Mains Test Series - 2021

Test - 2, Paper-II, Answer Key

Modern Algebra, Real Analysis & Complex Analysis and LPP

1(a) If G is an infinite group, what can you say about the number of elements of order 21 in a group, Generalize.

Sol^n: Given that $(G,*)$ is an infinite group.

Let $a \in G$ such that $O(a) = 21$

then $a^{21} = e$.

$\therefore \langle a \rangle = \{e, a, a^2, a^3, a^4 \dots a^{19}, a^{20}\} \leq G$

Here $21 = 7 \times 3$

by Euler ϕ- function,

$$\phi(21) = 21\left(1-\tfrac{1}{7}\right)\left(1-\tfrac{1}{3}\right)$$

$$= 21\left(\tfrac{6}{7}\right)\left(\tfrac{2}{3}\right)$$

$$\phi(21) = \underline{12}$$

$\therefore$ The number of elements of order 21 are 12 in $\langle a \rangle$.

If there is any element other than 12 elements with order 21 then there is cyclic subgroup of order 21 and it will have 12 generators.

$\therefore$ The number of elements with order 21 will always be an integral multiple of $\underline{12}$.

i.e. number of elements of order 21 are $= 12k : k \in \mathbb{Z}^+$.

1(b) Prove that every field is an integral domain. Is the converse true.

Proof: Let R be any field.

Let $ab = ac$, where $a, b, c \in R$ and $a \neq 0$.

Since $a \neq 0 \in R$, $a^{-1} \in R$ exists and $aa^{-1} = a^{-1}a = 1$.

Now $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, by associative law.

$\Rightarrow 1b = 1c \Rightarrow b = c$.

Thus R is a commutative ring in which

$ab = ac \Rightarrow b = c \ (a, b, c \in R, a \neq 0)$

Hence R is an integral domain.

The converse of the above theorem is not true.

for example, the set of integers Z is an integral domain which is not a field, since $a \neq 0 \in Z$ does not have the multiplicative inverse in Z.

1(c) Test for convergence the series

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \cdots\cdots$$

Sol's Omitting the first term of the series because it will not affect the convergence or divergence of the series, we have

$$u_n = \frac{2^2\cdot 4^2\cdot 6^2\cdots\cdots(2n)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots(2n+1)(2n+2)}x^{2n+2}$$

$$\therefore u_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2\cdots\cdots(2n)^2(2n+2)^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdots\cdots(2n+1)(2n+2)(2n+3)(2n+4)}x^{2n+4}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x^2}\frac{(2n+3)(2n+4)}{(2n+2)^2}$$

$$= \frac{1}{x^2}\cdot\frac{4n^2+14n+12}{4n^2+8n+4}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ by ratio test, $\sum u_n$ is convergent

if $\frac{1}{x^2} > 1$ i.e, $x^2 < 1$

and $\sum u_n$ is divergent if $\frac{1}{x^2} < 1$ i.e, $x^2 > 1$

For $\frac{1}{x^2} = 1$ i.e, $x^2 = 1$, the ratio test fails and we apply Raabe's Test in this case.

when $x^2 = 1$, we have $\frac{u_n}{u_{n+1}} = \frac{4n^2+14n+12}{4n^2+8n+4}$

$$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{4n^2+14n+12}{4n^2+8n+4}-1\right)$$

$$= \frac{6n^2+8n}{4n^2+8n+4}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} \frac{6n^2+8n}{4n^2+8n+4}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ by Raabe's test, Σu_n is convergent for $x^2=1$.

Hence the given series is convergent if $x^2 \leq 1$ and is divergent if $x^2 > 1$.

1(d). Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. Also find v such that $f(z) = u + iv$ is analytic.

Sol^n: (a) $\frac{\partial u}{\partial x} = (e^{-x})\sin y + (-e^{-x})(x\sin y - y\cos y)$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y)$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (2)}$$

Adding (1) & (2) yields. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

(b) from the Cauchys Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (3)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

Integrate (3) w.r.to y,

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \quad \text{—— (5)}$$

where $F(x)$ is arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y - xe^{-x}\cos y - ye^{-x}\sin y$$

$\Rightarrow F'(x) = 0$ and $F(x) = C$, a constant.

Then from (5) $v = e^{-x}(y\sin y + x\cos y) + C$.

1(e) Let $x_1=2$, $x_2=4$, and $x_3=1$ be a feasible solution to the system of equations

$2x_1 - x_2 + 2x_3 = 2$

$x_1 + 4x_2 = 18$

Reduce the given feasible solution to a basic feasible solution.

Sol: The given system of equations can be written as $AX=B$, where

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 4 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$$

Let the columns of A be denoted by

$$A_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Here $\rho(A)=2$

$\therefore$ A basic solution to the given system of equations exist with not more than two variables different from zero.

Also the column vectors A_1, A_2, A_3 are linear dependent.

$\therefore \exists$ scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0$$

$$\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}\lambda_1 + \begin{bmatrix} -1 \\ 4 \end{bmatrix}\lambda_2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\lambda_3 = 0$$

$$\Rightarrow 2\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$

$$\lambda_1 + 4\lambda_2 + 0\lambda_3 = 0$$

This is a system of two equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$.
Let us choose one of the λ's arbitrarily say $\lambda_3 = -1$.
Then $2\lambda_1 - \lambda_2 = 2$
& $\lambda_1 + 4\lambda_2 = 0$
Solving, we get $\lambda_1 = 8/9$, $\lambda_2 = -2/9$
Now to reduce the no. of +ve variables to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \min_i \left\{ \frac{x_i}{\lambda_i} \Big/ \lambda_i > 0 \right\}$$
$$= \min \left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \frac{x_3}{\lambda_3} \right\}$$
$$= \min \left\{ \frac{2}{8/9} \right\} = \frac{9}{4}.$$

Thus, we can remove vector A_1 for which $\frac{x_1}{\lambda_1} = \frac{9}{4}$ and obtain new solution with not more than two non-negative (non-zero) variables.
The values of new variables are given by

$$\hat{x}_1 = x_1 - \frac{x_1}{\lambda_1}\lambda_1 = 2 - \frac{9}{4} \times \frac{8}{9} = 0$$
$$\hat{x}_2 = x_2 - \frac{x_1}{\lambda_1}\lambda_2 = 4 - \frac{9}{4} \times \left(-\frac{2}{9}\right) = \frac{9}{2}$$
$$\hat{x}_3 = x_3 - \frac{x_1}{\lambda_1}\lambda_3 = 1 - \frac{9}{4}(-1) = 13/4.$$

Obviously, columns of A_2 & A_3 of A corresponding to these non-zero variables are L.I.
Hence the basic feasible solution to given system of equations is given by
$x_2 = 9/2$ and $x_3 = 13/4$ with $x_1 = 0$.

Note: If we let $\alpha_3 = 1$, another basic feasible solution is obtained as $x_1 = 26/9$ & $x_2 = 34/9$ with $x_3 = 0$.

2.(a)(i) Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of complex numbers defined by $f_1(z) = z$, $f_2(z) = 1-z$, $f_3(z) = \frac{z}{z-1}$, $f_4(z) = \frac{1}{z}$, $f_5(z) = \frac{1}{1-z}$ and $f_6(z) = \frac{z-1}{z}$ is a non-abelian group of order 6. with respect to composition of mappings.

sol^n: Let $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

suppose we denote ~~the~~ multiplicatively the composition known as the composite or product of two functions.

If $f: A \to B$ and $g: B \to C$ then by definition $(gf): A \to C$ such that $(gf)(x) = g[f(x)]$ $\forall x \in A$.

The function gf is called the composite of the functions g and f.

we prepare the composition table as follows.

Since the function f_1 is the identity function, therefore

$f_1 f_1 = f_1$, $f_1 f_2 = f_2 = f_2 f_1$, $f_1 f_3 = f_3 = f_3 f_1$

$f_4 f_1 = f_4 = f_1 f_4$, $f_1 f_5 = f_5 = f_5 f_1$, $f_1 f_6 = f_6 = f_6 f_1$.

Now, $f_2 f_2(z) = f_2(1-z) = 1-(1-z) = z = f_1(z)$.

$\therefore f_2 f_2 = f_1$.

