Date : 3/9/19

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

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MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

210/250

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-13: IAS(M)/18-AUG.-2019

Time: 3 Hours | Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 32 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: KATTA RAVI TEJA

Roll No.: 0830234

Test Centre: ORN

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			08	
	(c)			08	40
	(d)			08	
	(e)			08	
2	(a)				
	(b)				
	(c)				
	(d)				
3	(a)			08	
	(b)			08	44
	(c)			13	
	(d)			15	
4	(a)				
	(b)				
	(c)				
	(d)				
5	(a)			09	
	(b)			09	
	(c)			08	42
	(d)			08	
	(e)			08	
6	(a)			09	
	(b)			11	
	(c)			12	40
	(d)			08	
7	(a)				
	(b)				
	(c)				
	(d)				
8	(a)			11	
	(b)			12	44
	(c)			06	
	(d)			15	
Total Marks					

210/250

P.T.O.

SECTION – A

1. (a) Let A a non-singular, n × n square matrix. Show that A. (adj A) = $|A|I_n$. Hence show that

$|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$ **[10]**

Adj A = is an transpose of a matrix that contains the cofactors of the elements of matrix A in the corresponding location

i.e $\boxed{A_{ij} = (-1)^{i+j} \text{ minor of } a_{ij}}$

We know that

$$\sum a_{ij} A_{ij} = |A| \quad \text{if } i=j$$
$$= 0 \quad \text{if } i \neq j$$

$$\Rightarrow A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & & A_{2n} \\ \vdots & & \vdots \\ A_{n1} & & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum a_{1j}A_{j1} & \sum a_{1j}A_{j2} & \cdots & \sum a_{1j}A_{jn} \\ & \cdots & \sum a_{rj}A_{jr} & \\ \sum a_{nj}A_{j1} & \cdots & & \sum a_{ij}A_{jn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & & & & |A| \end{bmatrix}_{n\times n} = |A| I_{n\times n}$$

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$\therefore \boxed{A \cdot \text{adj } A = |A| I_{n\times n}}$ — (1)

From (1) $|A| \cdot |\text{adj } A| = |A|^n \Rightarrow \boxed{|\text{adj } A| = |A|^{n-1}}$ — (2)

replace A with adj A in (2)

$\Rightarrow |\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1} = (|A|^{n-1})^{n-1} = |A|^{(n-1)^2}$

$\Rightarrow \boxed{|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}}$

1. (b) Let S be space generated by the vectors {(0,2,6), (3,1,6), (4, −2, −2)}. What is the dimension of the space S? find a basis for S. **[10]**

The space generated by the given vectors can be represented by a matrix containing these vectors as rows

$$\Rightarrow \begin{pmatrix} 0 & 2 & 6 \\ 3 & 1 & 6 \\ 4 & -2 & -2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 4 & -2 & -2 \\ 3 & 1 & 6 \\ 0 & 2 & 6 \end{pmatrix}$$

$$\cancel{R_2 \rightarrow R_2 - \tfrac{3}{4}R_1 \begin{pmatrix} 4 & -2 & -2 \\ 0 & 5/2 & 15/2 \\ 0 & 2 & 6 \end{pmatrix}}$$

$$\xrightarrow[R_3 \rightarrow R_3/2]{R_1 \rightarrow R_1/2} \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 6 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 5/2 & 15/2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow 2R_2} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - \frac{1}{5}R_2} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\xrightarrow{R_2 \rightarrow \frac{1}{5}R_2} \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1/2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \text{ R.R.E form}$$

$$\Rightarrow \boxed{\text{dimension} = 2 \text{ and basis} = \{(1,0,1),(0,1,3)\}}$$

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1. (c) If $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$, show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin. **[10]**

Given $f(x,y) = \begin{cases}\dfrac{x^3+y^3}{x-y} & x\neq y \\ 0 & x=y\end{cases}$

(i) let us approach the origin through $y = x - mx^3$

here $\lim_{x\to 0}(y) = \lim_{x\to 0}(x-mx^3) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}(f(x, x-mx^3)) = \lim_{x\to 0}\dfrac{x^3+(x-mx^3)^3}{mx^3}$

$= \lim_{x\to 0}\dfrac{1+(1-mx^2)^3}{m}$

$= 2/m$ which depends on value of m

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$\Rightarrow$ $\boxed{f(x,y) \text{ is discontinuous at origin as limit does not exist.}}$

(ii) $f(x,y)$ is a continuous function and derivable at every other point except $(x=y)$

$f_x(x,y) = \lim_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h} = \dfrac{\dfrac{(x+h)^3+y^3}{x+h-y} - \dfrac{(x^3+y^3)}{x-y}}{h}$

$f_x(0,0) = \lim_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{\frac{h^3}{h}-0}{h} = 0$

$f_y(0,0) = \lim_{k\to 0}\dfrac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\dfrac{\frac{k^3}{-k}-0}{k} = 0$

$\boxed{\therefore f_x, f_y \text{ exist at origin}}$

& In general everywhere

?? .

