1(a) Let $T: R^3 \to R^3$ be a Linear operator on $R^3$ defined by
$$T(x,y,z) = (2y+z,\ x-4y,\ 3x).$$

Find the matrix of $T$ in the basis $\{(1,1,1),\ (1,1,0),\ (1,0,0)\}$. 8

$$T(1,1,1) = (3,-3,3)$$
$$= 3(1,1,1) - 6(1,1,0) + 6(1,0,0)$$

$$T(1,1,0) = (2,-3,3) = 3(1,1,1) - 6(1,1,0) + 5(1,0,0)$$

$$T(1,0,0) = (0,1,3) = 3(1,1,1) - 2(1,1,0) - 1(1,0,0)$$

$$[T] = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

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1(b) The eigenvalues of a real symmetric matrix A are $-1, 1$ and $-2$. The corresponding eigenvectors are
$\frac{1}{\sqrt{2}}(-1,1,0)^T$, $(0,0,1)^T$ and $\frac{1}{\sqrt{2}}(-1,-1,0)^T$
respectively. Find the matrix $A^4$.

If a matrix A is diagonalizable, then

$$P^{-1}AP = D$$

$$\therefore \quad A = PDP^{-1}$$

$$A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$= PD^4P^{-1}$$

Where P is diagonalizing matrix consisting of eigenvectors of A and D is diagonal matrix containing eignvalues of A at diagonal entries.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \frac{-1}{\sqrt{2}} \begin{bmatrix} +1 & 0 & +1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \frac{1}{2}\begin{bmatrix} 17 & 15 & 0 \\ 15 & 17 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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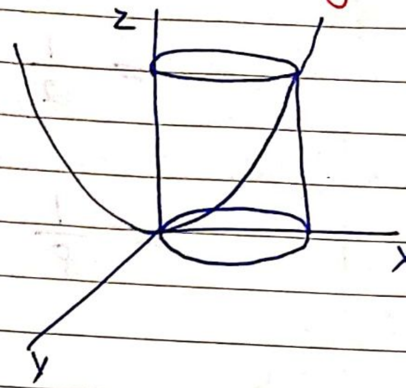
1.c) Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by xy-plane.

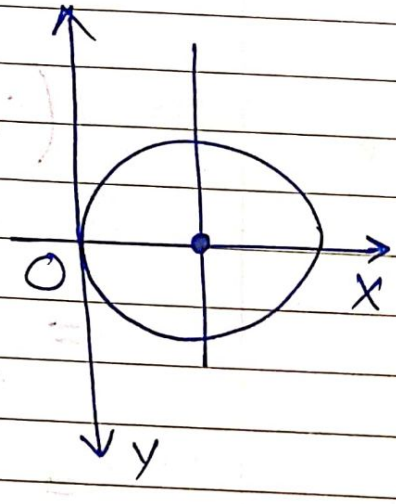



The required volume is found by integrating $z=\frac{1}{2}(x^2+y^2)$ over the circle $x^2+y^2=2x$

Changing to polar coordinates in the xy-plane,

$x=r\cos\theta$ , $y=r\sin\theta$

$\therefore \quad z=\frac{1}{2}(x^2+y^2)=\frac{r^2}{2}$




Polar eqn of circle is

$r^2\cos^2\theta+r^2\sin^2\theta=2r\cos\theta$

$r=2\cos\theta.$

To cover this circle,
r varies from 0 to $2\cos\theta$ and
$\theta$ varies from 0 to $\pi$.

$\therefore$ Required volume

$$V=\int_0^{\pi}\int_0^{2\cos\theta} z.\, r\,dr\,d\theta=\int_0^{\pi}\int_0^{2\cos\theta}\frac{r^3}{2}\,dr\,d\theta$$



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$$V = \frac{1}{2}\int_0^{\pi}\int_0^{2\cos\theta} r^3 \, dr \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left. \frac{r^4}{4} \right|_0^{2\cos\theta} d\theta$$

$$= \frac{1}{\cancel{8}}\int_0^{\pi} \overset{2}{\cancel{16}} \cos^4\theta \, d\theta$$

$$= 2 \cdot \times 2 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$\left( \because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(2a-x) = f(x) \right)$$

$$= \cancel{4} \times \frac{3}{\cancel{4}} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{3}{4}\pi$$

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1(d) Justify by using Rolle's theorem or MVT that there is no number K for which the equation $x^3 - 3x + K = 0$ has two distinct solutions in $[-1, 1]$. (8)

$$f(x) = x^3 - 3x + K$$

Let if possible $f(x)$ has two distinct roots $a$ and $b$ in $[-1, 1]$

ie. $$f(a) = 0 = f(b) \quad , \quad -1 \le a, b \le 1$$
$$a \neq b.$$

$f(x)$ is continuous and differentiable over the interval $[a, b]$.

