COMPLEX ANALYSIS

1. ANALYTIC FUNCTIONS, HARMONIC FN

1. 4a 2020

If $v(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0,$

then find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ 15

2. 1d 2019

Suppose $f(z)$ is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation $Im\, f(z) = (Re\, f(z))^2$, $Z \in D$. Show that $f(z)$ is constant in D. 10

3. 2c 2019 IFoS

If f(z) is analytic in a domain D and $|f(z)|$ is a non-zero constant in D, then show that f(z) is constant in D. 15

4. 1c 2018

Prove that the function : $u(x, y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. 10

5. 1c 2018 IFoS

(c) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. 10

6. 3b 2017

Let $f = u + iv$ be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

at all points of D. 15

7. 1c 2017 IFoS

If $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ and $u + 2v = x^3 - 2y^3 + 3xy(2x - y)$ then find $f(z)$ in terms of z. 8

8. 1d 2016

Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function $u(x, y)$ and hence obtain the analytic function whose real and imaginary parts are u and v respectively. *10*

9. 1d 2016 IFoS

Find the analytic function of which the real part is

$e^{-x}\left\{\left(x^2 - y^2\right)\cos y + 2xy\sin y\right\}$. 8

10. 1d 2015

Show that the function $v(x, y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z. 10

11. 1c 2015 IFoS

Let $u(x, y) = \cos x \sinh y$. Find the harmonic conjugate $v(x, y)$ of u and express $u(x, y) + i\,v(x, y)$ as a function of $z = x + iy$. *12*

12. 1c 2014

Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2},\quad z \neq 0;\ f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist. 10

13. 2e 2014 IFoS

Find the constants a, b, c such that the function

$$f(z) = 2x^2 - 2xy - y^2 + i(ax^2 - bxy + cy^2)$$

is analytic for all $z (= x + iy)$ and express $f(z)$ in terms of z. 8

14. 1c 2013 IFoS

Construct an analytic function

$f(z) = u(x, y) + iv(x, y)$, where

$v(x, y) = 6xy - 5x + 3$.

Express the result as a function of z. *10*

15. 1c 2012

(c) Show that the function defined by

$$f(z)=\begin{cases}\dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. 12

16. 3b 2012 IFoS

Show that the function

$$u(x, y) = e^{-x}(x\cos y + y\sin y)$$

is harmonic. Find its conjugate harmonic function v(x, y) and the corresponding analytic function f(z). 13

17. 1c 2011

If f(z) = u + iv is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, find f(z) subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$. 12

18. 1e 2010

Show that

$u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function.

Find a harmonic conjugate of $u(x, y)$. Hence find the analytic function f for which $u(x, y)$ is the real part. 12

19. 1d 2010 IFoS

Determine the analytic function $f(z) = u + iv$ if $v = e^x(x\sin y + y\cos y)$. 10

2. COMPLEX INTEGRATION

1. 1d 2020

Evaluate the integral $\int_C (z^2+3z)dz$ counterclockwise from (2, 0) to (0, 2) along the curve C, where C is the circle $|z|=2$. 10

2. 3c 2019

Evaluate the integral $\int_c Re(z^2)dz$ from 0 to $2+4i$ along the curve C where C is a parabola $y=x^2$. 10

3. 1e 2019 IFoS

Evaluate the integral $\int_C \text{Re}(z^2)dz$ from 0 to $2+4i$ along the curve $C: y=x^2$. 8

4. 1c 2012 IFoS

Evaluate the integral

$$\int_{2-i}^{4+i} (x+y^2-ixy)\,dz$$

along the line segment AB joining the points A(2, −1) and B(4, 1). 10

5. 4a 2010

(i) Evaluate the line integral $\int_c f(z)\,dz$ where $f(z)=z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.