1. (d) Find $\partial w/\partial x$ at the point $(x, y, z) = (1, 1, 1)$ if $w = \cos uv$, $u = xyz$, $v = \pi/(4(x^2 + y^2))$. **[10]**

$$w = \cos uv = f(u,v)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} \quad -ⓐ$$

$$u = xyz \Rightarrow \frac{\partial u}{\partial x} = yz\Big|_{(1,1,1)} = 1 \quad -①$$

$$v = \frac{\pi}{4(x^2+y^2)} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\pi(2x)}{4(x^2+y^2)^2}$$

$$= \frac{-\pi x}{2(x^2+y^2)^2}\Big|_{(1,1,1)} = -\pi/8 \quad -②$$

$$\frac{\partial w}{\partial u} = -[\sin uv][v]$$

$$\frac{\partial w}{\partial v} = -[\sin uv][u]$$

$$v\big|_{(1,1,1)} = \pi/8 \; ; \; u(1,1,1) = 1$$

$$\Rightarrow \frac{\partial w}{\partial u}\Big|_{(1,1,1)} = -\sin(\pi/8)(\pi/8) = -\pi/8 \sin\pi/8 \quad -③$$

$$\frac{\partial w}{\partial v}\Big|_{(1,1,1)} = -(\sin\pi/8)(1) = -\sin\pi/8 \quad -④$$

Putting ①, ②, ③, ④ in ⓐ

$$\Rightarrow \frac{\partial w}{\partial x} = \left(-\pi/8 \sin\pi/8\right)(1) + \left(-\sin\pi/8\right)(-\pi/8) = 0$$

$$\therefore \boxed{\frac{\partial w}{\partial x} = 0}$$

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1. (e) Show that the plane x + y – 2z = 3 cuts the sphere $x^2 + y^2 + z^2 - x + y = 2$ in a circle of radius 1 and find the equation of the sphere which has this circle as a great circle. **[10]**

Given sphere: $x^2+y^2+z^2-x+y=2$.

(i) $(x-1/2)^2+(y+1/2)^2+z^2 = 2+1/4+1/4 = 5/2$

Centre $(1/2, -1/2, 0)$ radius $=\sqrt{5/2}=r$

The perpendicular distance of the plane $x+y-2z=3$ from $(1/2, -1/2, 0)$ is given by

$$d = \left|\frac{(1/2)+(-1/2)-2(0)-3}{\sqrt{1^2+1^2+2^2}}\right| = \left|\frac{3}{\sqrt{6}}\right| = \left|\sqrt{3/2}\right|$$

The radius of the intersecting circle AB is given by $AB^2+d^2=r^2$

$\Rightarrow AB^2+(3/2)=5/2 \Rightarrow AB^2=1 \therefore \boxed{AB=1}$

(ii) Any sphere which passes through this circle $S+\lambda P=0$ is given by

$(x^2+y^2+z^2-x+y-2)+\lambda(x+y-2z-3)=0$

$\Rightarrow (x^2+y^2+z^2+(\lambda-1)x+(\lambda+1)y+(-2\lambda)z+(-2-3\lambda)=0$

For it to be great circle, centre must be on the plane

$\Rightarrow \left(\frac{1-\lambda}{2}\right)+\left(-\frac{(\lambda+1)}{2}\right)-2(\lambda)=3 \Rightarrow -\lambda-2\lambda=3 \Rightarrow \lambda=-1$

$\therefore$ The required sphere is

$$\boxed{x^2+y^2+z^2-2x+2z+1=0}$$

3. (a) The vectors $V_1 = (1,1,2,4)$, $V_2 = (2,-1,-5,2)$, $V_3 = (1,-1,-4,0)$ and $V_4 = (2,1,1,6)$ are linearly independent. Is it true? Justify your answer. **[10]**

Let us create the matrix which contains these vectors as rows.

$$M = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - \frac{1}{3}R_2]{R_3 \to R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

here $\mathcal{S}(M) = 2 < 4$

so the given vector are <u>linearly dependent</u>.

They are part of 2-Dimensional vector space whose basis is $\{(1,1,2,4), (0,1,3,2)\}$.

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3. (b) Find the inverse of the matrix

$$A=\begin{bmatrix}1 & 3 & 1\\ 2 & -1 & 7\\ 3 & 2 & -1\end{bmatrix}$$

by using elementary row operations. Hence solve the system of linear equations

$x+3y+z=10$

$2x-y+7z=21$

$3x+2y-z=4$

[10]

According <u>Gauss Jordan Method</u>

$$[A|I] \xrightarrow[\text{operations}]{\text{row}} [I|A^{-1}]$$

$$\Rightarrow [A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0\\ 2 & -1 & 7 & 0 & 1 & 0\\ 3 & 2 & -1 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & -7 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + \frac{3R_2}{7}} \left[\begin{array}{ccc|ccc} 1 & 0 & 22/7 & 1/7 & 3/7 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & 0 & -9 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3(-1/9)]{R_2 \to R_2(-1/7)} \left[\begin{array}{ccc|ccc} 1 & 0 & 22/7 & 1/7 & 3/7 & 0 \\ 0 & 1 & -5/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 1/9 & 1/9 & -1/9 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 5/7 R_3]{R_1 \to R_1 - \frac{22}{7} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/63 & 5/63 & 22/63 \\ 0 & 1 & 0 & 23/63 & -4/63 & -5/63 \\ 0 & 0 & 1 & 1/9 & 1/9 & -1/9 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -13/63 & 5/63 & 22/63 \\ 23/63 & -4/63 & -5/63 \\ 1/9 & 1/9 & -1/9 \end{bmatrix}$$

