Mains Test Series - 2020

Test-12 (Paper-II) Batch-I

Answer Key

full Syllabus

1(a) Describe all finite abelian groups of order 2^6.

Sol^n: By known theorem of finite abelian groups the following list of abelian groups is a complete list of abelian groups of order 2^6. i.e any abelian group of order 2^6 is isomorphic to exactly one group of the following

Z_{2^6}

$Z_{2^5} \times Z_2$

$Z_{2^4} \times Z_{2^2}$

$Z_{2^4} \times Z_2 \times Z_2$

$Z_{2^3} \times Z_{2^3}$

$Z_{2^3} \times Z_{2^2} \times Z_2$

$Z_{2^3} \times Z_2 \times Z_2 \times Z_2$

$Z_{2^2} \times Z_{2^2} \times Z_{2^2}$

$Z_{2^2} \times Z_{2^2} \times Z_2 \times Z_2$

$Z_{2^2} \times Z_2 \times Z_2 \times Z_2 \times Z_2$

$Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$

1(b) Prove that the polynomial $1+x+\cdots+x^{p-1}$ where p is a prime number, is irreducible over the field of rational numbers.

Sol^n: Consider the polynomial

$$1+(x+1)+(x+1)^2+\cdots+(x+1)^{p-1}$$

First observe that $f(x)=1+x+\cdots+x^{p-1}$ reducible, if and only if $g(x)=1+(x+1)+\cdots+(x+1)^{p-1}=f(x+1)$ is reducible

$$1+(x+1)+\cdots+(x+1)^{p-1}=\frac{(x+1)^p-1}{(x+1)-1}$$

(in the field of fractions of $F[x]$).

$$\frac{(x+1)^p-1}{x}=\frac{x^p+\binom{p}{1}x^{p-1}+\binom{p}{2}x^{p-2}+\cdots+\binom{p}{p-1}x}{x}$$

$$=x^{p-1}+\binom{p}{1}x^{p-2}+\binom{p}{2}x^{p-3}+\cdots+\binom{p}{p-1}$$

$\binom{p}{p-1}=p$. The constant term of $f(x+1)$ is p and is not divisible by p^2, also p does not divide the leading coefficient of this polynomial but divides each of the remaining coefficients.

1(d). The function f defined by

$$f(z) = u+iv = \begin{cases} \frac{\operatorname{Im}(z^2)}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z=0 \end{cases}$$

Satisfies the C-R equations at the origin, yet it is not differentiable there.

Sol'n: The function f defined by

$$f(z) = u+iv = \begin{cases} \frac{\operatorname{Im}(z^2)}{\bar{z}} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$$

$$= \begin{cases} \frac{2xy}{x-iy} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$$

$z^2 = (x+iy)^2$
$= x^2-y^2+2ixy$
$\operatorname{Im}(z^2) = 2xy$

$$= \begin{cases} \frac{2xy(x-iy)}{x^2+y^2} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$$

Here $u = \frac{2x^2y}{x^2+y^2}$; $v = \frac{2xy^2}{x^2+y^2}$ where $x \neq 0, y \neq 0$.

To show that Cauchy - Riemann equations are satisfied at $z=0$:

Since $f(0)=0$

$\Rightarrow u(0,0)+iv(0,0)=0$

$\Rightarrow u(0,0) = v(0,0)=0$

Now $\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{y \to 0}{Lt} \frac{u(x,0)-u(0,0)}{x} = \underset{x \to 0}{Lt} \frac{0-0}{x} = 0$

$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y \to 0}{Lt} \frac{u(0,y)-u(0,0)}{y} = \underset{y \to 0}{Lt} \frac{0-0}{y} = 0$

Similarly $\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = 0$ and $\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = 0$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ at $z=0$

$\therefore$ Cauchy – Riemann equations are satisfied.

To Prove that $f(z)$ doesnot differentiable at $(0,0)$.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{(x,y)\to 0} \frac{2xy\,(x+iy)}{(x^2+y^2)(x+iy)}$$

Let $(x,y) \to (0,0)$ along $y = mx$

$$f'(0) = \lim_{x\to 0} \frac{2x^2 m}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{2m}{1+m^2}$$

Clearly which depends on m.

$\therefore f'(0)$ does not exist.

$\therefore f(z)$ does not differentiable at $(0,0)$

$\therefore$ The given statement is true i.e. the given function f satisfies C-R equations. although it is not differentiable at $(0,0)$

1(e) Make a graphical representation of the set of Constraints of the following LPP. Find the extreme points of the feasible region. Finally, solve the problem graphically.

Max. $Z = 2x_1 + x_2$

Subject to $x_1 + x_2 \geq 5$

$2x_1 + 3x_2 \leq 20$

$4x_1 + 3x_2 \leq 25$

$x_1, x_2 \geq 0$.

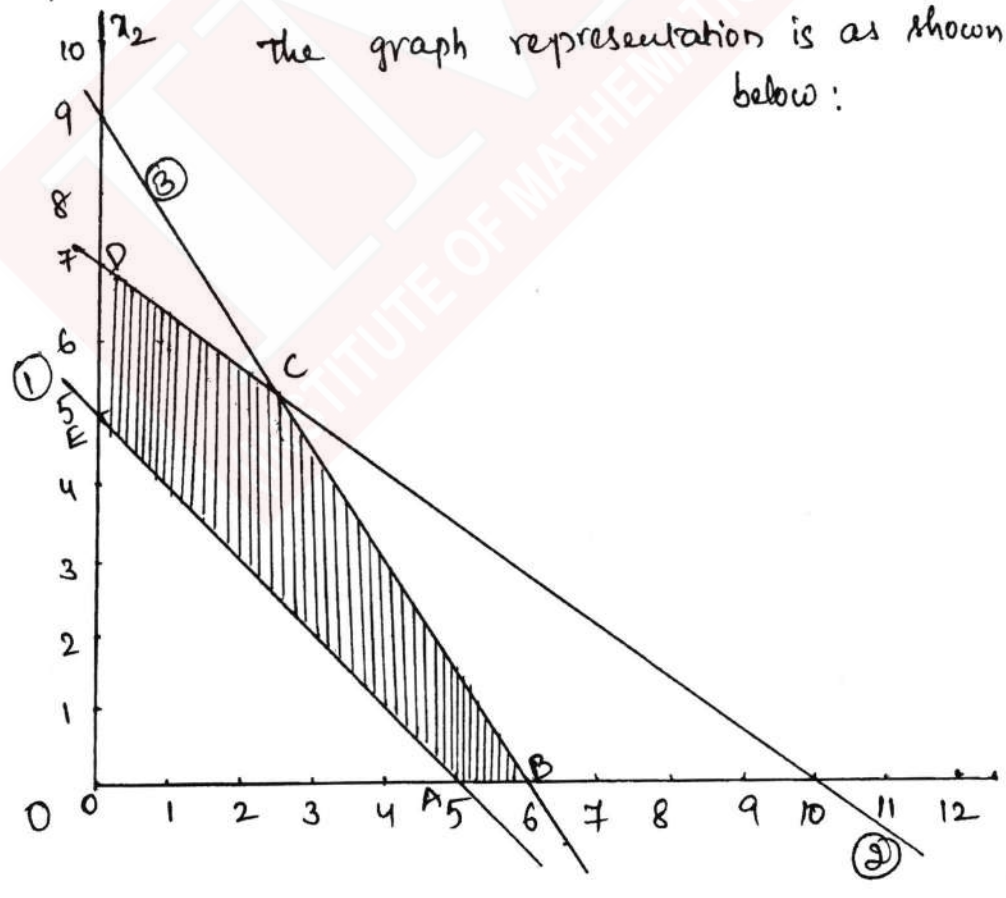
Sol^n: Let us consider the constraint $x_1 + x_2 \geq 5$ as equality $x_1 + x_2 = 5$ represents a line which passes through $(0,5)$ & $(5,0)$ in the XY-plane.

Similarly, we get

$2x_1 + 3x_2 = 20 \Rightarrow (10,0)$ & $(0, \frac{20}{3})$

$4x_1 + 3x_2 = 25 \Rightarrow (\frac{25}{4}, 0)$ & $(0, \frac{25}{3})$

The graph representation is as shown below:



The feasible region is given by ABCDE which satisfy all the constraints simultaneously coordinates of extreme points are
(Corner points)

$A(5,0)$, $B(\frac{25}{4},0)$, $C(5/2,5)$, $D(0,\frac{20}{3})$, $E(0,5)$

To maximize $z = 2x_1 + x_2$

The values of the objective function $z = 2x_1 + x_2$ at these extreme points are

$z(0,5) = 5$

$z(5,0) = 10$

$z(\frac{25}{4},0) = 25/2$

$z(5/2,5) = 10$

$z(0,20/3) = 20/3$

$\therefore$ The Maximum value of z is at the point
$(25/4, 10)$

and the maximum value is
$z = 25/2$

$\therefore$ The optimal solution is
$x_1 = 25/4$, $x_2 = 0$ and $z = 25/2$.

2(a) (i) Find all normal subgroups in S_4.

(ii) Give an example of a group G, subgroup H and an element $a \in G$ such that $aHa^{-1} \subset H$ but $aHa^{-1} \neq H$.

(iii) List all the conjugate classes in S_3 and verify the class equation.

Sol^n: (i) $\{(1)\}$, S_4,

$$V = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

$$A_4 = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,3,4), (2,3,4), (1,2,4), (2,4,3), (1,4,3), (1,3,2), (1,4,2)\}$$

So S_4 has four normal subgroups.

(ii) $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad-bc \neq 0,\ a,b,c,d \in R\right\}$

Let $H = \left\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in Z\right\}$, obviously $H \leq G$ and let

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Then $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2x & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2x & 1 \end{pmatrix}$

$$gHg^{-1} = \left\{\begin{pmatrix} 1 & 0 \\ 2x & 1 \end{pmatrix} \middle| x \in Z\right\}$$

So $gHg^{-1} \leq H$ but $gHg^{-1} \neq H$ as $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \notin gHg^{-1}$ but $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in H$.

