \text{ and } B$$

$$= \frac{1}{6} z_1 ab (\sin\beta\cos\alpha - \sin\alpha\cos\beta) = \frac{1}{6} ab\, z_1 \sin(\beta - \alpha)$$

$$= \frac{1}{6} ab \left[c \tan\frac{\alpha-\beta}{2}\right] \sin(\beta - \alpha), \text{ from (iv) of Ex. 11 Page 13}$$

$$= \frac{1}{6} a \tan\frac{\alpha-\beta}{2} \sin(\alpha-\beta), \text{ numerically}$$

$$= \frac{1}{6} abc, \text{ given}$$

$$\therefore \tan\frac{\alpha-\beta}{2}\sin(\alpha-\beta) = 1 \quad\text{or}\quad \tan\frac{\alpha-\beta}{2}\cdot\frac{2\tan\{(\alpha-\beta)/2\}}{1+\tan^2\{(\alpha-\beta)/2\}} = 1$$

or $\quad 2\tan^2\dfrac{\alpha-\beta}{2} = 1 + \tan^2\dfrac{\alpha-\beta}{2}$ or $\tan^2\dfrac{\alpha-\beta}{2} = 1$

or $\quad \tan\dfrac{\alpha-\beta}{2} = \pm 1$ or $\dfrac{z_1}{c} = \pm 1$, from (iv) of Ex. 11 Page 13

or $\quad z_1 = \pm c$.

$\therefore P(x_1, y_1, z_1)$ lies on one of the planes $z = \pm c$.

***Ex. 15. Show that the perpendicular from the origin on the generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ lie on the curve**

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}$$

Sol. We know [See Ex. 3 Page 6 of this chapter] that the equations to a generator of the hyperboloid through any point of the principal elliptic section

$(x^2/a^2)+(y^2/b^2)=1,\ z=0$ are $\dfrac{x-a\cos\theta}{a\sin\theta}=\dfrac{y-b\sin\theta}{-b\cos\theta}=\dfrac{z-0}{+c}$ ...(i)

Equations of any line through the origin are $\dfrac{x-0}{l}=\dfrac{y-0}{m}=\dfrac{z-0}{n}$ ...(ii)

If the line (ii) is perpendicular to the generator (i), then

$$al\sin\theta-bm\cos\theta+cn=0 \quad ...(iii)$$

Also if (i) and (ii) are coplanar, then

$$\begin{vmatrix} a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & +c \\ l & m & n \end{vmatrix}=0$$

...See chapter on Straight Lines

or $\quad a\cos\theta(-nb\cos\theta-mc)-b\sin\theta(an\sin\theta-lc)=0$

or $\quad -anb(\cos^2\theta+\sin^2\theta)-amc\cos\theta+lbc\sin\theta=0$

or $\quad bcl\sin\theta-acn\cos\theta-abn=0$ ...(iv)

Solving (iii) and (iv) simultaneously for $\sin\theta$ and $\cos\theta$, we get

$$\frac{\sin\theta}{ab^2nm+ac^2mn}=\frac{\cos\theta}{bc^2nl+a^2b\,ln}=\frac{1}{-a^2c\,lm+b^2c\,lm}$$

or $\quad \dfrac{\sin\theta}{amn(b^2+c^2)}=\dfrac{\cos\theta}{bnl(c^2+a^2)}=-\dfrac{1}{-clm(a^2-b^2)}$

$$\Rightarrow \sin\theta=\frac{an(b^2+c^2)}{-cl(a^2-b^2)},\ \cos\theta=\frac{bn(c^2+a^2)}{-cm(a^2-b^2)}$$

$$\Rightarrow\left[\frac{an(b^2+c^2)}{-cl(a^2-b^2)}\right]^2+\left[\frac{bn(c^2+a^2)}{-cm(a^2-b^2)}\right]^2=1,\quad \because\ \cos^2\theta+\sin^2\theta=1$$

$$\Rightarrow\frac{a^2(b^2+c^2)^2}{l^2}+\frac{b^2(c^2+a^2)^2}{m^2}=\frac{c^2(a^2-b^2)^2}{n^2}$$

This shows that the line (ii) lies on the curve

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}=\frac{c^2(a^2-b^2)^2}{z^2}$$ Proved.

Ex. 16. If the generator through a point of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ meet the principal elliptic section in two points such that the eccentric angle of one is three times that of other, prove that P lies on the curve of intersection of the hyperboloid with the cylinder $y^2(z^2+c^2)=4b^2z^2$.

In case the difference of the eccentric angle is 2θ, then the locus of P is the curve of intersection of the hyperboloid with the cone

$$(x^2/a^2)+(y^2/b^2)=z^2/(c^2\sin^2\theta)$$

Sol. Let P be (x_1, y_1, z_1) and let the generator through P of the given hyperbolid meet the principal elliptic section in A and B, where A is $(a\cos\alpha, b\sin\alpha, 0)$ and B is $(a\cos\beta, b\sin\beta, 0)$

Here $\alpha = 3\beta$ (given)

$\therefore$ From results (iii) and (iv) of Ex. 11 Page 13, we have

$$\frac{x_1}{a} = \frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\cos 2\beta}{\cos\beta} \quad \text{...(i)}$$

$$\frac{y_1}{b} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\sin 2\beta}{\cos\beta} = 2\sin\beta \quad \text{...(ii)}$$

and $$\frac{z_1}{c} = \pm\tan\left(\frac{\alpha-\beta}{2}\right) = \pm\tan\beta \quad \text{...(iii)}$$

From (iii), $\dfrac{z_1}{c} = \pm\left(\dfrac{\sin\beta}{\cos\beta}\right) = \pm\dfrac{(y_1/2b)}{\cos\beta}$, from (ii)

or $\cos\beta = \pm\dfrac{cy_1}{2bz_1}$ and from (ii), $\sin\beta = \dfrac{y_1}{2b}$

Squaring and adding these, β is eliminated and we get

$$\frac{c^2y_1^2}{4b^2z_1^2} + \frac{y_1^2}{4b^2} = 1 \quad \text{or} \quad y_1^2(c^2+z_1^2) = 4b^2z_1^2$$

$\therefore P(x_1, y_1, z_1)$ lies on the curve of intersection of the given hyperboloid and the cylinder $y^2(c^2+z^2) = 4b^2z^2$.

Again if $\alpha - \beta = 2\theta$, then from results (iii) and (iv) of Ex. 11 Page 13, we have

$$\frac{x_1}{a} = \frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\cos\{\cos(\alpha+\beta)/2\}}{\cos\theta}$$

$$\frac{y_1}{b} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\theta},$$

and $$\frac{z_1}{c} = \pm\tan\frac{\alpha-\beta}{2} = \pm\tan\theta$$

These give $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta = \left(1 + \frac{z_1}{c}\right)^2$$

or $$(x_1^2/a^2) + (y_1^2/b^2) - (z_1^2/c^2) = 1 \quad \text{...(iv)}$$

Also $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{\cos^2\theta} = \frac{1}{\cos^2\theta}\left(\pm\frac{z_1}{c\tan\theta}\right)^2 \quad \text{(Note)}$$

or $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{z_1^2}{c^2\sin^2\theta}$$

$\therefore$ From (iv) and (v) we conclude that the locus of $P(x_1, y_1, z_1)$ is the curve of intersection of the hyperbolid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ and $(x^2/a^2) + (y^2/b^2) = z^2/(c^2\sin^2\theta)$, which being a homogeneous equation of second degree in x, y and z represents a cone. Hence proved.

****Ex. 17. Prove that the shortest distance between generators of the same system drawn at the ends of diameters of the principal elliptic section**

of the hyperbolloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ lie on the surfaces whose equations are
$$\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}$$

Sol. We know that if any point on the elliptic section be $P(a\cos\alpha, b\sin\alpha, 0)$, then the equations of the generator through it are
$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z-0}{c} \qquad \text{...(i)}$$

The extremity of the diameter through this point P is $Q(-a\cos\alpha, -b\sin\alpha, 0)$ which is obtained by putting $\alpha+\pi$ for α in the coordinates of the point P and so the equations of the generator of the same system through Q is obtained by putting $\alpha+\pi$ for α in (i) and are
$$\frac{x+a\cos\alpha}{-a\sin\alpha}=\frac{y+b\sin\alpha}{b\cos\alpha}=\frac{z-0}{c} \qquad \text{...(ii)}$$

If l, m, n be the direction cosines of the S. D. then
$$la\sin\alpha-mb\cos\alpha+nc=0$$
and
$$-la\sin\alpha+mb\cos\alpha+nc=0$$

Solving these simultaneously for l, m, n, we get
$$\frac{l}{-2bc\cos\alpha}=\frac{m}{-2ac\sin\alpha}=\frac{n}{0} \quad \text{or} \quad \frac{l}{b\cos\alpha}=\frac{m}{a\sin\alpha}=\frac{n}{0} \qquad \text{...(iii)}$$

Also equation of plane containing the generator (i) and the line of S.D. is
$$\begin{vmatrix} x-a\cos\alpha & y-b\sin\alpha & z-0 \\ a\sin\alpha & -b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix}=0$$

and the equation of the plane containing the generator (ii) and the line of S.D. is
$$\begin{vmatrix} x+a\cos\alpha & y+b\sin\alpha & z-0 \\ -a\sin\alpha & b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix}=0$$

Expanding the above determinants w.r. to third column, we get
$$z(a^2\sin^2\alpha+b^2\cos^2\alpha)-c[a\sin\alpha(x-a\cos\alpha) - b\cos\alpha(y-b\sin\alpha)]=0 \qquad \text{...(iv)}$$
and
$$-z(a^2\sin^2\alpha+b^2\cos^2\alpha)-c[a\sin\alpha(x+a\cos\alpha) - b\cos\alpha(y+b\sin\alpha)]=0 \qquad \text{...(v)}$$

Eliminating α between (iv) and (v) we can find the locus of S.D. For this adding and subtracting (iv) and (v), we get
$$-2acx\sin\alpha+2bcy\cos\alpha=0 \Rightarrow \tan\alpha=(by)/(ax) \qquad \text{...(vi)}$$
and
$$2z(a^2\sin^2\alpha+b^2\cos^2\alpha)+2c(a^2-b^2)\sin\alpha\cos\alpha=0 \qquad \text{...(vii)}$$

From (vii), $z(a^2\tan^2\alpha+b^2)+c(a^2-b^2)\tan\alpha=0$

or $z[a^2(b^2y^2/a^2x^2)+b^2]+c(a^2-b^2)(by/ax)=0$, from (vi)

or
$$\frac{b^2z^2(x^2+y^2)}{x^2}+\frac{c(a^2-b^2)by}{ax}=0$$

or $$abz(x^2+y^2)+cxy(a^2-b^2)=0 \quad \text{or} \quad \frac{cxy}{x^2+y^2}=\frac{-abz}{(a^2-b^2)}$$

In a similar manner if we consider the generator of the other system, we can find that the locus of S.D. is $\dfrac{cxy}{x^2+y^2}=\dfrac{abz}{a^2-b^2}$

Hence the required locus of S. D. is $\dfrac{cxy}{x^2+y^2}=\dfrac{\pm abz}{a^2-b^2}$.

****Ex. 18. Prove that the equations of the generating lines, through the point (θ, ϕ) on the hyperboloid of one sheet are**

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c}$$

(Garhwal 93, 92; Gorakhpur 97)

Sol. The point $P(\theta, \phi)$ on the hyperboloid of one sheet

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{...(i)}$$

is $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$ **(Remember)**

Now we know that the generating lines through P are the lines of intersection of the hyperboloid (i) with the tangent plane at P whose equation is

$$(x/a)\cos\theta\sec\phi+(y/b)\sin\theta\sec\phi-(z/c)\tan\phi=1 \quad \text{...(ii)}$$

This plane (ii) meets the plane $z=0$ in the line given by

$$(x/a)\cos\theta\sec\phi+(y/b)\sin\theta\sec\phi=1,\ z=0$$

or $$(x/a)\cos\theta+(y/b)\sin\theta=\cos\phi,\ z=0 \quad \text{...(iii)}$$

Also the section of the hyperboloid (i) by the plane $z=0$ is

$$(x^2/a^2)+(y^2/b^2)=1,\ z=0 \quad \text{...(iv)}$$

Let the line (iii) meet the section (iv) of the hyperboloid (i) in the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$. Then the equation of AB is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right),\ z=0 \quad \text{...(v)}$$

Comparing (iii) and (v), we get $\theta=\dfrac{\alpha+\beta}{2}$ and $\phi=\dfrac{\alpha-\beta}{2}$

Adding and substracting, these, we get

$$\alpha=\theta+\phi \quad \text{and} \quad \beta=\theta-\phi \quad \text{...(vi)}$$

Hence the two generators through P are AP and BP.

Now the direction ratios of AP are

$a(\cos\alpha-\cos\theta\sec\phi),\ b(\sin\alpha-\sin\theta\sec\phi),\ c(0-\tan\phi)$ **(Note)**

or $$a\left[\frac{\cos(\theta+\phi)\cos\phi-\cos\theta}{\cos\phi}\right],\ b\left[\frac{\sin(\theta+\phi)\cos\phi-\sin\theta}{\cos\phi}\right],\ \frac{-c\sin\phi}{\cos\phi},$$

from (vi)

or $$a\left[\frac{\cos\theta\cos^2\phi-\sin\theta\sin\phi\cos\phi-\cos\theta}{\cos\phi}\right],$$

$$b\left[\frac{\sin\theta\cos^2\phi+\cos\theta\sin\phi\cos\phi-\sin\theta}{\cos\phi}\right],\ \frac{-c\sin\phi}{\cos\phi}$$

or $$a\left[\frac{-\cos\theta(1-\cos^2\phi)-\sin\theta\sin\phi\cos\phi}{\sin\phi}\right],$$

$$b\left[\frac{\cos\theta\sin\phi\cos\phi-\sin\theta(1-\cos^2\phi)}{\sin\phi}\right],-c.$$

multiplying each term by $(\cos\phi)/\sin\phi$

or $a(-\cos\theta\sin\phi-\sin\theta\cos\phi), b(\cos\theta\cos\phi-\sin\theta\sin\phi), -c$

or $a\sin(\theta+\phi), -b\cos(\theta+\phi), c,$

where $\theta+\phi$ is constant from (vi) for all points on the generator AP, whose equations therefore are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c} \quad \text{...(vii)}$$

Similarly we can show that the equations of the generator BP are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta-\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta-\phi)}=\frac{z-c\tan\phi}{c}. \quad \text{...(viii)}$$

where $\theta-\phi$ is constant from (vi) for all points on the generator.

Combining (vii) and (viii) we get the required result.

***Ex. 19. The normals to $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ at points of a generator meet the plane $z=0$ at points lying on a straight line and for different generators of the same system this line touches a fixed conic.**

Sol. We know that the equation of any generator through a point (θ,ϕ) of the given hyperboloid is

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c}, \quad \text{...(i)}$$

where $\theta+\phi=\text{constant}=\alpha$, say [See Ex. 18 above] ...(ii)

The equation to the tangent plane to the given hyperboloid at the point (θ,ϕ) *i.e* $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$ is

$$\frac{x}{a}\cos\theta\sec\phi+\frac{y}{b}\sin\theta\sec\phi-\frac{z}{c}\tan\phi=1$$

or $$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-\frac{z}{c}\sin\phi=\cos\phi$$

$\therefore$ The direction ratios of the normal at (θ,ϕ) to the given hyperboloid are $(\cos\theta)/a, (\sin\theta)/b, -(\sin\phi)/c$

$\therefore$ The equation to the above normal are

$$\frac{x-a\cos\theta\sec\phi}{(\cos\theta)/a}=\frac{y-b\sin\theta\sec\phi}{(\sin\theta)/b}=\frac{z-c\tan\phi}{-(\sin\phi)/c}$$

This normal meets the plane $z=0$, where

$$x=\frac{\cos\theta}{a}\cdot\frac{c^2\tan\phi}{\sin\phi}+a\cos\theta\sec\phi=\left(\frac{a^2+c^2}{a}\right)\cos\theta\sec\phi,$$

$$y=\frac{\sin\theta}{b}\cdot\frac{c^2\tan\phi}{\sin\phi}+b\sin\theta\sec\phi=\left(\frac{b^2+c^2}{b}\right)\sin\theta\sec\phi,\ z=0$$

Putting $\theta = \alpha - \phi$, from (ii) in above, we get

$$x = \frac{a^2+c^2}{a}\frac{\cos(\alpha-\phi)}{\cos\phi} = \frac{a^2+c^2}{a}[\cos\alpha + \sin\alpha\tan\phi],$$

$$y = \frac{b^2+c^2}{b}\frac{\sin(\alpha-\phi)}{\cos\phi} = \frac{b^2+c^2}{b}[\sin\alpha - \cos\alpha\tan\phi]$$

and $z = 0$

or $$\frac{ax}{a^2+c^2} = \cos\alpha + \sin\alpha\tan\phi. \quad ...(iii)$$

$$\frac{by}{b^2+c^2} = \sin\alpha - \cos\alpha\tan\phi \quad ...(iv)$$

and $$z = 0 \quad ...(v)$$

Multiplying (iii) by $\cos\alpha$ and (iv) by $\sin\alpha$ and adding, ϕ is eliminated and we get $$\frac{ax\cos\alpha}{a^2+c^2} + \frac{by\sin\alpha}{b^2+c^2} = 1,\ z = 0 \quad ...(vi)$$

which are the equations of the required line.

The envelope of this line, for different generators of the same system is obtained by differentiating (vi) with respect to α and then eliminating α.

Differentiating (vi) w.r. to α, we get $$\frac{-ax\sin\alpha}{a^2+c^2} + \frac{by\cos\alpha}{b^2+c^2} = 0 \quad ...(vii)$$

Squaring and adding (vi) and (vii), we get

$$\frac{a^2x^2}{(a^2+c^2)^2} + \frac{b^2y^2}{(b^2+c^2)^2} = 1,\quad z = 0, \text{ which is a fixed conic.}$$

****Ex. 20. Find the locus of the poing of intersection of perpendiculars of a hyperboloid of one sheet.**

Sol. As in Ex. 18 Page 20 we can obtain the equations of the generators through any point (θ, ϕ) *i.e.* $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$ or (x_1, y_1, z_1) as

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)} = \frac{z - c\tan\phi}{\pm c}$$

$\therefore$ If these generators are mutually perpendicular, then

'$l_1l_2 + m_1m_2 + n_1n_2 = 0$'

i.e. $a\sin(\theta+\phi)\,.\,a\sin(\theta-\phi) + \{-b\cos(\theta+\phi)\}\{-b\cos(\theta-\phi)\} + (c)(-c) = 0$

or $a^2(\sin^2\theta - \sin^2\phi) + b^2(\cos^2\theta - \sin^2\phi) - c^2 = 0$

or $a^2(\cos^2\phi - \cos^2\theta) + b^2(\cos^2\phi - \sin^2\theta) - c^2 = 0$,

$\because \sin^2\alpha = 1 - \cos^2\alpha$, and $\cos^2\alpha = 1 - \sin^2\alpha$

or $a^2\cos^2\phi - a^2\cos^2\theta + b^2\cos^2\phi - b^2\sin^2\theta - c^2 = 0$

or $a^2\cos^2\phi - (x_2\cos\phi)^2 + b^2\cos^2\phi - (y_1\cos\phi)^2 - c^2 = 0$,

$\because x_1 = a\cos\theta\sec\phi,\ y_1 = b\sin\theta\sec\phi$

$$(x_1^2+y_1^2)\cos^2\phi = a^2\cos^2\phi + b^2\cos^2\phi - c^2$$
$$x_1^2+y_1^2 = a^2+b^2-c^2\sec^2\phi = a^2+b^2-c^2(1+\tan^2\phi)$$
$$x_1^2+y_1^2 = a^2+b^2-c^2-(c\tan\phi)^2, \text{ where } z_1 = c\tan\phi$$
$$x_1^2+y_1^2+z_1^2 = a^2+b^2-c^2$$

$\therefore$ The required locus of the point (x_1, y_1, z_1) is curve of intersection of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ and the director sphere

$$x^2+y^2+z^2 = a^2+b^2-c^2.$$

****Ex. 21. Prove that the angle between the generators through any point P on the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ is given by**

$$\tan\alpha = 2abc/[p(a^2+b^2-c^2-OP^2)].$$

where p is the perpendicular from the centre on the tangent plane at P.

Hence or otherwise find the locus of the point of intersection of perpendicular generators.

