COMPLEX ANALYSIS BACK YEARS SOLUTIONS

Q(4) that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad (z \neq 0), \; f(0) = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ doesnot exist there.

Sol^n: Here $u = \dfrac{x^3 - y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$ where $z \neq 0$.

Here we see the both u and v are rational and finite for all values of $z \neq 0$.

so u and v are continuous at all those points for which $z \neq 0$.

Hence $f(z)$ is continuous where $\neq 0$.

At the origin $u = 0, v = 0$ [since $f(0) = 0$]

Hence u and v are both continuous at the origin. consequently $f(z)$ is continuous at the origin.

$\therefore f(z)$ is continuous every-where.

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \left(\frac{-y}{y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \left(\frac{y}{y}\right) = 1.$$

$\therefore$ we see that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence Cauchy-Riemann conditions are satisfied at $z = 0$.

Again $f'(0) = \lim_{z\to 0} \dfrac{f(z)-f(0)}{z}$

$$= \lim_{z\to 0}\left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}\right]$$

Now let $z \to 0$ along $y=x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2}\cdot\frac{1}{x+ix}$$

$$= \lim_{x\to 0} \frac{2i}{2(1+i)}$$

$$= \frac{1}{2}(1-i)$$

Again let $z\to 0$ along $y=0$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3(1+i)}{x^3} = 1+i$$

$\therefore$ we see that $f'(0)$ is not unique.

i.e, the values of $f'(0)$ are not the same as $z \to 0$ along different curves.

$\therefore$ $f'(z)$ doesnot exist at the origin.

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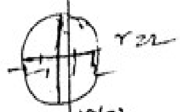
2012

→ Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4} dz$, where C is the circle $|z|=2$.

Sol: Comparing the given integral with $\int_C \frac{f(z)}{(z-z_0)^n} dz$,



we get

$f(z) = e^{3z}$, $z = -1$

· Since $f(z)$ is analytic in $|z|=2$

and $z_0 = -1$ is a point inside $|z|=2$

∴ we apply Cauchy's integral formula

$$\int_C \frac{dz}{(z-z_0)^4} = \frac{2\pi i}{n!} f^3(z_0) \quad \text{—①}$$

Now $f(z) = e^{3z}$

$\Rightarrow f'(z) = 3e^{3z} \Rightarrow f'(-1) = 3e^{-3}$ $\quad (\because z_0 = -1)$

$\Rightarrow f''(z) = 9e^{3z} \Rightarrow f''(-1) = 9e^{-3}$

$\Rightarrow f'''(z) = 27e^{3z} \Rightarrow f'''(-1) = 27e^{-3}$

∴ from ①, we have

$$\int_C \frac{dz}{(z+1)^4} = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{6} (27) e^{-3}$$

$$= 9\pi i e^{-3}$$

IAS-2012

Ex. 9. *Examine the nature of the function*

$$f(z)=\frac{x^2y^5(x+iy)}{x^4+y^{10}}\quad z\neq 0$$

$$f(0)=0$$

in the region including the origin. (Meerut 68)

Sol. Here $u+iv=+\dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$

$$\therefore\quad u=\frac{x^3y^5}{x^4+y^{10}},\quad v=\frac{x^2y^5}{x^4+y^{10}}$$

At the origin,

$$u_x=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$u_y=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

Similarly $v_x=0,\ v_y=0$

Hence Cauchy-Riemann equations are satisfied at the origin.

But $\displaystyle\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{x\to 0,\ y\to 0}\left[\frac{x^2y^5(x+iy)}{x^4+y^{10}}-0\right]\frac{1}{x+iy}$ since $f(0)=0$ and $z=x+iy$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^5}{x^4+y^{10}}$$

$$=\lim_{x\to 0}\frac{x^2m^5x^5}{x^4+m^{10}x^{10}}$$

if $z\to 0$ along the radius vector $y=mx$

$$=\lim_{x\to 0}\frac{m^5x^3}{1-m^{10}x^6}=0$$

and $\displaystyle=\lim_{x\to 0}\frac{x^2x^2}{x^4+x^4}$ if $z\to 0$ along the curve $y^5=x^2$

$$=\tfrac{1}{2}$$

showing that $f'(0)$ does not exist.

Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

IAS 2013 → Using Cauchy's residue theorem, evaluate integral $I=\int_0^{\pi}\sin^4\theta\,d\theta$

$\because 1-\cos 2\theta = 2\sin^2\theta$

$\dfrac{(1-\cos 2\theta)}{2}\dfrac{(1-\cos 2\theta)}{2}$

take $2\theta=t$

Sol $\displaystyle I=\int_0^{2\pi}\frac{(1-\cos t)(1-\cos t)}{4}\frac{dt}{2}$

$\displaystyle=\frac{1}{8}\int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt$

let $z = e^{it}$

$dz = i e^{it} dt$

$dz = iz\, dt$

$$\oint_C \frac{(z^2 - 2z + 1)^2}{iz\, 4z^2}\, dz = \oint_C f$$

C.

C

$e^f C$

If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, subject to the condition $f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{2}$.

Sol. We have $u+iv=f(z)$. $\therefore$ $iu-v=if(z)$.

On adding, we have

$$u-v+i(u+v)=(1+i)f(z)=F(z) \text{ say.}$$

i.e., $$(u-v)+i(u+v)=F(z).$$

Let $U=u-v$, and $V=u+v$, then $U+iV=F(z)$ is an analytic function.

Here $$U=\frac{e^y-\cos x+\sin x}{\cosh y-\cos x}=\frac{\cosh y+\sinh y-\cos x+\sin x}{\cosh y-\cos x}$$

$$=1+\frac{\sinh y+\sin x}{\cosh y-\cos x}=\left\{1-\frac{\sin x+\sinh y}{\cos x-\cosh y}\right\}.$$

$$\therefore \quad \frac{\partial U}{\partial x}=\frac{-1-\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_1(x,y)$$

and $$\frac{\partial U}{\partial y}=\frac{1-\sin x\sinh y-\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_2(x,y)$$

By Milne's method we have

$$F'(z)=[\phi_1(z,0)-i\phi_2(z,0)]$$

$$=-\frac{1}{1-\cos z}-i\frac{1}{1-\cos z}$$

$$=-(1+i)\frac{1}{1-\cos z}=-\tfrac{1}{2}(1+i)\operatorname{cosec}^2\frac{z}{2}$$

Integrating it, we get

$$F(z)=-\tfrac{1}{2}(1+i)\int\operatorname{cosec}^2 z/2+c=(1+i)\cot z/2+c$$

i.e. $$(1+i)f(z)=(1+i)\cot z/2+c$$

or $$f(z)=\cot z/2+c_1.$$

But when $z=\dfrac{\pi}{2}, f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{2}$ $\therefore$ $c_1=\dfrac{3-i}{2}-1=\dfrac{1-i}{2}$.

Hence $f(z)=\cot z/2+\frac{1}{2}(1-i)$.

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$|z-a| < R$, prove that when $0 < r < R$,

2011

2015

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$. (Agra 1957, 70)

Solution. Since $f(z)$ is analytic in $|z-a| < R$ and $r < R$, it follows that $f(z)$ is also analytic inside the circle C defined by

$$|z-a| = r.$$

Hence by Cauchy's formula for the derivative, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz = f'(a). \quad \ldots(1)$$

Also $f(z)$ can be expanded as a Taylor's series about $z=a$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m (z-a)^m.$$

Putting $z-a = re^{i\theta}$, we have

$$f(z) = f(a+re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{mi\theta}.$$

Then $\overline{f(z)} = \sum_{m=0}^{\infty} \bar{a}_m r^m e^{-mi\theta}$

$$\therefore \quad \frac{1}{2\pi i}\int \frac{\overline{f(z)}}{(z-a)^2}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \bar{a}_m r^m e^{-mi\theta}}{r^2 e^{2i\theta}} \cdot r\, ie^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi} \sum_{m=0}^{\infty} \bar{a}_m r^{m-1} \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta$$

$$= 0 \left[\because \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta = 0\right]. \quad \ldots(2)$$

Adding (1) and (2), we have

$$0 + f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\text{real part of } (a+re^{i\theta})}{r^2 e^{2i\theta}} \cdot rie^{i\theta}\, d\theta$$

$$[\because \quad z = a + re^{i\theta}]$$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

Show that $e^{\frac{\lambda}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \lambda\sin\theta)\, d\theta$.

$f(z) = \exp\left[\lambda/2\left(z-\frac{1}{z}\right)\right]$.

$f(z)$ is analytic at every point except at $z=0$ & $z=\infty$

i.e. it is analytic in the ring shaped region

$r \le |z| \le R$ where $r < R$.

