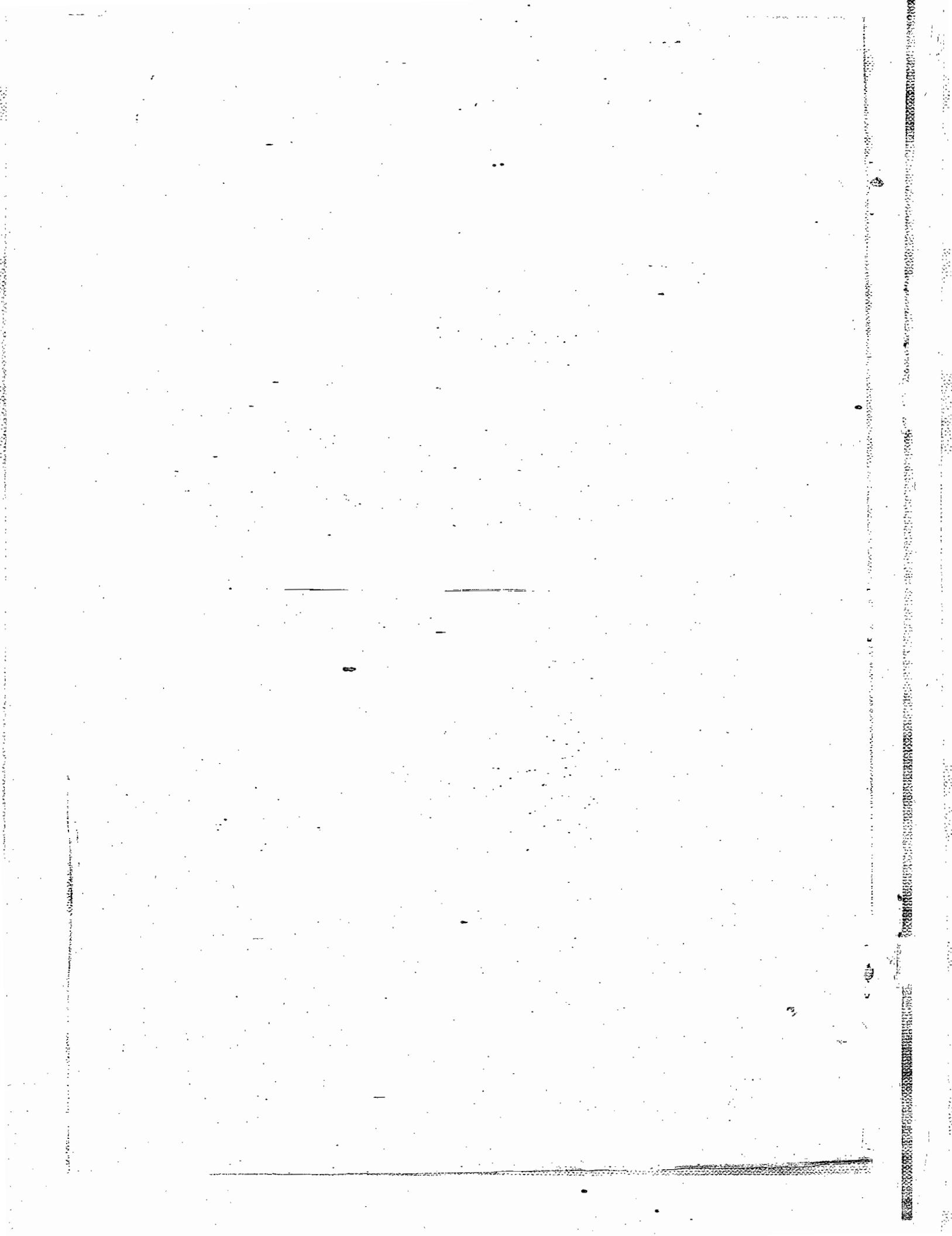


IMS
MATHS
BOOK - 14



Set IV
The Sphere

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Sphere:

Defn: A sphere is the locus of a point which moves so that its distance from a fixed point always remains constant.

The fixed point is called the centre and the constant (b), Central form:
distance is called the radius of the sphere.

Equations of a Sphere in

- different forms:

(a) Standard form:

To show that the eqn of the sphere whose centre is the origin and radius 'a' is

$$x^2 + y^2 + z^2 = a^2$$

proof: Let $P(x, y, z)$ be any point on the sphere

Join OP .

Then

$$OP = \text{radius of sphere} = a \quad (\text{given}) \quad (1)$$

By distance formula

$$OP = \sqrt{x^2 + y^2 + z^2}. \quad (2)$$

from (1) & (2), we have

$$\sqrt{x^2 + y^2 + z^2} = a$$

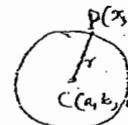
$$\Rightarrow x^2 + y^2 + z^2 = a^2$$

which is the required eqn of the sphere.

To find the eqn of a sphere whose centre is (a, b, c) and radius is r .

proof: Let $C(a, b, c)$ be the centre of the sphere.

Let $P(x, y, z)$ be the point on the sphere.



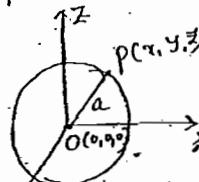
Join CP . Then

$$CP = \text{radius of sphere} = r \quad (\text{given})$$

$$\Rightarrow CP^2 = r^2$$

$$\Rightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

which is the required eqn.



(C) General form:

To prove that the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere and find its centre and radius.

proof: The given eqn is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

This can be written as

$$(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz)$$

$$+ d = 0$$

Adding $u^2 + v^2 + w^2$ to both sides we get

$$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = -d + u^2 + v^2 + w^2$$

$$\Rightarrow (x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - d.$$

$$\Rightarrow [x - (-u)]^2 + [y - (-v)]^2 + [z - (-w)]^2 = [\sqrt{u^2 + v^2 + w^2 - d}]^2 \quad (2)$$

which is clearly of the central form of the sphere.

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \quad (3)$$

$\therefore (1)$ represents a sphere.

NOW Comparing (2) & (3). we have

$$a = -u, b = -v, c = -w \text{ and}$$

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

\therefore Centre of the Sphere (1)

is $(-u, -v, -w)$ and the

radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Note: If $u^2 + v^2 + w^2 - d < 0$

then the radius of the sphere imaginary and the centre $(-u, -v, -w)$ is real.

In this case the sphere is called pseudo-sphere (or) a virtual sphere.

Working rule for finding the centre and radius of the sphere:

(1) First of all make the coefficients of $x^2, y^2, z^2 = 1$ if they are not so.

(2) Centre is

$$[-\frac{1}{2} \text{ coeff. of } x, -\frac{1}{2} \text{ coeff. of } y, -\frac{1}{2} \text{ coeff. of } z]$$

and radius is

$$\sqrt{\left(\frac{1}{2} \text{ coeff. of } x\right)^2 + \left(\frac{1}{2} \text{ coeff. of } y\right)^2 + \left(\frac{1}{2} \text{ coeff. of } z\right)^2} \quad \text{Constant term.}$$

(1) Conditions for a Sphere

The given eqn represents a sphere if

(i) it is a second degree in x, y, z

(ii) coeff of x^2 = coeff of y^2 =
coeff of z^2 ,

and

(iii) it does not contain the terms involving the products xy , xz and yz .

(2) Since the general eqn of the sphere -

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Contains four unknown constants u, v, w, d .

So the Sphere can be found to satisfy four conditions.

Four-point form:

To find the eqn of a sphere passing through the given points.

Sol: Let (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , (x_4, y_4, z_4) be the given four points.

Let the required eqn of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{①}$$

Since it passes through

(x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4)

$$\therefore (x_1^2 + y_1^2 + z_1^2) + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$$(x_2^2 + y_2^2 + z_2^2) + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$$

$$(x_3^2 + y_3^2 + z_3^2) + 2ux_3 + 2vy_3 + 2wz_3 + d = 0$$

$$(x_4^2 + y_4^2 + z_4^2) + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$$

eliminating u, v, w, d from

(1), (2), (3), (4) & (5) with the help of determinants, we have

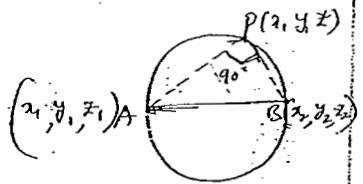
$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is the required equation.

Diameter form:

To find the equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter.

Sol: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points.



Let $P(x, y, z)$ be any point on the sphere. Join AP and BP.

Since AB is diameter of the sphere then

- $\angle APB = \text{angle in the semi-circle} = 90^\circ$
i.e., $AP \perp PB$ ①

Now the d.r.'s of AP are

$$x - x_1, y - y_1, z - z_1 \quad \text{and}$$

the d.r.'s of BP are

$$x - x_2, y - y_2, z - z_2$$

Since $AP \perp BP$

$$\therefore (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

which is the required eqn
of the sphere.

Problems:

→ find the radius and centre of the sphere

$$x^2 + y^2 + z^2 - 2x + 4y - 6z = 2$$

Sol: This is comparing with the general eqn of the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$$

we have

$$2a = -2 \quad || \quad 2b = 4 \quad || \quad 2c = -6 \\ \Rightarrow a = -1 \quad || \quad b = 2 \quad || \quad c = -3$$

$$\text{and } d = -2$$

∴ centre is $(-1, 2, -3)$

$$= (-1, 2, 3)$$

$$\text{radius} = \sqrt{a^2 + b^2 + c^2 - d}$$

$$= \sqrt{1 + 4 + 9 + 2}$$

$$= \sqrt{16} = 4$$

→ find the centres and radii of the following spheres.

$$(i) x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0$$

$$(ii) x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$$

$$(iii) x^2 + y^2 + z^2 - 2x + 4y + 2z + 3 = 0$$

→ Obtain the eqn of the sphere described on the join of the points $A(2, -3, 4)$

Sol: Now let the required eqn of sphere on the join of two

points (x_1, y_1, z_1) & (x_2, y_2, z_2) as diameter be

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

$$(x - 2)(x + 5) + (y + 3)(y - 6) + (z - 4)(z + 1) = 0$$

$$x^2 + y^2 + z^2 + 2x - 3y + 3z - 56 = 0$$

(3)

→ find the eqn of the sphere through the points $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Solⁿ: let the eqn of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

Since it passes through $(0,0,0)$

$$\therefore d = 0$$

$$(1) \Rightarrow x^2 + y^2 + z^2 + 2ua + 2vb + 2wc = 0 \quad (2)$$

Since it passes through $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$.

$$\therefore \text{we have } a^2 + 2ua = 0 \Rightarrow u = \frac{a}{2}$$

$$b^2 + 2vb = 0 \Rightarrow v = -\frac{b}{2}$$

$$c^2 + 2wc = 0 \Rightarrow w = -\frac{c}{2}$$

Putting these values in (2)
we get

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

which is the required equation.

Note: Eqn of the sphere ABC , where $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are three points on the axis if $x^2 + y^2 + z^2 - ax - by - cz = 0$.

→ find the eqn of the sphere circumscribing the tetrahedron $x=0$, $y=0$, $z=0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solⁿ: Three plane out of the given four planes taken at a time determine one vertex of the tetrahedron.

Hence the vertices of the tetrahedron are $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Remaining solution similar to previous problem.

→ find the eqn of the sphere passing through the three points $(3,0,2)$, $(-1,1,1)$, $(2,-5,4)$ and having its centre on the plane $2x + 3y + 4z = 6$ if $x^2 + y^2 + z^2 + 6x + 6y + 6z = 1$.

Solⁿ: let the eqn of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Since it passes through $(3,0,2)$, $(-1,1,1)$ and $(2,-5,4)$.

$$\therefore 9 + 4 + 6u + 4v + 4w + d = 0$$

$$\Rightarrow 6u + 4v + 4w + 13 + d = 0 \quad (2)$$

$$1+1+1-2u+2v+2w+d=0$$

$$\Rightarrow -2u+2v+2w+3+d=0 \quad \text{--- (3)}$$

$$4+25+16+4u-10v+8w+d=0$$

$$\Rightarrow 4u-10v+8w+45+d=0 \quad \text{--- (4)}$$

Also centre (u, v, w) lies

$$\text{on the plane } 2x+3y+4z-6=0$$

$$\therefore 2u+3v+4w+6=0 \quad \text{--- (5)}$$

Solving the above eqns ②, ③, ④ & ⑤ we get u, v, w, d values
putting these values in eqn ①

which is the required sphere

- find the eqn to a sphere passing through the points $(1, -3, 4)$, $(1, -5, 2)$, $(1, -3, 0)$ and having centre on the plane $x+y+z=0$

- Obtain the sphere having its centre on the line $5y+2z=0 = 2x-3y$ and passing through the two points $(0, -2, -4)$, $(2, -1, -1)$.

Sol: Let the eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

Since its centre lies on

$$\text{the line } 5y+2z=0 = 2x-3y$$

$$\therefore 5(-v)+2(-w)=0 = 2(-u)-3(-v)$$

$$\text{i.e., } 5v+2w=0 \quad \text{--- (2)}$$

$$\& 2u-3v=0 \quad \text{--- (3)}$$

Since the sphere passes

through the points $(0, -2, -4)$ and $(2, -1, -1)$

$$\therefore u+1+1+4u-2v-8w+d=0$$

$$\Rightarrow 5u-8w+d+16=0 \quad \text{--- (4)}$$

$$\& u+1+1+4u-2v-2w+d=0$$

$$\Rightarrow 6u-2v-2w+d+6=0 \quad \text{--- (5)}$$

Solving the eqns ②, ③, ④ & ⑤

we get

$$u=-3, v=-2, w=5$$

$$d=12$$

∴ ① is

$$x^2+y^2+z^2-6x-4y+10z+12=0$$

which is the required

equation

(4)

→ P.T. the eqn $a^2x^2 + a^2y^2 + a^2z^2 + 2ax + 2ay + 2az + d = 0$
 represents a sphere. find its
 radius and centre.

$$\text{Ans: } \frac{\sqrt{a^2 - ad}}{a}, \left(\frac{-a}{a}, \frac{-a}{a}, \frac{-a}{a} \right)$$

→ Find the eqn to the sphere
 through the points $(0,0,0)$,
 $(0,1,-1)$, $(-1,2,0)$, $(1,2,3)$.

→ Find the eqn of the sphere
 through the four points
 $(4,-1,2)$, $(0,-2,3)$, $(1,-5,-1)$,
 $(2,0,1)$.

→ Find the eqn of the sphere
 through the four points
 $(0,0,0)$, $(-a,b,c)$, $(a,-b,c)$,
 $(a,b,-c)$. and determine its
 radius.

→ Find the eqn of the sphere
 inscribed in the tetrahedron
 whose faces are $x=0$, $y=0$,
 $z=0$ and $x+2y+2z=1$.

Sol: The given faces are

$$x=0, y=0, z=0 \text{ and}$$

$$x+2y+2z=0$$



Let (α, β, γ) be the centre
 and 'r' the radius of the
 inscribed sphere. Then
 distances of the centre from
 all the four faces are equal
 and each equal to radius.

$$\therefore \frac{d}{l} = \frac{R}{l-1} = \frac{l-\alpha-2\beta-2\gamma}{l+4+r} =$$

$$\therefore \alpha = \beta = \gamma = r \text{ and}$$

$$l-\alpha-2\beta-2\gamma = r$$

Eliminating α, β, γ , we get

$$l-\alpha-2\beta-2\gamma = r$$

$$\Rightarrow 8r = l$$

$$\Rightarrow r = \frac{l}{8}$$

$$\therefore \alpha = \beta = \gamma = \frac{l}{8}$$

∴ The centre is

$$(\alpha, \beta, \gamma) = \left(\frac{l}{8}, \frac{l}{8}, \frac{l}{8} \right) \text{ and}$$

$$\text{the radius } (l) = \frac{l}{8}$$

∴ The eqn of the sphere with
 centre $\left(\frac{l}{8}, \frac{l}{8}, \frac{l}{8} \right)$ and
 radius $\frac{l}{8}$ is

$$(x - \frac{l}{8})^2 + (y - \frac{l}{8})^2 + (z - \frac{l}{8})^2 = (\frac{l}{8})^2$$

$$\Rightarrow x^2 - \frac{1}{4}x + \frac{1}{64} + y^2 - \frac{1}{4}y + \frac{1}{64} + z^2 - \frac{1}{4}z + \frac{1}{64} =$$

$$\Rightarrow x^2 + y^2 + z^2 - \frac{1}{4}(x+y+z) + \frac{1}{32} = 0$$

$$\Rightarrow 32(x^2 + y^2 + z^2) - 8(x+y+z) + 1 = 0$$

→ A sphere is inscribed in the tetrahedron whose faces are $x=0, y=0, z=0$
 $2x+6y+3z=14$.

Find its centre, radius and write down its equation.

2002 → find the co-ordinates of the centre of the sphere.

Inscribed in the tetrahedron formed by the planes whose equations are $x=0, y=0, z=0$
 $x+y+z=a$.

→ Find the eqn of the sphere inscribed in the tetrahedron whose faces are

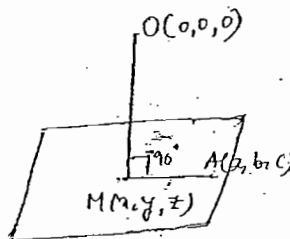
(i) $x=0, y=0, z=0, 2x+6y+3z+6=0$

2007 (ii) $x=0, y=0, z=0, 2x+3y+6z=6$

→ A plane passes through a fixed point (a, b, c) , show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2+y^2+z^2+ax+by+cz=0$.

Sol: Let $A(a, b, c)$ be the fixed point on the variable plane α .

and let $M(x, y, z)$ be the foot of \perp from the origin to the plane α .



$$\therefore OM \perp MA$$

Now the d.r.'s of OM are x, y, z and the d.r.'s of MA are $a-x, b-y, c-z$.

Since $OM \perp MA$:

$$\therefore x(a-x) + y(b-y) + z(c-z) = 0 \\ \Rightarrow x^2 + y^2 + z^2 - ax - by - cz = 0$$

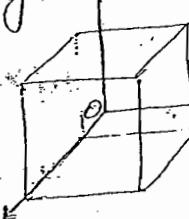
which is the required locus and it represents a sphere.

→ A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that

the locus is a sphere.

Take the centre of the cube as the origin and

the planes through
the centre parallel
to its faces as
co-ordinate planes.



Let each of the edges of the cube be equal to $2a$.

Then the eqns of the three pairs of parallel faces of the cube are

$$x=a, x=-a, y=a, y=-a$$

$$\text{and } z=a, z=-a.$$

Now let (α, β, r) be any point in the locus.

Now the sum of squares of distances of P from the six faces is constant $= 6k^2$ (say).

$$\left(\frac{\alpha-a}{1}\right)^2 + \left(\frac{\alpha+a}{1}\right)^2 + \left(\frac{\beta-a}{1}\right)^2 + \left(\frac{\beta+a}{1}\right)^2 + \left(\frac{r-a}{1}\right)^2 + \left(\frac{r+a}{1}\right)^2 = 6k^2$$

$$\Rightarrow 2(\alpha^2 + \beta^2 + r^2) + 6a^2 = 6k^2$$

$$\Rightarrow \alpha^2 + \beta^2 + r^2 = 3(k^2 - a^2)$$

\therefore Locus of $P(\alpha, \beta, r)$ is

$$x^2 + y^2 + z^2 = 3(k^2 - a^2).$$

which clearly represents a sphere.

(5) $\rightarrow OA, OB, OC$ are three mutually perpendicular lines through the origin whose direction cosines are $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$. If $OA = a, OB = b, OC = c$, show that the eqn of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - x(al_1 + bl_2 + cl_3) - y(am_1 + bm_2 + cm_3) - z(an_1 + bn_2 + cn_3) = c^2$$

Solⁿ's Since l_1, m_1, n_1 are the actual d.c's of OA and $OA = a$.

The co-ordinates of A are $(l_1 a, m_1 a, n_1 a)$. Using (l_1, m_1, n_1)

Similarly, the co-ordinates of B & C are

$$(l_2 a, m_2 a, n_2 a) \quad \text{by front fig. } \cos \theta = \frac{x}{a}$$

$$\text{and } (l_3 a, m_3 a, n_3 a) \quad \Rightarrow l = \frac{x}{a}$$

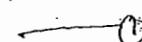
respectively.

Also O is $(0, 0, 0)$.

Now let the required eqn

of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$



Since it passes through $O(0,0,0)$

$$\text{①} \Rightarrow x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad (2)$$

Since it passes through

$$A(l_1, m_1, n_1)$$

$$\therefore l_1^2 + m_1^2 + n_1^2 + 2ul_1 + 2vm_1 + 2wn_1 = 0$$

$$\Rightarrow u + 2ul_1 + 2vm_1 + 2wn_1 = 0 \quad (3)$$

$$(l_1^2 + m_1^2 + n_1^2 = 0)$$

Similarly for B and C

$$b + 2ul_2 + 2vm_2 + 2wn_2 = 0 \quad (4)$$

$$\text{and } c + 2ul_3 + 2vm_3 + 2wn_3 = 0 \quad (5)$$

Since the lines OA, OB, OC are mutually perpendicular.

$\therefore l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$ are the d.c's of ox, oy, oz

referred to OA, OB, OC as axes.

$$\text{So } l_1^2 + l_2^2 + l_3^2 = 1, m_1^2 + m_2^2 + m_3^2 = 1$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \text{ and}$$

$$lm_1 + bm_2 + cm_3 = m_1m_2 + m_2m_3 + m_3m_1$$

$$= n_1l_1 + n_2l_2 + n_3l_3 = 0$$

Now multiplying (3) by l_1 ;

(4) by l_2 ; (5) by l_3 and

adding, we get

$$\begin{aligned} al_1 + 2ul_1^2 + 2vl_1m_1 + 2wl_1n_1 \\ + bl_2 + 2ul_2^2 + 2vl_2m_2 + 2wl_2n_2 \\ + cl_3 + 2ul_3^2 + 2vl_3m_3 + 2wl_3n_3 = 0 \end{aligned}$$

$$\Rightarrow al_1 + bl_2 + cl_3 + 2u(l_1) + 2v(l_2) \\ + 2w(l_3) = 0$$

$$\Rightarrow u = -\frac{1}{2}(al_1 + bl_2 + cl_3)$$

$$\text{Similarly } v = -\frac{1}{2}(am_1 + bm_2 + cm_3)$$

$$w = -\frac{1}{2}(an_1 + bn_2 + cn_3)$$

Substituting in eqn (2),

we get

$$x^2 + y^2 + z^2 - x(al_1 + bl_2 + cl_3)$$

$$-y(am_1 + bm_2 + cm_3)$$

$$-z(an_1 + bn_2 + cn_3) = 0$$

which is the required equation

1996 find the eqn of the

sphere which passes through

the points $(1, 0, 0), (0, 1, 0)$

$(0, 0, 1)$ and has its radius

as small as possible.

Sol: Let the eqn of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

Since it passes through $(1, 0, 0)$

$$\therefore 1 + 2u + d = 0$$

(6)

Since (1) passes through $(0,1,0)$

$$\therefore u+2v+d=0$$

Since (1) passes through $(0,0,1)$

$$\therefore u+2w+d=0$$

from these, we have

$$u=v=w=-\frac{(1+d)}{2}$$

$$\therefore (1) \equiv x^2+y^2+z^2-(1+d)x-(1+d)y$$

$$-(1+d)z+cd=0 \quad (2)$$

Centre $\therefore (-u, -v, -w)$

$$=\left(\frac{1+d}{2}, \frac{1+d}{2}, \frac{1+d}{2}\right)$$

If R' is the radius of the sphere, then

$$R'^2 = u^2 + v^2 + w^2 - d$$

$$= 3\left(\frac{1+d}{2}\right)^2 - d$$

$$= \frac{3}{4}(1+d^2+2d) - d$$

$$= \frac{1}{4}(3+3d^2+6d-4d)$$

$$= \frac{1}{4}(3+3d^2+2d)$$

$$< \frac{3}{4}(d^2+\frac{2}{3}d+1)$$

$$= \frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2 + \left(1-\frac{1}{9}\right)\right]$$

$$= \frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2 + \frac{8}{9}\right]$$

If $(d+\frac{1}{3})=0$ then R'^2 isleast (i.e., R is least)

$$\Rightarrow d = -\frac{1}{3}$$

$$\therefore (2) \equiv x^2+y^2+z^2-\left(1-\frac{1}{3}\right)x-\left(1-\frac{1}{3}\right)y$$

$$-\left(1-\frac{1}{3}\right)z-\frac{1}{3}=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}$$

$$\Rightarrow 3(x^2+y^2+z^2)-2(x+y+z)-1=0$$

which is the required eqn of the sphere.

1885

A variable plane through a fixed point (a, b, c) cuts the co-ordinate axes in the points A, B, C . Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ Soln: Let the eqn of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ since it passes through (a, b, c) ,

$$\therefore \frac{a}{a} + \frac{b}{b} + \frac{c}{c} = 1 \quad (2)$$

Since (1) cuts the axes in

 A, B, C .The co-ordinates of $A(a, 0, 0), B(0, b, 0), C(0, 0, c)$ and $(0, 0, 0)$.

Let the eqn of the sphere

OABC be

$$x^2+y^2+z^2+2ax+2by+2cz+d=0 \quad (3)$$

Since it passes through $O(0,0,0)$.

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad (4)$$

and since it passes through $A(\alpha, 0, 0)$

$$\therefore \alpha^2 + 2u\alpha = 0 \\ \Rightarrow 2u = -\alpha$$

Similarly $2v = -\beta$, $2w = -\gamma$

putting these values in (4), we get,

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0 \quad (5)$$

If (x_1, y_1, z_1) is the centre then $x_1 = \frac{\alpha}{2}$, $y_1 = \frac{\beta}{2}$, $z_1 = \frac{\gamma}{2}$

$$\Rightarrow d = 2x_1, \beta = 2y_1, \gamma = 2z_1$$

$$\therefore (2) \Rightarrow \frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1$$

$$\Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$$

Locus of (x_1, y_1, z_1) is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

→ A plane through a fixed point $(1, 1, 1)$ cuts the axes in A, B, C . Find the locus of the centre of the sphere $OABC$ where O is origin.

→ A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C . Find the locus of the centroid of the tetrahedron $OABC$.

Sol: Let co-ordinates of

the points A, B, C be

$(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The eqn of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Radius of this sphere

$$= \sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2 + (\frac{c}{2})^2} = 2k \text{ (given)}$$

squaring on both sides, we get

$$a^2 + b^2 + c^2 = 16k^2 \quad (1)$$

Let (x_1, y_1, z_1) be the co-ordinates of the tetrahedron $OABC$, then

$$x_1 = \frac{a+x_1}{4} \Rightarrow x_1 = \frac{a}{4} \\ \Rightarrow a = 4x_1$$

$$\text{Similarly } b = 4y_1 \text{ & } c = 4z_1$$

putting these values of

a, b, c in ①, we get

$$16x^2 + 16y^2 + 16z^2 = 16k^2$$

$$\Rightarrow x^2 + y^2 + z^2 = k^2.$$

∴ Locus of the centroid.

(x_1, y_1, z_1) is

$$x_1^2 + y_1^2 + z_1^2 = k^2$$

1988, A Sphere of constant radius k passes through the origin and meets the axes in A, B, C . Prove that the centroid of the triangle ABC lies on the sphere

$$9(x^2 + y^2 + z^2) = 4k^2.$$

→ A variable sphere passes through the origin 'O' and meets the axes in A, B, C so that the volume of the tetrahedron $OABC$ is constant. Find the locus of the centre of the sphere.

2003 A Sphere of constant radius 'r' passes through the origin 'O' and cuts the axes in A, B, C . Find the

(7)
locus of the foot of the perpendicular from 'O' to the plane ABC .

Solⁿ: Let the co-ordinates of the points A, B, C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Then the eqn of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

$$\text{Its radius} = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2} \\ = r \text{ (given)}$$

$$\Rightarrow x^2 + y^2 + z^2 = 4r^2. \quad \text{--- } \textcircled{1}$$

NOW the eqn of the plane

ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad \text{--- } \textcircled{2}$$

D.r.'s of the \perp to this plane $\textcircled{2}$ are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

∴ Eqs of the line through $O(0, 0, 0)$ and \perp to the plane $\textcircled{2}$ are

$$\frac{x-0}{1/a} = \frac{y-0}{1/b} = \frac{z-0}{1/c}$$

$$\Rightarrow ax = by = cz \quad (3)$$

To find the locus of foot of

i. from 'O' on the plane (2),
i.e., the locus of the point of
intersection of the plane (2)
and line (3), we have to
eliminate the unknown constants
a, b, c from (1), (2) & (3).

Now from (3),

$$\text{Let } ax = by = cz = \lambda \quad (say)$$

$$\Rightarrow a = \frac{\lambda}{x}, b = \frac{\lambda}{y}, c = \frac{\lambda}{z}$$

Putting these values in (1),

we get

$$x^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) = 4r^2$$

$$\Rightarrow x^2(x^2 + y^2 + z^2) = 4r^2 \quad (4)$$

And putting the values of
a, b, c in (2), we get

$$\frac{1}{\lambda} (x^2 + y^2 + z^2) = 1.$$

$$\Rightarrow \frac{1}{\lambda^2} (x^2 + y^2 + z^2)^2 = 1 \quad (5)$$

Multiplying (4) & (5), we get

$$(x^2 + y^2 + z^2)(x^2 + y^2 + z^2) = 4r^2$$

which is the required

locus.

Plane section of a sphere:

To prove that the section of a sphere by a plane is a circle.

Proof: Let 'C' be the centre of the sphere, 'a' its radius and α be the plane.

Draw $CO \perp$ from 'C' on the plane α and let $CO = p$.

'O' is the fixed point and p is a fixed length.

Let 'P' be any point on the section of the sphere by the plane α . Join CP & OP .

$\therefore CO \perp OP$

In the right angled $\triangle COP$ we have

$$OP^2 = CP^2 - CO^2 \\ = a^2 - p^2$$

(or)

$$OP = \sqrt{a^2 - p^2} \text{ which is constant.}$$

and 'O' is fixed point.

(S) $\therefore P$ lies on a circle whose centre is 'O' and radius is $\sqrt{a^2 - p^2}$.

\therefore The section of the sphere by a plane is a circle.

Equation of a circle

Since the intersection of a sphere with a plane is a circle.

In general a circle can be represented by the eqns of a sphere and a plane taken together.

i.e., the two eqns
 $x^2 + y^2 + 2ax + 2by + 2cz + d = 0$
 and $lx + my + nz = p$. taken together represent a circle.

Note:

I (1) The centre of the circle is the foot of \perp from the centre of the sphere on the plane and

(2) Radius of the circle
 $= \sqrt{a^2 - p^2}$, where 'a' is the radius of the sphere and 'p' the length of \perp from

the centre of the sphere
on the plane.

II. The section of a sphere
by a plane passing through
the centre of the sphere
is called a great circle.
Its centre and radius
is the same as that of
the sphere.

problems :-

→ find the centre and the
radius of the circle.

$$x^2 + y^2 + z^2 - 2y - 4z = 11,$$

$$x + 2y + 2z = 15.$$

Sol: The given sphere is

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 \quad (1)$$

Its centre is

$$(-u, -v, -w) = (0, 1, 2) \text{ and}$$

$$\text{radius} = \sqrt{1+4+11} = \sqrt{16} = 4$$

The given plane is

$$x + 2y + 2z = 15 \quad (2)$$



Eqns (1) & (2) taken together
represent a circle.

Now the centre of the
circle is the foot of \perp

from the centre of the
sphere (1) on the plane (2).

Now the dir's of the
normal to the plane (2)
are $1, 2, 2$.

∴ Eqns of the \perp line CA
through C and \perp to
plane (2) are:

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r \text{ (say)}$$

Any point on the line is

$$(r, 2(r)+1, 2(r)+2) \quad (3)$$

Let it be A

Since it lies on the
plane (2):

$$(r)(1) + 2(2(r)+1) + 2(2(r)+2) = 15$$

$$\Rightarrow 9r = 9$$

$$\Rightarrow r = 1$$

$$\therefore (3) \ni A(1, 2(1)+1, 2(1)+2)$$

$$= (1, 3, 4)$$

which is required
centre of the circle.

Again $p = CA = \text{distance}$
from C(0,1,2)

to the plane $x + 2y + 2z = 15$

$$\therefore p = \frac{10+2+4}{\sqrt{1+4+4}} \\ = \frac{9}{3} = 3$$

\therefore The radius of the circle is

$$AP = \sqrt{a^2 - pr} \\ = \sqrt{(4)^2 - (3)^2} \\ = \sqrt{16-9} \\ = \sqrt{7}$$

→ find the eqn of the circle circumscribing the triangle formed by the three points

$(a, 0, 0), (0, b, 0), (0, 0, c)$

Obtain also the co-ordinates of the centre of the circle.

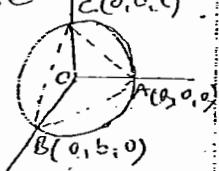
Solⁿ: Let the given points be

$A(a, 0, 0), B(0, b, 0), C(0, 0, c)$

Then the circumcircle of $\triangle ABC$ is the intersection

of the plane ABC and

the sphere $OABC$.



Now the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{ and the}$$

①

eqn of the sphere $OABC$

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \quad (2)$$

\therefore The eqns of the circle of $\triangle ABC$ are

$$x^2 + y^2 + z^2 - ax - by - cz = 0,$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

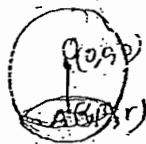
remaining solution

similar to previous problem

→ find the eqn of that plane which cuts the sphere $x^2 + y^2 + z^2 = a^2$ in a circle whose centre is (α, β, γ)

Solⁿ: Since 'O' is the centre of the sphere and $A(\alpha, \beta, \gamma)$ is the centre of the circle.

$\therefore OA \perp$ to the required plane of the circle.



Now the d.r.s of OA

are $\alpha=0, \beta=0, \gamma=0$

$\Rightarrow \alpha, \beta, \gamma$

The coeff of x, y, z in the eqn of the plane are α, β, γ . ($\because OA \perp$ to the plane of circle).

\therefore Eqn of the plane of the circle is

$$\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0 \quad \text{--- if 'r' be the radius}$$

1989 Show that the centres of all sections of the sphere $x^2 + y^2 + z^2 = a^2$ by planes through a point (α, β, γ) lie on the sphere

$$\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0.$$

Sol: Let (x_1, y_1, z_1) be the centre of one of the

sections, then the eqn of the plane is

$$x_1(x-\alpha) + y_1(y-\beta) + z_1(z-\gamma) = 0 \quad \text{--- } \textcircled{1}$$

Since it passes through the point (α, β, γ)

$$\therefore x_1(\alpha-\alpha) + y_1(\beta-\beta) + z_1(\gamma-\gamma) = 0.$$

$$\Rightarrow x_1(x_1-\alpha) + y_1(y_1-\beta) + z_1(z_1-\gamma) = 0.$$

\therefore Locus of (x_1, y_1, z_1) is

$$x(x-\alpha) + y(y-\beta) + z(z-\gamma) = 0$$

which is the required

eqn of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

$$dx + my + nz = 0.$$

Prove that

$$(x+d)(u^2+m^2+n^2) = (nu-mv)^2$$

$$+(nv-lw)^2 + (lv-mu)^2$$

So: The equations of the circles are

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{--- } \textcircled{1}$$

$$\text{and } dx + my + nz = 0 \quad \text{--- } \textcircled{2}$$

The centre of the

sphere $\textcircled{1}$ is

$C(-u, -v, -w)$ and

$$\text{its radius } CP = \sqrt{u^2 + v^2 + w^2 - d}$$

Also $CA = 1^r$ distance of $C(-u, -v, -w)$ from the plane $\textcircled{2}$.



$$= \frac{|l(-u) + m(-v) + n(-w)|}{\sqrt{l^2 + m^2 + n^2}}$$

$$= \frac{|lu + mv + nw|}{\sqrt{l^2 + m^2 + n^2}}$$

By the right angled $\triangle CAP$

$$AP^2 = CP^2 - CA^2$$

$$\Rightarrow s^2 = (u^2 + v^2 + w^2 + d^2) - \frac{(lu + mv + nw)^2}{l^2 + m^2 + n^2}$$

$$\Rightarrow (s^2 + d^2)(l^2 + m^2 + n^2) = \\ - (u^2 + v^2 + w^2)(l^2 + m^2 + n^2) \\ - (lu + mv + nw)^2$$

(by using the Lagrange's identity)

$$= (mw - nv)^2 + (nu - lw)^2 \\ + (lv - mu)^2$$

Hence the result

Lagrange's identity

$$(l^2 + m^2 + n^2)(l_1^2 + m_1^2 + n_1^2) \\ - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 \\ = (m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2 \\ + (l_1 m_2 - m_1 l_2)^2$$

(10)
Note: The four points are said to be concyclic if the circle through any three points passes through the fourth point.

→ Show that the following sets of points are concyclic

- (i) $(5, 0, 2), (2, -6, 0), (-7, -3, 8), (4, -9, 6)$
- (ii) $(-8, 5, 2), (-5, 2, 2), (-7, 6, 6), (-4, 3, 6)$

Sol: (i) Let the four given points be

$$A(5, 0, 2), B(2, -6, 0), C(-7, -3, 8), D(4, -9, 6)$$

Let us find the eqns of the circle ABC:

To find the eqn of the plane ABC:

ABC :

Any plane through A is

$$l(x-5) + m(y) + n(z-2) = 0 \quad \text{--- (1)}$$

Since it passes through B & C

we get

$$3l + 6m + 2n = 0 \quad \text{--- (2)}$$

$$and. 2l - 3m + 6n = 0 \quad \text{--- (3)}$$

Solving (2) & (3), we get

$$\frac{l}{6} = \frac{m}{-2} = \frac{n}{-3}$$

$$\therefore \textcircled{1} \Leftrightarrow 6x - 2y - 3z - 24 = 0 \quad \textcircled{4}$$

Now to find the eqn of the sphere OABC.

Let the eqn of the sphere through OABC be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \textcircled{5}$$

Since it passes through O(0,0,0)

$$\therefore x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad \textcircled{6}$$

and it passes through the points A, B, C.

$$29 + 10u + 4w = 0 \quad \textcircled{7}$$

$$10 + u - 3v = 0 \quad \textcircled{8}$$

$$61 + 7u - 3v + 8w = 0 \quad \textcircled{9}$$

Subtracting $\textcircled{8}$ from $\textcircled{9}$

we get

$$51 + 6u + 8w = 0 \quad \textcircled{10}$$

Multiplying $\textcircled{7}$ by 2 and subtract $\textcircled{10}$ from it.

we get

$$7 + 14u = 0$$

$$\Rightarrow u = -\frac{1}{2}$$

$$\textcircled{8} \Leftrightarrow 3v = u + 10$$

$$= -\frac{1}{2} + 10$$

$$\Rightarrow v = \frac{19}{6}$$

$$\textcircled{7} \Leftrightarrow w = -6.$$

Putting u, v, w in $\textcircled{6}$, we get

$$x^2 + y^2 + z^2 - x + \frac{19}{3}y - 12z = 0 \quad \textcircled{11}$$

\therefore The eqn of the circle through A, B, C are
(i.e. intersection of sphere OABC and plane ABC)

$$x^2 + y^2 + z^2 - x + \frac{19}{3}y - 12z = 0 \quad \textcircled{12}$$

$$\text{and } 6x - 2y - 3z - 24 = 0 \quad \textcircled{13}$$

The fourth point D(4, -9, 6) lies on circle ABC, if it lies both on the sphere $\textcircled{12}$ and plane $\textcircled{13}$:

Now D(4, -9, 6) lies on sphere $\textcircled{12}$ if

$$16 + 81 + 36 - 4 - 57 - 72 = 0$$

$$\Rightarrow 0 = 0 \text{ which is true}$$

Similarly D(4, -9, 6) lies on plane $\textcircled{13}$ if

$$25 + 18 - 18 - 24 = 0$$

$$\Rightarrow 0 = 0 \text{ which is true.}$$

\therefore D lies on the circle through A, B, C.

\therefore The points A, B, C, D are concyclic.

Intersection of two Spheres:

We now consider two spheres and assume that the given spheres have points in common, i.e., intersect.

Assuming that two given spheres intersect, we show that the locus of the points of intersection of two spheres is a circle.

The co-ordinates of points, if any, common to the two spheres

$$S_1 = x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1 = 0$$

$$S_2 = x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0$$

satisfy both these eqns and, therefore, also satisfy the eqn:

$$S_1 - S_2 = 2x(u_1 - u_2) + 2y(v_1 - v_2) + 2z(w_1 - w_2) + d_1 - d_2 = 0$$

which being a linear eqn in x, y, z , represents a plane.

(11)

Now the points of intersection of the two spheres $S_1 = 0, S_2 = 0$ are the same as those of a one of these spheres and the plane $S_1 - S_2 = 0$ and so it is a circle.

Note: The eqns of two spheres taken together also represents a circle.

→ Show that the sphere

$$S_1 = x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1 = 0 \text{ cuts}$$

$$S_2 = x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0 \text{ in a great circle if}$$

$$2(u_1^2 + v_1^2 + w_1^2) - d_1 =$$

$$2(u_2^2 + v_2^2 + w_2^2) - d_2 =$$

$$\Rightarrow 2(u_1 u_2 + v_1 v_2 + w_1 w_2) =$$

$$\therefore r_2^2 = r_1^2 + d_1^2 + d_2^2$$

where r_2 is the radius

of the second sphere.

Ex. The plane of the circle,

i.e. the plane in which

their circle of intersection

$$\text{if } S_1 - S_2 = 0$$

$$\Rightarrow 2(u_1 - u_2)x + 2(v_1 - v_2)y +$$

$$2(w_1 - w_2)z + d_1 - d_2 = 0 \quad (1)$$

The circle of intersection

is the great circle of the sphere S_2 only when the

above plane passes through

the centre of the sphere S_2 ,

i.e; the plane passes through

$$(-u_2, -v_2, -w_2)$$

$$\begin{aligned} \therefore 2(u_1 - u_2)(-u_2) + 2(v_1 - v_2)(-v_2) \\ + 2(w_1 - w_2)(-w_2) + d_1 - d_2 = 0 \end{aligned}$$

$$\Rightarrow \frac{2(u_1^2 + v_1^2 + w_1^2) - d_2}{2(u_1 u_2 + v_1 v_2 + w_1 w_2) - d_1} =$$

$$\begin{aligned} \Rightarrow 2[(u_1^2 + v_1^2 + w_1^2) - d_2] + d_2 \\ = 2(u_1 u_2 + v_1 v_2 + w_1 w_2) - d_1 \end{aligned}$$

$$\Rightarrow 2r_2^2 + d_1 + d_2 = 2(u_1 u_2 + v_1 v_2 + w_1 w_2)$$

(12)

Spheres through a given circle:

Let the circle be given by the eqns

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

$$P = lx + my + nz - p = 0 \quad (2)$$

then the eqn $S + \lambda P = 0$

$$\text{i.e., } x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d + \lambda(lx + my + nz) = 0 \quad (3)$$

The equation (3) Clearly

represents a sphere

\therefore (i) it is a second degree equation.

(ii) coefficients of x^2, y^2, z^2 are equal.

and (iii) it does not contain the product terms xy, yz, zx .

Also the co-ordinates of the points which satisfy (1) & (2) both, also satisfy (3).

Hence (3) represents a sphere through the curve of intersection of (1) & (2).

i.e., the given circle.

\therefore The set of spheres through the circle $S=0, P=0$ is

$\{S + \lambda P = 0; \lambda \text{ is parameter}\}$

Similarly if the circle is given by the intersection of two spheres.

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$S' = x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

then any sphere through the circle is $S + kS' = 0$

\therefore The set of spheres through the circle $S=0, S'=0$ is

$\{S + kS' = 0; k \text{ is the parameter}\}$

\rightarrow The eqn of the plane of the circle through the two spheres $S=0, S'=0$ is

$$S - S' = 2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0$$

from this we see that the eqn of any sphere through the circle $S=0, S'=0$ is of the form $S + k(S - S') = 0$

where k is parameter.

This form is sometimes more convenient.

Note: The general eqn of the sphere through the circle.

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

or

$$x^2 + y^2 + z^2 + 2x + 2y + 2kz + c = 0$$

where k is the parameter

→ find the equation of the sphere through the circle

$$x^2 + y^2 + z^2 = 9, 2x + 3y + 4z = 5$$

and the point $(1, 2, 3)$.

Sol: Given equations of the circle

$$x^2 + y^2 + z^2 = 9$$

$2x + 3y + 4z = 5$ and the point $P(1, 2, 3)$.

Let the required eqn of sphere through a circle

$$(x^2 + y^2 + z^2 - 9) + \lambda(2x + 3y + 4z - 5) = 0 \quad (3)$$

Since it passes through $P(3, 2, 1)$

$$(1+4+9-9) + \lambda(2+6+12-5) = 0$$

$$5 + \lambda(15) = 0$$

$$\boxed{\lambda = -\frac{1}{3}}$$

(3) \Rightarrow

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow 3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$$

which is the required eqn of the sphere.

2000 → find the eqn of the sphere through the circle

$$x^2 + y^2 + z^2 = 4, x + 2y - z = 2$$

and the point $(1, -1, 1)$.

→ find the eqn to the sphere which passes through the point (α, β, γ) and the circle $x^2 + y^2 = a^2, z = 0$

$$(Ans: (x^2 + y^2 + z^2 - a^2) + (a^2 - \alpha^2 - \beta^2 - \gamma^2)z = 0)$$

$$\text{Hint: } (x^2 + y^2 + z^2 - a^2) + \lambda z = 0$$

→ Show that the eqn of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle with equations

$$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0,$$

$$x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$$

(13)

Sol^b: The given circle is

$$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0 \quad (1)$$

$$x^2 + y^2 + z^2 + 4x + 5y - 8z + 2 = 0 \quad (2)$$

$$(1) - (2) \Rightarrow 3x + 4y - 5z - 3 = 0 \quad (3)$$

Now circle represented by

(1) & (2) is same as the circle given by (1) & (3).

Now any sphere through the circle given by (1) & (3)

is

$$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 + \lambda(3x + 4y - 5z - 3) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (3\lambda - 2)x + (4\lambda - 7)y - (5\lambda + 4)z + (8 - 3\lambda) = 0$$

Its centre $(-u, -v, -w)$

$$= \left(\frac{2-3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{5\lambda+4}{2} \right)$$

Since it lies in the plane $4x - 5y - z = 3$,

$$\boxed{\lambda = 3}$$

$$(1) \Sigma x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$$

→ find the eqn. of the sphere through the circle

$$x^2 + y^2 + z^2 + 2x + 3y + 6 = 0,$$

$$x - 2y + 4z - 9 = 0.$$

and the centre of the sph

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$$

$$(\text{Ans}: x^2 + y^2 + z^2 + 7y - 8z + 24 =$$

→ Show that the two circ

$$x^2 + y^2 + z^2 - y + 2z = 0, x - y + z - 2 =$$

$$x^2 + y^2 + z^2 + x - 8y + z - 5 = 0,$$

$$2x - y + 4z - 1 = 0;$$

lie on the same sphere and find its equation.

Sol^b: The given circles are

$$x^2 + y^2 + z^2 - y + 2z = 0, x - y + z - 2 =$$

$$x^2 + y^2 + z^2 + x - 8y + z - 5 = 0,$$

$$2x - y + 4z - 1 = 0$$

$$\rightarrow$$

Any sphere through (1) is

$$x^2 + y^2 + z^2 - y + 2z + \lambda_1(x - y + z - 2) =$$

and any sphere through

(2) is

$$x^2 + y^2 + z^2 + x - 8y + z - 5 + \lambda_2(2x - y + 4z - 1) =$$

$$\rightarrow$$

The circles (1) & (2) will lie on same sphere

if the eqns (3) & (4)

represent the same sphere

for some values of λ_1, λ_2

Clearing the coefficients
of like terms in (3) & (4),
we get

$$\lambda_1 = 2\lambda_2 + 1, \quad -1 - \lambda_1 = -\lambda_2 - 3$$

(5) (6)

$$2 + \lambda_1 = 4\lambda_2 + 1, \quad -2\lambda_1 = -\lambda_2 - 5$$

(7) (8)

Solving (5) & (6), we get

$$\lambda_1 = 3, \quad \lambda_2 = 1.$$

and these values clearly satisfy
remaining two eqns (7) & (8).

∴ Two circles (1) & (2) lie
on the same sphere whose
eqn is (putting $\lambda_1 = 3, \lambda_2 = 1$)
(from (3) & (4))

we get

$$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$$

→ Show that the two circles
 $2(x^2 + y^2 + z^2) + 8x - 13y + 17z - 17 = 0$
 $2x + y - 3z + 1 = 0;$
 $x^2 + y^2 + z^2 + 3x - 4y + 5z = 0,$
 $x - y + 2z - 4 = 0;$

lie on the same sphere and
find its equation.

→ Prove that the circles
 $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0,$
 $5y + 6z + 1 = 0;$
 $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,$
 $x + 2y - 7z = 0$ lie

on the same sphere and
find its eqn.

→ prove that the plane
 $x + 2y - z = 4$ cuts the
sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$
in a circle of radius
unity and find the eqn
of sphere which has this
circle for one of its great
circle.

Sol:

The given sphere
 $x^2 + y^2 + z^2 - x + z - 2 = 0$ (1)

and the plane

$$x + 2y - z - 4 = 0 \quad (2)$$

Centre of the sphere (1) is

$$C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$$

and its radius



$$CP = \sqrt{\frac{1}{4} + 0 + \frac{1}{4} + 2}$$

$$= \sqrt{5/2}$$

CA = 1^r distance from

$$C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$$
 to the plane (2)

$$= \left| \frac{1}{2} + 2(0) - \left(-\frac{1}{2}\right) - 4 \right|$$

$$\sqrt{1 + 4 + 1}$$

$$= \frac{3}{\sqrt{6}} = \sqrt{3/2}$$

Radius of circle

$$\begin{aligned} AP &= \sqrt{CP^2 - CA^2} \\ &= \sqrt{\frac{5}{2} - \frac{3}{2}} \\ &= \sqrt{1} = 1. \end{aligned}$$

The plane (2) meets the sphere (1) in a circle of radius unity.

Now any sphere through the intersection of (1) & (2) is

$$x^2 + y^2 + z^2 - x + z - 2 + k(x + 2y - z - 4) = 0$$

If the circle of intersection of (1) & (2) is a great circle

of sphere (3), then the

centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ lies on the plane (2).

$$\frac{1-k}{2} + 2\left(-k\right) - \left(\frac{k-1}{2}\right) - 4 = 0$$

$$\Rightarrow k = -1$$

$$(3) \equiv x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0$$

Obtain the eqn of the sphere having the circle

$$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$$

as the great circle.

Find the eqn to the sphere which passes through the circle $x^2 + y^2 = 4, z = 0$ and is

(14)
cut by the plane $x + 2y + 2z =$
in a circle of radius 3.

Sol: The given circle is

$$x^2 + y^2 - 4 = 0, z = 0.$$

The eqns of this circle can be written as

$$x^2 + y^2 + z^2 - 4 = 0, z = 0$$

Any sphere through this

circle is

$$(x^2 + y^2 + z^2 - 4) + \lambda z = 0 \quad (1)$$

$$\text{Its centre} = (0, 0, -\frac{\lambda}{2})$$

$$\text{and radius} = \sqrt{\frac{\lambda^2}{4} + 4} = CP$$

Now the sphere (1)

cut by the plane

$$x + 2y + 2z = 0 \quad (2)$$

in a circle of the radius 3.

Draw $CA \perp$ to the plane (2)

from C.

$\therefore CA = 1^r$ distance from

$$(0, 0, -\frac{\lambda}{2})$$

on the plane (2).

$$CA = \sqrt{0 + 0 + \lambda^2} = \frac{\lambda}{3}$$

Now from the right \triangle

$$\triangle CAP, CA^2 + AP^2 = CP^2$$

$$\Rightarrow \frac{\lambda^2}{9} + 9 = \frac{\lambda^2}{4} + 4$$

$$\Rightarrow \lambda = \pm 6$$

$$(1) \equiv x^2 + y^2 + z^2 + 6z - 4 = 0$$

2008 → A Sphere 'S' has points $(0, 1, 0)$, $(3, -5, 2)$ at opposite ends of a diameter. Find

the equation of the sphere S with the plane $5x - 2y + 4z + 7 = 0$ as a great circle.

2009 → find the equation of the sphere having its centre on the plane

$4x - 5y - z = 3$, and passing through the circle —

$$x^2 + y^2 + z^2 - 12x - 3y + 4z + 8 = 0$$

$$3x + 4y - 5z + 3 = 0$$

(15)

Tangent plane (Line)

Property:

If a plane (line) touches a sphere, then \perp^r distance from the centre of the sphere on the plane (line) must be equal to its radius of the sphere.



Intersection of a Sphere by a straightline:

To find the points where the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ meets the sphere

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$$

Soln: The given line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \quad (say) \quad ①$$

and sphere

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0 \quad ②$$

Any point on the line ① is

$$(lx+x_1, mx+y_1, nx+z_1) \quad ③$$

If it lies on the sphere ②

$$\therefore (lx+x_1)^2 + (mx+y_1)^2 + (nx+z_1)^2 + 2a(lx+x_1) + 2b(mx+y_1) + 2c(nx+z_1) + d = 0 \quad ④$$

which is a quadratic in r hence it gives two values.

These values putting in ③ we get two points of intersection.

Note:

1. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$.

2. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$$

$$x^2 + y^2 + z^2 + 2x(x+x_1) + 2y(y+y_1) + 2z(z+z_1) + d = 0$$

Power of a point w.r.t a Sphere:

Let l, m, n be the actual d.c.s of the line ①,

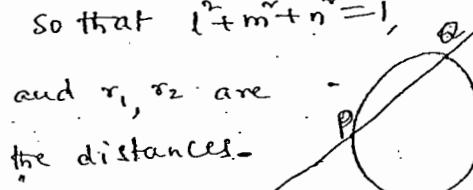
$$\text{so that } l^2 + m^2 + n^2 = 1,$$

and r_1, r_2 are

the distances

of the point $A(x_1, y_1, z_1)$

from the



points of intersections P and Q.

NOW the eqn (4) Reduces to

$$x^2 + y^2 + z^2 + d(x+u) + m(y+v) + n(w+z) + d = 0 \\ \left[\because l^2 + m^2 + n^2 = 1 \right]$$

and $\pi_1 = AP$, $\pi_2 = AQ$ are
its two roots.

$$\therefore AP \cdot AQ = \pi_1 \pi_2 \\ = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d.$$

$$\left[\because ax^2 + bx + c = 0 \text{ has two roots} \right. \\ \left. = \frac{-b}{a} \right]$$

which is independent of the
d.c.'s. l, m, n and is thus
constant.

i.e., if from a fixed point A,
lines are drawn in any
direction to intersect a given
Sphere in P and Q, then
 $AP \cdot AQ$ is constant. This
constant $AP \cdot AQ$ is called
the power of the point A
w.r.t. the sphere.

Note: The power of a point
is obtained by substituting
the co-ordinates of the point
in the eqn of the sphere after
making the R.H.S. zero.

→ Find the co-ordinates

of the points where the
line (i) $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5}$

intersects the sphere

$$x^2 + y^2 + z^2 + 2x - 10y - 23 = 0$$

(ii) $\frac{x+2}{4} = \frac{y+9}{3} = \frac{z-8}{-5}$ meets the
sphere $x^2 + y^2 + z^2 = 49$.

Sol: (i) The given line is

$$-\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = r \quad (say) \quad (1)$$

and the sphere is

$$x^2 + y^2 + z^2 + 2x - 10y - 23 = 0 \quad (2)$$

Any point on the line (1) is

$$(4r-3, 3r-4, -5r+8) \quad (3)$$

It lies on the sphere (2).

∴ (3)

$$(4r-3)^2 + (3r-4)^2 + (-5r+8)^2 + 2(4r-3) \\ - 10(3r-4) - 23 = 0$$

$$\Rightarrow 50r^2 - 150r + 100 = 0$$

$$\Rightarrow r^2 - 3r + 2 = 0$$

$$\Rightarrow (r-1)(r-2) = 0 \Rightarrow r=1, 2$$

putting, $r=1, r=2$ in (3).

∴ The required points of
intersection are $(1, -1, 3) \times (5, 2, -1)$

→ find the locus of the middle point of the system of parallel chords of the sphere

$$(i) x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$(ii) x^2 + y^2 + z^2 = a^2$$

(con)

show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chord.

Sol: (i)

Let all chords of the system be parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. (1)

Where l, m, n are actual d.c.s.

Let (x_1, y_1, z_1) be the mid point of one of the chords.

Then its equations are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

Any point on this line is $(lx+x_1, my+y_1, nz+z_1)$

This lies on the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$\Rightarrow (lx+x_1)^2 + (my+y_1)^2 + (nz+z_1)^2 + 2u(lx+x_1) + 2v(my+y_1) + 2w(nz+z_1) + d = 0$$

$$\Rightarrow r^2(l^2+m^2+n^2) + 2r[l(u+x_1) + v(y+y_1) + w(z+z_1)] + d = 0$$

which is a quadratic in r .

Since (x_1, y_1, z_1) is the mid-point of the chord.

The roots of (2) must be equal and opposite.

i.e., the sum of roots is zero.

i.e., the co-efficient of r

$$\therefore l(u+x_1) + v(y+y_1) + w(z+z_1) = 0$$

Locus of the mid point

$$(x_1, y_1, z_1)$$
 is

$$l(u+x_1) + m(y+y_1) + w(z+z_1) = 0$$

$$\Rightarrow l(x+u) + m(y+v) + w(z+w) = 0$$

which is clearly a plane through the centre (u, v, w) and \perp to the line (1).

$$(ii) Ans: luy + mvz + nwz = 0$$

→ Show that the sum of the squares of the intercepts made by a given sphere on any three mutually \perp straight lines through a fixed point is constant.

Sol: Let the given sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

and the three mutually \perp lines through the fixed point $(0,0,0)$ (say), be

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} ; \quad \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad (2) \quad (3)$$

$$\text{and } \frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3} \quad (4)$$

where l, m, n , etc. are the actual dir's, so that

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1, \quad l_1^2 + l_2^2 + l_3^2 = 1 \\ l_1m_1 + l_2m_2 + l_3m_3 &= 0, \\ l_1n_1 + l_2n_2 + l_3n_3 &= 0 \quad \text{etc.} \end{aligned} \quad (5)$$

To find the intercept on line (2).

Any point on line (2) is (l_1r, m_1r, n_1r) .

If it lies on the sphere (1), then

$$\begin{aligned} r^2(l_1^2 + m_1^2 + n_1^2) + 2r(l_1u + m_1v + n_1w) + d &= 0 \\ \Rightarrow r^2 + 2r(l_1u + m_1v + n_1w) + d &= 0 \quad (\because r \neq 0) \end{aligned}$$

It is a quadratic in r , let the two roots be r_1, r_2 , which are the distances from O of the two points

of intersection say A_1, A_2 of the line and the sphere

∴ If L_1 is the length of intercept on the first line, then

$$L_1 = A_1 A_2 = OA_2 - OA_1 = r_2 - r_1$$

$$\begin{aligned} L_1^2 &= (r_2 - r_1)^2 \\ &= (r_1 + r_2)^2 - 4r_1r_2 \end{aligned}$$

$$= 4(u^2l_1^2 + v^2m_1^2 + w^2n_1^2) - 4d.$$

$$\left(\because r_1 + r_2 = \frac{l_1u + m_1v + n_1w}{l_1} \right)$$

$$\begin{aligned} &= 4(u^2l_1^2 + v^2m_1^2 + w^2n_1^2 + 2ul_1m_1 \\ &\quad + 2vm_1n_1 + 2wn_1l_1) - 4d \end{aligned}$$

Similarly,

$$\begin{aligned} L_2 &= 4(u^2l_2^2 + v^2m_2^2 + w^2n_2^2 + \\ &\quad 2ul_2m_2 + 2vm_2n_2 + 2wn_2l_2) - 4d. \end{aligned}$$

$$L_3^2 = 4(u^2 l_3^2 + v^2 m_3^2 + w^2 n_3^2 + 2uvw l_3 m_3 + 2vuw m_3 n_3 + 2wuv l_3 n_3) - 4d. \quad (16)$$

Adding, the sum of square of the intercepts

$$\begin{aligned}
&= L_1^2 + L_2^2 + L_3^2 \\
&= 4[u^2(l_1^2 + l_2^2 + l_3^2) + v^2(m_1^2 + m_2^2 + m_3^2) \\
&\quad + w^2(n_1^2 + n_2^2 + n_3^2) + 2uvw(l_1 m_1 + l_2 m_2 + l_3 m_3) \\
&\quad + 2vuw(m_1 n_1 + m_2 n_2 + m_3 n_3) + \\
&\quad 2wuv(l_1 n_1 + l_2 n_2 + l_3 n_3)] - 4d \\
&= 4[u^2(1) + v^2(1) + w^2(1) + 2uvw(0) + \\
&\quad 2vuw(0) + 2wuv(0)] - 12d. \\
&= 4(u^2 + v^2 + w^2) - 12d. \quad (\text{from } (5))
\end{aligned}$$

which is free from l_1, m_1, n_1 , etc and
is therefore constant for any set of lines.

Hence the result

(OR)

Let the equation of the given sphere be
 $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and
take the fixed point 'O' as the origin and
any three mutually perpendicular lines through
it as the co-ordinate axes.

The 'x-axis' ($y, z = 0 \Rightarrow x$) meets the sphere
in points given by

$$x^2 + 2ux + d = 0,$$

so that if x_1, x_2 be its roots, the points
of intersection are $(x_1, 0, 0), (x_2, 0, 0)$.

Also we have

$$x_1 + x_2 = -uv ; \quad x_1 x_2 = d.$$

$$\begin{aligned} 1 & (\text{Intercept on } x\text{-axis})^2 = (x_1 - x_2)^2 \\ & = (x_1 + x_2)^2 - 4x_1 x_2 \\ & = 4u^2 - 4d \\ & = 4(u^2 - d). \end{aligned}$$

Similarly,

$$(\text{Intercept on } y\text{-axis})^2 = 4(v^2 - d)$$

$$(\text{Intercept on } z\text{-axis})^2 = 4(w^2 - d).$$

The sum of the squares of the intercepts

$$= 4(u^2 + v^2 + w^2 - 3d).$$

$$= 4(u^2 + v^2 + w^2) - 12d.$$

→ Show that the plane
 $lx+my+nz=p$ will touch
 the sphere

$$x^2+y^2+z^2+2ax+2by+2cz+d=0$$

if

$$(ul+mr+nw+p) = (l+m+n)(a+b+c+d)$$

2004 → find the tangent planes

to the sphere

$$x^2+y^2+z^2-4x+2y-6z+5=0$$

which are parallel to
 the plane $2x+2y-z=0$

Sol: equation of sphere is

$$x^2+y^2+z^2-4x+2y-6z+5=0$$

its centre $(2, -1, 3)$.

$$\text{and radius} = \sqrt{4+1+9-5} = 3$$

Any plane \parallel to the plane

$$2x+2y-z=0$$

$$2x+2y-z=k \quad \text{--- (1)}$$

If it touches the sphere,
 then length of \perp from
 the centre of sphere must
 be equal to the radius
 of the sphere.

$$\therefore \frac{|2(2)+2(-1)-(+3)-k|}{\sqrt{4+4+1}} = 3 \quad (1)$$

$$\Rightarrow |4-2-3-k| = 3\sqrt{9}$$

$$\Rightarrow -1-k = \pm 9$$

$$\Rightarrow k = -1 \pm 9$$

$$\Rightarrow k = -10 \text{ or } 8$$

from (1), we have

$$2x+2y-z = -10 \text{ and}$$

$$2x+2y-z = 8$$

∴ The required tangent
 planes are

$$2x+2y-z+10=0$$

$$2x+2y-z-8=0$$

2004 → find the equations of

tangent planes to the sphere

$$x^2+y^2+z^2-4x+2y-6z+5=0$$

which are parallel to the

$$\text{plane } 2x+y-z=4$$

→ find the equation of the
 tangent plane to the

sphere

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

at the point $(1, 2, 3)$.

→ find the value of 'a' for
 which the plane $x+y+z=a$ touches the sphere
 $a^2+y^2+z^2-2x-2y-2z-6=0$

→ find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z=3$ at the point $(1, 1, -1)$ and passes through the origin.

Sol:- The given sphere is

$$x^2+y^2+z^2-x+3y+2z=3 \quad \text{--- (1)}$$

Equation of tangent plane at $(1, 1, -1)$ to the sphere (1)

is

$$x(1)+y(1)+z(-1)-\frac{1}{2}(x+1)+\frac{3}{2}(y+1) + (z-1)-3=0$$

$$\Rightarrow \frac{1}{2}x+\frac{5}{2}y-3=0$$

$$\Rightarrow x+5y-6=0 \quad \text{--- (2)}$$

The required sphere touching (1) at $(1, 1, -1)$ is the sphere through the point circle of intersection of (1) and the tangent plane at $(1, 1, -1)$ to the sphere i.e., the plane (2).

Now any sphere through the circle of intersection of (1) & (2) is

$$x^2+y^2+z^2+x+3y+2z-3+k(x+5y-6)=0$$

If it passes through the origin $(0, 0, 0)$,

$$\text{then } -3-6k=0$$

$$\Rightarrow k=-\frac{1}{2}$$

∴ (3) is

$$x^2+y^2+z^2-x+3y+2z-3-\frac{1}{2}(x+5y-6)=0$$

$$2(x^2+y^2+z^2)-3x+y+4z=0$$

which is the required equation.

→ Show that the equation of the sphere which touches the sphere

$$4(x^2+y^2+z^2)+10x-25y-2z=0$$

at the point $(1, 2, -2)$ and passes through the point

$$(-4, 0, 0)$$

$$x^2+y^2+z^2+2x-6y+1=0$$

→ find the equations of the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$.

→ find the equations of the sphere which passes through the circle $x^2+y^2+z^2-2x+2y+4z=3$, $2x+y+z=4$ and touches the plane $3x+4y=14$.

→ Show that the plane
 $2x - 2y + 2z + 12 = 0$ touches
 the sphere
 $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$
 and find the point of
 contact.

Sol: The given plane is
 $2x - 2y + 2z + 12 = 0 \quad \text{--- (1)}$
 and the sphere is
 $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$
 The centre of the sphere
 is $(1, 2, -1)$. and its
 radius is

$$\sqrt{1+4+1+3} = 3.$$

Also the fr distance of the
 centre $(1, 2, -1)$ from the
 plane (1)

$$= \frac{2(1) - 2(2) + (-1) + 12}{\sqrt{4+4+1}}$$

$$\begin{aligned} &= \frac{2 - 4 - 1 + 12}{\sqrt{4+4+1}} \\ &= \frac{9}{\sqrt{9}} = 3. \end{aligned}$$

Since fr distance of the
 centre from the plane (1)
 = radius of the
 sphere
 \therefore the plane (1) touches
 the sphere (2).

(18) The point of contact is the
 foot of perpendicular from
 the centre of the sphere
 on the plane.

Now the equations of the
 line through the centre
 $(1, 2, -1)$ and \perp to the
 plane (1) are

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1}.$$

Any point on this line
 is $(2r+1, -2r+2, r-1)$ (3)
 If it lies on the plane (1)

$$2(2r+1) - 2(-2r+2) + r-1 + 12 = 0$$

$$\Rightarrow 9r + 9 = 0$$

$$\Rightarrow r = -1$$

$$(3) \in (2(-1)+1, -2(-1)+2, -1)$$

$$= (1, 4, -2)$$

which is the
 required point of
 contact.

→ find the co-ordinates
 of the points on the sphere
 $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$

the tangent planes at which are parallel to the plane

$$2x - y + 2z = 1$$

$$\text{Ans: } (4, -2, 2), (0, 5, 3)$$

²⁰⁰⁸ → find the equation of the tangent line to the circle

$$x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0,$$

$$3x - 2y + 4z + 3 = 0$$

at the point $(-3, 5, 4)$.

Soln:

Note: The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of circle.

The given sphere is

$$x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0 \quad (1)$$

and the plane of the

$$\text{circle } 3x - 2y + 4z + 3 = 0 \quad (2)$$

Now equation of the tangent plane at $P(-3, 5, 4)$ to the

sphere (1) is

$$\begin{aligned} x(-3) + y(5) + z(4) + \frac{5}{2}(x-3) - \frac{7}{2}(y+5) \\ + \frac{1}{2}(z+4) - 8 = 0 \end{aligned}$$

$$\Rightarrow -x + 3y + 10z - 58 = 0$$

$$\Rightarrow x - 3y - 10z + 58 = 0.$$

→ (3)

The eqns (2) & (3) taken together represent the equation of the tangent line to the circle given by (1) & (2).

To find the d.r.'s of the tangent line.

Omitting the constant term in (2) & (3), the equations are

$$3x - 2y + 12 = 0$$

$$x - 3y - 10z = 0$$

$$\therefore \frac{x}{20+12} = \frac{y}{4+30} = \frac{z}{-9+2}$$

$$\Rightarrow \frac{x}{32} = \frac{y}{34} = \frac{z}{-7}$$

∴ The d.r.'s of the tangent line are $32, 34, -7$.

Also the tangent line passes through the given point

$P(-3, 5, 4)$.

