CHAPTER 11

Differential Equations of First Order

1. Definitions. 2. Practical approach to differential equations. 3. Formation of a differential equation. 4. Solution of a differential equation—Geometrical meaning—5. Equations of the first order and first degree. 6. Variables separable. 7. Homogeneous equations. 8. Equations reducible to homogeneous form. 9. Linear equations. 10. Bernoulli's equation. 11. Exact equations. 12. Equations reducible to exact equations. 13. Equations of the first order and higher degree. 14. Clairut's equation. 15. Objective Type of Questions.

11.1 DEFINITIONS

(1) *A* **differential equation** *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x\,dx + e^y\,dy = 0$ (*ii*) $\dfrac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ (*iv*) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2} = c$

(*v*) $\dfrac{dx}{dt} - wy = a\cos pt,\ \dfrac{dy}{dt} + wx = a\sin pt$ (*vi*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(*vii*) $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (*i*) to (*v*) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (*vi*) and (*vii*) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(*i*) is of the first order and first degree ; (*ii*) is of the second order and first degree ;

(*iii*) written as $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (*iv*) written as $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = c^2\left(\dfrac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

(*i*) *formulation of differential equation from the given physical situation, called* **modelling.**

(*ii*) *solutions of this differential equation, evaluating the arbitrary constants from the given conditions,* and

(*iii*) *physical interpretation of the solution.*

11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11.1. *Form the differential equation of simple harmonic motion given by* $x = A\cos(nt + \alpha)$.

Solution. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA\sin(nt+\alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha) = -n^2x$$

Thus $$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11.2. *Obtain the differential equation of all circles of radius a and centre* (h, k).

(*Andhra, 1999*)

Solution. Such a circle is $(x-h)^2 + (y-k)^2 = a^2$ ...(*i*)

where h and k, the coordinates of the centre, and a are the constants.

Differential it twice, we have

$$x - h + (y-k)\frac{dy}{dx} = 0 \quad \text{and} \quad 1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and $$x - h = -(y-k)\,dy/dx = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$$

Substituting these in (*i*) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$ ...(*ii*)

as the required differential equation

Writing (*ii*) in the form $\dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$,

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. *Obtain the differential equation of the coaxial circles of the system* $x^2 + y^2 + 2ax + c^2 = 0$ *where c is a constant and a is a variable.* (*J.N.T.U., 2003*)

Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

Differentiating w.r.t. x, $2x + 2y dy/dx + 2a = 0$

or $$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2(x + y\,dy/dx)x + c^2 = 0$

or $$2xy\,dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example, $x = A\cos(nt + \alpha)$ ...(1)

is a solution of $\dfrac{d^2x}{dt^2} + n^2x = 0$ [Example 11.1] ...(2)

The **general** (*or* **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example, $x = A\cos(nt + \pi/4)$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent.*

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), *then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of* (3).

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1e^x + c_2(e^x + xe^x)$$

$\therefore$ $$y - y_1 = c_2e^x \qquad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2e^x \qquad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx (= m) when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring

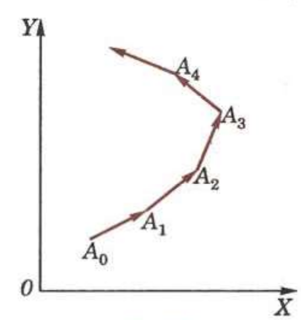


Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point A_0 (x_0, y_0). Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

1. $y = ax^3 + bx^2$.
2. $y = C_1 \cos 2x + C_2 \sin 2x$ (*Bhopal, 2008*)
3. $xy = Ae^x + Be^{-x} + x^2$. (*U.P.T.U., 2005*)
4. $y = e^x (A \cos x + B \sin x)$. (*P.T.U., 2003*)
5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

6. A family of circles passing through the origin and having centres on the x-axis. (*J.N.T.U., 2006*)
7. All circles of radius 5, with their centres on the y-axis.
8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
10. Determine the differential equation whose set of independent solutions is $[e^x, xe^x, x^2 e^x]$ (*U.P.T.U., 2002*)
11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through (0, 0), (1, 1), (0, 1) and (1, – 1) respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

(*i*) Equations where variables are separable, (*ii*) Homogeneous equations,
(*iii*) Linear equations, (*iv*) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable.* Thus the general form of such an equation is $f(y)\, dy = \phi(x)\, dx$

Integrating both sides, we get $\int f(y)\, dy = \int \phi(x)\, dx + c$ as its solution.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. (*V.T.U., 2008*)

Solution. Given equation is $x(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2\int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or $$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. *Solve* $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\dfrac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y}\,dy = (e^{3x} + x^2)\,dx$

Integrating both sides, $\int e^{2y}\,dy = \int (e^{3x} + x^2)\,dx + c$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. *Solve* $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$. *(V.T.U., 2005)*

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\dfrac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \displaystyle\int \frac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad [\text{Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\,d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan\frac{1}{2}(x + y)\right] + c$.

Example 11.8. *Solve* $dy/dx = (4x + y + 1)^2$, *if* $y(0) = 1$.

Solution. Putting $4x + y + 1 = t$, we get $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\dfrac{dt}{dx} - 4 = t^2$ or $\dfrac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\displaystyle\int \frac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1 \quad \therefore \quad \dfrac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\dfrac{y}{x}\dfrac{dy}{dx} + \dfrac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$. *(V.T.U., 2003)*

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{1}{2x}\dfrac{dt}{dx} - 1$.

Therefore the given equation becomes $\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or $$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}dt$$

or $$2x\,dx = \left(2 - \frac{3}{t+2}\right) dt$$

Integrating, we get $$x^2 = 2t - 3\log(t+2) + c$$

or $$x^2 + 2y^2 - 3\log(x^2 + y^2 + 2) + c = 0 \qquad [\because \; t = x^2 + y^2]$$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0.$
3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ (*P.T.U., 2003*)
4. $\frac{y}{x}\frac{dy}{dx} = \sqrt{(1 + x^2 + y^2 + x^2y^2)}.$ (*V.T.U., 2011*)
5. $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ (*V.T.U., 2009*)
6. $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$. (*J.N.T.U., 2006*)
7. $x\frac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$.
8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$
9. $\frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}.$
10. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$
11. $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}.$ (*Madras, 2000 S*)
12. $(x-y)^2\frac{dy}{dx} = a^2.$
13. $(x+y+1)^2\frac{dy}{dx} = 1.$ (*Kurukshetra, 2005*)
14. $\sin^{-1}(dy/dx) = x + y$ (*V.T.U., 2010*)
15. $\frac{dy}{dx} = \cos(x+y+1)$ (*V.T.U., 2003*)
16. $\frac{dy}{dx} - x\tan(y-x) = 1.$
17. $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (*i*) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0.$

Solution. Given equation is $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ which is homogeneous in x and y. ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\frac{dv}{dx} = \frac{1-v^2}{v}$

or $$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}.$$

Separating the variables, $\frac{v}{1-2v^2}\,dv = \frac{dx}{x}$

Integrating both sides, $\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

or $$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

or $$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \qquad [\text{Put } v = y/x]$$

or $$x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.11. *Solve* $(x \tan y/x - y \sec^2 y/x)\,dx - x \sec^2 y/x\, dy = 0.$ *(V.T.U., 2006)*

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2 y/x \qquad \ldots(i)$$

which is a homogeneous equation. Putting $y = vx$, (i) becomes $v + x\frac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$

or $$x\frac{dv}{dx} = v - \tan v \cos^2 v - v$$

Separating the variables $\frac{\sec^2 v}{\tan v}\,dv = -\frac{dx}{x}$

Integrating both sides $\log \tan v = -\log x + \log c$

or $$x \tan v = c \quad \text{or} \quad x \tan y/x = c.$$

Example 11.12. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0.$
(P.T.U., 2006 ; Rajasthan, 2005 ; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad \ldots(i)$$

which is a homogeneous equation. Putting $x = vy$ so that (i) becomes

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v+e^v) + c$

or $$y(v+e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad \text{(say)}$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2 - y^2)\,dx = 2xy\,dy$
2. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ *(Bhopal, 2008)*
3. $x^2y\,dx - (x^3 + y^3)\,dy = 0.$ *(V.T.U., 2010)*
4. $y\,dx - x\,dy = \sqrt{x^2+y^2}\,dx.$ *(Raipur, 2005)*
5. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}.$
6. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0.$ *(S.V.T.U., 2009)*

[Equations solvable like homogeneous equations : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. (V.T.U., 2000 S)
8. $ye^{x/y}\, dx = (xe^{x/y} + y^2)\, dy$. (V.T.U., 2006)
9. $xy\, (\log x/y)\, dx + [y^2 - x^2 \log (x/y)]\, dy = 0$.
10. $x\, dx + \sin^2 (y/x)\, (ydx - xdy) = 0$.
11. $x \cos\frac{y}{x}\, (ydx + xdy) = y \sin\frac{y}{x}\, (xdy - ydx)$.

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting $x = X + h,\ y = Y + k$, (h, k being constants)

so that $dx = dX,\ dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \quad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say)

$\therefore$ $a' = am,\ b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \quad ...(4)$$

Put $ax + by = t$, so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) \quad \therefore \text{ (4) becomes } \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. *Solve* $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. (*Raipur, 2005*)

Solution. Given equation is $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right]$ ...(*i*)

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX, dy = dY$, (*i*) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \quad ...(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1, k = 3$.

$\therefore$ (*ii*) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y. ...(*iii*)

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (*iii*) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or $$\frac{v-1}{1+2v-v^2}\,dv = \frac{dX}{X}.$$

Integrating both sides, $-\dfrac{1}{2}\displaystyle\int \frac{2-2v}{1+2v-v^2}\,dv = \int \frac{dX}{X} + c.$

or $$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or $$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or $$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \quad ...(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (*iv*) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. *Solve* $(3y + 2x + 4)\,dx - (4x + 6y + 5)\,dy = 0.$ *(Madras, 2000 S)*

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(*i*)

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\quad\therefore$ (*i*) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or $$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\,dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\,dt = \int dx + c$

or $$\int\left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t + 22}\right)dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + 49c$

or $\quad 21x - 42y + 9\log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2)\,dx + (x - 2y - 3)\,dy = 0.$ *(Rajasthan, 2006)*
2. $(2x + y - 3)\,dy = (x + 2y - 3)\,dx.$ *(V.T.U., 2009 S ; Madras, 2000)*
3. $(2x + 5y + 1)\,dx - (5x + 2y - 1)\,dy = 0.$ *(J.N.T.U., 2000)*
4. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}.$
6. $(4x - 6y - 1)\,dx + (3y - 2x - 2)\,dy = 0.$
7. $(x + 2y)(dx - dy) = dx + dy.$ *(Bhopal, 2002 S ; V.T.U., 2001)*

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,*** *is*

$$\frac{dy}{dx} + Py = Q \qquad \text{where, } P, Q \text{ are the functions of } x. \qquad \text{...(1)}$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y\,(e^{\int P dx} P) = Qe^{\int P dx} \quad \text{i.e.,} \quad \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get $ye^{\int P dx} = \int Q e^{\int P dx}\, dx + c$ as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F.** $= e^{\int P dx}$

and the solution is y **(I.F.)** $= \int Q(\text{I.F.})\, dx + c.$

Example 11.15. *Solve* $(x + 1) \dfrac{dy}{dx} - y = e^{3x} (x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x} (x + 1) \text{ which is Leibnitz's equation.} \qquad \text{...}(i)$$

Here $\qquad P = -\dfrac{1}{x+1}$ and $\int P dx = -\int \dfrac{dx}{x+1} = -\log (x + 1) = \log (x + 1)^{-1}$

$\therefore \qquad \text{I.F.} = e^{\int P dx} = e^{\log (x+1)^{-1}} = \dfrac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x} (x + 1)](\text{I.F.})\, dx + c$

or $\qquad \dfrac{y}{x+1} = \int e^{3x}\, dx + c = \dfrac{1}{3} e^{3x} + c$ or $y = \left(\frac{1}{3} e^{3x} + c\right)(x + 1).$

Example 11.16. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1.$

Solution. Given equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $\qquad$ ...(i)

$\therefore \qquad \text{I.F.} = e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (i) is $y\,(\text{I.F.}) = \int \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} (\text{I.F.})\, dx + c$

or $\qquad ye^{2\sqrt{x}} = \int \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$

or $\qquad ye^{2\sqrt{x}} = \int x^{-1/2}\, dx + c$ or $ye^{2\sqrt{x}} = 2\sqrt{x} + c.$

* See footnote p. 139.

Example 11.17. *Solve* $3x(1-x^2)\,y^2\,\dfrac{dy}{dx}+(2x^2-1)\,y^3=ax^3$ (*Rajasthan, 2006*)

Solution. Putting $y^3=z$ and $3y^2\,\dfrac{dy}{dx}=\dfrac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\,\frac{dz}{dx}+(2x^2-1)\,z=ax^3, \quad \text{or} \quad \frac{dz}{dx}+\frac{2x^2-1}{x-x^3}\,z=\frac{ax^3}{x-x^3} \quad ...(i)$$

which is Leibnitz's equation in z

$$\therefore \quad \text{I.F.}=\exp\left(\int\frac{2x^2-1}{x-x^3}\,dx\right)$$

Now $\displaystyle\int\frac{2x^2-1}{x-x^3}\,dx=\int\left(-\frac{1}{x}-\frac{1}{2}\,\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1-x}\right)dx=-\log x-\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)$

$$=-\log\left[x\sqrt{(1-x^2)}\right]$$

$$\therefore \quad \text{I.F.}=e^{-\log[x\sqrt{(1-x^2)}]}=\left[x\sqrt{(1-x^2)}\right]^{-1}$$

Thus the solution of (*i*) is

$$z(\text{I.F.})=\int\frac{ax^3}{x-x^3}\,(\text{I.F.})\,dx+c$$

or

$$\frac{z}{[x\sqrt{(1-x^2)}]}=a\int\frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}\,dx+c=a\int x(1-x^2)^{-3/2}\,dx$$

$$=-\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}\,dx+c=a\,(1-x^2)^{-1/2}+c$$

Hence the solution of the given equation is

$$y^3=ax+cx\sqrt{(1-x^2)}. \quad [\because \; z=y^3]$$

Example 11.18. *Solve* $y\,(\log y)\,dx+(x-\log y)\,dy=0.$ (*U.P.T.U., 2000*)

Solution. We have $\dfrac{dx}{dy}+\dfrac{x}{y\log y}=\dfrac{1}{y}$ ... (*i*)

which is a Leibnitz's equation in x

$$\therefore \quad \text{I.F.}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$$

Thus the solution of (*i*) is $x\,(\text{I.F.})=\displaystyle\int\frac{1}{y}(\text{I.F.})\,dy+c$

$$x\log y=\int\frac{1}{y}\log y\,dy+c=\frac{1}{2}(\log y)^2+c$$

i.e.,

$$x=\frac{1}{2}\log y+c\,(\log y)^{-1}.$$

Example 11.19. *Solve* $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy.$ (*Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005*)

Solution. This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2)\,\frac{dx}{dy}=\tan^{-1}y-x \quad \text{or} \quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \quad \text{I.F.}=e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Thus the solution is $x\,(\text{I.F.})=\displaystyle\int\frac{\tan^{-1}y}{1+y^2}\,(\text{I.F.})\,dy+c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + c \qquad \left[\begin{array}{l}\text{Put } \tan^{-1} y = t \\ \therefore \ \dfrac{dy}{1+y^2} = dt\end{array}\right]$$

$$= \int te^t\, dt + c = t \cdot e^t - \int 1 \cdot e^t\, dt + c \qquad \text{(Integrating by parts)}$$

$$= t \cdot e^t - e^t + c = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Example 11.20. *Solve $r \sin\theta\, d\theta + (r^3 - 2r^2 \cos\theta + \cos\theta)\, dr = 0$.*

Solution. Given equation can be rewritten as

$$\sin\theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2)\cos\theta = -r^2 \qquad \text{...(i)}$$

Put $\cos\theta = y$ so that $-\sin\theta\, d\theta/dr = dy/dr$

Then (*i*) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = -r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r)dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y\left(\dfrac{1}{r} e^{r^2}\right) = \int r^2 \cdot e^{r^2} \cdot \dfrac{1}{r}\, dr + c$

or $$y e^{r^2}/r = \frac{1}{2}\int e^{r^2}\, 2r\, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2e^{r^2}\cos\theta = re^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2\cos\theta.$$

PROBLEMS 11.5

Solve the following differential equations :

1. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.
2. $x \log x \dfrac{dy}{dx} + y = \log x^2$. (*V.T.U., 2011*)
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (*V.T.U., 2003*)
4. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2\cosh^2 x \sinh x$. (*J.N.T.U., 2003*)
5. $(1 - x^2) \dfrac{dy}{dx} - xy = 1$ (*V.T.U., 2010*)
6. $(1 - x^2) \dfrac{dy}{dx} + 2xy = x\sqrt{(1 - x^2)}$ (*Nagpur, 2009*)
7. $\dfrac{dy}{dx} = -\dfrac{x + y\cos x}{1 + \sin x}$.
8. $dr + (2r \cot\theta + \sin 2\theta)\, d\theta = 0$. (*J.N.T.U., 2003*)
9. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ (*P.T.U., 2005*)
10. $(x + 2y^3) \dfrac{dy}{dx} = y$. (*Marathwada, 2008*)
11. $\sqrt{(1 - y^2)}\, dx = (\sin^{-1} y - x)\, dy$.
12. $ye^y\, dx = (y^3 + 2xe^y)\, dy$.
13. $(1 + y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0$. (*V.T.U., 2006*)
14. $e^{-y} \sec^2 y\, dy = dx + x\, dy$.