Hence it can be expanded as a Laurent's series in the form

$$e^{\frac{1}{2}\lambda\left[z-\frac{1}{z}\right]} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

where $a_n = \frac{1}{2\pi i}\int_C e^{\frac{\lambda}{2}(z-\frac{1}{z})} \cdot \frac{dz}{z^{n+1}}$.

and $b_n = \frac{1}{2\pi i}\int_C e^{\frac{\lambda}{2}(z-\frac{1}{z})} \cdot \frac{dz}{z^{-n+1}}$.

where C is any circle with origin as centre

Let us take C as $|z|=1$ so that $e^{i\theta} = z$.

$dz = ie^{i\theta} d\theta$.

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{\lambda}{2}(e^{i\theta}-e^{-i\theta})} \cdot \frac{ie^{i\theta}}{e^{(n+1)i\theta}}\, d\theta.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} i e^{\lambda i\sin\theta} \cdot e^{-ni\theta}\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\cos(n\theta - \lambda\sin\theta) - i\sin(n\theta - \lambda\sin\theta)\right] d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \lambda\sin\theta)\,d\theta.$$

Since $\int_0^{2\pi} \sin(n\theta - \lambda\sin\theta)\,d\theta$ vanishes by the property of definite integrals.

Now replace z by $-\frac{1}{z}$.

$$f(z) = e^{\frac{\lambda}{2}\left(\frac{-1}{z}+z\right)} = e^{\frac{\lambda}{2}\left(z-\frac{1}{z}\right)}$$

$$= \sum_{n=0}^{\infty} a_n\left(\frac{-1}{z}\right)^n + \sum_{n=1}^{\infty} b_n(-z)^n.$$

$$= \sum_{n=0}^{\infty} a_n(-1)^n . z^{-n} + \sum_{n=1}^{\infty} b_n (-1)^n . z^n.$$

$$\therefore \quad b_n = (-1)^n a_n.$$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} (-1)^n a_n z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a_n z^n. \qquad n \in I.$$

Q-4) Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$.

Assume that the zeros of the denominator are simple. Show that the sum of the residues of $f(z)$ at its poles is equal to $\dfrac{a_{n-1}}{b_n}$.

Sol^n, Let $f(z) = \dfrac{a_0}{b_0 + b_1 z}$, where $b_1 \neq 0$

$$= \frac{a_0}{b_1\left[\frac{b_0}{b_1} + z\right]}$$

$z = -\frac{b_0}{b_1}$ is a pole of order 1.
i.e, simple pole.

The residue at $z = -\frac{b_0}{b_1}$ is

$$= \lim_{z \to -\frac{b_0}{b_1}} \left(\frac{b_0}{b_1} + z\right) \frac{a_0}{b_1\left(\frac{b_0}{b_1} + z\right)}$$

$$= \frac{a_0}{b_1}$$

Now let us assume ~~that~~ $f(z) = \dfrac{a_0 + a_1 z}{b_0 + b_1 z + b_2 z^2}$

$$= \frac{a_0 + a_1 z}{b_2\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)}$$

Let α, β be the simple poles of

$\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)$. roots of $z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2}$

Since α, β ~~are the~~

Now

$$z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2} = (z-\alpha)(z-\beta)$$
$$= z^2 - (\alpha+\beta) z + \alpha\beta$$

Residue at $z=\alpha$ is $= \lim_{z\to\alpha} (\cancel{z-\alpha}) \frac{a_0+a_1 z}{b_2(\cancel{z-\alpha})(z-\beta)}$

$$= \frac{a_0+a_1\alpha}{b_2(\alpha-\beta)}$$

Residue at $z=\beta$ is $= \lim_{z\to\beta} (\cancel{z-\beta}) \frac{a_0+a_1 z}{b_2(z-\alpha)(\cancel{z-\beta})}$

$$= \frac{a_0+a_1\beta}{b_2(\beta-\alpha)}$$

$$= -\frac{a_0+a_1\beta}{b_2(\alpha-\beta)}$$

$\therefore$ Sum of the residues of $f(z) = \frac{a_0+a_1 z}{b_2\left(z^2+\frac{b_1}{b_2}z+\frac{b_0}{b_2}\right)}$

$$= \frac{1}{b_2}\left[\frac{a_0+a_1\alpha}{\alpha-\beta} - \frac{a_0+a_1\beta}{\alpha-\beta}\right]$$

$$= \frac{1}{b_2}\left[\frac{a_1(\cancel{\alpha-\beta})}{(\cancel{\alpha-\beta})}\right] = \frac{a_1}{b_2}.$$

Let $f(z) = \frac{a_0+a_1 z+a_2 z^2}{b_0+b_1 z+b_2 z^2+b_3 z^3}$

$$= \frac{a_0+a_1 z+a_2 z^2}{b_3[(z-\alpha)(z-\beta)(z-\gamma)]}$$

where α, β, γ are three simple poles.

Residue at $z=\alpha$ is $\lim_{z\to\alpha} (z-\alpha) \frac{a_0+a_1 z+a_2 z^2}{b_3(\cancel{z-\alpha})(z-\beta)(z-\gamma)}$

$$= \frac{a_0+a_1\alpha+a_2\alpha^2}{b_3(\alpha-\beta)(\alpha-\gamma)}$$

Residue at $z=\beta$ is $\lim_{z\to\beta} (z-\beta) \frac{a_0+a_1 z+a_2 z^2}{b_3(z-\alpha)(\cancel{z-\beta})(z-\gamma)}$

$$b_3(\beta-\alpha)(\beta-\gamma)$$

Residue at $z=\gamma$ is $= \lim_{z\to\gamma} \cancel{(z-\gamma)} \dfrac{a_0+a_1z+a_2z^2}{b_3(z-\alpha)(z-\beta)\cancel{(z-\gamma)}}$

$$= \frac{a_0+a_1\gamma+a_2\gamma^2}{b_3(\gamma-\alpha)(\gamma-\beta)}$$

$\therefore$ Sum of the residues of $f(z)$ at its poles α, β, γ

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} + \frac{a_0+a_1\beta+a_2\beta^2}{(\beta-\alpha)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} - \frac{a_0+a_1\beta+a_2\beta^2}{(\alpha-\beta)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\alpha-\gamma)(\beta-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{(a_0+a_1\alpha+a_2\alpha^2)(\beta-\gamma)-(a_0+a_1\beta+a_2\beta^2)(\alpha-\gamma) + (a_0+a_1\gamma+a_2\gamma^2)(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0(0)+a_1(0)+a_2\left(\alpha^2(\beta-\gamma)-\beta^2(\alpha-\gamma)+\gamma^2(\alpha-\beta)\right)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_2(\alpha^2\beta-\gamma\alpha^2-\beta^2\alpha+\gamma\beta^2+\gamma^2\alpha-\gamma^2\beta)}{\alpha^2\beta-\cancel{\alpha\gamma\beta}-\alpha^2\gamma+\alpha\gamma^2-\beta^2\alpha+\beta^2\gamma+\cancel{\alpha\beta\gamma}-\gamma^2\beta}\right]$$

$$= \frac{a_2}{b_3}, \quad (b_3\neq 0)$$

$\therefore$ we conclude that

$$f(z) = \frac{a_0+a_1z+a_2z^2+\cdots+a_{n-1}z^{n-1}}{b_0+b_1z+b_2z^2+\cdots+b_nz^n}$$

then the sum of the residues of $f(z)$ at its poles is $= \dfrac{a_{n-1}}{b_n}$, where $b_n\neq 0$

The poles of $\dfrac{2}{(\gamma+\beta i)\left[z^2+\left(\dfrac{2\alpha i}{\gamma+\beta i}\right)z+\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$

are the simple poles.

$$z = \frac{-\dfrac{2\alpha i}{\gamma+\beta i} \pm \sqrt{\dfrac{-4\alpha^2}{(\gamma+\beta i)^2}-4(1)\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)}}{2(1)}$$

$$= \frac{-\dfrac{2\alpha i}{\gamma+\beta i} \pm \dfrac{2}{\gamma+\beta i}\sqrt{-\alpha^2-(-\gamma+\beta i)(\gamma+\beta i)}}{2}$$

$$= \frac{\dfrac{-2\alpha i}{\gamma+\beta i} \pm \dfrac{2i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)}}{2}$$

$$= \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)}. \qquad \text{—(2)}$$

Given that $\alpha^2 > \beta^2+\gamma^2$

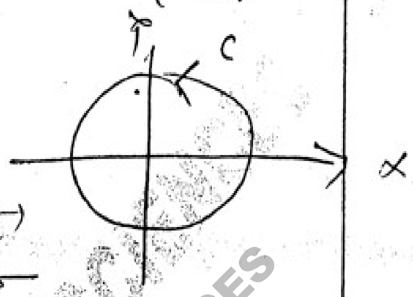
$\therefore z = \dfrac{-\alpha i + i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}{\gamma+\beta i} = z_1$ say. lies inside 'C' because $|z|<1$.