∴ The eqns of the tangent line to the circle at $P(-3, 5, 4)$

$$\text{are } \frac{x+3}{32} = \frac{y-5}{34} = \frac{z-4}{-7}$$

Find the eqn of tangent line to the circle $x^2 + y^2 + z^2 + 3x - 2y - 4z - 22 = 0$, $3x + 4y + 5z - 26 = 0$ at the point $(1, 2, 3)$.

→ find the equations of the two tangent planes to the sphere $x^2 + y^2 + z^2 = 9$ which pass through the line $x+y=6, x-2z=3$.

Sol: The given line is $x+y=6, x-2z=3$.

Any plane through this line is

$$x+y-6+k(x-2z-3)=0 \quad \text{--- (1)}$$

If it touches the sphere $x^2 + y^2 + z^2 = 9 = 0$, then the

distance from the centre $(0,0,0)$ on the plane (1) must be equal to the radius ($=3$) of the sphere.

$$\therefore \frac{-6-3k}{\sqrt{(1+k)^2 + 1 + 4k^2}} = 3$$

$$\Rightarrow -2-k = \sqrt{5k^2+2k+2}$$

$$\Rightarrow (2+k)^2 = 5k^2+2k+2$$

$$\Rightarrow k^2+4k+4 = 5k^2+2k+2$$

$$\Rightarrow 4k^2-2k-2=0$$

$$\Rightarrow 2k^2-k-1=0$$

$$\Rightarrow 2k^2-2k+k-1=0$$

$$\Rightarrow 2k(k-1)+(k-1)=0$$

$$(2k+1)(k-1)=0 \quad (15)$$

$$\Rightarrow k=\frac{1}{2}, 1.$$

Putting these values in (1) the required planes are

$$2x+y-2z=9 \quad \text{and}$$

$$\cancel{x+y+2z=9}$$

→ Obtain the equations of the tangent planes to the sphere.

(i) $x^2 + y^2 + z^2 = 9$ which can be drawn through the line $\frac{x-5}{2} = \frac{y-1}{-2} = \frac{z-1}{1}$

(ii) $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which pass through the line

$$3(16-x) = 3z = 2y+30. \quad \cancel{\underline{\underline{}}}$$

(Hint: If the given line is symmetrical form then convert it into very symmetrical form.)

→ find the equations of spheres that pass through the points $(4, 1, 0)$, $(2, -3, 4)$, $(1, 0, 0)$ and touch the plane $2x + 2y - z = 11$.

Sol: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

Since it passes through $(4, 1, 0)$

$$\therefore 8u + 2v + d + 17 = 0 \quad (2)$$

Since (1) passes through $(2, -3, 4)$ & $(1, 0, 0)$

∴ we have

$$4u - 6v + 8w + d + 29 = 0 \quad (3)$$

$$2u + d + 1 = 0 \quad (4)$$

centre of the sphere (1) is

$$(-u, -v, -w)$$

and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

Since the sphere touches the plane $2x + 2y - z = 11$.

∴ length of the \perp from the centre of the sphere to the plane $2x + 2y - z = 11 = 0$ must be equal to the radius of the sphere

$$\therefore 2(-u) + 4(-v) - (-w) - 11 = \sqrt{u^2 + v^2 + w^2 - d}$$

$$\Rightarrow (-2u - 4v + w - 11)^2 = 9(u^2 + v^2 + w^2 - d)$$

$$\Rightarrow 5u^2 + 8v^2 + 8w^2 - 8uv - 4vw + 4uw - 4uv + 2vw - 9d - 121 = 0 \quad (5)$$

$$\text{from (1)} \quad u = -\frac{1}{2}(d+1) \quad (6)$$

$$\text{from (2)} \quad 2v = -8u - d - 17 \quad (7)$$

From (1) & (2), we get

$$v = \frac{1}{2}(3d - 13) \quad (8)$$

$$\text{from (3)} \quad w = \frac{5d - 33}{4} \quad (9)$$

Substituting these values of u, v, w in (5), we get

$$72d^2 - 747d + 1935 = 0$$

$$8d^2 - 83d + 215 = 0$$

$$\Rightarrow d = 5, \frac{43}{8}$$

Substituting $d = 5$ in (6) & (8), we get

$$u = -3, v = 1, w = -2$$

∴ the required eqn of the sphere is

$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$$

Also, sub. $d = \frac{43}{8}$ in (1) & (9), we get $u = \frac{v}{2} = \frac{w}{2}$

and the required eqn of the sphere is

$$16(x^2 + y^2 + z^2) - 102x + 50y - 492 + 86 = 0$$

→ find the locus of the centre of the sphere of constant radius which passes through a given point and touches the given line.

Soln: Take x -axis to be given line and perpendicular from the given point on the x -axis as the z -axis, then the co-ordinates of the given point on the z -axis are of the form $(0, 0, c)$.

Let the eqn of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

If passes through $(0, 0, c)$.

$$\therefore c^2 + 2wc + d = 0 \quad (2)$$

Given that the radius of the sphere is constant say ' λ '.

$$\therefore u^2 + v^2 + w^2 - d = \lambda^2 \quad (3)$$

The sphere meets the x -axis. ($y = 0, z = 0$).

where

$$u^2 + v^2 + w^2 + d = 0 \quad (4)$$

Since the line i.e., x -axis touches the sphere, then the two values of z given by (4) must be equal i.e., the discriminant of (4) is zero ($b^2 - 4ac = 0$)

$$4u^2 - 4d = 0$$

$$\Rightarrow u^2 = d \quad (5)$$

Eliminating d from

(2), (3) & (4), we get

$$u^2 + 2wc + c^2 = 0 \quad (\text{by adding } (2) \text{ & } (3))$$

$$\text{and } v^2 + w^2 = \lambda^2 \quad [\text{Subtracting } (3) \text{ from } (2)]$$

∴ The locus of the centre $(-u, -v, -w)$ of the sphere (1) is

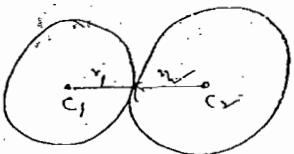
$$x^2 - 2xz + c^2 = 0$$

$$y^2 + z^2 = \lambda^2$$

which represents a cone of intersection of two surfaces.

Touching Spheres:

- (i) Two spheres touch externally if the distance between their centres is equal to the sum of their radii.
- (ii) Two spheres touch internally if the distance between their centres is equal to the difference of their radii.

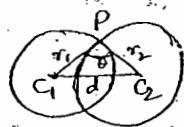


Angle of intersection of two spheres:

The angle of intersection of two spheres is the angle between their tangent planes at a common point of intersection. Since the radii of the spheres to a common point are \perp to the tangent planes at the point, so the angle between the radii of spheres at the common point is equal to the angle between the tangent planes.

i.e., the angle of intersection of the spheres.

To find the angle:



Let C_1, C_2 be the centres of the spheres of radii r_1, r_2 .

Let P be their common point of intersection.

Let $C_1C_2 = d$.

(2)

The angle of intersection i.e., the angle between the tangent planes at P is the angle between the radii of the two spheres.

i.e., $\angle C_1PC_2 = \theta$, then

$$\cos \theta = \frac{(C_1P)^2 + (C_2P)^2 - (C_1C_2)^2}{2(C_1P) \cdot (C_2P)}$$

[Cosine formula]

$$= \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\therefore \theta = \cos^{-1} \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \right)$$

Orthogonal Spheres:

Two spheres are said to be orthogonal if the angle of intersection of two spheres is a right angle.

i.e., If the two spheres cut orthogonally then the square of the distance between the centres of two spheres

= sum of squares of

i.e. $d^2 = r_1^2 + r_2^2$ radii.

Condition of orthogonality
of two spheres

To find the condition that the spheres :

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1^2 = 0$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2^2 = 0$$

to be orthogonal.

Sol": Two given spheres,

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1^2 = 0 \quad (1)$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2^2 = 0 \quad (2)$$

If the spheres cut orthogonally then square of distance between their centres =

sum of the squares of their radii.

Now the centres of the spheres (1) & (2) are

$C_1(u_1, -v_1, -w_1)$ and

$C_2(u_2, -v_2, -w_2)$. and their

are

$$\sqrt{u_1^2 + v_1^2 + w_1^2 - d_1^2}, \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2^2}$$

$$(3) \Rightarrow (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2$$

$$\Rightarrow = (u_1^2 + v_1^2 + w_1^2 - d_1^2) + (u_2^2 + v_2^2 + w_2^2 - d_2^2)$$

$$\Rightarrow 2u_1 u_2 + 2v_1 v_2 + 2w_1 w_2 = d_1^2 + d_2^2$$

which is the required condition.

→ Two spheres of radii r_1 and r_2 cut orthogonally prove that the radius of the common circle is

$$\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

Sol": Let the common circle be $x^2 + y^2 = a^2$; $z=0$. (1)

where 'a' is radius of the circle.

Let the two given spheres through the circle (1) be

$$x^2 + y^2 + z^2 - a^2 + 2\lambda_1 z = 0 \quad (2)$$

$$\text{and } x^2 + y^2 - a^2 + 2\lambda_2 z = 0 \quad (3)$$

Since r_1, r_2 are radii of the given spheres

(2) & (3)

$$\left. \begin{aligned} r_1^2 &= \lambda_1^2 + a^2 \\ r_2^2 &= \lambda_2^2 + a^2 \end{aligned} \right\} \quad (4)$$

Since the spheres (2) & (3)
cut orthogonally.

$$\therefore 2\lambda_1 \lambda_2 = -a^2 - a^2$$

$$\Rightarrow \lambda_1 \lambda_2 = -a^2$$

$$\Rightarrow \lambda_1^2 \lambda_2^2 = a^4$$

$$\therefore (4) \Rightarrow (r_1^2 - a^2)(r_2^2 - a^2) = a^4$$

$$\Rightarrow r_1^2 r_2^2 - a^2(r_1^2 + r_2^2) + a^4$$

$$= a^4.$$

$$\Rightarrow a^2(r_1^2 + r_2^2) = r_1^2 r_2^2$$

$$\Rightarrow a^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}$$

$$\Rightarrow a = \boxed{\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}}$$

Hence, the result.

1995 → Two spheres of radii r_1 and r_2 cut orthogonally.
Prove that the area of
the common circle is

$$\frac{\pi r_1^2 r_2^2}{r_1^2 + r_2^2}$$

Sol: From the above problem
The radius of the
common circle is $a = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$

∴ The area of the common circle
 $= \pi a^2 = \pi \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}$

(2) → find the equation of
the sphere that passes
through the circle
 $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$
 $3x - 4y + 5z - 15 = 0$ and
cuts the sphere

$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$
orthogonally.

Sol: Given equations of
circle

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$$

$$3x - 4y + 5z - 15 = 0 \quad \text{--- (1)}$$

and given sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0 \quad \text{--- (2)}$$

Any sphere through the
circle (1) is

$$x^2 + y^2 + z^2 - 2x + 3y + 6 + \lambda(3x - 4y + 5z - 15) = 0 \quad \text{--- (3)}$$

$$\Rightarrow x^2 + y^2 + z^2 + (-2 + 3\lambda)x + (3 - 4\lambda)y + (-4 + 5\lambda)z + 6 - 15\lambda = 0$$

This will cut the sphere (2)
orthogonally iff

$$2 \cdot \frac{(-2 + 3\lambda)}{2} (1) + 2 \cdot \frac{(3 - 4\lambda)}{2} (2) + 2 \cdot \frac{(-4 + 5\lambda)}{2} = (6 - 15\lambda) + 11$$

[By using $2w_1 w_2 + 2w_3 w_4 + 2w_5 w_6 = d_{tot}$]

$$\Rightarrow \lambda = -\frac{1}{5}$$

Putting this value of λ in ③, we get

$$5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0.$$

→ Find the equation of the sphere that passes through the two points $(0, 3, 0), (-2, -1, -4)$ and cuts orthogonally the two spheres.

$$\begin{aligned} x^2 + y^2 + z^2 + x - 3z - 2 &= 0, \\ 2(x^2 + y^2 + z^2) + x + 3y + 4 &= 0. \end{aligned}$$

2006 → Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and also cuts orthogonally the sphere.

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$$

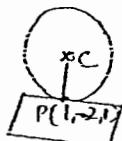
Sol'n: Given plane $3x + 2y - z + 2 = 0$ ①

and the given sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0 \quad ②$$

Since the required sphere touches the plane ① at $P(1, -2, 1)$.

Its centre lies on the normal to the plane at P .



Now the equations of normal to the plane ① through $P(1, -2, 1)$ are

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = \delta \text{ (say)} \quad ③$$

Any point on this line

$$(3\delta + 1, 2\delta - 2, -\delta + 1).$$

Let this point be the centre of the required sphere.

Now the radius of the required sphere.

$$\begin{aligned} CP &= \sqrt{(3\delta + 1 - 1)^2 + (2\delta - 2 + 2)^2 + (-\delta + 1 - 1)^2} \\ &= \sqrt{9\delta^2 + 4\delta^2 + \delta^2} \\ &= \delta\sqrt{14} \end{aligned}$$

Since the required sphere cuts the sphere ② orthogonally,
∴ Square of distance between the centres = Sum of squares of their radii. ④

Now the centre of the sphere

$$C'(2, -3, 0)$$

$$\text{and radius } = \sqrt{4 + 9 - 4}$$

$$\therefore 4 = 3.$$

$$\begin{aligned} (3\delta + 1 - 2)^2 + (2\delta - 2 + 3)^2 + (-\delta + 1 - 0)^2 &= 9 + 14\delta^2 \\ &= 9 + 14 \end{aligned}$$

$$\Rightarrow \delta = -3/2$$

\therefore the centre of the required sphere.

$$C \left(-\frac{7}{2}, -5, \frac{5}{2} \right).$$

and the radius $CP = \frac{-3}{2}\sqrt{14}$

$$\approx \frac{3}{2}\sqrt{14}$$

(numerically)

\therefore The required sphere is

$$(x + \frac{7}{2})^2 + (y + 5)^2 + (z - \frac{5}{2})^2 = \left(\frac{3\sqrt{14}}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$$

→ show that every sphere through the circle

$$x^2 + y^2 - 2ax + z^2 = 0, z=0$$

cuts orthogonally every sphere through the circle $x^2 + z^2 = r^2, y=0$.

Sol'n: Any sphere through the first circle.

$$x^2 + y^2 - 2ax + z^2 = 0, z=0$$

i.e. the circle

$$x^2 + y^2 + z^2 - 2ax + z^2 = 0, z=0$$

$$\text{is } x^2 + y^2 + z^2 - 2ax + z^2 + \lambda_1 z = 0$$

①

Again any sphere through the second circle.

$$x^2 + z^2 = r^2, y=0$$

$$\text{i.e. the circle } x^2 + y^2 + z^2 = r^2,$$

$y=0$ is

$$x^2 + y^2 + z^2 - r^2 + \lambda_2 y = 0 \quad \text{---} ②$$

(1) & (2) will cut orthogonally if $2u_1 u_2 + 2v_1 v_2 + 2w_1 w_2 = d_1 + c$

$$\text{if } 2(-a)(0) + 2(0)\left(\frac{\lambda_2}{2}\right) + 2\left(\frac{\lambda_1}{2}\right)(0) \\ = r^2 - r^2$$

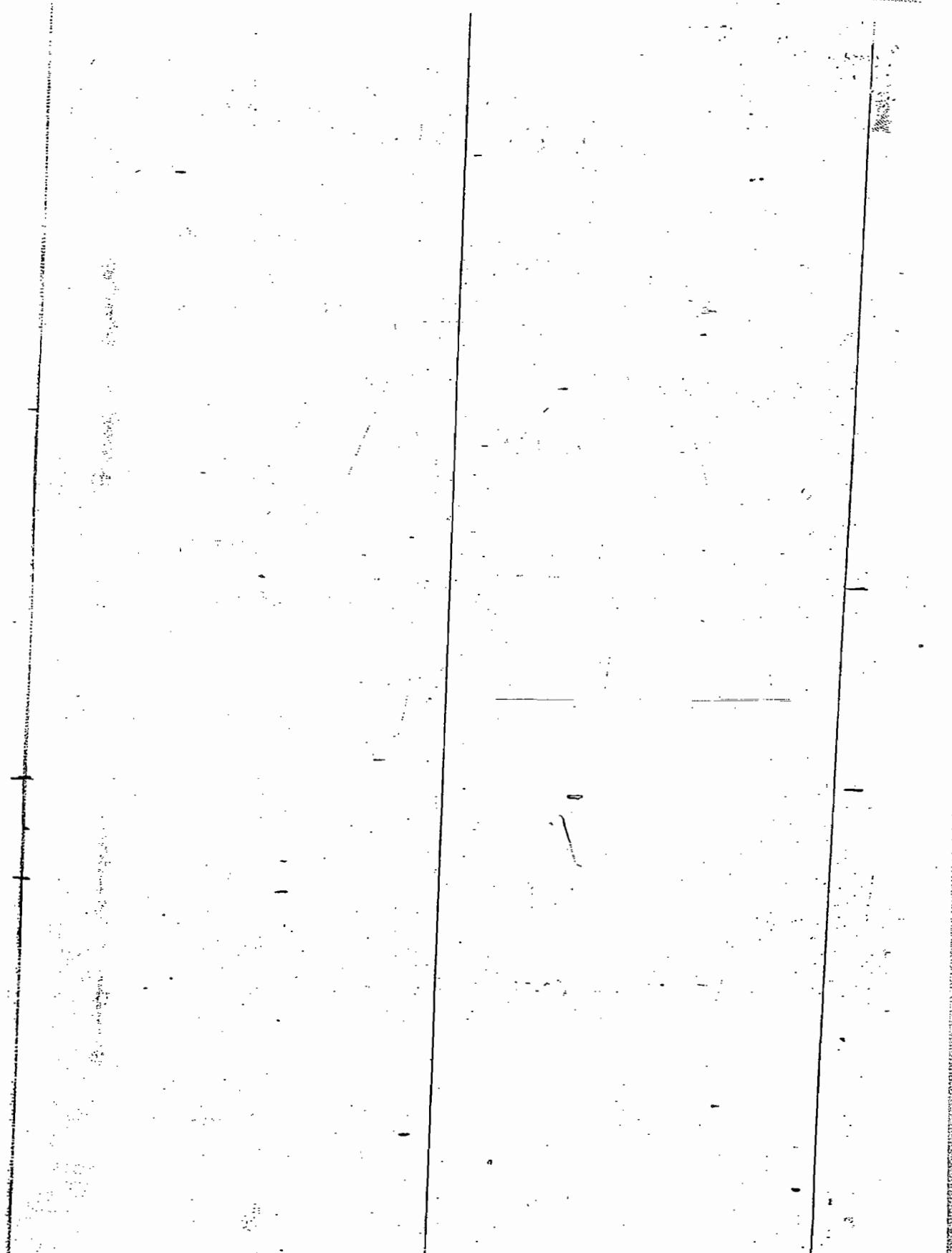
if $0 = 0$

which is true.

Hence the result.

2007 → show that the spheres $x^2 + y^2 + z^2 - x + z - 2 = 0$ and $3x^2 + 3y^2 + 3z^2 - 8x - 10y + 8z + 14 = 0$ cut orthogonally. find the centre and radius of their common circle

(23)



ME

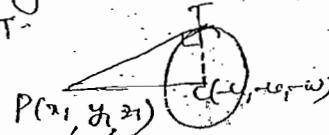
Length of Tangent

To find the length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Let $P(x_1, y_1, z_1)$ be a point outside the sphere
 $S: x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Its centre is $C(-u, -v, -w)$ and radius $r = \sqrt{u^2 + v^2 + w^2 + d}$

Now let the tangent from $P(x_1, y_1, z_1)$ to the sphere meet at T , then radius CT at T must be at right angles to the tangent PT .



$\therefore \triangle PTC$ is right \angle triangle.

$$\begin{aligned} PT^2 &= PC^2 - CT^2 \\ &= (x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2 - (u^2 + v^2 + w^2 - d) \\ &= x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d. \\ &= S_{11} \end{aligned}$$

(25)

→ find the length of the tangent drawn from the point $P(1, 2, 3)$ to the sphere $S: x^2 + y^2 + z^2 + 10x + 10y + 20z + 8 = 0$

Soln: Let $P(1, 2, 3)$ be the given point.

Let the tangent from $P(1, 2, 3)$ to the sphere meet at T .

$$\begin{aligned} PT^2 &= x_1^2 + y_1^2 + z_1^2 - 2ux_1 - 2vy_1 - 2wz_1 + d \\ &= 1 + 4 + 9 - 2\left(\frac{1}{10}\right)(1+2+1) \\ &\quad + 2\left(\frac{1}{2}\right)(3) + \frac{8}{5} \end{aligned}$$

$$PT^2 = \frac{157}{5}$$

$\Rightarrow PT = \sqrt{\frac{157}{5}}$
 which is the required length of the tangent

Radical plane of two spheres:

The locus of a point whose powers w.r.t two spheres are equal i.e., the locus of a point

where the square of the lengths of the tangents to the two spheres are equal, is a plane called the radical plane of the two spheres.

Equation of Radical plane of two spheres:

To find the equation of the radical plane of the

$$\text{spheres } x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + \\ 2w_1 z + d_1 = 0$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0$$

Sol: The given spheres are

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + \\ 2w_1 z + d_1 = 0 \quad (1)$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + \\ 2w_2 z + d_2 = 0 \quad (2)$$

Let $P(x, y, z)$ be any point on the radical plane.

Then the power of P w.r.t sphere (1) = the power of P w.r.t sphere (2)

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z \\ + d_1 = x^2 + y^2 + z^2 + 2u_2 x + \\ 2v_2 y + 2w_2 z + d_2$$

$$\Rightarrow 2(u_1 - u_2) + 2(v_1 - v_2) + 2(w_1 - w_2) \\ + d_1 - d_2 \rightarrow (3)$$

which is the required eqn

Note: The radical plane of two spheres $S_1 = 0, S_2 = 0$ (in both of which the coefficients of x^2, y^2, z^2 are equal to unity) is $S_1 - S_2 = 0$

→ The radical plane of two spheres is perpendicular to the line joining their centres.

Sol: Let the spheres be

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z \\ + d_1 = 0 \quad (1)$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + \\ d_2 = 0 \quad (2)$$



The centres of (1) & (2) are

$$C_1(-u_1, -v_1, -w_1) \text{ & } C_2(-u_2, -v_2, -w_2)$$

∴ d.r.'s of line joining the centres C_1, C_2 are

$$-u_1 - u_2, -v_1 - v_2, -w_1 - w_2$$

Also the d.r.'s of the norma

to the radical plane are proportional to

$$2(u_1 - u_2), 2(v_1 - v_2), 2(w_1 - w_2) \\ \text{or } u_1 - u_2, v_1 - v_2, w_1 - w_2 \quad (\because \text{Radical plane of two spheres is } 2(u_1 - u_2) + 2(v_1 - v_2) + 2(w_1 - w_2) = 0)$$

$$2(u_1 - u_2) + 2(v_1 - v_2) + 2(w_1 - w_2) = 0$$

The normal to the radical plane is parallel to the line C_1C_2 (or) the line C_1C_2 is \perp to the radical plane.

Note: If the spheres intersect then the plane of their common circle is their radical plane.

Radical line of three spheres:

The three radical planes of three spheres intersect in a line.

i.e., If $S_1=0, S_2=0, S_3=0$

be the three spheres then their radical planes

$$S_1-S_2=0, S_2-S_3=0, S_3-S_1=0$$

clearly meet in the

$$\text{line } S_1=S_2=S_3 \Leftrightarrow S_1-S_2=0, S_2-S_3=0$$

This line is called the radical line or radical axis of three spheres.

Radical centre of four spheres:

The four radical lines of four spheres taken three at

(26)
a time in a point which is called the radical centre of the four spheres.

Let the four spheres $S_1=0, S_2=0, S_3=0, S_4=0$.

Then the point common to the three planes

$$S_1=S_2=S_3=S_4=0$$

clearly common to the radical lines, taken three by three, of four spheres.

This point is the intersection of two lines

$$S_1-S_2=0, S_2-S_3=0;$$

$$S_1-S_3=0, S_2-S_4=0$$

This point is called radical centre.

Co-axial Spheres:

A system of spheres any two members of which have the same radical plane is called a co-axial system of spheres.

Equation of co-axial system of spheres determined by two given spheres:

If $s_1=0, s_2=0$ be two spheres then $s_1+\lambda s_2=0$ represents a system of spheres, where λ is a parameter, such that any two members of the system have the same radical plane.

Let $s_1+\lambda_1 s_2=0, s_1+\lambda_2 s_2=0$ by any two members of the system $s_1+\lambda s_2=0$

Making the co-efficients x^2, y^2, z^2 unity in the two equations,

we write them in the form

$$\frac{s_1+\lambda_1 s_2}{1+\lambda_1} = 0, \frac{s_1+\lambda_2 s_2}{1+\lambda_2} = 0$$

The radical plane of these two spheres is

$$\frac{s_1+\lambda_1 s_2}{1+\lambda_1} - \frac{s_1+\lambda_2 s_2}{1+\lambda_2} = 0$$

$$\Rightarrow (s_1+\lambda_1 s_2)(1+\lambda_2) - (s_1+\lambda_2 s_2)(1+\lambda_1) = 0$$

$$\Rightarrow \lambda_2(s_1-s_2) - \lambda_1(s_1-s_2) = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1)(s_1 - s_2) = 0$$

$$\Rightarrow s_1 - s_2 = 0 \quad (\because \lambda_1 \neq \lambda_2)$$

Since the radical plane is

independent of λ_1, λ_2 ,

we see that every two members of the system have the same radical plane.

$s_1 + \lambda s_2 = 0$ represents a system of co-axial spheres determined by two spheres

$$s_1 = 0, s_2 = 0$$

The co-axial system is also given by the eqn.

$$s_1 + \lambda (s_1 - s_2) = 0$$

Equation of co-axial system in the simplest form

To prove that the equation of a co-axial system of spheres can be put in the form $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ where 'u' is the parameter.

Soln: Let any two spheres of the system be

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1 = 0 \quad (1)$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0 \quad (2)$$

Now take the line joining the centres as the x -axis.

\therefore y & z co-ordinates of their centres become zero i.e. $v_1 = 0, w_1 = 0, v_2 = 0, w_2 = 0$ and the equations of the spheres (3) & (4) become

$$x^2 + y^2 + z^2 + 2u_1 x + d_1 = 0 \quad (3)$$

$$x^2 + y^2 + z^2 + 2u_2 x + d_2 = 0 \quad (4)$$

Now the equation of their radical plane is
 $2(u_2 - u_1)x + d_1 - d_2 = 0$

Let this be taken as the yz -plane i.e., $x = 0$.

$$\therefore d_1 - d_2 = 0$$

$$\Rightarrow d_1 = d_2 = d \text{ (say)}$$

\therefore Equations of spheres (3) & (4) become

$$x^2 + y^2 + z^2 + 2u_1 x + d = 0$$

$$x^2 + y^2 + z^2 + 2u_2 x + d = 0$$

The equations of the coaxial system can be put in the form

$$x^2 + y^2 + z^2 + 2u x + d = 0$$

where 'u' is parameter

(76) Limiting points of a co-axial system:

The centres of two spheres of a co-axial system which have zero radius are called the limiting points of the system.

To find the limiting points of a system of coaxial spheres $x^2 + y^2 + z^2 + 2ux + d = 0$

Sol: The given system of coaxial spheres is $x^2 + y^2 + z^2 + 2ux + d = 0$.

Its centre is $(-u, 0, 0)$ and radius $\sqrt{u^2 - d}$

Since for limiting points,

$$\text{radius} = 0$$

$$\therefore \sqrt{u^2 - d} = 0 \Rightarrow u^2 - d = 0$$

$$\Rightarrow u = \pm \sqrt{d}$$

\therefore The centre $(-u, 0, 0)$ becomes $(\pm \sqrt{d}, 0, 0)$ & $(-\sqrt{d}, 0, 0)$, which are the reqd. limiting points.

Find the limiting points of the coaxial system defined by the spheres

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0 \quad (1)$$

$$x^2 + y^2 + z^2 - 6x - 6y - 6z + 6 = 0 \quad (2)$$

Join the equation of any plane of circle through two given spheres is (1)-(2)

$$\therefore 3x + 3y + 6z = 0 \\ \Rightarrow x + y + 2z = 0$$

Now the eqn of co-axial system determined by the given spheres is

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + c_1 = 0$$

$$\quad (\because s_1 + \lambda(s_1 - s_2) = 0)$$

$$\Rightarrow x^2 + y^2 + z^2 + (3+\lambda)x + (\lambda-3)y + 2\lambda z + 6 = 0 \quad (3)$$

where ' λ ' is parameter.

$$\text{Its centre} = \left(\frac{-(3+\lambda)}{2}, \frac{-(\lambda-3)}{2}, -\lambda \right) \quad (4)$$

and radius

$$= \sqrt{\left(\frac{3+\lambda}{2}\right)^2 + \left(\frac{\lambda-3}{2}\right)^2 + \lambda^2 - 6}$$

For limiting point, equating

this to zero, we get

$$\left(\frac{3+\lambda}{2}\right)^2 + \left(\frac{\lambda-3}{2}\right)^2 + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

$$\textcircled{1} = (-1, 2, 1) \text{ & } (1, 1, -1).$$

which are the required limiting points.

→ Show that the spheres which cut two given spheres along great circles all pass through two fixed points.

Sol: Let the two given spheres be

$$x^2 + y^2 + z^2 + 2u_1 x + 2v_1 y + 2w_1 z + c_1 = 0$$

$$x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + c_2 = 0 \quad \textcircled{2}$$

The eqn of another sphere be

$$x^2 + y^2 + z^2 + 2u_3 x + 2v_3 y + 2w_3 z + c_3 = 0 \quad \textcircled{3}$$

where u_i, v_i, w_i, c_i are different for different spheres.

If $\textcircled{3}$ cuts $\textcircled{1}$ in a great circle then the centre $(-u_1, 0, 0)$ of $\textcircled{1}$ must be lie on the radical plane i.e., the plane of circle

$\textcircled{1} \cap \textcircled{3}$ is

$$2(u-u_1) + 2v_1 y + 2w_1 z + c-d = 0$$

$$\therefore 2(u-u_1)(-u_1) + 2v_1(0) + 2w_1(0) + c-d = 0$$

$$\Rightarrow 2uu_1 - 2u_1^2 + c-d = 0 \quad \textcircled{4}$$

Similarly $\textcircled{3}$ cuts $\textcircled{2}$ in a great circle if

$$2uu_2 - 2u_2^2 + c-d = 0 \quad \textcircled{5}$$

$$\textcircled{4} - \textcircled{5} \Leftrightarrow$$

$$2u(u_1 - u_2) - 2(u_1^2 - u_2^2) = 0$$

$$\Rightarrow u - (u_1 + u_2) = 0$$

$$\Rightarrow \boxed{u = u_1 + u_2}$$

$$\textcircled{4} \Leftrightarrow \boxed{c = 2u_1 u_2 + d}$$

u & c are constants dependent on only u_1, u_2, d , the given quantities.

The sphere (3) cuts x-axis where putting $y=0, z=0$ in (3).

$$\textcircled{3} \equiv x^2 + 2ux + c = 0$$

The roots of this equation are constant; depending upon the constants u & c . only,

Every sphere (3) cuts the x-axis at the same two points.

Hence the result.

→ Find the limiting points of the co-axial system of spheres

$$x^2 + y^2 + 2zx + 3oy - 4oz + 29 + \lambda(2x - 3y + 4z) = 0.$$

(Ans: $(2, -3, 4), (-2, 3, -4)$)

→ Prove that the every sphere that passes through the limiting points of a co-axial system cuts every sphere of the system orthogonally

Sol: Let the system of co-axial sphere be

$$x^2 + y^2 + z^2 + 2\lambda x + d = 0 \quad \textcircled{4}$$

The limiting points of system

are $(Jd, 0, 0) \& (-Jd, 0, 0)$.

Let the eqn of the sphere through the limiting points be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0 \quad \textcircled{5}$$

Since it passes through the limiting points $(Jd, 0, 0) \& (-Jd, 0, 0)$,

$$d + 2uJd + c = 0 \quad \textcircled{6}$$

$$d - 2uJd + c = 0 \quad \textcircled{7}$$

Solving these we get
 $u = 0 \quad \textcircled{8}$ & $c = -d$.

$$\textcircled{5} \equiv x^2 + y^2 + z^2 + 2vy + 2wz - d = 0 \quad \textcircled{9}$$

Since (3) & (9) cut orthogonally

if $2u_1 u_2 + 2v_1 v_2 + 2w_1 w_2 = d_1 + d_2$

$$\Rightarrow 0(2\lambda) + 2v(0) + 2w(0) = d - d = 0 \text{ which is true}$$

Hence the result.

→ Show that the eqn $x^2 + y^2 + z^2 + 2\mu x + 2\nu y + 2\omega z = d = 0$

where $\mu \& \nu$ are parameters.

Represents a system of spheres passing through limiting points of the system

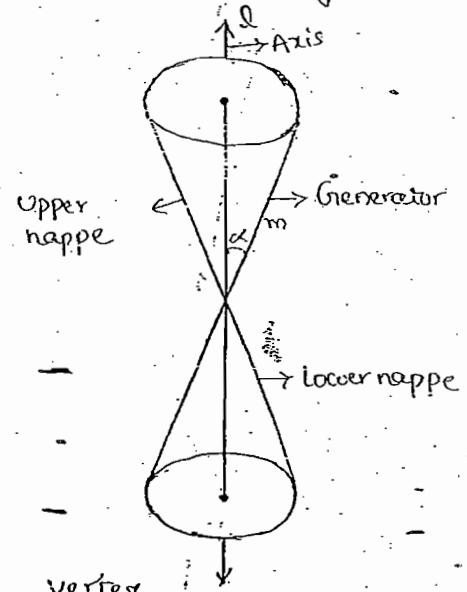
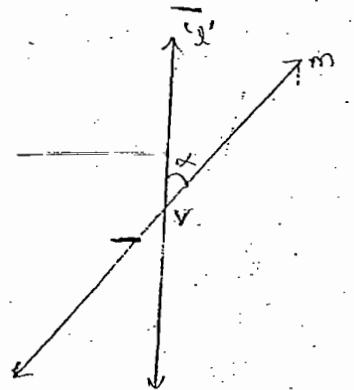
$$x^2 + y^2 + z^2 + 2\lambda x + d = 0$$

Cutting every member of the system at right angles.

Cone

Def Let 'l' be a fixed vertical line and 'm' be another line intersecting it at a fixed point V and inclined to it at an angle α .

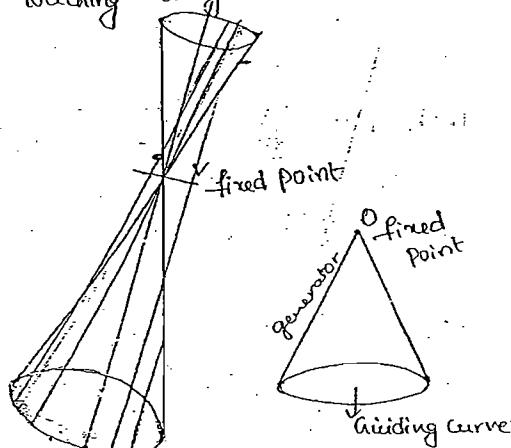
Suppose we rotate the line 'm' around the line 'l' in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone here in after referred as Cone and extending indefinitely far in both directions.



The point V is called vertex. The line 'l' is the axis of the cone. The rotating line 'm' is called a generator of the cone. The vertex separates the cone into two parts called nappes.

Definition

A cone is a surface generated by a straight line which passes through a fixed point and satisfying one more condition i.e. intersecting a given curve (or) touching a given surface.



A fixed point is called the vertex and the given curve (or) surface is called the guiding curve [or guiding surface] of the cone. The straight line is known as the generator of the cone.

→ A cone whose equation is of second degree is known as quadratic cone (or) quadratic cone.

* Equation of the Cone with Vertex at the Origin:

To show that the equation of

* Cone

Set - V

① A cone whose vertex is the origin homogeneous in x, y, z .

Sol'n: Let the equation of the cone with vertex as the origin be

$$f(x, y, z) = 0 \quad \text{--- (1)}$$

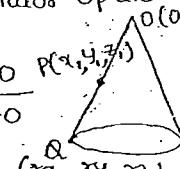
Let $P(x_1, y_1, z_1)$ be any point on the cone.

$$\therefore f(x_1, y_1, z_1) = 0 \quad \text{--- (2)}$$

Equations of the generator OP are

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0} \quad P(x_1, y_1, z_1)$$

$$\Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} \quad \text{--- (3)}$$



Any point on OP line is $Q(x_1, y_1, z_1)$. Since the generator completely lies on the cone, then the point Q lies on cone for all values of x, y, z . $\therefore f(x_1, y_1, z_1) = 0$ for all values of x, y, z .

from (2) & (4) we have

$f(x_1, y_1, z_1)$ is homogeneous equation in x_1, y_1, z_1 .

$\therefore f(x, y, z) = 0$ is homogeneous in x, y, z .

Conversely, any homogeneous equation in x, y, z represents a cone whose vertex is the origin.

Sol'n: Let the homogeneous equation be

$$f(x, y, z) = 0 \quad \text{--- (5)}$$

If $P(x_1, y_1, z_1)$ is any point on the above surface then $f(x_1, y_1, z_1) = 0$

Since the equation (5) is homogeneous, we have

$$f(rx_1, ry_1, rz_1) = 0 \quad \text{--- (7)}$$

for all values of r .

But the point $Q(rx_1, ry_1, rz_1)$ is any point on the line op.

\therefore Every point of the line op. lies on the surface (6).

The surface is generated by the line through '0'.

it represents a cone with vertex at the origin.

Note:— The second degree homogeneous

$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2tby = 0$
represents a cone with vertex at the origin.

Note:— Method to make both equations homogeneous, when none of the two equations is a linear in x, y, z —

(i) Make both equations homogeneous in x, y, z and t , by introducing proper power of it, where t stands for 1.

(ii) Eliminate t from the two equations so obtained.

→ find the equation to the cone with vertex at the origin and which pass through the curves given by the equations.

$$(i) x^2 + y^2 + z^2 - x - 1 = 0$$

$$x^2 + y^2 + z^2 - y - 2 = 0$$

$$(ii) x^2 + y^2 + z^2 + x - 2y + 3z = 4,$$

$$x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$$

Sol: (i) The given equations

$$x^2 + y^2 + z^2 - x - 1 = 0 \& x^2 + y^2 + z^2 + y - 2 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - x - t^2 = 0 \& x^2 + y^2 + z^2 + yt - t^2 = 0$$

$$\text{--- (1)} \quad \text{--- (2)}$$

To eliminate t from (1) & (2)

$$\text{Now } (2) - (1) \equiv -ty - tx + t^2 = 0$$

$$\Rightarrow t(t - x - y) = 0$$

$$\Rightarrow t - x - y = 0 \quad (\because t \neq 0)$$

$$\Rightarrow t = x + y$$

$$\therefore (1) \equiv x^2 + y^2 + z^2 - x(x+y) - (x+y)^2 = 0$$

$$\Rightarrow x^2 + 2xy - z^2 = 0$$

which is the required equation of the cone.

→ find the equations to the cone with vertex at the origin which pass through the curve.

$$ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = p$$

Sol: The given equations are

$$ax^2 + by^2 + cz^2 = 1 \quad \text{--- (1)}$$

$$\text{and } lx + my + nz = p \quad \text{--- (2)}$$

$$(2) \equiv \frac{lx + my + nz}{p} = 1$$

$$① \equiv ax^2 + by^2 + cz^2 = 1^2$$

$$② \equiv ax^2 + by^2 + cz^2 = \left(\frac{lx+my+nz}{P}\right)^2$$

$\Rightarrow P^2(ax^2 + by^2 + cz^2) = (lx+my+nz)^2$
which is the required equation
of the cone.

To find the equation to the cone
at the origin which passes the
curve $ax^2 + by^2 = 2z$, $lx+my+nz=P$

$$\text{Sol'n} - ③ \equiv \frac{lx+my+nz}{P} = 1$$

$$④ \equiv ax^2 + by^2 = 2z$$

$$\Rightarrow ax^2 + by^2 = 2z \left[\frac{lx+my+nz}{P} \right]$$

$\Rightarrow P(ax^2 + by^2) = 2z(lx+my+nz)$
which is the required equation
of the cone.

* Equation of a cone with
a given vertex and a given
base conic:

To find the equation to the
cone whose vertex is the point
(α, β, γ) and base the conic

$$f(x, y) = ax^2 + by^2 + 2hxy + 2fy + 2gx + c = 0, z=0$$

Sol'n - The equations of the conic
are $ax^2 + by^2 + 2hxy + 2fy + 2gx + c = 0$,

$$z=0 \quad (1)$$

The equations of any line through
(α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad (2)$$

This line meets the plane $z=0$.

$$\therefore \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{0-\gamma}{n}$$

$$\Rightarrow \frac{x-\alpha}{l} = \frac{-\gamma}{n} \quad / \quad \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$\Rightarrow x = \alpha - \frac{l\gamma}{n}, y = \beta - \frac{m\gamma}{n}$$

If this point $(x, y, 0)$ lies on the given
conic then

$$a\left[\alpha - \frac{l\gamma}{n}\right]^2 + b\left[\beta - \frac{m\gamma}{n}\right]^2 + 2h\left[\alpha - \frac{l\gamma}{n}\right]$$

$$- \left[\beta - \frac{m\gamma}{n}\right] + 2g\left[\alpha - \frac{l\gamma}{n}\right] + c = 0 \quad (3)$$

This is the condition for line (2) to
intersect the conic (1).

Now eliminating l, m, n from (2) & (3)

Now putting the values of l, m, n
from (2) in (3) we have

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\right)^2 + b\left(\beta - \frac{y-\beta}{z-\gamma}\right)^2$$

$$+ 2h\left[\alpha - \frac{x-\alpha}{z-\gamma}, \gamma\right]\left(\beta - \frac{y-\beta}{z-\gamma}\right) +$$

$$2f\left(\beta - \frac{y-\beta}{z-\gamma}\right) + 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}\right) + c = 0$$

$$\Rightarrow a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) + 2f(\beta z - \gamma y)(z - \gamma) + 2g(\alpha z - \gamma x)(z - \gamma) + c(z - \gamma)^2 = 0 \quad (4)$$

which is the required equation of
the cone.

Note! - (i) The equation of the cone is satisfied by the coordinates of the vertex (α, β, γ) . i.e. putting α, β, γ for x, y, z in (4) we have

$$a(\alpha^2 - \gamma^2) + b(\beta\gamma - \gamma\beta) + 2h(\alpha\gamma - \gamma\alpha) \\ + 2l(\beta\gamma - \gamma\beta)(\gamma - \gamma) + 2g(\alpha\gamma - \gamma\alpha)(\gamma - \gamma) \\ + c(\gamma - \gamma)^2 = 0$$

$\Rightarrow 0 = 0$ which is true.

ii. The equation of the cone (4) also satisfied by the equation of the base cone.

Putting $z=0$ in (4) we have

$$a\gamma^2 z^2 + b\gamma^2 y^2 + 2h\gamma^2 yz + 2\gamma^2 y^2 z + 2g\gamma^2 z^2 \\ + cy^2 = 0$$

through dividing with γ^2

$$az^2 + by^2 + 2hzy + 2\gamma^2 y + 2g^2 z + c = 0$$

→ find the equation of the cone whose vertex is (α, β, γ) and whose base is

$$(i) az^2 + by^2 = 1, z=0.$$

$$(ii) \gamma^2 = 4az^2, z=0. \quad \text{vertex } (1, 0, 0)$$

Soln (i) The given base-conic is

$$az^2 + by^2 = 1, z=0. \quad (1)$$

Now equation of any line through $w(\alpha, \beta, \gamma)$

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad (2)$$

it meets the plane $z=0$ where

(Ex-209) the section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1), z=0$ by the plane $2x=0$ is a rectangular

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

$$\Rightarrow \frac{x+\alpha}{l} = \frac{y+\beta}{m} = \frac{z+\gamma}{n}, \quad y-\beta = -\frac{m}{n}\gamma$$

This point lies on the conic (1)

$$a\left(\alpha - \frac{l}{n}\gamma\right)^2 + b\left(\beta - \frac{m}{n}\gamma\right)^2 = 1 \quad (3)$$

Now eliminating l, m, n from (2)&(3) we have -

$$a\left[\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right]^2 + b\left[\beta - \frac{y-\beta}{z-\gamma}\gamma\right]^2 = 1$$

$$\Rightarrow a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 = (z-\gamma)^2$$

∴ which is the required equation of the cone.

How → obtain the locus of the lines which pass through a point (α, β, γ) and through points of the conic.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z=0$$

$$\text{Ans: } \left(\frac{xz - \gamma x}{a}\right)^2 + \left(\frac{\beta z - \gamma y}{b}\right)^2 = (z-\gamma)^2$$

→ find the equation of the cone whose vertex is the point $(1, 1, 0)$ and whose guiding curve is $y=0$,

$$x^2 + z^2 = 4$$

$$\text{Ans: } x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$$

hyperbola. Show that the locus of P is $\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1$

Soln:- Let the vertex P be (α, β, γ)

and given geodetic curve the

$$\text{ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z=0 \quad \text{--- (1)}$$

Now the equation of any line through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (2)}$$

it meets the plane $z=0$.

$$\therefore \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{\alpha-\gamma}{n}$$

$$\Rightarrow x-\alpha = -\frac{l}{n}\gamma, \quad y-\beta = -\frac{m}{n}\gamma$$

$$\Rightarrow x = \alpha - \frac{l\gamma}{n}, \quad y = \beta - \frac{m\gamma}{n}, \quad z=0$$

this point lies on the ellipse (1)

$$\therefore \frac{1}{a^2} \left[\alpha - \frac{l\gamma}{n} \right]^2 + \frac{1}{b^2} \left[\beta - \frac{m\gamma}{n} \right]^2 = 1 \quad \text{--- (3)}$$

Now eliminating l, m, n from (2) & (3) we have.

$$\frac{1}{a^2} \left(\alpha - \frac{x-\alpha}{z-\gamma} \cdot \gamma \right)^2 + \frac{1}{b^2} \left(\beta - \frac{y-\beta}{z-\gamma} \cdot \gamma \right)^2 = 1$$

$$\Rightarrow \frac{1}{a^2} (\alpha z - \gamma x)^2 + \frac{1}{b^2} (\beta z - \gamma y)^2 = (z-\gamma)^2 \quad \text{--- (4)}$$

which is required equation of cone.

This meets the plane $x=0$

$$\therefore (4) \equiv \frac{1}{a^2} (\alpha z - 0)^2 + \frac{1}{b^2} (\beta z - \gamma y)^2 = (z-\gamma)^2$$

$$\Rightarrow \frac{\alpha^2 z^2}{a^2} + \frac{\beta^2 z^2 + \gamma^2 y^2 - 2\beta\gamma yz}{b^2} = z^2 - \gamma^2$$

this will be a rectangular hyperbola in yz -plane.

if Coefficient of y^2 + Coefficient of z^2

$$\text{if } \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{r^2}{b^2} - 1 = 0$$

\therefore the locus of $P(\alpha, \beta, \gamma)$ is

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1$$

show that the equation of the cone whose vertex is the origin and whose base is the circle through the three points $(a, 0, 0), (0, b, 0), (0, 0, c)$ is $\sum a(b^2+c^2)yz=0$

(or)

the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz \left(\frac{b}{c} + \frac{c}{b} \right) + zx \left(\frac{c}{a} + \frac{a}{c} \right) + xy \left(\frac{a}{b} + \frac{b}{a} \right) - = 0$$

Soln:- The equation of the plane is

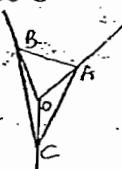
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{--- (1)}$$

since it meets the coordinate axes in A, B, C, the

coordinates of A, B, C are

$$(a, 0, 0), (0, b, 0), (0, 0, c)$$

Now the circle through A, B, C is the intersection of plane through A, B, C i.e.



i.e. Plane ① and any sphere through the points A, B, C say the sphere OABC.

Now the Sphere OABC through the points O(0,0,0), A(a,0,0), B(0,b,0) C(0,0,c) is $x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$

∴ The guiding curve is the circle given by ① & ②

$$\text{i.e. } x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$② \equiv x^2+y^2+z^2-(\alpha x+\beta y+\gamma z)(1)=0$$

$$\Rightarrow x^2+y^2+z^2-(\alpha x+\beta y+\gamma z)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$$

$$(1)=0.$$

$$\Rightarrow x^2+y^2+z^2-x^2-\frac{b}{a}xy-\frac{c}{a}zx-\frac{a}{b}yz$$

$$-y^2-\frac{c}{b}yz-\frac{a}{c}zx-\frac{b}{c}yz-z^2=0$$

$$\Rightarrow -y+\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)$$

$$xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+$$

$$xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow \sum a(b^2+c^2)yz=0. \quad \text{---} ③$$

which is required equation of the cone.

Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve the circle

$$x^2+y^2+z^2=4, x+y+z=1.$$

Sol'n. — Any generator through (1, 2, 3) is

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{x+y+z-6}{l+m+n}$$

if it meets the plane $x+y+z=1$ then from ①, we have

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{1-6}{l+m+n}$$

$$\Rightarrow \frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{-5}{l+m+n}$$

$$\Rightarrow x=1-\left[\frac{5l}{l+m+n}\right], y=2-\left[\frac{5m}{l+m+n}\right]$$

$$\text{and } z=3-\left[\frac{5n}{l+m+n}\right].$$

i.e. the generator ① meets the plane $x+y+z=1$ in the point

$$\left[\frac{m+n-4l}{l+m+n}, \frac{2l-3m-2n}{l+m+n}, \frac{3l+3m-2n}{l+m+n}\right]$$

If this point lies on $x^2+y^2+z^2=4$ we get

$$(m+n-4l)^2 + (2l-3m-2n)^2 + (3l+3m-2n)^2$$

$$= 4(l+m+n)^2 \quad \text{---} ④$$

Eliminating l, m, n between ③ & ④ we get:

$$\begin{aligned} & [(y-2)+(z-3)-4(x-1)]^2 + \\ & [2(y+1)-3(z-2)+2(x-3)]^2 + \\ & [3(x+1)+3(y-2)-2(z-3)]^2 \end{aligned}$$

$$= 4[(x-1)+(y-2)+(z-3)]^2$$

$$\Rightarrow (y+2-4x-1)^2 + (2z-3y+2z-2)^2$$

$$+ (3x+3y-2z-3)^2 = 4(x+y+z-6)^2$$

$$\Rightarrow 5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y$$

+10z -

which is the required equation.

To show that the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$
where $l^2 + 3m^2 - 3n^2 = 0$, is a generator
of the cone $x^2 + 3y^2 - 3z^2 = 0$.

Sol: The given line is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{--- (1)}$$

$$\text{where } l^2 + 3m^2 - 3n^2 = 0 \quad \text{--- (2)}$$

We can eliminate l, m, n from (1) & (2)

$$l = x, m = y, n = z$$

$$(3) \equiv x^2 + 3y^2 - 3z^2 = 0 \quad \text{--- (3)}$$

which is the required cone.

\therefore (1) lies on the cone (3).

To show that the lines through
the point (α, β, γ) whose d.c's
satisfies $a\alpha^2 + b\beta^2 + c\gamma^2 = 0$
generate the cone.

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

Sol: Any line through the point
 (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (1)}$$

$$\text{where } al^2 + bm^2 + cn^2 = 0 \quad \text{--- (2)}$$

Eliminate l, m, n from (1) & (2)
we have

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

which is the required cone.

Hence the result.

(4)

To show that the equation of the
cone whose vertex is at the origin
and the d.c's of whose generator
satisfy the relation $3l^2 - 4m^2 + 5n^2 = 0$
is $3x^2 - 4y^2 + 5z^2 = 0$.

* Enveloping Cone of a Sphere:

Definition:- The locus of the tangent from a given point to sphere is a cone called the enveloping cone or tangent cone from the point to the sphere.

(or)

The cone formed by the tangent lines to a surface, drawn from a given point is called the enveloping cone of the surface with given point as its vertex.

→ To find Equation of the enveloping cone from the point

(x_1, y_1, z_1) to the Sphere

$$x^2 + y^2 + z^2 = a^2.$$

Sol'n :- The given equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ————— ①

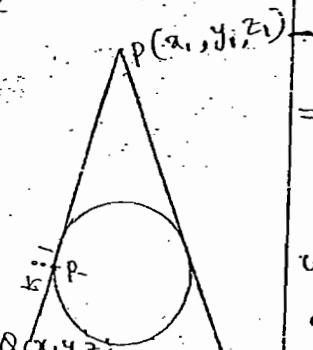
Let $P(x_1, y_1, z_1)$ be

any given point.

Let $Q(x, y, z)$ be

any point on a tangent from P to the given sphere.

Let PQ be divided by the point of contact R in the ratio $K:1$.



∴ The coordinates of R are

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1} \right)$$

Since this point R lies in the sphere ①

$$\left(\frac{kx+x_1}{k+1} \right)^2 + \left(\frac{ky+y_1}{k+1} \right)^2 + \left(\frac{kz+z_1}{k+1} \right)^2 = a^2$$

$$\Rightarrow k^2x^2 + x_1^2 + 2kx_1x + k^2y^2 + y_1^2 + 2ky_1y + k^2z^2 + z_1^2 + 2kz_1z = a^2(k^2 + 2k + 1)$$

$$\Rightarrow k^2(x^2 + y^2 + z^2 - a^2) + 2k(x_1x + y_1y + z_1z - a^2) + (x_1^2 + y_1^2 + z_1^2 - a^2) = 0 \quad \text{--- ②}$$

which is quadratic in k .

Since the line PQ touches the sphere, the two values of k must be equal

∴ Discriminant of ② = 0

$$\text{i.e. } b^2 - 4ac = 0$$

$$\therefore 4[x_1x + y_1y + z_1z - a^2] - 4[x^2 + y^2 + z^2 - a^2]$$

$$[x_1^2 + y_1^2 + z_1^2 - a^2] = 0$$

$$\Rightarrow [x^2 + y^2 + z^2 - a^2][x_1^2 + y_1^2 + z_1^2 - a^2]$$

$$= [x_1^2 + y_1^2 + z_1^2 - a^2]^2$$

which is the required equation of the enveloping cone.

Note:- If $s = x^2 + y^2 + z^2 - a^2$

so that $s=0$ is the equation

of the sphere then

$S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$ i.e. S_1 is the result of substituting the point (x_1, y_1, z_1) in S .

and $T = 2x_1 + yy_1 + zz_1 - a^2$ the expression of the tangent plane at (x_1, y_1, z_1) to the sphere. Then the enveloping cone is $SS_1 = T^2$.

→ find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with vertex at $(1, 1, 1)$.

Soln : The equation of the sphere is $x^2 + y^2 + z^2 - 2x + 4z = 1$ —①

and the given vertex is $P(1, 1, 1)$

Let $S = x^2 + y^2 + z^2 - 2x + 4z - 1$

and $x_1 = 1, y_1 = 1, z_1 = 1$

$$\therefore S_1 = (1)^2 + (1)^2 + (1)^2 - 2(1) + 4(1) \\ = 4$$

and $T = 2x_1 + yy_1 + zz_1 - (x+x_1)$

$$+ 2(z+z_1) - 1$$

$$= 2(1) + y(1) + z(1) - (1+1) \\ + 2(z+1) - 1$$

$$= x + y + z - 1 + 2z + 2 - 1$$

$$= y + 3z$$

Equation of the enveloping cone is $SS_1 = T^2$.

$$\Rightarrow (x^2 + y^2 + z^2 - 2x + 4z - 1)(4) = (y + 3z)$$

$$\Rightarrow 4x^2 + 4y^2 + 4z^2 - 8x + 16z - 4 = y^2 + 9z^2 + 1$$

$$\Rightarrow 4x^2 + 3y^2 - 5z^2 - 8x + 16z - 6yz - 4 = 0$$

→ Show that the plane $z=0$ cuts the enveloping cone of the sphere

$x^2 + y^2 + z^2 = 11$ which has its vertex at $(2, 4, 1)$ in a rectangular hyperbola

Given : The given equation of the sphere is $x^2 + y^2 + z^2 = 11$ —①

and given vertex $(2, 4, 1)$:

Let $S = x^2 + y^2 + z^2 - 11$ and $x_1 = 2$,

$$y_1 = 4, z_1 = 1.$$

$$\therefore S_1 = 4 + 16 + 1 - 11 = 10$$

$$T = 2x_1 + yy_1 + zz_1 - 11$$

$$= 2x + 4y + z - 11$$

∴ the equation of the enveloping cone is $SS_1 = T^2$

$$\Rightarrow (x^2 + y^2 + z^2 - 11)(10) = (2x + 4y + z - 11)^2$$

This meets the plane $z=0$.

$$\therefore (x^2 + y^2 + 0 - 11)(10) = (2x + 4y + 0 - 11)^2$$

$$\Rightarrow (x^2 + y^2 - 11)10 - (2x + 4y - 11)^2 = 0.$$

This represents a rectangular hyperbola in the xy -plane.

If coefficient of x^2 + coefficient of $y^2 = 0$

$$\therefore (10-4) + (10-16) = 0$$

$$\Rightarrow 6 - 6 = 0$$

$$\Rightarrow 0 = 0 \text{ which is true.}$$

Hence the result.

* Quadratic Cone through the axes:

→ Show that the general equation of a cone of second degree which pass through the axes is
 $fyz + gzx + hay = 0$.

where f, g, h are parameters.

Sol: - The general equation of the cone with its vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hay = 0 \quad (1)$$

Since it passes through x-axis

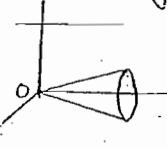
∴ the d.c's of x-axis are 1, 0, 0 must satisfy (1)

$$\therefore a(1)^2 + b(0)^2 + c(0)^2 - 2f(0) + 2g(0) + 2h(0) = 0$$

$$\Rightarrow a = 0$$

Similarly the cone passes through the axes of y & z.

We have $b=0, c=0$.



$$\therefore (1) \equiv a(z^2) + 0(y^2) + 0(z^2) + 2fyz + 2gzx + 2hay = 0$$

$$\Rightarrow 2fyz + 2gzx + 2hay = 0$$

which is the required condition.

→ Show that a cone can be found so as to contain any two given sets of three mutually perpendicular concurrent lines as generators.
 (or)

Show that a cone of second degree can be found to pass through any

two sets of rectangular axes through the same origin.

Sol: - Take the three lines of one set as coordinate axes (i.e. ox, oy, oz).

Let the lines ox', oy', oz'

of the second set be $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$

$$\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}, \quad \frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3}$$

Now, general equation of the cone through axes (i.e. ox, oy, oz)

$$is fyz + gzx + hay = 0 \quad (1)$$

If it passes through ox' & oy'

then the d.c's l_1, m_1, n_1 & l_2, m_2, n_2 of ox', oy', oz' satisfy (1)

$$\therefore fm_1n_1 + gn_1l_1 + nl_1m_1 = 0 \quad (2)$$

$$fm_2n_2 + gn_2l_2 + nl_2m_2 = 0 \quad (3)$$

Adding (2) & (3) we have

$$f(m_1n_1 + m_2n_2) + g(n_1l_1 + n_2l_2) + h(l_1m_1 + l_2m_2) = 0. \quad (4)$$

But l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the d.c's of three mutually \perp lines.

$$\therefore m_1n_1 + m_2n_2 + m_3n_3 = 0 \Rightarrow m_1n_1 + m_2n_2 = -m_3n_3$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \Rightarrow n_1l_1 + n_2l_2 = -n_3l_3$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0 \Rightarrow l_1m_1 + l_2m_2 = -l_3m_3$$

Putting these values in (4) we have

$$\therefore fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$$

$$\Rightarrow fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$$

i.e. ① is satisfied by the d.c's l_3, m_3, n_3 of Oz.

\therefore The cone passes through the Oz! i.e. the cone passes through OX, OY, Oz and OX', OY', Oz' i.e. two sets of rectangular axes.

7 find the equation of the cone which contains the three coordinate axes and the lines through the origin having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2

Sol'n - The equation of any cone through the three coordinate axes is $fgz + gzx + hxy = 0$ — ①

Since it passes through lines with d.c's l_1, m_1, n_1 and l_2, m_2, n_2 and the d.c's of the generators satisfy the equation of the cone.

$$\therefore fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \quad \text{--- ②}$$

$$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0 \quad \text{--- ③}$$

\therefore Eliminating f, g, h from ①, ②, ③

we have

$$\begin{vmatrix} yz & zx & xy \\ m_1n_1 & n_1l_1 & l_1m_1 \\ m_2n_2 & n_2l_2 & l_2m_2 \end{vmatrix} = 0$$

$$\Rightarrow yz[n_1l_1m_2 - m_2l_1n_1] - zx[m_1n_1l_2 - m_2n_1l_1] + xy[m_1n_2l_2 - m_2n_2l_1] = 0$$

$$\Rightarrow l_1l_2yz[n_1m_2 - n_2m_1] + m_1m_2xz[n_2l_1 - n_1l_2] + n_1n_2xy[m_1l_2 - m_2l_1]$$

which is required equation.

\rightarrow find the equation to the cone which passes through the three coordinate axes as well as the two lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}; \frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$

$$\text{--- ④} \qquad \text{--- ⑤}$$

Sol'n - the equation of any cone through the three coordinate axes

$$fyz + gzx + hxy = 0 \quad \text{--- ⑥}$$

Since it passes through the lines

④ & ⑤ and d.c's of the generators satisfy the equation of the cone.

$$\therefore f(-2)(3) + g(3)(1) + h(1)(-2) = 0$$

$$\& f(-1)(1) + g(1)(3) + h(3)(-1) = 0$$

$$\Rightarrow f(-6) + g(3) + h(-2) = 0 \quad \text{--- ⑦}$$

$$\& f(-1) + g(3) + h(-3) = 0 \quad \text{--- ⑧}$$

Eliminating f, g, h from ⑥, ⑦ & ⑧ we get

$$\begin{vmatrix} yz & zx & xy \\ -6 & 3 & -2 \\ -1 & 3 & -3 \end{vmatrix} = 0.$$

$$2y(-9+6) - zx(18-2) + xy(-18+3) = 0$$

$$\Rightarrow 3yz + 16zx + 15xy = 0$$

→ Find the equation of the quadric cone which passes through the 3 coordinate axes and three mutually perpendicular lines.

$$\frac{1}{2}x = y = -z, x = \frac{1}{3}y = \frac{1}{5}z, \frac{1}{8}x = -\frac{1}{11}y = \frac{1}{5}z$$

Soln: Now the equation of any cone through the coordinate axes is

$$fyz + gyz + hxy = 0 \quad \text{--- (1)}$$

Since it passes through the line.

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$$

$$\therefore f(1)(-1) + g(-1)(2) + h(2)(1) = 0$$

$$\Rightarrow -f - 2g + 2h = 0$$

$$\Rightarrow f + 2g - 2h = 0 \quad \text{--- (2)}$$

Similarly (1) passes through $\frac{x}{1} = \frac{y}{3} = \frac{z}{5}$

$$\therefore f(3)(5) + g(5)(1) + h(1)(3) = 0$$

$$\Rightarrow 15f + 5g + 3h = 0 \quad \text{--- (3)}$$

Solving (2) & (3)

$$\frac{f}{6+10} = \frac{9}{-30-3} = \frac{h}{5-30}$$

$$\Rightarrow \frac{f}{16} = \frac{9}{-33} = \frac{h}{-25}$$

∴ putting these values of f, g, h in

(1) we get

$$16(yz) + (-33)zx + (-25)xy = 0$$

$$\Rightarrow 16yz - 33zx - 25xy = 0$$

which is the required equation of the cone and the generator

line $\frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$ also satisfy the equation of this cone.

Planes through ox & oy include an angle α , show that their line of intersection lies on the cone $z^2(x^2+y^2+z^2) = x^2y^2 \tan^2 \alpha$.

Soln: The equation of any plane through ox ($y=0, z=0$) is $y+\lambda z=0$ (1)

and the equation of any plane through oy ($x=0, z=0$) is $x+\mu z=0$ (2)

The angle between the two planes

(1) & (2) is

$$\cos \alpha = \frac{0.1 + 1.0 + \mu \lambda}{\sqrt{1+\lambda^2} \cdot \sqrt{1+\mu^2}} = \frac{\mu \lambda}{\sqrt{1+\lambda^2} \cdot \sqrt{1+\mu^2}}$$

$$= \frac{\mu \lambda}{\sqrt{1+\lambda^2+\lambda^2+\mu^2}}$$

$$\sec \alpha = \frac{1}{\cos \alpha} = \frac{\sqrt{1+\lambda^2+\lambda^2+\mu^2}}{\mu \lambda}$$

$$\tan^2 \alpha = \sec^2 \alpha - 1$$

$$= \frac{1+\lambda^2+\lambda^2+\mu^2}{\mu^2 \lambda^2} - 1$$

$$= \frac{1+\lambda^2+\mu^2}{\lambda^2} \quad \text{--- (3)}$$

Eliminating λ, μ from (1), (2) & (3)

we get

$$\tan^2 \alpha = \frac{1 + \frac{y^2}{z^2} + \frac{z^2}{y^2}}{\frac{y^2}{z^2} + \frac{z^2}{y^2}}$$

$$= \frac{\left(\frac{y^2}{z^2}\right)\left(\frac{z^2}{y^2}\right)}{\frac{y^2}{z^2} + \frac{z^2}{y^2}}$$

$$= \frac{z^2(x^2+y^2+z^2)}{x^2y^2}$$

$$\therefore z^2(x^2+y^2+z^2) = x^2y^2 \tan^2 \alpha$$

which is the required eqn of the cone.

* Condition for general second degree equation to represent a cone :-

To find the condition that the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux$$

$$+ 2vy + 2wz + d = 0.$$

may represent a cone.

Sol:- The given equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$+ 2ux + 2vy + 2wz + d = 0 \quad (1)$$

If it represents a cone with vertex at (x_1, y_1, z_1) say.

then shifting the

origin to the point

so that we change $x = x + x_1$,

$$y = y + y_1 \text{ and } z = z + z_1.$$

\therefore the transformed equation is

$$a(x+x_1)^2 + b(y+y_1)^2 + c(z+z_1)^2$$

$$+ 2f(y+y_1)(z+z_1) + 2g(z+z_1)(x+x_1)$$

$$+ 2h(x+x_1)(y+y_1) + 2u(x+x_1)$$

$$+ 2v(y+y_1) + 2w(z+z_1) + d = 0$$

$$\Rightarrow ax^2 + by^2 + cz^2 + 2fyz + 2gzx +$$

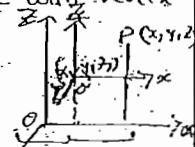
$$2hxy + 2x(ax_1 + hy_1 + gz_1 + u)$$

$$+ 2y(hx_1 + by_1 + fz_1 + v) +$$

$$2z(gx_1 + fy_1 + cz_1 + w) +$$

$$(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 +$$

$$2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d) = 0 \quad (2)$$



Since (2) represents a cone with vertex at the origin, so it must be homogeneous in x, y, z .

\therefore Coefficient of $x=0$, coefficient of coefficient of $z=0$ and Constant term

$$i.e. ax_1 + hy_1 + gz_1 + u = 0 \quad (3)$$

$$hx_1 + by_1 + fz_1 + v = 0 \quad (4)$$

$$gx_1 + fy_1 + cz_1 + w = 0 \quad (5)$$

$$\text{and } ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \quad (6)$$

Now (6) can be written as

$$x_1(ax_1 + hy_1 + gz_1 + u) + y_1(hx_1 + by_1 + fz_1 + v) + z_1(gx_1 + fy_1 + cz_1 + w) + ux_1 + vy_1 + wz_1 + d = 0$$

$$\Rightarrow ux_1 + vy_1 + wz_1 + d = 0 \quad (7)$$

(using (3), (4) & (5))

Eliminating x_1, y_1, z_1 from (3), (4),

(5) & (7) we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

which is the required condition.

Note:- The vertex of the cone is obtained by solving any three of the four equations (3), (4), (5) and (7) for x_1, y_1, z_1 .

Method for Numerical Questions

- Make the given equation homogeneous in x, y, z, t by introducing proper powers of t where $t=1$.
- Let this be denoted by $F(x, y, z, t) = 0$.
- Then the four equations $\textcircled{3}, \textcircled{4}, \textcircled{5}$ & $\textcircled{6}$ are obtained by equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ and $\frac{\partial F}{\partial t} = 0$ where ultimately $t=1$.
- Solve any three of the above four equations for x, y, z .
- Substitute these values of x, y, z in the fourth equation and if it is satisfied then the given equation represents a cone and values of x, y, z found in (iv) are the coordinates of the vertex.

→ Show that the equation

$$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$$

represents a cone with vertex $(-1, 2, -3)$.

