The simplex method.

Introduction.

The graphical method of solving an L.P.P. was discussed and it was shown that this method is giving difficulty when we are dealing with a LPP involving three or more variables. Also we know the meanings of various types of a LPP have been explained namely feasible solutions, basic solutions, basic feasible solutions and optimal solutions.

It was shown that the set of all feasible solutions for a LPP forms a convex set and that the optimal solution, if it exists, occurs at one of the extreme points of the convex set. Further that every extreme point of a convex set of feasible solution of a L.P.P. corresponds to a basic feasible solutions of the problem.

How to find that extreme point which corresponds to the optimal solution? In other words, how to identify an optimal solution out of the basic feasible solutions of a L.P.P?

To answer this question, we use an algebraic method popularly called simplex method.

— The simplex method is a computational process suitable for a numerical solution of a linear programming problem.

It was given by a famous American mathematician G.B. Dantzig in 1947. The word "simplex" has nothing to do with the method as such. Its origin can be traced back to a special problem that was studied in the early development of its algorithm. During world war-II, a group worked on allocation problems for the U.S. Air Force. A few models were developed by this group to allocate resources in such a way so as to maximize or minimize some linear objective function.

However, it was Dantzing, a member of this group, who ultimately formulated the general linear programming problem and devised the simplex method for its solution. problems of linear ~~problem~~ programming type were formulated and discussed even before the method was developed by Dantzig. However, the simplex method is the most efficient and reliable procedure that is generally used to solve a L.P.P. The method is extensively applied with the help of modern computers when the LPP involves a large number of constraints and variables.

But before we discuss the method, let us first understand the meaning of an Algorithm.

Meaning of ~~An~~ Algorithm:

An algorithm is an iterative solution procedure. It is simply a process in which the steps are repeated (iterated) over and over again until the desired result is achieved

Thus, an algorithm is a procedure starting with the first step known as the initial step, developing a criteria to know when and where to stop and ~~the~~ reach the last step where the desired result is obtained.

This can be summarized in the following way:

(i) Structure of a General Algorithm:

First step: Ready to start the iterations

Subsequent steps: performing the iterations

Concluding steps: Has the desired result been achieved?

If yes : stop

If NO : Repeat the iterations.

for the algorithm of the simplex method, we have the following similar structure:

(ii) Structure of the simplex Algorithm:

The simplex Algorithm is an algebraic procedure in which each iteration involves a system of equations to obtain a new trial solution for the optimality test. According to the procedure, we have to take the following three steps:

(I) Initial step: Start with a basic feasible solution of a given LPP.

(II) Iterative step: Move to a better basic feasible solution.

(III) optimality Test step: The current basic feasible solution is optimal.

(IV) If yes : stop
If no : Repeat the iterative step.

we give a complete description of the general algorithm through the following example of a LPP:

Example:

Maximize $Z = 3x_1 + 5x_2$

subject to: $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1 \ge 0,\ x_2 \ge 0.$

Sol By introducing the slack variables, the problem becomes:

Maximize $Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$.

subject to $x_1 + x_3 = 4$ —— (1)

$2x_2 + x_4 = 12$. —— (2)

$3x_1 + 2x_2 + x_5 = 18$ —— (3)

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0$. —— (4)

where x_3, x_4, x_5 are slack variables.

Note that this problem is identical to the original (given problem) and its form is much more convenient for algebraic manipulation and for identification of the feasible solutions. While dealing with this problem, it is easier to manipulate the equation of the objective function along with the constraint equations simultaneously.

Therefore, before we use the steps of the algorithm, we rewrite the problem once again in an equivalent way in the equations form as follows:

Maximize Z

subject to

$Z - 3x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5 = 0$

$0Z + x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 = 4$

$0Z + 0x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 = 12$

$0Z + 3x_1 + 2x_2 + 0x_3 + 0x_4 + x_5 = 18.$

where $x_i \geq 0$; $i = 1, 2, 3, 4, 5$.

Now to use the algorithm, we have to find answers to the following corresponding questions:

(I) Initial step: How is the basic feasible solution selected?

(II) Iterative step: while seeking a better basic feasible solution, how is the direction of the movement chosen? where do we stop? How to identify the new solution?

(III) optimality Test: How to determine whether the latest basic feasible solution is the optimal solution.?

for the step I, we can start with any basic feasible solution which is convenient to us. when the problem is in equality form, the obvious choice is the origin i.e all the variables are taken as equal to zero

i.e $x_1 = 0, x_2 = 0$ is the starting basic feasible solution.

But after introducing the slack or surplus variables, we take original variables viz x_1, x_2 as the non-basic variables and the slack variables viz x_3, x_4, x_5 as basic variables for the initial (starting) basic feasible solution.

Therefore from ①, ② and ③,

we get the basic variables $x_3 = 4, x_4 = 12$
$x_5 = 18$

with the non-basic variables x_1, x_2 chosen as $x_1 = 0, x_2 = 0$.

Therefore, the initial basic feasible solution is $(0, 0, 4, 12, 18)$

for which $\boxed{Z = 0}$.

For step (II):

We move from the current basic feasible solution to the next better basic feasible solution. This is done by replacing one non-basic variable (called the entering basic variable) by a basic variable (called the Leaving Basic variable) and identifying the new basic feasible solution.

Now the question arises: How to identify the entering Basic variable?

Since the objective (aim) is to increase the value of the objective function Z, therefore the variable that has the largest coefficient

in the equation for Z would be the one to increase Z and hence should be chosen as the entering basic variable.
Here the choice of the entering variable is $\underline{\underline{x_2}}$.

How to identify the Leaving Basic Variable?

The possibilities for the leaving basic variable is that it is one of the non-basic variables x_3, x_4, x_5.

Here one of these x_3, x_4, x_5 for which the entering variable x_2 achieves the maximum value and none of x_3, x_4, x_5 becomes negative. This is done as follows:

Table-I

Basic variable	eqn		Maximum value of x_2
x_3:	$x_1 + x_3 = 4 \Rightarrow x_3 = 4 - x_1 \Rightarrow x_3 = 4 \; (\because x_1 = 0)$	:	No limit
x_4:	$2x_2 + x_4 = 12 \Rightarrow x_4 = 12 - 2x_2$	:	$x_2 \leq \frac{12}{2} = 6 \quad (\because x_4 \geq 0)$
x_5:	$3x_1 + 2x_2 + x_5 \Rightarrow x_5 = 18 - 3x_1 - 2x_2$	:	$x_2 \leq \frac{18}{2} = 9 \quad (\because x_1 = 0, \; x_5 \geq 0)$

Since x_4 (slack variable for the constraint $2x_2 \leq 12$) gives $x_2 = 6$, which satisfies the two conditions viz that it is the maximum value for which none of x_3, x_4, x_5 becomes negative, therefore, x_4 is the leaving basic variable.

Note that $x_2 = 9$ is greater than $x_2 = 6$ but then if we take $x_2 = 9$ then x_4 becomes negative which we do not want.