Now residue at z_1

$$= \lim_{z\to z_1} (z-z_1)\frac{2}{(\gamma+\beta i)\left[z^2+\left(\dfrac{2\alpha i}{\gamma+\beta i}\right)z+\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$$

IAS 2009. If α, β, γ are real nos s.t. $\alpha^2 > \beta^2 + \gamma^2$. Show that

$$\int_0^{2\pi} \frac{d\theta}{(\alpha + \beta\cos\theta + \gamma\sin\theta)} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$$



sol let $z = e^{i\theta}$ then

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \; ; \; dz = ie^{i\theta}d\theta$$

$$\Rightarrow \boxed{dz = iz\, d\theta}$$

$$\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \oint_C \frac{dz}{iz\left[\alpha + \beta\left(\frac{z+z^{-1}}{2}\right) + \gamma\left(\frac{z-z^{-1}}{2i}\right)\right]}$$

$$= \oint_C \frac{2\,dz}{z\left[2\alpha i + \beta i(z+z^{-1}) + \gamma(z - z^{-1})\right]}$$

$$= \oint \frac{2dz}{2\alpha zi + z^2\beta i + \beta i + \gamma z^2 - \gamma}$$

$$= \oint_C \frac{2\,dz}{(\gamma + \beta i)z^2 + 2\alpha zi + (-\gamma + \beta i)}$$

$$= \oint_C \frac{2\,dz}{(\gamma + \beta i)\left[z^2 + \left(\frac{2\alpha i}{\gamma + \beta i}\right)z + \left(\frac{-\gamma + \beta i}{\gamma + \beta i}\right)\right]}$$

where C is the circle of ~~unit~~ (1) radius with centre at the origin

$$= \underset{z \to z_1}{Lt} \frac{2}{(\gamma+\beta i)\left[\frac{z + \alpha i + i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}{\gamma+\beta i}\right]}$$

$$= \frac{1}{i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}$$

$$\therefore (1) \equiv \oint_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = 2\pi i \begin{pmatrix}\text{residue} \\ \text{at } z_1\end{pmatrix}.$$

$$= \frac{2\pi}{\sqrt{\alpha^2-(\beta^2+\gamma^2)}}.$$

Let $f(z)$ be an entire function satisfying $|f(z)| \le K|z|^2$ for some constant K and all z. Show that $f(z) = az^2$ for some constant 'a'.

Soln: Given that f is an entire function.

$\therefore f$ has a Taylor series expansion around $z=0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

By Cauchy estimates, $|f^n(z)| \le \frac{n!\,M}{R^n}$

M is bound of $f(z)$.

for $n=3$, $|f^{(3)}(z)| \le \frac{3!\,R^2}{R^3}$

$|f(z)| \le K|z|^2$.

$\le K R^2 < \frac{3}{R}$

Since f is entire, we can let $R \to \infty$

$\Rightarrow f^{(3)}(z) = 0 \ \forall z$

Hence the claim.

$f(z) = f(0) + f'(0) + f''(0)z^2$

Using the inequality, we have

$|f(z)| \le K|z|^2$.

Clearly $f(0) = 0$ & $f'(0) = 0$.

Using the inequality once again

$|f(0)| \le 0$ and $f'(z) f(0) + \frac{z^2}{2} f''(0)$

$\therefore f(z) = K.z^2$

.5 Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

Solution We have, as in Method 2 of Problem 7.4(b),

$$F(z)\frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)}$$

$$= \frac{\left(1-\frac{z^4}{6}+\cdots\right)}{z^5\left(1-\frac{z^4}{90}+\cdots\right)} = \frac{1}{z^5}\left(1-\frac{7z^4}{45}+\cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z\to 0}\frac{1}{4!}\frac{d^4}{dz^4}\left\{z^5\,\frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

2007 → Evaluate (by using residue theorem)

$$\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta}$$

Sol^n: Let $I = \int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \int_0^{2\pi} \frac{d\theta}{5+4\cos 2\theta}$

Let the contour C be the unit circle $|z|=1$ with centre at the origin.

Let $z = e^{i\theta}$ then $\cos 2\theta = \frac{1}{2}(z^2+z^{-2}) = \frac{(z^4+1)}{2z^2}$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$I = \int_0^{2\pi} \frac{d\theta}{5+4\cos 2\theta} = \int_C \frac{1}{5+4\frac{(z^4+1)}{2z^2}} \frac{dz}{iz} = \int_C \frac{1}{5+\frac{2(z^4+1)}{z^2}} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{z\,dz}{2z^4+5z^2+2}$$

$$= \frac{1}{i}\int_C \frac{z\,dz}{(2z^2+1)(z^2+2)} = \frac{1}{2i}\int \frac{z\,dz}{(z^2+\frac{1}{2})(z^2+2)}$$

$$= \frac{1}{2i}\int_C f(z)\,dz \quad \text{——— (1)}$$

where $f(z) = \dfrac{z}{(z^2+\frac{1}{2})(z^2+2)}$

$\therefore f(z)$ has poles at $z = \pm\sqrt{2}i, \pm\frac{i}{\sqrt{2}}$ of orders 1 respectively.

But only the poles are $z = \pm\frac{i}{\sqrt{2}}$ inside C.

$$\therefore \int_C f(z)dz = 2\pi i\left[\left(\text{Residue at } z=\tfrac{i}{\sqrt{2}}\right) + \left(\text{Residue at } z = \tfrac{-i}{\sqrt{2}}\right)\right.$$

Residue at $z = \frac{i}{\sqrt{2}}$ is

$$\lim_{z \to \frac{i}{\sqrt{2}}} \left(z - \frac{i}{\sqrt{2}}\right) f(z) = \lim_{z \to \frac{i}{\sqrt{2}}} \cancel{\left(z - \frac{i}{\sqrt{2}}\right)} \frac{z}{(z^2+2)\cancel{\left(z - \frac{i}{\sqrt{2}}\right)}\left(z + \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{i/\sqrt{2}}{\left(\frac{-1}{2}+2\right)\left(\frac{2i}{\sqrt{2}}\right)} = \frac{1}{2(3/2)} = \frac{1}{3}$$

Residue at $z = \frac{-i}{\sqrt{2}}$ is

$$\lim_{z \to -\frac{i}{\sqrt{2}}} \left(z + \frac{i}{\sqrt{2}}\right) f(z) = \lim_{z \to -\frac{i}{\sqrt{2}}} \frac{\cancel{\left(z + i/\sqrt{2}\right)} \cdot z}{(z^2+2)\left(z - \frac{i}{\sqrt{2}}\right)\cancel{\left(z + \frac{i}{\sqrt{2}}\right)}}$$

$$= \frac{-i/\sqrt{2}}{\left(-\frac{1}{2}+2\right)\left(\frac{-2i}{\sqrt{2}}\right)}$$

$$= \frac{1}{(3/2)(2)} = \frac{1}{3}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{1}{3}+\frac{1}{3}\right)$$

$$= 2\pi i\left(\frac{2}{3}\right)$$

$$= \frac{4\pi i}{3}$$

$\therefore$ from ①, we have

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos 2\theta} = \frac{1}{2i}\left(\frac{4\pi i}{3}\right)$$

$$= \frac{2\pi}{3}.$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \frac{2\pi}{3}$$

$$\int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\,d\theta$$

$$\int_0^{2\pi} \frac{1}{5+4\cos 2\theta}\,d\theta$$

$$\int_0^{4\pi} \frac{1}{5+4\cos\alpha}\,\frac{d\alpha}{2}$$

$$= 2 \times \frac{1}{2}\int_0^{2\pi} \frac{1}{5+4\cos\alpha}\,d\alpha$$

Prove that the function 'f' defined by f(z) as

$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

is not differentiable at $z=0$.

Solution:

Given function is.

$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

Consider,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0} \frac{z^5/|z|^4}{z-0}$$

$$= \lim_{z\to 0} \frac{z^4}{|z|^4}$$

$$= \lim_{x,y\to 0} \frac{(x+iy)^4}{(x^2+y^2)^2}$$

put $z=x$, $y=0$
ie Along X-axis, we get

$$\lim_{z\to 0} \frac{(x+iy)^4}{(x^2+y^2)^2} = \frac{x}{x} = 1$$

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Along St. line ie $y = x$.

$$\lim_{x,y\to 0} \frac{(x+iy)^4}{(x^2+y^2)^2} \text{ becomes}$$

$$= \lim_{y\to 0} \frac{(y+iy)^4}{(y^2+y^2)^4}$$

$$= \lim_{y\to 0} \frac{y^4(1+i)^4}{y^4(4)} = \frac{(1+i)^4}{4}$$

$$= \frac{1}{4}[4i^2] = i^2 = -1.$$

Since $1 \neq -1$ ∴ limit does not exist.

Hence function $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

is not differentiable at $z=0$.

ie $f'(z)$ does not exist at $z=0$.

* for Other Two Methods, Refer Notes of Complex, Page No- 11.

IAS-2006

Evaluate by Contour integration $I = \int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}$, $a^2<1$.