Sol. Let the point P be $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$

Then $OP^2 = a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi + c^2\tan^2\phi$ ...(i)

Also equation of the tangent plane to the given hyperboloid at P is

$$\frac{x}{a}\cos\theta\sec\phi + \frac{y}{b}\sin\theta\sec\phi - \frac{z}{c}\tan\phi = 1$$

$\therefore$ If p be the length of the perpendicular from origin O on the above tangent plane, then

$$p = \frac{1}{\sqrt{[\{(\cos\theta\sec\phi)/a\}^2+\{(\sin\theta\sec\phi)/b\}^2+\{(\tan\phi)/c\}^2]}}$$

or $$\frac{1}{p} = \sqrt{\left[\frac{\cos^2\theta\sec^2\phi}{a^2}+\frac{\sin^2\theta\sec^2\phi}{b^2}+\frac{\sin^2\phi\sec^2\phi}{c^2}\right]} \quad \text{(Note)}$$

or $$\frac{abc\cos\phi}{p} = \sqrt{[b^2c^2\cos^2\theta + c^2a^2\sin^2\theta + a^2b^2\sin^2\phi]} \quad \text{...(ii)}$$

Also as in Ex. 18 Page 20 we can find that the direction ratios of the two generators through $P(\theta, \phi)$ are

$$a\sin(\theta+\phi), -b\cos(\theta+\phi), c$$
and $$a\sin(\theta-\phi), -b\cos(\theta-\phi), -c$$

$\therefore$ If α be the angle between these generators, then

$$\tan\alpha = \frac{\sqrt{[(l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2]}}{(l_1l_2+m_1m_2+n_1n_2)}$$

$$= \frac{\begin{gathered}[\{-ab\sin(\theta-\phi)\cos(\theta-\phi)+ab\cos(\theta+\phi)\sin(\theta-\phi)\}^2 \\ +\{bc\cos(\theta+\phi)+bc\cos(\theta-\phi)\}^2 \\ +\{ac\sin(\theta-\phi)+ac\sin(\theta+\phi)\}^2]^{1/2}\end{gathered}}{a^2\sin(\theta+\phi)\sin(\theta-\phi)+b^2\cos(\theta+\phi)\cos(\theta-\phi)-c^2} \quad \text{...(iii)}$$

Numerator of R. H. S. of (iii)

$= \sqrt{[a^2b^2\{\sin(\theta+\phi)\cos(\theta-\phi)-\cos(\theta+\phi)\sin(\theta-\phi)\}^2}$
$\quad + b^2c^2\{\cos(\theta+\phi)+\cos(\theta+\phi)\}^2 + a^2c^2\{\sin(\theta+\phi)+\sin(\theta-\phi)\}^2$
$= \sqrt{[a^2b^2(\sin 2\phi)^2 + b^2c^2(2\cos\theta\cos\phi)^2 + a^2c^2(2\sin\theta\cos\phi)^2]}$
$= \sqrt{[4\cos^2\phi\{a^2b^2\sin^2\phi + b^2c^2\cos^2\theta + a^2c^2\sin^2\theta\}]}$
$= (2\cos\phi)[(abc\cos\phi)/p]$, from (ii)
$= 2(abc/p)\cos^2\phi$

And denominator of R. H. S. of (iii)
$= a^2(\sin^2\theta - \sin^2\phi) + b^2(\cos^2\theta - \sin^2\phi) - c^2$
$= a^2(\cos^2\phi - \cos^2\theta) + b^2(\cos^2\phi - \sin^2\theta) - c^2, \quad \because \quad \cos^2\alpha + \sin^2\alpha = 1$
$= (a^2+b^2)\cos^2\phi - (a^2\cos^2\theta + b^2\sin^2\theta) - c^2$
$= [a^2+b^2-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi) - c^2\sec^2\phi]\cos^2\phi$
$= [a^2+b^2-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi) - c^2(1+\tan^2\phi)]\cos^2\phi$
$= [(a^2+b^2-c^2)-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi + c^2\tan^2\phi)]\cos^2\phi$
$= [a^2+b^2-c^2-OP^2]\cos^2\phi$, from (i) ...(v)

$\therefore$ From (iii) with the help of (iv) and (v), we get

$$\tan\alpha = \frac{2(abc/p)\cos^2\phi}{(a^2+b^2-c^2-OP^2)\cos^2\phi} = \frac{2abc}{p(a^2+b^2-c^2-OP^2)}$$ Proved.

Again if $\alpha = 90^\circ$ (*i.e.* the generators are perpendicular), then we have $\tan\alpha = \tan 90^\circ = \infty$ and so $p(a^2+b^2-c^2-OP^2)=0$
or $\quad OP^2 = a^2+b^2-c^2 \quad$ or $\quad x_1^2+y_1^2+z_1^2 = a^2+b^2-c^2$,
if P be (x_1, y_1, z_1)

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ in this case is

$$x^2+y^2+z^2 = a^2+b^2-c^2.$$ Ans.

Ex. 22. If A and A′ are the extremities of the major axis of the principal elliptic section and any generator meets two generators of the same system through A and A′ in P and P′ respectively, then prove that

$$\mathbf{AP \cdot A'P' = b^2 + c^2}$$

Sol. We know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ are given by

$$x = \frac{a(1+\lambda\mu)}{\lambda+\mu},\ y = \frac{b(\lambda-\mu)}{\lambda+\mu},\ z = \frac{c(1-\lambda\mu)}{\lambda+\mu} \quad \text{...(i)}$$

[See Ex. 9 Page 11]

The extremities of the major axis of the principal elliptic section are $A(a, 0, 0)$ and $A'(-a, 0, 0)$

$\therefore$ At A and A' from (i) we have $\lambda - \mu = 0,\ 1 - \lambda\mu = 0$
$\Rightarrow \lambda = \mu \quad$ and $\quad 1 - \lambda^2 = 0 \quad$ or $\quad \lambda = \pm 1$

Now consider the generator through A $(a, 0, 0)$ corresponding to $\lambda = +1$ and then its point of intersection P with a generator of μ-system is obtained from (i) by putting $\lambda = +1$ and is

$$\left(a, \frac{b(1-\mu)}{1+\mu}, \frac{c(1-\mu)}{1+\mu}\right) \quad \text{or} \quad (a, bt, ct), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore AP^2 = (a-a)^2 + (bt-0)^2 + (ct-0)^2 = (b^2+c^2)\,t^2 \qquad \text{...(ii)}$$

Again the generator through A' $(-a, 0, 0)$ corresponding to $\lambda = -1$ meets the generator of μ-system at P', whose coordinates are obtained from (i) by putting $\lambda = -1$ and is

$$\left(-a, \frac{b(1+\mu)}{1-\mu}, \frac{c(1+\mu)}{-(1-\mu)}\right) \quad \text{or} \quad \left(-a, \frac{b}{t}, -\frac{c}{t}\right), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore (A'P')^2 = (-a-a)^2 + \left(\frac{b}{t}-0\right)^2 + \left(-\frac{c}{t}-0\right)^2 = \frac{b^2+c^2}{t^2} \qquad \text{...(iii)}$$

$\therefore$ From (ii) and (iii) we get $AP^2 \cdot (A'P')^2 = (b^2+c^2)\,t^2 \cdot [(b^2+c^2)/t^2]$

or $\qquad AP^2 \cdot (A'P')^2 = (b^2+c^2)^2 \quad$ or $\quad AP \cdot A'P' = b^2 + c^2$ Proved.

***Ex. 23. Show that the equations $y - \lambda z + \lambda + 1 = 0$, $(\lambda + 1)x + y + \lambda = 0$ represent for different values of λ generators of one system of the hyperboloid $yz + zx + xy + 1 = 0$ and find the equations to the generators of the other system.**

Sol. Given $y + 1 = \lambda(z-1)$ and $x + 1 = -\dfrac{x+y}{\lambda}$ ...(i)

Multiplying, λ is eliminated and we get $(y+1)(x+1) = -(z-1)(x+y)$

or $\quad yx + y + x + 1 = -(zx + zy - x - y)$

or $\quad xy + yz + zx + 1 = 0$, which is the given surface.

Also generators of the other system [with the help of (i)] are

$$y + 1 = \mu(x+y), \; x + 1 = -(1/\mu)(z-1) \qquad \text{(Note)}$$

or $\qquad \mu x + \mu y - y - 1 = 0, \; \mu x + \mu + z - 1 = 0$ Ans.

Ex. 24. Prove that any point on the lines $x + 1 = \mu y = -(\mu + 1)z$ lies on the surface $yz + zx + xy + y + z = 0$ and find equations to determine the other system of lines which lies on the surface.

Sol. Given $\qquad x + 1 = \mu y = -(\mu+1)z$

which gives $\qquad x + 1 = \mu y$ and $\mu(y+z) = -z$

or $\qquad x + 1 = \mu y$ and $y + z = -z/\mu$ ...(i)

Eliminating μ (by multiplying), we get the surface

$$(x+1)(y+z) = -yz \quad \text{or} \quad xy + xz + y + z = -yz$$

or $$xy + yz + zx + y + z = 0$$

Also the other system of generators with the help of (i) can be written as

$$x + 1 = \lambda z, \; y + z = -y/\lambda \qquad \text{(Note)}$$

or $\qquad x + 1 = \lambda z = -(\lambda+1)y$ Ans.

****Ex. 25. Obtain the conditions for the line given by equations $l_1x + m_1y + n_1z + p_1 = 0$, $l_2x + m_2y + n_2z + p_2 = 0$ to be a generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.**

Sol. If the given line is a generator, then any plane through it must be a tangent plane to the given hyperboloid. **(Remember)**

Now the equation of any plane through the given line (generator) is

$$(l_1x + m_1y + n_1z + p_1) + k\,(l_2x + m_2y + n_2z + p_2) = 0$$

or $$(l_1 + kl_2)\,x + (m_1 + km_2)\,y + (n_1 + kn_2)\,z + (p_1 + kp_2) = 0$$

This plane, for all values of k, will be a tangent plane to the given hyperboloid if $a\,(l_1 + kl_2)^2 + b\,(m_1 + km_2)^2 + c\,(n_1 + kn_2)^2 = (p_1 + kp_2)^2$ **(Note)**

or $$k^2\,(al_2^2 + bm_2^2 + cn_2^2 - p_2^2) + 2k\,(al_1l_2 + bm_1m_2 + cn_1n_2 - p_1p_2) + (al_1^2 + bm_1^2 + cn_1^2 - p_1^2) = 0$$

As this relation holds goods for all values of k, so we have

$$al_2^2 + bm_2^2 + cn_2^2 = p_2^2,\quad al_1l_2 + bm_1m_2 + cn_1n_2 = p_1p_2$$

and $al_1^2 + bm_1^2 + cn_1^2 = p_1^2$, which are the required conditions. **Ans.**

****§ 13.06. Generating Lines of a hyperbolic paraboloid.**

The equation of a hyperbolic paraboloid is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$ ...(i)

Consider any line whose equations are

$$\frac{x}{a} - \frac{y}{b} = \lambda z,\quad \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}, \qquad \text{...(ii)}$$

where λ is a constant.

If λ is eliminated from these, by multiplying them, we get $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$, which is the equation (i) of the hyperbolic paraboloid.

$\therefore$ We conclude that all those points which satisfy the equation (ii) *i.e.* which lie on the line (ii) must also satisfy the equation (i) of the hyperbolic paraboloid.

Hence the line given by (ii) lies on the hyperbolic paraboloid given by (i).

As λ can take different values, so the equation (ii) represents an infinite number of straight lines all lying wholly on the hyperbolic paraboloid given by (i) *i.e.* these lines cover the whole of the surface (i) and are called the generators or the generating lines of the hyperbolic paraboloid given by (i).

In a similar manner we can show that the system of lines given by the equations

$$\frac{x}{a} - \frac{y}{b} = \frac{2}{\mu},\quad \frac{x}{a} + \frac{y}{b} = \mu z \qquad \text{...(iii)}$$

lie wholly on the paraboloid (i) and so are its generators.

Then we find that as λ and μ vary, we get two families of straight lines such that every member of each system lies wholly on the paraboloid given by (i) and these two systems of lines given by (ii) and (iii) are known as the two systems of generating lines.

****§ 13.07. Properties of generating lines of hyperbolic paraboloid.**

Let the equation of the hyperbolic paraboloid be

$$(x^2/a^2) - (y^2/b^2) = 2z \quad \text{...(i)}$$

From § 13.06 above we know that the systems of generators of (i) are given by the equations

$(x/a) - (y/b) = \lambda z$ ...(ii), $(x/a) + (y/b) = 2/\lambda$ ...(iii)

and $(x/a) - (y/b) = 2/\mu$ ...(iv) $(x/a) + (y/b) = \mu z$ ...(v)

Prop I. *One generator of each system passes through every point of the hyperbolic paraboloid.*

Let $P(\alpha, \beta, \gamma)$ be any point on the hyperbolic paraboloid (i), then

$$(\alpha^2/a^2) - (\beta^2/b^2) = 2\gamma \quad \text{...(vi)}$$

Now the generators of λ-system of (i) given by (ii) and (iii) will pass through the point $P(\alpha, \beta, \gamma)$ if and only if λ has a value equal to each of the fractions $\dfrac{(\alpha/a) - (\beta/b)}{\gamma}, \dfrac{2}{(\alpha/a) + (\beta/b)}$ ...(vii)

obtained from (ii) and (iii).

$\dfrac{(\alpha/a) - (\beta/b)}{\gamma} = \dfrac{2}{(\alpha/a) + (\beta/b)}$, equating above two values of λ

$[(\alpha/a) - (\beta/b)][(\alpha/a) + (\beta/b)] = 2\gamma$

$(\alpha^2/a^2) - (\beta^2/b^2) = 2\gamma$, which is true by virtue of (vi).

Thus if λ is chosen equal to the values given by either of the fractions (vii) the corresponding generator of the system of generators given by (ii) and (iii) will pass through the point $P(\alpha, \beta, \gamma)$.

In a similar manner we can show that if μ is equal to either of the fractions $\dfrac{2}{(\alpha/a) - (\beta/b)}$ or $\dfrac{(\alpha/a) + (\beta/b)}{\gamma}$ [obtained by evaluating μ from the equations given by (iv) and (v)], then a member of the μ-system of generators given by (iv) and (v) corresponding to either of equal values of μ will pass through the point $P(\alpha, \beta, \gamma)$.

Prop. II. *No two generators of the same system intersect.*

Consider two generators of the λ-system given by (ii) and (iii) corresponding to two distinct values λ_1, λ_2 of λ.

$(x/a) - (y/b) = \lambda_1 z$ ...(viii) $(x/a) + (y/b) = 2/\lambda_1$ ...(ix)

and $(x/a) - (y/b) = \lambda_2 z$ ...(x) $(x/a) + (y/b) = 2/\lambda_2$ ...(xi)

Substracting (x) from (viii) we get $(\lambda_1 - \lambda_2) z = 0$ or $z = 0$, $\because \lambda_1 \neq \lambda_2$

Similarly subtracting (xi) from (ix) we get $(1/\lambda_1) - (1/\lambda_2) = 0$

or $\lambda_2 - \lambda_1 = 0$ or $\lambda_2 = \lambda_1$ which contradicts $\lambda_1 \neq \lambda_2$.

Thus we find that the four equations giving two generators of the same system are inconsistent and so we conclude that the two generators of the same system do not intersect.

Prop. III. *Any two generators of the different systems intersect.*

Here we consider two generafros, one of each system, given by (ii), (iiii) and (iv), (v).

Solving (ii) and (iv), we get $\lambda z = 2/\mu$ or $z = 2/(\lambda\mu)$.

Adding (iii) and (iv) we have $2(x/a) = (2/\lambda) + (2/\mu)$

or $\quad x/a = (\lambda+\mu)/\lambda\mu \quad$ or $\quad x = a(\lambda+\mu)/\lambda\mu$

Subtracting (iv) from (iii) we get $2(y/b) = (2/\lambda) - (2/\mu)$

or $$y/b = (\mu-\lambda)/\mu\lambda \quad \text{or} \quad y = b(\mu-\lambda)/\lambda\mu$$

Hence the point of intersection of two generators, one of each system is

$$\left(\frac{a(\lambda+\mu)}{\lambda\mu}, \frac{b(\mu-\lambda)}{\lambda\mu}, \frac{2}{\lambda\mu}\right) \quad \text{...(xii)}$$

Here we observe that for all values of λ and μ the coordinates of this point satisfy the equation of the hyperbolic paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$ and so the parametric equations of this hyperbolic paraboloid can be taken as

$$x = \frac{a(\lambda+\mu)}{\lambda\mu}, \quad y = \frac{b(\mu-\lambda)}{\lambda\mu}, \quad z = \frac{2}{\lambda\mu} \quad \text{...(xiii)}$$

Prop. IV. *The tangent planes at any point meet the hyperboloid in two generators through that point.*

Left as an exercise for the students.

Solved Examples on Generating lines of a hyperbolic paraboloid.

Ex. 1. Find the locus of the perpendicular from the vertex of the paraboloid $x^2/a^2 - (y^2/b^2) = 2z$ to the generators of the one system.

Sol. The equations for a generator of λ-system are given by

$$(x/a) - (y/b) = \lambda z \quad \text{and} \quad (x/a) + (y/b) = 2/\lambda$$

The symmetrical form of the above generator is

$$\frac{x-(a/\lambda)}{a\lambda} = \frac{y-(b/\lambda)}{-b\lambda} = \frac{z-0}{2} \quad \textbf{(Note)} \quad \text{...(i)}$$

Equations of any line through the origin are $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n} = r$, say ...(ii)

If (ii) is perpendicular to (i), then $a\lambda l - b\lambda m + 2n = 0$ ...(iii)

Again as (i) and (ii) intersect, so they are coplanar and the condition for the same is
$$\begin{vmatrix} 0-(a/\lambda) & 0-(b/\lambda) & 0-0 \\ a\lambda & -b\lambda & 2 \\ l & m & n \end{vmatrix} = 0$$

or $$-\frac{a}{\lambda}(-bn\lambda - 2m) + \frac{b}{\lambda}(an\lambda - 2l) = 0 \quad \text{or} \quad abn + \frac{2ma}{\lambda} + abn - \frac{2bl}{\lambda} = 0$$

or $\quad 2abn = 2(bl - ma)/\lambda \quad$ or $\quad \lambda = (bl - am)/(abn)$

Substituting this value of λ in (iii) we get $\dfrac{(al-bm)(bl-am)}{abn} + 2n = 0$

or $\quad abl^2 - a^2lm - b^2lm + abm^2 + 2abn^2 = 0$

or $\quad l^2 - \dfrac{alm}{b} - \dfrac{blm}{a} + m^2 + 2n^2 = 0$; dividing by ab

or $\quad l^2 + m^2 + 2n^2 - \left(\dfrac{a}{b} + \dfrac{b}{a}\right) lm = 0$

or $\quad l^2 + m^2 + 2n^2 - [(a^2+b^2)/ab]\, lm = 0$

Hence the locus of the line (ii) is given by

$$x^2 + y^2 + 2z^2 - [(a^2+b^2)/ab]\, xy = 0$$

Similarly if we consider the generator of μ-system [given by (iii) of § 13.06 Page 26] then the locus of the line (ii) is

$$x^2+y^2+2z^2+[(a^2+b^2)/ab]\,xy=0.$$

Ex. 2. Find the point of intersection of the generators of the paraboloid $xy=az$ and in case the generators include an angle α, their point of intersection lies on the curve of intersection of the paraboloid and $x^2+y^2-z^2\tan^2\alpha+a^2=0$.

Sol. Let the two generators of λ and μ-system be $x=\lambda z,\ y=a/\lambda$ ...(i)

and $$x=a/\mu,\quad y=\mu z \qquad ...(ii)$$

Solving these we get $x=a/\mu,\ y=a/\lambda$ and $z=x/\lambda=a/\lambda\mu$

$\therefore$ The point of intersection of (i) and (ii) is $\left(\dfrac{a}{\mu},\dfrac{a}{\lambda},\dfrac{a}{\lambda\mu}\right)$ ...(iii)

The direction ratios of these two generators given by (i) and (ii) can be calculated as in Ex. 1 above to be $\lambda, 0, 1$ and $0, \mu, 1$.