$(f_2 f_3)(z) = f_2(f_3(z)) = f_2\left(\frac{z}{z-1}\right) = 1 - \frac{z}{z-1} = \frac{-1}{z-1} = \frac{1}{1-z}$
$= f_5(z)$

$\therefore f_2 f_3 = f_5$

$(f_2 f_4)(z) = f_2(f_4(z)) = f_2\left(\frac{1}{z}\right) = 1 - \frac{1}{z} = \frac{z-1}{z} = f_6$

$\therefore f_2 f_4 = f_6$

$(f_2 f_5)(z) = f_2(f_5(z)) = f_2\left(\frac{1}{1-z}\right) = 1 - \frac{1}{1-z} = \frac{-z}{1-z} = \frac{z}{z-1}$
$= f_3(z)$

$\therefore f_2 f_5 = f_3$

$(f_2 f_6)(z) = f_2(f_6(z)) = f_2\left(\frac{z-1}{z}\right) = 1 - \frac{z-1}{z} = f_4$

$\therefore f_2 f_6 = f_4$

Similarly calculating the other products we get the composition table as given below

composition of two functions	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

we make the following observations

(i) All the entries in the composition table are elements of the set G, therefore G is closed w.r.t to the given composition.

(ii) we know that the composite of functions is an associative composition.

i.e. If $f: A \to B$, $g: B \to C$, $h: C \to D$

then $h(gf) = (hg)f$.

(iii) The identity function f_1 is the identity element.

(iv) Each function possesses inverse.

Thus $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$,

$f_5^{-1} = f_6$, $f_6^{-1} = f_5$.

(v) The composition is not commutative

Since $f_2 f_3 = f_5$ and $f_3 f_2 = f_6$.

Thus $f_2 f_3 \neq f_3 f_2$.

The set G contains 6 elements.

Hence G is a finite non-abelian group of order 6 with respect to the composite composition.

Note: Here we see in the composition table that the entries in the second row donot coincide with the corresponding entries in the second column. Thus $f_2 f_3 \neq f_3 f_2$.

Therefore the composition is not commutative.

2(a)(ii) Let $\beta = (1\ 3\ 5\ 7\ 9\ 8)(2\ 4\ 10)$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

Sol'n: $|\beta| = \text{LCM}(7, 3)$

$= 21$

$\therefore \beta^{21} = I$

$\beta^{16}\beta^{5} = I$

$\beta^{16} = \beta^{-5}$

$\Rightarrow \boxed{n = 16}$

2(b) → (i) Prove that $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is absolutely convergent for any real x.

(ii) Prove that the sequence $\{a_n\}$ recursively defined by $a_1=\sqrt{5}$, $a_{n+1}=\sqrt{5+a_n}$, $n\geq 1$ converges to the +ve root of the equation $x^2-x-5=0$.

Sol^n: (i) Here $1+a_n=\left(1+\frac{x}{n}\right)e^{-x/n}$

$$=\left(1+\frac{x}{n}\right)\left[1-\frac{x}{n}+\frac{x^2}{2!n^2}-\frac{x^3}{3!n^3}+\cdots\right]$$

$$\Rightarrow 1+a_n=1-\frac{x^2}{n^2}+\frac{x^2}{2n^2}+\frac{x^3}{2n^3}-\frac{x^3}{6n^3}+\cdots$$

$$\Rightarrow a_n=\frac{-x^2}{2n^2}+\frac{x^3}{2n^3}+\cdots$$

$$=\frac{1}{n^2}\left(-\frac{x^2}{2}+\frac{x^3}{3n}+\cdots\right)$$

$$\Rightarrow |a_n|=\frac{1}{n^2}\left|\frac{-x^2}{2}+\frac{x^3}{3n}+\cdots\right|$$

comparing $\sum|a_n|$ with $\sum\frac{1}{n^2}$, we have

$\lim_{n\to\infty}\frac{|a_n|}{1/n^2}=\frac{x^2}{2}$, a finite quantity but

$\sum\frac{1}{n^2}$ is convergent, therefore $\sum|a_n|$ is convergent.

$\Rightarrow \sum a_n$ is absolutely convergent.

$\Rightarrow \prod_{n=1}^{\infty}(1+a_n)=\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is

absolutely convergent for all values of x.

2(b)(ii) Given that $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5+x_n}$

$a_2 = \sqrt{5+x_1} = \sqrt{5+\sqrt{5}} > \sqrt{5} = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$

Suppose $x_{n+1} > x_n$ for some 'n'

$\Rightarrow 5 + x_{n+1} > 5 + x_n$

$\Rightarrow \sqrt{5+x_{n+1}} > \sqrt{5+x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$

$\therefore$ By mathematical induction $x_{n+1} > x_n \ \forall n$

$\therefore (x_n)$ is monotonically increasing

Now $x_1 = \sqrt{5} < 5$

$x_2 = \sqrt{5+\sqrt{5}} < 5$

Similarly $x_3 < 5$

Suppose $x_n < 5$

$\Rightarrow 5 + x_n < 10$

$\Rightarrow \sqrt{5+x_n} < \sqrt{10} < \sqrt{25} = 5$

$\therefore x_{n+1} < 5$

$\therefore$ By mathematical induction $x_n < 5 \ \forall n$

$\therefore (x_n)$ is bounded above.

Since (x_n) is monotonically increasing and bounded above.

$\therefore$ It is convergent.

$\lim_{n\to\infty} x_n = l$ & $\lim_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{5+x_n} \Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{5 + \lim_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{5+l}$

$\Rightarrow l^2 - l - 5 = 0$

$\Rightarrow l = \frac{1 \pm \sqrt{21}}{2}$ but $l = \frac{1-\sqrt{21}}{2} < 0$

whereas $x_n > 0 \ \forall n$

$\therefore l \neq \frac{1-\sqrt{21}}{2}$, $\therefore x_n$ cgs to $\frac{1+\sqrt{21}}{2}$ which is +ve root of the eqn $x^2 - x - 5 = 0$

2(C) Using Contour integration method evaluate

$$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} \quad \text{where } a>0.$$

Sol^n: Let $I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$

Then $I = \int_0^{\pi} \frac{2a\,d\theta}{2a^2+2\sin^2\theta} = \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$

$= \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\cos t}$, Putting $2\theta = t$

$= \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\frac{1}{2}(e^{it}+e^{-it})}$

putting $z = e^{it}$ so that $dz = ie^{it}$, we get

$I = \int_C \frac{2a}{2(2a^2+1)-(z+z^{-1})} \cdot \frac{dz}{iz}$, where C is unit circle $|z|=1$.

$\Rightarrow I = \frac{2a}{i} \int_C \frac{dz}{2(2a^2+1)z-z^2-1} = 2ai \int_C \frac{dz}{z^2-2(2a^2+1)z+1}$

$\Rightarrow I = 2ai \int_C f(z)\,dz$ ——— (1)

$f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$

poles of $f(z)$ are given by

$z^2-2(2a^2+1)z+1=0$

$z = \frac{2(2a^2+1) \pm \sqrt{[4(2a^2+1)^2-4]}}{2}$

$= 2a^2+1 \pm \sqrt{[(2a^2+1)^2-1]} = 2a^2+1 \pm 2a\sqrt{(a^2+1)}.$

Taking $\alpha = 2a^2+1+2a\sqrt{(a^2+1)}$

$\beta = 2a^2+1-2a\sqrt{(a^2+1)}$

we get $z = \alpha, \beta$. Evidently $|\alpha| > 1$ and $|\beta| < 1$.

$f(z)$ has only one Simple pole $z = \beta$ lying within C.

$$\text{Res}(z=\beta) = \lim_{z \to \beta} (z-\beta) f(z) = \lim_{z \to \beta} \frac{(z-\beta) \cdot 1}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\beta - \alpha} = \frac{1}{-4a\sqrt{[(a^2+1)]}}$$

By Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i \text{ (Sum of residues within C)} = \frac{2\pi i}{-4a(a^2+1)^{1/2}}$$

Using this in ①, we get

$$I = \frac{2ia \cdot 2\pi i}{-4a(a^2+1)^{1/2}}$$

$$= \frac{\pi}{(1+a^2)^{1/2}}$$

3(a) (i) Let R be the ring of all the real-valued continuous functions on the closed unit interval. show that $M = \{f \in R : f(1/3) = 0\}$ maximal ideal of R.

Sol^n: Let $\mathbb{R}$ denote the set of all real numbers.

The given ring is

$$R = \{f \mid f : [0,1] \to \mathbb{R} \text{ is continuous on } [0,1]\}$$

Notice that R is a ring w.r.t the compositions:

$$\left.\begin{array}{l}(f+g)(x) = f(x) + g(x) \\ (fg)(x) = f(x) \cdot g(x)\end{array}\right\} \forall\, x \in [0,1] \text{ and } f, g \in R$$

we proceed to show that M is an ideal of R.

Let $f, g \in M$. Then $f(1/3) = 0 = g(1/3)$.

we have $(f-g)(1/3) = f(1/3) - g(1/3) = 0$ and so $f-g \in M$.