Soln: Let $I = \int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}$. — (1)

Let the contour C be the unit circle $|z|=1$ with centre at the origin.

Let $z=e^{i\theta}$ then $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

and $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) = \frac{z^4+1}{2z^2}$

$\therefore \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{z^4+1/2z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} = \frac{z^4+1}{2z\left[z(1-az)+a^2z-a\right]}$

Since $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$\Rightarrow dz = iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$.

$\therefore (1) \equiv \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_C \frac{(z^4+1)\frac{dz}{iz}}{2z\left[z(1-az)+a^2z-a\right]}$

$= \frac{1}{4i}\int_C \frac{z^4+1}{z^2\left[z(1-az)+a(az-1)\right]}dz$

$= \frac{1}{4i}\int_C \frac{z^4+1}{z^2(z-a)(1-az)}dz$

$= \frac{1}{4i}\int_C f(z)\,dz$ — (2)

where $f(z) = \dfrac{z^4+1}{z^2(z-a)(1-az)}$

$\therefore f(z)$ has poles at $z=0,\ z=a,\ z=1/a$ of

orders 2, 1 and 1 respectively.

But only the poles are $z=0$ and $z=a$ inside C.

$$\therefore \int_C f(z)\,dz = 2\pi i\left[(\text{Residue at } z=0) + (\text{Residue at } z=a)\right] \quad (3)$$

Now Residue at $z=0$ of order 2 is

$$\lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\frac{d}{dz}\,\frac{z^4+1}{(z-az^2-a+a^2z)}$$

$$= -\frac{(1+a^2)}{a^2}$$

and Residue at $z=a$ is

$$\lim_{z\to a}(z-a)f(z) = \frac{a^4+1}{a^2(1-a^2)}$$

$\therefore$ from (3).

$$\int_C f(z)\,dz = 2\pi i\left[-\frac{(1+a^2)}{a^2} + \frac{(a^4+1)}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left[\frac{-(1-a^4)+a^4+1}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left(\frac{2a^2}{1-a^2}\right) = \frac{4a^2 i\pi}{1-a^2}$$

$\therefore$ from (2), we have

$$\int_0^{\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{4i}\left(\frac{4i\pi a^2}{1-a^2}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

between the circles $|z|=1$ *and* $|z|=2$.

Solution. Let C_1, C_2 denote respectively the circles $|z|=1$ and $|z|=2$. We write $f(z)=12$ and $g(z)=z^7-4z^3$. Then on C_1, $f(z)$ and $g(z)$ are analytic and

$$\left|\frac{g(z)}{f(z)}\right|=\left|\frac{z^7-5z^3}{12}\right|$$

$$\leq \frac{|z|^7+|-5z^3|}{12}=\frac{|z|^7+5|z|^3}{12}=\frac{1+5}{12}=\frac{1}{2}$$

$[\because \ |z|=1 \text{ on } C]$

Thus $\left|\frac{g(z)}{f(z)}\right|<1$, or $|g(z)|<|f(z)|$ on C_1.

Hence by Rouche's theorem, $f(z)+g(z)=z^7-5z^3+12$ has the same number of zeros inside C_1 as $f(z)=12$. But $f(z)=12$ has no zeros inside C_2. It follows that z^7-5z^3+12 has no zeros inside C_1.

We next consider the circle C_2. We write $F(z)=z^7$ and $\phi(z)=12-5z^3$. Evidently $F(z)$ and $\phi(z)$ are analytic within and on C_2. Further we have on C_2,

$$\left|\frac{\phi(z)}{f(z)}\right|=\frac{|12-5z^3|}{|z|^7}$$

$$\leq \frac{|12|+5|z|^3}{|z^7|}=\frac{12+5\cdot 2^8}{2^7}=\frac{52}{128}<1$$

$[\because \ |z|=2 \text{ on } C_2]$

Thus $|\phi(z)|<|F(z)|$ on C_2. Hence by Rouche's theorem $F(z)+\phi(z)=z^7-5z^3+12$ has the same number of zeros as $F(z)=z^7$ inside C_2. But all the seven zeros of z^7 lie inside $|z|=2$ since they are all located at the origin. It follows that all the seven zeros of z^7-5z^3+12 lie inside $|z|=2$.

Thus we have shown that the equation $z^7-5z^3-12=0$ has no roots inside $|z|=1$ but nas all the seven roots inside $|z|=2$. It follows that all the roots of this equation lie between the circles $|z|=1$ and $|z|=2$ as required.

IAS-2004

If all zeros of a polynomial $P(z)$ lie in a half plane, then show that zeros of the derivative $P'(z)$ also lie in the same half plane.

Solution. We can assume without loss of generality that the zeros of $P(z)$ lie in the half plane $\operatorname{Re} z < 0$. Let $P(z) = \prod_{j=1}^{n}(z - \alpha_j)$ where $\alpha_j = x_j + iy_j, x_j < 0$.

If $\operatorname{Re} z \geq 0$, then $P(z) \neq 0$ and

$$\begin{aligned}\frac{P'(z)}{P(z)} &= \sum_{j=1}^{n} \frac{1}{z - \alpha_j} \\ &= \sum_{j=1}^{n} \frac{1}{x - x_j + i(y - y_j)} \\ &= \sum_{j=1}^{n} \frac{x - x_j - i(y - y_j)}{(x - x_j)^2 + (y - y_j)^2}\end{aligned}$$

Since $x_j < 0, 1 \leq j \leq n$, it follows that

$$\operatorname{Re}\left(\frac{P'(z)}{P(z)}\right) = \sum_{j=1}^{n} \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2} > 0$$

whenever $\operatorname{Re} z = x \geq 0$. Thus $\frac{P'(z)}{P(z)}$ and therefore $P'(z)$ has no zeros in the right half plane $\operatorname{Re} z \geq 0$. Hence all zeros of $P'(z)$ lie in the same half plane in which the zeros of $P(z)$ lie. ∎

$$= \pi \frac{1-p+p^2}{1-p}, \quad 0 < p < 1.$$

Sol. Let $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p \cos 2\theta + p^2}$

$$= \frac{1}{2} \int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p \cos 2\theta + p^2} d\theta$$

$$= \frac{1}{2} \text{ real part of } \int_0^{2\pi} \frac{1 + e^{6i\theta}}{1 - p\,(e^{2i\theta} + e^{-2i\theta} + p^2)} d\theta$$

$$= \frac{1}{2} \text{ real part of } \int_C \frac{1 + z^6}{1 + p\left(z^2 + \frac{1}{z^2}\right) + p^2} \cdot \frac{dz}{iz} \text{ write } e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$$

where C denotes the unit circle $|z| = 1$.

$$= \frac{1}{2} \text{ real part of } \frac{1}{i} \int_C \frac{z(1 + z^6)}{(1 - pz^2)(z^2 - p)} dz$$

$$= \frac{1}{2} \text{ real part of } \int_C f(z)\, dz \text{ where } f(z) = \frac{z(1 + z^6)}{(1 - pz^2)(z^2 - p)}$$

Poles of $f(z)$ are given by $(1 - pz^2)(z^2 - p) = 0$.

Thus $z = \pm\sqrt{p}$ and $z = \pm \frac{1}{\sqrt{p}}$ are the simple poles.

The only poles which lie within C are $z = \pm\sqrt{p}$, as $p < 1$.

Residue at $z = \sqrt{p}$ is $\lim_{z \to \sqrt{p}} (z - \sqrt{p}) \cdot \frac{z(1 + z^6)}{(1 - pz^2)(z^2 - p)}$

$$= \frac{1}{2} \cdot \frac{1 + p^3}{1 - p^2}$$

and residue at $z = -\sqrt{p}$ is $\lim_{z \to -p} (z + \sqrt{p}) \cdot \frac{z(1 + z^6)}{(1 - pz^3)(z^2 - p)}$

$$= \frac{1}{2} \cdot \frac{1 + p^3}{1 - p^2}$$

so that sum of the residues $= \frac{1 + p^3}{1 - p^2}$.

Hence by Cauchy's residue theorem we have

$\int_C f(z)\, dz = 2\pi i \times$ sum of residues within the contour

$$= 2\pi i \times \frac{1 + p^3}{1 - p^2}$$

$\therefore \quad I = \frac{1}{2}$ real part of $\frac{1}{i} \int_C f(z)\, dz$

$$= \frac{1}{2} \text{ real part of } \frac{1}{i} \cdot 2\pi i \cdot \frac{1 + p^3}{1 - p^2}$$

$$= \frac{1}{2} \text{ real part of } 2\pi \cdot \frac{1 + p^3}{1 - p^2}$$

$$= \pi \frac{1 + p^3}{1 - p^2} = \pi \frac{1 - p + p^2}{1 - p}$$

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IAS-2003

Evaluate $\int_0^{\pi} \frac{a\,d\phi}{a^2+\sin^2\phi}$

Let $I = \int_0^{\pi} \frac{a\,d\phi}{a^2+\sin^2\phi}$

$= \int_0^{\pi} \frac{2a\,d\phi}{2a^2+1-\cos 2\phi}$

$= \int_0^{2\pi} \frac{a\,d\theta}{2a^2+1-\cos\theta}$ putting $2\phi=\theta \Rightarrow 2d\phi = d\theta$.