11.10 BERNOULLI'S EQUATION

The equation $$\frac{dy}{dx} + Py = Qy^n \qquad \text{...(1)}$$

*where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation**.

*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$ or $\dfrac{dz}{dx} + P(1-n)\,z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. *Solve* $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution. Dividing throughout by xy^6, $y^{-6}\dfrac{dy}{dx} + \dfrac{y^{-5}}{x} = x^2$ ...(*i*)

Put $y^{-5} = z$, so that $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\therefore$ (*i*) becomes $-\dfrac{1}{5}\dfrac{dz}{dx} + \dfrac{z}{x} = x^2$

or $\dfrac{dz}{dx} - \dfrac{5}{x}z = -5x^2$ which is Leibnitz's linear in z. ...(*ii*)

$$\text{I.F.} = e^{-\int (5/x)dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ the solution of (*ii*) is $z\,(\text{I.F.}) = \int (-5x^2)(\text{I.F.})\,dx + c$ or $zx^{-5} = \int (-5x^2)\,x^{-5}\,dx + c$

or $y^{-5}x^{-5} = -5.\,\dfrac{x^{-2}}{-2} + c$ $[\because\ z = y^{-5}]$

Dividing throughout by $y^{-5}x^{-5}$, $1 = (2.5 + cx^2)\,x^3y^5$ which is the required solution.

Example 11.22. *Solve* $xy\,(1 + xy^2)\dfrac{dy}{dx} = 1$. (*Nagpur, 2009*)

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \qquad \text{...}(i)$$

Putting $x^{-1} = z$ so that $-x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$ (*i*) becomes

$\dfrac{dz}{dy} + yz = -y^3$ which is Leibnitz's linear in z.

Here I.F. $= e^{\int y\,dy} = e^{y^2/2}$

$\therefore$ the solution is $z\,(\text{I.F.}) = \int (-y^3)(\text{I.F.})\,dy + c$

or $ze^{y^2/2} = -\int y^2 \,.\, e^{\frac{1}{2}y^2}\,.\,y\,dy + c$ | Put $\frac{1}{2}y^2 = t$ so that $y\,dy = dt$

$= -2\int t\,.\,e^t\,dt + c$ [Integrate by parts]

$= -2\,[t\,.\,e^t - \int 1\,.\,e^t\,dt] + c = -2\,[te^t - e^t] + c = (2 - y^2)\,e^{y^2/2} + c$

or $z = (2 - y^2) + ce^{-\frac{1}{2}y^2}$ or $1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}$.

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(*A*)

where P, Q are functions of x. To solve it, put $f(y) = z$.

Example 11.23. *Solve* $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. *(V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005)*

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$

or $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ which is of the form (*A*) above. ...(*i*)

$\therefore$ put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$ $\therefore$ (*i*) becomes $\frac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2}(x^2 - 1)e^{x^2} + c$.

Replacing z by $\tan y$, we get $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. *Solve* $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x}(\log z)^2$.

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \qquad ...(i)$$

Put $\log z = t$ so that $\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$. $\therefore$ (*i*) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \qquad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (*ii*) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\therefore$ I.F. $= e^{-\int 1/x\,dx} = 1/x$

$\therefore$ the solution is $v \cdot \frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + c = \frac{1}{x} + c$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$ which is the required solution.

PROBLEMS 11.6

Solve the following equations :

1. $\frac{dy}{dx} = y \tan x - y^2 \sec x$. *(P.T.U., 2005)*
2. $r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$. *(V.T.U., 2005)*
3. $2xy' = 10x^3y^5 + y$.
4. $(x^3y^2 + xy)\,dx = dy$. *(B.P.T.U., 2005)*
5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$. *(Bhillai, 2005)*
6. $x(x - y)\,dy + y^2\,dx = 0$. *(I.S.M., 2001)*
7. $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)e^x \sec y$. *(Bhopal, 2009)*
8. $e^y\left(\frac{dy}{dx} + 1\right) = e^x$. *(V.T.U., 2009)*
9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3$.
10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$. *(Sambalpur, 2002)*
11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}$. *(V.T.U., 2011)*
12. $(y \log x - 2)\,ydx - xdy = 0$. *(V.T.U., 2006)*

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\,\mathbf{dx} + \mathbf{N}(x, y)\,\mathbf{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad \text{...(1)}$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{...(2)}$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{(Assumption)}$$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \; M = \frac{\partial u}{\partial x}$$

$\therefore$ $$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \left\{\begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y} \end{array}\right. \qquad \text{...(3)}$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \qquad \text{...(4)}$$

$\therefore$ $$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy \qquad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + f(y)\,dy = du + f(y)\,dy = d\left[u + \int f(y)dy\right] \qquad \text{...(5)}$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y)dy] = 0$

Integrating $\quad u + \int f(y)dy = 0.$

But $\quad u = \int_{y\text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y\text{ cons.})} \mathbf{M\,dx} + \int (\textbf{terms of N not containing x})\,\mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0.$ *(V.T.U., 2006)*

Solution. Here $\quad M = y^2 e^{xy^2} + 4x^3 \quad$ and $\quad N = 2xy\, e^{xy^2} - 3y^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y\, e^{xy^2} + y^2\, e^{xy^2} . \, 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y\ \text{const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2)\, dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\, dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $\ M = y\,(1 + 1/x) + \cos y \quad$ and $\quad N = x + \log x - x \sin y$

$$\therefore \qquad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y\ \text{const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\, y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy \quad$ and $\quad N = \sin x^2 - x^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y\ \text{const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 . \, 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y \quad$ and $\quad N = \sin x + x \cos y + x.$

$$\therefore \qquad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y\ \text{const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\,xdx - (3x^2 + 2y^2 - 8)\,ydy = 0.$ *(U.P.T.U., 2005)*

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1} \qquad \text{[By componendo \& dividendo]}$$

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c' \qquad [\text{Writing } c = \log c']$$

or $$x^2 + y^2 - 3 = c'\,(x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$ *(Kurukshetra, 2005)*
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$ *(V.T.U., 2008)*
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$
8. $\dfrac{2x}{y^3}\,dx + \dfrac{y^2 - 3x^2}{y^4}\,dy = 0$
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$ *(Marathwada, 2008)*
11. $(2xy + y - \tan y)\,dx + x^2 - x \tan^2 y + \sec^2 y)\,dy = 0.$ *(Nagpur, 2009)*

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -\,d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy.$ *(Kurukshetra, 2005)*

Solution. It is easy to note that the terms ye^xdx and e^xdy should be put together.

$$\therefore \qquad (ye^xdx - e^xdy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2x dx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2x dx = 0$$

Integrating, we get $\frac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor $(Mx + Ny \neq 0)$.*

Example 11.31. *Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.* *(Osmania, 2003 S)*

Solution. This equation is homogeneous in x and y.

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ or $\frac{x}{y} - 2\log x + 3\log y = c$.

(3) I.F. for an equation of the type $\mathbf{f_1(xy)ydx + f_2(xy)xdy = 0}$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$.* *(S.V.T.U., 2008)*

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y, N = (1 - xy)\,x.$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ the solution is $\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

or
$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $\mathbf{Mdx + Ndy = 0}$,

(a) *if* $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only $= f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.*

(b) *if* $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only $= F(y)$ say, then $e^{\int F(y)dy}$ is an integrating factor.*

Example 11.33. *Solve $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$.* *(S.V.T.U., 2009 ; Mumbai, 2007)*

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\frac{y^2}{x^3} - \frac{1}{4^4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y,\ N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y \text{ const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2} x^2y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y}(1 - \log y - 1) = -\frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}(x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \frac{\partial}{\partial y}(\log y) = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)\right.$

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \quad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b (my\,dx + nx\,dy) + x^{a'} y^{b'} (m'y\,dx + n'x\,dy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. *Solve* $y(xy + 2x^2y^3)\,dx + x\,(xy - x^2y^2)\,dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^ay^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$.

$\therefore$ $$\text{I.F.} = x^hy^k.$$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e. $$\frac{1+h+1}{1} = \frac{1+k+1}{1},\ \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or $$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\displaystyle\int_{(y\ const)} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0.$
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}.$ (U.P.T.U., 2005)
3. $ydx - xdy + \log x\,dx = 0.$
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}.$
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0.$
6. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)\,xdy = 0.$
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)dy = 0$ (Mumbai, 2006)
9. $x^4\dfrac{dy}{dx} + x^3y + \text{cosec}\,(xy) = 0.$
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ (Mumbai, 2006)
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0.$ (Kurukshetra, 2006)
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0.$ (Rajasthan, 2005)
13. $2ydx + x(2\log x - y)\,dy = 0.$ (P.T.U., 2005)

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)]\,\ldots\,[p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y),\ p = f_2(x, y),\ \ldots,\ p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0,\ \ldots,\ F_n\,(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,.......\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0 \quad ...(ii)$

From (i), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (ii), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm\, y \operatorname{cosec} x$

i.e., $\quad p = y(-\cot x + \operatorname{cosec} x) \quad ...(i)$

or $\quad p = y(-\cot x - \operatorname{cosec} x) \quad ...(ii)$

From (i), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec}\, x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y\,(1 + \cos x) = c \quad ...(iii)$

From (ii), $\quad \dfrac{dy}{dx} = -y\,(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad ...(iv)$

Thus combining (iii) and (iv), the required general solution is

$$y\,(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0.$
2. $p(p + y) = x\,(x + y).$ *(V.T.U., 2011)*
3. $y = x\,[p + \sqrt{(1 + p^2)}]$.
4. $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + xy = 0.$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ *(Madras, 2003)*

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad ...(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$. ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c),\ y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\cdot\frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int\frac{dx}{x} + 2\int\frac{dp}{p}$ = a constant

or $$\log x + 2\log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \qquad \text{or} \quad p = \sqrt{(c/x)} \qquad ...(ii)$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. *(Bhopal, 2009)*

Solution. Given equation is $y = 2px + p^n$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int\frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2}) \,.\, (\text{I.F.})\, dp + c \quad \text{or} \quad xp^2 = -n\int p^n\, dp + c = -\frac{np^{n+1}}{n+1} + c$$

or
$$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad \text{...(iii)}$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}\, p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a\tan^{-1} p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad \text{...(1)}$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve $y = 2px + y^2p^3$.* (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2} \cdot \dfrac{p\left(1 - 2y \,.\, p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or
$$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or
$$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p\,(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or
$$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\qquad py = c.$ ...(*i*)

Thus eliminating from the given equation and (*i*), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. *(Kanpur, 1996)*
2. $p^3y + 2px = y$.
3. $x - yp = ap^2$. *(Andhra, 2000)*
4. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$. *(S.V.T.U., 2008)*

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

or $$[x + f'(p)]\frac{dp}{dx} = 0 \quad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.,* (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$$\sin^{-1} p = y - xp \text{ or } y = px + \sin^{-1} p \text{ which is the } \textit{Clairaut's equation}.$$

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = \sqrt{(x^2 - 1)}/x$$

Now substituting this value of c in (*i*), we get

$$y = \sqrt{(x^2 - 1)} + \sin^{-1}\{\sqrt{(x^2 - 1)}/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p$.* *(V.T.U., 2011 ; J.N.T.U., 2006)*

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$

$$\therefore \quad p = \frac{dy}{dx} = \frac{dv}{y}\bigg/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}.x-y\right)\left(\frac{xP}{y}.y+x\right)=a^2\frac{xP}{y}$

or $(uP-v)(P+1)=a^2P$ or $uP-v=\dfrac{a^2P}{P+1}$

or $v=uP-a^2P/(P+1)$, which is Clairaut's form.

∴ its solution is $v=uc-a^2c/(c+1)$, *i.e.*, $y^2=cx^2-a^2c/(c+1)$.

PROBLEMS 11 12

1. Find the general and singular solution of the equations :
   (*i*) $xp^2-yp+a=0$. (*J.N.T.U., 2006*) (*ii*) $p=\log(px-y)$.
   (*iii*) $y=px+\sqrt{(a^2p^2+b^2)}$ (*W.B.T.U., 2005*) (*iv*) $\sin px\cos y=\cos px\sin y+p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.
3. $(y-px)(p-1)=p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-y=0$.
5. $x^2(y-px)=yp^2$.
6. $(px+y)^2=py^2$.
7. $(px-y)(x+py)=2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y=cx-c^2$, is the general solution of the differential equation
   (*i*) $(y')^2-xy'+y=0$ (*ii*) $y''=0$ (*iii*) $y'=c$ (*iv*) $(y')^2+xy'+y=0$.
2. The differential equation having a basis for its solution as sinh $6x$ and cosh $6x$ is
   (*i*) $y''+36y=0$ (*ii*) $y''-36y=0$ (*iii*) $y''+6y=0$ (*iv*) none of these.
3. The differential equation $(dx/dy)^2+5y^{1/3}=x$ is
   (*i*) linear of degree 3 (*ii*) non-linear of order 1 and degree 6
   (*iii*) non-linear of order 1 and degree 2.
4. The differential equation $ydx/dy+1=y$, $y(0)=1$, has
   (*i*) a unique solution (*ii*) two solutions
   (*iii*) infinite number of solutions (*iv*) no solution
5. Solution of $(x^2+y^2)\,dy=xy\,dx$ is ....
6. Solution of $(3x-2y)\,dx=xdy$ is ....
7. Solution of $dy/dx-y=2xy^2e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2}+6x)\,dx+(2xye^{xy^2}-4y)\,dy=0$ is
   (*i*) linear, homogeneous and exact (*ii*) non-linear, homogeneous and exact
   (*iv*) non-linear, non-homogeneous and exact (*iv*) non-linear, non-homogeneous and inexact.
9. Solution of $xdx+ydy+\dfrac{xdy-ydx}{x^2+y^2}$ is ....
10. Solution of $dy/dx=\dfrac{x^3+y^3}{xy^2}$ is ......
11. The differential equation $(x+x^8+ay^2)\,dx+(y^8-y+bxy)\,dy=0$ is exact if
    (*i*) $b=2a$ (*ii*) $a=b$ (*iii*) $a\neq 2b$ (*iv*) $a=1, b=3$.
12. Solution of $xy\,(1+xy^2)\,dy=dx$ is ....
13. Solution of $xp^2-yp+a=0$ is ....
14. The differential equation $p=\log(px-y)$ has the solution ....
15. Solution of $\dfrac{dy}{dx}-\dfrac{dx}{dy}=\dfrac{x}{y}-\dfrac{y}{x}$ is ....

16. The order of the differential equation $(1 + y_1^2)^{3/2}/y_2 = c$ is ......

17. The general solution of $\dfrac{1}{x^2y^2}(xdy + ydx) = 0$ is ....

18. Integrating factor of the differential equation $\dfrac{dx}{dy} + \dfrac{3x}{y} = \dfrac{1}{y^2}$ is

(a) e^{y^3} (b) y^3 (c) x^3 (d) $-y^3$. (V.T.U., 2009)

19. Solution of the equation $\dfrac{dy}{dx} = \dfrac{y}{x} - \operatorname{cosec}\dfrac{y}{x}$ is

(a) $\cos(y/x) - \log x = c$ (b) $\cos(y/x) + \log x = c$
(c) $\cos^2(y/x) + \log x = c$ (d) $\cos^2(y/x) - \log x = c$. (V.T.U., 2010)

20. Solution of $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\ dy/dx = 0$ is ....

21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is ....

22. The substitution that transforms the equation $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$ to homogeneous form is ....

23. Integrating factor of $xy' + y = x^3y^6$ is ....

24. Solution of the exact differential equation $Mdx + Ndy = 0$ is ....

25. Solution of $(2x^3y^2 + x^4)\,dx + (x^4y + y^4)\,dy = 0$ is ....

26. The general solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = \tan 2x$ is ....

27. Degree of the differential equation $\left(\dfrac{d^2y}{dx^2}\right)^2 + x\left(\dfrac{dy}{dx}\right)^5 x^2y = 0$ is

(a) 2 (b) 0 (c) 3 (d) 5 (Bhopal, 2008)

28. Integrating factor of the differential equation $\dfrac{dy}{dx} + y\cos x = \dfrac{\sin 2x}{2}$ is

(a) $e^{\sin^2 x}$ (b) $e^{\sin^3 x}$ (c) $e^{\sin x}$ (d) $\sin x$

29. The differential equation of the family of circles with centre as origin is ...... (Nagarjuna, 2008)

30. Solution of $x\,e^{-x^2}\,dx + \sin y\,dy = 0$ is .... (Nagarjuna, 2008)

31. Solution of $p = \sin(y - xp)$ is

(a) $y = \dfrac{c}{x} + \sin^{-1} c$ (b) $y = cx + \sin c$ (c) $y = cx + \sin^{-1} c$ (d) $y = x + \sin^{-1} c$ (V.T.U., 2011)

32. Differential equation obtained by eliminating A and B form $y = A\cos x + B\sin x$ is $d^2y/dx^2 - y = 0$ (True or False)

33. $(x^3 - 3xy^2)\,dx + (y^3 - 2x^2y)\,dy = 0$ is an exact differential equation. (True or False)

CHAPTER 12

Applications of Differential Equations of First Order

1. Introduction. **2.** Geometric applications. **3.** Orthogonal trajectories. **4.** Physical applications. **5.** Simple electric circuits. **6.** Newton's law of cooling. **7.** Heat flow. **8.** Rate of decay of radio-active materials. **9.** Chemical reactions and solutions. **10.** Objective Type of Questions.