(iii) Conjugacy class of (1) is $[1] = \{(1)\}$

Conjugacy Class of $(1,2)$ is $[(1,2)] = \{(1,2),(1,3),(2,3)\}$

Conjugacy class of $(1,2,3)$ is $[(1,2,3)] = \{(1,2,3),(1,3,2)\}$.

Class equation:

$|G| = \sum_{x\in A} |G : C_G(x)| = \sum_{x\in A} |[x]|$ where A is a complete set of representatives of all Conjugacy classes of G

$6 = 1+3+2$

2(b) (i) The series $\sum u_n = 1-\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^3- \cdots$

and $\sum v_n = 1+\left[2+\left(\frac{1}{2}\right)^2\right]+\frac{3}{2}\left[2^2+\left(\frac{1}{2}\right)^3\right]+\cdots$

are divergent but their product series $\sum\left(\sum u_r v_{n-r+1}\right)$ converges absolutely.

(ii) If $f(x)$ be defined on $[a,b]$ such that for $\delta>0$.
$|f(x)-f(y)| \le |x-y|^{1+\delta}$ $\forall$ $x,y\in[a,b]$, then $f(x)$ is constant on $[a,b]$.

Sol'n: (i) Since $-\left(\frac{3}{2}\right)^n < -1$ $\forall$ $n\ge 2$ and $1\le v_n$ $\forall$ n, therefore $\sum u_n$ diverges to $-\infty$ and $\sum v_n$ diverges to ∞. And $\left|\sum_{r=1}^{n} u_r v_{n-r+1}\right|$

$$=\left(\frac{3}{2}\right)^{n-2}\left[2^{n-1}+\left(\frac{1}{2}\right)^n\right]-\left(\frac{3}{2}\right)^{n-2}\left[2^{n-2}+\left(\frac{1}{2}\right)^{n-1}\right]-\cdots-\left(\frac{3}{2}\right)^{n-2}\left[2+\left(\frac{1}{2}\right)^2\right]-\left(\frac{3}{2}\right)^{n-1}$$

$$=\left(\frac{3}{2}\right)^{n-2}\left[2^{n-1}+\left(\frac{1}{2}\right)^n-\left\{2^{n-2}+2^{n-3}+\cdots+2+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-2}}+\cdots+\frac{1}{2^2}\right)+\frac{3}{2}\right\}\right]$$

$$=\left(\frac{3}{2}\right)^{n-2}\left[2^{n-1}+\left(\frac{1}{2}\right)^n-\left(2^{n-1}-2\right)-\left(\frac{1}{2}-\frac{1}{2^{n-1}}\right)-\frac{3}{2}\right]$$

$$=\left(\frac{3}{2}\right)^{n-2}\left[\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{n-1}\right]=\left(\frac{3}{4}\right)^{n-1}$$

Thus $\sum\left(\sum u_r v_{n-r+1}\right)=1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots$ Converges absolutely.

(ii) For $\varepsilon>0$, when $|x-y|<\varepsilon^{1/\delta}$

$$|f(x)-f(y)|\le|x-y|^{1+\delta}$$

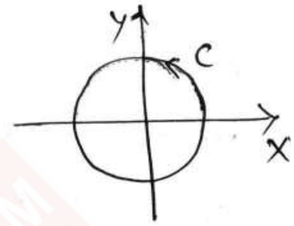
$$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|}<\varepsilon \quad \text{i.e. } f'(x)=0 \ \forall x\in[a,b]$$

Hence, $f(x)$ is constant on $[a,b]$.

Q2(c) Let a, b, c be real with $a^2 > b^2 + c^2$. Show that

$$\int_0^{2\pi} \frac{dt}{a + b\cos t + c\sin t} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}.$$

Sol'n: Let $z = e^{it}$ then



$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2} \; ; \; dz = ie^{it}dt$$

$$\Rightarrow \boxed{dz = iz\,dt}$$

$$\therefore \int_0^{2\pi} \frac{dt}{a + b\cos t + c\sin t} = \oint_C \frac{dz}{iz\left[a + b\left(\frac{z + z^{-1}}{2}\right) + c\left(\frac{z - z^{-1}}{2i}\right)\right]}$$

$$= \oint_C \frac{2dz}{z\left[2ai + bi(z + z^{-1}) + c(z - z^{-1})\right]}$$

$$= \oint_C \frac{2dz}{2azi + z^2 bi + bi + cz^2 - c}$$

$$= \oint_C \frac{2dz}{(c + bi)z^2 + 2azi + (-c + bi)}$$

$$= \oint_C \frac{2dz}{(c + bi)\left[z^2 + \left(\frac{2ai}{c + bi}\right)z + \left(\frac{-c + bi}{c + bi}\right)\right]} \quad \text{—(1)}$$

where C is the circle of unit radius with centre at the origin.

The poles of $\dfrac{2}{(c + bi)\left[z^2 + \left(\frac{2ai}{c + bi}\right)z + \left(\frac{-c + bi}{c + bi}\right)\right]}$

are the simple poles.

$$z = \frac{-\frac{2ai}{c+bi} \pm \sqrt{\frac{-4a^2}{(c+bi)^2} - 4(1)\left(\frac{-c+bi}{c+bi}\right)}}{2(1)}$$

$$= \frac{\frac{-2ai}{c+bi} \pm \frac{2}{c+bi}\sqrt{-a^2-(-c+bi)(c+bi)}}{2}$$

$$= \frac{\frac{-2ai}{c+bi} \pm \frac{2i}{c+bi}\sqrt{a^2-(c^2+b^2)}}{2}$$

$$= \frac{-ai}{c+bi} \pm \frac{i}{c+bi}\sqrt{a^2-(c^2+b^2)}$$

Given that $a^2 > b^2+c^2$

$\therefore z = \dfrac{-ai + i\sqrt{a^2-(b^2+c^2)}}{c+bi} = z_1$ say lies

inside 'c' because $|z|<1$.

Now residue at z_1

$$= \lim_{z\to z_1} (z-z_1) \frac{2}{(c+bi)\left[z^2+\left(\frac{2ai}{z+bi}\right)z + \left(\frac{-c+bi}{c+bi}\right)\right]}$$

$$= \lim_{z\to z_1} \frac{2}{(c+bi)\left[z + \frac{ai+i\sqrt{a^2-(b^2+c^2)}}{c+bi}\right]} = \frac{1}{i\sqrt{a^2-(b^2+c^2)}}$$

$$\therefore ① \equiv \oint_C^{2\pi} \frac{dt}{a+b\sin t+c\cos t} = 2\pi i\,(\text{residue at } z_1)$$

$$= \frac{2\pi i}{\sqrt{a^2-(b^2+c^2)}}$$

3(a) (i) Let R be a ring with unit element, R not necessarily commutative, such that the only right ideals of R are (0) and R. Prove that R is a division ring.

(ii) Prove that any homomorphism of a field is either a monomorphism or takes each element into 0.

Sol'n: (i) Now consider aR for any $a \neq 0$.

$aR = \{ar \mid r \in R\}$ is a right ideal of R.

Indeed if ar_1 and ar_2 be two elements in aR,

then $ar_1 - ar_2 = a(r_1 - r_2) \in aR$.

moreover for any $r \in R$ and $ax \in aR$

$(ax)r = axr \in aR$.

Hence aR is a right ideal and

$aR \neq \{0\}$ as $a \in aR$.

Hence $aR = R$, this means that there exists $y \in R$ such that $ay = 1$.

But then for every $a \neq 0$ $\exists$ $y \in R$ such that $ay = 1$.

Now consider $ya = y(ay)a$.

so $ya = (ya)(ya)$
$= (ya)^2$.

Since there exists $t \in R$ such that

$(ya)t = 1$

we get $yat = (ya)^2 t$

$\Rightarrow 1 = ya$ as required.

3(a)(ii) Sol'n: Let F be a field and α be a homomorphism of F to F.

Assume that α is a non-zero homomorphism.

If $\alpha(a)=0$ for some $a\in F$.

then $\alpha(1)=\alpha(aa^{-1})$
$$= \alpha(a)\,\alpha(a^{-1})$$
$$= 0$$

Then for any $b\in F$,

$$\alpha(b)=\alpha(1)\alpha(b)=0$$

but this is impossible as α is a non-zero map.

Therefore α is a one to one homomorphism.

3(b) (i) If $f(x) = \sqrt{1-x^2}$ when x is rational
$= 1-x$ when x is irrational

then $\overline{\int_0^1} f(x)\,dx = \frac{\pi}{4}$, and $\underline{\int_0^1} f(x)\,dx = \frac{1}{2}$.

(ii) The sequence $\alpha_n \equiv 1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n, \forall n$ is monotonically decreasing and bounded between 0, 1 and converges to a non-zero limit between 0 and 1.