Let us now calculate the new values of variables. Since we put only one non-basic variable $x_1 = 0$.

from the first row in Table I, we get $x_3 = 4$.

from the second row $x_4 = 0$ (i.e $x_4 = 12 - 2(6) = 0$)

from the third row $x_5 = 6$ (i.e $x_5 = 18 - 3(0) - 2(6) = 6$)

$\therefore$ new basic feasible solution is

$x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 6,$

$\boxed{Z = 30.}$ and x_2, x_3 and x_5 are new basic variables.

Next, we shall check whether our solution is optimal.

for this, we need to write our objective function in terms of the current non-basic variables ~~x_1 and x_4~~. So, we need ~~to need~~ to write x_1 and x_4. So, we need to write the current non-basic variables x_2 in terms of the current x_1 and x_4 and use these two to eliminate x_2

Also to find the leaving variable we need to write all the current basic variables x_2, x_3 and x_5 in terms of non-basic variables x_1 and x_4

$①\equiv \boxed{x_3 = 4 - x_1}$ —— 1ⓐ

$② \equiv 2x_2 = 12 - x_4$

$\boxed{x_2 = 6 - \frac{x_4}{2}}$ —— 2ⓐ

$③ \equiv x_5 = 18 - 3x_1 - 2x_2$

$= 18 - 3x_1 - 2\left(6 - \frac{x_4}{2}\right)$ (from (2ⓐ))

$= 18 - 3x_1 - 12 + x_4$

$\boxed{x_5 = 6 - 3x_1 + x_4}$ —— 3ⓐ

∴ The objective function becomes

$$Z = 3x_1 + 5x_2$$

$$= 3x_1 + 5\left(6 - \frac{x_4}{2}\right)$$

$$\boxed{Z = 3x_1 - \frac{5x_4}{2} + 30.}$$

In the above eqn, the coefficient x_1 is +ve, so we can improve Z by increasing x_1 (It is entering variable)

since the coefficient of x_4 is −ve, Z will decrease if we increase x_4. So, we keep x_4 at '0' level (i.e $x_4 = 0$)

we now have the following situation:

Table – II

Basic variables		Eqn	Maximum value of x_1
x_2	:	$x_2 = 6 - \frac{x_4}{2}$. $= 6$ ($\because x_4 = 0$)	No limit
x_3	:	$x_3 = 4 - x_1$	$x_1 \leq 4$ ($\because x_3 \geq 0$)
x_5	:	$x_5 = 6 - 3x_1 + x_4$ $= 6 - 3x_1$	$x_1 \leq \frac{6}{3} = 2$ ($\because x_5 \geq 0$)

since x_5, gives $x_1 \leq 2$ and this satisfies both $x_1 \leq 4$ and $x_1 \leq 2$, we let $\boxed{x_1 = 2}$, then $\boxed{x_5 = 0}$ Hence the x_5 is the leaving variable.

from the first row of Table-II, we get $\boxed{x_2 = 6}$

from the second row of Table-II, we get $x_3 = 4 - x_1$
$= 4 - 2$
$\therefore \boxed{x_3 = 2}$

from the third row of Table-II $\boxed{x_5 = 0}$

∴ the new basic feasible solution is

$x_1 = 2,\ x_2 = 6,\ x_3 = 2,\ x_4 = 0,\ x_5 = 0,$

$Z = 36$

i.e $(2, 6, 2, 0, 0)$ for which $\boxed{Z = 36}$

Now let us check whether our solution is optimal. Once again, we need to write (52) our objective function in terms of the current non-basic variables x_4 and x_5. So, we need to write x_1 in terms of x_4 and x_5 to eliminate x_1.

Also, to find the leaving variable we need to write all the current basic variables x_1, x_2 and x_3 in terms of non-basic variables x_4 and x_5. We have from (1c), (2c) and (3c),

$$\left.\begin{array}{l} x_1 + x_3 = 4 \quad \text{(i)} \\ x_2 + \frac{x_4}{2} = 6 \quad \text{(ii)} \\ 3x_1 + x_4 + x_5 = 6 \quad \text{(iii)} \end{array}\right\}$$

$$\text{(iii)} \equiv x_1 = \frac{6 - x_4 - x_5}{3} \Rightarrow \boxed{x_1 = 2 - \frac{x_4}{3} - \frac{x_5}{3}} \quad \text{(1d)}$$

$$\text{(i)} \equiv x_3 = 4 - x_1$$

$$\Rightarrow x_3 = 4 - \left(2 - \frac{x_4}{3} - \frac{x_5}{3}\right)$$

$$\Rightarrow \boxed{x_3 = 2 + \frac{x_4}{3} + \frac{x_5}{3}} \quad \text{(2d)}$$

$$\text{(ii)} \equiv \boxed{x_2 = 6 - \frac{x_4}{2}} \quad \text{(3d)}$$

$\therefore$ The objective function becomes,

$$Z = 3x_1 - \frac{3x_4}{2} + 30$$

$$= 3\left(2 - \frac{x_4}{3} - \frac{x_5}{3}\right) - \frac{5x_4}{2} + 30$$

$$\boxed{Z = 36 - \frac{7x_4}{2} - x_5}$$

In the above eqn the coefficients of all the non-basic variables are all negative. So, the solution can not be improved further and we have obtained the optimal solution.

* Working procedure of the simplex Method:

Assuming the existence of an initial basic feasible solution, an optimal solution to any L.P.P. by simplex method is found as follows:

step(1): (i) check whether the objective function is to be maximized (or) minimized.

If $Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$ is to be minimized, then convert it into a problem of maximization by writing

$$\text{Maximize } Z^* = \text{Minimize}(-Z) = -\text{Minimize}(Z).$$

(ii) check whether all b's are positive.

If any of the b_i's is negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

step(2): Express the problem in the standard form.

convert all inequalities of constraints into equations by introducing slack/surplus variables in the constraints giving eqns of the form $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + S_1 + 0S_2 + 0S_3 + \cdots \cdots = b_1$

step(3): find an initial basic feasible solution.

If there are 'm' equations involving 'n' unknowns, then assign zero values to any $(n-m)$ of the variables for finding a solution. Starting with a basic solution for which a_j : $j = 1, 2, 3, \dots (n-m)$ are each zero, find all s_i.

If all s_i's are > 0, the basic solution is feasible and non-degenerate.

If one or more of the s_i values are zero, then the solution is degenerate.

The above information can be expressed as:

	C_j	c_1	c_2	$c_3 \dots$	0	0	0 $\dots$	
C_B	Basis	x_1	x_2	$x_3 \dots$	s_1	s_2	$s_3 \dots$	b
0	s_1	a_{11}	a_{12}	$a_{13} \dots$	1	0	0 $\dots$	b_1
0	s_2	a_{21}	a_{22}	$a_{23} \dots$	0	1	0 $\dots$	b_2
0	s_3	a_{31}	a_{32}	$a_{33} \dots$	0	0	1 $\dots$	b_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
		BODY MATRIX			UNIT MATRIX			

Note:

→ The variables s_1, s_2, s_3 etc are called basic variables and variables x_1, x_2, x_3 etc are called non-basic variables.