12.1 INTRODUCTION

In this chapter, we shall consider only such practical problems which give rise to differential equations of the first order. The fundamental principles required for the formation of such differential equations are given in each case and are followed by illustrative examples.

12.2 GEOMETRIC APPLICATIONS

(*a*) *Cartesian coordinates.* Let $P(x, y)$ be any point on the curve $f(x, y) = 0$ (Fig. 12.1), then [as per 4.6 §(1) & 4.11(1) & (4)], we have

(*i*) slope of the tangent at $P(= \tan \psi) = dy/dx$

(*ii*) equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x-intercept $(= OT)$

$$= x - y \,.\, dx/dy$$

and y-intercept $(= OT') = y - x \,.\, dy/dx$

(*iii*) equation of the normal at P is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) length of the tangent $(= PT) = y\sqrt{[1 + (dx/dy)^2]}$

(*v*) length of the normal $(= PN) = y\sqrt{[1 + (dy/dx)^2]}$

(*vi*) length of the sub-tangent $(= TM) = y \,.\, dx/dy$

(*vii*) length of the sub-normal $(= MN) = y \,.\, dy/dx$

(*viii*) $\dfrac{ds}{dx} = [1 + (dy/dx)^2]$; $\dfrac{ds}{dy} = \sqrt{[1 + (dx/dy)^2]}$

(*ix*) differential of the area $= ydx$ or xdy

(*x*) ρ, radius of curvature at $P = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$

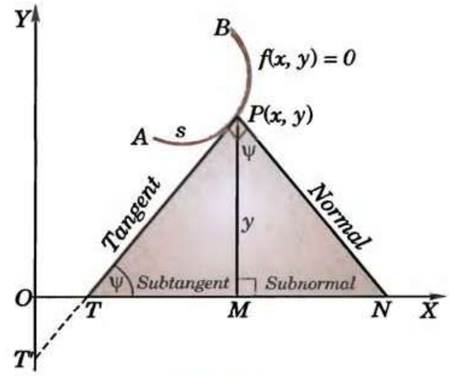


Fig. 12.1

(*b*) *Polar coordinates*. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ (Fig. 12.2), then [as per § 4.7, 4.9 (2) & 4.11 (4)], we have

(*i*) $\psi = \theta + \phi$

(*ii*) $\tan\phi = r d\theta/dr$, $p = r\sin\phi$

(*iii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*iv*) polar sub-tangent $(= OT) = r^2 d\theta/dr$

(*v*) polar sub-normal $(ON) = dr/d\theta$

(*vi*) $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]}; \dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

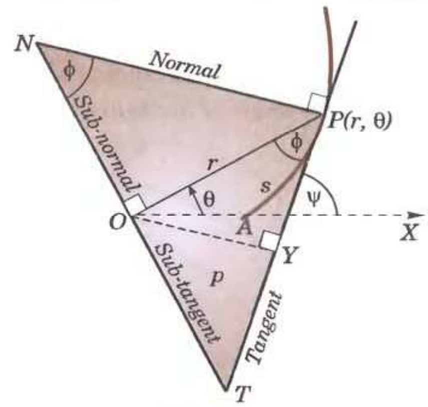


Fig. 12.2

Example 12.1. *Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.*

Solution. Let the tangent at any point $P(x, y)$ of a curve cut the axes at T and T' (Fig. 12.3).

We know that its x-intercept $(= OT) = x - y \,.\, dx/dy$

and y-intercept $(= OT') = y - x \,.\, dy/dx$

$\therefore$ the co-ordinates of T and T' are

$$(x - y \,.\, dx/dy, 0), (0, y - x\, dy/dx)$$

Since P is the mid-point of TT'

$$\therefore \quad \frac{[x - y.dx/dy] + 0}{2} = x$$

or $\qquad x - y \,.\, dx/dy = 2x$ or $x\,dy + y\,dx = 0$

or $\qquad d(xy) = 0$ Integrating, $xy = c$

which is the equation of a rectangular hyperbola, having x and y axes as its asymptotes.

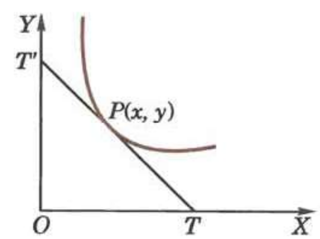


Fig. 12.3

Example 12.2. *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

Solution. Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

$$\tan\phi = r\, d\theta/dr$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP \text{ (Fig. 12.4).}$$

$$\therefore \quad \theta = \angle PON = 180° - (180° - 2\phi) = 2\phi$$

or $\qquad \theta/2 = \phi \quad \therefore \quad \tan\dfrac{\theta}{2} = \tan\phi = r\dfrac{d\theta}{dr}.$

Here the variables are separable.

$$\therefore \quad \frac{dr}{r} = \frac{\cos\theta/2}{\sin\theta/2} d\theta$$

Integrating both sides $\log r = 2\log\sin\theta/2 + \log c$

or $\qquad r = c\sin^2\theta/2 = \dfrac{1}{2}c(1 - \cos\theta)$

Thus the curve is the cardioid $r = a(1 - \cos\theta)$.

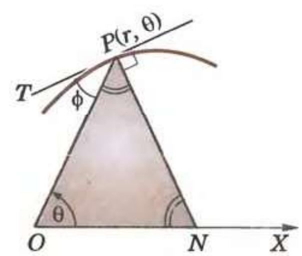


Fig. 12.4

Example 12.3. *Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.*

Solution. Taking the fixed source of light as the origin and the X-axis parallel to the reflected rays; the reflector will be a surface generated by the revolution of a curve $f(x, y) = 0$ about X-axis (Fig. 12.5).

In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.

If TPT' be the tangent at P, then

$\because$ angle of incidence = angle of reflection,

$\therefore$ $\phi = \angle OPT = \angle P'PT' = \angle OTP = \psi$

i.e., $$p = \frac{dy}{dx} = \tan \angle XOP = \tan 2\phi$$

$$= \frac{2\tan\phi}{1-\tan^2\phi} = \frac{2p}{1-p^2}$$

or $$2x = \frac{y}{p} - yp \text{ which is solvable for } x \qquad ...(i)$$

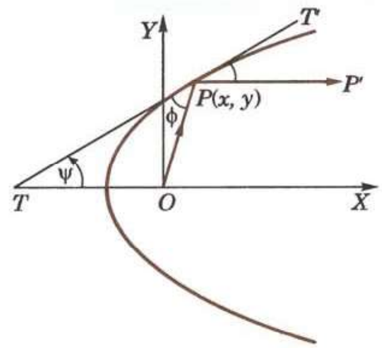


Fig. 12.5

$\therefore$ differentiating (i) w.r.t. y, $\dfrac{2}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - p - y\dfrac{dp}{dy}$

i.e., $$\left(\frac{1}{p}+p\right)+\left(\frac{1}{p^2}+1\right)y\frac{dp}{dy} = 0 \quad \text{or} \quad \left(\frac{1}{p}+p\right)\left(1+\frac{y}{p}\frac{dp}{dy}\right) = 0$$

This gives $dp/p = -dy/y$

Integrating, $\log p = \log c - \log y$, *i.e.,* $p = c/y$ ...(ii)

Thus eliminating p from (i) and (ii), we have family of curves $y^2 = 2cx + c^2$.

Hence the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2cx + c^2$.

PROBLEMS 12.1

1. Find the equation of the curve which passes through
   (i) the point $(3, -4)$ and has the slope $2y/x$ at the point (x, y) on it.
   (ii) the origin and has the slope $x + 3y - 1$.
2. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through (0, 1). Find the equation of the curve.
3. A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is
   $$y^2 - 2xy\frac{dy}{dx} - x^2 = 0, \text{ and hence find the curve.}$$
4. A plane curve has the property that the tangents from any point on the y-axis to the curve are of constant length a. Find the differential equation of the family to which the curve belongs and hence obtain the curve.
5. Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point (1, 2). *(Sambalpur, 1998)*
6. Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.
7. The tangent at any point of a certain curve forms with the coordinate axes a triangle of constant area A. Find the equation to the curve.
8. Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the x-axis is twice the cube of that ordinate.
9. Find the curve whose (i) polar sub-tangent is constant.
   (ii) polar sub-normal is proportional to the sine of the vectorial angle.
10. Determine the curve for which the angle between the tangent and the radius vector is twice the vectorial angle. *(Kanpur, 1996)*
11. Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.
12. Find the curve for which the tangent, the radius vector r and the perpendicular from the origin on the tangent form a triangle of area kr^2.

12.3 (1) ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called **orthogonal trajectories** *of each other (Fig. 12.6).*

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and *vice versa*. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

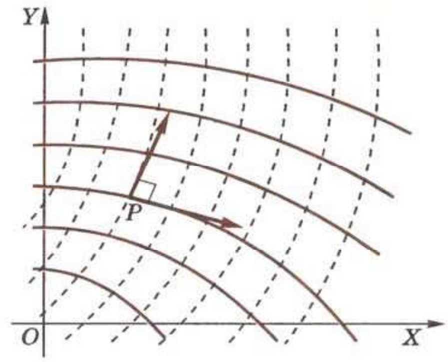


Fig. 12.6

(2) To find the orthogonal trajectories of the family of curves F(x, y, c) = 0.

(*i*) *Form its differential equation in the form $f(x, y, dy/dx) = 0$ by eliminating c.*

(*ii*) *Replace, in this differential equation, dy/dx by $-dx/dy$*, (so that the product of their slopes at each point of intersection is -1).

(*iii*) *Solve the differential equation of the orthogonal trajectories i.e., $f(x, y, -dx/dy) = 0$.*

Example 12.4. *If the stream lines (paths of fluid particles) of a flow around a corner are xy = constant find their orthogonal trajectories (called equipotential lines-§ 20.6)* (*Marathwada, 2008*)

Solution. Taking the axes as the walls, the stream lines of the flow around the corner of the walls is

$$xy = c \quad ...(i)$$

Defferentiating, we get, $$x\frac{dy}{dx} + y = 0 \quad ...(ii)$$

as the differential equation of the given family (*i*).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (*ii*), we obtain $x\left(-\frac{dx}{dy}\right) + y = 0$

or $$xdx - ydy = 0 \quad ...(iii)$$

as the differential equation of the orthogonal trajectories.

Integrating (*iii*), we get $x^2 - y^2 = c'$ as the required orthogonal trajectories of (*i*) *i.e.*, the *equipotential lines*, shown dotted in Fig. 12.7.

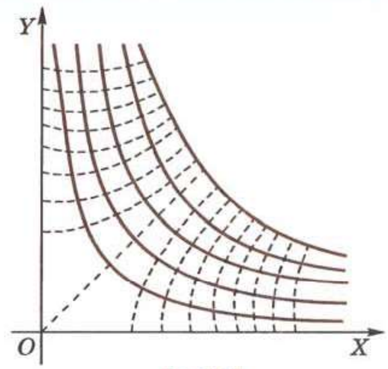


Fig. 12.7

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ is the parameter.* (*V.T.U., 2009 S*)

Solution. Differentiating the given equation, we get $\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$

or $$\frac{y}{a^2+\lambda} = -\frac{x}{a^2\,(dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2+\lambda} = \frac{-xy}{a^2\,(dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2\,(dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \quad ...(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\,dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int ydy = \int \frac{a^2 - x^2}{x}\,dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2\log x - \frac{1}{2}x^2 + c$$

or $$x^2 + y^2 = 2a^2\log x + c' \quad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \quad ...(i)$$

Differentiating, $$y\frac{dy}{dx} = 2a \quad ...(ii)$$

Substituting the value of a from (ii) in (i), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ as the differential equation of the family.} \quad ...(iii)$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (iii), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ which is the same as } (iii).$$

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.*, each member of the family (i) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(i) *Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.*

(ii) *Replace in this differential equation,*

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[∵ for the given curve through $P(r, \theta)$ $\tan\phi = rd\theta/dr$

and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

(iii) *Solve the differential equation of the orthogonal trajectories*

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

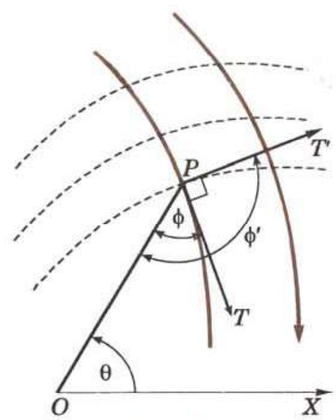


Fig. 12.8

Example 12.7. *Find the orthogonal trajectory of the cardioids $r = a(1 - \cos\theta)$.* (*Kurukshetra, 2005*)

Solution. Differentiating $$r = a(1 - \cos\theta). \quad ...(i)$$

with respect to θ, we get $$\frac{dr}{d\theta} = a\sin\theta \quad ...(ii)$$

Eliminating a from (i) and (ii), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2} \text{ which is the differential equation of the given family.}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2 \log \cos \theta/2 + \log c$

or $$r = c \cos^2 \theta/2 = \frac{1}{2} c(1 + \cos \theta) \quad \text{or} \quad r = a'(1 + \cos \theta)$$

which is the required orthogonal trajectory.

Example 12.8. *Find the orthogonal trajectory of the family of curves $r^n = a \sin n\theta$.* *(V.T.U., 2006)*

Solution. We have $n \log r = \log a + \log \sin n\theta$.

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad \tan n\theta.\ d\theta - \frac{dr}{r} = 0$$

Integrating, $$\int \frac{dr}{r} + \int \frac{\sin n\theta}{\cos n\theta}\, d\theta = c,$$

i.e., $$\log r - \frac{1}{n} \log \cos n\theta = c \quad \text{or} \quad \log (r^n/\cos n\,\theta) = nc = \log b. \text{ (say)}$$

or $r^n = b \cos n\theta$, which is the required orthogonal trajectory.

PROBLEMS 12.2

Find the orthogonal trajectories of the family of :

1. Parabolas $y^2 = 4ax$. *(Marathwada, 2009)*
2. Parabolas $y = ax^2$. *(J.N.T.U., 2006)*
3. Semi-cubical parabolas $ay^2 = x^3$. *(J.N.T.U., 2005)*
4. Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, λ being the parameter. *(J.N.T.U., 2006)*
5. Confocal conics $\dfrac{x^2}{a^2 + \lambda} + \dfrac{y^2}{b^2 + \lambda} = 1$, λ being the parameter. *(Kurukshetra, 2006)*
6. Cardioids $r = a(1 + \cos \theta)$. *(J.N.T.U., 2003)*
7. $r = 2a(\cos \theta + \sin \theta)$ *(V.T.U., 2010 S)*
8. Confocal and coaxial parabolas $r = 2a/(1 + \cos \theta)$. *(Nagpur, 2008)*
9. Curves $r^2 = a^2 \cos 2\theta$. *(V.T.U., 2009 S)*
10. $r^n \cos n\theta = a^n$. *(V.T.U. 2011)*
11. Show that the family of parabolas $x^2 = 4a(y + a)$ is self orthogonal. *(Kerala, 2005)*
12. Show that the family of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ are orthogonal. *(Mumbai, 2005)*
13. The *electric lines of force* of two opposite charges of the same strength at $(\pm 1, 0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their *equipotential lines* (orthogonal trajectories).

    [**Isogonal trajectories.** *Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (Say), are called* ***isogonal trajectories*** *of each other.* The slopes m, m' of the tangents to the corresponding curves at each point, are connected by the relation $\dfrac{m \square m'}{1 + mm'} = \tan a = \text{const.}$]
14. Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$ which intersect at 45°.

12.4 PHYSICAL APPLICATIONS

(1) Let a body of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P where $OP = x$, then

(*i*) its velocity $(v) = \dfrac{dx}{dt}$

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{vdv}{dx}$ or $\dfrac{d^2x}{dt^2}$

If, however, the body be moving along a curve, then

(*i*) its velocity $(v) = ds/dt$ and

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}, v\dfrac{dv}{ds}$ or $\dfrac{d^2s}{dt^2}$.

The quantity mv is called the *momentum*.

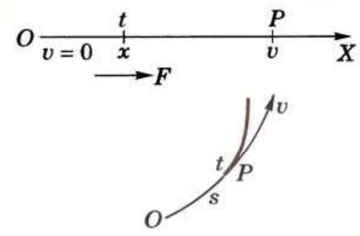


Fig. 12.9

(2) Newton's second law *states that* $F = \dfrac{d}{dt}(mv)$.

If m is constant, then $F = m\dfrac{dv}{dt} = ma$, *i.e., net force = mass × acceleration.*

(3) Hooke's law* *states that tension of an elastic string (or a spring) is proportional to extension of the string (or the spring) beyond its natural length.*

Thus $\quad\quad \mathbf{T = \lambda e/1},$

where e is the extension beyond the natural length l and λ is the *modulus of elasticity.*

Sometimes for a spring, we write $\mathbf{T = ke}$,

where e is the extension beyond the natural length and k is the *stiffness of the spring*.

(4) Systems of units

I. **F.P.S.** [foot (*ft.*) pound (*lb.*), second (*sec.*)] **system**. If mass m is in *pounds* and acceleration (a) is in *ft/sec*^{2}, then the force $F(= ma)$ is in *poundals*.

