2018 IFoS MATHS P-2

(SECTION-A)

1 (a) Prove that a non commutative group of order 2n, where n is an odd prime, must have a subgroup of order n. [8]

Solution-

Let $|G| = 2n$, and

let $\Omega = \{x \in G \mid x \neq x^{-1}\} \subseteq G$,

i.e., Ω is the set of all elements of G which do not have order 1 or 2.

Note that if $x \in \Omega$, then $x^{-1} \in \Omega$, and $x \neq x^{-1}$ by the hypothesis that $x \in \Omega$.

Hence, Ω has even order, so G/Ω has even order.

Since $e \in G/\Omega$, G/Ω has at least order 2, whence G must have an element of order 2.

1(b) A function $f:[0,1] \rightarrow [0,1]$ is continuous on [0,1]. Prove that there exists a point c in [0,1] such that $f(c) = c$. [10]

Solution-

Let, $g(x) = f(x) - x$

Let's apply the Intermediate Value Theorem (IVT) to the function g, on the interval $[a,b] = [0,1]$

Here, $g(0) = f(0) - 0 = f(0)$

Also $g(1) = f(1) - 1$

But since f has codomain [0,1], it must be $f(1) \leq 1$.

If the equality stands, then the proof is over;

the point x=1 is such that $f(x) = x$.

If the inequality is strict, that is, $f(1) < 1$, then $g(1)<0$.
Similarly, $g(0) = f(0) \geq 0$, because f has codomain [0,1]. If the equality stands, we are done, just as bedore: $f(0) = 0$Otherwise, $g(0)$ is positive and finally we can apply the IVT:
The function g is such that $g(0)>0$ and $g(1)<0$, and is continuous because it is sum of continuous functions, therefore a number c must exist, such that $0<c<1$, and $g(c)=0$
But if $g(c)=0$, then by the definition of g, $f(c) - c = 0$, and rearranging, $f(c)=c$.

1(c) if $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that u+iv is a regular function of x+iy. [10]

Solution-
Here, $u = (x-1)^3 - 3xy^2 + 3y^2$

We find $f(z) = u + iv$ using Milne's Method

$$\emptyset_1 (z,0) = \frac{\partial u}{\partial x}\Big|_{x=z,\, y=0} = 3(z-1)^2$$

$$\emptyset_2 (z,0) = \frac{\partial u}{\partial y}\Big|_{x=z,\, y=0} = 0$$

$$f(z) = \int [\, \emptyset_1(z,0) - i\, \emptyset_2 (z,0)]\, dz + c$$

$$= \int [\, 3(z-1)^2\, dz + c$$

$$= (z-1)^2 + c$$

Being a polynomial function in z, f(z) is regular.

1(d) Solve by simplex method the following Linear Programming Problem:

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to the constraints

$$x_1 + 2x_2 + x_3 \leq 430$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 4x_2 \leq 420$$
$$x_1,\ x_2, x_3 \geq 0.$$

[12]

Solution-

By introducing slack variables convert the problem into standard from and inequality into equlity; we have

Max $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

S.T. $x_1 + 2x_2 + x_3 + S_1 = 430$

$3x_1 + 0x_2 + 2x_3 + S_2 = 460$

$x_1 + 4x_2 + 0x_3 + S_3 = 420$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Initial basic feasible solution is given by

$x_1 = x_2 = x_3 = 0$

$S_1 = 430,\ S_2 = 460,\ S_3 = 420$

Now prepare initial simplex table, Initial simplex table

C_B	B	C_j X_B	3 x_1	2 x_2	5 x_3	0 S_1	0 S_2	0 S_3
0	S_1	430	1	2	1	1	0	0
$\leftarrow 0$	S_2	460	3	0	$\boxed{2}$	0	1	0
0	S_3	420	1	4	0	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	-3	-2	-5 $\uparrow$	0	0	0

Here all values of $Z_j - C_j$ are not positive. hence, solution is not optimum. To find optimum solution select the most negative number. Here -5 is the most negative number, it will enter into the basis. Corresponding column will be treated as key column. To find key row, find

$$\min\left(\frac{X_B}{x_3}\right) = \min = \left(\frac{430}{1}, \frac{460}{2}, \frac{420}{0}\right) = 230$$

∴ The basic variable S_2 will leave the basis.

$\boxed{2}$ is the key element. Make it unity and other element of key column zero by matrix row transformation. Now we have first simplex table. First simplex table

C_B	B	C_j X_B	3 x_1	2 x_2	5 x_3	0 S_1	0 S_2	0 S_3
0	S_1	200	-1/2	$\boxed{2}$	0	1	1/2	0
5	x_3	230	3/2	0	1	0	1/2	0
0	S_3	420	1	4	0	0	0	1
	Z_j	1150	15/2	0	5	0	5/2	0
		Z_j - C_j	9/2	-2 ↑	0	0	5/2	0

Further, all values of Z_j - C_j are not positive. Hence, repeat the above process.

$\boxed{2}$ is the key element make it unity and other element of key column zero by applying matrix row transformation, we have the second simplex table.

Second simplex table

C_B	B	C_j X_B	3 x_1	2 x_2	5 x_3	0 S_1	0 S_2	0 S_3
2	x_2	100	-1/4	1	0	1/2	-1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	S_3	20	2	0	0	-2	1	1
	Z_j	1350	7	2	5	1	2	0
		Z_j - C_j	4	0	0	1	2	0

Since all values of Z_j - $C_j \geq 0$.

Hence, the solution is optimum, it is given by

$x_1 = 0$, $x_2 = 100$,

$x_3 = 230$, Max $Z = 1350$.

Q2 (a) Find all the homomorphisms from the group $(\mathbb{Z},+)$ to $(\mathbb{Z}_4, +)$.

[10]

Solution –

A homomorphism $\emptyset : \mathbb{Z}\rightarrow\mathbb{Z}_4$ is determined by $\emptyset(1)$

since $\emptyset(n)=n.\,\emptyset(1)$ for every $n\epsilon\mathbb{Z}$.

Also, for any $a \in \mathbb{Z}_4$, we can get a homomorphism $\mathbb{Z}\rightarrow\mathbb{Z}_4$ taking 1 to 'a' by sending n to the reducing mod 4 of an. So, there are four homomorphisms $\emptyset : \mathbb{Z}\rightarrow\mathbb{Z}_4$ one for each value in $\mathbb{Z}_4$.

If $\emptyset(1)=0$, we get the zero map. Its kernel is all of $\mathbb{Z}$ and its image is $\{0\}$.