$= \int_0^{2\pi} \frac{2a\,d\theta}{4a^2+2-(e^{i\theta}+e^{-i\theta})}$

$= \int_C \frac{2a}{4a^2+2-\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$; writing $e^{i\theta}=z$, $d\theta = \frac{dz}{iz}$

where C is unit circle $|z|=1$

$= \int_C \frac{2ai\,dz}{(z-\alpha)(z-\beta)}$

where $\alpha = (1+2a^2)+2a\sqrt{1+a^2}$

$\beta = (1+2a^2)-2a\sqrt{1+a^2}$.

Clearly $|\alpha| >$ and as $|\alpha\beta|=1 \Rightarrow |\beta|<1$.

$\therefore$ Only β lies inside the contour C.

Residue at $z=\beta$ is

$= \lim_{z\to\beta} (z-\beta)\frac{2ai}{(z-\alpha)(z-\beta)}$

$= \frac{2ai}{\beta-\alpha} = \frac{2ai}{-4a\sqrt{1+a^2}} = \frac{-i}{2\sqrt{1+a^2}}$.

Hence by Cauchy's residue theorem

$I = 2\pi i$ (sum of residues with in the contour C

$= 2\pi i \frac{-i}{2\sqrt{1+a^2}} = \frac{\pi}{\sqrt{1+a^2}}$.

I.A.S. 1995 → The function $f(z)$ has a double pole at $z=0$ with residue 2, a simple pole at $z=1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If $f(2)=5$ and $f(-1)=2$, find $f(z)$.

Solution. Since $f(z)$ has a simple pole at $z=1$ with residue 2, and double pole at $z=0$ with residue 2, the principal part of $f(z)$ will be of the form $\frac{2}{z-1}+\frac{2}{z}+\frac{b}{z^2}$ and so $f(z)$ will have a Laurent's expansion of the form

$$f(z)=\sum_{n=0}^{\infty} a_n z^n+\frac{2}{z-1}+\frac{2}{z}+\frac{b}{z^2} \quad \ldots(1)$$

As $f(z)$ is bounded when $|z| \to \infty$, there exists a positive constant M such that $|f(z)| \leq M$ for all values of z. It implies that $f(z)$ has no singularity at $z=\infty$. Hence $f\left(\frac{1}{\zeta}\right)$ has no singularity at $\zeta=0$. Therefore principal part of $f\left(\frac{1}{\zeta}\right)$ will have no term.

Now $f\left(\frac{1}{\zeta}\right)=a_0+\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\ldots+\frac{2\zeta}{1-\zeta}+2\zeta+b\zeta^2.$

$\therefore$ Principal part of $f\left(\frac{1}{\zeta}\right)$

$$=\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\frac{a_3}{\zeta^3}+\ldots$$

Hence the principal part of $f\left(\frac{1}{\zeta}\right)$ will contain no term if

$$a_1=a_2=a_3=\ldots=0.$$

Then (1) takes the form

$$f(z)=a_0+\frac{2}{z-1}+\frac{2}{z}+\frac{b}{z^2} \quad \ldots(2)$$

Now by hypothesis,

$$f(2)=5 \text{ and } f(-1)=2.$$

Substituting these values in (2), we get

$$5=a_0+2+1+\frac{b}{4} \text{ and } 2=a_0-1-2+b.$$

These give $a_0=1$ and $b=4$.

$$f(z)=1+\frac{2}{z-1}+\frac{2}{z}+\frac{4}{z^2}$$

$$=\frac{z^3+3z^2+2z-4}{z^2(z-1)}$$

Thus $f(z)$ is determined.

1999

What kind of singularity have the following functions:

(i) $\frac{1}{1-e^z}$ at $z=2\pi i$

(ii) $\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ at $z=0$ and $z=\infty$.

Sol. (i) $f(z)=\frac{1}{1-e^z}$

Poles of $f(z)$ are given by putting the denominator equal to zero, *i.e.* by $1-e^z=0$ or $e^x=1=e^{2n\pi i}$

$\therefore \quad z=2n\pi i \qquad (n=0, \pm 1, \pm 2, \ldots)$

Obviously $z=2\pi i$, is a simple pole.

(ii) Here $f(z)=\frac{1}{\sin z-\cos z}$.

Poles of $f(z)$ are given by putting the denominator equal to zero.

i.e. by $\sin z-\cos z=0$ or $\tan z=1$

or $z=n\pi+\frac{\pi}{4} \qquad (n=0, 1, 2, \ldots\ldots)$

Obviously $z=\frac{\pi}{4}$ is a simple pole.

(iii) $f(z)=\frac{\cot \pi z}{(z-a)^2}$.

Poles of $f(z)$ are given by putting the denominator equal to zero,

i.e. by $\{\sin(\pi z)\}(z-a)^2=0$

or by $\sin \pi z=0$ and $(z-a)^2=0$

Now $\sin \pi z=0$ gives $\pi z=n\pi$

or $z=n \qquad (n=0, \pm 1, \pm 2, \ldots)$

Obviously $z=\infty$ is the limit point of these poles, hence $z=\infty$ is a non-isolated essential singularity.

And $(z-a^2)=0$ gives $z=a$ repeated twice.

Hence $z=a$ is a double pole.

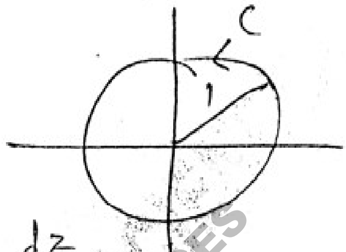
Evaluate

IFoS 1998 $\rightarrow$ $$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}.$$

Sol let $z=e^{i\theta}$. Then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$

$dz = iz\,d\theta$



$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz}{iz\left(3-2\left(\frac{z+z^{-1}}{2}\right)+\left(\frac{z-z^{-1}}{2i}\right)\right)}$$

$$= \oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} \quad \text{--- (1)}$$

where 'C' is the circle of unit radius with centre at the origin.

The poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2-4(1-2i)(-1-2i)}}{2(1-2i)}.$$

$$= \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ \frac{2-i}{5}$$

Only $\frac{2-i}{5}$ lies inside 'C'

Residue at $\frac{2-i}{5} = \lim_{z\to\left(\frac{2-i}{5}\right)} \left[z-\left(\frac{2-i}{5}\right)\right] \left[\frac{2}{(1-2i)z^2+6iz-1-2i}\right]$

$$= \lim \frac{2}{2(1-2i)z+6i} = \frac{1}{2i}$$

$$\oint_C \frac{2i\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right)$$

$$= \pi$$

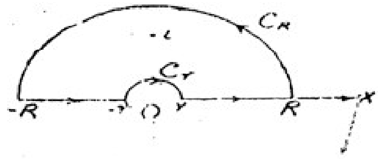
is the reqd value

IAS-1984. $\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = \frac{\pi^3}{8}$

Sol. Consider the integral $\int_C f(z)\, dz$

where $f(z) = \frac{(\log z)^2}{1+z^2}$

taken round the contour C, consisting of the upper half of a large circle $|z| = R$, the upper half of a small circle $|z| = r$, and the lines joining their ends.



The singularities of $f(z)$ are given by

$$1+z^2=0 \quad i.e. \quad z = \pm i$$

thus the only singularity within the contour is a simple pole at $z = i$, the residue at which is

$$\lim_{z \to i} \left\{ (z-i) \frac{(\log z)^2}{1+z^2} \right\}$$

$$= \lim_{z \to i} \left\{ \frac{(\log z)^2}{z+i} \right\}$$

$$= \frac{(\log i)^2}{2i} = \frac{(\log e^{i\pi/2})^2}{2i} = \frac{\left(i\frac{\pi}{2}\right)^2}{2i} = -\frac{\pi^2}{8i}.$$

Hence by Cauchy's residue theorem we have

$\int_C f(z)\, dz = 2\pi i \times$ sum of the residue wthin C.

$$\int_{-R}^{-r} x\,(dx) + \int_{C_r} f(z)\, dz + \int_r^R f(x)\, dx + \int_{C_R} f(z)\, dz$$

$$= 2\pi i \cdot \left(\frac{-\pi^2}{8i} \right)$$

Now, we see that

$$\lim_{z \to \infty} z f(z) = \lim_{z \to \infty} \frac{z(\log z)^2}{1+z^2}$$

$$= \lim_{z \to \infty} \frac{z^3 \left(\frac{\log z}{z} \right)^2}{1+z^2} = \lim_{z \to \infty} \frac{\left(\frac{\log z}{z} \right)^2}{\frac{1}{z^3} + \frac{1}{z}} = 0$$

since $\lim_{z \to \infty} \frac{\log z}{z} = 0$.