Let $f \in M$ and $h \in R$. Then $f(1/3) = 0$ and

$$(fh)(1/3) = f(1/3)\, h(1/3) = 0 \cdot h(1/3) = 0 \Rightarrow fh \in M.$$

Similarly, $hf \in M$ and so M is an ideal of R. Finally, we show that M is a maximal ideal of R. Let U be any ideal of R such that

$$M \subset U \subset R \text{ and } M \neq U.$$

we need to show that $U = R$

Since $M \subset U$ and $M \neq U$, there exists a function $g \in U$ such that $g \notin M$, i.e, $g(1/3) \neq 0$.

i.e, $\alpha \neq 0$, where $\alpha = g(1/3)$ ——— (1)

(Notice that $g(1/3) = 0 \Rightarrow g \in M$, a contradiction)

we define a function $h : [0,1] \to \mathbb{R}$ as

$h(x) = g(x) - \alpha, \ \forall x \in [0,1]$ —— ②

$\Rightarrow h(1/3) = g(1/3) - \alpha = 0$, using ①

$\Rightarrow h \in M \Rightarrow h \in U$, since $M \subset U$.

Since U is an ideal of R, therefore

$g \in U$ and $h \in U \Rightarrow g - h \in U \Rightarrow \alpha \in U$, by ②

Since $\alpha \neq 0$, $\alpha^{-1} \in R$ exists. The constant function

$\alpha^{-1} : [0,1] \to R$ defined as $\alpha^{-1}(x) = \alpha^{-1} \ \forall x \in [0,1]$

is a continuous function on $[0,1]$ and as such $\alpha^{-1} \in R$.

Since U is an ideal of R, so

$\alpha \in U$ and $\alpha^{-1} \in R \Rightarrow \alpha\alpha^{-1} \in U \Rightarrow 1 \in U$.

$\Rightarrow 1.f \in U \ \forall f \in R \Rightarrow f \in U \ \forall f \in R \Rightarrow R = U$

Hence M is a maximal ideal of R.

(ii) Let $U = \langle 12 \rangle = 12Z = \{ \ldots, -36, -24, -12, 0, 12, 24, \ldots \}$

The only ideals of Z which contain U are Z, and $12Z$

and $2Z = \{ \ldots, -4, -2, 0, 2, 4 \ldots \}$

$3Z = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$

$4Z = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

$6Z = \{ \ldots, -12, -6, 0, 6, 12 \ldots \}$

Hence all the ideals of Z/U are $Z/U = \{ \bar{0}, \bar{1}, \ldots \bar{11} \}$

$U/U = \bar{0} = U$, $2Z/U = \{ \bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10} \}$, $3Z/U = \{ \bar{0}, \bar{3}, \bar{6}, \bar{9} \}$

$4Z/U = \{ \bar{0}, \bar{4}, \bar{8} \}$, $6Z/U = \{ \bar{0}, \bar{6} \}$. (Here $\bar{n} = n + U$)

Here $M = \{ \bar{0}, \bar{3}, \bar{6}, \bar{9} \}$ is a maximal ideal of $Z/(12) = \{ \bar{0}, \bar{1}, \bar{2}, \ldots \bar{11} \}$

Since there does not exist any proper ideal between

$M = \{ \bar{0}, \bar{3}, \bar{6}, \bar{9} \}$ and $Z/(12) = \{ \bar{0}, \bar{1}, \ldots \bar{11} \}$.

$\Rightarrow$ [Notice that $\{ \bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10} \}$ is also a maximal ideal of $Z/(12)$

However $\{ \bar{0} \}$, $\{ \bar{0}, \bar{4}, \bar{8} \}$, $\{ \bar{0}, \bar{6} \}$ are not maximal

ideals of $Z/(12)$, since for example,

$\{ \bar{0}, \bar{6} \} \subset \{ \bar{0}, \bar{3}, \bar{6}, \bar{9} \} \subset Z/(12)$.

3(b). Let the function f be defined on $[0,1]$ as follows:

$f(x) = 2rx$ when $\frac{1}{r+1} < x \le \frac{1}{r}$, $r = 1,2,3\ldots$

Prove that f is R-integrable in $[0,1]$ and evaluate $\int_0^1 f(x)\,dx$.

Sol'n: Given that the function f defined on $[0,1]$ as $f(x) = 2rx$ if $\frac{1}{r+1} < x < \frac{1}{r}$, $r \in N$

i.e. $f(x) = 2x$, when $\frac{1}{2} < x < 1$

$= 4x$, when $\frac{1}{3} < x < \frac{1}{2}$

$= 6x$, when $\frac{1}{4} < x < \frac{1}{3}$

$\vdots$

$= 2(n-1)x$ when $\frac{1}{n} < x < \frac{1}{n-1}$

$\vdots$

Note that f is bounded and continuous on $[0,1]$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on $[0,1]$ has a finite number of limit points, therefore, f is integrable on $[0,1]$.

Now

$$\int_{1/n}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \cdots + \int_{1/n}^{1/n-1} f(x)\,dx$$

$$= \sum_{r=1}^{n-1} \int_{1/r+1}^{1/r} 2rx\,dx$$

$$= \sum_{r=1}^{n-1} \left[rx^2 \right]_{1/r+1}^{1/r}$$

$$= \sum_{r=1}^{n-1} r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n-1} \frac{2r+1}{r(r+1)^2}$$

$$= \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n-1}\left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1} \frac{1}{(r+1)^2}$$

$$= \left[\left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] + \left[\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \cdots + \frac{1}{n^2}\right]$$

$$= \left(1-\tfrac{1}{n}\right) + \left(\frac{\pi^2}{6} - 1\right)$$

($\because$ the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ convergent and its sum is $\pi^2/6$)

$$= \frac{\pi^2}{6} - \frac{1}{n}$$

proceeding to the limit as $n \to \infty$

$$\int_0^1 f(x)\,dx = \frac{\pi^2}{6}.$$

3(C) Solve the following LPP

$\text{Max } z = 2x_1 + x_2$

subject to $4x_1 + 3x_2 \le 12$, $4x_1 + x_2 \le 8$, $4x_1 - x_2 \le 8$ and $x_1, x_2 \ge 0$.

Sol^n: Introducing the slack variables $s_1, s_2, s_3 \ge 0$, the problem becomes

$\text{Max } z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints

$$4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 8$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

Now the initial basic feasible solution is given by setting $x_1 = x_2 = 0$ (non-basic)

$s_1 = 12$, $s_2 = 8$, $s_3 = 8$ (basic)

$\therefore$ The IBFS is $(0, 0, 12, 8, 8)$ for which $z = 0$.

put the above information in tableau form

	c_j	2	1	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	4	3	1	0	0	12	$12/4 = 3$
0	s_2	4	1	0	1	0	8	$8/4 = 2$
0	s_3	$\boxed{4}$	-1	0	0	1	8	$8/4 = 2$ →
$z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0	
$C_j = c_j - z_j$		2 ↑	1	0	0	0		

from the table,

x_1 is the incoming variable. But the two rows have the same ratio under θ-column.

This is an indication of degeneracy.

Using the procedure of degeneracy, compute

min [elements of first column of unit matrix / corresponding element of key column]

only for second and 3rd rows

min $\left[-, \frac{0}{4}, \frac{0}{4}\right]$ which is not unique.

So again compute

min $\left[\frac{\text{elements of } 2^{nd} \text{ column of unit matrix}}{\text{corresponding element of key column}}\right]$.

only for 2nd and 3rd rows.

$\therefore$ Min $\left[-, \frac{1}{4}, \frac{0}{4}\right] = 0$ which occurs corresponding to the 3rd row. and therefore S_2 is outgoing variable the key element is 4.

