IAS MATHEMATICS (OPT.)

PAPER - I : VECTOR ANALYSIS (2007)

IAS-2007

5(f). If $\vec{r}$ denotes the position vector of a point and if $\hat{r}$ be the unit vector in the direction of $\vec{r}$, $r = |\vec{r}|$, determine grad(r^{-1}) in terms of $\hat{r}$ and r. 10

Sol^n: Let $\vec{r}$ denotes the position vector a point (x, y, z).

i.e, $\vec{r} = xi + yj + zk$.

Given that $\hat{r}$ be unit vector in the direction

i.e, $\hat{r} = \frac{1}{r}\vec{r}$ where $r = |\vec{r}|$.

$= \sqrt{x^2+y^2+z^2}$.

now $\text{grad}(r^{-1}) = \text{grad}\left(\frac{1}{r}\right)$

$$= i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + j\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + k\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left(i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left\{i\frac{1}{2\sqrt{x^2+y^2+z^2}}2x + j\frac{1}{2\sqrt{x^2+y^2+z^2}}2y + k\frac{1}{2\sqrt{x^2+y^2+z^2}}2z\right\}$$

$$= -\frac{1}{r^2}\frac{2}{2\sqrt{x^2+y^2+z^2}}\{xi + yj + zk\}$$

$$= -\frac{1}{r^2}\cdot\frac{1}{r}\vec{r}$$

$$= -\frac{1}{r^2}\left(\frac{1}{r}\vec{r}\right)$$

$$= -\frac{1}{r^2}\hat{r}.$$

$$\therefore \boxed{\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r}}$$

2007 **8(a).**
15M Find the curvature and torsion at any
1991 point of the curve $x = a\cos 2t$, $y = a\sin 2t$, $z = 2a\sin t$

Sol: The position vector for any point on the curve is

$$\vec{r} = a\cos 2t\, i + a\sin 2t\, j + 2a\sin t\, k$$

$$\frac{d\vec{r}}{dt} = -2a\sin 2t\, i + 2a\cos 2t\, j + 2a\cos t\, k$$

$$\frac{d^2\vec{r}}{dt^2} = -4a\cos 2t\, i - 4a\sin 2t\, j - 2a\sin t\, k$$

and $$\frac{d^3\vec{r}}{dt^3} = 8a\sin 2t\, i - 8a\cos 2t\, j - 2a\cos t\, k$$

Now $$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ -2a\sin 2t & 2a\cos 2t & 2a\cos t \\ -4a\cos 2t & -4a\sin 2t & -2a\sin t \end{vmatrix}$$

$$= i(-4a^2\sin t\cos 2t + 8a^2\cos t\sin 2t)$$
$$+ j(-8a^2\cos t\cos 2t + 4a^2\sin t\sin 2t)$$
$$+ k(-8a^2\sin^2 2t - 8a^2\cos^2 2t)$$

$$= i a^2(-4\sin t\cos 2t + 8\cos t\sin 2t)$$
$$+ j a^2(-8\cos t\cos 2t + 4\sin t\sin 2t)$$
$$- 8a^2 k.$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = a^2\sqrt{(-4\sin t\cos 2t + 8\cos t\sin 2t)^2 + (-8\cos t\cos 2t + 4\sin t\sin 2t)^2 + (8)^2}.$$

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$$= a^2\sqrt{16\sin^2 t\cos^2 2t + 64\cos^2 t\sin^2 2t - \cancel{64\sin t\cos t\cos 2t\sin 2t} + 64\cos^2 t\cos^2 2t + 16\sin^2 t\sin^2 2t + \cancel{64\sin t\cos t\cos 2t\sin 2t} + 64}$$

$$= a^2\sqrt{16\sin^2 t(\cos^2 2t + \sin^2 2t) + 64\cos^2 t(\sin^2 2t + \cos^2 2t) + 64}$$

$$= a^2\sqrt{16\sin^2 t(1) + 64\cos^2 t(1) + 64}$$

$$= a^2\sqrt{16(\sin^2 t + \cos^2 t) + 48\cos^2 t + 64}$$

$$= a^2\sqrt{48\cos^2 t + 80}$$

$$= 4a^2\sqrt{3\cos^2 t + 5}.$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\times\frac{d^3\vec{r}}{dt^3}$$