$$\therefore \lim_{R \to 0} \int_{C_R} f(z)\, dz = 0$$

Also, $\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z(\log z)^2}{1+z^2}$

$$= \lim_{\zeta \to \infty} \frac{\zeta (\log \zeta)^2}{1+\zeta^2} \quad \text{putting } z = \frac{1}{\zeta}$$

$= 0$ as before

Hence making $R \to \infty$, $r \to 0$, relation (1) becomes

$$\int_{-\infty}^0 \frac{(\log x)^2}{1+x^2}\, dx + \int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = 2\pi i \left(-\frac{\pi^2}{8i} \right)$$

$$\int_0^{\infty} \frac{\{\log(-x)\}^2}{1+x^2}\, dx + \int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = -\frac{\pi^3}{4}$$

putting $-x$ for x in first integral

or $$\int_0^{\infty} \frac{\{\log(xe^{i\pi})\}^2}{1+x^2}\, dx + \int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = -\frac{\pi^2}{4}$$

since $e^{i\pi} = -1$

or $$\int_0^{\infty} \frac{(\log x + i\pi)^2}{1+x^2}\, dx + \int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = -\frac{\pi^2}{4}.$$

or $$\int_0^{\infty} \frac{2(\log x)^2 - \pi^2 + i2\pi \log x}{1+x^2}\, dx = -\frac{\pi^2}{4}.$$

Equating real parts we have

$$2\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx - \pi^2 \int_0^{\infty} \frac{dx}{1+x^2}\, dx = -\frac{\pi^3}{4}$$

i.e. $$2\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx - \pi^2 \frac{\pi}{2} = -\frac{\pi^3}{4}$$

or $$2\int_0^{\infty} \frac{(\log x)^2}{(1+x^2)}\, dx = -\frac{\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{4}$$

$$\therefore \int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = \frac{\pi^3}{8}$$

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IAS-1997 . *Find a function $f(z)$ which is analytic throughout the circle C and its interior, whose centre is at origin and whose radius is unity and has the value*

$$\frac{a-\cos\theta}{a^2-2a\cos\theta+1}+\frac{i\sin\theta}{a^2-2a\cos\theta+1}$$

where $a>1$ and θ is the vertical angle at points on the circumference of C.

Solution. Since $f(z)$ is analytic within and on the circle

$$|z|=1,$$

we may expand it as a Taylor's series at any point inside and on the circle so that we have

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

where $$a_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz.$$

Now on C, $z=e^{i\theta}$, $dz=i\,e^{i\theta}\,d\theta$

and $$f(z)=\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}$$

$$=\frac{a-e^{-i\theta}}{a^2-a(e^{i\theta}+e^{-i\theta})+1}=\frac{a-e^{-i\theta}}{(a-e^{-i\theta})(a-e^{i\theta})}$$

$$=\frac{1}{a-e^{i\theta}}$$

Hence $$a_n=\frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{a-e^{i\theta}}\cdot\frac{ie^{i\theta}\,d\theta}{e^{(n+1)i\theta}}$$

$$=\frac{1}{2\pi a}\int_0^{2\pi} e^{-ni\theta}\left(1-\frac{e^{i\theta}}{a}\right)^{-1}d\theta$$

$$=\frac{1}{2\pi a}\int_0^{2\pi} e^{-ni\theta}\left[1+\frac{e^{i\theta}}{a}+\frac{e^{2i\theta}}{a^2}+\dots\right]d\theta$$

$$=\frac{1}{2\pi a}\int_0^{2\pi} e^{-ni\theta}\cdot\frac{e^{ni\theta}}{a^n}\,d\theta,$$

as the other integrals vanish

$$=\frac{1}{2\pi a^{n+1}}\int_0^{2\pi}d\theta=\frac{1}{2\pi a^{n+1}}\cdot 2\pi=\frac{1}{a^{n+1}}$$

Hence $$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}$$

$$=\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\dots=\frac{1/a}{1-z/a}=\frac{1}{a-z}.$$

... show that

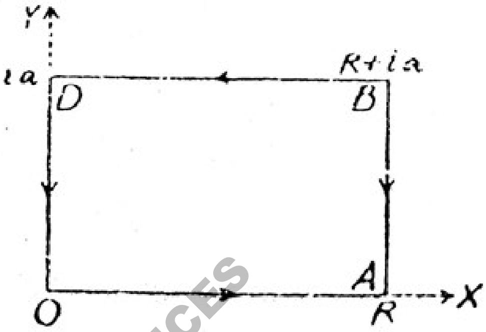
IAS-1997, 1988 (i) $\int_0^\infty e^{-x^2} \cos 2ax \, dx = \frac{e^{-a^2}}{2} \sqrt{\pi}$

and (ii) $\int_0^\infty e^{-x^2} \sin 2ax \, dx = e^{-a^2} \int_0^a e^{y^2} \, dy$

Sol. Consider the integral

$$\int_C f(z) \, dz, \text{ where } f(z) = e^{-z^2}$$

taken round closed contour C which is the perimeter of the given rectangle $OABD$.



Since $f(z)$ is analytic within and on the contour C (*i.e.* there is no singularity within the contour) hence by Cauchy's residue theorem we have

$$\int_C f(z) \, dz = 0 \text{ i.e. } \int e^{-z^2} \, dz = 0$$

i.e. $$\left[\int_{OA} + \int_{AB} + \int_{BD} + \int_{DO}\right] e^{-z^2} \, dz = 0$$

Since $\left[\begin{array}{l} \text{On } OA, z = x, dz = dx; \text{ On } AB, z = R + iy, dz = i dy \\ \text{On } BD, z = x + ia, dz = dx; \text{ On } DO, z = iy, dz = i dy \end{array}\right.$

Hence the above equation becomes

$$\int_0^R e^{-x^2} \, dx + \int_0^a e^{-(R+iy)^2} . i \, dy + \int_R^0 e^{-(x+ia)^2} \, dx$$
$$+ \int_a^0 e^{-(iy)^2} . i dy = 0 \qquad \ldots(1)$$

Now, $\left|\int_0^a e^{-(R+iy)^2} . i dy\right| < \int_0^a |e^{-(R+iy)^2}| \, |i dy|$

$$= \int_0^a e^{-R^2 + y^2} \, dy$$

$$\leqslant \int_0^a e^{-R^2 + a^2} \, dy \qquad \text{since } y \leqslant a \text{ on } AB$$

$= e^{-(R^2+a^2)} \, a$ which $\to 0$ as $R \to \infty$.

Hence by making $R \to \infty$, equation (1) becomes

$$\int_0^\infty e^{-(x+ia)^2} \, dx = \int_0^\infty e^{-x^2} \, dx - i \int_0^a e^{y^2} \, dy$$

i.e. $$\int_0^\infty e(-x^2 + a^2 - 2iax) \, dx = \frac{\sqrt{\pi}}{2} - i \int_0^a e^{y^2} \, dy$$

$$\text{since } \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

or $$\int_0^\infty (e^{-x^2 + a^2}) (\cos 2ax - i \sin 2ax) \, dx = \frac{\sqrt{\pi}}{2} - i \int_0^a e^{y^2} \, dy$$

Equating real and imaginary parts, we have

$$\int_0^\infty e^{-x^2} \cos 2ax \, dx = \frac{e^{-a^2}}{2} \sqrt{\pi}$$

and $$\int_0^\infty e^{-x^2} \sin 2ax \, dx = e^{-a^2} \int_0^a e^{y^2} \, dy$$

1997. *Find the Laurent series for the function* $e^{\frac{1}{z}}$ *in* $0 < |z| < \infty$. *Using the expansion show that*

$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

$n = 1, 2, \ldots$

Solution. Clearly $e^{\frac{1}{z}}$ is analytic in $0 < |z| < \infty$ and satisfies requirements of Laurent's expansion, and we have

$$e^{\frac{1}{z}} = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{\frac{1}{z}}}{z^{n+1}}\,dz \qquad (*)$$

~~Note~~ — $z = 0$ is an essential singularity, therefore we have infinitely many terms with negative exponents. In the expression for a_n we could have taken any disc, we have taken $|z| = 1$ for convenience.

Put $z = e^{i\theta}$ in $(*)$, $dz = ie^{i\theta}\,d\theta$, we get

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\cos\theta - i\sin\theta}}{e^{i(n+1)\theta}}\, ie^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} e^{-i\sin\theta - in\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\,d\theta \end{aligned}$$

Let $g(\theta) = e^{\cos\theta}[\sin(\sin\theta + n\theta)]$, then $g(2\pi - \theta) = -e^{\cos\theta}[\sin(\sin\theta + n\theta)] = -g(\theta)$. Thus $\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\,d\theta = 0$.

Thus $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\,d\theta$.

In particular, $a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\,d\theta$ for $n = 1, 2, ..$

But we know that $e^{\frac{1}{z}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n}$.