Given $AX = B$

$\Rightarrow X = A^{-1}B$

$$\therefore X = \begin{bmatrix} -13/63 & 5/63 & 22/63 \\ 23/63 & -4/63 & -5/63 \\ 1/9 & 1/9 & 1/9 \end{bmatrix} \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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$\therefore$ solution of linear equations is $\boxed{\begin{array}{l} x=1 \\ y=2 \\ z=3 \end{array}}$

P.T.O.

3. (c) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, and $z = x + y$. **[14]**

The given problem can be framed as

$$F(x,y,z) = x^2+y^2+z^2+\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right] + \mu(x+y-z)$$

For minima or maxima,

$\frac{\partial F}{\partial x} = 0$ and; $\frac{\partial F}{\partial y} = 0$; $\frac{\partial F}{\partial z} = 0$

$\Rightarrow \frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda\left(\frac{x}{2}\right) + \mu = 0 \Rightarrow x = \frac{-2\mu}{\lambda+4}$ ①

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda\left(\frac{2y}{5}\right) + \mu = 0 \Rightarrow y = \frac{-5\mu}{2(\lambda+5)}$ ②

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda\left(\frac{2z}{25}\right) - \mu = 0 \Rightarrow z = \frac{25\mu}{2(\lambda+25)}$ ③

Substituting ①, ②, ③ in $z = x+y$.

$\Rightarrow \frac{-2}{\lambda+4} - \frac{5}{2(\lambda+5)} = \frac{25}{2(\lambda+25)}$

[$\mu \neq 0$ because $\mu = 0 \Rightarrow x = 0 = y = z$ which doesn't satisfy given condition]

$\Rightarrow \frac{4}{\lambda+4} + \frac{5}{\lambda+5} + \frac{25}{\lambda+25} = 0$

$\Rightarrow 4(\lambda^2+30\lambda+125) + 5(\lambda^2+29\lambda+100) + 25(\lambda^2+9\lambda+20) = 0$

$\Rightarrow \lambda^2(4+5+25) + \lambda(120+145+225) + (500+500+500) = 0$

$\Rightarrow 34\lambda^2 + 490\lambda + 1500 = 0.$

$\Rightarrow \boxed{\lambda = -75/17 \,;\, -10}$

Substituting ①, ②, ③ in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$

$\Rightarrow \frac{\mu^2}{4}\left[\frac{1}{4}\left(\frac{4}{\lambda+4}\right)^2 + \frac{1}{5}\left(\frac{5}{\lambda+5}\right)^2 + \frac{1}{25}\left(\frac{25}{\lambda+25}\right)^2\right] = 1.$

$\Rightarrow \frac{\mu^2}{4}\left[\frac{4}{(\lambda+4)^2} + \frac{5}{(\lambda+5)^2} + \frac{25}{(\lambda+25)^2}\right] = 1.$ maxima

$\Rightarrow \mu = \pm\sqrt{\frac{180}{19}} \,;\, \pm 0.324 \quad \Rightarrow (x,y,z) = (-1.086, -1.53, 2.565), (1.085, 1.53, -2.565), (-1.57, 1.37, 0.196), (1.57, -1.37, -0.196)$ minima

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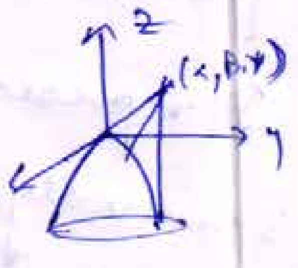
$\boxed{F_{max} = 4.75/17} \quad \boxed{f_{min} = 10}$

3. (d) Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2+y^2+2z=0$ is $x^2+y^2+4z=1$. **[16]**

Let (α, β, γ) be the point from which 3 mutually perpendicular tangent lines are drawn.



Let the tangent line be given as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r. \quad ①$$

any point on this line is given by.

$P: (\alpha + lr, \beta + mr, \gamma + nr)$

Since tangent intersects given paraboloid, this point lies on it.

