2019

1. (5c) Let the function be $\phi = xy^2 + yz^2 + zx^2$

$$\text{grad } \phi = \nabla\phi = (y^2 + 2xz)\hat{i} + (2xy + z^2)\hat{j} + (x^2 + 2yz)\hat{k}$$

At $(1,1,1)$, $\text{grad}\phi = 3\hat{i} + 3\hat{j} + 3\hat{k}$ — (1)

Let $\vec{r} = t\hat{i} + t^2\hat{j} + \cancel{3}t^3\hat{k}$ be given curve

Tangent to this curve is given by $\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

at $(1,1,1)$, ~~directi~~

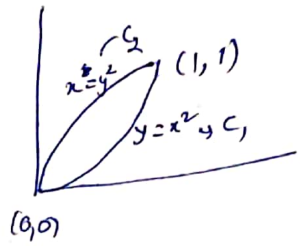
tangent is along $(\vec{T})$ – $\hat{i} + 2\hat{j} + 3\hat{k}$ $[t=1]$

Now, directional derivative of ϕ along this curve is—

$$\left(\frac{(\text{grad } \phi)\cdot \vec{T}}{|\vec{T}|}\right)$$

$$= \frac{(3\hat{i} + 3\hat{j} + 3\hat{k})\cdot(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$ Ans

2. (6b) $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$



Along the Curve C_1,

$$I_1 = \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_1} [(2x+y^2)dx + (3y-4x)\,dy]$$

Putting $y = x^2$, $dy = 2x\,dx$

$$\Rightarrow I_1 = \int_0^1 (2x+x^4)dx + \int_0^1 (2x)[3x^2-4x]\,dx.$$

$$= 1/30.$$

Along the Curve C_2,

$$I_2 = \oint_{C_2} \vec{F}\cdot d\vec{r} = \oint_{C_2} [(2x+y^2)dx + (3y-4x)\,dy$$

Putting $x = y^2$, $dx = 2y\,dy$

$$\Rightarrow I_2 = \int_1^0 (3y^2)\,2y\,dy + \int_1^0 (3y-4y^2)\,dy = -\frac{3}{2} - \frac{3}{2} + \frac{4}{3} = -\frac{5}{3}$$

$$\Rightarrow \quad I = \oint \bar{F}\cdot d\bar{r} = I_1 + I_2 = \frac{-49}{30}.$$

3. (7b) $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$

Let $\bar{r} = a\cos u\,\hat{i} + a\sin u\,\hat{j} + au\tan\alpha\,\hat{k}$

$$\frac{d\bar{r}}{du} = -a\sin u\,\hat{i} + a\cos u\,\hat{j} + a\tan\alpha\,\hat{k}$$

$$\frac{d^2\bar{r}}{du^2} = -a\cos u\,\hat{i} + -a\sin u\,\hat{j}$$

$$\frac{d^3 r}{du^3} = a\sin u\,\hat{i} + (-a\cos u)\hat{j}$$

$$K = \frac{\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right|}{\left|\frac{dr}{du}\right|^3}, \quad \tau = \frac{\left[\frac{dr}{du}\ \frac{d\bar{r}}{du^2}\ \frac{d^3r}{du^3}\right]}{\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right|^2}$$

$$\frac{dr}{du}\times\frac{d^2r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \end{vmatrix} = a^2\tan\alpha\sin u\,\hat{i} + (-a^2\tan\alpha\cos u)\hat{j} + a^2\hat{k}$$

$$\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right| = \sqrt{(a^2\tan\alpha\sin u)^2 + (a^2\tan\alpha\cos u)^2 + (a^2)^2}$$
$$= a^2\sqrt{1+\tan^2\alpha} = a^2\sec\alpha.$$

$$\left|\frac{dr}{du}\right| = \sqrt{(-a\sin u)^2 + (a\cos u)^2 + (a\tan\alpha)^2} = a\sec\alpha.$$

$$\left[\frac{dr}{du}\ \frac{d^2r}{du^2}\ \frac{d^3r}{du^3}\right] = \frac{d^3r}{du^3}\cdot\left[\frac{dr}{du}\times\frac{d^2r}{du^2}\right] = [a\sin u\,\hat{i} + (-a\cos u)\hat{j}]\cdot\left[a^2\tan\alpha\sin u\,\hat{i} + (-a^2\tan\alpha\cos u)\hat{j} + a^2\hat{k}\right]$$