If $\emptyset(1) = 1$, our map is just reduction mod4, which is clearly surjective: that is, its image is all of $\mathbb{Z}_4$. We see that an element is sent to zero if and only if it's a multiple of 4, so $\ker(\emptyset)=4\mathbb{Z}$.

If $\emptyset(1) = 2$, our map takes n to the reduction of 2n mod 4. The image is generated by 2, which is $<2> = \{0,2\}$. The kernel is the set of elements of n such that 2n is a multiple of 4. This is the set of even integers, $2\mathbb{Z}$.

If $\emptyset(1) = 3$, our map takes n to the reduction of 3n mod4. The image is generated by $\emptyset(1) = 3$ and so is all of $\mathbb{Z}_4$ (so it's surjective). Since 3n is a multiple of 4 if and only if n is a multiple of 4, the kernel is $4\mathbb{Z}$.

2(b) Consider the function f defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{where } x^2+y^2 \neq 0 \\ 0 \quad , & \text{where } x^2+y^2 = 0 \end{cases}.$$

Show that $f_{xy} \neq f_{yx}$ at (0,0). **[10]**

Solution- By definition,

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=0$$

$$f_y(x,0) \quad =\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}$$

$$=\lim_{k\to 0}\frac{1}{k}\left(xk.\frac{x^2-k^2}{x^2+k^2}-0\right)=x$$

$$f_x(0,y) \quad =\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(hy.\frac{h^2-y^2}{h^2+y^2}-0\right)=-y$$

$$f_{xy}(0,0) \quad =\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{1}{k}(-k-0)=-1 \quad \text{......(1)}$$

$$f_{yx}(0,0) \quad =\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}(h-0)=1 \quad \text{......(2)}$$

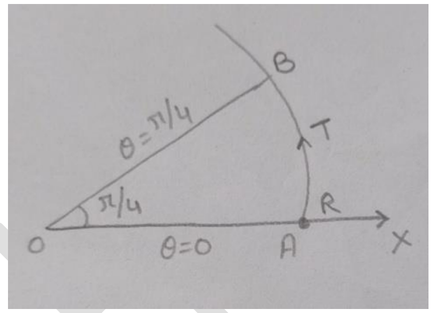
Here, $f_{xy} \neq f_{yx}$ at (0,0)

2(c) Prove that

$$\int_0^\infty \boldsymbol{cosx^2\, dx} = \int_0^\infty \boldsymbol{sinx^2\, dx} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

[10]

Solution –



Consider $\int_c f(z)dz = \int_c e^{-z^2}$dz

Where c is the contour as shown in the diagram.
Since f(z) is regular within and on the boundary of c, therefore by cauchy' residue theorem,

$$0 = \int_c f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BO} f(z)dz$$

Or $\int_0^R e^{-x^2}dx + \int_T e^{-z^2}\,dz + \int_R^0 e^{-r^2 e^{i\pi/2}}.e^{i\pi/4}dr = 0$ ......(1)

$$\left|\int_T e^{-z^2}\,dz\right| = \left|\int_0^{\pi/4} e^{-R^2 e^{i2\theta}}.iR\,e^{i\theta}\,d\theta\right|$$

$$\leq \int_0^{\pi/4} Re^{-R^2 cos2\theta}\,d\theta$$

$$\leq \frac{1}{2}\int_0^{\pi/2} Re^{-R^2 sin\emptyset}\,d\emptyset, \qquad \text{(Jordon's Inequality)}$$

$$= \frac{\pi}{4R}(1 - e^{-R^2}) \to 0 \text{ as } R \to \infty$$

(Jordon's Inequality :- $\frac{2\theta}{\pi} \leq \sin\theta \leq \theta$, where $0 \leq \theta \leq \frac{\pi}{2}$)

Hence as $R \to \infty$, we have from (1)

$$\int_0^\infty e^{-x^2}\,dx + 0 + (-1)\int_0^\infty e^{-ir^2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)dr = 0$$

$$\int_0^\infty (cosx^2 - i\,sinx^2)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)dx$$

$$= \int_0^\infty e^{-x^2}\,dx = \frac{\pi}{2}$$

Equating real and imaginary part on both sides

$\int_0^\infty (cosx^2 + \ sinx^2)\ dx \ = = \frac{\sqrt{\pi}}{2}$ and

$\int_0^\infty (cosx^2 - \ sinx^2)\ dx \ = 0$

Adding, we get the result: $\int_0^\infty cosx^2\ dx \ = \frac{1}{2}\ \frac{\sqrt{\pi}}{2}$

Subtracting, $\int_0^\infty sinx^2\ dx \ = \frac{1}{2}\ \frac{\sqrt{\pi}}{2}$

Q2(d) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. [10]

Solution-

Suppose that P is a prime ideal.
By definition, if a,b ϵ R-P, then ab ∈R-P.
Passing to the quotient ring, we see that $\bar{a}$ and $\bar{b}$ are not $\bar{0}$ in R/P implies that $\overline{ab} \neq 0$ in R/P.

Thus, R/P has no zero divisors. The quotient ring R/P has no zero divisors. The quotient ring R/P contains an identity $\bar{1}$ and inherits commutativity from R. Hence, R/P is an integral domain.

Conversely, suppose that R/P is an integral domain. Let nonzero elements in R/P correspond to $\bar{a}, \bar{b}$ with a,b ϵ R –P. Since R/P is an integral domain, $\overline{ab} \neq \bar{0}$ so $\overline{ab} \notin$ P. We have shown that a ∉ P and b∉P implies that ab ∉ P.

The contrapositive to this is ab ϵ P implies that a ϵ P and b ϵ P. This proves that P is a prime ideal.

