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UPSC CSE 2022 Mathematics Paper 1 – Solutions

S.No	UPSC Question	Topic	SuccessClap Test Series 2022
1	Paper 1-5c	Analytic Geometry	Full Length Test 7 -2b
2	Paper 1 8a-II	ODE	Full Length Test 1-7c
3	Paper 2-4b	Real Analysis	Full Length Test 4-3c

S.No	UPSC Question	Topic	SuccessClap Question Bank Source
1	Paper 1-1d	Calculus	SC B10 Qn 52
2	1e	Calculus	SC B05 Qn 6
3	2c	Analytic Geometry	SC C08 Qn 32
4	3b	Calculus	SC B15 Qn 26
5	Paper 2-1b	Complex Analysis	SC H01 Qn 43
6	1c	Real Analysis	SC B10 Qn 55
7	2a	Real Analysis	SC B26 Qn 3
8	2b	Algebra	SC G07 Qn 17
9	3b	Real Analysis	SC B07 Qn 4
10	5a	PDE	SC I01 Qn 16
11	6a	PDE	SC I05 Qn 2
12	7a	PDE	SC I04 Qn 20
13	8a	PDE	SC I07 Qn 4

1a) **Prove that any set of n linearly independent vectors in a vector space V of dimension n constitutes a basis for V.**

Given a) Set of "n" + Linear independent vectors

b) Finite dimension - n

Let $S=\{\alpha_1, \alpha_2 .. \alpha_n\}$ be given set of n + Linear independent vectors

<u>To prove</u> : S is basis of V

Given S: <u>If S is basis</u> → No need to proceed

↓

<u>If S is not basis</u> (<u>If assumed</u>)

then it can be extended to form basis

↳ so lets add p-terms to S to form basis (Add atleast one term)

Here $p \geq 1$ (Add atleast one term)

$S' = S \cup \{\beta_1, \beta_2 \dots \beta_p\}$ such that S' is Basis

$\boxed{\text{BUT}}$

Thm: Vector space V of dimension n must contain <u>exact</u> n vectors

But S' has more than n vectors

So S' cannot be basis

So assumption of [S is not basis] is false

and S is Basis.

1b) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3\\2\\8\end{pmatrix}$. Find $T\begin{pmatrix}2\\4\end{pmatrix}$

$$(2,4) = x(1,0) + y(1,1)$$
$$= (x,0) + (y,y) = (x+y, y)$$
$$\Rightarrow y = 4 \quad x = -2$$
$$(2,4) = -2(1,0) + 4(1,1)$$
$$T(2,4) = -2T(1,0) + 4T(1,1)$$
$$= -2(1\ 2\ 3) + 4(-3\ \ 2\ \ 8)$$
$$= (-2,-4,-6) + (-12\ \ 8\ \ 32)$$
$$= (-14, 4, 26)$$

1c) Evaluate $\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}$

$$y = \underset{x\to\infty}{Lt}\,(e^x+x)^{1/x}$$

$$\log y = \underset{x\to\infty}{Lt}\,\frac{\log(e^x+x)}{x} \quad \frac{\infty}{\infty} \text{ so use LH Rule}$$

$$= \underset{x\to\infty}{Lt}\,\frac{\frac{1}{e^x+x}(e^x+1)}{1} \quad \frac{\infty}{\infty} \text{ so again LH}$$

$$= \underset{x\to\infty}{Lt}\,\frac{e^x}{e^x+1} = \frac{1}{1+e^{-x}}$$

$$= \frac{1}{1+0} = 1$$

$$\boxed{y=e}$$

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1d) Examine the convergence of $\int_0^2 \frac{dx}{(2x-x^2)}$.

SuccessClap - Question Bank
Calculus & Real SC-B10 Qn.52

$2x-x^2 = x(2-x)$

$x=0, 2$ are points to be checked

$I = \int_0^c \frac{dx}{2x-x^2} + \int_c^2 \frac{dx}{2x-x^2}$ $\quad 0<c<2$

$\boxed{I_1}$ At $x=0$ $\quad I_1 = \int_0^c dx\, f(x)$ $\quad f(x) = \frac{1}{x(2-x)}$

μ-Test:

$\int_a^b f(x)\,dx$ converge iff $\mu<1$ $\rightarrow \underset{x\to a+}{Lt} (x-a)^{\mu} f(x)$, $\underset{x\to b-}{Lt} (b-x)^{\mu} f(x)$ [exists, non zero finite]

$\mu=1$ $\quad \underset{x\to 0+}{Lt} x\, f(x) = \underset{x\to 0}{Lt} \frac{1}{2-x} = \frac{1}{2}$

Since $\mu \not< 1$ So Diverge

$\boxed{II}$ At $x=2$ $\quad \mu=1$

$\underset{x\to 2-}{Lt} (2-x) f(x) = \underset{x\to 2-}{Lt} \frac{1}{x} = \frac{1}{2}$

$\mu \not< 1$ So Diverge

1e) A variable plane passes through a fixed point (a, b, c) and meets the axes at points A, B and C respectively. Find the locus of the centre of the sphere passing through the points O, A, B and C, O being the origin.

