6 Numerical Analysis and Computer Programming

1. Solution of Algebraic and Transcendental Equation of One Variable

1.1 Develop an algorithm for Regula-Falsi method to find a root of $f(x) = 0$ starting with two initial iterates x_0 and x_1 to the root such that sign $(f(x_0)) \neq \sin(f(x_1))$. Take n as the maximum number of iterations allowed and eps be the prescribed error.

(2009 : 30 Marks)

Solution:

1. Read x_0, x_1, eps, n

   Remarks : x_0 and x_1 are two initial guesses to the root such that $\sin(f(x_0)) \neq$ sign $(f(x_1))$. The prescribed precision is eps and n is the maximum number of iterations. Steps 2 and 3 are initialization steps.
2. $f_0 \leftarrow x\ f(x_0)$
3. $f_1 \leftarrow f(x_1)$
4. For $i = 1$ to n in steps of 1 do.
5. $x_2 \leftarrow (x_0 f_1 - x_1 f_0)/(f_1 - f_0)$
6. $f_2 \leftarrow f(x_2)$
7. if $|f_2| \leq$ eps then
8. begin write 'convergent solution', x_2, f_2
9. stop end
10. if sign $(f_2) \neq$ sign (f_0)
11. Then begin $x_1 \leftarrow x_2$
12. $f_1 \leftarrow f_2$ end
13. else begin $x_0 \leftarrow x_2$
14. $f_0 \leftarrow f_2$ end

    end for
15. Write 'Does not converge in n iterations'.
16. Write x_2, f_2
17. Stop

1.2 Find the positive root of the equation

$$10xe^{-x^2} - 1 = 0$$

correct upto 6 decimal places by using Newton-Raphson method. Carry out computations only for three iterations.

(2010 : 12 Marks)

Solution:

Given :
$$f(x) = 10xe^{-x^2} - 1$$

$\therefore$
$$f'(x) = 10e^{-x^2} + 10xe^{-x^2} \times (-2x)$$
$$= 10e^{-x^2}(1 - 2x^2)$$

Let $x_0 = 0$

Using Newton-Raphson's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - (-0.1) = 0.1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.101026$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.101026$$

∴ The positive root is 0.101026 (approx.)

1.3 Use Newton-Raphson method to find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

(2012 : 12 Marks)

Solution:

The given equation is

$$f(x) = 3x - \cos x - 1 = 0$$

$f(0.60) < 0$ and $f(0.61) > 0$

Hence, a real root of $f(x)$ lies in the interval (0.60, 0.61)

$$f'(x) = 3 + \sin x$$

From Newton-Raphson's method, we have

$$x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

Taking $x_0 = 0.60$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0}$$

$$= 0.60 - \frac{3(0.60) - \cos(0.60) - 1}{3 + \sin(0.60)}$$

$$= 0.60701$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1}$$

$$= 0.60701 - \frac{3(0.60701) - \cos(0.60701) - 1}{3 + \sin(0.60701)}$$

$$= 0.60710$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= x_2 - \frac{3x^2 - \cos x_2 - 1}{3 + \sin x_2}$$

$$= 0.60710 - \frac{3(0.60710) - \cos(0.60710) - 1}{3 + \sin(0.60710)}$$

$$= 0.60710$$

Since $x_2 = x_3$, $\therefore x = 0.60710$ is a root of $f(x)$.

1.4 Apply Newton-Raphson method to determine a root of the equation $\cos x - xe^x = 0$. Correct upto four decimal places.

(2014 : 10 Marks)

Solution:

$$f(x) = \cos x - xe^x$$

So that, $f(0) = 1$

and $f(s) = \cos 1 - e = -2.17798$

$\therefore$ $f(0)f(i) < 0$

Hence the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$

$\therefore$ $f(x, 0) = 1$ and $f(x_1) = -2.17798$

By the method of fase position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1(x_0)}{f(x_1) - f(x_0)} \quad \text{. . .(i)}$$

$$= \frac{0(-2.17798) - 1(1)}{-2.1779.8 - 1} = 0.31467$$

$\therefore$ The first approximation to the root is

$$x_2 = 0.31467$$

Now $f(x_2) = 0.51987 > 0$

$$f(x_2)f(x_1) < 0$$

$\therefore$ The root lies between 0.31467 and 1.

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore$ $f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (v), $$x_3 = \frac{(0.3146)(-2.17798) - 1(0.51987)}{-2.17798 - 0.5187} = 0.44673$$

The 2nd approximation to the root is,

$$x_3 = 0.44673$$

Now repeating this process, the successive approximations are

$$x_4 = 0.49402, x_5 = 0.50995$$

$$x_6 = 0.51520, x_7 = 0.51692, x_8 = 0.51748$$

$$x_9 = 0.51767, x_{10} = 0.51775 \text{ etc.}$$

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places.

1.5 Apply Newton-Raphson method, to find a real root of transcendental equation $x \log_{10} x = 1.2$, correct to three decimal places.

(2019 : 10 Marks)

Solution:

Here, $x \log x = 1.2$

i.e., $x \log x - 1.2 = 0$

Let $f(x) = x \log x - 1.2$

$\therefore$ $f(x) = \log x + 1$

Here,

x	1	2	3
$f(x)$	−1.2	−0.6	0.23

Here, $f(2) = -0.6 < 0$ and $f(3) = 0.23 > 0$

$\therefore$ Root lies between (2) and (3)

$$x_0 = \frac{2+3}{2} = 2 \Rightarrow x_0 = 2.5$$

1st Iteration :

$$f(x_0) = f(2.5) = (2.5)\log(2.5) - 1.2 = 0.21$$
$$f(x_0) = f'(2.5) = \log(2.5) + 1 = 1.4$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{(-0.21)}{1.4} = 2.650$$

2nd Iteration :

$$f(x_1) = f(2.65) = (2.65)\log(2.65) - 1.2 = -0.08$$
$$f'(x_1) = f'(2.65) = \log(2.65) + 1 = 1.42$$
$$x_2 = 2.65 - \frac{(-0.08)}{1.42} = 2.65 + 0.056 = 2.706$$

3rd Iteration :

$$f(x_2) = f(2.706) = 2.706 \log 2.706 - 1.2$$
$$= 1.16987 - 1.2 = -0.0301$$
$$f'(x_2) = f'(2.706) = \log(2.706) + 1 = 1.432$$
$$x_3 = 2.706 - \frac{(-0.0301)}{1.432} = 2.706 + 0.021$$
$$x_3 = 2.727$$

4th Iteration :

$$f(x_3) = f(2.727) = 2.727 \log 2.727 - 1.2$$
$$= 1.188 - 1.2 = -0.012$$
$$f'(x_3) = f'(2.727) = \log(2.727) + 1 = 1.436$$
$$x_4 = 2.727 - \frac{(-0.012)}{1.436} = 2.727 + 0.008$$
$$x_4 = 2.735$$

5th Iteration :

$$f(x_4) = f(2.735) = 2.735 \log(2.735) - 1.2$$
$$= 1.195 - 1.2 = -0.005$$
$$f'(x_4) = f'(2.735) = \log(2.735) + 1 = 1.437$$
$$x_5 = 2.735 - \frac{-0.005}{1.437} = 2.735 + 0.003$$
$$x_5 = 2.738$$

6th Iteration :

$$f(x_5) = f(2.738) = -0.003$$
$$f'(x_5) = f'(2.738) = 1.438$$
$$x_6 = 2.738 - \frac{(-0.003)}{1.438} = 2.738$$

Hence from fifth and sixth iterations

$$x_5 = 2.738 \text{ and } x_6 = 2.738$$

So, 2.738 is the real root of the given equation corrected upto 3 decimal.

2. Solutions of System of Linear Equations

2.1 The equation $x_2 + ax + b = 0$ has two real roots α and β. Show that the iterative method given by :

$x_{k+1} = -\dfrac{(ax_k + b)}{x_k}$, $k = 0, 1, 2$ ..... is convergent near $x = \alpha$, if $|\alpha| > |\beta|$.

(2009 : 6 Marks)

Solution:

Let the iterations are given by

$$x_{k+1} = -\frac{(ax_k + b)}{x_k} = g(x_k) \text{ (say)} \qquad \text{...(i) } (k = 0, 1, 2, ...)$$

Now by the known theorem, if $g(x)$ and $g'(x)$ are continuous in an interval about a root a of the equation $x = g(x)$ and if $|g'(x)| < 1$ for all x in the internal, then the successive approximations $x_1, x_2,$ ..... are given by $x_k = g(x_{k-1})$ $(k = 1, 2, ...)$ converges to the root α provided that the initial approximation x_0 is choosen in the interval.

$\therefore$ The iterations converges to α if

$$|g'(x)| = \left|\frac{-b}{x_{k^2}}\right| < 1 \qquad \text{(from (i))}$$

Note that $g'(x)$ is continuous near 'α' if the iteration converges to $x = \alpha$.

So, we require

$$|g'(\alpha)| = \left|\frac{b}{\alpha^2}\right| < 1$$

or $$|b| < |\alpha|^2 \qquad \text{...(ii)}$$

Now, given that α and β are roots of the equation

$$x^2 + ax + b = 0$$

So, $$\alpha + \beta = -a$$
$$\alpha\beta = b \qquad \text{...(iii)}$$

From (ii) and (iii) $$|\alpha\beta| < |\alpha|^2$$
$$|\alpha|(|\alpha| - |\beta|) > 0$$
$$|\alpha| > 0$$

So, $$|\alpha| > |\beta|. \text{ Hence Proved.}$$

2.2 Given the system of equations

$$2x + 3y = 1$$
$$2x + 4y + z = 2$$
$$2y + 6z + Aw = 4$$
$$Az + Bw = C$$

State the solubility Δ uniqueness conditions for the system. Give the solution when it exists.

(2010 : 20 Marks)

Solution:

From numerical analysis and computer programming.

The given system of equation can be written as

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 2 & 4 & 1 & 0 & 2 \\ 0 & 2 & 6 & A & 4 \\ 0 & 0 & 4 & B & C \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$, we get

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 6 & A & 4 \\ 0 & 0 & 4 & B & C \end{array}\right]$$

$$\downarrow R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & A & 2 \\ 0 & 0 & 4 & B & C \end{array}\right]$$

$$\downarrow R_4 \to R_4 - R_3$$

$$\left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & A & 2 \\ 0 & 0 & 0 & B-A & C-2 \end{array}\right]$$

Unique solution if

$$B - A = 0 \text{ and } C - 2 \neq 0$$

or

$$B \neq A \text{ and } C \neq 2$$

Infinite solution if

$$B - A = 0 \quad \text{or} \quad B = A$$

and

$$C - 2 = 0 \quad \text{or} \quad C = 2$$

No solution if

$$B = A \text{ and } C \neq 2 \quad \text{or} \quad B \neq A \text{ and } C = 2$$

Unique solution:

$$(B-A)w = C-2$$

$$\Rightarrow \quad w = \frac{C-2}{B-A}$$

$$4z + Aw = 2$$

$$\Rightarrow \quad z = \frac{2 - \dfrac{A(C-2)}{B-A}}{4} = \frac{2B - AC}{4(B-A)}$$

$$y + z = 1 \;\Rightarrow\; y = 1 - z = 1 - \frac{2B - AC}{4(B-A)} = \frac{2B - 4A + AC}{4(B-A)}$$

$$2x + 3y = 1 \;\Rightarrow\; x = \frac{1}{2}(1 - 3y)$$

$$\Rightarrow \quad x = \frac{1}{2}\left\{1 - 3 \times \frac{(2B - 4A + AC)}{4(B-A)}\right\} = \frac{1}{2}\frac{(8A - 2B - 3AC)}{4(B-A)}$$

$$\therefore \quad x = \frac{1}{2}\frac{(8A - 2B - 3AC)}{4(B-A)}, \; y = \frac{2B - 4A + AC}{4(B-A)}$$

$$z = \frac{2B - AC}{4(B-A)}, \; w = \frac{C-2}{B-A}$$

Many solutions:

$$B = A,\ C = 2$$

$$4z + Aw = 2 \Rightarrow z = \frac{2 - Aw}{4} \qquad (A \in R)$$

$$y = 1 - z = 1 - \left(\frac{2 - Aw}{4}\right) \Rightarrow y = \frac{2 + Aw}{4}$$

$$2x = 1 - 3y \Rightarrow x = \left(1 - \frac{6 + 3Aw}{4}\right) \Rightarrow x = \frac{1}{2}\frac{(-2 - 3Aw)}{4}$$