Sol^n: (i) Since $1-x \le \sqrt{1-x^2}\ \forall x \in [0,1]$

$$\therefore \overline{\int_0^1} f(x)\,dx = \inf_P \sum \sqrt{1-x_r^2}\,\delta_r$$

$$= \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}, \text{ by usual evaluation.}$$

$$\underline{\int_0^1} f(x)\,dx = \sup_P \sum (1-x_{r-1})\,\delta_r$$

$$= \int_0^1 (1-x)\,dx = \frac{1}{2}, \text{ by usual evaluation.}$$

(ii) Here $\sum_{r=1}^{n-1} \int_r^{r+1} \frac{1}{r+1}\,dx < \sum_{r=1}^{n-1} \int_r^{r+1} \frac{1}{x}\,dx < \sum_{r=1}^{n-1} \int_r^{r+1} \frac{1}{r}\,dx$

$$\Rightarrow \sum_{r=1}^{n-1} \frac{1}{r+1} < \int_1^n \frac{1}{x}\,dx < \sum_{r=1}^{n-1} \frac{1}{r}$$

$$\Rightarrow \alpha_n - 1 < \log n - \log n < \alpha_n - \frac{1}{n}$$

$$\Rightarrow \frac{1}{n} < \alpha_n < 1 \ \forall n \ge 2$$

and since,

$$\alpha_n - \alpha_{n+1} = \log\frac{n+1}{n} - \frac{1}{n+1}$$

$$= \int_n^{n+1} \frac{1}{x}\,dx - \int_n^{n+1} \frac{1}{n+1}\,dx > 0,$$

$$\therefore 0 < 1/(n+1) < \alpha_{n+1} < \alpha_n < 1 \ \forall n \ge 2.$$

Hence, α_n is a monotonically decreasing sequence bounded between 0, 1 and converges to a non-zero limit between 0 and 1, for

$$0 < u_n \equiv \int_0^1 \frac{t}{n(n+t)}\,dt = \frac{1}{n} - \log\frac{n+1}{n} < \frac{1}{n^2}$$

implies that Σu_n is convergent and

$$1 > \alpha_i > \lim \alpha_n = \lim\left\{1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \log n\right\}$$

$$= \lim\left\{\sum_{m=1}^{n} u_m + \log\frac{n+1}{n}\right\}$$

$$= \sum_{n=1}^{\infty} u_n > 0, \ \forall\ i \geq 2.$$

The limit of the sequence α_n as above is denoted by 'γ' and is known as Euler's constant, its value correct to six decimal places is 0.577216.

3(c)(I) Obtain the dual of the following LP problem Max $z = 2x_1+3x_2+x_3$

subject to $4x_1+3x_2+x_3=6$

$x_1+2x_2+5x_3=4$ and $x_1, x_2, x_3 \geq 0$.

Sol. The equality constraints can be written as

$4x_1+3x_2+x_3 \leq 6$, $4x_1+3x_2+x_3 \geq 6$

and $x_1+2x_2+5x_3 \leq 4$, $x_1+2x_2+5x_3 \geq 4$.

Since the problem is of maximization type, all constraints should be of $\leq$ type.

Now multiply 2nd and 4th constraint through by -1

we may rewrite the formal.

Max $Z = 2x_1+3x_2+x_3$

Subject to

$4x_1+3x_2+x_3 \leq 6$

$-4x_1-3x_2-x_3 \leq -6$

$x_1+2x_2+5x_3 \leq 4$

$-x_1-2x_2-5x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

Let y_1, y_2, y_3, y_4 be the dual variables associated with the above four constraints

Therefore the dual is given by

Min $Z_D = 6y_1-6y_2+4y_3-4y_4$

Subject to

$4y_1-4y_2+y_3-y_4 \geq 2$

$3y_1-3y_2+2y_3-2y_4 \geq 3$

$y_1-y_2+5y_3-5y_4 \geq 1$

$y_1, y_2, y_3, y_4 \geq 0$

Again, the dual can be written as

Minimize $W = 6w_1+4w_2$

Subject to the constraints

$4w_1+w_2 \geq 2$

$3w_1+2w_2 \geq 3$

$w_1+5w_2 \geq 1$ where $w_1 = y_1-y_2$, $w_2 = y_3-y_4$

w_1, w_2 are unrestricted.

3(c)(ii) Solve the problem by Simplex method.

Max $z = 6x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \le 30$

$3x_1 + 2x_2 \le 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Is the solution unique? If not, give two different solutions.

Sol^n: Introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$, surplus variable $s_3 \geq 0$ and artificial variable $a_1 \geq 0$. The problem becomes

Max $z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$ (By using Big-M method)

Subject to the constraints

$2x_1 + 3x_2 + s_1 = 30$

$3x_1 + 2x_2 + s_2 = 24$

$x_1 + x_2 - s_3 + A = 3$

$x_1, x_2, s_1, s_2, s_3, A \geq 0$

Now the IBFS is

$x_1 = x_2 = s_3 = 0$ (non-basic)

$s_1 = 30$, $s_2 = 24$, $A = 3$ (basic) for which $z = -3M$

	$C_j \to$	6	4	0	0	0	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	A	b	θ
0	s_1	2	3	1	0	0	0	30	15
0	s_2	3	2	0	1	0	0	24	8
$-M$	A	1	1	0	0	-1	1	3	3 $\rightarrow$
	$z_j = \Sigma c_B a_{ij}$	$-M$	$-M$	0	0	M	$-M$	$-3M$	
	$C_j - c_j - z_j$	$6+M \uparrow$	$4+M$	0	0	$-M$	0		

from the table, x_1 is the entering variable A is the outgoing variable and (1) is the key element and all other elements in its column equal to zero

Then the revised simplex table is

	C_j	6	4	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	0	1	1	0	2	24	12
0	S_2	0	-1	0	1	(3)	15	5 $\longrightarrow$
6	x_1	1	1	0	0	-1	3	—
$z_j = \Sigma C_B a_{ij}$		6	6	0	0	-6	18	
$C_j = c_j - z_j$		0	-2	0	0	6 $\uparrow$		

from the table S_3 is entering variable, S_2 is the outgoing variable, (3) is the key element and make it unity and all other elements in its column equal to zero

Then the revised simplex table is

	C_j	6	4	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	0	$5/3$	1	$-2/3$	0	14	$42/5$ $\rightarrow$
0	S_3	0	$-1/3$	0	$1/3$	1	5	—
6	x_1	1	$2/3$	0	$1/3$	0	8	12
$z_j = \Sigma C_B a_{ij}$		6	4	0	2	0	48	
$C_j = c_j - z_j$		0	0 $\uparrow$	0	-2	0		

from the above table, all C_j's ≤ 0. There remains no artificial variable in the basis

$\therefore$ The solution is an optimal BFS to the problem and is given by

$x_1 = 8$, $x_2 = 0$ and $\text{Max } Z = 48$

Since net evaluation $(c_j - z_j)$ corresponding to non basic variable x_2 is obtained zero, thus indicates that the alternative solutions also exist.

Therefore, the solution as obtained above is not unique.

Thus we can bring x_2 into the basis in place of S_1.

$\therefore$ Introducing x_2 into the basis in place of S_1 (which satisfies the exit criterion) the new optimum

	c_j	6	4	0	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b
4	x_2	0	1	3/5	$-2/5$	0	42/5
0	S_3	0	0	1/5	1/5	1	39/5
6	x_1	1	0	$-2/5$	3/5	0	12/5
	$z_j = \Sigma C_B a_{ij}$	6	4	0	2	0	
	$C_j = c_j - z_j$	0	0	0	-2	0	

From the above table, the alternative optimum solution is

$$x_1 = 12/5, \quad x_2 = 42/5, \quad \text{Max } z = 48$$

4(a) Find the greatest common divisor of the following polynomial over F, the field of rational numbers.
x^2+x-2 and $x^5-x^4-10x^3+10x^2+9x-9$.

Sol'n: $x^5-x^4-10x^3+10x^2+9x+9$

$$= (x^3-2x^2-6x+12)(x^2+x-2)+(-15x+15)$$

and $x^2+x-2 = (-15x+15)\left(-\frac{1}{15}x-\frac{2}{15}\right)$

Hence $-15x+15$ is a greatest common divisor of $x^5-x^4-10x^3+10x^2+9x-9$ and x^2+x-2. Since any associate of a greatest common divisor is again a greatest common divisor. we may say that $x-1$ is also greatest common divisor of the above polynomials.

4(b) Test for uniform convergence and term by term integration of the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$. Also prove that $\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2} \right) dx = \frac{1}{2}$.

Sol'n: Here $f_n(x) = \frac{x}{(n+x^2)^2}$ ——— (1)

$$\frac{df_n(x)}{dx} = \frac{(n+x^2)^2 \cdot 1 - x \cdot 2(n+x^2) \cdot 2x}{(n+x^2)^4} = \frac{n-3x^2}{(n+x^2)^3} \quad \text{——— (2)}$$

For max. or min,

$$\frac{df_n(x)}{dx} \Rightarrow n - 3x^2 = 0 \Rightarrow x = (n/3)^{1/2}$$

from (2),
$$\frac{d^2 f_n(x)}{dx} = \frac{(n+x^2)^3(-6x) - (n-3x^2) \cdot 3(n+x^2) \cdot 2x}{(n+x^2)^6}$$

$$= \frac{-6x(n+x^2+n-3x^2)}{(n+x^2)^4}$$

So when $x = (n/3)^{1/2}$,

$$\frac{d^2 f_n(x)}{dx^2} = \frac{-6(n/3)^{1/2}(n+n/3)}{(n+n/3)^4} = -\frac{27\sqrt{3}}{32 n^{5/2}} < 0,$$

showing that $f_n(x)$ is maximum when $x = (n/3)^{1/2}$ and from (1),

the maximum value of $f_n(x) = \frac{(n/3)^{1/2}}{(n+n/3)^2} = \frac{3\sqrt{3}}{16 n^{3/2}}$

Hence $|f_n(x)| \leq \frac{3\sqrt{3}}{16 n^{3/2}} < \frac{1}{n^{3/2}} = M_n$ (say)

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, by weistrass M-test, the given series is uniformly convergent for all values of x.

Also, $\forall\, n \in N$, f_n is integrable on $[0,1]$. Hence the series can be integrated term by term.

$$\therefore \int_0^1 \sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}\, dx = \sum_{n=1}^{\infty} \int_0^1 x\,(n+x^2)^{-2}\, dx$$

$$= \sum_{n=1}^{\infty} \left[\frac{(n+x^2)^{-1}}{-2}\right]_0^1$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}.$$

4(c)(1) Let $f(z) = u+iv$ be an analytic function. Find $f(z)$ (as a function of z), when $2u+3v = 13(x^2-y^2)+2x+3y$.