Let eqn of plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$

plane meets $O(0,0,0)$, $A(\alpha,0,0)$, $B(0,\beta,0)$, $C(0,0,\gamma)$

↳ Eqn of sphere through OABC is

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$$

↓

Its centre is $\left(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}\right)=(c_1,c_2,c_3)$

$\alpha=2c_1$

$\beta=2c_2$

$\gamma=2c_3$

Let plane pass thru (a,b,c) ⇒

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$$

$$\frac{a}{2c_1}+\frac{b}{2c_2}+\frac{c}{2c_3}=1$$

Locus of Centre is $(c_1,c_2,c_3)\rightarrow(x,y,z)$

$$\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=1$$

$$\boxed{\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2}$$

SuccessClap : Question Bank : Geometry
SC-CO5-Qn-6

2a) **Find all solutions to the following system of equations by row-reduced method:**

$$x_1 + 2x_2 - x_3 = 2$$
$$2x_1 + 3x_2 + 5x_3 = 5$$
$$-x_1 - 3x_2 + 8x_3 = -1$$

Row-reduction method

$$\left|\begin{matrix} 1 & 2 & -1 & 2 \\ 2 & 3 & 5 & 5 \\ -1 & -3 & 8 & -1 \end{matrix}\right| = [A \mid I] \qquad \begin{matrix} R_2 \to R_2 - R_1(2) \\ R_3 \to R_3 + R_1 \end{matrix}$$

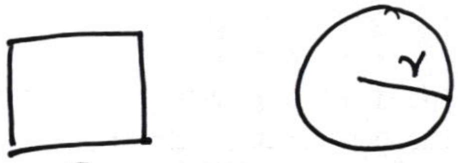
$$\left|\begin{matrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 7 & 1 \\ 0 & -1 & 7 & 1 \end{matrix}\right| \xrightarrow{R_3 \to R_3 - R_2} \left|\begin{matrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}\right|$$

$$\xrightarrow[R_1 \to R_1 + 2R_2]{} \left|\begin{matrix} 1 & 0 & 13 & 4 \\ 0 & -1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}\right| \Rightarrow \begin{bmatrix} x + 13z = 4 \\ -y + 7z = 1 \end{bmatrix}$$

Let $z = \alpha \Rightarrow y = 7\alpha - 1$
$x = -13\alpha + 4$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -13\alpha + 4 \\ 7\alpha - 1 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -13 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

2b) **A wire of length l is cut into two parts which are bent in the form of a square and a circle respectively. Using Lagrange's method of undetermined multipliers, find the least value of the sum of the areas so formed.**



radus = r

Side = a

Given $l = 4a + 2\pi r$

To find min value of $u = a^2 + \pi r^2$

$$\left.\begin{aligned} l &= 4a + 2\pi r \\ U &= a^2 + \pi r^2 \end{aligned}\right]$$ Lagrange multiplier

$dl = 4da + 2\pi dr$

$dU = 2ada + 2\pi r dr$

$du + \lambda dl = 0$

$$\Rightarrow \left.\begin{aligned} 2a + 4\lambda &= 0 \\ 2\pi r + 2\pi\lambda &= 0 \end{aligned}\right] \Rightarrow \left.\begin{aligned} \lambda &= -\frac{a}{2} \\ \lambda &= -r \end{aligned}\right] \Rightarrow r = \frac{a}{2}$$

$$l = 4a + 2\pi r = 4a + a\pi = (4+\pi)a \Rightarrow a = \frac{l}{4+\pi}$$

$$U = a^2 + \pi r^2 = a^2 + \pi\frac{a^2}{4} = \frac{4+\pi}{4}a^2 = \frac{4+\pi}{4}\frac{l^2}{(4+\pi)^2}$$

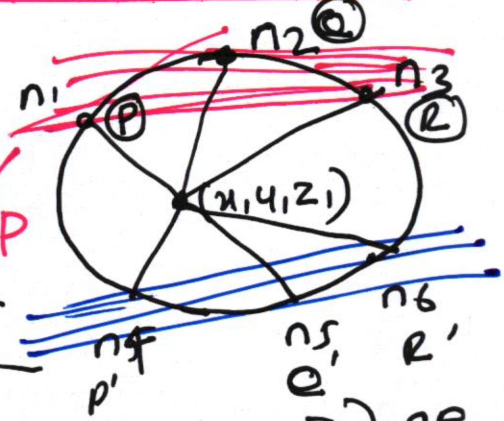
$$= \left(\frac{4+\pi}{4}\right)\left(\frac{l}{4+\pi}\right)^2 = \frac{l^2}{4(\pi+4)}$$

2c) If $P, Q, R; P', Q', R'$ are feet of the six normals drawn from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane PQR is represented by $lx+my+nz=p$, show that the plane $P'Q'R'$ is given by

$$\frac{x}{a^2 l}+\frac{y}{b^2 m}+\frac{z}{c^2 n}+\frac{1}{p}=0$$

SuccessClap : Queston Bank : Geometry
SC-C08 Qn-32

Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



Plane $lx+my+nz=p$

Find this plane eqn

→ Let (x_1,y_1,z_1) be any point
4 Normal to the ellipsoid thrue (x_1,y_1,z_1) are given by points (P,Q,R) $(P'Q'R')$

