IFoS 2019

(a) 1

$(D^2+1)y = x^2\sin 2x$

Auxillary equation of given DE

$$m^2+1=0$$

$$m=\pm i$$

$$y_c = c_1\cos x + c_2\sin x$$

$$y_p = \frac{1}{D^2+1}x^2\sin 2x$$

$$= \operatorname{Img} \frac{1}{D^2+1} x^2 e^{i2x}$$

$$= \operatorname{Img}\left(e^{i2x}\frac{1}{(D+2i)^2+1}x^2\right)$$

$$= \operatorname{Img}\left(e^{i2x}\frac{1}{D^2+4Di-3}x^2\right)$$

$$= \operatorname{Img}\left[-\frac{e^{i2x}}{3}\left(1-\left(\frac{D^2+4Di}{3}\right)\right)^{-1}x^2\right]$$

(using binomial expansion)

$$= \operatorname{Img}\left[-\frac{e^{i2x}}{3}\left(1+\frac{D^2+4Di}{3}+\frac{16}{9}D^2i^2\right)x^2\right]$$

[neglecting higher order term]

$$= \operatorname{Img}\left[-\frac{e^{i2x}}{3}\left(x^2+\frac{2}{3}+\frac{4i}{3}(2x)-\frac{16}{9}\times 2\right)\right]$$

$$= -\frac{1}{3}\left[\cos 2x\left(\frac{8x}{3}\right)+\sin 2x\left(x^2+\frac{2}{3}-\frac{32}{9}\right)\right]$$

$$= -\frac{1}{27}\left[24x\cos 2x + (9x^2-26)\sin 2x\right]$$

$y = y_c + y_p$

$$\boxed{y = c_1 \cos x + c_2 \sin x - \frac{1}{27}\left[24x\cos 2x + (9x^2 - 26)\sin 2x\right]}$$

(b) 2

$(px - y)(py + x) = h^2 p$ where $p = y'$

$p^2 xy + px^2 - py^2 - xy = h^2 p$ — (i)

Let

$X = x^2 \quad Y = y^2$ — (ii) $\quad P = \frac{dY}{dX} = \frac{y}{x} p \Rightarrow p = \frac{Px}{y} = P\sqrt{\frac{X}{Y}}$ (iii)

Combining (i) & (ii) & (iii), we have

$$P^2 \frac{X}{Y}\sqrt{XY} + P\sqrt{\frac{X}{Y}}\,X - P\sqrt{\frac{X}{Y}}\,Y - \sqrt{XY} = h^2 P\sqrt{\frac{X}{Y}}$$

$$P^2 X + PX - PY - Y = h^2 P$$

$$X(P^2 + P) - Y(P+1) = h^2 P$$

$$PX - Y = \frac{h^2 P}{P+1}$$

$$Y = PX - \frac{h^2 P}{P+1} \quad \text{— (iv)}$$

Equation (iv) is of Clauriet form

$$Y = PX + f(P)$$

So, replacing P by c

we have general solution

$$Y = cX - \frac{h^2 c}{c+1}$$

Substituting $Y = y^2$, $X = x^2$ in above

required solution of differential equation:-

$$\boxed{y^2 = cx^2 - \frac{h^2 c}{c+1}}$$

(3)

$$x''(t) - \frac{2x(t)}{t^2} = t$$

$$\left[D^2 - \frac{2}{t^2}\right] x(t) = t$$

$$\left[t^2D^2 - 2\right] x(t) = t^3$$

let $t = e^u$ $\Rightarrow u = \log t$

$tD = \theta$ where $\theta = \frac{\partial}{\partial u}$

$t^2D^2 = \theta(\theta - 1)$

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$$[\theta(\theta-1)-2]x(u) = e^{3u}$$

$$[\theta^2-\theta-2]\, x(u) = e^{3u}$$

$$m^2-m-2=0$$

$$(m-2)(m+1)=0$$

$$m=2,\ m=-1$$

$$x = c_1 e^{2u} + c_2 e^{-u}$$

$$\boxed{x_c = c_1 t^2 + \frac{c_2}{t}}$$

by method of variation of parameter,

$$x_p = At^2 + \frac{B}{t}$$

& $u = t^2 \quad v = \frac{1}{t}$

$$W = uv' - vu'$$

$$= t^2\left(-\frac{1}{t^2}\right) - \frac{1}{t}\cdot 2t$$

$$= -1-2$$

$$= -3$$

$$\neq 0$$

(so u, v are independent)