→ Basis refers to the basic variables $s_1, s_2, s_3, \dots$

→ C_j – row denotes the coefficients of the variables in the objective function.

→ C_B – column denotes coefficients of the basic variables in the objective function.

→ b- column denotes the values of the basic variables while remaining variables will always be zero.

→ The coefficients of the x's (decision variables) in the constaint equation constitute the body matrix. while the coefficients of the slack variables constitute the unit matrix.

step(4): Apply optimality test.

compute $C_j = c_j - Z_j$ where $Z_j = \sum c_B a_{ij}$

(C_j - row is called net evaluation row and indicates the per unit increase in the objective function if the variable heading the column is brought into the solution).

— If all C_j are negative, then the initial basic feasible solution is optimal.

— If even one C_j is +ve, then the current feasible solution is not optimal (i.e can be improved) and proceed to the next step.

step(5): (i) Identify the incoming and outgoing variables.

— If there are more than one +ve C_j, then the incoming variable is the one that heads the column containing maximum C_j. The column containing it is known as the key column (put one arrow at bottom)

— If more than one variable has the same maximum C_j, any of these variables may be selected arbitrarily as the incoming variable.

— Now Divide the elements under b-column by the corresponding elements of key column and choose the row containing the minimum ratio $\theta\ (\geq 0)$

— Then replace the corresponding basic variable (by making its value zero). It is termed as the outgoing variable. The corresponding row is called the key row (put an arrow on its right end).

— The element at the intersection of the key row and key column is called the key element (which is shown bracketted).
It is also called pivotal elt or leading elt.

— If all these ratio's are ≤ 0, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an unbounded solution and no further iteration is required.

(ii) Iteration towards an optimal solution:

— Drop the outgoing variable and introduce the incoming variable along with its associated value under C_B-column.

— convert the key element to unity by dividing the key row by key element.

— Then make all other elements of the Key column zero by subtracting proper multiples of key row from other rows.

Step (6): Go to step(4) and the computional procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

Problems.

2002 → Use simplex method to solve the following LPP:

Maximize $Z = 5x_1 + 3x_2$

subject to

$x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12$

$x_1, x_2 \ge 0.$

Sol

Step(1): The problem is of maximization type and all b's ≥ 0

Step(2): Express the problem in the standard form. By introducing the slack variables s_1, s_2, s_3,

∴ The problem in the standard form becomes:

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\left.\begin{aligned} x_1 + x_2 + s_1 + 0s_2 + 0s_3 &= 2 \\ 5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 &= 10 \\ 3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 &= 12 \end{aligned}\right\} \text{---} (1)$$

$x_1, x_2, s_1, s_2, s_3 \ge 0.$

step(3): Find an initial basic feasible solution.

There are three equations involving 5 unknowns and for obtaining a solution, we assign zero values any $5-3=2$ of the variables

Let us start from $x_1=0$, $x_2=0$.

from ①, we get the basic solution

$s_1=2$, $s_2=10$, $s_3=12$

since all s_1, s_2, s_3 are +ve

∴ The basic solution is feasible and non-degenerate.

The basic feasible solution is :

$x_1=x_2=0$ (non-basic)

and $s_1=2$, $s_2=10$, $s_3=12$ (basic).

∴ Initial basic feasible solution is given by the following table:

	c_j	5	3	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	(1)	1	1	0	0	2	$\frac{2}{1}=2$ →
0	s_2	5	2	0	1	0	10	$\frac{10}{5}=2$
0	s_3	3	8	0	0	1	12	$\frac{12}{3}=4$
$Z_j=\Sigma C_B a_{ij}$		0	0	0	0	0	0	
$C_j=c_j-Z_j$		5 ↑	3	0	0	0		

[for x_1 - column $(j=1)$, $Z_j = \Sigma C_B a_{i1} = 0(1)+0(5)+0(3)=0$

and for x_2 - column $(j=2)$, $Z_j = \Sigma C_B a_{i2} = 0(1)+0(2)+0(8)=0$.

similarly $Z_j(b) = \Sigma C_B b = 0(2)+0(10)+0(12)=0$]

Step (4): Apply optimality test:

As C_j's +ve under some columns,

$\therefore$ the initial basic feasible solution is not optimal and we proceed to the next step.

Step (5):- (i) Identify the incoming and outgoing variable

The above table shows that x_1 is the incoming variable as C_j $(=5)$ is maximum and the column in which it appears is the key column.

Dividing the elts under b-column by the corresponding elts of key-column, we find minimum +ve ratio θ is 2 in two rows.

$\therefore$ Arbitrarily we choose the row containing s_1 as the key row (shown marked by arrow on its right end)

The element at the intersection of key row and key column i.e (1), is the key element.

$\therefore$ s_1 is the outgoing variable which will now become non-basic variable (i.e $s_1=0$)

So, removing s_1 and the basis will contain x_1, s_2 and s_3 as the basic variables.

(ii) Iterate towards the optimal solution:

To transform the initial set of eqns (contraint eqns(1)) with a basic feasible solution into an equivalent set of equations with different basic feasible solution,

we make the key element unity and make all other elements of the key column zero, subtracting proper multiples of key row from the other rows.

Here subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the third row. Also change the corresponding value under C_B column from 0 to 5, while replacing S_1 by x_1 under the basis.

$\therefore$ The second basic feasible solution is given by the following table.

	C_j	5	3	0	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b
5	x_1	1	1	1	0	0	2
0	S_2	0	-3	-5	1	0	0
0	S_3	0	5	-3	0	1	6
$Z_j = \sum C_B a_{ij}$		5	5	5	0	0	10
$C_j = c_j - Z_j$		0	-2	-5	0	0	

As C_j is either zero or negative (i.e $C_j \leq 0$) under all columns, the above table gives the optimal basic feasible solution.

$\therefore$ The optimal solution is $x_1 = 2,\ x_2 = 0$.

and maximum $Z = 10$.

→ Use the simplex method to solve the following LPP:

Maximize $Z = x_1 + 2x_2$

subject to

$-x_1 + 2x_2 \leq 8,$

$x_1 + 2x_2 \leq 12,$

$x_1 - x_2 \leq 3; \quad x_1, x_2 \geq 0$

Sol The objective function of the given LPP is of maximization type and R.H.S of all constraints are ≥ 0.

Now we write the given LPP in the standard form:

$\max Z = x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

subject to

$$\left.\begin{aligned} -x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 &= 8 \\ x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 &= 12 \\ x_1 - x_2 + 0S_1 + 0S_2 + S_3 &= 3 \end{aligned}\right\} \longleftarrow (1);$$

$x_1, x_2, S_1, S_2, S_3 \geq 0$

where S_1, S_2, S_3 are slack variables.