3(a) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. **(10)**

Solution : - Let $u = x^2 + y^2 + z^2$ ... (i)

Where $\emptyset(x, y, z) = ax + by + cz - p = 0$ ... (ii)

Lagrange's equations for maximum or minima are

$$\frac{\partial u}{\partial x} + \frac{\lambda(\partial \emptyset)}{\partial x} = 2x + \lambda a = 0 \quad ...(iii)$$

$$\frac{\partial u}{\partial y} + \frac{\lambda(\partial \emptyset)}{\partial y} = 2y + \lambda b = 0 \quad ...(iv)$$

$$\frac{\partial u}{\partial z} + \frac{\lambda(\partial \emptyset)}{\partial z} = 2z + \lambda c = 0 \quad ...(v)$$

Multiplying (iii) $by\ x, (iv)\ by\ y\ and\ (v)\ by\ z\ and\ adding, we\ get$

$2(x^2+y^2 + z^2) + \lambda(ax + by + cz) = 0$

$$or \qquad 2u + \lambda p = 0$$

[using (i) and (ii)]

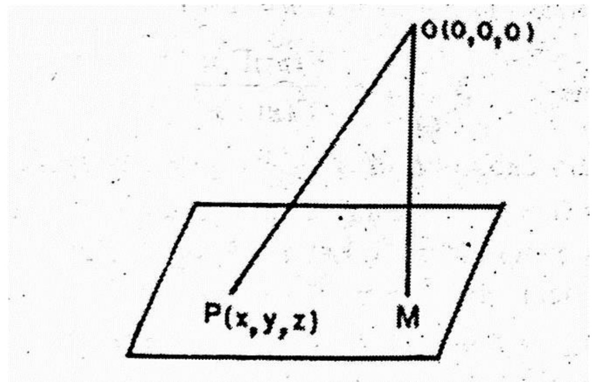
$$\therefore \qquad \lambda = -\frac{2u}{p}$$

From $(iii), (iv)\ and\ (v)$,

$$x = \frac{au}{p}, y = \frac{bu}{p}, z = \frac{cu}{p}$$

$$\therefore \quad from(1), \quad u = \frac{(a^2 + b^2 + c^2)u^2}{p^2}$$

$$or \qquad u = \frac{p^2}{a^2 + b^2 + c^2}$$



This is the maximum value or minimum value of u. now u is the square of the distance of any point $P(x, y, z)$ on the plane (ii) from the origin. Also, the length of perpendicular from O on the plane $= \frac{p}{\sqrt{a^2+b^2+c^2}}$

Clearly, OP is least when P coincides with M, the foot of perpendicular from

O on the plane. Hence, minimum value of $u = \frac{p^2}{a^2+b^2+c^2}$

3(b). Show by an example that in a finite commutative ring, every maximal ideal need not be prime. (10)

Solution : - Consider the ring $R = \{0, 2, 4, 6\}$ under addition and multiplication modulo 8.

Let $M = \{0, 4\}$ then M is easily seen to be an ideal of R.

Again as $2 \otimes 6 = 4 \in M$ but 2,6 $do\ not\ belongs\ to\ M$, we find M is not a prime ideal. We show M is maximal.

Let $M \subseteq N \subseteq R$, where N is an ideal of R.

Since $< M, + >$ will be a subgroup of $< N, + >$, by Lagrange's theorem $o(M)|o(N)$. Similarly, $o(N)|o(R) = 4$

i.e. , $2|o(N), o(N)|4$ i.e. , $o(N) = 2\ or\ 4$

if $o(N) = 2,\ then\ M = N\ as\ M \subseteq N$

if $o(N) = 4, them\ N = R\ as\ N \subseteq R$, Hence, M is maximal ideal of R.

Remark : In case the finite commutative ring contains unity, then every prime ideal is maximal.

3.(c) Evaluate the integral $\int_0^{2\pi} cos^{2n}\theta\, d\theta,$ where n is a positive integer.

(10)

Solution : - $2^n = cos^n x = \left(e^{ix} + e^{-ix}\right)^n$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^{ix})^k (e^{-ix})^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)ix} \qquad \text{.........}(*)$$

Since, $\int_0^{2\pi} e^{ilx}\, dx = \{2\pi \quad (l = 0)\ 0 \qquad (l \neq 0)$

At most one term on the rigt of $(*)$ contributes to the integral

$J_n = \int_0^{2\pi} cos^n x\, dx.$

When n is an odd then $2k - n \neq 0\ for\ all\ k$ in (*), therefore $J_n = 0$ in this case.
When n is even then $k = \frac{n}{2}$ gives the only contribution to the integral, and we get

$$\int_0^{2\pi} cos^n x\, dx = \frac{2\pi}{2^n} \binom{n}{\frac{n}{2}}.$$

4(a). Show that $\iint \; x^{m-1}y^{n-1}(1-x-y)^{l-1}dxdy = \frac{\tau(l)\tau(m)\tau(n)}{\tau(l+m+n)}; \quad l,m,n > 0$

Taken over R : the triangle bounded by $x = 0, y = 0, x + y = 1.$ **(10)**

Solution : -

In the definitions of Beta and Gamma Functions we substitute

$x = cos^2\theta \qquad => \qquad dx = -2\cos\cos\theta \sin\theta d\theta \quad or \;\; \beta(m,n)$

$$\therefore \quad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= -2\int_{-\pi/2}^{0} cos^{2m-2}\theta sin^{2n-2}\theta\, d\theta \qquad \text{... (1)}$$

Now we substitute $t = x^{\,2}$ $in\; G(m) = \int_0^\infty t^{m-2}e^{-1}\,dt$

$$\therefore \quad \tau(m) = \int_0^\infty x^{2m-2}e^{-x^2}.2xdx$$

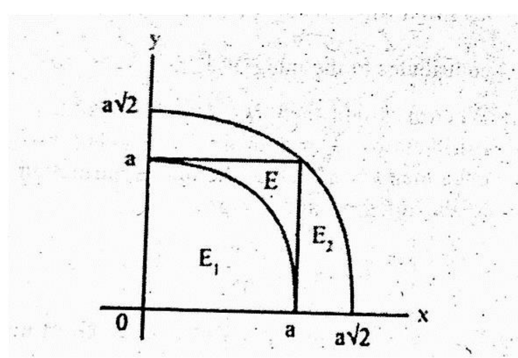
$$= 2\int_0^\infty x^{2m-1}e^{-x^2} \qquad \text{... (2)}$$

Now, $\tau(m)\tau(n)$

$$= 4\int_0^\infty x^{2m-1}e^{-x^2}\,dx\int_0^\infty y^{2n-1}e^{-y^2}\,dy$$

$$= 4\int_0^a x^{2m-1}e^{-x^2}\,dx\int_0^a y^{2n-1}e^{-y^2}\,dy \qquad or$$

$$\tau(m)\tau(n) = 4\iint_0^a x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dxdy \qquad \text{... (3)}$$



Where E is a square of side a.

It is clear that each diagonal of the square E is of length $\sqrt{a^2+a^2} = \sqrt{2a}$

Let $E_1\; and\; E_2$ be quarter circles of radii $a\; and \sqrt{2a}$ respectively.