Therefore, comparing the two expansions we get for $n = 1, 2, .$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\,d\theta = \frac{1}{n!}$$

Since $e^{\cos 2\pi - \theta}\cos(\sin(2\pi - \theta) - n(2\pi - \theta)) = e^{\cos\theta}[\cos(\sin\theta - n\theta)]$, we can double the integral and halve the limit to obtain

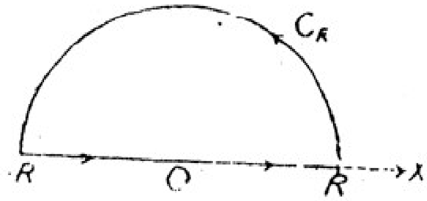
$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

1996

$\therefore \int_C \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2$ (Agra 52)

Sol. Consider the integral

$$\int_C f(z)\, dz,$$



where $f(z) = \frac{\log(z+i)}{z^2+1}$,

taken round the closed contour C consisting of real a six from $-R$ to R and the upper half of the circle $|z| = R$.

Poles of $f(z)$ are given by $1+z^2=0$ or $z=\pm i$ are simple poles, only $z=i$ lies within the contour.

The residue at $(z=i) = \underset{z\to i}{\text{Lt}} \frac{(z-i)\log(z+i)}{(z-i)(z+i)}$

$$= \frac{\log(2i)}{2i} = \frac{\log 2 + i\frac{1}{2}\pi}{2i}$$

Hence by Cauchy's theorem we have

$$\int_C f(z)\, dz = 2\pi i \times \text{sum of the residues within } C.$$

i.e. $\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \times \frac{\log 2 + i\frac{1}{2}\pi}{2i}$

or $\int_{-R}^{R} \frac{\log(x+i)}{x^2+1} dx + \int_{C_R} \frac{\log(z+i)\, dz}{z^2+1} = \pi(\log 2 + i\tfrac{1}{2}\pi)$ ...(1)

Now, $\underset{z\to\infty}{\text{Lim}} \frac{z\log(z+i)}{z^2+1}$

$$= \underset{z\to\infty}{\text{Lim}} \frac{z}{z-i} \cdot \frac{\log(z+i)}{z+i}.$$

But $\underset{z\to\infty}{\text{Lim}} \frac{z}{z-i} = 1$ and $\underset{z\to\infty}{\text{Lim}} \frac{\log(z+i)}{z+i} = 0.$

Hence $\underset{z\to\infty}{\text{Lim}} \int_{C_R} \frac{z\log(z+i)}{z^2+1} = 0$

$\therefore \underset{R\to\infty}{\text{Lim}} \int_{C_R} \frac{\log(z+i)}{z^2+1} = 0.$

Hence making $R\to\infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1} dx = \pi(\log 2 + \tfrac{1}{2}i\pi)$$

Equating real parts we have

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(x^2+1)}{x^2+1} dx = \pi \log 2.$$

or $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2.$

SEGMENTWISE UPSC PAPER-II SOLUTIONS (COMPLEX ANALYSIS)

→ Find the residues of $f(z) = e^z \operatorname{cosec} z$ at all its poles in the finite plane.

Sol. $f(z) = e^z \csc^2 z = e^z/\sin^2 z$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e. $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

Method 1. Residue at $z = m\pi$ is

$$\lim_{z\to m\pi} \frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2 \frac{e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi} \frac{e^z[(z-m\pi)^2 \sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2 \cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{\sin^3 u}\right\} = e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

$$\lim_{u\to 0}\frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$$

and thus write the limit as

$$e^{m\pi}\lim_{u\to 0}\left(\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right) = e^{m\pi}\lim_{u\to 0}\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit, we can instead use the series expansions $\sin u = u - u^3/3! + \ldots$, $\cos u = 1 - u^2/2! + \ldots$.

Method 2. (using Laurent's series).

In this method, we expand $f(z) = e^z \csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the coefficient of $1/(z - m\pi)$ as the required residue. To make the calculation easier, let $z = u + m\pi$. Then the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi+u}\csc^2(m\pi + u) = e^{m\pi}e^u \csc^2 u$. Using the Maclaurin expansions for e^u and $\sin u$, we find using long division

$$e^{m\pi}e^u\csc^2 u = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)}{\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right)^2} = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2}+\cdots\right)}{u^2\left(1-\frac{u^2}{6}+\frac{u^4}{120}-\cdots\right)^2}$$

$$= \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\cdots\right)}{u^2\left(1-\frac{u^2}{3}+\frac{2u^4}{45}+\cdots\right)} = e^{m\pi}\left(\frac{1}{u^2}+\frac{1}{u}+\frac{5}{6}+\frac{u}{3}+\cdots\right)$$

and so the residue is $e^{m\pi}$.

Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$.

IAS-1995

Solution Consider $\oint_C dz/(z^6+1)$, where C is the closed contour of Fig. 7.5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z=e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles at $1/(z^6+1)$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Then, using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}} \left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus

$$\oint_C \frac{dz}{z^6+1} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-5\pi i/2} + \frac{1}{6}e^{-25\pi i/6}\right\} = \frac{2\pi}{3}$$

That is,

$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3} \qquad (1)$$

Taking the limit of both sides of (1) as $R\to\infty$ and using Problems 7.7 and 7.8, we have

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \qquad (2)$$

Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^{\infty} \frac{dx}{x^6+1}$$

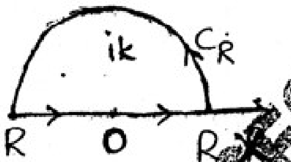
the required integral has the value $\pi/3$.

Use the method of Contour integration to prove that

1992, 1998: $\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-ma}, \quad (m \geq 0)$

Sol^n: Consider the integral $\int_C f(z)\,dz$,

where $f(z) = \frac{e^{imz}}{a^2+z^2}$ taken round



the Contour C, Consisting of the upper half of a large circle $|z| = R$ and real axis from $-R$ to R.

poles of $f(z)$ are given by $a^2+z^2 = 0$

i.e. $z = ia$, $z = -ia$ are the two simple poles, only $z = ia$ lies within the Contour.

Residue at $z = ia = \lim_{z\to ia}(z-ia)f(z)$

$$= \lim_{z\to ia}(z-ia)\frac{e^{imz}}{(z-ia)(z+ia)} = \frac{e^{-ma}}{2ia}$$

Hence by Cauchy's residue theorem, we have

$\int_C f(z)\,dz = 2\pi i \times$ Sum of the residue within the Contour.

i.e. $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \cdot \frac{e^{-ma}}{2ia} \quad \text{——— (1)}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{imx}}{a^2+x^2}\,dx + \int_{C_R}\frac{e^{imz}}{a^2+z^2}\,dz = \frac{1}{a}\pi e^{-ma}$$

Now, $\left|\int_{C_R}\frac{e^{imz}}{a^2+z^2}\,dz\right|$

$$\leq \int_{C_R} \frac{|e^{imx}||dz|}{|a^2+z^2|}$$

$$\leq \int_0^{\pi} \frac{e^{-mR\sin\theta} \cdot R\,d\theta}{(|z|^2-a^2)}$$

$$= \frac{R}{R^2-a^2}\int_0^{\pi} e^{-2mR\sin\theta}\,d\theta$$

$$\leq \frac{R}{R^2-a^2}\, 2\int^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$$

$$= \frac{\pi}{m(R^2-a^2)}\left[1-e^{-mR}\right] \text{ which } \to 0 \text{ as } R\to\infty$$

Hence by making $R\to\infty$ relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Equating real parts, we have

$$\int_0^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{a}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{2a}$$

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[IAS-1991]

→ Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0)=0$ is not analytic at $z=0$, although Cauchy-Riemann equations are satisfied at the point. How would you explain this:

Sol^n: Here $u+iv = e^{-(x+iy)^{-4}}$

$$= e^{\frac{-1}{(x+iy)^4}}$$

$$= e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left\{(x^4+y^4-6x^2y^2) - i\,4xy(x^2-y^2)\right\}}.$$

$$u+iv = e$$

$$\Rightarrow \quad u = e^{-\left(\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right)} \cos 4xy\,(x^2-y^2).$$

and $v = -e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \sin 4xy\,(x^2-y^2)$

At $z=0$, $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{e^{-x^{-4}}}{x} = 0$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0}{y} = 0.$$

Hence Cauchy-Riemann conditions are satisfied at $z=0$.

But $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \frac{e^{-z^{-4}}}{z}$

$$= \lim_{z\to 0} \frac{e^{-r^{-4}}}{re^{i\pi/4}} \quad , \text{ if } z\to 0 \text{ along } z = re^{i\pi/4}$$

IAS-1988

Show by the method of contour integration that $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$.