Sol'n: Let $f(z) = u+iv$

we have $2u+3v = 13(x^2-y^2)+2x+3y$

$$2\frac{\partial u}{\partial x}+3\frac{\partial v}{\partial x} = 13(2x)+2 = 26x+2 \quad \text{---(1)}$$

and $$2\frac{\partial u}{\partial y}+3\frac{\partial v}{\partial y} = 13(-2y)+3 = -26y+3$$

$$\Rightarrow 2\left(-\frac{\partial v}{\partial x}\right)+3\left(\frac{\partial u}{\partial x}\right) = -26y+3 \quad \text{---(2)}$$

($\because$ by Cauchy Riemann eqns $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

solving (1) & (2), we get

$$\frac{\partial u}{\partial x} = 4x-6y+1 = \phi_1(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x} = 6x+4y = \phi_2(x,y)$$

Now $f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$= \phi_1(z,0)+i\phi_2(z,0)$ by Milne's method.

$\therefore f(z) = \int[\phi_1(z,0)+i\phi_2(z,0)]dz+C$

$= \int\{(4z+1)+i(6z)\}dz+C$

$= 2z^2+z+i3z^2+C$

$= (2+3i)z^2+z+C.$

4(c)(ii). Classify the nature of the singularity of the function $f(z) = \frac{e^{-z}}{(z-2)^4}$ and compute the residue.

Sol^n.

The singularities of $f(z)$ are given by

$(z-2)^4 = 0 \Rightarrow z = 2$.

and write the Laurent expansion

in $0 < |z-2| < R$,

$$f(z) = \frac{e^{-2}\, e^{-(z-2)}}{(z-2)^4}$$

$$= \frac{e^{-2}}{(z-2)^4}\left[1 - \frac{(z-2)}{1!} + \frac{(z-2)^2}{2!} - \frac{(z-2)^3}{3!} + \cdots\right]$$

$$= e^{-2}\left[\frac{1}{(z-2)^4} - \frac{1}{(z-2)^3} + \frac{1}{2!(z-2)^2} - \frac{1}{3!(z-2)} + \cdots\right]$$

$\therefore$ $z=2$ is a pole of order 4 and the residue at this point is given by the coefficient of $\frac{1}{(z-2)}$ in the expansion.

Therefore, the residue of $f(z)$ at $z=2$ is $-\frac{1}{6e^2}$.

4 (d) → Solve the following transportion problem:

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	12	3	11	2
O_4	6	8	11	2	2	10	9
	4	4	6	2	4	2	22 (Total)

Sol^n: consists of following steps:

Step 1. Transportation Table. The total supply and total demand being equal, the transportation problem is balanced.

Step 2: Find the initial basic feasible solution. Using VAM, the initial basic feasible solution is as shown in table 1.

Step 3: Apply optimality check. Since the number of basic cells is 8 which is less than $m+n-1=9$, the basic solution degenerates. In order to complete the basis and there by remove degeneracy, we require only one more +ve basic variable we select the variable x_{23} and allocate a small the quantity ε to the cell (2,3).

Table 1

9	12	[5] 9	6	9	10	5
7	[4] 3	[ε] 7	7	5	[2] 5	$6+\varepsilon=6$
[1] 6	5	[1] 9	11	3	11	2
[3] 6	8	11	[2] 2	[4] 2	10	9
4	4	$6+\varepsilon=6$	2	4	2 +	

We now compute the net evaluations.

$w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited table 2

Since all the net evaluations are ≤ 0, the current solutions is optimal. Hence the optimal allocation is

$x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{33} = 1$, $x_{41} = 3$,

$x_{44} = 2$ and $x_{45} = 4$.

$\therefore$ The minimum (optimal) transportation cost

$= 5\times 9 + 4\times 3 + \varepsilon\times 7 + 2\times 5 + 1\times 6 + 1\times 9 + 3\times 6$
$+ 2\times 2 + 4\times 2$

$= 112 + 7\varepsilon = \text{Rs. } 112$ as $\varepsilon \to 0$

Table 2

u_i \ v_j	4	3	7	0	0	5
2	(−) 9	(−) 12	$\boxed{5}$ 9	(−) 6	(−) 9	(−) 10
0	(−) 7	$\boxed{4}$ 3	$\boxed{\varepsilon}$ 7	(−) 7	(−) 5	$\boxed{2}$ 5
2	$\boxed{1}$ 6	(0) 5	$\boxed{1}$ 9	(−) 11	(−) 3	(−) 11
2	$\boxed{3}$ 6	(−) 8	(−) 11	$\boxed{2}$ 2	$\boxed{4}$ 2	(−) 10

5(a) Find complete integral of $(x^2-y^2)pq - xy(p^2-q^2) = 1$

Sol'n: Here $f(x,y,z,p,q) = (x^2-y^2)pq - xy(p^2-q^2) - 1$ — ①

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2pqx - z(p^2-q^2)} = \frac{dq}{-2pqy - x(p^2-q^2)} = \frac{dx}{-(x^2-y^2)y + 2pxy}$$
$$= \frac{dy}{-(x^2-y^2)p - 2pxy}$$

Using x, y, p, q as multipliers, each fraction

$$= \frac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \frac{d(xp) + d(yq)}{0}$$

$\Rightarrow d(xp+qy) = 0 \Rightarrow xp + yq = 0 \Rightarrow p = (a-qy)/x$ — ②

Using ②, ① $\Rightarrow (x^2-y^2)\left(\frac{a-qx}{x}\right)q - xy\left[\left(\frac{a-qy}{x}\right)^2 - q^2\right] - 1 = 0$

$\Rightarrow \frac{a-qy}{x}\left\{(x^2-y^2)q - (a-qy)y\right\} + xyq^2 - 1 = 0$

$\Rightarrow \{(a-qy)/x\}(x^2q - ay) + xyq^2 - 1 = 0$

$\Rightarrow (a-qy)(x^2q - ay) + x^2yq^2 - x = 0 \Rightarrow aq(x^2+y^2) = a^2y + x$

$\therefore q = \frac{a^2y + x}{a(x^2+y^2)}$ and $p = \frac{1}{x}\left[a - \frac{(a^2y+x)y}{a(x^2+y^2)}\right] = \frac{a^2x - y}{a(x^2+y^2)}$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)dx + (a^2y + x)dy}{a(x^2+y^2)} = a\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{x\,dy - y\,dx}{a(x^2+y^2)}$$

Integrating $z = (a/2)\log(x^2+y^2) + 1/a \tan^{-1}(y/x) + b.$

5(b) Find a surface satisfying the equation $D^2z = 6x+2$ and touching $z = x^3+y^3$ along its section by the plane $x+y+1=0$.

Soln Given $D^2z = 6x+2$ ie $\frac{\partial p}{\partial x} = 6x+2$ —— ①

Integrating ① w.r.to x — $\quad (p = \frac{\partial z}{\partial x})$

$$p = 3x^2+2x+f(y)$$

and $\frac{\partial z}{\partial x} = 3x^2+2x+f(y)$ —— ②

Integrating ② w.r.to x —

$$z = x^3+x^2+xf(y)+F(y) \quad \text{—— ③}$$

where $f(y)$ and $F(y)$ are arbitrary functions.

The given surface is $z = x^3+y^3$ —— ④

and the given plane is $x+y+1=0$ —— ⑤

Since ③ and ④ touch each other along their section by ⑤, the values of p and q at any point on ⑤ must be equal. Thus we must have

$$3x^2+2x+f(y) = 3x^2 \quad \text{—— ⑥}$$

and $\quad xf'(y)+F'(y) = 3y^2$ —— ⑦

from ⑤ and ⑥, $f(y) = -2x = 2(y+1)$ —— ⑧

from ⑧, $f'(y) = 2$.

Using this value, ⑦ gives -

$$2x+F'(y) = 3y^2$$

$$F'(y) = 3y^2-2x$$

$$F'(y) = 3y^2+2(y+1) \quad \text{using ⑤}$$

Integrating it,

$$F(y) = y^3+y^2+2y+c \quad \text{—— ⑨}$$

where c is an arbitrary constant.

Using ⑧ and ⑨, ③ gives.

$$z = x^3+x^2+2x(y+1)+y^3+y^2+2y+c$$

Now at the point of contact of ④ and ⑩ values of z must be the same and hence we have.

$$x^3+x^2+2x(y+1)+y^3+y^2+2y+c = x^3+y^3$$

$$x^2+2x(y+1)+y^2+2y+c=0$$

$$x^2+2x(-x)+(x+1)^2-2(x+1)+c=0$$

$[\because$ from (5), $y+1=-x$ and $y=-(x+1)]$

which gives $c=1$. Putting this in ⑩ the required surface is -

$$z = x^3+x^2+2x(y+1)+y^3+y^2+2y+1$$

or $\quad z = x^3+y^3+(x+y+1)^2$

5(c) → Given that $f(0)=1$, $f(1)=3$, $f(3)=55$, find the unique polynomial of degree 2 or less, which fits the given data. find the bound on the error

Sol^n: we have $x_0=0$, $x_1=1$, $x_2=3$,

$f_0=1$; $f_1=3$, $f_2=55$.

The Lagrange's fundamental polynomials are given by

$$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-1)(x-3)}{(-1)(-3)}=\frac{1}{3}(x^2-4x+3)$$

$$l_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{x(x-3)}{(1)(-2)}=\frac{1}{2}(3x-x^2)$$

$$l_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{x(x-1)}{3(2)}=\frac{1}{6}(x^2-x)$$

Hence, the Lagrange's quadratic interpolating polynomial is given by

$$P_2(x)=l_0(x)f_0+l_1(x)f_1+l_2(x)f_2$$

$$=\frac{1}{3}(x^2-4x+3)+\frac{3}{2}(3x-x^2)+\frac{55}{6}(x^2-x)$$

$$=8x^2-6x+1.$$

we have

$$|E_2(f;x)|\le\frac{1}{6}M_3\left[\max_{0\le x\le 3}|x(x-1)(x-3)|\right]$$

$$=\frac{1}{6}(2.1126)M_3=0.3521M_3$$

where $M_3=\max\limits_{0\le x\le 3}|f'''(x)|$ and since the maximum of $|x(x-1)(x-3)|$ occurs at $x=2.2152$.