Then, $= 4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-2(x^2+y^2)}dxdy$

$$= 4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx$$

$$= \iint_{E_2} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dxdy \ldots (4)$$

Putting $x = rcos\theta,\ \ y = rsin\theta$ we obtain

$$= 4\iint_{E_1} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dxdy$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^a r^{2m-1} cos^{2m-1}\theta^{2n-1} sin^{2n-1}\theta e^{-r^2}\, rdrd\theta$$

$$= 4\int_0^{\pi/2} cos^{2m-1}\theta sin^{2n-1}\theta\, d\theta \int_0^a r^{2(m+n)^{-1}} e^{-r^2}\, dr$$

$$= 2\beta(m,n)\int_0^a r^{2m-1} e^{-r^2}, using\ (1) \qquad \ldots (5)$$

Similarly,

$$= 4\iint_{E_2} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dxdxy$$

$$= 2\beta(m,n)\int_0^{\sqrt{2a}} r^{2(m+n)^{-1}} e^{-r^2}\, dr \qquad \ldots (6)$$

Substituting (5) and (6) in (4), we obtain

$$= 2\beta(m,n)\int_0^a r^{2(m+n)^{-1}} e^{-r^2}\, dr$$

$$\leq 4\iint_E x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dxdy$$

$$\leq 2\beta\int_0^{\sqrt{2a}} r^{2(m+n)^{-1}} e^{-r^2}\, dr$$

Letting $a \to \infty$ and using (2) and (3), we get

$$\beta(m,n)\tau(m+n) \leq \tau(m)\tau(n) \leq \beta(m,n)\tau(m+n)$$

or $$\beta(m,n)\tau(m+n) = \tau(m)\tau(n)$$

Hence, $$\beta(m,n) = \frac{\tau(m)\tau(n)}{\tau(m+n)}$$

4(b). Let $f_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$. Show that the sequence $< f_n >$ is uniformly convergent on [0,1]. **(8)**

Solution :- Point wise limit function

$$f(x) = f_n(x) = 0$$

Consider, $|f_n(x) - f(x)| = \left|\frac{x}{n+x^2} - 0\right| = \frac{x}{n+x^2} \leq \frac{x}{n} \leq \frac{1}{n} \quad on\ x \in (0,1]$

$$\{since, x^2 + n \geq n => \frac{1}{x^2+n} < \frac{1}{n}\ \}$$

$$sup\,|f_n(x) - f(x)| \leq \frac{1}{n} = 0$$

$$i.e.\ \ sup\,|f_n(x) - f(x)| \leq 0$$

But $|f_n(x) - f(x)| \geq 0$

$$= sup\,|f_n(x) - f(x) = 0$$

By M_n test $\quad M_n = sup\ sup\,|f_n(x) - f(x)|\ = 0$

$\therefore \quad f_n(x) = \frac{x}{x^2+n}$ is uniformly convergent on $[0,1]$.

4(c). Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. (10)

Solution :-

Every subgroup of a cyclic group is cyclic. Let $G =< g >$ be any cyclic group. Let H be any subgroup of G. Let H be any subgroup of G. Let $a \in H$ be an arbitrary element. Hence $a = g^m$ for some $g \in N$.

Let $r = \{m \mid g^m \in H\}$. let $x = g^r$ and $K =< x >$.

Let $y \in H$ be any element such that $y \notin K$. Again, $y = g^m$ for some

$m \in N$ and $r \leq m$. Hence, $g^m = g^{rv} g^u$.

Again, $g^{m-rv} = g^u \in H$, a contradiction. Hence $K = H$.

If N is a cyclic normal group subgroup of a finite group G, then show that every subgroup H of N is normal in G.

Define $K =< h >$ and $N =< k >$ where $h = k^r$.

Again, $ghg^{-1} = hk^r g^{-1} = (gkg^{-1})^r = k^{rn} = h^n \in H$ for all $g \in G$ where $n \in N$ on G.

4(d). The capacities of three production facilities $S_{1,}$, $S_2, and\ S_3$ and the requirements of four destinations $D_1, D_2, D_3\ and\ D_4\ and$ transportation costs in rupees are given in the following tables :

	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Find the minimum transportation cost using the Vogel's Approximation Method (VAM). (12)

Solution :- The following matrix gives data concerning the transportation times t_{ij}.

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
01	25	30	20	40	45	37	37
02	30	25	20	30	40	20	22
03	40	20	40	35	45	22	32
04	25	24	50	27	30	25	14
Demand	15	20	15	25	20	10	

We compute an initial basic feasible solution by north west corner rule which is shown in table 1.

Table 1

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 15	30 20	20 2	40	45	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	45 16	22	32
O4	25	24	50	27	30 4	25 10	14
Demand	15	20	15	25	20	10	

Here $t_{11} = 25, t_{12} = 30,\ t_{13} = 20,\ t_{23} = 20,\ t_{24} = 30,\ t_{34} = 35,\ t_{34} = 35,$ $t_{35} = 45,\ t_6 = 25.$

Choose maximum from t_{ij}, $i.e. T_1 = 45$. Now, cross out all the unoccupied cells that are $\geq T_{1.}$ The unoccupied cells(O3D6) enters into the basis as shown in table 2.

Table 2

	Destination						
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	− 45 (16)	+ 22	32
O4	25	24	~~50~~	27	+ 30 (4)	− 25 (10)	14
Demand	15	20	15	25	20	10	

Choose the smallest value with a negative position on the closed path i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Table 3

	Destination						
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	45 (6)	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

Here, $T_1 = (25,\ 30,\ 20,\ 20,\ 20,\ 35,\ 45,\ 22,\ 30) = 45$. Now cross out all the unoccupied cells that are $\leq T_2$.

Table 4

	Destination						
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	- 30 (9)	+ 40	20	22
O3	40	20	40	+ 35 (16)	- 45 (6)	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

By following the same procedure as explained above, we get the following revised matrix.

Table 5

	Destination						
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	~~40~~	~~45~~	37	37
O2	30	25	20 (13)	30 (3)	40 (6)	20	22
O3	~~40~~	20	~~40~~	35 (22)	~~45~~	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

$$T_3 = Max(25, 30, 20, 20,20,30,40,35,22,30) = 40$$

Now cross out all the unoccupied cells that are $\geq T_3$.

Now we cannot form any other closed loop with T_3. Hence, the solution obtained at this stage is optimal. Thus, all the shipments can be mate in 40 units.