Hence, by Rolle's theorem, there exist some $c \in (a, b)$ s.t.

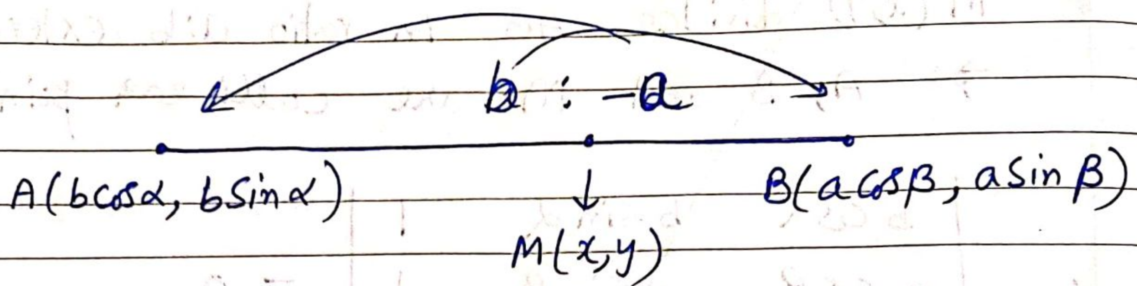
$$f'(c) = 0$$

ie $$3c^2 - 3 = 0 \Rightarrow c = \pm 1$$

which is contradiction to the fact that $a$ and $b$ lies within $[-1, 1]$

Hence $f(x)$ cannot have two distinct roots in $[-1, 1]$ for any value of 'K'.

1(e) Point $M(x,y)$ divides the line-segment AB in the ratio $b:a$ externally. We take it as $b:(-a)$.




$$x = \frac{ab\cos\beta - ab\cos\alpha}{b-a}$$

$$= \frac{ab}{b-a}(\cos\beta - \cos\alpha)$$

$$= \frac{ab}{b-a}\left(-2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \quad \text{—(1)}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$y = \frac{b\cdot(a\sin\beta) - a(b\sin\alpha)}{b-a}$$

$$= \frac{ab}{b-a}(\sin\beta - \sin\alpha)$$

$$= \frac{ab}{b-a}\left(2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \quad \text{—(2)}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\frac{x}{y} = \frac{-\sin\left((\alpha+\beta)/2\right)}{\cos\left((\alpha+\beta)/2\right)}$$

$$\Rightarrow x\cdot\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0$$

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2(a) Determine the extreme values of the fn $f(x,y) = 3x^2 - 6x + 2y^2 - 4y$ in the region $\{(x,y) \in \mathbb{R}^2 : 3x^2 + 2y^2 \le 20\}$. (10)

First we find the critical points $f(x,y)$

$f_x = 0 \Rightarrow 6x - 6 = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1$

$\therefore$ $P(1,1)$ is the only critical point.

as $3(1)^2 + 2(1)^2 = 5 < 20$

$\Rightarrow$ $P(1,1)$ lies in the given elliptical region.

$f(1,1) = 3 - 6 + 2 - 4 = -5$ —— ①

$f_{xx} f_{yy} - f_{xy}^2 = (6)(4) - 0^2 = 24 > 0$

and $f_{xx} = 6 > 0$ at $P(1,1)$

Hence Point $(1,1)$ is a point of local minima.