Sol^n: Let $I = \int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta \quad \text{—(1)}$$

write $z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta}d\theta$ & $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$= iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$\therefore$ from (1)

$$I = \frac{1}{2i}\int_C \frac{1+z+\frac{1}{z}}{5+2(z+\frac{1}{z})}\,dz.$$

where C is the unit circle $|z|=1$.

$$= \frac{1}{2i}\int_C \frac{z^2+z+1}{(2z^2+5z+2)}\,dz.$$

$$= \frac{1}{4i}\int_C \frac{z^2+z+1}{z(z+\frac{1}{2})(z+2)}\,dz = \frac{1}{4i}\int_C f(z)\,dz$$

The integrand has simple poles at $z=0$, $z=-\frac{1}{2}$, $z=-2$ of which first two lie with in C.

Residue at $z=0$ is

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} z\frac{(z^2+z+1)}{z(z+\frac{1}{2})(z+2)}$$

$$= 1.$$

Residue at $z = -\frac{1}{2}$ is

$$\lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)\left(z^2+z+1\right)}{z\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{\frac{1}{4}+\frac{1}{2}}{\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)} = -\frac{3/4}{3/4} = -1.$$

$$\therefore \frac{1}{4i}\int_C f(z)\,dz = \frac{2\pi i}{4i}\ (\text{sum of residues})$$

$$= 2\pi i \cdot \frac{1}{4i}(1-1)$$

$$= 0.$$

$$\therefore \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$$

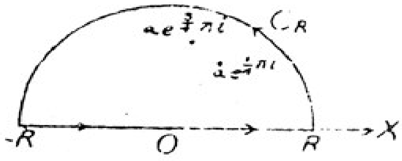
20.. '98
19..

$\int_0 \dfrac{}{x^4+a^4}$ ... (Punjab 58)

Sol. Consider the integral

$$\int_C f(z)\,dz, \text{ where } f(z)=\frac{1}{z^4+a^4},$$

taken round a closed contour C consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.



The poles of $f(z)$ are given by

$$z^4+a^4=0$$

i.e. $\quad z^4=-a^4=a^4e^{\pi i}=a^4e^{2n\pi i+\pi i}$

or $\quad z=ae^{\frac{1}{4}(2n+1)\pi i}$

$\therefore \quad z=ae^{\frac{1}{4}\pi i}$ and $z=ae^{\frac{3}{4}\pi i}$ (for $n=0$, and 1)

are the only two poles which lie within the contour.

Let α denote any one of these poles than $-a^4=\alpha^4$.

Residue of $f(z)$ at $ae^{\frac{1}{4}\pi i}$ is $\left[\dfrac{1}{\frac{d}{dz}(z^4+a^4)}\right]_{z=ae^{\frac{1}{4}\pi i}}$

$$=\frac{-1}{4a^3}e^{\frac{1}{4}i\pi}$$

Similarly residue at $ae^{3i\pi/4}=\dfrac{e^{-i\pi/4}}{4a^3}$.

$\therefore$ Sum of residues $=-\dfrac{1}{2a^3}\dfrac{e^{i\pi/4}-e^{i\pi/4}}{2}$

$$=-\frac{1}{2a^3}i\sin\frac{\pi}{4}=-\frac{i}{2\sqrt{2}a^3}$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz=2\pi i\times\text{sum of residues within } C$$

i.e. $\displaystyle\int_{-R}^{R} f(x)\,dx+\int_{C_R} f(z)\,dz=2\pi i\times\left(-\frac{i}{2\sqrt{2}a^3}\right)$

or $\displaystyle\int_{-R}^{R}\frac{dx}{x^4+a^4}+\int_{C_R}\frac{dz}{z^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}$ ...(1)

Now, $\displaystyle\left|\int_{C_R}\frac{1}{z^4+a^4}\,dz\right|$

$$\leq\int_{C_R}\frac{|dz|}{|z^4+a^4|}$$

$$\leq\int_{C_R}\frac{|dz|}{|z|^4-|a^4|}$$

$$=\int_0^{\pi}\frac{R\,d\theta}{R^4-a^4}$$

$$=\frac{\pi R}{R^4-a^4}\quad\text{which}\to 0 \text{ as } R\to\infty.$$

Hence by making $R\to\infty$, relation (1) becomes

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}$$

or $\displaystyle\int_0^{\infty}\frac{dx}{R^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}=\frac{\pi\sqrt{2}}{4a^3}$

Particularly, $\displaystyle\int_0^{\infty}\frac{dx}{x^4+1}=\frac{\pi\sqrt{2}}{4}$.

2001 → Show that $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^4}=\frac{\pi}{\sqrt{2}}$

19 Similar to 1999

$$\int_0^\infty \frac{x \sin mx}{x^4+a^4}\,dx = \frac{\pi}{2a^2}\, e^{-ma/\sqrt{2}} \cdot \sin\frac{ma}{\sqrt{2}}.$$

Sol. Consider the integral $\int_C f(z)\,dz$,

where $f(z) = \dfrac{ze^{imz}}{z^4+a^4}$,

taken round a closed contour C, consisting of the upper half of a large circle $|z| = R$, and the real axis from $-R$ to R.

Poles of $f(z)$ are given by $z^4+a^4=0$, or $z^4 = -a^4$

or $z^4 = e^{(2n\pi+\pi)i}\, a^4$

or $z = e^{1/4(2n\pi+\pi)i} \cdot a$ where $n = 0, 1, 2, 3,$

out of which $z = ae^{1/4\pi i}$ and $z = ae^{3/4\pi i}$ (for $n=0$ and 1)

are the only poles which lie within the contour.

The residue of $f(z)$ at $z = ae^{i\pi/4}$ is $\dfrac{ze^{imz}}{\left[\dfrac{d}{dz}(z^4+a^4)\right]}_{z=ae^{i\pi/4}}$

$$= \frac{e^{-ma/\sqrt{2}} \cdot e^{-ima/\sqrt{2}}}{4ia^2}$$

Similarly residue of $f(z)$ at $z = ae^{3/4i\pi}$ is

$$= \frac{e^{-ma/\sqrt{2}} \cdot e^{-ima/\sqrt{2}}}{-4ia^2}$$

So that sum of these two residues

$$= \frac{e^{-ma/\sqrt{2}}}{4ia^2}\left(e^{ima/\sqrt{2}} - e^{-ima/\sqrt{2}}\right) = \frac{e^{-ma/\sqrt{2}}}{2a^2} \sin\frac{ma}{\sqrt{2}}$$

Hence by Cauchy's residue theorem we have

$$\int_C f(z) = 2\pi i \times \text{sum of residues within the contour}$$

i.e. $\displaystyle\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times$ sum of the residues

or $\displaystyle\int_{-R}^{R} \frac{xa^{imx}}{x^4+a^4}\,dx + \int_{C_R} \frac{ze^{imz}}{z^4+a^4}\,dz$

$$= 2\pi i \times \frac{e^{-ma/\sqrt{2}}}{2a^2} \sin\frac{ma}{\sqrt{2}} \qquad \ldots(1)$$

Now, $\displaystyle\left|\int_{C_R} \frac{ze^{imz}}{z^4+a^4}\,dz\right|$

$$\leq \int_{C_R} \frac{|z|\,|e^{imz}|\,|dz|}{|z^4+a^4|} \leq \int_{C_R} \frac{|z|\,|e^{imz}|\,|dz|}{|z|^4-a^4}$$

$$= \int_0^\pi \frac{R \cdot e^{-mR\sin\theta} \cdot R\,d\theta}{R^4-a^4}$$

$$\leq \frac{2R^2}{R^4-a^4}\int_0^{\pi/2} e^{-mR\theta/\pi}\,d\theta \qquad \text{by Jordan's inequality}$$

$$= \frac{\pi R}{m(R^4-a^4)}\left[1 - e^{-2mR\theta/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi R}{m(R^4-a^4)}\left[1 - e^{-mR}\right] \text{ which} \to \text{ as } R \to \infty.$$

Hence by making $R \to \infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{xe^{imx}}{x^4+a^4}\,dx = 2\pi i \times \frac{e^{-ma/\sqrt{2}}}{2a^2} \sin\frac{ma}{\sqrt{2}}$$

Equating imaginary parts we have

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^4+a^4}\,dx = \frac{\pi}{a^2} e^{-ma/\sqrt{2}} \sin\frac{ma}{\sqrt{2}}$$

or
$$\int_0^\infty \frac{x \sin mx}{x^4+a^4}\,dx = \frac{\pi}{2a^2}\, e^{-ma/\sqrt{2}} \sin\frac{ma}{\sqrt{2}}$$

Particular Case $\displaystyle\int_0^\infty \frac{x \sin x\,dx}{x^4+a^4} = \frac{\pi}{2a^2} \cdot e^{-a/\sqrt{2}} \sin\frac{a}{\sqrt{2}}.$

1999 → Using residue theorem show that

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\, dx = \frac{\pi}{2} e^{-a} \sin a \quad (a>0).$$

Sol^n: put $a=\sqrt{2}$ in the previous problem.