$\alpha=\frac{a^2x_1}{a^2+\lambda}$ $\beta=\frac{b^2y_1}{b^2+\lambda}$ $\gamma=\frac{c^2z_1}{c^2+\lambda}$ (Feet points)

Put (α,β,γ) in Ellipsoid eqn $\Rightarrow$ $\frac{a^2x_1^2}{(a^2+\lambda^2)^2}+\frac{b^2y_1^2}{(b^2+\lambda^2)^2}+\frac{c^2z_1^2}{(c^2+\lambda^2)^2}=1$ — ①

Six $\lambda \Rightarrow$ Six points

→ PQR is $lx+my+nz=p$

$$l\left(\frac{a^2x_1}{a^2+\lambda}\right)+m\left(\frac{b^2y_1}{b^2+\lambda}\right)+n\left(\frac{c^2z_1}{c^2+\lambda}\right)=p \quad \text{—②}$$

→ $P'Q'R'$ be let $l'x+m'y+n'z=p'$

$$l'\left(\frac{a^2x_1}{a^2+\lambda}\right)+m'\left(\frac{b^2y_1}{b^2+\lambda}\right)+n'\left(\frac{c^2z_1}{c^2+\lambda}\right)=p' \quad \text{—③}$$

Eqn 2 → 3 points] Combined gives 6 points
Eqn 3 → 3 points which is eqn 1 (Six points

Product of Eqn 2 & Eqn 3 gives Eqn 1
↓
Comparing coefficients

$$\frac{ll'a^4x_1^2}{(a^2+\lambda)^2} = \frac{a^2x_1^2}{(a^2+\lambda)^2} \Rightarrow ll' = \frac{1}{a^2}$$

Similarly $mm' = \frac{1}{b^2}$, $nn' = \frac{1}{c^2}$ $pp' = 1$

$$\Rightarrow l' = \frac{1}{a^2 l},\ m' = \frac{1}{b^2 m},\ n' = \frac{1}{c^2 n}\quad p' = 1$$

Put in $P'Q'R'$ eqn we get

$$\boxed{\frac{x}{a^2 l} + \frac{y}{b^2 m} + \frac{z}{nc^2} = \frac{1}{p}}$$

3a) Let the set $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \begin{matrix} x - y - z = 0 \text{ and} \\ 2x - y + z = 0 \end{matrix} \right\}$ be the collection of vectors of a vector space $\mathbb{R}^3(\mathbb{R})$. Then
(i) prove that P is a subspace of $\mathbb{R}^3$.
(ii) find a basis and dimension of P.

$\rightarrow$ P is soln of $x-y-z=0$ & $2x-y+z=0$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 3 & -2 & 0 \\ 2 & -1 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 3x = 2y \\ 2x - y + z = 0 \end{bmatrix} \Rightarrow \text{Let } y = 3\alpha \quad \alpha \in R$$

$\Rightarrow 3x = 2y \Rightarrow 3x = 2.3\alpha \Rightarrow x = 2\alpha$

$2x - y + z = 0 \Rightarrow 4\alpha - 3\alpha + z = 0 \Rightarrow z = -\alpha$

P is of format $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 3\alpha \\ -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Clearly Basis is $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and dimension is 1

(a) To show subspace:

Let $l, m \in P \Rightarrow l = \alpha_1 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad m = \alpha_2 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Let $a, b \in R$

$$al + mb = a\alpha_1 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha_2 b \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= (a\alpha_1 + b\alpha_2) \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$a \in R, b \in R, \alpha_1 \in R, \alpha_2 \in R \Rightarrow a\alpha_1 + b\alpha_2 \in R$

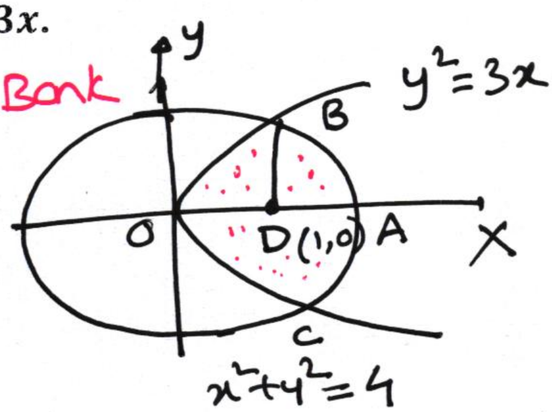
Let $a\alpha_1 + b\alpha_2 = \alpha_3$
$\alpha_3 \in R$

So $al + mb = \alpha_3 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \alpha_3 \in R$

$\Rightarrow al + mb \in P$ so subspace

3b) Use double integration to calculate the area common to the circle $x^2+y^2=4$ and the parabola $y^2=3x$.