$\therefore$ As α is the angle between these two generators, so

$$\cos\alpha=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{(\Sigma l_1^2)}\cdot\sqrt{(\Sigma l_2^2)}}=\frac{\lambda.0+0.\mu+1.1}{\sqrt{(\lambda^2+0+1)}\cdot\sqrt{(0+\mu^2+1)}}$$

or $$\cos\alpha=\frac{1}{\sqrt{(\lambda^2+1)}\sqrt{(\mu^2+1)}}\quad\text{or}\quad \sec^2\alpha=(\lambda^2+1)(\mu^2+1)$$

or $$1+\tan^2\alpha=\lambda^2\mu^2+\lambda^2+\mu^2+1\quad\text{or}\quad\tan^2\alpha=\lambda^2\mu^2+\lambda^2+\mu^2 \qquad ...(iv)$$

If (x_1, y_1, z_1) be the point of intersection of the generators (i) and (ii), then from (iii) we have $x_1=a/\mu,\ y_1=a/\lambda,\ z_1=a/\lambda\mu$ ...(v)

$\therefore\ \mu=a/x_1,\ \lambda=a/y_1$ and $\lambda\mu=a/z_1$

Substituting these values in (iv), we get

$$\tan^2\alpha=(a^2/z_1^2)+(a^2/y_1^2)+(a^2/x_1^2)$$

or $$\tan^2\alpha=\frac{a^2}{(x_1y_1/a)^2}+\frac{a^2}{y_1^2}+\frac{a^2}{x_1^2}\quad\because\ x_1y_1=az_1,\text{ as the point }(x_1,y_1,z_1)$$

lies on the given paraboloid

or $$x_1^2y_1^2\tan^2\alpha=a^4+a^2x_1^2+a^2y_1^2=a^2(a^2+x_1^2+y_1^2)$$

or $$(az_1)^2\tan^2\alpha=a^2(a^2+x_1^2+y_1^2)\quad\because x_1y_1=az_1,\text{ as before}$$

or $$x_1^2+y_1^2-z_1^2\tan^2\alpha+a^2=0$$

$\therefore$ The required locus of the point of intersection (x_1, y_1, z_1) of the generators (i) and (ii) is $x^2+y^2-z^2\tan^2\alpha+a^2=0$. Proved.

****Ex. 3. Find the locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid.**

Sol. Let the equation of the hyperbolic paraboloid be

$$(x^2/a^2)-(y^2/b^2)=2z$$

Its generators of λ and μ-systems are given by

$$(x/a)-(y/b)=\lambda z,\quad (x/a)+(y/b)=2/\lambda \qquad ...(i)$$

and $\quad (x/a)-(y/b)=2\mu, \quad (x/a)+(y/b)=\mu z$ ...(ii)

Equations (i) can be re-written as

$$(x/a)-(y/b)-\lambda z=0, \quad (x/a)+(y/b)+0\,.\,z-(2/\lambda)=0$$

$\therefore$ If l_1, m_1, n_1 be the direction ratios of this generator, then

$$\frac{l_1}{a}-\frac{m_1}{b}-\lambda n_1=0, \quad \frac{l_1}{a}+\frac{m_1}{b}+0\,.\,n_1=0$$

Solving these simultaneously, we get

$$\frac{l_1}{\lambda/b}=\frac{m_1}{-\lambda/a}=\frac{n_1}{2/ab} \quad \text{or} \quad \frac{l_1}{a\lambda}=\frac{m_1}{-b\lambda}=\frac{n_1}{2} \qquad \text{...(iii)}$$

Again equations (ii) can be rewritten as

$$(x/a)-(y/b)+0\,.\,z-(2/\mu)=0, \quad (x/a)+(y/b)-\mu\,.\,z=0$$

$\therefore$ If l_2, m_2, n_2 be the direction ratios of this generator then

$$\frac{l_2}{a}-\frac{m_2}{b}+0\,.\,n_2=0, \quad \frac{l_2}{a}+\frac{m_2}{b}-\mu\,.\,n_2=0$$

Solving these simultaneously, we get

$$\frac{l_2}{\mu/b}=\frac{m_2}{\mu/a}=\frac{n_2}{2/ab} \quad \text{or} \quad \frac{l_2}{a\mu}=\frac{m_2}{b\mu}=\frac{n_2}{2} \qquad \text{...(iv)}$$

As the two generators given by (i) and (ii) are perpendicular, so

$$l_1l_2+m_1m_2+n_1n_2=0 \quad \text{or} \quad a\lambda.a\mu+(-b\lambda)\,b\mu+2.2=0$$

or $$a^2\lambda\mu-b^2\lambda\mu+4=0 \qquad \text{...(v)}$$

Also we know [See § 13.07 Prop. III Page 27] that the coordinates of the point of intersection (x_1, y_1, z_1) of generators (i) and (ii) are given by

$$x_1=a\,(\lambda+\mu)/\lambda\mu, \quad y_1=b\,(\mu-\lambda)/\lambda\mu, \quad z_1=2/\lambda\mu$$

$\therefore$ From (v) viz. $(a^2-b^2)\,\lambda\mu+4=0$ we get $(a^2-b^2)\,(2/z_1)+4=0$

$\therefore$ The locus of the point of intersection (x_1, y_1, z_1) of the generators (i) and (ii) is $\quad (a^2-b^2)\,(2/z)+4=0 \quad$ or $\quad a^2-b^2+2z=0$ **Ans.**

***Ex. 4. Prove that the projections of the generators of a hyperbolic paraboloid on any principal plane are tangent to the section by the plane.**

Sol. Let the hyperbolic paraboloid be $(x^2/a^2)-(y^2/b^2)=2z$.

Its generators of λ and μ-system are given by

$$(x/a)-(y/b)=\lambda z, \quad (x/a)+(y/b)=2/\lambda \qquad \text{...(i)}$$

and $$(x/a)-(y/b)=2/\mu, \quad (x/a)+(y/b)=\mu z \qquad \text{...(ii)}$$

The projection of the paraboloid and its generators on the principal plane $x=0$ are $y^2=-2b^2z$ (a parabola) and from (i) we get $(y/b)=-\lambda z$, $(y/b)=2/\lambda$

or $$2\,(y/b)=(2/\lambda)-\lambda z, \text{ on adding}$$

or $$b\lambda^2z+2\lambda y-2b=0 \qquad \text{...(iii)}$$

And from (ii) we get $(y/b)=-2/\mu$, $y/b=\mu z$

or $$2y/b=\mu z-(2/\mu), \text{ on adding}$$

or $$b\mu^2z-2\mu y-2b=0 \qquad \text{...(iv)}$$

Envelope of (iii) or (iv) where λ or μ are parameters is given by "$B^2-4AC=0$" *i.e.* $4y^2+8b^2z=0$ or $y^2=-2b^2z$ and $x=0$. *i.e.* the envelope of the projections of the generators (i) or (ii) on the plane $x=0$ is the projection of the paraboloid on the plane $x=0$.

Hence the projection of generators on the plane $x=0$ is the tangent to the projection of paraboloid on the plane $x=0$.

In a similar manner we can consider projections on the planes $y=0$ and $z=0$.

***Ex. 5. Find the generators of the paraboloid $(x^2/a^2)-(y^2/b^2)=4z$ drawn through the point $(\alpha, 0, \gamma)$ and prove that the angle between them is**

$$\cos^{-1}[(a-b+\gamma)/(a+b+\gamma).]$$

Sol. Equations of any line through the point $(\alpha, 0\,\gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-0}{m}=\frac{z-\gamma}{n}=r, \text{ say} \qquad \text{...(i)}$$

Any point on it is $(\alpha+lr, mr, \gamma+nr)$, so intersection of the line (i) with the given paraboloid is given by $\dfrac{(\alpha+lr)^2}{a}-\dfrac{(mr)^2}{b}=4(\gamma+nr)$

or
$$\left(\frac{l^2}{a}-\frac{m^2}{b}\right)r^2+2\left(\frac{l\alpha}{a}-2n\right)r+\left(\frac{\alpha^2}{a}-4\gamma\right)=0 \qquad \text{...(ii)}$$

If the line (i) is a generator of the given paraboloid, then the line (i) lies wholly on the paraboloid and the conditions for the same from (ii) are

$$\frac{l^2}{a^2}-\frac{m^2}{b}=0,\ \frac{l\alpha}{a}-2n=0,\ \frac{\alpha^2}{a}-4\gamma=0 \qquad \text{...(iii)}$$

$$\Rightarrow \frac{l}{\sqrt{a}}=\frac{m}{\pm\sqrt{b}}=\frac{2n\sqrt{a}}{\alpha} \Rightarrow \frac{l}{2a}=\frac{m}{\pm 2\sqrt{(ab)}}=\frac{n}{\alpha}$$

$\Rightarrow$ direction ratios of the two generators are

$$2a,\ 2\sqrt{(ab)},\ \alpha \text{ and } 2a,\ -2\sqrt{(ab)},\ \alpha$$

$\therefore$ If θ be the angle between these two generators, then

$$\cos\theta=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{(\Sigma l_1^2)}\,.\,\sqrt{(\Sigma l_2^2)}}=\frac{4a^2-4ab+\alpha^2}{\sqrt{(4a^2+4ab+\alpha^2)}\,.\,\sqrt{(4a^2+4ab+\alpha^2)}}\,.$$

$$=\frac{4a^2-4ab+\alpha^2}{4a^2+4ab+\alpha^2}=\frac{4a^2-4ab+4a\gamma}{4a^2+4ab+4a\gamma},\quad \because\ \alpha^2=4a\gamma \text{ from (iii)}$$

or $\cos\theta=\dfrac{4a(a-b+\gamma)}{4a(a+b+\gamma)}$ or $\theta=\cos^{-1}\left(\dfrac{a-b+\gamma}{a+b+\gamma}\right)$ Proved.

Ex. 6. Prove that the angle θ between generating lines of the hyperbolic paraboloid $(x^2/a^2)-(y^2/b^2)=2z$ through (x, y, z) is given by

$$\tan\theta=ab\left(1+\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{1/2}\left(z+\frac{a^2-b^2}{2}\right)^{-1} \qquad (\textit{Gorakhpur } 95)$$

Sol. As in Ex. 3 Page 29 we can prove that the direction ratios of the generators, one each of λ and μ-systems, are $a\lambda, -b\lambda, 2$ and $a\mu, b\mu, 2$ and the co-ordinates of the point of intersection of these two generators [See § 13.07 Prop. III Page 27] are given by $x = \dfrac{a(\lambda+\mu)}{\lambda\mu},\ y = \dfrac{b(\mu-\lambda)}{\lambda\mu},\ z = \dfrac{2}{\lambda\mu}$ ...(i)

$\therefore$ If θ be the angle between these two generators, then

$$\tan\theta = \frac{\sqrt{\{\Sigma(m_1n_2 - m_2n_1)^2\}}}{l_1l_2 + m_1m_2 + n_1n_2}$$

or
$$\tan\theta = \frac{\sqrt{[(-2b\lambda - 2b\mu)^2 + (2a\mu - 2a\lambda)^2 + (ab\lambda\mu + ab\lambda\mu)^2]}}{a\lambda \cdot a\mu + (-b\lambda)(b\mu) + 2.2}$$

$$= \frac{\sqrt{[4b^2(\lambda+\mu)^2 + 4a^2(\mu-\lambda)^2 + 4a^2b^2\lambda^2\mu^2]}}{(a^2-b^2)\lambda\mu + 4}$$

$$= \frac{\sqrt{\left[4b^2\left(\dfrac{\lambda+\mu}{\lambda\mu}\right)^2 + 4a^2\left(\dfrac{\mu-\lambda}{\lambda\mu}\right)^2 + 4a^2b^2\right]}}{(a^2-b^2) + [4/\lambda\mu]}, \text{ dividing by } \lambda\mu$$

$$= \frac{\sqrt{[4b^2(x^2/a^2) + 4a^2(y^2/b^2) + 4a^2b^2]}}{(a^2-b^2) + 2z}, \text{ from (i)}$$

$$= \frac{2ab\sqrt{[(x^2/a^4) + (y^2/b^4) + 1]}}{2[z + \{(a^2-b^2)/2\}]}$$

$$= ab\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + 1\right)^{1/2}\left(z + \frac{a^2-b^2}{2}\right)^{-1}$$ Hence proved.

***Ex. 7. Find the equation to the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$, which pass through the point (1, 1, 1).**

Sol. The equations of the two generators of λ-μ system can be written as

$$x + y + z = 6\lambda,\ 2x + y - z = z/\lambda \quad \text{...(i)}$$

and
$$x + y + z = z/\mu,\ 2x + y - z = 6\mu \quad \text{...(ii)}$$

If these pass through the point (1, 1, 1) then

$$3 = 6\lambda \text{ and } 2 = 6\mu \Rightarrow \lambda = 1/2,\ \mu = 1/3 \quad \textbf{(Note)}$$

$\therefore$ From (i) and (ii) the generators are given by

$$x + y + z = 3,\ 2x + y - z = 2z$$

and
$$x + y + z = 3z,\ 2x + y - z = 2$$

i.e.
$$x + y + z = 3;\ 2x + y - 3z = 0 \quad \text{...(iii)}$$

and
$$x + y - 2z = 0,\ 2x + y - z = 2 \quad \text{...(iv)}$$

As in Ex. 3 Page 29 we can find that the direction ratios of the generators given by (iii) and (iv) are $4, -5, -1$ and $1, -3, -1$ respectively and as they pass through the given point (1, 1, 1), so their equations are

$$\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z-1}{-1} \text{ and } \frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}$$ **Ans.**

***Ex. 8. Planes are drawn through the origin O and the generators through any point P of the paraboloid $x^2 - y^2 = az$. Prove that the angle between them is $\tan^{-1}(2r/a)$, where r is the length of OP.**

Sol. The equations of the two generators, of λ-μ systems can be taken as

$$x - y = \lambda z, \quad x + y = a/\lambda \qquad \text{...(i)}$$

and

$$x - y = a/\mu, \; x + y = \mu z \qquad \text{...(ii)}$$

Equation of any plane through the first generator is

$$(x - y - \lambda z) + k\,[x + y - (a/\lambda)] = 0 \qquad \text{...(iii)}$$

If its passes through the origin (0, 0, 0) then $k = 0$ and so from (iii), the equation of the plane through generator (i) is $\quad x - y - \lambda z = 0 \qquad$ ...(iv)

Similarly we can find that the equation of the plane through the generator (ii) is

$$x + y - \mu z = 0 \qquad \text{...(v)}$$

$\therefore$ If α be the angle between the planes (iv) and (v), then

$$\cos\alpha = \frac{1.1 - 1.1 + \lambda\mu}{\sqrt{(1^2 + 1^2 + \lambda^2)}\,.\,\sqrt{(1^2 + 1^2 + \mu^2)}} = \frac{\lambda\mu}{\sqrt{[(2 + \lambda^2)(2 + \mu^2)]}}$$

or

$$\sec^2\alpha = \frac{(2 + \lambda^2)(2 + \mu^2)}{\lambda^2\mu^2} = \frac{4}{\lambda^2\mu^2} + \frac{2}{\lambda^2} + \frac{2}{\mu^2} + 1$$

or

$$\tan^2\alpha = \frac{4}{\lambda^2\mu^2} + \frac{2}{\lambda^2} + \frac{2}{\mu^2} = \frac{4 + 2(\lambda^2 + \mu^2)}{\lambda^2\mu^2} \qquad \text{...(vi)}$$

Now the point of intersection P of these generators is

$$\left(\frac{a(\lambda + \mu)}{2\lambda\mu}, \frac{a(\mu - \lambda)}{2\lambda\mu}, \frac{a}{\lambda\mu}\right)$$ ...See § 13.07 Prop. III P. 27

$$\therefore\; r^2 = OP^2 = \frac{a^2(\lambda + \mu)^2}{4\lambda^2\mu^2} + \frac{a^2(\mu - \lambda)^2}{4\lambda^2\mu^2} + \frac{a^2}{\lambda^2\mu^2}$$

$$= \frac{a^2}{4\lambda^2\mu^2}[(\lambda + \mu)^2 + (\mu + \lambda)^2 + 4] = \frac{a^2}{4\lambda^2\mu^2}[2(\lambda^2 + \mu^2) + 4]$$

$\therefore$ From (vi), we have $r^2 = (a^2/4)\tan^2\alpha \quad$ or $\quad \tan^2\alpha = 4r^2/a^2$

or $\qquad \tan\alpha = 2r/a \quad$ or $\quad \alpha = \tan^{-1}(2r/a) \qquad$ Proved.

****Ex. 9. Show that the equations to the generators through the point (r, θ) on the hyperbolic paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$ are**

$$\frac{x - ar\cos\theta}{a} = \frac{y - br\sin\theta}{\pm b} = \frac{z - (1/2)r^2\cos 2\theta}{r(\cos\theta \mp \sin\theta)}$$

Sol. The point (r, θ) on the given paraboloid can be taken as

$$x = ar\cos\theta, \; y = br\sin\theta, \; z = (1/2)r^2\cos 2\theta \qquad \text{(Remember)}$$

The equations of any generator through this point are

$$\frac{x - ar\cos\theta}{l} = \frac{y - br\sin\theta}{m} = \frac{z - (1/2)r^2\cos 2\theta}{n} = \rho, \text{ say} \qquad \text{...(i)}$$

$\therefore$ Its intersection with the given paraboloid is given by

$$\frac{1}{a^2}(l\rho + ar\cos\theta)^2 - \frac{1}{b^2}(m\rho + br\sin\theta)^2 = 2\left[n\rho + \frac{1}{2}r^2\cos 2\theta\right]$$

or $$\left(\frac{l^2}{a^2} - \frac{m^2}{b^2}\right)\rho^2 + 2\left(\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n\right)\rho$$

$$+ r^2(\cos^2\theta - \sin^2\theta - \cos 2\theta) = 0$$

or $$\left(\frac{l^2}{a^2} - \frac{m^2}{b^2}\right)\rho^2 + 2\left(\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n\right)\rho = 0$$

If the line (i) is a generator, it lies wholly on the paraboloid and the conditions for the same are $\frac{l^2}{a^2} - \frac{m^2}{b^2} = 0$, $\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n = 0$ (Note)

$\therefore$ $\frac{l^2}{a^2} = \frac{m^2}{b^2} \Rightarrow \frac{l}{a} = \frac{m}{\pm b} = k$ (say) $\Rightarrow l = ak,\ m = \pm bk$

Also from $\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n = 0$ we have

$$n = r(k\cos\theta \mp k\sin\theta) \quad \text{or} \quad n = kr(\cos\theta \mp \sin\theta)$$

$\therefore$ From (i), the required equations of the generators are

$$\frac{x - ar\cos\theta}{ak} = \frac{y - br\sin\theta}{\pm bk} = \frac{z - (1/2)r^2\cos 2\theta}{kr(\cos\theta \mp \sin\theta)}$$

or $$\frac{x - ar\cos\theta}{a} = \frac{y - br\sin\theta}{\pm b} = \frac{z - (1/2)r^2\cos 2\theta}{r(\cos\theta \mp \sin\theta)}$$

Ex. 10. Show that the polar lines with respect to the sphere $x^2 + y^2 + z^2 = a^2$ of generators of the quadric $x^2 - y^2 = 2az$ all lie on the quadratic $x^2 - y^2 = -2az$.

Sol. Any generator of λ-system of the paraboloid $x^2 - y^2 = 2az$ is

$$x - y = \lambda z,\ x + y = 2a/\lambda$$

or $$x - y - \lambda z = 0,\ x + y + 0.z - 2(a/\lambda) = 0$$

Let the direction ratios of this generator be l, m, n, then

$$l - m - \lambda n = 0 \text{ and } l + m + 0.n = 0$$

Solving these simultaneously, we get

$$\frac{l}{0 + \lambda} = \frac{m}{-\lambda - 0} = \frac{n}{1 + 1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{-1} = \frac{n}{2/\lambda}$$

$\therefore$ The direction ratios of this generator are $1, -1$ and $2/\lambda$.