2.3 Solve the following system of simultaneous equations, using Gauss-Seidel iterative method :

$$3x + 20y - z = -18$$
$$20x + y - 2z = 17$$
$$2x - 3y + 20z = 25$$

(2012 : 20 Marks)

Solution:

Solving each equation for the unknown having the largest coefficient, the new equations are :

$$x = \frac{1}{20}(17 - y + 2z) \qquad \text{...(i)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad \text{...(ii)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad \text{...(iii)}$$

Choosing $y = 0$, $z = 0$ to start with, we get,

$$x^{(1)} = \frac{1}{20}(17 - 0 + 0) = 0.85 \text{ (first approximation)}$$

Putting $x = x^{(1)} = 0.85$, $z = 0$ in (ii)

$$y^{(1)} = \frac{1}{20}(-18 - 2.55) = -1.0275$$

Putting $x = x^{(1)} = 0.85$, $y = y^{(1)} = -1.0275$ in (iii), we get

$$z^{(1)} = \frac{1}{20}(25 - 1.7 - 3.0825) = 1.0109$$

$$x^{(2)} = \frac{1}{20}(17 + 1.0275 + 2.0218) = 1.0025$$

$$y^{(2)} = \frac{1}{20}(-18 - 3.0075 + 1.0109) = -0.9998$$

$$z^{(2)} = \frac{1}{20}(25 - 2.0050 - 2.9994) = 0.9998$$

$$x^{(3)} = \frac{1}{20}(17 + 0.9998 + 1.9996) = 0.99997$$

$$y^{(3)} = \frac{1}{20}(-18 - 2.99991 + 0.9998) = -1.0000$$

$$z^{(3)} = \frac{1}{20}(25 - 1.99994 - 3.0000) = 1.0000$$

$$x^{(4)} = \frac{1}{20}(17 + 1.0000 + 2.0000) = 1.0000$$

$$y^{(4)} = \frac{1}{20}(-18-3.0000+1.0000) = -1.0000$$

$$z^{(4)} = \frac{1}{20}(25-2.0000-3.0000) = 1.0000$$

Since $x^{(4)}$, $y^{(4)}$, $z^{(4)}$ are sufficiently close to $x^{(3)}$, $y^{(3)}$, $z^{(3)}$ respectively, so that values 1.0000, −1.0000, 1.0000 can be taken as the solution of the given system.

2.4 Solve the system of equations :

$$2x_1 - x_2 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$-x_2 + 2x_3 = 1$$

Using Gauss-Seidel iteration method (Perform three iterations).

(2014 : 15 Marks)

Solution:

The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(7+x_2) \\ x_2 &= \frac{1}{2}(1+x_1+x_3) \\ x_3 &= \frac{1}{2}(1+x_2) \end{aligned}\right\} \quad \text{...(i)}$$

By the Gauss-Seidel method, system (i) can be written as

$$x_1^{k+1} = \frac{1}{2}\left(7+x_2^{(k)}\right)$$

$$x_2^{k+1} = \frac{1}{2}\left(1+x_2^{(k+1)}+x_3^{k}\right)$$

$$x_3^{k+1} = \frac{1}{2}\left(1+x_2^{(k+1)}\right)$$

where, $k = 0, 1, 2, 3 \ldots$

Now taking, $x(0) = 0$, we obtain the following iterations

$k = 0$:

$$x_1^{(1)} = \frac{1}{2}(7+0) = \frac{7}{2} = 3.5$$

$$x_2^{(1)} = \frac{1}{2}(1+3.5+0) = \frac{4.5}{2} = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1+2.25) = \frac{1}{2}(3.25) = 1.625$$

$k = 1$:

$$x_1^{(2)} = \frac{1}{2}\left(7+x_2^{(1)}\right) = \frac{1}{2}(7+2.25) = \frac{9.25}{2} = 4.625$$

$$x_2^{(2)} = \frac{1}{2}\left(1+x_1^{(2)}+x_3^{(1)}\right) = \frac{1}{2}(1+4.625+1.625) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1+x_2^{(2)}\right) = \frac{1}{2}\left(1+4.625+1.625\right) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1+x_2^{(2)}\right) = \frac{1}{2}(1+3.625) = 2.3125$$

$k = 2$:

$$x_1^{(3)} = \frac{1}{2}(7+x_2^{(2)}) = \frac{1}{2}(7+3.625) = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}(1+x_1^{(4)}+x_3^{(2)}) = \frac{1}{2}(1+5.3125+2.3125) = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}\left(1+x_2^{(3)}\right) = \frac{1}{2}(1+4.3125) = 2.6563$$

$k = 3$:

$$x_1^{(4)} = \frac{1}{2}\left(7+x_2^{(3)}\right) = \frac{1}{2}(7+4.3125) = 5.6563$$

$$x_2^{(4)} = \frac{1}{2}\left(1+x_1^{(4)}+x_3^{(3)}\right) = \frac{1}{2}(5+5.6563+2.6565) = 5.6563$$

$$x_3^{(4)} = \frac{1}{2}\left(1+x_2^{(4)}\right) = \frac{1}{2}(1+4.6563) = 2.8282$$

$k = 4$:

$$x_1^{(5)} = \frac{1}{2}\left(7+x_2^{(4)}\right) = \frac{1}{2}(7+4.6563) = 5.8282$$

$$x_2^{(5)} = \frac{1}{2}\left(1+x_1^{(5)}+x_3^{(4)}\right) = \frac{1}{2}(4+5.8282+2.8282) = 4.8282$$

$$x_3^{(5)} = \frac{1}{2}\left(1+x_1^{(5)}\right) = \frac{1}{2}(1+4.8282) = 2.9141$$

which is the good approximation to the exact solution,

$$x = (6\,5\,3)^T.$$

2.5 Find the solution of the system

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

Using Gauss-Seidel method (make four iterations).

(2015 : 15 Marks)

Solution:

We firstly form Gauss Siedel equations from given set of equations.

Given equations are :

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

$$x_1^i = \frac{3+2x_2^{i-1}+x_3^{i-1}+x_4^{i-1}}{10} \quad \text{...(i)}$$

$$x_2^i = \frac{15+2x_2^{i}+x_3^{i-1}+x_4^{i-1}}{10} \quad \text{...(ii)}$$

$$x_3^i = \frac{27+x_1^{i}+x_2^{i}+2x_4^{i-1}}{10} \quad \text{...(iii)}$$

$$x_4^i = \frac{-9+x_1^{i-1}+x_2^{i-1}+2x_3^{i-1}}{10} \quad \text{...(iv)}$$

Let

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From (i), (ii), (iii) and (iv), we get

$$\begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 1.56 \\ 2.886 \\ -0.1368 \end{bmatrix} \cdot \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} = \begin{bmatrix} 0.88692 \\ 1.83492 \\ 2.9448 \\ -0.038856 \end{bmatrix}$$

$$\begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \\ x_4^3 \end{bmatrix} = \begin{bmatrix} 0.95758 \\ 1.9821 \\ 2.3862 \\ 8.792 \times 10^{-3} \end{bmatrix}$$

$$\begin{bmatrix} x_1^4 \\ x_2^4 \\ x_3^4 \\ x_4^4 \end{bmatrix} = \begin{bmatrix} 0.9959 \\ 1.99868 \\ 3.0012 \\ -3.02 \times 10^{-4} \end{bmatrix}$$

$\therefore$ the solution is

$$x_1 = 0.9959 \text{ (approx.)}$$
$$x_2 = 1.9987 \text{ (approx.)}$$
$$x_3 = 3.001 \text{ (approx.)}$$
$$x_4 = -3.02 \times 10^{-4} \text{ (approx.)}$$

2.6 Explain the main steps of the Gauss-Jordan method and apply this method to find the inverse of the matrix $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$.

(2017 : 10 Marks)

Solution:

We write, $A = I_n A$

Using elementary row operations on matrices A (on LHS) and I_n, the matrix A is transformed to Identity matrix and in the process I_n gets reduced to A^{-1}.

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 6 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_2 - 3R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

i.e.,
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$\therefore$
$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

2.7 Apply Gauss-Seidel iteration method to solve the following system of equations :

$$2x + y - 2z = 17,$$
$$3x + 20y - z = -18,$$
$$2x - 3y + 20z = 25 \text{ correct to three decimal places.}$$

(2019 : 15 Marks)

Solution:

Given equations are :

$$2x + y - 2z = 17,$$
$$3x + 20y - z = -18,$$
$$2x - 3y + 20z = 25$$

which can be rewritten as :

$$x = \frac{1}{2}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

$$(x_0, y_0, z_0) = (0, 0, 0)$$

1st Iteration :

$$x_1 = \frac{1}{2}[17 - (0) + 2(0)] = \frac{1}{2} \times 17 = 8.5$$

$$y_1 = \frac{1}{20}[-18 - 3(8.5) + 0] = \frac{1}{20}[-43.5] = -2.175$$

$$z_1 = \frac{1}{20}[25 - 2(8.5) + 3(-2.175)] = \frac{1}{20}[1.475]$$

$$z_1 = 0.07375$$

$$(x_1, y_1, z_1) = (8.5, -2.175, 0.07375)$$

or $$(x_2, y_2, z_2) = (8.5, -2.175, 0.074)$$

2nd Iteration :

$$x_2 = \frac{1}{2}[17 - (-2.175) + 2(0.0734)] = \frac{1}{2}[19.323] = 9.661$$

$$y_2 = \frac{1}{20}[-18 - 3(9.661) + (0.0734)] = \frac{1}{20}[-46.91] = -2.345$$

$$z_2 = [25 - 2(9.661) + 3(-2.345)] = \frac{1}{20}[-1.359] = -0.068$$

$$(x_2, y_2, z_2) = (9.661, -2.345, -0.068)$$

3rd Iteration :

$$x_3 = \frac{1}{2}[17-(-2.345)+2(-0.068)] = \frac{1}{2}[19.345-0.136] = 9.6045$$

$$y_3 = \frac{1}{20}[-18-3(9.6045)+(-0.068)] = \frac{1}{20}[-46.8135-0.068]$$

$$= \frac{1}{20}[46.8815] = -2.344$$

$$z_3 = \frac{1}{20}[25-2(9.6045)+3(-2.344)]$$

$$= \frac{1}{20}[25-19.209-7.032] = \frac{1}{20}[25-26.241]$$

$$z_3 = -0.062$$

$$(x_3, y_3, z_3) = (9.6045, -2.344, -0.062)$$

4th Iteration :

$$x_4 = \frac{1}{2}[17-(-2.344)+2(-0.062)] = \frac{1}{2}[19.344-0.124]$$

$$x_4 = \frac{19.22}{2} = 9.609$$

$$y_4 = \frac{1}{20}[-18-3(9.609)-0.062] = \frac{-1}{20}[46.891]$$

$$y_4 = -2.344$$

$$z_4 = \frac{1}{20}[25-2\times 9.609+2\times(-2.344)] = \frac{1}{20}[-1.253]$$

$$z_4 = -0.063$$

5th Iteration :

$$(x_4, y_4, z_4) = (9.609, -2.344, -0.063)$$

$$x_5 = \frac{1}{2}[17-(-2.344)+2(-0.063)] = 9.609$$

$$y_5 = \frac{1}{20}[-18-3(4.609)+(0.063)] = -2.344$$

$$z_5 = \frac{1}{20}[25-2\times 9.609+3(-2.344)] = -0.063$$

∴ Solution by Gauss-Seidel method

$$x = 9.609,\ y = -2.344,\ z = -0.063 \text{ (Required Solution)}$$

3. Interpolation

3.1 Using Lagrange interpolation formula, calculate the value of $f(3)$ from the following table of values of x and $f(x)$:

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

(2009 : 15 Marks)

Solution:

Usually Lagrange interpolation formula, first we will form the equation of $f(x)$.

Now, as per the formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)f(x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)f(x_5)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}$$

Now, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 5$, $x_5 = 6$ and $f(x_0) = 1$, $f(x_1) = 14$, $f(x_2) = 15$, $f(x_3) = 5$, $f(x_4) = 6$, $f(x_5) = 19$.