SuccessClap Question Bank
Similar Qn
SC B15 - Qn - 26



$y^2=3x$ & $x^2+y^2=4$

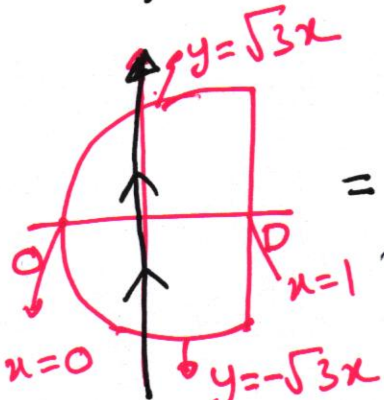
$x^2+3x-4=0$

$\Rightarrow x=1,\ x=-4 \Rightarrow D(1,0)$

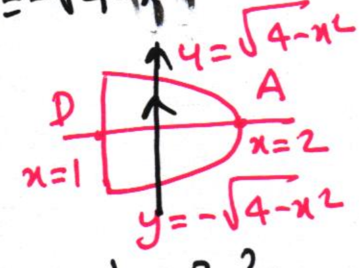
$B(1,\sqrt{3}),\ C(1,-\sqrt{3})$

$y=\pm\sqrt{3}$

Area = Area OBC + Area BCA



$$=\int_{x=0}^{x=1}\int_{y=-\sqrt{3x}}^{y=\sqrt{3x}} dy\,dx + \int_{x=1}^{x=2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx$$



$$\text{Area}=\int_{x=0}^{x=1} 2\sqrt{3}\,x^{1/2}dx + \int_{x=1}^{x=2} 2\sqrt{4-x^2}\,dx$$

$$=2\sqrt{3}\left.\frac{x^{3/2}}{3/2}\right|_0^1 \qquad + 2\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2$$

$$=\frac{4}{\sqrt{3}} \qquad + 2\left[2\sin^{-1}1-\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{6}\right)\right]$$

$$+2\left[2\times\frac{\pi}{2}-\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right]$$

$$+2\pi-\sqrt{3}-\frac{2\pi}{3}=\frac{4\pi}{3}-\sqrt{3}$$

$$\text{Area}=\frac{4}{\sqrt{3}}-\sqrt{3}+\frac{4\pi}{3}=\frac{4}{\sqrt{3}}-\frac{3}{\sqrt{3}}+\frac{4\pi}{3} \qquad \sqrt{3}=\frac{3}{\sqrt{3}}$$

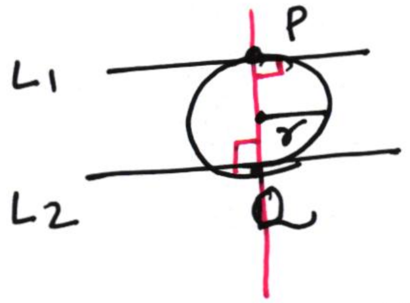
$$=\frac{1}{\sqrt{3}}+\frac{4\pi}{3}$$

3c) **Find the equation of the sphere of smallest possible radius which touches the straight lines :** $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ **and** $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ $\quad L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$

→ Clearly L_1 and L_2 is not parallel

Smallest possible radius between two lines ⇒ Perpendicular line to both Lines L_1 and L_2 pass through circle centre.



$P \Rightarrow P(3r_1+3, -r_1+8, r_1+3)$

$Q \Rightarrow Q(-3r_2-3, 2r_2-7, 4r_2+6)$

Direction ratio of PQ is

$((3r_1+3)-(-3r_2-3), (-r_1+8)-(2r_2-7), (r_1+3)-(4r_2+6))$

$= (3r_1+3r_2+6, -r_1-2r_2+15, r_1-4r_2-3)$

$\perp$ to $(3,-1,1)$ L_1 $\qquad$ $\perp$ to $(-3,2,4)$ L_2

so $3(3r_1+3r_2+6) - 1(-r_1-2r_2+15) + 1(r_1-4r_2-3) = 0$

$11r_1 + 7r_2 = 0$

so $-3(3r_1+3r_2+6) + 2(-r_1-2r_2+15) + 4(r_1-4r_2-3) = 0$

$-7r_1 - 29r_2 = 0$

$\Rightarrow r_1 = r_2 = 0$

$\Rightarrow P(3,8,3)$ $\quad Q(-3,-7,6)$

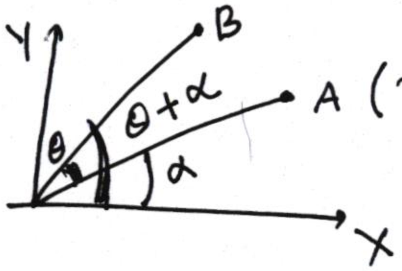
sphere eqn $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

$\Rightarrow x^2+y^2+z^2-y-9z-47=0$

4a) **Find a linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector of $\mathbb{R}^2$ by an angle θ. Also, prove that for $\theta = \frac{\pi}{2}$, T has no eigenvalue in $\mathbb{R}$.**

