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6(a) Solve the DE

$$(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$$ (10)

Compare with, $M\,dx + N\,dy = 0$

$M = y^2 + 2x^2y$ $\qquad \dfrac{\partial M}{\partial y} = 2y + 2x^2$

$N = 2x^3 - xy$ $\qquad \dfrac{\partial N}{\partial x} = 6x^2 - y$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\therefore$ DE is not exact.

Rearranging the terms :

$$(y^2dx - xy\,dy) + (2x^2y\,dx + 2x^3dy) = 0$$

$$y(y\,dx - x\,dy) + x^2(2y\,dx + 2x\,dy) = 0$$

Let $x^\alpha y^\beta$ be an integrating factor for

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0$$

then $\dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n}$; $\dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q}$

$\therefore \dfrac{0+\alpha+1}{1} = \dfrac{1+\beta+1}{-1}$; $\dfrac{2+\alpha+1}{2} = \dfrac{0+\beta+1}{2}$

$\Rightarrow \alpha + \beta = -3$

$\alpha - \beta = -2$ $\qquad \Rightarrow \alpha = -\dfrac{5}{2}$, $\beta = \dfrac{-1}{2}$

$\therefore x^{-5/2}y^{-1/2}$ is an integrating factor

Multiplying it ~~with~~ in given DE.

$$\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2} + x^{-3/2}y^{1/2}\right)dy = 0$$

General Solution: $\displaystyle\int_{y\text{-constant}} \left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \int 0\,dy = 0$

(Terms of N without x)

$$\boxed{-\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = C}$$

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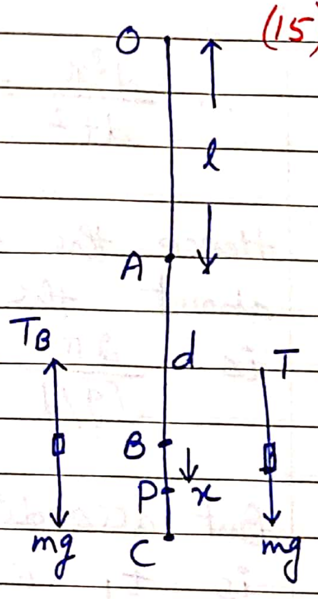
6(b) Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and c_1 and c_2 are the statical extensions due to these weights and g is the acceleration due to gravity. Show that

$$g = \frac{4\pi^2 (c_1 - c_2)}{T_1^2 - T_2^2}$$

(15)



One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m. Let AB = d.

In the equilibrium position of the particle of mass m at B, the tension $T_B = \lambda\left(\frac{d}{l}\right)$ in string OB balances the weight mg of the particle.

$$\therefore \lambda \frac{d}{l} = mg \quad \text{or} \quad \frac{\lambda}{lm} = \frac{g}{d} \quad \text{— (1)}$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where BP = x. When the particle is at P, the tension T_P in the string at P is $\lambda\left(\frac{d+x}{l}\right)$, acting vertically downwards.

By Newton's second law of motion, the equation of motion of particle at P is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{(d+x)}{l} + mg$$

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(Here weight mg is taken with +ve sign because it is acting downwards i.e. in the direction of x increasing.)

$$m\frac{d^2x}{dt^2} = -\frac{\lambda d}{l} - \frac{\lambda x}{l} + mg$$

$$= -\frac{\lambda x}{l} \qquad \left[\because \frac{\lambda d}{l} = mg \text{ \_using (1)}\right]$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{lm}x = -\frac{g}{d}x$$

Hence the motion of particle is SHM about the centre B and its period is $\frac{2\pi}{\sqrt{g/d}}$ i.e. $2\pi\sqrt{\frac{d}{g}}$.

But according to the question, the period is t_1 when $d = c_1$ and period is t_2 when $d = c_2$

i.e. $t_1 = 2\pi\sqrt{\frac{c_1}{g}}$, $t_2 = 2\pi\sqrt{\frac{c_2}{g}}$

$$\Rightarrow t_1^2 - t_2^2 = \left(2\pi\sqrt{\frac{c_1}{g}}\right)^2 - \left(2\pi\sqrt{\frac{c_2}{g}}\right)^2$$

$$\Rightarrow g = \frac{4\pi^2(c_1 - c_2)}{t_1^2 - t_2^2}$$

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6(c) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2 k$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$.

$$\text{Curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= i(0-0) + j(3z^2 - 3z^2) + k(2x - 2x)$$

$$= 0$$

Hence, $\vec{F}$ is a conservative force.

Let ϕ be the scalar potential

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= (2xy + z^3)i + x^2 j + 3xz^2 k$$

$$\Rightarrow \quad \phi(x,y,z) = x^2 y + xz^3$$

Work done in moving particle from $(1,-2,1)$ to $(3,1,4)$

$$= \phi(3,1,4) - \phi(1,-2,1)$$

$$= (9\cdot1 + 3\cdot64) - (1(-2) + 1\cdot1)$$

$$= 201 + 1 = 202$$

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