II. **C.G.S.** [centimetre (*cm.*), gram (*g*), second (*sec*)] **system**. If mass m is in *grams* and acceleration a is in *cm/sec*^{2} then the force $F(= ma)$ is *dynes*.

III. **M.K.S.** [metre (*m*), kilogram (*kg.*), second (*sec*)] **system**. If mass m is in kilograms and acceleration a in *m/sec*^{2}, then the force $F(= ma)$ is in *newtons* (*nt*).

These are called *absolute units*. If g is the acceleration due to gravity and w is the weight of the body, then w/g is the mass of the body in *gravitational units.*

$$g = 32 \text{ ft/sec}^2 = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 \text{ approx.}$$

Example 12.9. Motion of a boat across a stream. *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.*

Solution. Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 12.10.

At any time t after its start from O, let the boat be at $P(x, y)$, so that

dx/dt = velocity of the current $= ky(a - y)$

dy/dt = velocity with which the boat is being rowed $= u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \quad \quad ...(i)$$

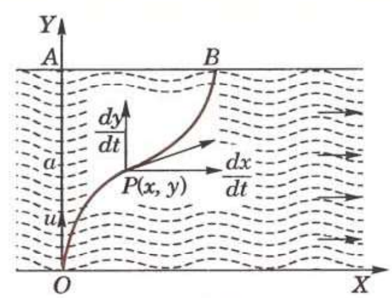


Fig. 12.10

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now (*i*) is of variables separable form and we can write it as

$$y(a - y)dy = \frac{u}{k}\, dx$$

Integrating, we get $\quad \dfrac{ay^2}{2} - \dfrac{y^3}{3} = \dfrac{u}{k}x + c$

Since $y = 0$ when $x = 0$, $\therefore\ c = 0$.

Hence the equation to the path of the boat is $x = \dfrac{k}{6u}y^2(3a - 2y)$

Putting $y = a$, we get the distance AB, down stream where the boat lands $= ka^3/6u$.

*Named after an English physicist *Robert Hooke* (1635–1703) who had discovered the law of gravitation earlier than Newton.

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* *(Marathwada, 2008)*

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or $$v\frac{dv}{dx} + bv^2 = -cx \qquad ...(i)$$

This is Bernoulli's equation. $\therefore$ Put $v^2 = z$ and $2v\, dv/dx = dz/dx$, so that (i) becomes

$$\frac{dz}{dx} + 2bz = -2cx \qquad ...(ii)$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

$\therefore$ the solution of (ii) is $\quad ze^{2bx} = -\displaystyle\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x \cdot \frac{e^{2bx}}{2b} - \int 1 \cdot \frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or $$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \qquad ...(iii)$$

Initially $v = 0$ when $x = 0$ $\therefore$ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (iii), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Example 12.11. Resisted vertical motion. *A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.* *(U.P.T.U., 2003)*

Solution. After falling a distance s in time t from rest, let v be velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance $m\lambda v$ upwards.

$\therefore$ equating of motion is $\quad m\dfrac{dv}{dt} = mg - m\lambda v$

or $$\frac{dv}{dt} = g - \lambda v \quad \text{or} \quad \frac{dv}{g - \lambda v} = dt$$

Integrating, $\displaystyle\int \frac{dv}{g - \lambda v} = \int dt + c$ or $\displaystyle -\frac{1}{\lambda}\log(g - \lambda v) = t + c$

Since $v = 0$ when $t = 0$, $\therefore$ $c = -\dfrac{1}{\lambda}\log g$

Thus $$\frac{1}{\lambda}\log\left[\frac{g}{g - \lambda v}\right] = t \quad \text{or} \quad \frac{g - \lambda v}{g} = e^{-\lambda t}$$

or $$\frac{ds}{dt} = v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \qquad ...(i)$$

Integrating, $$s = \frac{g}{\lambda}\int (1 - e^{-\lambda t})dt + c' \quad \text{or} \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c'$$

Since $s = 0$ when $t = 0$, $\therefore$ $c' = -g/\lambda^2$

Thus $$s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}(e^{-\lambda t} - 1) \qquad ...(ii)$$

Eliminating t from (i) and (ii), we get

$$s = \frac{g}{\lambda^2}\log\left(\frac{g}{g - \lambda v}\right) - \frac{v}{\lambda}$$

which is the desired relation between s and v.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv \frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv \frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{v dv}{a^2 - v^2} = \int \frac{k}{m} dx + c$

or $$-\frac{1}{2} \log (a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0, v = 0$. $\therefore -\frac{1}{2} \log a^2 = c$ ...(iii)

Subtracting (*iii*) from (*ii*), we have $\frac{1}{2} [\log a^2 - \log (a^2 - v^2)] = kx/m$

or $$\frac{2kx}{m} = \log \left(\frac{a^2}{a^2 - v^2} \right)$$

Obs. *When the resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the* **limiting** *or* **terminal** *velocity.* From (*i*), it follows that the acceleration will become zero when $v = a$. *Thus, the limiting velocity, i.e., the maximum velocity which the particle can attain is a.*

Example 12.13. Velocity of escape from the earth. *Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.*

Solution. According to Newton's law of gravitation, the acceleration α of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$$\alpha = v \frac{dv}{dr} = -\frac{\mu}{r^2}$$

where v is the velocity when at a distance r from the earth's centre. The acceleration is negative because v is decreasing. When $r = R$, the earth's radius then $\alpha = -g$, the acceleration of gravity at the surface.

i.e., $$-g = -\mu/R^2, \text{ i.e., } \mu = gR^2 \quad \therefore \quad v \frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain $\int v dv = -gR^2 \int \frac{dr}{r^2} + c$

i.e., $$v^2 = \frac{2gR^2}{r} + 2c \qquad ...(i)$$

On the earth's surface $r = R$ and $v = v_0$ (say), the initial velocity. Then

$$v_0^2 = 2gR + 2c, \quad i.e., \quad 2c = v_0^2 - 2gR$$

Inserting this value of c in (*i*), we get $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected fron the earth with this velocity will escape from the earth. Hence the minimum such velocity of projection $v_0 = \sqrt{(2gR)}$ *is called the velocity of escape* from the earth [See Problem 9, page 454].

Example 12.14. Rotating cylinder containing liquid. *A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the free surface of the water by a plane through the axis, is the curve*

$$y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h.$$

Solution. Let the figure represent an axial section of the cylindrical tank. Forces acting on a particle of mass m at $P(x, y)$ on the curve, cut out from the free surface of water, are :

(*i*) the weight mg acting vertically downwards,

(*ii*) the centrifugal force $m\omega^2x$ acting horizontally outwards.

As the motion is steady, P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. Thus the resultant R of mg and $m\omega^2x$ is along the outward normal to the curve.

$\therefore \quad R \cos \psi = mg$ and $R \sin \psi = m\omega^2 x$

whence $$\frac{dy}{dx} = \tan \psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad ...(i)$$

This is the differential equation of the surface of the rotating liquid.

Integrating (*i*), we get

$$\int dy = \frac{\omega^2}{g}\int x\,dx + c$$

i.e., $$y = \frac{\omega^2 x^2}{2g} + c \qquad ...(ii)$$

To find c, we note that the volume of the liquid remains the same in both cases (Fig. 12.11).

When $x = 0$ in (*ii*), $OA\ (= y) = c$. When $x = r$ in (*ii*), $h'\ (= y) = \dfrac{\omega^2 r^2}{2g} + c \qquad ...(iii)$

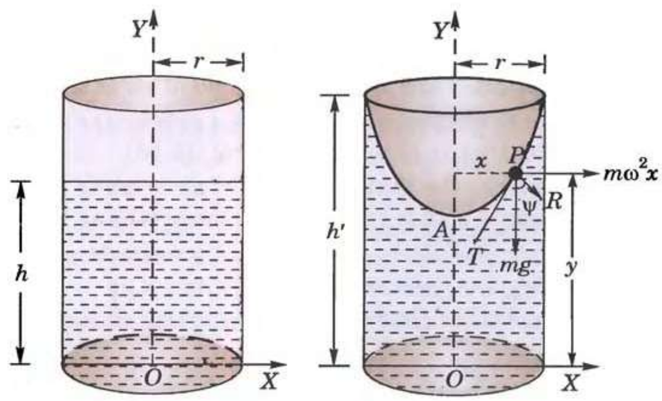


Fig 12.11

Now the volume of the liquid in the non-rotational case $= \pi r^2 h$, and the volume of the liquid in the rotational case

$$= \pi r^2 h' - \int_{OA}^{h'} \pi x^2 dy = \pi r^2 h' - \frac{2\pi g}{\omega^2}\int_c^{h'} (y - c)\,dy \qquad \text{[From (ii)]}$$

$$= \pi r^2 h' - \frac{\pi g}{\omega^2}(h' - c)^2 = \pi r^2\left(\frac{\omega^2 r^2}{4g} + c\right) \qquad \text{[By (iii)]}$$

Thus $$\pi r^2 h = \pi r^2\left(\frac{\omega^2 r^2}{4g} + c\right) \text{ whence } c = h - \frac{\omega^2 r^2}{4g}$$

$\therefore$ (*ii*) becomes, $$y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 x^2}{4g} \quad \text{or} \quad y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$$

which is the desired equation of the curve.

Example 12.15. Discharge of water through a small hole. *If the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$ where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area A_0 in the base.*

Solution. At any time t, let the height of the water level be y and radius of its surface be r (Fig. 12.12) so that

$$\frac{h - y}{r} = \frac{h}{a} \quad \text{or} \quad r = a(h - y)/h$$

$\therefore$ surface area of the liquid $= \pi r^2 = \pi a^2 (1 - y/h)^2$

Volume of water drained through the hole per unit time

$$= 0.6 \sqrt{(2gy)} A_0 = 4.8 \sqrt{y} A_0 \qquad [\because g = 32$$

$\therefore$ rate of fall of liquid level $= 4.8 A_0 \sqrt{y} \div \pi a^2 (1 - y/h)^2$

i.e., $\dfrac{dy}{dt} = -\dfrac{4.8 A_0 \sqrt{y}}{\pi a^2 (1 - y/h)^2}$ (– ve is taken since the water level decreases)

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2 (1 - y/h)^2}{4.8 A_0 \sqrt{y}} dy = \frac{\pi a^2}{4.8 A_0} \int_0^h (y^{-1/2} - 2y^{1/2}/h + y^{3/2}/h^2) dy$$

$$= \frac{\pi a^2}{4.8 A_0} \left| 2y^{1/2} - \frac{4}{3h} y^{3/2} + \frac{2}{5h^2} y^{5/2} \right|_0^h = 0.2 \pi a^2 \sqrt{h} / A_0.$$

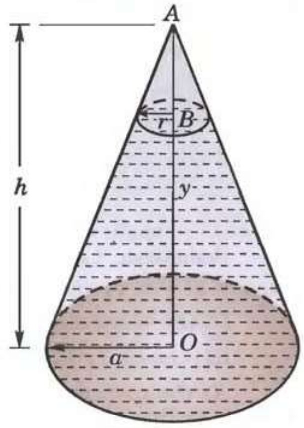


Fig. 12.12

Example 12.16. Atmospheric pressure. *Find the atmospheric pressure p lb. per ft. at a height z ft. above the sea-level, both when the temperature is constant or variable.*

Solution. Take a vertical column of air of unit cross-section.

Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height $z + \delta z$. Let ρ be the density at a height z. (Fig. 12.13)

Now since the thin column δz of air is being pressured upwards with pressure p and downwards with $p + \delta p$, we get by considering its equilibrium;

$$p = p + \delta p + g\rho\delta z.$$

Taking the limit, we get $\quad dp/dz = -g\rho \qquad$ ...(*i*)

which is the differential equation giving the atmospheric pressure at height z.

(*i*) *When the temperature is constant,* we have by Boyle's law*, $p = k\rho \qquad$ ...(*ii*)

$\therefore$ Substituting the value of ρ from (*ii*) in (*i*), we get

$$\frac{dp}{dz} = -gp/k \quad \text{or} \quad \int \frac{dp}{p} = -\frac{g}{k} \int dz + c \quad \text{or} \quad \log p = -\frac{g}{k} z + c$$

At the sea-level, where $z = 0$, $p = p_0$ (say) then $c = \log p_0$

$$\therefore \qquad \log p - \log p_0 = -\frac{g}{k} z \text{ i.e., } \log p/p_0 = -gz/k$$

Hence p is given by $p = p_0 e^{-gz/k}$.

(*ii*) *When the temperature varies,* we have $p = k\rho^n$.

Proceeding as above, we shall find that p is given by $\dfrac{n}{n-1}(p_0^{1-1/n} - p^{1-1/n}) = gk^{-1/n} z$.

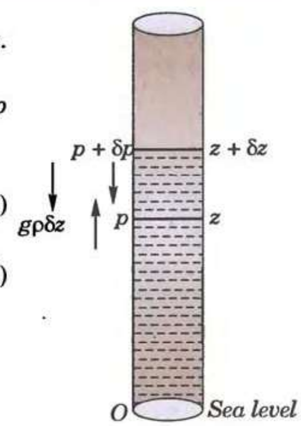


Fig. 12.13

PROBLEMS 12.3

1. A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\dfrac{m}{k} \log_e \left(1 + \dfrac{kv}{mg}\right)$.
2. A body of mass m falls from rest under gravity and air resistance proportional to square of velocity. Find velocity as function of time. (*Marathwada, 2008*)
3. A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.
4. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\dfrac{1}{\mu v}(\sqrt{1 + 2\mu v^2 t} - 1)$.

*Named after the English physicist *Robert Boyle* (1627–1691) who was one of the founders of the Royal Society.

5. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?
6. A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch x feet.
7. A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv \dfrac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{(x/3)}$ ft/sec.
8. A toboggan weighing 200 lb., descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the coefficient of friction is 1/10 and the air resistance varies as the square of the velocity and is 3 lb. weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity cannot exceed 44 ft per sec. [**Hint.** Fig. 12.14. Equation of motion is
$$\frac{W}{g} \cdot v\frac{dv}{dx} = -\mu R - kv^2 + W \sin \alpha]$$

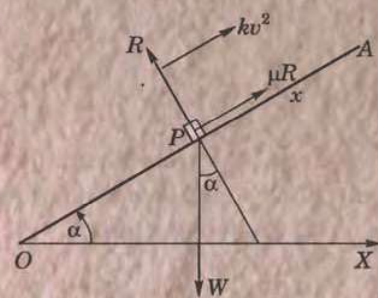


Fig. 12.14

9. Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth. [Take earth's radius = 3960 miles and g = 32.17 ft/sec²].
10. A cylindrical tank 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the times at which the tank is one-half full, one quarter full, and empty.

    [**Hint.** Take g = 980 cm/sec² in $v = 0.6\sqrt{(2gy)}$]
11. The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.
12. A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downwards. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $(\pi h \tan^2 \alpha)/k$ seconds.
13. Upto a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $(n > 1)$. If this relation continued to hold upto any height, show that the density would vanish at a finite height.

12.5 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

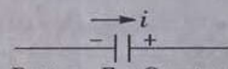
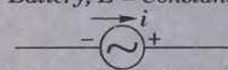
(*i*) three passive elements—resistance, inductance, capacitance and

(*ii*) an active element—voltage source which may be a battery or a generator.

(1) Symbols

Element	*Symbol*	*Unit**
1. *Quantity of electricity*	q	coulomb
2. *Current* (= time rate flow of electricity)	i	ampere (A)
3. *Resistance, R*	$+ \xrightarrow{i} -$ R	ohm (Ω)
4. *Inductance, L*	$+ \xrightarrow{i} -$ L	henry (H)
5. *Capacitance, C*	$+ \xrightarrow{i} -$ C	farad (F)

*These units are respectively named after the French engineer and physicist *Charles Augustin de Coulomb* (1736–1806); French physicist *Andre Marie Ampere* (1775–1836); German physicist *George Simon Ohm* (1789–1854); Italian physicist *Joseph Henry* (1797–1878); American physicist *Michael Faraday* (1791–1867) and the Italian physicist *Alessandro Volta* (1745–1827).

Element	*Symbol*	*Unit*
6. *Electromotive force (e.m.f.) or voltage, E*	 *Battery, E = Constant*  *Generator, E = Variable*	volt (V)

7. *Loop* is any closed path formed by passing through two or more elements in series.
8. *Nodes* are the terminals of any of these elements.

(2) Basic relations

(*i*) $i = \frac{dq}{dt}$ or $q = \int i dt$ [$\because$ current is the rate of flow of electricity]

(*ii*) Voltage drop across resistance $R = Ri$ [Ohm's Law]

(*iii*) Voltage drop across inductance $L = L\frac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \frac{q}{C}$.

(3) Kirchhoff's laws*. The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. *The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.*

II. *The algebraic sum of the currents flowing into (or from) any node is zero.*

(4) Differential equations

(*i*) *R, L series circuit.* Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. (Fig. 12.15).

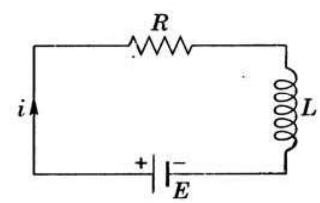


Fig. 12.15

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and $L = E$

i.e., $$Ri + L\frac{di}{dt} = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad ...(1)$$

This is a Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{Rt/L} \text{ and therefore, its solution is } i(\text{I.F.}) = \int \frac{E}{L}(\text{I.F.})dt + c$$

or $$i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\, dt + c = \frac{E}{L} \cdot \frac{L}{R} \cdot e^{Rt/L} + c \text{ whence } i = \frac{E}{R} + ce^{-Rt/L} \qquad ...(2)$$

If initially there is no current in the circuit, *i.e.*, $i = 0$, when $t = 0$, we have $c = -E/R$.