Given equation is

$$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$$

Making given equation homogeneous, we get

$$(x, y, z, t) = 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0$$

Now $\frac{\partial F}{\partial x} = 0$

$$\Rightarrow 8x + 2y + 12t = 0$$

$$\Rightarrow \boxed{4x + y + 6t = 0}$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow -2y + 2x - 3z - 11t = 0$$

$$\Rightarrow \boxed{2x - 2y - 3z - 11t = 0}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \boxed{4z - 3y + 6t = 0}$$

and $\frac{\partial F}{\partial t} = 0 \Rightarrow \boxed{12x - 11y + 6z + 8t = 0}$

Putting $t=1$ in above equations, we get $11y + 6 = 0$ — $\textcircled{3}$

$$2x - 2y - 3z - 11 = 0 \quad \textcircled{4}$$

$$3y - 4z - 6 = 0 \quad \textcircled{5}$$

$$12x - 11y + 6z + 8 = 0 \quad \textcircled{6}$$

$$\textcircled{3} \times 2 \equiv 4x - 4y - 6z - 22 = 0 \quad \textcircled{7}$$

Now $\textcircled{6} - \textcircled{7} \quad 5y + 6z + 28 = 0$

$$\Rightarrow 10y + 12z + 56 = 0 \quad \textcircled{8}$$

$$\textcircled{4} \times 3 \quad 9y - 12z - 18 = 0 \quad \textcircled{9}$$

$$\textcircled{8} + \textcircled{9} \equiv 19y + 38 = 0$$

$$\Rightarrow \boxed{y = -2}$$

$$\textcircled{4} \equiv 3(-2) - 4z - 6 = 0$$

$$\Rightarrow -4z = 12$$

$$\Rightarrow \boxed{z = -3}$$

$$\textcircled{3} \equiv 4x - 2 + 6 = 0$$

$$\Rightarrow 4x = -4$$

$$\Rightarrow \boxed{x = -1}$$

∴ These values of x, y & z as
 $x=1, y=-2$ & $z=-3$
 satisfy ⑤

∴ The equation represents
 a cone and its vertex is
 $(1, -2, -3)$.

- Show that the equation
 $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$, represents
 a cone with vertex $(1, -2, 3)$.
- Show that the equation
 $2y^2 - 8yz - 4zx - 8xy + 6x^2 - 4y - 2z + 5 = 0$
 represents a cone whose vertex
 is $(\frac{7}{6}, \frac{1}{3}, \frac{5}{6})$.

* Angle between two lines
 in which a plane through
 the vertex cuts a cone:

Find the angle between the
 lines of intersection of the plane
 $x - 3y + z = 0$ and the cone
 $x^2 - 5y^2 + z^2 = 0$.

Sol'n :- The given plane is
 $x - 3y + z = 0$ — ①

and given cone is $x^2 - 5y^2 + z^2 = 0$ — ②

Let the line of section be

$$\frac{x}{1} = \frac{y}{m} = \frac{z}{n} = \text{---} \quad ③$$

Since it lies on the plane ①

∴ It is \perp lar to the normal
 to the plane.

$$∴ al + bm + cn = 0 \\ \Rightarrow l - 3m + n = 0 \quad \text{---} ④$$

Also the line ③ lies on the cone

∴ Its d.c's satisfies the equation
 of the cone.

$$∴ l^2 - 5m^2 + n^2 = 0 \quad \text{---} ⑤$$

$$④ \equiv [l = 3m - n]$$

$$⑤ \Rightarrow (3m - n)^2 - 5m^2 + n^2 = 0 \\ \Rightarrow 9m^2 - 6mn + n^2 - 5m^2 + n^2 = 0 \\ \Rightarrow 4m^2 - 6mn + 2n^2 - 5m^2 + n^2 = 0 \\ \Rightarrow 4m^2 + 2n^2 - 4mn - 2mn = 0 \\ \Rightarrow 4m(m - n) - 2n(m - n) = 0 \\ \Rightarrow (4m - 2n)(m - n) = 0 \\ \Rightarrow m - n = 0 \quad | \quad 4m - 2n = 0 \\ \Rightarrow m = n \quad | \quad 4m = 2n \Rightarrow m = \frac{1}{2}n \\ \Rightarrow m - n = 0 \quad | \quad 0! + 4m - 2n = 0 \\ \Rightarrow 0l + m - n = 0 \quad | \quad \text{also } l - 3m + n = 0 \\ \text{from } ④ l - 3m + n = 0 \\ \text{Solving}$$

$$\frac{l}{1-3} = \frac{m}{-1-0} = \frac{n}{0-1} \quad | \quad \frac{l}{4-6} = \frac{m}{-2-0} = \frac{n}{0-4}$$

$$\frac{l}{-2} = \frac{m}{-1} = \frac{n}{-1} \quad | \quad \Rightarrow \frac{l}{-2} = \frac{m}{-2} = \frac{n}{-4} \\ \Rightarrow \frac{l}{2} = \frac{m}{1} = \frac{n}{1} \quad | \quad \Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{2}$$

Putting these values of l, m, n in
 ③, the required lines of section
 are $\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$ & $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$

Q. If θ is angle between two lines of section then $\cos\theta = \frac{2(1)+1(1)+2(1)}{\sqrt{4+1+1} \sqrt{1+1+4}}$ → find the angles between the lines of section of the planes & cones.

$$\cos\theta = \frac{(5)}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\therefore \theta = \cos^{-1}(5/6)$$

Formulae :

Let the plane be $ax+by+cz=0$ and the cone be

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (2)$$

then the angle between the lines cutting by plane (1) in the cone (2) is given by

$$\tan\theta = \frac{2P\sqrt{u^2+v^2+w^2}}{(a+b+c)(u^2+v^2+w^2) - F(u, v, w)}$$

where

$$P^2 = \begin{vmatrix} a & b & c & u \\ b & b & f & v \\ c & f & c & w \\ u & v & w & 0 \end{vmatrix} \text{ and}$$

$$F(u, v, w) = au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv.$$

Note : (1) The lines are \perp lar, if

$$F(u, v, w) = (a+b+c)(u^2+v^2+w^2)$$

(2) If the lines are coincident

$$\begin{vmatrix} a & b & c & u \\ b & b & f & v \\ c & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0.$$

(i) $10x+7y-6z=0$ and $20x^2+y^2-108z^2=0$

(ii) $4x-y-5z=0$ and $8yz+3zx-5zy=0$

(iii) $x+y+z=0$ and $6xy+3yz-2zx=0$

(iv) $x+y+z=0$ and $x^2yz+xy-3z^2=0$

Ans (i) $\cos^{-1}(6/6)$ (ii) $\pi/2$

(iii) $\pi/3$ (iv) $\pi/6$

→ find the equations to the lines in which plane $2x+4y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$

Soln :- The given plane is

$$2x+4y-z=0 \quad (1)$$

and cone is $4x^2-y^2+3z^2=0 \quad (2)$

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be the equations of any one of the two lines in which the given plane meets the given cone.

∴ we have

$$2l+m-n=0 \quad (3), 4l^2-m^2+3n^2=0 \quad (4)$$

$$(3) \equiv n=2l+m=0$$

$$(4) \equiv 4l^2-m^2+3(2l+m)^2=0$$

$$\Rightarrow 4l^2-m^2+3(4l^2+4lm+m^2)=0$$

$$\Rightarrow 16l^2-m^2+12lm+3m^2=0$$

$$\Rightarrow 16l^2+3m^2+12lm=0$$

$$\Rightarrow 8l^2+m^2+6lm=0$$

$$\Rightarrow 8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0$$

$$\Rightarrow l/m = \frac{-6 \pm \sqrt{36-32}}{16} = -\frac{1}{4} (\text{or}) -\frac{1}{2}$$

$$\therefore l/m = -\frac{1}{4} \text{ & } l/m = -\frac{1}{2}$$

$$\Rightarrow l/m + \frac{1}{4} = 0 \text{ & } l/m + \frac{1}{2} = 0$$

$$\Rightarrow 4l+m=0 \text{ & } 2l+m=0$$

From (3) we have

(16)

$$\begin{array}{l|l} 2l+m-n=0 & 2l+m-n=0 \\ \rightarrow 4l+2m+0n=0 & 2l+m+n=0 \\ & \& 2l+m-n=0 \end{array}$$

Solving

$$\begin{array}{l|l} \frac{l}{-1} = \frac{m}{4} = \frac{n}{4-2} & \frac{l}{-1} = \frac{m}{2} = \frac{n}{2-2} \\ \frac{l}{-1} = \frac{m}{4} = \frac{n}{2} & \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{0} \end{array}$$

 \therefore The required lines are

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2} ; \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}$$

→ find the equations of the lines of intersection of the following planes and cones.

(i) $x+3y-2z=0$ and $x^2+9y^2-4z^2=0$

(ii) $3x+4y+z=0$ and $15x^2-32y^2-7z^2=0$

2003 → (iii) $x+7y-5z=0$ and $3yz+14zx-30xy=0$

Ans: - (i) $x=2z, y=0; 3y=2z, x=0$

(ii) $\frac{x}{-3} = \frac{y}{2} = \frac{z}{1}; \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$

(iii) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}; \frac{x}{3} = \frac{y}{1} = \frac{z}{2}$

→ Show that the equation of the quadratic cone which contains the three coordinate axes and the lines in which the plane $x-5y-3z=0$, cuts the cone $7x^2+5y^2-3z^2=0$ is

$$yz + 10xz + 18xy = 0.$$

* Mutually Perpendicular generators of a Cone :-

→ The necessary & sufficient condition for cone

$x^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to have three mutually perpendicular generators is that sum of coefficient of x^2, y^2, z^2 is zero. i.e. $a+b+c=0$.

→ If the general equation of the second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + dx^2 + 2vy + 2wz + d = 0$$

represents a cone, then the condition that it may have three mutually perpendicular generators is $ab+bc+ca=0$.

This result follows on shifting the origin to vertex.

The coefficients of the second degree term remain unaffected.

→ Problems

(Q) Prove that the plane $ax+by+cz=0$ cuts the cone $y^2 + zx + xy = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

Soln :- The equation of the plane is $ax+by+cz=0$ — (1)

and the cone is $y^2 + zx + xy = 0$ — (2)

Comparing (2) with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\therefore a=0, b=0, c=0$$

$$\Rightarrow ab+bc+ca = 0+0+0 = 0$$

∴ The cone (2) has three mutually perpendicular generators.

The plane (1) will cut the cone (2)

in \perp lines if the normal to the plane (1) through the vertex $(0,0,0)$

[whose d.c's are proportional to

$a,b,c]$ lies on the cone (2).

If $bc+ca+ab=0$ (\because d.c of the generator satisfy the equation of the cone).

$$\text{if } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

(on dividing throughout by abc)

which is the required condition

→ Prove that the plane

$lx+my+nz=0$ cuts the cone

$$(b-c)x^2 + (c-a)y^2 + (a-b)z^2 + 2fyz$$

+ 2gzx + 2hxy = 0

$$\text{in perpendicular lines if } (b-c)l^2 + (c-a)m^2 + (a-b)n^2 + 2fmn +$$

$$2gin + 2hlm = 0.$$

Soln : The given plane is $lx+my+nz=0$ — (1)

and cone is

$$(b-c)x^2 + (c-a)y^2 + (a-b)z^2 + 2fyz$$

$$+ 2gzx + 2hxy = 0 — (2)$$

Here the sum of the coefficients of $x^2, y^2, z^2 = (b-c) + (c-a) + (a-b) = 0$.

The Cone ① has three mutually ^{per}perpendicular generators.

Now if the plane ④ cuts the cone ② in perpendicular lines then normal to the plane ④ through vertex (0,0,0) i.e. $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

is the generator of the cone ②

Since the d.c's of the generator satisfy the cone equation.

i.e. l,m,n must satisfy ②

$$\therefore (b-c)l^2 + (c-a)m^2 + (a-b)n^2 + 2lmn + 2gnl + 2hlm = 0$$

which is the required condition.

→ If $x = ly = z$ represents one of a set of three mutually perpendiculars of the cone $5yz + 6zx - 4xy = 0$, find the equations of other two.

2008 → If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ represents one of a set of three mutually

perpendicular generators of the cone, $5yz - 8zx - 3xy = 0$ find the equations of the other two.

Sol'n:- The given cone is

$$5yz - 8zx - 3xy = 0 \quad \text{--- ①}$$

and one of its three ^{per}perpendicular generators is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{--- ②}$$

(1) the other two ^{per}perpendicular generators are the lines which plane thru the vertex (0,0,0), and \perp to line i.e. the plane $x + 2y + 3z = 0$. Let a line of section of ① & ③

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{--- ④}$$

Since ④ lies in the plane ③

∴ It is \perp to the normal to the plane

$$\therefore l + 2m + 3n = 0 \quad \text{--- ⑤}$$

Also ④ lies on Cone ①

∴ i.e. the d.c's of ④ satisfies the equation of cone.

$$\therefore 5mn + 8nl + 3lm = 0 \quad \text{--- ⑥}$$

$$\textcircled{5} \equiv l \equiv - (2m + 3n)$$

$$\therefore \textcircled{6} \equiv 5mn + 8n(-2m - 3n) + 3m(2m + 3n) = 0$$

$$\Rightarrow m^2 + 5mn + 4n^2 = 0$$

$$\Rightarrow (m+n)(m+4n) = 0$$

$$\Rightarrow m+n=0 \quad \left| \begin{array}{l} m+4n=0 \\ \Rightarrow 0+l+m+n=0 \end{array} \right.$$

$$\text{Also } \textcircled{5} \equiv l + 2m + 3n = 0 \quad \left| \begin{array}{l} l+2m+3n=0 \\ \Rightarrow 0+l+m+n=0 \end{array} \right.$$

Solving:

$$\frac{l}{3-2} = \frac{m}{1-0} = \frac{n}{0-1} \quad \left| \begin{array}{l} l = m \\ \Rightarrow \frac{l}{3-2} = \frac{m}{4-0} = \frac{n}{0-1} \end{array} \right.$$

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-1} \quad \left| \begin{array}{l} l = m \\ \Rightarrow \frac{l}{-5} = \frac{m}{4} = \frac{n}{-1} \end{array} \right.$$

$$\therefore \textcircled{4} \equiv \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \& \frac{x}{-5} = \frac{y}{4} = \frac{z}{-1}$$

which are the other two generators.

→ show that the cone whose vertex is origin and which passes through the curve of intersection of the surface $3x^2 - y^2 + z^2 = 3a^2$ and any plane at a distance 'a' from the origin, has three mutually perpendicular generators.

→ show that the cone whose vertex at the origin and which passes through the curve of intersection of the sphere $x^2 + y^2 + z^2 = 3a^2$ and any plane at a distance 'a' from the origin has three mutually perpendicular generators.

sol b:- Given sphere is

$$x^2 + y^2 + z^2 = 3a^2 \quad \text{--- (1)}$$

Any plane at a distance 'a' from the origin is $lx + my + nz = a \quad \text{--- (2)}$

Where l, m, n are d.c's of normal to the plane.

Making (1) homogeneous with the help of (2),

the equation of the cone whose vertex is the origin and base, the curve of intersection of (1) & (2) is

$$\begin{aligned} x^2 + y^2 + z^2 &= 3(lx + my + nz)^2 \\ \Rightarrow x^2(1-3l^2) + y^2(1-3m^2) + z^2(1-3n^2) &- 6lmxy - 6nlxz - 6myz = 0 \end{aligned} \quad \text{--- (3)}$$

which is the required cone vertex at the origin.

Now in (3), we have

$$\begin{aligned} \text{Coefficient of } x^2 + \text{Coefficient of } y^2 + \\ \text{Coefficient of } z^2 &= (1-3l^2) + (1-3m^2) + (1-3n^2) \\ &= 3 - 3(l^2 + m^2 + n^2) \\ &= 3 - 3(1) (\because l^2 + m^2 + n^2 = 1) \\ &= 0 \end{aligned}$$

∴ The cone (3) has three mutually perpendicular generators.

→ find the locus of the points from which three mutually perpendicular lines can be drawn to intersect the conic

$$z=0, ax^2 + by^2 = 1$$

sol b:- The given conic is

$$z=0, ax^2 + by^2 = 1 \quad \text{--- (1)}$$

Let (α, β, γ) be the point from which three mutually perpendicular lines can be drawn to intersect the conic (1).

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{--- (2)}$$

Since it meets the plane $z=0$

$$\therefore \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{0-\gamma}{n}$$

$$\Rightarrow x = \alpha - \frac{l}{n}, y = \beta - \frac{m}{n}$$

∴ this point lies on (1) if

$$a\left(\alpha - \frac{1}{n}v\right)^2 + b\left(\beta - \frac{m}{n}v\right)^2 = 1 \quad \text{--- (3)}$$

Now eliminate l, m, n from (2) & (3)

we have

$$a\left[\alpha - \frac{\alpha - \epsilon}{z-v}v\right]^2 + b\left[\beta - \frac{y - \beta}{z-v}v\right]^2 = 1$$

$$\Rightarrow a[\alpha z - \alpha v]^2 + b[\beta z - \beta v]^2 = [z-v]^2$$

$$\Rightarrow a(\alpha z - \alpha v)^2 + b(\beta z - \beta v)^2 - (z-v)^2 = 0$$

This cone has three mutually
flat generators if

Coefficient of x^2 + Coefficient of y^2

$$+ \text{Coefficient of } z^2 = 0$$

$$\text{if } ay^2 + bv^2 + (\alpha^2 + b\beta^2 - 1) = 0$$

$$\text{if } \alpha^2 + b\beta^2 + (a+b)v^2 = 1$$

\therefore Locus of the point (α, β, v) is

$$\alpha x^2 + bv^2 + (a+b)z^2 = 1$$

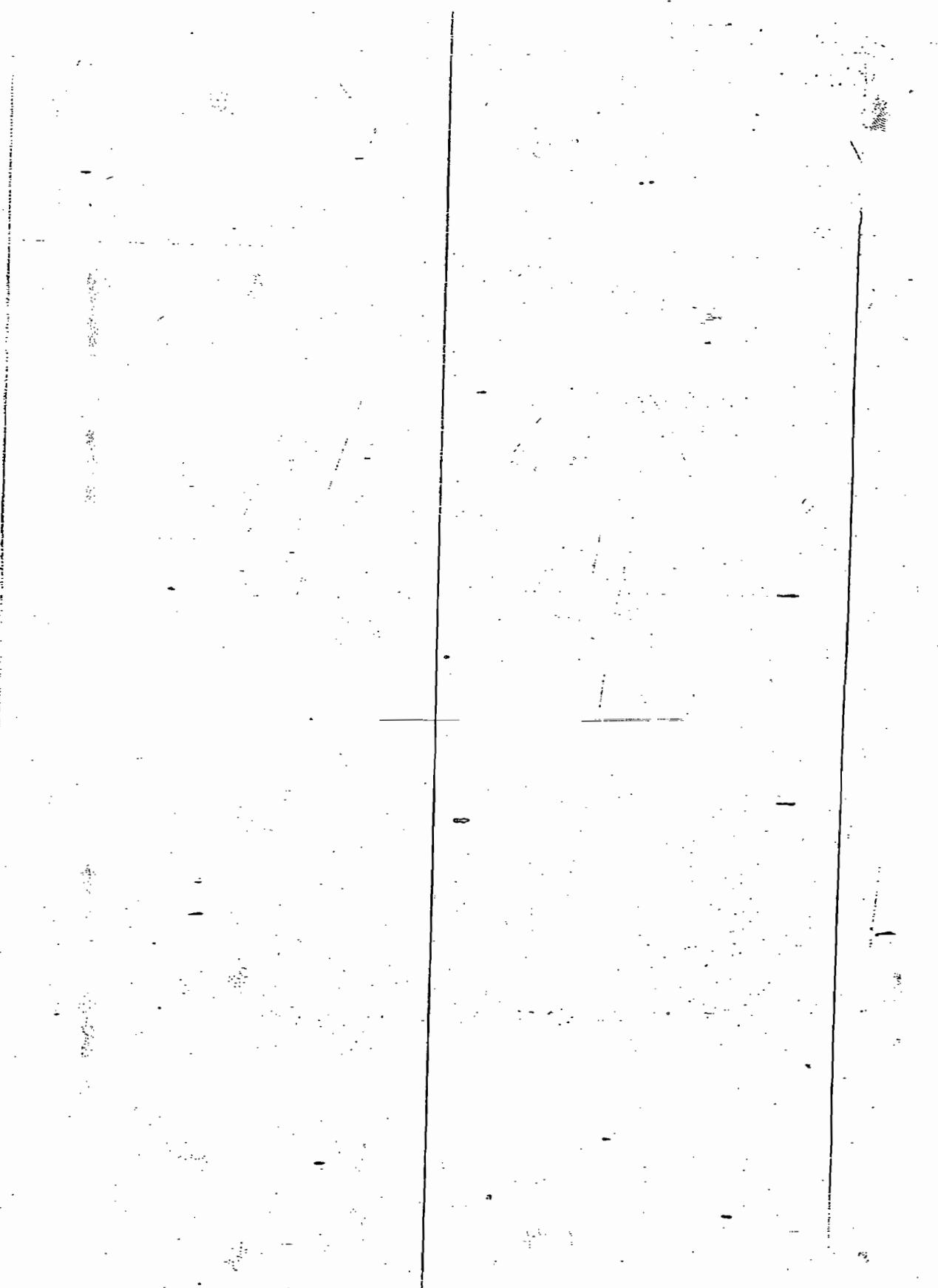
Hence the result.

Ques. Show that the plane

$$2x - y + 2z = 0$$
 cuts the cone

$$xy + yz + zx = 0$$
 in perpendicular
lines.

We find the intersection of
the plane with the cone
which we get two lines.
These two lines are
perpendicular to each other.



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* Tangent Plane :-

To find the equation of the tangent plane at the point (x_1, y_1, z_1) to the cone.

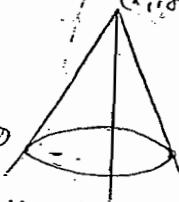
$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hyz = 0$$

Sol'n :- The given equation of the cone is $ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hyz = 0$

Equations of any line through $\stackrel{=0}{\textcircled{1}}$

(x_1, y_1, z_1) are (x_1, y_1, z_1)

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$



Any point on this line is

$$(x_1 + lr, y_1 + mr, z_1 + nr)$$

If it lies on the Cone $\textcircled{1}$ then

$$a(x_1 + lr)^2 + b(y_1 + mr)^2 + c(z_1 + nr)^2 + 2f(mr + y_1)(nr + z_1) + 2g(nr + z_1) + 2h(lr + z_1)(mr + y_1) = 0$$

$$\Rightarrow r^2 [al^2 + bl^2 + cl^2 + 2fml + 2gnl + 2hlm] + 2r[l(ax_1 + by_1 + cz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1)] + ax_1^2 + by_1^2 + cz_1^2 + 2fyz_1 + 2gz_1x_1 + 2hx_1y_1 = 0$$

which is a quadratic equation in r .

Since (x_1, y_1, z_1) lies on the cone $\textcircled{1}$.

$$\therefore ax_1^2 + by_1^2 + cz_1^2 + 2fyz_1 + 2gz_1x_1 + 2hx_1y_1 = 0$$

(13)

$$\therefore \textcircled{3} \equiv r^2 (al^2 + bl^2 + cl^2 + 2fml + 2gnl + 2hlm) + 2r [l(ax_1 + by_1 + cz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1)] +$$

$$\Rightarrow r[r(l) + 2(A)] = 0 \quad \text{(A)}$$

\Rightarrow This equation has one root as zero.
If the line $\textcircled{2}$ touches the cone, then the two values of r in (A) must be equal.

But since one root is zero.

\therefore Other root is also zero.

i.e. the coefficient of $r=0$.

$$\therefore l(ax_1 + by_1 + cz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1) = 0 \quad \text{(5)}$$

which is the condition for the line $\textcircled{2}$, to touch the cone $\textcircled{1}$ at (x_1, y_1, z_1) .

To find the locus of tangent line $\textcircled{2}$ -

we have to eliminate l, m, n from $\textcircled{2}$ & $\textcircled{5}$

\therefore putting the values of l, m, n from

$\textcircled{2}$ in $\textcircled{5}$,

we have

$$(x-x_1)(ax_1 + by_1 + cz_1) + (y-y_1)(hx_1 + by_1 + fz_1) + (z-z_1)(gx_1 + fy_1 + cz_1) = 0$$

$$\Rightarrow x(ax_1 + by_1 + cz_1) + y(hx_1 + by_1 + fz_1)$$

$$+ z(gx_1 + fy_1 + cz_1) =$$

$$ax_1^2 + by_1^2 + cz_1^2 + 2fyz_1 + 2gz_1x_1 + 2hx_1y_1$$

$$\Rightarrow x(ax_1 + by_1 + gz_1) + \\ y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0 \quad \text{--- (6)}$$

which is the required (\because from (4)) equation of the tangent plane.

Working Rule to Tangent

Plane at (x_1, y_1, z_1) :

In the equation of given cone or any surface change x^2 to ax_1 , y^2 to by_1 , z^2 to cz_1 , xy to $\frac{1}{2}(x_1y_1 + xy)$, xz to $\frac{1}{2}(x_1z_1 + xz)$, yz to $\frac{1}{2}(y_1z_1 + yz)$, x to $\frac{1}{2}(x_1 + x)$, y to $\frac{1}{2}(y_1 + y)$, z to $\frac{1}{2}(z_1 + z)$.

The equation obtained by this method will be same as equation (6).

Note:-

\rightarrow The tangent plane at any point of a cone passes through its vertex.

\rightarrow The vertex of the cone (1) is $(0, 0, 0)$ and it clearly lies on the tangent plane. (6)

\rightarrow The tangent plane at any point P' of a cone touches the cone along the generator through P .

Sol'n :- Let $P(x_1, y_1, z_1)$ be any point. The equation of the tangent plane at $P(x_1, y_1, z_1)$

is $x(ax_1 + by_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0 \quad \text{--- (1)}$
the equations of OP the generator through P are

$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0} \quad \left| \begin{array}{l} \text{using} \\ x-x_1 = y-y_1 \\ z-z_1 = y_1-y \\ = z-z_1 \\ z_1-z_1 \end{array} \right. \\ \Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} = r \quad (\text{say})$$

Any point on OP is $Q(ox_1, oy_1, oz_1)$.
The equation of the tangent plane at Q is

$$x(ax_1 + by_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0.$$

Dividing throughout by r .

$$x(ax_1 + by_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0.$$

which is the same as the equation (1) of the tangent plane at P.

\therefore The tangent plane at P also touches the cone at any point of OP.

i.e. the generator through P.

\therefore it touches the cone along OP.
This OP is called the generator of contact.

Note :- In the equation of the cone $ax^2 + by^2 + cz^2 + 2fyx + 2gzx + 2hxy = 0$,

We generally use the following notations:

$$(1) D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(2). A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in D.

$$\text{so that } A = bc - f^2, B = ca - g^2,$$

$$C = ab - h^2, F = gh - af, G = hf - bg,$$

$$H = fg - ch$$

$$(3) BC - F^2 = D.a$$

$$\text{Similarly } CA - G^2 = D.b, AB - H^2 = D.c$$

$$GH - AF = fD, HF - BG = gD,$$

$$FG - CH = hD.$$

$$\text{where } D = abc + 2fgh + af^2 - bg^2 - ch^2$$

$$(4) \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

$$= -(Au^2 + Bu^2 + Cu^2 + 2Fvw + 2Gwu + 2Huv)$$

* Condition of tangency

of a plane and cone:-

The condition that the plane

$$lx + my + nz = 0 \text{ may touch the}$$

$$\text{cone } ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\text{is } Al^2 + Bm^2 + Cn^2 + 2fmn + 2lm + 2hn = 0$$

* Reciprocal Cone : (14)

the locus of the normals to the tangent planes through vertex of the cone is another cone called the reciprocal cone.

* The equation of reciprocal

Cone of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\text{is } Ax^2 + By^2 + Cz^2 + 2fyz + 2gzx + 2hxy = 0$$

where A, B, C, D, F, G, H are cofactors of a, b, c, f, g, h in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Problems :-

Show that the locus of the mid points of chords of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

drawn parallel to the line.

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ is the plane}$$

$$x(Af + bm + cn) + y(hl + bm + fn) + z(gl + fm + cn) = 0.$$

Sol'n :- Let P(x₁, y₁, z₁) be the

midpoint of one of the chords.

drawn parallel to the $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Then equation of this

chord is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (1)$$



Any point on this line is

$$(lx_1 + x_1, my_1 + y_1, nz_1 + z_1)$$

If it lies on the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\text{then } a(lx_1 + x_1)^2 + b(my_1 + y_1)^2 + c(nz_1 + z_1)^2 \\ + 2f(mx_1 + y_1)(ny_1 + z_1) + 2g(nx_1 + z_1)(lx_1 + x_1) \\ + 2h(lx_1 + x_1)(my_1 + y_1) = 0$$

$$\Rightarrow \sigma^2 (al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) \\ + 2\sigma [l(ax_1 + by_1 + fz_1) + m(hx_1 + by_1 + fz_1) \\ + n(gx_1 + fy_1 + cz_1)] +$$

$$(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hxy_1) = 0$$

which is a quadratic in σ .

Since $P(x_1, y_1, z_1)$ is the midpoint of the chord:

\therefore the two values of σ should be equal in magnitude but opposite in sign.

\therefore sum of roots = 0 (or)

the coefficient of σ = 0.

$$\text{i.e. } l(ax_1 + by_1 + fz_1) + m(hx_1 + by_1 + fz_1) \\ + n(gx_1 + fy_1 + cz_1) = 0.$$

$$\Rightarrow \sigma(a(l + hm + gn) + Y(hl + bn + fn) \\ + Z(gl + fm + cn)) = 0.$$

$$\therefore \text{The locus of } P(x_1, y_1, z_1) \text{ is} \\ Z(alhm + gn) + Y(hl + bn + fn) + \\ Z(gl + fm + cn) = 0 \quad \text{--- (1)}$$

which is the required plane.

\rightarrow find the locus of the chords of the cone which are bisected at a fixed point.

Soln: Let $P(x_1, y_1, z_1)$ be the given fixed point and let any chord through P

which is bisected at P be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \text{--- (2)}$$

Any point on this line is

$$(lx_1 + x_1, my_1 + y_1, nz_1 + z_1)$$

If it lies on the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\text{then } a(lx_1 + x_1)^2 + b(my_1 + y_1)^2 + c(nz_1 + z_1)^2 \\ + 2f(mx_1 + y_1)(ny_1 + z_1) + 2g(nx_1 + z_1)(lx_1 + x_1) \\ + 2h(lx_1 + x_1)(my_1 + y_1) = 0$$

$$\Rightarrow \sigma^2 (al^2 + bm^2 + cn^2 + 2fmn + 2gnl \\ + 2hlm) \\ + 2\sigma [l(ax_1 + by_1 + fz_1) + m(hx_1 + by_1 + fz_1) \\ + n(gx_1 + fy_1 + cz_1)]$$

$$+ (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hxy_1) = 0.$$

which is a quadratic in σ .

Since $P(x_1, y_1, z_1)$ is the mid point of the chord (1)

\therefore the two values of σ should be equal in magnitude but opposite in sign.

\therefore Coefficient of σ = 0.

$$\therefore l(ax_1 + by_1 + cz_1) +$$

$$m(hx_1 + fy_1 + gz_1) + n(gx_1 + ly_1 + fz_1) = 0 \quad (1)$$

Eliminating l, m, n from (1) & (2)
the locus of the chords which
are bisected at P, is

$$(x-x_1)(ax_1 + by_1 + cz_1) + (y-y_1)$$

$$(hx_1 + fy_1 + gz_1) + (z-z_1)(gx_1 + ly_1 + fz_1) = 0$$

which is the required equation.

→ Prove that the cones

$$ax^2 + by^2 + cz^2 = 0 \text{ and } \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

are reciprocal.

Sol'n: The given first of the

$$\text{cone is } ax^2 + by^2 + cz^2 = 0 \quad (1)$$

Comparing with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we have $a=a, b=b, c=c$

$$-f=0, g=0, h=0.$$

$$\therefore A = bc - f^2 = bc - 0 = bc$$

$$\text{Similarly } B = ca - g^2 = ca - 0 = ca$$

$$C = ab - h^2 = ab - 0 = ab$$

$$F = gh - af = 0 - 0 = 0,$$

$$G = hf - bg = 0, H = fg - ch = 0$$

∴ the reciprocal cone of (1) is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx +$$

$$2hxy = 0$$

$$\Rightarrow bcx^2 + cay^2 + abz^2 = 0$$

(on dividing through by abc) (1)

$$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

which is the second cone.

Note:- The condition for the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

to have three mutually perpendicular tangent planes, if the reciprocal cone

$$-Ax^2 - By^2 - Cz^2 + 2fyz + 2gzx + 2hxy = 0$$

has three mutually perpendicular generators for which $A+B+C=0$

$$\text{i.e. } f^2 + g^2 + h^2 = bc + ca + ab$$

→ Prove that the perpendiculars

drawn from the origin to the

tangent plane to the cone

$$ax^2 + by^2 + cz^2 = 0 \text{ lie on the}$$

$$\text{cone } \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$$

Sol'n: The given cone is

$$-ax^2 - by^2 - cz^2 = 0 \quad (1)$$

We required to find the reciprocal cone of (1)

Comparing (1) with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we have $a=a, b=b, c=c$

$$f=0, g=0, h=0$$

$$\therefore A = bc - f^2 = bc - 0 = bc$$

$$B = ca - g^2 = ca - 0 = ca$$

$$C = ab - h^2 = ab - 0 = ab$$

$$F = gh - af = 0 - 0$$

$$G = hf - bg = 0 - 0$$

$$H = fg - ch = 0 - 0$$

∴ The reciprocal cone is

$$Ax^2 + By^2 + Cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow bcx^2 + cay^2 + abz^2 + f+g+h=0$$

$$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

which is the required equation.

→ show that the general equation of the cone which touches the three coordinate planes is.

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

Sol'n :- The general equation of a cone through the coordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{--- (1)}$$

Its reciprocal cone is

$$Ax^2 + By^2 + Cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\text{where } A = bc - f^2 = 0 - f^2 = -f^2 \quad \text{--- (2)}$$

$$B = ca - g^2 = -g^2$$

$$C = ab - h^2 = -h^2$$

$$F = gh - af = gh$$

$$G = hf - bg = hf \quad & H = fg - ch \\ = fg$$

$$\text{--- (3)} \\ = -f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz \\ + 2hgzx + 2fgxy = 0$$

$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hgzx \\ + 2fgxy = 4fgxy$$

$$\Rightarrow (fx + gy - hz)^2 = 4fgxy$$

$$\Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$$

$$\Rightarrow fx \pm 2\sqrt{fgxy} + gy = hz$$

$$\Rightarrow (fx \pm \sqrt{gy})^2 = hz$$

$$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

Find the equation of the cone which touches three coordinate planes and the planes

$$x+2y+3z=0, 2x+3y+4z=0$$

Sol'n :- Required cone which touches the three coordinate planes and the planes $x+2y+3z=0$, $2x+3y+4z=0$ is reciprocal line of a cone which passes through normals through the origin i.e. which passes through the three coordinate axes and two normals

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad \text{--- (1)} \quad \& \quad \frac{x}{2} = \frac{y}{3} = \frac{z}{4} \quad \text{--- (2)}$$

Now, any cone equation through the coordinate axis is

$$fyz + gzx + hxy = 0 \quad \text{--- (3)}$$

If this cone passes through the lines (1) & (2)

∴ d.c's of these lines satisfy the equation of cone (3).

$$6f + 3g + 2h = 0 \text{ and}$$

$$12f + 8g + 6h = 0$$

$$\therefore \frac{f}{2} = \frac{g}{-12} = \frac{h}{12} \Rightarrow \frac{f}{1} = \frac{g}{-6} = \frac{h}{6}$$

$$\therefore ③ \equiv yz - 6zx + 6xy = 0$$

$$\Rightarrow 2yz - 12zx + 12xy = 0 \quad (4)$$

The required cone is the reciprocal cone of (4)

Comparing (4) with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we have

$$a = b = c = 0, f = 1, g = -6, h = 6$$

$$A = bc - f^2 = -1, B = ca - g^2 = -36$$

$$C = ab - h^2 = -36$$

$$F = gh - af = -36, G = hf - bg = -6$$

$$H = fg - ch = -6$$

∴ The reciprocal cone is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$-x^2 - 36y^2 - 36z^2 - 72yz + 12zx - 12xy = 0$$

$$\Rightarrow x^2 + 36y^2 + 36z^2 + 72yz - 12zx + 12xy = 0$$

which is the required equation of the cone which touches the three coordinate planes and the two given planes.

→ Prove that the cones

$$ayz + bzx + cxg = 0$$

$$(ax)^2 + (by)^2 + (cz)^2 = 0 \text{ are}$$

reciprocal.

Sol'n → The given cones are (18)

$$ayz + bzx + cxg = 0 \quad (1)$$

$$(ax)^2 + (by)^2 + (cz)^2 = 0 \quad (2)$$

We required to find the reciprocal cone of (2) is (1)

$$③ \equiv \sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0$$

$$\Rightarrow \sqrt{ax} + \sqrt{by} = -\sqrt{cz}$$

$$\Rightarrow (\sqrt{ax} + \sqrt{by})^2 = cz$$

$$\Rightarrow ax + by + 2\sqrt{ax}\sqrt{by} = cz$$

$$\Rightarrow ax + by - cz = -2\sqrt{abxy}$$

$$\Rightarrow (ax + by - cz)^2 = 4abxy$$

$$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 + 2abxy - 2bcyz - 2acxz = 4abxy$$

$$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 - 2abxy - 2bcyz - 2acxz = \quad (3)$$

for the reciprocal cone

This is comparing with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$a = a^2, b = b^2, c = c^2, f = -bc, g = -ac = h = -ab.$$

$$\therefore A = bc - f^2 = b^2c^2 - b^2c^2 = 0$$

$$B = ca - g^2 = c^2a^2 - a^2c^2 = 0$$

$$C = ab - h^2 = a^2b^2 - a^2b^2 = 0$$

$$F = gh - af = (-ac)(-ab) + a^2bc$$

$$= a^2bc + a^2bc = 2a^2bc$$

$$G = hf - bg = 2a^2bc, H = 2abc^2$$

The reciprocal cone is

$$a^2x^2 + b^2y^2 + c^2z^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow 0 + 0 + 0 + 4a^2bcyz + 4ab^2czx + 4abc^2xy = 0$$

$$\Rightarrow ayz + bzx + cxy = 0$$

which is required equation.

→ prove that the tangent planes to the cone $x^2 + y^2 + 2z^2 - 3yz + 4zx - 5xy = 0$ are perpendicular to the generator of the cone $17x^2 + 8y^2 + 29z^2 + 2fyz + 28yz - 46zx - 16xy = 0$

Sol'n : The given first cone is $x^2 + y^2 + 2z^2 - 3yz + 4zx - 5xy = 0$ (1)

We are required to find the reciprocal cone of (1)

Comparing (1) with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we have $a=1, b=1, c=2,$
 $f=-3, g=4, h=-5$

Continue this we get the solution

* The Right Circular Cone :-

Definition :- The surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point is known as the right circular cone.

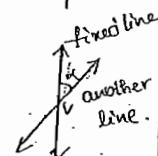
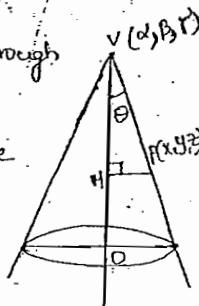
→ the fixed point is called the vertex.

→ the constant angle is called the semi-vertical angle.

→ The fixed line through the fixed point (i.e. vertex) is called the axis of the cone.

Note :- The section of a right circular cone

by a plane perpendicular to its axis is a circle.



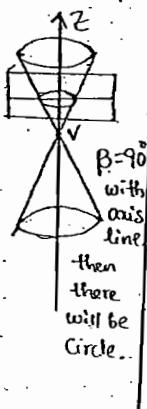
Equations of a right Circular Cone :-

(a) Standard form

To show that the equation of the right circular cone whose vertex is the origin, axis oz and semi-

vertical angle α is.

$$x^2 + y^2 = z^2 \tan^2 \alpha$$



Let $P(x, y, z)$ be any point on the line.

Draw $PM \perp oz$

$$\therefore \angle MOP = \alpha$$

Now, in the right angled

$\triangle OMP$,

$$\frac{OM}{OP} = \cos \alpha \quad \text{--- (1)}$$

Now, OM = Projection of OP on oz

whose d.c's are $0, 0, 1$

$$= 0(x-0) + 0(y-0) + 1(z-0)$$

$$= z \quad [\text{using } l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)]$$

$$\text{Also } OP = \sqrt{x^2 + y^2 + z^2}$$

$$(1) \equiv \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \cos \alpha$$

$$\Rightarrow z^2 = (x^2 + y^2 + z^2) \cos^2 \alpha$$

$$\Rightarrow z^2 \sec^2 \alpha = (x^2 + y^2 + z^2)$$

$$\Rightarrow z^2 (1 + \tan^2 \alpha) = x^2 + y^2 + z^2$$

$$\Rightarrow x^2 + y^2 = z^2 \tan^2 \alpha$$

which is a required equation.

(b) General Form :-

To find the equation of a right circular cone whose vertex is (α, β, γ) , semi vertical angle θ , and axis has d.c's l, m, n .

Sol'n :- Let $P(x, y, z)$ be any point on cone and AB , the axis of

Cone whose d.c's are l, m, n and passes through the vertex $A(\alpha, \beta, \gamma)$

Draw $DM \perp AB$.

$\therefore \angle PAM = \theta$, the semi vertical angle of right angle $\triangle AMP$,

$$\frac{AM}{AP} = \cos \theta \quad \text{--- (1)}$$

AM = projection of AP on the AB line whose d.c's are l, m, n :

$$= l(x-\alpha) + m(y-\beta) + n(z-\gamma)$$

$$\text{and } AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$$

$$(1) \equiv [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$

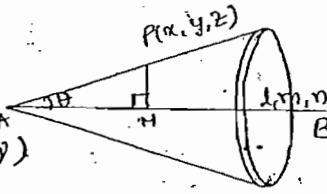
$$= [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] \cos^2 \theta \quad \text{--- (2)}$$

which is the required equation of the cone.

Note :- (c) Put $\alpha = \beta = \gamma = 0$ in (2). then the equation of the right circular cone whose vertex is origin and axis with the d.c's l, m, n and semi vertical angle θ is.

$$(lx+my+nz)^2 = (x^2+y^2+z^2) \cos^2 \theta$$

If OZ is the axis of cone and $(0, 0, 0)$ as the vertex and θ , the semi vertical angle,



then putting $\alpha = \beta = \gamma = 0, l = 0, m, n$

$\eta = 1$

$$\begin{aligned} (2) &\equiv z^2 = (x^2+y^2+z^2) \cos^2 \theta \\ &\Rightarrow z^2 \sec^2 \theta = x^2+y^2+z^2 \\ &\Rightarrow z^2 (1+\tan^2 \theta) = x^2+y^2+z^2 \\ &\Rightarrow z^2 \tan^2 \theta = x^2+y^2. \end{aligned}$$

(iii) The semi vertical angle of a right circular cone admitting sets of three mutually perpendicular generators is $\tan^{-1}\sqrt{2}$.

For this, the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero and this means that $1+1-\tan^2 \theta = 0 \Rightarrow \theta = \tan^{-1}\sqrt{2}$.

→ Find the equation to the right circular cone whose vertex is $P(2, -3, 5)$; axis PQ which makes equal angles with the axes and semi vertical angle θ is 30° .

Sol:- Since the d.c's of the axis PQ which makes equal angles with the coordinate axes are l, m, n .

If a line PQ makes angles α, β, γ with x, y, z axes;

the value

$$\alpha = \beta = \gamma$$

$$\Rightarrow 180^\circ - 3\alpha = 180^\circ$$

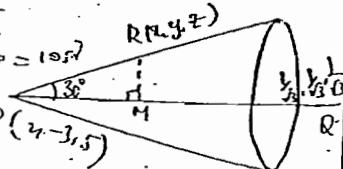
$$\Rightarrow \alpha = 60^\circ$$

$$\Rightarrow l = m = n$$

$$\text{since } l^2 + m^2 + n^2 = 1$$

$$\therefore \Rightarrow 3\alpha = 180^\circ$$

$$\Rightarrow l = \pm \frac{1}{\sqrt{3}}$$



we take the slant height. Given that the axis the line

$$l = m = n = \frac{1}{\sqrt{3}}$$

Let $R(2, 4, 2)$ be any pt on the surface of the cone.

Draw $RM \perp PR$

$$\therefore \angle MPQ = 30^\circ$$

In the rt. angled $\triangle PRM$

$$\cos 30^\circ = \frac{MP}{PR}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{MP}{PR} \quad (1)$$

MP = projection of PR on the axis PQ . (2)

and d.r's of PR are $2-2$,

and d.c's of PQ are $\frac{y+3}{\sqrt{3}}, \frac{z-5}{\sqrt{3}}$

$$\therefore (2) \Rightarrow MP = \frac{1}{\sqrt{3}}(2-2) + \frac{1}{\sqrt{3}}(y-5) + \frac{1}{\sqrt{3}}(z-5) \\ = \frac{1}{\sqrt{3}}(x+y+z-6)$$

$$\text{and } PR = \sqrt{(x-2)^2 + (y+3)^2 + (z-5)^2}$$

$$\therefore (1) \Rightarrow \frac{1}{\sqrt{3}}(x+y+z-6) = \frac{1}{\sqrt{3}}\sqrt{(x-2)^2 + (y+3)^2 + (z-5)^2}$$

→ find the equation of the circular cone which passes through the point $(1, 1, 2)$ and has its vertex at the origin and the axis

$$\text{the line } \frac{x}{2} = \frac{y}{-4} = \frac{z}{3}.$$

Sol'n:- Let the d.c's of axis be l, m, n .

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$$

∴ the d.c's of $(0, 0, 0)$

the OQ are

proportional to $2, -4, 3$

∴ the actual d.c's of OQ are

$$\frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}, \frac{3}{\sqrt{29}}$$

Let α be the semi-vertical angle of the cone.

Since $A(1, 1, 2)$ lies on the cone.

∴ The d.c's of OA are proportional to $1, 1, 2$.

∴ The actual d.c's are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$
The semi vertical angle α of a right circular cone is the angle between the axis & the generator of the cone.

∴ α is the angle between OQ & OA .

$$\cos \alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$= \left(\frac{2}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{-4}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{3}{\sqrt{29}}\right)\left(\frac{2}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{29}} \cdot \frac{1}{\sqrt{6}} [2 - 4 + 6]$$

$$\cos \alpha = \frac{4}{\sqrt{29} \cdot \sqrt{6}}$$

Let $P(x, y, z)$ be any point on the cone.

Draw $PM \perp OQ$

In the right angle $\triangle RMO$

$$\therefore \cos\alpha = \frac{MO}{PO}$$

$$\Rightarrow (MO)^2 = (PO)^2 \left(\frac{16}{29 \times 6} \right) \quad (1)$$

Now $MO = \text{projection of } PO \text{ on } OQ$

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{2}{\sqrt{29}}(x - 0) + \left(\frac{-4}{\sqrt{29}}\right)(y - 0) + \frac{3}{\sqrt{29}}(z - 0)$$

$$= \frac{1}{\sqrt{29}}[2x - 4y + 3z]$$

$$\text{and } PO = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore (1) \equiv \frac{1}{29}(2x - 4y + 3z)^2 = (x^2 + y^2 + z^2) \frac{16}{29 \times 6}$$

$$\Rightarrow 3(4x^2 + 16y^2 + 9z^2 - 16xy - 24yz + 12xz)$$

$$= 8x^2 + 8y^2 + 8z^2$$

$$\Rightarrow 4x^2 + 40y^2 + 19z^2 + 48xy - 72yz + 36xz = 0$$

→ Lines are drawn from the origin with the d.c's proportional $(1, 2, 2), (2, 3, 6), (3, 4, 12)$; find the direction cosines of the axis of right circular cone through them, and prove that the semi vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Sol'n = Let l, m, n be the d.c's of the axis of the right circular cone.

Let O be the origin and P, Q, R be the given points.

Now the d.r's of OP, OQ, OR are $(1, 2, 2), (2, 3, 6), (3, 4, 12)$.

The d.c's of OP, OQ and OR are $\frac{l}{\sqrt{29}}, \frac{2l}{\sqrt{29}}, \frac{2l}{\sqrt{29}}$, $\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{6}{\sqrt{3}}$ and $\frac{3}{\sqrt{13}}, \frac{4}{\sqrt{13}}, \frac{12}{\sqrt{13}}$.

Let α be the semi-vertical angle of the cone, then

$$\cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{3}{3}m + \frac{6}{3}n$$

$$= \frac{3}{13}l + \frac{4}{13}m + \frac{12}{13}n \quad (A)$$

Now take first two members.

$$\frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n$$

$$\Rightarrow 7l + 14m + 14n = 6l + 9m + 18n$$

$$\Rightarrow l + 5m - 4n = 0 \quad (1)$$

From first & last we get

$$2l + 7m - 5n = 0 \quad (2)$$

Solving, we get

$$\frac{l}{-1} = \frac{m}{1} = \frac{n}{1} = \pm \sqrt{\frac{l^2 + m^2 + n^2}{13}} = \pm \frac{1}{\sqrt{13}}$$

∴ The d.c's of the axis are

$$\frac{-1}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{1}{\sqrt{13}}$$

∴ Putting these values in (A) we get

$$\cos\alpha = \frac{1}{3}\left(-\frac{1}{\sqrt{13}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{13}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{13}}\right)$$

$$= \frac{1}{3\sqrt{13}}(-1 + 2 + 2) = \frac{3}{3\sqrt{13}}$$

$$\cos\alpha = \frac{1}{\sqrt{13}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$$

→ find the equation of the right circular cone generated by the straight lines drawn from the origin to cut the circle through the three points $(1, 2, 2)$, $(2, 1, -2)$ and $(2, -2, 1)$.

Sol'n: Let $A(1, 2, 2)$, $B(2, 1, -2)$, $C(2, -2, 1)$ be the given point.

Let l, m, n be the actual d.c's of the axis OX .

Then OA, OB, OC

make

the same

angle α

with the axis OX , where α is the semi - vertical angle.

The direction ratios of OA, OB, OC are $(1, 2, 2)$, $(2, 1, -2)$, $(2, -2, 1)$

∴ The d.c's of OA, OB, OC are

$$\frac{1}{3}, \frac{2}{3}, \frac{2}{3}; \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}; \frac{2}{3}, \frac{2}{3}, \frac{1}{3}$$

$$\therefore \cos \alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m + \frac{2}{3}n$$

$$\frac{2}{3}n = \frac{2}{3}l + \left(\frac{2}{3}\right)m + \frac{1}{3}n \quad \textcircled{1}$$

from first two members we have

$$\frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m + \frac{2}{3}n$$

$$\Rightarrow l + 2m + 2n = 2l + m - 2n$$

$$\Rightarrow l - m - 4n = 0 \quad \textcircled{2}$$

from last two members we have

$$3m - 3n = 0 \Rightarrow 3l + m - n = 0 \quad \textcircled{3}$$

Solving $\textcircled{2}$ & $\textcircled{3}$ we have

$$\frac{l}{5} = \frac{m}{1} = \frac{n}{1} = \pm \frac{\sqrt{27m^2+n^2}}{\sqrt{27+1+1}} = \pm \frac{1}{\sqrt{27}}$$

$$\therefore l = \frac{5}{\sqrt{27}}, m = \frac{1}{\sqrt{27}}, n = \frac{1}{\sqrt{27}}$$

$$\therefore \textcircled{1} \equiv \cos \alpha = \frac{1}{3}\left(\frac{5}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right)$$

$$= \frac{1}{\sqrt{27}}\left(\frac{5+2+2}{3}\right)$$

$$= \frac{1}{\sqrt{27}} \times 9 = \frac{9}{9\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \boxed{\cos \alpha = \frac{1}{\sqrt{3}}}$$

Let $P(x, y, z)$ be any point on the cone.

Draw $PM \perp OX$

$$\therefore \angle MOP = \alpha$$

In the right angle $\triangle OMP$,

$$\frac{OM}{OP} = \cos \alpha$$

$$(OM)^2 = (OP)^2 \cdot \frac{1}{3} \quad \textcircled{4}$$

OM = projection of OP on OX .

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

Continue this solution.

If α is the semi - vertical angle of the right circular cone which passes through the lines

OX, OY , $x = y = z$, show that

$$\cos \alpha = (2 - 4/\sqrt{3})^{-1/2}$$

Sol'n: Let l, m, n be the d.c's of the axis of the cone. Since the

axis makes the same angle α with each of the lines ox, oy and $x=y=z$.

Now the d.r.s of $ox, oy, x=y=z$ are $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 1)$

\therefore the d.r.s of ox, oy and $x=y=z$ are $(1, 0, 0), (0, 1, 0)$ and $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$$\begin{aligned} \cos \alpha &= l(1) + m(0) + n(0) = l(0) + m(1) + n(0) \\ &= l(\frac{1}{\sqrt{3}}) + m(\frac{1}{\sqrt{3}}) + n(\frac{1}{\sqrt{3}}) \end{aligned}$$

from first two nos. $\rightarrow \textcircled{1}$

$$\text{we have } l=m \Rightarrow l-m+n=0 \quad \text{---} \textcircled{2}$$

from last two nos.

$$\text{we have } m=\frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

$$\Rightarrow \frac{l}{\sqrt{3}} + \left(\frac{1-\sqrt{3}}{\sqrt{3}}\right)m + \frac{n}{\sqrt{3}} = 0$$

$$\Rightarrow l+(1-\sqrt{3})m+n=0 \quad \text{---} \textcircled{3}$$

solving $\textcircled{2}$ and $\textcircled{3}$.

$$\frac{l}{-1+0} = \frac{m}{0-1} = \frac{n}{1-\sqrt{3}+1} \Rightarrow \frac{l}{-1} = \frac{m}{-1} = \frac{n}{2-\sqrt{3}}$$

$$\therefore \frac{l}{-1} = \frac{m}{-1} = \frac{n}{2-\sqrt{3}} = \pm \sqrt{l^2+m^2+n^2}$$

$$= \pm \frac{1}{\sqrt{2+4+3-4\sqrt{3}}}$$

$$= \pm \frac{1}{\sqrt{9-4\sqrt{3}}}$$

$$\frac{l}{-1} = \frac{-1}{\sqrt{9-4\sqrt{3}}} \Rightarrow l = \frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$m = \frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$n = \frac{-(2-\sqrt{3})}{\sqrt{9-4\sqrt{3}}} = \frac{\sqrt{3}-2}{\sqrt{9-4\sqrt{3}}}$$

from $\textcircled{1}$

$$\cos \alpha = \frac{-1}{\sqrt{9-4\sqrt{3}}}$$

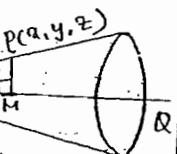
$$\alpha = \cos^{-1} \left(\frac{-1}{\sqrt{9-4\sqrt{3}}} \right)$$

\rightarrow show that the equation of the right circular cone with vertex $(2, 3, 1)$, axis parallel to the line $-x = \frac{y}{2} = z$ and one of its generators having d.r.s proportional to $(1, -1, 1)$ is

$$x^2 - 8y^2 + z^2 + 12xy - 12yz + 16zx - 46x + 36y + 22z - 19 = 0$$

Soln:- Let l, m, n be the d.r.s of the axis of the right circular cone.

The given line $\frac{x}{-1} = \frac{y}{2} = \frac{z}{1}$ is parallel to the axis.



\therefore The d.r.s of the axis are $(-1, 2, 1)$

proportional to $-1, 2, 1$.

\therefore The actual d.r.s are $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

$$\therefore l = \frac{-1}{\sqrt{6}}, m = \frac{2}{\sqrt{6}}, n = \frac{1}{\sqrt{6}}$$

Now, the d.r.s of its generator are proportional to $1, -1, 1$.

The actual d.c's are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Now let α be the semi-vertical angle.

Then the semi vertical angle α of a right circular cone is the angle between the axis and the generator of the cone.

$$\therefore \cos \alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$= \left(\frac{-1}{\sqrt{6}} \right) \left(\frac{1}{\sqrt{3}} \right) + \left(\frac{2}{\sqrt{6}} \right) \left(\frac{-1}{\sqrt{3}} \right) + \left(\frac{1}{\sqrt{6}} \right) \left(\frac{1}{\sqrt{3}} \right)$$

$$= \frac{1}{\sqrt{18}} [-1 - 2 + 1] = \frac{-2}{3\sqrt{2}}$$

Let $P(x, y, z)$ be any point on the cone.

- Draw $PH \perp AQ$

$$\therefore \angle MAP = \alpha$$

In right angle $\triangle AMP$,

$$\cos \alpha = \frac{AM}{AP}$$

$$\Rightarrow (AM)^2 = (AP)^2 \cos^2 \alpha \quad \text{--- (1)}$$

Now $AM = \text{projection of } AP \text{ on } AQ$

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \left(\frac{-1}{\sqrt{6}} \right) (x_2 - x_1) + \left(\frac{2}{\sqrt{6}} \right) (y_2 - y_1) + \left(\frac{1}{\sqrt{6}} \right) (z_2 - z_1)$$

$$= \frac{1}{\sqrt{6}} [-x_2 + 2y_2 + z_2 - x_1 - 2y_1 - z_1]$$

$$= \frac{1}{\sqrt{6}} [-x + 2y + z - 5]$$

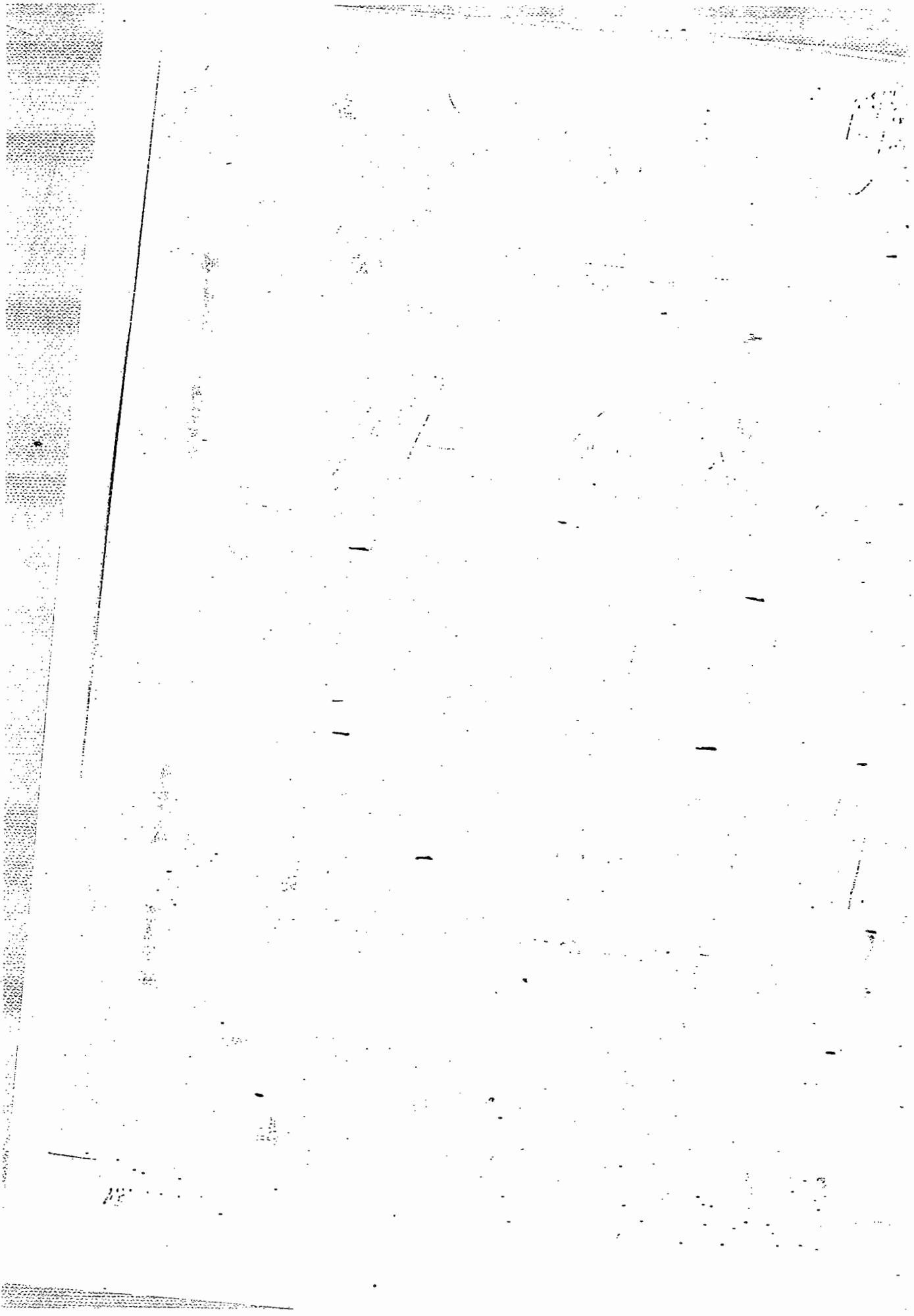
$$\text{and } (AP)^2 = \sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2}$$

$$① \equiv \frac{1}{8} [(-x+2y)+(z-2)]^2 = [(x-2)^2 + (y-3)^2 + (z-1)^2] \times \frac{1}{3}$$

$$\Rightarrow 3[2x^2 + 4y^2 - 4xy + z^2 + 4 - 4z + 2(-x+2y)(z-2)] = 4[x^2 + y^2 + z^2 - 4x - 4y - 2z + 4 + 9 + 1]$$

$$\Rightarrow 3x^2 + 12y^2 - 12xy + 3z^2 + 12 - 12z + 6(-x^2 + 2x + 2yz - 4y) = 4x^2 + 4y^2 - 16x - 24y - 2z + 56$$

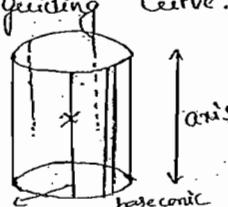
$$x^2 + 8y^2 + z^2 + 12xy - 12yz + 6z^2 - 46x + 36y + 22z - 19 = 0$$



Definition

The surface generated by a variable line which is always parallel to a fixed line and intersects a given curve (or touches a given surface) is called the cylinder.

The variable line is called the generator, the fixed line the axis and the given curve (or surface) the guiding curve.



* Equation of a Cylinder:

To find the equation of the cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

and the base conic is

$$f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

$z=0$

Sol'n :- The given line is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{--- (1)}$$

and the base conic is

$$f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, z=0$$

--- (2)

The Cylinder * Set-VI

(2)

Let (x_1, y_1, z_1) be any point on a generator of the cylinder and parallel to the line (1). Then equations of generator line (i.e. a line through (x_1, y_1, z_1) and parallel to (1)) are:

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{--- (3)}$$

It meets the plane $z=0$.

∴ Putting $z=0$ in (3) we get,

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{0-z_1}{n}$$

$$\therefore x = x_1 - \frac{l}{n}z_1, \quad y = y_1 - \frac{m}{n}z_1$$

∴ the point $(x_1 - \frac{l}{n}z_1, y_1 - \frac{m}{n}z_1)$ if this point lies on the conic (then $a[x_1 - \frac{l}{n}z_1]^2 + b[y_1 - \frac{m}{n}z_1]^2 + 2h(x_1 - \frac{l}{n}z_1)(y_1 - \frac{m}{n}z_1) + 2g(x_1 - \frac{l}{n}z_1) + 2f(y_1 - \frac{m}{n}z_1) + c = 0$)

$$\therefore a(x - \frac{l}{n}z)^2 + b(y - \frac{m}{n}z)^2 + 2h(x - \frac{l}{n}z)(y - \frac{m}{n}z) + 2g(x - \frac{l}{n}z) + 2f(y - \frac{m}{n}z) + c = 0$$

$$\Rightarrow a(nx - lz)^2 + b(ny - mz)^2 + 2h(nx - lz)(ny - mz) + 2ng(nx - lz) + 2nf(ny - mz) + cn^2 = 0$$

which is the required equation of the cylinder.

Note:- If the generators are parallel to z -axis, then $l=0, m=0$ and $n=1$.

∴ the equation of the cylinder becomes $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$. which is free from z .

→ If we required to find the equation of the cylinder whose generators are parallel to z-axis, and intersect a given conic then eliminate z from the equations of the conic.

∴ If given the equation of the cylinder:

→ If the generators are parallel to x-axis then eliminate z and if the generators are parallel to y-axis then eliminate y from the equations of the conic to get equations of the cylinder.

Problems

- find the equation of a cylinder whose generating lines have the dir's (l,m,n) and which passes through the circle $x^2+y^2=a^2, y=0$.
- find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, and whose guiding curve is the ellipse $x^2+2y^2=1, z=3$.
- find the equation of the cylinder whose generators intersect the curve $ax^2+by^2=2z, lx+my+nz=p$ and are parallel to z-axis.

Sol: → The given base conic is

$$ax^2+by^2=2z, lx+my+nz=p \quad \text{①}$$

Since the generators of the cylinder are parallel to the z-axis.

∴ The required equation of the cylinder free from the z-coordinate.

Now eliminate z from the equations ① to get the required cylinder.

From first equation of ① we have,

$$z = \frac{ax^2+by^2}{2}$$

Putting in the second equation of ①,

$$lx+my+n\left(\frac{ax^2+by^2}{2}\right)=p -$$

$$\Rightarrow 2lx+2my+n(ax^2+by^2)=2p$$

$$\Rightarrow n(ax^2+by^2)+2lx+2my-2p=0$$

which is the required cylinder.

→ find the equation of the cylinder with generators parallel to x-axis and passing through the curve

$$ax^2+by^2+cz^2=1,$$

$$lx+my+nz=p.$$

* Enveloping cylinder of a sphere:

To find the equation to the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = a^2$ and are parallel to the line

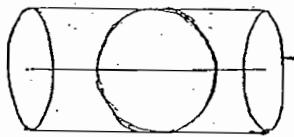
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad (or)$$

To find the locus of the tangent lines drawn to a sphere and parallel to a given line.

Sol'n :- The given sphere $x^2 + y^2 + z^2 = a^2$

and the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad (2)$$



Let (α, β, γ) be

any point on the cylinder.

\therefore Any line through (α, β, γ) parallel to (2) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (say).

Any point on this line is

$(\alpha+r, m\beta+r, n\gamma+r)$.

This point lies on the sphere (1), then

$$(\alpha+lr)^2 + (\beta+mr)^2 + (\gamma+nr)^2 = a^2$$

$$\Rightarrow r^2(l^2+m^2+n^2) + 2r(\alpha l + \beta m + \gamma n) + (\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0 \quad (3)$$

Clearly, which is a quadratic in r .

Since the generator (3) is

(2g)

a tangent line of the given sphere.

\therefore the two values of r given in (3) must be equal.

\therefore the discriminant of (3) = 0.

$$i.e. b^2 - 4ac = 0.$$

$$[2(l\alpha + m\beta + n\gamma)]^2 = 4(l^2 + m^2 + n^2)(l^2 + m^2 + n^2 - a^2)$$

The locus of (α, β, γ) is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)$$

which is required equation of the cylinder and is known as the Enveloping cylinder of a sphere.

Problem

Find the enveloping cylinder of a sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generators parallel to the line $x = y = z$.

$$(Ans: x^2 + y^2 + z^2 - xy - yz - zx - 4x + 5y - z - 2 = 0)$$

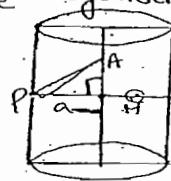
Sol'n :- Let (α, β, γ) be any point on the cylinder.

Continue in this way.

* Right Circular Cylinder :-

A surface generated by a line which intersects a fixed circle (is called guiding curve) and is parallel to the plane of the circle is called right circular cylinder.

→ the normal to the plane of the circle through its centre is called the axis of the cylinder and the radius of the circle is the radius of the cylinder.



Equation of Right Circular Cylinder

(a) Standard form :-

Show that the equation of the right circular cylinder whose axis is the z -axis and radius is $x^2 + y^2 = a^2$.

Let $P(x, y, z)$ be any point on the cylinder.

Draw $PM \perp z$ -axis.

$\therefore OM = z$ and the coordinates of $M(0, 0, z)$.

$\therefore MP = \text{radius of the cylinder}$ (given).

$$\text{But } MP = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$

$$= \sqrt{x^2 + y^2}$$

$$\therefore \sqrt{x^2 + y^2} = a$$

$$\Rightarrow x^2 + y^2 = a^2$$

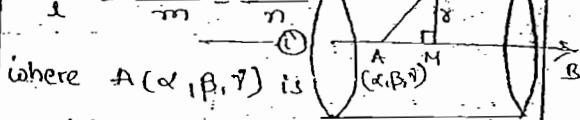
which is required equation.

(b) General Form :-

To find the equation to the right circular cylinder whose radius is a and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

Sol :- Let AB be the axis of the cylinder whose equations are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad (1)$$



where $A(\alpha, \beta, \gamma)$ is a point on it.

The d.s. of AB are l, m, n .

\therefore the actual d.c's are

$$\frac{l}{\sqrt{l^2}}, \frac{m}{\sqrt{m^2}}, \frac{n}{\sqrt{n^2}}$$

Let $P(x, y, z)$ be any on the cylinder.

Draw $PM \perp AB$ axis.

and join PA .

$PM = \text{radius of the cylinder}$

In the right angled $\triangle PAM$,

$$AP^2 = AM^2 + PM^2 \quad (2)$$

$$(AP)^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$

$AM = \text{Projection of } AP \text{ on } AB \text{-axis}$.

$$= \frac{l}{\sqrt{\sum l^2}}(x-\alpha) + \frac{m}{\sqrt{\sum m^2}}(y-\beta) + \frac{n}{\sqrt{\sum n^2}}(z-\gamma)$$

$$= \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

$$\textcircled{D} = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$

$$= \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{l^2+m^2+n^2}$$

which is the required equation of the cylinder.