Similar question on R^3 – rotation + dilation asked in SuccessClap Test Series 2022-Qn-7

→ Same qn on R^2 axis rotation is present in SuccessClap Question Bank - Set 15-LT-2 Qn2



A $(r\cos\alpha, r\sin\alpha)$

A: $x = r\cos\alpha$
$y = r\sin\alpha$

↓ rotated by θ

New coordinate B

$B \to (x', y') \to x' = r\cos(\theta+\alpha) = r[\cos\alpha\cos\theta - \sin\theta\sin\alpha]$

$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$

$= x\cos\theta - y\sin\theta$

$y' = r\sin(\theta+\alpha) = r(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$

$= x\sin\theta + y\cos\theta$

$T(x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Part 2: $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\theta = 90^\circ} T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Eigen $|T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$

$\lambda = \pm i$ (No real solution)

Eigen values not present in Real R

But Eigen values present in Complex system

4b) Trace the curve $y^2x^2 = x^2 - a^2$, where a is a real constant.

→ X-axis symetry : power of y is even ✓

→ Y-axis symetry : power of x is even ✓

→ Symetry about $y=x$ → $y^2x^2 = x^2 - a^2$ —(1)

↓ Xchage
$y \to x$
$x \to y$

$x^2y^2 = y^2 - a^2$ —(2)

(1) ≠ (2)

Not symetric about $y=x$

→ Put $x=0, y=0 \Rightarrow 0 = -a^2$ Eqn do not satisfy

origin do not lie on curve

→ pt of Intersection

$y=0 \Rightarrow x = \pm a$ $(a,0), (-a,0)$

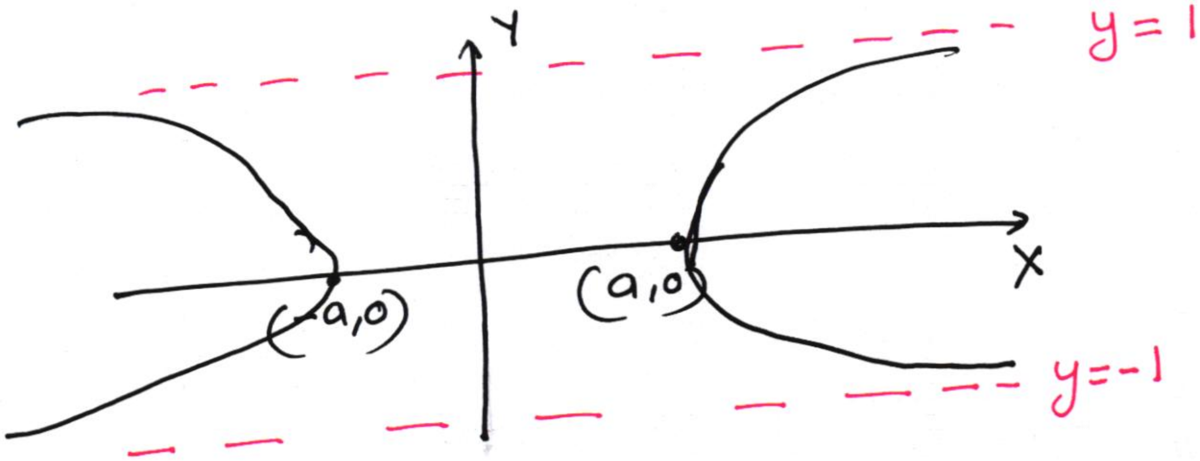
$x=0 \Rightarrow$ No relation

→ Asymptote :

→ ||el to x-axis $\Rightarrow$ $x^2 \underbrace{(y^2-1)}_{\text{Equate to zero}} = -a^2$

$y^2 - 1 = 0 \Rightarrow y = \pm 1$

→ ||el to y-axis $y^2 \underbrace{x^2}_{\text{Equate to zero}} = x^2 - a^2 \Rightarrow x = 0$



Region:
$0<x<a$ Let $x=\frac{a}{2}$ $\Rightarrow$ $y^2x^2=x^2-a^2$

$$y^2\frac{a^2}{4}=\frac{a^2}{2}-a^2$$
$$=-\frac{1}{2}a^2$$

$y^2=-2$
No soln
$\downarrow$
do not lie in $0<x<a$

→ Symetry ⇒ Do not lie in $-a<x<0$

→ Check $y>1$ ⇒ $y=2$ ⇒ $y^2x^2=x^2-a^2$

$$4x^2=x^2-a^2$$
$$3x^2=-a^2$$
$\downarrow$
Not possible

$y>1$ do not lie

Similarly by symetry
$y<1$ do not lie

4c) If the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators, then prove that $(b + c)u^2 + (c + a)v^2 + (a + b)w^2 = 0$

SuccessClap Test Series 2022
Full Length Test – 7 Paper 1- 2(b)

Let the equations of a line of section of the cone $ax^2 + by^2 + cz^4 = 0$ by the plane $ux + vy + wz = 0$ be given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad ..(1)$$

$$Then \ \ al^2 + bm^2 + cn^2 = 0 \quad ..(2)$$

$$and \ \ vl + vm + wn = 0 \quad ..(3)$$

Eliminating n between (2) and (3), we get

$$al^2 + bm^2 + c\{-ul + vm)/w\}^2 = 0$$

$$(aw^2 + cu^2)l^2 + 2cuvlm + (bw^2 + cv^2)m^2 = 0$$

$$(aw^2 + cu^2)(l/m)^2 + 2cuv(l/m) + (bw^2 + cv^2) = 0$$

This is a quadratic equation in l/m and hence it shows that the plane $ux + vy + wz = 0$ cuts the given cone in two lines (or generators). Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of these lines. Then l_1/m_1 and l_2/m_2 are the roots of the equation (4).