So,

$$f(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)\times 1}{(0-1)(0-2)(0-4)(0-5)(0-6)}$$
$$+\frac{(x-0)(x-2)(x-4)(x-5)(x-6)\times 14}{(1-0)(1-2)(1-4)(1-5)(1-6)}$$
$$+\frac{(x-0)(x-1)(x-4)(x-5)(x-6)\times 15}{(2-0)(2-1)(2-4)(2-5)(2-6)}$$
$$+\frac{(x-0)(x-1)(x-2)(x-5)(x-6)\times 5}{(4-0)(4-1)(4-2)(4-5)(4-6)}$$
$$+\frac{(x-0)(x-1)(x-2)(x-4)(x-6)\times 6}{(5-0)(5-1)(5-2)(5-4)(5-6)}$$
$$+\frac{(x-0)(x-1)(x-2)(x-4)(x-5)\times 19}{(6-0)(6-1)(6-2)(6-4)(6-5)}$$

Now,

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{-1\times -2\times -4\times -5\times -6}$$
$$+\frac{(3-0)(3-2)(3-4)(3-5)(3-6)\times 14}{1\times(-1)\times(-3)(-4)(-5)}$$
$$+\frac{(3-0)(3-1)(3-4)(3-5)(3-6)\times 15}{2\times 1\times(-2)(-3)(-4)}$$
$$+\frac{(3-0)(3-1)(3-2)(3-5)(3-6)\times 5}{4\times 3\times 2\times(-1)(-2)}$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-6)\times 6}{5\times 4\times 3\times 1\times(-1)}$$
$$+\frac{(3-0)(3-1)(3-2)(3-4)(3-5)\times 19}{6\times 5\times 4\times 2\times 1}$$

$$f(3) = \frac{2\times 1\times(+1)\times(-2)\times(-3)}{2\times 4\times 5\times(+6)} + \frac{3\times 1\times(-1)\times(+2)\times(-3)\times 14}{1\times 3\times 4\times 5}$$

$$+\frac{3\times2\times(-1)(-2)(-3)\times15}{2\times2\times3\times(-4)}+\frac{3\times2\times1\times(-2)(-3)\times5}{4\times3\times2\times2}$$

$$+\frac{3\times2\times1\times(-1)(-3)\times6}{5\times4\times3\times(-1)}+\frac{3\times2\times1\times(-1)\times(-2)\times19}{6\times5\times4\times2}$$

$$=\frac{1}{20}-\frac{21}{5}+\frac{45}{4}+\frac{15}{4}-\frac{9}{5}+\frac{19}{20}$$

$$=\frac{1-84+180+75-36+19}{20}$$

$$=\frac{275-120}{20}=\frac{155}{20}$$

$$=\frac{31}{4}$$

So, $$f=\frac{31}{4}$$

3.2 Find the value of $y(1.2)$ using Runge-Kutta fourth order method with step size $h = 0.2$ from the initial value problem.

$$y' = xy$$
$$y(1) = 2$$

(2009 : 15 Marks)

Solution:

As per Runge-Kutta forth order method,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

and next y will be

$$y_1 = y_0 + k$$

and $$f(x, y) = y'$$

Here $$f(x, y) = xy$$

Now, as per the given problem

$$y_0 = y(1) = 2$$
$$y' = f(x, y) = xy$$
$$h = 0.2$$

Now, $$k_1 = hf(x_0, y_0)$$
$$= 0.2f(1, 2)$$
$$= 0.2\times1\times2 = 0.4$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= 0.2f\left(1+\frac{0.2}{2}, 2+\frac{0.4}{2}\right)$$

$$= 0.2f(1.1, 2.2)$$
$$= 0.2 \times 1.1 \times 2.2 = 0.484$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.2f\left(1 + \frac{0.2}{2}, 2 + \frac{0.484}{2}\right)$$
$$= 0.2f(1.1, 2.242)$$
$$= 0.2 \times 1.1 \times 2.242$$
$$= 0.49324$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.2f(1 + 0.2, 2 + 0.49324)$$
$$= 0.2f(1.2, 2.49324)$$
$$= 0.2 \times 1.2 \times 2.49324$$
$$= 0.598$$

So,
$$k_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.4 + 2(0.484) + 2(0.49324) + 0.598)$$
$$= \frac{1}{6}(0.4 + 0.968 + 0.98648 + 0.598)$$
$$= \frac{2.95248}{6} = 0.49208$$

$\Rightarrow$
$$f(1.2) = 2 + 0.49208$$

3.3 Find $\frac{dy}{dx}$ at $x = 0.1$ from the following data :

x :	0.1	0.2	0.3	0.4
y :	0.9975	0.9900	0.9776	0.9604

(2012 : 20 Marks)

Solution:

We will use Newton Forward interpolation formula.

The table of finite differences is given by :

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.1	0.9975			
		–0.0075		
0.2	0.9900		–0.0049	
		–0.0124		0.0001
0.3	0.9776		–0.0048	
		–0.0172		
0.4	0.9604			

By Newton Forward interpolation formula, we have

$$f(a+mh) = f(a)+msf(a)+\frac{m(m-1)}{2!}\Delta^2 f(a)+\frac{m(m-1)(m-2)}{3!}s^3 f(a)+..... \quad ...(i)$$

Here, $a = 0.1$, $h = 0.1$

$$\therefore \quad m = \frac{x-a}{h} = \frac{x-0.1}{0.1}$$

$\therefore$ from (i),

$$f(x) = 0.9975+\frac{(x-0.1)}{0.1}\cdot(-0.0075)+\frac{(x-0.1)}{0.1}\cdot\left(\frac{x-0.1}{0.1}-1\right)\cdot\frac{1}{2}\cdot(-0.0049)$$
$$+\left(\frac{x-0.1}{0.1}\right)\left(\frac{x-0.1}{0.1}-1\right)\left(\frac{x-0.1}{0.1}-2\right)\cdot\frac{1}{6}(0.0001)$$

$$\Rightarrow \quad y = f(x) = 0.9975+\frac{(x-0.1)}{0.1}\cdot(-0.0075)+\frac{(x-0.1)}{0.1}\left(\frac{x-0.2}{0.1}\right)\cdot\frac{1}{2}(-0.0049)$$
$$+\frac{(x-0.1)}{0.1}\cdot\frac{(x-0.2)}{0.1}\cdot\frac{(x-0.3)}{0.1}\cdot\frac{1}{6}(0.0001)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{0.1}(-0.0075)+\frac{1}{0.1}\left(\frac{x-0.2}{0.1}\right)\cdot\frac{1}{2}(-0.0049)+\frac{(x-0.1)}{0.1}\cdot\frac{1}{0.1}\cdot\frac{1}{2}(-0.0049)$$
$$+\frac{(3x^2-1.2x-0.07)}{6\times(0.1)^3}\cdot(0.0001)$$

$$\therefore \quad \frac{dy}{dx}(\text{at } x=0.1) = -\frac{0.0075}{0.1}+\frac{(-0.1)}{(0.1)^2}\cdot\frac{1}{2}(-0.0049)+\frac{(3(0.1)^2-1.2\times 0.1-0.07)}{6\times(0.1)^3}(0.0001)$$
$$= -0.0532$$

3.4 In an examination, the number of students who obtained marks between certain limits are given in the following table :

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Using Newton's forward interpolation formula, find the number of students whose marks be between 45 and 50.

(2013 : 10 Marks)

Solution:

We define the function $f(x)$ as

$$f(x) = \text{Number of students who scored marks less than or equal to } x$$

The table for $f(x)$ is

x_i	30	40	50	60	70	80
$f(x_i)$	0	31	73	124	159	190

We use Newton's forward interpolation to evaluate $f(45)$.

Here, $x_0 = 30$, $h = 10$, $x = 45$

$$s = \frac{x-x_0}{h} = 1.5$$

We construct the table for Newton's forward differences.

$$\begin{array}{ccccccccccccc}
x_i & 30 & & 40 & & 50 & & 60 & & 70 & & 80 \\
f(x_i) & 0 & & 31 & & 73 & & 124 & & 159 & & 190 \\
\Delta f & & 31 & & 42 & & 51 & & 35 & & 31 & \\
\Delta^2 f & & & 11 & & 9 & & -16 & & -4 & & \\
\Delta^3 f & & & & -2 & & -25 & & 12 & & & \\
\Delta^4 f & & & & & -23 & & 37 & & & & \\
\Delta^5 f & & & & & & 60 & & & & &
\end{array}$$

Newton's interpolating polynomial

$$P_n(s) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f_0$$

$$P_0(1.5) = \binom{1.5}{0} f_0 = 0$$

$$P_1(1.5) = P_0(1.5) + \binom{1.5}{1} \Delta f_0 = 31 \times 1.5 = 46.5$$

$$P_2(1.5) = P_1(1.5) + \binom{1.5}{2} \Delta^2 f_0$$

$$= 46.5 + \frac{1.5 \times (1.5-1)}{2} \times 11 = 50.625$$

$$P_3(1.5) = P_2(1.5) + \binom{1.5}{3} \Delta^3 f_0 = 50.750$$

$$P_4(1.5) = P_3(1.5) + \binom{1.5}{4} \Delta^4 f_0$$

$$= 50.750 + \frac{1.5 \times (1.5-1) \times (1.5-2)(1.5-3)}{4!} \times -23$$

$$= 50.2109$$

$$P_5(1.5) = P_4(1.5) + \binom{1.5}{5} \Delta^5 f_0$$

$$= 49.5078$$

$\therefore$ $f(4.5) = 49.5078$ correct upto 4 places.

$\therefore$ Students with marks between 45 and 50

$$= f(50) - f(45) = 23.4922$$

3.5 Find a Lagrange interpolating polynomial that fits the following data :

x :	-1	2	3	4
$f(x)$:	-1	11	31	69

(2015 : 20 Marks)

Solution:

Given data is

$$\begin{array}{ccccc}
x: & \underline{x_0} & \underline{x_1} & \underline{x_2} & \underline{x_4} \\
 & -1 & 2 & 3 & 4 \\
f(x): & -1 & 11 & 31 & 69
\end{array}$$

Lagrange polynomial,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$\times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_4)(x_3-x_2)} \times f(x_3)$$

Using given data, we get

$$f(x) = \frac{(x-2)(x-3)(x-4)}{(-1-2)(-1-3)(-1-4)} \times (-1) + \frac{(x+2)(x-3)(x-4)}{(2+1)(2-3)(2-4)} \times 11$$

$$+ \frac{(x+1)(x-2)(x-4)}{(3+1)(3-2)(3-4)} \times 31 + \frac{(x+1)(x-2)(x-3)}{(4+1)(4-2)(4-3)} \times 69$$

$\Rightarrow$

$$f(x) = \frac{(x^2-5x+6)(x-4)}{-60} \times 1 + \frac{(x^2-2x-3)(x-4)}{6} \times 11$$

$$+ \frac{(x^2-x-2)(x-4)}{-4} \times 31 + \frac{(x^2-x-2)(x-3)}{10} \times 69$$

$$= \frac{x^3-9x^2+26x-24}{60} + \frac{x^3-6x^2+5x+12}{6} \times 11$$

$$+ \frac{x^3-5x^2+2x+8}{-4} \times 31 + \frac{x^3-4x^2+x+6}{10} \times 69$$

$$= \frac{60x^3+0x^2+60x+60}{60} = x^3+x+1$$

$\therefore$ $f(x) = x^3 + x + 1$

So, $f(1.5) = (1.5)^3 + 1.5 + 1$

$= 5.875$

3.6 Let $f(x) = e^{2x} \cos 3x$ for $x \in [0, 1]$. Estimate the value of $f(0.5)$ using Lagrange interpolating polynomial of degree 3 over the modes $x = 0$, $x = 0.3$, $x = 0.6$ and $x = 1$. Also compute the error bound over the interval [0, 1] and actual error $E(0.5)$.