3. CAUCHY INTEGRAL FORMULA

1. 2c 2020 IFoS

Using Cauchy theorem and Cauchy integral formula, evaluate the integral

$$\oint_C \frac{e^z}{z^2(z+1)^3}\,dz$$

where C is $|z|=2$. 15

2. 1d 2019 IFoS

Using Cauchy's Integral formula, evaluate the integral $\oint_c \frac{dz}{(z^2+4)^2}$ where $c: |z-i| = 2$. 8

3. 1c 2014 IFoS

Using Cauchy integral formula, evaluate

$$\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz$$

where C is the circle $|z-i|=2$. 8

4. 2c 2012

(c) Use Cauchy integral formula to evaluate

$$\int_c \frac{e^{3z}}{(z+1)^4}\,dz,$$ where c is the circle $|z|=2$.

15

4. SINGULARITIES, TAYLOR'S AND LAURENT'S SERIES

1. 4b 2019

Obtain the first three terms of the Laurent series expansion of the function $f(z)=\dfrac{1}{(e^z-1)}$ about the point $z=0$ valid in the region $0<|z|<2\pi$. 10

2. 4b 2019 IFoS

Classify the singular point $z = 0$ of the function $f(z)=\dfrac{e^z}{z+\sin z}$ and obtain the principal part of the Laurent series expansion of $f(z)$. 15

3. 4b 2018

Find the Laurent's series which represent the function $\dfrac{1}{(1+z^2)(z+2)}$ when

(i) $|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$ 15

4. 1d 2017

Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f\left(\dfrac{1}{z}\right)$. 10

5. 4a 2017

For a function $f : \mathbb{C} \to \mathbb{C}$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\dfrac{1}{k}\right) = 0$ for all $k = 1, 2, 3, \ldots$. Show that f is a polynomial. 15

6. 4c 2016

Prove that every power series represents an analytic function inside its circle of convergence. 20

7. 4b 2016 IFoS

Find the Laurent series for the function $f(z)=\dfrac{1}{1-z^2}$ with centre $z=1$. 10

8. 2c 2015

Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about the point $z = 0$. 20

9. 1d 2014

Expand in Laurent series the function $f(z) = \dfrac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$. 10

10. 4b 2014 IFoS

Find the Laurent series expansion at $z = 0$ for the function

$$f(z) = \frac{1}{z^2(z^2+2z-3)}$$

in the regions *(i)* $1 < |z| < 3$ and *(ii)* $|z| > 3$. 15

11. 1d 2013

Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has n zeroes in the unit circle. *10*

12. 4b 2013 IFoS

Find Laurent series about the indicated singularity. Name the singularity and give the region of convergence.

$\dfrac{z - \sin z}{z^3}$; $z = 0$. *13*

13. 3c 2012

(c) Expand the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1 < |z| < 3$ (ii) $|z| > 3$ (iii) $0 < |z+1| < 2$
(iv) $|z| < 1$ **15**

14. 3d 2011

Find the Laurent series for the function

$f(z) = \dfrac{1}{1-z^2}$, with centre $z = 1$. 15

15. 2c 2011

If the function f(z) is analytic and one valued in $|z - a| < R$, prove that for $0 < r < R$,

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta) e^{-i\theta}\, d\theta,$$ where $P(\theta)$ is the real part of $f(a + re^{i\theta})$. 15

16. 1d 2011 IFoS

Expand the function

$$f(z) = \frac{2z^2 + 11z}{(z+1)(z+4)}$$

in a Laurent's series valid for $2 < |z| < 3$. 10

17. 4b 2010

Find the Laurent series of the function

$$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right] \text{ as } \sum_{n=-\infty}^{\infty} C_n z^n$$

for $0 < |z| < \infty$

where $C_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \lambda \sin\phi)\, d\phi$,

$n = 0, \pm 1, \pm 2, \ldots..$

with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}\ (-\pi \le \phi \le \pi)$ as contour in this region. 15

18. 4b 2010 IFoS

Obtain Laurent's series expansion of the function

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in the region $0 < |z+1| < 2$. 13

5. RESIDUES, CAUCHY'S RESIDUE THEOREM

1. 2d 2019

Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non zero at z_0.