The product of the roots of the equation (4) is given by

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bw^2 + cv^2}{aw^2 + cu^2}.$$

$$\therefore \quad \frac{l_1 l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{cu^2 + aw^2} = \frac{n_1 n_2}{av^2 + bu^2}$$

writing the third fraction by symmetry.

Now the generators (I.e. the lines) will be perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$(bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2) = 0$$

$$(b + c)u^2 + (c + a)v^2 + (a + b)w^2 = 0$$

Wrong Printing

5a) Show that the general solution of the differential equation $\frac{dy}{dx} + Py =$ Q can be written in the form $y = \frac{Q}{P} - e^{-\int P dx}\left\{C + \int e^{\int P dx} d\left(\frac{Q}{P}\right)\right\}$, where P, Q are non-zero functions of x and C, an arbitrary constant.

$\rightarrow \frac{dy}{dx} + Py = Q \Rightarrow (Py - Q)dx + dy = 0$

Exact $\Rightarrow \frac{\partial}{\partial y}(Py - Q) = P$, $\frac{\partial 1}{\partial x} = 0$

$P \neq 0$

$\Rightarrow$ Not exact

Exact
$Mdx + Ndy = 0$
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\rightarrow$ Let $IF = e^{\int P dx}$ $\rightarrow$ Check if it is exact

$\underbrace{(Py - Q)IF}_{M'}\, dx + \underbrace{IF}_{N'}\, dy = 0$

$\frac{\partial M'}{\partial y} = P.IF$ $\qquad$ $\frac{\partial N'}{\partial x} = \frac{\partial}{\partial x} e^{\int P dx} = e^{\int P dx} \frac{d}{dx}\left(\int P dx\right)$

$= e^{\int P dx} \cdot P$

$= P.IF$

Same

So Exact

$(Py - Q)IF\, dx + IF\, dy = 0$

$Py.(IF)dx + (IF)dy - Q(IF)dx = 0$ — (1)

$d(y.IF) = (IF)dy + y\, d\left(e^{\int P dx}\right)$

$= (IF)dy + y \cdot e^{\int P dx} \cdot \underbrace{d\left(\int P dx\right)}_{= P}$

$= (IF)dy + Py \cdot IF$

$d(y.IF) = Q.(IF)dx$

Integrating

$$y.(IF) = \int Q.(IF)\,dx + C$$

$$y.(IF) = Q\int (IF)\,dx - \int \underbrace{\frac{dQ}{dx}\int (IF)\,dx} + C$$

$$\int IF = \int e^{\int P\,dx} = \frac{e^{\int P\,dx}}{P} = \frac{(IF)}{P}$$

$$y.IF = Q.\frac{IF}{P} - \int \frac{dQ}{dx}\frac{(IF)}{P}\,dx + C$$

$$y = \frac{Q}{P} - \left(\frac{1}{IF}\right)\left[\int \left((IF).\left(\frac{dQ}{dx}\right).\frac{1}{P}\right)dx + C\right]$$

$$= \frac{Q}{P} = e^{-\int P\,dx}\left[C + \int e^{\int P\,dx}.\frac{dQ}{dx}.\frac{1}{P}\,dx\right]$$

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5b) Show that the orthogonal trajectories of the system of parabolas $x^2 = 4a(y+a)$ belong to the same system.

→ To prove self-orthogonal → <u>do not solve</u> <u>Completely</u>

→ $x^2 = 4a(y+a) \Rightarrow 2x\dfrac{dx}{dy} = 4a \Rightarrow a = \dfrac{x}{2}\dfrac{dx}{dy}$

$\downarrow$

$x^2 = 2x\dfrac{dx}{dy}\left[y + \dfrac{x}{2}\dfrac{dx}{dy}\right]$

$p = \dfrac{dy}{dx}$

$x = 2\dfrac{dx}{dy}\left(y + \dfrac{x}{2}\dfrac{dx}{dy}\right)$

$x = \dfrac{2}{p}\left(y + \dfrac{x}{2p}\right) = \dfrac{2}{p}\left(\dfrac{2py + x}{2p}\right) = \dfrac{2py + x}{p^2}$