5(d) Simplify the boolean expression: $(a+b)(\bar{b}+c)+b.(\bar{a}+\bar{c})$ by using the laws of boolean algebra. From its truth table write it in minterm normal form.

Sol^n: $(a+b)(\bar{b}+c)+b(\bar{a}+\bar{c})$

$$=a\bar{b}+ac+b\bar{b}+bc+b\bar{a}+b\bar{c}$$
$$=a\bar{b}+ac+0+b(c+\bar{c})+b\bar{a} \quad [\because b\bar{b}=0]$$
$$=a\bar{b}+ac+b[0]+b\bar{a} \quad [\because c+\bar{c}=1]$$
$$=a\bar{b}+ac+b[1+\bar{a}]$$
$$=a\bar{b}+ac+b \quad [\because 1+\bar{a}=1]$$

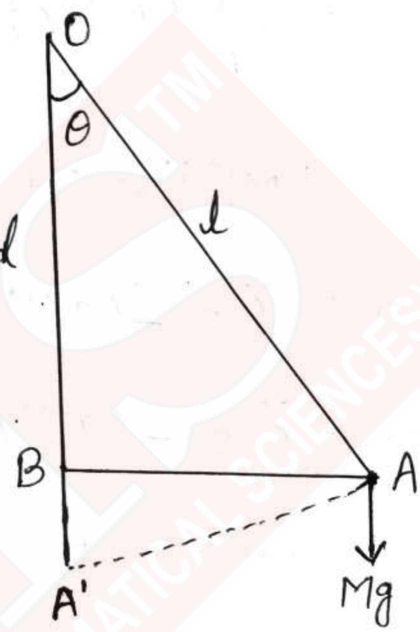
a	b	c	$\bar{b}$	$a.\bar{b}$	ac	$a\bar{b}+ac$	$z=a\bar{b}+ac+b$	minterm.
0	0	0	1	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
0	0	1	1	0	0	0	0	$\bar{a}\bar{b}c$
0	1	0	0	0	0	0	1	$\bar{a}b\bar{c}$
0	1	1	0	0	0	0	1	$\bar{a}bc$
1	0	0	1	1	0	1	1	$a\bar{b}\bar{c}$
1	0	1	1	1	1	1	1	$a\bar{b}c$
1	1	0	0	0	0	0	1	$ab\bar{c}$
1	1	1	0	0	1	1	1	abc

From truth table, minterm normal form is

$$\bar{a}b\bar{c}+\bar{a}bc+a\bar{b}\bar{c}+a\bar{b}c+ab\bar{c}+abc.$$

5(e) For a simple pendulum (i) find the Lagrangian function and (ii) obtain an equation describing its motion.

Sol: Let l be the length of the simple pendulum and θ the angle made by the string with the vertical at time t. Thus θ is the only generalised coordinate. Then the velocity of mass M at A will be $v = l\dot{\theta}$.



$\therefore$ Total K.E.,

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$$

And the potential function

$$V = Mg(A'B) = Mg(l - l\cos\theta)$$
$$= Mgl(1-\cos\theta)$$

(i) $\therefore$ The Lagrangian function

$$L = T - V$$
$$= \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$$

(ii) Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

i.e. $\frac{d}{dt}(Ml^2\dot{\theta}) + Mgl\sin\theta = 0$

or $Ml^2\ddot{\theta} + Mgl\sin\theta = 0$

or $\ddot{\theta} = -(g/l)\sin\theta$

or $\ddot{\theta} = -(g/l)\theta$, since θ is small.

which is the required equation of motion.

6(a)i Form partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2-y)+g(x^2+y)$.

Sol^n Given $z = f(x^2-y)+g(x^2+y)$ —— ①

differentiating ① partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = 2xf'(x^2-y) + 2xg'(x^2+y).$$

$$= 2x\{f'(x^2-y)+g'(x^2+y)\} \quad \text{—— ②}$$

and $\dfrac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y)$ —— ③

Differentiating ② and ③ w.r.to x and y respectively, we get -

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2-y)+g'(x^2+y)\} + 4x^2\{f''(x^2-y)+g''(x^2+y)\} \quad \text{—— ④}$$

and $\dfrac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y)$ —— ⑤

Again ② $\Rightarrow f'(x^2-y)+g'(x^2+y)$

$$= \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right) \quad \text{—— ⑥}$$

Substituting the values of $f''(x^2-y)+g''(x^2+y)$ and $f'(x^2-y)+g'(x^2+y)$ from ⑤ and ⑥ in ④ we have.

$$\frac{\partial^2 z}{\partial x^2} = 2\left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2}$$

$$x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

Which is the required partial differential eq^n.

6(a)ii Find the general integral of the partial differential equation $(2xy-1)p+(z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x=1, y=0$:

Sol'n: Given $(2xy-1)p+(z-2x^2)q = 2(x-yz)$ —— ①

Given line is $x=1, y=0$. —— ②

Here the Lagrange's auxiliary equations of ① are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2x-2yz} \quad \text{—— ③}$$

Taking $z, 1, x$ as multipliers, each fraction of ③

$$= \frac{z\,dx + 1.dy + x\,dz}{0} \quad \text{so that } z\,dx + dy + x\,dz = 0$$

$\Rightarrow d(xz) + dy = 0$ and hence $xz + y = c_1$ — ④.

Again, taking $x, y, \frac{1}{2}$ as multipliers, each fraction of ③

$$= \frac{x\,dx + y\,dy + \frac{1}{2}dz}{0} \quad \text{so that } x\,dx + y\,dy + \tfrac{1}{2}dz = 0$$

$\Rightarrow 2x\,dx + 2y\,dy + dz = 0$ and so $x^2+y^2+z = c_2$.

Since the required curve given by ④ and ⑤ passes through the line ②, so putting $x=1$ and $y=0$ in ④ and ⑤, we get

$z = c_1$ and $1+z = c_2$ so that $1+c_1 = c_2$ —— ⑥

Substituting the values of c_1 and c_2 from ④ and ⑤ in ⑥, the equation of the required surface is given by

$$1 + xz + y = x^2+y^2+z$$

$$\Rightarrow x^2+y^2+z-xz-y = 1$$

6(b)(i) The equation $x^2+ax+b=0$ has two real roots α and β show that the iteration method $x_{k+1} = -\frac{(ax_k+b)}{x_k}$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$ and that $x_{k+1} = \frac{-b}{x_k+a}$ is convergent near $x=\alpha$ if $|\alpha|<|\beta|$.
show also that iteration method $x_{k+1} = -\frac{(x_k^2+b)}{a}$ is convergent near $x=\alpha$ if $2|\alpha|<|\alpha+\beta|$.

Sol'n: The iterations are given by

$$x_{k+1} = -\frac{(ax_k+b)}{x_k} = g(x_k) \text{ (say)}$$
$$k=0,1,2 \ldots$$

By the known theorem
If $g(x)$ and $g'(x)$ are continuous in an interval about a root α of the equation $x=g(x)$ and if $|g'(x)|<1$ for all x in the interval, then the successive approximations $x_1, x_2, \ldots$ given by

$$x_k = g(x_{k-1}), \quad k=1,2,3\ldots$$

converges to the root α provided that the initial approximation x_0 is chosen in the in the interval.
$\therefore$ These iterations converge to α if

$$|g'(x)|<1 \text{ near } \alpha.$$

i.e. $|g'(x)| = \left|\frac{-b}{x^2}\right| < 1$

Note that $g'(x)$ is continuous near α.
If the iterations converge to $x=\alpha$, then we require $|g'(\alpha)| = \left|\frac{-b}{\alpha^2}\right|<1$

Thus $|b|<|\alpha|^2$

i.e. $|\alpha|^2>|b|$ —— (1)

Given that α and β are roots of the equation $x^2+ax+b=0$

then $\alpha+\beta=-a$ and $\alpha\beta=b \Rightarrow |b|=|\alpha||\beta|$ —— ③

Substituting ③ in ①, we get

$$|\alpha|^2 > |b| = |\alpha||\beta|$$

$$\Rightarrow |\alpha|^2 > |\alpha||\beta|$$

$$\Rightarrow |\alpha| > |\beta|$$

Now, if $x = \dfrac{-b}{x+a}$

The iteration $x_{k+1} = \dfrac{-b}{x_k+a} = g(x_k)$ (Say)

Converges to α if

$$|g'(x)| = \left|\frac{b}{(x+a)^2}\right| < 1 \text{ in an interval containing } \alpha.$$

Inparticular we require

$$|g'(\alpha)| = \left|\frac{b}{(\alpha+a)^2}\right| < 1$$

$$\Rightarrow (\alpha+a)^2 > |b|$$

But we have $\alpha+\beta=-a$ & $\alpha\beta=b$

$$\Rightarrow \beta^2 > |b| = |\alpha||\beta|$$

$$\Rightarrow |\beta|^2 > |\alpha||\beta|$$

$$\Rightarrow |\beta| > |\alpha|$$

$\therefore x_{k+1} = \dfrac{-b}{x_k+a}$ is convergent near

$x=\alpha$ if $|\beta| > |\alpha|$.

6(b)(iii) Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using Gauss – Legendre three – point formula.

Sol^n: First we transform the interval $[0,1]$ to the interval $[-1,1]$. Let $t = ax+b$.