Now the initial basic feasible solution is given by

setting $x_1 = x_2 = 0$ (non-basic)

$S_1 = 8, S_2 = 12, S_3 = 3$ (basic)

∴ The IBFS is $(0, 0, 8, 12, 3)$

for which $Z = 0$.

Now we move from the current basic feasible solution to the next better basic feasible soln.

put the above information in tableau form: (57)

	C_j	1	2	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	-1	(2)	1	0	0	8	$\frac{8}{2}=4$ →
0	S_2	1	2	0	1	0	12	$\frac{12}{2}=6$
0	S_3	1	-1	0	0	1	3	—
$Z_j=\sum C_B a_{ij}$		0	0	0	0	0	0	
$C_j=c_j-Z_j$		1	2 ↑	0	0	0		

from the above table,

x_2 is incoming variable as C_j (=2) is maximum and the corresponding column is known as key column.

The minimum +ve ratio θ occurs in the first row

∴ S_1 is outgoing variable and the common intersection element (2) is the key element.

Now convert the key element to unity and all other elements in its column to zero. Then we obtain a new iterated simplex tableau as

	c_j	1	2	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
2	x_2	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	0	4	
0	S_2	(2)	0	-1	1	0	4	$\frac{4}{2} = 2 \rightarrow$
0	S_3	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	7	$\frac{7}{1/2} = 14$
$Z_j = \sum c_B a_{ij}$		-1	2	1	0	0	8	
$C_j = c_j - Z_j$		2 ↑	0	-1	0	0		

from the above tableau, x_1 is the incoming variable, S_2 is the outgoing variable and (2) is the key elt.

Now convert the key element to unity and all other elements in its column to zero. Then we get the new iterated simplex tableau as

	c_j	1	2	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
2	x_2	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	5	
1	x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	2	
0	S_3	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	1	6	
$Z_j = \sum c_B a_{ij}$		1	2	0	1	0	12	
$C_j = c_j - Z_j$		0	0	0	-1	0		

As C_j is either zero or negative (i.e. $C_j \le 0$) under all columns, the above tableau gives the optimal ~~solution~~ basic feasible solution.

$\therefore$ The optimal solution is $x_1 = 2, x_2 = 5$ and maximum $Z = 12$

Note! (i) from the optimum tableau, we observe that the net evaluation corresponding to non-basic variable s_1 is zero.

This is an indication for the existence of an alternate basic feasible solution.

Thus we can bring s_1 into basis in place of s_3 which satisfies the exist criterion. (It is giving minimum +ve ratio 0).

Therefore by introducing s_1 into the basis in place of s_3, the new optimum simplex tableau is as follows:

	c_j	1	2	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b
2	x_2	0	1	0	1/3	-1/3	3
1	x_1	1	0	0	1/3	2/3	6
0	s_1	0	0	1	-1/3	4/3	8
$z_j = \sum C_B a_{ij}$		1	2	0	1	0	12
$C_j = c_j - z_j$		0	0	0	-1	0	

from the above table, the alternative optimum sol^n is $x_1 = 6$, $x_2 = 3$

$$\boxed{\text{Max } Z = 12}.$$

we observe that the basic feasible solution (BFS) has been changed but the optimum solution remains the same.

$5-\frac{1}{2}$

$4-\frac{1}{3}$

$-\frac{1}{6}+\frac{1}{2}$

$\frac{-1+3}{6}$ $\frac{2}{6}$

$\frac{1}{2}-\frac{1}{8}$

$\frac{2}{3}$

$4+2$

$\frac{1}{4}+\frac{1}{12}$

$\frac{3+1}{12}$ $\frac{4}{12}$

→ Solve the following LPP by simplex method:

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Minimize $Z = x_1 - 3x_2 + 3x_3$

subject to

$$+3x_1 - x_2 + 2x_3 \le 7$$
$$2x_1 + 4x_2 \ge -12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10,$$
$$x_1, x_2, x_3 \ge 0.$$

Sol The objective function of the given is of minimization type.

converting it to the maximization type

we have $\text{Max} Z' = \text{Min}(-Z)$

$$= -x_1 + 3x_2 - 3x_3.$$

As the R.H.S of the second constraint is negative, we change it into +ve

we have $-2x_1 - 4x_2 \le 12$

Now we write the given LPP in standard form

$$\text{Max} Z' = -x_1 + 3x_2 - 3x_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$\left.\begin{array}{l} 3x_1 - x_2 + 2x_3 + S_1 + 0S_2 + 0S_3 = 7 \\ -2x_1 - 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12 \\ -4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10 \end{array}\right\} — ①$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

where S_1, S_2 & S_3 are slack variables.

procede in this way, we obtain the optimal solution

$x_1 = \frac{31}{5}$, $x_2 = \frac{58}{5}$, $x_3 = 0$ (non-basic)

and $Z'_{max} = 143/5$

i.e $\max(Z') = \frac{143}{5}$

Hence $\min Z = \max(-Z')$

$= -\frac{143}{5}$

→ Solve the following LPP by simplex method:

$\max Z = 107x_1 + x_2 + 2x_3$

subject to the constraints:

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2, x_3, x_4 \geq 0$.

Sol The objective function of the given LPP is of maximization type.

Now we write the given LPP in the standard form

$\max Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0S_1 + 0S_2$

subject to

$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0S_1 + 0S_2 = \frac{7}{3}$

$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + S_1 + 0S_2 = 5$

$3x_1 - x_2 - x_3 + 0x_4 + 0S_1 + S_2 = 0$.

$x_1, x_2, x_3, x_4, S_1, S_2 \geq 0$

where x_4, S_1, S_2 are the slack variables.

Now the IBFS is given by

setting $x_1 = x_2 = x_3 = 0$ (non-basic)

$x_4 = 7/3$, $s_1 = 5$, $s_2 = 0$ (basic).

$\therefore$ The IBFS is $(0, 0, 7/3, 5, 0)$

for which $Z = 0$.

Now we move from the current BFS to the next better BFS.

put the above information in tableau form:

	c_j	107	1	2	0	0	0		
C_B	Basis	x_1	x_2	x_3	x_4	s_1	s_2	b	θ
0	x_4	$\frac{14}{3}$	$1/3$	-2	1	0	0	$7/3$	$7/3 / \frac{14}{3} = 1/2$
0	s_1	16	$1/2$	-6	0	1	0	5	$5/16$
0	s_2	(3)	-1	-1	0	0	1	0	$0 \longrightarrow$
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	0	
$C_j = c_j - z_j$		107 $\uparrow$	1	2	0	0	0		

From the above table, x_1 is the entering variable, s_2 is outgoing variable and (3) is key element.

convert the key element to unity and all other elements in its column to zero. Then we obtain the new iterated simplex tableau:

	c_j	107	1	2	0	0	0		
C_B	Basis	x_1	x_2	x_3	x_4	s_1	s_2	b	θ
0	x_4	0	$17/9$	$-4/9$	1	0	$14/9$	$7/3$	$-\frac{21}{4}$
0	s_1	0	$35/6$	$-2/3$	0	1	$-\frac{16}{3}$	5	$-15/2$
107	x_1	1	$-1/3$	$-1/3$	0	0	$1/3$	0	0
$Z_j = \sum c_B a_{ij}$		107	$-107/3$	$-107/3$	0	0	$107/3$	0	
$C_j = c_j - Z_j$		0	$110/3$	$113/3$	0	0	$-107/3$		

As C_j is +ve under some columns, the solution is not optimal. Here $\frac{113}{3}$ being the largest +ve value of C_j, x_3 is the incoming variable. But all the values of θ being ≤ 0 x_3 will not entre the basis.