→ Find the equation of the right circular cylinder of radius 2 whose axis is the line

$$\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-2}{2}$$

→ The axis of the a right circular cylinder of radius 2 is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

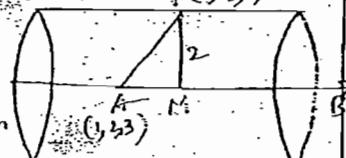
Show that its equation is

$$10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4zx - 8x + 30y - 14z + 59 = 0.$$

→ Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has d.c's proportional to (2, -3, 6).

Let AB be

Axis of the cylinder which



(28)
passes through the point P
and has d.c's proportional to
Dividing each by

$$\sqrt{4+9+36} = \sqrt{49} = 7$$

∴ Actual d.c's are $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}$

Let P(x, y, z) be any point on cylinder:

Draw PH ⊥ AB

∴ In right angled $\triangle APM$

$$AP^2 = AM^2 + PM^2$$

Continue this solution.

→ find equation to the right circular cylinder whose guiding circle is

$$x^2+y^2+z^2=9, x-y+z=3.$$

Note: → The axis of the cylinder is the line through centre of sph and \perp to the plane of the circle and radius of the cylinder is equal to radius of circle.

Sol'ns:- The sphere is $x^2+y^2+z^2=9$
and plane is $x-y+z=3$ (2)

The centre of the

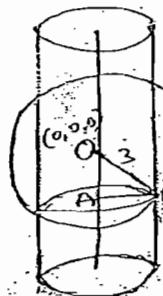
sphere is $O(0,0,0)$

and its radius is

$$OB=3.$$

$OA = \perp$ distance of $O(0,0,0)$
from the plane (2)

$$= \frac{|6-0+0-3|}{\sqrt{1+1+1}} = \frac{|3|}{\sqrt{3}} = \sqrt{3}$$



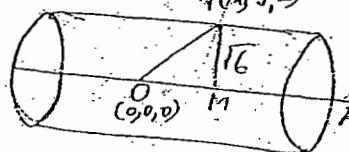
$\therefore AB = \text{radius of the circle}$

$$= \sqrt{OB^2 - OA^2} = \sqrt{9 - 3} = \sqrt{6}$$

Again equation of the line through the centre $O(0,0,0)$ of the sphere and \perp to plane α are

$$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$$

which is the axis of the cylinder and radius $\sqrt{6}$.



The d.e.'s of the axis are proportional to $1, -1, 1$.

The actual d.e.'s are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Let $P(x,y,z)$ be any point on the cylinder.

Join OP and draw $MP \perp OA$.

$$(OP)^2 = (OM)^2 + (MP)^2 \quad \text{--- (3)}$$

$$\text{Now } (OP)^2 = \sqrt{x^2 + y^2 + z^2}$$

$$(MP)^2 = 6$$

& $OM = \text{Projection of } OP \text{ on } OA$.

$$= \frac{1}{\sqrt{3}}(x) - \frac{1}{\sqrt{3}}(y) + \frac{1}{\sqrt{3}}(z)$$

$$= \frac{1}{\sqrt{3}}(x-y+z)$$

$$\text{--- (3)} \quad (x^2 + y^2 + z^2) = \frac{(x-y+z)^2}{3} + 6$$

$$\Rightarrow x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$$

which is the required equation of the cylinder.

Find the equation of the right circular cylinder whose guiding curve is the circle through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Sol'n Let $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ be the given points.

Then the circle through A, B, C is the intersection of the plane ABC and the sphere $OABC$.

Now the equation of the plane ABC is

$$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1 \quad \text{[intercept form]}$$

$$x+y+z=1 \quad \text{--- (1)}$$

and the sphere $OABC$ is

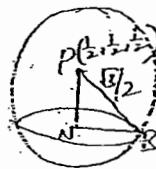
$$x^2 + y^2 + z^2 - x - y - z = 0 \quad \text{--- (2)}$$

(using $x^2 + y^2 + z^2 - ax - by - cz = 0$)

The centre of the sphere is

$$P(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

$$\text{and radius} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \\ = \frac{\sqrt{3}}{2}$$



From the right angle $\triangle PMB$,

$$NB^2 = \sqrt{PB^2 - NP^2} \quad \text{--- (3)}$$

$NP = \text{distance from } P \text{ to the plane}$

$$= \sqrt{|y_2 + b_2 + c_2 - 1|} \\ = \sqrt{1+1+1} = \frac{\sqrt{3}}{2}$$

$$\textcircled{2} \equiv NB \quad \sqrt{\frac{3}{4}} = \frac{1}{4 \times 3}$$

$$= \sqrt{\frac{9-1}{4 \times 3}} = \sqrt{\frac{8}{4 \times 3}} = \sqrt{\frac{2}{3}}$$

which is the radius of the circle.

\therefore This is also radius of the cylinder.

Now the equations of PN are

[i.e. through $P(y_1, y_2, y_2)$ and

parallel to the plane (1)]

$$\frac{x-y_2}{1} = \frac{y-y_2}{1} = \frac{z-y_2}{1}$$

which is the axis of the cylinder.

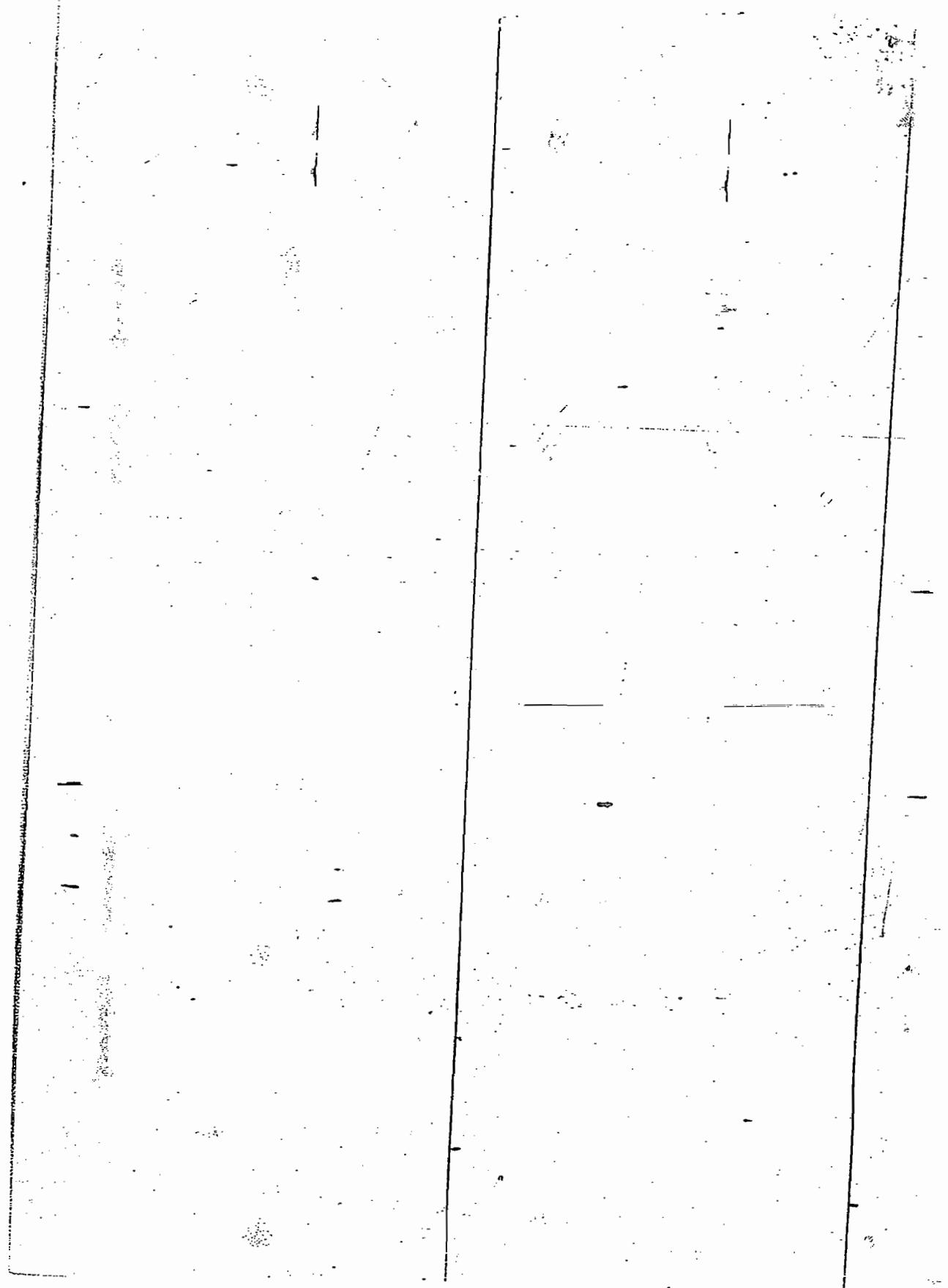
\Rightarrow Now the d.c's of the axis are proportional to 1, 1, 1.

The actual d.c's are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

and the radius is $\sqrt{2}/2$.

Continue in this way we
get the solution.

Ques. find the right circular cylinder whose guiding curve is the circle through three points $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$.
find also the axis of the cylinder



$$(5) \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z^2}{c^2}; \text{ Hyperbolic paraboloid.}$$

* Shapes of surfaces.

$$(1) \text{ Ellipsoid: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(i) Centre: If (α, β, r) is a

point on the ellipsoid,

then $(-\alpha, -\beta, -r)$ is also a point on it.

The middle point of the join

of these points is $(0, 0, 0)$, the origin.

Hence $(\alpha, \beta, r), (-\alpha, -\beta, -r)$ are the points on a straight line through the origin and are equidistant from the origin.

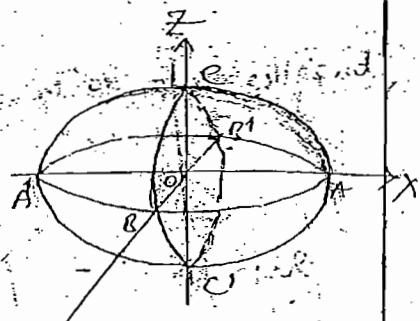
Hence origin bisects every chord which passes through it and is therefore the centre of the surface.

(ii) Symmetry: Since there are only even powers of x ,

the surface is symmetrical about $y=$ -plane.

Similarly, the surface is symmetrical about xz and xy planes.

If the point (α, β, r) satisfies the eqn, then $(\alpha, \beta, -r)$ also satisfies it. The line joining (α, β, r) , $(\alpha, \beta, -r)$ is bisected at right-angle by the xy -plane. It follows that the xy -plane bisects all chords perpendicular to it. Similarly other co-ordinate planes also bisect chords \perp to them.



The conicoid

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A surface whose equation is of the second degree in x, y, z is called the conicoid i.e., the general equation of second degree in x, y, z

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ua + 2vy + 2wz + d = 0$
represents a locus called a conicoid or Quadric.

The above equation contains ten unknown constants which can be reduced to nine effective constants by dividing the equation throughout by 'a'.

Hence a conicoid can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

By suitable transformation of axes, the above general equation can be reduced to one of the following standard forms.

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \text{ Ellipsoid}$$

$$(2) \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1; \text{ Hyperboloid of one sheet}$$

$$(3) \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1; \text{ Hyperboloid of two sheets}$$

$$(4) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}; \text{ Elliptic paraboloid}$$

These three planes are called principal planes.

The three lines of intersection of three principal planes taken in pairs are called principal axes.

In the present case co-ordinate axes are the principal axes.

(iii) Intersection with axes:

The surface meets x-axis ($y=0, z=0$)

$$\text{if we have } \frac{x^2}{a^2} + 1 \Rightarrow x = \pm a$$

i.e. the surface meets the x-axis in the points $A(a, 0, 0)$ and $A'(-a, 0, 0)$.

Similarly it meets y-axis ($x=0, z=0$) at $B(0, b, 0)$ and $B'(0, -b, 0)$

and z-axis ($x=0, y=0$) at $C(0, 0, c)$ and $C'(0, 0, -c)$

(iv) sections by co-ordinate planes: The surface meets the yz -plane i.e., $x=0$.

$$\text{we have } \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

which is an ellipse in the plane. (Fig 17)

Similarly ; it meets the zx -plane ($y=0$) in the

$$\text{ellipse } \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \text{ in that plane}$$

and it meets the xy -plane ($z=0$) in the

$$\text{ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ in that plane}$$

(v) Generated by a variable curve:

The surface meets the plane $z=k$ in a curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}; z = k, -c \leq k \leq c$$

The surface is generated by the variable ellipse ① in which k takes different values and whose plane is \parallel to the xy -plane ($z=0$) and centre $(0,0,k)$ moves on the z -axis.

The ellipse ① is real only if $1 - \frac{k^2}{c^2} > 0$
i.e. $k^2 < c^2$
i.e. $|k| < c$.

i.e. k lies between $-c$ and c .

Similarly x and y can not be both numerically greater than a & b respectively.

So that we have for every point (x, y, z) on the surface $-a \leq x \leq a, -b \leq y \leq b, -c \leq z \leq c$.

Hence, the surface lies between the planes $x=a, x=-a; y=b, y=-b;$
 $z=c, z=-c$.

and therefore is a closed surface.

Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(1) The origin bisects all chords which pass through it and is therefore the centre of the surface.

(2) It is symmetrical about each of the coordinate planes. For only even powers x, y, z occur in its eqn. Coordinate planes are the

(iii) It meets the x -axis at $A(a, 0, 0)$, $A'(-a, 0, 0)$
 the y -axis at $B(0, b, 0)$, $B'(0, -b, 0)$; and the
 z -axis in imaginary points $(\pm \frac{a}{c} \cos \theta, \pm \frac{b}{c} \cos \theta, \pm \frac{c}{c})$
 we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(iv) Its section by the yz -plane ($x=0$)

is the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (i.e., DE, D'E')

- Its section by the zx -plane ($y=0$)

is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ (i.e., FG, F'G')

- Its section by the xy -plane ($z=0$)

is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(v) The sections by the planes $z=k$ which
 are parallel to the xy -plane are the
 similar ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, z=k \quad \text{--- (1)}$$

- whose centre lie on z -axis and will
 increase in size as k increases.
 There is no limit to the increase of k .

The surface may, therefore, be generated
 by the variable ellipse (1)

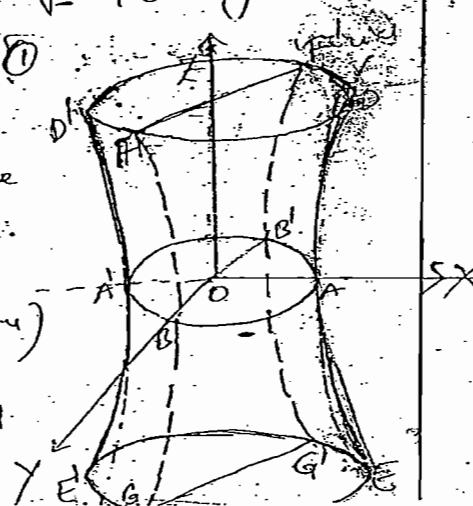
where k varies from 0 to $+\infty$.

The shape of the surface
 as shown in the figure.

(which is like Juggler's dome)

[i.e. friend - Hwala]

It is known as hyperboloid
 of one sheet.



The hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

- (i) origin is the centre; co-ordinate planes are the principal planes and co-ordinate axes are the principal axes of the surface.
- (ii) it is symmetrical about each of the co-ordinate planes for only even powers of x, y, z occurs in its equation.
- (iii) It meets the x -axis at $A(a, 0, 0), A'(-a, 0, 0)$ and the y and z -axes in imaginary points.
- (iv) Its section by the xy -plane ($z=0$) is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (i.e. $AB, A'B'$)
- Its section by the yz -plane ($x=0$) is the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (i.e. $DAE, D'A'E'$)
- Its section by the xz -plane ($y=0$) is the imaginary ellipse $\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$.
- (v) The surface cuts the plane xzk in an ellipse.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = k^2 - 1, \quad x \neq 0$$

which increases in size as k^2 increases,

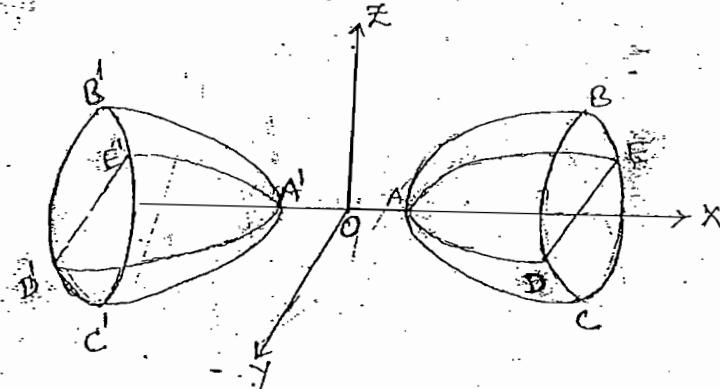
but is real when $\frac{k^2}{a^2} - 1 > 0$.

i.e., $|k| > a$

i.e., when k does not lie between $-a$ and a .

Thus no portion of the surface lies between the planes $x = \pm a$.

This surface thus consists of two detached portions as shown in the figure.
It is known as hyperboloid of two sheets.



Its shape is like that of two tables placed as shown by the figure.

* central conicoid:

A conicoid whose all chords through the origin are bisected at the origin is called a central conicoid.

The equation of a hyperboloid:

In general, represents a central conicoid.
All the above three equations

$$\left[\text{viz. } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right]$$

are covered by these equations.

- (i) When a, b, c are all +ve, (1) represents an ellipsoid.
- (ii) When two are +ve and one -ve, it represents a hyperboloid of one sheet.

and (iii) when two are +ve and one is -ve
it represents a hyperboloid of two sheets.

The above equation for all values of a, b, c ($-ve$ or $+ve$) represents a surface whose centre is origin and coordinate planes, the three principal planes.

The equation $ax^2 + by^2 + cz^2 = 1$ is called the standard form of central conicoid.

* Intersection of a line and a conicoid

To find the points of intersection of the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ with the conicoid $ax^2 + by^2 + cz^2 = 1$.

Sol: The given line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad (1)$$

and the conicoid is (x_1, y_1, z_1)
 $ax^2 + by^2 + cz^2 = 1 \quad (2)$

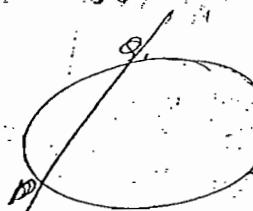
Any point on the line (1) is -

$$(lr+x_1, mr+y_1, nr+z_1) \quad (3)$$

If it lies on the conicoid (2), then
 $a(lr+x_1)^2 + b(mr+y_1)^2 + c(nr+z_1)^2 = 1$.

$$\Rightarrow a^2(l^2r^2 + bmr^2 + cnr^2) + 2ar(lmr^2 + bmr + cnr) + (a^2r^2 + b^2r^2 + c^2r^2 - 1) = 0.$$

which is a quadratic in r , giving two values of r .



putting these values of σ in (3), we get the two points of intersection P and Q.

Hence, every line meets a central conicoid in two points.

The two values r_1 and r_2 of σ obtained from the equation (4) are the measures of the distances of the points of intersection P and Q from the point (x_1, y_1) if (l, m, n) are the direction cosines of the line.

Note: The equations (4) will frequently be used in what follows.

Def → A chord of a central conicoid which passes through the centre called a diameter.

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→ Prove that the sum of the squares of the reciprocals of any three mutually perpendicular diameters of an ellipsoid is constant.

Sol: Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let $d_1, d_2, d_3, d_4, d_5, d_6$ be the actual d.c.s of three mutually perpendicular diameters say PCP' , QCQ' , RCR' and let $2r_1, 2r_2, 2r_3$ be the lengths of the diameters.

Since the diameters of the ellipsoid are bisected at the centre

$$C(0,0,0), \quad (P \equiv C P') = r_1; \quad (Q \equiv C Q') = r_2 \\ (R \equiv C R') = r_3$$

Now as P is at a distance r_1
from $C(0,0,0)$ and d.c's of CP are
 a_1, m_1, n_1

i.e. co-ordinates of P are
($a_1 r_1, m_1 r_1, n_1 r_1$)

Since P lies on ellipsoid (1)

$$\frac{a_1^2}{a^2} + \frac{m_1^2 r_1^2}{b^2} + \frac{n_1^2 r_1^2}{c^2} = 1$$

$$\Rightarrow \frac{1}{r_1^2} = \frac{a_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}$$

or

$$\therefore \frac{1}{(P.P.)^2} = \frac{1}{r_1^2} = \frac{1}{4r_1^2}$$

$$= \frac{1}{4} \left(\frac{a_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2} \right) \quad (2)$$

$$\text{Similarly } \frac{1}{(Q.Q)^2} = \frac{1}{4} \left(\frac{a_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2} \right)$$

$$\frac{1}{(R.R)^2} = \frac{1}{4} \left(\frac{a_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2} \right) \quad (3)$$

Adding (2) (3) and (4), we have

$$\frac{1}{(P.P.)^2} + \frac{1}{(Q.Q)^2} + \frac{1}{(R.R)^2} = \frac{1}{4} \left[\frac{1}{a^2} (a_1^2 + a_2^2 + a_3^2) \right]$$

$$+ \frac{1}{b^2} (m_1^2 + m_2^2 + m_3^2)$$

$$+ \frac{1}{c^2} (n_1^2 + n_2^2 + n_3^2) \right]$$

$$= \frac{1}{4} \left[\frac{1}{a^2} (l_1^2 + l_2^2 + l_3^2) + \frac{1}{b^2} (m_1^2 + m_2^2 + m_3^2) + \frac{1}{c^2} (n_1^2 + n_2^2 + n_3^2) \right]$$

$$= \frac{1}{4} (l_1^2 + l_2^2 + l_3^2)$$

= constant. (a_1, m_1, n_1, \dots etc.

are the d.c's
of three mutually f.lines)

→ A line through a given point A meets the central conicoid in P, Q . If $R'OR$ is the diameter parallel to APQ , prove that $AP \cdot AQ : OR^2$ is constant.

Soln: Let $A(x_1, y_1, z_1)$ be the given point and let the conicoid be $ax^2 + by^2 + cz^2 = 1$. (1)

Let l, m, n be the actual d.c.'s of the line through A which meets the conicoid in P and Q .

∴ Equations of the line APQ passing through $A(x_1, y_1, z_1)$ and with d.c.'s l, m, n are

$$\frac{xy}{l} - \frac{y-z}{m} = \frac{z-x}{n} \quad \text{--- (2)}$$

Any point on this line is $(lx+x_1, my+y_1, nz+z_1)$.

If it lies on the conicoid (1), then

$$a(lx+x_1)^2 + b(my+y_1)^2 + c(nz+z_1)^2 = 1$$

$$\Rightarrow r^2(a^2l^2 + b^2m^2 + c^2n^2) + 2r(lamx + bmy_1 + cnz_1) + (a^2x_1^2 + b^2y_1^2 + c^2z_1^2 - 1) = 0 \quad \text{--- (3)}$$

which is a quadratic in r .

Since l, m, n are the actual d.c.'s of the line (2).

∴ The two values of r in (2) are the lengths AP and AQ .

$$AP \cdot AQ = \frac{\text{product of roots}}{\text{coeff. of } r^2}$$

$\begin{aligned} & \text{coeff. of } r^2 \\ & \text{roots of } ax^2 + by^2 + cz^2 - 1 \\ & \quad x_1, y_1, z_1 \\ & \quad \text{coeff. of } r \\ & \quad x_1 \cdot y_1 = \frac{c}{a} \end{aligned}$

Now the equations of the diameter OR through $O(0,0,0)$ and \parallel to line ② are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

If $OR = r$, then the coordinates of R are (lr^2, mr^2, nr^2) .

Since R lies on the conicoid ①, then

$$al^2r^2 + bm^2r^2 + cn^2r^2 = 1$$

$$\Rightarrow r^2 (al^2 + bm^2 + cn^2) = 1$$

$$\Rightarrow r^2 = OR^2 = \frac{1}{al^2 + bm^2 + cn^2} \quad \textcircled{3}$$

Dividing ④ & ③, we get

$$\frac{AP \cdot AQ}{OR^2} = \frac{OR^2 + bP^2 + cQ^2 - 1}{al^2 + bm^2 + cn^2} \times \frac{al^2 + bm^2 + cn^2}{1}$$

$$= al^2 + bm^2 + cn^2 - 1$$

$$= \text{constant}$$

Hence the result.

\rightarrow All given point and PQ any diameter of a central conicoid. If OQ and OQ' are the diameters parallel to AP and AP' ,

prove that $\frac{AP^2}{OQ^2} + \frac{AP'^2}{OQ'^2}$ is constant.

Sol: Let the central conicoid be $ax^2 + by^2 + cz^2 = 1$ ①

Let A be the point (α_1, β_1, r) and $P(x_1, y_1, z_1)$, $P'(x_2, y_2, z_2)$ extremities of diameter PQ .

The d.c.'s of AP are proportional to

$$x_1 - \alpha, y_1 - \beta, z_1 - \gamma \quad | \text{ using } x_2 - x_1, y_2 - y_1, z_2 - z_1$$

Dividing each by $\sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2} = AP$,

∴ the actual d.c.'s of AP are

$$l = \frac{x_1 - \alpha}{AP}, m = \frac{y_1 - \beta}{AP}, n = \frac{z_1 - \gamma}{AP}. \quad (2)$$

∴ d.c.'s of OQ (|| to AP) are also l, m, n.

then if OQ = r, the co-ordinates of Q
are (lr, mr, nr).

Since Q lies on the conicoid (1).

$$\therefore (1) \Rightarrow a(x_1 - \alpha)^2 + b(y_1 - \beta)^2 + c(z_1 - \gamma)^2 = 1$$

$$\Rightarrow r^2(a(l)^2 + b(m)^2 + c(n)^2) = 1$$

$$\Rightarrow (OQ)^2 \left[a \left(\frac{x_1 - \alpha}{AP} \right)^2 + b \left(\frac{y_1 - \beta}{AP} \right)^2 + c \left(\frac{z_1 - \gamma}{AP} \right)^2 \right] = 1 \quad (\because \text{from (2) & OQ} = r)$$

$$\Rightarrow \frac{OQ^2}{AP^2} \left[a(x_1 - \alpha)^2 + b(y_1 - \beta)^2 + c(z_1 - \gamma)^2 \right] = 1$$

$$\Rightarrow \frac{AP^2}{OQ^2} = a(x_1 - \alpha)^2 + b(y_1 - \beta)^2 + c(z_1 - \gamma)^2. \quad (3)$$

Now changing (x_1, y_1, z_1) to $(-x_1, -y_1, -z_1)$,
i.e., p changes to p' and
Q changes Q' .

$$\begin{aligned} \frac{AP'^2}{OQ'^2} &= a(-x_1 - \alpha)^2 + b(-y_1 - \beta)^2 + c(-z_1 - \gamma)^2 \\ &= a(x_1 + \alpha)^2 + b(y_1 + \beta)^2 + c(z_1 + \gamma)^2 \end{aligned} \quad (4)$$

Adding (3) and (4), we have

$$\begin{aligned}
 \frac{AP^2}{OQ^2} + \frac{AP'^2}{OQ'^2} &= a[(x_1+\alpha)^2 + (x_1-\alpha)^2] + b[(y_1+\beta)^2 + (y_1-\beta)^2] \\
 &\quad + c[(z_1+\gamma)^2 + (z_1-\gamma)^2]. \\
 &= 2a(x_1^2 + \alpha^2) + 2b(y_1^2 + \beta^2) + 2c(z_1^2 + \gamma^2) \\
 &= 2(ax^2 + b\beta^2 + cz^2) + 2(ax_1^2 + b\beta_1^2 + cz_1^2). \\
 &= 2(ax^2 + b\beta^2 + cz^2) + 2(1) \\
 &= 2(ax^2 + b\beta^2 + cz^2 + 1) \quad (\because P(x_1, y_1, z_1) \text{ lies on } \textcircled{1}) \\
 &\quad \cdot \quad \because ax^2 + by^2 + cz^2 = 1 \\
 &= \text{constant}.
 \end{aligned}$$

Hence the result

Tangent plane.

To find the equation of tangent plane at the point (x_1, y_1, z_1) of the central conicoid $ax^2 + by^2 + cz^2 = 1$.

The given conicoid is $ax^2 + by^2 + cz^2 = 1$ (1)

Equation of a line through (x_1, y_1, z_1) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{--- (2)}$$

Any point on (2) is (lx_1+m, my_1+n, nz_1)

If it lies on (1), then -

$$a(lx_1+m)^2 + b(my_1+n)^2 + c(nz_1)^2 = 1.$$

$$\Rightarrow r^2(a^2l^2 + b^2m^2 + c^2n^2) + 2r(a^2x_1 + b^2y_1 + c^2z_1) +$$

$$(ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \text{--- (3)}$$

But (x_1, y_1, z_1) lies on (1)

$$\therefore a^2x_1^2 + b^2y_1^2 + c^2z_1^2 - 1 = 0 \quad \text{--- (4)}$$

\therefore (3) becomes

$$r(al^2 + bm^2 + cn^2) + 2s(alx_1 + bmy_1 + cnz_1) = 0$$

$$r[al^2 + bm^2 + cn^2] + 2(alx_1 + bmy_1 + cnz_1) = 0 \quad (4)$$

$$\Rightarrow r=0.$$

Since the line (2) touches the conicoid (1)

\therefore it cuts (1) at two coincident points

which is so if the two values of 'r'
in (5) are equal.

Since one root of (5) is zero,

\therefore the other must also be zero.

Coefficient of $r=0$.

$$\text{i.e., } alx_1 + bmy_1 + cnz_1 = 0 \quad (6)$$

eliminating l, m, n from (2) & (6),

the locus of line (2) is

$$ax_1(x-x_1) + by_1(y-y_1) + cz_1(z-z_1) = 0$$

$$\Rightarrow ax_1 + by_1 + cz_1 = ax_1 + by_1 + cz_1 \quad (\because by(4))$$

$$\Rightarrow ax_1 + by_1 + cz_1 = 1$$

which is the required equation of
tangent plane at (x_1, y_1, z_1)

Condition of Tangency

To find the condition that the plane
 $lx+my+nz=p$ should touch the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

Sol: The given plane is $lx+my+nz=p$ — (1)

and the "conicoid" is $ax^2 + by^2 + cz^2 = 1$ — (2)

Let the plane (1) touch the conicoid (2)

at the point (x_1, y_1, z_1) .

Then ① should be identical with the tangent plane at (x_1, y_1, z_1) to ②.

NOW the equation of tangent plane at (x_1, y_1, z_1) to ② is

$$ax_1x + by_1y + cz_1z = 1 \quad \text{--- } ③$$

Comparing ① and ③, we have

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p}$$

$$\therefore x_1 = \frac{l}{ap}, y_1 = \frac{m}{bp}, z_1 = \frac{n}{cp} \quad \text{--- } ④$$

but since (x_1, y_1, z_1) being the point of contact lies on the conicoid ②.

$$\therefore a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1$$

$$\Rightarrow \boxed{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2}$$

which is the required condition of tangency

Note: From ④, the point of contact is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

Find the equations of two tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ which are parallel to the plane $lx + my + nz = 0$.

sol: The given conicoid is $ax^2 + by^2 + cz^2 = 1$ --- ①

and any plane \parallel to $lx + my + nz = 0$ is

$$lx + my + nz = p \quad \text{--- } ②$$

If ② touches ①, then

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$p = \pm \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}.$$

\therefore (2) the required tangent planes are

$$lx + my + nz = \pm \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}} \quad \text{--- (3)}$$

Note: The equation (3) represents tangent planes for all values of l, m, n .

Thus any tangent plane to conicoid (1) is

$$lx + my + nz = \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}.$$

Director sphere:

To find the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid $ax^2 + by^2 + cz^2 = 1$

Sol: The given conicoid is

$$ax^2 + by^2 + cz^2 = 1 \quad \text{--- (1)}$$

$$\text{Let } l_1x + m_1y + n_1z = \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}} \quad \text{--- (2)}$$

$$l_2x + m_2y + n_2z = \sqrt{\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}} \quad \text{--- (3)}$$

$$\text{and } l_3x + m_3y + n_3z = \sqrt{\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}} \quad \text{--- (4)}$$

be three mutually ~~perpendicular~~ tangent planes

so that $l_1l_2 + m_1m_2 + n_1n_2 = 0$, etc. and

and $l_2l_3 + m_2m_3 + n_2n_3 = 0$, etc. and

$l_1l_3 + m_1m_3 + n_1n_3 = 0$, etc. and $l_1^2 + m_1^2 + n_1^2 = 1$, etc. } (5)

The co-ordinates of the point of intersection satisfy the three equations (2), (3), (4) and its locus is therefore obtained by eliminating l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 from equations squaring and adding (2), (3), (4). we have

$$\begin{aligned}
 & (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2) \\
 & = \left(\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2} \right) + \left(\frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2} \right) + \left(\frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2} \right) \\
 \Rightarrow & x^2(l_1^2 + l_2^2 + l_3^2) + y^2(m_1^2 + m_2^2 + m_3^2) + z^2(n_1^2 + n_2^2 + n_3^2) \\
 & + 2xy(l_1m_1 + l_2m_2 + l_3m_3) + 2yz(l_1n_1 + m_2n_2 + m_3n_3) \\
 & + 2zx(l_1n_1 + n_2l_2 + n_3l_3) = \frac{1}{a^2}(l_1^2 + l_2^2 + l_3^2) + \\
 & \quad \frac{1}{b^2}(m_1^2 + m_2^2 + m_3^2) + \frac{1}{c^2}(n_1^2 + n_2^2 + n_3^2) \\
 \Rightarrow & x^2(1) + y^2(1) + z^2(1) + 2xy(0) + 2yz(0) + 2zx(0) \\
 & = \frac{1}{a^2}(1) + \frac{1}{b^2}(1) + \frac{1}{c^2}(1) . \quad (1 \text{ from } 3)
 \end{aligned}$$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

which is the required locus and is a sphere concentric with the ellipsoid and is known as the director sphere.

Show that the length of the perpendicular from the origin on the tangent plane at the point (x_1, y_1, z_1) of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is given by } \frac{1}{P} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}$$

Sol: The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{(1)}$

The equation of the tangent plane at (x_1, y_1, z_1) to (1)

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1 = 0 \quad \text{(2)}$$

$$\text{Let } l_1x + m_1y + n_1z = \sqrt{\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}} \quad \dots(2)$$

$$l_2x + m_2y + n_2z = \sqrt{\frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}} \quad \dots(3)$$

$$\text{and } l_3x + m_3y + n_3z = \sqrt{\frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}} \quad \dots(4)$$

be three mutually \perp tangent planes so that

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= 0 \text{ etc. and } l_1m_1 + l_2m_2 + l_3m_3 = 0 \text{ etc.} \\ \text{and } l_1^2 + m_1^2 + n_1^2 &= 1 \text{ etc. and } l_2^2 + l_3^2 + l_3^2 = 1 \text{ etc.} \end{aligned} \quad \dots(5)$$

The co-ordinates of the point of intersection satisfy the three equations (2), (3), (4) and its locus is therefore obtained by eliminating $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ from equations. Squaring and adding (2), (3), (4), we have

$$(l_1x + m_1y + n_1z)^2 + (l_2x + m_2y + n_2z)^2 + (l_3x + m_3y + n_3z)^2 = \left(\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}\right) + \left(\frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}\right) + \left(\frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}\right)$$

$$\text{or } x^2(l_1^2 + l_2^2 + l_3^2) + y^2(m_1^2 + m_2^2 + m_3^2) + z^2(n_1^2 + n_2^2 + n_3^2) = 2x(l_1m_1 + l_2m_2 + l_3m_3) + 2xy(m_1n_1 + m_2n_2 + m_3n_3) + 2xz(l_1n_1 + l_2n_2 + l_3n_3)$$

$$= \frac{1}{a^2}(l_1^2 + l_2^2 + l_3^2) + \frac{1}{b^2}(m_1^2 + m_2^2 + m_3^2) + \frac{1}{c^2}(n_1^2 + n_2^2 + n_3^2)$$

$$\text{or } x^2(1) + y^2(1) + z^2(1) + 2xy(0) + 2yz(0) + 2zx(0) = \frac{1}{a^2}(1) + \frac{1}{b^2}(1) + \frac{1}{c^2}(1) \quad \text{Using (5)}$$

$$\text{or } x^2 + y^2 + z^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

which is the required locus and is a sphere concentric with the conoid and is known as the director sphere.

Example 1. Show that the length of the perpendicular from the origin on the tangent plane at the point (x', y', z') of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is given by}$$

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}$$

Sol. The given ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

The equation of the tangent plane at (x', y', z') to (1) is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1$$

$$\text{or } \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} - 1 = 0 \quad \dots(2)$$

If p is the \perp distance from the origin $(0, 0, 0)$ on (2), we have

$$p = \sqrt{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}}$$

$$\text{or } \frac{1}{p} = \sqrt{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}}$$

$$\text{Squaring, } \frac{1}{p^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}$$

which proves the required result.

Example 2. If P, Q are any two points on the ellipsoid, the plane through the centre and the line of intersection of the tangent planes at P, Q bisects PQ .

Sol. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots (1)$$

$$\left. \begin{aligned} \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} &= 1 \\ \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} &= 1 \end{aligned} \right\} \quad \dots (2)$$

and

$$\left. \begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} &= 1 \end{aligned} \right\} \quad \dots (3)$$

$\therefore P, Q$ lie on (1)
Now equations of the tangent planes at P and Q to (1) are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad \text{or} \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1 = 0 \quad \dots (3)$$

$$\text{and } \frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} = 1 \quad \text{or} \quad \frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} - 1 = 0 \quad \dots (4)$$

Now any plane through the line of intersection of (3) and (4) is

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1 \right) + k \left(\frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} - 1 \right) = 0 \quad \dots (5)$$

If it passes through the centre $(0, 0, 0)$ of the ellipsoid, then

$$0 - 1 + k(0 - 1) = 0 \quad \text{or} \quad k = 1$$

From (5), $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1$

$$- \left(\frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} - 1 \right) = 0$$

$$\text{or } \frac{x(x_1 - x_2)}{a^2} + \frac{y(y_1 - y_2)}{b^2} + \frac{z(z_1 - z_2)}{c^2} = 0 \quad \dots (6)$$

Now mid-point of PQ is

$$M \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

It lies on (6) if

$$\frac{(x_1+x_2)(x_1-x_2)}{2a^2} + \frac{(y_1+y_2)(y_1-y_2)}{2b^2} + \frac{(z_1+z_2)(z_1-z_2)}{2c^2} = 0$$

or if

$$\frac{x_1^2 - x_2^2}{2a^2} + \frac{y_1^2 - y_2^2}{2b^2} + \frac{z_1^2 - z_2^2}{2c^2} = 0$$

or if

$$\frac{1}{2} \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} \right) - \frac{1}{2} \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} \right) = 0$$

or if

$$\frac{1}{2}(1) - \frac{1}{2}(1) = 0 \quad [\text{Using (2)}]$$

or if $\frac{1}{2} - \frac{1}{2} = 0$ which is true. Hence the result.

Example 3: (a) A tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

meets the co-ordinate axes in A, B and C. Find the locus of the centroid of the (i) triangle ABC, (ii) tetrahedron OABC.

(Agra 1985, 87 ; Kanpur 1983)

(b) If P be the point of contact of a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

which meets the axes in A, B, C and PD, PE, PF are perpendiculars drawn from P to the axes, prove that

$$OD \cdot OA = a^2, OE \cdot OB = b^2, OF \cdot OC = c^2.$$

Sol. (a) Let P(x_1, y_1, z_1) be any point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad (2)$$

Equation of tangent plane at P(x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad (3)$$

This meets X-axis ($y=0, z=0$)

$$\text{where } \frac{xx_1}{a^2} = 1 \quad \therefore x = \frac{a^2}{x_1}$$

Thus (3) meets X-axis in the point A $\left(\frac{a^2}{x_1}, 0, 0 \right)$. Similarly it

meets Y-axis in B $\left(0, \frac{b^2}{y_1}, 0 \right)$, and Z-axis in C $\left(0, 0, \frac{c^2}{z_1} \right)$.

(i) Then if G(α, β, γ) be the centroid of $\triangle ABC$.

$$\alpha = \frac{\frac{a^2}{x_1} + 0 + 0}{3} = \frac{a^2}{3x_1} \quad \text{similarly } \beta = \frac{b^2}{3y_1}, \gamma = \frac{c^2}{3z_1}$$

$$\text{which give } x_1 = \frac{a^2}{3\alpha}, y_1 = \frac{b^2}{3\beta}, z_1 = \frac{c^2}{3\gamma}$$

Putting these values of (x_1, y_1, z_1) in (2), we get

$$\frac{1}{a^2} \cdot \frac{a^2}{9\alpha^2} + \frac{1}{b^2} \cdot \frac{b^2}{9\beta^2} + \frac{1}{c^2} \cdot \frac{c^2}{9\gamma^2} = 1,$$

or

$$\frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2} = 9.$$

\therefore Locus of $G(x, \beta, \gamma)$ is [changing (α, β, γ) to (x, y, z)]

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9.$$

(ii) Please try yourself.

$$[\text{Ans. } \frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 16]$$

(b) Let $P(x_1, y_1, z_1)$ be the point of contact.

Then the equation of the tangent plane at $P(x_1, y_1, z_1)$ to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1. \quad (1)$$

If PD, PE, PF are drawn from P on the axes, then $OD = x_1$, $OE = y_1$, $OF = z_1$. (Def. of co-ordinates)

Now the plane (1) meets X-axis ($y=0, z=0$) in the point A

$$\frac{xx_1}{a^2} = 1 \quad \text{or} \quad xx_1 = a^2$$

or

$$OA \cdot OD = a^2 \quad \therefore x = OA \cdot x_1 = OD$$

Similarly (1) meets Y-axis ($z=0, x=0$) in the point B

where

$$\frac{yy_1}{b^2} = 1 \quad \text{or} \quad yy_1 = b^2$$

or

$$OB \cdot OE = b^2$$

Similarly we can prove that $OC \cdot OF = c^2$.

Hence the result:

Example 4. The tangent plane to the surface $x^2 + 12y^2 + 4z^2 = 8$ at the point $(1, \frac{1}{2}, 1)$ meets the co-ordinate axes at A, B, C. Find the centroid of $\triangle ABC$. (Agra, 1986)

Sol. The tangent plane to the given surface at $(1, \frac{1}{2}, 1)$ is

$$x(1) + 12y(\frac{1}{2}) + 4z(1) = 8$$

or $x + 6y + 4z = 8$

which meets the co-ordinate axes at A, B and C.

$$\Rightarrow A(8, 0, 0), \quad B(0, 4/3, 0), \quad C(0, 0, 2)$$

\therefore Centroid of $\triangle ABC$ is,

$$\frac{8+0+0}{3}, \quad \frac{0+\frac{4}{3}+0}{3}, \quad \frac{0+0+2}{3}$$

i.e.,

$$\left(\frac{8}{3}, \frac{4}{9}, \frac{2}{3} \right).$$

Example 5. A tangent plane to the conoid $ax^2 + by^2 + cz^2 = 1$ meets the co-ordinate axes in P, Q and R . Find the locus of the centroid of the $\triangle PQR$.

Sol.

Any tangent plane to the given conoid is

$$lx + my + nz = \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

This plane meets the x -axis at P ; so the co-ordinates of P , are

$$\left[\left(\frac{1}{l} \right) \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}, 0, 0 \right]$$

putting $y=0=z$ in (1),

Similarly Q and R are

$$\left[0, \frac{1}{m} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}, 0 \right]$$

and

$$\left[0, 0, \frac{1}{n} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}} \right]$$

If (x_1, y_1, z_1) be the centroid of $\triangle PQR$, then

$$x_1 = \frac{1}{3} \left[\frac{1}{l} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}} + 0 + 0 \right] \\ = \frac{1}{3l} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

$$y_1 = \frac{1}{3m} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

$$z_1 = \frac{1}{3n} \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

or

$$(3l)^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$$

$$\frac{9l^2}{a} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \right) \cdot \frac{1}{a^2}$$

Similarly, $\frac{9m^2}{b} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \right) \cdot \frac{1}{b^2}$

$$\text{and } \frac{9n^2}{c} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \right) \cdot \frac{1}{c^2}$$

Addition, $\frac{9(l^2 + m^2 + n^2)}{a+b+c} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} \right) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$

$$\text{or } \frac{9(l^2 + m^2 + n^2)}{a+b+c} = 9 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$$

The required locus is

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 9.$$

Example 6. Show that the plane $3x^2 - 6y^2 + 9z^2 + 17 = 0$ touches the conoid $x^2 - 2y^2 + 3z^2 = 2$.

(b) $3x^2 + 12y^2 - 6z^2 - 17 = 0$ touches the conoid

(Bundelkhand 1985)

Find also the points of contact in each case.

Sol. (a) Let the plane

$$3x^2 - 6y^2 + 9z^2 + 17 = 0 \quad \dots(1)$$

touch the conoid

$$x^2 - 2y^2 + 3z^2 = 2 \quad \dots(2)$$

at the point (x_1, y_1, z_1) .

The equation of tangent plane at (x_1, y_1, z_1) to (2) is

$$x_1 x - 2y_1 y + 3z_1 z = 2 \quad \dots(3)$$

Since (3) and (1) are identical, i.e., comparing (3) and (1), we get

$$\frac{x_1}{3} = -2 \frac{y_1}{6} = 3 \frac{z_1}{9} = \frac{2}{17}$$

Now the plane (1) will touch (2) if the point of contact

$$(x_1, y_1, z_1)$$
 i.e., (1, -1, 1), lies on the conoid (2), i.e., if

$$(1)^2 - 2(-1)^2 + 3(1)^2 = 2 \quad \text{or, } 1 + 2 + 3 = 2 \text{ or, } 2 = 2$$

which is true.

Hence the plane (1) touches the conoid (2) and the point of contact is (x_1, y_1, z_1) i.e., (1, -1, 1).

(b) Please try yourself.

Example 7. Find the equations to the tangent planes to the surface

$$4x^2 - 3y^2 + 7z^2 + 13 = 0, \text{ parallel to the plane}$$

$$4x + 20y - 21z = 0.$$

Sol. (a) Any plane || to $4x + 20y - 21z = 0$

$$4x + 20y - 21z = p \quad \text{Type } 4x + 20y + nz = p$$

The given conoid is

$$4x^2 - 3y^2 + 7z^2 + 13 = 0$$

$$4x^2 - 3y^2 + 7z^2 = -13$$

or

$$-\frac{4}{3}x^2 + \frac{5}{3}y^2 - \frac{7}{3}z^2 = 13$$

or

$$-\frac{4}{3}x^2 + \frac{5}{3}y^2 - \frac{7}{3}z^2 = -13$$

[Form $a_1 x^2 + b_1 y^2 + c_1 z^2 = 1$]

(1) will touch the conoid (1) if

$$\frac{4}{3}x^2 + \frac{5}{3}y^2 + \frac{7}{3}z^2 = p^2$$

$$\text{or } \text{If } \left(-\frac{4}{13}\right)^2 + \left(\frac{20}{13}\right)^2 + \left(\frac{(-2)^2}{13}\right) = K^2$$

$$\text{or } \text{If } -52 + 1040 - 819 = K^2$$

$$\text{or } \text{If } K^2 = 1040 - 871 = 169$$

Putting these values of K in (i), we required tangent planes are

$$4x + 20y - 21z = \pm 13.$$

(b) Please try yourself.

Example 8. Find the coordinates of the point of contact of the plane $4x - 6y + 3z = 5$ and the conicoid $2x^2 - 6y^2 + 3z^2 = 5$. (Bundelkhand 1984)

Sol. Let the plane

$$4x - 6y + 3z = 5 \\ \text{touch the boncioid} \quad 2x - 6y + 3z = 5 \\ \text{at } (x_1, y_1, z_1).$$

The tangent plane to (i) at (x_1, y_1, z_1) is

$$2x_1 - 6y_1 + 3z_1 = 5$$

As (i) and (ii) represent the same plane, so comparing them, we have

$$\frac{2x_1}{4} - \frac{-6y_1}{-6} = \frac{3z_1}{3} = \frac{5}{3}$$

which gives $x_1 = 2, y_1 = 1, z_1 = 1$.

Required point is $(2, 1, 1)$.

Example 9. (a) Find the equations to the two tangent planes which contain the line given by $\begin{matrix} 7x - 10y - 30 = 0, \\ 2x - 6y + 3z = 5 \end{matrix}$ and touch the conicoid $\begin{matrix} 7x^2 - 6y^2 + 3z^2 = 0 \\ 2x^2 - 6y^2 + 3z^2 = 5 \end{matrix}$

(b) Find the equations of the tangent plane to which pass through the line $\begin{matrix} x + 9y - 3z = 0 \\ 2x - 6y + 3z = 5 \end{matrix}$ (Agra 1986, 84; M.D.U. 1984)

(c) Find the equations of the tangent planes to which pass through the line $\begin{matrix} 7x - 3y - 2z = 0 \\ 7x + 10y - 30 = 0, \\ 2x - 6y + 3z = 0 \end{matrix}$ (M.D.U. 1986; I.I.T. 1983)

(Imp.) (M.D.U. 1986; I.I.T. 1983)

(d) Find the equations of the tangent planes to

$$7x - 3y - 2z = 0$$

which pass through the line

$$7x - 6y + 9 = 0, z = 3.$$

Sol. (e) The given line is

$$7x - 3y - 2z = 0$$

An plane through this line is

$$7x + 5(k+2)y - 3z = 30.$$

Form $lx + my + nz = p$... (1)

$$\text{Using } \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$$

$$\text{The plane (1) touches (2) if } \left(\frac{l+3k}{2}\right)^2 + \left(\frac{5(k+2)}{6}\right)^2 + \left(\frac{9(1-2k)}{3}\right)^2 = (3k)^2$$

$$\text{Using } \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$$

$$\text{The given ellipsoid is } 7x^2 + 5y^2 + 3z^2 = 60$$

$$\text{or } \frac{7}{60}x^2 + \frac{5}{60}y^2 + \frac{3}{60}z^2 = 1 \quad \text{[Form } ax^2 + by^2 + cz^2 = 1 \text{]}$$

$$\text{The plane (1) touches the ellipsoid (2) if } \left(\frac{l+3k}{\sqrt{60}}\right)^2 + \left(\frac{5(k+2)}{\sqrt{60}}\right)^2 + \left(\frac{9(1-2k)}{\sqrt{60}}\right)^2 = (3k)^2$$

$$\text{or } \frac{(l+3k)^2}{60} + \frac{25(k+2)^2}{60} + \frac{81(1-2k)^2}{60} = (3k)^2$$

$$\text{or } 49k^2 + 300(k^2 + 4k + 4) + 180k^2 = 900$$

$$\text{or } 480k^2 + 1200k + 720 = 0 \quad \text{or. } (2k+3)(6k+1) = 0$$

$$\therefore k = -\frac{3}{2}, \text{ or } k = -\frac{1}{6}$$

$$\text{Putting these values of } k \text{ in (1), the required tangent planes are}$$

$$7x + 5\left(-\frac{3}{2} + 2\right)y + \frac{9}{2}z = 30 \quad \text{and} \quad 7x + 5(-1+2)y + 3z = 30$$

$$\text{or } 7x + \frac{5}{2}y + \frac{9}{2}z = 30 \quad \text{and} \quad 7x + 5y + 6z = 30$$

$$\text{or } 14x + 5y + 9z = 60 \quad \text{and} \quad 7x + 5y + 3z = 20.$$

(b) The given line is

$$x + 9y - 3z = 0 = 3x + 3y + 6z - 5$$

Any plane through this line is

$$x + 9y - 3z + k(3x + 3y + 6z - 5) = 0$$

$$\text{or } x(1+3k) + 3(3-k)y + 3(1-2k)z = 5k \quad \text{[Type } lx + my + nz = p \text{]}$$

and thus given conicoid is

$$2x^2 - 6y^2 + 3z^2 = 5$$

$$\text{The plane (1) touches (2) if } \left(\frac{1+3k}{2}\right)^2 + \left(\frac{5(1-k)}{6}\right)^2 + \left(\frac{9(1-2k)}{3}\right)^2 = (5k)^2$$

$$\text{or } \frac{2}{4}x^2 - \frac{5}{6}y^2 + \frac{3}{2}z^2 = 1 \quad \text{[Form } ax^2 + by^2 + cz^2 = 1 \text{]}$$

$$pr \quad \frac{5}{2} (1+3k)^2 - \frac{45}{8} (3-k)^2 + \frac{45}{3} (1-2k)^2 = 250 \\ \text{or} \quad (1+3k)^2 - 3(3-k)^2 + 6(1-2k)^2 = 10k^2$$

or

$$1+9k^2+6k-3(9+k^2-6k)+6(1+4k^2-9k)-10k^2=0$$

or

$$1+9k^2+6k-27-3k^2+18k+9+24k-24k=10k^2=0$$

or

$$20k^2-20=0 \quad \text{or} \quad k=\pm 1 \quad \therefore k=\pm 1$$

Putting these values of k in (1), the required tangent planes are

$$(1+3)x+3(3+1)y-3(1-2)z=5$$

and

$$4x+6y+3z-5=0 \quad \text{and} \quad -2x+12y-9z+5=0$$

or

$$4x+6y+3z-5=0 \quad \text{and} \quad 2x-12y+9z-5=0.$$

(c) Please try yourself.
(Ans. $7x-6y-4z+21=0$, $14x-12y-2+21=0$,
 $7x-6y+9z+17=0$ parallel to the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{2}=0$.)

Sol. Any plane parallel to the given plane is
 $x+4y-2z=p$.
If this plane touches the ellipsoid
 $3x^2-6y^2+9z^2+17=0$

or
 $3x^2-6y^2+9z^2=-17$
or
 $(-\frac{3}{17})x^2+(\frac{6}{17})y^2-(\frac{9}{17})z^2=1$.

then the condition of tangency is
 $\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$

or
 $(\frac{-1}{17})x^2 + (\frac{6}{17})y^2 + (\frac{-9}{17})z^2 = p^2$

or
 $p^2 = (-\frac{1}{17}) + (\frac{136}{17}) - (\frac{68}{9})$

or
 $p^2 = -51 + 408 - 68 = 289$

or
 $p = \pm \sqrt{\frac{289}{9}} = \pm (\frac{17}{3})$

∴ From (1) the required tangent planes are

$$x+4y-2z = \pm (\frac{17}{3})$$

$$3x+12y-6z = \pm 17.$$

Example 11. Find the equations to the tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which pass through the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, $z=3$.
(Ans. $\frac{7x-6y-4z+21}{a^2} = 0$, $\frac{14x-12y-2+21}{b^2} = 0$, $\frac{7x-6y+9z+17}{c^2} = 0$)

Example 12. Find the condition that the line $\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c}$ may touch the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
(Ans. 1983)

Sol. Given line is ρ
 $\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c}$
Given ellipsoid is
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow \frac{3}{a^2} x^2 + \frac{8}{b^2} y^2 + \frac{1}{c^2} z^2 = 1$.
The condition for becoming a tangent line is
 $a^2l + b^2m + c^2n = 0$.

$\frac{3}{a^2} \cdot (2), l + \frac{8}{b^2} \cdot (1), m + \frac{1}{c^2} \cdot 3, n = 0$.

Ex. 13. If P is the point of contact of a tangent plane ABC to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and PD, PE, PF are perpendiculars from P on the axes, prove that $OD \cdot OA = a^3$, $OE \cdot OB = b^3$, $OF \cdot OC = c^3$; A, B, C being the points where the tangent plane at P meets the coordinate axes.

Sol. Let $P=(x, y, z)$ so that the equation of tangent plane ABC is
 $\frac{ax}{a^2} + \frac{by}{b^2} + \frac{cz}{c^2} = 1$
It meets the co-ordinates axes in A, B, C .
OA = $\frac{a^2}{a} = a$, OB = $\frac{b^2}{b} = b$ and OC = $\frac{c^2}{c} = c$
Also PD, PE, PF are perpendiculars from P on the axes,
 $OD = a$, $OB = b$ and $OP = r$
Hence $OD \cdot OA = a \cdot a = a^2$

OB . OB = $b \cdot b = b^2$

OC . OC = $c \cdot c = c^2$

and
 $OD \cdot OB = a \cdot b = ab$
or
 $OC \cdot OC = c \cdot c = c^2$

Example 14. Show that the tangent planes at the extremities of any diameter of an ellipsoid are parallel.

Sol. Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

As its centre is $(0, 0, 0)$ so any diameter of this ellipsoid is a line through $(0, 0, 0)$ and its equation is given by

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r$$

Any point on this diameter is (r, br, cr) .

If this point lies on (i), then

$$\frac{r^2}{a^2} + \frac{b^2 r^2}{b^2} + \frac{c^2 r^2}{c^2} = 1$$

$$r^2 \left[\frac{1}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right] = 1$$

$$r = \pm \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{n}{c}\right)^2} = \pm k$$

A. The extremities of the diameter (ii) are

(k, mk, nk) and $(-k, -mk, -nk)$.

Now the equation of the tangent plane to the ellipsoid (i) at (k, mk, nk) is given by

$$\frac{x}{a} + \frac{my}{b} + \frac{nk}{c} = 1$$

or

$$\frac{ax}{a^2} + \frac{bmy}{b^2} + \frac{cnk}{c^2} = \frac{1}{k}$$

Similarly the equation of the tangent plane to (i) at the other extremity $(-k, -mk, -nk)$ of (ii) is

$$\frac{x}{a} - \frac{my}{b} - \frac{nk}{c} = -\frac{1}{k}$$

Since the equations (ii) and (iii) differ in the constant terms in x, y, z , so these represent parallel planes (each being a linear equation).

Example 15. Through a fixed point $(k, 0, 0)$ pairs of perpendicular lines are drawn to the coaxial $a_1x + b_1y + c_1z = 1$. Show that the

cone

$$(b+c)(ak^2-1) + \frac{a}{c}(ck^2-1) + \frac{b}{c}(ak^2-1) - \frac{a^2}{c} = 0$$

Sol. Any line through the point $(k, 0, 0)$ is

$$\frac{x-k}{m} = \frac{y}{n} = \frac{z}{p}$$

Eliminating m, n, p between (i) and (ii), we get

$$\frac{x-k}{m} = \frac{y}{n} = \frac{z}{p} = \frac{2}{m_3}$$

and that the normal to the plane containing the lines, given by (i)

$$(x-k)^2(b+c)(ak^2-1) + y^2(c(ck^2-1) - a) + z^2(c(ck^2-1) - d) = 0$$

Any point on (i) is $(k+r, mr, nr)$, which is at a distance r from the origin.

The distance of the points where the line (i) meets the given coaxial

$$a_1x + b_1y + c_1z = 1$$

$$d((k+r)^2 + b(mr)^2 + c(nr)^2) = 1$$

$$r^2(a^2 + b^2 + c^2) + 2kr + (ck^2 - 1) = 0 \quad \dots(iii)$$

$$\text{Given by (iii), if the line (i) touches (ii) at } (k, 0, 0), \text{ then the two values of } r \text{ must be coincident and this condition for the same is}$$

$$\frac{(2rk)^2}{B^2} = \frac{4(a^2 + b^2 + c^2)(ck^2 - 1)}{4A^2C^2} \quad \dots(iv)$$

$$\text{or } (al^2 + bl^2 + cl^2)(ak^2 - 1) = a^2k^2/2 \quad \dots(v)$$

Now let the two perpendicular tangent lines through $(k, 0, 0)$ be

$$\frac{x-k}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \quad \dots(vi)$$

$$\text{and } \frac{x-k}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad \dots(vii)$$

Then from (vi), we get

$$(al^2 + bl^2 + cl^2)(al^2 - 1) = a^2k^2/l_1^2 \quad \dots(viii)$$

$$\text{and } (al^2 + bl^2 + cl^2)(dk^2 - 1) = a^2k^2/l_2^2 \quad \dots(ix)$$

Adding these, we get

$$[l_1^2(l_1^2 + l_2^2) + bl^2m_1^2 + cl^2n_1^2] + [l_2^2(l_1^2 + l_2^2) + al^2m_2^2 + cn^2l_2^2] = a^2k^2(l_1^2 + l_2^2) \quad \dots(x)$$

If the line $\frac{(x-k)^2}{l_1^2} = \frac{(y-0)^2}{m_1^2} = \frac{(z-0)^2}{n_1^2}$ be the normal to the

plane containing the tangent lines given by (vi), then we obtain a set of three mutually perpendicular lines, for which we have the relations

$$\frac{l_1^2}{a^2} + \frac{k^2}{b^2} + \frac{l_2^2}{c^2} = 1 = \frac{l_1^2}{a^2} + \frac{k^2}{b^2} + \frac{m_1^2}{n_1^2} \quad \dots(x)$$

$$\text{Similarly } m_1^2 + m_2^2 = n_1^2 + n_2^2, n_1^2 + n_2^2 = l_1^2 + l_2^2 \quad \dots(x)$$

Substituting these values in (xi), we get

$$[a(m_1^2 + n_1^2) + b(l_1^2 + l_2^2) + c(l_1^2 + m_1^2)](ck^2 - 1) = a^2k^2(m_1^2 + n_1^2) \quad \dots(xii)$$

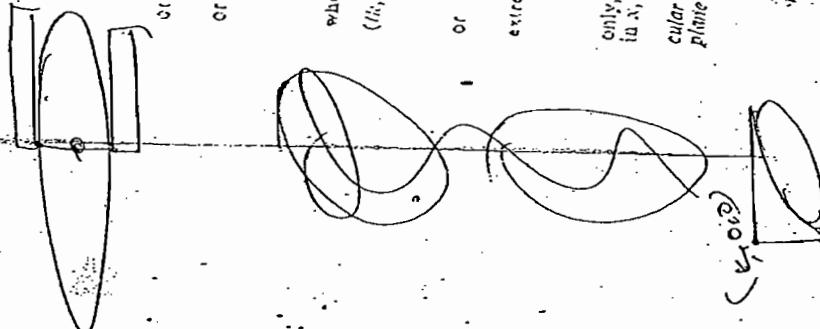
$$l_1^2[(b+c)(ak^2 - 1)] + m_1^2[(a+c)(ak^2 - 1) - d^2/l_1^2] = 0 \quad \dots(xiii)$$

$$l_1^2(b+c)(ak^2 - 1) + m_1^2(b(ak^2 - 1) - d) = 0 \quad \dots(xiv)$$

$$\text{or } l_1^2(b+c)(ak^2 - 1) + \frac{a^2}{c}(ck^2 - 1) - d = 0 \quad \dots(xv)$$

$$\text{and that the normal to the plane containing the lines, given by (v)}$$

$$(x-k)^2(b+c)(ak^2 - 1) + y^2(c(ck^2 - 1) - a) + z^2(c(ck^2 - 1) - d) = 0 \quad \dots(xvi)$$



and the plane itself touches the reciprocal cone

$$\frac{(x-k)^2}{(k+c)(ak-b)} + \frac{y^2}{c(ak-b)} - \frac{z^2}{a} = 0.$$

Example 16. Prove that the equation to the two tangent planes to the conoid $\frac{ax^2+by^2+cz^2}{abc}=1$

which pass through the line

$$u^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) - 2un \left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp' \right) + un^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) = 0$$

Sol. Any plane through the line

$$u=0, u'=0 \text{ is } u+ku'=0$$

i.e.,

$$(l+k)x + (m+k)y + (n+k)z = p + kp'$$

or

$$(l+k)x + (m+k)y + (n+k)z = p + kp' \quad \text{[From } lx+my+nz=p]$$

This will be tangent plane to the conoid

$$ax^2+by^2+cz^2=1$$

or

$$\frac{(l+k)^2}{a} + \frac{(m+k)^2}{b} + \frac{(n+k)^2}{c} = (p+kp')^2$$

$$\text{Using } \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

or if

$$k^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) + 2k \left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp' \right) + \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) = 0$$

$$\text{Putting } k = \frac{-u}{u'}, \text{ from (1), the required equation is}$$

$$\frac{u^2}{u'^2} \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) - 2 \frac{u}{u'} \left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp' \right) + \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) = 0$$

$$\text{or } u^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) - 2un \left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp' \right) + un^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2 \right) = 0$$

$$\text{Hence the result.}$$

(b) Tangent planes are drawn to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{[V. Imp.]}$$

through the point (x_1, y_1, z_1) . Prove that the perpendiculars to them from the origin generate the cone

$$(ax+by+cz)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \quad \text{[Top. Imp.]}$$

(c) Tangent planes are drawn to the conoid

$$ax^2+by^2+cz^2=1 \quad \text{[Top. Imp.]}$$

through the point (x_1, y_1, z_1) . Prove that the perpendiculars to them from the origin generate the cone

$$(ax+by+cz)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \quad \text{[K.U. 1986 ; Kanpur 1981]}$$

Sol.: (a) The given plane is $lx+my+nz=p$ [Imp.]

and the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ [1]

Let the plane (1) touch the ellipsoid (2) at (x_1, y_1, z_1) . Then plane (1) will be identical with the tangent plane at (x_1, y_1, z_1) to the surface (2), i.e.,

$$\frac{x_{11}}{a^2} + \frac{y_{11}}{b^2} + \frac{z_{11}}{c^2} = 1 \quad \text{[1]}$$

Comparing (1) and (3), we get

$$\frac{x_1}{a^2} = \frac{y_1}{b^2} = \frac{z_1}{c^2} \quad \text{or}$$

$$\frac{x_1}{a^2} = \frac{y_1}{b^2} = \frac{z_1}{c^2} = \frac{1}{p} \quad \text{or}$$

$$\frac{x_1}{a^2} = \frac{y_1}{b^2} = \frac{z_1}{c^2} = \frac{1}{p} \quad \text{or}$$

$$x_1 = \frac{a^2}{p}, y_1 = \frac{b^2}{p}, z_1 = \frac{c^2}{p} \quad \text{[4]}$$

Since (x_1, y_1, z_1) being the point of contact lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{or } \frac{1}{a^2} \left(\frac{\alpha \beta \gamma}{\rho} \right)^2 + \frac{1}{b^2} \left(\frac{b^2 m}{\rho} \right)^2 + \frac{1}{c^2} \left(\frac{c^2 n}{\rho} \right)^2 = 1 \quad | \text{ Using (4).}$$

which is the required condition.

(b) Any plane through (α, β, γ) is

$$(\alpha - \sigma) + m(\gamma - \beta) + n(\epsilon - \gamma) = 0$$

$\alpha + m\gamma + n\epsilon = \lambda(\alpha + m\beta + n\gamma)$

This will touch the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Using } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = p^2 \quad | \text{ Form } \alpha x + m\gamma + n\epsilon = p$$

Now equations of the normal to (1) through $(0, 0, 0)$ are

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{p}$$

Locus of the line (3) is [eliminating l, m, n from (2) and (3)]

$$\alpha^2 x^2 + b^2 y^2 + c^2 z^2 = (\alpha + m\beta + n\gamma)^2$$

which being a second degree homogeneous equation in x, y, z represents a cone.

(c) Any plane through (α, β, γ) is

$$(\alpha - \sigma) + m(\gamma - \beta) + n(\epsilon - \gamma) = 0$$

$\alpha + m\gamma + n\epsilon = \lambda(\alpha + m\beta + n\gamma)$

If it is the tangent plane to the conicoid

$$\alpha^2 + b^2 y^2 + c^2 z^2 = 1 \quad | \text{ Form } \alpha x + m\gamma + n\epsilon = p$$

then

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = (\alpha + m\beta + n\gamma)^2 \quad | \text{ Using } \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$$

Now d.c.'s of the normal to plane (1) are proportional to l, m, n [Co-effs. of x, y, z in (2)]

(1) through the origin are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad \text{or } \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

To find the locus of line (3) we have to eliminate l, m, n from (3) and (2). Putting the values of l, m, n from (3) in (2), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\alpha + m\beta + n\gamma)^2$$

which is the required equation of cone.

I.C.T.M.

Example 18. Obtain the tangent planes for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

which are parallel to $\alpha x + m\gamma + n\epsilon = 0$.
Given the distance between the planes, show that a line through

$$\alpha^2 + b^2 + c^2 = r^2$$

is perpendicular to the planes lies on the cone

$$(\alpha^2 - r^2) + y(b^2 - r^2) + z(c^2 - r^2) = 0.$$

[V. Imp.] (K.S.U. 1983; Rohilkhand 1982)

Sol: Any plane \parallel to $\alpha x + m\gamma + n\epsilon = 0$ is

$$\alpha x + m\gamma + n\epsilon = p \quad | \text{ This will touch the ellipsoid}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad | \text{ If}$$

$$\alpha^2 + b^2 m^2 + c^2 n^2 = p^2$$

Putting these values of p in (1), the required tangent planes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \sqrt{\alpha^2 + b^2 m^2 + c^2 n^2} \quad | \text{ and}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2} \quad | \text{ Now one point on the plane (2) is (spurting } x=0, y=0)$$

$$P = \frac{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r} \quad | \text{ Now distance between two } \parallel \text{ planes is the } \perp \text{ distance of a point}$$

on one plane from the other.