We have $-1 = b$, $1 = a+b$

$\Rightarrow a = 2,\ b = -1$ and $t = 2x-1$

$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{dt}{t+3}$$

Using Gauss–Legendre three-point rule (corresponding to $n=2$), we get

$$I = \frac{1}{9}\left[8\left(\frac{1}{0+3}\right) + 5\left(\frac{1}{3+\sqrt{3/5}}\right) + 5\left(\frac{1}{3-\sqrt{3/5}}\right)\right]$$

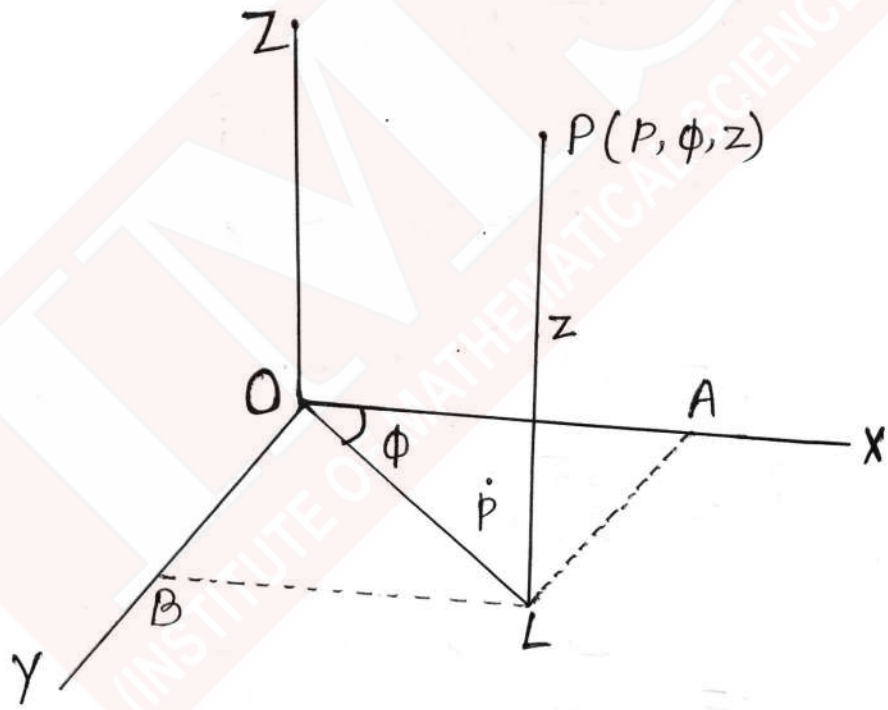
$$= \frac{131}{189}$$

$$= 0.0693122.$$

The exact solution is $I = \ln 2 = 0.693147$.

6(c) → A particle of mass m moves in a conservative forces fields. Find (i) the Lagrangian function and (ii) the equation of motion in cylindrical co-ordinates (ρ, ϕ, z).

Sol: Let P be the position of the particle of mass m whose cylindrical coordinates reffered to axes OX, OY, OZ are (ρ, ϕ, z).



∴ If (x, y, z) are its cartesian coordinates, then

$$x = OA = \rho\cos\phi,$$

$$y = OB = \rho\sin\phi, \quad z = z.$$

If $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along OX, OY, OZ respectively, then

$$\vec{OP} = \vec{r} = p\cos\phi\,\vec{i} + p\sin\phi\,\vec{j} + z\vec{k}$$

If $\hat{P}_1$ and $\hat{\phi}_1$ are the unit vectors in the directions of p and ϕ increasing respectively, then

$$\hat{P}_1 = \frac{\partial\vec{r}}{\partial p} \Big/ \left|\frac{\partial\vec{r}}{\partial p}\right|$$

$$= \frac{\cos\phi\,\vec{i} + \sin\phi\,\vec{j}}{\sqrt{(\cos^2\phi + \sin^2\phi)}}$$

$$= \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$$

$$\hat{\phi}_1 = \frac{\partial\vec{r}}{\partial\phi} \Big/ \left|\frac{\partial\vec{r}}{\partial\phi}\right|$$

$$= \frac{-p\sin\phi\,\vec{i} + p\cos\phi\,\vec{j}}{\sqrt{(p^2\sin^2\phi + p^2\cos^2\phi)}}$$

$$= -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}.$$

Now $v = \dot{\vec{r}} = (\dot{p}\cos\phi - p\sin\phi\,\dot{\phi})\vec{i} + (\dot{p}\sin\phi + p\cos\phi\,\dot{\phi})\vec{j} + \dot{z}\vec{k}$

$$= \dot{p}(\cos\phi\vec{i} + \sin\phi\vec{j}) + p\dot{\phi}(-\sin\phi\vec{i} + \cos\phi\vec{j}) + \dot{z}\vec{K}$$

$$= (\dot{p})\hat{P}_1 + (p\dot{\phi})\hat{\phi}_1 + \dot{z}\vec{K}$$

$$\therefore\ v^2 = \dot{p}^2 + (p\dot{\phi})^2 + \dot{z}^2$$

Total K.E.,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{p}^2 + p^2\dot{\phi}^2 + \dot{z}^2)$$

Let $V = V(p, \phi, z)$ be the potential function.

$\therefore$ (i) Lagrangian function,

$$L = T - V$$

i.e. $$L = \frac{1}{2}m(\dot{p}^2 + p^2\dot{\phi}^2 + \dot{z}^2) - V(p, \phi, z)$$

(ii) Lagrange's p equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0$$

or $$\frac{d}{dt}(m\dot{p}) - \left(mp\dot{\phi}^2 - \frac{\partial V}{\partial p}\right) = 0$$

i.e. $$m\ddot{p} - mp\dot{\phi}^2 = -\frac{\partial V}{\partial p} \quad \text{——— (1)}$$

Lagrange's ϕ equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

or $$\frac{d}{dt}(mp^2\dot{\phi}) - \left(-\frac{\partial V}{\partial \phi}\right) = 0$$

or $$\frac{d}{dt}(mp^2\dot{\phi}) = -\frac{\partial V}{\partial \phi} \quad \text{—— (2)}$$

and Lagrange's z equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

or $$\frac{d}{dt}(m\dot{z}) - \left(-\frac{\partial V}{\partial z}\right) = 0$$

or $$m\ddot{z} = -\frac{\partial V}{\partial z} \quad \text{—— (3)}$$

7(a). Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the following boundary conditions: $u(x,0)=u(x,b)=0$ for $0\le x\le a$, $u(0,y)=0$ and $u(a,y)=f(y)$ for $0\le y\le b$.

Sol'n: Given Laplace's equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ — (1)

Also given $u(x,0)=u(x,b)=0$ for $0\le x\le a$ — (2)

$u(0,y)=0$ for $0\le y\le b$ — (3a)

and $u(a,y)=f(y)$ for $0\le y\le b$ — (3b)

Let a solution of (1) be of the form

$$u(x,y)=X(x)\,Y(y) \quad — (4).$$

Using (4), (1) reduces to

$$X''Y+XY''=0 \Rightarrow \left(\tfrac{1}{X}\right)X''=-\left(\tfrac{1}{Y}\right)Y'' \quad — (5)$$

Since the L.H.S of (5) depends only on x and the R.H.S depends only on y, each side of (5) must be equal to the same constant say μ. Then (5) leads to $X''-\mu X=0$ — (6)

and $Y''+\mu Y=0$ — (7)

Using (2), (4) gives $X(x)Y(0)=0$ and $X(x)Y(b)=0$

So that $Y(0)=0$ and $Y(b)=0$ — (8)

where we have taken $X(x)\neq 0$, since otherwise $u=0$ does not satisfy 3(b).

We now solve (7) under boundary conditions (8).

Three cases arise:

Case I: Let $\mu=0$. Then, solution of (7) is $Y(y)=Ay+B$ — (9)

Using B.C (8), (9) gives $0=B$ and $0=Ab+B$.

These give $A=B=0$ so that $Y(y)\equiv 0$. This leads to $u=0$ which does not satisfy 3(b). So we reject $\mu=0$.

Case II: Let $\mu=-\lambda^2$ (-ve). Here $\lambda\neq 0$. Then solution of (7) is $Y(y)=Ae^{y\lambda}+Be^{-y\lambda}$ —— (10)

Using B.C (8), (10) gives $0=A+B$ and $0=Ae^{b\lambda}+Be^{-b\lambda}$ — (11)

(11) $\Rightarrow A=B=0$. So that $X(x)=0$. This leads to $u=0$ which does not satisfy 3(b). So we reject $\mu=-\lambda^2$.

Case III: Let $\mu=\lambda^2$ (+ve). Here $\lambda\neq 0$. Then solution of (7) is $Y(y)=A\cos\lambda y+B\sin\lambda y$ — (12)

Using B.C (8), (12) gives $0=A$ and $0=A\cos\lambda b+B\sin\lambda b$ so that $A=0$ and $\sin\lambda b=0$, where we have taken $B\neq 0$, since otherwise we shall get $Y(y)=0$. This leads to $u=0$ which does not satisfy 3(b).

Now $\sin\lambda b=0 \Rightarrow \lambda b=n\pi \Rightarrow \lambda=n\pi/b$, $n=1,2,3\ldots$

Hence, non-zero solutions $Y_n(y)$ of (7) are given by
$$Y_n(y)=B_n\sin(n\pi y/b),\ n=1,2,3\ldots \quad (13)$$

Also, then $\mu=-\lambda^2=-n^2\pi^2/b^2$. Hence (6) reduces to $x''-(n^2\pi^2/b^2)x=0$, $n=1,2,3\ldots$

whose solutions are $X_n(x)=C_ne^{n\pi x/b}+D_ne^{-n\pi x/b}$ — (15)

Using 3(a), (4) gives $0=X(0)Y(y)$ so that $X(0)=0$, where we have taken $Y(y)\neq 0$, since otherwise $u=0$ which does not satisfy 3(b).