$\Rightarrow (\alpha + l r)^2 + (\beta + m r)^2 + 2(\gamma + n r) = 0$

~~replacing value of r from (1)~~

$\Rightarrow$ ~~$\alpha + l \frac{x-\alpha}{l}$~~

$\Rightarrow r^2(l^2 + m^2) + 2r[l\alpha + m\beta + n] + [\alpha^2 + \beta^2 + 2\gamma] = 0$

Since the tangent intersects at only one point, the above quadratic equation has discriminant zero

$\Rightarrow [l\alpha + m\beta + n]^2 - [l^2 + m^2][\alpha^2 + \beta^2 + 2\gamma] = 0$

replacing values of l, m, n from (1), we get

$[(x-\alpha)\alpha + (y-\beta)\beta + (z-\gamma)]^2 = [(x-\alpha)^2 + (y-\beta)^2][\alpha^2 + \beta^2 + 2\gamma]$

It represents a cone and for it to have 3 mutually perpendicular ~~tangent lines~~ generators the sum of coefficients of x^2, y^2, z^2 must be zero

$\Rightarrow (\alpha^2 + \beta^2 + 2\gamma - \alpha^2) + (\alpha^2 + \beta^2 + 2\gamma - \beta^2) + (-1) = 0$

$\Rightarrow \alpha^2 + \beta^2 + 4\gamma - 1 = 0$

$\therefore$ the locus of (α, β, γ) is $\boxed{x^2 + y^2 + 4z = 1}$

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SECTION – B

5. (a) Show that the family of parabolas $y^2 = 4cx + 4c^2$ is self-orthogonal. [10]

Given $y^2 = 4cx + 4c^2$

differentiating the equation w.r.t x, we get

$2yy' = 4c. \Rightarrow c = \frac{yy'}{2}.$

substituting it in given equation, we get.

$$y^2 = 4x\left(\frac{yy'}{2}\right) + 4\left(\frac{yy'}{2}\right)^2.$$

$$\boxed{y^2 = 2xyy' + y^2y'^2}$$ is the required —(1)

differential equation.

For orthogonal trajectory, we must replace y' with $\frac{-1}{y'}$ in ①

$\Rightarrow y^2 = 2xy\left(\frac{-1}{y'}\right) + y^2\left(\frac{-1}{y'}\right)^2$

$\Rightarrow (y')^2 y^2 = -2xy\,y' + y^2$

$\Rightarrow \boxed{(y')^2 y^2 + 2xyy' = y^2}$ — ②

It is clear that ① = ②

∴ The given family of parabolas $y^2 = 4cx + 4c^2$ is self orthogonal.

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5. (b) Solve the differential equation:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2).$$

[10]

Given differential equation can be written as

$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2$

Comparing it with standard form, we get

$y'' + Py' + Qy = R.$

$P = -1/x$; $Q = -4x^2$; $R = 8x^2\sin x^2$.

we replace independent variable 'x' with another variable 'z' such that

$Q_1 = \frac{Q}{\left(\frac{\partial z}{\partial x}\right)^2} = -1$ $\Rightarrow \frac{-4x^2}{\left(\frac{\partial z}{\partial x}\right)^2} = -1$ $\Rightarrow \frac{\partial z}{\partial x} = 2x$ $\Rightarrow \boxed{z = x^2}$

$$P_1 = \frac{m\left(\frac{\partial z}{\partial x}\right) - \frac{\partial^2 z}{\partial x^2}}{\left(\frac{\partial z}{\partial x}\right)^2} = 0 \;;\; R_1 = \frac{R}{\left(\frac{\partial z}{\partial x}\right)^2} = \frac{8x^2 \sin x^2}{4x^2} = 2\sin x^2 = 2\sin z$$

$$\Rightarrow \quad y'' - y = 2\sin z.$$

$$\Rightarrow (D^2 - 1)y = 2\sin z.$$

Complementary solution

auxillary equation $m^2 - 1 = 0$

$$\Rightarrow \boxed{y_c = C_1 e^{z} + C_2 e^{-z}}$$

Particular integral

$$y_p = \frac{2}{D^2-1}\sin z = \frac{2}{-1-1}\sin z = -\sin z. \quad \Rightarrow y = y_c + y_p = C_1 e^{z} + C_2 e^{-z} - \sin z$$

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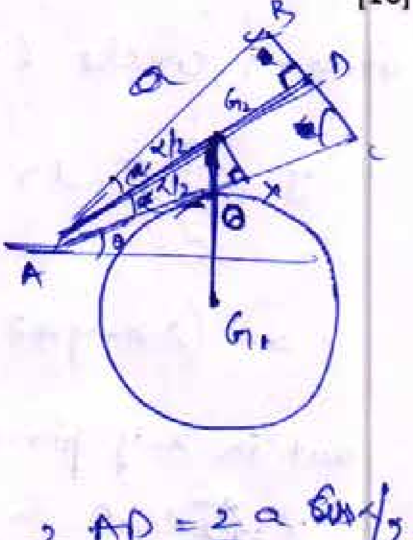
$$\therefore \boxed{y = C_1 e^{x^2} + C_2 e^{-x^2} - \sin x^2}$$

5. (c) A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if sin α < 3r/a, where a is one of the equal sides of the triangle. [10]