(SECTION-B)

Q 5. (a) Find the partial differential equation of all planes which are at a constant distance from the origin. (10)

Soln . Let the required equation of the plane is

z = lx + my + n

lx + my - z + n = 0 ...(1)

Now the plane (1) is at a constant distance a from the origin

$$\therefore a = \frac{|n|}{\sqrt{l^2 + m^2 + 1}}$$

$$\Rightarrow a = \frac{\pm n}{\sqrt{l^2 + m^2 + 1}}$$

Here, $p = \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$

$$\Rightarrow n = \pm a\sqrt{l^2 + m^2 + 1}$$

$\therefore$ (1) becomes $lx + my + nz \pm a\sqrt{l^2 + m^2 + 1} = 0$ ...(2)

Differentiating (2) w.r.t. x and y , we get

$$l - \frac{\partial z}{\partial x} = 0, m - \frac{\partial z}{\partial y} = 0$$

or $p = l, q = m$

$\therefore$ (2) reduces to

$$px + qy - z \pm a\sqrt{p^2 + q^2 + 1} = 0$$

or $z = px + qy \pm a\sqrt{p^2 + q^2 + 1}$ is required differential equation.

5 (b) A solid of revolution is formed by rotating about the x-axis, the area between x-axis, the line x = 0 and a curve through the points with the following coordinates.

x	**y**
0.0	**1.0**
0.25	**0.9896**
0.50	**0.9589**
0.75	**0.9089**
1.00	**0.8415**
1.25	**0.8029**
1.50	**0.7635**

Estimate the Volume of the solid formed using Weddle's rule. (10)

Sol.

Here , h =0.25 , $y_0 = 1.0000$, $y_1 = 0.9896$, $y_2 = 0.9589$, $y_3 = 0.9089$,

$y_4 = 0.08415$,

Volume of solid revolution,

$$V = \pi\int_0^1 y^2 dx = \pi\frac{h}{3}\left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2\right]$$

$$= \frac{0.25}{3} \times \pi[1.7081 + 3.9172 + 3.3044 + 1.839]$$

$$= \frac{0.25\times\pi}{3} \times 10.7687 = 0.89739\pi = 2.81924.$$

5 (c) Write a program in BASIC to multiply two matrices (checking for consistency for multiplication is required) (10)

Soln.
```
#include <stdio.h>
 int main()
{
      int m, n, p, q, c, d, k, sum = 0;
      int first [10][10], second [10][10], multiply [10][10]
      printf("Enter number of rows and columns of the  matrix\n");
      scanf ("%d%d", &m, &n);
      printf ("Enter elements of first matrix\n");
      for (c = 0; c < m; C++)
            for (d = 0; d < n; d++)
                  scanf ("%d", &first[c][d]);
      printf ("Enter number of rows and columns second matrix/n");
      scanf("%d%d", &p, &q);



      if(n!=p)
            printf("The multiplicatation isn't possible. /n) else
             {
           printf("Enter elements of second matrix\n");
                 for (c = 0; c < p; C++)
                      for (d = 0; d < q; d++)
```

```
                    scanf("%d", &second[c][d]];
            for (c = 0; c < m; c++) {
                for (d = 0; d < q; d++) {
                    for (k = 0; k <p; k++) {
                     sum = sum + first[c][k]*second[k][d]
                     }
                      multiply[c][d] = sum;
                      sum = 0;
                 }
      }
      printf("Product of the matrices:In");


      for (c = 0; c < m; C++) {
         for (d = 0; d < q; d++)
            printf("%d\t", multiply[c][d]);
         printf("\n");
      }
   }
    return 0;
   }
```

5(d) Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if p be the density and v be the velocity at a distance x from a fixed point at time i, then $\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2 + k)\}$ **(10)**

Soln. Let p be the pressure and v be the velocity at a distance x from the end of the tube at any time t. The equation of motion and the equation of continuity is given by

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \ldots.(1)$$

and $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad \ldots.(2)$$

Since the air obeys Boyle's law, then

$$p = k\rho \Rightarrow dp = kd\rho \qquad \ldots.(3)$$

From (1) and (3), we have

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad \ldots.(4)$$

Differentiating (2) partially w.r.t. t, we have

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}(\rho v)\right\} = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t}\right\} = 0$$

From (2) and (4), we have

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\partial\left(-v\frac{\partial v}{\partial t} - \frac{k}{\rho}\frac{\partial \rho}{\partial x}\right) - v\frac{\partial}{\partial x}(\rho v)\right\} = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v.v) + k\frac{\partial \rho}{\partial x}\right\} = \frac{\partial^2}{\partial x^2}\{\rho(v^2 + k)\}$$ Proved.

Q 6. (a) Find the complete integral of the partial differential equation $(p^2+q^2)x = zp$ **and deduce the solution which passes through the curve** $x=0, z^2=4y$**.**

(12)

Sol: The given PDE can be written as :

$f(x,y,z,p,q)=0$ where $f(x,y,z,p,q)=(p^2+q^2)x-pz$ ....(1)

We find that, $f_x = p^2+q^2$

$f_y=0$, $f_z=-p$

$f_p=2px-z, f_q=2qx$

The charpit auxiliary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

$$\frac{dx}{2px-z}=\frac{dy}{2qx}=\frac{dz}{2(p^2+q^2)x-pz}=\frac{dp}{-q^2}=\frac{dq}{pq} \quad ....(2)$$

Taking the last two fractions of (2), we have

$$\frac{dp}{-q^2}=\frac{dq}{pq} \text{ or } \frac{dp}{-q}=\frac{dq}{p}$$

$$i.e., pdp+qdq=0 \Rightarrow 2pdp+2qdq=0$$

Integrating, we get $p^2+q^2=a^2$, where a is arbitrary constant ....(3)

Solving the equations (1) and (3), we get

$$p=\frac{a^2x}{z}, q=\frac{a\sqrt{z^2-a^2x^2}}{z} \quad ....(4)$$

Substituting the values of p and q in the integrable equation

$$dz = pdx+qdy,$$

we get, $dz = \frac{a^2 x}{z} dx + \frac{a\sqrt{z^2 - a^2x^2}}{z} dy$

which can be rearranged as $\frac{zdz - a^2 xdx}{\sqrt{z^2 - a^2x^2}} = ady$ ....(5)

substituting $u^2 = z^2 - a^2x^2$, we get

$$udu = zdz - a^2 xdx$$

Thus equation (5) can be simplified as du=ady

Integrating, we get complete integral of given PDE as

$$u = ay + b$$

$\sqrt{z^2 - a^2x^2} = ay + b$, where b is arbitrary constant.