Now $X(0)=0 \Rightarrow X_n(0)=0$. Putting $n=0$ in (15) and using $X_n(0)=0$, we get $0=C_n+D_n$ so that $D_n=-C_n$.

Then (13) reduces to
$$X_n(x)=C_n\left(e^{n\pi x/b}-e^{-n\pi x/b}\right)=2C_n\sinh\left(\frac{n\pi x}{b}\right). \quad (16)$$
$[\because e^{\theta}-e^{-\theta}=2\sinh\theta]$

from (4), (13) and (16), we see that a solution $u_n(x,y)$ of (7) is

$$u_n(x,y) = X_n(x)Y_n(y) = E_n \sinh(n\pi x/b)\sin(n\pi y/b),\ n=1,2,3 \quad \text{—— (17)}$$

where $E_n (= 2B_nC_n)$ are new arbitrary constants. In order to obtain a solution also satisfying B.C 3(b). we consider more general solution

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} \quad \text{—— (18)}$$

putting $x=a$ in (18) and using B.C (3b), we have

$$f(y) = \sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b},$$

which is the half range Fourier Sine series of $f(y)$ in $(0,b)$. Hence

$$E_n \sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

$$E_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\frac{n\pi y}{b} \quad \text{—— (19)}$$

Hence, (18) is the required solution wherein E_n is given by (19).

7(b)(ii) Using fourth order Runge-kutta method find the solution of the initial value problem.

$$y' = \frac{1}{(x+y)}, \; y(0) = 1 \text{ in the range}$$

$0.5 \le x \le 2.0$, by taking $h = 0.5$.

Sol'n: Runge kutta 4th order

$$f(x,y) = \frac{1}{x+y}, \; y(0)=1 \Rightarrow x_0 = 0, \; y_0 = 1$$
$$h = 0.5$$

$$y_1 = y_0 + \tfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = hf(x_0, y_0) = 0.5(1) = 0.5$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2) = 0.5f(0.25, 1.25) = 0.333$$

$$K_3 = hf(x_0 + h/2, y_0 + K_2/2) = 0.5f(0.25, 1.1666) = 0.35294$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.5f(0.5, 1.35294) = 0.2698$$

$$y(0.5) = 1 + \tfrac{1}{6}[0.5 + 0.666 + 0.70588 + 0.2698]$$

$$\boxed{y(0.5) = 1.3569}$$

Now, $y_0 = 1.3569$, $x_0 = 0.5$, $h = 0.5$

$$K_1 = 0.5f(0.5, 1.3569) = 0.2692$$

$$K_2 = 0.5f(0.75, 1.4915) = 0.2230$$

$$K_3 = 0.5f(0.75, 1.4684) = 0.2253$$

$$K_4 = 0.5f(1, 1.5822) = 0.1936$$

$$y(1) = 1.3569 + \tfrac{1}{6}[0.2692 + 0.446 + 0.4506 + 0.1936]$$

$$\boxed{y(1) = 1.5834}$$

Now, $y_0 = 1.5834$, $x_0 = 1$, $h = 0.5$

$$K_1 = 0.5f(1, 1.5834) = 0.1935$$

$$K_2 = 0.5f(1.25, 1.6801) = 0.1706$$

$K_3 = 0.5f\,(1.25,\ 1.6687) = 0.1713$

$K_4 = 0.5f\,(1.5,\ 1.7547) = 0.1536$

$y(1.5) = 1.5834 + \frac{1}{6}\left[0.1935 + 0.3412 + 0.3426 + 0.1536\right]$

$\boxed{y(1.5) = 1.7552}$

Now $y_0 = 1.7554,\ x_0 = 1.5,\ h = 0.5$

$K_1 = 0.5f\,(1.5,\ 1.7554) = 0.1536$

$K_2 = 0.1396$

$K_3 = 0.1399$

$K_4 = 0.1283$

$y(1.75) = 1.89555$

7(b)(ii) Given the number 59.625 in decimal system. writes its binary system.

(iii) Given the number 3898 in decimal system. writes its equivalent in system base 8.

Sol^n: Taking the integral part

$$\begin{array}{r|l}
2 & 59 \\ \hline
2 & 29 - 1 \\ \hline
2 & 14 - 1 \\ \hline
2 & 7 - 0 \\ \hline
2 & 3 - 1 \\ \hline
2 & 1 - 1 \\ \hline
 & 0 - 1
\end{array} \Rightarrow (59)_{10} = (111011)_2$$

Taking the fractional part:

fraction	fraction×2	Remainder in new fraction	Integer
0.625	1.25	0.25	1
0.25	0.5	0.5	0
0.5	1.0	0.0	1

$\therefore (0.625)_{10} = (.101)_2$

$\therefore (59.625)_{10} = (111011.101)_2$.

(iii)

$$\begin{array}{r|l} 8 & 3898 \\ \hline 8 & 487-2 \\ \hline 8 & 60-7 \\ \hline 8 & 7-4 \\ \hline & 0-7 \end{array}$$

$\therefore (3898)_{10} = (7472)_8$

7(c) An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{1/2} c^{5/4}$.

Sol: Let v be the velocity, p the pressure at a distance x from the origin, then the equations of motion and continuity are respectively.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and $\quad x^2 v = F(t)$ so that

$$v = \frac{F(t)}{x^2}, \quad \frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

$$\therefore \quad \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v\right)^2 = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating,

$$\frac{F'(t)}{x} + \frac{1}{2}v^2 = \frac{2\mu}{\sqrt{x}} - \frac{p}{\rho} + C \quad \text{——} (1)$$

Boundary condition are

2. When $x = \infty$, $v = 0$, $p = 0$

3. When $x = r$, (radius of cavity), $p=0$, $v = \dot{r}$

4. When $r = c$, $v = 0$ so that $F(t) = 0$,

5. Let T be the required time of filling the cavity.

Subjecting (1) to the conditions (2) and (3),

$$0 + 0 = 0 - 0 + C$$

and $$\frac{-F'(t)}{r} + \frac{1}{2}(\dot{r}^2) = \frac{2\mu}{\sqrt{r}} - 0 + C$$

or $$\frac{-F'(t)}{r} + \frac{1}{2}\dot{r}^2 = \frac{2\mu}{\sqrt{r}}.$$

since $r^2(\dot{r}) = F(t)$, $r^2 dr = F(t)\,dt$.

Multiplying by $2F(t)\,dt$ or $2r^2 dr$,

$$\frac{-2F'(t)\,F(t)\,dt}{r} + \frac{F^2(t)}{r^4}\cdot r^2 dr$$

$$= \frac{4\mu}{\sqrt{r}}\cdot r^2 dr$$

or $$d\left[\frac{-F^2(t)}{r}\right] = 4\mu\, r^{3/2}\, dr$$

Integrating, $$\frac{-F^2(t)}{r} = 4\mu\cdot\frac{2}{5}r^{5/2} + A \quad \text{—— (6)}$$

Subjecting ⑥ to ④,

$$0 = \frac{8\mu}{5} c^{5/2} + A$$

Now ⑥ $\Rightarrow$ $$\frac{-(r^2\dot{r})^2}{r} = \frac{8\mu}{5}(r^{5/2} - c^{5/2})$$

$$\Rightarrow \frac{dr}{dt} = -\left[\frac{8\mu}{5r^3}\cdot(c^{5/2} - r^{5/2})\right]^{1/2}$$

[negative sign is taken as velocity increases when r decreases.]

$$-\int_c^0 \frac{r^{3/2}}{[c^{5/2} - r^{5/2}]^{1/2}} dr = \int_0^T \left(\frac{8\mu}{5}\right)^{1/2} dt$$

or $$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\, dr}{[c^{5/2} - r^{5/2}]^{1/2}} \quad \text{—— ⑦}$$

Putting $r^{5/2} = c^{5/2}\sin^2\theta$,

$$\frac{5}{2} r^{3/2} dr = c^{5/2}\cdot 2\sin\theta\cos\theta\, d\theta.$$

$$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/2}\cdot \frac{\sin\theta\cos\theta\, d\theta}{c^{5/4}\cos\theta}$$

$$= \left(\frac{5}{8\mu}\right)^{1/2}\cdot\frac{4}{5} c^{5/4}(-\cos\theta)_0^{\pi/2}$$

or $$T = \left(\frac{2}{5\mu}\right)^{1/2}\cdot c^{5/4}$$ Proved.

8(a) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$

Sol^n: Given that $z = p^2 - q^2$

$\Rightarrow f(x, y, z, p, q) = p^2 - q^2 - z = 0$ — ①

Now we are to find the integral surface of ① which is passing through the parabola $y = 0,\ 4z + x^2 = 0$

whose parametric equations are

$y = 0,\ x = \lambda,\ z = -\frac{\lambda^2}{4}$.

i.e. $x = f_1(\lambda),\ y = f_2(\lambda),\ z = f_3(\lambda)$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = f_1(\lambda) = \lambda,\ y_0 = f_2(\lambda) = 0,\ z_0 = f_3(\lambda) = -\frac{\lambda^2}{4}$.