$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

here $R \to \infty$; $r = r$

$$\Rightarrow \frac{1}{h} > \frac{1}{r}$$

08 $\Rightarrow r > h = OG_2$

$$AG_2 = \frac{2}{3} AD = \frac{2}{3} a \cos\alpha/2$$

$$OG_2 = G_2 x = AG_2 \sin\alpha = \frac{AG_2}{3}\sin\alpha/2$$

$r > \left(\frac{2}{3}a\cos\alpha/2\right)\sin\alpha/2$

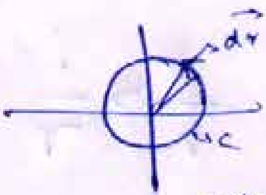
$r > \frac{a}{3}\sin\alpha$

$\boxed{\frac{3r}{a} > \sin\alpha}$ required for equilibrium

5. (d) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field F is given by

$\mathbf{F} = (2x - y + 2z)\mathbf{i} + (x + y - z)\mathbf{j} + (3x - 2y - 5z)\mathbf{k}$. **[10]**

Given: circle C: $\boxed{x^2+y^2=4}$



$I = \oint_C F \cdot dr$

$= (2x-y+2z)dx + (x+y-z)dy + (3x-2y-5z)dz$

but in x-y plane $\boxed{z=0}$ $\boxed{dz=0}$

$x = 2\cos\theta \Rightarrow dx = -2\sin\theta\, d\theta$

$y = 2\sin\theta \Rightarrow dy = 2\cos\theta\, d\theta$

$$I = \int_0^{2\pi} \Big(2(2\cos\theta) - 2\sin\theta\Big)(-2\sin\theta)\,d\theta + (2\cos\theta + 2\sin\theta)(2\cos\theta)\,d\theta$$

$$I = 4\int_0^{2\pi} \left(-2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta\right) d\theta$$

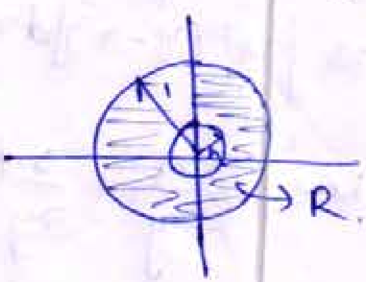
$$I = 4\int_0^{2\pi} \left(1 - \frac{\sin 2\theta}{2}\right) d\theta$$

$$I = 4(2\pi) - 4(0)$$

08 $\therefore \boxed{I = 8\pi}$ units

5. (e) Verify the Green's theorem for $M = \dfrac{-y}{x^2+y^2}, N = \dfrac{x}{x^2+y^2}$ $R\{(x,y)/h^2 \le x^2 + y^2 \le 1\}$, where $0 < h < 1$. **[10]**

Consider



$$I_1 = \oint_C M\,dx + N\,dy.$$

$$= \oint_C \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$$

$$= \oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \oint_C \frac{d(y/x)}{1+(y/x)^2} = \left[\tan^{-1}(y/x)\right]_{(x_1,y_1)}^{(x_1,y_1)}$$

$\Rightarrow \boxed{I_1 = 0}$ —(1) since $M\,dx + N\,dy$ is an exact integral.

$$I_2 = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint \left[\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right] - \left[\frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}\right] dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=h}^{1} \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{(y^2-x^2)}{(x^2+y^2)^2}\right] dx\,dy$$

$$= 0 \Rightarrow \boxed{I_2 = 0} \quad —②$$

Hence Green's theorem is verified.

-08

From ① & ②

$$\boxed{\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy}$$

6. (a) Solve: $\frac{dy}{dx} = \frac{y^2(x-y)}{3xy^2 - x^2y - 4y^3}, y(0) = 1$ **[10]**

It can be represented as

$$\Rightarrow \left[y^2(y-x)\right]dx + (3xy^2 - x^2y - 4y^3)\,dy = 0$$

here $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^3 - y^2x) = (3y^2 - 2yx)$

$$\frac{\partial N}{\partial x} = 3y^2 - 2xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact integral.}$$

$$I = \int M\,dx + \int N\,dy$$

considering y as constant   after removing terms containing x

$$= \int y^2(y-x)\,dx + \int (-4y^3)\,dy.$$

$$\boxed{C = y^3x - \frac{x^2y^2}{2} - y^4}$$

$y(0)=1$

$$\Rightarrow C = (1)^3(0) - \frac{(0)^2(1)^2}{2} - (1)^4$$

$$= 0 - 0 - 1 \Rightarrow \boxed{C=-1}.$$

$$\therefore \boxed{xy^3 - \frac{x^2y^2}{2} - y^4 + 1 = 0}$$ is the required answer.

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6. (b) Show that the differential equation

$(3y^2 - x) + 2y(y^2 - 3x)y' = 0$

admits an integrating factor which is a function of $(x + y^2)$. Hence solve the equation. **[12]**

Given $(3y^2-x)dx + [2y(y^2-3x)]dy = 0.$

here $M = 3y^2 - x \Rightarrow \frac{\partial M}{\partial y} = 6y$

$N = 2y(y^2-3x) \Rightarrow \frac{\partial N}{\partial x} = -6y$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (not exact)