Q 6(b) Apply fourth-order Runge-Kutta method to compute y at x = 0.1 and x =0.2, given that $\frac{dy}{dx} = x + y^2$, y=1 at x=0. (12)

Sol. Let $h = 0.1$

Here, $x_0 = 0, y_0 = 1$

$$f(x, y) = x + y^2$$

Now, $k_1 = hf(x_0, y_0) = 0.1(0 + 1^2) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0 + 0.05, 1 + 0.05)$$

$$= 0.1[0.05 + (1.05)^2] = 0.11525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[0.05 + (1.05763)^2] = 0.11686$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0 + 0.1, 1 + 0.11686)$$

$$= 0.1[0.1 + (1.11686)^2] = 0.13474$$

According to Range-kutta (fourth order) formula

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{0.1} = 1 + \frac{1}{6}[0.1 + 2(0.11525) + 2(0.11686) + 0.13474] = 1.11649$$

For the second step $x_1 = 0.1, y_1 = 1.649$

$$k_1 = hf(x_1, y_1) = 0.1f[0 + 0.1, 1.11649]$$

$$= 0.1(0.1 + 1.24655) = 0.13466$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f\left(0.1 + \frac{0.1}{2}, 1.11649 + \frac{0.13466}{2}\right)$$

$$= 0.1(0.15 + 1.40143) = 0.15514$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f\left(0.1 + \frac{0.1}{2}, 1.11649 + \frac{0.15514}{2}\right)$$

$$= 0.1f(0.15, 1.19406) = 0.1[0.15 + (1.19406)^2]$$

$$= 0.15758$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.1 + 0.1, 1.11649 + 0.15758) = 0.1f(0.2, 1.27407)$$

$$= 0.1[0.2 + (1.27404)^2] = 0.18233$$

$$y_{0.2} = y_{0.1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.11649 + \frac{1}{6}(0.13466 + 2(0.15514) + 2(0.15758) + 0.18233)$$

$$= 1.27356$$

Q 6 (c) For a particle having charge q and moving in an electromagnetic field, the potential energy is $U = q(\phi - \vec{v}.\vec{A})$, where ϕ and $\vec{A}$ are respectively known as scalar and vector potentials.Derive the expression for Hamiltonian for the particle in the electromagnetic field. (8)

Sol. In an accelerator beam line, we are interested in charged particles moving through electromagnetic fields. If we are to apply Hamiltonian mechanics to the problem, we need an expression for the Hamiltonian for a charged particle in an electromagnetic field.

Since we already know that the equations of motion for such a particle are given by (1) and (2), all we need to do is find a Hamiltonian that, when substituted into Hamiltonian equations (3) and (4), gives the correct equation of motion, i.e., equations of motion consistent with the Lorentz force (2) in Newtonian mechanics. In the non-relativistic case, an appropriate Hamiltonian is:

$$H = \frac{(p - qA)^2}{2m} + q\phi \qquad \text{....(11)}$$

where ϕ is the scalar potential and A is the vector potential, which are related to the electric and magnetic fields by the usual equations:

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \qquad \text{....(12)}$$

$$B = \nabla \times A \qquad \text{....(13)}$$

In general, ϕ and A are functions of the coordinates x,y and z , and the time t. Note that with the Hamiltonian (11), the relationship between the particle velocity and momentum, given by (3) is :

$$p = mv + qA \qquad \text{....(14)}$$

The momentum p defined in this way is known as the canonical momentum, to emphasise that it is distinct from the mechanical momentum mv.

Particles in high energy acclerators tend to be moving with relativistic velocities. We therefore need to generalise the Hamiltonian (11) to the relativistic case. In the special relativity, the total energy E and momentum p for a particle in free space (i.e., with zero magnetic and electric fields) are related by :

$$E^2 = p^2c^2 + m^2c^4 \qquad \text{....(15)}$$

where c is speed of light in free space.

Equation (15) follows from the expressions for the energy and momentum in terms of the mass and velocity of the particle :

$$E = \gamma mc^2 \qquad \text{....(16)}$$

$$p = \beta\gamma mc \qquad \text{....(17)}$$

$$\text{where } \beta = v/2, \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{....(18)}$$

In an electromagnetic field, we simply include the contributions of the scalar and vector potentials in the same way as we would for a non-relativistic particle, so that the total energy and canonical momentum become :

$$E = \gamma mc^2 + q\phi \qquad \text{....(19)}$$

$$P = \beta\gamma mc + qA \qquad \text{....(20)}$$

It then follows from the relationship (18) between $\gamma and \beta$ that the relationship between total energy and canonical momentum can be written :

$$(E - q\phi)^2 = (p - qA)^2c^2 + m^2c^4 \qquad \text{....(21)}$$

We saw above that, for particle performing simple harmonic motion, the Hamiltonian could be expressed as the total energy of the particle. If we assume that the same is true for a relativistic particle moving in an electromagnetic field, then we propose the following form of Hamiltonian :

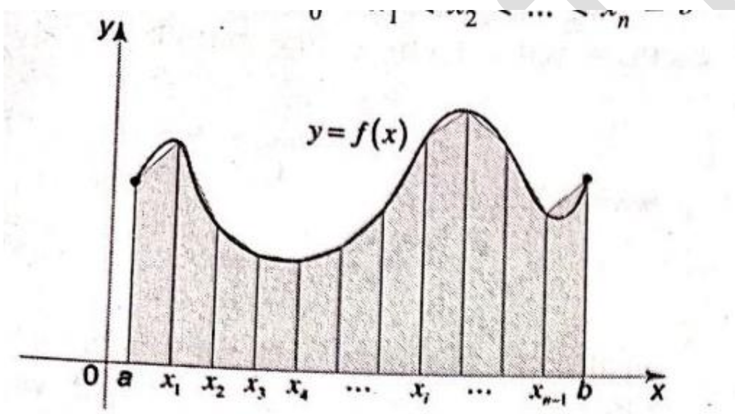
$$H = c\sqrt{(p - qA)^2 + (mc)^2} + q\phi \qquad \text{....(22)}$$

Whether or not this is correct Hamiltonian (that is, whether it gives the correct equations of motion when substituted into Hamiltonian's equations) must ultimately be tested by experiment. It turns out (22) is indeed correct Hamiltonian for a relativistic charged particle moving in an electromagnetic field.