Thus the new simplex table:

	c_j	2	1	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	4	1	0	-1	4	1
0	s_2	0	$\boxed{2}$	0	1	-1	0	$0/2 = 0 \rightarrow$
2	x_1	1	$-1/4$	0	0	$1/4$	2	$-$
$z_j' = \sum C_B a_{ij}$		2	$-1/2$	0	0	$1/2$	4	
$C_j' = C_j - z_j$		0	$3/2$ $\uparrow$	0	0	$-1/2$		

From the above table,

x_2 is the incoming variable, S_2 is the outgoing variable and (2) is the key element and making it into unity and all other elements in its column to zero.

	c_j	2	1	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	0	1	-2	(1)	4	4 →
1	x_2	0	1	0	1/2	-1/2	0	-
2	x_1	1	0	0	1/8	1/8	2	16
$z_j = \sum c_B a_{ij}$		2	1	0	3/4	-1/4	4	
$c_j' = c_j - z_j$		0	0	0	-3/4	1/4 ↑		

from the above table,
s_3 is incoming variable, s_1 is the outgoing variable. and 1 is the key element and making all other elements in its column to zero.

	c_j	2	1	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b
0	s_3	0	0	1	-2	1	4
1	x_2	0	1	1/2	-1/2	0	2
2	x_1	1	0	-1/8	3/8	0	3/2
$z_j = \sum c_B a_{ij}$		2	1	1/4	1/4	0	5
$c_j' = c_j - z_j$		0	0	-1/4	-1/4	0	

from the above table,
all $c_j' \leq 0$.
So, the table gives the optimal solution. Hence an optimal basic feasible solution is
$x_1 = 3/2$, $x_2 = 2$ and Max $Z = 5$

4(a) Show that $Z[\sqrt{3}] = \{m + n\sqrt{3} \,/\, m, n \in Z\}$ is a Euclidean domain.

Sol: It is easy to verify that $Z[\sqrt{3}]$ is an integral domain with unity $1 = 1 + \sqrt{3}(0)$.

Let $a = m + n\sqrt{3} \neq 0$, $b = m_1 + n_1\sqrt{3} \neq 0 \in Z[\sqrt{3}]$.

we define

$$d(a) = d(m + n\sqrt{3}) = |m^2 - 3n^2| \quad \text{—} (1)$$

obviously, $d(a)$ is a positive integer, for each $a \neq 0 \in Z[\sqrt{3}]$.

Indeed $d(a) \geq 1$, $\forall a \neq 0 \in Z[\sqrt{3}]$. — (2)

we have

$$ab = (mm_1 + 3nn_1) + (mn_1 + m_1 n)\sqrt{3}.$$

and $d(ab) = |(mm_1 + 3nn_1)^2 - 2(mn_1 + m_1 n)^2|$ [$\because$ by (1)]

$$= |m^2 m_1^2 + 9n^2 n_1^2 - 3(m^2 n_1^2 + n^2 m_1^2)|$$

$$= |(m^2 - 3n^2)(m_1^2 - 3n_1^2)|$$

$$= |m^2 - 3n^2|\,|m_1^2 - 3n_1^2|$$

$$\geq |m^2 - 3n^2| \quad (\because |m_1^2 - 3n_1^2| \geq 1)$$

$$= d(a)$$

$\therefore d(ab) \geq d(a)$.

Now, we have

$$\frac{a}{b} = \frac{m+n\sqrt{3}}{m_1+n_1\sqrt{3}} = \frac{(m+n\sqrt{3})(m_1-n_1\sqrt{3})}{(m_1+n_1\sqrt{3})(m_1-n_1\sqrt{3})}$$

$$= \left(\frac{mm_1-3nn_1}{m_1^2-3n_1^2}\right) + \left(\frac{m_1n-mn_1}{m_1^2-3n_1^2}\right)\sqrt{3}$$

$$= p+q\sqrt{3}.$$

where $p = \frac{mm_1-3nn_1}{m_1^2-3n_1^2}$ and $q = \frac{m_1n-mn_1}{m_1^2-3n_1^2}$ are rational numbers

corresponding to the rational numbers p & q, we can find two integers p' and q' such that $|p'-p| \le \frac{1}{2}$ and $|q'-q| \le \frac{1}{2}$.

Let $t = p'+q'\sqrt{3}$.

Then $t \in Z[\sqrt{3}]$

we have $\frac{a}{b} = \lambda$, where $\lambda = p+q\sqrt{3}$

$$\Rightarrow a = \lambda b = (\lambda - t)b + tb$$

$$= tb + r, \text{ where } r = (\lambda - t)b$$

Now $a, b, t \in Z[\sqrt{3}]$

$$\Rightarrow a - tb \in Z[\sqrt{3}]$$

$$\Rightarrow r \in Z[\sqrt{3}].$$

$\therefore \exists\, t, r \in Z[\sqrt{3}]$ such that $a = tb + r$, where

$r = 0$ or $d(r) = d\{(\lambda - t)b\}$

$$= d\{(p+q\sqrt{3}) - (p'+q'\sqrt{3})\}\, d(b)$$

$$= d\{(p-p') + (q-q')\sqrt{3}\}\, d(b)$$

$$= |(p-p')^2 - 3(q-q')^2|\, d(b).$$

$$\le \left|\frac{1}{4} - \frac{3}{4}\right| d(b)$$

$$= \frac{1}{2} d(b) < d(b).$$

$\therefore Z[\sqrt{3}]$ is Euclidean domain.

4(c) Show that the function f defined by $f(z) = |\text{Re}\, z\, \text{Im}\, z|^{1/2}$ satisfies the C-R equations at the origin. Is it differentiable at this point? Justify your answer.

Sol'n: Given function can be written as

$$f(z) = \frac{|z^2 - \bar{z}^2|^{1/2}}{2} \text{ (or) } f = u + iv \text{ with}$$

$$u(x,y) = |xy|^{1/2} \text{ and } v(x,y) = 0.$$

Note that f is identically zero on the real and imaginary axes.

$\therefore$ It is trivial to see that

$$u_x(0,0) = u_y(0,0) = v_x(0,0) = v_y(0,0) = 0.$$

For example,

$$u_x(0,0) = \lim_{\delta \to 0} \frac{u(\delta,0) - u(0,0)}{\delta} = 0.$$

Thus, the C-R equations hold at $z=0$.

However, taking $h = re^{i\theta} \neq 0$ with $r \to 0$, we find that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{r\to 0} \frac{|r^2 \cos\theta \sin\theta|^{1/2}}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{e^{-i\theta}|\sin 2\theta|^{1/2}}{\sqrt{2}}$$

which is clearly depending upon θ (e.g. take $\theta = 0$ and $\theta = \pi/4$). We conclude that f is not differentiable at $z=0$ even though f satisfies the C-R equations at the origin.

4(d) Consider the problem of assingning the operators to the machines. The assignment cost in rupees are given in the table operator O_2 cannot be assigned to machine M_2 and operator O_5 cannot be assigned to machine M_4. find the optimal cost of assignment.

	M_1	M_2	M_3	M_4	M_5
O_1	8	4	2	6	1
O_2	0	–	5	5	4
O_3	3	8	9	2	6
O_4	4	3	1	0	3
O_5	9	5	8	–	5

Sol:

The problem indicates that two assignments are impossible. Therefore, we put a very large cost 10 in the cells (2,2) and (5,4), then proceed as usual. performing first row reduction and then column reduction, we get Table-I.

8	4	2	6	1
0	10	5	5	4
3	8	9	2	6
4	3	1	0	3
9	5	8	10	5

$\longrightarrow$

7	3	1	5	0
0	10	5	5	4
1	6	7	0	4
4	3	1	0	3
4	0	3	5	0

Table I

7	3	0	5	0
0	10	4	5	4
1	6	6	0	4
4	3	0	0	3
4	0	2	5	0

Draw the minimum no. of horizontal and vertical lines to cover all zeros of table - I. which shows that the no. of lines $(n) = 4 = m =$ order of matrix.

7	3	~~0~~	5	0
0	10	4	5	4
1	6	6	0	4
4	3	0	~~0~~	3
4	0	2	5	~~0~~

$\therefore$ an optimum assignment has been reached.

The assignment is given by

$O_1 \to M_5$

$O_2 \to M_1$

$O_3 \to M_4$

$O_4 \to M_3$

$O_5 \to M_2$

$\therefore$ Minimum cost $= 1+0+2+1+5$

$= 9$

5(a) Let H be a subgroup of a group G. If $x^2 \in H \ \forall\ x \in G$, then Prove that H is a normal subgroup of G and G/H is commutative.

Sol^n: Let $g \in G$ and $h \in H$. consider ghg^{-1} and note that

$$ghg^{-1} = ghghh^{-1}g^{-2} = (gh)^2 h^{-1} g^{-2}.$$

Now $h^{-1} \in H$ and by our hypothesis $(gh)^2; g^{-2} \in H$.

this implies that $ghg^{-1} \in H$

which in turn shows that $gHg^{-1} \subseteq H$.

Hence H is a normal subgroup of G.

To show G/H is commutative,

let $xH, yH \in G/H$.

we show that $xHyH = yHxH$

$$\Rightarrow xyH = yxH$$

$$\Rightarrow (yx)^{-1}(xy) \in H.$$

Now, $(yx)^{-1}(xy) = (x^{-1}y^{-1})(xy)$

$$= (x^{-1}y^{-1})^2 (yxy^{-1})^2 y^2.$$

Since $a^2 \in H \ \forall\ a \in G$,

it follows $(x^{-1}y^{-1})^2 (yxy^{-1})^2 y^2 \in H$

and so $(yx)^{-1}(xy) \in H$.