Since $2r$ is given to be the distance between two \parallel planes (2) and (3).

$$2r = \perp \text{ distance of } P \text{ from the plane (3)}$$

$$0 + 0 + n \cdot \frac{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r} = \frac{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r} + \frac{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r}$$

$$= \frac{2\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r} \quad | \text{ or}$$

$$\frac{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}}{r} = \sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}$$

$$\text{Squaring } \sqrt{\alpha^2 + b^2 m^2 + c^2 n^2} = \alpha^2 + b^2 m^2 + c^2 n^2$$

$$\alpha^2 + b^2 + c^2 = r^2 \quad | \text{ or}$$

$$\text{Now equations of the line through } (0, 0, 0) \text{ and } \perp \text{ to the tangent planes (2) or (3) are}$$

$$\frac{x-0}{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}} = \frac{y-0}{r} = \frac{z-0}{r} \quad | \text{ or } \frac{x}{\sqrt{\alpha^2 + b^2 m^2 + c^2 n^2}} = \frac{y}{r} = \frac{z}{r} \quad | \text{ ... (5)}$$

Eliminating l, m, n from (5) and (4), by putting the values of

$$x^2(a^2 - r^2) + y^2(b^2 - r^2) + z^2(c^2 - r^2) = 0$$

which is a cone, being a homogeneous equation in x, y, z .

Example 19. If the line of intersection of two perpendicular

axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

passes through the fixed point $(q, 0, k)$, show that it lies on the cone

$$x^2(b^2 + c^2 - k^2) + y^2(a^2 + c^2 - k^2) + z^2(a^2 + b^2 - k^2) = 0.$$

Sol. The equation of the ellipsoid is [imp.]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

Ay line through $(0, 0, k)$ is

$$\frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} \quad \text{or} \quad \frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} \quad \dots(2)$$

and any plane through this line (2) is

$$Lx + My + Nz - k = 0 \quad \dots(3)$$

If the plane (3) i.e., $Lx + My + Nz - k = 0$ touches the ellipsoid (1), then

$$L^2a^2 + M^2b^2 + N^2c^2 = \frac{k^2}{a^2} \quad \text{Using } l^2a^2 + m^2b^2 + n^2c^2 = p^2$$

or $\frac{l^2a^2 + M^2b^2 + N^2c^2}{p^2} = \frac{k^2}{a^2} \quad \dots(4)$

If the tangent planes are \perp , their normals are also

(4) and (5) are normals to the planes (3).

Now the lines whose d.c.s L, M, N are given by the equations

From (4), $L = \pm \frac{Ma+Nn}{Na+Mb}$. Putting this in (5), we get

$$\frac{(Ma+Nn)^2}{p^2} a^2 + M^2b^2 + N^2(c^2 - k^2) = 0 \quad \dots(6)$$

or $M^2(a^2 + b^2) + 2MNa + N^2(c^2 - k^2) = 0$

Dividing throughout by N^2 , we get

$$\frac{M^2}{N^2} (a^2n^2 + b^2n^2) + 2Mna + \frac{1}{N^2} + (c^2 - k^2) = 0 \quad \dots(6)$$

which is a quadratic in $\frac{M}{N}$. If L_1, M_1, N_1 and L_2, M_2, N_2 are the

d.c.'s of the two lines, then, $\frac{M_1}{N_1}, \frac{M_2}{N_2}$ are the roots of (6), so that

$$\frac{M_1 M_2}{N_1 N_2} = \frac{a^2p^2 + b^2p^2 - k^2p^2}{a^2n^2 + b^2n^2} = \frac{(c^2 - k^2)p^2 + n^2p^2}{a^2n^2 + b^2n^2} \quad \dots(7)$$

$$\frac{M_1 M_2}{N_1 N_2} = \frac{(c^2 - k^2)p^2 + n^2p^2}{a^2n^2 + b^2n^2} \quad \dots(7)$$

Similarly, eliminating M between (4) and (5), we have

$$\frac{N_1 N_2}{L_1 L_2} = \frac{1}{a^2n^2 + b^2n^2} \quad \dots(8)$$

Since the two normals with d.c.'s L_1, M_1, N_1 and L_2, M_2, N_2 are \perp , From (7) and (8)

$$\frac{L_1 L_2}{N_1 N_2} + M_1 M_2 + N_1 N_2 = 0 \quad \dots(9)$$

$$l^2(a^2 + b^2 + c^2 - k^2) + m^2(a^2 + b^2 + c^2 - k^2) + n^2(a^2 + b^2 - k^2) = 0 \quad \dots(9)$$

Hence the result.

Example 20. Find the locus of the feet of perpendiculars from the origin to the tangent planes to the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which cuts off from the axes, intercepts the sum of whose reciprocals is equal to constant $\frac{1}{K}$. (K.U. 1983)

Sol. The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $\dots(1)$

Let $P(x_1, y_1, z_1)$ be the foot of \perp from $O(0, 0, 0)$ to any tangent plane to (1).

d.r.'s of OP are $x_1 - 0, y_1 - 0, z_1 - 0$ $| x_1 - x_1, y_1 - y_1, z_1 - z_1$

or x_1, y_1, z_1

OP is \perp to the tangent plane,

i.e., d.r.'s of OP are co-effs. of x, y, z in the equation of tangent plane.

Equation of tangent plane (1) is

$$\frac{xx_1 + yy_1 + zz_1}{a^2} = p \quad \dots(2)$$

Plane (2) touches (1) $|$ Using $a^2x^2 + b^2y^2 + c^2z^2 = p^2$ $\dots(3)$

Plane (2) meets X-axis ($y=0, z=0$) where

$$xx_1 = p \quad \text{or} \quad x = \frac{p}{x_1}$$

True the plane cuts off intercept from the X-axis which is

The sum of reciprocals of intercepts = $\frac{1}{k}$ (given)

$$\frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} = \frac{1}{k}$$

$$\frac{x_1 + y_1 + z_1}{a+b+c} = \frac{1}{k}$$

Eliminating p [By putting this value of p in (3)], we get

$$a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = k(x_1 + y_1 + z_1)^2$$

Locus of foot of \perp (x_1, y_1, z_1) is

$$a^2x^2 + b^2y^2 + c^2z^2 = k^2(x+y+z)^2$$

Example 21. Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

to any of its tangent planes is $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$. [Imp.]
(Kanpur 1981; K.U. 1986, 82; M.D.U. 1983; Allahabad 1986)

Sol. The given conicoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$... (1)

Let $P(x_1, y_1, z_1)$ be the foot of \perp from centre $O(0, 0, 0)$ of (1) to any tangent plane to (1)

d.r.'s of OP are x_1, y_1, z_1

Equation of tangent plane is $\frac{x_1}{a}x + \frac{y_1}{b}y + \frac{z_1}{c}z = p$... (2) [\because Tangent plane is \perp to OP , and hence co-effs. of x, y, z in tangent plane are d.r.'s of OP]

Plane (2) touches ellipsoid (1)
 $\therefore a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = p^2$... (3) [Using $a^2p^2 + b^2m^2 + c^2n^2 = p^2$]

Also $P(x_1, y_1, z_1)$ lies on (2) [$\therefore (0, 0, 0)$ to the tangent plane (2)]

$\therefore x_1^2 + y_1^2 + z_1^2 = p^2$... (4)

Eliminating p from (3) and (4), by equating its values, we get
 $a^2x_1^2 + b^2y_1^2 + c^2z_1^2 = (x_1^2 + y_1^2 + z_1^2)^2$

\therefore Locus of (x_1, y_1, z_1) , the foot of \perp is

$$a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$$

Example 22. If P is the point on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, such that the perpendicular from the origin on the tangent plane at P is of unit length, show that P lies on one or other of the planes $3y = \pm z$:

Sol. Let P be the point (x_1, y_1, z_1) .

Since P lies on the ellipsoid

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad \dots (1)$$

$$\frac{x_1^2 + y_1^2 + z_1^2}{a^2 + b^2 + c^2} = 1 \quad \dots (2)$$

Now equation of tangent plane at $P(x_1, y_1, z_1)$ to (1) is

$$x_1x + 2y_1y + 2z_1z = 1 \quad \dots (3)$$

Now the \perp distance of P from $(0, 0, 0)$ to (3) = 1
or $\frac{|x_1x_1 + 2y_1y_1 + 2z_1z_1 - 1|}{\sqrt{a^2 + b^2 + c^2}} = 1$

$$\therefore |x_1^2 + 4y_1^2 + 2z_1^2 - 1| = 0 + 0 + 0 - 3 \quad \dots (4)$$

Example 23. (a) Prove that the locus of the points of intersection of three mutually perpendicular tangent planes to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is

(b) Prove that the locus of points from which three mutually perpendicular tangent planes can be drawn to touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$

Sol. (a) Please try yourself as in Art. 9.

(b) Let

$$x + my + nz = p$$

be a tangent plane to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$

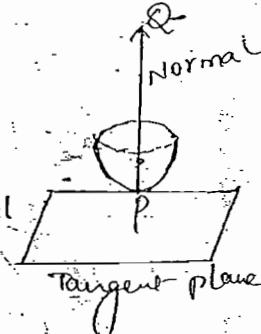
Is the sphere.

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2$$

Example 24. If P is the point on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, such that the perpendicular from the origin on the tangent plane at P is of unit length, show that P lies on one or other of the planes $3y = \pm z$:

NORMALS

The normal at any point P of a surface (quad.) is a line through the point of contact P and perpendicular to the tangent plane at P.

EQUATIONS OF THE NORMAL:

To find the equations of the normal at the point (x_1, y_1, z_1) of the conicoid $ax^2 + by^2 + cz^2 = 1$.

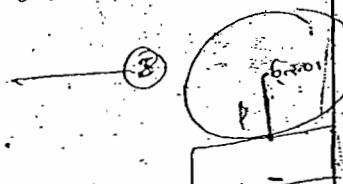
Sol: The given conicoid is $ax^2 + by^2 + cz^2 = 1$ (1)
equation of tangent plane at (x_1, y_1, z_1) to (1)

$$ax_1x + by_1y + cz_1z = 1 \quad (2)$$

The d.c.'s of the normal to this plane are proportional to ax_1, by_1, cz_1 .

Equations of the normal at $P(x_1, y_1, z_1)$ to (1) [i.e., a line through (x_1, y_1, z_1) and \perp to the tangent plane (2)] are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{cz_1}$$

Actual d.c.'s form:

If p is the length of the perpendicular distance from the centre $(0, 0, 0)$ to the tangent plane (2)

then $p = \sqrt{\frac{1}{a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2}}$

The d.c's of the normal at (x_1, y_1, z_1) to (1) are proportional to ax_1, by_1, cz_1 .

Dividing each by $\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}$

\therefore the actual direction cosines are

$$\frac{ax_1}{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}}, \frac{by_1}{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}}, \frac{cz_1}{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}}$$

$$\Rightarrow ax_1 p, by_1 p, cz_1 p \quad (\because p = \frac{1}{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}})$$

The equations of normal at (x_1, y_1, z_1) in actual direction cosines form are

$$\frac{x-x_1}{ax_1 p} = \frac{y-y_1}{by_1 p} = \frac{z-z_1}{cz_1 p}$$

SF. The equations of the normal at the point

$$(x_1, y_1, z_1) \text{ of the ellipsoid } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x-x_1}{a^2} = \frac{y-y_1}{b^2} = \frac{z-z_1}{c^2}$$

Notes. The equations of the normal at (x_1, y_1, z_1)

in the actual d.c's form are

$$\frac{x-x_1}{p x_1} = \frac{y-y_1}{p y_1} = \frac{z-z_1}{p z_1}$$

where $p = \text{length of 1' distance from}$
 $\text{the centre } (0, 0, 0) \text{ to the tangent}$
 $\text{plane of the ellipsoid.}$

The normal at a point P of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meets the principal planes G_1, G_2, G_3 .

(i) Show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$

(ii) If $PG_1^2 + PG_2^2 + PG_3^2 = k^2$, find the locus of P .

Soln: The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (1)

Let $P(x_1, y_1, z_1)$ be any point on the surface.

Then the equations of the normal at $P(x_1, y_1, z_1)$ to (1) in vector form are

$$\frac{\alpha \cdot x}{P \cdot x} = \frac{\gamma \cdot y}{P \cdot y} = \frac{z \cdot z}{P \cdot z} = r \text{ (say)} \quad (2)$$

where ' r ' denotes the distance of any point on the normal from $P(x_1, y_1, z_1)$.

Any point on the normal is

$$\left(x_1 + \frac{r \alpha x}{a^2}, y_1 + \frac{r \gamma y}{b^2}, z_1 + \frac{r z}{c^2} \right)$$

If it lies on the yz plane, i.e., $x=0$,

$$\text{then } x_1 + \frac{r \alpha x}{a^2} = 0 \Rightarrow r \frac{a^2}{a^2} = 0$$

$$\therefore r = -\frac{a^2}{\alpha}$$

$$\text{i.e. } PG_1 = \frac{-a^2}{\alpha}$$

$$\text{Similarly } PG_2 = \frac{-b^2}{\gamma} \text{ & } PG_3 = \frac{-c^2}{z}$$

$$\therefore PG_1 : PG_2 : PG_3 = \frac{-a^2}{\alpha} : \frac{-b^2}{\gamma} : \frac{-c^2}{z} \\ = a^2 : b^2 : c^2$$

(ii) We are given that, $PG_1^2 + PG_2^2 + PG_3^2 = k^2$

$$\Rightarrow \frac{a^4}{\alpha^2} + \frac{b^4}{\gamma^2} + \frac{c^4}{z^2} = k^2$$

$$\Rightarrow \frac{1}{\alpha^2} = \frac{k^2}{a^4 + b^4 + c^4} \quad (3)$$

Put $b = 1$. distance from $(0,0,0)$

on the tangent plane

$$\frac{x_1}{a^2} + \frac{y_1}{b^2} + \frac{z_1}{c^2} = 1 \text{ at } (x_1, y_1, z_1)$$

$\boxed{[-1]} \dots \text{to (1)}$

$$\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}}$$

$$\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}} = \frac{k^2}{a^2 + b^2 + c^2} \quad (\text{say})$$

Locus of $P(x_1, y_1, z_1)$ is [changing (x_1, y_1, z_1) to

$$\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} = \frac{k^2}{a^2 + b^2 + c^2}$$

Also P lies on (1), $\quad \quad \quad (4)$

Thus P lies on the curve of intersection
of two ellipsoids (1) and (4).

→ find the length of the normal chord through
 $P(x_1, y_1, z_1)$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and
prove that if it is equal to $4PG_3$, where G_3 is the
point in which the normal chord meets the plane
 XOY , then P lies on the cone

$$\frac{x^2}{a^2}(2c^2 - a^2) + \frac{y^2}{b^2}(2c^2 - b^2) + \frac{z^2}{c^2} = 0$$

Soln: The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$

Equations of the normal at $P(x_1, y_1, z_1)$ to (1)

in the actual dir's form are

$$\frac{x-x_1}{\frac{px_1}{a^2}} = \frac{y-y_1}{\frac{py_1}{b^2}} = \frac{z-z_1}{\frac{pz_1}{c^2}} = r \quad (\text{say})$$

$$\text{where } p = \frac{1}{\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}}} \quad (2)$$

Any point on the normal at a distance 'r' from P is

$$A \left(x_1 + \frac{px_1}{a^2} r, y_1 + \frac{py_1}{b^2} r, z_1 + \frac{pz_1}{c^2} r \right)$$

If 'r' is the length of the normal chord, then the point must lie on the ellipsoid ①

$$\begin{aligned} & \frac{1}{a^2} \left(x_1 + \frac{px_1}{a^2} r \right)^2 + \frac{1}{b^2} \left(y_1 + \frac{py_1}{b^2} r \right)^2 + \frac{1}{c^2} \left(z_1 + \frac{pz_1}{c^2} r \right)^2 \\ \Rightarrow & p^2 r \left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} \right) + 2pr \left(\frac{x_1}{a^2} + \frac{y_1}{b^2} + \frac{z_1}{c^2} \right) + \\ & \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 = 0 \quad \text{--- ③} \end{aligned}$$

But since $P(x_1, y_1, z_1)$ lies on ①

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

∴ ③ becomes

$$p^2 r \left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} \right) + 2pr \left(\frac{x_1}{a^2} + \frac{y_1}{b^2} + \frac{z_1}{c^2} \right) = 0$$

$$\Rightarrow rp \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right) + 2\left(\frac{r}{p}\right) = 0 \quad (\because \text{from ③})$$

$$\Rightarrow r^2 = \frac{-2}{p^2 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)}$$

which is the length of the required normal chord.

Again the normal meets the xy -plane in $z=0$.

$$\text{where } z_1 + \frac{pz_1}{c^2} r = 0$$

$$\Rightarrow r = -\frac{c^2}{p}$$

$$\therefore PG_3 = -\frac{c^2}{p}$$

Now if length of normal = $4PG_3$; then

$$\frac{2}{P^2 \left[\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right]} = -\frac{4C^2}{P}$$

$$\Rightarrow \frac{1}{P^2} = 2C^2 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right)$$

$$\Rightarrow \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} = 2C^2 \left(\frac{x_1^2}{a^6} + \frac{y_1^2}{b^6} + \frac{z_1^2}{c^6} \right) \quad (\text{from } ②)$$

$$\Rightarrow \frac{x_1^2}{a^6} (2C^2 - a^2) + \frac{y_1^2}{b^6} (2C^2 - b^2) + \frac{z_1^2}{c^6} = 0$$

∴ Locus of $P(x_1, y_1, z_1)$ is

$$\frac{x_1^2}{a^6} (2C^2 - a^2) + \frac{y_1^2}{b^6} (2C^2 - b^2) + \frac{z_1^2}{c^4} = 0$$

which is the required surface

Example 3. The normal at a variable point P of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

meet the plane XOY in A and AQ is drawn parallel to OZ and equal to AP . Prove that the locus of Q is given by

$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} + \frac{z^2}{c^2} = 1.$$

Find the locus of R if OR is drawn from the centre equal and parallel to AP .

Sol. The given ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

Let $P(x_1, y_1, z_1)$ be the variable point on (1).

Then equations of the normal at $P(x_1, y_1, z_1)$ to (1) in the actual d.c.'s form are

$$\frac{x - x_1}{px_1} = \frac{y - y_1}{py_1} = \frac{z - z_1}{pz_1} = r \text{ (say)}$$

$$\text{where } p = \sqrt{\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}\right)} \quad \dots(2)$$

Any point on the normal at a distance r from P is

$$A\left(x_1 + \frac{px_1}{a^2}r, y_1 + \frac{py_1}{b^2}r, z_1 + \frac{pz_1}{c^2}r\right) \quad \dots(3)$$

The normal meets the XOY plane, i.e., $z=0$, in A where

$$z_1 + \frac{pz_1}{c^2}r = 0 \quad \text{or} \quad r = -\frac{c^2}{p}$$

$$AP = r = -\frac{c^2}{p}$$

Putting this value of r in (3), the co-ordinates of A are

$$\left(x_1 - \frac{c^2 x_1}{a^2}, y_1 - \frac{c^2 y_1}{b^2}, 0\right)$$

∴ Equations of line AQ through A and \parallel to OZ are

$$\frac{x - \left(x_1 - \frac{c^2 x_1}{a^2}\right)}{0} = \frac{y - \left(y_1 - \frac{c^2 y_1}{b^2}\right)}{0} = \frac{z - 0}{1} \Rightarrow \sigma \text{ (say)}$$

where $\sigma = \frac{c^2}{p}$

If each member $= AQ = AP = \frac{-c^2}{p}$, then the co-ordinates of Q are given by

$$x = x_1 - \frac{c^2 x_1}{a^2}, \quad y = y_1 - \frac{c^2 y_1}{b^2}, \quad z = \frac{-c^2}{p}$$

$$\text{or} \quad x = \frac{(a^2 - c^2)}{a^2} x_1, \quad y = \frac{b^2 - c^2}{b^2} y_1, \quad z = \frac{-c^2}{p} \quad \text{Eq. (4)}$$

The locus of Q is obtained by eliminating (x_1, y_1, z_1) from the equations (4). Now

$$\frac{z^2}{c^2} = \frac{c^2}{p^2} = c^2 \left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} \right) \quad | \text{ Using (2)}$$

$$= \frac{c^2 x_1^2}{a^4} + \frac{c^2 y_1^2}{b^4} + \frac{z_1^2}{c^2} \quad | \text{ P lies on (1)}$$

$$= \frac{c^2 x_1^2}{a^4} + \frac{c^2 y_1^2}{b^4} \left(1 - \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} \right) \quad | \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

$$= \frac{(c^2 - a^2)x_1^2}{a^4} + \frac{c^2 - b^2}{b^4} y_1^2 + 1 \quad | \text{ From (4)}$$

$$= \frac{c^2 - a^2}{a^4} \left(\frac{a^2 x}{a^2 - c^2} \right)^2 + \frac{c^2 - b^2}{b^4} \left(\frac{b^2 y}{b^2 - c^2} \right)^2 + 1$$

$$= \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} + 1$$

$$\text{or} \quad \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} + \frac{z^2}{c^2} = 1$$

which is the required locus of Q.

Second part. Equations of OR, a line through O(0, 0, 0) and parallel to normal at P, are

$$\frac{x-0}{\frac{px_1}{a^2}} = \frac{y-0}{\frac{py_1}{b^2}} = \frac{z-0}{\frac{pz_1}{c^2}} = AP = -\frac{c^2}{p} \quad \text{for R.}$$

Then if R be (x, y, z)

$$x = x_1 - \frac{c^2}{a^2}, \quad y = y_1 - \frac{c^2}{b^2}, \quad z = z_1$$

$$\text{so that} \quad x_1 = \frac{a^2 x}{c^2}, \quad y_1 = \frac{-b^2 y}{c^2}, \quad z_1 = -z$$

$$\text{But } (x_1, y_1, z_1) \text{ lies on (1)} \quad \therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

$$\text{or} \quad \frac{1}{a^2} \cdot \frac{a^2 x^2}{c^4} + \frac{1}{b^2} \cdot \frac{b^2 y^2}{c^4} + \frac{z^2}{c^2} = 1$$

$$\text{or} \quad a^2 x^2 + b^2 y^2 + c^2 z^2 = c^4$$

which is the required locus of R.

Example 4. The normals to an ellipsoid at the points P, P' meet a principal plane in G, G' ; show that the plane which bisects PP' at right angles, bisects GG' .

Sol. Let the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

and let the principal plane be $x=0$ (2)

Let the points P, P' be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then since P, P' lie on the ellipsoid (1),

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1$$

$$\text{Subtracting, } \frac{1}{a^2}(x_1^2 - x_2^2) + \frac{1}{b^2}(y_1^2 - y_2^2) + \frac{1}{c^2}(z_1^2 - z_2^2) = 0 \quad \dots(3)$$

The normal at $P(x_1, y_1, z_1)$ to (1) is

$$\frac{x-x_1}{a^2} = \frac{y-y_1}{b^2} = \frac{z-z_1}{c^2}$$

This meets the plane $x=0$, where

$$\frac{0-x_1}{a^2} = \frac{y-y_1}{b^2} = \frac{z-z_1}{c^2}$$

or

$$-a^2 = \frac{y-y_1}{b^2} = \frac{z-z_1}{c^2}$$

$$\therefore y = y_1 - \frac{a^2}{b^2}, \quad z = z_1 - \frac{a^2}{c^2}.$$

Thus the point G is $\left(0, y_1 - \frac{a^2}{b^2} y_1, z_1 - \frac{a^2}{c^2} z_1\right)$

Similarly G' is $\left(0, y_2 - \frac{a^2}{b^2} y_2, z_2 - \frac{a^2}{c^2} z_2\right)$

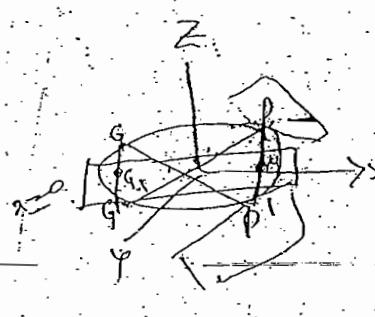
The mid-point of GG' is

$$G_1 \left[0, \frac{y_1+y_2}{2} - \frac{a^2}{b^2} \left(\frac{y_1+y_2}{2} \right), \frac{z_1+z_2}{2} - \frac{a^2}{c^2} \left(\frac{z_1+z_2}{2} \right) \right]$$

$$\text{i.e., } G_1 \left[0, \frac{y_1+y_2}{2} \left(1 - \frac{a^2}{b^2} \right), \frac{z_1+z_2}{2} \left(1 - \frac{a^2}{c^2} \right) \right]$$

Now mid-point of PP' is $M \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right]$

and the d.c.'s of PP' are proportional to $x_1 - x_2, y_1 - y_2, z_1 - z_2$



Equation of the plane through M , the mid-point of PP'
and \perp to PP' is

$$(x_1 - x_2) \left(x - \frac{x_1 + x_2}{2} \right) + (y_1 - y_2) \left(y - \frac{y_1 + y_2}{2} \right) + (z_1 - z_2) \left(z - \frac{z_1 + z_2}{2} \right) = 0.$$

This passes through G_1 if

$$(x_1 - x_2) \left(0 - \frac{x_1 + x_2}{2} \right) + (y_1 - y_2) \left[\frac{y_1 + y_2}{2} \left(1 - \frac{a^2}{b^2} \right) - \frac{y_1 + y_2}{2} \right] \\ + (z_1 - z_2) \left[\frac{z_1 + z_2}{2} \left(1 - \frac{a^2}{c^2} \right) - \frac{z_1 + z_2}{2} \right] = 0$$

$$\text{or if } -\frac{1}{2} (x_1^2 - x_2^2) - \frac{a^2}{2b^2} (y_1^2 - y_2^2) - \frac{a^2}{2c^2} (z_1^2 - z_2^2) = 0$$

$$\text{or if } \frac{1}{a^2} (x_1^2 - x_2^2) + \frac{1}{b^2} (y_1^2 - y_2^2) + \frac{1}{c^2} (z_1^2 - z_2^2) = 0$$

which is true by (3). Hence the result.

Example 5. The normals at P and P' , points of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ meet the plane $z=0$ in G_3 and G'_3 and make angles θ and θ' with PP' . Show that $PG_3 : \cos \theta + P'G'_3 : \cos \theta' = 0$.

Sol. Let $P \rightarrow (\alpha, \beta, \gamma)$ and

$P' \rightarrow (\alpha', \beta', \gamma')$

Equations of normal at P are

$$\frac{x-\alpha}{\frac{px}{a^2}} = \frac{y-\beta}{\frac{py}{b^2}} = \frac{z-\gamma}{\frac{py}{c^2}} = \gamma \text{ (say)}$$

It meets the plane $z=0$ where

$$\frac{x-\alpha}{\frac{px}{a^2}} = \frac{y-\beta}{\frac{py}{b^2}} = \frac{0-\gamma}{\frac{py}{c^2}} = \gamma \\ \Rightarrow \gamma = -\frac{c^2}{p} = PG_3$$

Similarly

$$P'G'_3 = -\frac{c^2}{p}$$

D.C.'s of normal at P are

$$\frac{px}{a^2}, \frac{py}{b^2}, \frac{py}{c^2}$$

D.C.'s of normal at P' are

$$\frac{p'x'}{a^2}, \frac{p'y'}{b^2}, \frac{p'z'}{c^2}$$

D.R.'s of PP' are $\alpha' - \alpha, \beta' - \beta, \gamma' - \gamma$

\therefore D.C.'s of PP' are

$$\frac{\alpha' - \alpha}{PP'}, \frac{\beta' - \beta}{PP'}, \frac{\gamma' - \gamma}{PP'}$$

Since θ is the angle between the normal at P and the line PP' , we have

$$\begin{aligned} \cos \theta &= \frac{p\alpha}{a^2} \cdot \frac{\alpha' - \alpha}{PP'} + \frac{p\beta}{b^2} \cdot \frac{\beta' - \beta}{PP'} + \frac{p\gamma}{c^2} \cdot \frac{\gamma' - \gamma}{PP'} \\ \therefore PG_3 \cos \theta &= -\frac{c^2}{p} \cdot \frac{p}{PP'} \left[\frac{\alpha(\alpha' - \alpha)}{a^2} + \frac{\beta(\beta' - \beta)}{b^2} + \frac{\gamma(\gamma' - \gamma)}{c^2} \right] \\ &= -\frac{c^2}{PP'} \left[\frac{\alpha\alpha' + \beta\beta' + \gamma\gamma'}{a^2 + b^2 + c^2} - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \right) \right] \\ &= -\frac{c^2}{PP'} \left[\frac{\alpha\alpha' + \beta\beta' + \gamma\gamma'}{a^2 + b^2 + c^2} - 1 \right] \end{aligned}$$

$P(\alpha, \beta, \gamma)$ lies on the given ellipsoid

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1$$

Similarly $P'G_3 \cos \theta'$

$$\begin{aligned} &\text{INSTITUTE OF INVESTIGATIONAL SCIENCE} \\ &\text{INSTITUTE FOR AS/IS EXAMINATION} \\ &\text{NEW DELHI-110037} \\ &\text{MSB: 09999999999} \end{aligned}$$

$$\begin{aligned} &PG_3 \cos \theta' = -\frac{c^2}{PP'} \left[\left(\frac{\alpha\alpha' + \beta\beta' + \gamma\gamma'}{a^2 + b^2 + c^2} \right) - 1 \right] \\ &= -PG_3 \cos \theta \end{aligned}$$

$$\Rightarrow P'G_3 \cos \theta' + PG_3 \cos \theta = 0$$

Example 6. Prove that two normals to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

lie in the plane

$$lx + my + nz = 0$$

and the line joining their feet has direction cosines proportional to $a^2(b^2 - c^2)mn, b^2(c^2 - a^2)nl, c^2(a^2 - b^2)lm$.

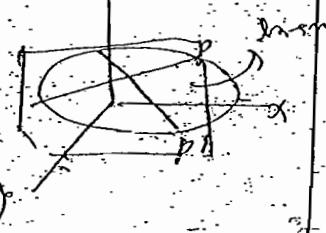
Also obtain the co-ordinates of these points.

Sol: The given ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

Let $P(x_1, y_1, z_1)$ be any point on (1). The normal at this point P is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} = \frac{z - z_1}{z_1/c^2}$$



If It lies on the plane $lx+my+nz=0$ (2)
 and $lx_1+my_1+nz_1=0$
 $l\left(\frac{x_1}{a^2}\right) + m\left(\frac{y_1}{b^2}\right) + n\left(\frac{z_1}{c^2}\right) = 0$ (3)

Solving (2) and (3) by cross-multiplication,

$$\begin{aligned} \frac{x_1}{\frac{nm}{c^2}} &= \frac{y_1}{\frac{nl}{a^2}} = \frac{z_1}{\frac{lm}{b^2}} \\ \text{or } \frac{x_1}{\frac{nm(a^2-b^2)}{c^2(b^2-a^2)}} &= \frac{y_1}{\frac{nlb^2(a^2-c^2)}{a^2(c^2-b^2)}} = \frac{z_1}{lmc^2(b^2-a^2)} \\ \text{or } \frac{x_1}{\frac{nma(b^2-c^2)}{a^2(b^2-c^2)}} &= \frac{y_1}{\frac{nlb(c^2-a^2)}{b^2(c^2-a^2)}} = \frac{z_1}{lmc(a^2-b^2)} \\ &= \sqrt{\frac{\sum \frac{x_1^2}{a^2}}{\sum m^2 n^2 a^2 (b^2 - c^2)^2}} = \sqrt{\frac{\pm 1}{\sum a^2 m^2 n^2 (b^2 - c^2)^2}} \\ &= \pm \frac{1}{d} \quad (\text{say}) \quad | P(x_1, y_1, z_1) \text{ lies on (1),} \\ &\qquad\qquad\qquad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \end{aligned}$$

where $d = \sqrt{\sum a^2 m^2 n^2 (b^2 - c^2)^2}$

The required two points are

$$\begin{aligned} \text{and } & \left[\frac{a^2 nm(b^2 - c^2)}{d}, \frac{b^2 nl(c^2 - a^2)}{d}, \frac{c^2 lm(a^2 - b^2)}{d} \right] \\ & \left[\frac{a^2 nm(b^2 - c^2)}{d}, -\frac{b^2 nl(c^2 - a^2)}{d}, -\frac{c^2 lm(a^2 - b^2)}{d} \right] \quad | \text{On taking } -ve \text{ sign} \end{aligned}$$

The d.c.'s of the line joining these two points are proportional

$$\text{to } \frac{a^2 nm(b^2 - c^2)}{d}, \frac{a^2 nm(b^2 - c^2)}{d} \quad | \text{Using } x_2 - x_1, y_2 - y_1, z_2 - z_1$$

$$\text{i.e., } a^2 nm(b^2 - c^2), b^2 nl(c^2 - a^2), c^2 lm(a^2 - b^2)$$

Hence the result.

Number of normals from a given point.

(I)

To prove that there are six points on an ellipsoid the normals at which pass through a given point (α, β, γ) .

Sol: Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. — (1)

Equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2} = \frac{z-z_1}{z_1/c^2} = \lambda$$

If it passes through (α, β, γ) , then

$$\frac{\alpha-x_1}{x_1/a^2} = \frac{\beta-y_1}{y_1/b^2} = \frac{\gamma-z_1}{z_1/c^2} = \lambda \text{ (say)}$$

From first and last members, we have:

$$\alpha - x_1 = \frac{\lambda x_1}{a^2}$$

$$\Rightarrow \alpha = x_1 \left(1 + \frac{\lambda}{a^2} \right)$$

$$= x_1 \left(\frac{a^2 + \lambda}{a^2} \right)$$

$$\Rightarrow x_1 = \frac{\alpha a^2}{a^2 + \lambda}$$

Similarly, $\frac{\beta - y_1}{y_1/b^2} = \frac{\gamma - z_1}{z_1/c^2}$ $\therefore z_1 = \frac{\gamma c^2}{c^2 + \lambda}$

Since (x_1, y_1, z_1) lies on (1), we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

$$\Rightarrow \frac{1}{a^2} \left[\frac{\alpha^2}{a^2 + \lambda} \right]^2 + \frac{1}{b^2} \left[\frac{\beta^2}{b^2 + \lambda} \right]^2 + \frac{1}{c^2} \left[\frac{\gamma^2}{c^2 + \lambda} \right]^2 = 1. \quad (\because \text{from (1)})$$

$$\Rightarrow \frac{a^2 x^2}{(a^2 + \lambda)^2} + \frac{b^2 y^2}{(b^2 + \lambda)^2} + \frac{c^2 z^2}{(c^2 + \lambda)^2} = 1$$

$$\Rightarrow a^2 x^2 (a^2 + \lambda)^2 (c^2 + \lambda)^2 + b^2 y^2 (a^2 + \lambda)^2 (c^2 + \lambda)^2 + c^2 z^2 (a^2 + \lambda)^2 (b^2 + \lambda)^2 = (a^2 + \lambda)^2 (b^2 + \lambda)^2 (c^2 + \lambda)^2$$

which, being an equation of the sixth degree, gives six values of λ , to each of which there corresponds a point (x_1, y_1, z_1) , as obtained from (2).

∴ There are six points on a central quadric (i.e., ellipsoid) the normals at which pass through a given point.

i.e., through a given point, six normals, in general can be drawn to a central quadric.

Note & Foot of normal:

$$\text{from (2)} \quad \left(\frac{a^2 x}{a^2 + \lambda}, \frac{b^2 y}{b^2 + \lambda}, \frac{c^2 z}{c^2 + \lambda} \right)$$

are the co-ordinates of the foot of normal.

Cubic curve through the feet of six normals from a point:

To show that the feet of the normals from (x_1, y_1, z_1) to the ellipsoid are the six points of intersection of the ellipsoid and a certain cubic curve.

Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (1)

If the normal at (x_1, y_1, z_1) to the ellipsoid

→ Show that in general six normals can be drawn from a given point (x_1, y_1, z_1) to the conicoid $ax^2 + by^2 + cz^2 = 1$. prove also that the six feet of the normals from (x_1, y_1, z_1) to the conicoid are the intersections of the conicoid with a cubic curve.

* Quadruple cone through six concurrent normals:

To show that, the six normals from (α, β, γ) to the ellipsoid lie on a cone of second degree

say. Let the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. — (1)

Now since the normal at (x_1, y_1, z_1) passes through (α, β, γ) we have

$$x_1 = \frac{a^2 \alpha}{a^2 + \lambda}, \quad y_1 = \frac{b^2 \beta}{b^2 + \lambda}, \quad z_1 = \frac{c^2 \gamma}{c^2 + \lambda} \quad \begin{matrix} \text{(from (1))} \\ \text{equation (2)} \end{matrix}$$

Let the equations of the normal from (α, β, γ) to the ellipsoid be

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = \frac{z - \gamma}{c} \quad (2)$$

But w.r.t the equations of the normal at (x_1, y_1, z_1) in the actual d.c.s form

$$\text{are } \frac{x - x_1}{a^2} = \frac{y - y_1}{b^2} = \frac{z - z_1}{c^2} \quad \begin{matrix} \text{(from (2))} \\ \text{(is - 1 back) note condition} \end{matrix}$$

where a^2 length of 1^r distance from the centre $(0, 0, 0)$ to the tangent plane of (1).

passes through the given point (x₁, y₁, z₁),

then $x_1 = \frac{ax}{a^2 + \lambda}$, $y_1 = \frac{b\beta}{b^2 + \lambda}$, $z_1 = \frac{c\gamma}{c^2 + \lambda}$ (from eqn)

The feet of the normals (x₁, y₁, z₁) lie on the curve (changing (x, y, z) to (x₁, y₁, z₁)).

$$x = \frac{ax}{a^2 + \lambda}, y = \frac{b\beta}{b^2 + \lambda}, z = \frac{c\gamma}{c^2 + \lambda}$$

Where λ is "a" parameter. \rightarrow ②

To prove that the curve ② is a cubic curve.

To test the degree of curve we see its intersection with any arbitrary plane.

The curve ② meets an arbitrary plane

$$ux + vy + wz + d = 0 \rightarrow ③$$

$$\Rightarrow u \frac{ax}{a^2 + \lambda} + v \frac{b\beta}{b^2 + \lambda} + w \frac{c\gamma}{c^2 + \lambda} + d = 0$$

$$\Rightarrow ua^2x(b^2 + \lambda)(c^2 + \lambda) + vb^2\beta(a^2 + \lambda)(c^2 + \lambda) + wc^2\gamma(a^2 + \lambda)(b^2 + \lambda) + d(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) = 0$$

which is a cubic in λ , giving three values of λ .

Thus the curve ② is a cubic curve.

Since feet of the normals also lie on the ellipsoid ①, we can conclude that feet of the six normals from a given point are the six points of intersection of the ellipsoid and a cubic curve.

$$\text{Then } l = \frac{px_1}{a^2}$$

$$= \frac{p}{a^2} - \frac{a^2\alpha}{a^2 + \lambda} \quad (\text{from } ②)$$

$$= \frac{p\alpha}{a^2 + \lambda}$$

$$\Rightarrow a^2 + \lambda = \frac{p\alpha}{l} \quad ④$$

$$\text{similarly } b^2 + \lambda = \frac{p\beta}{m}; c^2 + \lambda = \frac{p\gamma}{n} \quad ⑤ \quad ⑥$$

multiplying ④, ⑤ & ⑥ by $b^2 - c^2, c^2 - a^2, a^2 - b^2$

and adding,

we get

$$(b^2 - c^2)(a^2 + \lambda) + (c^2 - a^2)(b^2 + \lambda) + (a^2 - b^2)(c^2 + \lambda) \\ = (b^2 - c^2) \frac{p\alpha}{l} + (c^2 - a^2) \frac{p\beta}{m} + (a^2 - b^2) \frac{p\gamma}{n}$$

$$\Rightarrow \alpha + \lambda(0) = \frac{p\alpha}{l}(b^2 - c^2) + \frac{p\beta}{m}(c^2 - a^2) + \frac{p\gamma}{n}(a^2 - b^2)$$

$$\Rightarrow \frac{\alpha(b^2 - c^2)}{l} + \frac{\beta(c^2 - a^2)}{m} + \frac{\gamma(a^2 - b^2)}{n} = 0 \quad ⑦$$

eliminating l, m, n from ③ and ⑦,

the locus of the normals ③ is

$$\frac{\alpha(b^2 - c^2)}{y - \beta} + \frac{\beta(c^2 - a^2)}{z - \gamma} + \frac{\gamma(a^2 - b^2)}{x - \alpha} = 0$$

$$\Rightarrow \alpha(b^2 - c^2)(y - \beta)(z - \gamma) + \beta(c^2 - a^2)(x - \alpha)(z - \gamma) \\ + \gamma(a^2 - b^2)(x - \alpha)(y - \beta) = 0$$

which is a cone of second degree.

Hence the result.

1983

Prove that the feet of the six normals from (α, β, r) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie on the curve of intersection of the ellipsoid

$$\text{and the cone } \frac{a^2(b^2 - c^2)\alpha}{x} + \frac{b^2(c^2 - a^2)\beta}{y} + \frac{c^2(a^2 - b^2)r}{z} = 0$$

Sol: The ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Equations of at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a} = \frac{y-y_1}{y_1/b} = \frac{z-z_1}{z_1/c}$$

If it passes through (α, β, r) then

$$\frac{\alpha-x_1}{x_1/a} = \frac{\beta-y_1}{y_1/b} = \frac{r-z_1}{z_1/c} = \lambda \text{ (say)}$$

The six feet of the normals from (α, β, r) are given by

$$x_1 = \frac{a^2\alpha}{a^2 + \lambda}$$

$$y_1 = \frac{b^2\beta}{b^2 + \lambda} \text{ and } z_1 = \frac{c^2r}{c^2 + \lambda}$$

These give

$$a^2 + \lambda = \frac{a^2\alpha}{x_1}, b^2 + \lambda = \frac{b^2\beta}{y_1}$$

$$\text{and } c^2 + \lambda = \frac{c^2r}{z_1}$$

Multiplying these equations by $b^2 - c^2$, $c^2 - a^2$, $a^2 - b^2$, and adding, we get

$$0 + \lambda(0) = \frac{a^2\alpha(b^2 - c^2)}{x_1} + \frac{b^2\beta(c^2 - a^2)}{y_1} + \frac{c^2\gamma(a^2 - b^2)}{z_1}$$

[Note this step]

(x_1, y_1, z_1) i.e., the feet of the normals, lie on the cone

$$\frac{a^2\alpha(b^2 - c^2)}{x} + \frac{b^2(c^2 - a^2)\beta}{y} + \frac{c^2(a^2 - b^2)\gamma}{z} = 0$$

Also the feet (x_1, y_1, z_1) of the normals lie on the ellipsoid (1). Thus the feet of the six normals lie on the curve of intersection of the ellipsoid and the above cone.

Note. In the equation of the cone through the feet of six normals from a point to an ellipsoid,

Co-eff. of $x^2 = 0$; co-eff. of $y^2 = 0$; co-eff. of $z^2 = 0$,

constant term = 0.

[Remember]

Example 2. If $P, Q, R; P', Q', R'$ are the feet of six normals from a point to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the plane PQR is given by

$$lx + my + nz = p,$$

then the plane $P'Q'R'$ is given by

$$\frac{x}{a^2l} + \frac{y}{b^2m} + \frac{z}{c^2n} + \frac{1}{p} = 0. \quad (\text{K.U. 1984})$$

[Imp.]

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad (1)$$

and that the plane PQR is $lx + my + nz = p$ (2)

Let the required equation of plane $P'Q'R'$ be

$$l'x + m'y + n'z - p' = 0 \quad (3)$$

The joint equation of the planes PQR and $P'Q'R'$ is

$$(lx + my + nz - p)(l'x + m'y + n'z - p') = 0 \quad (4)$$

The equation of conicoid through the points of intersection of the ellipsoid (1) and pair of planes (4) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) + k(lx + my + nz - p)(l'x + m'y + n'z - p') = 0 \quad (5)$$

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If it is the same as the equation of the cone through the feet P, Q, R; P', Q', R' of the six normals from the given point to the ellipsoid, then

$$\text{Co-eff. of } x^2 = 0 \quad \text{i.e., } \frac{1}{a^2} + kll' = 0 \quad \text{or, } l' = -\frac{1}{kla^2}$$

$$\text{Co-eff. of } y^2 = 0 \quad \text{i.e., } \frac{1}{b^2} + knm' = 0 \quad \therefore m' = -\frac{1}{kmb^2}$$

$$\text{Co-eff. of } z^2 = 0 \quad \text{i.e., } \frac{1}{c^2} + knn' = 0 \quad \therefore n' = -\frac{1}{knc^2}$$

$$\text{Constant term} = 0 \quad \text{i.e., } -1 + kpp' = 0 \quad \therefore p' = \frac{1}{kp}$$

Putting these values of l' , m' , n' , p' in (3), the required plane P'Q'R' is

$$\frac{x}{kla^2} + \frac{y}{kmb^2} + \frac{z}{knc^2} - \frac{1}{kp} = 0$$

$$\text{or, } \frac{x}{la^2} + \frac{y}{mb^2} + \frac{z}{nc^2} + \frac{1}{p} = 0.$$

Hence the result.

Article 14. Plane of Contact.

To find the equation of plane of contact of the point (x_1, y_1, z_1) with respect to conicoid $ax^2 + by^2 + cz^2 = 1$.

Let (x', y', z') be the point of contact any tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$. (1)

Tangent plane at (x', y', z') to (1) is

$$ax'x + by'y + cz'z = 1$$

If it passes through the given point (x_1, y_1, z_1) , then

$$ax_1x' + by_1y' + cz_1z' = 1$$

Locus of the points of contact (x', y', z') is

$$ax_1x + by_1y + cz_1z = 1$$

$$\text{or, } ax_1x + by_1y + cz_1z = 1$$

which is the required plane of contact.

Article 15. Polar plane of a point.

To find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the central conicoid $ax^2 + by^2 + cz^2 = 1$.

(M.D.U. 1985; K.U. 1986, 85; Manipur 1983)

The given conicoid is

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(1)$$

Let $P(x_1, y_1, z_1)$ be the given point and let PQR be any line through P which meets (1) in Q and R .

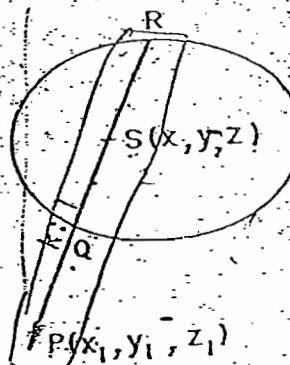
Also the points C & D are called conjugate of P w.r.t. Q and R .

Harmonic conjugates Let Q divide PS in the ratio $k : 1$.

w.r.t A and B .

Then co-ordinates of Q are

$$\text{Thus } C \text{ is harmonic conjugate of } D \text{ w.r.t. } A \text{ and } B \\ \text{vice versa } \left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1} \right)$$



Harmonic division:

If the line AB is divided externally &

internally in the same same ratio at C & D then C and D are said to divide harmonically.

Since Q lies on the conicoid (1),

$$\therefore a\left(\frac{kx+x_1}{k+1}\right)^2 + b\left(\frac{ky+y_1}{k+1}\right)^2 + c\left(\frac{kz+z_1}{k+1}\right)^2 = 1$$

or

$$a(kx+x_1)^2 + b(ky+y_1)^2 + c(kz+z_1)^2 - (k+1)^2 = 0$$

or

$$k^2(ax^2+by^2+cz^2-1) + 2k(axx_1+byy_1+czz_1-1) + (ax_1^2+by_1^2+cz_1^2-1) = 0 \quad \dots(2)$$

which is a quadratic equation in k.

Since PS is divided harmonically, i.e., internally and externally in the same ratio at Q and R, the quadratic (2) has equal and opposite roots.

\therefore Sum of roots = 0 i.e., coeff. of k = 0

or

$$axx_1+byy_1+czz_1-1=0$$

or

$$axx_1+byy_1+czz_1=1$$

C.T.M.

which is the equation of required polar plane of P,

Cor. : If P lies on the conicoid, the polar plane at P becomes the tangent plane at P.

Article 16. Pole of a given plane.

To find the pole of the plane $lx+my+nz=p$, w.r.t. the conicoid $ax^2+by^2+cz^2=1$.

Let (x_1, y_1, z_1) be the required pole.

Then the polar plane of (x_1, y_1, z_1) w.r.t. conicoid

$$ax^2+by^2+cz^2=1$$

i.e.,

$$axx_1+byy_1+czz_1=1$$

must be identical with the given plane

$$lx+my+nz=p$$

Comparing (1) and (2), we have

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = 1$$

$$x_1 = \frac{1}{ap}, y_1 = \frac{m}{bp}, z_1 = \frac{n}{cp}$$

Thus the pole is $\left(\frac{1}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

Example. Prove that the locus of the poles of the tangent planes to $a^2x^2+b^2y^2-c^2z^2=1$ with respect to

$$\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1$$

is the hyperboloid of one sheet. Find its equation.

Sol. Let $lx+my+nz=p \dots(1)$ be a tangent plane

to $a^2x^2+b^2y^2-c^2z^2=1 \dots(2)$

$$a^2x^2+\frac{m^2}{b^2}+\frac{n^2}{c^2}=p^2 \dots(3)$$

$$\text{Using } \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$$

Condition of tangency

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(2)

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Let (x_1, y_1, z_1) be the pole of plane (1) w.r.t.

$$\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 = 1 \quad \dots(4)$$

∴ Equation of polar plane of (x_1, y_1, z_1) w.r.t. (4) is

$$\alpha^2 x x_1 + \beta^2 y y_1 + \gamma^2 z z_1 = 1 \quad \dots(5)$$

Comparing (1) and (5),

$$\frac{\alpha^2 x_1}{l} = \frac{\beta^2 y_1}{m} = \frac{\gamma^2 z_1}{n} = \frac{1}{p}$$

From first and fourth members

$$l = \alpha^2 p x_1, \text{ similarly } m = \beta^2 p y_1 \text{ and } n = \gamma^2 p z_1$$

Putting these values of l, m, n in (3) [To eliminate l, m, n], we have

$$\frac{\alpha^4 p^2 x_1^2}{a^2} + \frac{\beta^4 p^2 y_1^2}{b^2} + \frac{\gamma^4 p^2 z_1^2}{c^2} = p^2$$

$$\text{Cancelling } p^2, \frac{\alpha^4 x_1^2}{a^2} + \frac{\beta^4 y_1^2}{b^2} + \frac{\gamma^4 z_1^2}{c^2} = 1$$

∴ Locus of (x_1, y_1, z_1) [pole of (1) w.r.t. (4)] is

$$\frac{\alpha^4}{a^2} x^2 + \frac{\beta^4}{b^2} y^2 + \frac{\gamma^4}{c^2} z^2 = 1$$

which is a hyperboloid of one sheet. [∴ co-effs. of x^2 and y^2 are positive, but co-eff. of z^2 is negative]

Article 17. Conjugate points and conjugate planes.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let the conicoid be

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(1)$$

Then polar plane of (x_1, y_1, z_1) w.r.t. (1) is

$$ax_1 x + by_1 y + cz_1 z = 1$$

If it passes through $Q(x_2, y_2, z_2)$, then

$$ax_1 x_2 + by_1 y_2 + cz_1 z_2 = 1$$

The symmetry of this result shows that the polar plane of Q also passes through P .

The two points such that the polar plane of each passes through the other are called the conjugate points.

Similarly it can be easily shown that if the pole of a plane S_1 lies on another plane S_2 , then pole of S_2 must lie on S_1 . Two such planes (as S_2 and S_1 here) are called conjugate planes.

Article 18. Polar lines

Two lines such that the polar plane of any point on one line passes through the other line are called conjugate lines or polar lines.

Polar of a line.

To find the equations of the polar of the line

$$\frac{x - x_1}{m} = \frac{y - y_1}{n} = \frac{z - z_1}{l} \text{ w.r.t. the conicoid } ax^2 + by^2 + cz^2 = 1.$$

(K.U. 1986)

The given conicoid is $ax^2 + by^2 + cz^2 = 1$... (1)
 and the given line is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$... (2)

Any point on line (2) is $(lr+x_1, mr+y_1, nr+z_1)$.

It's polar plane w.r.t. conicoid (1) is

$$ax(lr+x_1) + by(mr+y_1) + cz(nr+z_1) = 1$$

or,

$$axx_1 + byy_1 + czz_1 - 1 + r(ax+by+cz) = 0.$$

This passes through the line

$$\left. \begin{array}{l} axx_1 + byy_1 + czz_1 - 1 = 0 \\ ax+by+cz = 0 \end{array} \right\}$$

for all values of r .

Hence the equations of the polar line of (2) are

$$axx_1 + byy_1 + czz_1 = 1; ax+by+cz = 0.$$

Method to write down the polar of

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

w.r.t. a central conicoid (equation in the standard form).

1. Write down the polar plane of (x_1, y_1, z_1) w.r.t. conicoid thus getting $axx_1 + byy_1 + czz_1 = 1$.
2. Write down the polar plane of (l, m, n) and omit the constant term thus getting $ax+by+cz = 0$.
3. The above two equations are the required equations of the polar.

Example 1. Show that the equations of the polar of the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

w.r.t. quadric $x^2 - 2y^2 + 3z^2 = 4$ are $\frac{x+6}{3} = \frac{y-2}{3} = \frac{z-2}{4}$. (Kanpur 58)

Sol. The given line is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$... (1)

$$x^2 - 2y^2 + 3z^2 = 4$$
 ... (2)

and the conicoid is

Any point on (1) is $(2r+1, 3r+2, 4r+3)$.

Polar plane of this point w.r.t. (2) is

$$x(2r+1) - 2y(3r+2) + 3z(4r+3) = 4$$

or,

$$x+2rx-6yr-4y+12rz+yz-4=0$$

or,

$$(x-4y+9z-4) + 2r(x-3y+6z) = 0$$

which passes through the line

$$x-4y+9z-4=0 \quad \dots (3) \text{ for all values of } r$$

$$x-3y+6z=0 \quad \dots (4)$$

\therefore Equations (3) and (4) are the equations of the polar line of (1) w.r.t. conicoid (2).

To reduce the line given by (3) and (4) in symmetrical form.

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To find d.r.'s of this line [omitting constant terms in (3) and (4)], we get

$$x - 4y + 9z = 0$$

$$x - 3y + 6z = 0$$

$$\frac{x}{-24+27} = \frac{y}{9-6} = \frac{z}{-3+4} \text{ or } \frac{x}{3} = \frac{y}{3} = \frac{z}{1}$$

Thus the d.r.'s of polar line are 3, 3, 1.

For any point put $z=2$ in (3) and (4).

(As suggested by the question)

$$\therefore x - 4y + 14 = 0 \text{ and } x - 3y + 12 = 0$$

Solving, we have $x = -6$, $y = 2$. Also $z = 2$.

Hence one point on the polar is (-6, 2, 2).

Thus the equations of the polar line in the symmetrical form are

$$\frac{x+6}{3} = \frac{y-2}{3} = \frac{z-2}{1}$$

Hence the result.

Example 2. Find the condition that the lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ and } \frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$$

should be polar with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Sol. The polar of the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$ is given by

$$a\alpha x + b\beta y + c\gamma z - 1 = 0, a\alpha l + b\beta m + c\gamma n = 0 \quad (1)$$

[See Article 18 above]

But

$$\frac{x-\alpha}{l'} = \frac{y-\beta}{m'} = \frac{z-\gamma}{n'} \quad (2)$$

is given to be polar. Hence (2) should be identical with (1), i.e., line (2) should lie on both the planes given by (1).

For this the point $(\alpha', \beta', \gamma')$ should lie on both the planes and the line (2) should be \perp to the normal of each of the planes in (1).

The required conditions are

$$a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' = 1$$

$$a\alpha l' + b\beta m' + c\gamma n' = 0$$

$$a\alpha' l + b\beta' m + c\gamma' n = 0$$

$$a\alpha' l' + b\beta' m' + c\gamma' n' = 0$$

Example 3. Find the locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to

$$ax^2 + by^2 + cz^2 = 1.$$

(M.D.U. 1984; Kanpur 1982, 88; Lucknow 1980)

Sol. Any line through (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$... (1)

The polar of (1) w.r.t. the given conicoid is

$$\left. \begin{array}{l} a\alpha x + b\beta y + c\gamma z = 1 \\ alx + bmy + cnz = 0 \end{array} \right\} \quad \dots (2)$$

Omitting the constant terms in (2), the d.c.'s of line (2) are, given by

$$a\alpha x + b\beta y + c\gamma z = 0$$

$$alx + bmy + cnz = 0$$

$$\therefore \frac{x}{bc(n\beta - m\gamma)} = \frac{y}{ca(l\gamma - n\alpha)} = \frac{z}{ab(m\alpha - l\beta)}$$

Thus the d.c.'s of line (2) are proportional to

$$bc(n\beta - m\gamma), ca(l\gamma - n\alpha), ab(m\alpha - l\beta).$$

The lines (1) and (2) are \perp

$$\therefore lbc(n\beta - m\gamma) + mca(l\gamma - n\alpha) + nab(m\alpha - l\beta) = 0$$

$$\text{or } \alpha mna(b-c) + \beta nlb(c-a) + \gamma lmc(a-b) = 0$$

$$\text{or } \sum \frac{\alpha}{l} \left(\frac{1}{c} - \frac{1}{b} \right) = 0 \quad \dots (3)$$

on dividing by $lmnabc$.

To find the locus of (1) — eliminating l, m, n from (1) and (3), we have

$$\sum \left(\frac{\alpha}{x-\alpha} \right) \left(\frac{1}{c} - \frac{1}{b} \right) = 0 \quad \text{or} \quad \sum \left(\frac{\alpha}{x-\alpha} \right) \left(\frac{1}{b} - \frac{1}{c} \right) = 0.$$

Example 4. If P, Q are the points, (x_1, y_1, z_1) , (x_2, y_2, z_2) , the polar of PQ w.r.t. $ax^2 + by^2 + cz^2 = 1$ is given by

$$axx_1 + byy_1 + czz_1 = l, \quad axx_2 + byy_2 + czz_2 = l.$$

Sol. Equations of line PQ are $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

The polar of this line w.r.t. the given conicoid is

$$axx_1 + byy_1 + czz_1 = 1 \quad \dots (1)$$

$$\text{and } ax(x_2 - x_1) + by(y_2 - y_1) + cz(z_2 - z_1) = 0 \quad \dots (2)$$

$$\text{Adding (1) and (2), we have } axx_2 + byy_2 + czz_2 = 0 \quad \dots (3)$$

Hence (1) and (3) are the required equations of the polar.

Observations. It is clear from the equations (1) and (3) that the polar of PQ is the line of intersection of polar planes of P and Q .

Example 5. Find the polar plane of the point $(2, -3, 4)$ with respect to the conicoid $x^2 + 2y^2 + z^2 = 4$. (Bundelkhand 1984)

Sol. Required polar plane is

$$x(2) + 2y(-3) + z(4) = 4$$

or

$$x - 3y + 2z = 2.$$

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Example 6. Find the locus of straight line through a fixed point (α, β, γ) whose polar lines with respect to the quadratics $ax^2 + by^2 + cz^2 = 1$ and $a'x^2 + b'y^2 + c'z^2 = 1$ are coplanar.

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = \gamma \text{ (say)} \quad \dots(i)$$

The equations of the polar line of (i) w.r.t.

$$ax^2 + by^2 + cz^2 = 1 \text{ are}$$

$$a\alpha x + b\beta y + c\gamma z = 1 \quad \dots(ii)$$

$$alx + bmy + cnz = 0 \quad \dots(iii)$$

And the equations of polar line of (i) w.r.t.

$$a'x^2 + b'y^2 + c'z^2 = 1 \text{ are}$$

$$a'\alpha x + b'\beta y + c'\gamma z = 1 \quad \dots(iv)$$

$$a'l x + b'm y + c'n z = 0 \quad \dots(v)$$

From (ii) and (iv), we have

$$(a-a')\alpha x + (b-b')\beta y + (c-c')\gamma z = 0 \quad \dots(vi)$$

From (iii) and (v), solving simultaneously, we have

$$\frac{lx}{(bc'-b'c)} = \frac{my}{(ca'-c'a)} = \frac{nz}{(ab'-a'b)} \quad \dots(vii)$$

Eliminating x, y, z between (vi) and (vii), we get

$$\frac{(a-a')\alpha(bc'-b'c)}{l} + \frac{(b-b')\beta(ca'-c'a)}{m} + \frac{(c-c')\gamma(ab'-a'b)}{n} = 0 \quad \dots(viii)$$

Eliminating l, m, n between (i) and (viii), we get the locus of the line as

$$\sum \frac{(a-a')\alpha(bc'-b'c)}{(x-\alpha)} = 0$$

Ex 7. Prove that the locus of the poles of the tangent planes of $ax^2 + by^2 + cz^2 = 1$ with respect to $a'x^2 + b'y^2 + c'z^2 = 1$ is the conicoid $\frac{(a'x)^2}{a} + \frac{(b'y)^2}{b} + \frac{(c'z)^2}{c} = 1$. (Allahabad 1982; Kanpur 1986)

Sol. Let $lx + my + nz = p$ be the tangent plane to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(ii)$$

$$\text{Then } \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2 \quad \dots(iii)$$

Let (α, β, γ) be the pole of the plane (i) w.r.t. to $a'x^2 + b'y^2 + c'z^2 = 1$. Then we have $a'\alpha^2 + b'\beta^2 + c'\gamma^2 = 1 \quad \dots(iv)$

Comparing (ii) and (iv), we get

$$\frac{a'\alpha}{l} = \frac{b'\beta}{m} = \frac{c'\gamma}{n} = \frac{1}{p} \quad \dots(v)$$

Eliminating l, m, n between (iii) and (v), we get

$$\frac{(a'\alpha p)^2}{a} + \frac{(b'\beta p)^2}{b} + \frac{(c'\gamma p)^2}{c} = p^2$$

$$\text{or } \frac{(a'\alpha)^2}{a} + \frac{(b'\beta)^2}{b} + \frac{(c'\gamma)^2}{c} = 1$$

The required locus of (α, β, γ) is

$$\frac{(a'x)^2}{a} + \frac{(b'y)^2}{b} + \frac{(c'z)^2}{c} = 1.$$

Example 8. Show that the locus of the pole of the plane $lx+my+nz=p$ with respect to the system of conicoids $\Sigma[x^2/(a^2+k)=1]$ is a straight line perpendicular to the given plane, where k is a parameter.

Sol. Let (α, β, γ) be the pole of the plane

$$lx+my+nz=p$$

with respect to the conicoid

$$\frac{x^2}{(a^2+k)} + \frac{y^2}{(b^2+k)} + \frac{z^2}{(c^2+k)} = 1 \quad \dots (ii)$$

The polar plane of (α, β, γ) w.r.t. this conicoid is

$$\frac{\alpha x}{(a^2+k)} + \frac{\beta y}{(b^2+k)} + \frac{\gamma z}{(c^2+k)} = 1 \quad \dots (iii)$$

Since (i) and (iii) represents the same plane, therefore comparing them, we get

$$\frac{\alpha}{a(a^2+k)} = \frac{\beta}{b(b^2+k)} = \frac{\gamma}{c(c^2+k)} = \frac{1}{p}$$

where

$$\alpha = (a^2+k) \frac{l}{p}, \beta = (b^2+k) \frac{m}{p}, \gamma = (c^2+k) \frac{n}{p}$$

or

$$\frac{\alpha - (a^2 l/p)}{l} = \frac{k}{p} = \frac{\beta - (b^2 m/p)}{m} = \frac{\gamma - (c^2 n/p)}{n}$$

The locus of (α, β, γ) is

$$\frac{x - (a^2 l/p)}{l} = \frac{y - (b^2 m/p)}{m} = \frac{z - (c^2 n/p)}{n}$$

which is a straight line and its direction cosines being l, m, n is perpendicular to the plane (i).

Article 19. Enveloping Cone

To find the equation of enveloping cone from the point (x_1, y_1, z_1) to the central conicoid $ax^2 + by^2 + cz^2 = 1$. (M.D.U. 1984)

The given conicoid is

$$ax^2 + by^2 + cz^2 = 1 \quad \dots (1)$$

Let P be the point (x_1, y_1, z_1) .

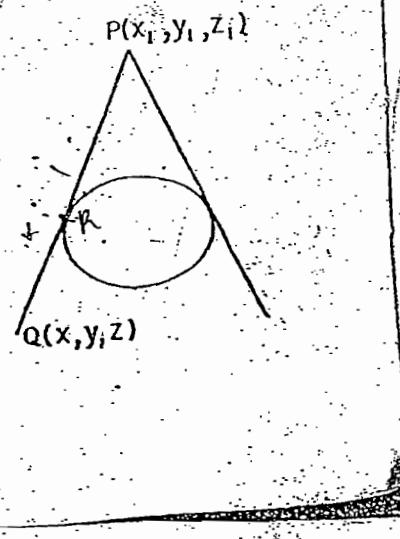
Let $Q(x, y, z)$ be any point on a tangent from P to the conicoid.

The point which divides PQ in the ratio $k : 1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1} \right).$$

If it lies on (1), then

$$a \left(\frac{kx+x_1}{k+1} \right)^2 + b \left(\frac{ky+y_1}{k+1} \right)^2 + c \left(\frac{kz+z_1}{k+1} \right)^2 = 1$$



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which simplifies to

$$k^2(ax^2 + by^2 + cz^2 - 1) + 2k(axy_1 + bzy_1 + czz_1 - 1) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \dots(2)$$

which is a quadratic in k .

Since the line PQ touches the conicoid (1), \therefore (2) must have equal roots.

$$\begin{aligned} & 4(axy_1 + bzy_1 + czz_1 - 1)^2 \\ &= 4(ax^2 + by^2 + cz^2 - 1)(ax_1^2 + by_1^2 + cz_1^2 - 1) \quad | \text{ Using } b^2 = 4ac \\ \text{or} \quad & (axy_1 + bzy_1 + czz_1 - 1)^2 \\ &= (ax^2 + by^2 + cz^2 - 1)(ax_1^2 + by_1^2 + cz_1^2 - 1) \end{aligned}$$

which is the required equation.

Remember. If $S=0$ is the given surface, then with usual notations the enveloping cone is given by $SS_1=T^2$.

Example 1. Find the locus of points from which three mutually perpendicular tangents can be drawn to the surface $ax^2 + by^2 + cz^2 = 1$. [Imp.]

Sol. Let $P(x_1, y_1, z_1)$ be the point.

Then the three mutually \perp tangents drawn from P will be three mutually \perp generators of the enveloping cone with P as vertex. The equation of the enveloping cone is $SS_1=T^2$.

$$\text{or } (ax^2 + by^2 + cz^2 - 1)(ax_1^2 + by_1^2 + cz_1^2 - 1) = (axy_1 + bzy_1 + czz_1 - 1)^2$$

Since this cone has three mutually \perp generators,

$$\begin{aligned} & \text{Co-eff. of } x^2 + \text{co-eff. of } y^2 + \text{co-eff. of } z^2 = 0 \\ \text{i.e., } & a(by_1^2 + cz_1^2 - 1) + b(ax_1^2 + cz_1^2 - 1) + c(ax_1^2 + by_1^2 - 1) = 0 \end{aligned}$$

$$\text{or } a(b+c)x_1^2 + b(c+a)y_1^2 + c(a+b)z_1^2 = a+b+c$$

$$\begin{aligned} & \text{Locus of } P(x_1, y_1, z_1) \text{ is [changing } (x_1, y_1, z_1) \text{ to } (x, y, z)] \\ & a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = a+b+c, \end{aligned}$$

Example 2. The section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ whose vertex is } P \text{ by the plane } z=0 \text{ is (i) a parabola,}$$

(ii) a rectangular hyperbola. Find the locus of P . [Imp.]

(M.D.U. 1985)

Sol. Let $P(x_1, y_1, z_1)$ be the vertex of enveloping cone of the ellipsoid $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ (1)

The enveloping cone of (1) is $SS_1=T^2$

$$\begin{aligned} \text{i.e., } & \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) \\ &= \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1 \right)^2 \end{aligned}$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2$$

$$\text{or } \frac{x^2}{a^2} \left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) + \frac{y^2}{b^2} \left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1 \right) - \frac{2x_1 y_1}{a^2 b^2} xy + \dots = 0 \quad \dots (2)$$

(i) and (2) represent a parabola in the XY plane if

$$\frac{1}{a^2} \left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right), \frac{1}{b^2} \left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1 \right) = \frac{x_1^2 y_1^2}{a^4 b^4}$$

[Using $ab = h^2$ if the equation is $ax^2 + 2hxy + by^2 + \dots = 0$]

$$\text{or } \frac{1}{a^2 b^2} \left(\frac{x_1^2 y_1^2}{a^2 b^2} + \frac{y_1^2 z_1^2}{b^2 c^2} - \frac{y_1^2}{b^2} + \frac{z_1^2 x_1^2}{a^2 c^2} + \frac{z_1^4}{c^4} - \frac{z_1^2}{c^2} - \frac{x_1^2}{a^2} - \frac{z_1^2}{c^2} + 1 \right) = \frac{x_1^2 y_1^2}{a^4 b^4}$$

$$\text{or } \left(\frac{y_1^2 z_1^2}{b^2 c^2} + \frac{z_1^2 x_1^2}{c^2 a^2} + \frac{z_1^4}{c^4} - \frac{z_1^2}{c^2} \right) - \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = 0$$

$$\text{or } \frac{z_1^2}{c^2} \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) - \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = 0$$

$$\text{or } \left(\frac{z_1^2}{c^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = 0$$

∴ Locus of P(x₁, y₁, z₁) is $\left(\frac{z^2}{c^2} - 1 \right) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = 0$

$$\therefore \text{Either } \frac{z^2}{c^2} - 1 = 0 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

Rejecting the second equation [as it is the given ellipsoid and

P does not lie on it], the locus is $\frac{z^2}{c^2} - 1 = 0$ or $z = \pm c$.