Q6 (d) Write a program in BASIC to implement trapezoidal rule to compute $\int_0^{10} e^{-x^2} dx$ with 10 subdivisions. (8)

Sol. Trapezoidal rule: Under this rule, the area under a curve is evaluated by dividing the total area into little trapezoids rather than rectangles. let f(x) be continous on [a,b]. we partition the interval [a,b] into n equal subintervals, each of width

$$\Delta x = \frac{b-a}{n} \text{ such that } a = x_0 < x_1 \ldots\ldots < x_n = b$$



The trapezoidal Rule for approximating

$$\int_a^b f(x)dx = T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots.. + f(x_n)]$$

where $\frac{\Delta x}{1} = \frac{b-a}{n} and x_i = a + i\Delta x$

As $n \to \infty$, the right hand side of the expression approaches the definite integral $\int_a^b f(x)dx$

7(a) Solve $(z^2 - 2yz - y^2)\, p + (xy + zx)\, q = xy - zx$
If the solution of the above equation represents a sphere , what will be the coordinates of its centre?

(8)

Sol: Here Lagrange's auxiliary equations for given equation are
$dx/(z^2 - 2yz - y^2) = dy/x(y+z) = dz/x(y-z)$ ………… (1)

Taking the last two fraction of (1) we have
$(y-z)\, dy = (y+z)\, dz$

Or $2y\, dy - 2z\, dz - 2(z dy + y dz) = 0$
Integrating, $y^2 - z^2 - 2yz = c_1$, ……… (2)
c_1 being an arbitrary constant

Choosing x y, z as multiplier, each fraction of (1)

$$= \frac{\mathrm{xdx} + \mathrm{ydy} + \mathrm{zdz}}{\mathrm{x}(z^2 - 2\mathrm{yz} - y^2) + \mathrm{xy}(\mathrm{y}+\mathrm{z}) + \mathrm{xz}(\mathrm{y}-\mathrm{z})}$$

$$= \frac{\mathrm{xdx} + \mathrm{ydy} + \mathrm{zdz}}{0}$$

$$xdx + ydy + zdz = 0$$

$$2x\, dx + 2y\, dy + 2z\, dz = 0$$

So that $x^2 + y^2 + z^2 = c_2$ ……… (3)

From (2) and (3) ,

Solution is $f(y^2 - z^2 - 2yz,\ x^2 + y^2 + z^2) = 0$
f, being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be (0,0,0), i.e. Origin .

Q7(b) The velocity v (km/min) of a moped is given at fixed interval of time (min) as below :

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7
v	1.00	1.104987	1.219779	1.34385	1.476122	1.615146	1.758819

t	0.8	0.9	1	1.1
v	1.904497	2.049009	2.18874	2.31977

Estimate the distance covered during the time (Use Simpson's one-third rule).
(10)

Sol: If 's' be the distance, then $s = \int_0^{1.1} v dt$

By Simpson's 1/3 rd rule,

$$s = \int_0^{1.1} v dt$$

=h3 ($V_0 + V_{10}$ +4($V_1+V_3+V_5+V_7+V_9$) +2($V_2+V_4+V_6+V_8$)

=0.13 3.319770+48.15722+26.503729
=0.13 3.319770+32.62888+13.007958
=1.63187

Q 7(c) Assuming a 16-bit computer representation of signed integers, represent - 44 in 2's complement representation.
(10)

Sol: 16 bit representation of 44 is,
44 = 00000000 00101100
1's complement of 44 is ,
11111111 11010011

So, the 2's complement representation is
(-44) = 11111111 11010011 + 1
= 11111111 11010100

Q7(d) In the case of two-dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction, where u is a variable. Show that the maximum value of the velocity at any time of the field is 2u . Prove that the force necessary to hold the disc is 2mu, where m is the mass of the fluid displaced by the disc. (12)

Sol: The velocity potential of a liquid streaming past a fixed circular disc is given by

= u(r+ a2r)cos .... (1)

Differentiating (1) w.r. to r and , respectively, we have

r= u(1- a2r2)cos

= -u(r+ a2r)sin

Let q be the velocity at point P (r,), then

q2=- r2+-1r r2

= u21-2 a2r2cos2 + a4r4 ----------- (2)

The velocity q will be maximum with regard to , if

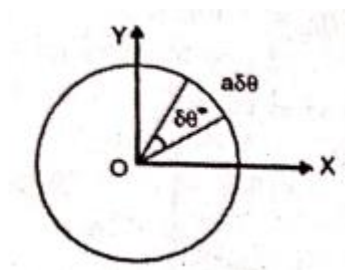
cos2 = -1 2 =

Then, q2=u21+2 a2r2cos2 + a4r4

q=u(1+ a2r2) ---------------------- (3)

Max. value of (q)r=a=u(1+ a2r2) = 2u

Hence, the maximum value of the velocity at any point of the fluid is 2u . Proved



By Bernoulli’s equation, we have

p= f(t) - 1 2q2+ t

or

p= f(t) - 1 2u21-2 a2r2cos2 + a4r4+u(r+ a2r)cos

At the boundary of the disc r=a, the pressure becomes

p= f(t) -2u2sin2+2au cos

Let a be an element on the surface of the disc.

The resultant pressure on the disc is

P = 02(-p cos) a d

P= -a 02f(t) -2u2sin2+2au cos cos d

P= -2a2u 02cos2 d

P= -2a2u =-2mu, where m=a2

Hence, the force necessary to hold the disc at rest is 2mu.

8(a) Find a real function V of x and y ,satisfying

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2)$$

and reducing to zero, when $y = 0$. **(10)**

Sol: A Real function V satisfying the given equation can be obtained by finding the particular integral of the given equation.

$$P.I. = \frac{1}{D^2 + D'^2}(-4\pi(x^2 + y^2))$$

$$= \frac{1}{D^2}\left[1 + \frac{D'^2}{D^2}\right]^{-1} \{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2}\left[1 - \frac{D'^2}{D^2} \dots \dots\right]\{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2}\{-4\pi(x^2 + y^2)\} - \frac{1}{D^4}D'^2\{-4\pi(x^2 + y^2)\}$$

$$= -4\pi\left\{\frac{x^4}{12} + \frac{x^2y^2}{2}\right\} + 4\pi\left\{2.\frac{x^4}{24}\right\} = -2\pi x^2 y^2$$

Which reduces to zero when $y = 0$.

$$\therefore \quad V = -2\pi x^2 y^2$$

Note that here the roots of the AE are imaginary and hence CF does not give a real value of $V(x, y)$.

8(b) The Equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between 1.4 and 1.5 . Find the root to four places of decimal by regula-falsi method.