Hence G/H is commutative.

5.(b) Prove that b/w any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

Sol Let $x = a$ & $x = b$ be the roots of $e^x \sin x + 1 = 0$ then

$e^a \sin a + 1 = 0$ & $e^b \sin b + 1 = 0$ ——— (1)

Let $f(x) = e^x \sin x + 1 \quad \forall x \in [a, b]$

Since e^x & $\sin x$ are continuous and differentiable for all reals.

$\therefore f(x) = e^x \sin x + 1$ is continuous and differentiable in $[a, b]$

and $f(a) = f(b) = 0$ (by (1))

$\therefore$ f has been satisfied the conditions of Rolle's theorem.

$\therefore \exists$ at least one $x \in (a, b)$ s.t $f'(x) = 0$ ——— (2)

$\because f'(x) = e^x \sin x + e^x \cos x$.

$\Rightarrow f'(x) = e^x (\sin x + \cos x)$

$\therefore$ (1) $\equiv e^x (\sin x + \cos x) = 0 \quad \forall x \in (a, b)$

$\Rightarrow \sin x + \cos x = 0 \quad (\because e^x \neq 0 \; \forall x)$

$\Rightarrow \tan x + 1 = 0 \; ; \; x \in (a, b)$.

(5)(c) A function $f: R \to R$ is continuous on R and $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all $x, y \in R$, prove that $f(x) = ax + b$ ($a, b \in R$) for all $x \in R$

Sol^n. Given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ —— ①

Putting $2x$ in place of x and $2y$ in place of $y \to$

$$f\left(\frac{2x+2y}{2}\right) = \frac{f(2x)+f(2y)}{2}$$

$$\Rightarrow f(x+y) = \frac{f(2x)+f(2y)}{2} \quad \text{—— ②}$$

Now taking $y=0$

$$f(x+0) = \frac{f(2x)+f(0)}{2}$$

$$\Rightarrow f(x) = \frac{f(2x)+f(0)}{2}$$

$$\Rightarrow f(2x) = 2f(x) - f(0) \quad \text{—— ③}$$

Since f is continuous so -

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$\Rightarrow f'(x) = \lim_{h\to 0} \dfrac{\frac{f(2x)+f(2h)}{2} - f(x)}{h}$ [using (2)]

$\Rightarrow f'(x) = \lim_{h\to 0} \dfrac{f(2x)+f(2h)-2f(x)}{2h}$

$\Rightarrow f'(x) = \lim_{h\to 0} \dfrac{2f(x)-f(0)+f(2h)-2f(x)}{2h}$ [using (3)]

$\Rightarrow f'(x) = \lim_{h\to 0} \dfrac{f(2h)-f(0)}{2h}$

$\Rightarrow f'(x) = \lim_{2h\to 0} \dfrac{f(0+2h)-f(0)}{2h}$

$\Rightarrow f'(x) = f'(0)$ [using definition]

Integrating w.r.t. $x \rightarrow$

$\Rightarrow f(x) = f'(0)x + b$

where b is integrating constant

Let $f'(0) = a \in R$

then $\boxed{f(x) = ax + b}$

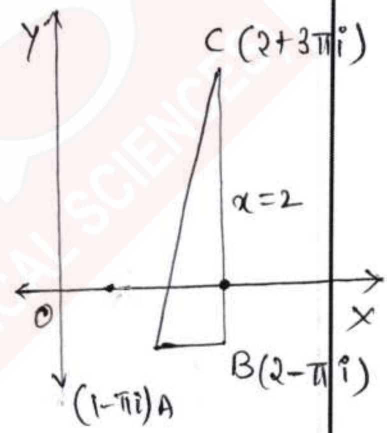
Hence proved.

5(d) Show that $\int_C e^{-2z}\,dz$ is independent of the path C joining the points $1-\pi i$ to $2+3\pi i$ and determine its value.

Sol^n: Let $I = \int_C f(z)\,dz$, where $f(z) = e^{-2z}$ and C is a straight line joining points $1-\pi i$ to $2+3\pi i$.

Evidently $f(z)$ is differentiable every where in z-plane. Hence $f(z)$ is analytic in entire z-plane.

$\therefore$ By corollary of Cauchy's theorem $\int_C f(z)\,dz$ is independent of path of integration.



Also $\int_C f(z)dz = \int_{1-\pi i}^{2+3\pi i} e^{-2z}\,dz$

$$= -\frac{1}{2}\left(e^{-2z}\right)_{1-\pi i}^{2+3\pi i}$$

$$= -\frac{1}{2}\left[e^{-2(2+3\pi i)} - e^{-2(1-\pi i)}\right]$$

$$= -\frac{1}{2}\left[e^{-4} - e^{-2}\right]$$

as $e^{-6\pi i} = 1 = e^{2\pi i}$

$$\therefore \int_C f(z)\,dz = \frac{1}{2}\left(e^{-2} - e^{-4}\right) \quad \text{—— (1)}.$$

Let $I = \int_C f(z)\,dz = I_1(\vec{AB}) + I_2(\vec{BC})$

For I_1: equation of AB is $y = -\pi$, $z = x+iy = x - i\pi$, $dx = dz$

$$I_1 = \int_1^2 e^{-2(x-i\pi)}\,dx = e^{2i\pi}\int_1^2 e^{-2x}\,dx = 1\cdot\int_1^2 e^{-2x}\,dx$$

$$= \left(\frac{e^{-2x}}{-2}\right)_1^2 = -\frac{1}{2}\left(e^{-4} - e^{-2}\right)$$

For I_2: equation of BC is $x = 2$, $z = 2+iy$, $dz = i\,dy$

$$I_2 = \int_{-\pi}^{3\pi} e^{-2(2+iy)}\,i\,dy = \frac{ie^{-4}}{-2}\left(e^{-i2y}\right)_{-\pi}^{3\pi}$$

$$= -\frac{i}{2e^4}\left(e^{-6i\pi} - e^{-2i\pi}\right) = 0 \text{ as } e^{-6i\pi} = 1 = e^{-2\pi}$$

$$I = \int_C f(z)\,dz = I_1(\overrightarrow{AB}) + I_2(\overrightarrow{BC})$$

$$= \frac{1}{2}\left(e^{-2} - e^{-4}\right) + 0 \qquad \text{———— (2)}$$

This integral is the same along two different paths:

(i) line $\overrightarrow{AC}$

(ii) line $\overrightarrow{AB}$ + line $\overrightarrow{BC}$

This $\Rightarrow$ Integral is independent of path.

5.(e) → A firm manufactures two products A & B on which the profit earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M_1 & M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 while processing of product M_2 requires 1 minute on M_1 and 1 minute on M_2. M_1 is not available for more than 7 hours 30 minutes while M_2 is available for 10 hours during any working day. Find the number of units of products A and B need to be manufactured to get maximum profit. Formulate this as an LP problem model and solve by graphical method.

Solution:– A firm manufactures two products 'A' and 'B' on which the profit earned per unit are ₹ 3 and ₹ 4 respectively.

i.e $\boxed{\text{Max. } Z = 3A + 4B}$

Working time of Machine one (M_1) is not more than 7 hours 30 minutes i.e 450 minutes

Product A takes 1 minute and so 1 minute for B.

i.e. $M_1 \Rightarrow A + B \leq 450$ —(1)

Working time of Machine two (M_2) is 10 hours i.e 600 minutes.

A takes 2 minutes and B takes 1 minute on M_2.

$\therefore M_2 \Rightarrow 2A + B = 600$ —(2)

and No. of manufactured product should

be more than zero;

i.e $A, B \geq 0$ — (3)

Hence, The LPP is

$$\text{Max. } z = 3A + 4B$$

$$\text{s.c.} \Rightarrow \quad A + B \leq 450$$
$$2A + B = 600$$
$$A, B \geq 0$$

For graphical Method

$A+B=450$ is the line and below it the region it will cover i.e optimal region.

$2A+B=600$ — is a line.

Hence from these two equation.