(Kanpur 1988)

(ii) The equation (2) represents a rectangular hyperbola in the XY plane if co-eff. of x^2 + co-eff. $y^2 = 0$

$$\text{i.e., if } \frac{1}{a^2} \left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) + \frac{1}{b^2} \left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1 \right) = 0$$

∴ Locus of P(x₁, y₁, z₁) is

$$\frac{1}{a^2} \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) + \frac{1}{b^2} \left(\frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 \right) = 0$$

$$\text{or } \frac{x^2}{a^2 b^2} + \frac{y^2}{a^2 b^2} + \frac{z^2}{c^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) - \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = 0$$

$$\text{or } \frac{x^2 + y^2}{a^2 b^2} + \frac{z^2 (a^2 + b^2)}{a^2 b^2 c^2} - \frac{a^2 + b^2}{a^2 b^2} = 0$$

$$\text{or } \frac{x^2 + y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1.$$

Example 3. Find the locus of luminous point if the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ casts a circular shadow on the plane } z=0.$$

(Kanpur 1988)

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Sol. Let $P(x_1, y_1, z_1)$ be the luminous point.

The enveloping cone of the given ellipsoid with vertex at P is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) \\ = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1 \right)^2 \quad | \text{ Using } SS_1 = T^2$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2$$

This will be a circle if the co-eff. of $xy = 0$

and

$$\text{co-eff. of } x^2 = \text{co-eff. of } y^2$$

i.e., if

$$\frac{xy_1}{a^2 b^2} = 0 \quad \dots(1)$$

and

$$\frac{1}{a^2} \left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = \frac{1}{b^2} \left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1 \right) \quad \dots(2)$$

From (1) either $x_1 = 0$ or $y_1 = 0$.

Case I. If $x_1 = 0$ from (2), we have

$$\frac{1}{a^2} \left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 \right) = \frac{1}{b^2} \left(\frac{z_1^2}{c^2} - 1 \right) \\ \frac{y_1^2}{a^2 b^2} + \frac{z_1^2(b^2 - a^2)}{a^2 b^2 c^2} = \frac{b^2 - a^2}{a^2 b^2}$$

or

$$\frac{y_1^2}{b^2 - a^2} + \frac{z_1^2}{c^2} = 1.$$

Thus the locus of $P(x_1, y_1, z_1)$ is $x=0, \frac{y^2}{b^2-a^2} + \frac{z^2}{c^2} = 1$ which is an ellipse in the YZ plane.

Case II. If $y_1 = 0$, (2) gives

$$\frac{1}{a^2} \left(\frac{z_1^2}{c^2} - 1 \right) = \frac{1}{b^2} \left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1 \right)$$

Locus of $P(x_1, y_1, z_1)$ is

$$y=0, \frac{1}{a^2} \left(\frac{z^2}{c^2} - 1 \right) = \frac{1}{a^2} \left(\frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 \right)$$

or

$$y=0, \frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1.$$

Article 20. Enveloping Cylinder

To find the equation of the enveloping cylinder of the central conicoid $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

(Garhwal 1986)

The given conicoid is
 $ax^2 + by^2 + cz^2 = 1 \quad \dots (1)$

and the given line is $\frac{x}{l} = \frac{y}{m}$
 $= \frac{z}{n} \quad \dots (2)$

Let $P(x_1, y_1, z_1)$ be any point on a tangent \parallel to the line (2). | Note this step

The equations of the tangent line through (x_1, y_1, z_1) and \parallel to (2) are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say).}$$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$.

If it lies on (1), then $a(lr+x_1)^2 + b(mr+y_1)^2 + c(nr+z_1)^2 = 1$
or $r^2(a^2 + bm^2 + cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \dots (3)$

Since the line (2) touches the conicoid (1), \therefore (3) has equal roots.

$$\begin{aligned} & 4(alx_1 + bmy_1 + cnz_1)^2 \\ & - 4(a^2 + bm^2 + cn^2)(ax_1^2 + by_1^2 + cz_1^2 - 1) = 0. \text{ Using } b^2 - 4ac = 0 \\ \text{or} \quad & (alx_1 + bmy_1 + cnz_1)^2 = (a^2 + bm^2 + cn^2)[ax_1^2 + by_1^2 + cz_1^2 - 1] \end{aligned}$$

$$\therefore \text{Locus of } (x_1, y_1, z_1) \text{ is } (alx + bmy + cnz)^2 = (a^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1).$$

which is the required equation of enveloping cylinder.

Method to write down the enveloping cylinder.

If $S = ax^2 + by^2 + cz^2 - 1$, so that $S=0$ is the equation of central conicoid then $s_1 = a^2 + bm^2 + cn^2$, i.e., s_1 is obtained by putting (l, m, n) in S and neglecting the constant term.

$t = alx + bmy + cnz$, where t is the expression for the tangent plane at (l, m, n) after omitting the constant term, then the enveloping cylinder is $Ss_1 = t^2$. [Remember]

Example 1. Prove that the enveloping cylinder of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ whose generators are parallel to the lines}$$

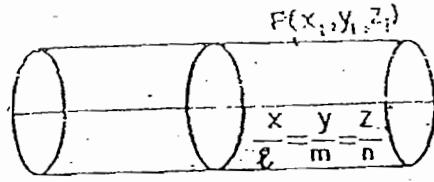
$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}, \text{ meet the plane } z=0 \text{ in circles.}$$

$$\text{Sol. The given ellipsoid is } S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \dots (1)$$

$$\text{and the given lines are } \frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c} \quad \dots (2)$$

The equation of enveloping cylinder is $Ss_1 = t^2$

$$\begin{aligned} \text{i.e., } & \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \left[\frac{1}{a^2}(0)^2 + \frac{1}{b^2}(\pm\sqrt{a^2 - b^2})^2 + \frac{1}{c^2}(c)^2 \right] \\ & = \left[\frac{1}{a^2}(0)x + \frac{1}{b^2}(\pm\sqrt{a^2 - b^2})y + \frac{1}{c^2}cz \right]^2 \end{aligned}$$



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$$\text{or } \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \left(\frac{a^2 - b^2}{b^2} + 1 \right) = \left(\pm \frac{\sqrt{a^2 - b^2}}{b^2} y + \frac{z}{c} \right)^2$$

This meets the plane $z=0$ where

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{a^2 - b^2}{b^2} + 1 \right) = \left(\pm \frac{\sqrt{a^2 - b^2}}{b^2} y \right)^2$$

$$\text{or } \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \cdot \frac{a^2}{b^2} = \frac{(a^2 - b^2) y^2}{b^4}$$

$$\text{or } x^2 + \frac{a^2}{b^2} y^2 - a^2 = \frac{(a^2 - b^2) y^2}{b^2} = \frac{a^2}{b^2} y^2 - y^2$$

$$\text{or } x^2 + y^2 = a^2, \text{ which is a circle.}$$

Example 2. Show that the enveloping cylinder of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ with generators parallel to Z-axis meet the plane $z=0$ in ellipse. *CY in para 21.*

Sol. Please try yourself as above.

[Hint. Remember that an equation in x, y represents a parabola in XY plane if its second degree terms form a perfect square.]

Article 21. Section with a given centre. [V. Imp.]

To find the locus of chords of the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

which are bisected at the given point (x_1, y_1, z_1) .

$$\text{The given conicoid is } ax^2 + by^2 + cz^2 = 1 \quad \dots (1)$$

$$\text{Any chord through } (x_1, y_1, z_1) \text{ is } \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \dots (2)$$

Any point on this chord is $(lr + x_1, mr + y_1, nr + z_1)$

If it lies on (1), then

$$a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$$

$$\text{or } r^2(a l^2 + b m^2 + c n^2) + 2r(alx_1 + bmy_1 + cnz_1) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \dots (3)$$

which is a quadratic in r .

If l, m, n are the actual d.c.'s of line (2), then here r is the distance of any point common to the conicoid (1) and the chord (2) from the given point (x_1, y_1, z_1) .

If (x_1, y_1, z_1) is the middle point of chord (2), the points of intersection of (1) and (2) should be equidistant and on either side of (x_1, y_1, z_1) , i.e., the two values of r should be equal and opposite or the sum of roots in (3) is zero.

$$\text{Co-eff. of } r=0 \text{ giving } alx_1 + bmy_1 + cnz_1 = 0 \quad \dots (4)$$

Eliminating l, m, n from (2) and (4), the locus of chords (2) is

$$ax_1(x - x_1) + by_1(y - y_1) + cz_1(z - z_1) = 0$$

$$\text{or } ax_1x + by_1y + cz_1z = ax_1^2 + by_1^2 + cz_1^2$$

$$\text{or } axx_1 + byy_1 + czz_1 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1 \quad \dots(5)$$

which is of the form $T = S_1$. [Remember]

Note. The plane (5) meets the given conicoid in a conic whose centre is (x_1, y_1, z_1) .

Article 22. To find the locus of middle points of a system of chords of the conicoid $ax^2 + by^2 + cz^2 = 1$ which are parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

(M.D.U. 1983).

Any chord through (x_1, y_1, z_1) drawn \parallel to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \dots(1)$$

Any point on this chord is $(lr+x_1, mr+y_1, nr+z_1)$

This lies on the given conicoid $ax^2 + by^2 + cz^2 = 1$ if

$$a(lr+x_1)^2 + b(mr+y_1)^2 + c(nr+z_1)^2 = 1$$

$$\text{or } r^2(a(l^2+bm^2+cn^2) + 2l(alx_1+bmy_1+cnz_1) + (ax_1^2+by_1^2+cz_1^2-1)) = 0 \quad \dots(2)$$

If (x_1, y_1, z_1) is the mid-point of (1), then the two values of r in (2) must be equal in magnitude but opposite in sign, i.e., its sum of two roots is zero or the co-eff. of $r=0$

$$alx_1 + bmy_1 + cnz_1 = 0$$

Locus of (x_1, y_1, z_1) the mid-point is

$$alx + bmy + cnz = 0$$

which is a plane through the centre of the conicoid.

Example 1. Find the equation to the plane which cuts the surface (a) $2x^2 + 3y^2 + 5z^2 = 4$ in a conic whose centre is at the point $(1, 2, 3)$.

(b) $x^2 + 4y^2 - 5z^2 = 1$ in a conic whose centre is at the point $(2, 3, 4)$.

Sol: (a) The given conicoid is $S = 2x^2 + 3y^2 + 5z^2 - 4 = 0$. [Make R.H.S. = 0]

$$\text{Here } S_1 = 2(1)^2 + 3(2)^2 + 5(3)^2 - 4 \quad | \text{ Putting } (1, 2, 3) \text{ in } S \\ = 2 + 12 + 45 - 4 = 55$$

$$\text{and } T = 2x(1) + 3y(2) + 5z(3) - 4 \\ = 2x + 6y + 15z - 4.$$

The required plane is $T = S_1$, i.e.,

$$2x + 6y + 15z - 4 = 55 \quad \text{or} \quad 2x + 6y + 15z = 59.$$

(b) Please try yourself as above. [Ans. $x + 6y - 10z + 20 = 0$]

Example 2. Show that centre of the conic given by

$$ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$$

is the point $\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2} \right)$

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where $I^2 + m^2 + n^2 = 1$ and $p_0 = \sqrt{\sum \frac{I^2}{a}}$.

Sol. Let (x_1, y_1, z_1) be the centre of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$.

Then the plane with (x_1, y_1, z_1) as the centre of section is $(T=S_1)$

$$\text{i.e., } ax_1^2 + by_1^2 + cz_1^2 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1$$

$$\text{or } ax_1^2 + by_1^2 + cz_1^2 = ax_1^2 + by_1^2 + cz_1^2 \quad \dots(1)$$

$$\text{The plane should be identical with } lx + my + nz = p \quad \dots(2)$$

Comparing the co-efficients in (1) and (2), we have

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{ax_1^2 + by_1^2 + cz_1^2}{p} = k \text{ (say),}$$

$$\text{From these we have, } x_1 = \frac{lk}{a}, y_1 = \frac{mk}{b}, z_1 = \frac{nk}{c} \quad \dots(3)$$

$$\text{and } ax_1^2 + by_1^2 + cz_1^2 = pk \quad \dots(4)$$

Putting the values of x_1, y_1, z_1 from (3) in (4), we have

$$a\left(\frac{l^2k^2}{a^2}\right) + b\left(\frac{m^2k^2}{b^2}\right) + c\left(\frac{n^2k^2}{c^2}\right) = pk$$

$$\text{or } \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)k = p \text{ or } p_0^2 k = p \therefore k = \frac{p}{p_0^2}.$$

Putting this value of k in (3), the centre of section (x_1, y_1, z_1)

$$\text{is } \left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right). \text{ Hence the result.}$$

Example 3. Find the locus of centres of all plane sections of a conicoid

- (a) which pass through a fixed point.
- (b) which are at a constant distance from the centre.
- (c) which are parallel to a given line.
- (d) which pass through a given line.

Sol. Let (x_1, y_1, z_1) be the centre of plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$. $\dots(1)$

Then equation of the plane with (x_1, y_1, z_1) as centre is

$$ax_1x + by_1y + cz_1z - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1 \quad | \text{ Using } T=S_1$$

$$\text{or } ax_1x + by_1y + cz_1z = ax_1^2 + by_1^2 + cz_1^2 \quad \dots(2)$$

(a) The plane (2) passes through a fixed point say (α, β, γ) , then

$$a\alpha x + b\beta y + c\gamma z = ax_1^2 + by_1^2 + cz_1^2$$

\therefore Locus of (x_1, y_1, z_1) is $ax^2 + by^2 + cz^2 = a\alpha x + b\beta y + c\gamma z$ which is a conicoid.

(b) The plane (2) is at a constant distance k (say) from the centre $(0, 0, 0)$.

$$\frac{ax_1^2 + by_1^2 + cz_1^2}{\sqrt{a^2x_1^2 + b^2y_1^2 + c^2z_1^2}} = k$$

or

$$(ax_1^2 + by_1^2 + cz_1^2)^2 = k^2(a^2x_1^2 + b^2y_1^2 + c^2z_1^2)$$

\therefore Locus of (x_1, y_1, z_1) is $(ax^2 + by^2 + cz^2)^2 = k^2(a^2x^2 + b^2y^2 + c^2z^2)$.

(c) Let the given line be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (3)

The plane (2) will be \parallel to line (3) if its normal is \perp to (3), i.e.,

if

$$l(ax_1) + m(by_1) + n(cz_1) = 0$$

or

$$lx_1 + bmy_1 + cnz_1 = 0.$$

Locus of (x_1, y_1, z_1) is $lx_1 + bmy_1 + cnz_1 = 0$ which is a plane.

(d) Let the given line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ (4)

The line (4) will lie on plane (2) if (i) the line (4) is \perp to normal to the plane (2) i.e.,

$$lx_1 + bmy_1 + nc_1 = 0$$

and (ii) one point (α, β, γ) on (4) lies on the plane (2) i.e.,

$$ax_1 + b\beta y_1 + c\gamma z_1 = ax_1^2 + by_1^2 + cz_1^2$$

From (5) and (6), the locus of (x_1, y_1, z_1) is

$$alx + bmy + cnz = 0, ax^2 + by^2 + cz^2 = a\alpha x + b\beta y + c\gamma z$$

which being the intersection of a plane and a conicoid represents a conic.

Example 4. Find the centre of the conic given by the equations

$$2x - 2y - 5z + 5 = 0, 3x^2 + 2y^2 - 15z^2 = 4. \quad (1)$$

Sol. The conicoid is $S \equiv 3x^2 + 2y^2 - 15z^2 - 4 = 0$

Let (x_1, y_1, z_1) be the centre of the given conic. Then equation of the plane which cuts (1) in a conic with centre (x_1, y_1, z_1) is $T = S_1$

$$i.e., - 3xx_1 + 2yy_1 - 15zz_1 - 4 = 3x_1^2 + 2y_1^2 - 15z_1^2 - 4$$

$$or, \quad 3xx_1 + 2yy_1 - 15zz_1 - (3x_1^2 + 2y_1^2 - 15z_1^2) = 0 \quad (2)$$

Now this is the same as the given plane ... (3)

$$2x - 2y - 5z + 5 = 0$$

Comparing (2) and (3), we get

$$\frac{3x_1}{2} = \frac{2y_1}{-2} = \frac{-15z_1}{-5} = \frac{(3x_1^2 + 2y_1^2 - 15z_1^2)}{5} = k \text{ (say)}$$

$$\therefore x_1 = \frac{2}{3}k, y_1 = -k, z_1 = \frac{k}{3}. \quad (4)$$

and

$$3x_1^2 + 2y_1^2 - 15z_1^2 = -5k.$$

Putting the values of x_1, y_1, z_1 in the last equation, we get

$$3\left(\frac{4}{9}k^2\right) + 12k^2 - 15\left(\frac{k^2}{9}\right) = -5k$$

or

$$\frac{4}{3}k^2 + 2k^2 - \frac{5}{3}k^2 = -5k$$

or

$$4k^2 + 6k^2 - 5k^2 = -15k \quad \text{or} \quad 5k^2 = -15k \quad \therefore k = -3.$$

\therefore From (4), the centre is (x_1, y_1, z_1) i.e., $(-2, 3, -1)$.

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Example 5. Prove that the centres of sections of

$$ax^2 + by^2 + cz^2 = 1$$

by the planes which are at a constant distance p from the origin lie on the surface

$$(ax^2 + by^2 + cz^2)^2 = p^2(a^2x^2 + b^2y^2 + c^2z^2).$$

Sol. If (α, β, γ) be the centre of the section of the given ellipsoid then equation of this section of the sphere is " $T=S_1$ "

i.e. $(\alpha xx + \beta \beta y + \gamma \gamma z - 1) = (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$

or $a\alpha x + b\beta y + c\gamma z + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad \dots(i)$

The distance of this plane (i) from the origin $(0, 0, 0)$ is given as p .

$$p = \frac{a\alpha^2 + b\beta^2 + c\gamma^2}{\sqrt{(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2}}$$

or $p^2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) = (a\alpha^2 + b\beta^2 + c\gamma^2)^2$

The locus of the centre (α, β, γ) is

$$p^2(a^2x^2 + b^2y^2 + c^2z^2) = (ax^2 + by^2 + cz^2)^2.$$

Example 6. Prove that the centre of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane ABC whose equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the centroid of the triangle ABC .

Sol. The equation of the ellipsoid is

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \dots(i)$$

and the equation of the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(ii)$$

Let (α, β, γ) be the centre of the section (i) by the plane (ii) then the equation of this section is " $T=S_1$ "

i.e. $\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} - 1 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1$

or $\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \quad \dots(iii)$

The equations (ii) and (iii) represent the same plane, so comparing them, we get

$$\frac{\left(\frac{\alpha}{a^2}\right)}{\left(\frac{1}{a}\right)} \cdot \frac{\left(\frac{\beta}{b^2}\right)}{\left(\frac{1}{b}\right)} \cdot \frac{\left(\frac{\gamma}{c^2}\right)}{\left(\frac{1}{c}\right)} = \frac{\left(\frac{\alpha^2}{a^2}\right) + \left(\frac{\beta^2}{b^2}\right) + \left(\frac{\gamma^2}{c^2}\right)}{1} = k \quad (\text{say})$$

$$\therefore \alpha = ak, \beta = bk, \gamma = ck \text{ and}$$

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = k$$

$$\left(\frac{a^2 k^2}{a^2}\right) + \left(\frac{b^2 k^2}{b^2}\right) + \left(\frac{c^2 k^2}{c^2}\right) = k$$

or $3k^2 = k \quad \text{or} \quad k = \frac{1}{3}$

$$\alpha = ak = \frac{1}{3}a, \quad \beta = bk = \frac{1}{3}b, \quad \gamma = ck = \frac{1}{3}c.$$

∴ The centre of the section of (i) by the plane (ii) is (α, β, γ)
or $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

Also the co-ordinates of the vertices of $\triangle ABC$ are

$$A(a, 0, 0), B(0, b, 0), C(0, 0, c)$$

∴ The co-ordinates of the centroid of $\triangle ABC$ are $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

Hence the centre of the section of (i) by (ii) is the centre of $\triangle ABC$.

Example 7. Find the locus of the mid-points of the chords of the conicoid $ax^2 + by^2 + cz^2 = 1$ which passes through (α, β, γ) .
(Allahabad 1981; Kanpur 1979; Lucknow 1982)

Sol. Let (x_1, y_1, z_1) be the mid-point of the chord of the given conicoid. Then the locus of the chords of the given conicoid with (x_1, y_1, z_1) as mid-point is "T=S₁".

where $T = ax_1 + by_1 + cz_1 - 1 \quad \text{and}$

$$S_1 = ax_1^2 + by_1^2 + cz_1^2 - 1$$

i.e. $axx_1 + byy_1 + czz_1 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1$

or $axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2$

If it passes through (α, β, γ) , we have

$$axx_1 + b\beta y_1 + c\gamma z_1 = ax_1^2 + by_1^2 + cz_1^2$$

∴ The required locus of the mid-point (x_1, y_1, z_1) of the chords of the given conicoid is

$$ax^2 + by^2 + cz^2 = a\alpha x + b\beta y + c\gamma z$$

or $a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$.

Example 8. Show that the line joining a point P to the centre of a conicoid $ax^2 + by^2 + cz^2 = 1$ passes through the centre of the section of the conicoid by the polar plane of P.

Sol. Let (x', y', z') be the co-ordinates of the point P. Then the polar plane of P(x', y', z') with respect to the given conicoid is

$$axx' + byy' + czz' = 1 \quad \dots(i)$$

Let (α, β, γ) be the centre of the section of the given conicoid by the plane (i), then equation of this plane section can also be written as

$$T = S_1 \text{ or}$$

$$a\alpha x + b\beta y + c\gamma z - 1 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$$

or $a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 \quad \dots(ii)$

Since the equations (i) and (ii) represent the same plane, so comparing them, we get

$$\frac{\alpha}{x'} = \frac{\beta}{y'} = \frac{\gamma}{z'} \quad \dots(iii)$$

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Also the equations of the line joining the point $P(x', y', z')$ to the centre $(0, 0, 0)$ of the given conicoid is

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}$$

If this line passes through the centre (α, β, γ) of the section of given conicoid be the plane (i), then

$$\frac{\alpha}{x'} = \frac{\beta}{y'} = \frac{\gamma}{z'}$$

which is true by virtue of (iii).

Hence the line joining $P(x', y', z')$ to the centre of the given conicoid passes through the centre (α, β, γ) of the section of the conicoid by the polar plane (i) of P .

Example 9. Prove that the section of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose centre is at the point $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ passes through the extremities of the axes. (Rohilkhand 1985)

Sol. The ellipsoid is

$$S: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \dots(i)$$

The equation of the section of this ellipsoid with

$$\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right) \text{ as its centre is "T=S_1"}$$

$$\text{i.e., } \frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1$$

$$= \frac{\left(\frac{1}{3a}\right)^2}{a^2} + \frac{\left(\frac{1}{3b}\right)^2}{b^2} + \frac{\left(\frac{1}{3c}\right)^2}{c^2} - 1$$

$$\text{or } \frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

$$\text{or } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

which is the plane evidently passing through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, the three extremities of the axes of the ellipsoid given by (i).

Example 10. Find the locus of centres of sections of $ax^2 + by^2 + cz^2 = 1$ which touch $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$. (Rohilkhand 1983)

Sol. Let (x_1, y_1, z_1) be the centre of the section of conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(i)$$

The equation of the section is $T = S_1$

$$\text{or } ax_1x + by_1y + cz_1z - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1 \quad \dots(i)$$

$$\text{or } ax_1x + by_1y + cz_1z = (ax_1^2 + by_1^2 + cz_1^2) \quad \dots(ii)$$

If the plane (i) touches the conicoid $ax^2 + by^2 + cz^2 = 1$, then we must have:

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$\therefore \frac{(ax_1)^2}{a} + \frac{(by_1)^2}{b} + \frac{(cz_1)^2}{c} = (ax_1^2 + by_1^2 + cz_1^2)^2$$

The required locus of (x_1, y_1, z_1) is

$$\frac{a^2x^2}{\alpha} + \frac{b^2y^2}{\beta} + \frac{c^2z^2}{\gamma} = (ax^2 + by^2 + cz^2)^2.$$

Example 11. Prove that the middle point of the chords of $ax^2 + by^2 + cz^2 = 1$ which are parallel to $x=0$ and touch $x^2 + y^2 + z^2 = \gamma^2$ lies on the surface

$$by^2(bx^2 + by^2 + cz^2 - bx^2) + cz^2(-cx^2 + by^2 + cz^2 - cy^2) = 0. \quad (\text{Kanpur 1982; Rohilkhand 1983})$$

Sol. The equation of any line having (α, β, γ) as mid-point and parallel to the plane $x=0$ is

$$\frac{x-\alpha}{o} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = \lambda \text{ (say)} \quad \dots(i)$$

where m and n are variables.

Any point on this line is $(\alpha, \beta + m\gamma, \gamma + n\lambda)$. If this point lies on the conicoid $ax^2 + by^2 + cz^2 = 1$, then we have

$$a\alpha^2 + b(\beta + m\gamma)^2 + c(\gamma + n\lambda)^2 = 1$$

$$\text{or } \lambda(bm^2 + cn^2) + 2\lambda(b\beta m + c\gamma n) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad \dots(ii)$$

(α, β, γ) is the mid-point of the chord (i) of the given conicoid, so that sum of the roots of equation (ii), which is a quadratic in λ , must be zero

$$\text{i.e., } bni\beta + cn\gamma = 0 \quad \dots(iii)$$

Also the line (i) touches the sphere

$$x^2 + y^2 + z^2 = r^2$$

The length of perpendicular from the centre $(0, 0, 0)$ of the sphere to (i) must be equal to the radius r of the sphere

$$\text{i.e., } \left[\frac{-\alpha}{o} + \frac{-\beta^2}{m} + \frac{-\gamma^2}{n} + \frac{-\alpha^2}{n} \right] \div (m^2 + n^2) = r^2$$

$$\text{or } m^2\alpha^2 + (n\beta - m\gamma)^2 + \alpha^2n^2 = r^2(m^2 + n^2)$$

$$\text{or } (r^2 - \alpha^2)(m^2 + n^2) = (n\beta - m\gamma)^2$$

$$\text{or } (r^2 - \alpha^2) \left[\left(\frac{m}{n} \right)^2 + 1 \right] = \left[\beta - \left(\frac{m}{n} \right) \gamma \right]^2 \quad \dots(iv)$$

$$\text{Also from (iii), we have } \frac{m}{n} = \frac{(-\gamma)}{(b\beta)}$$

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Substituting the value in (iv), we get

$$(r^2 - \alpha^2) \left[\left(\frac{c^2 r^2}{b^2 \beta^2} \right) + 1 \right] = \left[\beta + \left(\frac{c \gamma}{b \beta} \right) \right]^2$$

or

$$(r^2 - \alpha^2)[c^2 \gamma^2 + b^2 \beta^2] = [b \beta^2 + \gamma^2 c]^2$$

∴ The required locus of (α, β, γ) is

$$(r^2 - x^2)(c^2 z^2 + b^2 y^2) = (b y^2 + c z^2)^2$$

or $c^2 r^2 z^2 + b^2 r^2 y^2 - c^2 x^2 z^2 - b^2 y^2 x^2 = b^2 y^4 + c^2 z^4 - 2 b c y^2 z^2$

or $b y^2(b x^2 + b y^2 + c z^2 - b x^2) + c z^2(c z^2 + b y^2 - c x^2 - c i^2) = 0$

CONE

Article 23. To trace the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

The given surface is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$..(1)

(i) Symmetry. Since the equation (1) contains even powers of x, y, z , so the surface is symmetrical about the YZ, ZX, and XY planes.

(ii) Axes intersection. The cone meets X-axis ($y=0, z=0$) where $\frac{x^2}{a^2} = 0$ or $x=0, 0$, i.e., in two coincident points.

Thus cone meets X-axis at the origin. Similarly, it meets Y and Z-axis also at the origin.

(iii) Sections by co-ordinate planes. The cone (1) meets the YZ plane ($x=0$), where $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ or $y = \pm \frac{b}{c} z$ which are two straight lines in that plane [on opposite sides of Z-axis and making equal angles with it].

Similarly, the cone (1) meets ZX plane ($y=0$) in two lines $x = \pm \frac{a}{c} z$ which are equally inclined to Z-axis and on opposite sides of it.

Again it meets the XY plane ($z=0$), where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ which is a point ellipse in that plane.

(iv) Generated by a variable curve. The surface (1) meets the plane $z=k$ [where putting $z=k$ in (1)].

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{k^2}{c^2} = 0 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$$

Thus the cone (i) is generated by the variable ellipse

$$z=k, \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} \dots (2) \quad (k \text{ varies})$$

whose plane is \parallel to XY plane and centre $(0, 0, k)$ moves on Z-axis. The ellipse (2) is real for all values of k +ve or -ve and the semi-axes $\frac{ak}{c}, \frac{bk}{c}$ increase as k increases numerically and $\rightarrow \infty$ as $k \rightarrow \infty$.

The cone extends to infinity both above and below the XY-plane.

Hence the shape is as shown in the adjoining figure.

Note 1. The standard equation of the cone is of the form

$$ax^2 + by^2 + cz^2 = 0.$$

Note 2. A cone can be regarded as a central conicoid whose centre is the vertex.

Article 24. Some important results about the cone

$$ax^2 + by^2 + cz^2 = 0.$$

(i) The tangent plane at (x_1, y_1, z_1) and the plane of contact of (x_1, y_1, z_1) and polar plane of (x_1, y_1, z_1) w.r.t. given cone is

$$axx_1 + byy_1 + czz_1 = 0.$$

(ii) The plane $lx + my + nz = 0$ touches the cone if

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0.$$

(iii) The equation of plane which cuts the cone in a conic with centre (x_1, y_1, z_1) is given by $T = S_1$.

~~The student is advised to prove these results as in Arts. 7, 8, 14, 15, 21.~~

Example 1. Find the equation of the normal plane of the cone

$$ax^2 + by^2 + cz^2 = 0,$$

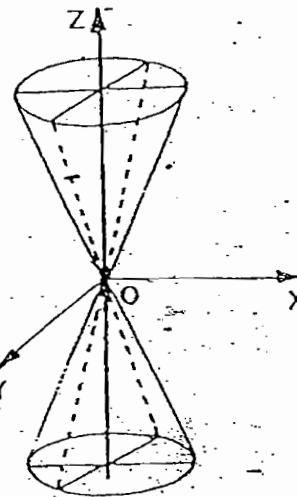
through the generator

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Sol. [The normal plane through the generator OP of a cone (vertex O) is the plane through OP and \perp to the tangent plane at any point of OP.] [Remember]

The given cone is $ax^2 + by^2 + cz^2 = 0 \dots (1)$

and the generator is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \dots (2)$



THE CONICOID

Any plane through the line (2) is

$$Ax + By + Cz = 0 \quad \dots(3)$$

where

$$Al + Bm + Cn = 0 \quad \dots(4)$$

Any point on (2) is $Q(lr, mr, nr)$. The tangent plane at Q to (1) is

$$ax(lr) + by(mr) + cz(nr) = 0 \text{ or } alx + bmy + cnz = 0 \quad \dots(5)$$

If (3) is the normal plane through (2), then (3) is \perp to (5).

$$Aal + Bbm + Ccn = 0 \quad \dots(6)$$

Solving (4) and (6) by cross-multiplication, we have

$$\frac{A}{mn(c-b)} = \frac{B}{nl(a-c)} = \frac{C}{lm(b-a)}$$

Putting these values of A, B, C in (3) and taking out -ve sign common, we have

$$mn(b-c)x + nl(c-a)y + lm(a-b)z = 0.$$

Dividing throughout by lmn , we get

$$\frac{(b-c)x}{l} + \frac{(c-a)y}{m} + \frac{(a-b)z}{n} = 0,$$

which is the required normal plane.

Example 2. Lines are drawn through the origin perpendicular to normal planes of the cone

$$ax^2 + by^2 + cz^2 = 0.$$

Show that they generate the cone

$$\frac{a(b-c)^2}{x^2} + \frac{b(c-a)^2}{y^2} + \frac{c(a-b)^2}{z^2} = 0.$$

Sol. Let the line OP given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

be any generator of the cone

$$ax^2 + by^2 + cz^2 = 0 \quad \dots(1)$$

Since the d.c.'s of the generator satisfy the equation of the cone,

$$al^2 + bm^2 + cn^2 = 0 \quad \dots(2)$$

Also equation of the normal plane through OP to (1) is

$$\left(\frac{b-c}{l}\right)x + \left(\frac{c-a}{m}\right)y + \left(\frac{a-b}{n}\right)z = 0 \quad \dots(3)$$

[See Example 1, above]

Equations of the line through $(0, 0, 0)$ \perp to (3) are

$$\frac{x}{\left(\frac{b-c}{l}\right)} = \frac{y}{\left(\frac{c-a}{m}\right)} = \frac{z}{\left(\frac{a-b}{n}\right)}$$

or

$$\frac{l}{(b-c)} = \frac{m}{(c-a)} = \frac{n}{(a-b)} \quad \dots(4)$$

To find the locus of line (4), we have to eliminate l, m, n from (4) and (2). Putting the values of l, m, n from (4) in (2), we get

$$a\left(\frac{b-c}{x}\right)^2 + b\left(\frac{c-a}{y}\right)^2 + c\left(\frac{a-b}{z}\right)^2 = 0$$

$$\text{or } \frac{a(b-c)^2}{x^2} + \frac{b(c-a)^2}{y^2} + \frac{c(a-b)^2}{z^2} = 0$$

which is the required cone.

Example 3. Prove that if a plane cuts the cone

$$ax^2 + by^2 + cz^2 = 0$$

in perpendicular generators, it touches the cone

$$\frac{x^2}{b+c} + \frac{y^2}{c+a} + \frac{z^2}{a+b} = 0 \quad \dots(1)$$

Sol. Let the plane be $ux + vy + wz = 0$

$$\text{and the cone is } ax^2 + by^2 + cz^2 = 0 \quad \dots(2)$$

Let a line of section of (1) and (2) be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Since it lies on (1) and (2) both,

$$\therefore ul + vm + wn = 0 \text{ and } al^2 + bm^2 + cn^2 = 0 \quad \dots(3)$$

The two lines given by (3) are \perp if

$$u^2(b+c) + v^2(c+a) + w^2(a+b) = 0 \quad \dots(4)$$

Now the plane (1) will touch the cone

$$\frac{x^2}{b+c} + \frac{y^2}{c+a} + \frac{z^2}{a+b} = 0$$

if

$$\frac{u^2}{\left(\frac{1}{b+c}\right)} + \frac{v^2}{\left(\frac{1}{c+a}\right)} + \frac{w^2}{\left(\frac{1}{a+b}\right)} = 0$$

$$\Rightarrow \text{Using } \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0$$

or if $u^2(b+c) + v^2(c+a) + w^2(a+b) = 0$ which is true by (4).

Hence the result.

Remember. Two lines given by

$$ul + vm + wn = 0 \text{ and } al^2 + bm^2 + cn^2 = 0$$

are \perp if $u^2(b+c) + v^2(c+a) + w^2(a+b) = 0$.

Example 4. Show that the perpendicular tangent planes to

$$ax^2 + by^2 + cz^2 = 0$$

intersect in generators of the cone

$$a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0$$

[Imp.]

Sol. The given cone is $ax^2 + by^2 + cz^2 = 0$ $\dots(1)$

THE CONOID

Any tangent plane to (1) is $lx+my+nz=0$... (2)

where $\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = 0$... (3)

Let the line of intersection of two tangent planes, (through origin) be

$$\frac{x}{L} = \frac{y}{M} = \frac{z}{N} \quad \dots (4)$$

Since it lies on (2), $\therefore LI+Mm+Nn=0$... (5)

The two lines given by (5) and (3) are \perp ,

$$L^2\left(\frac{1}{b} + \frac{1}{c}\right) + M^2\left(\frac{1}{c} + \frac{1}{a}\right) + N^2\left(\frac{1}{a} + \frac{1}{b}\right) = 0$$

Eliminating L, M, N from this and (4), the required locus is

$$x^2\left(\frac{1}{b} + \frac{1}{c}\right) + y^2\left(\frac{1}{c} + \frac{1}{a}\right) + z^2\left(\frac{1}{a} + \frac{1}{b}\right) = 0$$

or $a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0$.

Example 5. (a) The locus of the asymptotes drawn from the origin to the conoid

$$ax^2 + by^2 + cz^2 = 1$$

is the asymptotic cone

$$ax^2 + by^2 + cz^2 = 0$$

(b) Prove that the hyperboloids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

have the same asymptotic cone.

Sol. [Def. An asymptote meets the given surface at two points an infinity]. [Remember]

(a) The conoid is $ax^2 + by^2 + cz^2 = 1$... (1)

Let the asymptote through (0, 0, 0) be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \dots (2)$$

Any point on (2) is (lr, mr, nr) .

If it lies on (1), then

$$al^2r^2 + bm^2r^2 + cn^2r^2 = 1$$

or $al^2 + bm^2 + cn^2 = \frac{1}{r^2}$

Since the asymptote (2) meets (1) an infinity $\therefore r = \infty$

$$\therefore al^2 + bm^2 + cn^2 = \frac{1}{\infty} = 0 \quad \dots (3)$$

Eliminating l, m, n from (2) and (3), the locus of (2) is

$$ax^2 + by^2 + cz^2 = 0$$

which is a cone.

(b) Please try yourself as in part (a).

Example 6. Any plane whose normal lies on the cone
 $(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = 0$
cuts the surface

$$ax^2 + by^2 + cz^2 = 1$$

in a rectangular hyperbola.

[Imp.]

Sol. Let the plane be $ux + vy + wz = 0$

... (1)

This cuts the surface $ax^2 + by^2 + cz^2 = 1$
in rectangular hyperbola.

Let the asymptote of this hyperbola be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \dots (2)$$

The asymptote (2) lies on plane (1)

$$ul + vm + wn = 0 \quad \dots (3)$$

Also any point on (2) is (lr, mr, nr) . This point will lie on the surface $ax^2 + by^2 + cz^2 = 1$, if

$$r^2(al^2 + bm^2 + cn^2) = 1 \text{ or } al^2 + bm^2 + cn^2 = \frac{1}{r^2}$$

But $r \rightarrow \infty$, as the asymptote cuts the surface at ∞

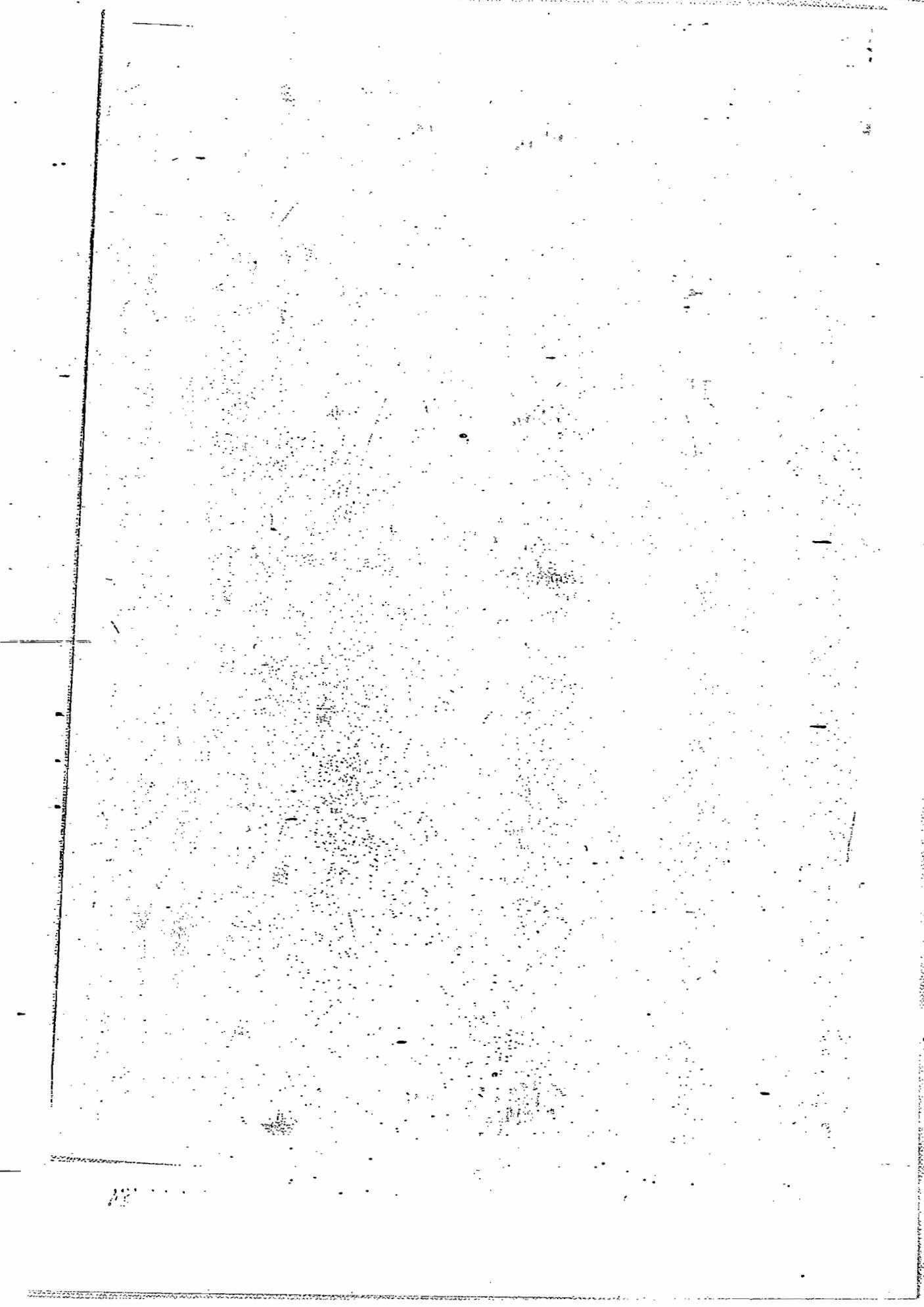
$$\therefore \text{We have } al^2 + bm^2 + cn^2 = 0 \quad \dots (4)$$

The asymptote of a rectangular hyperbola are \perp . Thus the two lines given by (3) and (4) are \perp .

$$\therefore u^2(b+c) + v^2(c+a) + w^2(a+b) = 0 \quad | \text{ Refer Ex. 15 (i), page 35}$$

This shows that the normal $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$
to plane (1) lies on the cone

$$(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = 0$$



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Similarly it touches X -axis at $(0, 0, 0)$. The the surface (1) touches the XY -plane. Also (1) meets Z -axis at $(0, 0, 0)$.
(iii) Sections by co-ordinate planes, (1) meets the YZ plane, $(x=0)$

where

$$\frac{y^2}{b^2} = \frac{2z}{c}, \text{ or } y^2 = \frac{2b^2}{c} z$$

which is an upward parabola in that plane.

Similarly (1) meets the ZX plane, $(y=0)$ in an upward para-
boloid

$$x^2 = \frac{2a^2}{c} z \text{ in that plane.}$$

Again (1) meets the XY plane, $(z=0)$ in

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

which is a point ellipse in that plane.

(iv) Generated by a variable curve. The surface (1) meets the plane $z=k$, where

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c} \text{ or } \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = \frac{2k}{c}$$

Thus the surface is generated by a variable ellipse

$$z=k, \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = \frac{2k}{c} \quad \dots(2)$$

where k varies.

Its plane is \parallel to XY -plane and the centre $(0, 0, k)$ moves on Z -axis. Now the ellipse (2) is real if k is +ve. Thus the surface lies only above the XY -plane.

PARABOLOID

Article 24: (r) To trace elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2x$$

The given paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$... (1)

(i) Symmetry: Since (1) contains even powers of x and y , so it is symmetrical about the Z -axis and XY planes.

(ii) Intersection with axes, The surface (1) meets X -axis where $\frac{x^2}{a^2} = 0$, or $x^2 = 0$, or $x=0$.

The surface (1) touches the X -axis at $(0, 0, 0)$.

Also the semi-axis of the Ellipse (2) are $a\sqrt{\frac{2k}{c}}$, $b\sqrt{\frac{2k}{c}}$ which increases as $k > 0$ increases and $\rightarrow \infty$ as $k \rightarrow \infty$. Thus the surface extends to ∞ above the XY -plane.

The shape is as shown in the adjoining figure.

Article 24: (d) To trace the hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$$

The equation of the surface is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$$

(i) Symmetry. Since the equation (1) contains even powers of x and y , so the surface is symmetrical about the YZ and ZX -planes.

(ii) Axes-intersection. The surface (1) meets X -axis ($y=0, z=0$) where $\frac{x^2}{a^2} = 0$ or $x^2 = 0$ or $x = 0, 0$. Thus the surface (1) touches X -axis at the origin. Similarly it touches Y -axis at the origin.

Thus (1) touches XY -plane at $O(0, 0, 0)$.

It meets Z -axis ($x=0, y=0$), where $\frac{z^2}{c^2} = 0$ or $z=0$ i.e., at the origin.

(iii) Sections by co-ordinate planes. The surface (1) meets the YZ -planes ($x=0$) where

$$\frac{-y^2}{b^2} = \frac{2z}{c} \quad \text{or} \quad y^2 = -2 \frac{b^2}{c} z$$

which is a downward parabola in that plane (assuming c to be +ve).

Similarly (1) meets the ZX -plane in the upward parabola

$$x^2 = \frac{2a^2}{c} z \text{ in that plane.}$$

It meets the XY -plane ($z=0$) where

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y = \pm \frac{b}{a} x$$

which are two straight lines in that plane equally inclined to X -axis, (iv) Generated by a variable curve. The surface (1) meets the plane $z=k$ where [putting $z=k$ in (1)],

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2k}{c} = 0 \quad \text{or} \quad \left(\frac{2ak}{c}\right)^2 - \left(\frac{2bk}{c}\right)^2 = 1.$$

Thus the surface is generated by a variable hyperbola

$$\left(\frac{2ak}{c}\right)^2 - \left(\frac{2bk}{c}\right)^2 = 1, z = k \quad \dots(2) \quad [\text{as } k \text{ varies}]$$

whose plane is \parallel to the XY -plane, and centre $(0, 0, k)$ moves on the Z -axis.

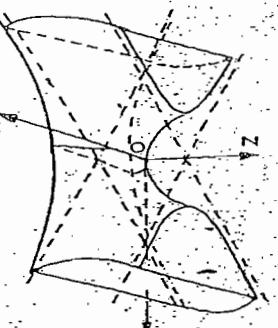
The hyperbola (2) has transverse axis \parallel to X -axis if k is +ve and \parallel to Y -axis if k is -ve. [$\because c$ is assumed to be +ve]

Also the transverse semi-axis is $a \sqrt{\frac{2k}{c}}$ which increases as $k(+ve)$ increases and $\rightarrow \infty$ as $k \rightarrow \infty$.

Thus surface extends to infinity above the XY -plane.

Similarly the surface extends to infinity below the XY -plane. The surface extends to infinity both above and below the XY -plane. Hence the shape of the surface is as shown in the figure.

THE CONOID



Note. The general equation of the paraboloid is of the form $ax^2 + by^2 = 2z$, which is an elliptic or hyperbolic paraboloid according as a and b are of the same or opposite signs.

Article 25. Intersection of a line with the paraboloid.

Let the line be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \dots(1)$$

and the paraboloid be $ax^2 + by^2 = 2z$ $\dots(2)$

Any point on (1) is $P(l(x_1 + x) + y_1, m(x_1 + y) + z_1)$. If it lies on (2) then $a(l(x_1 + x) + y_1)^2 + b(m(x_1 + y) + z_1)^2 = 2(z_1 + z)$ $\dots(3)$

or $a(l^2x_1 + 2lx + y_1^2 + 2my_1 + z_1^2) + b(m^2x_1 + 2mx + y_1^2 + 2yz_1 + z_1^2) = 0$ $\dots(3')$

which gives two values of z . This shows that every line meets the paraboloid in two points.

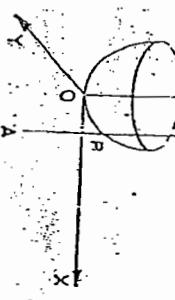
Again if $l=m=0, n=1$ from (3), one value of z is infinite showing that any line \parallel to the Z -axis meets the paraboloid in one point at an infinite distance from $A(x_1, y_1, z_1)$ and in a finite point P whose distance from A is given by

$$= \sqrt{ax_1^2 + by_1^2 - 2z_1}.$$

Such a line drawn through a point A , which meets the paraboloid in one point at an infinite distance from A and in a point P is called a diameter of the paraboloid and P is called the extremity of the diameter.

Thus a line \parallel to OZ is a diameter of the paraboloid $ax^2 + by^2 = 2z$.

Def. The diameter of a paraboloid which is \perp to the tangent plane at its extremity is called the axis of the paraboloid and its extremity is called the vertex of the paraboloid.



Thus OZ is the axis and O the vertex of the paraboloid

$$ax^2+by^2=2z.$$

Cor. A line \parallel to the axis of a paraboloid is a diameter.

Article 26. Some standard results about the paraboloid

Following are some of the results about a paraboloid which can be easily proved. The student is advised to prove these results for himself.

Let the paraboloid be

$$\frac{ax^2}{a} + \frac{by^2}{b} = 2z. \quad \dots(1)$$

(i) The tangent plane to (1) at (x_1, y_1, z_1) is

$$ax_1x+by_1y=2z_1. \quad (\text{K.U. 1970})$$

[See Art. 1 (e) prove by General Methods]

(ii) The condition of tangency for a given plane

$$lx+my+nz=p. \quad \dots(2)$$

and the paraboloid (1) is

$$\frac{x^2}{a} + \frac{y^2}{b} = \frac{2z}{p}. \quad \dots(3)$$

[For proof see page (xxv) of general methods. Art. 2(d)]

and the point of contact is

$$\left(\frac{-al}{pn}, \frac{-bm}{pn}, \frac{-c}{n} \right)$$

and any tangent plane to (1) \parallel to $lx+my+nz=0$ is

$$2n(lx+my+nz)+\frac{l^2}{a}+\frac{m^2}{b}=0.$$

(iii) The plane of contact and the polar plane of (x_1, y_1, z_1) w.r.t. (1) is $ax_1x+by_1y=2z_1$.

(iv) The enveloping cone of (1) is given by $SS_1=T^2$

$$(ax^2+by^2-2z)(ax^2+by^2-2z_1)=(lx(x_1+by_1)-(z_1+z))^2.$$

(v) The plane section of (1) with given centre (x_1, y_1, z_1) is given by

$$lxx_1+mxy_1-(z_1-y_1)=ax_1^2+by_1^2=2z_1, \quad \dots(4)$$

Example 1. Find the condition that the plane $lx+my+nz=1$ may be a tangent plane to the paraboloid $x^2+y^2=2z$.

Sol. Reproduce Art. 2(a) page (xxv) General Methods replacing a, b, p by 1 each.

Example 2. (a) Show that the plane $8x-6y-2z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ and find the co-ordinates of the point of contact.

(b) Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find the co-ordinates of the point of contact.

(Agra 1988)
(Agra 1987 : Madurai 1983)

THB CONOID

Sol. (a) Let the plane $8x-6y-2z=5$ touch the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ or $3x^2-2y^2=6z$ at the point (x_1, y_1, z_1) . Then the tangent plane to (2) at (x_1, y_1, z_1) is

$$3xx_1-2yy_1=3(z_1+z_2), \quad \dots(3)$$

Now this plane is identical with (1). Comparing the co-effs. in (1) and (3), we have

$$\frac{3x_1}{8}=\frac{-2y_1}{6}=\frac{-3}{-1}=\frac{3}{5},$$

which gives $x_1=8, y_1=9, z_1=5$.

The plane (1) touches the paraboloid (2) i.e. if $3(6z)-2(8)=6(5)$ or $192-162=30$ or $30=30$ which is true.

Hence (1) touches (2) and the point of contact is $(8, 9, 5)$. (b) Please try yourself.

Example 3. Prove that the paraboloids $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c_1}$ and $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_2}$ have a common tangent plane if

$$\begin{vmatrix} a^2 & b^2 & c \\ a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \end{vmatrix} = 0.$$

Sol. The given paraboloids are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c_1}$ and $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_2}$... (1)

$$\frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}=\frac{2z}{c_1} \dots (2) \text{ and } \frac{a^2}{a_1^2}+\frac{b^2}{b_1^2}=\frac{2z}{c_2} \dots (3)$$

Let the common tangent plane be $lx+my+nz=p$ Since it touches the paraboloid (1)

$$\frac{c_1}{a_1^2}x^2+\frac{c_1}{b_1^2}y^2=2z, \quad \dots(4)$$

$$\left(\frac{c_1}{a_1^2} \right)^2 + \left(\frac{c_1}{b_1^2} \right)^2 = -2np \quad \text{Using } \frac{c_1^2}{a_1^2} + \frac{c_1^2}{b_1^2} = -2np$$

$$\text{or } \frac{c_1^2}{a_1^2}+m^2b_1^2+2nc_1p=0 \quad \dots(5)$$

$$\text{Similarly (4) touches (2) and (3), } \frac{c_1^2}{a_1^2}+m^2b_1^2+2nc_1p=0 \quad \dots(6)$$

$$\text{and } \frac{c_1^2}{a_1^2}+m^2b_1^2+2nc_1p=0 \quad \dots(7)$$

$$\text{Eliminating } l^2, m^2, 2np \text{ from (5), (6), (7) by determinants, we have } \begin{vmatrix} a^2 & b^2 & c \\ a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix} = 0 \quad \dots(8)$$

Example 4. Show that the equation to two tangent planes to the surface $ax^2 + by^2 = 2z$ which pass through the line $u = kx + my + nz - p = 0$, where u' is $\frac{\partial u}{\partial x}$, etc.

$$u^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + 2np' \right) - 2u u' \left(\frac{lp'}{a} + \frac{mp'}{b} + np' + np' \right) + u'^2 \left(\frac{p^2}{a} + \frac{m^2}{b} + 2np' \right) = 0.$$

Sol. Any plane through the line
 $u=0, u'=0$ is, $uk + wp = 0$

$$\text{or } (l+kp)x + (m+kp)y + (n+kp)z - p = 0$$

If it touches the paraboloid $ax^2 + by^2 = 2z$, then

$$\frac{(l+kp)^2}{a} + \frac{(m+kp)^2}{b} = -2(n+kp), (p+kp)$$

$$\text{Using } \frac{l^2}{a} + \frac{m^2}{b} = -2np$$

$$\begin{aligned} & \left(\frac{l^2}{a} + \frac{m^2}{b} + 2np' \right) + 2k \left(\frac{lp'}{a} + \frac{mp'}{b} + np' \right) \\ & + \left(\frac{p^2}{a} + \frac{m^2}{b} + 2np' \right) = 0. \end{aligned}$$

Putting $k = -\frac{u}{u'}$, from (1) in this, we get the required result.

Example 5. Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid

$$(i) ax^2 + by^2 + 2z = 0$$

or
 $ax^2 + by^2 + 2z = 0$

or
 $(ax^2 + by^2 + 2z)(mx_1 + my_1 + nz_1) = SS_1 = T^2$

If the three lines drawn from P to touch the given paraboloid are mutually \perp , then the conic must have three mutually \perp genera-

tors, i.e. sum of coeffs. of x_1^2, y_1^2, z_1^2 in (1) is zero.

$a(by_1^2 + 2z_1) + b(ax_1^2 + 2z_1) + 1 = 0$.

Locus of $P(x_1, y_1, z_1)$ is
 $ab(x_1^2 + y_1^2) + 2(a+b)z_1 - 1 = 0.$

(ii) Please try yourself. [Ans. $ab(x^2 + y^2) - 2(a+b)z - 1 = 0$]

Example 6. (a) Find the equation of the plane which cuts the paraboloid $x^2 + 2y^2 = 2z$ in a cone with its centre at $(2, \frac{3}{2}, \frac{7}{2})$.

(b) Find the centre of the cone $ax^2 + by^2 = 2z$, $lx + my + nz = p$, and the centre (x_1, y_1, z_1) is $(2, \frac{3}{2}, 4)$.

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Then the plane which has the centre $P(2, \frac{3}{2}, \frac{7}{2})$ is given by zero, i.e. by

$$T = S_1, \text{ where } T \text{ is the expression for tangent plane at } P \text{ with R.H.S.}$$

$$x(2) - 2y \left(\frac{3}{2} \right) - \frac{1}{2}(z(4)) = (2)^2 - 2 \left(\frac{3}{2} \right)^2 - 4.$$

$$\text{or } 2x - 3y - \frac{1}{2}z - 2 = 4 - \frac{9}{2} + 4.$$

$$4x - 6y - z + 5 = 0$$

$$(b) Let (x_1, y_1, z_1) be the centre of the conic given by$$

$$ax^2 + by^2 - 2z = 0$$

and Then equation of the plane which cuts (1) in a conic with a

$$ax_1^2 + by_1^2 - (z + z_1) = ax_1^2 + by_1^2 - 2z_1$$

$$ax_1^2 + by_1^2 - z - (ax_1^2 + by_1^2 - z_1) = 0 \quad \dots(2)$$

$$\text{Now (2) and (3) are identical. Comparing coeffs., we have}$$

$$\frac{ax_1}{1} = \frac{by_1}{m} = \frac{-1}{n} = \frac{ax_1^2 + by_1^2 - z_1}{p} \quad \dots(3)$$

which give $x_1 = \frac{-1}{n}, y_1 = \frac{-m}{p}$, and $ax_1^2 + by_1^2 - z_1 = \frac{-p}{n}$. Putting the values of x_1, y_1 from first two equations in the third equation, we get

$$a \cdot \frac{l^2}{a} + b \cdot \frac{m^2}{b} - z_1 = -\frac{p}{n}$$

$$\text{or } z_1 = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{p}{n}.$$

Thus the centre (x_1, y_1, z_1) of the conic is

$$\left(\frac{-1}{n}, \frac{-m}{p}, \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{p}{n} \right).$$

Example 7. Show that the locus of centres of a system of parallel plane sections of a paraboloid is a diameter.

Prove also that the tangent plane at the extremity of the diameter is parallel to the plane sections.

Sol. Let the paraboloid be $ax^2 + by^2 = 2z$ and let (x_1, y_1, z_1) be the centre of one of the plane sections of (1)

$lx + my + nz = p$ drawn parallel to a given plane

(x_1, y_1, z_1) is the equation of the plane section of (1), whose centre is

$$ax_1^2 + by_1^2 - (z + z_1) = ax_1^2 + by_1^2 - 2z_1 \quad \dots(2)$$

Now (2) is \parallel to plane (3), $|T=S_1|$

$$\frac{dx_1}{d_1} = \frac{dy_1}{m} = \frac{-1}{n}$$

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\therefore Locus of (x_1, y_1, z_1) is $\frac{ax}{l} = \frac{by}{m} = \frac{-z}{n}$... (4)

which is the line of intersection of the planes

$nar + l^2 = 0$ [from first and third members of (4)]
and $nby + m^2 = 0$ [from second and third members of (4)]

which are respectively \perp to planes $x=0$ and $y=0$.

Thus the line (4) is \perp to the Z-axis ($x=0, y=0$) which is the axis of the paraboloid, and, consequently, (3), is a diameter of (1). Hence the first part.

Second Part. To find the extremity of diameter (3), we have to solve (4) and (1).

From (4), $x = \frac{l}{na}, y = -\frac{m}{nb}$

Putting these values of x, y in (1), we get

$$a\left(-\frac{l}{na}\right)^2 + b\left(-\frac{m}{nb}\right)^2 = 2z$$

$$\text{or } z^2 = \frac{l^2}{2n^2a^2} + \frac{m^2}{2n^2b^2}.$$

Hence the extremity of the diameter is

$$\left(-\frac{l}{na}, -\frac{m}{nb}, \frac{l^2}{2n^2a^2} + \frac{m^2}{2n^2b^2}\right).$$

\therefore Equation of the tangent plane to (1) at this extremity is

$$ax\left(-\frac{l}{na}\right) + by\left(-\frac{m}{nb}\right) = z + \left(\frac{l^2}{2n^2a^2} + \frac{m^2}{2n^2b^2}\right)$$

[Using $ax_1 + by_1 + cz_1 = 0$]

$$\text{or } lx + my + nz + \frac{l^2}{2n^2a^2} + \frac{m^2}{2n^2b^2} = 0$$

which is clearly \parallel to the given plane (2). Hence the result.

Article 27. To find the locus of intersection of three mutually perpendicular tangent planes to the paraboloid

$$ax^2 + by^2 = 2z$$

[V. Imp.]

The given paraboloid is

$$ax^2 + by^2 + nz = p_1$$

Let $lx + my + nz = p_1$ (l, m, n being the actual d.c.'s) be one of the three mutually \perp tangent planes so that

$$\frac{l^2}{a} + \frac{m^2}{b} = -2np_1$$

Condition of tangency

$$p_1 = -\frac{1}{2n}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)$$

Putting the value of p_1 in the equation of one of the three mutually \perp tangent planes is

$$(lx + my + nz)^2 = -\frac{1}{4n^2}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)^2$$

or $(lx + my + nz)^2 = -\frac{1}{4n^2}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)^2$

or $\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{-1}$

(ii) Please try yourself.

$$\text{Ans. } \frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{-\frac{1}{4n^2}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)^2}$$

Multiplying both sides by n_1
or $n_1(lx + my + nz) + \frac{l^2}{2a} + \frac{m^2}{2b} = 0$ (Note this step) ... (2)

Similarly the equations of other two tangent planes is

$$n_2(lx + my + nz) + \frac{l^2}{2a} + \frac{m^2}{2b} = 0 \quad \dots (3)$$

$$n_3(lx + my + nz) + \frac{l^2}{2a} + \frac{m^2}{2b} = 0 \quad \dots (4)$$

The locus of the point of intersection of (2), (3), (4) is given by eliminating l, m, n_1 etc. from these equations. Adding (2), (3), (4), we get,

$$x\Sigma l_1 n_1 + y\Sigma m_1 n_1 + z\Sigma n_1^2 + \frac{1}{2a} \Sigma l_1^2 + \frac{1}{2b} \Sigma m_1^2 = 0$$

$$\text{or } x(0) + y(0) + z(1) + \frac{1}{2a}(1) + \frac{1}{2b}(1) = 0$$

$$\text{or } 2 + \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = 0$$

$\therefore l_1, m_1, n_1$ etc. are the d.c.'s of three mutually \perp lines,
 $\Sigma l_1^2 = 2m_1^2 = \Sigma n_1^2 = 1; \Sigma mn_1 = 0$ etc.

which is the required locus.

It is clearly a plane \parallel to XY plane, i.e. \perp to the Z-axis, the axis of the paraboloid.

Article 28. Normal to the paraboloid.

To find the equations of the normal at the point (x_1, y_1, z_1) of the paraboloid

$$(i) ax^2 + by^2 = 2z$$

$$(ii) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$$

(i) The given paraboloid is $ax^2 + by^2 = 2z$... (1)

The tangent plane at (x_1, y_1, z_1) to (1) is $ax_1x + by_1y - 2z_1^2 = 0$. Using the rule of tangent plane

or $ax_1 + by_1 = z_1^2$... (2)

The d.c.'s of normal to this tangent plane are proportional to $ax_1, by_1, -1$.

Equations of the normal at (x_1, y_1, z_1) , [i.e. a line through (x_1, y_1, z_1) and \perp to (2)] are

$$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{-1}$$

Article 29. Number of normals

To prove that there are five points on an elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$, the normals at which pass through a given point (α, β, γ)

[V. Imp] (Allahabad 1982; L.N.M., 1982)

The given paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$

The normal at (x_1, y_1, z_1) is $\frac{x_1}{a^2} + \frac{y_1}{b^2} = 2z$

This passes through (α, β, γ) , if

$$\frac{x_1}{a^2} + \frac{y_1}{b^2} = \frac{\gamma - z_1}{1} = \lambda \text{ (say)}$$

From first and last members,

$$\alpha - x_1 = \frac{y_1 \lambda}{a^2} \text{ or } \alpha = x_1 \left(1 + \frac{\lambda}{a^2}\right) = \frac{\lambda + a^2}{a^2} x_1$$

$$x_1 = \frac{a^2 \alpha}{a^2 + \lambda} \quad \dots(2)$$

Similarly $y_1 = \frac{b^2 \beta}{b^2 + \lambda}$ and $z_1 = \gamma + \lambda$.

But since (x_1, y_1, z_1) lies on (1), i.e. $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 2z_1$

$$\text{or } \frac{1}{a^2} \cdot \frac{a^4 \alpha^2}{(a^2 + \lambda)^2} + \frac{1}{b^2} \cdot \frac{b^4 \beta^2}{(b^2 + \lambda)^2} = 2(\gamma + \lambda) \quad \dots(3)$$

This equation being of fifth degree in λ gives five values of λ the normals at which pass through (α, β, γ) . Hence the result.

Ques: From (2), the root of normal is

$$\left(\frac{a^2 \alpha}{a^2 + \lambda}, \frac{b^2 \beta}{b^2 + \lambda}, \gamma + \lambda \right)$$

Article 30. Prove that the feet of normals from a given point (α, β, γ) to an elliptic paraboloid are the five points of intersection of the elliptic paraboloid and a certain cubic curve.

Let us an exercise for the student.

Proceed exactly as in Article 12 in the case of an ellipsoid.

Example 1. Prove that the normals from (α, β, γ) to the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ lie on the cone

$$\frac{x}{a} - \frac{\beta}{b} + \frac{a^2 - b^2}{z - \gamma} = 0.$$

[M.D.U. J.966, 85]

Sol. The given paraboloid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$$

Let any line through (α, β, γ) be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \dots(2)$$

be the normal at (x_1, y_1, z_1) to (1).

The equation of the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - (z + z_1) = 0 \quad \dots(3)$$

Since (2) is normal to (3) it is \perp to the normal to (3)

$$\frac{x_1}{a^2} - \frac{y_1}{b^2} = \frac{n}{l} = k \text{ (say)} \quad \dots(4)$$

Again if the normal at (x_1, y_1, z_1) to (1) passes through (α, β, γ) , then $x_1 = \frac{a^2 \alpha}{a^2 + \lambda}$, $y_1 = \frac{b^2 \beta}{b^2 + \lambda}$, $z_1 = \gamma + \lambda$. [From Eqn. (2) of Art. 29]

$$\text{From (4), } l = k \frac{x_1}{a^2} = \frac{k}{a^2} \cdot \frac{a^2 \alpha}{a^2 + \lambda} = \frac{k \alpha}{a^2 + \lambda} \quad \dots(5)$$

$$\text{or } a^2 + \lambda = \frac{k \alpha}{l} \quad \dots(6)$$

$$m = k \frac{y_1}{b^2} = \frac{k}{b^2} \cdot \frac{b^2 \beta}{b^2 + \lambda} = \frac{k \beta}{b^2 + \lambda} \text{ or, } b^2 + \lambda = \frac{k \beta}{m} \quad \dots(7)$$

$$n = -k \quad \dots(8)$$

Subtracting (7) from (6), we get

$$\alpha^2 - b^2 = k \left(\frac{\alpha}{l} - \frac{\beta}{m} \right) = -n \left(\frac{\alpha}{l} - \frac{\beta}{m} \right) \quad \dots(9)$$

To find the locus, we have to eliminate l, m, n from (2) and (9). Using (8),

Putting the value of l, m, n from (2) in (9), we have

$$\alpha^2 - b^2 = -(l - \gamma) \left(\frac{\alpha}{l} - \frac{\beta}{m} \right)$$

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$$\text{or } \frac{\alpha^2 - \beta^2}{x^2 - y^2} = -\frac{\alpha}{x - y} + \frac{\beta}{y - \alpha}$$

$$\text{or } \frac{\alpha}{x - \alpha} - \frac{\beta}{y - \beta} + \frac{\alpha^2 - \beta^2}{x^2 - y^2} = 0$$

which is the required result.

Example 2. Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2az$. Prove also that if the point lies on the surface

$$27a(x^2 + y^2) + 8(a - z)^3 = 0$$

then two of the three normal coincide.

Sol. The given paraboloid is $x^2 - y^2 = 2az$

$$\text{or } \frac{x^2}{a} + \frac{y^2}{a} = 2z \quad \dots(1)$$

The normal at (x_1, y_1, z_1) to (1) is

$$\frac{x_1^2 - x_1}{a} + \frac{y_1^2 - y_1}{a} = \frac{z_1 - z_1}{a}$$

If it passes through the given point (s, β, γ) , then

$$\frac{s - x_1}{a} + \frac{\beta - y_1}{a} = \frac{z_1 - z_1}{a} = \lambda \quad (\text{say}) \quad \dots(3)$$

which gives $x_1 = \frac{ax}{a+\lambda}$, $y_1 = \frac{a\beta}{a+\lambda}$, $z_1 = \gamma - \lambda$

$$\text{or } \frac{1}{a} \cdot \frac{a^2 x^2}{(a+\lambda)^2} + \frac{1}{a} \cdot \frac{a^2 \beta^2}{(a+\lambda)^2} = 2(\gamma - \lambda) \quad \dots(4)$$

$$\text{or } ax^2 + a\beta^2 = 2(\gamma + \lambda)(a + \lambda) \quad \dots(5)$$

Putting a third degree equation in λ , gives three values of λ . Paraboloid (1) at which the normals pass thro' (s, β, γ) . Thus from a given point three normals can be drawn to (1).