$$A = -\int \frac{vR}{uv'-vu'}\, dt$$

$$= \frac{1}{3}\int \frac{1}{t}\cdot t\, dt = t/3$$

$$B = \int \frac{uR}{uv'-vu'}\, dt$$

$$= \int \frac{t^2\cdot t}{-3}\, dt$$

$$= -\frac{t^4}{3\times 4} = -\frac{t^4}{12}$$

$$x_p = t^3/3 - t^3/12 = \frac{t^3}{4}$$

$$x = x_c + x_p$$

$$\boxed{x = c_1 t^2 + \frac{c_2}{t} + \frac{t^3}{4}}$$

(6)

$x'' + 4x = \sin^2 2t$

$(D^2+4)\, x = \sin^2 2t$

Auxiliary eqn

$m^2+4=0$

$m = \pm 2i$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

$x_p = \frac{1}{(D^2+4)} \sin^2 2t$

$= \frac{1}{D^2+4} \left[\frac{1-\cos 4t}{2}\right]$

$= \frac{1}{2}\left[\frac{1}{D^2+4}(1) - \frac{1}{D^2+4}(\cos 4t)\right]$

$= \frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} 1 - \frac{\cos 4t}{4-16}\right]$

$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{12}\cos 4t\right]$

$x_p = \frac{1}{24}[3+\cos 4t]$

$x = x_c + x_p$

$= c_1 \cos 2t + c_2 \sin 2t + \frac{1}{24}[3+\cos 4t]$

at $t = \pi/8$ $x = 0$

$0 = \frac{c_1 + c_2}{\sqrt{2}} + \frac{1}{8}$ $\Rightarrow c_1 + c_2 = -\frac{1}{4\sqrt{2}}$

at $t = \pi/8$ $\dot{x} = 0$

$\frac{-2c_1 + 2c_2}{\sqrt{2}} - \frac{4}{24} = 0 \Rightarrow$ $c_2 - c_1 = \frac{1}{6\sqrt{2}}$

$$2C_2 = \left(\frac{1}{6\sqrt{2}} - \frac{1}{4\sqrt{2}}\right) \Rightarrow C_2 = -\frac{1}{24\sqrt{2}}$$

$$C_1 = \frac{-5}{24\sqrt{2}}$$

$$\boxed{x = -\frac{1}{24\sqrt{2}}(5\cos 2t + \sin 2t) + \frac{1}{24}(3 + \cos 4t)}$$

(5) Ex.

$$y'' - \frac{(4x-7)}{(x-2)}y' + \frac{4x-6}{(x-2)}y = 0$$

comparing to $y'' + Py' + Q = 0$

$$P = -\frac{(4x-7)}{x-2} \quad Q = \frac{4x-6}{x-2}$$

$$a^2 + aP + Q = 0$$

$$a^2 - a\frac{(4x-7)}{x-2} + \frac{4x-6}{x-2}$$

$$x(a^2 - 4a + 4) = 2a^2 - 7a + 6$$

$$a = 2.$$

$\therefore$ one solution $\boxed{u = e^{2x}}$

$$\frac{d^2v}{dx^2} + \left(-\frac{(4x-7)}{x-2} + \frac{2}{e^{2x}} \cdot 2e^{2x}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} - \frac{1}{x-2}\frac{dv}{dx} = 0$$

$z = \frac{dv}{dx}$, $\frac{dz}{dx} = \frac{z}{x-2}$

$$\ln z - \ln(x-2) = C_1$$

$$z = C_1(x-2)$$

$$\frac{dv}{dx} = C_1(x-2)$$

$$v = C_1\left(\frac{x^2}{2} - 2x\right) + C_2$$

$$\therefore \boxed{y = u.v = e^{2x}\left[C_1(x^2/2 - 2x) + C_2\right]}$$