Again any point on this generator is $(a/\lambda,\ a/\lambda,\ 0)$

$\therefore$ The equations of above generator are

$$\frac{x - (a/\lambda)}{1} = \frac{y - (a/\lambda)}{-1} = \frac{z - 0}{2/\lambda} = r, \text{ (say)}$$

and so any point on it is $\left(r+\frac{a}{\lambda}, -r+\frac{a}{\lambda}, \frac{2r}{\lambda}\right)$

The polar plane of this point with respect to the given sphere $x^2+y^2+z^2=a^2$ is

$$x\left(r+\frac{a}{\lambda}\right)+y\left(-r+\frac{a}{\lambda}\right)+z\left(\frac{2r}{\lambda}\right)=a^2 \quad \text{or} \quad r\left(x-y+\frac{2z}{\lambda}\right)+a\left(\frac{x}{\lambda}+\frac{y}{\lambda}-a\right)=0$$

$\therefore$ The polar line of λ-generator is given by the planes

$$x-y+2(z/\lambda)=0 \quad \text{and} \quad x+y-a\lambda=0 \qquad \textbf{(Note)}$$

or

$$\frac{x-y}{-2z}=\frac{1}{\lambda} \quad \text{and} \quad \frac{x+y}{a}=\lambda$$

Eliminating λ, the required locus of the polar lines is

$$\left(\frac{x-y}{-2z}\right)\cdot\left(\frac{x+y}{a}\right)=1 \quad \text{or} \quad x^2-y^2=-2az. \qquad \text{Proved.}$$

***Ex. 11. Prove that the equations**

$$\mathbf{2x=ae^{2\phi},\ y=be^{\phi}\cosh\theta,\ z=c\,e^{\phi}\sinh\theta}$$

determine a hyperbolic paraboloid, and that $\theta+\phi$ is constant for points of given generator of one system and $\theta-\phi$ is constant for a given generator of the other system. *(Gorakhpur 96)*

Sol. From the given equations, we have

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=e^{2\phi}(\cosh^2\theta-\sinh^2\theta)=e^{2\phi}(1)=\frac{2x}{a}$$

$\therefore$ On eliminating θ and ϕ from the given equations, we get

$$(y^2/b^2)-(z^2/c^2)=2(x/a), \qquad \text{...(i)}$$

which is a hyperbolic paraboloid. Proved.

The generators of λ and μ systems of the above hyperboloid are given by

$$\frac{y}{b}-\frac{z}{c}=2\lambda,\ \frac{y}{b}+\frac{z}{c}=\frac{x}{a\lambda} \qquad \text{...(ii)}$$

and

$$\frac{y}{b}-\frac{z}{c}=\frac{x}{a\mu},\quad \frac{y}{b}+\frac{z}{c}=2\mu \qquad \text{...(iii)}$$

Both of these generators pass through a given point.

Again from (ii) on substituting

$$x=\tfrac{1}{2}a\,e^{2\phi},\ y=b\,e^{\phi}\cosh\theta,\ z=c\,e^{\phi}\sinh\theta$$

we get $\quad e^{\phi}(\cosh\theta-\sinh\theta)=2\lambda,\ e^{\phi}(\cosh\theta+\sinh\theta)=(1/2\lambda)\,e^{2\phi}$

i.e. $\quad e^{\phi}\cdot e^{-\theta}=2\lambda,\ e^{\phi}\cdot e^{\theta}=e^{2\phi}/(2\lambda)$

i.e. $\quad e^{\phi-\theta}=2\lambda$, from both the equations.

$\therefore$ For a given generator of λ-system, λ being constant, we find that $\phi-\theta$ or $\theta-\phi$ is constant. Proved.

In a similar manner from (iii), we get

$e^{\phi}(\cosh\theta - \sinh\theta) = (1/2\mu)\, e^{2\phi}$, $e^{\phi}(\cosh\theta + \sinh\theta) = 2\mu$

or $\quad e^{\phi}\, e^{-\theta} = e^{2\phi}/(2\mu)$, $e^{\phi} . e^{\theta} = 2\mu$

or $\quad e^{\phi+\theta} = 2\mu$, from both the equations.

$\therefore$ As before for a given generator of μ-system, μ being constant, we find that $\theta + \phi$ is constant. Proved.

Exercises on Chapter XIII

Ex. 1. A variable generator meets two generators of the same system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid in P and P', prove that $BP \quad B'P' = b^2 + c^2$.

[Hint : See Ex. 22 Page 24]

***Ex. 2.** Prove that the tangent planes to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ which are parallel to the tangent planes to

$$\frac{b^2c^2x^2}{c^2-b^2} + \frac{c^2a^2y^2}{c^2-a^2} + \frac{a^2b^2z^2}{a^2+b^2} = 0$$

meet the surface in perpendicular generators.

***Ex. 3.** Prove that the equations $4x = a(1 + \cos 2\theta)$, $y = b\cosh\phi\cos\theta$, $z = c\sinh\phi\cos\theta$ determine a hyperbolic paraboloid and show that the angle between the generators through (θ, ϕ) is given by

$$\sec^{-1}\left[\frac{\{(b^2+c^2)^2 + a^4\cos^4\theta + 2a^2(b^2+c^2)\cos^2\theta\cosh 2\phi\}^{1/2}}{a^2\cos^2\theta + b^2 - c^2}\right]$$

skew-quadrilateral with two opposite angles, right angles and the other diagonal of which is a generator of the cylinder

$$\frac{x^2(a^2+c^2)}{a^2}+\frac{y^2(b^2+c^2)}{b^2}=(a^2+b^2).$$

25. Prove that the perpendicular from the origin on the generators of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

lies on the cone

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(a^2+c^2)^2}{y^2}=\frac{c^2(a^2-b^2)^2}{z^2}.$$

26. Show that the most general quadric surface which has the lines

$$x=0,\ y=0;\ x=0,\ z=c;\ y=0,\ z=-c,$$

as generators is

$$fy(z-c)+gx(z+c)+hxy=0$$

where f, g, h are arbitrary constants.

27. Find equations in symmetrical form for the line of intersection of the two planes whose equations are

$$x+y=2(\lambda+\mu)(z-1),\ (x-y)=2(\lambda-\mu)(z-1),$$

where λ and μ are constants. Find also the co-ordinates of the point in which this line meets the plane $z=0$.

If now λ and μ are taken to be variable parameters connected by the relation $\lambda^2+\mu^2=1$, show that the line traces out a right circular cone.

11

General Equation of the Second Degree

11.1. REDUCTION TO CANONICAL FORMS AND CLASSIFICATION

A quadric has been defined as the locus of a point satisfying an equation of the second degree. Thus, a quadric is the locus of a point satisfying an equation of the type

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

which we may rewrite as

$$\Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0 \qquad ...(1)$$

splitting the set of all terms into three homogeneous subsets.

We have considered so far special forms of the equations of the second degree in order to discuss *geometrical* properties of the various types of quadrics. In this chapter we shall see how the general equation of a second degree by means of an appropriate change of co-ordinate system can be reduced to simpler forms and also thus classify the types of quadrics.

Equations of various loci connected with a given quadric. We proceed to determine the equations of various loci associated with a quadric given by a general second degree equation. In this connection, we shall start obtaining a quadric in r, which will play a very important role in connection with the determination of the equations of these loci.

Consider a point (α, β, γ) and a line through the same with direction cosines (l, m, n). The co-ordinates of the point on this line at a distance r from (α, β, γ) are

$$(lr + \alpha, mr + \beta, nr + \gamma).$$

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$

This point will lie on the quadric

$$F(x, y, z) \equiv \Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

for values of r satisfying the equation

$$\Sigma[a(lr + \alpha)^2 + 2f(mr + \beta)(nr + \gamma)] + 2\Sigma u(lr + \alpha) + d = 0$$

$$\Leftrightarrow \quad r^2\Sigma(al^2 + 2fmn) + 2r[l(a\alpha + h\beta + g\gamma + u) + m(h\alpha + b\beta + f\gamma + v)$$
$$+ n(g\alpha + f\beta + c\gamma + w) + F(\alpha, \beta, \gamma) = 0 \qquad ...(2)$$

which is a quadric in r. Thus, if r_1, r_2 be the roots of this quadric, the two points of intersection of the line with the quadric are

$$(lr_1 + \alpha, mr_1 + \beta, nr_1 + \gamma), (lr_2 + \alpha, mr_2 + \beta, nr_2 + \gamma).$$

Note. It may be noted that the equation (2) can be rewritten as

$$r^2\Sigma(al^2 + 2fmn) + r\left(l\frac{\partial F}{\partial \alpha} + m\frac{\partial F}{\partial \beta} + n\frac{\partial F}{\partial \gamma}\right) + F(\alpha, \beta, \gamma) = 0$$

where $\partial F/\partial\alpha$, $\partial F/\partial\beta$, $\partial F/\partial\gamma$ denote the values of the partial derivatives of F, w.r.t. x, y, z respectively at the point (α, β, γ).

11.1.1. The tangent plane at a point.

Suppose that the point (α, β, γ) lies on the quadric so that we have

$$F(\alpha, \beta, \gamma) = 0$$

and one root of the quadric equation (2) is zero. The vanishing of value of r is also a simple consequence of the fact that one of the two points of intersection of the quadric with every line through a point of the quadric coincides with the point in question.

A line through the point (α, β, γ) on the quadric with direction cosines (l, m, n) will be a tangent line the second point of intersection also coincides with (α, β, γ), *i.e.*, if the second value of r, as given by (2) is also zero. This will be so if the coefficient of r is also zero, *i.e.*, if

$$l(a\alpha + h\beta + g\gamma + u) + m(h\alpha + b\beta + f\gamma + v) + n(g\alpha + f\beta + c\gamma + w) = 0 \quad ...(3)$$

which is thus the condition for the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(4)$$

to be a tangent line at the point (α, β, γ). The locus of the tangent lines through (α, β, γ), obtained on eliminating l, m, n between (3) and (4) is

$$\Sigma(x-a)(a\alpha + h\beta + g\gamma + u) = 0$$

$$\Leftrightarrow \quad \Sigma x(a\alpha + h\beta + g\gamma + u) = \Sigma\alpha(a\alpha + h\beta + g\gamma + u)$$

Adding $u\alpha + v\beta + w\gamma + d$ to both sides, we get

$$\Sigma x(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + g\gamma + d) = F(\alpha, \beta, \gamma) = 0$$

Thus, the locus of the tangent lines (α, β, γ) is

$$\Sigma x(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + w\gamma + d) = 0$$

which is a plane called the *tangent plane* at (α, β, γ).

11.1.2. The normal at a point

The line through (α, β, γ), perpendicular to the tangent plane thereat, *viz.*,

$$\frac{x-\alpha}{a\alpha + h\beta + g\gamma + u} = \frac{y-\beta}{h\alpha + b\beta + f\gamma + v} = \frac{z-\gamma}{g\alpha + f\beta + c\gamma + w}$$

is the normal at the point (α, β, γ).

11.1.3. Enveloping cone from a point

Suppose now that (α, β, γ) is a point not necessarily on the quadric. Then any line through (α, β, γ) with direction cosines (l, m, n) will touch the quadric, *i.e.*, meet the same in two coincident points, if the two roots of the quadric equation in r, are equal. The condition for this is

$$[\Sigma l(a\alpha + h\beta + g\gamma + u)]^2 = \Sigma(al^2 + 2fmn)F(\alpha, \beta, \gamma) \quad ...(5)$$

The locus of the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(6)$$

through (α, β, γ) touching the quadric, obtained on eliminating l, m, n between (5) and (6), is

$$[\Sigma(x-\alpha)(a\alpha + h\beta + g\gamma + u)]^2 = [\Sigma a(x-\alpha)^2 + 2f(y-\beta)(z-\gamma)]F(\alpha, \beta, \gamma) \quad ...(7)$$

To put this equation in a convenient form, we write

$$S = F(x, y, z),\ S_1 = F(\alpha, \beta, \gamma),\ T = \Sigma x(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + w\gamma + d)$$

Then (7) can be written as

$$(T - S_1)^2 = S_1(S + S_1 - 2T)$$

$$\Leftrightarrow \quad SS_1 = T^2$$

which is the *equation of the Enveloping cone of the quadric $S = 0$ with the point (α, β, γ) as its vertex.*

11.1.4. Enveloping cylinder

Suppose now that (l, m, n) are given and we require the locus of tangent lines with direction cosines (l, m, n). If (α, β, γ) be a point on any such tangent line, we have the condition

$$[\Sigma l\,(a\alpha + h\beta + g\gamma + u)]^2 = [\Sigma\,(al^2 + 2fmn)]\,F(\alpha, \beta, \gamma)$$

as obtained in § 11.1.3 above. Thus, the required locus is

$$[\Sigma l\,(ax + hy + gz + u)]^2 = \Sigma\,(al^2 + 2fmn)\,F(x, y, z)$$

known as Enveloping Cylinder.

This is the equation of the enveloping cylinder of the quadric $F(x, y, z) = 0$ with generators parallel to the line with direction cosines (l, m, n).

11.1.5. Section with a given centre

Suppose now that (α, β, γ) is a given point. Then a chord with direction cosines (l, m, n) through the point (α, β, γ) will be bisected thereat if the sum of the two roots of the r-quadratic (2) is zero. This will be so, if and only if

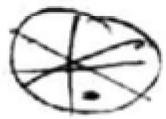
$$\Sigma l\,(a\alpha + h\beta + g\gamma + u) = 0 \qquad \text{...(8)}$$

so that the locus of the chord

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(9)}$$

through the point (α, β, γ) and bisected thereat, obtained on eliminating l, m, n from the relations (8) and (9) is

$$\Sigma\,(x-\alpha)\,(a\alpha + h\beta + g\gamma + u) = 0$$



which, we may rewrite as,

$$T = S_1.$$

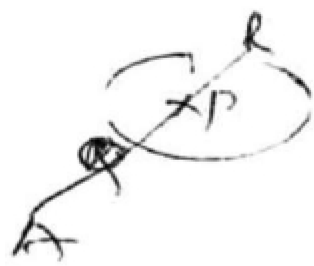
The *plane* $T = S_1$, *meets the quadric in a conic with its centre at* (α, β, γ).

11.2. POLAR PLANE OF A POINT

If a line through a point $A\,(\alpha, \beta, \gamma)$ meets the quadric in points Q, R and a point P is taken on the line such that the points A and P divide the segment QR internally and externally in the same ratio, then the locus of P for different lines through A is a plane called the *Polar plane* of the point A with respect to the quadric. It is easily seen that if the points A and P divide the segment QR internally and externally in the same ratio, then the points Q and R also divide the segment AP internally and externally in the same ratio.

Consider a line through the point $A\,(\alpha, \beta, \gamma)$ and let P be the point (x, y, z). The point dividing the segment AP in the ratio $\lambda : 1$ is

$$\left(\frac{\lambda x+\alpha}{\lambda+1}, \frac{\lambda y+\beta}{\lambda+1}, \frac{\lambda z+\gamma}{\lambda+1}\right).$$



This point will lie on the quadric

$$\Sigma\,(ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

if
$$\sum\left[a\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2 + 2f\left(\frac{\lambda y+\beta}{\lambda+1}\right)\left(\frac{\lambda z+\gamma}{\lambda+1}\right)\right] + 2\sum u\left(\frac{\lambda x+\alpha}{\lambda+1}\right) + d = 0$$

$$\Leftrightarrow \quad \lambda^2 F(x, y, z) + 2\lambda\,[x\,(a\alpha + h\beta + g\gamma + u) + y\,(h\alpha + b\beta + f\gamma + v) + z\,(g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d)] + F(\alpha, \beta, \gamma) = 0$$

The two values of λ give the two ratios in which the points Q and R divide the segment AP. In order that the points Q and R may divide the segment AP internally and externally in the same ratio, the sum of the two values of λ should be zero, *i.e.*,

$$x(a\alpha + h\beta + g\gamma + u) + y(h\alpha + b\beta + f\gamma + v) + z(g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0 \quad ...(10)$$

which is the required locus of the point $P(x, y, z)$.

Thus, (10) is the required equation of the polar plane.

Note. The notions of *Conjugate points*, *Conjugate planes*, *Conjugate lines* and *Polar lines* can be introduced as in the case of particular forms of equations in the preceding chapters.

11.3. DIAMETRAL PLANE CONJUGATE TO A GIVEN DIRECTION

We know [Refer equation (2), page 207] that if (l, m, n) be the direction cosines of a chord and (x, y, z) the mid-point of the same, then we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0 \quad ...(1)$$

Thus, if l, m, n be supposed to be given, then the equation of the locus of the mid-point (x, y, z) of parallel chords with direction cosines (l, m, n) is given by (1) above. This locus is a plane called the *Diametral plane conjugate to the direction cosines* (l, m, n). We can rewrite *the equation* (1) *of the diametral plane conjugate to* l, m, n *as*

$$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) + (ul + vm + wn) = 0 \quad ...(2)$$

Note. In this connection we should remember that there does not necessarily correspond a diametral plane conjugate to *every* given direction. Thus, we see from above that there is no diametral plane conjugate to the direction cosines (l, m, n) if l, m, n are such that the coefficients of x, y, z in the equation (2) are all zero. Thus, there will be no diametral plane corresponding to a direction whose direction cosines (l, m, n) satisfy the three relations

$$al + hm + gn = 0$$
$$hl + bm + fn = 0$$
$$gl + fm + cn = 0$$

These three homogeneous linear equations in l, m, n will have a non-zero solution, if and only if

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

We also denote by A, B, C, the co-factors of a, b, c in this determinant so that we have $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$.

11.4. PRINCIPAL DIRECTIONS AND PRINCIPAL PLANES

A direction l, m, n is said to be a *Principal direction*, if it is perpendicular to the diametral plane conjugate to the same. Also then the corresponding conjugate diametral plane is called a *Principal plane* in that the chords perpendicular to itself are bisected by it.

Thus, l, m, n will be a principal direction if and only if the direction ratios

$$al + hm + gn,\ hl + bm + fn,\ gl + fm + cn$$

of the normal to the corresponding conjugate diametral plane are proportional to l, m, n, *i.e.*, if and only if there exists a number λ such that

$$al + hm + gn = l\lambda$$
$$hl + bm + fn = m\lambda$$
$$gl + fm + cn = n\lambda$$

We rewrite these as

$$(a - \lambda)\, l + hm + gn = 0 \qquad ...(1)$$
$$hl + (b - \lambda)\, m + fn = 0 \qquad ...(2)$$
$$gl + fm + (c - \lambda)\, n = 0 \qquad ...(3)$$

These three linear homogeneous equations in l, m, n will possess a non-zero solution in l, m, n, if and only if

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0.$$

On expanding this determinant, we see that λ must be a root of the cubic

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (A + B + C) - D = 0 \qquad ...(4)$$

This cubic is known as the **Discriminating cubic** and each root of the same is called a **Characteristic root.**

The eqn. (4) has three roots which may not all be real or distinct. Also to each real root of (4) corresponds at least one principal direction l, m, n obtained on solving any two of the eqns. (1), (2) and (3).

Note 1. If l, m, n be a principal direction corresponding to a real root λ of the discriminating cubic, then we may easily see that the equation of the corresponding principal plane takes the form

$$\lambda (lx + my + nz) + (ul + vm + wn) = 0.$$

This equation shows that we shall have no principal plane corresponding to $\lambda = 0$ if $\lambda = 0$ is a root of the discriminating cubic. In spite of this, however, we shall find it useful to say that l, m, n is a principal direction corresponding to $\lambda = 0$. Thus, every direction l, m, n satisfying the eqns. (1), (2), (3) corresponding to a root λ of the discriminating cubic (4) will be called a *Principal direction*.

Note 2. In the following, we shall prove some important results concerning the nature of the roots of the discriminating cubic and the existence of principal directions and principal planes.

Before taking up this consideration, we give a few preliminary results of algebraic character in the following section.

11.5. SOME PRELIMINARIES TO REDUCTION AND CLASSIFICATION

In this section we shall state some points which will prove useful in relation to the problem of reduction and classification.

In the following discussion, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

to be denoted by D will play an important part.

We may verify that

$$D = abc + 2fgh - af^2 - bg^2 - ch^2.$$

As usual, A, B, C, F, G, H will denote the co-factors of a, b, c, f, g, h respectively in the determinant D, so that we have

$$A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2;$$
$$F = gh - af, \quad G = hf - bg, \quad H = fg - ch$$

It can be easily verified that

$$\left.\begin{array}{l} BC - F^2 = aD, \quad CA - G^2 = bD, \quad AB - H^2 = cD; \\ GH - AF = fD, \quad HF - BG = gD, \quad FG - CH = hD \end{array}\right\} \quad ...(i)$$

Also we have

$$aA + bH + gG = D, \quad hA + bH + fG = 0, \quad gA + fH + cG = 0;$$
$$aH + hB + gF = 0, \quad hH + bB + fF = D, \quad gH + fB + cF = 0;$$
$$aG + hF + gC = 0, \quad hG + bF + fC = 0, \quad gG + fF + cC = D.$$

11.5.1. If $D = 0$, then from (i), we have

$$BC = F^2, \quad CA = G^2, \quad AB = H^2,$$
$$GH = AF, \quad HF = BG, \quad FG = CH.$$

Ex. Show that

(i) $D = 0$ and $A = 0 \Rightarrow H = 0, G = 0$,

(ii) $D = 0$ and $H = 0 \Rightarrow A = 0, H = 0, C = 0$ or $H = 0, B = 0, F = 0$.

Further prove that if $D = 0, A = 0, B = 0$, then F, G, H must all be zero but G may or may not be zero.