Rewriting (5) as

$$f(\lambda) = 2(\alpha + \lambda)(\alpha - \lambda)^2 + (\beta + \lambda)^2 = 0 \quad \dots(6)$$

$$f'(\lambda) = 2(\alpha + \lambda)^2 + 4(\gamma + \lambda)(\alpha + \lambda) = 0 \quad \dots(7)$$

If two of the normals coincide, then (6) must have two equal roots showing that $f(\lambda)$ and $f'(\lambda)$ must have a common linear factor. From (7), $2(\alpha + \lambda)(\alpha + 2\gamma + \lambda) = 0$

$$\text{or } \lambda = -\frac{\alpha + 2\gamma}{3} \quad \therefore \alpha + \lambda \neq 0$$

Putting this value of λ in (6), we get

$$\begin{aligned} a(\alpha^2 + \beta^2) - 2 & \left[\left(\gamma - \frac{\alpha + 2\gamma}{3} \right) \left(\alpha - \frac{\alpha + 2\gamma}{3} \right)^2 \right] \\ & = 2 \left(\frac{\alpha - \alpha}{3} \right) \left(\frac{2\alpha - 2\gamma}{3} \right)^2 = -\frac{8}{27} (\alpha - \gamma)^3 \end{aligned}$$

$$a(\alpha^2 + \beta^2) + \frac{8}{27} (\alpha - \gamma)^3 = 0$$

$$27a(\alpha^2 + \beta^2) + 8(\alpha - \gamma)^3 = 0$$

$$\text{or } 27a(\alpha^2 + \beta^2) = 0$$

$$\text{or } 27a(\alpha^2 + \beta^2) = 0$$

$$\text{Hence } (\alpha, \beta, \gamma) \text{ lies on}$$

$$27a(x^2 + y^2) + 8(a - z)^3 = 0.$$

Example 3. Show that the feet of the normals from the points (α, β, γ) to the paraboloid $x^2 + y^2 = 2az$ lie on the sphere $x^2 + y^2 + z^2 - z(a + \gamma) - \frac{2\beta}{2\alpha} (\alpha^2 + \beta^2) = 0$.

Sol. If (x_1, y_1, z_1) be the foot of normal through (α, β, γ) to the paraboloid

$$x^2 + y^2 = 2az \quad \text{or} \quad \frac{x^2}{a} + \frac{y^2}{a} = 2z, \quad \dots(1)$$

$$\text{or } \frac{x_1}{a + \lambda} = \frac{\alpha}{a + \lambda}, \quad y_1 = \frac{a\beta}{a + \lambda}, \quad z_1 = \gamma + \lambda \quad \dots(2)$$

But (x_1, y_1, z_1) lies on the paraboloid, so we have

$$\frac{1}{a} \cdot \frac{a^2 x_1^2}{(a + \lambda)^2} + \frac{1}{a} \cdot \frac{a^2 y_1^2}{(a + \lambda)^2} = 2(\gamma + \lambda) \quad \dots(1)$$

$$\text{or } \frac{a(x_1^2 + y_1^2)}{(a + \lambda)^2} = 2(\gamma + \lambda) \quad \dots(2)$$

Now (x_1, y_1, z_1) will lie on the given sphere

$$x^2 + y^2 + z^2 - z(a + \gamma) - \frac{2\beta}{2\alpha} (\alpha^2 + \beta^2) = 0$$

$$\text{if } \frac{a^2 x_1^2}{(a + \lambda)^2} + \frac{a^2 y_1^2}{(a + \lambda)^2} + (x_1 + y_1)^2 + (x_1 + y_1)(a + \gamma) - \frac{2\beta}{2\alpha} (\alpha^2 + \beta^2) = 0$$

$$\text{or if } \frac{a^2 x_1^2}{(a + \lambda)^2} + (x_1 + y_1)^2 - (x_1 + y_1)(a + \gamma) - \frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)} = 0$$

$$\text{or if } \frac{a(x_1^2 + y_1^2)}{(a + \lambda)^2} [2a - (a + \lambda)] + (x_1 + y_1)(\gamma + \lambda - a - \gamma) = 0$$

$$\text{or if } \frac{a(x_1^2 + y_1^2)}{2(a + \lambda)^2} - (a - \lambda) + (x_1 + y_1)(\gamma + \lambda - a) = 0$$

$$\text{or if } (a - \lambda) \left[\frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)^2} - \gamma - \lambda \right] = 0$$

or if

$$\frac{a(\alpha^2 + \beta^2)}{2(\alpha + \beta)} = \gamma + \lambda, \quad a - \lambda \neq 0$$

which is true by (2). Hence the result.

Example 4. Prove that the perpendicular from (α, β, γ) to the polar plane w.r.t. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ lies on the cone

$$\frac{a}{x-a} - \frac{\beta}{y-\beta} + \frac{\gamma^2 - b^2}{z-\gamma} = 0.$$

Sol. The polar place of (α, β, γ) w.r.t. the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z \text{ is } \frac{\alpha x}{b^2} + \frac{\beta y}{b^2} = z + \gamma \quad \dots(1)$$

Any line through (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$... (2)

If (2) is \perp to the polar place (1), then it is \parallel to the normal to (1).

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{-1}{n} = k \text{ (say)}$$

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = k \text{ and } k = \frac{-1}{n}$$

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma^2 - b^2}{n} = \frac{-(a^2 - b^2)}{n} \quad \dots(3)$$

This shows that line (2) lies on the cone [putting l, m, n from (2) in (3)]

$$\frac{\alpha}{x-a} - \frac{\beta}{y-\beta} = -\frac{(a^2 - b^2)}{z-\gamma}.$$

Hence the result.

CONJUGATE DIAMETERS

(Vikram 1984)

1. We know that if l, m, n be proportional to the dirs. of a system of parallel chords of the conicoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ and (α, β, γ) be the dir. of any one of them, then the locus of the chord passing through the centre $(0, 0, 0)$ of the conicoid

$$al^2 + bl^2 + cl^2 = 0.$$

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This plane is called the *diametral plane conjugate to the direction* l, m, n . Conversely, any plane $Ax + By + Cz = 0$... (ii) through the centre is the diametral plane conjugate to the direction l, m, n given by $\frac{dl}{A} = \frac{bm}{B} = \frac{cn}{C}$ [identifying (i) and (ii)]

Thus every central plane is a diametral plane conjugate to some direction.

If P be any point on the conicoid, then the plane bisecting chords parallel to OP is called the *diametral plane of OP* .

Note. In what follows, we shall confine our attention to the ellipsoid only.

2. Definitions

Conjugate Semi-diameters. Any three semi-diameters are called the diametral planes of the third.

Conjugate Planes. Any three diametral planes are called *Conjugate Planes* if each is the diametral plane of the line of intersection of the other two.

3. Relations between the coordinates of the extremes of a system of conjugate diameters of an ellipsoid.

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \dots(i)$$

Then the diametral plane of OP (i.e. the plane bisecting chords parallel to OP) is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} = 0. \quad \dots(ii)$$

Let $Q(x_2, y_2, z_2)$ be any point on the section of (i) by the plane (ii)

$$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} = 0 \quad \dots(iii)$$

which shows that Q lies on the diametral plane of OP .

The equation (iii) is also the condition that the diametral plane $\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} = 0$ or OQ passes through P .

Thus if the diametral plane of OP passes through P , then the diametral plane of OQ also passes through P .

Let the line of intersection of the diametral planes of OP and OQ meet the surface (i) in $R(x_3, y_3, z_3)$.

Since R lies on the diametral plane of OP, the plane $\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} = 0$ should pass through P and Q.

(The three semi-diameters OP, OQ, OR are such that the plane containing any two of them is the diametral plane of the third. Hence they are called conjugate semi-diameters.)

Since the points P, Q, R lie on (I)

$$\begin{aligned} \therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} &= 1, \\ \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} &= 1, \end{aligned} \quad \dots (II)$$

Since the diametral plane of OP passes through Q and R, the diametral plane of OQ passes through P and Q. Hence OR passes through P and Q.

$$\therefore \frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} = 0, \quad \frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0. \quad \dots (III)$$

By virtue of the relations (II), we observe that

$\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}; \frac{y_1}{a}, \frac{y_2}{b}, \frac{y_3}{c}; \frac{z_1}{a}, \frac{z_2}{b}, \frac{z_3}{c}$ can be regarded as the direction cosines of any three lines.

(If $b^2 + m^2 + n^2 = 1$, then l, m, n are d.c.s. of a line.)

Also, by virtue of the relation (III), these lines are mutually perpendicular. Hence, we have $\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c}; \frac{y_1}{a}, \frac{y_2}{b}, \frac{y_3}{c}$ and $\frac{z_1}{a}, \frac{z_2}{b}, \frac{z_3}{c}$ as the d.c.s. of another set of three mutually perpendicular lines.

If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the d.c.s. of three mutually perpendicular lines, then $\frac{l_1}{a}, \frac{m_1}{b}, \frac{n_1}{c}; \frac{l_2}{a}, \frac{m_2}{b}, \frac{n_2}{c}; \frac{l_3}{a}, \frac{m_3}{b}, \frac{n_3}{c}$ are also the d.c.s. of three mutually perpendicular lines.

Hence we have

$$\begin{aligned} \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} &= 1 \\ \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} &= 1 \\ \frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} + \frac{z_3^2}{c^2} &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{x_1y_2 + x_2y_1}{ab} + \frac{x_2y_3 + x_3y_2}{bc} + \frac{x_3y_1 + x_1y_3}{ca} &= 0 \\ \frac{y_1z_2 + y_2z_1}{ab} + \frac{y_2z_3 + y_3z_2}{bc} + \frac{y_3z_1 + y_1z_3}{ca} &= 0 \\ \frac{z_1x_2 + z_2x_1}{ab} + \frac{z_2x_3 + z_3x_2}{bc} + \frac{z_3x_1 + z_1x_3}{ca} &= 0 \end{aligned} \quad \dots (IV)$$

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The coordinates of the conjugate semi-diameters are connected by the relations (I), (II), (III) and (IV) above.

4. The sum of the squares of three conjugate semi-diameters of an ellipsoid is constant.

Let OP, OQ, OR be three conjugate semi-diameters of the ellipsoid.

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1.$$

Let the coordinates of P, Q, R be (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) . Then

$$\begin{aligned} &= (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) + (x_3^2 + y_3^2 + z_3^2) \\ &= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) + (z_1^2 + z_2^2 + z_3^2) \\ &\quad - \frac{x_1^2 + y_1^2 + z_1^2}{a^2} - b^2 + c^2 \end{aligned}$$

which is constant.

Now $OP^2 + PQ^2 + OR^2$.

5. The volume of the parallelopiped formed by three conjugate semi-diameters of an ellipsoid as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) . Then

$$\left. \begin{aligned} x_1^2 + x_2^2 + x_3^2 &= a^2 \\ y_1^2 + y_2^2 + y_3^2 &= b^2 \\ z_1^2 + z_2^2 + z_3^2 &= c^2 \end{aligned} \right\} \quad \begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= 0 \\ y_1z_1 + y_2z_2 + y_3z_3 &= 0 \\ z_1x_1 + z_2x_2 + z_3x_3 &= 0 \end{aligned} \quad \dots (A)$$

Volume of the parallelopiped with OP, OQ, OR as coterminous edges = $V = \sqrt{abc}$ volume of tetrahedron OPR.

$$\begin{aligned} &= 6x_1 \begin{vmatrix} 0 & 0 & 0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad (\text{numerically}) \\ &= 6x_1 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad \dots (A) \\ &= abc^2 \end{aligned}$$

$\therefore V = abc$, which is constant.

6. The sum of the squares of the areas of the faces of the parallelopiped with any three conjugate semi-diameters as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities

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Since diametral plane of OP passes through Q and R.

$$\begin{cases} \frac{x_1}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 0 \\ \frac{x_2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 0 \end{cases}$$

$$\begin{cases} \left(\frac{x_1}{a} \right) \frac{x_1}{a} + \left(\frac{y_1}{b} \right) \frac{y_1}{b} + \left(\frac{z_1}{c} \right) \frac{z_1}{c} = 0 \\ \left(\frac{x_2}{a} \right) \frac{x_2}{a} + \left(\frac{y_2}{b} \right) \frac{y_2}{b} + \left(\frac{z_2}{c} \right) \frac{z_2}{c} = 0 \end{cases}$$

$$\begin{aligned} & \frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} = 0 \\ & \frac{x_2}{a} + \frac{y_2}{b} + \frac{z_2}{c} = 0 \end{aligned}$$

$$\frac{x_1}{a} = -\frac{y_1}{b} - \frac{z_1}{c}$$

$$\frac{y_1}{b} = -\frac{x_1}{a} - \frac{z_1}{c}$$

$$\frac{z_1}{c} = -\frac{x_1}{a} - \frac{y_1}{b}$$

$$\frac{x_2}{a} = -\frac{y_2}{b} - \frac{z_2}{c}$$

$$\frac{y_2}{b} = -\frac{x_2}{a} - \frac{z_2}{c}$$

$$\frac{z_2}{c} = -\frac{x_2}{a} - \frac{y_2}{b}$$

perpendicular straight lines.

$$\therefore \frac{x_1}{a} = \pm \frac{2x_3 - 2y_3}{bc}, \quad \frac{y_1}{b} = \pm \frac{2x_3 - 2z_3}{ca}, \quad \frac{z_1}{c} = \pm \frac{2y_3 - 2z_3}{ab}$$

Let A_1, A_2, A_3 be the areas of the triangles OQR, ORP and OQP and l_1, m_1, n_1 , ($r=1, 2, 3$) be the d.c.s. of normals to these planes.

Projecting the $\triangle OQR$ on the plane $x=0$, we get a triangle with vertices $(0, 0, 0)$, $(0, j_3, z_3)$ and $(0, j_3, z_3)$ having area $\frac{1}{2}(j_3 z_3 - j_3 z_3)$. The area of the projection is also $A_1 l_1$.

Using (A)

$$\begin{aligned} \text{Similarly } A_1 l_1 &= \pm \frac{2y_3}{b}, \\ \text{Squaring and adding} & \quad A_1 l_1 = \pm \frac{abz_3}{2c} \end{aligned}$$

$$A_1^2 = \frac{b^2 c^2 z_3^2}{4c^2} + \frac{c^2 a^2 z_3^2}{4b^2} + \frac{a^2 b^2 z_3^2}{4c^2}$$

$$(i) \quad l_1^2 + m_1^2 + n_1^2 = 1$$

which is constant.

Similarly, projecting the areas ORP and QPQ on the coordinate planes, we get

$$\begin{aligned} A_2^2 &= \frac{b^2 c^2 x_3^2}{4a^2} + \frac{c^2 a^2 y_3^2}{4b^2} + \frac{a^2 b^2 z_3^2}{4c^2} \\ A_3^2 &= \frac{b^2 c^2 x_3^2}{4a^2} + \frac{c^2 a^2 y_3^2}{4b^2} + \frac{a^2 b^2 z_3^2}{4c^2} \end{aligned}$$

Adding, we get

$$\begin{aligned} A_1^2 + A_2^2 + A_3^2 &= \frac{b^2 c^2}{4a^2} \Sigma x_3^2 + \frac{c^2 a^2}{4b^2} \Sigma y_3^2 + \frac{a^2 b^2}{4c^2} \Sigma z_3^2 \\ &= \frac{b^2 c^2}{4a^2} (a^2) + \frac{c^2 a^2}{4b^2} (b^2) + \frac{a^2 b^2}{4c^2} (c^2) \\ &= \frac{1}{4}(b^2 a^2 + c^2 a^2 + a^2 b^2), \end{aligned}$$

which is constant.

7. The sum of the squares of the projections of any given line on three conjugate semi-diameters on any line is constant.

Let l, m, n be the d.c.s. of any given line and OP, OQ, OR be the three conjugate semi-diameters.

Projection of OP on this line is,

$$l_1 + m_1 + n_1$$

Projection of OQ on this line is,

$$l_2 + m_2 + n_2$$

Projection of OR on this line is,

$$l_3 + m_3 + n_3$$

the sum of the squares of the projections of OP, OQ and OR on this line.

$$\begin{aligned} &= ((l_1 + m_1 + n_1)^2 + (l_2 + m_2 + n_2)^2 + (l_3 + m_3 + n_3)^2) \\ &= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2) \\ &= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2) \\ &= l^2 + m^2 + n^2, \end{aligned}$$

which is constant.

Let l', m', n' be the d.c.s. of the normal to any given plane conju-

gated to semi-diameters on any plane, which is constant.

Sum of the squares of the projections of OP, OQ and OR on this plane is

$$\begin{aligned} &= [OP^2 - (l_1^2 + m_1^2 + n_1^2)] + [OQ^2 - (l_2^2 + m_2^2 + n_2^2)] \\ &\quad + [OR^2 - (l_3^2 + m_3^2 + n_3^2)] \\ &= (OP^2 + OQ^2 + OR^2) - (l_1^2 + m_1^2 + n_1^2) \\ &\quad - (l_2^2 + m_2^2 + n_2^2) - (l_3^2 + m_3^2 + n_3^2) \\ &= a^2 + b^2 + c^2 - (a^2 l^2 + b^2 m^2 + c^2 n^2) \quad | \text{ using Arts. 4 and 7} \\ &= a^2(1 - l^2) + b^2(1 - m^2) + c^2(1 - n^2) \\ &= a^2(m^2 + n^2) + b^2(n^2 + l^2) + c^2(l^2 + m^2) \\ &= l^2 + m^2 + n^2 = 1 \end{aligned}$$

Example 1. If (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) be the extremes of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Find the equation of the plane through these points.
 Sol. Let P, Q, R be the extremities (x_r, y_r, z_r) , $r = 1, 2, 3$ of the three conjugate semi-diameters OP, OQ, OR.

Let the equation of the plane PQR be
 $lx + my + nz = p$

$$\begin{aligned} \text{Since } P, Q, R \text{ all lie on (i),} \\ \therefore & lx_1 + my_1 + nz_1 = p \\ & lx_2 + my_2 + nz_2 = p \\ & lx_3 + my_3 + nz_3 = p \end{aligned}$$

$$\begin{aligned} \text{Multiplying (ii) by } x_1, \text{ (iii) by } y_1 \text{ and (iv) by } z_1 \text{ and adding,} \\ \text{we get} \\ & l(x_1^2 + m\sum x_i y_i + n\sum x_i z_i) = p(x_1 + x_2 + x_3) \end{aligned}$$

$$\begin{aligned} \text{or} \\ & l(a^2 + m(\sum x_i^2) + n(0)) = p(x_1 + x_2 + x_3) \\ & \therefore l = \frac{p}{a^2 + m(\sum x_i^2) + n(0)} = \frac{p}{a^2 + m(x_1^2 + x_2^2 + x_3^2)} \end{aligned}$$

Similarly multiplying (ii), (iii), (iv) by y_1, y_2, y_3 respectively and adding

$$m = \frac{p}{b^2} (y_1 + y_2 + y_3)$$

and multiplying (ii), (iii), (iv) by z_1, z_2, z_3 respectively and adding

$$n = \frac{p}{c^2} (z_1 + z_2 + z_3)$$

Substituting the values of l, m, n in (i), equation of plane PQR is

$$\begin{aligned} & \frac{p}{a^2} (x_1 + x_2 + x_3) + \frac{p}{b^2} (y_1 + y_2 + y_3) + \frac{p}{c^2} (z_1 + z_2 + z_3) = p \\ & \text{or} \\ & \frac{N}{a^2} (x_1 + x_2 + x_3) + \frac{N}{b^2} (y_1 + y_2 + y_3) + \frac{N}{c^2} (z_1 + z_2 + z_3) = 1, \end{aligned}$$

Example 2. Show that the plane PQR, where F, O, R are the extremes of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, touches the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{l}{3}$ at the centroid of the triangle PQR.

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Sol. Proceeding as in Ex. 1, the equation of plane PQR is
 $\frac{x}{a^2} (x_1 + x_2 + x_3) + \frac{y}{b^2} (y_1 + y_2 + y_3) + \frac{z}{c^2} (z_1 + z_2 + z_3) = 1 \quad \dots(i)$

The centroid of ΔPQR is

$$G \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

The tangent plane to the ellipsoid

$$\begin{aligned} & \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{3} \text{ at } G \text{ is} \\ & x \left(\frac{x_1 + x_2 + x_3}{3} \right) + y \left(\frac{y_1 + y_2 + y_3}{3} \right) + z \left(\frac{z_1 + z_2 + z_3}{3} \right) = \frac{1}{3} \\ & \text{or} \\ & \frac{x}{a^2} (x_1 + x_2 + x_3) + \frac{y}{b^2} (y_1 + y_2 + y_3) + \frac{z}{c^2} (z_1 + z_2 + z_3) = 1, \end{aligned}$$

which is the same as (i).

Hence the plane PQR touches the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{3} \text{ at } G.$$

Example 3. Prove that the pole of the plane PQR lies on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3$, where OP, OQ, OR are three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol. Equation of the plane PQR is

$$\frac{x}{a^2} (x_1 + x_2 + x_3) + \frac{y}{b^2} (y_1 + y_2 + y_3) + \frac{z}{c^2} (z_1 + z_2 + z_3) = 1 \quad \dots(i)$$

| See Ex. 1

Let (x', y', z') be the pole of the plane PQR.

Equation to the polar plane of (x', y', z') w.r.t. the ellipsoid

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1 \text{ is}$$

$$\frac{x'}{a^2} + \frac{y'}{b^2} + \frac{z'}{c^2} = 1 \quad \dots(ii)$$

Comparing (i) and (ii), we have

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Also from (ii)

$$\begin{aligned} & \left(\frac{x'}{\alpha}\right)^2 + \left(\frac{y'}{\beta}\right)^2 + \left(\frac{z'}{\gamma}\right)^2 \\ &= \frac{1}{\alpha^2} (x_1 + x_2 + x_3)^2 + \frac{1}{\beta^2} (y_1 + y_2 + y_3)^2 + \frac{1}{\gamma^2} (z_1 + z_2 + z_3)^2 \\ &= \frac{1}{\alpha^2} \Sigma x_i^2 + \frac{1}{\beta^2} \Sigma y_i^2 + \frac{1}{\gamma^2} \Sigma z_i^2 + \frac{2}{\alpha^2} 2x_1x_2 + \frac{2}{\beta^2} 2y_1y_2 \\ &\quad + \frac{2}{\gamma^2} 2z_1z_2 \\ &= \frac{1}{\alpha^2} (\sigma^2) + \frac{1}{\beta^2} (\tau^2) + \frac{1}{\gamma^2} (\lambda^2) + \frac{2}{\alpha^2} (0) + \frac{2}{\beta^2} (0) \\ &= 3 \end{aligned}$$

Hence (x', y', z') lies on the surface

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 3.$$

Example 4. Prove that the locus of the foot of the perpendicular through the extremities of three conjugate semi-diameters is

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 3(x^2 + y^2 + z^2).$$

Sol. Equation of the plane POR through the extremities of three conjugate semi-diameters OP, OQ, OR is

$$\frac{x}{\alpha^2} (x_1 + x_2 + x_3) + \frac{y}{\beta^2} (y_1 + y_2 + y_3) + \frac{z}{\gamma^2} (z_1 + z_2 + z_3) = 1 \quad \dots(i)$$

Let $D(x, y, z)$ be the foot of the perpendicular from the centre

x, y, z , the d.c.s. of OD, the normal to (i) are proportional to

$$\frac{x_1 + x_2 + x_3}{\alpha^2} = \frac{y_1 + y_2 + y_3}{\beta^2} = \frac{z_1 + z_2 + z_3}{\gamma^2}.$$

Since (x, y, z) lies on (i)

$$\frac{x}{\alpha^2} + \frac{y}{\beta^2} + \frac{z}{\gamma^2} = 1.$$

From (i), we have

$$\begin{aligned} & \frac{x_1 + x_2 + x_3}{\alpha^2} + \frac{y_1 + y_2 + y_3}{\beta^2} + \frac{z_1 + z_2 + z_3}{\gamma^2} = 1 \\ & \Sigma \frac{a(x_1 + x_2 + x_3)}{\alpha^2} = 1 \end{aligned}$$

From (ii), we have

$$\Sigma \frac{a(x_1 + x_2 + x_3)}{\alpha^2} = 1.$$

From (iii), we have

$$\Sigma \frac{a(x_1 + x_2 + x_3)}{\alpha^2} = 1.$$

From (iv), we have

$$\Sigma \frac{a(x_1 + x_2 + x_3)}{\alpha^2} = 1.$$

Similarly $\frac{b^2 y^2}{\lambda^2} = 1, \frac{c^2 z^2}{\lambda^2} = 1$

$$\begin{aligned} & \text{Adding } \frac{a^2 x^2}{\alpha^2} + \frac{b^2 y^2}{\lambda^2} + \frac{c^2 z^2}{\lambda^2} = 3 \\ & \text{or } a^2 x^2 + b^2 y^2 + c^2 z^2 = 3(x^2 + y^2 + z^2). \end{aligned}$$

Hence the locus of (x, y, z) is

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 3(x^2 + y^2 + z^2);$$

Example 5. Find the locus of the equal conjugate diameters of the ellipsoid

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1. \quad (\text{Vikram 1984; L.N.M. 1982})$$

Then we have

$$\text{OP}^2 + \text{OQ}^2 + \text{OR}^2 = a^2 + b^2 + c^2$$

$$\text{OP}^2 = \text{OQ}^2 = \text{OR}^2$$

$$\text{each} = \frac{1}{3}(a^2 + b^2 + c^2)$$

Let P be the point (x_1, y_1, z_1) . Equations of OP are

$$\frac{x}{\alpha^2} = \frac{y}{\beta^2} = \frac{z}{\gamma^2} \quad \dots(ii)$$

where $x_1^2 + y_1^2 + z_1^2 = \text{OP}^2 = \frac{1}{3}(a^2 + b^2 + c^2)$

using (i)

$$\text{Also } \frac{x_1^2}{\alpha^2} + \frac{y_1^2}{\beta^2} + \frac{z_1^2}{\gamma^2} = 1 \quad \dots(iii)$$

(iii) P lies on the ellipsoid.

From (ii) and (iv), we have

$$\frac{x_1^2}{\alpha^2} + \frac{y_1^2}{\beta^2} + \frac{z_1^2}{\gamma^2} = \frac{3(x_1^2 + y_1^2 + z_1^2)}{a^2 + b^2 + c^2} \quad \dots(iv)$$

(each = 1)

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THE CONOID

Eliminating x_1, y_1, z_1 from (ii) and (v), the required locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3(x^2 + y^2 + z^2)}{a^2 + b^2 + c^2}.$$

Example 6. Prove that the plane through a pair of equal conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ touches the cone $\frac{x^2}{a^2(2a^2 - b^2 - c^2)} + \frac{y^2}{b^2(2b^2 - a^2 - c^2)} + \frac{z^2}{c^2(2c^2 - a^2 - b^2)} = 0$.

Sol. Let P, Q, R be the extremities of three equal-conjugate semi-diameters OP, OQ, OR.

Let the coordinates of P be (x_1, y_1, z_1) . Also let

The plane OQR through the conjugate semi-diameters OQ and OR is the diameter plane of OP.

The equation of the plane OQR is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} = 0 \quad \dots(i)$$

The plane (i) will touch the cone $\Sigma \frac{x^2}{a^2(2a^2 - b^2 - c^2)} = 0$

$$\text{If } \Sigma \frac{x_1^2}{a^2} + a^2(2a^2 - b^2 - c^2) = 0$$

∴ The plane $lx + my + nz = p$ touches $a^2 + by^2 + cz^2 = 1$
if $\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = p^2$.

$$\text{or if } \Sigma \frac{x_1^2}{a^2} [3a^2 - (a^2 + b^2 + c^2)] = 0$$

$$\text{or if } 3x_1^2 - \Sigma \frac{x_1^2}{a^2} (a^2 + b^2 + c^2) = 0$$

$$\text{or if } 3\Sigma x_1^2 = (a^2 + b^2 + c^2) \Sigma \frac{x_1^2}{a^2}$$

$$\text{or if } 3 : OP^2 = (a^2 + b^2 + c^2)(1) \quad \therefore \text{P lies on the ellipsoid}$$

$$\text{or if } 3r^2 = a^2 + b^2 + c^2$$

$$\text{which is true} \because OP + OQ + OR^2 = a^2 + b^2 + c^2$$

$$\text{and } OP = OQ = OR = r$$

Hence the result.

THE ELLIPSOID

Example 7. Prove that the locus of the section of the ellipsoid

$$\sum \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 1$$

by the plane PQR is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{\beta^2}.$$

Sol. The equation of the plane PQR is

$$\left(\frac{x_1 + x_2}{a^2} + \frac{y_1 + y_2}{b^2} + \frac{z_1 + z_2}{c^2} \right) x + \left(\frac{2x_1 + 2y_1 + 2z_1}{a^2 + b^2 + c^2} \right) y + \left(\frac{z_1 + z_2 + z_3}{c^2} \right) z = 1 \quad \dots(ii)$$

If (x, y, z) be the centre of the section of the given ellipsoid by the plane PQR, then the equation of PQR can be written as "T = S."

$$\text{i.e. } \frac{ax}{a^2} + \frac{by}{b^2} + \frac{cz}{c^2} = \frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} \quad \dots(iii)$$

Comparing (i) and (ii) represent the same plane, therefore,

$$\frac{x_1 + x_2 + x_3}{a^2} = \frac{2x_1 + 2y_1 + 2z_1}{a^2 + b^2 + c^2} = \frac{z_1 + z_2 + z_3}{c^2} = \frac{1}{\beta^2} \quad \frac{a^2}{a^2 + b^2 + c^2}$$

$$\text{where } \frac{x}{a} = \left(\frac{x_1 + x_2 + x_3}{a^2} \right) \left(\frac{a^2}{a^2 + b^2 + c^2} \right).$$

Similarly,

$$\frac{y}{b} = \left(\frac{2x_1 + 2y_1 + 2z_1}{a^2 + b^2 + c^2} \right) \left(\frac{a^2}{a^2 + b^2 + c^2} \right)$$

$$\text{and } \frac{z}{c} = \left(\frac{z_1 + z_2 + z_3}{c^2} \right) \left(\frac{a^2}{a^2 + b^2 + c^2} \right)$$

Squaring and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{x_1 + x_2 + x_3}{a^2} \right)^2 \left[\left(\frac{x_1 + x_2 + x_3}{a^2} \right)^2 + \left(\frac{2x_1 + 2y_1 + 2z_1}{a^2 + b^2 + c^2} \right)^2 + \left(\frac{z_1 + z_2 + z_3}{c^2} \right)^2 \right]$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 \left(\frac{a^2}{a^2 + b^2 + c^2} \right)^2$$

$$\frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$$

or

The required locus of (α, β, γ) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{3}$$

Example 3. Prove that the locus of the point of intersection of three tangent planes to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to conjugate diametral planes of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}$$

Sol. Let P(x_1, y_1, z_1), Q(x_2, y_2, z_2) and R(x_3, y_3, z_3) be the extremities of the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Then the diametral plane of P w.r.t. to (i) is

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} + \frac{z z_1}{c^2} = 0$$

Any plane parallel to (ii) is

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} + \frac{z z_1}{c^2} = p_1$$

If (iii) is a tangent plane to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then

$$\gamma^2 = a^2 + b^2 m^2 + c^2 n^2$$

or

$$\rho_1^2 = a^2 \left(\frac{x_1}{a^2} \right)^2 + b^2 \left(\frac{y_1}{b^2} \right)^2 + c^2 \left(\frac{z_1}{c^2} \right)^2 \quad \dots(iv)$$

Similarly the equation of other planes parallel to OQ and OR are

$$\frac{x x_2}{a^2} + \frac{y y_2}{b^2} + \frac{z z_2}{c^2} = p_2 \quad \dots(v)$$

and

$$\frac{x x_3}{a^2} + \frac{y y_3}{b^2} + \frac{z z_3}{c^2} = p_3 \quad \dots(vi)$$

where

$$p_1^2 = \sum \left[a^2 \left(\frac{x_1}{a^2} \right)^2 \right] \quad \dots(vii)$$

and

$$p_2^2 = \sum \left[b^2 \left(\frac{y_1}{b^2} \right)^2 \right] \quad \dots(viii)$$

and

$$p_3^2 = \sum \left[c^2 \left(\frac{z_1}{c^2} \right)^2 \right] \quad \dots(ix)$$

Now for the locus of the point of intersection of (iii), (iv) and (vii), we square and add (iv), (vii), (ix) and get

$$\sum \frac{x^2}{a^2} (x_1^2 + x_2^2 + x_3^2) + \sum \frac{2xy}{a^2 b^2} (x_1 y_1 + x_2 y_2 + x_3 y_3) \\ = p_1^2 + p_2^2 + p_3^2$$

$$\text{or} \quad \sum \left[\frac{x^2}{a^2} (x_1^2) \right] + \sum \frac{2xy}{a^2 b^2} (0) = \sum \frac{\alpha^2}{a^2} (x_1^2 + x_2^2 + x_3^2)$$

$$\text{from (iv) and (vii) using } \sum x_i^2 = a^2 \text{ and } x_i y_i = 0 \text{ etc.}$$

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\alpha^2}{a^2} (x_1^2) + \frac{\beta^2}{b^2} (y_1^2) + \frac{\gamma^2}{c^2} (z_1^2)$$

Since $2x_1^2 = \alpha^2$ etc.

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}$$

Reduction of General Equation of Second degree

General equation of the second degree

The most general equation of second degree is written as $F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fxy + 2az + 2bz + d = 0$ (1)

(or)

$$f(x, y, z) \equiv f(x, y, z) + 2ax + 2ay + 2az + d = 0 \quad (2)$$

where $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2fzy$ (2)

The equation (1) contains ten unknowns which can be reduced to nine effective constants by dividing the equation throughout by λ .

Then a surface can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

Note :- In all discussions in this chapter, we shall take $f(x, y, z)$ and $f(x, y, z)$ as defined in (1) and (2) above i.e. $f(x, y, z)$ will be taken as the homogeneous part of $F(x, y, z)$.

Note :- Here $\frac{\partial f}{\partial x} = 2(ax + by + fz)$

$$\frac{\partial f}{\partial y} = 2(bx + dy + fz)$$

$$\frac{\partial f}{\partial z} = 2(gx + fy + cz); \text{ and}$$

$$\frac{\partial F}{\partial x} = 2(ax + by + fz)$$

$$\frac{\partial F}{\partial y} = 2(bx + dy + fz), \frac{\partial F}{\partial z} = 2(gx + fy + cz)$$

* Determination of the centre
of surface $f(x, y, z) = 0$:-

Let (x_1, y_1, z_1) be the centre of the surface $f(x, y, z) = 0$.

Shifting the origin to the centre (x_1, y_1, z_1) ,

the transformed equation of the surface

$$f(x+x_1, y+y_1, z+z_1) = 0$$

$$\begin{aligned} \text{i.e. } & a(x+x_1)^2 + b(y+y_1)^2 + c(z+z_1)^2 + 2f(x+x_1)(y+y_1) \\ & + 2g(x+x_1)(z+z_1) + 2h(y+y_1)(z+z_1) \\ & + 2u(x+x_1) + 2v(y+y_1) + 2w(z+z_1) \\ & + d = 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow & f(x, y, z) + 2a(x_1 + hy_1 + gz_1 + u) + 2y(hx_1 + by_1 + fz_1) \\ & + 2z(gx_1 + fy_1 + cz_1 + v) + 2x_1^2 + by_1^2 + cz_1^2 \\ & + 2xy_1z_1 + 2yz_1x_1 + 2hx_1y_1 + 2uy_1 + \\ & 2wz_1 + d = 0. \end{aligned}$$

(1)

where

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz \\ + 2gzx + 2hxy.$$

Now as the centre of (1) is origin, so it should be homogeneous in (x, y, z) .

(Since if (x_1, y_1, z_1) is a point on it, $(-x_1, -y_1, -z_1)$ must also lie on it as $(0, 0, 0)$, the mid-point of the chord joining (x_1, y_1, z_1) and $(-x_1, -y_1, -z_1)$, is the centre of the surface $f(x, y, z) = 0$ and therefore only second degree terms must exist in (1).)

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$$\begin{aligned} \text{From (I) we have } & ax_1 + by_1 + cz_1 + u = 0 \\ & hx_1 + iy_1 + fz_1 + v = 0 \\ & gx_1 + dy_1 + cz_1 + w = 0 \end{aligned}$$

(Also constant term in (II) can be rewritten as
 $x_1(ax_1 + by_1 + cz_1 + u) + y_1(bx_1 + cy_1 + v) + z_1(cx_1 + fz_1 + w) = 0$

$$= ux_1 + vy_1 + wz_1 + d, \text{ with the help of (II), (III), (IV).}$$

$$= d' \quad (\text{say})$$

where

$$\text{Then (I) reduces to } d' = (ax_1 + by_1 + cz_1) + d' = 0$$

$$d' = ux_1^2 + vy_1^2 + wz_1^2 + d,$$

$$\text{And } x_1, y_1, z_1 \text{ is obtained from (II), (III) and (IV),}$$

which can be obtained from

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0 \text{ replacing } x, y, z \text{ by } x_1, y_1, z_1.$$

Hence centre of the surface $F(x, y, z) = 0$, is given by solving

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0 \text{ for } x, y, z$$

and the equation of the surface referred to centre as origin is
 $f(x, y, z) + (ux_1 + vy_1 + wz_1 + d) = 0, \quad (\text{VII})$

where (x_1, y_1, z_1) is the centre of the surface. (Remember)

Note. The equations (II), (III), (IV) may or may not give a unique centre.

There may be more than one centre, a line of centres or a plane of centres depending upon the nature of solutions of the above three situations.

From (II), (III), (IV) we find that the centre (x_1, y_1, z_1) lies on the planes

$$ax + by + cz + u = 0$$

$$hx + iy + fz + v = 0$$

$$gx + dy + cw + w = 0 \quad (\text{VIII})$$

and

These planes are known as central planes and any point common to these planes is a centre.

Transformation of $f(x, y, z)$.

To show that by the rotation of axes the expression $f(x, y, z) = ax^2 + by^2 + cz^2 + 2gx + 2iy + 2jz = 0$, transforms to $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic

$$\begin{vmatrix} a-\lambda & b-\lambda & c-\lambda \\ h & i & j \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\text{or } \lambda^3 - \lambda^2(a+b+c) + \lambda(ab+bc+ca - j^2 - g^2 - h^2) = 0$$

$$\text{or } \lambda^3 - \lambda^2(11 + h + v) + \lambda(ab + bc + ca - j^2 - g^2 - h^2) = 0$$

$$= (abc + 2ghf - af^2 - bg^2 - ch^2) = 0$$

$$\text{or } \lambda^3 - \lambda^2(a+b+c) + \lambda(\lambda + B + C) - D = 0.$$

Reduction-of-General Equation of Second Degree

where A, B, C are the cofactors of a, b, c respectively in the determinant

$$D = \begin{vmatrix} a & b & c \\ h & i & j \\ g & f & c \end{vmatrix}$$

We know that the expression $x^2 + y^2 + z^2$ is an invariant when the rectangular axes are rotated through the same origin. (See chapter on Change of axes)

If we put $x = (ax + my + nz, y = (bx + mz + ny, z = (cx + px + mz)$
 $\text{and } z = (hx + \frac{ny}{m}, iy + fz) \text{ in } x^2 + y^2 + z^2, \text{ then by the relations}$

$$l_1^2 + m_1^2 + n_1^2 = 1, l_2^2 + m_2^2 + n_2^2 = 1, l_3^2 + m_3^2 + n_3^2 = 1;$$

$$l_1^2 + l_2^2 + l_3^2 = 1, m_1^2 + m_2^2 + m_3^2 = 1, n_1^2 + n_2^2 + n_3^2 = 1,$$

$$l_1m_1 + lm_1 + nm_1 = 0, l_2n_2 + mn_2 + nm_2 = 0, l_3l_1 + nl_1 + nl_3 = 0,$$

$$l_1n_1 + bn_1 + cn_1 = 0, m_1n_1 + mn_1 + nn_1 = 0,$$

$$l_1m_1 + bm_1 + cm_1 = 0, nl_1 + nl_2 + nl_3 = 0,$$

$$l_1l_2 + ml_2 + nl_2 = 0, l_2l_3 + ml_3 + nl_3 = 0,$$

$$l_1l_3 + bl_3 + cl_3 = 0, nl_1 + nl_2 + nl_3 = 0,$$

$$l_1l_2 + ml_2 + nl_2 = 0, l_2l_3 + ml_3 + nl_3 = 0,$$

$$l_1l_3 + bl_3 + cl_3 = 0, nl_1 + nl_2 + nl_3 = 0,$$

$$l_1l_2 + ml_2 + nl_2 = 0, l_2l_3 + ml_3 + nl_3 = 0,$$

$$l_1l_3 + bl_3 + cl_3 = 0, nl_1 + nl_2 + nl_3 = 0,$$

$$l_1l_2 + ml_2 + nl_2 = 0, l_2l_3 + ml_3 + nl_3 = 0,$$

$$l_1l_3 + bl_3 + cl_3 = 0, nl_1 + nl_2 + nl_3 = 0,$$

Now if (I) i.e. if $(a - \lambda_1)x^2 + (b - \lambda_1)y^2 + (c - \lambda_1)z^2 + 2fx + 2gy + 2hz = 0$ is the product of two linear factors then we must have

$$\begin{vmatrix} a-\lambda_1 & b-\lambda_1 & c-\lambda_1 \\ h & i & j \\ g & f & c \end{vmatrix} = 0 \quad (\text{III})$$

$$\text{or } \lambda^3 - \lambda^2(a+b+c) + \lambda(ab+bc+ca - j^2 - g^2 - h^2) = 0$$

$$\text{or } \lambda^3 - \lambda^2(a+b+c) + \lambda(A + B + C) - D = 0,$$

where A, B, C are the cofactors of corresponding small letters in the determinant

$$D = \begin{vmatrix} a & b & c \\ h & i & j \\ g & f & c \end{vmatrix}$$

And the expression $(\lambda_1 - \lambda)(\lambda_2 - \lambda)^2 + (\lambda_3 - \lambda)^2$ in (III) will be the product of two linear factors, if

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{vmatrix} = 0 \text{ i.e., } (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$$

i.e. when $\lambda = 2\lambda_1$ or λ_2 or λ_3

The same values of λ should be obtained from (iii) also.

Hence $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic (iii) in λ , which is called discriminating cubic.

Let λ be any root (real or imaginary) of the discriminating cubic (iii) and that l, m, n be the principal direction cosines (which may also be real or imaginary) corresponding to this value of λ , then

$$al + hm + gn = \lambda l$$

$$hl + bm + fn = \lambda m$$

(See chapter on Change of axes)

$$gl + fm + cn = \lambda n$$

$$(a - \lambda) l + hm + gn = 0$$

$$hl + (b - \lambda) m + fn = 0$$

$$gl + fm + (c - \lambda) n = 0$$

where λ is to be replaced by $\lambda_1, \lambda_2, \lambda_3$ to get the corresponding direction cosines of the axes.

§ 12.04. Various Forms of General Equation of Second Degree.
The general equation of second degree viz. $F(x, y, z) = 0$, as given in § 12.01

S. No. 1. Page 1 of this chapter can be reduced to any one of the following forms :—

Equation

Name of the surface

Ellipsoid.

1. $Ax^2 + By^2 + Cz^2 = 1$

2. $Ax^2 + By^2 - Cz^2 = 1$

3. $Ax^2 - By^2 - Cz^2 = 1$

4. $Ax^2 + By^2 + Cz^2 = 0$

5. $Ax^2 + By^2 + 2kz = 0$

6. $Ax^2 - By^2 + 2kz = 0$

7. $Ax^2 + By^2 + d = 0$

8. $Ax^2 - By^2 + d = 0$

9. $Ax^2 - By^2 = 0$

If second degree terms form a perfect square, then

10. $Ax^2 + Bx + C = 0$

11. $y^2 = Ax$

12. $-A(x^2 + y^2) + Bz = 0$

13. $A(x^2 + y^2) + Cz^2 = 1$

14. $A(x^2 - y^2) + Cz^2 = 1$

Note. In discussion to follow we shall take
 $f(l, m, n) = al^2 + bm^2 + cn^2 + 2fml + 2gnl + 2hmn$,

$$\frac{\partial f}{\partial l} = 2(al + bm + gn);$$

whence

Reduction of General Equation of Second Degree

$$\frac{\partial f}{\partial m} = 2(hl + bn + fn), \quad \frac{\partial f}{\partial n} = 2(gl + fm + cn)$$

§ 12.05. Equation of surface referred to centre as origin.

From § 12.02, result (VII) on Page 2 of this chapter we know that the centre (x_1, y_1, z_1) of $F(x, y, z) = 0$; lies on the planes given by

$$ax + hy + gz + u = 0$$

$$hx + by + fz + v = 0$$

Multiplying (i), (ii) and (iii) by A, H, G , respectively and adding we get

where A, B, C, F, G, H are the coefficients of the determinant $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

viz. a, b, c, f, g, h in the determinant $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Similarly multiplying (i), (ii), (iii) by H, B, F and G, F, C , respectively and adding separately, we get $D_x + (Hu + Bv + Fw) = 0$

From (iv), (v) and (vi) the coordinates of the centre are given by

$$\frac{x}{Au + Bv + Cw} = \frac{Hu + Bv + Fw}{Gv + Fy + Cw} = \frac{1}{2} \quad \dots(A)$$

$$\frac{y}{Au + Bv + Cw} = \frac{Hv + Bu + Fw}{Gv + Fy + Cw} = -\frac{1}{2} \quad \dots(B)$$

Cor. 1. The equation of a diametral plane of the surface (conoid) $F(x, y, z) = 0$ is

$$\frac{\partial F}{\partial x} + ni \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0 \quad \dots(C)$$

and so any diametral plane passes through the centre or centres.

Cor. 2. From (ii), (iii), (iv) and (vi) of § 12.02 Page 2 of this chapter we have

$$ax_1 + hy_1 + gz_1 + u = 0$$

$$hx_1 + by_1 + fz_1 + v = 0$$

$$gx_1 + fy_1 + cz_1 + w = 0$$

which on eliminating (x_1, y_1, z_1) gives $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d + d' \end{vmatrix} = 0$

$$\text{or } \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \quad \dots(D)$$

$$\text{or } \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = P \quad \text{or } d' = P/D,$$

$$\text{and } D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Hence referred to centre (x, y, z) [See result (VII) of § 12.02, Page 2 of this chapter] the equation of the surface $F(x, y, z) = 0$ is

$$f(x, y, z) + (P/D) = 0 \quad \dots(\text{E})$$

* § 12.06. Some properties of determinant D.

We know $D = \begin{vmatrix} a & b & c \\ h & i & g \\ f & j & d \end{vmatrix}$ and $a, b, c, f, g, h, i, j, d$ denote the cofactors of

the corresponding small letters in this determinant D.

$$A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2,$$

$$F = gh - qf, \quad G = hf - bg, \quad H = fg - ch.$$

$$\text{Also } BC - F^2 = (ca - g^2)(ab - h^2) - (gh - qf)^2$$

$$= a^2(bc - abg^2 - ach^2 + g^2h^2) - g^2h^2 - qf^2 + 2qfh$$

$$= a(a(bc + qf) - af^2 - bg^2 - ch^2) = aD. \quad \dots(\text{I})$$

$$\text{Similarly } CA - G^2 = bd - cD, \quad AB - H^2 = cd.$$

$$GH - AF = fD, \quad HF - BG = gD, \quad FG - CH = hD.$$

And from the properties of determinants (See Author's Algebra or Matrices) we know that

$$Af + Hh + Gg = D, \quad Bf + Bh + Fg = 0, \quad Ca + Fh + Gg = 0$$

and similar other results.

$$(i) \quad \text{If } D \neq 0, \text{ from above we have}$$

$$BC = F^2, \quad CA = G^2, \quad AB = H^2, \quad GH = AF, \quad HF = BG, \quad FG = CH$$

$$(ii) \quad \text{If } D = 0 \text{ and } A = 0, \text{ then we have } G = 0, \quad H = 0,$$

$$(iii) \quad \text{If } D = 0 \text{ and } A = 0, \quad B = 0, \text{ then } F = 0, \quad G = 0, \quad H = 0 \text{ but } C \text{ may or}$$

may not be zero.

$$(iv) \quad \text{If } D = 0 \text{ and } H = 0, \text{ then either } A = 0, \quad C = 0 \text{ or } B = 0, \quad F = 0.$$

$$(v) \quad \text{If } D = 0 \text{ and } A + B + C = 0, \text{ then}$$

$$A = B = C = F = G = H = 0.$$

since, A, B, C have the same sign when $D \neq 0$ and so $A + B + C = 0$ gives

$$aix + biy + ciw + di = 0 \quad \dots(\text{II})$$

$$a2x + b2y + c2z + d2 = 0 \quad \dots(\text{III})$$

and each representing a plane. Let there be two equations

$$aix + biy + ciw + di = 0 \quad \dots(\text{II})$$

$$a2x + b2y + c2z + d2 = 0 \quad \dots(\text{III})$$

These two equations will represent the same plane; if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} = 0 \quad \dots(\text{IV})$$

i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} = 0$$

The planes given by (IV) will be parallel but not the same provided

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \neq \frac{d_1}{d_2} \quad \dots(\text{IV})$$

i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} \neq 0 \quad \dots(\text{V})$$

and will intersect in a line provided

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \quad \dots(\text{VI})$$

i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} \neq 0 \quad \dots(\text{VII})$$

In this case the coordinates of the centre as obtained from (A) of § 12.05

of this chapter are finite and unique.

Here we shall consider the various cases which depend on the solution of the equations (I), (II), (III) of § 12.05 on Page 5 of this chapter.

Case I. $D \neq 0$.

In this case the coordinates of the centre as obtained from (A) of § 12.05

of this chapter are infinite, provided

$Au + Hv + Cv, \quad Hu + Bv + Fw$ and $Gv + Fy + Cw$ are not zero.

Thus the surface $F(x, y, z) = 0$ has a single centre at infinity.

Case II. $D = 0, \quad Au + Hv + Cv = 0$.

In this case the coordinates of the centre as obtained from (A) of § 12.05

of this chapter by $S_1 = 0, \quad S_2 = 0$ and $S_3 = 0$ respectively then we can see that

$AS_1 + HS_2 + GS_3 = 0$.

If we denote the equations (I), (II) and (III) of § 12.05 on Page 5 of this chapter by $S_1 = 0, \quad S_2 = 0$ and $S_3 = 0$ respectively then we can see that

have a common line of intersection.

Also if $A = bc - f^2 \neq 0$, then the planes $S_2 = 0$ and $S_3 = 0$ are neither identical nor parallel. So there is a definite line of intersection and the surface $F(x, y, z) = 0$ in this case possesses a line of centres at a finite distance.

We can easily see that when $D = 0$ and $Au + Hv + Cv = 0$ but $A \neq 0$, then $Hu + Bu + Fw = 0$ and $Cv + Fv - Cw = 0$ since in this case from § 12.06 (I) Page 6 we have $F = \sqrt{(BC)}, \quad G = \sqrt{(CA)}, \quad H = \sqrt{(AB)}$ $\dots(\text{a})$

$\therefore Au + Hv + Cv = 0 \Rightarrow \sqrt{A}(\sqrt{Au} + \sqrt{Bv} + \sqrt{Cw}) = 0$

$$\Rightarrow \sqrt{Au} + \sqrt{Bv} + \sqrt{Cw} = 0, \quad A \neq 0, \quad \dots(\text{b})$$

$$\text{Now } Hu + Bu + Fw = \sqrt{(AB)}u + Bv + \sqrt{(BC)}w$$

$$= \sqrt{B}(\sqrt{Au} + \sqrt{Bv} + \sqrt{Cw}) = 0, \text{ from (b)}$$

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Similarly we can prove that $G_u + F_v + C_w = 0$

[Also we can see from above that if $D=0$, $A=0$, then from (c) we get

$G=0$, $H=0$.

Hence in this case $Au+Bu+Cw=0$ but $Hu+Bv+Fw=0$

$Cu+Fv+Cw$ may or may not be zero].

Case IV. A, B, C, E, G, H are all zero.

As in case III, above, the central planes have a common line of intersection. But these planes are parallel as is evident from § 12.07 Page 6 of this chapter.

Also we assume that $f_1 = g_1 = h_1 = 0$; because otherwise the two-planes given by (i) and (ii) of § 12.05 Page 5 of this chapter would be identical and similarly this chapter would be identical.

Hence in this case central planes [given by (i), (ii) and (iii) of § 12.05 Page 5 of this chapter] are parallel but not coincident and so the surface $F(x, y, z)=0$ has line of centres at an infinite distance.

In Case IV, A, B, C, E, G, H are all zero and $f_1 = g_1 = h_1 = 0$.

In this case if f_1, g_1, h_1 are not zero, the central planes (as discussed above) are identical and so the surface $F(x, y, z)=0$ has a plane of centres.

§ 12.09. Reduction of general equation.

In § 12.04 Page 4 of this chapter, we have seen the various forms of the surfaces represented by the general equation of second degree. Now we shall discuss in articles to follow the reduction to the standard forms depending upon the various cases as given in § 12.08 on Pages 7-8 Ch. XII.

In this case there is a unique centre at a finite distance. Also none of the roots of the discriminating cubic (or λ -cubic) vanishes and so $D \neq 0$.

Here the forms to any one of which the given equation can reduce are:

(I) $Ax^2 + By^2 + Cz^2 = 1$ (Ellipsoid)

(II) $Ax^2 + By^2 - Cz^2 = 1$ (Hyperboloid of one sheet)

(III) $Ax^2 - By^2 + Cz^2 = 1$ (Hyperboloid of two sheets)

(IV) $Ax^2 + By^2 + Cz^2 = 0$ (Cone)

Method of Procedure.

(i) Find the coordinates (x_1, y_1, z_1) of the centre of the given surface $F(x, y, z) = 0$ by solving the equations

$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

(ii) Shift the origin to the centre (x_1, y_1, z_1) and then the equation of the surface referred to centre as origin is

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$f(x, y, z) + d' = 0$, where $d' = ux_1 + vy_1 + wz_1 + d$.

By rotation of axes, transform the given equation to the form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic, which can be reduced to any one of the forms given above.

(iv) The direction ratios of axes can be obtained by solving any two of the following three equations:

$$(a - \lambda)x + hy + gz = 0$$

$$hl + (b - \lambda)y + fm = 0$$

$$gl + fm + (c - \lambda)z = 0$$

Putting the three values of λ , the direction ratios of the three axes can be obtained and their equations can be obtained i.e. we can find the equations of three lines through the centre and having above direction-ratios.

(v) The principal planes are given by $\lambda(x + my + nz) + (ul + vm + wn) = 0$.

(vi) If $d' = 0$, then the surface is a cone.

Note: For the solution of a cubic equation, students should go through the section on solution of cubic equations from Author's Theory of Equations. It is not always possible to solve a cubic equation when all its roots are real, but with the help of Descarte's Rule of Signs, we can find the number of its positive and negative roots.

Solved Examples on § 12.10.

*Ex. 1. Reduce the equation $3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz - 4x - 8z + 5 = 0$ to the standard form. Find the nature of the conicoid, its centre and equations of its axes.

Sol. Let $F(x, y, z) = 3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz - 4x - 8z + 5 = 0$

Then the coordinates of the centre are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$,

$$6x + 2y + 2z - 4 = 0 \quad \text{or} \quad 3x + y + z - 2 = 0$$

$$10y + 2z + 2x = 0 \quad \text{or} \quad x + 5y + 3z - 4 = 0$$

$$6z + 2y + 2x - 8 = 0 \quad \text{or} \quad x + y + 3z - 4 = 0$$

Solving the equations of (1) we get the centre (x_1, y_1, z_1) as $(1/3, -1/3, 4/3)$ i.e. $x_1 = 1/3, y_1 = -1/3, z_1 = 4/3$.

Shifting the origin to the centre $(1/3, -1/3, 4/3)$, the equation of the surface reduces to

$f(x_1, y_1, z_1) + d' = 0$, where $d' = ux_1 + vy_1 + wz_1 + d$,

$$= (-2)(1/3) + (0)(-1/3) + (-4)(4/3) + 5 = -1$$

∴ From (1), the reduced equation of the surface is $(3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz + 5) + (-1) = 0$.

(III) $(3x^2 + 5y^2 + 3z^2 + 2xy + 2xz + 2yz + 5) + (-1) = 0$ (III)

as

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$f(x, y, z)$ is the homogeneous part of $F(x, y, z)$.
Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b-\lambda & g \\ h & c-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

or

$$(3-\lambda)[(5-\lambda)(3-\lambda)-1] - [(3-\lambda)-1] + [1-(5-\lambda)] = 0$$

or

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

L.H.S. of (IV) and so we can rewrite (IV) as

$$(\lambda-2)(\lambda^2-9\lambda+18) = 0 \quad \text{or} \quad (\lambda-2)(\lambda-3)(\lambda-6) = 0.$$

\therefore The roots of the discriminating cubic (IV) are 2, 3, 6.

\therefore By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{i.e.} \quad 2x^2 + 3y^2 + 6z^2 - 1 = 0. \quad \text{(V)}$$

substituting values of $\lambda_1, \lambda_2, \lambda_3$ and d' .

Then equation (V) can be rewritten as $2x^2 + 3y^2 + 6z^2 = 1$, which represents an ellipsoid.

The direction-ratios of axes can be obtained by solving two of the following three equations

$$\text{or} \quad (a-\lambda)x + hm + hn = 0, \quad hm + (b-\lambda)m + fn + (c-\lambda)n = 0.$$

$$\text{Taking } \lambda = 2, \text{ we have } l+m+n=0, \quad l+(5-\lambda)m+n=0. \quad \text{(VI)}$$

$$\text{Solving } l+m+n=0, \quad l+3m+n=0, \quad l+m+n=0,$$

$$\frac{l}{l-3} = \frac{m}{1-1} = \frac{n}{3-1} \quad \text{or} \quad \frac{l}{1-1} = \frac{m}{0-1}$$

\therefore The equations of the axis, corresponding to $\lambda = 2$, are

$$\frac{x-(1/3)}{1-3} = \frac{y-(1/3)}{1-1} = \frac{z-(4/3)}{3-1}.$$

Similarly corresponding to $\lambda = 3$ and $\lambda = 6$ the direction ratios of the axes (i.e. the principal directions) are

$$\frac{l}{1-1} = \frac{m}{1-1} \quad \text{and} \quad \frac{l}{1-1} = \frac{m}{-2} = \frac{n}{1}$$

As the equation of the corresponding axes are

$$\frac{x-(1/3)}{1-1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1},$$

$$\frac{x-(1/3)}{1-1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1};$$

$$\therefore \frac{x-(1/3)}{1-1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1}.$$

Ex. 2: Reduce the equation $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$ to its standard form. Also find its centre and the equation referred to centre as origin.

Solution: Given $F(x, y, z) = 3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$. (Ans. $x^2 + y^2 + z^2 + 2yz + 2zx - 2xy - 6x - 6y - 2z - 2 = 0$)

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$$\begin{aligned} \frac{\partial F}{\partial x} &= 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \\ \text{i.e.,} \quad \delta x - 6 &= 0 \quad \text{or} \quad x-1 = 0, \quad \text{or} \quad x = 1 \\ -2y + 6z + 6 &= 0 \quad \text{or} \quad y-3z-3 = 0 \quad \dots(i) \\ -2z-2 &= 0 \quad \text{or} \quad z+1 = 0 \quad \text{or} \quad z=-1 \quad \dots(ii) \end{aligned}$$

$$\begin{aligned} \therefore \text{Solving (i), (ii) and (iii) we get the centre } (x_1, y_1, z_1) \text{ as } (1, 0, -1). \quad \text{Ans.} \\ \text{Shifting the origin to the centre } (1, 0, -1) \text{ the equation of the surface} \\ \text{reduces to} \quad f(x, y, z) + d' = 0, \end{aligned}$$

$$\begin{aligned} \text{where} \quad d' &= ux_1 + vy_1 + wz_1 + d \\ &= (-3)(1) + (3)(0) + (-1)(-1) = 2 = -4. \\ \therefore \text{From (i), the equation of the surface referred to centre as origin is} \\ (3x^2 - y^2 - z^2 + 6yz) + (-4) &= 0 \quad \text{or} \quad 3x^2 - y^2 - z^2 + 6yz - 4 = 0. \quad \text{Ans.} \\ \text{Now the discriminating cubic is} \end{aligned}$$

$$\begin{vmatrix} a-\lambda & b-\lambda & g \\ h & c-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0, \quad \text{putting values of } a, b, c, f, g, h \\ \text{or} \quad (3-\lambda)[(5-\lambda)(3-\lambda)-1] - [(3-\lambda)-1] + [1-(5-\lambda)] = 0 \\ \therefore \quad (3-\lambda)(4\lambda-2)(\lambda-4) = 0, \quad \text{or} \quad \lambda = 2, 3, -4. \\ \therefore \text{By rotation of axes, the given equation transforms to} \\ "3x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0" \quad \text{i.e.} \quad 2x^2 + 3y^2 - 4z^2 - 4 = 0 \\ \text{or} \quad 2x^2 + 3y^2 - 4z^2 = 4, \text{ which represents a hyperboloid of one sheet} \\ [\because \text{it is of the form } Ax^2 + By^2 - Cz^2 = 1]. \quad \text{Ans.} \\ \text{Ex. 3: Show that the equation } x^2 + y^2 + z^2 - 6yz - 2zx - 2xy - 6x - 2y - 2z + 2 = 0 \text{ represents a hyperboloid of two sheets.} \\ \text{Solution: Comparing the given equation } F(x, y, z) = 0 \text{ with the equation} \\ ax^2 + by^2 + cz^2 + 2fyx + 2gzx + 2zy + 2ux + 2vy + 2wz + d = 0, \text{ we have } a = 1, \\ b = 1, c = 1, f = -3, g = -1, h = -1, u = -3, v = -1, w = -1, d = 2. \quad \text{... (i)} \\ \text{Now coordinates of the centre } (x_1, y_1, z_1) \text{ of the given surface are given by} \\ \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0. \end{math>$$

$$\begin{aligned} \text{i.e.,} \quad 2x_1 - 2y_1 - 2z_1 - 6 &= 0 \quad \text{or} \quad x_1 - y_1 - z_1 = 3 \quad \dots(ii) \\ -2y_1 - 6z_1 - 2x_1 - 2 &= 0 \quad \text{or} \quad x_1 - y_1 + 3z_1 = -2 \quad \dots(iii) \\ 2z_1 - 6y_1 - 2x_1 - 2 &= 0 \quad \text{or} \quad x_1 + 3y_1 - 2z_1 = -2 \quad \dots(iv) \end{aligned}$$

$$\therefore \text{Solving (ii), (iii) and (iv) we get } x_1 = 1/2, y_1 = -5/4, z_1 = -5/4.$$

$$\therefore \text{centre of the given surface is } (1/2, -5/4, -5/4).$$

$$\begin{aligned} \text{Also } d' &= ux_1 + vy_1 + wz_1 + d \\ &= (-3)(1/2) + (-1)(-5/4) + (-1)(-5/4) + 2 = 3 \quad \dots(v) \end{aligned}$$

Solid Geometry.

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ b & a-\lambda & -1 \\ c & -1 & a-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 1-\lambda & -4 \\ -1 & -4 & \lambda \end{vmatrix} = 0, \quad \text{applying } C_3 - C_2$$

or $\lambda^3 - 3\lambda^2 - 8\lambda + 16 = 0$

\therefore Either $\lambda = 4$ or $\lambda^2 + \lambda - 4 = 0$

Now $\lambda^2 + \lambda - 4 = 0$ gives $\lambda = [-1 \pm \sqrt{(1+16)}]/2$.

Thus we find that two values of λ are +ve and one -ve.

\therefore By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0, \quad \text{i.e. } -\frac{\lambda_1}{3}x^2 - \frac{\lambda_2}{3}y^2 - \frac{\lambda_3}{3}z^2 = 1, \quad \therefore d' = 3.$$

Now two values of λ being positive and one negative, from above the equation of the surface transforms to the form $Ax^2 + By^2 + Cz^2 = 1$, where A, B, C are negative and third positive, so that the given surface is a hyperboloid of two sheets.

Ex. 4. Reduce the equation $2x^2 - 7y^2 + 2z^2 - 10xz - 8xy - 10xy + 6x + 12y - 6z + 5 = 0$ to the standard form. What does it represent?

Sol. Comparing the given equation $F(x, y, z) = 0$ with the equation $ax^2 + by^2 + cz^2 + 2fxy + 2gxz + 2hy + 2ux + 2wy + dz = 0$,

we have $a = 2, b = -7, c = 2, f = -5, g = -4, h = -5, u = 3, v = 6, w = -3, d = 5$.

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \text{and} \quad \frac{\partial F}{\partial z} = 0.$$

i.e. $4x_1 - 8y_1 - 10z_1 + 6 = 0$ or $-14y_1 - 10z_1 - 10x_1 + 12 = 0$

$4z_1 - 10y_1 - 8x_1 - 6 = 0$ or $5x_1 + 7y_1 + 5z_1 - 6 = 0$

Solving (i), (ii) and (iii) we get $x_1 = 1/3, y_1 = -1/3, z_1 = 4/3$.

\therefore Centre of the given surface is $(1/3, -1/3, 4/3)$.

Also $d' = ux_1 + vy_1 + wz_1 + d'$
 $= 3(1/3) + 6(-1/3) + (-1/3)(4/3) + 5 = 1 - 2 - 4 + 5 = 0$

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ b & a-\lambda & f \\ c & f & a-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & 2-\lambda & -5 \\ -5 & -5 & 2-\lambda \end{vmatrix} = 0$$

or $(2-\lambda)[-(7+\lambda)(2-\lambda) - 25] + 5[-5(2-\lambda) - 20] - 4(25 - 4(7+\lambda)) = 0$

or $\lambda^3 + 3\lambda^2 - 9\lambda + 216 = 0$, on simplifying

Ans.

$$\begin{vmatrix} a-\lambda & b & c \\ b & a-\lambda & f \\ c & f & a-\lambda \end{vmatrix} = 0$$

(Note)

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or $(\lambda - 3)(\lambda^2 + 6\lambda - 72) = 0$ or $(\lambda - 3)(\lambda + 12) = 0$

or $\lambda = 3, 6, -12$

\therefore Let $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = -12$

\therefore By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

or $3x^2 + 6y^2 - 12z^2 + 0 = 0$, substituting values of $\lambda_1, \lambda_2, \lambda_3, d'$

or $x^2 + 2y^2 - 4z^2 = 0$, which is the required standard form and represents a cone.

Also the vertex of the cone (and not centre) is $(1/3, -1/3, 4/3)$ as calculated above.

Exercises on § 12.10 (Case I).

Ex. 1. Reduce the equation $11x^2 + 11y^2 + 6z^2 - 8yz + 4xz - 12xy + 72x$

- $72y + 36z + 150 = 0$ to the standard form and show that it represents an ellipsoid and find the equations of the axes. (Avadh 9; Garhwa 94, 92)

Ans. Centre $(-2, 2, -1)$; $3x^2 + 6y^2 + 18z^2 = 12$ (ellipsoid)

dr's of the axes are $1, 1, 2, 2, 1, -2, 2, -1$. Ex. 2. Reduce $3x^2 + 6y^2 - y^2 - z^2 - 6x + 6y - 2z - 2 = 0$ to the standard form. What surface does it represent?

Ex. 3. Reduce $2x^2 - y^2 - 4z^2 = 4$; Hyperboloid of one sheet. $= 0$ to the standard form. What does it represent?

Ex. 4. For the conicoid $ax^2 + by^2 + cz^2 + 2xyz + 2yz + 2zx + 2xy + 2u$

+ $2wz + d = 0$, show that

(i) all the roots of the discriminating cubic are real,
 and (ii) principal directions are perpendicular to each other. (Rohilkhand 91)

Ex. 5. Reduce $x^2 + 3y^2 + 3z^2 - 2yz + 6z + 3 = 0$ to the standard form and show that the surface represented by it is an ellipsoid.

§ 12.11. Case II. D = 0 and Au + Hv + Gw ≠ 0. Ans. $x^2 + 2y^2 + 4z^2 = 1$. $D = 0 \Rightarrow$ one root of the discriminating cubic is zero. Here the forms to any one of which the given equation can reduce are

$$\begin{vmatrix} Ax^2 + By^2 + Cz = 0 \\ Ax^2 - By^2 + Cz = 0 \\ (Hypothetic Paraboloid) \end{vmatrix}$$

Method of Procedure.

(i) Find the discriminating cubic viz. $\begin{vmatrix} a-\lambda & b & c \\ b & a-\lambda & f \\ c & f & a-\lambda \end{vmatrix} = 0$

One root of this cubic will be zero in this case.

(ii) Put $\lambda = 0$ in the above determinant and associate each row with

l_3, m_3, n_3 ,

$$al_3 + bm_3 + cm_3 = 0,$$

$$bl_3 + cl_3 + fl_3 = 0,$$

$$cl_3 + fl_3 + nl_3 = 0.$$

Solve any two of these, which will give the direction ratios of the axis corresponding to $\lambda = 0$.

(iii) Evaluate $k = ul_3 + vm_3 + wn_3$,

where l_3, m_3, n_3 are actual direction cosines.

If $k \neq 0$, then reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2xz = 0, \quad \text{where } \lambda_1, \lambda_2 \text{ are non-zero roots of the}$$

discriminating cubic.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1 and λ_2 have the same or opposite signs.