Let us check at boundaries of the ellipse

ie $3x^2 + 2y^2 = 20$

$\therefore$ $f(x,y) = 3x^2 - 6x + 2y^2 - 4y$

$= 20 - 6x - 4y$

$= 20 - 6x \pm 2\sqrt{2}\sqrt{20 - 3x^2}$

~~$= g(x)$ (say).~~

$\left( y = -\frac{1}{\sqrt{2}} \sqrt{20-3x^2} \right)$

Let $g(x) = 20 - 6x + 2\sqrt{2}\sqrt{20-3x^2}$

$$g'(x) = -6 + 2\sqrt{2}\frac{(-6x)}{2\sqrt{20-3x^2}}$$

$g'(x) = 0$ gives $x = \pm 2 \Rightarrow y = \mp 2$

At $(2,-2)$,

$$f(x,y) = 20 - 6(2) - 4(-2)$$
$$= 20 - 12 + 8 = 16 \quad \text{— ②}$$

At $(-2,2)$, $f(x,y) = 20 - 6(-2) - 4(2)$
$$= 12 + 12 - 8 = 16 \quad \text{— ③}$$

Again let $h(x) = 20 - 6x - 2\sqrt{2}\sqrt{20-3x^2}$

$$\left( y = \frac{1}{\sqrt{2}}\sqrt{20-3x^2} \right)$$

$$h'(x) = -6 + \cancel{2}\sqrt{2}\frac{6x}{\cancel{2}\sqrt{20-3x^2}}$$

$h'(x) = 0 \Rightarrow x = \pm 2 \Rightarrow y = \pm 2$

At $(2,2) \Rightarrow f(x,y) = 20 - 12 - 8 = 0$ — ④

At $(-2,-2)$ $f(x,y) = 20 + 12 + 8 = 40$ — ⑤

from ①, 2, 3, 4 and 5, we get

Max at $(-2,-2)$, $f(x,y) = 40$

Min at $(1,1)$, $f(x,y) = -5$.

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(METHOD-2) using polar coordinates (elliptical)

$3x^2+2y^2=20 \Rightarrow \dfrac{x^2}{\frac{20}{3}}+\dfrac{y^2}{10}=1$

Let $x=2\sqrt{\frac{5}{3}}\, r\cos\theta$, $y=\sqrt{10}\, r\sin\theta$

for $0\le r\le 1$, it gives elliptical region.

$$f(x,y)=3x^2-6x+2y^2-4y$$

$$=20r^2-12\frac{\sqrt5}{\sqrt3}r\cos\theta-4\sqrt{10}\,r\sin\theta$$

$$=20r^2-4\sqrt5 r\left(\sqrt3\cos\theta-\sqrt2\sin\theta\right)$$

$$=20r^2-20r\left(\sqrt{\tfrac{3}{5}}\cos\theta-\sqrt{\tfrac{2}{5}}\sin\theta\right)$$

$$=20r^2-20r\left(\sin(A-\theta)\right) \text{ where}$$

$$\left(\sin A=\sqrt{\tfrac{3}{5}},\ \cos\theta=\sqrt{\tfrac{2}{5}}\right)$$

$$f(r,\theta)=20r\left[r-\sin(A-\theta)\right]$$

Max value of $f(r,\theta)$ will occur where $\sin(A-\theta)=-1$ and $r=1$

$$f(r,\theta)=20(1)\left(1-(-1)\right)=\underline{\underline{40}}$$

for minimum, $\sin(A-\theta)=1$

$$f(r,\theta)=20r(r-1)=20(r^2-r)$$

$$f'(r,\theta)=20(2r-1)\Rightarrow r=\frac{1}{2}$$

Min value $20\times\frac{1}{2}\left(\frac{1}{2}-1\right)=\underline{\underline{-5}}$

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2(b) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of $A$ is 4 and one eigenvector that does not correspond to this eigenvalue is $(1,1,0,0)^T$. Find all the eigenvalues of $A$ other than 4 and hence also find the real numbers $p, q, r$ that satisfy the matrix equation

$$A^4 + pA^3 + qA^2 + rA = 0. \qquad (15)$$

Let $\lambda_1 = 4$, $v_2 = (1,1,0,0)^T$

$$Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(2, 2, 0, 0) = (\lambda_2, \lambda_2, 0, 0)$$

$$\Rightarrow \lambda_2 = 2.$$

Let the other two eigenvalues be $\lambda_3$ and $\lambda_4$.

