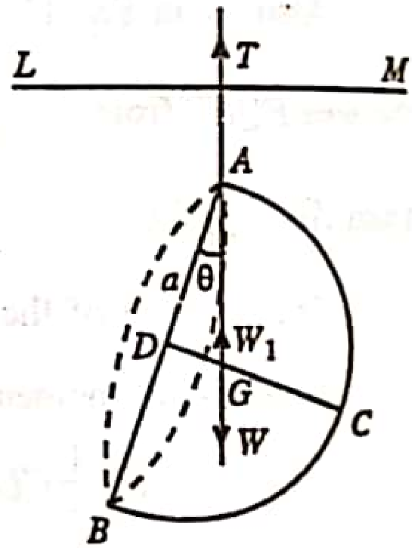
or $\qquad T \cdot 2a + W \cdot b = W \cdot a$ or $T = \dfrac{W(a-b)}{2a}$.

IFS 2018 8(a)

Ex. 14. *A solid hemisphere floats in a liquid completely immersed with a point of the rim joined to a fixed point by means of a string. Prove that the inclination of the base to the vertical is* $\tan^{-1}\frac{3}{8}$. *Also prove that the tension of the string is* $\frac{2}{3}\pi(\rho-\sigma)a^3 g$ *where* ρ *and* σ *are the densities of the solid and the liquid respectively and* a *is the radius of the hemisphere.*

Sol. Let the point A of the rim be joined to a fixed point by means of a string, and the base AB be inclined at an angle θ to the vertical. The forces acting on the hemisphere are :



(i) Wt. of the hemisphere $W = \frac{2}{3}\pi a^3 \rho g$, acting vertically downwards at G such that $DG = \frac{3}{8}a$.

(ii) The force of buoyancy *i.e.*, the weight of the liquid displaced $W_1 = \frac{2}{3}\pi a^3 \sigma g$, acting vertically upwards at G.

(iii) The tension T at A.

Since W and W_1 are acting in the same vertical line, therefore, for the equilibrium of the body the third force T will also act along the line of action of W and W_1 *i.e.*, the line of action of T will pass through G.

Here in ΔADG, $\tan\theta = \dfrac{DG}{DA} = \dfrac{3a}{8a} = \dfrac{3}{8}$.

$\therefore \qquad \theta = \tan^{-1}\frac{3}{8}$.

Also $\qquad T + W_1 = W$ or $T = W - W_1 = \frac{2}{3}\pi a^3 \rho g - \frac{2}{3}\pi a^3 \sigma g = \frac{2}{3}\pi a^3(\rho-\sigma)g$.

IFoS 2018 ODE

8(b). A snowball of radius $r(t)$ melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process.

Let $\frac{dr}{dt} = K$ (uniform), density $= \rho$, fixed

$$M = \left(\frac{4}{3}\pi r^3\right)\rho \Rightarrow \frac{dM}{dt} = (4\pi\rho) r^2 \frac{dr}{dt}$$

$$\frac{dM}{dt} = 4\pi\rho\left(\frac{3M}{4\pi\rho}\right)^{2/3} \cdot K \Rightarrow$$

$$\Rightarrow \frac{dM}{dt} = k_1 M^{2/3}, \quad \text{where } k_1 = \frac{4\pi\rho \cdot 3^{2/3} \cdot K}{(4\pi\rho)^{2/3}}$$

$$M^{-2/3}\, dM = k_1\, dt$$

Integrating, $3M^{1/3} = k_1 t + k_2$

Let M_0 be initial mass of snowball

$\therefore M(0) = M_0$ and $M(1) = \frac{M_0}{2}$

$\Rightarrow k_2 = 3M_0^{1/3}$ & $k_1 = 3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}$

We want to calculate time t, when $M = 0$.

i.e. $0 = \left[3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}\right]t + 3M_0^{1/3}$

$$t = \frac{-3}{3/2^{1/3} - 3} = \frac{-1}{2^{-1/3} - 1} = 4.85 \text{ hours}$$

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8(c) For a curve lying on a sphere of radius 'a' and such that the torsion is never 0, show that

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2 \tau}\right)^2 = a^2. \quad (10)$$

Let vector point $r(s)$ lies on a sphere with centre r_0 and radius a.

$$\therefore \quad |(r - r_0)| = a$$

$$|r - r_0|^2 = a^2 \Rightarrow (r - r_0)\cdot(r - r_0) = a^2 \quad \text{— } \star$$

Differentiating w.r.t s

$$\frac{dr}{ds}\cdot(r - r_0) + (r - r_0)\cdot\frac{dr}{ds} = 0$$

$$2\frac{dr}{ds}\cdot(r - r_0) = 0$$

$$\Rightarrow (r - r_0)\cdot t = 0 \quad \text{— } (1)$$

Again differentiating w.r.t s

$$\frac{dr}{ds}\cdot t + (r - r_0)\cdot\frac{dt}{ds} = 0$$

$$t\cdot t + (r - r_0)\cdot(\kappa n) = 0 \quad \text{(serret frenet)}$$

$$1 + (r - r_0)\cdot(\kappa n) = 0$$

$$(r - r_0)\cdot n = \frac{-1}{\kappa} \quad \text{— } (2)$$

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Again differentiating w.r.t. s

$$\frac{dr}{ds}\cdot n + (r-r_0)\cdot\frac{dn}{ds} = \frac{1}{k^2}\cdot k'$$

$$t\cdot n + (r-r_0)\cdot(\tau b - k t) = \frac{k'}{k^2}$$

(Serret-Frenet)

$$0 + (r-r_0)\cdot(\tau b) - (r-r_0)\cdot(k t) = \frac{k'}{k^2}$$

$$(r-r_0)\cdot b = \frac{k'}{k^2\tau} \quad [\text{using } ①] \quad \text{—③}$$

From ①, ②, ③, we see that ~~Thus~~ the components of $(r-r_0)$ with respect to t, n, b are 0, $-\frac{1}{k}$ and $\frac{k'}{k^2\tau}$

Hence,

$$r-r_0 = \frac{-1}{k}n + \left(\frac{k'}{k^2\tau}\right)b$$

$\therefore$ from (A) we get

$$a^2 = (r-r_0)\cdot(r-r_0) = \left(\frac{-1}{k}n + \frac{k'}{k^2\tau}b\right)\cdot\left(\frac{-1}{k}n + \frac{k'}{k^2\tau}b\right)$$

$$= \frac{1}{k^2}n\cdot n - \frac{k'}{k^3\tau}b\cdot n - \frac{k'}{k^3\tau}n\cdot b + \left(\frac{k'}{k^2\tau}\right)^2 b\cdot b$$

$$= \frac{1}{k^2} + \left(\frac{k'}{k^2\tau}\right)^2 \qquad \left[\because \begin{array}{l} n\cdot n = 1 = b\cdot b \\ n\cdot b = 0 \end{array}\right.$$