(iv) Vertex. The coordinates of the vertex of the paraboloid in this case are obtained by solving any two of the three equations

$$\left[\frac{\partial F}{\partial x} \right] = \left[\frac{\partial F}{\partial y} \right] = \left[\frac{\partial F}{\partial z} \right] = 2k,$$

with the equation $k(l_3x + m_3y + n_3z) + ux + vy + wz + d = 0$

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Ex. 1. Determine completely what is represented by the equation

$$2x^2 + 2y^2 + z^2 + 2xz + 2xy + x + y = 0.$$

Find the coordinates of its vertex and the equations to its axis.

Solution: Here 'a' = 2, 'b' = 2, 'c' = 1, 'f' = 1, 'g' = -1. (Garrett 93)

'H' = -2, 'u' = 1/2, 'v' = 1/2, 'w' = 0 and 'd' = 0

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & f \\ h & b-\lambda & g \\ f & g & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 2-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0 \quad \dots(1)$$

or $(2-\lambda)((2-\lambda)(1-\lambda) - 1) + 2[-2(1-\lambda) + 1] - [-2 + (2-\lambda)\lambda] = 0$,

or $\lambda^3 - 5\lambda^2 + 2\lambda = 0$ or $\lambda(\lambda^2 - 5\lambda + 2) = 0$ or $\lambda = 0, \frac{1}{2}(5 \pm \sqrt{17})$

Let $\lambda_1 = \frac{1}{2}(5 + \sqrt{17}), \lambda_2 = \frac{1}{2}(5 - \sqrt{17}), \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3 , we have

$$2l_3 - l_3 = 0, -2l_3 - 2m_3 + m_3 = 0, -l_3 + m_3 + n_3 = 0$$

Solving last two equations simultaneously for l_3, m_3, n_3 we get

$$\frac{l_3}{2-1} = \frac{m_3}{-1+2} = \frac{n_3}{-2+2}$$

$$\text{i.e., } \frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{0} = \frac{\sqrt{l_3^2 + m_3^2 + n_3^2}}{\sqrt{l_3^2 + m_3^2 + n_3^2}} = \frac{1}{\sqrt{2}}$$

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$$\therefore l_3 = 1/\sqrt{2}, m_3 = 1/\sqrt{2}, n_3 = 0.$$

These gives d.c.'s of the axis corresponding to $\lambda = 0$.

$$\text{Now } k = 'ul_3 + vm_3 + wn_3 = \left(\frac{1}{2}\right)(1/\sqrt{2}) + \left(\frac{1}{2}\right)(1/\sqrt{2}) + 0 = 1/\sqrt{2}$$

The required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kxz = 0$

$$\text{or } \frac{1}{2}[5 + \sqrt{17}]x^2 + \frac{1}{2}[5 - \sqrt{17}]y^2 + 2(1/\sqrt{2})z = 0$$

which represents an elliptic paraboloid as both λ_1, λ_2 are positive.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the three equations

$$\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = 2k,$$

along with

$$k(l_3x + m_3y + n_3z) + ux + vy + wz + d = 0$$

i.e., any two of the equations $4x - 2z - 4y + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$

$$4y + 2z - 4x + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$$

$$2z + 2y + 2x - 2 = 2(0)(1/\sqrt{2})$$

$$4y + 2z - 4x + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$$

$$2x - 2y - z = 0$$

$$x - y - z = 0$$

with

$$\frac{1}{2}\left[\frac{1}{2}x + \frac{1}{2}y + 0z\right] + \frac{1}{2}x + \frac{1}{2}y + 0 + 0 = 0$$

i.e.,

$$2x - 2y - z = 0, x - y - z = 0, x + y = 0$$

Solving these we get $x = 0, y = 0, z = 0$ i.e., the coordinates of the vertex are (0, 0, 0).

\therefore The equations to its axis are

$$\begin{vmatrix} l_3 & m_3 & n_3 \\ l_3 & m_3 & n_3 \end{vmatrix} = \frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{0}$$

i.e.,

$$x = y, z = 0,$$

Solving these we get $x = 0, y = 0, z = 0$ i.e., the coordinates of the vertex are (0, 0, 0).

\therefore The equations to its axis are

$$\begin{vmatrix} l_3 & m_3 & n_3 \\ l_3 & m_3 & n_3 \end{vmatrix} = \frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{0}$$

i.e.,

$$x = 0, y = 0, z = 0,$$

Ansl.

*Ex. 2. Reduce the equation $3x^2 - 6yz - 6zx - 7xy - 5y + 3z = 0$ to standard form and find its nature. (Ans. Hyperbolic paraboloid)

Sol. Here 'a' = 0, 'b' = 0, 'c' = 3, 'f' = -3, 'g' = -3, 'u' = 0, 'v' = -7/2.

$$v' = -5/2, w' = 3$$

The discriminating cubic is

$$\begin{vmatrix} d-\lambda & h & f \\ h & b-\lambda & g \\ f & g & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & -3 \\ 0 & 0-\lambda & -3 \\ -3 & -3 & 3-\lambda \end{vmatrix} = 0 \quad \dots(1)$$

or

$$-\lambda(-\lambda(3-\lambda)-9) + 0 - 3(-3\lambda) = 0 \quad \text{or} \quad \lambda^3 - 3\lambda^2 - 18\lambda = 0$$

or

$$\lambda(\lambda^2 - 3\lambda - 18) = 0 \quad \text{or} \quad \lambda(\lambda - 6)(\lambda + 3) = 0 \quad \text{or} \quad \lambda = 0, 6, -3$$

Let $\lambda_1 = 6, \lambda_2 = -3, \lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3 we have

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$$0, l_1 + 0, m_1 - 3n_1 = 0, 0, l_2 + 0, m_2 - 3n_2 = 0, 0, l_3 + 0, m_3 - 3n_3 = 0.$$

These gives $n_1 = 0, l_2 + m_2 = 0, l_3 + m_3 = 0$

$$\text{L.C. } \frac{l_1}{l_1} = \frac{m_1}{m_1} = \frac{n_1}{n_1} = \frac{\sqrt{(l_2^2 + m_2^2 + n_2^2)}}{\sqrt{(l_2^2 + (-l_1)^2 + 0^2)}} = \frac{1}{\sqrt{2}}$$

$$\therefore l_3 = 1/\sqrt{2}, m_3 = -1/\sqrt{2}, n_3 = 0$$

These gives c's of the axis corresponding to $\lambda = 0$.

$$\text{Now } k = ul_3 + vnl_3 + wnl_3 = -\frac{7}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{5}{2}\left(\frac{1}{\sqrt{2}}\right) + 3(0) = -\frac{1}{\sqrt{2}}$$

\therefore Required reduced equation is
 $\lambda_1 l_1^2 + \lambda_2 l_2^2 + 2kz = 0 \quad \text{or} \quad 6x^2 - 3y^2 - \sqrt{2}z = 0,$

which represents a hyperbolic paraboloid as λ_1 and λ_2 are of opposite signs.
 Ans.

**Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid $2x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$.
 Ans.

Solution. Here 'a' = 4, 'b' = -2, 'c' = 4 and 'd' = -2. (Rohilkhand 95).

'w' = -4, 'v' = -2, 'u' = 4 and 'd' = -2.
 \therefore The discriminating cubic is

$$\begin{vmatrix} a-\lambda & b-\lambda & c-\lambda \\ b & a-\lambda & f \\ c-\lambda & f & g \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\text{or} \quad (4-\lambda)[(1+\lambda)^2 - 1] = 0 \quad \text{or} \quad \lambda(\lambda+2)(\lambda-4) = 0 \quad \text{or} \quad \lambda = 0, -2, 4.$$

\therefore Let $\lambda_1 = -2, \lambda_2 = 4, \lambda_3 = 0$.
 Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_1, m_1, n_3 , we have

$$4l_3 = 0, -m_1 + n_3 = 0, m_3 - n_3 = 0 \Rightarrow l_3 = 0, m_3 = n_3;$$

\therefore We have $l_3 = 0, m_3 = 1, so 0 + m_3^2 + m_3^2 = 1 or m_3 = 1/\sqrt{2} = \eta_3$

$$\text{Now } k = ul_3 + vnl_3 + wnl_3 = -4(0) - 2(1/\sqrt{2}) + 4(1/\sqrt{2}) = \sqrt{2}$$

\therefore Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$.

or $2x^2 + y^2 + 2\sqrt{2}z = 0$ or which represents a hyperbolic paraboloid as λ_1 is $-ve$ and λ_2 is $+ve$.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations.

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 2k$$

$$\text{and } k((x+m_3y+n_3z) + ly + mz + dz = 0, \dots \text{See § 12.11 (iv) Page 14.}$$

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i.e., any two of the equations

$$\begin{aligned} 8x - 8 &= 2(\sqrt{2})(0) \quad i.e., x = 1; \\ -2y + 2z - 4 &= 2(\sqrt{2})(1/\sqrt{2}) \quad i.e., y - z + 3 = 0 \\ -2x + 2y + 8 &= 2(\sqrt{2})(1/\sqrt{2}) \quad i.e., y - z + 3 = 0 \\ \sqrt{2}\left[0, x + \frac{1}{\sqrt{2}}, y + \frac{1}{\sqrt{2}}, z\right] - 4x - 2y + 4z - 2 &= 0 \end{aligned}$$

with

$$i.e., x = 1, y - z + 3 = 0, 4x + y - 5z + 2 = 0.$$

Solving these we get $x = 1, y = -9/4, z = 3/4$.

\therefore Coordinates of the vertex are $(1, -9/4, 3/4)$. Ans.

And the equations of its axis are $\frac{x-1}{\sqrt{3}} = \frac{y-(-9/4)}{m_3} = \frac{z-(-3/4)}{n_3}$

$$\text{or} \quad \frac{x-1}{0} = \frac{y+9/4}{1/\sqrt{2}} = \frac{z-(-3/4)}{-1/\sqrt{2}} \quad i.e., \frac{x-1}{0} = \frac{4y+9}{1} = \frac{z-(-3/4)}{1} \quad \text{Ans.}$$

*Ex. 4. Show that the following equation represents a paraboloid.

Find its vertex and equations to the axis.

$$4y^2 + 4x^2 + 4yz - 2x - 14y - 22z + 33 = 0. \quad (\text{Rohilkhand 92, 90})$$

Solution. Here 'a' = 0, 'b' = 4, 'c' = 4, 'd' = 2, 'g' = 0, 'h' = 0, 'w' = -7, 'u' = -11 and 'd' = 33.

\therefore The discriminating cubic is

$$\begin{vmatrix} a-\lambda & b-\lambda & g \\ b & a-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\text{or} \quad -\lambda[(4-\lambda)^2 - 4] = 0 \quad \text{or} \quad \lambda[(\lambda^2 - 8\lambda + 12) = 0]$$

$$\text{or} \quad \lambda(\lambda-2)(\lambda-6) = 0 \quad \text{or} \quad \lambda = 0, 2, 6$$

\therefore Let $\lambda_1 = 2, \lambda_2 = 6$ and $\lambda_3 = 0$.
 Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_1, m_1, n_3 , we have

$$4l_3 = 0, -m_1 + n_3 = 0, m_3 - n_3 = 0 \Rightarrow l_3 = 0, m_3 = n_3;$$

\therefore We have $l_3 = 0, m_3 = 1, so 0 + m_3^2 + m_3^2 = 1 or m_3 = 1/\sqrt{2} = \eta_3$

$$\text{Now } k = ul_3 + vnl_3 + wnl_3 = -4(0) - 2(1/\sqrt{2}) + 4(1/\sqrt{2}) = \sqrt{2}$$

\therefore Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$ or

$$2x^2 + 6y^2 + 2(-1)z = 0 \quad \text{or} \quad x^2 + 3y^2 - z = 0,$$

which represents an elliptic paraboloid as both λ_1 and λ_2 are positive.
 Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 2k$$

and $k((x+m_3y+n_3z) + ly + mz + dz = 0, \dots \text{See § 12.11 (iv) P. 14.}$
 $\therefore 2 = 2(-1)(0)$ which is absurd.

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$$8x + 4y - 14 = 2(-1)0 \Rightarrow 4y + 2z = 7;$$

$$\text{with } (-1)(x+0, y+0, z) - x - 7y - 11z + 33 = 0 \quad \text{or} \quad 2x + 7y + 11z = 33$$

$$\text{i.e. } 4y + 2z = 7, \quad 2x + 7y + 11z = 33$$

Solving these we get $x = 1, y = 1/2, z = 5/2$

\therefore Coordinates of the vertex are $(1, 1/2, 5/2)$

And the equations of its axis are $\frac{x-1}{l_3} = \frac{y-(1/2)}{m_3} = \frac{z-(5/2)}{n_3}$

$$\text{or } \frac{x-1}{1}, \frac{y-(1/2)}{1}, \frac{z-(5/2)}{0} = 0 \quad \text{or} \quad \frac{x-1}{1} = \frac{2y-1}{0} = \frac{2z-5}{0}$$

$$\text{or } 2y - 1 = 0, \quad 2z - 5 = 0 \quad \text{or} \quad y = 1/2, \quad z = 5/2.$$

*Ex. 5. Prove that $ax + by + cz = 0$ represents a paraboloid and the equations to the axis are

$$ax + by + 2cz = 0, \quad (a^2 + b^2)z + ax + b\beta = 0. \quad (\text{Rohilkhand 93})$$

Sol. Given equation is $cx^2 + by^2 + az^2 + \beta z = 0$

Here 'a' = 0, 'b' = 0, 'c' = 0, 'y' = 1/2, 'z' = 5/2.

$w = p/2, w^2 = 0$ and 'd' = 0.

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & a/2 \\ 0 & b/2 & 0 \\ a/2 & b/2 & c-\lambda \end{vmatrix} = 0 \quad \text{(i)}$$

$$\text{or } -\lambda[-\lambda(c-\lambda) - (b^2/4)] + (a/2)[(a\lambda/2) - a^2] = 0$$

$$\text{or } 4k^3 - 4a\lambda^2 - (a^2 + b^2)\lambda = 0 \quad \text{or} \quad \lambda[a^2 - 4a\lambda - a^2 - b^2] = 0$$

$$\text{or } 4k^3 - 4a\lambda^2 - (a^2 + b^2)\lambda = 0 \quad \text{or} \quad \lambda[a^2 - 4(a^2 + b^2)] = 0$$

$$\text{or } \lambda = 0 \quad \text{and} \quad \lambda = \frac{4a \pm \sqrt{(16a^2 + 16b^2)}}{8} = \frac{4a \pm \sqrt{2(a^2 + b^2)}}{2}$$

$$\text{Let } \lambda_1 = \frac{1}{2}[a + \sqrt{(2a^2 + b^2)}], \quad \lambda_2 = \frac{1}{2}[a - \sqrt{(2a^2 + b^2)}], \quad \lambda_3 = 0$$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each

$$\text{row with } l_3, m_3, n_3, \text{ we have }$$

$$0 \cdot l_3 + 0, \quad m_3 + (a/2), \quad n_3 = 0, \quad l_3 + 0, \quad m_3 + (b/2), \quad n_3 = 0$$

These gives $n_3 = 0$ and $a\lambda_3 + b\beta_3 = 0$

$$\text{i.e. } \frac{l_3 + m_3 + n_3}{a} = \frac{\sqrt{(b^2 + a^2 + 0)}}{0} = \frac{1}{\sqrt{(a^2 + b^2)}}$$

$$\therefore l_3 = b/\sqrt{(a^2 + b^2)}, \quad m_3 = -a/\sqrt{(a^2 + b^2)}, \quad n_3 \neq 0.$$

Now $k = l_3\alpha + m_3\beta + n_3\gamma$

$$= \frac{a}{2} \cdot \frac{b}{\sqrt{(a^2 + b^2)}} + \frac{\beta}{2} \left[\frac{-a}{\sqrt{(a^2 + b^2)}} \right] + 0 = \frac{ba - a\beta}{2\sqrt{(a^2 + b^2)}} \neq 0$$

Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

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$$\text{or } \frac{1}{2} [a + \sqrt{(2a^2 + b^2)}] x^2 + \frac{1}{2} [a - \sqrt{(2a^2 + b^2)}] y^2 + \frac{b\alpha - a\beta}{\sqrt{(a^2 + b^2)}} z = 0 \quad \text{... (ii)}$$

Now, as $a + \sqrt{(2a^2 + b^2)} > 0$, and $a - \sqrt{(2a^2 + b^2)} < 0$, so (ii) represents a hyperbolic paraboloid.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

and $k(l_3x + m_3y + n_3z) + \alpha x + \beta y + d = 0$ See § 12.11 (iv) P. 14 Ch. XII

$$\text{or } \frac{\alpha + a\beta}{\sqrt{(a^2 + b^2)}} = \frac{\beta + b\alpha}{\sqrt{(a^2 + b^2)}} = \frac{\alpha x + \beta y + 2cz}{0} = \frac{2(b\alpha - a\beta)}{2\sqrt{(a^2 + b^2)}}$$

Thus we have $\frac{\alpha + a\beta}{b} = \frac{\beta + b\alpha}{a}$, $\alpha x + \beta y + 2cz = 0$

$$\text{and } \frac{bx - a\beta}{2\sqrt{(a^2 + b^2)}} \left[\frac{bx - a\beta}{\sqrt{(a^2 + b^2)}} \right] + \frac{1}{2} (ax + \beta y) = 0$$

$$\text{i.e. } (a^2 + b^2)x^2 + \alpha x + \beta y + 2cz = 0$$

$$\text{and } (b\alpha - a\beta)x^2 + \alpha x + \beta y + (ax + \beta y)(a^2 + b^2) = 0$$

Now if (x_1, y_1, z_1) be the vertex of the paraboloid then x, y, z satisfies

$$\text{above three equations}$$

$$(a^2 + b^2)x^2 + \alpha x + \beta y + 2cz = 0 \quad \text{... (iii)}$$

$$\alpha x + \beta y + 2cz = 0 \quad \text{... (iv)}$$

$$(ax + \beta y)(a^2 + b^2) = 0 \quad \text{... (v)}$$

And the equations of the axis are $\frac{x-x_1}{l_3} = \frac{y-y_1}{m_3} = \frac{z-z_1}{n_3}$

$$\text{Or, } \frac{x-x_1}{l_3} = \frac{y-y_1}{m_3} = \frac{z-z_1}{n_3} \quad \text{... (vi)}$$

$$b\sqrt{(2a^2 + b^2)} = \frac{a\sqrt{(2a^2 + b^2)}}{0}, \quad \text{substituting values of } l_3, m_3, n_3$$

These give $z - z_1 = 0$, or $z = z_1 = \frac{a\alpha + b\beta}{(a^2 + b^2)}$, from (vi)

$$(a^2 + b^2)z + \alpha x + \beta y = 0 \quad \text{... (vii)}$$

Again from first two fractions of (vi), we get $a(x - x_1) + b(y - y_1) = 0$

$$\alpha x + \beta y = \alpha x_1 + \beta y_1 = -2cz, \text{ from (v)}$$

$$= -2cz, \text{ from (vii)}$$

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or
Hence from (vii) and (viii) the equations of the axis of the paraboloid are
 $(\alpha + \beta^2)^2 + \alpha\gamma\beta = 0, \alpha x + \beta y + 2cz = 0$

Exercises on § 12.11 (Case II)

Ex. 1. Prove that the surface represented by the equation $3x^2 + 4y^2 + 9z^2 + 6xz + 4xy + 6y + 2z + 1 = 0$ is an elliptic paraboloid.
Ans. Reduced form is $[8 + (3/8)x]^2 + [8 - \sqrt{(3/8)}y]^2 - [10/\sqrt{13}z]^2 = 0$.

*Ex. 2. Find the coordinates of the vertex and equation to the axis of the elliptic paraboloid $4x^2 + y^2 + z^2 - 2xz - 2xy + x + y - 4z - 6 = 0$.

Ans. $(-1, 2, -1); x + 1 = \frac{1}{2}(y - 2) = \frac{1}{2}(z + 1)$.
Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid.

$5x^2 - 16y^2 + 5z^2 + 8yz - 14xz + 8xy + 4x + 20y + 4z - 24 = 0$.
Ans. $(1, 1, 1), \frac{1}{2}(x - 1) = y - 1 = \frac{1}{2}(z - 1)$.
Ex. 12.12. Case III. $D = 0, Ax + Hy + Gw = 0; A \neq 0$.
In this case the forms (o) or (n) one of which the given equation can reduce

- (i) $Ax^2 + By^2 + C = 0$ (Elliptic cylinder)
- (ii) $Ax^2 - By^2 + C = 0$ (Hyperbolic cylinder)
- (iii) $Ax^2 - By^2 = 0$ (Pair of planes)

In this case there is a line of centres at a finite distance and the discriminating cubic has one root zero, say λ_3 .

The line of centres is given by any two of $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$, where $F(x, y, z) = 0$ is the equation of the given surface.

Let (α, β, γ) be the coordinates of any point lying on this line. Then shifting the origin to (α, β, γ) and rotating the axes in such a manner that these coincide with a set of mutually perpendicular principal directions, the given equation reduces to the form $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, where $k = \alpha\alpha + \beta\beta + \gamma\gamma + d$.

Nature : If $k = 0$, this represents a pair of planes.
If $k \neq 0$, this represents an elliptic or hyperbolic cylinder according as both of the same or opposite signs.

The line of intersection of the principal planes corresponding to non-zero values of λ is the axis of the cylinder. It is parallel to the principal direction corresponding to λ_3 which is zero and is also the line of the centres.

Solved Examples on § 12.12.

*Ex. 1. Show that the surface $26x^2 + 20y^2 + 10z^2 - 4yz - 16xz - 36xy + 52x - 3y - 16z + 25 = 0$ represents an elliptic cylinder. Also find the equations to its axis.

Solution. Here ' a' ' = 5, ' b' ' = 5, ' c' ' = 8, ' f ' = 4, ' g ' = 4, ' h ' = -1, ' n' ' = 6.

Solution. Here the discriminating cubic is given by

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$$\left| \begin{array}{ccc} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{array} \right| = 0 \quad \text{or} \quad \left| \begin{array}{ccc} 26-\lambda & 8 & -8 \\ -8 & 20-\lambda & -2 \\ -8 & -2 & 10-\lambda \end{array} \right| = 0$$

$$\text{or} \quad (26-\lambda)[(20-\lambda)(10-\lambda)-4] + 18[-18(10-\lambda)-16]$$

$$\text{or} \quad \lambda^3 - 56\lambda^2 + 588\lambda = 0 \quad \text{or} \quad \lambda(\lambda^2 - 56\lambda + 588) = 0$$

$$\text{or} \quad \lambda(\lambda-14)(\lambda-42) = 0 \quad \text{or} \quad \lambda = 0, 14, 42.$$

Let $\lambda_1 = 14, \lambda_2 = 42$ and $\lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with (α, β, γ) , we have $26\alpha_3 - 18\beta_3 - 8\gamma_3 = 0, -18\beta_3 + 20\gamma_3 = 0, -8\beta_3 - 2m_3 + 10\gamma_3 = 0$.

Solving first and third of these simultaneously, we have

$$\frac{2x}{1} - \frac{m_3}{4} = \frac{\gamma_3}{1} = \frac{\sqrt{(\beta_3^2 + m_3^2 + \alpha_3^2)}}{\sqrt{1^2 + 4^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

$$\text{or} \quad \frac{2x}{1} - \frac{1}{4}\sqrt{3} = \frac{\gamma_3}{1} = \frac{\sqrt{(\beta_3^2 + m_3^2 + \alpha_3^2)}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

The line of centres is given by any two of $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 52x - 36y - 16z + 32 = 0 \quad \text{i.e. } 13x - 9y - 4z + 15 = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 40y - 4z - 36x - 36 = 0 \quad \text{i.e. } 9x - 10y + z + 9 = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 20z - 4y - 16x - 16 = 0 \quad \text{i.e. } 4x + y - 5z + 4 = 0.$$

Let (α, β, γ) be any point on the line of centres.

Choosing $\alpha = -1, \beta = 0, \gamma = 0$ we find $(-1, 0, 0)$ is a point on the line of centres.

[Note. The method of choosing α, β, γ is not unique]

Now $k = \alpha\alpha + \beta\beta + \gamma\gamma + d = 26(-1)^2 + (-18)(0)^2 + (-8)(0)^2 + 25 = 1 \neq 0$. Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$

i.e. $14x^2 + 42y^2 - 1 = 0$, which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign.

[See § 12.12 Page 20 Ch. XII]

$$\frac{x - (-1)}{l_3} = \frac{y - 0}{m_3} = \frac{z - 0}{n_3} \quad \text{or} \quad \frac{x+1}{1} = \frac{y}{1} = \frac{z}{1}, \text{ from (iii)}$$

*Ex. 2. Prove that the surface represented by the equation

$$5x^2 + 5y^2 + 8z^2 + 8yz + 8xy + 2zx - 12y + 6 = 0$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$ and find the equations to its axis.

Solution. Here ' a' ' = 5, ' b' ' = 5, ' c' ' = 8, ' f ' = 4, ' g ' = 4, ' h ' = -1, ' n' ' = 6.

∴ The discriminating cubic is

Equation of General Equation of Second Degree

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -1 & 4 \\ -1 & 5-\lambda & 4 \\ 4 & 4 & 8-\lambda \end{vmatrix} = 0$$

or $(5-\lambda)(5-\lambda)(8-\lambda)-16)+[-(8-\lambda)-16]+4[-4-4(5-\lambda)]=0$
 or $\lambda^2-18\lambda^2+72\neq 0$ or $\lambda(\lambda^2-48\lambda+72)=0$
 or $\lambda(\lambda-6)(\lambda-12)=0$ or $\lambda=0, 6, 12$.

∴ Let

$$\lambda_1 = 6, \lambda_2 = 12, \lambda_3 = 0.$$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with b_3, m_3, n_3 , we have

$$5_3 - m_3 + 4n_3 = 0, \quad \text{or} \quad b_3 + 5m_3 + 4n_3 = 0, \quad 4b_3 + 4m_3 + 8n_3 = 0.$$

Solving last two equations simultaneously, we get

$$\frac{b_3}{4-10} = \frac{m_3}{-2-4} = \frac{n_3}{5+1}, \quad \text{or} \quad \frac{b_3}{-1} = \frac{m_3}{-1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \quad \text{... (iii)}$$

Also the line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 10x + 8z - 2y + 12 = 0 \quad \text{or} \quad 5x - y + 4z + 6 = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 10y + 8z - 2x - 12 = 0 \quad \text{or} \quad x - 5y - 4z + 6 = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 16z + 8y + 8x = 0 \quad \text{or} \quad x + y + 2z = 0.$$

Let (α, β, γ) be any point on the line of centres. Choosing $y = 0, \beta = 1, \alpha = -1$ we find $(-1, 1, 0)$ is a point on the line of centres.

$$\text{Now } k = ux + v\beta + w\gamma + d = (-\frac{1}{2})(-2) + (-1)(-\frac{1}{2}) + (\frac{3}{2})(0) - 2 = -\frac{1}{2} \neq 0$$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, i.e., $6x^2 + 12y^2 - 6 = 0$ i.e., $x^2 + 2y^2 - 1 = 0$

which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign.

Also (iv) can be rewritten as $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$.

And so if e be the required eccentricity, then

$$e^2 = b^2(1 - e^2) \Rightarrow 1/2 = (1 - e^2) \text{ or } e = 1/\sqrt{2}.$$

Also the equations of the axis of cylinder are,

$$\frac{x+(-1)}{b_3} = \frac{y-1}{m_3} = \frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{-1} = \frac{y-1}{-1} = \frac{z}{1} \quad \text{Ans.}$$

*Ex. 3. Determine completely the surface represented by

$$2y^2 - 2xz + 2xy - x - 2y + 3z - 2 = 0.$$

Sol. Here 'a' = 0, 'b' = 2, 'c' = 0, 'f' = -1, 'g' = 1, 'h' = -1, 'i' = -1/2.

'w' = -1, 'w' = 3/2 and 'd' = -2.

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -2 & -7 \\ -1 & 2-\lambda & -1 \\ 4 & -7 & 5-\lambda \end{vmatrix} = 0 \quad \text{... (i)}$$

$$\begin{aligned} *Ex. 4 (a). \text{ Prove that the equation } 5x^2 - 4y^2 + 5z^2 + 4yz - 14xz + 4xy \\ + 16x + 16y + 32z + 8 = 0 \text{ represents a pair of planes which pass through the} \end{aligned}$$

line $x+2=y=1=z$ and are inclined at an angle $2 \tan^{-1}(1/\sqrt{2})$.

Solution. Here 'a' = 5, 'b' = -4, 'c' = 5, 'f' = 2, 'g' = -7, 'h' = 2, 'i' = 8, 'w' = 8, 'w' = -16 and 'd' = 8.

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -2 & -7 \\ -1 & 2-\lambda & -1 \\ 4 & -7 & 5-\lambda \end{vmatrix} = 0 \quad \text{... (ii)}$$

$$\begin{aligned} \text{From these on solving we get } b_3 = m_3 \pm n_3 = 1/\sqrt{3} \\ \text{Further the line of centres is given by any two of } \end{aligned}$$

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \quad \text{(Note)}$$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 2z - 2y - 1 = 0$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 4y - 2z - 2x - 2 = 0 \quad \text{or} \quad x - 2y + z + 1 = 0;$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow -2y + 2x + 3 = 0 \quad \text{or} \quad 2x - 2y + 3 = 0$$

Let (α, β, γ) be any point on the line of centres. Choosing $y = 0, \beta = -1/2, \alpha = -2$ we find that $(-2, -1/2, 0)$ is a point on the line of centres.

$$\text{Now } k = ux + v\beta + w\gamma + d = (-\frac{1}{2})(-2) + (-1)(-\frac{1}{2}) + (\frac{3}{2})(0) - 2 = -\frac{1}{2} \neq 0$$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, i.e., $3x^2 - y^2 - (1/2) = 0$

which represents a hyperboloid cylinder as λ_1, λ_2 are of different signs.

Also the equations of the axis of the cylinder are

$$\frac{x+(-2)}{b_3} = \frac{y-(-1/2)}{m_3} = \frac{z-0}{n_3} \quad \text{or} \quad \frac{x+2}{-1} = \frac{y+(1/2)}{-1} = \frac{z}{1} \quad \text{Ans.}$$

*Ex. 4 (a). Prove that the equation $5x^2 - 4y^2 + 5z^2 + 4yz - 14xz + 4xy$

+ 16x + 16y + 32z + 8 = 0 represents a pair of planes which pass through the

line $x+2=y=1=z$ and are inclined at an angle $2 \tan^{-1}(1/\sqrt{2})$.

Solution. Here 'a' = 5, 'b' = -4, 'c' = 5, 'f' = 2, 'g' = -7, 'h' = 2, 'i' = 8, 'w' = 8.

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -2 & -7 \\ -1 & 2-\lambda & -1 \\ 4 & -7 & 5-\lambda \end{vmatrix} = 0 \quad \text{... (ii)}$$

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$$\text{or } \lambda(\lambda^2 - 6\lambda - 72) = 0 \quad \text{or } \lambda(\lambda + 6)(\lambda - 12) = 0 \quad \text{or } \lambda = 0, -6, 12.$$

Let $\lambda_1 = 12, \lambda_2 = -6, \lambda_3 = 0$. Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with n_1, n_2, n_3 , we have

$$5n_1 + 2n_2 - 7n_3 = 0, \quad 2n_1 - 4n_2 + 2n_3 = 0, \quad 7n_1 + 2n_2 + 5n_3 = 0.$$

Solving first two equations simultaneously, we get

$$\frac{n_1}{4-28} = \frac{-14-10}{-20-4} = \frac{-7}{-20} \quad \text{or} \quad \frac{n_1}{4} = \frac{n_2}{-1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \quad (\text{Note}) \dots (\text{ii})$$

Further the line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0.$$

$$\text{Now } \frac{\partial F}{\partial x} = 0 \Rightarrow 10x - 12z + 4y + 16 = 0 \quad \text{or} \quad 5x + 2y - 7z + 8 = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow -8y + 4z + 4x + 16 = 0 \quad \text{or} \quad x - 2y + z + 4 = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 10z + 4y - 14z - 32 = 0 \quad \text{or} \quad 7x - 2y - 5z + 16 = 0.$$

Choosing $y = 0, z = 0$, we find that $(-2, 1, 0)$ is a point on the line of the centres.

Now $k = ux + vy + wz + d = 8(-2) + 8(1) - 16(0) + 8 = 0$.

Hence the reduced equation of the given surface is $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$ or $12x^2 - 6y^2 + 0 = 0$ or $2x^2 - y^2 = 0$... (iii)

which represents a pair of planes whose line of section is the line through $(-2, 1, 0)$ and direction ratios from (ii) are $1, 1, 1$.

\therefore The equations of this line through which the two planes given by (ii) pass are

$$\frac{x - (-2)}{1} = \frac{y - 1}{1} = \frac{z - 0}{1} \quad \text{or} \quad x + 2 = y - 1 = z.$$

Again the planes represented by (iii) are $y^2 = 2x^2$.

i.e., $y = \pm x\sqrt{2}$ and $y = -x\sqrt{2}$ i.e., $x\sqrt{2} - y = 0$ and $x\sqrt{2} + y = 0$

\therefore The direction ratios of their normals are $\sqrt{2}, -1, 0$ and $\sqrt{2}, 1, 0$.

\therefore If θ be the angle between these planes, then

$$\cos \theta = \frac{|(\sqrt{2})^2 + (-1)^2 + 0^2|}{\sqrt{[(\sqrt{2})^2 + (-1)^2 + 0^2] \cdot [(1)^2 + 0^2]}} = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1}{3} \quad \text{or} \quad 3 - 3 \tan^2 \frac{\theta}{2} = 1 + \tan^2 \frac{\theta}{2} \quad \text{or} \quad 2 \tan^2 \frac{\theta}{2} = 1$$

$$\text{or} \quad \tan(\theta/2) = 1/\sqrt{2} \quad \text{or} \quad \theta = 2 \tan^{-1}(1/\sqrt{2}). \quad \text{Hence proved.}$$

In Ex. 4 (a), the two planes pass through the line $x + 3 = y = z + 1$ and the angle between them is $\tan^{-1}(2/\sqrt{2})$.

Reduction of General Equation of Second Degree

Hint. Proceed exactly as in Ex. 4 (a) above.

Here prove that if (α, β, γ) be any point on the line of centres then choosing $\beta = 0, \gamma = -1, \alpha = -3$, we find that $(-3, 0, -1)$ is a point on the line of centres.

\therefore The equations of the line through which the planes given by (iii) of

Ex. 4 (a) above pass, are $\frac{x - (-2)}{1} = \frac{y - 0}{1} = \frac{z - (-1)}{1}$ or $x + 3 = y = z + 1$

Also in Ex. 4 (a) above $\cos \theta = 1/3$ i.e., $\sec \theta = 3$

$$\Rightarrow \tan^2 \theta + 1 = \sec^2 \theta = 9 \Rightarrow \tan^2 \theta = 8 \Rightarrow \tan \theta = 2\sqrt{2}$$

$\Rightarrow \theta = \tan^{-1}(2\sqrt{2})$.

Hence proved.

Exercises on § 12.12. (Case III)

Ex. 1. Reduce $2x^2 + 5y^2 + 2z^2 - 2yz + 4xz - 2xy + 14x - 16y + 14z + 26 = 0$ to the standard form. What does it represent?

Ans. $2x^2 + y^2 = 1$, elliptic cylinder whose axis is $\frac{x+3}{-1} = \frac{y-1}{0} = \frac{z-1}{1}$

Ex. 2. Reduce $x^2 - y^2 + 4yz + 4xz - 6x - 2y - 8z + 5 = 0$ to the standard form. What does it represent?

Ans. Hyperbolic cylinder $x^2 - y^2 = 1$, axis is $\frac{x-1}{-2} = \frac{y-1}{0} = \frac{z-1}{1}$

Ex. 3. Find the condition that the homogeneous equation of second degree in x, y, z represent a pair of planes.

§ 12.13. Case IV. A, B, C, F, G, H are all zero; f, u, v, w are not zero.

In this case there is a line of centres at infinity and the two roots of discriminating cubic are zero, say λ_2 and λ_3 . Also third root $\lambda_1 \neq 0$.

If the axes through the same origin is so rotated that they are parallel to a set of three mutually perpendicular principal directions then the transformed equation is $\lambda_1 x^2 + 2x(u_1 + vw_1) + 2y(w_1 + vw_2 + vw_3) + 2z(u_1 + vw_1 + vw_2 + vw_3) + d = 0$... (i)

The direction cosines l_1, m_2, n_2 and l_3, m_3, n_3 corresponding to zero roots λ_2 and λ_3 satisfy the equation $al_1 + bm_2 + cn_3 = 0$... (ii)

Choose l_2, m_2, n_2 such that $al_2 + bm_2 + cn_2 = 0$... (iii)

Then from (i), (iii) we have $\lambda_1 x^2 + 2px + 2rz + d = 0$, where $p = u_1 + vw_1 + vw_2$, $r = u_1 + vw_1 + vw_3 + vw_2$... (iv)

From (iv), $\lambda_1 \left(x^2 + \frac{2p}{\lambda_1} x + \frac{p^2}{\lambda_1^2} \right) + 2rz + \left(d - \frac{p^2}{\lambda_1} \right) = 0$... (v)

or $\lambda_1 \left(x + \frac{p}{\lambda_1} \right)^2 + 2r \left[z + \frac{1}{2r} \left(d - \frac{p^2}{\lambda_1} \right) \right] = 0$... (vi)

(Note)

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Reduction of General Equation of Second Degree

Shifting the origin to the point $\left[-\frac{P}{\lambda_1}, 0, -\frac{1}{2r}\left(d-\frac{P^2}{\lambda_1}\right)\right]$

the equation (vi) transforms to $\lambda_1 x^2 + 2rz = 0$ or $x^2 + (2z/\lambda_1)z = 0$,
which is the required reduced form and represents a parabolic cylinder.

The law's rectum of a normal section is

$$2/r\lambda_1 \text{ i.e. } (2/\lambda_1)(u(r \pm v m_3 + w n_3)) \text{ from (v).}$$

Alternative method.

$\therefore A = 0 = B = C$, so we have $bc - f^2 = 0$, $ca - g^2 = 0$, $ab - h^2 = 0$.
These imply that a, b, c have the same sign, say positive.

Also, $F = 0$, $G = 0$, $H = 0$ give $gh - af = 0$, $fg - bg = 0$, $fg - ch = 0$
and so either f, g, h are all positive or two negative and one positive.
(Note)

$$\therefore f(x, y, z) = ax^2 + by^2 + cz^2 + 2px + 2qy + 2rz$$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $f \neq g \neq h$, then we proceed as follows :—
General equation $F(x, y, z) = 0$ can be rewritten as

$$(ax + \sqrt{by} + \sqrt{cz} + \lambda)^2 = 2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v)$$

Now choose λ in such a way that the planes $\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda = 0$ and
 $2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda^2 - d) = 0$ are at right angles
so $\sqrt{a}(\lambda \sqrt{a} - u) + \sqrt{b}(\lambda \sqrt{b} - v) + \sqrt{c}(\lambda \sqrt{c} - w) = 0$,
or $\lambda(a + b + c) = u\sqrt{a} + v\sqrt{b} + w\sqrt{c}$
 $\lambda = (u\sqrt{a} + v\sqrt{b} + w\sqrt{c})/(a + b + c)$

The equation (I) with the help of (II) can be rewritten as

$$\frac{\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda}{\sqrt{(a + b + c)}} = \frac{2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda^2 - d)}{2\sqrt{(a \sqrt{a} - u)^2 + (b \sqrt{b} - v)^2 + (c \sqrt{c} - w)^2}}$$

where $k = 2\sqrt{(a \sqrt{a} - u)^2 + (b \sqrt{b} - v)^2 + (c \sqrt{c} - w)^2}/(a + b + c)$.

The above equation takes the form $y^2 = k^2$,
where $y = (\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda)/(a + b + c)$
and

$$y = \frac{2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda^2 - d)}{2\sqrt{(a \sqrt{a} - u)^2 + (b \sqrt{b} - v)^2 + (c \sqrt{c} - w)^2}}$$

This represents a parabolic cylinder.

In this case there is a plane of centres and two roots λ_2, λ_3 (say) of the

discriminating cubic are zero.

If l_1, m_1, n_1 be the principal direction cosines corresponding to the non-zero root λ_1 of the discriminating cubic, then

$$\frac{al_1 + bm_1 + cn_1}{\lambda_1} = \frac{hl_1 + blm_1 + fln_1 + glm_1 + fml_1 + cn_1}{\lambda_1} \quad \dots(i)$$

But $f^2 = bc$, $g^2 = ca$ and $h^2 = ab$, so

$$al_1 + bm_1 + cn_1 = al_1 + \sqrt{(abc)m_1 + \sqrt{(cab)m_1 + (bac)n_1}}$$

Similarly $hl_1 + blm_1 + fln_1 = \lambda_1 [\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$

$$glm_1 + fml_1 + cn_1 = \lambda_1 [cl_1 + blm_1 + cn_1]$$

\therefore From (i) we have

$$\Rightarrow (bc)l_1 = \sqrt{(ca)v} = \sqrt{(ab)m_1} \quad \dots(ii)$$

Also here $fu = gv = hw$

$$\Rightarrow (bc)l_1 = \sqrt{(ca)v} = \sqrt{(ab)m_1} \quad \dots(iii)$$

From (ii), $w/l_1 = v/m_1 = w/n_1$.

Now if l_2, m_2, n_2 and l_3, m_3, n_3 be the principal direction cosines

corresponding to zero roots λ_2 and λ_3 , then

$$4l_2^2 + 4m_2^2 + 4n_2^2 = l_2^2 + m_2^2 + n_2^2 + 4l_3^2 + m_3^2 + n_3^2 = 0$$

Now as in § 12.13, Page 23 Ch. XII rotating the axes we find that the

transformed equations is $\lambda_1 x^2 + 2x(wl_1 + vnl_1 + um_1) + d = 0$ (Note)

$$\text{or } \lambda_1 x^2 + 2px + d = 0 \text{ where } p = wl_1 + vnl_1 + um_1$$

$$\text{or } \lambda_1 \left(\frac{x+P}{\lambda_1}\right)^2 + \left(d - \frac{P^2}{\lambda_1}\right) = 0 \quad \text{or } \lambda_1 x^2 + k = 0,$$

changing the origin to $(-p/\lambda_1, 0, 0)$ and where $k = d - (p^2/\lambda_1)$.

This equation represents a pair of planes which are identical or parallel according as $k = 0$ or $k \neq 0$.

Alternative method.

A, B in the alternative method given in § 12.13 on Page 23, If A, B, C, F, G and H are zero, we can prove that $f(x, y, z) = \{(\lambda x \pm \sqrt{by} \pm \sqrt{cz})^2\}$,

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu = gv = hw$, then as above we can get

Also the general equation $F(x, y, z) = 0$ in this case can be written as

$$(ax + \sqrt{by} + \sqrt{cz})^2 - 2(ax + by + cz) + d = 0 \quad \text{(Note)}$$

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$$\begin{aligned} \text{or } & (\sqrt{ax + by + cz})^2 + 2\mu(\sqrt{ax + by + cz}) + d = 0, \text{ from (iv)} \\ \text{or } & \sqrt{ax + by + cz} = \pm \sqrt{(1^2 - \lambda)}, \text{ solving as a quadratic equation in} \end{aligned}$$

This represents a pair of parallel planes.

Solved Examples on § 12.13 — § 12.14 (Chise IV and V).

*Ex. 1. Reduce the equation $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz + x - 4y - 2 + 1 = 0$ to the standard form and find the latus rectum of the principal section.

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x - y + z)^2 = -x + 4y - z - 1$

or $(x - y + z + \lambda)^2 = (2\lambda - 1)x - 2(\lambda - 2)yz + (2\lambda - 1)z + (\lambda^2 - 1)$ adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on the R.H.S.

Now choose λ in such a way that the planes $x - y + z + \lambda = 0$ and $(2\lambda - 1)x - 2(\lambda - 2)yz + (2\lambda - 1)z + (\lambda^2 - 1) = 0$ are at right angles,

Then 1. $(2\lambda - 1) + (-1) - 2(\lambda - 2) + 1; (2\lambda - 1) = 0 \Rightarrow \lambda = 1$

∴ From (i), the given equation of the surface can be rewritten as,

$$(x - y + z + 1)^2 = x + 2y + z;$$

$$\text{or } \lambda \left[\frac{x - y + z + 1}{\sqrt{3}} \right]^2 = \sqrt{6} \left[\frac{x + 2y + z}{\sqrt{6}} \right]^2$$

(Note)

or $3x^2 = \sqrt{6}y$ or $x^2 = (1/3)y^2$, which represents a parabolic cylinder Ans.

**Ex. 2. Show that the equation $x^2 + 4y^2 + 9z^2 + 12xy + 6xz + 4xy - 52x - 52y + 62z + 113 = 0$ represents a parabolic cylinder, and that the foci of the normal parabolic section lie on the line

$$x + 2y + 3z + 1 = 0 = x + y - z - 5.$$

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x + 2y + 3z)^2 = 54x + 52y - 62z - 113$

or $(x + 2y + 3z + \lambda)^2 = 2(\lambda + 27)x + 4(\lambda + 13)y + 2(\lambda^2 - 113)$, adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on R.H.S.

Now choose λ in such a way that the planes $x + 2y + 3z + \lambda = 0$ and $2(\lambda + 27)x + 4(\lambda + 13)y + 2(\lambda^2 - 113) = 0$ are at right angles.

Then 1. $[2(\lambda + 27)] + 2[4(\lambda + 13)] + 3(2(\lambda^2 - 113)) = 0$ or $2\lambda + 54 + 8\lambda + 104 + 18\lambda - 186 = 0$ or $\lambda = 1$.

∴ From (i), the given equation of the surface reduces to

Reduction of General Equation of Second Degree

$$(x + 2y + 3z + 1)^2 = 56x + 56y - 56z - 112$$

$$(x + 2y + 3z + 1)^2 = 56(x + y - z - 2)$$

$$\text{or } \frac{14}{\sqrt{(1^2 + 2^2 + 3^2)}} \left[\frac{x + 2y + 3z + 1}{\sqrt{(1^2 + 2^2 + 3^2)}} \right]^2 = 56/\sqrt{14} \left[\frac{x + y - z - 2}{\sqrt{(1^2 + 1^2 + (-1)^2)}} \right]^2$$

$$\text{or } y^2 = 4/3x,$$

which represents a parabolic cylinder and the latus rectum of the normal parabolic section is $4\sqrt{3}$.

[Note : The vertex of the parabolic cylinder lie on the line of intersection of the planes $x + 2y + 3z + 1 = 0$, $x + y - z - 2 = 0$, the latter being a tangent plane which touches the cylinder along the vertices.]

The foci evidently lie on the line of intersection of the plane $x + 2y + 3z + 1 = 0$ i.e. the plane through the axis and a plane parallel to the tangent plane $x + y - z - 2 = 0$ but at a distance $(1/4)$ of latus rectum ($i.e. \sqrt{3}$) from it.

Now any plane parallel to the tangent plane $x + y - z - 2 = 0$ and it should be at a distance $\sqrt{3}$ from the tangent plane.

Now any point on the tangent plane is $(2, 0, 0)$, putting $y = 0, z = 0$ in $x + y - z - 2 = 0$.

∴ distance of the plane $x + y - z + k = 0$ from $(2, 0, 0)$ must be $\sqrt{3}$.

$$\text{i.e. } \frac{2 + 0 - 0 + k}{\sqrt{[1^2 + 1^2 + (-1)^2]}} = \sqrt{3} \quad \text{or } 2 + k = 3 \quad \text{or } k = 1.$$

∴ Foci lie on the line of intersection of the planes

$$x + 2y + 3z + 1 = 0 \text{ and } x + y - z + 1 = 0, \text{ from (iii).}$$

Hence proved.

**Ex. 4. Show that the equation $4x^2 + 9y^2 + 36z^2 - 36yz + 24xz - 12xy - 10x + 15y - 30z + 6 = 0$ represents a pair of parallel planes and find the reduced equation.

Solution. As the second degree terms of the given equation form a perfect square, so it can be rewritten as

$$(2x - 3y + 6z)^2 = 10x - 15y + 30z - 6 = 5(2x - 3y + 6z) - 6 \quad \text{... (i)}$$

$$\text{or } (2x - 3y + 6z)^2 - 5(2x - 3y + 6z) + 6 = 0$$

$$\text{or } x^2 - 5x + 6 = 0, \text{ where } x = 2x - 3y + 6z$$

$$\text{or } (x - 2)(x - 3) = 0 \quad \text{or } x = 2, x = 3$$

Hence the given equation represents a pair of parallel planes given by (i), Also from (i) we have

$$\frac{49}{\sqrt{(2^2 + 3^2 + 6^2)}} \left[\frac{2x - 3y + 6z}{\sqrt{(2^2 + 3^2 + 6^2)}} \right]^2 = 7 \left[\frac{5(2x - 3y + 6z)}{\sqrt{(2^2 + 3^2 + 6^2)}} \right] - 6 \quad \text{... (ii)}$$

Now choose $2x - 3y + 6z = 0$ as $x = 2, y = 3, z = 6$. If (x, y, z) be the coordinates of any point, then $x = \frac{2x - 3y + 6z}{\sqrt{(2^2 + 3^2 + 6^2)}}$

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Then (iii) reduces to $49x^2 - 35x + 6 = 0$, which is the required reduced equation.

Exercises on § 12.13 — § 12.14 (Cases IV—V).

Ans.

**Ex. 1. Reduce the equation $3ax^2 + 4y^2 + z^2 - 4yz - 12xz + 24xy + 4x + 16y - 26z - 3 = 0$ to the standard form. Show that it represents a parabolic cylinder and find the latus rectum of a normal section. Also show that the foci of the normal parabolic sections lie on the line $6x + 2y - z + 1 = 0 = 2x - 3y + 6z + (9/4)$.

Ans. $41y^2 = 28x$; latus rectum = $28/41$

Ex. 2. Reduce the equation $9x^2 + 4y^2 + 4z^2 + 8yz + 12xz - 12xy + 4x + y + 10z + 1 = 0$.

Ans. $17y^2 = 7x$; a parabolic cylinder

Ex. 3. What surface is represented by the equation

$$x^2 + 4y^2 + z^2 + 2xz - 4yz - 2x + 4y - 22 - 3 = 0?$$

Reduce it to the standard form.

Ans. A pair of parallel planes, $6x^2 - 2\sqrt{6}x - 3 = 0$

Ex. 4. Show that $(3x - 4y + 2)^2 + 9x - 12y + 3z - 10 = 0$ represents a pair of parallel planes. Also reduce it to the standard form $26x^2 - 3\sqrt{26}x - 10 = 0$.

§ 12.15. Conicoids of revolution.

Here two cases arise viz.

(i) Two roots of the discriminating cubic are equal and third root not equal to zero.

(ii) Two roots of the discriminating cubic are equal and third root equal to zero.

Under (i) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz^2 = 1. \quad (\text{Ellipsoid of revolution})$$

and

$$\Lambda(x^2 - y^2) + Bz^2 = 1. \quad (\text{Hyperboloid of revolution})$$

Under (ii) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz = 0. \quad (\text{Paraboloid of revolution})$$

and

$$A(x^2 + y^2) + D = 0. \quad (\text{Right circular cylinder})$$

We conclude that if the two roots of the discriminating cubic are equal, then surface $F(x, y, z) = 0$ represents a surface (or conicoid) of revolution.

Here we proceed in the usual way and the direction ratios of the axis of rotation are obtained from the usual equations by taking that value of λ which is different from the equal values.

Solved Examples on § 12.15.

**Ex. 1. Show that the equation $x^2 + y^2 + z^2 + yz + zx + xy + 3x + y + 4z + 4 = 0$ represents a surface of revolution and determine the equations of its axis of rotation.

Reduction of General Equation of Second Degree

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 1/2 & 1/2 \\ 1/2 & 1-\lambda & 1/2 \\ 1/2 & 1/2 & 1-\lambda \end{vmatrix} = 0 \quad \dots(1)$$

$$\text{or} \quad (1-\lambda)(1-\lambda)^2 - (1/4) - (1/2)[(1/2)(1-\lambda) - (1/4)] + (1/2)[(1/4) - (1/2)(1-\lambda)] = 0$$

$$\text{or} \quad (1-\lambda)^3 - (3/4)(1-\lambda) + (1/4) = 0$$

$$\text{or} \quad 4(1-\lambda)^3 - 3(1-\lambda) + 1 = 0 \quad \text{or} \quad 4\lambda^3 - 12\lambda^2 + 9\lambda - 2 = 0$$

$$\text{or} \quad (\lambda - 2)(2\lambda - 1)^2 = 0 \quad \text{or} \quad \lambda = 2, 1/2, 1/2.$$

We observe that two roots of discriminating cubic are equal and the third is different from zero so the given equation represents a surface of revolution [either ellipsoid or hyperboloid of revolution]. See § 12.15 (i), Page 30 Ch. XII.

The central planes are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

$$\text{i.e. } 2x + y + z + 3 = 0, x + 2y + z + 1 = 0, y + 2z + 4 = 0.$$

$$\text{Solving these we get } x = -1, y = 1, z = -2.$$

Centre of the given surface is $(-1, 1, -2)$.

$$\therefore d' = ux + vy + wz + d = (3/2)(-1) + (1/2)(1) + (2)(-2) + 4 = -1$$

$$\therefore \text{The reduced equation is } \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

$$\text{or} \quad (1/2)x^2 + (1/2)y^2 + 2z^2 - 1 = 0, \quad \text{or} \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{1/2} = 1$$

which is an ellipsoid (of revolution), the squares of whose semiaxes are $2, 2, 1/2$.

Now putting $\lambda = 2$ in the determinant given by (i) and associating each row with l, m, n , the direction cosines of the principal axis (or axis of revolution), we have

$$-1 + (1/2)m + (1/2)n = 0, \quad (1/2)l + (1/2)m - n = 0,$$

$$\text{i.e. } -2l + m + n = 0, \quad l - 2m + 1 = 0, \quad l + m - 2n = 0. \quad (1/2)l + (1/2)m - n = 0$$

and these give $l = m = n = 1/\sqrt{3}$. $\sqrt{l^2 + m^2 + n^2} = 1$.

Now the required axis of rotation (or principal axis) is a line through the centre $(-1, 1, -2)$ of the surface of revolution and direction cosines $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ or direction ratios $1, 1, 1$.

The required equations of the axis of rotation are

$$\frac{x - (-1)}{1} = \frac{y - 1}{1} = \frac{z - (-2)}{1} \quad \text{or} \quad x + 1 = y - 1 = z + 2. \quad \text{Ans.}$$

*Ex. 2. Reduce to standard form the equation
 $7x^2 + y^2 + z^2 + 16yz + 8zx - 8xy + 2x + 4y - 40z - 14 = 0$

and find the principal axis.

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Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ b & \lambda & f \\ c & f & \lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 7-\lambda & -4 & 4 \\ -4 & 1-\lambda & 8 \\ 4 & 8 & 1-\lambda \end{vmatrix} = 0 \quad \dots(1)$$

or $(7-\lambda)[(1-\lambda)^2 - 64] + 4[-4(1-\lambda) - 32] + 4[-32 - 4(1-\lambda)] = 0$

or $\lambda^3 - 9\lambda^2 - 8[\lambda + 729 = 0]$ i.e. $(\lambda - 9)(\lambda^2 + 81) = 0$

or $(\lambda - 9)(\lambda + 9) = 0$ i.e. $\lambda = 9, -9$

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

i.e. $14x - 8y + 2 = 0, -8x + 2y + 16z + 4 = 0, 8x + 16y + 2z - 40 = 0$

i.e. $7x - 4y + 4z + 1 = 0, -4y + y + 8z + 2 = 0, 4x + 8y + z - 20 = 0$

Solving these we get $x = 1, y = 2, z = 0$.

∴ Centre of the given surface is $(1, 2, 0)$.

N. d' = $\alpha\lambda + \beta\lambda^2 + \gamma\lambda^3 + d = (1)(1) + (2)(2) + (-2)(0) - 14 = -9$

The reduced equation of the given surface is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad 9x^2 + 9y^2 - 9z^2 - 9 = 0$$

or $x^2 + y^2 - z^2 = 1$, which represents a hyperboloid of revolution, the squares of whose semi-axes are $1, 1, 1$.

Now putting $\lambda = -9$ in the determinant given by (1) and associating each row with (l, m, n) , the d.c.'s of the principal axis, we have

$16l - 4m + 4n = 0, -4l + 4m + 8n = 0, 4l + 8m + 10n = 0$

and these gives $\frac{l}{1} = \frac{m}{2} = \frac{n}{-2} = \frac{1}{3}$, $\beta^2 + m^2 + n^2 = 1$.

∴ The equations of the principal axis passing through the centre $(1, 2, 0)$

and d.r.s $1, 2, -2$ are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-0}{-2}.$$

Ex. 3. Show that the equation $x^2 + 2yz = 1$ represents a surface of revolution and find the axis of revolution.

Solution. Given $F(x, y, z) = x^2 + 2yz - 1 = 0$

Here $a^2 = 1, b = c = 0, f = 1, g = 0, m = h = \bar{u} = v = w, \alpha = 1, \beta = 0, \gamma = 0$

∴ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ b & \lambda & f \\ c & f & \lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \quad \dots(1)$$

or $(1-\lambda)(\lambda^2 - 1) = 0 \quad \text{or} \quad \lambda = 1, -1, 0$

∴ The direction ratios of the axis arc given by

$$i.e. l = \frac{1}{2}, m = \frac{1}{2}, n = 0, -l + m - \frac{1}{2}n = 0, -\frac{1}{2}l + m + n = 0$$

$$i.e. l = 1, m = n = 0, -l + 2m - n = 0, -l - m + 2n = 0$$

These give $l = m = n = 1/\sqrt{3}$

Now $k = ml + mn + mn$

$$or k = (-3/2)(1/\sqrt{3}) + (-3/2)(1/\sqrt{3}) + (-9/2)(1/\sqrt{3}) = -3\sqrt{3} = 0.$$

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i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

i.e. $2x = 0, 2z = 0, 2y = 0$ i.e. $x = 0, z = 0, y = 0$

∴ Centre of the given surface is $(0, 0, 0)$.

i.e. $\alpha\lambda + \beta + \gamma\lambda^2 + d = 0 + 0 + 0 - 1 = -1$.

∴ Reduced equation of the given surface is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad x^2 + y^2 - z^2 - 1 = 0$$

or $x^2 + y^2 = z^2 + 1$, which represents a hyperboloid of revolution.

Now putting $\lambda = -1$ in the determinant given by (1) and associating each row with l, m, n , the d.c.'s of the axis of revolution (or principal axis) we have

$$2l = 0, m + n = 0, m + n = Q \Rightarrow \frac{l}{0} = \frac{m}{1} = \frac{n}{-1}$$

∴ The equations of required axis of revolution which passes through the centre $(0, 0, 0)$ and whose d.r.s are $0, 1, -1$ are

$$\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{-1} \quad i.e. x = 0, y = z = 0. \quad \text{Ans.}$$

**Ex. 4. Show that the surface represented by the equation

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - xy - 3x - 6y - 9z + 21 = 0$$

is a paraboloid of revolution, the coordinates of the focus being $(1, 2, 3)$ and the equations to axis are $x = y - 1 = z - 2$. (Avadh 95; Rohlkhand 97, 96, 94)

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b & c \\ b & \lambda & f \\ c & f & \lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$or \quad ((1-\lambda)(1-\lambda)^2 - (\frac{1}{4})) + (\frac{1}{2})(1-\lambda)(-\frac{1}{4}) - (\frac{1}{2})(\frac{1}{4}) + (\frac{1}{2})(1-\lambda) + (\frac{1}{2})(1-\lambda) - (\frac{1}{2})(1-\lambda) = 0$$

$$or \quad (1-\lambda)(8(1-\lambda)^2 - 2) - 2(2(1-\lambda) + 1) = 0 \quad \text{or} \quad 4\lambda^3 - 12\lambda^2 + 9\lambda = 0$$

$$or \quad \lambda(4\lambda^2 - 12\lambda + 9) = 0 \quad \text{or} \quad \lambda(2\lambda - 3)^2 = 0 \quad \text{or} \quad \lambda = 3/2, 3/2, 0$$

so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The direction ratios of the axis arc given by

$$i.e. l = \frac{1}{2}, m = \frac{1}{2}, n = 0, l + m + f/n = 0, b/f + m/n + c/f = 0$$

$$i.e. l = 1, m = n = 0, -l + 2m - n = 0, -l - \frac{1}{2}m + n = 0$$

These give $f = m = n = 1/\sqrt{3}$

Now $k = ml + mn + mn$

$$or k = (-3/2)(1/\sqrt{3}) + (-3/2)(1/\sqrt{3}) + (-9/2)(1/\sqrt{3}) = -3\sqrt{3} = 0.$$

The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

(Note)

or $(1/2)x^2 + (3/2)y^2 + 2(-3\sqrt{3})z = 0$

or $x^2 + y^2 = 4\sqrt{3}z^2$, which represents a paraboloid of revolution.

Also the coordinates of the vertex of the paraboloid are obtained by solving any two of the three equations

$$\left[\frac{\partial F}{\partial x} \right] = \left[\frac{\partial F}{\partial y} \right] = \left[\frac{\partial F}{\partial z} \right]$$

$$\frac{2x}{m} = \frac{2y}{n} = 2k \quad \text{... See § 12.11 (iv) Page 14 Ch. XII}$$

$$\text{or } \frac{2x - 2y - 3}{1/\sqrt{3}} = \frac{2y - 2z - 6}{1/\sqrt{3}} = \frac{2z - 2y + x - 9}{1/\sqrt{3}} = -6\sqrt{3}.$$

$$\text{or } 2x - y - z - 3 = 2y - z - x - 6 = 2z - y - x - 9 = -6$$

$$\text{or } 2x - y - z + 3 = 0, \quad x - 2y + z = 0, \quad x + y - 2z + 3 = 0$$

$$\text{with the equation: } k(x + my + nz) + ux + vy + wz + d = 0$$

$$\text{i.e. } -3\sqrt{3}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right) + \left(-\frac{3}{2}\right)x + \left(-\frac{9}{2}\right)z + 21 = 0$$

$$\text{i.e. } 3x + 4y + 5z - 14 = 0 \quad \text{... (10)}$$

$$\text{Solving } 2x - y - z + 3 = 0, \quad x - 2y + z = 0, \quad 3x + 4y + 5z - 14 = 0,$$

$$\text{we get: } x = 0, y = 1, z = 2.$$

$$\therefore \text{The required vertex is } (0, 1, 2).$$

$$\therefore \text{Equations of the axis are } \frac{x-0}{1} = \frac{y-1}{-1} = \frac{z-2}{-1} \quad \text{Ans.}$$

$$\text{or } x = y - 1 = z - 2.$$

Also the focus will be a point on the axis whose actual direction cosines are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and will be at a distance $(1/\sqrt{2})\sqrt{3} i.e. \sqrt{3}$ from the vertex $(0, 1, 2)$.

Coordinates of the focus are given by

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = \sqrt{3} \quad (\text{Note})$$

$$\text{or } x = 1, y = 2, z = 3$$

$$\therefore \text{The required focus is } (1, 2, 3).$$

*Ex. 5. Show that $13x^2 + 45y^2 + 40z^2 + 12xy + 36xz - 24yz - 49 = 0$ represents a right circular cylinder whose axis is $x/5 = y/2 = z/3$ and radius 1.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & b-\lambda & c-\lambda \\ b-\lambda & c-\lambda & a-\lambda \\ c-\lambda & a-\lambda & b-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 13-\lambda & -12 & 18 \\ -12 & 45-\lambda & 6 \\ 18 & 6 & 40-\lambda \end{vmatrix} = 0$$

$$\text{or } (13-\lambda)(45-\lambda)(40-\lambda) - 36[-12(40+\lambda) - 108] + 12[-12(40+\lambda) - 108]$$

$$\text{or } \lambda(\lambda-49)^2 = 0 \quad \text{or} \quad \lambda = 0, 49, 49.$$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The d. ratios of the axes are given by $a : b : c = 13 : 45 : 40$, $m : n : l = 5 : 2 : 3$, $g : f : h = 0$.

$al + bm + cn = 0$, $bl + dm + fn = 0$, $cl + bm + gn = 0$.

$|3l - 12m + 18n| = 0 = |12l + 18m + fn| = 0$, $|8l + 6m + 40n| = 0$.

Solving these we get, $\frac{n}{6} = \frac{m}{12} = \frac{l}{3} = \frac{1}{\sqrt{7}}$. (Note)

Nowhere $k = ul + vn + wn = 0$, $u = 0 \Rightarrow v = w$.

Also the line of centres is given by $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

or $26x - 24y + 36z = 0 \Rightarrow 24x + 9y + 12z = 0$, $36x + 12y + 80z = 0$

or $33x - 12y + 18z = 0$, $8x - 3y - 3z = 0$, $42x + 9y + 20z = 0$,

which gives $x = 0, y = 0, z = 0$.

Any point on the line of centres is $(0, 0, 0)$.

Also $d = ux + vy + wz + d = 0 + 0 + 49 = 49$.

The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$.

or $49x^2 + 45y^2 - 49z^2 = 0$ or $x^2 + y^2 - z^2 = 1$.

which is a right circular cylinder of radius 1, as any section of this surface by a plane $z = k$ is a circle $x^2 + y^2 = 1$, whose radius is 1.

And the equations of the axis are

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-0}{1/\sqrt{3}} = \frac{z-0}{1/\sqrt{3}} \quad \text{i.e. } \frac{x}{\sqrt{3}} = \frac{y}{\sqrt{3}} = \frac{z}{\sqrt{3}}$$

**Ex. 6. Prove that the equation $2y^2 + 4xz + 2x - 4y + 6z + 5 = 0$ represents a right circular cone. Show also that the semi-vertical angle of this cone is $\pi/4$ and its axis is given by $x+z+2=0$, $y=1$. (Cartesian 96)

Solution. The discriminating cubic is

$$\begin{vmatrix} a-\lambda & b-\lambda & c-\lambda \\ b-\lambda & c-\lambda & a-\lambda \\ c-\lambda & a-\lambda & b-\lambda \end{vmatrix} = 0 \quad \text{... (i).}$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \text{... (ii).}$$

$$\text{or } (2-\lambda)(2-\lambda)(2-\lambda) - 2(1-\lambda)(2-\lambda) + 2(-2(2-\lambda)) = 0$$

$$\text{or } (2-\lambda)(2-\lambda)(2-\lambda) = 0 \quad \text{i.e. } 2-\lambda = 0 \Rightarrow \lambda = 2, 2, -2.$$

As two roots of this cubic are equal and third is not zero, so the given surface is a surface of revolution.

Also the line of centres is given by any two of $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

$$4z+2 = 0 \Rightarrow z = -1/2, \quad 4x+6 = 0 \Rightarrow x = -3/2, \quad 4y+2 = 0 \Rightarrow y = -1/2.$$

Any point on the line of centres is $(-3/2, -1/2, -1/2)$.

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$$\therefore d' = u\alpha + v\beta + w\gamma + d = 1(-3/2) - 2(1) + 3(-1/2) + 5 = 0.$$

\therefore The reduced form of the equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad 2x^2 + 2y^2 - 2z^2 + 0 = 0$$

or

$$x^2 + y^2 - z^2 = 0 \quad \text{or} \quad x^2 + y^2 = z^2 \tan^2 45^\circ$$

which represents a right circular cone of semi-vertical angle $\pi/4$.

Now putting the unequal values of λ , viz., -2 in the determinant of (i) and associating each row with l, m, n we have $2l + 2n = 0$, $4m = 0$, $2l + 2n = 0$.

These gives

$$l = 0, m = 1, n = -1/\sqrt{2}$$

The equations of its axis are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \frac{x+1}{1} = \frac{z-1}{-1/\sqrt{2}} \quad \text{or} \quad \frac{x+3/2}{1} = \frac{y-1}{0} = \frac{z+1}{1}$$

or

$$x + 3/2 = z + 1/2, \quad y - 1 = 0, \quad x + 2 + 2 = 0, \quad y = 1$$

Hence proved.

Exercises on § 12.15

Ex. 1. Prove that the equation $2x^2 + 5y^2 + 2z^2 - 4x - 8x + 14y + 3 = 0$ is a surface of revolution. Also find the equations of its principal axis;

Ans. Reduced equation is $x^2 + y^2 + 6z^2 = 8$, axis $x + y - 1 = 0 = z$.

Ex. 2. Find the reduced equation of the surface $x^2 - y^2 + 2yz - 2xz - x - y + z = 0$. Also find its axis.

Ans. 3 ($x^2 - y^2$) $\pm z; x - (1/3) = y + (1/3) = z$.
Ans. A hyperboloid of revolution ; reduced equation is

$$2x^2 - y^2 - z^2 = 2a^2, \text{ axis is } x = y = z$$

Exercises on Chapter XII.

Ex. 1. Reduce the surface $40x^2 + 10y^2 - 9z^2 - 8yz - 16zx + 26xy + 4xz + 20y - 28z - 3 = 0$ into the standard form and find the latus rectum of a normal section.

Ex. 2. Reduce the equation $12x^2 + 10y^2 + 8z^2 - 9yz + 2x - 13xy + 75z + 77y - 28z + 100 = 0$ into the standard form and also describe the nature of the surface and find the equations of its axes.

(Ans. 1) $40x^2 + 10y^2 - 9z^2 - 8yz - 16zx + 26xy + 4xz + 20y - 28z - 3 = 0$

(Ans. 2) $12x^2 + 10y^2 + 8z^2 - 9yz + 2x - 13xy + 75z + 77y - 28z + 100 = 0$