Now we find the initial values p_0 and q_0 by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ and

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

i.e. $f(\lambda, 0, -\frac{\lambda^2}{4}, p_0, q_0) = 0$

$\Rightarrow -\frac{\lambda}{2} = p_0(1) + q_0(0)$ and $p_0^2 - q_0^2 + \frac{\lambda^2}{4} = 0$

$\Rightarrow \boxed{-\frac{\lambda}{2} = p_0}$ —) $\Rightarrow q_0^2 = \frac{\lambda^2}{4} + p_0^2$

$\Rightarrow q_0^2 = \frac{\lambda^2}{4} + \frac{\lambda^2}{4}$

$\Rightarrow \boxed{q_0 = \frac{\lambda}{\sqrt{2}}}$

$\therefore x_0 = \lambda,\ y_0 = 0,\ z_0 = \frac{-\lambda^2}{4},\ p_0 = -\frac{\lambda}{2},\ q_0 = \frac{\lambda}{\sqrt{2}}$ & $t = t_0 = 0$ — ②

Now the characteristic equations of ① are

$x'(t) = \frac{\partial f}{\partial p} = 2p$ ——③

$y'(t) = \frac{\partial f}{\partial q} = -2q$ ——④

$z'(t) = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = p(2p) + q(-2q) = 2(p^2 - q^2) = 2z$ ——⑤ $(\because z = p^2 - q^2)$

$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -0 - p(-1) = p$ ——⑥

$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -0 - q(-1) = q$ ——⑦

from ③ & ⑥:

$\frac{x'(t)}{2} = p'(t) \Rightarrow \frac{dx}{2} = dp$

$\Rightarrow dx = 2dp$

$\Rightarrow x = 2p + c_1$

Using the initial values

$x_0 = 2p_0 + c_1$

$\Rightarrow \lambda = 2(-\frac{\lambda}{2}) + c_1 \Rightarrow c_1 = 2\lambda$

$\therefore \boxed{x = 2p + 2\lambda}$ ——⑧

from ④ & ⑦:

$-\frac{y'(t)}{2} = q'(t) \Rightarrow -\frac{dy}{2} = dq \Rightarrow dy = -2q$

$\Rightarrow y = -2q + c_2$

using the initial values

$y_0 = -2q_0 + c_2$

$\Rightarrow 0 = -2(\frac{\lambda}{\sqrt{2}}) + c_2$

$\Rightarrow c_2 = \sqrt{2}\lambda$

$\therefore \boxed{y = -2q + \lambda\sqrt{2}}$ ——⑨

from ⑥: $\frac{dp}{dt} = p \Rightarrow \frac{dp}{p} = dt \Rightarrow \log p = t + \log c_3$

$\Rightarrow p = c_3 e^t$

from ⑦: $\frac{dq}{dt} = q \Rightarrow \frac{dq}{q} = dt \Rightarrow \log q = t + \log c_4$

$\Rightarrow q = c_4 e^t$

Using the initial values

$p_0 = c_3 e^{t_0}$ and $q_0 = c_4 e^{t_0}$

$\Rightarrow \frac{-\lambda}{2} = c_3$ and $\frac{\lambda}{\sqrt{2}} = c_4$

$\therefore \boxed{p = \frac{-\lambda}{2} e^t}$ and $\boxed{q = \frac{\lambda}{\sqrt{2}} e^t}$

$\therefore$ from (8) & (9),

$x = 2p + 2\lambda = 2\left(-\frac{\lambda}{2}\right) + 2\lambda = -\lambda e^t + 2\lambda$

i.e., $\boxed{x = -\lambda e^t + 2\lambda}$

and $y = -2\left(\frac{\lambda}{\sqrt{2}}\right)e^t + \sqrt{2}\lambda = -\sqrt{2}\lambda e^t + \sqrt{2}\lambda$

$\Rightarrow \boxed{y = \lambda\sqrt{2}(1 - e^t)}$

from (5), $z'(t) = 2z$

$\Rightarrow \frac{dz}{z} = 2dt$

$\Rightarrow \log z = 2t + \log c_5$

$\Rightarrow$ Using initial values, we get

$\log z_0 = 2t_0 + \log c_5 \Rightarrow c_5 = z_0$

$\therefore \log z = 2t + \log z_0$

$\Rightarrow z = z_0 e^{2t}$

$\Rightarrow \boxed{z = -\frac{\lambda^2}{4} e^{2t}}$

$\therefore$ The required characteristics of (1) are given by

$\boxed{x = \lambda(2 - e^t)}$ — (i), $\boxed{y = \lambda\sqrt{2}(1 - e^t)}$ — (ii) ; $\boxed{z = -\frac{\lambda^2}{4} e^{2t}}$ — (iii)

Now eliminating e^t and λ from (i), (ii) & (iii)

from (i) ; $\lambda = \frac{x}{2 - e^t}$

from (ii) $y = \frac{x}{(2-e^t)}\sqrt{2}(1-e^t) = \frac{\sqrt{2}x(1-e^t)}{2-e^t}$

$\Rightarrow y(2-e^t) = \sqrt{2}x(1-e^t)$

$\Rightarrow 2y - ye^t = \sqrt{2}x - \sqrt{2}xe^t$

$\Rightarrow (\sqrt{2}x-y)e^t = \sqrt{2}x - 2y$

$\Rightarrow e^t = \frac{\sqrt{2}x-2y}{\sqrt{2}x-y}$

from (i) $\lambda = \frac{x}{2-\left(\frac{\sqrt{2}x-2y}{\sqrt{2}x-y}\right)} = \frac{(\sqrt{2}x-y)x}{2\sqrt{2}x-2y-\sqrt{2}x+2y}$

$\Rightarrow \lambda = \frac{(\sqrt{2}x-y)x}{\sqrt{2}x} =$

$\Rightarrow \boxed{\lambda = \frac{\sqrt{2}x-y}{\sqrt{2}}}$

from (iii):

$z = \frac{-\lambda^2 e^{2t}}{4}$

$= -\frac{1}{4}\left(\frac{\sqrt{2}x-y}{\sqrt{2}}\right)^2\left(\frac{\sqrt{2}x-2y}{\sqrt{2}x-y}\right)^2$

$= \frac{-(\sqrt{2}x-2y)^2}{8}$

$= -\frac{[2x^2+4y^2-4\sqrt{2}xy]}{8}$

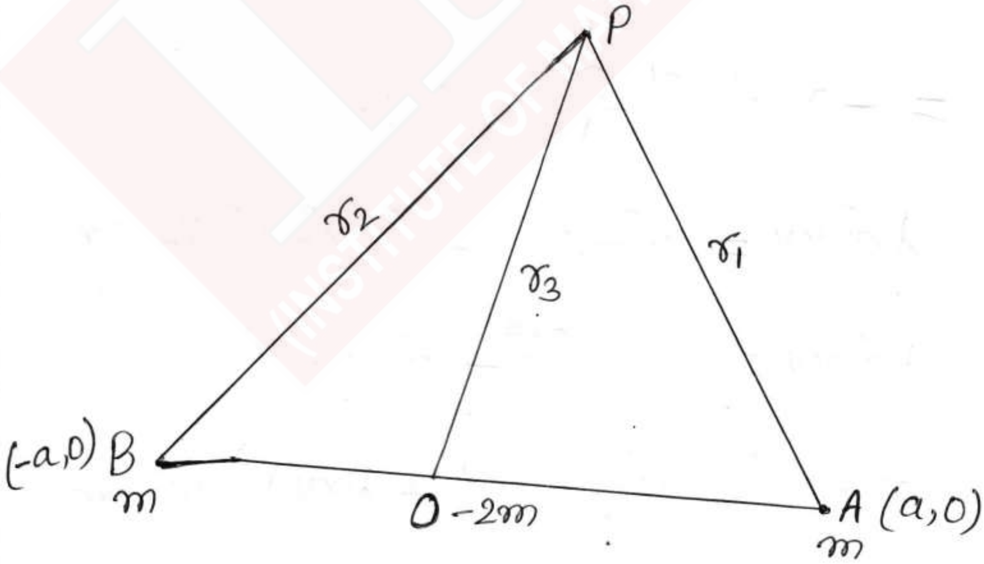
$= \frac{-[x^2-2\sqrt{2}xy+2y^2]}{4}$

$\Rightarrow \boxed{4z+(x-\sqrt{2}y)^2=0}$ which is the required integral surface of (1)

8(c) Two sources, each of strength m, are placed at the point $(-a,0)$ and $(a,0)$ and a sink of strength $2m$ is placed at the origin. Show that the stream lines are curves

$$(x^2+y^2)^2 = a^2[x^2-y^2+\lambda xy],$$ where λ is a parameter. Show also that the fluid speed at any point is $2ma^2/r_1 r_2 r_3$ where r_1, r_2, r_3 are respectively the distances of the point from the source and the sink.

Sol: The complex potential at any point $P(z)$ is given by



$$w = -m\log(z-a) - m\log(z+a) + 2m\log(z-0)$$

or $w = -m \log(z^2 - a^2) + m \log z^2$ ——— (1)

or $\phi + i\psi = -m \log(x^2 - a^2 - y^2 + 2ixy) + m \log(x^2 - y^2 + 2ixy)$.

Equating imaginary parts,

$$\psi = -m \tan^{-1} \frac{2xy}{x^2 - a^2 - y^2} + m \tan^{-1} \frac{2xy}{x^2 - y^2}$$

$$= -m \tan^{-1} \frac{2a^2 xy}{(x^2-y^2)(x^2-a^2-y^2) + 4x^2y^2}$$

stream lines are given by $\psi = \text{const.}$, i.e.,

$$-m \tan^{-1} \frac{2a^2 xy}{(x^2-y^2)(x^2-a^2-y^2) + 4x^2y^2}$$

$$= -m \tan^{-1}\left(\frac{2}{\lambda}\right), \text{ say.}$$

or $\lambda a^2 xy = (x^2-y^2)^2 - a^2(x^2-y^2) + 4x^2y^2$

or $\lambda a^2 xy = (x^2+y^2)^2 - a^2(x^2-y^2)$

or $(x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ where

λ is a variable parameter.

This completes the first part of the problem.

Flow speed $= \left|\dfrac{dw}{dz}\right| = \left| -\dfrac{2mz}{z^2-a^2} + \dfrac{2mz}{z^2}\right|$

$$= \frac{2ma^2}{|z(z^2-a^2)|}$$

$$= \frac{2ma^2}{|z|\cdot|z-a|\cdot|z+a|}$$

$$= \frac{2ma^2}{r_1 r_2 r_3}.$$

This concludes the problem.