(2016 : 20 Marks)

Solution:

Given: $f(x) = e^{2x} \cos 3x$

At $x = 0$, $f(0) = 1$

At $x = 0.3$, $f(0.3) = 1.1326$

At $x = 0.6$, $f(0.6) = -0.754$

At $x = 1$, $f(1) = -7.315$

$$f(x) = \frac{(x-0.3)(x-0.6)(x-1)}{(0-0.3)(0-0.6)(0-1)} \times f(0) + \frac{(x-0)(x-0.6)(x-1)f(0.3)}{(0.3-0)(0.3-0.6)(0.3-1)}$$

$$+ \frac{(x-0)(x-0.3)(x-1)}{(0.6-0)(0.6-0.3)(0.6-1)} \times f(0.6)$$

$$+ \frac{(x-0)(x-0.3)(x-0.6) \times f(1)}{(1-0)(1-0.3)(1-0.6)}$$

(Using Lagrange's interpolation formula)

which is a polynomial of degree 3.

$$f(0.5) = \frac{(0.5-0.3)(0.5-0.6)(0.5-1)}{(0-0.3)(0-0.6)(0-1)} \times 1 + \frac{(0.5-0)(0.5-0.6)(0.5-1)}{(0.3-0)(0.3-0.6)(0.3-1)} \times f(0.3) + \frac{(0.5-0)(0.5-0.3)(0.5-1)}{(0.6-0)(0.6-0.3)(0.6-1)} \times f(0.6) + \frac{(0.5-0)(0.5-0.3)(0.5-0.6)}{(1-0)(1-0.3)(1-0.6)} \times f(1)$$

$$\Rightarrow \quad f(0.5) = \frac{0.2\times-0.1\times-0.5}{-0.3\times-0.6\times-1} \times 1 + \frac{0.5\times-0.1\times-0.5}{0.3\times-0.3\times-0.7} \times 1.1326 + \frac{0.3\times0.2\times-0.5}{0.6\times0.3\times-0.4} \times -0.754 + \frac{0.5\times0.2\times-0.1}{1\times0.7\times0.4} \times -7.315$$

$$\Rightarrow \quad f(0.5) = 0.1315277 \text{ (approx.)}$$

Actual value of $f(0.5) = e^{2\times0.5} \times \cos(3 \times 0.5)$

$$= 0.19228$$

$$\therefore \quad \text{Actual error} = 10.19228 - 0.13152771 = 0.06081$$

Error bound over the interval [0, 1] or trunctation error ($\in$).

$$\in_x = \left|\frac{(x-0)(x-0.3)(x-0.6)(x-1)}{4!} f^{IV}(\xi)\right|$$

$$\in_{0.5} = \left|\frac{(0.5-0)(0.5-0.3)(0.5-0.6)(0.5-1)}{24} f^{IV}(\xi)\right|$$

Now,

$$f^{I}(x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$$

$$f^{II}(x) = 4e^{2x}\cos 3x - 6e^{2x}\sin 3x - 6e^{2x}\sin 3x - 3e^{2x}\cos 3x$$

$$\Rightarrow \quad f^{II}(x) = -5e^{2x}\cos 3x - 12e^{2x}\sin 3x$$

$$f^{III}(x) = -10e^{2x}\cos 3x + 15e^{2x}\sin 3x - 24e^{2x}\sin 3x - 36e^{2x}\cos 3x$$

$$\Rightarrow \quad f^{III}(x) = -46e^{2x}\cos 3x - 9e^{2x}\sin 3x$$

$$f^{IV}(x) = -92e^{2x}\cos 3x + 138e^{2x}\sin 3x - 18e^{2x}\sin 3x - 27e^{2x}\cos 3x$$

$$\Rightarrow \quad f^{IV}(x) = 120e^{2x}\sin 3x - 119e^{2x}\cos 3x$$

$f^{IV}(x)$ is maximum at $x = 0.98. \in [0, 1]$

$$\therefore \quad \in_{0.5} \le \left|\frac{0.5\times0.2\times-0.1\times-0.5\times998.25}{24}\right|$$

$$\Rightarrow \quad \in_{0.5} \le 0.208$$

$\therefore$ Error bound over the interval is 0.208 (approx.)

3.7 For given equidistance values u_{-1}, u_0, u_1 and u_2, a value is interpolated by Lagrange's formula. Show that it may be written in the form $u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$ where $x + y = 1$.

(2017 : 15 Marks)

Solution:

$$\Delta^2 u_1 = (E-1)^2 u_{-1} = (E^2 - 2E + 1)u_{-1} = u_1 - 2u + u_{-1}$$

$$\Delta^2 u_0 = (E^2 - 2E + 1)u_0 = u_2 - 2u_1 + u_0$$

$$\text{R.H.S.} = (1-x)u_0 + xu_1 + \frac{1}{3!}(1-x)[(1-xx)^2 - 1](u_1 - 2u_0 + u_{-1}) + \frac{1}{3!}x(x^2-1)(u_2 - 2u_1 + u_0)$$

$$= \frac{-x(x-1)(x-2)}{6}u_{-1} + \frac{(x-2)(x-1)(x+1)}{2}u_0$$

$$= \frac{-x(x-1)(x-2)}{2}u_1 + \frac{(x+1)(x)(x-1)}{6}u_2 \quad ...(i)$$

Applying Lagrange's formula for arguments -1, 0, 1, 2

$$u_x = \frac{x(x-1)(x-2)}{(-1)(-2)(-3)}u_{-1} + \frac{(x+1)(x-1)(x-2)}{2}u_0 + \frac{x(x+1)(x)(x-2)}{-2}u_1 + \frac{(x+1)(x)(x-1)}{6}u_2$$

$$= \frac{-x(x-1)(x-2)}{6}u_{-1} + \frac{(x-2(x-1)(x+1)}{2}u_0 + \frac{(x+1)x(x-2)}{2}u_1 + \frac{(x+1)(x)(x-1)}{6}u_2 \quad ...(ii)$$

From (i) and (ii),

L.H.S. = R.H.S. Hence Proved

3.8 Using Newton's forward difference formula find lowest degree polynomial ux when it is given that $u_1 = 1$, $u_2 = 9$, $u_3 = 25$, $u_4 = 55$ and $u_5 = 105$.

(2018 : 10 Marks)

Solution:

Newton's forward difference table is computer program.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
1	1			
2	9	8		
3	25	16	8	
4	55	30	14	6
5	105	50	20	6

$$y = \frac{x - x_0}{n} = \frac{x-1}{1} = x - 1$$

$$f(x) = u_0 + y\Delta u_0 + \frac{y(y-1)}{2!}\Delta^2 y_0 + \frac{y(y-1)(y-2)}{3!}\Delta^3 y_0 + ...$$

$$= 1 + (x-1)8 + \frac{(x-1)(x-2)}{2}\times 8 + \frac{(x-1)(x-2)(x-3)}{6}\times 6$$

$$= 1 + 8(x-1) + 7(x-1)(x-2) + (x-1)(x-2)(x-3)$$

$$= x^3 - 2x^2 + 7x - 5$$

4. Numerical Integration

4.1 Find the value of the integral

$$\int_1^5 \log_{10} x\, dx$$

by using Simpson's $\frac{1}{3}$-rule correct up to 4 decimal places. Take 8 sub-intervals in your computation.

(2010 : 20 Marks)

Solution:

Analysis and computer programming:

Given, Integral $= \int_1^5 \log_{10} x\, dx$

$$v = \log_{10} x$$

x	1	1.5	2	2.5	3	3.5	4	4.5	5
v	0	0.1761	0.3010	0.3979	0.4771	0.5441	0.6021	0.6532	0.699
	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8

$n = 0.5$ = length of sub interval

By Simpson's 1/3-rule,

$$\int_1^5 \log_{10} x\,dx = \frac{n}{3}[(v_0+v_8)+2(v_2+v_4+v_6)+4(v_1+v_3+v_5+v_7)]$$

$$= \frac{n}{3}[(0+0.699)+2(0.3010+0.4771+0.6021)+4(0.1761+0.3979+0.5441+0.6532)]$$

$$= 1.7574 \text{ (approx.)}$$

4.2 Calculate $\int_2^{10} \frac{dx}{1+x}$ (upto 3 places of decimal) by dividing the range into 8 equal parts by Simpson's 1/3rd rule.

(2011 : 12 Marks)

Solution:

Here the range of integration is (2, 10). Dividing it into 8 equal parts each of width $\frac{10-2}{8} = \frac{8}{8} = 1$, we get $h = 1$.

x	$y = \frac{1}{1+x}$
$x_0 = 2$	$\frac{1}{3}$
$x_0 + h = 3$	$\frac{1}{4}$
$x_0 + 2h = 4$	$\frac{1}{5}$
$x_0 + 3h = 5$	$\frac{1}{6}$
$x_0 + 4h = 6$	$\frac{1}{7}$
$x_0 + 5h = 7$	$\frac{1}{8}$
$x_0 + 6h = 8$	$\frac{1}{9}$
$x_0 + 7h = 9$	$\frac{1}{10}$
$x_0 + 8h = 10$	$\frac{1}{11}$

By Simpson's 1/3 rule, we get

$$\int_2^{10} \frac{dx}{1+x} = \frac{h}{3}[y_0 + y_8 + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)]$$

$$= \frac{1}{3}\left[\frac{1}{3}+\frac{1}{11}+4\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}\right)+2\left(\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\right]$$

$$= \frac{1}{3}\times 3.8988453 = 1.2996151$$

$$= 1.300$$

4.3 A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line x = 0 and x = 1 and a curve through the points with the following co-ordinates :

x	0	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Find the volume of the solid.

(2011 : 20 Marks)

Solution:

x	0.00	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415
$z = y^2$	1	0.9793	0.9195	0.8261	0.7081

The required volume is $= \pi\int_0^1 y^2 dx$

$$= \pi\int_0^1 z dx$$

By Simpson's 1/3rd rule,

$$\int_0^1 z dx = \frac{h}{3}[z_0 + z_4 + 4(z_1 + z_3) + 2z_2]$$

$$= \frac{0.2}{3}[(1+0.708) + 4(0.9793+0.8261) + 2\times 0.9195]$$

$$\left[\because h = \frac{1-0}{5} = 0.2\right]$$

$$= 0.3570$$

$\therefore$ Required volume $= \pi\times\int_0^1 z dx$

$$= 3.14\times 0.3570$$

$$= 1.12098 \text{ units cube}$$

4.4 The velocity of a train which starts from rest is given in the following table. The time is in minutes and velocity is in km/hr.

t	2	4	6	8	10	12	14	16	18	20
v	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance run in 30 minutes by using composite Simpson's 1/3 rules.

(2013 : 15 Marks)

Solution:

The distance travelled by the train is given by

$$x = \int_0^t v\,dt$$

where v is the velocity at any time t. We estimate this integral by Simpson's rule.
First converting the time to hours

$$h = 2 \text{ min.} = \frac{2}{60} = 0.033 \text{ hrs.}$$

$$x = \int_0^{20} v\,dt = \int_0^{4/60} v\,dt + \int_{4/60}^{8/60} v\,dt + \int_{8/60}^{12/60} v\,dt + \int_{12/60}^{16/60} v\,dt + \int_{16/60}^{20/60} v\,dt$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \frac{h}{3}[f_7 + 4f_8 + f_9]$$

where f_i denotes the velocity at the i^{th} serial number, i.e.,

i	0	1	2	3	4	5	6	7	8	9
f_i	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

$\therefore$
$$x = \frac{h}{3}[(f_0 + f_9) + 2(f_2 + f_4 + f_6 + f_8) + 4(f_1 + f_3 + f_5 + f_7)]$$

$$= \frac{2}{60 \times 3}[(16 + 0) + 2(40 + 51.2 + 17.6 + 3.2) + 4(28.8 + 46.4 + 32.0 + 8)]$$

$$= 7.78667 \text{ kms}$$

4.5 Use five subintervals to integrate $\int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rule.

(2014 : 10 Marks)

Solution:

Here

$$f(x) = \frac{1}{1+x^2}$$

$$a = 0,\ b = 1, \text{ and } n = 5$$

$\therefore$
$$h = \frac{1-0}{5} = \frac{1}{5} = 0.5$$

x	0.0	0.2	0.4	0.6	0.8	1
$y = f(x)$	1.000000	0.961538	0.735294	0.735294	0.609756	0.500000
	y_0	y_1	y_2	y_3	y_4	y_5

Using trapezoidal rule we get

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_4)]$$

$$= \frac{0.2}{2}[(1.000000 + .500000) + 2(0.961538 + 0.862069 + 0.735294 + 0.609756)]$$

$$= 0.7837314$$

$\therefore$ $I = 0.78373$, correct to five significant figures.

The exact value $= \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} = 0.7853981$$

$$\int_0^1 \frac{1}{1+x^2} = 0.78540$$

correct to five significant figures.