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Trace (A) = Sum of Eigenvalues

$4 + 2 + \lambda_3 + \lambda_4 = -1 + 5 + (-10) + 8$

$\lambda_3 + \lambda_4 = -4$

Also, Product of eigenvalues = Det (A)

$4 \cdot 2 \cdot \lambda_3 \cdot \lambda_4 = 0 \Rightarrow \lambda_3 \cdot \lambda_4 = 0.$

ie $\lambda_3(-4-\lambda_3) = 0$

$\Rightarrow \lambda_3 = 0$ or $\lambda_3 = -4$

$\therefore \lambda_4 = -4$ or $\lambda_4 = 0$

Characteristic polynomial

$\prod (x - \lambda_i) = 0$

$(x-4)(x-2)(x+4)(x-0) = 0.$

$(x^2-16)(x-2)x = 0$

$(x^3 - 16x - 2x^2 + 32)x = 0$

$x^4 - 16x^2 - 2x^3 + 32x = 0$

Since, Every square matrix satisfies its characteristic eqn (Cayley Hamilton Thm)

$\therefore A^4 - 2A^3 - 16A^2 + 32A = 0$

$\therefore p = -2, \quad q = -16, \quad r = 32.$



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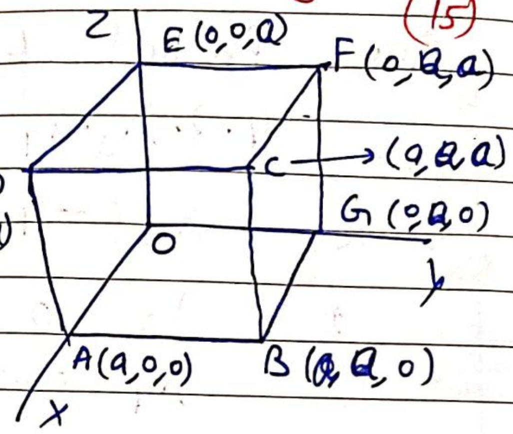
2(c) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

(15)




The d.r. of four diagonals

$AF = (-a, a, a)$
$\quad = (-1, 1, 1)$

$BE = (-a, -a, a)$
$\quad = (1, 1, -1)$

$CO = (-a, -a, -a) = (1, 1, 1)$

$DG = (-a, a, -a) = (1, -1, 1)$

Let the d.r. of line are $\langle l, m, n \rangle$

$$\cos\alpha = \frac{-l+m+n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}}, \quad \cos\beta = \frac{l+m-n}{\sqrt{3}\sqrt{l^2+m^2+n^2}}$$

$$\cos\gamma = \frac{l+m+n}{\sqrt{3}\sqrt{l^2+m^2+n^2}}, \quad \cos\delta = \frac{l-m+n}{\sqrt{3}\sqrt{l^2+m^2+n^2}}$$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3(l^2+m^2+n^2)}\left[(-l+m+n)^2 + (l+m-n)^2 + (l+m+n)^2 + (l-m+n)^2\right]$$

$$= \frac{4(l^2+m^2+n^2)}{3(l^2+m^2+n^2)} = \frac{4}{3}.$$

(After simplification)

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3(a). $x_1 = (1, 2, 1, -1)$, $x_2 = (2, 4, 1, 1)$
$x_3 = (-1, -2, 0, -2)$, $x_4 = (3, 6, 2, 0)$

We find span $\{x_1, x_2, x_3, x_4\}$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -2 \\ 3 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= \text{Span}\{(1, 2, 0, 2), (0, 0, 1, -3)\}$

~~Given subspace~~ $= \{a(1, 2, 0, 2) + b(0, 0, 1, -3);\ a, b \in \mathbb{R}\}$

$= \{(a, 2a, b, 2a - 3b)\}$

$= \{(x, y, z, w) : x = a, y = 2a, z = b, w = 2a - 3b$
ie $y = 2a$, $w = 2x - 3z\}$

If we take $a = \alpha$, $b = \beta$ then above subspace can be written as
$\{\alpha, 2\alpha, \beta, 2\alpha - 3\beta\}$, Dim $= 2$.

as $\alpha$ and $\beta$ are linearly independent.