Consider a function $f(x+y^2)$ and multiply to given equation

$$\Rightarrow \left[(3y^2-x)f(x+y^2)\right]dx + \left[2y(y^2-3x)f(x+y^2)\right]dy = 0$$

here $M = (3y^2 - x)\, f(x+y^2)$

$$\Rightarrow \frac{\partial M}{\partial y} = 6y\, f(x+y^2) + 2y(3y^2-x)\, f'(x+y^2) \quad \text{①}$$

$$N = 2y(y^2-3x)\, f(x+y^2)$$

$$\frac{\partial N}{\partial x} = -6y\, f(x+y^2) + 2y(y^2-3x)\, f'(x+y) \quad \text{②}$$

For it to be exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ equating ① & ②, we get

$$12y\, f(x+y^2) = f'(x+y^2)\left[2y^3 - 6xy - 6y^3 + 2xy\right]$$

$$\Rightarrow 12y\, f(x+y^2) = f'(x+y^2)\left[-4y(x+y^2)\right]$$

$$\Rightarrow -3 = \frac{f'(x+y^2)\,(x+y^2)}{f(x+y^2)}$$

$$\Rightarrow \frac{-3\, d(x+y^2)}{(x+y^2)} = \frac{d f(x+y^2)}{f(x+y^2)} \xrightarrow[\text{we get}]{\text{integrating}} -3\ln(x+y^2) + \ln c = \ln f(x+y^2)$$

$$\therefore \boxed{f(x+y^2) = c(x+y^2)^{-3}}$$

$$\therefore \text{I.F} = \frac{1}{(x+y^2)^3}$$

$$\Rightarrow \int \frac{3y^2-x}{(x+y^2)^3}\,dx + \frac{2y(y^2-3x)}{(x+y^2)^3}\,dy = \int\left[\frac{4y^2}{(x+y^2)^3} - \frac{1}{(x+y^2)^2}\right]dx$$

considering 'y' as constant

$$\therefore \boxed{I = \frac{-2y^2}{(x+y^2)^2} + \frac{1}{(x+y^2)} + C}$$

6. (c) Solve $(1+x)^2 y'' + (1+x) y' + y = \sin 2[\log(1+x)]$. [13]

Given $(1+x)^2 y'' + (1+x) y' + y = \sin 2[\log(1+x)]$

Let us take $x+1 = e^z \Rightarrow dx = e^z dz = (x+1) dz$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1}\frac{dy}{dz} \Rightarrow (x+1)\frac{dy}{dx} = \frac{dy}{dz} \quad -①$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{(x+1)^2}\frac{dy}{dz} + \frac{d^2y}{dz^2}\cdot\frac{1}{(x+1)^2}$$

$$\Rightarrow (x+1)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad -②.$$

Substituting ① & ② in given equation.

$$\frac{d^2y}{dz^2} - \cancel{\frac{dy}{dz}} + \cancel{\frac{dy}{dz}} + y = \sin 2z$$

$$\Rightarrow (D^2+1)y = \sin 2z.$$

Complementary solution

auxiliary equation. $m^2+1=0$

$m = \pm i$

$$\Rightarrow \boxed{y_c = C_1 \sin z + C_2 \cos z}$$

Particular integral

$$y_p = \frac{1}{D^2+1}\sin 2z$$

we know that $\boxed{\frac{1}{f(D^2)}\sin az = \frac{1}{f(-a^2)}\sin az}$

$\Rightarrow y_p = \frac{1}{(-4+1)} \sin 2z = \frac{-1}{3} \sin 2z.$

$\therefore y = y_c + y_p = C_1 \sin z + C_2 \cos z - \frac{1}{3} \sin 2z$

$\therefore \boxed{y(x) = C_1 \sin[\log(1+x)] + C_2 \cos[\log(1+x)] - \frac{1}{3} \sin 2[\log(1+x)]}$

12

6. (d) Using Laplace transform, solve the initial value problem $y'' - 3y' + 2y = 4t + e^{3t}$ with $y(0) = 1,\ y'(0) = -1$. **[15]**

Given, $y'' - 3y' + 2y = 4t + e^{3t}$

Applying Laplace transform on above equation, we get

$L(y'') - 3L(y') + 2L(y) = L(4t + e^{3t})$

$[p^2 L(y) - p y(0) - y'(0)] - 3[pL(y) - y(0)] + 2[L(y)]$
$= 4L(t) + L(e^{3t})$

08

$$\Rightarrow L(y)[p^2-3p+2]-p+1+3=\frac{4}{p^2}+\frac{1}{p-3}$$

$$L(y)[(p-1)(p-2)] = p-4+\frac{4}{p^2}+\frac{1}{p-3}$$

$$L(y)=\frac{1}{(p-1)(p-2)}\left[(p-4)+\frac{4}{p^2}+\frac{1}{p-3}\right]$$

$$=\left[\frac{1}{p-2}-\frac{1}{p-1}\right]\left[p-4+\frac{4}{p^2}+\frac{1}{p-3}\right] = \frac{p-4}{p-2}+\frac{4}{p^2(p-2)}+\frac{1}{(p-3)(p-2)}-\frac{p-4}{p-1}-\frac{4}{p^2(p-1)}-\frac{1}{(p-1)(p-3)}$$

$$=1-\frac{2}{p-2}+$$

Please complete

$$y(t)=\frac{1}{2}e^{3t}-2e^{2t}-\frac{1}{2}e^{t}+2t+3.$$

7. (a) One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C is contact with the wall below A, and the angle between the rods is $2\sin^{-1}(3/4)$. **[16]**

8. (a) (i) If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.