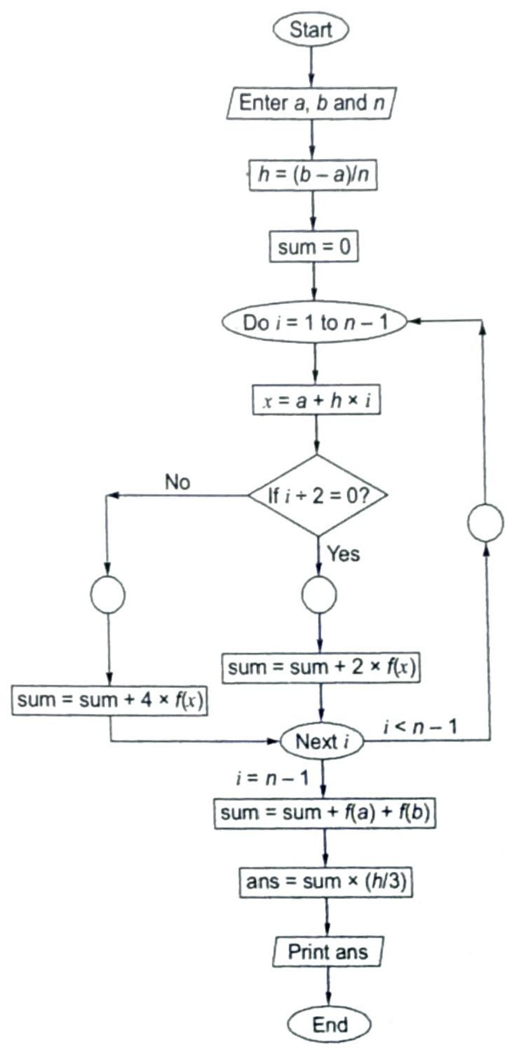
$\therefore$ The error is $= 0.78540 - 0.78373 = 0.00167$

$\therefore$ Absolute error $= 0.00167$

4.6 Draw a flowchart for Simpson's one-third rule.

(2014 : 15 Marks)

Solution:



4.7 For an integral $\int_1^1 f(x)dx$, show that the two-point Gauss quadrative rule is given by $\int_{-1}^1 f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)$. Using this rule, estimate $\int_2^4 2xe^x dx$.

(2016 : 15 Marks)

Solution:

Let $$f(x) = C_0 + C_1x + C_2x^2 + ...$$ (C_0, C_1, ... are constants)

$\therefore$ $$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (C_0 + C_1x + C_2x^2 + ...)dx$$

$\Rightarrow$ $$\int_{-1}^{1} f(x)dx = C_0[x]_{-1}^{1} + C_1\left[\frac{x^2}{2}\right]_{-1}^{1} + \left[C_2\frac{x^3}{3}\right]_{-1}^{1} + ...$$

$\Rightarrow$ $$\int_{-1}^{1} f(x)dx = 2C_0 + \frac{2}{3}C_2 + \frac{2}{5}C_4 + ... \quad ...(i)$$

Also, for two-point quadrature formula,

$$\int_{-1}^{1} f(x)dx = w_1f(x_1) + w_2f(x_2),$$

where w_1 and w_2 are weights corresponding to points x_1 and x_2 respectively.

$\Rightarrow$ $$\int_{-1}^{1} f(x)dx = w_1(c_0 + c_1x_1 + c_2x_1^2 + ...) + w_2(c_0 + c_1x_2 + c_2x_2^2 + ...) + ... \quad ...(ii)$$

Comparing (i) and (ii), we get

$$w_1 + w_2 = 2 \quad ...(iii)$$

$$w_1x_1 + w_2x_2 = 0 \quad ...(iv)$$

$$w_1x_1^2 + w_2x_2^2 = \frac{2}{3} \quad ...(v)$$

$$w_1x_1^3 + w_2x_2^3 = 0 \quad ...(vi)$$

Solving (iii), (iv), (v) and (vi), we get

$$w_1 = 1$$

$$w_2 = 1$$

$$x_1 = \frac{-1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

$\therefore$ $$\int_{-1}^{1} f(x)dx = 1\cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1\cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Now, consider $$I = \int_{2}^{4} 2ne^x dx$$

Since, limits are from 2 to 4, we first charnge the limits by substitution.

Let $u = ax + b$ such that

$$u(4) = 1 = 4a + b$$

$$u(2) = -1 = 2a + b$$

$\Rightarrow$ $$a = 1, b = -3$$

$\therefore$ $$u = x - 3 \Rightarrow x = u + 3$$

$\therefore$ $$du = dx$$

So, $$I = \int_{-1}^{1} 2(u+3)e^{u+3}du$$

$\therefore$ $$I = \int_{-1}^{1} f(u)du, \ f(u) = 2(u+3)e^{u+3}$$

Using two-point Gauss quadrature formula, we get

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 2\left(\frac{3-1}{\sqrt{3}}\right)e^{3-\frac{1}{\sqrt{3}}} + 2\left(\frac{3+1}{\sqrt{3}}\right)e^{3+\frac{1}{\sqrt{3}}}$$

$\Rightarrow$ $$I = 54.634 + 255.98 = 310.614$$

$\therefore$ $$\int_2^4 2xe^x dx = 310.614 \text{ (approx.)}$$

4.8 Derive the formula :

$$\int_a^b y dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots.. + y_{n-1}) + 2(y_3 + y_6 + \ldots.. + y_{n-3})]$$

Is there any restriction on n? State that condition. What is the error bound in the case of Simpson's 3/8 rule?

(2017 : 20 Marks)

Solution:

Let $I = \int_a^b y dx$, where the interval $[a, b]$ is divided into 3 equal sub-intervals of width $h = \frac{b-a}{3}$ so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h = b$.

$$I = \int_{x_0}^{x_0+3h} f(x)dx = h\int_0^3 f(x_0 + rh)dr \text{ [Taking } xx = x_0 + rh \Rightarrow dx = hdr]$$

$$= h\int_0^3 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0\right]dr$$

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0\right]_0^3$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}[(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] \quad \ldots(i)$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6] \quad \ldots(ii)$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n] \quad \ldots(iii)$$

(i) + (ii) + (iii) gives

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots$$
$$+ y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

The intervals must be divided into sub-intervals which are multiple of 3, i.e., n must be multiple of 3.

The error (principal part) of Simpson's 3/8 rule is $-\frac{3h^5}{80}y^{(4)}$ in the interval $[x_0, x_3]$.

4.9

Time (min.)	2	4	6	8	10	12	14	16	18	20
Speed (km/h)	10	18	25	29	32	20	11	5	2	8.5

Starting from rest in the beginning, the speed (km/h) of a train at different times (in min) is given by above table.
Using Simpson's 1/3rd rule, find approximate distance travelled (in km) in 20 minutes from beginning.
(2018 : 10 Marks)

Solution:

Velocities in km/min are . $v_0 = 0$, $v_1 = 0.17$, $v_2 = 0.3$, $v_4 = 0.48$, $v_5 = 0.53$, $v_6 = 0.33$, $v_7 = 0.18$, $v_8 = 0.08$, $v_9 = 0.03$, $v_{10} = 0.14$

Here $n = 2$

$$\text{Distance travelled} = \int_0^{20} vdt$$

By Simpson's 1/3rd rule,

$$\int_0^{20} vdt = \frac{h}{3}[(v_0 + v_1) + 2(v_2 + v_8 + v_6 + v_8) + 4(v_1 + v_3 + v_5 + v_7 + v_9)]$$

$$= \frac{2}{3}[(0 + 0.14) + 2(0.3 + 0.48 + 0.33 + 0.08) + 4(0.17 + 0.42 + 0.53 + 0.18 + 0.03)]$$

$$= \frac{2}{3}[0.14 + 4(1.33) + 2(1.19)]$$

$$\Rightarrow \int_0^{20} vdt = 5.2267 \text{ kms. (approx.)}$$

4.10 Find the values of constants a, b, c such that the quadrature formula $\int_0^h f(x)dx = h\left[af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right]$ is exact for polynomials of as high degree as possible and hence find the truncation error.
(2018 : 15 Marks)

Solution:

Given, the quadrature formula is

$$\int_0^h f(x)dx = h\left[af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right]$$

If it is exact, then it should hold true for polynomials 1, x, x^2, ... Plugging these into formula, we obtain

$f(x) = x^0$: $$\int_0^h x^0 dx = h = h[a+b+c] \quad ...(i)$$

$f(x) = x^1$: $$\int_0^h x dx = \frac{h^2}{2} = h\left[\frac{bh}{3} + ch\right] \quad ...(ii)$$

$f(x) = x^2$: $$\int_0^h x^2 dx = \frac{h^3}{3} = h\left[\frac{bh^2}{9} + ch^2\right] \quad ...(iii)$$

Solving (i), (ii) and (iii), we get

$$a = 0, b = \frac{3}{4}$$

∴ The quadrative formula is

$$\int_0^h f(x)dx = h\left[\frac{3}{4}f\left(\frac{h}{3}\right) + \frac{1}{4}f(h)\right]$$

The truncation error, E is given by

$$E = \frac{c_1}{3!}f'''(\xi), \text{ where } 0 < \xi < h$$

and $$c_1 = \int_0^h x^3 dx - h\left[\frac{bh^3}{27} + ch^3\right] = \frac{-h^7}{36}$$

∴ $$E, \text{truncation error} = \frac{-h^4}{36}f'''(\xi) = 0(h^4)$$

5. Ordinary Differential Equation

5.1 Use Euler's method with step size $h = 0.15$ to compute the approximate value of $y(0.6)$ correct upto few decimal places from the initial value problem

$$y' = x(y + x) - 2$$
$$y(0) = 2$$

(2013 : 15 Marks)

Solution:

We use Euler's predictor equation

$$y_n = y_{n-1} + hf(x_0 + (n-1)h, y_{n-1})$$

to predict value at a point and then the equation

$$y_n^i = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{i-1})]$$

to refine it upto 5 decimal places

$$x_0 = 0, h = 0.15, y_0 = 0$$

x	$y' = x(y+x) - z$	Mean Slope (M.S.)	$y^{(k)} = y^{k-1} + h \times$ M.S.
0	–2	–	–0.30000
0.25	–2.02250	–2.01125	–0.30169
0.15	–2.02275	–2.01138	–0.30171
0.15	–2.02276	–2.01138	–0.30171

∴ $$y(0.15) = -0.30171$$

0.15	−2.02276	−	−0.60512
0.30	−2.09154	−2.05715	−0.61028
0.30	−2.09308	−2.05792	−0.61039
0.3	−2.09312	−2.05794	−0.61040
0.3	−2.09312	−2.05794	−0.61040

$\therefore \quad y(0.3) = -0.61040$

0.3	−2.09312	−	−0.92437
0.45	−2.21345	−2.15329	−0.93339
0.45	−2.21753	−2.15532	−0.93370
0.45	−2.21766	−2.15539	−0.93371
0.45	−2.21767	−2.15539	−0.93371

$\therefore \quad y(0.45) = -0.93371$

0.45	−2.21767	−	−1.26636
0.60	−2.39981	−2.30874	−1.28002
0.60	−2.40801	−2.31283	−1.28063
0.60	−2.40838	−2.31302	−1.28066
0.60	−2.40840	−2.31303	−1.28066

$\therefore \quad y(0.6) = -1.28066$

5.2 Use Runge-Kutta formula of fourth order to find the value of y at $x = 0.8$, where $\frac{dy}{dx} = \sqrt{x+y}$, $y(0.4) = 0.41$. Take the step length $h = 0.2$.

(2014 : 20 Marks)

Solution:

Given that $\quad \frac{dy}{dx} = \sqrt{x+y}$

To find $y(0.6)$:

Here $\quad x_0 = 0.4,\ y_0 = 0.41,\ h = 0.2$

$$f(x_0, y_0) = \sqrt{0.81}$$

$$\therefore \quad K_1 = hf(x_0, y_0) = (0.2)\sqrt{0.81} = 0.18$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.2)f(0.5, 0.51) = 0.2$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.2)f(0.5, 0.51) = 0.20099$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = (0.2)f(0.6, 0.61099) = 0.220099$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}[0.18 + 2(0.2) + 2(0.20099) + 0.220089]$$

$$= \frac{1}{6}(1.202069982)$$

$$= 0.2003449$$

$$y_1 = y(0.6) = y_0 + K = 0.41 + 0.2003449 = 0.6103449$$

To find $y(0.8)$:

Here, $x_1 = 0.6,\ y_1 = 0.6103449,\ h = 0.2$

$$K_1 = hf(x_1, y_1) = (0.2)f(0.6, 0.6103449) = 0.220031$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0.2)f(0.7, 0.72036) = 0.23836$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = (0.2)f(0.7, 0.72952) = 0.23913$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = (0.2)f(0.8, 0.85257) = 0.257105$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.238686$$

$$y_2 = y(0.8) = y_1 + K = 0.6103449 + 0.238686$$

$$= 0.8490309$$

5.3 Solve the initial value problem $\frac{dy}{dx} = x(y-x)$, $y(2) = 3$ in the interval [2, 2.4] using the Raunge-Kutta fourth-order method with step size $h = 0.2$.