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3(b) Given

$\frac{dl}{dt} = 2$ cm/sec $\quad \frac{dw}{dt} = 2 \quad \frac{dh}{dt} = -3$

$l = 2t + l_0 \quad , \quad w = 2w + w_0 \quad , \quad h = -3t + h_0$

$l = 2t + 10$ $\quad$ using $l(0) = 10,$

$w = 2t + 8$ $\quad w(0) = 8, \quad h(0) = 20.$

$h = -3t + 20$

At $t = 5$ sec,

$l = 20$ cm, $\quad w = 18$ cm, $\quad h = 5$ cm

$$V = lwh$$
$$= (2t+10)(2t+8)(-3t+20)$$

$$\frac{dV}{dt} = 2(2t+8)(-3t+20) + 2(2t+10)(-3t+20)$$
$$(-3)(2t+10)(2t+8)$$

$$\left.\frac{dV}{dt}\right|_{t=5} = 2(18)(5) + 2(20)(5) - 3(20)(18)$$
$$= 180 + 200 - 1080$$
$$= -700 < 0 \quad \text{(Decreasing V)}$$

$$S = 2(lw + wh + hl)$$

$\cancel{S = 2[(2t+10)(2t+8)}$

$$\frac{dS}{dt} = 2\left[(w+h)\frac{dl}{dt} + (l+h)\frac{dw}{dt} + (l+w)\frac{dh}{dt}\right]$$
$$= 2[23(2) + 25(2) + 38(-3)]$$
$$= 2(46 + 50 - 114) = -36 < 0$$

(S is decreasing).





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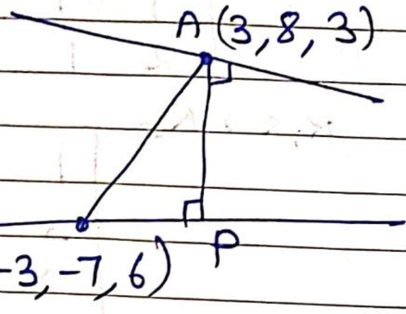
3(c). $L_1 : \dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1}$

$L_2 : \dfrac{x+3}{-3} = \dfrac{y+7}{2} = \dfrac{z-6}{4}$

D.r. of line which is $\perp$ to $L_1$ and $L_2$ both (ie. shortest distance line)




$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3}$$

$$\langle 2, 5, -1 \rangle$$

D.r. of AB $= \langle 3+3, 8+7, 3-6 \rangle$

$= \langle 6, 15, -3 \rangle$

$\cancel{\pm \langle 2, 5, -1 \rangle}$

$$\therefore S.D = \frac{1}{\sqrt{4+25+1}} \left( 2.6 + 5.15 + (-1)(-3) \right)$$

$$= \frac{1}{\sqrt{30}} \left( 12 + 75 + 3 \right) = \frac{90}{\sqrt{30}} = 3\sqrt{30}$$

Since, S.D. line is parallel to AB Hence taking A(3,8,3) as one point its eqn is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z+1}{-1}$$



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4(a) $A = IA$

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 3 & 0 \\ 2 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

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$R_4 \to R_4 - \frac{R_3}{2}$ , $R_3 \to R_3/-4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} A$$

$R_4 \to -R_4$ , $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A$$

$R_3 \to R_3 + 2R_4$, $R_2 \to R_2 - 2R_4$, $R_1 \to R_1 - R_4$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1/2 & -1/2 & 1 \\ 0 & -1 & -1 & 2 \\ -1/4 & 3/4 & 9/4 & -2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & -1 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -5/4 & -11/4 & 3 \\ -1/4 & -1/4 & 5/4 & 0 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A.$$

$R_1 \to R_1 - R_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -4 & 3 \\ -1/4 & -1/4 & 5/4 & 0 \\ -1/4 & 3/4 & 9/4 & -2 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & -1 \end{bmatrix} A$$

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$$AX = b$$

$$X = A^{-1} b$$

$$x = \frac{1}{4}\begin{bmatrix} 4 & -4 & -16 & 12 \\ -1 & -1 & 5 & 0 \\ -1 & 3 & 9 & -8 \\ -2 & 2 & 6 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8-4+48 \\ -2-1+0 \\ -2+3-32 \\ -4+2-16 \end{bmatrix} = \begin{bmatrix} 13 \\ -3/4 \\ -31/4 \\ -9/2 \end{bmatrix}$$