(ii) If $F = (x + y + az)\,\mathbf{I} + (bx + 2y - z)\,\mathbf{J} + (x + cy + 2z)\,\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = 0$, then find ϕ such that $\mathbf{F} = \nabla\phi$. **[12]**

(i) Given $\nabla^2\phi = 0$

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = 0$$

$$\Rightarrow \boxed{\nabla\phi \text{ is solenoidal}}$$

We know that $\nabla\phi(r) = \phi'(r)\nabla r = \phi'(r)\dfrac{\vec{r}}{r}$

$$\Rightarrow \nabla\times(\nabla\phi) = \nabla\times\left(\phi'(r)\frac{\vec{r}}{r}\right)$$

$$\nabla\times(\psi\vec{A}) = \left[(\nabla\psi)\times\vec{A}\right] + \left[\nabla\psi(\nabla\times\vec{A})\right]$$

$$\Rightarrow \nabla\times\left(\phi'(r)\frac{\vec{r}}{r}\right) = \left[\left(\nabla\frac{\phi'(r)}{r}\right)\times\vec{r}\right] + \left[\frac{\phi'(r)}{r}[\nabla\times\vec{r}]\right]$$

$$= \left[\left[\frac{r\phi''(r) - \phi'(r)}{r^2}\right]\frac{\vec{r}}{r}\times\vec{r}\right] + \left[\frac{\phi'(r)}{r}[\nabla\times\vec{r}]\right]$$

$$= 0$$

$$\therefore \boxed{\nabla\times(\nabla\phi) = 0} \quad \therefore \boxed{\nabla\phi \text{ is irrotational}}$$

(ii)
$$\text{Curl}\,F = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix} = 0$$

$$= i(c+1) - j(1-a) + k(b-1) = 0$$

$$\therefore \boxed{a=1;\ b=1;\ c=-1}$$

$F = \nabla\phi \Rightarrow \dfrac{\partial\phi}{\partial x} = x+y+az \Rightarrow \phi = \dfrac{x^2}{2} + xy + azx + f(y,z)$

$\dfrac{\partial\phi}{\partial y} = x+2y-z = x + f'(y,z) \Rightarrow f(y,z) = y^2 - zy + h(z) \Rightarrow \dfrac{\partial\phi}{\partial z} = x-y+2z = x-y+h'(z) \Rightarrow h(z) = z^2$

$\therefore \boxed{\phi = \dfrac{x^2}{2} + y^2 + xy + zx - zy + z^2 + C}$ where 'c' is constant.

P.T.O.

$$\therefore \boxed{\phi = \frac{x^2}{2} + y^2 + xy + zx - zy + z^2 + C}$$

8. (b) (i) A person going east wards with a velocity of 4 km per hour, finds the the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.

(ii) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? **[15]**

(i) Let the wind speed be $v_\omega = v_1\hat{i} + v_2\hat{j}$

Case(i) $v_\omega - v_{man} = -v_a\hat{j}$

Case(ii) $v_\omega - v_{man} = -v_b\hat{i} - v_b\hat{j}$

$\Rightarrow v_1\hat{i} + v_2\hat{j} = 4\hat{i} - v_a\hat{j} \Rightarrow v_1 = 4,\ v_2 = -v_a$

$\Rightarrow v_1\hat{i} + v_2\hat{j} = 8\hat{i} - v_b\hat{i} - v_b\hat{j} = 4\hat{i} - v_a\hat{j}$

$\Rightarrow 8 - v_b = 4 \Rightarrow v_b = v_a = 4$

$\Rightarrow v_b = 4$

$\therefore \boxed{\vec{v}_\omega = 4\hat{i} - 4\hat{j}}\ \boxed{\text{kmph.}}$

(ii) $\phi = xy^2 + yz^3$

the directional derivative is given by

$\nabla\phi = y^2\hat{i} + (2xy + z^3)\hat{j} + (3z^2y)\hat{k}$

$\nabla\phi(2,-1,1) = \hat{i} + (-3)\hat{j} + (-3)\hat{k}$ — (1)

The direction of normal of $S_2 : x\log z - y^2 + 4 = 0$

is given by $\nabla S = (\log z)\hat{i} + (-2y)\hat{j} + (x/z)\hat{k}$

$@ (-1,2,1) = -4\hat{j} - \hat{k}$ — (2)

we need to find vector (1) in direction of (2)

$$\Rightarrow \left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|v_2|^2}\right)\vec{v_2}$$

$$\Rightarrow \left[\frac{(1,-3,-3).(0,-4,-1)}{0^2+4^2+1^2}\right](0,-4,-1)$$

$$= \left[\frac{0+12+3}{17}\right](0,-4,-1)$$

$$\boxed{Ans = \frac{15}{17}(-4\hat{j}-\hat{k})}$$

$15/\sqrt{17}$

12

8. (c) A curve in space is defined by the vector equation $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Determine the angle between the tangents to this curve at the points t = +1 and t = −1. **[07]**