(2015 : 15 Marks)

Solution:

Given : $\frac{dy}{dx} = x(y-x) = f(x, y)$

Step size $h = 0.2$

$y(2) = 3$

Taking $y_0 = y(z)$, $x_0 = z$

$$K_1 = hf(x_0, y_0) = 0.2f(2, 3)$$

$$= 0.4$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2f(2.1, 3.2)$$

$$= 0.462$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2f(2.1., 3.231)$$

$$= 0.475$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2f(2.2, 3.475) = 0.561$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}(0.4 + 0.462\times 2 + 0.475\times 2 + 0.561)$$

$$= 0.4725$$

$$y_1 = y_0 + K = y(2.2) = y(2) + K = 3 + 0.4725$$

$$= 3.4725 \text{ (approx.)}$$

Taking $y_1 = y(2.2) = 3.4725, x_1 = 2.2$

$$K_1 = hf(x_1, y_1) = 0.2f(2.2, 3.4725) = 0.5599$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2f(2.3, 3.75245) = 0.6681$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2f(2.4, 4.1655) = 0.84744$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.6883$$

$$y_2 = y(2.4) = y_1 + K = 3.4725 + 0.6883 = 4.1608 \text{ (approx.)}$$

$$\therefore \quad y(2.2) = 3.4725 \text{ (approx.)}$$

$$y(2.4) = 4.1608 \text{ (approx.)}$$

5.4 Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$. Use four decimal places for calculation and step length 0.2.

(2019 : 10 Marks)

Solution:

Given : $$y' = \frac{y^2 - x^2}{y^2 + x^2}$$

at $x = 0$, $y(0) = 1$, $h = 0.2$, $y(0.2) = ?$

Using Runge-Kutta method for fourth order

$$y(0.2) = y(0) + k$$

where $$k = \frac{1}{6}[k_1 + k_4 + 2(k_2 + k_3)]$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h_1, y_0 + k_3)$$

$$\therefore \quad k_1 = 0.2f(0, 1) = 0.2 \times 1 = 0.2$$

$$k_2 = 0.2f(0.1, 1.1) = 0.2 \times 0.98361 = 0.19672$$

$$k_3 = 0.2f(0.1, 1.09836) = 0.2 \times 0.98356 = 0.19672$$

$$k_4 = 0.2f(0.2, 1.19671) = 0.2 \times 0.94566 = 0.18913$$

$$y(0.2) = 1 + \frac{1}{6}\{(0.2 + 0.18913 + 2(0.19672 + 0.19671)\}$$

$$y(0.2) = 1 + \frac{1}{6}(0.38913 + 0.78686)$$

$$y(0.2) = 1 + \frac{1}{6}(1.17599)$$

$$y(0.2) = 1 + 0.195998$$

$$y(0.2) = 1.960 \text{ (Required solution)}$$

6. Computer Programming

6.1 Find the values of two valued boolean variables A, B, C, D by resolving the following simultaneous equations :

$$\bar{A} + AB = 0$$
$$AB = AC$$
$$AB + A\bar{C} + CD = \bar{C}D$$

where $\bar{x}$ denotes the complement of x.

(2009 : 6 Marks)

Solution:

$$\bar{A} + AB = 0$$

$\Rightarrow$ $(\bar{A} + A)(A + B) = 0$ $\quad [\because x + yz = (x + y)(x + z)]$

$\Rightarrow$ $1 \cdot (\bar{A} + B) = \bar{A} + B = 0$ $\quad [\because x + \bar{x} = 1 \;\&\; 1 \cdot x = x]$

$\Rightarrow$ $\bar{A} + B = 0$

It means $\bar{A} = 0$ and $B = 0$

$\Rightarrow$ $A = 1$ and $B = 0$

Put this values in second equation, i.e.,

$$AB = AC$$
$$1.B = 1.C$$
$$B = C = 0$$

Put values of A, B & C in third equation, i.e.,

$$AB + A\bar{C} + CD = \bar{C}D$$
$$1.0 + 1.\bar{0} + 0.D = \bar{0}.D$$
$$0 + 1 + 0 = 1.D = D$$
$$D = 1$$

6.2 (i) Realize the following expression by using NAND gates only : $g = (\bar{a} + \bar{b} + c)\bar{d}(\bar{a} + e)f$ where $\bar{x}$ denotes the complement of x.

(2009 : 6 Marks)

(ii) Find the decimal equivalent of $(357.32)_8$.

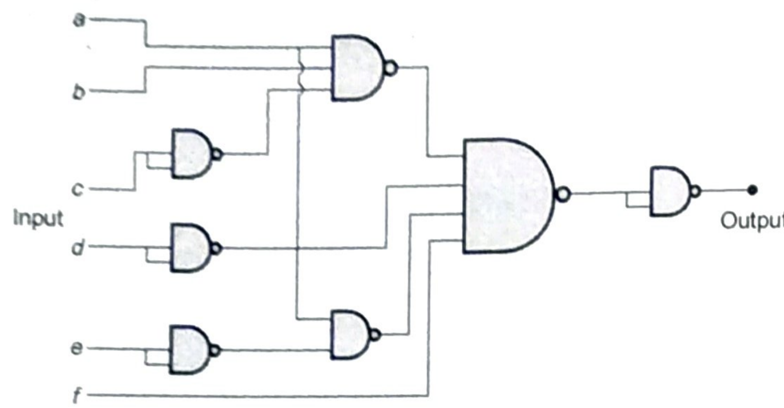
(2009 : 6 Marks)

Solution:

(i) So $g(a, b, c, d) = (\bar{a} + \bar{b} + c)\bar{d}(\bar{a} + e)f$

$$= \overline{a \cdot b \cdot \bar{c}}\; \bar{d}\; \overline{a \cdot \bar{e}} f = ABCD = \overline{\overline{ABCD}}$$

Now the expression by NAND gates will be



This is the required circuit.

(ii) $(357.32)_8$ to decimal conversion

$$
\begin{aligned}
(357.32)_8 &= 3\times 8^2 + 5\times 8^1 + 7\times 8^0 + 3\times 8^{-1} + 2\times 8^{-2}\\
&= (192 + 40 + 7 + 0.375 + 0.03125)_{10}\\
&= (239.40625)_{10}
\end{aligned}
$$

6.3 If $A \oplus B = AB' + A'B$. Find the value of $x \oplus y \oplus z$.

(2010 : 6 Marks)

Solution:

Given : $A \oplus B = AB' + A'B$

$$
\begin{aligned}
\therefore \quad x \oplus y \oplus z &= (xy' + x'y) \oplus z\\
&= xy'z' + x'yz' + (xy')'.(x'y)'z && \text{(Demorgen's law)}\\
&= xy'z' + x'yz' + (x' + (y')').((x')' + y')z\\
&= xy'z' + x'yz' + (x' + y).(x + y')z\\
&= xy'z' + x'yz' + (x'x + x'y' + yx + y.y')z && \text{(Distributive law)}\\
&= xy'z' + x'yz' + (0 + x'y' + xy + 0)z\\
&= xy'z' + x'yz' + x'y'z + xyz && \text{(Distributive law)}\\
&= xyz + xy'z' + x'y'z + x'yz'
\end{aligned}
$$

6.4 (a) Find the hexadecimal equivalent of the decimal number $(587632)_{10}$.

(2010 : 5 Marks)

(b) For the given set of data points

$$(x_1, f(x_1)), (x_2, f(x_2)), \ldots (x_n, f(x_n))$$

Write an algorithm to find the value of $f(x)$ by using Lagrange's interpolation formula.

(2010 : 10 Marks)

(c) Using Boolean algebra, simplify the following expressions:

(i) $a + a'b + a'b'c + a'b'c'd + \ldots$

(ii) $x'yz + yz + xz$

where x' represents the complement of x.

(2010 : 5 Marks)

Solution:

(a) Given number is $(587632)_{10}$.

16	587632	0
16	36727	7
16	2295	7
16	143	F
16	8	8
	0	

$\therefore \quad (587632)_{10} = (8F770)_{16}$

(b) The algorithm is written below:

1. Read $x_1 n$
2. For $i = 1$ to $(n + 1)$ in steps of 1 do read x_i, $f(x_i)$ end for
3. For $i = 1$ $(n + 1)$ in steps of 1 do
5. Prod func $\leftarrow 1$
6. For $j = 1$ to (n + 1) in steps of 1 do
7. If $(j \neq i)$ then

   prod func $\leftarrow$ prod func $\times \dfrac{(x - x_j)}{(x_i - x_j)}$

   end for
8. Sum $\leftarrow$ sum + $f(x_i) \times$ Prod func

   end for
9. Write x, sm
10. Stop

(c) (i) $a + a'b + a'b'c + a'b'c'd + \ldots$

Now, $\quad x + x'y = x(1 + y) + x'y = x + xy + x'y$

$= x + y(x + x') = x + y \cdot 1$

$\therefore \quad x + x'y = x + y$

Similarly, $\quad x + x'y + x'y'z = x + y + z$

$\therefore \quad a + a'b + a'b'c + a'b'c'd + \ldots = a + b + c + d + \ldots$

(ii) $x'y'z + yz + xz$

$$
\begin{aligned}
&= z(x'y' + y) + xz \\
&= z(x'y' + y(x + x')) + xz \\
&= z(x'y' + xy + x'y) + xz \\
&= xyz + z(x'y' + x'y) + xz \\
&= xyz + x'z + xz \\
&= xyz + z(x + x') \\
&= xyz + z \\
&= z(1 + xy) \\
&= z
\end{aligned}
$$

6.5 (a) Compute $(3205)_{10}$ to the base 8.

(b) Let A be an arbitrary but fixed Boolean algebra with operations $\wedge$, $\vee$ and $'$ and the zero and the unit element denoted by 0 and 1 respectively. Let $x, y, z, \ldots\ldots$ be elements of A.
If $x, y \in A$ be such that $x \wedge y = 0$ and $x \vee y = 1$ then prove that $y = x'$.

(2011 : 12 Marks)

Solution:

$(3205)_{10}$ to be base 8.

8	3205	
8	400	Remainder = 5
8	50	Remainder = 0
8	6	Remainder = 2 ↑
	0	Remainder = 6

$\therefore \qquad (3205)_{10} = (6205)_8$

6.6 Find the logic circuit that represents the following Boolean function. Find also an equivalent simpler circuit :

x	y	z	$f(x, y, z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

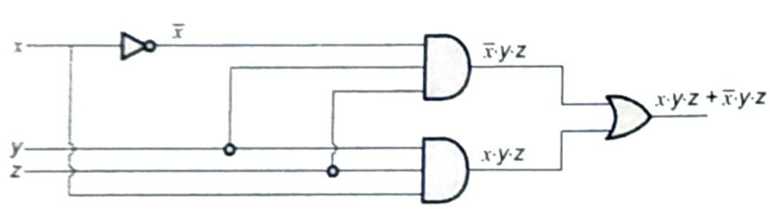
(2011 : 20 Marks)

Solution:

The given Boolean function is

$$f(x, y, z) = x \cdot y \cdot z + \bar{x} \cdot y \cdot z$$

The logic circuit is :

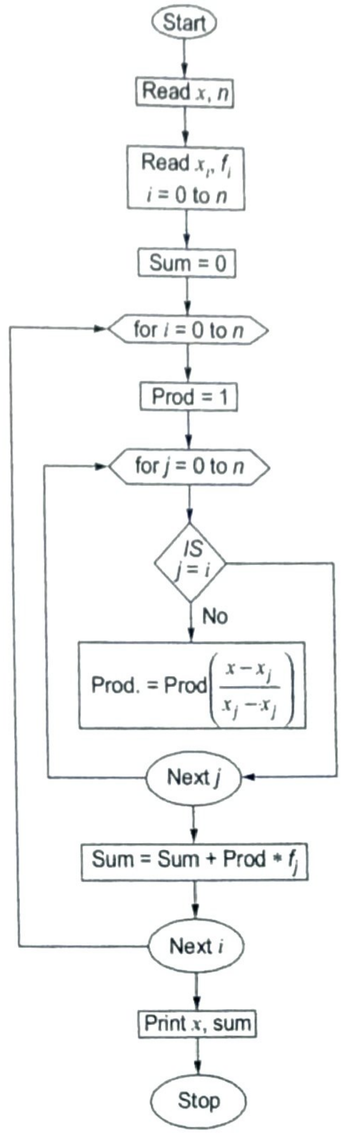


6.7 Draw a flow chart for Lagrange's interpolation formula.

(2011 : 20 Marks)

Solution:

Flow chart for Lagrange's interpolation formula :



6.8 Provide a computer algorithm to solve an ordinary differential equation $\frac{dy}{dx} = f(x, y)$ in the interval [a, b] for n number of discrete points, where the initial value is $y(a) = \alpha$, using Euler's method.