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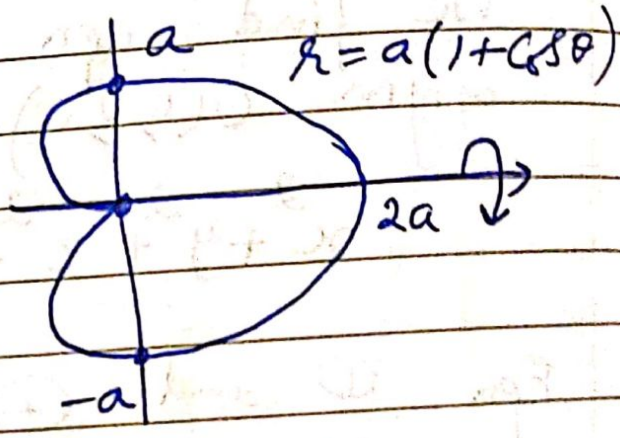
4(b) Find the centroid of the solid generated by revolving the upper half of the cardiod $r = a(1+\cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform. 10.




As the solid of revolution is symmetric about initial line (x-axis), the centroid will lie on it.

ie. y-coordinate will be zero.

$\bar{y} = 0.$

x-coordinate

$$\bar{x} = \frac{\int x \, dV}{\int dV}$$

[ In polar-coordinates $x = r\cos\theta$

$$dV = \frac{2}{3}\pi r^2 \sin\theta \, d\theta \, dr$$

$\theta$ varies from 0 to $\pi$ (upper part) ]

$\pi$

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$$V = \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr\, d\theta$$

$$= 2\pi\int_0^{\pi} \frac{r^3}{3}\Big|_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\cdot\frac{(1+\cos\theta)^4}{-4}\Big|_0^{\pi}$$

$$= \frac{2\pi a^3}{3}\cdot\frac{\cancel{16}\,4}{\cancel{-4}} = \frac{8\pi}{3}a^3$$

$$\int x\, dV = \int_0^{\pi}\int_0^{a(1+\cos\theta)} (r\cos\theta)(2\pi r^2 \sin\theta)\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3 \cos\theta \sin\theta \, dr\, d\theta$$

$$= \frac{2\pi}{4}\int_0^{\pi} a^4(1+\cos\theta)^4 \cos\theta \sin\theta\, d\theta$$

$+1-1$

$$= \frac{\pi}{2}a^4\int_0^{\pi} (1+\cos\theta)^5\sin\theta - (1+\cos\theta)^4 \sin\theta\, d\theta$$

$$= \frac{\pi a^4}{2}\left(\frac{64}{6} - \frac{32}{5}\right) = \frac{\pi a^4 \times 32}{15}$$

$$\therefore \bar{x} = \frac{\cancel{32}^{\,4}\,\cancel{\pi}\, a^4}{\cancel{15}_{\,5}} \times \frac{\cancel{3}}{\cancel{8\pi}\, a^3} = \frac{4a}{5}$$



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4(C). Let the eqn of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=p \quad \text{--- (1)}$$

It meets the axis at points

$A(ap,0,0)$, $B(0,bp,0)$, $C(0,0,cp)$

We find eqn of sphere passing through origin $O(0,0,0)$ and $A, B, C$.

$$x^2+y^2+z^2-apx-bpy-cpz=0. \quad \text{--- (2)}$$

Eqn (1) and (2) together gives the equation of circle ABC.

If we homogenize eqn (2) with help of eqn (1), we will get the eqn of cone with ~~centre~~ vertex, at origine.

$$x^2+y^2+z^2-(apx+bpy+cpz)\left(\frac{x}{ap}+\frac{y}{bp}+\frac{z}{cp}\right)=0$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\left(\cancel{x^2}+\frac{b}{a}xy+\frac{c}{a}zx+\frac{a}{b}xy+\cancel{y^2}+\frac{c}{b}yz\right.$$

$$\left.+\frac{a}{c}xz+\frac{b}{c}zy+\cancel{z^2}\right)=0.$$

$$\therefore\ yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$