11.5.2. If $D = 0$ and $A + B + C = 0$, then

$$A, B, C, F, G, H$$

are all zero.

Now $\quad D = 0 \Rightarrow BC = F^2, CA = G^2, AB = H^2$

$\Rightarrow \quad A, B, C$ are all of the same sign.

Now A, B, C being all of the same sign,

$$A + B + C = 0 \Rightarrow A = 0, B = 0, C = 0.$$

Further A, B, C being zero

$$F^2 = BC, B = 0, C = 0$$

$\Rightarrow \quad F = 0$

Similarly $\quad G = 0, H = 0.$

Note. Three homogeneous linear equations

$$a_1x + b_1y + c_1z = 0, \; a_2x + b_2y + c_2z = 0, \; a_3x + b_3y + c_3z = 0$$

will possess a non-zero solution, *i.e.*, a solution wherefor x, y, z are not all zero, if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

11.6. THEOREM I

The roots of the discriminating cubic are all real.

We, of course, suppose that the coefficients of the equation $F(x, y, z) = 0$ are all real.

Suppose that λ is a root of the discriminating cubic (4), page 363, and l, m, n is any non-zero set of values satisfying the corresponding eqns. (1), (2), (3), page 363.

Here it should be remembered that we cannot regard l, m, n as real, for λ is not yet proved to be real.

In the following, the complex conjugate of any number will be expressed by putting a bar over the same. Thus, $\bar{l}, \bar{m}, \bar{n}$ will denote the complex conjugate of the numbers l, m, n respectively.

Now, we have

$$al + hm + gn = l\lambda,\ hl + bm + fn = m\lambda,\ gl + fm + cn = n\lambda.$$

Multiplying these by $\bar{l}, \bar{m}, \bar{n}$ respectively and adding, we obtain

$$\Sigma al\bar{l} + \Sigma f(\bar{m}n + m\bar{n}) = \lambda\Sigma l\bar{l} \qquad \text{...(1)}$$

Now, a, b, c, f, g, h are real. Also

$$l\bar{l},\ m\bar{m},\ n\bar{n}$$

being the products of pairs of conjugate complex numbers, are real.

Also we notice that $m\bar{n}$ is the conjugate complex of $\bar{m}n$ so that

$$m\bar{n} + \bar{m}n$$

is real.

Similarly $\qquad n\bar{l} + \bar{n}l,\ l\bar{m} + \bar{l}m$

are real.

Finally $\Sigma l\bar{l}$ is a non-zero real number.

Thus, λ, being the ratio of two real numbers from (1), is necessarily a real number.

Hence, the roots of the discriminating cubic are all real. Also, therefore, the numbers l, m, n corresponding to each λ are real.

11.6.1. Theorem II

The two principal directions corresponding to any two distinct roots of the discriminating cubic are perpendicular.

Suppose that λ_1, λ_2 are two distinct roots of the discriminating cubic, and

$$l_1, m_1, n_1;\ l_2, m_2, n_2$$

are the two corresponding principal directions.

We then have

(2) $al_1 + hm_1 + gn_1 = \lambda_1 l_1$, $\qquad$ (5) $al_2 + hm_2 + gn_2 = \lambda_2 l_2$,

(3) $hl_1 + bm_1 + fn_1 = \lambda_1 m_1$, $\qquad$ (6) $hl_2 + bm_2 + fn_2 = \lambda_2 m_2$,

(4) $gl_1 + fm_1 + cn_1 = \lambda_1 n_1$, $\qquad$ (7) $gl_2 + fm_2 + cn_2 = \lambda_2 n_2$.

Multiplying (2), (3), (4) by l_2, m_2, n_2 respectively and adding, we obtain

$$\Sigma al_1l_2 + \Sigma f(m_1n_2 + m_2n_1) = \lambda_1\Sigma l_1l_2 \qquad \text{...(8)}$$

Also multiplying (5), (6), (7) by l_1, m_1, n_1 respectively and adding, we obtain

$$\Sigma al_1l_2 + \Sigma f(m_1n_2 + m_2n_1) = \lambda_1\Sigma l_1l_2 \qquad \text{...(9)}$$

From (8) and (9), we obtain

$$\lambda_1\Sigma l_1l_2 = \lambda_2\Sigma l_1l_2$$

$$\Rightarrow \qquad (\lambda_1 - \lambda_2)\,\Sigma l_1l_2 = 0$$

$$\Rightarrow \qquad \Sigma l_1l_2 = 0, \text{ for } \lambda_1 - \lambda_2 \neq 0.$$

Thus, the two directions are perpendicular. Hence, the theorem.

11.6.2. Theorem III

For every quadric, there exists at least one set of three mutually perpendicular principal directions.

We have to consider the following three cases :

(A) The roots of the discriminating cubic are all distinct.

(B) Two of the roots are equal and the third is different from these.

(C) The three roots are all equal.

These three cases will be considered one by one.

(A) Case of three distinct roots. The roots being distinct, there will correspond a principal direction l, m, n satisfying the equation (1), (2), (3) on page 312 to each of these.

Also by Theorem II, these three directions will be mutually perpendicular. The three principal directions are unique in this case.

Thus, there exist three principal directions in this case. Moreover, these directions are as well unique in this case.

(B) Case of two equal roots. Let the discriminating cubic have two equal roots and let the third root be different from the same.

Suppose λ is a root of the D-cubic repeated twice so that λ satisfies the equation

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (A + B + C) - D = 0 \qquad ...(10)$$

and the equation

$$3\lambda^2 - 2\lambda (a + b + c) + (A + B + C) = 0 \qquad ...(11)$$

obtained on differentiating the cubic (10) with respect to λ. We write these as

$$(a - \lambda)(b - \lambda)(c - \lambda) + 2fgh - (a - \lambda) f^2 - (b - \lambda) g^2 - (c - \lambda) h^2 = 0$$

$$[(b - \lambda)(c - \lambda) - f^2] + [(c - \lambda)(a - \lambda) - g^2] [(a + \lambda)(b - \lambda) - h^2] = 0$$

Here we have two relations corresponding to

$$D = 0, A + B + C = 0$$

obtained on replacing a, b, c by $a - \lambda, b - \lambda, c - \lambda$ respectively.

Thus, we conclude that (Refer § 11.5.2, page 313)

$$\begin{cases} (b - \lambda)(c - \lambda) = f^2, & (c - \lambda)(a - \lambda) = g^2, & (a - \lambda)(b - \lambda) = h^2 \\ (a - \lambda) f - gh, & (b - \lambda) g = hf, & (c - \lambda) h = fg \end{cases} \qquad ...(12)$$

corresponding to $A = 0, B = 0, C = 0; F = 0, G = 0, H = 0$. These relations show that the equation

$$(a - \lambda) l + hm + gn = 0$$
$$hl + (b - \lambda) m + fn = 0$$
$$gl + fm + (c - \lambda) n = 0$$

for the determination of l, m, n are all equivalent.

Thus, we see that if λ is a twice repeated root, the direction cosines (l, m, n) satisfy the single relation

$$(a - \lambda) l + hm + gn = 0 \qquad ...(13)$$

Suppose now that l_1, m_1, n_1 is any direction satisfying equation (13).* Further we determine a direction l_2, m_2, n_2 satisfying equation (13) and perpendicular to the direction l_1, m_1, n_1. Thus, l_2, m_2, n_2 are determined from

$$(a - \lambda) l_2 + hm_2 + gn_2 = 0$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

* If desired, l_1, m_1, n_1 may be selected further so as to satisfy some additional suitable condition.

The principal direction corresponding to the third root of the discriminating cubic will, of course, be perpendicular to each of the two principal directions

$$l_1, m_1, n_1; l_2, m_2, n_2.$$

Thus, in this case also we have a set of three mutually perpendicular principal directions. Of course, they are not unique in this case.

(C) **Case of all three roots equal.** Suppose now that all the three roots are equal to λ. The root λ satisfies the three equations

$$\lambda^3 - \lambda^2 (a+b+c)\, \lambda\, (A+B+C) - D = 0 \qquad \text{[by eqn. (10)]}$$

$$3\lambda^2 - 2\lambda (a+b+c) + (A+B+C) = 0 \qquad \text{[by eqn. (11)]}$$

$$3\lambda - (a+b+c) = 0 \qquad ...(14)$$

In this case also the relation (12), page 366 as deduced from (10) and (11) are true.

We rewrite (14) as

$$(a-\lambda) + (b-\lambda) + (c-\lambda) = 0 \qquad ...(15)$$

Also, we have

$$(b-\lambda)(c-\lambda) = f^2,\ (c-\lambda)(a-\lambda) = g^2,\ (q-\lambda)(b-\lambda) = h^2 \qquad ...(16)$$

From (16), we see that

$$a-\lambda,\ b-\lambda,\ c-\lambda$$

must all have the same sign so that with the help of (15), it follows that

$$a-\lambda = 0,\ b-\lambda = 0,\ c-\lambda = 0$$

$$\Rightarrow \qquad \lambda = a = b = c.$$

Also then it follows that $f = 0$, $g = 0$, $h = 0$.

We now see that in thisj case the equations

$$(a-\lambda)\, l + hm + gn = 0,\ hl + (b-\lambda)\, m + fn = 0,\ gl + fm + (c-\lambda)\, n = 0$$

for the determination of the principal directions are identically satisfied, *i.e.*, they are true for arbitrary values of l, m, n, so that *every direction is a principal direction* in this case.

Thus, in this case also a quadric has a set of three mutually perpendicular principal directions. In fact, any set of three mutually perpendicular directions is a set of three mutually perpendicular principal directions in this case.

The reader may observe that the quadric is a sphere in the last case.

EXAMPLE

Find a set of three mutually perpendicular principal directions for the following conicoids :

1. $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy + 2z = 0.$
2. $8x^2 + 7y^2 + 3z^2 - 8yz + 4zx - 12xy + 2x - 8y + E = 0.$
3. $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy - 3y + 5z = 0.$

Sol. 1. We have $a = 3, b = 5, c = 3, f = -1, g = 1, h = -1$.

Therefore, the discriminating cubic is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \qquad -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

Its root are $$\lambda = 2, 3, 6$$
so that the characteristic roots are all different.

The principal direction corresponding to $\lambda = 2$ is given by

$$l - m + n = 0$$
$$-l + 3m - n = 0$$
$$l - m + n = 0$$

$\Rightarrow$ $$l : m : n = 1 : 0 : -1$$

Thus, the principal direction corresponding to $\lambda = 2$ is given by

$$1/\sqrt{2}, 0, -1/\sqrt{2}.$$

Again the principal direction corresponding to $\lambda = 3$ is given by

$$0 \cdot l - m + n = 0,$$
$$-l + 2m - n = 0,$$
$$l - m + 0 \cdot n = 0.$$

$\Rightarrow$ $$l : m : n = 1 : 1 : 1$$

and we have the corresponding principal direction

$$1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}.$$

Finally the principal direction corresponding to $\lambda = 6$ is given by

$$-3l - m + n = 0$$
$$-l - m - n = 0$$
$$l - m - 3n = 0$$

wherefrom we may see that this principal direction is

$$1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}.$$

The principal planes corresponding to the characteristic root λ being

$$\lambda (lx + my + nz) + (ul + vm + wn) = 0.$$

We see that the three principal planes are

$$2x - 2z - 1 = 0,\ 3x + 3y + 3z + 1 = 0,\ 6x - 12y + 6z + 1 = 0.$$

2. We have $a = 8, b = 7, c = 3, f = -4, g = 2, h = -6.$

Therefore, the discriminating cubic is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$\Rightarrow$ $$\lambda = 0, 3, 15.$$

Thus, 0, 3, 15 are three distinct characteristic roots.

The principal direction l, m, n corresponding to $\lambda = 0$ is given by

$$8l - 6m + 2n = 0$$
$$-6l + 7m - 4n = 0$$
$$2l - 4m + 3n = 0$$

Solving these, we see that $$l : m : n = 1 : 2 : 2.$$

Thus, the principal direction corresponding to $\lambda = 0$ is given by

$$1/3, 2/3, 2/3.$$

Again the principal direction corresponding to $\lambda = 3$ is given by

$$5l - 6m + 2n = 0$$
$$-6l + 4m - 4n = 0$$
$$2l - 4m + 0 \cdot n = 0$$

These give $\qquad l : m : n = 2 : 1 : -2$

so that the corresponding principal direction is given by

$$2/3,\ 1/3,\ -2/3.$$

Finally the principal direction corresponding to $\lambda = 15$ is given by

$$-7l - 6m + 2n = 0$$
$$-6l - 8m - 4n = 0$$
$$2l - 4m - 12n = 0$$

which give $\qquad l : m : n = 2 : -2 : 1$

so that the corresponding principal direction is given by

$$2/3,\ -2/3,\ 1/3.$$

The reader may verify that the three directions are mutually perpendicular.

The principal plane corresponding to the characteristic root λ being

$$\lambda\,(lx + my + nz) + (ul + vm + wn) = 0$$

we may see that the two principal planes corresponding to the non-zero values 3, 15 of λ are

$$3\,(2x + y - 2z) + (-2) = 0 \iff 6x + 3y - 6z - 2 = 0$$

and $\qquad 15\,(2x - 2y + z) + 10 = 0 \iff 6x - 6y + 3z + 2 = 0.$

3. We have $a = 6, b = 3, c = 3, f = -1, g = 2, h = -2$.

The discriminating cubic is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \qquad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

whose roots are 2, 2, 8. Thus, two of the characteristic roots are equal. Firstly we consider the non-repeated root 8. The principal direction corresponding to this is given by

$$-2l - 2m + 2n = 0$$
$$-2l - 5m - 2n = 0$$
$$2l - m - 5n = 0$$

$\Rightarrow \qquad l : m : n = 2 : -1 : 1$

so that the principal direction corresponding to $\lambda = 8$ is given by

$$2/\sqrt{6},\ -2/\sqrt{6},\ 1/\sqrt{6}.$$

Again the principal direction corresponding to $\lambda = 2$ is given by

$$4l - 2m + 2n = 0$$
$$-2l + m - n = 0$$
$$2l - m + n = 0$$

These three equations for the determination of l, m, n are all equivalent as has been shown for the general case.

Thus, every l, m, n satisfying the single equation

$$2l - m + n = 0 \qquad ...(1)$$

determines a principal direction. Consider any set of values of l, m, n satisfying (1), say

$$-1, -1, 1$$

We write $\qquad l_1 : m_1 : n_1 = -1 : -1 : 1.$

Then we determine l_2, m_2, n_2 satisfying (1) and perpendicular to l_1, m_1, n_1.

Thus,

$$-2l_2 - m_2 + n_2 = 0$$
$$-l_2 - m_2 + n_2 = 0$$

$\Rightarrow$ $$l_2 : m_2 : n_2 = 0 : 1 : 1$$

Thus, we have obtained a set of three mutually perpendicular principal directions given by

$$1/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}; 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}; 0, 1/\sqrt{2}, 1/\sqrt{2}.$$

The choice of principal directions is not unique in the present case as two of the characteristic roots are equal.

Note. It may be verified that every direction perpendicular to the principal direction corresponding to the non-repeated root 8 is a principal direction for the twice repeated root 2.

EXERCISES

Examine the following quadrics for principal directions and principal planes :

1. $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z = 0.$
2. $x^2 + 2yz - 4x + 6y + 2z = 0.$
3. $4y^2 - 4yz - 4zx - 4xy - 2x + 2y - 1 = 0.$
4. $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0.$

ANSWERS

1. Principal directions : $1, 0, 0; 0, 1/\sqrt{2}, -1/\sqrt{2}; 0, 1/\sqrt{2}, 1/\sqrt{2}.$
   Principal planes : $x = 1, y - z + 3 = 0.$
2. Principal directions : $0, 1/\sqrt{2}, -1/\sqrt{2}$ and every direction perpendicular to it.
   Principal planes : $y - z - 2 = 0$ and every plane through the line, $y + z + 4 = 0, x = 2.$
3. Principal directions : $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}; 1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}; 1/\sqrt{2}, 0, -1/\sqrt{2}.$
   Principal planes : Any plane at right angle to $x = y = z - 1/2$, $2(x - 2y + z) = 1$, $2(x - z) + 1 = 0.$
4. Principal directions : $0, 1/\sqrt{2}, 1/\sqrt{2}; 1, 0, 0; 0, 1/\sqrt{2}, -1/\sqrt{2}.$
   Principal planes : $y + z + 1 = 0, x = 1, y - z = 1.$

11.6.3. Centre

We know that if a point (x, y, z) is the mid-point of a chord with direction cosines (l, m, n) of a quadric

$$F(x, y, z) = 0$$

then we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0 \qquad ...(1)$$

This shows that if (x, y, z) is such that

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0$$

hen the condition (1) is satisfied, whatever values l, m, n may have, implying that every chord rough (x, y, z) is bisected thereat. Such a point is known as a Centre of the quadric. We can ewrite these equations as

$$ax + hy + gz + u = 0 \quad ...(2)$$
$$hx + by + fz + v = 0 \quad ...(3)$$
$$gx + fy + cz + w = 0 \quad ...(4)$$

It should be remembered that a quadric may or may not have a centre; also it may have more than one centre — *a line of centres* or *a plane of centres*, depending upon the nature of the solutions of the three equations (2), (3), (4).

In the following, we shall consider the different cases regarding the possible solutions of these equations. This discussion will be facilitated a good deal, if regarding x, y, z as variables, we consider the three planes represented by these equations. We have thus to examine the nature of the points of intersection, if any, of these three planes to be called *Central planes*.

Before we proceed to consider the problem of the existence of the centre of a quadric, we state a preliminary result.

11.6.4. The two planes

$$p_1x + q_1y + r_1z + s_1 = 0$$
$$p_2x + q_2y + r_2z + s_2 = 0$$

will be

(*i*) same if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0, \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} = 0, \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} = 0$$

(*ii*) parallel but not same if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0, \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} = 0, \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} \neq 0$$

(*iii*) neither parallel nor same, *i.e.*, will intersect in a straight line if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \neq 0.$$

11.7. CASE OF A UNIQUE CENTRE

Multiplying the equations (2), (3), (4) by A, H, G respectively and adding, we obtain

$$Dx + (Au + Hv + Gw) = 0$$ (Refer § 11.5.2, page 313)

Again, on multiplying (2), (3), (4) by H, B, F and by G, F, C and adding separately, we obtain

$$Dy + (Hu + Bv + Fw) = 0$$
$$Dz + (Gu + Fv + Cw) = 0$$

If $D \neq 0$, we obtain from these

$$x = -(Au + Hv + Gw)/D, \; y = -(Hu + Bv + Fw)/D, \; z = -(Gu + Fv + Cw)/D$$

Substituting these in (2), (3), (4) we may easily verify that the same are satisfied.

Thus, if $D \neq 0$, the quadric has a unique centre (x, y, z) where (x, y, z) have the values given above.

11.7.1. Case of No Centre

Now suppose that $D = 0$. Then, we have

$$A\,(ax + hy + gz + u) + H\,(hx + by + fz + v) + G\,(gx + fy + cz + w) \equiv Au + Hv + Gw$$

(Refer § 11.5.1, page 364)

This shows that the three equations cannot have a common solution, *i.e.*, the quadric will not have a centre if

$$Au + Hv + Gw \neq 0.$$

Considering H, B, F and G, F, C as sets of multipliers instead of A, H, G, we may similarly see that the quadric will not have a centre if

$$Hu + Bv + Fw \neq 0 \text{ or if } Gu + Fv + Cw \neq 0.$$

Thus, we see that *the quadric will not have a centre if $D = 0$ and any one of*

$$\underline{Au + Hv + Gw},\ \underline{Hu + Bv + Fw},\ \underline{Gu + Fv + Cw}$$

is not zero.

11.7.2. Case of a Line of Centres

We now suppose that $D = 0$ as well as $Au + Hv + Gw = 0$.

Then we have

$$A\,(ax + hy + gz + u) + H\,(hx + by + fz + v) + G\,(gx + fy + cz + w) = 0.$$

(*i*) Thus, if $A \neq 0$, we have

$$ax + hy + gz + u = -\frac{H}{A}(hx + by + fz + v) - \frac{G}{A}(gx + fy + cz + w)$$

(*ii*) Also, if $A \neq 0$, the two planes

$$hx + by + fz + v = 0$$
$$gx + fy + cz + w = 0$$

are neither the same nor parallel so that they intersect in a line. This is because

$$\begin{vmatrix} b & f \\ f & c \end{vmatrix} = A \neq 0$$

[Refer § 11.6.4, page 371]

From (*i*) and (*ii*), we deduce that the plane

$$ax + hy + gz + u = 0$$

passes through the line of intersection of the two intersecting planes

$$hx + by + fz + v = 0,\ gx + fy + cz + w = 0$$

Thus, in case

$$D = 0,\ Au + Hv + Gw = 0,\ A \neq 0$$

the three central planes all pass through one line so that we have a line of centres.