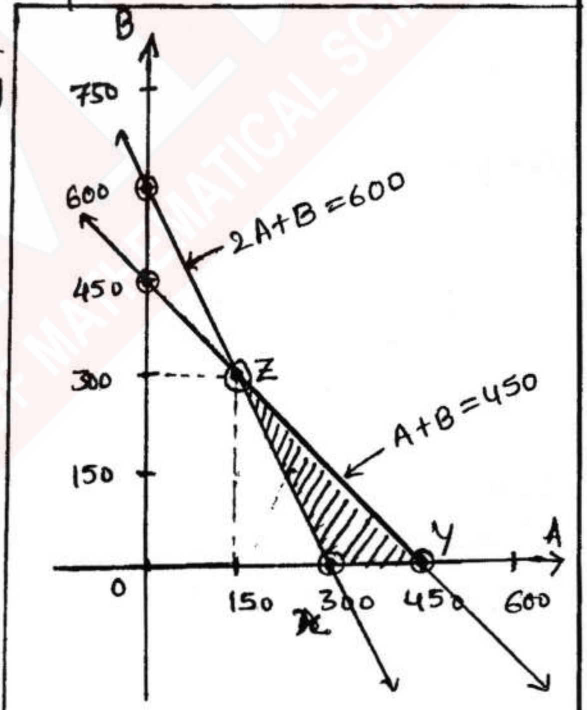
The graphical representation, shows the shaded optimal region.

This consist of three optimal points

$x \to (300, 0)$

$y \to (450, 0)$

$z \to (150, 300)$



$z_{max \text{ at } x} = 3 \times 300 = 900$

$z_{max \text{ at } y} = 3 \times 450 = 1350$

$z_{max \text{ at } z} = 3 \times 150 + 4 \times 300$
$= 450 + 1200 = 1650$

$\therefore$ z_{max} at $(150, 300)$ gives us maximum value.

$\boxed{\therefore \; z_{max} (\text{Profit}) = 1650 \; ; \text{ with } A = 150, B = 300}$

6(a) In a Group G if $a^5 = e$ and $aba^{-1} = b^m$ for some positive Integer m, and some $a, b \in G$, then prove that $b^{m^5-1} = e$.

Sol^n Given - $aba^{-1} = b^m$ —— ①

$(aba^{-1})(aba^{-1}) = b^m \cdot b^m$

$\Rightarrow aba^{-1}aba^{-1} = b^{m+m}$ [In G, if $b \in G$ then $b^{m+n} = b^m \cdot b^n$]

$\Rightarrow abeba^{-1} = b^{2m}$ [$\because a^{-1}a = e = aa^{-1}$ $\forall a \in G$]

$\Rightarrow ab^2a^{-1} = b^{2m}$

Similarly -

$ab^ma^{-1} = b^{m^2}$

$\Rightarrow a(aba^{-1})a^{-1} = b^{m^2}$ [using equation ①]

$\Rightarrow a^2ba^{-2} = b^{m^2}$

Again

$(a^2ba^{-2})(a^2ba^{-2}) = b^{m^2} \cdot b^{m^2}$

$\Rightarrow a^2ba^{-2}a^2ba^{-2} = b^{2m^2}$

$\Rightarrow a^2b^2a^{-2} = b^{2m^2}$

Similarly

$a^2b^ma^{-2} = b^{m^3}$

$\Rightarrow a^2(aba^{-1})a^{-2} = b^{m^3}$ [using eqn ①]

$\Rightarrow a^3ba^{-3} = b^{m^3}$

Therefore; repeating above steps-
we get $a^5ba^{-5} = b^{m^5}$

But given $a^5 = e \Rightarrow a^{-5} = e$

Using.-

$ebe = b^{m^5}$

$\Rightarrow b = b^{m^5}$

$\Rightarrow \boxed{b^{m^5-1} = e}$

Hence proved

6(b) (i) Suppose a group element a and b such that $|a|=4$, $|b|=2$ and $a^3b=ba$. Find $|ab|$

(ii) Suppose a and b are group elements such that $|a|=2$, $b\neq e$ and $aba=b^2$. Determine $|b|$.

(iii) Find three elements σ in S_9 with the property that $\sigma^3=(1\ 5\ 7)(2\ 8\ 3)(4\ 6\ 9)$

Sol^n: (i) Let $(G,*)$ be a group

$a,b\in G$ such that $|a|=4$, $|b|=2$

then $a^4=e$ & $b^2=e$ —— (1)

given that $a^3b=ba$ —— (2)

Now we have

$$(ab)^2=(ab)(ab)$$
$$=a(ba)b \quad (\text{by associative of } G)$$
$$=a(a^3b)b \quad (\text{by } (2))$$
$$=a^4\cdot b^2 \quad (\text{by associate prop. of } G)$$
$$=e\cdot e \quad (\text{by } (1))$$

$\therefore (ab)^2=e$

$\therefore o(ab)=2$ i.e $\boxed{|ab|=2}$

(ii) Given $a,b\in G$ such that $o(a)=2$ and $aba=b^2$.

Now $aba=b^2$

$$\Rightarrow b^4=b^2b^2$$
$$=(aba)(aba)$$
$$=aba^2ba \quad (\because a^2=e)$$
$$=ab^2a$$

$\Rightarrow b^4=b$

$\Rightarrow b^3=e$

$\therefore o(b)=3$ & $b\neq e$.

(iii) Observe that if we start with a 9-cycle

σ (say) $= (a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9)$

and cube it

we get $\sigma^3 = (a_1\ a_4\ a_7)\ (a_2\ a_5\ a_8)\ (a_3\ a_6\ a_9)$

$\therefore$ we can take $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$

But $(1\ 5\ 7)\ (2\ 8\ 3)\ (4\ 6\ 9) = (1\ 5\ 7)\ (4\ 6\ 9)\ (2\ 8\ 3)$

$= (2\ 8\ 3)\ (1\ 5\ 7)\ (4\ 6\ 9)$

So these given by

$(1\ 2\ 4\ 5\ 8\ 6\ 7\ 3\ 9)$, $(1\ 4\ 2\ 5\ 6\ 8\ 7\ 9\ 3)$

and $(2\ 1\ 4\ 8\ 5\ 6\ 3\ 7\ 9)$

6(c) (i) Prove or disprove that subring of a non-commutative ring is commutative.

(ii) If R is a ring with unity 1 and f is a homomorphism of R into an integral domain R' with $\ker f \neq R$, Prove that $f(1)$ is the unity of R'.

Sol'n: (i) A subring of a non-commutative ring may be commutative.

The ring M_2 of 2×2 matrices over integers is non-commutative, since

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

and so $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \neq \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

The set $S = \left\{\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix} : a \in I\right\}$ is a subring of M_2, which is commutative,

Since $\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}ab & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$.

(ii) Let $a' \in R'$ be arbitrary. We shall prove that

$$f(1)a' = a'f(1) = a'.$$

Obviously $f(1)a' - f(1)a' = 0'$.

$\Rightarrow f(1.1)a' - f(1)a' = 0'$, since 1 is the unity of R.

$\Rightarrow f(1)f(1)a' - f(1)a' = 0'$, since f is a homomorphism.

$\Rightarrow f(1)[f(1)a' - a'] = 0'$

$\Rightarrow f(1) = 0'$ (or) $f(1)a' - a' = 0'$,

Since R is an integral domain

If $f(1) = 0'$, then $1 \in \ker f$, where $\ker f$ is an ideal of R

Consequently, $r = 1 . r \in \ker f$ $\forall r \in R$ and so $\ker f = R$

This is contrary to the given hypothesis and so

$$f(1)a' - a' = 0' \Rightarrow f(1)a' = a' \; \forall a' \in R'$$

Similarly, by considering $a'f(1) - a'f(1) = 0'$

we can prove

$$a'f(1) = a' \; \forall a' \in R'.$$

Hence $f(1)$ is the unity of R'.

6(d) → Show that 3 is an irreducible element of $Z[\sqrt{-5}]$.

Sol^n: Let $3 = (a+b\sqrt{5}i)(c+d\sqrt{5}i)$; $a,b,c,d \in Z\sqrt{-5}$

Taking conjugates on bothsides, we get

$$3 = [a-b\sqrt{5}i][c-d\sqrt{5}i]$$

On multiplying the respective sides of the above equations, we get

$$9 = (a^2+5b^2)(c^2+5d^2)$$

Both the sides of the above equation are positive integers.

Consequently, we have the following cases.

Case(1): $a^2+5b^2=1$ and $c^2+5d^2=9$

Case (2): $a^2+5b^2=9$ and $c^2+5d^2=1$

Case (3): $a^2+5b^2=3$ and $c^2+5d^2=3$

It is clear that case (3) is not possible in Z.

Case (1) is possible when $a=\pm1$, $b=0$

$$\Rightarrow a+b\sqrt{5}i = \pm 1$$

which are units in $Z[\sqrt{-5}]$.

Similarly, case (2) yields that $c+d\sqrt{5}i = \pm 1$, which are units in $Z[\sqrt{-5}]$

Hence 3 is an irreducible element of $Z[\sqrt{-5}]$.