This indicates that the solution to the problem is unbounded.

* Using the simplex method, solve the following L.P.P.

→ Max $Z = x_1 + 3x_2$
subject to
$x_1 + 2x_2 \leq 10$
$x_1 \leq 5$
$x_2 \leq 4$.
$x_1, x_2 \geq 0$

→ Max $Z = 4x_1 + 5x_2$
subject to
$x_1 - 2x_2 \leq 2$
$2x_1 + x_2 \leq 6$
$x_1 + 2x_2 \leq 5$
$-x_1 + x_2 \leq 2$;
$x_1, x_2 \geq 0$

→ Max $Z = 4x_1 + 10x_2$
Subject to
$2x_1 + x_2 \leq 50$
$2x_1 + 5x_2 \leq 100$
$2x_1 + 3x_2 \leq 90$
$x_1, x_2 \geq 0$.

→ Max $Z = 10x_1 + x_2 + 2x_3$
Subject to
$x_1 + x_2 - 2x_3 \leq 10$
$4x_1 + x_2 + x_3 \leq 20$
$x_1, x_2, x_3 \geq 0$.

→ $\text{Min } Z = 3x_1 + 5x_2 + 4x_3$

subject to

$2x_1 + 3x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1, x_2, x_3 \ge 0$

→ $\text{Min } Z = x_1 - 3x_2 + 2x_3$

subject to

$3x_1 - x_2 + 2x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \ge 0.$

* Artificial variable Techniques:

So far we have seen that the introduction of slack variables provided the initial basic feasible solution. But there are many problems wherein at least one of the constraints is of $(\ge)$ or $(=)$ type and slack variables fail to give such a solution.

Suppose the given LPP is of the form:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ge b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ge b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ge b_m$$

$$x_1, x_2, \ldots, x_n \ge 0.$$

Now we write in the standard form

$$\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - x_{n+2} = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - x_{n+m} = b_m.$$

$$x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m} \geq 0.$$

where $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are surplus variables.

Now we may obtain the initial basic feasible solution (IBFS) to this problem by setting $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

Thus we obtain $x_{n+1} = -b_1, x_{n+2} = -b_2, \ldots, x_{n+m} = -b_m$.

Now, the following relevent question arises: This basic solution is not feasible solution, it violates non-negativity restrictions, how can we apply simplex method to solve such a problem?

For such a problem a slight modification is required.

For the problems with $(\geq)$ or $(=)$ constraints, the slack variables cannot provide a starting feasible solution.

To find a starting feasible solution in such cases, we use the methods of "artificial variables." The methods have acquired the name "artificial variables" because in these methods, we take the help of some variables which are fictitious and have no physical meaning. These variables are eliminated from the simplex

Tableau as soon as they become non-basic.

There are two similar method for solving such problems which we explain below:

(i) The "big M-method" or "Method of penalties" due to A. Charnes and

(ii) "two phase method" due to Dantzig, orden and wolfe.

* The Big M-method or Method of Penalties:

The big M-method or method of penalties consists of the following basic steps:

Step(1): ~~Express~~ Express the problem in standard form.

Step(2): Introduce non-negative variables to L.H.S. of all the constraints of $(\geq)$ or $(=)$ type. such new variables are called artificial variables. The purpose of introducing artificial variables is just to obtain an IBFS. However, addition of these variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the optimum simplex table. To achieve this, we assign very large penalty '$-M$' to these artificial variables in the objective function, for maximization objective function.

Step (3): solve the modified L.P.P. by simplex method.

At any iteration of simplex method, one of the following three cases may arise:

(i) There remains no artificial variable in the basis and the optimality condition is satisfied. Then the solution is an optimal basic feasible solution to the problem.

(ii) There is atleast one artificial variable in the basis at zero level (with zero value in b-column) and the optimality condition is satisfied. Then the solution is a degenerate optimal basic feasible solution.

(iii) There is at least one artificial variable in the bases at non-zero level (with positive value in b-column) and the optimality condition is satisfied. Then the problem has no feasible solution. The final solution is not optimal, since the objective function contains an unknown quantity M. Such a solution satisfies the constraints but does not optimize the objective function and is therefore, called pseudo optimal solution

Step(4): Continue the simplex method until either an optimal basic feasible solution is obtained or an unbounded solution is indicated.

Note:- The artificial variables are only a computational device for getting a starting solution. Once an artificial variable leaves the basis, it has served its purpose and we forget about it i.e the column for this variable is omitted from the next simplex tableau.

problems

→ Use Charne's penalty method to

Minimize $Z = 2x_1 + x_2$

subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$; $x_1, x_2 \geq 0$.

Sol The objective function of the given LPP is of minimization type

So, we convert it into maximization type

$$\text{Max } Z' = \text{Min}(-Z)$$

$$= -2x_1 - x_2$$

Now we write the given LPP in the standard form

$$\text{Max } Z' = -2x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

subject to

$$3x_1 + x_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 3$$

$$4x_1 + 3x_2 - S_1 + 0S_2 + 0A_1 + A_2 = 6$$

$$x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 3;$$

$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

where S_1 is the surplus variable,

S_2 is the slack variable

A_1, A_2 are the artificial variable.

Now the surplus variable S_1 is not a basic variable since its value is -6. As negative quantities are not feasible, S_1 must be prevented from appearing in the initial solution. This is done by taking $\boxed{S_1 = 0}$.

By setting other non-basic variables
$$x_1 = x_2 = 0,$$
we obtain the IBFS as
$$x_1 = x_2 = 0,\ S_1 = 0,\ A_1 = 3,\ A_2 = 6,\ S_2 = 3.$$

Thus the initial simplex table is:

	c_j	-2	-1	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	A_1	A_2	b	θ
$-M$	A_1	(3)	1	0	0	1	0	3	$\frac{3}{3} = 1 \rightarrow$
$-M$	A_2	4	3	-1	0	0	1	6	$\frac{6}{4} = 3/2$
0	S_2	1	2	0	1	0	0	3	$3/1 = 3$
$Z_j = \sum C_B a_{ij}$		$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$	
$C_j = c_j - Z_j$		$+7M-2$ ↑	$4M-1$	$-M$	0	0	0		

From the above table,
the variable x_1 is entering variable,
A_1 is the outgoing variable and omit column for this variable in the next simplex table.