$$= (-8a\sin 2t)\left[-4a^2\sin t\cos 2t + 8a^2\cos t\sin 2t\right] - 8a\cos 2t\left[8a^2\cos t\cos 2t + 4a^2\sin t\sin 2t\right] - 2a\cos t\left[-8a^2\right]$$

$$= a^3\left[\cancel{32\sin t\cos 2t\sin 2t} - 64\cos t\sin^2 2t - 64\cos t\cos^2 2t - \cancel{32\sin t\sin 2t\cos 2t} + 16\cos t\right]$$

$$= a^3\left[-64\cos t(\sin^2 2t + \cos^2 2t) + 16\cos t\right]$$

$$= a^3\left[-64\cos t + 16\cos t\right]$$

$$= -48a^3\cos t.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4a^2\sin^2 2t + 4a^2\cos^2 2t + 4a^2\cos^2 t}$$

$$= 2a\sqrt{1+\cos^2 t}$$

. Curvature $(K) = \dfrac{\left| \dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2} \right|}{\left| \dfrac{d\vec{r}}{dt} \right|^3}$

$= \dfrac{4a^2\sqrt{3\cos^2 t+5}}{2a\sqrt{1+\cos^2 t}}$

$= \dfrac{2a\sqrt{3\cos^2 t+5}}{\sqrt{1+\cos^2 t}}$.

Torsion $(\tau) = \dfrac{\left[\dfrac{d\vec{r}}{dt} \; \dfrac{d^2\vec{r}}{dt^2} \; \dfrac{d^3\vec{r}}{dt^3} \right]}{\left| \dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2} \right|^2}$

$= \dfrac{-48a^3\cos t}{\left[4a^2\sqrt{3\cos^2 t+5}\right]^2}$

$= \dfrac{-48a^3\cos t}{16a^4(3\cos^2 t+5)}$.

$= -\dfrac{3\cos t}{3\cos^2 t+5}$.

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15M
2007
8(b). For any constant vector $\vec{a}$, show that the vector represented by curl $(\vec{a}\times\vec{r})$ is always parallel to the vector $\vec{a}$, $\vec{r}$ being the position vector of a point (x, y, z), measured from the origin.

Soln: Let $\vec{r}$ be the position vector of a point (x, y, z)

$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a constant vector

Then the scalars a_1, a_2, a_3 are all constants.

we have

$$\vec{r}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \hat{i}(a_3y - a_2z) + \hat{j}(a_1z - a_3x) + k(a_2x - a_1y)$$

$$\therefore \text{curl}(\vec{r}\times\vec{a}) = \nabla\times(\vec{r}\times\vec{a}) = \begin{vmatrix} i & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$$= \hat{i}(-a_1 - a_1) + \hat{j}(-a_2 - a_2) + k(-a_3 - a_3)$$

$$= -2a_1\hat{i} - 2a_2\hat{j} - 2a_3k = -2(a_1\hat{i} + a_2\hat{j} + a_3k)$$

$$= -2\vec{a}.$$

$\therefore$ curl $(\vec{r}\times\vec{a})$ is parallel to the vector $\vec{a}$.

2007 15M
2006 1994 repeated

8(c). If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value(s) of n in order that $r^n\vec{r}$ may be solenoidal or irrotational.

Soln: Let $F = r^n\vec{r}$.

The vector F is irrotational if

$$\nabla \times F = 0.$$

i.e., $\nabla \times (r^n\vec{r}) = 0$.

We know that $\text{curl}(\phi A) = (\nabla\phi)\times A + \phi\,\text{curl} A$ ——— (1)

putting $\phi = r^n$ and $A = \vec{r}$ in (1)

we get

$$\text{curl}(r^n\vec{r}) = (\nabla r^n)\times\vec{r} + r^n(\text{curl}\,\vec{r}) \quad \text{——— (2)}$$