Thus (2) becomes $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$ which shows that i increases with t and attains the maximum value E/R.

(*ii*) *R, L, C series circuit.* Now consider a circuit containing resistance R, inductance L and capacitance C all in series with a constant e.m.f. E (Fig. 12.16)

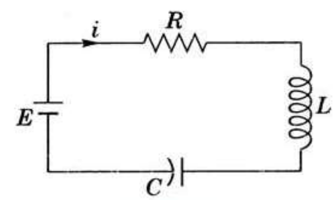


Fig. 12.16

If i be the current in the circuit at time t, then the charge q on the condenser $= \int i\, dt$, *i.e.*, $i = \frac{dq}{dt}$.

Applying Kirchhoff's law, we have, sum of the voltage drops across R, L and $C = E$.

i.e., $$Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

*Named after the German physicist *Gustav Robert Kirchhoff* (1824–1887).

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E.$$

This is the desired differential equation of the circuit and will be solved in § 14.5.

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(*Kurukshetra, 2005*)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e., $$Ri + L\frac{di}{dt} = E \sin \omega t.$$

This is the required differential equation which can be written as $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{E}{L}\sin \omega t$

This is a *Leibnitz's equation*. Its I.F. $= e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int \dfrac{E}{L}\sin \omega t \cdot (\text{I.F.})dt + c$

or $$ie^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\sin \omega t\, dt + c = \frac{E}{L}\frac{e^{Rt/L}}{\sqrt{[(R/L)^2 + \omega^2]}}\sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2L^2)}}\sin(\omega t - \phi) + ce^{-Rt/L} \quad \text{where } \tan \phi = L\omega/R \qquad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0 = \dfrac{E\sin(-\phi)}{\sqrt{(R^2 + \omega^2L^2)}} + c$, *i.e.,* $c = \dfrac{E\sin\phi}{\sqrt{(R^2 + \omega^2L^2)}}$

Thus (i) takes the form $i = \dfrac{E\sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2L^2)}} + \dfrac{E\sin\phi}{\sqrt{(R^2 + \omega^2L^2)}} \cdot e^{-Rt/L}$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2L^2)}}[\sin(\omega t - \phi) + \sin\phi \,.\, e^{-Rt/L}]$$ which gives the current at any time t.

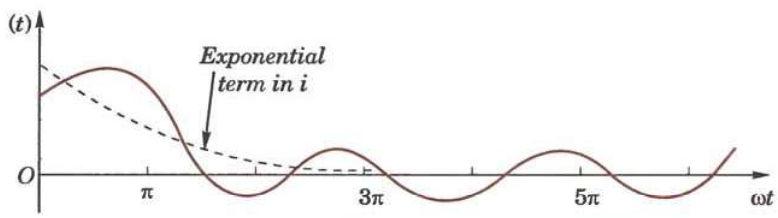


Fig. 12.17

Obs. As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only (Fig. 12.17).

PROBLEMS 12.4

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\, di/dt + Ri = E$.
   Find i as a function of t. How long will it be, before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?
2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\, di/dt + Ri = E$.
   If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ sec, if $i = 0$ at $t = 0$. *(Marathwada, 2008)*
4. The equation of electromotive force in terms of current i for an electrical circuit having resistance R and condenser of capacity C in series, is
$$E = Ri + \int \frac{idt}{C}$$
Find the current i at any time t when $E = E_m \sin \omega t$. *(S.V.T.U., 2008, P.T.U., 2006)*
5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. $E \sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations
$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = pE \cos pt.$$
Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $(E \sin pt)/R$.

12.6 NEWTON'S LAW OF COOLING*

According to this law, *the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$
Integrating,
$$\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c, \qquad \text{where } c \text{ is a constant.}$$
or
$$\log(\theta - 40) = -kt + \log c \quad i.e., \quad \theta - 40 = ce^{-kt} \qquad ...(i)$$
When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$. $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

Thus (i) becomes $\quad \theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$

When $t = 40$ min., $\quad \theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \frac{1}{4} = 50°C.$

12.7 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(i) *Heat flows from a higher temperature to the lower temperature.*
(ii) *The quantity of heat in a body is proportional to its mass and temperature.*
(iii) *The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.*

If q (cal./sec.) be the quantity of heat that flows across a slab of area α (cm²) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (iii) above
$$q = -k\alpha dT/dx \qquad ...(A)$$
where k is a constant depending upon the material of the body and is called the *thermal conductivity*.

*Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

Example 12.19. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.*

Solution. Let q cal./sec. be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. and length 1 cm (Fig. 12.18). Then the area of the lateral surface (belt) $= 2\pi x$.

$\therefore$ the equation (A) above gives

$$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have

$$T = -\frac{q}{2\pi k} \log_e x + c$$

Since $T = 150$, when $x = 10$. $\therefore$ $150 = -\frac{q}{2\pi k} \log_e 10 + c$ ...(*i*)

Again since $T = 40$, when $x = 15$, $40 = -\frac{q}{2\pi k} \log_e 15 + c$ ...(*ii*)

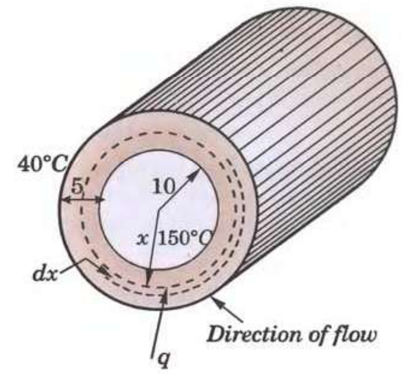


Fig. 12.18

Subtracting (*ii*) from (*i*), $110 = \frac{q}{2\pi k} \log_e 1.5$ ...(*iii*)

Let $T = t$, when $x = 12.5$ $\therefore$ $t = -\frac{q}{2\pi k} \log_e 12.5 + c$ ...(*iv*)

Subtracting (*i*) from (*iv*), $t - 150 = -\frac{q}{2\pi k} \log_e 1.25$ ...(*v*)

Dividing (*v*) by (*iii*), $\frac{t - 150}{110} = -\frac{\log_e 1.25}{\log_e 1.5}$, whence $t = 89.5$°C.

PROBLEMS 12.5

1. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.
2. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
3. Two friends A and B order coffee and receive cups of equal temperature at the same time. A adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, B waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?
4. A pipe 20 cm. in diameter contains steam at 200°C. It is covered by a layer of insulations 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from two metre length of the pipe.
5. A steam pipe 20 cm. in diameter contains steam at 150°C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

12.8 RATE OF DECAY OF RADIO-ACTIVE MATERIALS

This law states that disintegration at any instant is proportional to the amount of material present.

of material at any time t, then $\frac{du}{dt} = -ku$, where k is a constant.

Example 12.20. *Uranium disintegrates at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.*

Solution. Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is $\frac{dm}{dt} = -\mu m$, where μ is a constant.

Integrating, we get $\int \frac{dm}{dt} = -\mu \int dt + c$ or $\log m = c - \mu t$ ...(i)

Initially, when $t = 0$, $m = M$ (say) so that $c = \log M$ $\therefore$ (i) becomes, $\mu t = \log M - \log m$ ...(ii)

Also when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$ ...(iii)

$\mu T_2 = \log M - \log M_2$ ...(iv)

Subtracting (iii) from (iv), we get

$$\mu(T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2) \text{ whence } \mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T. *i.e.*, when $t = T$, $m = \frac{1}{2}M$.

$\therefore$ from (ii), we get $\mu T = \log M - \log (M/2) = \log 2$.

Thus $$T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1)\log 2}{\log (M_1/M_2)}.$$

12.9 CHEMICAL REACTIONS AND SOLUTIONS

A type of problems which are especially important to chemical engineers are those concerning either chemical reactions or chemical solutions. These can be best explained through the following example :

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* *(Andhra, 1997)*

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

= 2 gal. × 2 lb. = 4 lb./min.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

= 2 gal. × C lb. = $2C$ lb./min.

$$\therefore \quad \frac{du}{dt} = 4 - 2C \quad \text{...(i)}$$

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

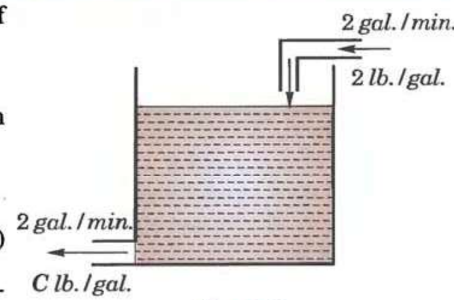


Fig. 12.19

$\therefore$ (i) becomes $\frac{du}{dt} = 4 - 2\frac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \quad \text{...(ii)}$$

Initially when $t = 0$, $u = 0$ $\therefore$ $0 = -25 \log_e 100 + k$ ...(iii)

Eliminating k from (ii) and (iii), we get $t = 25 \log_e \frac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$= 25 \log_e 3 = 25 \times 1.0986 = 27$ min. 28 sec.

PROBLEMS 12.6

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours ? *(Nagarjuna, 2008 ; J.N.T.U., 2003)*
2. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple ? *(Andhra, 2000)*
3. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years ?
4. If 30% of radio active substance disappeared in 10 days, low long will it take for 90% of it to disappear ? *(Madras, 2000 S)*
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm. at time $t = 0$, 8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a-x)(b-x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
7. A tank contains 1000 gallons of brine in which 500 lt. of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons /minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. of salt is left in the tank ?

   [**Hint.** If u be the amount of salt after t minutes, then $du/dt = -10u/1000$.]
8. A tank is initially filled with 100 gallons of salt solution containing 1 lb. of salt per gallon. Fresh brine containing 2 lb. of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.

12.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 12.7

Fill up the blanks or choose the correct answer in the following problems :

1. If a coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply then the current after 2 secs is .....
2. A tennis ball dropped from a height of 6 m, rebounds infinitely often. If it rebounds 80% of the distance that it falls, then the total distance for these bounces is .....
3. Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years then ......% will remain after 100 years.
4. The curve whose polar subtangent is constant is ......
5. The curve in which the length of the subnormal is proportional to the square of the ordinate, is ......
6. The curve in which the portion of the tangent between the axes is bisected at the point of contact, is ......
7. If the stream lines of a flow around a corner are $xy = c$, then the equipotential lines are ......
8. The orthogonal trajectories of a system of confocal and coaxial parabolas is ......
9. When a bullet is fired into a sand tank, its retardation is proportional to $\sqrt{\text{(velocity)}}$. If it enters the sand tank with velocity v_o, it will come to rest after ...... seconds.
10. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in two hours, then it will triple after ..... hours.
11. Ram and Sunil order coffee and receive cups simultaneously at equal temperature. Ram adds a spoon of cold cream but doesn't drink for 10 minutes, Sunil waits for 10 minutes and adds a spoon of cold cream and begins to drink. Who drinks the hotter coffee ?
12. The equation $y - 2x = c$ represents the orthogonal trajectories of the family

    (*i*) $y = ae^{-2x}$ (*ii*) $x^2 + 2y^2 = a$ (*iii*) $xy = a$ (*iv*) $x + 2y = a$.

13. In order to keep a body in air above the earth for 12 seconds, the body should be thrown vertically up with a velocity of

(a) $\sqrt{6}$ g m/sec (b) $\sqrt{12}$ g m/sec (c) 6 g m/sec (d) 12g m/sec.

14. The orthogonal trajectory of the family $x^2 + y^2 = c^2$ is

(a) $x + y = c$ (b) $xy = c$ (c) $x^2 + y^2 = x + y$ (d) $y = cx$. *(V.T.U., 2010)*

15. If a thermometer is taken outdoors where the temperature is 0°C, from a room having temperature 21°C and the reading drops to 10°C in 1 minute then its reading will be 5°C after ......minutes.

16. The equation of the curve for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = 2a \sin 2\theta$. This satisfies the differential equation

(a) $r\dfrac{dr}{d\theta} = \tan 2\theta$ (b) $r\dfrac{dr}{d\theta} = \cos 2\theta$ (c) $r\dfrac{d\theta}{dr} = \tan 2\theta$ (d) $r\dfrac{d\theta}{dr} = \cos 2\theta$.

17. Two balls of m_1 and m_2 grams are projected vertically upwards such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise be h_1 and h_2 respectively then

(a) $h_1 = 2h_2$ (b) $2h_1 = h_2$ (c) $h_1 = 4h_2$ (d) $4h_1 = h_2$.

18. Two balls are projected simultaneously with same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1 and t_2, then the height of the tower is

(a) $\dfrac{1}{2} g t_1 t_2$ (b) $\dfrac{1}{2} g(t_1^2 + t_2^2)$ (c) $\dfrac{1}{2} g(t_1^2 - t_2^2)$ (d) $\dfrac{1}{2} g(t_1 + t_2)^2$.

19. A particle projected from the earth's surface with a velocity of 7 miles/sec will return to the earth.
(Taking $g = 32.17$ and earth's radius = 3960 miles) (True/False)

20. If a particle falls under gravity with air resistance k times its velocity, then its velocity cannot exceed g/k.
(True/False)

CHAPTER 13

Linear Differential Equations

1. Definitions. 2. Complete solution. 3. Operator D. 4. Rules for finding the Complementary function. 5. Inverse operator. 6. Rules for finding the particular integral. 7. Working procedure. 8. Two other methods of finding P.I.—Method of variation of parameters ; Method of undetermined coefficients. 9. Cauchy's and Legendre's linear equations. 10. Linear dependence of solutions. 11. Simultaneous linear equations with constant coefficients. 12. Objective Type of Questions.

13.1 DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2 (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = 0 \qquad \ldots(1)$$

then $c_1 y_1 + c_2 y_2 \ (= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \ldots + k_n y_1 = 0 \qquad \ldots(2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \ldots + k_n y_2 = 0 \qquad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1}(c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \ldots + k_n (c_1 y_1 + c_2 y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \dots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + k_n y = 0 \qquad \dots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \dots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = X \qquad \dots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \dots + k_n v = X \qquad \dots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1 \dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \dots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13.3 OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \dots + k_n)y = X$, *i.e.*, $f(D)y = X$,
where $f(D) = D^n + k_1D^{n-1} + \dots + k_n$, *i.e.*, a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., $f(D)$ can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

13.4 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1} y}{dx^{n-1}} + k_2 \dfrac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = 0 \qquad \dots(1)$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = 0 \qquad \dots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2) \dots (D - m_n)y = 0 \qquad \dots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e., $m_1 = m_2$)*, then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \qquad [\because \ c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$y e^{-m_1 x} = \int c_1 e^{m_1 x} \, dx + c_2 = c_1 x + c_2 \quad \text{or} \quad y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3) \, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.,* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta$]

$$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.,* $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \ldots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\frac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\frac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0, x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ (*i*)

When $t = 0, dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (*i*) and (*ii*), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\, e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0 \quad \therefore \quad D = -1, \pm 2i.$

Hence the C.S. is $y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x.$

Example 13.4. *Solve* (*i*) $(D^4 - 4D + 4)\, y = 0$ *(Bhopal, 2008)*
(*ii*) $(D^2 + 1)^3\, y = 0$ *where* $D \equiv d/dx$.

Solution. (*i*) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x})$ [Roots being repeated]

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i.$

Hence the C.S. is $y = e^{0x}\, [(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x.$

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\, x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{t}(c_3 \cos t + c_4 \sin t)$.

PROBLEMS 13.1

Solve :

1. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 13x = 0, x(0), \dfrac{dx(0)}{dt} = 2.$ *(V.T.U., 2008)*
2. $y'' - 2y' + 10y = 0, y(0) = 4, y'(0) = 1.$
3. $4y''' + 4y'' + y' = 0.$
4. $\dfrac{d^3y}{dx^3} + y = 0.$ *(V.T.U., 2000 S)*
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$
6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ *(J.N.T.U., 2005)*
7. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0.$ *(V.T.U., 2008)*
8. $(D^2 + 1)^2 (D - 1)y = 0.$
9. If $\dfrac{d^4x}{dt^4} = m^4 x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.,
$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2)
$$\mathbf{\frac{1}{D}X = \int X dx}$$

Let
$$\frac{1}{D}X = y \qquad \text{...}(i)$$

Operating by D,
$$D\frac{1}{D}X = Dy \quad i.e., \; X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (i) does not contain any constant.

Thus
$$\frac{1}{D}X = \int X\,dx.$$

(3)
$$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.}$$

Let
$$\frac{1}{D-a}X = y \qquad \text{...}(ii)$$

Operating by $D - a$, $(D-a) \cdot \frac{1}{D-a}X = (D-a)y.$

or
$$X = \frac{dy}{dx} - ay, \; i.e., \; \frac{dy}{dx} - ay = X \text{ which is a Leibnitz's linear equation.}$$

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (ii) doesn't contain any constant.