Here (3) is the key element and convert it into unity and all other elements in its column to zero.
Then the new simplex table is:

	c_j	-2	-1	0	0	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	A_2	b	θ
-2	x_1	1	$1/3$	0	0	0	1	3
$-M$	A_2	0	$(5/3)$	-1	0	1	2	$6/5 \rightarrow$
0	s_2	0	$5/3$	0	1	0	2	$6/5$
$z_j = \Sigma c_B a_{ij}$		-2	$-\frac{2}{3}-\frac{5M}{3}$	M	0	$-M$	$-2-2M$	
$C_j = c_j - z_j$		0	$(-\frac{1}{3}+\frac{5M}{3})\uparrow$	$(-M)$	0	0		

From the above table,
x_2 is the entering variable,
A_2 is the outgoing variable and omit its column in the next simplex ~~table~~
Here $(5/3)$ is ~~the~~ key element and make it unity and all other elements in its column equal to zero.
Then the revised simplex table is:

	c_j	-2	-1	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	b
-2	x_1	1	0	$1/5$	0	$3/5$
-1	x_2	0	1	$-3/5$	0	$6/5$
0	s_2	0	0	1	1	0
$z_j = \Sigma c_B a_{ij}$		-2	-1	$1/5$	0	$-12/5$
$C_j = c_j - z_j$		0	0	$-1/5$	0	

From the above table, all C_j's ≤ 0
There remains no artificial variable in the basis.
$\therefore$ the solution is an optimal BFS to the problem and is given by
$x_1 = 3/5$, $x_2 = 6/5$ & $s_2 = 0$.
$\therefore$ MAX $Z' = -12/5$

Hence the optimal value of the objective function is $\text{Min } Z = -\text{Max } Z' = \frac{12}{5}$.

→ $\text{Max } Z = 3x_1 + 2x_2$

Subject to the

$2x_1 + x_2 \leq 2,$

$3x_1 + 4x_2 \geq 12, \; x_1, x_2 \geq 0.$

Sol The objective function of the given LPP is of maximization type.

Now we write the given LPP in standard form:

$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

Subject to

$2x_1 + x_2 + S_1 + 0S_2 + 0A_1 = 2$

$3x_1 + 4x_2 + 0S_1 - S_2 + A_1 = 12;$

$x_1, x_2, S_1, S_2, A_1 \geq 0.$

where S_1 is slack variable, S_2 is the surplus variable and A_1 is the artificial variable.

Now the IBFS is

$S_2 = x_1 = x_2 = 0$ (non-basic)

$S_1 = 2, A_1 = 12$ (basic)

for which $\boxed{Z = 0}$.

Now we put the above information in the simplex tableau:

	c_j	3	2	0	0	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	A_1	b	θ
0	S_1	2	(1)	1	0	0	2	2 →
$-M$	A_1	3	4	0	-1	1	12	3.
$Z_j = \Sigma C_B a_{ij}$		$-3M$	$-4M$	0	M	$-M$	$-12M$	
$C_j = c_j - Z_j$		$3+3M$	$2+4M$ ↑	0	$-M$	0		

from the above table,
x_2 is the entering variable,
S_1 is the outgoing variable and (1) is the key elt. and all other elements in its column equal to zero.
Then the revised simplex table is

	c_j	3	2	0	0	$-M$	
C_B	Basis	x_1	x_2	S_1	S_2	A_1	b
2	x_2	2	1	1	0	0	2
$-M$	A_1	-5	0	-4	-1	1	4
$Z_j = \Sigma C_B a_{ij}$		$4+5M$	2	$2+4M$	M	$-M$	$4-4M$
$C_j = c_j - Z_j$		$-(1+5M)$	0	$-(2+4M)$	$-M$	0	

from the above table
all C_j's are ≤ 0 and artificial variable appears in the basis at non-zero level.
Thus there exists a pseudo optimal solution to the problem.

* solve the following L.P. problems using M-method:

→ Maximize $Z = 3x_1 + 2x_2 + 3x_3$

subject to

$2x_1 + x_2 + x_3 \leq 2$

$3x_1 + 4x_2 + 2x_3 \geq 8$,

$x_1, x_2, x_3 \geq 0$.

→ Max $Z = 2x_1 + x_2 + 3x_3$

subject to

$x_1 + x_2 + 2x_3 \leq 5$

$2x_1 + 3x_2 + 4x_3 = 12$

$x_1, x_2, x_3 \geq 0$

→ Min $Z = 4x_1 + 3x_2 + x_3$

subj. to

$x_1 + 2x_2 + 4x_3 \geq 12$

$3x_1 + 2x_2 + x_3 \geq 8$,

$x_1, x_2 \geq 0$.

→ Max $Z = 8x_2$

subject to:

$x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$

x_1, x_2 unrestricted

→ Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

subj. to

$x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$,

$x_1, x_2, x_3, x_4 \geq 0$.

→ Min $Z = 4x_1 + 3x_2$

subject to

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$.

* Two-phase method:

The two-phase method is an alternative method to solve a given L.P.P. in which artificial variables are involved.

It is solved in two phases.

phase-I :

step(1): express the given LPP in the standard form by introducing slack, surplus and artificial variables.

step(2): formulate an artificial objective function $Z^* = -A_1 - A_2 - \cdots - A_m$ by assigning (-1) cost to each of the artificial variable A_i and zero cost to all other variables.

step(3): Max Z^* subject to the constraints of the original problem using the simplex method. Then three cases arise:

(a) Max $Z^* < 0$ and at least one artificial variable appears in the optimal basis at a positive level.
In this case, the original problem does not possess any feasible solution and the procedure comes to an end.

(b) Max $Z^* = 0$ and no artificial variable appears in the optimal basis.
In this case, a basic feasible solution is obtained and we proceed to phase-II for finding the optimal basic feasible solution to the original problem.

(c) Max $Z^* = 0$ and at least one artificial variable appears in the optimal basis at zero level.
Here a feasible solution to the auxilary L.P.P. is also a feasible solution to the original problem with all artificial variables set $= 0$.

To obtain a basic feasible solution, we prolong phase-I for pushing all the artificial variables out of the basis (without proceeding on to phase-II).

Phase-II :

The basic feasible solution found at the end of phase-I is used as the starting solution for the original problem in this phase i.e the final simplex table of phase-I is taken as the initial simplex table of phase-II and the artificial objective function is replaced by the original objective function.