Now $\nabla r^n = \sum i\dfrac{\partial}{\partial x} r^n$

$= \sum i\, n r^{n-1}\dfrac{\partial r}{\partial x}$

$= n r^{n-1}\sum i\dfrac{\partial r}{\partial x}$

$= n r^{n-1}\sum i\dfrac{x}{r}$ $\quad \left(\because r = \sqrt{x^2+y^2+z^2},\ \dfrac{\partial r}{\partial x} = \dfrac{1}{2\sqrt{x^2+y^2+z^2}}2x = \dfrac{x}{r}\right)$

$= n r^{n-1}\cdot\dfrac{1}{r}\sum ix$

$= n r^{n-1}\dfrac{1}{r}\vec{r}$

$= n r^{n-2}\vec{r}$. ——— (3)

Also $\text{curl}\,\vec{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$

$= i(0) + j(0) + k(0) = 0.$

$\therefore$ from (2), we have

$$\text{curl}(r^n\vec{r}) = (n r^{n-2}\vec{r}) \times \vec{r} + r^n(0)$$

$$= n r^{n-2}(\vec{r}\times\vec{r})$$

$$= 0.$$

$\therefore$ curl $(r^n\vec{r}) = 0$ for any value of n.

$\therefore$ $r^n\vec{r}$ is an irrotational vector for any value of n.

The vector F is solenoidal if $\text{div}F = 0$

i.e., $\text{div}\, r^n\vec{r} = 0$

We know that $\text{div}(\phi A) = \phi\, \text{div}A + A\cdot\text{grad}\phi$ —— (4)

putting $A = \vec{r}$ and $\phi = r^n$ in (4),

we get

$$\text{div}(r^n\vec{r}) = r^n\,\text{div}\,\vec{r} + \vec{r}\cdot\text{grad}\, r^n$$

$$= r^n(3) + \vec{r}\cdot n r^{n-2}\vec{r} \quad (\text{from } (3)) \quad (\because \text{div}\,\vec{r} = 3)$$

$$= 3r^n + n(r^{n-2})(\vec{r}\cdot\vec{r})$$

$$= 3r^n + n r^n$$

$$= (n+3) r^n$$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if $(n+3)r^n = 0$

i.e., only if $n+3 = 0$

$$\Rightarrow \boxed{n = -3}.$$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if $\boxed{n=-3}$ and irrotational for any value of n.

15M/2007 → Determine $\int (y\,dx + z\,dy + x\,dz)$ by using Stoke's theorem, where 'C' is the curve defined by $(x-a)^2+(y-a)^2+z^2=2a^2$, $x+y=2a$ that starts from the point $(2a,0,0)$ and goes at first below the z-plane.

8(d).

Sol^n: The centre of the sphere $x^2+y^2+z^2-2ax-2ay=0$ is the point $(a,a,0)$

Since the plane $x+y=2a$ passes through the point $(a,a,0)$.

$\therefore$ The circle C is great circle of this sphere.

$\therefore$ Radius of the circle = Radius of the sphere

$$= \sqrt{a^2+a^2}$$
$$= \sqrt{2a^2}$$
$$= \sqrt{2}\,a.$$

Now $\int_C y\,dx+z\,dy+x\,dz = \int (y\hat{i}+z\hat{j}+x\hat{k})\cdot dr$

$$= \iint_S [\text{curl}(y\hat{i}+z\hat{j}+x\hat{k})]\cdot\hat{n}\,ds$$

$\left(\because \int_C F\cdot dr = \iint_S \text{curl } F\cdot\hat{n}\,ds \text{ by Stoke's theorem}\right)$

where S is any surface of which circle C is boundary.

Now curl $(xi+yj+zk) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$

$= -i-j-k$

$= -(i+j+k)$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle 'C'.

Then a vector normal to S is

grad $(x+y) = i+j$

$\therefore \hat{n} =$ unit normal to S $= \frac{1}{\sqrt{2}}(i+j)$

$\therefore \int\limits_C y\,dx + z\,dy + x\,dz = \iint\limits_S -(i+j+k)\cdot\left(\frac{i}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)ds$

$= \frac{-2}{\sqrt{2}}\iint ds$

$= -\sqrt{2}$ (area of the circle of radius $a\sqrt{2}$)

$= -\sqrt{2}\left[\pi(a\sqrt{2})^2\right]$

$= -\sqrt{2}\,\pi a^2(2)$

$= -2\sqrt{2}\pi a^2$