7.(b) → Show that the sequence of functions f_n defined on $[0,1]$ by $f_n(x) = n(1-nx)$, $0 \le x < \frac{1}{n}$
$= 0$, $\frac{1}{n} \le x \le 1$
converges to the function 'f' given by $f(x)=0$, $x \in [0,1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$.
Is the convergence of the sequence uniform?

Solution:-

Given; $f_n(x) = \begin{cases} n(1-nx) & ; 0 \le x < \frac{1}{n} \\ 0 & ; \frac{1}{n} \le x \le 1. \end{cases}$

At $x=0$, the sequence is $\{1,1,1,\dots\}$
This converges to 'n'.

at $x=1$, the sequence ... is $\{0,0,0,0\}$. This converges to 0.

Let $c \in (0,1)$. By Archimedean property of $\mathbb{R}$ there exists a natural 'm' such that $0 < \frac{1}{m} < c$ and therefore $0 < \frac{1}{n} < c$ for all $n \ge m$.

$\therefore f_m(c) = 0$ and $f_n(c) = 0$ for all $n \ge m$.

This proves $\lim_{n\to\infty} f_n(c) = 0$.

There the sequence $\{f_n\}$ converges to the function f on $[0,1]$ given by.

$$f(x) = \begin{cases} 1 & , x=0 \\ 0 & , 0 < x \le 1 \end{cases}$$

Each f_n is continuous on $[0,1]$

But the limit function 'f' is not continuous on $[0,1]$.

Here; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n(1-nx)$ ($\infty \times 0$ form),

$= \lim_{n\to\infty} \frac{1-nx}{1/n}$ (Apply D.L rule.

$= \lim_{n\to\infty} x n^2 = 0$

Since; $n\to\infty$, $\frac{1}{n}\to 0$, as $x\to 0$

Hence; $\lim_{n\to\infty} f_n(x) = 0$ for $x\in[0,1]$

Now; $\int_0^1 f_n(x)\,dx = \int_0^1 n(1-nx)\,dx.$

Put $1-nx = t$ at $x=0$ $t=1$
$-n\,dx = dt$ $x=1$ $t=1-n.$

$\Rightarrow \int_1^{1-n} -t\,dt = \int_{1-n}^{1} t\,dt = \frac{1}{2}\left[t^2\right]_{1-n}^{1}.$

$= \frac{1}{2}\left[1-[1-n]^2\right] = \frac{1}{2}\left[2n-n^2\right]$

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \infty$ —— (A)

but $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ —— (B)

Clearly; from (A) and (B).

$$\boxed{\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx}$$

Therefore, the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$, since uniform convergence of the sequence $\{f_n\}$ of continuous functions on $[0,1]$ implies continuity of 'f' on $[0,1]$.

7(C) Prove that $\frac{x}{1+x} < \log(1+x) < x,\ x>0$. Deduce that $\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$, n being a positive integer.

Sol^n: Let $f(t) = \log(1+t)\ \forall\ t \in [0,x]$ where $x>0$.

and $f'(t) = \frac{1}{1+t}\ \forall\ t \in (0,x)$

By Lagrange's Mean value theorem $\exists\ c \in (0,x)$

such that $f'(c) = \frac{f(x)-f(0)}{x-0}$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{—— (1)}$$

Since $c \in (0,x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x} \Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$

i.e. $\frac{x}{1+x} < \log(1+x) < x$

Now, we have $\log(1+x) < x$

Let $x = \frac{1}{n+1}$

then $\log\left(1+\frac{1}{n+1}\right) < \frac{1}{n+1}$ i.e. $\log\left(\frac{n+2}{n+1}\right) < \frac{1}{n+1}$

$\log\left(1+\frac{1}{n+2}\right) < \frac{1}{n+2}$ i.e. $\log\left(\frac{n+3}{n+2}\right) < \frac{1}{n+2}$

similarly $\log\left(\frac{n+4}{n+3}\right) < \frac{1}{n+3}$

$$\log\left(\frac{n+5}{n+4}\right) < \frac{1}{n+4}$$

$$\vdots$$

$$\log\left(1+\frac{1}{n+n}\right) < \frac{1}{n+n} \text{ i.e. } \log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+n}$$

$\therefore$ Adding all the above, we get

$$\log\left(\frac{n+2}{n+1}\right) + \log\left(\frac{n+3}{n+2}\right) + \log\left(\frac{n+4}{n+3}\right) + \cdots + \log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$\Rightarrow \log\left(\frac{n+2}{n+1} \cdot \frac{n+3}{n+2} \cdot \frac{n+4}{n+3} \cdots \frac{2n}{2n-1} \cdot \frac{2n+1}{2n}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\Rightarrow \log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \quad \text{—— (2)}$$

Also we have $\frac{x}{1+x} < \log(1+x)$

let $x = 1/n$, then $\frac{1/n}{1+1/n} < \log(1+1/n) \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right)$

$x = \frac{1}{n+1}$ then $\frac{1/n+1}{1+\frac{1}{n+1}} < \log\left(1+\frac{1}{n+1}\right) \Rightarrow \frac{1}{n+2} < \log\left(\frac{n+2}{n+1}\right)$

$$\vdots$$

$x = \frac{1}{2n-1}$ then $\frac{1/2n-1}{1+\frac{1}{2n-1}} < \log\left(1+\frac{1}{2n-1}\right) \Rightarrow \frac{1}{2n} < \log\frac{2n}{2n-1}$

Adding we get

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} < \log\left(\frac{n+1}{n}\right) + \log\left(\frac{n+2}{n+1}\right) + \cdots \log\frac{2n}{2n-1}$$

$$= \log \frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdot \frac{n+3}{n+2} \cdots \frac{2n}{2n-1}$$

$$= \log\left(\frac{2n}{n}\right) = \log 2$$

$$\therefore \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2 \quad \text{—— (3)}$$

$\therefore$ from (2) & (3) $\log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2.$

7(d) → Prove that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = B(m,n)$

where m,n are both positive.

Sol'n: $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{—(1)}$$

In the second integral of R.H.S of (1), put $x = \frac{1}{t}$,

so that $$dx = \frac{-1}{t^2}\,dt$$

when $x=1$, $t=1$; when $x\to\infty$, $t=0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^\infty \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(\frac{-1}{t^2}\right)dt$$

$$= \int_0^1 \frac{1}{t^{m-1}}\cdot\frac{t^{m+n}}{(1+t)^{m+n}}\cdot\frac{1}{t^2}\,dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

$\therefore$ from (1)

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

8(a)(i) Using Cauchy's theorem / Cauchy integral formula evaluate $\int_C \frac{z\,dz}{z^2+1}$, where C is

(a) $\left|z+\frac{1}{z}\right|=2$, (b) $|z+i|=1$.

Sol'n: (a) Let $I=\int_C \frac{z\,dz}{z^2+1}$, where C is (a) $z+\frac{1}{z}=2$

$$\left|z+\frac{1}{z}\right|=2 \Rightarrow |z^2+1|=|2z| \Rightarrow |x^2-y^2+2ixy+1| = 2|x+iy|$$

$$\Rightarrow (x^2-y^2+1)^2+4x^2y^2=4(x^2+y^2)$$

$$\Rightarrow (x^2+y^2)^2+1=2x^2+6y^2$$

$$\Rightarrow (x^2+y^2-1)^2+2(x^2+y^2)=2x^2+6y^2$$

$$\Rightarrow (x^2+y^2-1)^2=4y^2 \Rightarrow x^2+y^2-1=\pm 2y$$

$$\Rightarrow (x-0)^2+(y\pm 1)^2=2$$

$\Rightarrow$ circle c_1, Centre $(0,1)$, $r_2=\sqrt{2}$ and circle c_2, Centre $(0,-1)$, $r_2=\sqrt{2}$

$\Rightarrow c_1, |z-i|=\sqrt{2}$ and $c_2: |z+i|=\sqrt{2}$

$$I=\int_C \frac{z\,dz}{(z+i)(z-i)}=\int_{C_1}\frac{z\,dz}{(z+i)(z-i)}+\int_{C_2}\frac{z\,dz}{(z+i)(z-i)}$$

Take $f=\frac{z}{z+i}$ for C_1 and $g=\frac{z}{z-i}$ for C_2

Then $I=\int_{C_1}\frac{f(z)\,dz}{z-i}+\int_{C_2}\frac{g(z)\,dz}{z+i}$

$= 2\pi i\,[f(i)+g(-i)]$, by Cauchy's integral formula

$$=2\pi i\left[\left(\frac{z}{z+i}\right)_{at\ z=i}+\left(\frac{z}{z-i}\right)_{at\ z=-i}\right]$$

$$=2\pi i\left[\frac{1}{2}+\frac{1}{2}\right]=2\pi i$$

(b) Let $I = \int_C \frac{z\,dz}{z^2+1}$, where C is $|z+i| = 1$.