Note :- Before initiating phase-II, remove all artificial variables from the table, which were non-basic at the end of phase-I.

problems → Use two-phase simplex method to

(1) $\text{Max } Z = 5x_1 + 3x_2$

subject to

$2x_1 + x_2 \leq 1,$

$x_1 + 4x_2 \geq 6,$

$x_1, x_2 \geq 0.$

sol phase-I

we write the given LPP in the standard form

$\text{Max } Z = 5x_1 + 3x_2 + 0S_1 + 0S_2 - MA$

subject to

$$\left.\begin{aligned} 2x_1 + x_2 + S_1 - 0S_2 + 0A &= 1 \\ x_1 + 4x_2 + 0S_1 - S_2 + A &= 6 \end{aligned}\right\} \text{—} (1).$$

$x_1, x_2, S_1, S_2, A \geq 0$

where s_1 is the slack variable

s_2 is the surplus variable and

A is an artificial variable.

Now we formulate an artificial objective function z^* by assigning (-1) cost to an artificial variable A and zero cost to all other variables x_1, x_2, s_1, s_2.

we have $\max z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 - A$

subject to

$$\left.\begin{array}{l} 2x_1 + x_2 + s_1 - 0s_2 + 0A = 1 \\ x_1 + 4x_2 + 0s_1 - s_2 + A = 6 \end{array}\right\} \text{--- (2)}$$

$x_1, x_2, s_1, s_2, A \geq 0$

Now the IBFS is given by

setting $x_1 = x_2 = s_2 = 0$ (non-basic)

$s_1 = 1$, $A = 6$ (Basic).

and $z^* = -6$ (< 0).

$\therefore$ Initial simplex table is:

	c_j	0	0	0	0	-1		
c_B	Basis	x_1	x_2	s_1	s_2	A	b	θ
0	s_1	2	(1)	1	0	0	1	$1 \rightarrow$
-1	A	1	4	0	-1	1	6	$\frac{6}{4} = \frac{3}{2}$
$z_j^* = \sum c_B a_{ij}$		-1	-4	0	1	-1	-6	
$C_j = c_j - z_j^*$		1	$4 \uparrow$	0	-1	0		

from the above table:

x_2 is the entering variable,

s_1 is the outgoing variable,

and (1) is the key element and we make

all other elements in its column equal to zero.

$\therefore$ The new simplex table is:

	c_j	0	0	0	0	-1	
c_B	Basis	x_1	x_2	s_1	s_2	A	b
0	x_2	2	1	1	0	0	1
-1	A	-7	0	-4	-1	1	2
$Z_j^* = \Sigma c_B a_{ij}$		7	0	4	1	-1	-2
$C_j = c_j - Z_j$		-7	0	-4	-1	0	

- from the above table all C_j's ≤ 0.

$\therefore$ an optimum BFS to the auxiliary LPP is obtained.

But $\max Z^* = -2 \,(< 0)$ and artificial variable A is in the basis at a +ve level.

$\therefore$ The original LPP doesnot posses any feasible solution.

→ Use two-phase method to

$\min Z = 7.5x_1 - 3x_2$

subject to

$3x_1 - x_2 - x_3 \geq 3,$

$x_1 - x_2 + x_3 \geq 2,$

$x_1, x_2, x_3 \geq 0.$

Sol The objective of the function of the given LPP is of minimization type.

so we convert it into maximization type

we have $\max Z_1 = \min(-Z)$

$\therefore \max Z_1 = -7.5x_1 + 3x_2$

We write the given LPP in the standard form:

$$\max Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$$

subject to

$$\left.\begin{aligned} 3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 &= 3 \\ x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 &= 2 \end{aligned}\right\} \quad (1)$$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

where S_1, S_2 are surplus variables and A_1, A_2 are artificial variables.

Now we formulate an artificial objective function Z_1^* by assigning (−1) cost to an artificial variables A_1 & A_2 and zero cost to all other variables x_1, x_2, x_3, S_1 & S_2.

∴ we have

$$\max Z_1^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1 - A_2$$

subject to

$$\left.\begin{aligned} 3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 &= 3 \\ x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 &= 2 \end{aligned}\right\} \quad (2)$$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$.

Now the IBFS is given by setting $x_1 = x_2 = S_1 = S_2 = 0$ (non-basic)

$A_1 = 3, A_2 = 2$ (basic)

and $Z_1^* = -5$ (<0)

∴ Initial simplex table is:

	C_j	0	0	0	0	0	-1	-1		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b	θ
-1	A_1	(3)	-1	-1	-1	0	1	0	3	1 →
-1	A_2	1	-1	1	0	-1	0	1	2	2
$z_j^* = \Sigma C_B a_{ij}$		-4	2	0	1	1	-1	-1	-5	
$C_j = c_j - z_j^*$		4 ↑	-2	0	-1	-1	0	0		

from the above ~~above~~

x_1 is the entering variable,

A_1 is the outgoing variable.

Here (3) is the key element and make it into unity and make all other elts in its column to zero.

∴ The new simplex table is:

	C_j	0	0	0	0	0	-1	-1		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b	θ
0	x_1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	1	-3
-1	A_2	0	$-2/3$	$(4/3)$	$1/3$	-1	$-1/3$	1	1	$\frac{3}{4}$ →
$z_j^* = \Sigma C_B a_{ij}$		0	$2/3$	$-4/3$	$-1/3$	1	$1/3$	-1	-1	
$C_j = c_j - z_j^*$		0	$-2/3$	$4/3$ ↑	$1/3$	-1	$-1/3$	0		

$+1+\frac{1}{3}$

from the above table,

x_3 is the entering variable,

A_2 is the outgoing variable,

Here $(4/3)$ is the key element and we convert it into unity and all other elements in its column equal to zero.

∴ The revised simplex table is:

	c_j	0	0	0	0	0	-1	-1	
c_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b
0	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	$3/4$
$z_{1j}^* = \sum c_B a_{ij}$		0	0	0	0	0	0	0	0
$C_j = c_j - z_{1j}^*$		0	0	0	0	0	-1	-1	

From the above table,

all C_j's ≤ 0,

this table gives the optimal solution.

Also $\max z_1^* = 0$.

and no artificial variable appears in the basis. Therefore $x_1 = 5/4$, $x_3 = 3/4$ is an OBFS to the auxiliary L.P.P.

To find OBFS to the original problem we proceed to phase-II

phase-II:

Considering the actual costs associated with the original variables, the objective function is

$$\max Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0S_1 + 0S_2 - 0A_1 - 0A_2$$

Subject to

$$3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 = 3$$

$$x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 = 2$$

$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \ge 0.$$

Using final table of phase-I, the initial simplex table of phase-II is as follows:

	c_j	$-15/2$	3	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	b
$-\frac{15}{2}$	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$3/4$
$z_j = \sum c_B a_{ij}$		$-15/2$	$15/4$	0	$15/8$	$15/8$	$-75/8$ ✓
$C_j = c_j - z_j$		0	$-3/4$	0	$-15/8$	$-15/8$	

from the above table,

all C_j's ≤ 0,

This gives optimal solution

Hence an OBFS to the given L.P.P is

$x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$.

and $\max Z_1 = -75/8$.

Hence $\min Z = -\max(-Z)$

$= -\max Z$

$= \frac{75}{8}$

* Use two phase method to solve the following L.P. problems.