We may similarly see that the quadric will have a line of centres if

$$D = 0,\ Hu + Bv + Fw = 0,\ B \neq 0$$

or if

$$D = 0,\ Gu + Fv + Cw = 0,\ C \neq 0$$

Note 1. We can show that if $D = 0$, and $A \neq 0$ and $Au + Hv + Gw = 0$, then we must also simultaneously have

$$Hu + Bv + Fw = 0,\ Gu + Fv + Cw = 0$$

In fact we have

$$A\,(Hu + Bv + Fw) \equiv H\,(Au + Hv + Gw)$$

and

$$A\,(Gu + Fv + Cw) \equiv G\,(Au + Hv + Gw)$$

the equalities holding for all values of u, v and w. Thus, if $A \neq 0$, we have

$$Hu + Bv + Fw = \frac{H}{A}(Au + Hv + Gw)$$

$$Gu + Fv + Cw = \frac{G}{A}(Au + Hv + Cw)$$

The result stated now follows.

It may be remembered that if $A = 0$ then also $H = 0$, $G = 0$, so that $Au + Hv + Gw = 0$. In this case when $A = 0$, $H = 0$, $G = 0$, we may not have

$$Hu + Bv + Fw = 0 \text{ or } Gu + Fv + Cw = 0.$$

For example, consider

$$x^2 + 2y^2 + 2xy + 2x + y + 2z + 3 = 0$$

Here $a = 1$, $b = 2$, $c = 0$, $f = 0$, $g = 0$, $h = 1$, $u = 1$, $v = 1/2$, $w = 1$,

so that $$A = 0, B = 0, C = 1, F = 0, G = 0, H = 0, D = 0.$$

Thus, we have $Au + Hv + Gw = 0$ but $Gu + Fv + Cw \neq 0$.

Note 2. The cases treated above cover the cases when $D = 0$ and one at least of A, B, C is not zero.

If we suppose that A, B, C are all zero, then it follows that F, G, H are also all zero, for

$$F^2 = BC, G^2 = CA, H^2 = AB$$

In the next sub-section we consider the case when A, B, C, F, G, H are all zero. The vanishing of D then follows from the vanishing of these co-factors in as much as we have

$$D = Aa + Hh + Gg,$$

so that $D = 0$ even if A, H, G only are known to be zero.

11.7.3. Case of no centre

Suppose now that A, B, C, F, G, H are all zero, so that $D = 0$ also.

We have in this case,

$$(1) \quad \begin{cases} f(ax + hy + gz + u) - g(hx + by + fz + v) = fu - gv \\ f(ax + hy + gz + u) - h(gx + fy + cz + w) = fu - hw \end{cases}$$

These show that if

$$fu - gv \neq 0 \text{ or } fu - hw \neq 0,$$

then the quadric cannot have a centre.

11.7.4. Case of a plane of centres

Suppose now that

$$fu - gv = 0 \text{ and } fu - hw = 0$$

$$\Leftrightarrow \quad fu = gv = hw.$$

Then if $g \neq 0$, $h \neq 0$, we have from (1) above in § 11.7.3 that

$$hx + by + fz + v = \frac{f}{g}(ax + hy + gz + u)$$

$$gx + fy + cz + w = \frac{f}{h}(ax + hy + gz + u)$$

so that every point of the plane

$$ax + hy + gz + u = 0$$

is also a point of the other two central planes. Thus, we have a plane of centres in this case.

Similarly we may show that if

$$fu = gv = hw$$

and some two of f, g, h are not zero, then the quadric has a plane of centres.

Note. It can be easily seen that if A, B, C, F, G, H are zero and one of f, g, h is known to be zero, then one more of f, g, h must also be zero. For instance, suppose that $f = 0$. Then, because

$$0 = F = gh - af$$

it follows that either g or h must also be zero. Thus, the case treated here can be stated as follows :

If A, B, C, F, G, H are all zero, none of f, g, h is zero and $fu = gv = hw$, then the quadric has a line of centres.

The case where one and, therefore, two of f, g, h are zero is treated below here.

11.7.5. Now *suppose that two of f, g, h are zero in addition to A, B, C, F, G, H* being all zero and $fu = gv = hw$.

Let $g = 0 = h$ and $f \neq 0$. In this case we see from (1) above, § 11.7.3, page 325 that

$$ax + hy + gz + u = 0$$

so that $a = 0$, $h = 0$, $g = 0$, $u = 0$.

The vanishing of u also follows from the fact that

$$fu = gv = hw \text{ and } g = 0,\ h = 0,\ f \neq 0.$$

Consider now the two central planes

$$hx + by + fz + v = 0$$
$$gx + fy + cz + w = 0$$

the coefficients of the third central plane being all zero. As h and g are both zero, we can rewrite these as

$$by + fz + v = 0$$
$$fy + cz + w = 0$$

Here $$\begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2 = A = 0, \begin{vmatrix} f & v \\ c & w \end{vmatrix} = fw - cv$$

Thus, if $fw - cv \neq 0$, the quadric has no centre and if $fw - cv = 0$, the quadric has a plane of centres.

We can obtain similar conditions, when

$$f = 0 = h,\ g \neq 0$$

or when $$f = 0 = g,\ h \neq 0.$$

11.7.6. *Now suppose* that f, g, h are all zero in addition to the vanishing of A, B, C, F, G, H.

In this case two of a, b, c must be zero. Suppose that $b = c = 0$ and $a \neq 0$. Then the first of the three central planes is

$$ax + u = 0$$

and the other two are

$$0x + 0y + 0z + v = 0$$
$$0x + 0y + 0z + w = 0$$

Thus, if $v \neq 0$ or $w \neq 0$ the quadric has no centre and if $v = 0 = w$, the quadric has a plane of centres.

SUMMARY OF THE VARIOUS CASES

1. $D \neq 0$. Unique centre.
2. $\begin{cases} D = 0,\ Au + Hv + Gw \neq 0. \text{ No centre.} \\ D = 0,\ Hu + Bv + Fw \neq 0. \text{ No centre.} \\ D = 0,\ Gu + Fv + Cw \neq 0. \text{ No centre.} \end{cases}$
3. $\begin{cases} D = 0,\ Au + Hv + Fw = 0,\ A \neq 0. \text{ Line of centres.} \\ D = 0,\ Hu + Bv + Gw = 0,\ B \neq 0. \text{ Line of centres.} \\ D = 0,\ Gu + Fv + Cw = 0,\ C \neq 0. \text{ Line of centres.} \end{cases}$
4. A, B, C, F, G, H all zero, $fu \neq gv$ or $gv \neq hw$. No centre.
5. A, B, C, F, G, H all zero, $fu = gv = hw$, $f \neq 0$, $g \neq 0$, $h \neq 0$. Plane of centres.
6. A, B, C, F, G, H all zero, $fu = gv = hw$, $g = 0$, $h = 0$, $f \neq 0$, $fw - cv \neq 0$. No centre.
7. A, B, C, F, G, H all zero, $fu = gv = hw$, $g = 0$, $h = 0$, $f \neq 0$, $fw - cv = 0$. Plane of centres.
   We may have results similar to (6) and (7), when $f = 0$, $g = 0$, $h \neq 0$ or when $h = 0$, $f = 0$, $g \neq 0$.
8. A, B, C, F, G, H all zero, f, g, h all zero. Then two of a, b, c must be zero and one non-zero. Then we have no centre if
$$a \neq 0,\ v \neq 0 \text{ or } w \neq 0$$
   and a plane of centres if
$$a \neq 0,\ v = 0 = w.$$
   We have similar results when $b \neq 0$ or $c \neq 0$.

Note. The results given above need not be committed to memory.

EXERCISE

Examine the following quadrics for centres :

1. $z^2 - yz + zx + xy - 2y + 2z + 2 = 0$. [**Ans.** Unique centre; $(1, 1, -1)$]
2. $2z^2 - 2yz - 2zx + 2xy + 3x - y - 2z + 1 = 0$.
   $\left[\textbf{Ans. } \text{Line of centres; } \dfrac{x}{1} = \dfrac{y+2}{1} = \dfrac{2z+1}{2}\right]$
3. $4x^2 + 9y^2 + 4z^2 + 12xy + 12yz + 8zx + 3x + 4y + z = 0$. [**Ans.** No centre]
4. $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - y + z = 0$.
   [**Ans.** Plane of centres; $2x - 2y + 2z + 1 = 0$]
5. $4x^2 - 2y^2 - 2z^2 + 5yz + 2zx + 2xy - x + 2y + 2z - 1 = 0$. [**Ans.** No centre]
6. $2x^2 + 2y^2 + 5z^2 - 2yz - 2zx - 4xy - 14x - 14y + 16z + 6 = 0$.
   [**Ans.** Line of centres; $x = 3 - y$, $z + 1 = 0$]
7. $18x^2 + 2y^2 + 20z^2 - 12zx + 12yz + x - 22y - 6z + 1 = 0$. [**Ans.** No centre]
8. $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$.
   [**Ans.** Unique centre; $(-1, -2, -3)$]

11.8. TRANSFORMATION OF CO-ORDINATES

Before we take up the problem of actual reduction and classification, we shall consider two important cases of transformation of co-ordinates.

11.8.1. The Form of the Equation of a Quadric Referred to a Centre as Origin

We suppose that the given quadric has a centre.

Let (α, β, γ) be a centre of the quadric with equation

$$F(x, y, z) \equiv \Sigma (ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

Consider now a new system of co-ordinate axes parallel to the given system and with its origin at (α, β, γ). The equation of the quadric, w.r.t. the new system, obtained on replacing x, y, z by $x + \alpha, y + \beta, z + \gamma$ respectively is

$$\Sigma [a (x + \alpha)^2 + 2f (y + \beta) (z + \gamma)] + 2\Sigma u (x + \alpha) + d = 0$$

$$\Leftrightarrow \quad \Sigma (ax^2 + 2fyz) + 2x (a\alpha + h\beta + g\gamma + u) + 2y (h\alpha + b\beta + f\gamma + v) + 2z (g\alpha + f\beta + c\gamma + w) + F (\alpha, \beta, \gamma) = 0$$

As (α, β, γ) is a centre, we have

$$a\alpha + h\beta + g\gamma + u = 0,\ h\alpha + b\beta + f\gamma + v = 0,\ g\alpha + f\beta + c\gamma + w = 0$$

Further, as may be easily seen

$$F (\alpha, \beta, \gamma) = \alpha (a\alpha + h\beta + g\gamma + u) + \beta (h\alpha + b\beta + f\gamma + v) + \gamma (g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d)$$

$$= u\alpha + v\beta + w\gamma + d$$

Thus, the required new equation is

$$\Sigma (ax^2 + 2fyz) + (u\alpha + v\beta + w\gamma + d) = 0.$$

It will be seen that the second degree homogeneous part of the equation has remained unchanged and the first degree terms have disappeared.

Note 1. The discussion above is applicable whether the quadric has one centre, a line of centres or a plane of centres. In case the quadric has more than one centre, (α, β, γ) may denote any one of them.

Note 2. The co-ordinates, w.r.t. the old as well as the new system of axes have both been denoted by the same symbols, x, y, z.

11.8.2. The Form of the Equation of a Quadric, When the Co-ordinate Axes are Parallel to a Set of Three Mutually Perpendicular Principal Directions

Suppose that

$$(l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3) \quad ...(1)$$

are the direction cosines of three mutually perpendicular principal directions corresponding to the three roots

$$\lambda_1, \lambda_2, \lambda_3$$

of the discriminating cubic. These roots may not be all different.

We take now a new co-ordinate system through the same origin such that the axes of the new system are parallel to the directions given by (1) above.

The equation referred to the new system of axes is obtained on replacing

$$x, y, z$$

by $$l_1x + l_2y + l_3z,\ m_1x + m_2y + m_3z,\ n_1x + n_2y + n_3z$$

respectively.

As homogeneous linear expressions are to be substituted for x, y, z, we may note that a homogeneous expression of any degree will be transformed to a homogeneous expression of the same degree.

Thus, we may separately consider the transforms of the homogeneous parts

$$\Sigma (ax^2 + 2fyz) \text{ and } 2\Sigma ux.$$

We shall show that the transform of the second degree homogeneous part

$$\Sigma(ax^2 + 2fyz) \qquad ...(2)$$

is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2.$$

On direct substitution, we may see that the coefficient of x^2 in the transform of (2) is

$$al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1$$
$$= l_1(al_1 + hm_1 + gn_1) + m_1(hl_1 + bm_1 + fn_1) + n_1(gl_1 + fm_1 + cn_1)$$
$$= l_1(\lambda_1 l_1) + m_1(\lambda_1 m_1) + n_1(\lambda_1 n_1) = \lambda_1(l_1^2 + m_1^2 + n_1^2) = \lambda_1.$$

Similarly the coefficients of y^2 and z^2 in the transform can be shown to be

$$\lambda_2 \text{ and } \lambda_3$$

respectively.

Again the coefficient of $2yz$ in the transform of (1) is

$$= al_2l_3 + bm_2m_3 + cn_2n_3 + f(m_2n_3 + m_3n_2) + g(n_2l_3 + n_3l_2) + h(l_2m_3 + l_3m_2)$$
$$= l_2(al_3 + hm_3 + gn_3) + m_2(hl_3 + bm_3 + fn_3) + n_2(gl_3 + fm_3 + cn_3)$$
$$= \lambda_3(l_2l_3 + m_2m_3 + n_2n_3) = 0.$$

Similarly the coefficients of zx and xy in the transform can be seen to be zero.

Thus, the transform of

$$\Sigma(ax^2 + 2fyz)$$

is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

Finally, we see that the transform of

$$\Sigma(ax^2 + 2fyz) + 2\Sigma ux + d$$

is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + 2u(l_1x + l_2y + l_3z) + 2y(m_1x + m_2y + m_3z) + 2w(n_1x + n_2y + n_3z) + d$

$$= \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2) + 2z(ul_3 + vm_3 + wn_3) + d$$

11.9. REDUCTION TO CANONICAL FORMS AND CLASSIFICATION

We shall now consider the several cases one by one.

11.9.1. Case I. *When $D \neq 0$.* In this case the quadric has a unique centre and no root of the discriminating cubic is zero.

Shifting the origin to the centre (α, β, γ), the equation takes the form

$$\Sigma(ax^2 + 2fyz) + (u\alpha + v\beta + w\gamma + d) = 0 \qquad (\S\ 11.8.1,\ \text{page } 376)$$

Now rotating the axes so that the axes of the new system are parallel to the set of three mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + (u\alpha + v\beta + w\gamma + d) = 0$$

which is the required canonical form.

Below we find an elegant form for the constant term.

We have

$$a\alpha + h\beta + g\gamma + u = 0 \qquad ...(1)$$
$$h\alpha + b\beta + f\gamma + v = 0 \qquad ...(2)$$
$$g\alpha + f\beta + c\gamma + w = 0 \qquad ...(3)$$

Also we write

$$u\alpha + v\beta + w\gamma + d = k$$

$$\Leftrightarrow \qquad u\alpha + v\beta + w\gamma + (d-k) = 0 \qquad ...(4)$$

Eliminating α, β, γ from (1), (2), (3) and (4), we obtain

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & (d-k) \end{vmatrix} = 0$$

$$\Leftrightarrow \qquad \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} - \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} k = 0$$

$$\Leftrightarrow \qquad k = \frac{\Delta}{D}, \; (D \neq 0)$$

where we have represented the fourth order determinant on the left by Δ.

We thus see that the new equation assumes the form–

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \Delta / D = 0.$$

The equation represents various types of surfaces as shown in the following table. It may be remembered that the word 'roots' refers to the characteristic roots :

Given		*Conclusion*
$\Delta = 0$	Roots all > 0 or < 0	Imaginary cone.
$\Delta = 0$	Two roots > 0 and one < 0	Real cone.
$\Delta = 0$	Two roots < 0 and one > 0	Real cone.
$\Delta/D > 0$	Roots all > 0	Imaginary ellipsoid.
$\Delta/D > 0$	Roots all < 0	Real ellipsoid.
$\Delta/D > 0$	Two roots > 0 and one < 0	Hyperboloid of two sheets.
$\Delta/D > 0$	Two roots < 0 and one > 0	Hyperboloid of one sheet.
$\Delta/D < 0$	Roots all > 0	Real ellipsoid.
$\Delta/D < 0$	Roots all < 0	Imaginary ellipsoid.
$\Delta/D < 0$	Two roots > 0 and one < 0	Hyperboloid of one sheet.
$\Delta/D < 0$	Two roots < 0 and one > 0	Hyperboloid of two sheets.

11.9.2. Case II. *When $D = 0$, $Au + Hv + Gw \neq 0$.* In this case the quadric has no centre and the discriminating cubic has one zero root and two non-zero roots.

We denote the non-zero roots by λ_1, λ_2. The third root λ_3 is 0.

We rotate the co-ordinate axes through the same origin so that new axes are parallel to the set of three mutually perpendicular principal directions.

The new equation takes the form

$$\lambda_1 x^2 + \lambda_2 y^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2) + 2z(ul_3 + vm_3 + wn_3) + d = 0 \qquad ...(1)$$

where l_3, m_3, n_3 correspond to $\lambda_3 = 0$.

Here we notice that $\qquad ul_3 + vm_3 + wn_3 \neq 0$

If possible, let $$ul_3 + vm_3 + wn_3 = 0 \quad ...(2)$$
We also have
$$hl_3 + bm_3 + fn_3 = 0 \quad ...(3)$$
$$gl_3 + fm_3 + cn_3 = 0$$
As l_3, m_3, n_3 are not all zero, we have from (2), (3), (4)
$$\begin{vmatrix} u & v & w \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$
$$\Leftrightarrow \quad Au + Hv + Gw = 0$$
which is contradictory to the given condition.

Denoting the coefficients of x, y, z by p, q, r, we rewrite (1) as
$$\lambda_1 x^2 + \lambda_2 y^2 + 2px + 2qy + 2rz + d = 0, \text{ where } r \neq 0.$$
$$\Leftrightarrow \quad \lambda_1\left(x+\frac{p}{\lambda_1}\right)^2 + \lambda_2\left(y+\frac{q}{\lambda_2}\right)^2 + 2r\left[z+\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}-\frac{q^2}{\lambda_2}\right)\right] = 0$$
so that shifting the origin to the point
$$\left[-\frac{p}{\lambda_1}, -\frac{q}{\lambda_2}, -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}-\frac{q^2}{\lambda_2}\right)\right],$$
we see that the equation takes the form
$$\lambda_1 x^2 + \lambda_2 y^2 + 2rz = 0$$
where $r = ul_3 + vm_3 + wn_3 \neq 0$.

This is the required canonical form in the present case.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1, λ_2 are of the same or opposite signs.

Cor. Axis and vertex of the paraboloid. It is known that Z-axis is the axis and (0, 0, 0) is the vertex of the paraboloid
$$\lambda_1 x^2 + \lambda_2 y^2 + 2rz = 0.$$

Also the principal directions of the paraboloid are those of the co-ordinate axes; the principal direction corresponding to the characteristic root zero being that of Z-axis and the principal direction corresponding to the non-zero roots λ_1, λ_2 being those of X-axis and Y-axis respectively. Further, it can be easily seen that the principal planes corresponding to the non-zero characteristic roots are the planes $x = 0, y = 0$ whose intersection Z-axis is the axis of the paraboloid. Thus, we have the following important and useful result :

The line of intersection of the principal planes corresponding to the non-zero characteristic roots is the axis and the point where the axis meets the paraboloid is the vertex. Also the axis is the line through the vertex parallel to the principal direction corresponding to the characteristic root zero.

11.9.3. Case III. *When $D = 0$, $Au + Hv + Gw = 0$, $A \neq 0$.* In this case the quadric has a line of centres and the discriminating cubic has one zero and two non-zero roots.

We may see that $A + B + C \neq 0$, for if it were so, then we would have A, B, C all zero and the condition $A \neq 0$, would be contradicted. Since $D = 0$ and $A + B + C \neq 0$, the discriminating cubic would have only one zero root.

Let (α, β, γ) be a centre. Shifting the origin to (α, β, γ) and rotating the axes so that the new axes are parallel to the set of mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + (u\alpha + v\beta + w\gamma + d) = 0$$

which is the required canonical form.

We may, as follows, obtain an expression for the constant term in a form free from α, β, γ. In this case the central planes all pass through one line.