Thus
$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = X.$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n}X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + \ldots + k_n)e^{ax} = (a^n + k_1a^{n-1} + \ldots + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}, \frac{1}{f(D)}f(D)e^{ax} = \frac{1}{f(D)}f(a)\,e^{ax}$ or $e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\mathbf{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}} \quad \textit{provided } f(a) \neq 0 \qquad \text{...(1)}$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\,\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(D)}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(a)}e^{ax} \qquad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)}\cdot\frac{1}{D-a}e^{ax} = \frac{1}{\phi(a)}\cdot e^{ax}\int e^{ax}\cdot e^{-ax}dx \qquad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)}e^{ax}\int dx = x\frac{1}{\phi(a)}e^{ax} \quad i.e., \quad \mathbf{\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}} \qquad \text{...(2)}$$

$$\left[\begin{array}{l} \because\ f'(D) = (D-a)\phi'(D) + 1\,.\,\phi(D) \\ \therefore\ f'(a) = 0 \times \phi'(a) + \phi(a) \end{array}\right.$$

If $f'(a) = 0$, then applying (2) again, we get $\mathbf{\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of* $(D^2 + 5D + 6)y = e^x$.

Solution. P.I. $= \frac{1}{D^2 + 5D + 6}e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6}e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of* $(D + 2)(D - 1)^2 y = e^{-2x} + 2\sinh x$.

Solution. P.I. $= \frac{1}{(D+2)(D-1)^2}[e^{-2x} + 2\sinh x] = \frac{1}{(D+2)(D-1)^2}[e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\cdot\left[\frac{1}{(D-1)^2}e^{-2x}\right]$$

$$= \frac{1}{D+2}\cdot\frac{1}{(-2-1)^2}e^{-2x} = \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x}$$

$$= \frac{1}{9}\cdot x\cdot\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x} \qquad \left[\because\ \frac{d}{dD}(D+2) = 1\right]$$

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{1}{1+2}\cdot\frac{1}{(D-1)^2}e^x = \frac{1}{3}\cdot x^2\cdot\frac{1}{2}e^x = \frac{x^2}{6}e^x \qquad \left[\because\ \frac{d^2}{dD^2}(D-1)^2 = 2\right]$$

and

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{e^{-x}}{4}$$

Hence, P.I. $= \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since

$$D\sin(ax + b) = a\cos(ax + b)$$
$$D^2\sin(ax + b) = -a^2\sin(ax + b)$$
$$D^3\sin(ax + b) = -a^3\cos(ax + b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e.,
$$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$
$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general
$$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$

$\therefore$
$$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)}\cdot f(D^2)\ \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or
$$\sin(ax+b) = f(-a^2)\,\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$. $\mathbf{\dfrac{1}{f(D^2)}\ \sin(ax+b) = \dfrac{1}{f(-a^2)}\ \sin(ax+b)}$ *provided* $f(-a^2) \neq 0$ ...(4)

If $\mathbf{f(-a^2) = 0}$, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$$\therefore \quad \frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)} e^{i(ax+b)} \qquad [\text{Since } f(-a^2) = 0 \quad \therefore \quad \text{by (2)}]$$

$$= \text{I.P. of } x\frac{1}{f'(D^2)} e^{i(ax+b)} \qquad \text{where } D^2 = -a^2$$

$\therefore$ $\mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = x\ \dfrac{1}{f'(-a^2)}\ \sin(ax+b)}$ *provided* $f'(-a^2) \neq 0$ ...(5)

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cdot\sin(ax+b) = x^2\ \dfrac{1}{f''(-a^2)}\ \sin(ax+b)}$, *provided* $f''(-a^2) \neq 0$, and so on.

Similarly, $\mathbf{\dfrac{1}{f(D^2)}\ \cos(ax+b) = \dfrac{1}{f(-a^2)}\ \cos(ax+b)}$, *provided* $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x\cdot\dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\ \dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2) \neq 0$ and so on.

Example 13.8. *Find the P.I. of* $(D^3 + 1)y = \cos(2x - 1)$.

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$$= \frac{1}{D(-4)+1}\ \cos(2x-1) \qquad [\text{Multiply and divide by } 1 + 4D]$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D)\cdot\frac{1}{1-16D^2}\cos(2x-1) \qquad [\text{Put } D^2 = -2^2 = -4]$$

$$= (1+4D)\frac{1}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)].$$

Example 13.9. *Find the P.I. of* $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.

Solution. Given equation in *symbolic form* is $(D^3 + 4D)y = \sin 2x$

$\therefore$ $$\text{P.I.} = \frac{1}{D(D^2+4)}\sin 2x$$ [$\because$ $D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (5) 477]

$$= x\frac{1}{3D^2+4}\sin 2x$$ $\left[\because \frac{d}{dD}[D^3+4D] = 3D^2+4\right]$

[Put $D^2 = -2^2 = -4$]

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x.$$

Case III. When $\mathbf{X = x^m}$.

Here $$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

Solution. Given equation in *symbolic form* is $(D^2 + D)y = x^2 + 2x + 4$.

$\therefore$ $$\text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1 - D + D^2 - \ldots)(x^2+2x+4) = \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int (x^2+4)dx = \frac{x^3}{3} + 4x.$$

Case IV. When $\mathbf{X = e^{ax} V}$, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $$D^n(e^{ax}u) = e^{ax}(D+a)^n u$$

$\therefore$ $$f(D)(e^{ax}u) = e^{ax} f(D+a)u$$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\mathbf{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.} \quad \ldots(6)$$

Example 13.11. *Find P.I. of* $(D^2 - 2D + 4)y = e^x \cos x$.

Solution. $$\text{P.I.} = \frac{1}{D^2-2D+4}e^x\cos x$$ [Replace D by $D+1$]

$$= e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x = e^x\frac{1}{D^2+3}\cos x$$ [Put $D^2 = -1^2 = -1$]

$$= e^x\frac{1}{-1+3}\cos x = \frac{1}{2}e^x\cos x.$$

Case V. When X is any other function of x.

Here $\quad \text{P.I.} = \dfrac{1}{f(D)}X.$

If $\quad f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$$\therefore \quad \text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right]X$$

$$= A_1 \frac{1}{D - m_1}X + A_2 \frac{1}{D - m_2}X + \ldots + A_n \frac{1}{D - m_n}X$$

$$= A_1 . e^{m_1 x} \int Xe^{-m_1 x}dx + A_2 . e^{m_2 x} \int Xe^{-m_2 x}dx + \ldots + A_n . e^{m_n x} \int Xe^{-m_n x}dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n)y = X.$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e., $\quad D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n = 0$ and solve it for D.

(*ii*) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two real and equal roots)	$(c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. $m_1, m_1, m_1, m_4 \ldots$ (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \ldots$ (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$ (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1}D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)}X$

(*i*) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x\frac{1}{f'(D)}e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)}e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $\quad f'(D) =$ diff. coeff. of $f(D)$ w.r.t. D

$f''(D) =$ diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin (ax + b)$ or $\cos (ax + b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin (ax + b) \text{ [or } \cos (ax + b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin (ax + b) \text{ [or } \cos (ax + b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin (ax + b) \text{ [or } \cos (ax + b)\text{], put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$ as in (*i*), (*ii*), and (*iii*).

(*v*) *When X is any function of x.*

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1 - e^x)^2$

(*i*) *To find C.F.*

Its A.E. is $D^2 + D + 1 = 0$, $\therefore \quad D = \frac{1}{2}(-1 \pm \sqrt{3}i)$

Thus C.F. $= e^{-x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + D + 1} (1 - 2e^x + e^{2x}) = \frac{1}{D^2 + D + 1} (e^{0x} - 2e^x + e^{2x})$$

$$= \frac{1}{0^2 + 0 + 1} e^{0x} - 2 . \frac{1}{1^2 + 1 + 1} e^x + \frac{1}{2^2 + 2 + 1} e^{2x} = 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$$

(*iii*) Hence the C.S. is $y = e^{-x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$, $y(0) = 1$ and $y'(0) = 0$.* (J.N.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

(*i*) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2x)e^{-2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x) = \frac{1}{-1 + 4D + 4}(3 \sin x + 4 \cos x)$$

$$= \frac{4D - 3}{16D^2 - 9}(3 \sin x + 4 \cos x) = \frac{(4D - 3)}{-16 - 9}(3 \sin x + 4 \cos x)$$

$$= \frac{-1}{25}\{3(4 \cos x - 3 \sin x) + 4(-4 \sin x - 3 \cos x)\} = \sin x$$

(*iii*) C.S. is $y = (c_1 + c_2)\, e^{-2x} + \sin x$

When $x = 0, y = 1$, $\therefore$ $1 = c_1$

Also $y' = c_2e^{-2x} + (c_1 + c_2x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0$, $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x)\, e^{-2x} + \sin x$.

Example 13.14. *Solve $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2x)\, e^{2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now $\dfrac{1}{(D-2)^2}e^{2x} = x^2\dfrac{1}{2(1)}e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2e^{2x}}{2}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{(-2^2) - 4D + 4}\sin 2x$$

$$= -\frac{1}{4}\int \sin 2x\, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

and

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}.x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right]x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right)x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(*iii*) Hence the C.S. is $y = (c_1 + c_2x)\, e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of $y'' - 2y' + 2y = x + e^x \cos x$.* (U.P.T.U., 2002)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x \cos x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \dfrac{2 \pm \sqrt{(4 - 8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x\,(c_1 \cos x + c_2 \sin x)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x\frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x\, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(*iii*) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

(V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2}(e^{3x} \cdot x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2 - 3(D+3) + 2}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x) = \frac{e^{3x}}{2} \cdot \left[1 + \left\{\frac{3D + D^2}{2}\right\}\right]^{-1} x - \frac{(3D-2)}{9(-4) - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2} \ldots\right)x + \frac{1}{40}(6 \cos 2x - 2 \sin 2x) = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3 \cos 2x - \sin 2x)$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3 \cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x$. *(Madras, 2000 S)*

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 4}x \sinh x = \frac{1}{D^2 - 4}x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 - 4}e^x \cdot x - \frac{1}{D^2 - 4}e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}x - e^{-x}\frac{1}{(D-1)^2 - 4}x\right] = \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3}x - e^{-x}\frac{1}{D^2 - 2D - 3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}.x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}\cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}+\ldots\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right)-e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

(*iii*) Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2-1)y = x\sin 3x + \cos x$.

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 1 = 0$, whence $D = \pm 1$. $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2-1}(x\sin 3x + \cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix}.x + \frac{1}{(-1)^2-1}\cos x = \text{I.P. of }\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right]-\frac{\cos x}{2}$$

[Replacing D by $D + 3i$]

$$= \text{I.P. of }\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right]-\frac{\cos x}{2} \qquad \text{[Expand by Binomial theorem]}$$

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1+\frac{3iD}{5}+\ldots\ldots\right)x\right]-\frac{\cos x}{2} = \text{I.P. of }\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of }\left[\frac{-1}{10}(\cos 3x + i\sin 3x)\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\text{ I.P. of }\left[\left(x\cos 3x - \frac{3\sin 3x}{5}\right)+i\left(x\sin 3x+\frac{3}{5}\cos 3x\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x+\frac{3}{5}\cos 3x\right)-\frac{\cos x}{2}.$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. (*S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005*)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 1)\,y = xe^x \sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 1 = 0$, *i.e.*, $(D-1)^2 = 0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \,.\, x \sin x = e^x \,.\, \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \qquad \text{[Integrate by parts]}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1 \,.\, (-\cos x)\, dx\right] = e^x \int [-x \cos x + \sin x]\, dx$$

$$= e^x\left[-\left\{x \sin x - \int 1 \,.\, \sin x \, dx\right\} - \cos x\right] = e^x[-x \sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2 \cos x).$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)\, e^x - e^x(x \sin x + 2 \cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$. *(Nagarjuna, 2008 ; Rajasthan, 2005)*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D^2 + 1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of } \left\{\frac{1}{(D^2+1)^2} e^{ix} \,.\, x^2\right\} = \text{Re.P. of } \left\{e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2\right\}$$

$$= \text{Re.P. of } \left\{e^{ix} \frac{1}{(D^2+2iD)^2} x^2\right\} = \text{Re.P. of} \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1 - \frac{i}{2}D\right)^{-2} x^2\right\}\right]$$

$$= \text{Re.P. of} \left[-\frac{1}{4} e^{ix} \,.\, \frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right\} x^2\right]$$

$$= \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} \,.\, \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of } \left\{-\frac{1}{4} e^{ix} \,.\, \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4} \text{ Re.P. of } \left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48} \text{ Re.P. of } \{(\cos x + i \sin x)(x^4 + 4ix^3 - 9x^2)\}$$

$$= -\frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x]$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + \dfrac{1}{48}$ $[4x^3 \sin x - x^2 (x^2 - 9) \cos x\}$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$. *(J.N.T.U., 2006 ; U.P.T.U., 2004)*

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-2)^2} (8x^2 e^{2x} \sin 2x) = 8e^{2x} \frac{1}{(D+2-2)^2} (x^2 \sin 2x)$$

$$= 8e^{2x} \frac{1}{D^2} (x^2 \sin 2x) = 8e^{2x} \,.\, \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 . \frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x \sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 . \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$.

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia$.

Thus C.F. $= c_1 \cos ax + c_2 \sin ax$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\quad \dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax . e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\quad \text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax . \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax . \cos ax .$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

PROBLEMS 13.2

Solve :

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (V.T.U., 2005)
2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.
3. $\frac{d^2x}{dt^2} + n^2x = k\cos(nt + \alpha)$.
4. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$.
5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$. (Bhopal, 2002 S)
6. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$. (Madras, 2000)
7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. (V.T.U., 2004)
8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. (Delhi, 2002)
9. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$. (Nagarjuna, 2008)
10. $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$. (Nagpur, 2009)
11. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$. (Mumbai, 2006)
12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$. (V.T.U., 2006)
13. $(D^2 + 1)^2 y = x^4 + 2\sin x \cos 3x$. (Madras, 2006)
14. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$. (Bhopal, 2008)
15. $(D^4 + D^2 + 1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$. (Rajasthan, 2006)
16. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x\cos x$. (V.T.U., 2010)
17. $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. (Raipur, 2005 ; Anna, 2002 S)
18. $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$.
19. $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.
20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$. (P.T.U., 2003)
21. $\frac{d^2y}{dx^2} + 16y = x\sin 3x$. (V.T.U., 2010 S)
22. $(D^2 + 2D + 1)y = x\cos x$. (Rajasthan, 2006)
23. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$.
24. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (S.V.T.U., 2009)
25. $(D^2 + a^2)y = \tan ax$. (V.T.U., 2005)

13.8 TWO OTHER METHODS OF FINDING P.I.

I. Method of variation of parameters. *This method is quite general and applies to equations of the form*

$$y'' + py' + qy = X \quad \text{...(1)}$$

where p, q, and X are functions of x. It gives P.I. $= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$ ...(2)

where y_1 *and* y_2 *are the solutions of* $y'' + py' + qy = 0$ ...(3)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ *is called the Wronskian* of* y_1, y_2.

Proof. Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as *parameters*) by unknown functions $u(x)$ and $v(x)$, let the P.I. be

$$y = uy_1 + vy_2 \quad \text{...(4)}$$

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$

*Named after the Polish mathematician and philosopher *Hoene Wronsky* (1778–1853).

$$= uy_1' + vy_2' \quad ...(5)$$

on assuming that $u'y_1 + v'y_2 = 0$ ...(6)

Differentiate (4) and substitute in (1). Then noting that y_1 and y_2, satisfy (3), we obtain

$$u'y_1' + v'y_2' = X \quad ...(7)$$

Solving (6) and (7), we get

$$u' = -\frac{y_2X}{W}, v' = \frac{y_1X}{W} \qquad \text{where } W = y_1y_2' - y_2y_1'$$

Integrating $u = -\int \frac{y_2X}{W}\,dx$, $v = \int \frac{y_1X}{W}\,dx$. Substituting these in (4), we get (2).

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x. \qquad \text{(V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S)}$$

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(*ii*) *To find P.I.*

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}\,dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\,dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters, $d^2y/dx^2 - y = 2/(1 + e^x)$.* (V.T.U., 2005 ; Hissar, 2005)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0$, $D = \pm 1$, $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

Here $y_1 = e^x$, $y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^xe^{-x} - e^xe^{-x} = -2.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^x \int \frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}dx + e^{-x}\int \frac{e^x}{-2}\cdot\frac{2}{1+e^x}dx$$

$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx - e^{-x}\log(1+e^x) = e^x\left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1}dx\right] - e^{-x}\log(1+e^x)$$

$$= e^x[-e^{-x} + \log(e^{-x}+1)] - e^{-x}\log(1+e^x) = -1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Hence C.S. is $y = c_1e^x + c_2e^{-x} - 1 + e^x\log(e^{-x}+1) - e^{-x}\log(e^x+1)$.

Example 13.25. *Solve by the method of variation of parameters $y'' - 6y' + 9y = e^{3x}/x^2$.*
(*Nagpur, 2009 ; S.V.T.U., 2009*)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D - 3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2x)e^{3x}$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2}\,dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2}\,dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\,dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\quad y = (c_1 + c_2x)e^{3x} - e^{3x}(\log x + 1).$

Example 13.26. *Solve, by the method of variation of parameters, $y'' - 2y' + y = e^x \log x$.*
(*V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003*)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\,y = e^x \log x$

(*i*) *To find C.F.*

Its A.E. is $(D - 1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)\,e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}}\,dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}}\,dx = -e^x \int x \log x\,dx + xe^x \int \log x\,dx$$

$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx\right) + x \cdot e^x \left(x \log x - \int \frac{1}{x} \cdot x\,dx\right)$$

$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4}\right) + x\ e^x (x \log x - x) = \frac{1}{4}\ x^2 e^x (2 \log x - 3)$$

Hence C.S. is $y = (c_1 + c_2x)\,e^x + \dfrac{1}{4}\ x^2 e^x (2 \log x - 3)$.