Given, $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$

tangent vector $\frac{d\vec{r}}{ds} = \frac{ds}{dt}\ \vec{T} = \frac{d\vec{r}}{dt}$

$$\boxed{\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=1} = 2\hat{i} + 2\hat{j} - 3\hat{k} = \vec{u}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=-1} = -2\hat{i} + 2\hat{j} - 3\hat{k} = \vec{v}.$$

the angle between vectors $\vec{U}$ & $\vec{V}$ is given by

$$\cos\theta = \frac{\vec{U}\cdot\vec{V}}{|\vec{U}||\vec{V}|}$$

$$= \frac{(2,2,-3)\cdot(-2,2,-3)}{\left(\sqrt{2^2+2^2+3^2}\right)^2} = \frac{\cancel{-4}+\cancel{4}+9}{17}$$

$$\cos\theta = 9/17$$

$$\therefore \boxed{\theta = \cos^{-1}(9/17)}$$ is the required angle.

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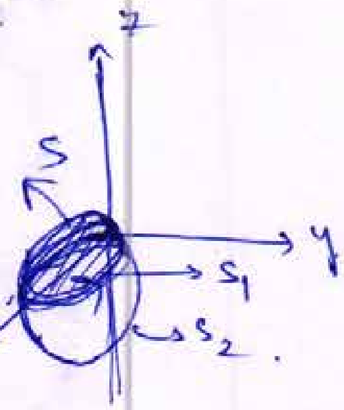
8. (d) If $F = (y^2 + z^2 - x^2)\mathbf{i} + (z^2 + x^2 - y^2)\mathbf{j} + (x^2 + y^2 - z^2)\mathbf{k}$, evaluate $\iint \text{curl } F \cdot \text{n}dS$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ above the plane $z = 0$, and verify Stoke's theorem. **[16]**

Given $(x-a)^2 + y^2 + (z+a/2)^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$



we know that

$$\iint_{S+S_1} (\nabla\times F)\cdot\hat{n}\,ds = \iiint_V \nabla\cdot(\nabla\times F)\,dV = 0$$

$$\Rightarrow \iint_S (\nabla\times F)\cdot\hat{n}\,ds + \iint_{S_1} (\nabla\times F)\cdot(-\hat{k})\,ds = 0$$

$$\therefore \boxed{\iint_S (\nabla\times F)\cdot\hat{n}\,ds = \iint_{S_1} (\nabla\times F)\cdot k\,ds}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$$= i(2y-2z) - j(2x-2z) + k(2x-2y)$$

$$I_1 = \iint (\nabla \times F)\cdot k\, dS = \iint (2x-2y)\cdot dx\,dy$$

$x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow r^2 - 2ar\cos\theta = 0 \Rightarrow r = 2a\cos\theta$

$$I_1 = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{r=2a\cos\theta} 2r(\cos\theta - \sin\theta)(r\,dr\,d\theta)$$

$$= \int (\cos\theta - \sin\theta)\left(\frac{2r^3}{3}\right)_0^{2a\cos\theta} \cdot d\theta$$

$$= \frac{2}{3}\int_{-\pi/2}^{\pi/2} (\cos\theta - \sin\theta)(8a^3\cos^3\theta)\cdot d\theta$$

$$= \frac{16a^3}{3}\int_{-\pi/2}^{\pi/2} (\cos^4\theta - \sin\theta\cos^3\theta)\,d\theta$$

$$= \frac{16a^3}{3}\left(2\int_0^{\pi/2} \cos^4\theta\,d\theta + \left[\frac{\cos^4\theta}{4}\right]_{-\pi/2}^{\pi/2}\right)$$

$$= \frac{16a^3}{3} \times 2 \times \left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \boxed{2\pi a^3 = I_1}$$

$$I_2 = \oint_C F\cdot dr = \oint (y^2 - x^2)\,dx + (x^2 - y^2)\,dy$$

$x = a(1+\cos\theta)$
$y = a\sin\theta$

$$= a^3 \oint (a1 + 2\cos\theta + \cos^2\theta - \sin^2\theta)(\cos\theta + \sin\theta)\,d\theta$$

$$= a^3 \int_0^{2\pi} (2\cos^2\theta - \sin^3\theta)\cdot d\theta = 2a^3 \times 4\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 2\pi a^3.$$

Hence stoke's theorem proved $\boxed{I_2 = 2\pi a^3}$.

i.e $\boxed{\iint_S (\nabla \times F)\cdot n\,ds = \oint_C F\cdot dr = 2\pi a^3}$

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ROUGH SPACE

$-4x^3 y^{-4x^3} = \sin x^2$

$-y' = -2x \cos x^2$

$x y'' = 2x \cos x^2 - 4x^3 \sin x^2$

$\propto 8x^{35}$.

$-4x^3 y = -4x^3 e^{x^2}$

$-y' = -2x e^{x^2}$

$x y''' = 4x^3 e^{x^2} + 2x e^{x^2}$