(10)

Sol: Let $f(x) = x^6 - x^4 - x^3 - 1$

$$f(1.4) = -0.056, \qquad f(1.41) = 0.102$$

Hence, the root lies between 1.4 and 1.41.
Using method of false position,

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_2)}\right\} f(x_2) \qquad \{replacing\ x_0\ by\ x_2\}$$

$$= 1.4035 - \left(\frac{1.41-1.4035}{0.102+0.0016}\right)(-0.056) \qquad .$$

$$\{Let,\ \ x_0 = 1.4\ \ and \qquad x_0 = 1.41\}$$

$$= 1.4 + \left(\frac{0.01}{0.158}\right)(0.056) = 1.4035$$

Now, $f(x_2) = -0.0016 (-)ve$
Hence, the root lies between 1.4035 and 1.41. Using the method of false position,

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2) \qquad \{replacing\ x_0\ by\ x_2\}$$

$$= 1.4035 - \left(\frac{1.41 - 1.4035}{0.102 + 0.0016}\right)(-0.0016) = 1.4036$$

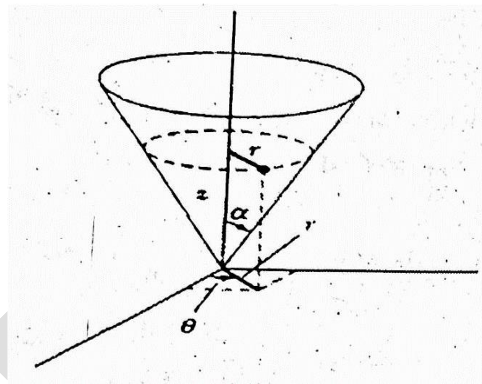
Now, $f(x_3) = -0.00003 (-)ve$
Hence, the root lies between 1.4036 and 1.41.
Using the method of false position.

$$x_4 = x_3 - \left\{\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right\} f(x_3)$$

$$= 1.4036 - \left(\frac{1.41 - 1.4036}{0.102 + 0.00003}\right)(0.00003) = 1.4036$$

Since, x_3 and x_4 are approximately the same upto four places of decimal, hence the required root of the given equation is 1.4036.

8(c) A particle of mass '*m*' is constrained to move on the inner surface of a cone of semi angle under the action of gravity. Write the equation of constraint and mention the generalized coordinates. Write down the Equation of motion. (10)



Sol: Here, we use the cylindrical coordinates $r, \theta, \&\ z$. The equation of constraint is $z = r\,cot\alpha$.
So we have 2-degrees of freedom and the generalized coordinates area $r\ \&\ \theta$. Now

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}(\dot{r}^2 + r^{2\dot{\theta}2} + \dot{r}^2 cot^2\alpha)$$

$1/2(\dot{r}^2 + csc^2\alpha^2 + r^2\dot{\square}^2)$

$U = mgz = mgrcot\alpha$

$$=> L = \frac{1}{2}m\left(\dot{r}^2 + csc^2\alpha^2 + r^2\dot{\theta}^2\right) - mgrcot\alpha$$

For the r-coordinate we have

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$=> mr\dot{\theta}^2 - mgcot\alpha - \frac{d}{dt}(m\dot{r}csc^2\alpha) = 0$$

$$=> \quad r\theta^2 - g\dot{c}ot\alpha - \ddot{r}csc^2\alpha = 0$$

For the $\theta - coordinate$ we have

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta}}\right) = 0$$

$$=> \quad 0 - \frac{d}{dt}\left(mr^{2\dot{\theta}}\right) = 0$$

$$=> \quad mr^2\dot{\theta} = constatnt$$

But $\quad L = I\omega = mr^2\dot{\theta} = constant$

8(d) Two sources , each of strength 'm' , are placed at the points $(-a,0),(a,0)$ and a sink of strength 2m at the origin. Show that the streamlines are the curve, $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$, where λ is a variable parameter. Show also that the fluid speed at any point is $\frac{(2ma^2)}{(r_1.r_2.r_3)}$, where r_1 ,r_2 ,r_3 the distances of the point from the sources and the sink are. **(10)**

Sol: ***First Part :***

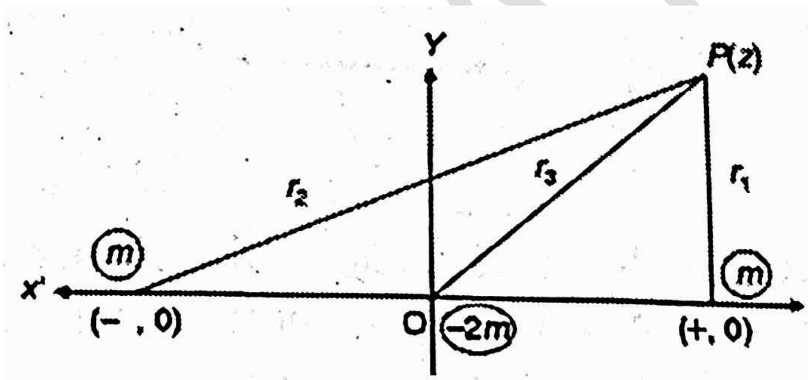
The complex potential w at any point P(z) is given by

$$w = -mlog(z-a) - mlog(z+a) + 2mlogz \qquad \text{... (1)}$$

or $$w = m[logz^2 - log(z^2-a^2)]$$

or $$\phi + i\psi = m[log(x^2-y^2+2ixy) - log(x^2-y^2-a^2+2ixy)]$$

as $z = x+iy$



Equating the imaginary parts, we have $\psi = m[\{\frac{2xy}{x^2-y^2}\} - \left\{\frac{2xy}{x^2-y^2-a^2}\right\}]$

$$\therefore \quad \psi = m\left[-\frac{2a^2xy}{[(x^2+y^2)^2 - a^2(x^2-y^2)]}\right]$$

On simplification.

The desired streamlines are given by $\psi = constant = m\left(-\frac{2}{\lambda}\right)$

Then we obtain $\left(-\frac{2}{\lambda}\right) = \frac{-2a^2xy}{[(x^2+y^2)^2-a^2(x^2-y^2)]}$

Or $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$

Second Part :

From (1), we have

$$\frac{dw}{dz} = -\frac{m}{z+a} - \frac{m}{z-a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)(z+a)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

Where, $r_1 = |z-a|$ $\quad r_2 = |z+a|$ $\quad$ *and* $\quad r_3 = |z|$.