Centre of circle C is at $z = -i$ and radius 1.

Take $f = \frac{z}{z-i}$, then $I = \int_C \frac{f(z)\,dz}{z+i}$

$\therefore\ I = 2\pi i f(-i) = 2\pi i \left(\frac{z}{z-i}\right)$ at $z = -i$ or

$I = \pi i$

(ii) If $a > e$, use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$.

Sol^n: Let C denote the circle $|z| = 1$, the given equation may be written as $az^n - e^z = 0$.

we write, $f(z) = az^n$ and $g(z) = -e^z$

Then evidently both $f(z)$ and $g(z)$ are analytic within and on C.

Also we have

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-e^z}{az^n}\right| = \frac{e^z}{|a||z^n|}$$

$$= \frac{e^z}{a|z|^n} \quad (\because a \text{ is real +ve})$$

$$= \frac{\left|1 + z + z^2/2! + z^3/3! + \cdots\right|}{a|z|^n}$$

$$\le \frac{1 + |z| + \frac{1}{2!}|z|^2 + \frac{1}{3!}|z|^3 + \cdots}{a|z|^n}$$

$$= \frac{1 + 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots}{a} \quad \text{as } |z| = 1 \text{ on } C.$$

$$= \frac{e}{a} < 1 \quad (\because a > e)$$

Thus $|g(z)| < |f(z)|$ on C

Hence all the conditions of Rouche's theorem are satisfied so that $f(z)+g(z) = az^n - e^z$ has the same number of zeros as $f(z) = az^n$ inside C.

But az^n has n zeros all located at the origin which is the centre of C.

It follows that $az^n - e^z$ has n zeros inside C as required.

8(b) Find the Taylor's or Laurent's series which represent the function
$$f(z) = \frac{1}{(1+z^2)(z+2)}$$
(i) when $|z|<1$
(ii) $1<|z|<2$
(iii) $|z|>2$.

Sol^n:
Resolving $f(z)$ into partial fractions, we obtain $f(z) = \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$

(i) For $|z|<1$, $f(z)$ is analytic and so we have
$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left[1+\frac{z}{2}\right]^{-1} - \frac{1}{5}(z-2)(1+z^2)^{-1}$$
$$= \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}+\cdots+(-1)^n\frac{z^n}{2^n}+\cdots\right]$$
$$-\frac{z-2}{5}\left[1-z^2+z^4-z^6+\cdots+(-1)^n z^{2n}+\cdots\right]$$
$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \frac{(z-2)}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$
This series being in the +ve powers of z represents Taylor's expansion for $f(z)$.

(ii) $1<|z|<2$, we have
$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{1}{2}z\right)^{-1} - \frac{z-2}{5}\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$
$$= \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right] - \frac{z-2}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right]$$
$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \frac{z-2}{5z^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}}$$
$\left(\because |z|>1, \Rightarrow \frac{1}{|z|}<1 \Rightarrow \frac{1}{|z|^2}<1\right.$ so the binomial expansion of $\left(1+\frac{1}{z^2}\right)^{-1}$ is valid$\Big)$

The above series being in positive and negative powers of z represents Laurent's expansion for $f(z)$ in the region $1<|z|<2$.

(iii) For $|z|>2$, we have
$$f(z) = \frac{1}{5}\cdot\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{1}{5}(z-2)\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$
$$= \frac{1}{5z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right] - \frac{z-2}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right]$$
$$= \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{z-2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}}$$

8.(c) Obtain the dual of the LP program:

Min. $Z = x_1 + x_2 + x_3$; subject to the constraints :

$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \leq 3$$
$$2x_2 - x_3 \geq 4$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Solve:- Transform the given LPP into the standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \geq 0$, $x_3'' \geq 0$.

$$\text{Max. } Z_x' = -x_1 - x_2 - (x_3' - x_3'') ; \quad Z_x' = -Z$$

Subject to constraints:

$$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$$
$$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$$
$$x_1 - 2x_2 \leq 3$$
$$-2x_2 + (x_3' - x_3'') \leq -4$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

Let ; w_1', w_1'', w_2, w_3 be the dual variables.

The dual problem of above standard primal is obtained as :

$$\text{Min } Z_W' = 5(w_1' - w_1'') + 3w_2 - 4w_3$$

Subject to constraints :

$$(w_1' - w_1'') + w_2 + 0w_3 \geq -1$$
$$-3(w_1' - w_1'') + (-2)w_2 - 2w_3 \geq -1$$
$$4(w_1' - w_1'') + 0w_2 + w_3 \geq -1.$$
$$-4(w_1' - w_1'') + 0w_2 - w_3 \geq 1.$$
$$w_1', w_1'', w_2, w_3 \geq 0$$

This dual can be written in more compact form as:

$$\text{Max } Z_W = -5w_1 - 3w_2 + 4w_3$$

Subject to constraints.

$$-w_1 - w_2 \leq 1$$
$$3w_1 + 2w_2 + 2w_3 \leq 1$$
$$-4w_1 - w_3 = 1$$

$w_2, w_3 \geq 0$ and w_1 is unrestricted

which is required dual.

8.(d) A product is produced by four factories F_1, F_2, F_3, F_4. The unit production costs in them are Rs 2, Rs 3, Rs 1 and Rs 5 respectively. Their production capacities are F_1 – 50 units, F_2 – 70 units, F_3 – 30 units, F_4 – 50 units. These factories supply the product to four stores S_1, S_2, S_3 and S_4, demands of which are 25, 35, 105, and 20 units respectively. Unit transport cost in rupees from each ~~factory~~ to each store is given in the table. Determine the extent of deliveries from each of the factories to each of the stores so that the total production and transportation cost is minimum.

	S_1	S_2	S_3	S_4
F_1	2	4	6	11
F_2	10	8	7	5
F_3	13	3	9	12
F_4	4	6	8	3

Sol: First of all we shall form a new table of unit costs which consists of both production and transportation costs. The new cost matrix is given below

	S_1	S_2	S_3	S_4	a_i
F_1	2+2	4+2	6+2	11+2	50
F_2	10+3	8+3	7+3	5+3	70
F_3	13+1	3+1	9+1	12+1	30
F_4	4+5	6+5	8+5	3+5	50
Demand b_j	25	35	105	20	

Since $\sum a_i \neq \sum b_j$

it is an unbalanced transportation problem with surplus capacity = 15 units.

Therefore, we create a dummy store S_5 with associated cost coefficients which are taken as zero.

Therefore, the starting cost matrix becomes:

	S_1	S_2	S_3	S_4	S_5	
F_1	4	6	8	13	0	50
F_2	13	11	10	8	0	70
F_3	14	4	10	13	0	30
F_4	9	11	13	8	0	50
	25	35	105	20	15	

Using the vogel's Approximation method, the initial basic feasible solution is given by

(25) 4	(5) 6	(20) 8	13	0
13	11	(70) 10	8	0
14	(30) 4	10	13	0
9	11	(15) 13	(20) 8	(15) 0

which is non-degenerate basic feasible solution.

since the no. of allocations $= m+n-1$
$= 4+5-1$
$= 8$ (basic variables)

<u>Now finding the values of u_i and v_j:-</u>

As the maximum no of basic cells exists in the first row and 3rd column.

putting either $u_1=0$ or $v_3=0$

Let $u_1=0$ and the values of u_i's and v_j's and also the net evaluations $\Delta_{ij} = u_i+v_j-c_{ij}$

for all unoccupied cells are exhibited as shown below.

(25) 4	(5) 6	(20) 8	(−) 13	(−) 0	0
(−) 13	(−) 11	(70) 10	(−) 8	(−) 0	2
(−) 14	(30) 4	(−) 10	(−) 13	(−) 0	−2
9 (0)	11 (0)	(15) 13	(20) 8	(15) 0	5
4	6	8	3	−5	

Since all the net evaluations are ≤ 0. and atleast one $\Delta_{ij} = 0$, the current initial basic feasible solution is optimal but not unique.

There exists alternate optimal solution.

Therefore one of the optimal solution becomes

$x_{11} = 25$, $x_{12} = 5$, $x_{13} = 20$

$x_{23} = 70$, $x_{32} = 30$, $x_{43} = 15$,

$x_{44} = 20$, $x_{45} = 15$

with optimum ~~transportation~~ transportation plus product cost = 1465 _