(2012 : 12 Marks)

Solution:

Computer algorithm to solve $\frac{dy}{dx} = f(x, y)$ by Euler's method.

1. Enter the initial values of x and y ($x_0 = a$, $y_0 = \alpha$)
2. Enter the value of x for which y to be determined.
3. Enter the width of the interval 'h'.
4. Do :

$$y = y_0 + h\frac{dy}{dx}_{(x_0, y_0)}$$

$$y_0 = y$$

$x_0 = x_0 + h$
Until $(x_0 \geq x)$

5. Print y, which is the solution.

6.9 Develop an algorithm for Newton-Raphson method to solve $f(x) = 0$ starting with initial iterate x_0, n be the number of iterations allowed, ε be the prescribed relative error and δ be the prescribed lower bound for $f'(x)$.

(2013 : 20 Marks)

Solution:

The required algorithm is

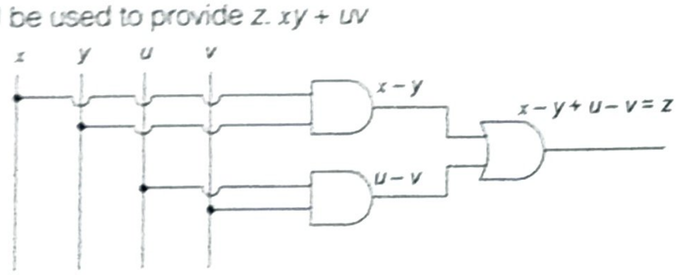
1. Input x_0, ε, δ, n # x_0 is the initial guess,
   ε error tolerance
   n number of iterations
   δ tolerance for f'
2. For $i = 1$ to n in steps of 1 do
3. $f_0 = f(x_0)$
4. $f'_0 = f'(x_0)$
5. If $|f'_0| < \delta$ go to 12
6. Else $x_1 = x_0 - \dfrac{f_0}{f'_0}$
7. If $|x_1 - x_0| < \varepsilon$ go to 14
8. Else $x_0 \leftarrow x_1$
9. End for
10. Write 'Does not converge in n iterations f_0, f'_0, x_0, x_1
11. Stop
12. Write slope too small x_0, f_0, f'_0, i
13. Stop
14. Converges to x_1, $f(x_1)$
15. Stop

6.10 Use only AND and OR logic gates to construct a logic circuit for the Boolean expression $z = xy + uv$.

(2014 : 10 Marks)

Solution:

Here, $x.y$ uses one AND gate
$u.v$ uses another AND gate
and one OR gate will be used to provide z. $xy + uv$



6.11 For any Boolean variables x and y, show that $x + xy = x$.

(2014 : 15 Marks)

Solution:

$$L(s) = x + xy = x \cdot 1 + xy$$
$$= x(1 + y)$$

$$= x \cdot 1 \qquad (\because 1 + y = 1)$$
$$= x$$
$$\therefore \quad x + xy = x$$

6.12 Find the principal (or canonical) disjunction normal form in three variables p, q, r for the Boolean expression $[(p \wedge q) \rightarrow r) \vee (p \wedge q) \rightarrow -r)$. If the given Boolean expression a contradiction or a tautology?

(2015 : 10 Marks)

Solution:

We first convert all symbols in $\vee$ or $\wedge$ and then solve it.

Given equation is

$[(p \wedge q) \rightarrow r) \vee (p \wedge q) \rightarrow -r)$. $\qquad (\wedge = \text{and}; \vee = \text{or})$

$$= (\sim(p \wedge q) \vee r) \vee (r(p \wedge q) \vee \sim r$$
$$= \sim(p \wedge q) \vee r \vee \sim(p \wedge q) \vee \sim r$$

which is the required disjunction normal form.

The truth table for above expression is

p	q	r	$p \wedge q$	$((p \wedge q) \rightarrow r) \vee ((p \wedge q) \rightarrow \sim r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T
F	T	F	F	T
T	F	F	F	T
F	F	F	F	T

As the result is always true, $\therefore$ it is tautology.

6.13 Convert the following decimal numbes to equivalent binary and hexadecimal numbers

(a) 4036

(b) 0.4375

(c) 2048.0625

(2016 : 10 Marks)

Solution:

(a) 4096

Binary :

2	4096	0
2	2048	0
2	1024	0
2	512	0
2	256	0
2	128	0
2	64	0
2	32	0
2	16	0
2	8	0
2	4	0
2	2	0
2	1	1
	0	

$$(4096)_{10} = (1000000000000)_2$$

Hexadecimal :

16	4096	0
16	256	0
16	16	0
16	1	1
	0	

$$(4096)_{10} = (1000)_{16}$$

(b) 0.4375

Binary :

	Remainder
$2 \times 0.4375 = 0.875$	0
$2 \times 0.875 = 1.75$	1
$2 \times 0.75 = 1.5$	1
$2 \times 0.5 = 1$	1

$\therefore$ $(0.4375)_{10} = (0.0111)_2$

Hexadecimal :

	Remainder
$16 \times 0.4375 = 7$	7

$\therefore$ $(0.4375)_{10} = (0.7)_{16}$

(c) 2048.0625

Binary :

2	2048	0
2	1024	0
2	512	0
2	256	0
2	128	0
2	64	0
2	32	0
2	16	0
2	8	0
2	4	0
2	2	0
2	1	1
	0	

$$(2048)_{10} = (100000000000)_2$$

	Remainder
$2 \times 0.0625 = 0.125$	0
$2 \times 0.125 = 0.25$	0
$2 \times 0.25 = 0.5$	0
$2 \times 0.5 = 1$	1

$\therefore$ $(0.0625)_{10} = (0.0001)_2$

$\therefore$ $(2048.0625)_{10} = (100000000000.0001)_2$

Hexadecimal :

16	2048	0
16	128	0
16	8	8
	0	

$\therefore$ $(2048)_{10} = (800)_{16}$

$$
\begin{array}{lrcll}
 & 16 \times 0.0625 & = & 1 & \quad\text{Remainder } 1 \\
\therefore & (0.0625)_{10} & = & (0.1)_{16} & \\
\text{So,} & (2048.0625)_{10} & = & (800.1)_{16} &
\end{array}
$$

6.14 Let A, B, C be boolean variables, $\bar{A}$ denote complement of A, $A + B$ is an expression for A or B and $A \cdot B$ is an expression for A and B. Then, simplify the following expression and draw a block diagram of the simplified expression, using AND and OR gates.

$$A \cdot (A+B+C) \cdot (\bar{A}+B+C) \cdot (A+\bar{B}+C) \cdot (A+B+\bar{C})$$

(2016 : 15 Marks)

Solution:

Given, the expression is

$$
\begin{aligned}
& A \cdot (A+B+C) \cdot (\bar{A}+B+C) \cdot (A+\bar{B}+C) \cdot (A+B+\bar{C}) \\
&= (AA+AB+AC)(\bar{A}A+\bar{A}B+\bar{A}C+BA+B\bar{B}+BC+C\bar{A}+\ldots\ldots) \\
&= (A\bar{A}\bar{B}+A\bar{A}C+AAB+ABC+AAC+A\bar{B}C+AC+AB\cdot\bar{A}\bar{B} \\
&\quad + AB\cdot\bar{A}C+AB\cdot AB+AB\cdot BC+AB\cdot AC+AB\cdot\bar{B}C+ABC \\
&\quad + AC\cdot C\,(A+B+\bar{C}) \\
&= (0+0+AB+ABC+AC+A\bar{B}C+AC+0+0+AB+ABC+AB \\
&\quad +0+ABC+0+0+ABC+ABC+AC+A\bar{B}C+AC)(A+B+\bar{C}) \\
&= (AB+AC+ABC+A\bar{B}C)(A+B+\bar{C}) \\
&= \{AB+AC+AC(B+\bar{B})\}\cdot(A+B+\bar{C}) \\
&= (AB+AC+AC)\cdot(A+B+\bar{C}) = (AB+AC)\cdot(A+B+\bar{C}) \\
&= AB\cdot A+AB\cdot B+AB\cdot\bar{C}+AC\cdot A+AC\cdot B+A\bar{C}C \\
&= AB+AB\bar{C}+AC+ABC \\
&= AB+AC+AB(C+\bar{C}) \\
&= AB+AC+AB \\
&= AB+AC \\
&= A\cdot(B+C)
\end{aligned}
$$

$\therefore$ The simplified expression is $A\cdot(B+C)$

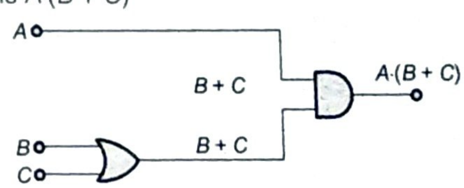


Figure is block diagram of simplified expression.

6.15 Write the Boolean expression, $z(y+z)(x+y+z)$ in its simplest form using Boolean postulate rules. Mention the rules used during simplification. Verify your result by constructing the truth table for the given expression and for its simplest form.

(2017 : 10 Marks)

Solution:

Let

$$
\begin{aligned}
A &= z(y+z)(x+y+z) \\
&= (zy+z^2)(x+y+z) \\
&= (zy+z)(x+y+z) \qquad (\because A^2 = A) \\
&= (zyx+zy^2+zyz+zx+zy+z^2)
\end{aligned}
$$

$$= zy(x+1) + z^2y + zx + z(y+1)$$
$$= zy + zy + zx + z \qquad (\because 1 + A = 1)$$
$$= z(1+y) + zy = z(1+y) = z$$

Truth Table :

x	y	z	$z(y+z)$	$x+y+z$	A	z
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	0	1	0	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	1	1	1	1
1	1	0	0	1	0	0
1	1	1	1	1	1	1

$$\therefore \qquad z(y+z)(x+y+z) = z$$

6.16 Write an algorithm in the form of a flow-chart for Newton-Raphson method. Describe the cases of failure of this method.

(2017 : 15 Marks)

Solution:

To find a root of the equation $f(x) = 0$, the iterative scheme for Newton-Raphson method is given by

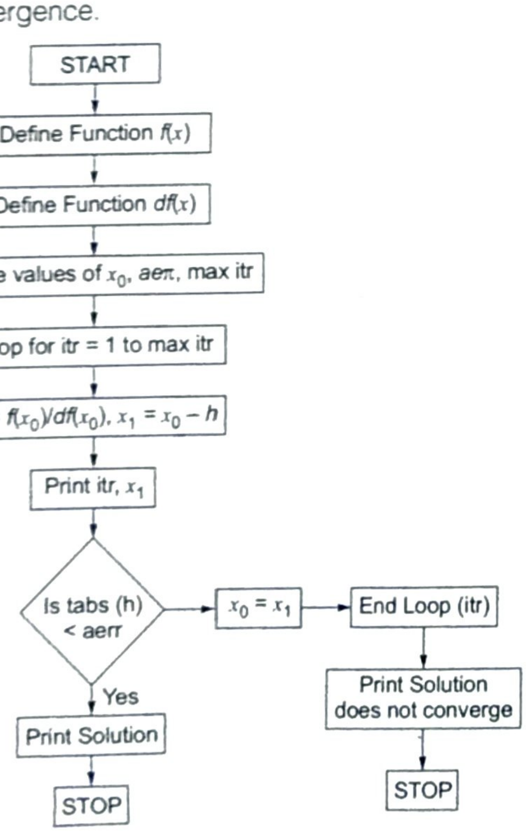
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method can fail in following cases :

(i) If $f'(x) = 0$

(ii) If $f(x)$ is not continuously differentiable.

(iii) If starting point, x_0 is outside the range of guaranteed convergence.



6.17 Write down the basic algorithm for solving equation : $ne^{x-1} = 0$ by bisection method, correct to 4 decimal places.

(2018 : 10 Marks)

Solution:

The basic algorithm is :

1. Start
2. Read x_1, x_2, E
   * Here x_1 and x_2 are initial values. E is absolute error, i.e., accuracy
   * Here, $F(x) = xe^x - 1 = 0$
3. Compute $F(x1)$ and $F(x2)$

$$F(1) = F(x1),\ F(2) = F(x2)$$

4. If $F(1) \cdot F(2) > 0$, then display initial values ..... wrong and go to (11). Otherwise continue.
5. $x = (x1 + x2)/2$
6. If $[(x1 - x2)/x] < \in$, then display x and go to (11).
   * Here [ ] refers to modulus sign.
7. Else; $F = F(x)$
8. If $F * F(1) > 0$, then $x1 = x$ and $F(1) = F$
9. Else; $x2 = x$ and $F(2) = F$
10. Go to (5)
    * Now the loop continues with new values upto correct 4 decimal places.
11. Stop

6.18 Find the equivalent of numbers given in a specified number system to the system mentioned against them.

(a) $(111011.101)_2$ to decimal system.
(b) $(1000111110000.00101100)_2$ to hexadecimal system.
(c) $(C4F2)_{16}$ to decimal system.
(d) $(418)_{10}$ to binary system.