*We select the following two equations of the centre giving

$$hx + by + fz + v = 0$$
$$gx + fy + cz + w = 0$$

so that they are different.

Now (α, β, γ) is a point satisfying these two equations. Taking $\alpha = 0$, we have

$$b\beta + f\gamma + v = 0$$
$$f\beta + c\gamma + w = 0$$

Also we write $$v\beta + w\gamma + (d - k) = 0.$$

These give

$$\begin{vmatrix} b & f & v \\ f & c & w \\ v & w & (d-k) \end{vmatrix} = 0 \Rightarrow k = \frac{1}{A}\begin{vmatrix} b & f & v \\ f & c & w \\ v & w & d \end{vmatrix}$$

Thus, the required canonical form is

$$\lambda_1 x^2 + \lambda_2 y^2 + k = 0.$$

The equation represents various types of surfaces as shown in the following table :

	Given	Conclusion
$k = 0$	Roots both > 0 or < 0	Imaginary pair of planes.
$k = 0$	One root > 0 and other < 0	Pair of intersecting planes.
$k > 0$	Roots both > 0	Imaginary cylinder.
$k > 0$	Roots both < 0	Elliptic cylinder.
$k > 0$	One root > 0 and other < 0	Hyperbolic cylinder.
$k < 0$	Roots both > 0	Elliptic cylinder.
$k < 0$	Roots both < 0	Imaginary cylinder.
$k < 0$	One root > 0 and other < 0	Hyperbolic cylinder.

Cor. 2. Axis of the Cylinder. The Z-axis is known to be the axis of the cylinder.

$$\lambda_1 x^2 + \lambda_2 y^2 + k = 0, \; k \neq 0.$$

As in the case of the paraboloid, we have the following result regarding the axis of the cylinder.

The axis of the cylinder is the line of intersection of the principal planes corresponding to the non-zero characteristic roots. Also, it is parallel to the principal direction corresponding to the characteristic root zero. The axis is also the line of centres.

Cor. 3. Planes bisecting the angles between two planes. It may be seen that planes bisecting the angles between the two planes

$$\lambda_1 x^2 + \lambda_2 y^2 = 0$$

are $$x = 0, \; y = 0.$$

Thus, we see that *the two principal planes corresponding to the two non-zero characteristic roots are the two bisecting planes.*

Cor. 4. The homogeneous second degree equation

$$\Sigma (ax^2 + 2fyz) = 0$$

will represent a pair of planes if $D = 0$.

11.9.4. Case IV. *When A, B, C, F, G, H are all zero and $fu \neq gv$.*

In this case the quadric has no centre and two roots of the discriminating cubic are zero and one non-zero.

We rotate the axes so that the new axes are parallel to the three mutually perpendicular principal directions. The new equation takes the form

$$\lambda_1 x^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2) + 2z(ul_3 + vm_3 + wn_3) + d = 0$$

As the two roots λ_2, λ_3 are equal, both being zero, we know that l_2, m_2, n_2 is *any* direction satisfying

$$al + hm + gn = 0 \quad ...(1)$$

We suppose that l_2, m_2, n_2 are so chosen that these satisfy (1) and

$$ul_2 + vm_2 + wn_2 = 0 \quad ...(2)$$

Then l_3, m_3, n_3 are chosen so as to satisfy (1) and

$$l_3 l_2 + m_3 m_2 + n_3 n_2 = 0$$

Denoting the coefficients of x and z by p, r, we rewrite the equation as

$$\lambda_1 x^2 + 2px + 2rz + d = 0 \quad ...(3)$$

the coefficient of y being zero by (2).

Again we rewrite (3) as

$$\lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + 2rz + \left(d - \frac{p^2}{\lambda_1}\right) = 0 \quad ...(4)$$

Also we may see that $r \neq 0$, for otherwise the quadric will have a centre. Again, we rewrite (4) as

$$\lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + 2r\left[z + \frac{1}{2r}\left(d - \frac{p^2}{\lambda_1}\right)\right] = 0$$

Shifting the origin to

$$\left[-\frac{p}{\lambda_1}, 0, -\frac{1}{2r}\left(d - \frac{p^2}{\lambda_1}\right)\right]$$

we see that the equation becomes

$$\lambda_1 x^2 + 2rz = 0$$

which is the required canonical form.

The equation represents a parabolic cylinder in this case.

11.9.5. Case V. *When A, B, C, F, G, H are all zero, $fu = gv = hw$, and no one of f, g, h is zero.*

In this case the quadric has a plane of centres and the discriminating cubic has two zero and one non-zero root.

Let (α, β, γ) be a centre. Shifting the origin to (α, β, γ) and rotating the axes so that the axes of the new system are parallel to a set of three mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1 x^2 + (u\alpha + v\beta + w\gamma + d) = 0$$

The equation represents a pair of parallel or coincident planes.

Note. The case when any two or all of f, g, h are zero can be easily considered and it can be shown that we shall have a parabolic cylinder in case the quadric does not have a centre and a pair of parallel planes if the quadric has a plane of centres.

11.10. QUADRICS OF REVOLUTION

Firstly, we shall prove a lemma concerning surfaces of revolution obtained on revolving a plane curve about an axis of co-ordinates.

Lemma. *The equation of a surface of revolution obtained on revolving a plane curve about X-axis is of the form*

$$\sqrt{y^2 + z^2} = f(x)$$

Consider the surface of revolution on revolving a curve about X-axis. Let the equations of the section of this surface by the plane $z = 0$ be

$$y = f(x),\ z = 0 \qquad ...(1)$$

If P be a point on the curve and M the foot of the perpendicular from P on X-axis, we have

$$OM = x,\ MP = y$$

so that we can rewrite

$$y = f(x) \text{ as } MP = f(OM). \qquad ...(2)$$

Now this relation remains unchanged as the curve revolves about the X-axis so that the point P describes a circle with M as its centre.

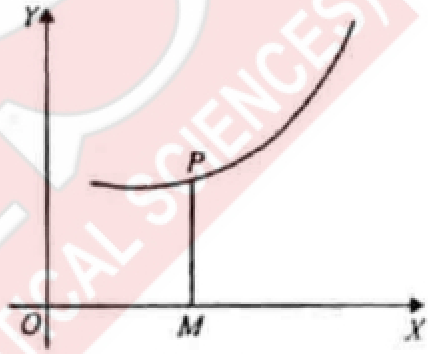


Fig. 44

In terms of the co-ordinates (x, y, z) of the point P in any position, we have

$$MP = \sqrt{y^2 + z^2},\ OM = x$$

so that we can rewrite (2) as

$$\sqrt{y^2 + z^2} = f(x)$$

Hence, the result.

Similarly the equations of the surfaces of revolution obtained on revolving plane curves about Y-axis and Z-axis are of the form

$$\sqrt{z^2 + x^2} = \phi(y),\ \sqrt{x^2 + y^2} = \psi(z)$$

respectively.

Cor. *A quadric is a surface of revolution, if and only if, it has equal non-zero characteristic roots.* To see the truth of this result, we examine the various canonical forms which we have obtained. These are as follows :

Case I $\qquad \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \Delta / D = 0 \qquad ...(1)$

Case II $\qquad \lambda_1 x^2 + \lambda_2 y^2 + 2rz = 0 \qquad ...(2)$

Case III $\qquad \lambda_1 x^2 + \lambda_2 y^2 + k = 0 \qquad ...(3)$

Case IV $\qquad \lambda_1 x^2 + 2rz = 0 \qquad ...(4)$

Case V $\qquad \lambda_1 x^2 + k = 0 \qquad ...(5)$

On comparison with the equations of the surfaces of revolution we see that for the surface (1) to be that of revolution we must have two of $\lambda_1, \lambda_2, \lambda_3$ equal and for the surfaces (2) and (3) to be of revolution we must have $\lambda_1 = \lambda_2$. The quadrics (4) and (5) cannot be surfaces of revolution.

It will be seen that a necessary condition for a quadric to be a surface of revolution is that two of the characteristic roots are equal.

Clearly the equations (1) will represent a sphere if the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ are all equal.

Hence, the result.

11.10.1. Condition for the general equation of the second degree to represent a quadric of revolution

Let the equation $\qquad \Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0$

represent a surface of revolution so that two of the characteristic roots are equal, so that as shown in § 11.6.2 (B), page 315, we have the following two sets of necessary conditions:

$$(b-\lambda)(c-\lambda) = f^2,\ (c-\lambda)(a-\lambda) = g^2,\ (a-\lambda)(b-\lambda) = h^2 \qquad \text{...(I)}$$

$$gh = (a-\lambda)f,\ hf = (b-\lambda)g,\ fg = (c-\lambda)h. \qquad \text{...(II)}$$

λ denoting the twice repeated root.

11.10.2. Case I. *Firstly, suppose that none of f, g, h is zero.*

We show that in this case the set of conditions I and deducible from the Set II so that the Set I is not an independent set of conditions and can as such be ignored. Let us assume the Set II. Now

$$gh = (a-\lambda)f,\ hf = (b-\lambda)g$$

$$\Rightarrow \quad fgh^2 = (a-\lambda)(b-\lambda)fg$$

$$\Rightarrow \quad (a-\lambda)(b-\lambda) = h^2;\ fg \text{ being not equal to zero.}$$

We may similarly deduce the other two conditions of the Set I from the set of conditions II.

Thus, if the given equation represents a surface of revolution and none of f, g, h is zero, we have

$$\lambda = a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}$$

$$\Rightarrow \quad a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}$$

$$\Leftrightarrow \quad \frac{F}{f} = \frac{G}{g} = \frac{H}{h} \qquad \text{...(III)}$$

Now we suppose that the conditions III are satisfied and show that the quadric is a surface of revolution.

Let $\qquad \dfrac{F}{f} = \dfrac{G}{g} = \dfrac{H}{h} = k$ (say)

$$\Leftrightarrow \quad \frac{gh - af}{f} = \frac{hf - bg}{g} = \frac{fg - ch}{h} = k$$

$$\Leftrightarrow \quad a = \frac{gh}{f} - k,\ b = \frac{hf}{g} - k,\ c = \frac{fg}{h} - k$$

Replacing a, b, c by $\dfrac{gh}{f} - k, \dfrac{hf}{g} - k, \dfrac{fg}{h} - k$, we get

$$F(x, y, z) = -k(x^2 + y^2 + z^2) + fgh\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right) + 2ux + 2vy + 2wz + d$$

$$= -k(x^2 + y^2 + z^2) + 2ux + 2wz + d + fgh\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right)^2$$

This form of the equation shows that every plane parallel to the plane

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0 \qquad ...(1)$$

cuts the surface in a circle. Thus, the equation represents a surface of revolution and the axis of revolution, being the locus of the centres of the circular sections, is the line through the centre of the sphere

$$-k(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

perpendicular to the plane (1). Thus, the axis of revolution is

$$\frac{x+\dfrac{u}{\lambda}}{1/f}=\frac{y+\dfrac{v}{\lambda}}{1/g}=\frac{z+\dfrac{w}{\lambda}}{1/h}.$$

We have thus shown that *if no one of f, g, h is zero, the necessary and sufficient condition for the equation*

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d=0$$

to be a surface of revolution is

$$\frac{F}{f}=\frac{G}{g}=\frac{H}{h}.$$

11.10.3. Case II. *When any one of f, g, h is zero and not each one of f, g, h is zero.*

Suppose that $f=0$.

Now $\quad gh=(a-\lambda)f$ and $f=0$

$\Rightarrow \quad gh=0 \Rightarrow g=0$ or $h=0$.

Since we have supposed that not each one of f, g, h is 0, both of g, h are not zero.

Let $g=0$ and $h\neq 0$.

Now $\quad f=0,\ g=0,\ h\neq 0$ and $fg=(c-\lambda)h$

$\Rightarrow \quad \lambda=c$.

Also $\quad (a-\lambda)(b-\lambda)=h^2$ and $\lambda=c$

$\Rightarrow \quad (a-c)(b-c)=h^2$.

We have thus shown that if the given quadric is a surface of revolution and

$$f=0,\ g=0,\ h\neq 0$$

then we necessarily have the relation

$$(a-c)(b-c)=h^2$$

We ~~now~~ show that this condition is also sufficient.

Since $\quad (a-c)(b-c)=h^2$

$a-c$ and $b-c$ must both be of the same sign. Suppose that they are both positive.

We have in this case

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=(a-c)x^2+(b-c)y^2+c(x^2+y^2+z^2)\pm\sqrt{2(a-c)(b-c)}\,xy$$
$$=(\sqrt{a-c}\,x\pm\sqrt{b-c}\,y)^2+c(x^2+y^2+z^2)$$

where actually we have no ambiguity of sign in that the sign is positive or negative according as h is positive or negative.

Thus, we see that planes parallel to the plane

$$\sqrt{(a-c)}\, x \pm \sqrt{(b-c)}\, y = 0$$

cut the surface in circular sections. Thus, the quadric is a surface of revolution, the axis of revolution being the line through the centre

$$(-u/c, -v/c, -w/c)$$

of the sphere

$$c(x^2+y^2+z^2) + 2ux + 2vy + 2wz + d = 0$$

perpendicular to the plane (4); *viz.*, the line

$$\frac{x+u/c}{\sqrt{(a-c)}} = \pm \frac{y+v/c}{\sqrt{(b-c)}},\ z+\frac{w}{c} = 0$$

We have thus shown *that if*

$$f = 0,\ g = 0,\ h \neq 0,$$

the necessary and sufficient condition for the equation

$$\Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

to represent a surface of revolution is

$$(a-c)(b-c) = h^2.$$

We may similarly consider the cases $g = 0$, $h = 0$, $f \neq 0$; and $f = 0$, $h = 0$, $g \neq 0$.

11.10.4. Case III. *When f, g, h are all zero.*

We have in this case from Set I,

$$(b-\lambda)(c-\lambda) = 0,\ (c-\lambda)(a-\lambda) = 0,\ (a-\lambda)(b-\lambda) = 0$$

These relations imply that we necessarily have

$$a = b \text{ or } b = c \text{ or } c = a$$

This can also be seen as follows. The three conditions imply that the three equations

$$\lambda^2 - \lambda(b+c) + bc = 0$$
$$\lambda^2 - \lambda(c+a) + ca = 0$$
$$\lambda^2 - \lambda(a+b) + ab = 0$$

are consistent. This means that we have the relations

$$\begin{vmatrix} 1 & b+c & bc \\ 1 & c+a & ca \\ 1 & a+b & ab \end{vmatrix} = 0$$

$$\Rightarrow \quad (a-b)(b-c)(c-a) = 0$$

$$\Rightarrow \quad a = b \text{ or } b = c \text{ or } c = a$$

Thus, we see that if $f = g = h = 0$ and the given equation represents a surface of revolution, we necessarily have

$$a = b \text{ or } b = c \text{ or } c = a$$

We now show that the condition is as well sufficient

Let $a = b$.

We have in this case

$$\Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

$$\Rightarrow \quad [a(x^2+y^2+z^2) + 2\Sigma ux + d] + (c-a)z^2 = 0$$

so that planes parallel to $z = 0$ cut the surface in circles implying that the given quadric is a surface of revolution, The equations of the axis of revolution may now be easily obtained.

The cases $b = c$, $c = a$ may now be similarly discussed.

We have thus shown that if f, g, h are all zero, a necessary and sufficient condition for the quadric to be a surface of revolution is that

$$a = b \text{ or } b = c \text{ or } c = a.$$

11.11. REDUCTION OF EQUATIONS WITH NUMERICAL COEFFICIENTS

The manner in which we actually proceed in any given case depends upon whether the second degree terms do or do not form a perfect square.

Case I. *Suppose that the second degree terms form a perfect square.*

Thus, the given equation is of the form

$$(px + qy + rz)^2 + 2(ux + vy + wz) + d = 0 \quad ...(1)$$

We rewrite it as

$$(px + qy + rz + t)^2 + 2(u - pt)x + 2(v - qt)y + 2(w - rt)z + (d - t^2) = 0 \quad ...(2)$$

t, being any number whatsoever. Consider now the two planes

$$px + qy + rz = 0$$

$$(u - pt)x + (v - pt)y + (w - rt)z = 0$$

We so choose t that these planes are perpendicular to each other. Thus, t is given by

$$p(u - pt) + q(v - qt) + r(w - rt) = 0$$

$$\Rightarrow \quad (pu + qv + rw) = (p^2 + q^2 + r^2)t$$

$$\Rightarrow \quad t = (pu + qv + rw)/(p^2 + q^2 + r^2)$$

Having thus chosen t, we rewrite (2) as

$$\left(\frac{px + qy + rz + t}{\sqrt{p^2 + q^2 + r^2}}\right)^2 = k\,\frac{2(u - pt)x + 2(v - qt)y + 2(w - rt)z + (d - t^2)}{2\sqrt{(u - pt)^2 + (v - qt)^2 + (w - rt)^2}}$$

where $k = -\dfrac{2\sqrt{(u - pt)^2 + (v - qt)^2 + (w - rt)^2}}{\sqrt{p^2 + q^2 + r^2}}$

Taking

$$\frac{px + qy + rz + t}{\sqrt{p^2 + q^2 + r^2}} = Y$$

$$\frac{2(u - pt)x + 2(v - qt)y + 2(w - rt)z + (d - t^2)}{2\sqrt{(u - pt)^2 + (v - qt)^2 + (w - rt)^2}} = X$$

we see that the given equation takes the form

$$Y^2 = kX$$

so that the surface is a parabolic cylinder.

Ex. Show that the second degree terms form a perfect square if A, B, C, F, G, H are all zero.

Case II. The following procedure is suggested for the reduction of numerical equations when the second degree terms do not form a perfect square.

1. Find the discriminating cubic and solve the same.
2. If no characteristic root is zero, then put down the centre-giving equations and solve them.

If (α, β, γ) is a centre and $\lambda_1, \lambda_2, \lambda_3$ are the characteristics roots, then the reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + (u\alpha + v\beta + w\gamma + d) = 0$$

3. If one characteristic root is zero, find the principal direction l, m, n corresponding to the zero characteristic root by solving two of the three equations

$$al + hm + gn = 0,\ hl + bm + fn = 0,\ gl + fm + cn = 0$$

Then find $ul + vm + wn$. If this is not zero, the reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2(ul + vm + wn)z = 0;$$

λ_1, λ_2 being the non-zero characteristic roots.

4. If $ul + vm + wn = 0$, find the centre-giving equations. In this case we have a line of centres and only two of the three centres-giving equations will be independent. Find any point (α, β, γ) satisfying two of the three equations. Then

$$\lambda_1 x^2 + \lambda_2 y^2 + (u\alpha + v\beta + w\gamma + d) = 0$$

is the required reduced equation.

Note. If one characteristic root is zero and two non-zero, then the line of intersection of the two principal planes corresponding to the non-zero roots is the axis, if the quadric is a parabolic or an elliptic or hyperbolic cylinder and the line of intersection of the planes, if the quadric is a pair of intersecting planes.

In the case of elliptic and hyperbolic cylinders, and a pair of intersecting planes, the line of centres is also the axis.

EXAMPLES

1. *Reduce the equation $2x^2 + 7y^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 2 = 0$ to a canonical form.*

Sol. The discriminating cubic is $\lambda^3 + 3\lambda^2 - 90\lambda + 216 = 0$.

This shows that $D = -216 \neq 0$. The roots of the discriminating cubic are 3, 6, -12.

Again the centre-giving equations are

$$2x - 5y - 4z + 3 = 0,\ 5x + 7y + 5z - 6 = 0,\ 4x + 5y - 2z + 3 = 0.$$

Solving these we see that the centre is

$$\left(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}\right).$$

Denoting this by (α, β, γ), we have

$$u\alpha + v\beta + w\gamma + d = -3$$

Thus, the canonical form of the equation is

$$3x^2 + 6y^2 - 12z^2 - 3 = 0 \iff x^2 + 2y^2 - 4z^2 - 1 = 0 \qquad ...(1)$$

which shows that the given quadric is a hyperboloid of one sheet.

The equation (1) represents the given quadric when the origin of co-ordinates is its centre and the co-ordinate axes are parallel to the principal directions, *i.e.*, (1) is an equation referred to principal axes as co-ordinate axes.

2. *Reduce to canonical form the equation $x^2 - y^2 + 4yz - 4xz - 3 = 0$ of a quadric.*

Sol. The discriminating cubic is

$$\lambda^3 - 9\lambda = 0$$

so that the characteristic roots are

$$0, 3, -3$$

Thus, $D = 0$.