$xp^2 - x - 2py = 0$

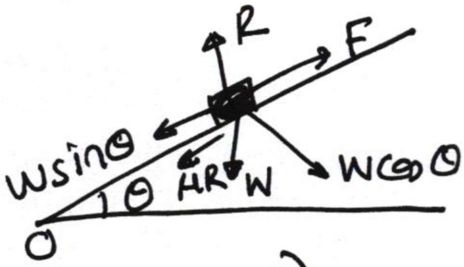
$\downarrow$ orthogonal $p \to -\dfrac{1}{p}$

$\dfrac{x}{p^2} - x - 2\left(-\dfrac{1}{p}\right)y = 0 \Rightarrow \dfrac{x - xp^2 + 2yp}{p^2} = 0$

$xp^2 - 2py - x = 0$

Same as above $\Rightarrow$ So self orthogonal

5c) **A body of weight w rests on a rough inclined plane of inclination θ, the coefficient of friction, μ, being greater than $\tan\theta$. Find the work done in slowly dragging the body a distance 'b' up the plane and then dragging it back to the starting point, the applied force being in each case parallel to the plane.**



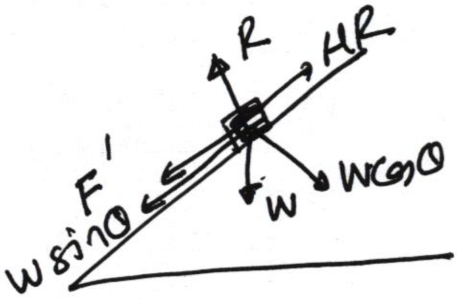
Case 1: up plane

$F = W\sin\theta + \mu R$
$R = W\cos\theta$

$F = W\sin\theta + \mu W\cos\theta = W\cos\theta(\mu + \tan\theta)$

workdone to drag "b" distance $W = bF$

$W = bW\cos\theta(\mu + \tan\theta)$



Case 2: Down plane

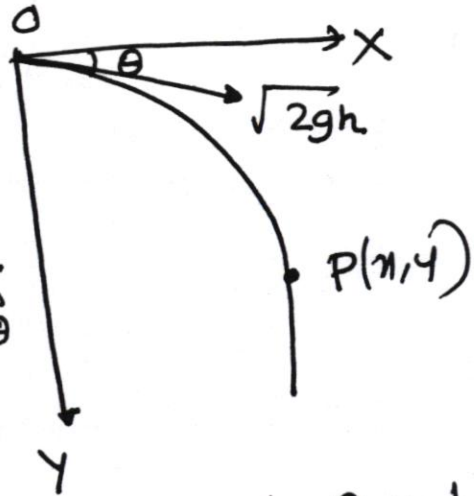
~~W+~~
$W\sin\theta + F' = \mu R$
$R = W\cos\theta$

$F' = \mu R - W\sin\theta = \mu W\cos\theta - W\sin\theta$
$= W\cos\theta(\mu - \tan\theta) > 0 \quad \mu > \tan\theta$

work done $W' = bF'$
$= bW\cos\theta(\mu - \tan\theta)$

Total workdone (Up and Down) (as asked)
$= W + W' = 2Wb\mu\cos\theta$

5d) A projectile is fired from a point O with velocity $\sqrt{2gh}$ and hits a tangent at the point $P(x, y)$ in the plane, the axes OX and OY being horizontal and vertically downward lines through the point O, respectively. Show that if the two possible directions of projection be at right angles, then $x^2 = 2hy$ and then one of the possible directions of projection bisects the angle POX.



$$\begin{cases} x = \sqrt{2gh}\cos\theta\, t \\ y = \sqrt{2gh}\sin\theta\, t + \frac{1}{2}gt^2 \end{cases}$$

Should eliminate $t \Rightarrow t = \frac{x}{\sqrt{2gh}\cos\theta}$

$y = x\tan\theta + \frac{x^2\sec^2\theta}{4h}$

$\Rightarrow \tan^2\theta + \frac{4h}{x}\tan\theta + 1 - \frac{4hy}{x^2} = 0$

Directions are at right angle $\tan\theta_1 \tan\theta_2 = -1$

~~tan~~ $1 - \frac{4hy}{x^2} = -1 \Rightarrow \frac{2hy}{x^2} = 1$

$\tan^2\theta + \frac{4h}{x}\tan\theta - 1 = 0$

$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{x}{2h} = \frac{y}{x}$

$\angle POX = \phi$

$\tan 2\theta = \frac{y}{x} = \tan\phi$

$\theta = \frac{\phi}{2}$ Bisects

5e) Show that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational.
Also find ϕ such that $\vec{A} = \nabla\phi$.