$$= a^3\tan\alpha.$$

$$\Rightarrow k = \frac{a^2 \sec\alpha}{(a\sec\alpha)^3} = \frac{\cos^2\alpha}{a}$$

$$\Rightarrow \boxed{\text{Radius of Curvature} = \frac{1}{k} = a\sec^2\alpha}$$

$$\text{and } \tau = \frac{a^3\tan\alpha}{(a^2\sec\alpha)^2} = \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\cos^2\alpha}{a} = \frac{\sin\alpha\cos\alpha}{a} = \frac{\sin 2\alpha}{2a}$$

$$\Rightarrow \boxed{\text{Radius of torsion} = \frac{1}{\tau} = 2a\,\operatorname{cosec} 2\alpha}$$

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4. (8c)

(i) $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

As per Gauss Divergence Theorem, $\iint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot F\,dV$

$$\text{Volume Integral} = \iiint_V \nabla\cdot\bar{F}\,dV = \iiint (4 - 4y + 2z)\,dV$$

$$= \cancel{\iint_{(x,y)}\int_{z=0}^{z=3}(4-4y+2z)\,dz\,dx\,dy} = \iint\left[\int_{z=0}^{3}(4-4y+2z)\,dz\right]dx\,dy$$

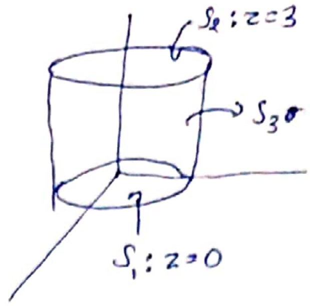
$$= \iint (21 - 12y)\,dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $dx\,dy = r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}(21r - 12r^2\sin\theta)\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left[42 - 32\sin\theta\right]d\theta$$

$$= \underline{84\pi}$$

Surface Integral $= \iint \vec{F}\cdot\hat{n}\, dS$



For S_1, $\quad \hat{n}_1 = -\hat{k}$ and $z=0$

$$\Rightarrow I_1 = \iint \vec{F}\cdot\hat{n}_1\, dS = 0$$

For S_2, $\quad \hat{n}_2 = \hat{k}$ and $z=3$

$$\Rightarrow I_2 = \iint \vec{F}\cdot\hat{n}_2\, dS = \iint 9\, dS = 9\cdot[\pi(2)^2]$$
$$= 36\pi$$

For S_3, $\quad \hat{n}_3 = \dfrac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{2}$

$$F\cdot\hat{n}_3 = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$
$$= 2x^2 - y^3$$

$$I_3 = \iint (2x^2 - y^3)\, dS$$ Let ~~$x = 2\cos\theta,\ y = 2\sin\theta,\ z = z$~~

$$\Rightarrow dS = \frac{dx\, dz}{|\hat{n}_3\cdot\hat{j}|} = \frac{dx\,dz}{(y/2)}$$

$$\Rightarrow I_3 = 2\iint\left(\frac{2x^2 - y^3}{y}\right)dx\,dz = 2\iint\left(\frac{2x^2}{y} - y^2\right)dx\,dz$$

$$= 2\iint\left[\frac{2x^2}{\sqrt{4-x^2}} - 4 + x^2\right]dx\,dz$$

Putting $x = 2\cos\theta$ and $dx = -2\sin\theta\, d\theta$

$$\Rightarrow I_3 = (-2)\int_{z=0}^{3}\int_{\theta=0}^{2\pi}(8\sin^3\theta - 8\cos^2\theta)\, d\theta\, dz = \int_{\theta=0}^{2\pi}(48)\left[\cos^2\theta - \sin^3\theta\right) d\theta$$

$$= 48\pi$$

Surface Integral $= I_1 + I_2 + I_3 = 84\pi$

which is the same as Volume Integral.

(ii) Let $\bar{F} = e^x \hat{i} + 2y\hat{j} - \hat{k}$, $C: x^2+y^2=4,\ z=2$

By Stoke's theorem, $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, dS$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix} = 0$$

$\Rightarrow \iint (\nabla\times\bar{F})\cdot\hat{n}\, dS = 0$

$$\Rightarrow \boxed{\oint_C e^x dx + 2y\, dy - dz = \iint (\nabla\times\bar{F})\hat{n}\, dS = 0}$$