The direction cosines (l, m, n) of the principal direction corresponding to $\lambda = 0$ are given by

$$2l + 4n = 0, -2m + 4n = 0, 4l + 4m = 0.$$

These give $$l : m : n = 2 : -2 : -1$$

Thus, in this case we have

$$ul + vm + wn = 0$$

so that we proceed to find the centre-giving equations.

These are

$$2x + 4z = 0, -2y + 4z = 0, 4y + 4x = 0.$$

These three planes meet in the line. Clearly (0, 0, 0) is a point on it. Denoting this by (α, β, γ), we have

$$u\alpha + v\beta + w\gamma + d = -3$$

Thus, the required canonical form of the equation is

$$3x^2 - 3y^2 = 0 \Leftrightarrow x^2 - y^2 = 0.$$

The given equation, therefore, represents a pair of intersecting planes.

Note. The fact that the given equation is free from first degree terms also shows that (0, 0, 0) is a centre of the given quadric.

3. *Show that $2x^2 + 2y^2 + z^2 + 2yz - 2zx - 4xy + x + y = 0$ represents a paraboloid. Obtain its reduced equation.*

Sol. The discriminating cubic is $\lambda^3 - 5\lambda^2 + 2\lambda = 0$.

Its roots are $0, \dfrac{5+\sqrt{21}}{2}, \dfrac{5-\sqrt{21}}{2}$.

This shows that $D = 0$. The direction cosines (l, m, n) of the principal direction corresponding to $\lambda = 0$ are given by

$$4l - 4m - 2n = 0 \quad ...(1)$$

$$-4l + 4m + 2n = 0 \quad ...(2)$$

$$-2l + 2m + 2n = 0 \quad ...(3)$$

Clearly (1) and (2) are the same. Solving (2) and (3), we obtain

$$l = \frac{1}{\sqrt{2}}, m = \frac{1}{\sqrt{2}}, n = 0$$

$$\Rightarrow \quad ul + vm + wn = \frac{1}{\sqrt{2}} \neq 0$$

Thus, the reduced equation is

$$\frac{5+\sqrt{21}}{2}x^2 + \frac{5-\sqrt{21}}{2}y^2 + \sqrt{2}\,z = 0.$$

4. *Discuss the nature of the surface whose equation is*

$$4x^2 - y^2 - z^2 + 2yz + 3z - 4y + 8z - 2 = 0$$

and find the co-ordinates of its vertex and equations to its axis.

Sol. It may be shown that the roots of the discriminating cubic are $0, -2, 4$.

The direction cosines (l, m, n) of the principal direction corresponding to the root 0; are given by

$$8l = 0, -2m + 2n = 0, -2m + 2n = 0$$

These give $$l = 0, m = 1/\sqrt{2}, n = 1/\sqrt{2}$$

Then $$ul + vm + wn = 2/\sqrt{2} \neq 0$$

Thus, the quadric is a paraboloid.

We now proceed to find the axis and the vertex.

The direction cosines (l, m, n) of the principal direction corresponding to $\lambda = -2$ are given by

$$6l + 0\cdot m + 0\cdot n = 0,\ 0\cdot l + m + n = 0,\ 0\cdot l + m + n = 0$$

These give $$l = 0,\ m = 1/\sqrt{2},\ n = -1/\sqrt{2}$$

so that the corresponding principal plane is

$$-2(y-z) + (-2-4) = 0,\ y - z + 3 = 0 \quad ...(1)$$

Again the direction cosines of the principal direction corresponding to $\lambda = 4$ are given by

$$0\cdot l + 0\cdot m + 0\cdot n = 0,\ 0\cdot l - 5m + n = 0,\ 0\cdot l + m - 5n = 0$$

These give $$l : m : n = 1 : 0 : 0$$

so that the corresponding principal plane is

$$4x - 4 = 0,\ x = 1 \quad ...(2)$$

Thus, $$y - z + 3 = 0,\ x = 1$$

is the required axis of the paraboloid.

The vertex is the point where the axis meets the paraboloid. Rewriting the equations of the axis in the form

$$x = 1,\ \frac{y+3}{1} = \frac{z}{1}$$

we see that $$(1,\ r-3,\ r)$$

are the co-ordinates of a point on this line; r being the parameter.

This point will lie on the surface for $r = 3/4$ so that *the vertex is the point*

$$\left(1, -\frac{9}{4}, \frac{3}{4}\right).$$

5. *Prove that $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$ represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$.* (*Garhwal, 1995*)

Find also the equations of the axis of the cylinder.

Sol. The discriminating cubic is

$$\lambda^3 - 18\lambda^2 + 72\lambda = 0$$

so that the values of λ are

$$0, 6, 12$$

The direction cosines (l, m, n) of the principal direction corresponding to $\lambda = 0$ are given by

$$l - 5m - 4n = 0$$
$$5l - m + 4n = 0$$

$$\Leftrightarrow \quad l = 1/\sqrt{3},\ m = 1/\sqrt{3},\ n = -1/\sqrt{3}$$

Thus, $$ul + vm + wn = 6/\sqrt{3} - 6/\sqrt{3} - 0/\sqrt{3} = 0$$

We have, therefore, to proceed to put down the centre-giving equations. These are

$$10x - 2y + 8z + 12 = 0 \quad ...(1)$$
$$-2x + 10y + 8z - 12 = 0 \quad ...(2)$$
$$8x + 8y + 16z = 0 \quad ...(3)$$

Clearly (3) can be obtained on adding (1) and (2) so that as expected, these three equations are equivalent to only two. Putting $z = 0$ in (1) and (2), we obtain

$$x = -1,\ y = 1,\ z = 0$$

so that $(-1, 1, 0)$ is a centre. Thus,

$$u\alpha + v\beta + w\gamma + d = -6 - 6 + 6 = -6$$

Hence, the reduced equation is

$$12x^2 + 6y^2 - 6 = 0 \Leftrightarrow 2x^2 + y^2 = 1$$

The cross-section is $2x^2 + y^2 = 1$, $z = 0$.

Its eccentricity is now easily seen to be $1/\sqrt{2}$.

The line of centres is the axis of the cylinder so that the equations of the axis are

$$5x - y + 4z + 6 = 0,\ x + y + 2z = 0.$$

6. *Show that the equation $x^2 + 2yz = 1$ represents a quadric of revolution. Also find the axis of revolution.*

Sol. The discriminating cubic is

$$(1-\lambda)(\lambda^2 - 1) = 0 \Leftrightarrow (\lambda + 1)(\lambda - 1)^2 = 0$$

so that the characteristic roots are

$$-1, 1, 1$$

Two of the characteristic roots being equal and non-zero, we see that the given equation represents a quadric of revolution.

Further rewriting the given equation as

$$(x^2 + y^2 + z^2) - (y - z)^2 = 1$$

$$\Leftrightarrow \quad (x^2 + y^2 + z^2 - 1) - (y - z)^2 = 0$$

we see that the planes parallel to the plane

$$y - z = 0 \qquad ...(1)$$

cut the quadric in circles. Thus, the axis of revolution which is the line through the centre of the sphere

$$x^2 + y^2 + z^2 = 1$$

perpendicular to the line (1) is

$$x = 0,\ y = z.$$

7. *Prove that $x^2 + y^2 + z^2 - yz - zx - zy - 3x - 6y - 9z + 21 = 0$ represents a paraboloid of revolution and find the co-ordinates of its focus.* (*Garhwal, 1997, 2000*)

Sol. The discriminating cubic is

$$-4\lambda^3 + 12\lambda^2 - 9\lambda = 0$$

so that the characteristic roots are

$$0,\ 3/2,\ 3/2$$

Two values of λ being equal, the given quadric is a surface of revolution.

The direction cosines (l, m, n) of the principal direction corresponding to $\lambda = 0$ are given by any two of the three equations

$$l - \frac{1}{2}m - \frac{1}{2}n = 0$$

$$-\frac{1}{2}l + m - \frac{1}{2}n = 0$$

$$-\frac{1}{2}l - \frac{1}{2}m + n = 0$$

These give $$l : m : n = 1 : 1 : 1$$

$\therefore$ $$l = 1/\sqrt{3},\ m = 1/\sqrt{3},\ n = 1/\sqrt{3}$$

Now we have

$$ul + vm + wn = -\frac{3}{2}\cdot\frac{1}{\sqrt{3}} - 3\cdot\frac{1}{\sqrt{3}} - \frac{9}{2}\cdot\frac{1}{\sqrt{3}} = -\frac{9}{\sqrt{3}} \neq 0$$

Thus, the quadric is a paraboloid of revolution and the reduced equation is

$$\frac{3}{2}x^2 + \frac{3}{2}y^2 - 2\cdot\frac{9}{\sqrt{3}}z = 0 \Leftrightarrow x^2 + y^2 = 4\sqrt{3}\,z.$$

This form of the equation shows that the latus rectum of the generating parabola is $4\sqrt{3}$.

With respect to the given system of co-ordinate axes, the direction ratios of the axis of the paraboloid which is also the axis of revolution are 1, 1, 1.

We rewrite the given equations in the form

$$x^2 + y^2 + z^2 - \frac{1}{2}[(x+y+z)^2 - (x^2+y^2+z^2)] - 3x - 6y - 9z + 21 = 0$$

$$\Leftrightarrow \quad \frac{3}{2}(x^2+y^2+z^2) - 3x - 6y - 9z + 21 - \frac{1}{2}(x+y+z)^2 = 0$$

$$\Leftrightarrow \quad x^2 + y^2 + z^2 - 2x - 4y - 6z + 14 - \frac{1}{3}(x+y+z)^2 = 0$$

Thus, the axis of revolution, being the line through the centre of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y - 6z + 14 = 0$$

and perpendicular to the plane

$$x + y + z = 0$$

is $$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1} \qquad \ldots(1)$$

which is the axis of the paraboloid.

The vertex is the point where this axis meets the paraboloid. It can be shown that any point

$$(r+1, r+2, r+3)$$

on the axis will be on the paraboloid if $r = -1$.

Thus, (0, 1, 2) is the vertex of the paraboloid.

The required focus is the point on the axis (1) at a distance $\sqrt{3}$ from (0, 1, 2). Rewriting the equations of the axis in the form

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}}$$

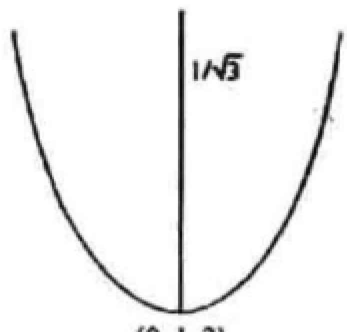


Fig. 45

we see that the point on the axis at a distance $\sqrt{3}$ from (0, 1, 2) is

$$(1, 2, 3).$$

Thus, (1, 2, 3) is the required focus.

8. *If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes, prove that the planes bisecting the angles between them are*

$$\begin{vmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \\ x & y & z \\ F^{-1} & G^{-1} & H^{-1} \end{vmatrix} = 0.$$

Sol. As the given equation represents a pair of planes, we must have $D = 0$.

The line of intersection of the two planes is parallel to the principal direction corresponding to the characteristic root 0 so that if (l, m, n) be the direction cosines of this line, we have

$$al + hm + gn = 0$$
$$hl + bm + fn = 0$$

These give
$$\frac{l}{G} = \frac{m}{F} = \frac{n}{C}$$

As $FG = CH$, we see on replacing C by FG/H, that l, m, n are proportional to F^{-1}, G^{-1}, H^{-1}.

The result can also be obtained if we regard the line of intersection as the line of centres.

Now we know that the two bisecting planes are the principal planes corresponding to the two non-zero characteristic roots.

Suppose that (x, y, z) is any point on either bisecting plane. Let this bisecting plane, as a principal plane, bisect chords with direction cosines (l_1, m_1, n_1) and perpendicular to the plane. The equation of the plane being

$$l_1 (ax + hy + gz) + m_1 (hx + by + fz) + n_1 (gx + fy + cz) = 0 \qquad ...(1)$$

we see that any point (x, y, z) on the bisecting plane satisfies this equation.

Further the plane being normal to the line with direction cosines (l_1, m_1, n_1), its equation is also

$$l_1 x + m_1 y + n_1 z = 0 \qquad ...(2)$$

so that (x, y, z) satisfies (2) also.

Finally, the principal direction l_1, m_1, n_1 corresponding to a non-zero characteristic root being perpendicular to that corresponding to the zero characteristic root, we have

$$l_1 F^{-1} + m_1 G^{-1} + n_1 H^{-1} = 0 \qquad ...(3)$$

From (1), (2) and (3), we have

$$\begin{vmatrix} ax + hy + gz & hx + by + fz & gx + fy + cz \\ x & y & z \\ F^{-1} & G^{-1} & H^{-1} \end{vmatrix} = 0$$

Hence, the result.

9. *Prove that if*

$$a^3 + b^3 + c^3 = 3abc \text{ and } u + v + w \neq 0$$

the equation

$$ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy + 2ux + 2vy + 2wz + d = 0$$

represents either a parabolic cylinder or a hyperbolic paraboloid.

Sol. The discriminating cubic of the given quadric is

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (ab + bc + ca - a^2 - b^2 - c^2) - (3abc - a^3 - b^3 - c^3) = 0$$

so that under one of the given conditions, one root of the cubic is zero.

We have

$$0 = a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

so that either
$$a + b + c = 0 \qquad ...(1)$$
or
$$a^2 + b^2 + c^2 - ab - bc - ca = 0 \qquad ...(2)$$

The condition (2) is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2=0 \Rightarrow a=b=c. \quad ...(3)$$

Assuming (2) to be satisfied, we see that the given equation takes the form

$$a(x+y+z)^2+2(ux+vy+wz)+d=0$$

which is a parabolic cylinder, if $u \neq v$ or $v \neq w$.

Suppose now that the condition (1) is satisfied so that onl one root of the discriminating cubic is zero.

The direction cosines (l, m, n) of the principal direction corresponding to the zero root are given by

$$al+cm+bn=0$$
$$cl+bm+an=0$$

so that

$$\frac{l}{ac-b^2}=\frac{m}{bc-a^2}=\frac{n}{ab-c^2}$$

As $\qquad a+b+c=0,$

we may see that $\qquad ac-b^2=bc-a^2=ab-c^2$

Thus, the principal direction corresponding to the zero root is given by

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Also $\qquad ul+vm+wn=\frac{1}{\sqrt{3}}(u+v+w) \neq 0$

Thus, in this case the quadric is a paraboloid. This paraboloid is hyperbolic for the two non-zero characteristic roots given by

$$\lambda^2+(ab+bc+ca-a^2-b^2-c^2)=0$$

are of opposite signs.

EXERCISES

1. Show that $4x^2-y^2-z^2+2yz-8x-4y+8z-2=0$ represents a paraboloid. Find the reduced equation and the co-ordinates of the vertex.
2. Reduce to its principal axes $2y^2-2yz+2zx-2xy-x-2y+3z-2=0$ and state the nature of the surface represented by the equation.
3. Find the nature of the surface represented by the equation
$$x^2+2y^2-3z^2-4yz+8zx-12xy+1=0.$$
4. Find the reduced equation of
   (i) $x^2+2yz-4x+6y+2z=0.$
   (ii) $x^2-y^2+2yz-2xz-x-y+z=0.$
   (iii) $yz+zx+xy-7x-6y-5z-25=0.$
   (iv) $4y^2-4yz+4zx-4xy-2x+2y-1=0.$
   (v) $2x^2+2y^2+z^2+2yz-2zx-4xy+x+y+z=0.$
   (vi) $(x\cos\alpha-y\sin\alpha)^2+(y\cos\alpha+z\sin\alpha)^2+2y=1.$
   (vii) $3x^2+6yz-y^2-z^2-6x+6y-2z-2=0$ *(Avadh, 2006)*
   (viii) $4x^2+y^2+z^2-4xy-2yz+4zx-12x+6y-6z+8=0.$
   (ix) $x^2+y^2+z^2-2xy-2yz+2zx+x-4y+z+1=0.$

5. Show that the equation

$$a(z-x)(x-y)+b(x-y)(y-z)+c(y-z)(z-x)=0$$

represents two planes whose line of intersection is equally inclined to the three co-ordinate axes.

6. Show that the equation $2yz+2zx+2xy=1$ represents a hyperboloid of revolution. Is this a hyperboloid of one or two sheets ?

7. Show that the quadric $2y^2+4zx-6x-8y+2z+5=0$ is a cone and obtain its reduced equation. Show further that this is a right circular cone with its axis of revolution parallel to the line $x+z=0=y$.

8. Show that the quadric with generators

$$y=1,\ z=-1;\ z=1,\ x=-1;\ x=1,\ y=-1$$

is a hyperboloid of revolution.

9. Find the reduced equation of the quadric with generators

$$x-1=0=y-1;\ x=0=y-z,\ x-2=0=z.$$

10. Prove that every quadric of the linear system determined by the two equations

$$y^2-zx+x=0,\ x^2+y^2+2xz=0$$

is a cone.

11. Discuss the nature of the quadrics represented by the equation

$$2x^2+(m^2+2)(y^2+z^2)-4(yz+zx+xy)=m^2-2m+2$$

as m varies from $-\infty$ to $+\infty$.

Obtain the reduced equation of the quadric corresponding to $m=1$.

12. Show that there is only one paraboloid in the system of quadrics

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d+\lambda(lx+my+nz+p)^2=0.$$

In particular, show that if f, g, h, u, v, w are all zero, the equation of this paraboloid is

$$ax^2+by^2+cz^2+d\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)-(lx+my+nz+p)^2=0$$

Further prove that its axis is parallel to the line

$$\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}.$$

13. If the general equation $\Sigma(ax^2+2fyz)+2\Sigma ux+d=0$ represents a right circular cylinder, prove that

$$\frac{a}{f}+\frac{h}{g}+\frac{g}{h}=0;\ \frac{h}{f}+\frac{b}{g}+\frac{f}{h}=0;\ \frac{g}{f}+\frac{f}{g}+\frac{c}{h}=0;\ \frac{u}{f}+\frac{v}{g}+\frac{w}{h}=0.$$

14. Show that the condition for the equation

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

to represent a cone is

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1.$$

15. Prove that the principal axes of the conicoid $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$ are given by the equations

$$x(f\lambda_r + F) = y(g\lambda_r + G) = z(h\lambda_r + H), \ (r = 1, 2, 3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

and $F = gh - af$, $G = hf - bg$, $H = fg - ch$.

Also show that the cone which touches the co-ordinate planes and the principal planes of the above conicoid is

$$\sqrt{[(gH - hG)x]} + \sqrt{[(hF - fH)y]} + \sqrt{[(fG - gF)z]} = 0.$$

16. If the feet of the six normals from P to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie upon a concentric conicoid of revolution, prove that the locus of P is the cone

$$\frac{y^2z^2}{a^2(b^2 - c^2)} + \frac{z^2x^2}{b^2(c^2 - a^2)} + \frac{x^2y^2}{c^2(a^2 - b^2)} = 0$$

and that the axes of symmetry of the conicoids lie on the cone

$$a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)y^2 + c^2(a^2 - b^2)z^2 = 0.$$

17. Prove that the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ will represent a right circular cone with vertical angle θ provided that

$$\frac{af - gh}{f} = \frac{bg - hf}{g} = \frac{ch - fg}{h} = \frac{(a + b + c)(1 + \cos\theta)}{(1 + 3\cos\theta)}.$$

18. Given the ellipsoid of revolution $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$, $(a^2 < b^2)$, show that the cone whose vertex is one of the foci of the ellipse $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and whose base is any plane section of the ellipsoid is a surface of revolution.

19. Prove that if $F(x, y, z) \equiv \Sigma(ax^2 + 2fyz) + 2\Sigma ux + d = 0$ represents a paraboloid of revolution, we have

$$agh + f(g^2 + h^2) = bhf + g(h^2 + f^2) = cfg + h(f^2 + g^2) = 0$$

and that if it represents a right circular cylinder, we have also

$$\frac{u}{f} + \frac{v}{g} + \frac{w}{h} = 0.$$

ANSWERS

1. $2x^2 - y^2 + \sqrt{2}z = 0, \left(1, -\frac{9}{4}, \frac{3}{4}\right)$.
2. $3x^2 - y^2 = \frac{1}{2}$. Hyperbolic cylinder.
3. $3x^2 + 6y^2 - 9z^2 + 1 = 0$. Hyperboloid of two sheets.
4. (i) $x^2 + y^2 - z^2 = 10$. (ii) $3x^2 - 3y^2 = z$. (iii) $2x^2 - y^2 - z^2 = 102$.

   (iv) $6x^2 - 2y^2 = 1$. (v) $\frac{5 + \sqrt{17}}{2}x^2 + \frac{5 - \sqrt{17}}{2}y^2 + \sqrt{2}z = 0$.