II. Method of undetermined coefficients

To find the P.I. of $f(D)\,y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when (*i*) $X = 2e^{3x}$, trial solution $= ae^{3x}$.

(*ii*) $X = 3 \sin 2x$, trial solution $= a_1 \sin 2x + a_2 \cos 2x$

(*iii*) $X = 2x^3$, trial solution $= a_1x^3 + a_2x^2 + a_3x + a_4$

However when $X = \tan x$ or $\sec x$, this method fails, since the number of terms obtained by differentiating $X = \tan x$ or $\sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present, appear in the C.F.

Example 13.27. *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$. *(V.T.U., 2008)*

Solution. Here C.F. $= e^{-x}(c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x)$

Assume P.I. as $y = a_1x^2 + a_2x + a_3 + a_4e^{-x}$

$\therefore$ $Dy = 2a_1x + a_2 - a_4e^{-x}$ and $D^2y = 2a_1 + a_4e^{-x}$

Substituting these in the given equation, we get

$$4a_1x^2 + (4a_1 + 4a_2)\, x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$4a_1 = 2,\ 4a_1 + 4a_2 = 0,\ 2a_1 + 2a_2 + 4a_3 = 0,\ 3a_4 = 3$$

Then $a_1 = \dfrac{1}{2}, a_2 = -\dfrac{1}{2}, a_3 = 0, a_4 = 1$. Thus P.I. $= \dfrac{1}{2}x^2 - \dfrac{1}{2}x + e^{-x}$

$\therefore$ C.S. is $y = e^{-x}(c_1 \cos \sqrt{3x} + c_2 \sin \sqrt{3x}) + \dfrac{1}{2}x^2 - \dfrac{1}{2}x + e^{-x}$.

Example 13.28. *Solve* $(D^2 + 1)\, y = \sin x$.

Solution. Here C.F. $= c_1 \cos x + c_2 \sin x$

We would normally assume a trial solution as $a_1 \cos x + a_2 \sin x$.

However, since these terms appear in the C.F., we multiply by x and assume the trial P.I. as

$$y = x\,(a_1 \cos x + a_2 \sin x)$$

$\therefore$ $Dy = (a_1 + a_2x) \cos x + (a_2 - a_1x) \sin x$ and $D^2y = (2a_2 - a_1x) \cos x - (2a_1 + a_2x) \sin x$

Substituting these in the given equation, we get $2a_1 \cos x - 2a_2 \sin x = \sin x$

Equating corresponding coefficients,

$$2a_1 = 0, \quad -2a_2 = 1 \quad \text{so that } a_1 = 0, a_2 = -\frac{1}{2}. \quad \text{Thus P.I.} = -\frac{1}{2}\, x \sin x$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \dfrac{1}{2}\, x \sin x$.

Example 13.29. *Solve by the method of undetermined coefficients,*

$$\frac{d^2y}{dx^2} - y = e^{3x} \cos 2x - e^{2x} \sin 3x.$$

Solution. Its A.E. is $D^2 - 1 = 0$, $\therefore$ $D = \pm 1$.

Thus C.F. $= c_1e^x + c_2e^{-x}$

Assume P.I. as $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$

$$\therefore \quad \frac{dy}{dx} = e^{3x}\{(3c_1 + 2c_2) \cos 2x + (3c_2 - 2c_1) \sin 2x\} - e^{2x}\{(2c_3 + 3c_4) \cos 3x + (2c_4 - 3c_3) \sin 3x\}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = e^{3x}\{(5c_1 + 12c_2) \cos 2x + (5c_2 - 12c_1) \sin 2x\} - e^{2x}\{(12c_4 - 5c_3) \cos 3x - (5c_4 + 12c_3) \sin 3x\}$$

Substituting these in the given equation, we get

$$e^{3x}\{(4c_1 + 12c_2) \cos 2x + (4c_2 - 12c_1) \sin 2x\} - e^{2x}\{(12c_4 - 6c_3) \cos 3x - (6c_4 + 12c_3) \sin 3x\}$$
$$= e^{3x} \cos 2x - e^{2x} \sin 3x$$

Equating corresponding coefficients,

$$4c_1 + 12c_2 = 1,\ 4c_2 - 12c_1 = 0\ ;\ 12c_4 - 6c_3 = 0,\ 6c_4 + 12c_3 = -1$$

whence $c_1 = 1/40, c_2 = 3/40, c_3 = -1/15, c_4 = -1/30$

Thus P.I. $= \dfrac{1}{40}\, e^{3x}(\cos 2x + 3 \sin 2x) + \dfrac{1}{30}\, e^{2x}(2 \cos 3x + \sin 3x)$

Hence C.S. is $y = c_1e^x + c_2e^{-x} + \dfrac{1}{30}\, e^{2x}(2 \cos 3x + \sin 3x) + \dfrac{1}{40}\, e^{3x}(\cos 2x + 3 \sin 2x)$.

PROBLEMS 13.3

Solve by the method of variation of parameters :

1. $\frac{d^2y}{dx^2} + a^2y = \text{cosec } ax.$

2. $\frac{d^2y}{dx^2} + y = \sec x.$ (*Bhopal, 2007*)

3. $\frac{d^2y}{dx^2} + y = \tan x.$ (*P.T.U., 2005 ; Raipur, 2004*)

4. $\frac{d^2y}{dx^2} + y = x \sin x.$ (*S.V.T.U., 2007 ; J.N.T.U., 2005*)

5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x/]x.$ (*V.T.U., 2006*)

6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}.$ (*V.T.U., 2010 S ; U.P.T.U., 2005*)

7. $y'' - 2y' + 2y = e^x \tan x.$ (*V.T.U., 2010*)

8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$ (*U.P.T.U., 2003*)

9. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}.$ (*V.T.U., 2004*)

Solve by the method of undetermined coefficients :

10. $(D^2 - 3D + 2)y = x^2 + e^x.$ (*V.T.U., 2003 S*)

11. $\frac{d^2y}{dx^2} + y = 2 \cos x.$ (*V.T.U., 2000 S*)

12. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x.$ (*V.T.U., 2008*)

13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x.$ (*V.T.U., 2010*)

14. $(D^2 - 2D + 3)y = x^3 + \cos x.$

15. $(D^2 - 2D)y = e^x \sin x.$ (*V.T.U., 2006*)

13.9 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^ny}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1} x \frac{dy}{dx} + k_n y = X \qquad \ldots(1)$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $\quad x^2\frac{d^2y}{dx^2} = D(D-1)\,y.$ Similarly, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y$ and so on.

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x.$ (*V.T.U., 2010*)

Solution. This is a Cauchy's homogeneous linear.

*See footnote p. 144.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\, y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\, t = (1-D)^{-2}\, t = (1 + 2D + 3D^2 + \ldots)\, t = t + 2.$

Hence the solution of (*i*) is $y = (c_1 + c_2t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\, x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. *(P.T.U., 2003)*

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\, dy/dx = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]\, y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\, y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2x)\, e^{-t} = (c_1 + c_2 \log x)\, \frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\frac{1}{(1-e^t)^2} = \frac{1}{D+1}\, u, \text{ where } u = \frac{1}{D+1}\cdot\frac{1}{(1-e^t)^2} \text{ i.e. } \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$$\therefore \text{ its solution is} \quad ue^t = \int \frac{e^t}{(1-e^t)^2}\, dt = \frac{1}{1-e^t} \quad \text{or} \quad u = \frac{e^{-t}}{1-e^t}$$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t}\int \frac{1}{1-e^t}\, dt = \frac{1}{x}\int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x}\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = \frac{1}{x}\,[\log x - \log(1-x)] = \frac{1}{x}\log\frac{x}{x-1}$$

Hence the solution is $$y = \left\{c_1 + c_2 \log x + \log\frac{x}{x-1}\right\}\frac{1}{x}.$$

Example 13.32. *Solve* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$$[D(D-1) + D + 1]\, y = t \sin t \quad \text{i.e.} \quad (D^2+1)\, y = t \sin t \qquad \text{...(i)}$$

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$$\therefore \quad \text{C.F.} = c_1 \cos t + c_2 \sin t$$

and $$P.I. = \frac{1}{D^2+1}\, t \sin t = \frac{1}{D^2+1}\, t\, (\text{I.P. of } e^{it})$$

$$= \text{I.P. of } e^{it}\, \frac{1}{(D+i)^2 + 1}\, t = \text{I.P. of } e^{it}\cdot\frac{1}{D^2 + 2iD}\, t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD(1+D/2i)} t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1-\frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1+\frac{iD}{2}+\ldots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t+\frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int\left(t+\frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2}+\frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4}t^2+\frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4}+\frac{t}{4}\right) = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

Hence the C.S. of (*i*) is $y = c_1 \cos t + c_2 \sin t - \frac{t^2}{4} \cos t + \frac{t}{4} \sin t$

or $\quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4}(\log x)^2 \cos(\log x) + \frac{1}{4} \log(\log x) \sin(\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = \log x \frac{\sin(\log x)+1}{x}$. *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$\{D(D-1) - 3D + 1\} y = t \frac{\sin t + 1}{e^t} \qquad \text{or} \qquad (D^2 - 4D + 1) y = e^{-t} t (\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$\therefore \qquad$ C.F. $= c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t})$

and $\qquad$ P.I. $= \frac{1}{D^2 - 4D + 1} e^{-t} t (\sin t + 1) = e^{-t} \frac{1}{(D-1)^2 - 4(D-1) + 1} t (\sin t + 1)$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6} t + \frac{1}{D^2 - 6D + 6} t \sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6} t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}(1 + D) t = \frac{1}{6}(t + 1)$$

$$\frac{1}{D^2 - 6D + 6} t \sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6} e^{it} \cdot t$$

$$= \text{I.P. of } e^{it} \frac{1}{(D+i)^2 - 6(D+i) + 6} t = \text{I.P. of } e^{it} \frac{1}{D^2 + (2i-6)D + (5-6i)} t$$

$$= \text{I.P. of } \frac{e^{it}}{5-6i}\left\{1 + \frac{(2i-6)D + D^2}{5-6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5-6i}\left(1 - \frac{2i-6}{5-6i} D\right) t$$

$$= \text{I.P. of } \frac{(5+6i)}{61} (\cos t + i \sin t)\left(t - \frac{2i-6}{5-6i}\right)$$

$$= \text{I.P. of } \frac{1}{61} \{(5 \cos t - 6 \sin t) + i(5 \sin t + 6 \cos t)\}\left(t + \frac{42+26i}{61}\right)$$

$$= \frac{26}{3721}(5 \cos t - 6 \sin t) + \frac{1}{61}(5 \sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5\sin t + \cos t) + \frac{2}{3721}(27\sin t + 191\cos t)$$

$\therefore$ $$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

Hence $$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

or $$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{3721}\{27\sin(\log x) + 191\cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^{x^x}$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\, y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\, y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and $$\text{P.I.} = \frac{1}{(D^2+3D+2)}e^{e^t} = \frac{1}{(D+1)(D+2)}e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^t}.$$

Now $$\frac{1}{D+1}e^{e^t} = \frac{1}{D+1}e^{-t}\,.\,e^t e^{e^t} = e^{-t}\frac{1}{(D-1)+1}e^t e^{e^t}$$

$$= e^{-t}\frac{1}{D}e^t e^{e^t} = e^{-t}\int e^{e^t}\,d(e^t) = x^{-1}\int e^x dx = x^{-1}e^x$$

$$\frac{1}{D+2}e^{e^t} = \frac{1}{D+2}e^{-2t}\,.\,e^{2t}e^{e^t} = e^{-2t}\frac{1}{(D-2)+2}e^{2t}e^{e^t}$$

$$= e^{-2t}\frac{1}{D}e^{e^t}e^{2t} = e^{-2t}\int e^{e^t}e^t\,d(e^t)$$

$$= x^{-2}\int e^x\,x\,dx \qquad [\because\ e^t = x]$$

$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$ $$\text{P.I.} = x^{-1}e^x - x^{-2}(xe^x - e^x) = x^{-2}e^x$$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n\frac{d^ny}{dx^n} + k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation.*

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax+b)$.

Then, if $$D = \frac{d}{dt}, \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b}\cdot\frac{dy}{dt} \quad \textit{i.e.}\ (ax+b)\frac{dy}{dx} = a\,Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

* A French mathematician *Adrien Marie Legender* (1752 – 1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)\,y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}$ $a^3D(D-1)(D-2)\,y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$. ...(*i*)

(*V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x = e^t$, *i.e.*, $t = \log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and $$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)\,y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)\,y + Dy + y = 2\sin t$

or $$(D^2+1)\,y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2 + 1 = 0$, whence $D = \pm i$ $\therefore$ C.F. $= c_1\cos t + c_2 \sin t$

and $$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t\,.\,\frac{1}{2D}\sin t$$

$$= t\int \sin t\,dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2,\ D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2 \dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$. (*V.T.U., 2006*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x - 1 = e^t$ *i.e.*, $t = \log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and $$(2x-1)^2\frac{d^2y}{dx^2} = 4D(D-1)\,y, \text{ where } D = \frac{d}{dt}.$$

Then the given equation becomes

$$4D(D-1)\,y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or $$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ C.F. $= c_1e^t + c_2e^{-t/2}$

and $$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\frac{t}{4D-1}e^t + 2\,.\,\frac{1}{2.0^2-0-1}e^{0t}$$

[$\because$ on putting $t = 1$, $2D^2 - D - 1 = 0$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\cdot\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1 (2x-1) + c_2 (2x-1)^{-1/2} + \frac{1}{5}(2x-1)^2 + \frac{1}{2}(2x-1)\log(2x-1) - 2.$$

which is the required solution.

PROBLEMS 13.4

Solve :

1. $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2.$
2. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4.$
3. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = (1+x^2).$ *(S.V.T.U., 2007)*
4. $x\dfrac{d^2y}{dx^2} - \dfrac{2y}{x} = x + \dfrac{1}{x^2}.$ *(V.T.U., 2005 S)*
5. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0,$ where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

Solve :

6. $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = \log x.$ *(Bhopal, 2009)*
7. $x^3 \dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x + \log x$ *(Bhopal, 2008)*
8. $x^2y'' + xy' + y = 2\cos^2(\log x).$ *(V.T.U., 2011)*
9. $x^3 \dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$ *(S.V.T.U., 2006 ; P.T.U., 2003)*
10. $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}.$ *(P.T.U., 2003)*
11. $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x.$ *(U.P.T.U., 2004)*
12. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3\log x.$ *(Bhopal, 2008)*
13. $(2x+3)^2 \dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x.$ *(V.T.U., 2007 ; Kerala, 2005 ; Anna, 2002 S)*
14. $(x-1)^3 \dfrac{d^3y}{dx^3} + 2(x-1)^2\dfrac{d^2y}{dx^2} - 4(x-1)\dfrac{dy}{dx} + 4y = 4\log(x-1).$ *(Nagpur, 2009)*
15. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin[2\log(1+x)]$ *(P.T.U., 2006 ; V.T.U., 2004)*
16. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2 + x + 1.$ *(Mumbai, 2006)*

13.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \qquad ...(1)$$

with variable coefficients $p(x)$ and $q(x)$ and two initial conditions $y(x_0) = k_0, y'(x_0) = k_1$ ...(2)

Let its general solution be $y = c_1y_1 + c_2y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2.*

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** *$y(x)$ on the interval I.*

* As in §2.12, y_1, y_2 are said to be *linearly dependent* in an interval I, if and only if there exist numbers λ_1, λ_2 not both zero such that $\lambda_1y_1 + \lambda_2y_2 = 0$ for all x in I.
If no such numbers other than zero exist, then y_1, y_2 are said to be *linearly independent.*

(2) Theorem. *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian*† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 *are linearly independent on I.*

Proof. If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \qquad ...(4)$$

Differentiating w.r.t. x, $c_1y_1' + c_2y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \end{aligned}\right\} \qquad ...(6)$$

which, on eliminating c_1, c_2, give $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1y_1(x) + c_2y_2(x)$

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $\mathbf{y = 0}$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{\mathbf{y}} = y$ *i.e.*, $c_1y_1 + c_2y_2 = 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Example 13.37. *Show that the two functions sin 2x, cos 2x are independent solutions of $y'' + 4y = 0$.*

Solution. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also
$$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

PROBLEMS 13.5

Solve :

1. Show that e^{-x}, xe^{-x} are independent solutions of $y'' + 2y' + y = 0$ in any interval.
2. Show that $e^x \cos x$, $e^x \sin x$ are independent solutions of the equation $xy'' - 2y' = 0$.
3. If y_1, y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$, show that the *Wronskian* can be expressed as $W(y_1, y_2) = ce^{-\int p dx}$

13.11 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Quite often we come across linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as *simultaneous linear equations.* Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Example 13.38. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. *(S.V.T.U., 2009 ; Kurukshetra, 2005)*

† See footnote on p. 486.