→ $\max Z = 5x_1 - 4x_2 + 3x_3$

subject to

$2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \le 76$

$8x_1 - 3x_2 + 6x_3 \le 50$

$x_1, x_2, x_3 \ge 0$.

→ $\max Z = 3x_1 + 2x_2$

Subj. to

$2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

$x_1, x_2 \ge 0$.

$\rightarrow$ Min $Z = x_1 + x_2$

Sub. to

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$.

$\rightarrow$ Max $Z = 5x_1 + 3x_2$

Sub. to. $2x_1 + x_2 \leq 1$,

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$.

* Exceptional cases.

(1) Tie for the incoming variable:
when more than one variable has the same largest +ve value in C_j row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the incoming variables arbitrarily. Such an arbitrary choice doesnot affect the optimal solution in any way.

(2) Tie for the outgoing variable:
when more than one variable has the same least +ve ratio under the θ- column, a tie for the choice of outgoing variable occurs. If the equal values of ratio are > 0, choose arbitrarily any one of them as leaving variable. Such an arbitrary choice doesnot affect the optimal solution.

If the equal values of ratios are zero, the simplex method fails. and we make use ~~the~~ following degeneracy technique.

(3) Degeneracy:

We know that a basic feasible solution is said to be degenerate if any of the basic variables vanishes. This phenomenon of getting a degenerate basic feasible solution is called degeneracy which may arise

(i) at the initial stage, when atleast one basic variable is zero in the initial basic feasible solution.

or (ii) at any subsequent stage, when the least +ve ratio's under θ - column are equal for two or more rows. In this case, an arbitrary choice of one of these basic variables may result in one or more basic variables becoming zero in the next iteration. At times, the same sequence of simplex iterations is repeated endlessly without improving the solution. These are termed as cycling type of problems. Cycling occurs very rarely. In fact, cycling has rarely occured in practical problems.

To avoid cycling, we apply the following procedure:

(i) Divide each element in the tied rows by the +ve coefficients of the key column in that row.

(ii) compare the resulting ratios (from left to right) first of unit matrix and then of the body matrix, column by column.

(iii) The outgoing variable lies in that row which first contains the smallest algebraic ratio.

problems

→ Max $Z = 5x_1 + 3x_2$

sub. to

$x_1 + x_2 \leq 2,$

$5x_1 + 2x_2 \leq 10,$

$3x_1 + 8x_2 \leq 12;$

$x_1, x_2 \geq 0.$

Sol We write the given L.P.P in standard form:

$\max Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

sub. to

$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$

$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$

$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12.$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

where s_1, s_2 & s_3 are slack variables.

The IBFS is $x_1 = x_2 = 0$ (non-basic)

$s_1 = 2, s_2 = 10, s_3 = 12$ (basic)

and $Z = 0$.

Initial simplex table is:

	c_j	5	3	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	1	1	1	0	0	2	$2/1 = 2$
0	s_2	(5)	2	0	1	0	10	$\frac{10}{5} = 2$ →
0	s_3	3	8	0	0	1	12	$\frac{12}{3} = 4$
$z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	
$C_j = c_j - z_j$		5 ↑	3	0	0	0		

from the above table;

x_1 is the incoming variable.

But the two rows have the same ratio under θ- column.

Now first column of unit matrix has 1 and 0 in the tied rows.

Dividing these by the corresponding elements of the key column, we get $\frac{1}{1}$ and $\frac{0}{5}$

$\therefore$ s_2-row gives the smaller ratio and therefore s_2 is the first outgoing variable and (5) is the key element

Thus the new simplex table:

	c_j	5	3	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	$(3/5)$	1	$-1/5$	0	0	0 $\longrightarrow$
5	x_1	1	$\frac{2}{5}$	0	$1/5$	0	2	5.
0	s_3	0	$34/5$	0	$-3/5$	1	6	$\frac{15}{17}$
$z_j = \sum c_B a_{ij}$		5	2	0	1	0	10	
$C_j = c_j - z_j$		0	1 $\uparrow$	0	-1	0		

$\frac{6\times 5}{34}$ 17

from the above table,

x_2 is the incoming variable.

s_1 is the outgoing variable.

Here $(\frac{3}{5})$ is the key element and making it into unity and all other elements in its column to zero.

∴ The revised simplex table is

	c_j	5	3	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b
3	x_2	0	1	$5/3$	$-1/3$	0	0
5	x_1	1	0	$-2/3$	$1/3$	0	2
0	s_3	0	0	$-34/3$	$5/3$	1	6
$Z_j = \sum c_B a_{ij}$		5	3	$5/3$	$2/3$	0	10
$C_j = c_j - Z_j$		0	0	$-5/3$	$-2/3$	0	

from the above table,
all C_j's ≤ 0.
So, the table gives the optimal solution.
Hence an optimal basic feasible solution
is $x_1 = 2$, $x_2 = 0$ and
Max $Z = 10$.

Problems

* solve the following degenerate L.P. problems.

→ $\max Z = 9x_1 + 3x_2$
sub. to
$4x_1 + x_2 \leq 8$
$2x_1 + x_2 \leq 4$,
$x_1, x_2 \geq 0$

→ $\max Z = 2x_1 + 3x_2 + 10x_3$
sub. to
$x_1 + 2x_3 = 0$,
$x_2 + x_3 = 1$;
$x_1, x_2, x_3 \geq 0$.

$\xrightarrow[12M]{2002}$ Using simplex method
$\max Z = 45x_1 + 80x_2$
sub. to
$5x_1 + 20x_2 \leq 400$
$10x_1 + 15x_2 \leq 450$, $x_1, x_2 \geq 0$.

2004 → 12M Use simplex method to solve the L.P.P.

$\text{Max } Z = 3x_1 + 2x_2,$

Sub. to

$x_1 + x_2 \le 4$

$x_1 - x_2 \le 2 \; ; \; x_1, x_2 \ge 0$

2005 → 30M Use simplex method to solve the following

$\text{Max } Z = 5x_1 + 2x_2$

Sub. to

$6x_1 + x_2 \ge 6$

$4x_1 + 3x_2 \ge 12$ and $x_1, x_2 \ge 0$.

2006 → 30M Use simplex method to solve the L.P.P.

$\text{Max } Z = 2x + 3y$

sub. to

$-2x + 3y \le 2$

$3x + 2y \le 5$

$x, y \ge 0$.

2007 → 30M solve the following by simplex method:

$\text{Max } U = x + y$

sub. to

$-x + y \le 1$

$x - 2y \le 4$

$x, y \ge 0$

2003 → An animal feed company must produce 200kg of mixture consisting of ingradients x_1 and x_2 daily. x_1 costs Rs. 3/- per kg and x_2 costs Rs. 8/- per kg. No more than 80 kg of x_1 can be used, and atleast 60 kg of x_2 must be used.

formulate linear programming model of the problem and use simplex method to determine the ingredients x_1 and x_2 to be used to minimize cost.