(2018 : 15 Marks)

Solution:

(a)

$$\begin{aligned}(111011.101)_2 &= 1\times 2^5 + 1\times 2^4 + 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 + 1\times 2^{-1} \\ &\quad + 0\times 2^{-2} + 1\times 2^{-3} \\ &= (59.625)_{10}\end{aligned}$$

(b)

$$(\underline{1000}\,\underline{1}\ \underline{1111}\ \underline{0000}\ \underline{0010}\ \underline{1100})_2 = (11FO\cdot 2C)_{16}$$

(c)

$$\begin{aligned}(C4F2)_{16} &= C\times 16^3 + 4\times 16^2 + F\times 16^1 + 2\times 160 \\ &= 12\times 16^3 + 4\times 16^2 + 15\times 16 + 2\times 1 \\ &= (50418)_{10}\end{aligned}$$

(d) $(418)_{10}$

2	418	0
2	209	1
2	104	0
2	52	0
2	26	0
2	13	1
2	6	0
2	3	1
2	1	1
	0	

So $$(418)_{10} = (110100010)_2$$

6.19 Simplify the boolean expression $(a+b)\cdot(\bar{b}+c)+b\cdot(\bar{a}+\bar{c})$ by using laws of boolean algebra. From its truth table, write it in minterm normal form.

(2018 : 15 Marks)

Solution:

Given : $(a+b)\cdot(\bar{b}+c)+b\cdot(\bar{a}+\bar{c})$

$$= a\bar{b}+ac+b\bar{b}+bc+b\bar{a}+b\bar{c}$$

$$= a\bar{b}+ac+bc+b\bar{a}+b\bar{c} \qquad (b\cdot\bar{b}=0)$$

$$= a\bar{b}+ac+b\bar{a}+b \qquad (c+\bar{c}=1)$$

$$= a\bar{b}+ac+b \qquad \text{(Absorption law)}$$

Truth Table :

a	b	c	$a\bar{b}$	ac	b	$z=a\bar{b}+ac+b$	Minterm
0	0	0	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
0	0	1	0	0	0	0	$\bar{a}\bar{b}c$
0	1	0	0	0	1	1	$\bar{a}b\bar{c}$
1	0	0	1	0	0	1	$a\bar{b}\bar{c}$
1	1	0	0	0	1	1	$ab\bar{c}$
1	0	1	1	1	0	1	$a\bar{b}c$
0	1	1	0	0	1	1	$\bar{a}bc$
1	1	1	0	1	1	1	abc

From truth table, minterm normal form is

$$z = \cancel{\bar{a}\bar{b}\bar{c}}^{\,0}+\cancel{\bar{a}\bar{b}c}^{\,0}+\bar{a}b\bar{c}+a\bar{b}\bar{c}+ab\bar{c}+a\bar{b}c+\bar{a}bc+abc$$

$$z = \bar{a}b\bar{c}+a\bar{b}\bar{c}+ab\bar{c}+a\bar{b}c+\bar{a}bc+abc$$

6.20 Draw a flow chart and write a basic algorithm (in FORTRAN/C/C++) for evaluating $y=\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule.

(2019 : 10 Marks)

Solution:

Given :

$$y = \int_0^6 \frac{dx}{1+x^2} \qquad ...(1)$$

$$f(x) = \frac{1}{1+x^2};\ a = 0,\ b = 6$$

Hence, algorithm for evaluating (1) using trapezoidal rule in C++.

C++ Program to important trapezoidal

\# include <stdio.h>

A sample function whose definite integral's approximate value is computed using trapezoidal rule

float y(float x)

{

Declaring the function $f(x) = \dfrac{1}{(1+x+x)}$

}

Declaring the function f(x) = $\dfrac{1}{(1+x+x)}$

return $\dfrac{1}{(1+x+x)}$

}

function to evaluate the value of integer float trapezoidal (float a, float b, float n)

{

Grid spacing

float h = $\dfrac{(b-a)}{n}$

Computing sum of first and last terms in above formula

Float S = y(a) + y(b)

Adding middle terms in above formula

for (int i = 1, i < n; i++)

S + = 2*y(a + i*h);

$n/2$ indicates $\dfrac{(b-a)}{2n}$. Multiplying $\dfrac{n}{2}$ with S.

return $\left(\dfrac{h}{2}\right) * S$

}

Driver program to test above function into main ( )

{

Range of definite integral

Float x = 0;

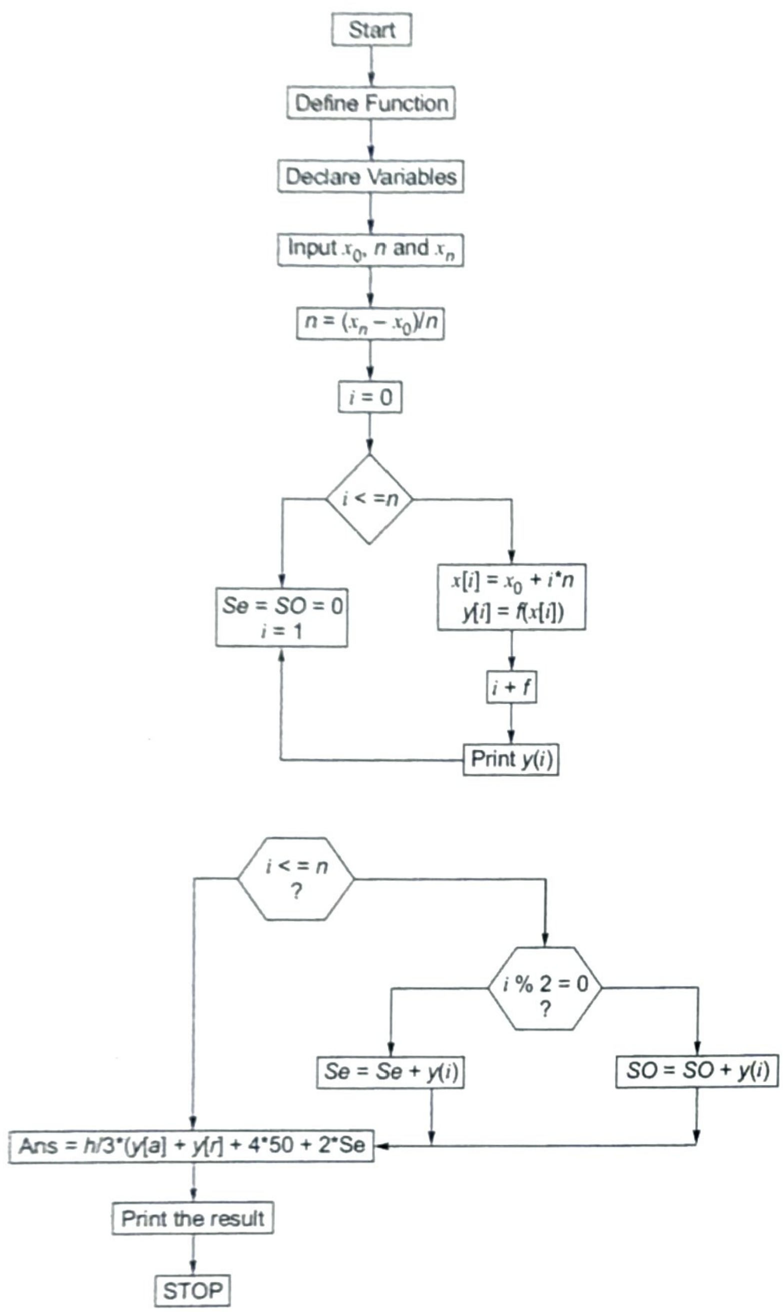
Float xn = 1;

Number of grids, higher value means more accuracy into n = 6;

Print f ("Value of integral is % 6.4f/n" trapezoidal (x0, xn, n));

Return 0;

{



6.21 Find the equivalent numbers given in a specified number to the system mentioned against them :

(a) Integer 524 in binary system.

(b) 101010110101·101101011 to cotal system.

(c) decimal number 5280 to hexadecimal system.

(d) Find the unknown number $(1101{\cdot}101)_8 \rightarrow (?)10$.

(2019 : 15 Marks)

Solution:

(a)

2	524	
2	262	0
2	131	0
2	65	1
2	32	1
2	16	0
2	8	0
2	4	0
2	2	0
	1	0

$\therefore$ $$(524)_{10} \rightarrow (1000001100)_2$$

(b) $(\underset{5}{\underline{101}}\ \underset{2}{\underline{010}}\ \underset{6}{\underline{110}}\ \underset{5}{\underline{101}}\ \underset{5}{\underline{101}}\ \underset{5}{\underline{101}}\ \underset{3}{\underline{011}})_2 \longleftrightarrow (\quad)_8$

$\therefore$ $(101\,010\,110\,101\,101\,101\,011)_2 \leftrightarrow (6265.553)_8$

(c) $$(5280)_{10} \leftrightarrow (\quad)_{16}$$

16	5280	
16	330	D
16	20	A
	1	4

$\therefore$ $$(5280)_{10} \leftrightarrow (14A0)_{16}$$

(d) $(1101\,.\,101)_8$

$$1101 = 1 \times 8^0 + 0 \times 8^1 + 1 \times 8^2 + 1 \times 8^3$$
$$= 1 + 0 + 64 + 512 = 577$$

For decimal (101)

$$= 1 \times 8^{-1} + 0 \times 8^{-2} + 1 \times 8^{-3}$$
$$= 0.125 + 0 + 0.001953125$$
$$= 0.126953125$$

$\therefore$ $$(1101.101)_8 \leftrightarrow (577.126953125)_{10}$$

6.22 Given the Boolean expression

$$X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$$

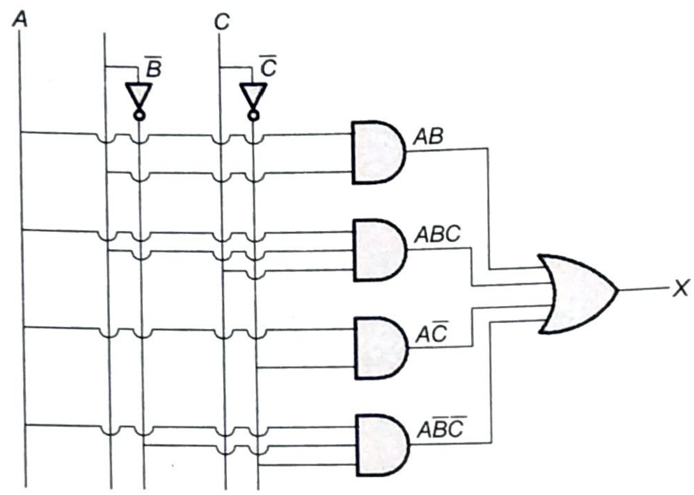
(a) Draw the logical diagram for the expression.
(b) Minimize the expression.
(c) Draw the logical diagram for the reduced expression.

(2019 : 15 Marks)

Solution:

Given $$X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$$

(a) ABC



(b) To minimize the expression :

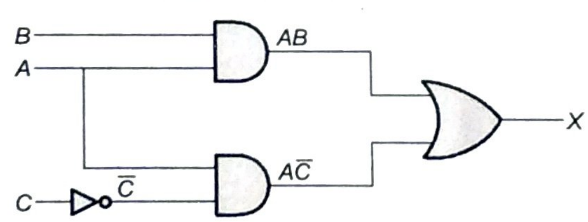
$$X = AB + ABC + A\overline{B}\overline{C} + A\overline{C}$$

$$X = AB(1+C) + A\overline{C}(\overline{B}+1)$$

$$X = AB + A\overline{C} \text{ or } X = A(B+\overline{C})$$ $[\because\ 1 + C = 1;\ \overline{B}+$

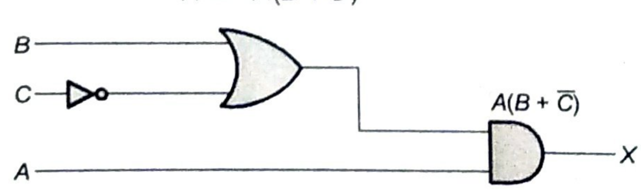
(c) Logical diagram of reduced expression :

$$X = AB + A\overline{C}$$



or

$$X = A(B+\overline{C})$$



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