Solution. Taking $d/dt = D$, the given equations become

$$(D + 5)\,x - 2y = t \qquad ...(i)$$

$$2x + (D + 1)\,y = 0 \qquad ...(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (*i*) by 2 and operating on (*ii*) by $D + 5$ and then subtracting, we get

$$[-4 - (D + 5)(D + 1)]\,y = 2t \text{ or } (D^2 + 6D + 9)\,y = -2t$$

Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$

whence $D = -3, -3 \quad \therefore \quad \text{C.F.} = (c_1 + c_2 t)\,e^{-3t}$

and
$$\text{P.I.} = \frac{1}{(D+3)^2}(-2t) = -\frac{2}{9}\left(1+\frac{D}{3}\right)^{-2} t = -\frac{2}{9}\left(1-\frac{2D}{3}+...\right) t = -\frac{2t}{9}+\frac{4}{27}$$

Hence
$$y = (c_1 + c_2 t)\,e^{-3t} - \frac{2t}{9} + \frac{4}{27} \qquad ...(iii)$$

Now to find x, either eliminate y from (*i*) and (*ii*) and solve the resulting equation or substitute the value of y in (*ii*). Here, it is more convenient to adopt the latter method.

From (*iii*), $\quad Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\,e^{-3t} - \dfrac{2}{9}$

$\therefore$ Substituting for y and Dy in (*ii*), we get

$$x = -\frac{1}{2}[Dy + y] = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2 t\right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \qquad ...(iv)$$

Hence (*iii*) and (*iv*) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (*iii*) and (*iv*) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0 \text{ whence } c_1 = -\frac{4}{27},\, c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1+6t)\,e^{-3t} + \frac{1}{27}(1+3t),\quad y = -\frac{2}{27}(2+3t)\,e^{-3t} + \frac{2}{27}(2-3t).$$

Example 13.39. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 2y + \sin t = 0,\ \dfrac{dy}{dt} - 2x - \cos t = 0$ *given that* $x = 0$ *and* $y = 1$ *when* $t = 0$.

Solution. Given equations are

$$Dx + 2y = -\sin t \quad ...(i)\,; \qquad -2x + Dy = \cos t \qquad ...(ii)$$

Eliminating x by multiplying (*i*) by 2 and (*ii*) by D and then adding, we get

$$4y + D^2 y = -2\sin t - \sin t \text{ or } (D^2 + 4)y = -3\sin t$$

Its A.E. is $D = \pm 2i \quad \therefore \quad \text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.} = -3\,\frac{1}{D^2+4}\sin t = -3\,\frac{1}{-1+4}\sin t = -\sin t$$

$\therefore$
$$y = c_1 \cos 2t + c_2 \sin 2t - \sin t \qquad ...(iii)$$

and
$$dy/dt = -2\sin 2t + 2c_2 \cos 2t - \cos t \qquad ...(iv)$$

Substituting (*iii*) in (*ii*), we get

$$2x = Dy - \cos t = -2c_1 \sin 2t + 2c_2 \cos 2t - 2\cos t$$

or
$$x = -c_1 \sin 2t + c_2 \cos 2t + -\cos t \qquad ...(v)$$

When $t = 0, x = 0, y = 1$, (*iii*) and (*v*) give $1 = c_1,\ 0 = c_2 - 1$

Hence $x = \cos 2t - \sin 2t - \cos t,\ y = \cos 2t + \sin 2t - \sin t$.

Example 13.40. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\quad \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \qquad \textit{(U.P.T.U., 2001)}$$

Solution. Given equations are

$$Dx + (D-2)y = 2\cos t - 7\sin t \quad ...(i)$$

$$(D+2)x - Dy = 4\cos t - 3\sin t \quad ...(ii)$$

Eliminate y by operating on (*i*) by D and (*ii*) by $(D-2)$ and then adding, we get

$$D^2x + (D-2)(D+2)x = -2\sin t - 7\cos t + 4(-\sin t - 2\cos t) - 3(\cos t - 2\sin t)$$

or $$2(D^2-2)x = -18\cos t \text{ or } (D^2-2)x = -9\cos t$$

Its A.E. is $$D^2 - 2 = 0 \text{ or } D = \pm\sqrt{2}, \quad \therefore \quad \text{C.F.} = c_1e^{\sqrt{st}} + c_2e^{-\sqrt{2}t}$$

$$\text{P.I.} = (-9)\frac{1}{D^2-2}\cos t = \frac{-9\cos t}{-1-2} = 3\cos t.$$

Hence $x = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t}\ 3\cos t.$

Now substituting this value of x in (*ii*), we get

$$Dy = (D+2)(c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t) - 4\cos t + 3\sin t$$

$$= c_1\sqrt{2}e^{\sqrt{2}t} + 2c_1e^{\sqrt{2}t} + c_2(-\sqrt{2}e^{-2t}) + 2c_2e^{-\sqrt{2}t} - 3\sin t + 6\cos t - 4\cos t + 3\sin t$$

$$= (2+\sqrt{2})c_1e^{\sqrt{2}t} + (2-\sqrt{2})c_2e^{-\sqrt{2}t} + 2\cos t$$

Hence $$y = (\sqrt{2}+1)c_1e^{\sqrt{2}t} - (\sqrt{2}-1)c_2e^{-\sqrt{2}t} + 2\sin t + c_3.$$

Example 13.41. *The small oscillations of a certain system with two degrees of freedom are given by the equations*

$$D^2x + 3x - 2y = 0$$

$$D^2x + D^2y - 3x + 5y = 0$$

where $D = d/dt$. If $x = 0$, $y = 0$, $Dx = 3$, $Dy = 2$ when $t = 0$, find x and y when $t = 1/2$.

Solution. Given equations are $(D^2+3)x - 2y = 0 \quad ...(i)$

$$(D^2-3)x + (D^2+5)y = 0 \quad ...(ii)$$

To eliminate x, operate these equations by $D^2 - 3$ and $D^2 + 3$ respectively and subtract (*i*) from (*ii*). Then

$$[(D^2+3)(D^2+5) + 2(D^2-3)]\,y = 0 \quad \text{or} \quad (D^4 + 10D^2 + 9)y = 0$$

Its auxiliary equation is $D^4 + 10D^2 + 9 = 0$ whence $D = \pm i, \pm 3i$

Thus $$y = c_1\cos t + c_2\sin t + c_3\cos 3t + c_4\sin 3t \quad ...(iii)$$

To find x, we eliminate y from (*i*) and (*ii*).

$\therefore$ operating (*i*) by $D^2 + 5$ and multiplying (*ii*) by 2 and adding, we get

$$(D^4 + 10D^2 + 9)x = 0. \text{ Thus } x = k_1\cos t + k_2\sin t + k_3\cos 3t + k_4\sin 3t \quad ...(iv)$$

To find the relations between the constants in (*iii*) and (*iv*), substitute these values of x and y either of the given equations, say (*i*). This gives

$$2(k_1 - c_1)\cos t + 2(k_2 - c_2)\sin t - 2(3k_3 + c_3)\cos 3t - 2(3k_4 + c_4)\sin 3t = 0$$

which must hold for all values of t.

$\therefore$ Equating to zero the coefficients of $\cos t$, $\sin t$, $\cos 3t$ and $\sin 3t$, we get

$$k_1 = c_1,\ k_2 = c_2,\ k_3 = -c_3/3,\ k_4 = -c_4/3$$

Thus $$x = c_1\cos t + c_2\sin t - \frac{1}{3}(c_3\cos 3t + c_4\sin 3t) \quad ...(v)$$

Hence (*iii*) and (*iv*) constitute the solutions of (*i*) and (*ii*).

Since $x = y = 0$, when $t = 0$; $\therefore$ (*iii*) and (*v*) give

$$0 = c_1 + c_3 \text{ and } c_1 - \frac{1}{3}c_3 = 0 \text{ i.e. } c_1 = c_3 = 0$$

Thus (*iii*) and (*v*) reduce to

and $$\left.\begin{aligned} y &= c_2\sin t + c_4\sin 3t \\ x &= c_2\sin t - \frac{c_4}{3}\sin 3t \end{aligned}\right\} \quad ...(vi)$$

$\therefore$ $Dx = c_2 \cos t - c_4 \cos 3t$ and $Dy = c_2 \cos t + 3c_4 \cos 3t$.

Since $Dx = 3$ and $Dy = 2$ when $t = 0$

$\therefore$ $3 = c_2 - c_4$ and $2 = c_2 + 3c_4$, whence $c_2 = 11/4$, $c_4 = -\dfrac{1}{4}$.

Hence equation (*vi*) becomes $x = \dfrac{1}{4}(11 \sin t + \dfrac{1}{3} \sin 3t)$, $y = \dfrac{1}{4}(11 \sin t - \sin 3t)$ ...(*vii*)

$\therefore$ when $t = 1/2$, $x = \dfrac{1}{4}\left[11 \sin(0.5) + \dfrac{1}{3}\sin(1.5)\right] = \dfrac{1}{4}\left[11(0.4794 + \dfrac{1}{3}(0.9975)\right] = 1.4015$

and $y = \dfrac{1}{4}[11 \sin(0.5) - \sin(1.5)] = 1.069$.

Example 13.42. *Solve the simultaneous equations:* $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x$.

(*S.V.T.U., 2006 S ; U.P.T.U., 2004*)

Solution. Differentiating first equation w.r.t. t, $\dfrac{d^2x}{dt^2} = 2\dfrac{dy}{dt} = 2(2z)$

Again differentiating w.r.t. t, $\dfrac{d^3x}{dt^2} = 4\dfrac{dz}{dt} = 4(2x)$ ...(*i*)

or $(D^3 - 8)x = 0$

Its A.E. is $D^3 - 8 = 0$ or $(D - 2)(D^2 + 2D + 4) = 0$

or $D = 2, -1 \pm i\sqrt{3}$

$\therefore$ the solution of (*i*) is $x = c_1e^{2t} + e^{-t}(c_2 \cos \sqrt{3}t + c_3 \sin \sqrt{3}t)$ ...(*ii*)

From the first equation, we have $y = \dfrac{1}{2}\dfrac{dx}{dt}$

$$\therefore \quad y = \frac{1}{2}[2c_1e^{2t} + (-1)e^{-t}(c_2 \cos \sqrt{3}t + c_3 \sin \sqrt{3}t) + e^{t}(-\sqrt{3}\,c_2 \sin \sqrt{3}t + \sqrt{3}\,c_3 \cos \sqrt{3}t)]$$

$$\text{or} \quad y = c_1e^{2t} + \frac{1}{2}e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3}t - (c_3 + \sqrt{3}c_2)\sin \sqrt{3}t\right\} \quad ...(iii)$$

From the second equation, we have $z = \dfrac{1}{2}\dfrac{dy}{dt}$

$$\therefore \quad z = \frac{1}{2}2c_1e^{2t} + \frac{1}{4}\Big[(-1)e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3}t - (c_3 + \sqrt{3}c_2)\sin \sqrt{3}t\right\}$$
$$+ e^{-t}\left\{\sqrt{3}(c_2 - \sqrt{3}c_3)\sin \sqrt{3}t - \sqrt{3}(c_3 + \sqrt{3}c_2)\cos \sqrt{3}t\right\}\Big]$$

$$= c_1e^{2t} + \frac{1}{4}e^{-t}\left\{(-2c_2 - 2\sqrt{3}c_3)\cos \sqrt{3}t - (2\sqrt{3}\,c_2 - 2c_3)\sin \sqrt{3}t\right\}$$

$$\text{or} \quad z = c_1e^{2t} - \frac{1}{2}e^{-t}\left\{(\sqrt{3}\,c_2 - c_3)\sin \sqrt{3}t + (c_2 + \sqrt{3}c_3)\cos \sqrt{3}t\right\} \quad ...(iv)$$

Hence the equations (*ii*), (*iii*) and (*iv*) taken together give the required solution.

PROBLEMS 13.6

Solve the following simultaneous equations :

1. $\dfrac{dx}{dt} = 5x + y, \dfrac{dy}{dt} = y - 4x$.
2. $\dfrac{dx}{dt} + y = \sin t, \dfrac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

(*Bhopal, 2009 ; J.N.T.U., 2006 ; Kerala, 2005*)

3. $\frac{dx}{dt} + 2x + 3y = 0,\ 3x + \frac{dy}{dt} + 2y = 2e^{2t}$. (*Delhi, 2002*)
4. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$.
5. $\frac{dx}{dt} + 2y = e^t,\ \frac{dy}{dt} - 2x = e^{-t}$. (*Bhopal, 2002 S*)
6. $\frac{dx}{dt} + 2x - 3y = t;\ \frac{dy}{dt} - 3x + 2y = e^{2t}$. (*Nagpur, 2009*)
7. $(D-1)x + Dy = 2t+1,\ (2D+1)x + 2Dy = t$.
8. $(D+1)x + (2D+1)y = e^t,\ (D-1)x + (D+1)y = 1$.
9. $Dx + Dy + 3x = \sin t,\ Dx + y - x = \cos t$. (*U.P.T.U., 2003*)
10. $t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0$ given $x(1) = 1, y(-1) = 0$.
11. $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y - x = \cos t$.
12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0$. (*U.P.T.U., 2005*)
13. $\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t$. (*U.P.T.U., 2004*)
14. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4;\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain expression for x and y in terms of t, given $x, y, dx/dt, dy/dt$ all vanish at $t = 0$.

13.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 13.7

Fill up the blanks or choose the correct answer in the following problems :

1. The complementary function of $(D^4 - a^4)y = 0$ is ....
2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....
3. P.I. of $y'' - 3y' + 2y = 12$ is ....
4. The Wronskian of x and e^x is ....
5. The C.F. of $y'' - 2y' + y = xe^x \sin x$ is
   (*a*) $C_1e^x + C_2e^{-x}$ (*b*) $(C_1x + C_2)e^x$ (*c*) $(C_1 + C_2x)e^{-x}$ (*d*) None of these. (*V.T.U., 2010*)
6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ....
7. The particular integral of $(D^2 + a^2)y = \sin ax$ is
   (*a*) $-\frac{x}{2a}\cos ax$ (*b*) $\frac{x}{2a}\cos ax$ (*c*) $-\frac{ax}{2}\cos ax$ (*d*) $\frac{ax}{2}\cos ax$.
8. The solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....
9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\pi/2) = 2$, is ....
10. $e^{-x}(c_1\cos\sqrt{3x} + c_2\sin\sqrt{3x}) + c_3e^{2x}$ is the general solution of
    (*a*) $d^3y/dx^3 + 4y = 0$ (*b*) $d^3y/dx^3 - 8y = 0$
    (*c*) $d^3y/dx^3 + 8y = 0$ (*d*) $d^3y/dx^3 - 2d^2y/dx^2 + dy/dx - 2 = 0$.
11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ....
12. The particular integral of $d^2y/dx^2 + y = \cos h\ 3x$ is ....
13. The solution of $x^2y'' + xy' = 0$ is ....
14. The general solution of $(D^2 - 2)^2, y = 0$ is.......
15. P.I. of $(D + 1)^2 y = xe^{-x}$ is ....
16. If $f(D) = D^2 - 2,\ \frac{1}{f(D)}e^{2x} =$ ....
17. If $f(D) = D^2 + 5,\ \frac{1}{f(D)}\sin 2x =$ ....
18. The particular integral of $(D + 1)^2 y = e^{-x}$ is ....
19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ....

20. P.I. of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{1}{2}\sin 2x$ (b) $\frac{1}{2}x\sin 2x$ (c) $\frac{1}{4}x\sin 2x$ (d) $\frac{1}{2}x\cos 2x$. (*Bhopal, 2008*)

21. By the method of undetermined coefficients y_p of $y'' + 3y' + 2y = 12x^2$ is of the form

(a) $a + bx + cx^2$ (b) $a + bx$ (c) $ax + bx^2 + cx^3$ (d) None of these. (*V.T.U., 2010*)

22. In the equation $\frac{dx}{dt} + y = \sin t + 1$, $\frac{dy}{dt} + x = \cos t$ if $y = \sin t + 1 + e^{-t}$, then $x =$ ....

23. $(x^2D^2 + xD + 7)y = 2/x$ converted to a linear differential equation with constant coefficients is ....

24. P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ is

(a) $\frac{x^2}{3} + 4x$ (b) $\frac{x^3}{3} + 4$ (c) $\frac{x^3}{3} + 4x$ (d) $\frac{x^3}{3} + 4x^2$.

25. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is given by

(a) $y = C_1e^{x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (b) $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}e^{3x}$

(c) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (d) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{-3x}$.

26. The particular integral of the differential equation $(D^3 - D)y = e^x + e^{-x}$, $D = \frac{d}{dx}$ is

(a) $\frac{1}{2}(e^x + e^{-x})$ (b) $\frac{1}{2}x(e^x + e^{-x})$ (c) $\frac{1}{2}x^2(e^x + e^{-x})$ (d) $\frac{1}{2}x^2(e^x - e^{-x})$.

27. The complementary function of the differential equation $x^2y'' - xy' + y = \log x$ is ....

28. The homogeneous linear differential equation whose auxiliary equation has roots $1, -1$ is ......

29. The particular integral of $(D^2 - 6D + 9)y = \log 2$ is .... (*V.T.U., 2011*)

30. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficients, put $x =$ ....

31. The particular integral of $(D^2 - 4)y = \sin 3x$ is

(a) 1/4 (b) – 1/13 (c) 1/5 (d) None of these. (*V.T.U., 2010*)

32. The solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$ is ....

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ....

34. Complementary function of $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 2x\log x$ is

(a) $(C_1 + C_2x)e^x$ (b) $(C_1 + C_2\log x)x$ (d) $(C_1 + C_2x)\log x$ (d) $(C_1 + C_2\log x)e^x$. (*Bhopal, 2008*)

35. The general solution of $(D^2 - D - 2)x = 0$ is $x = c_1e^t + c_2e^{-2t}$ (True or False)

36. $\frac{1}{f(D)}(x^2e^{ax}) = \frac{1}{f(D+a)}(e^{ax}x^2)$. (True or False)