Moreover $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geqslant 1$. 15

2. 3b 2018

Show by applying the residue theorem that $\int_0^\infty \dfrac{dx}{(x^2+a^2)^2} = \dfrac{\pi}{4a^3}$, $a > 0$. 15

3. 2c 2018 IFoS

Prove that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$. 10

4. 3c 2018 IFoS

Evaluate the integral $\int_0^{2\pi} \cos^{2n} \theta \, d\theta$, where n is a positive integer. 10

5. 4b 2017 IFoS

Find the sum of residues of $f(z) = \dfrac{\sin z}{\cos z}$ at its poles inside the circle $|z| = 2$. 8

6. 3a 2015

State Cauchy's residue theorem. Using it, evaluate the integral

$$\int_C \frac{e^z+1}{z(z+1)(z-i)^2} dz; \quad C: |z| = 2$$

15

7. 2c 2015 IFoS

Evaluate the integral $\int_\Gamma \dfrac{z^2}{(z^2+1)(z-1)^2} dz$, where Γ is the circle $|z| = 2$. 12

8. 3c 2014

Evaluate the integral $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues. 20

9. 2c 2014 IFoS

Evaluate : 15

$$\int_{|z|=1} \frac{z}{z^4 - 6z^2 + 1} dz$$

10. 4b 2013

Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^{\pi} \sin^4 \theta \, d\theta$$ 15

11. 3c 2013 IFoS

Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4} dz$ where c is the circle $|z| = 3$. 13

12. 4a 2012 IFoS

Using the Residue Theorem, evaluate the integral

$$\int_C \frac{e^z - 1}{z(z-1)(z+i)^2} dz,$$

where C is the circle $|z| = 2$. 13

13. 4b 2011 IFoS

State Cauchy's residue theorem. Using it, evaluate the integral

$$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)} dz$$

counterclockwise around the circle $C : |z+1| = 4$. 13

6. CONTOUR INTEGRATION

1. 2c 2020

Using contour integration, evaluate the integral $\int_0^{2\pi} \frac{1}{3+2\sin\theta} d\theta$. 20

2. 2b 2017

Using contour integral method, prove that

$$\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}.$$ 15

3. 4a 2017 IFoS

Prove by the method of contour integration that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$. 12

4. 3c 2016

Let

$\gamma : [0, 1] \to \mathbb{C}$ be the curve

$\gamma(t) = e^{2\pi i t}$, $0 \le t \le 1$.

Find, giving justifications, the value of the contour integral 15

$$\int_{\gamma} \frac{dz}{4z^2 - 1}$$

5. 4c 2016 IFoS

Evaluate by Contour integration $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$. 10

6. 4b 2015 IFoS

Show that $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$ by using contour integration and the residue theorem. 15

7. 3d 2012

(d) Evaluate by contour integration

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}, \quad a^2 < 1. \qquad 15$$

8. 3c 2011

Evaluate by Contour integration,

$$\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} \qquad 15$$

9. 2c 2010 IFoS

Using the method of contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} \qquad 14$$

7. CONFORMAL MAPPINGS & BILINEAR TRANSFORMATIONS

1. 4a 2020 IFoS

Show that the bilinear transformation

$$w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$$

z_0 being in the upper half of the z-plane, maps the upper half of the z-plane into the interior of the unit circle in the w-plane. If under this transformation, the point $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$, then find this transformation. 10

2. 3c 2014 IFoS

Find the bilinear transformations which map the points -1, ∞, i into the points—

(i) i, 1, $1+i$

(ii) ∞, i, 1

(iii) 0, ∞, 1 15

3. 2c 2011 IFoS

Sketch the image of the infinite strip $1 < y < 2$ under the transformation $w = \frac{1}{z}$. 14

4. 4a 2010

(ii) Find the image of the finite vertical strip $R:\ x = 5$ to $x = 9$, $-\pi \leq y \leq \pi$ of z-plane under exponential function. 15