→ Irrotational means $\nabla \times A = 0$ (Prove)

$$\nabla \times A = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \hat{\imath}(-1+1) - \hat{\jmath}(3z^2-3z^2) + \hat{k}(6x-6x) = 0$$

$\Rightarrow$ A is irrotational

→ $A = \nabla\phi$ Find ϕ

$$A = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy+z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y,z)$$

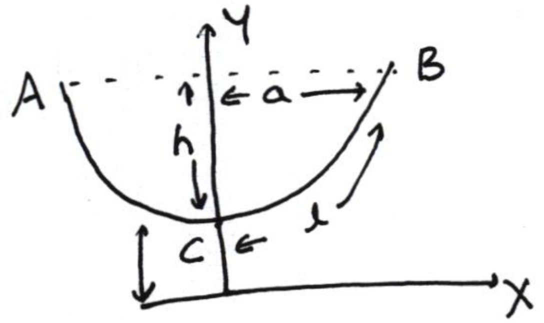
$$\frac{\partial\phi}{\partial y} = 3x^2-z \Rightarrow \phi = 3x^2y - yz + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2-y \Rightarrow \phi = xz^3 - yz + f_3(x,y)$$

Comparing $f_1(y,z) = -yz$, $f_2(x,z) = xz^3$
$f_3(x,y) = 3x^2y$

$$\boxed{\phi = 3x^2y + xz^3 - yz}$$

6a) **A cable of weight w per unit length and length $2l$ hangs from two points P and Q in the same horizontal line. Show that the span of the cable is $2l\left(1-\frac{2h^2}{3l^2}\right)$, where h is the sag in the middle of the tightly stretched position.**



$y^2 = c^2 + s^2$

$\rightarrow$ At B : $y = c+h$, $s = l$

$\rightarrow (c+h)^2 = c^2 + l^2$

$\Rightarrow c^2 + h^2 + 2ch = c^2 + l^2$

$c = \frac{l^2-h^2}{2h} \Rightarrow \frac{l}{c} = \frac{2lh}{l^2-h^2}$ —(1)

$\rightarrow s = c\sinh\frac{x}{c}$

At B $\quad l = c\sinh\left(\frac{a}{c}\right)$ $\qquad$ we want $2a$ = span

$\frac{a}{c} = \sinh^{-1}\frac{l}{c} = \log\left(\frac{l}{c} + \sqrt{\left(\frac{l}{c}\right)^2+1}\right)$ $\qquad \sinh^{-1}x = \log(x+\sqrt{x^2+1})$

$\left(\frac{l}{c}\right)^2 + 1 = \frac{(2lh)^2}{(l^2-h^2)^2} + 1 = \frac{4l^2h^2 + l^4 + h^4 - 2l^2h^2}{(l^2-h^2)^2}$

$= \frac{(l^2+h^2)^2}{(l^2-h^2)^2} \Rightarrow \sqrt{\left(\frac{l}{c}\right)^2+1} = \frac{l^2+h^2}{l^2-h^2}$

$\frac{l}{c} + \sqrt{\left(\frac{l}{c}\right)^2+1} = \frac{2lh}{l^2-h^2} + \frac{l^2+h^2}{l^2-h^2} = \frac{(l+h)^2}{l^2-h^2} = \frac{(l+h)^2}{(l-h)(l+h)}$

$= \frac{l+h}{l-h}$

$\frac{a}{c} = \log\frac{l+h}{l-h} \Rightarrow a = c\log\frac{l+h}{l-h}$

$a = \frac{l^2-h^2}{2h}\log\frac{l+h}{l-h}$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots$$

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \cdots\right)$$

$$\log\frac{l+h}{l-h} = \log\frac{1+h/l}{1-h/l} = 2\left(\frac{h}{l} + \frac{1}{3}\frac{h^3}{l^3} + \cdots\right)$$

$$a = \frac{l^2-h^2}{2h}\log\frac{l+h}{l-h} = \frac{l^2-h^2}{2h} \times 2\left(\frac{h}{l} + \frac{h^3}{3l^3} + \cdots\right)$$

$$= l^2 - h^2\left(\frac{1}{l} + \frac{h^2}{3l^3} + \cdots\right)$$

$$= \frac{l^2-h^2}{l} + (l^2-h^2)\left(\frac{h^2}{3l^3}\right) + \cdots$$

$$= l \underbrace{- \frac{h^2}{l} + \frac{h^2}{3l}} - \frac{h^4}{3l^3}$$

$$= \cancel{l - \frac{2h^2}{3l}} \quad l - \frac{2h^2}{3l} - \frac{h^4}{3l^3} + \cdots$$

$$= l\left(1 - \frac{2h^2}{3l^2} - \frac{1}{3}\left(\frac{h}{l}\right)^4 + \cdots\right.$$

$$= l\left(1 - \frac{2h^2}{3l^2}\right) \quad \text{neglecting } \left(\frac{h}{l}\right)^4 \text{ higher terms)}$$

$$\text{span} = 2a = 2l\left(1 - \frac{2h^2}{3l^2}\right)$$

6b) Solve the following differential equation by using the method of variation of parameters : $(x^2-1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2-1)^2$, given that $y = x$ is one solution of the reduced equation.

$y_2 + Py_1 + Qy = R \qquad P = \frac{-2x}{x^2-1} \quad Q = \frac{2}{x^2-1} \quad R = x^2-1$

$y = x$ is one soln , Let $y = uv = xv$

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0 \Rightarrow \frac{d^2v}{dx^2} + \left(\frac{-2x}{x^2-1} + \frac{2}{x}\right)\frac{dv}{dx} = 0$

$\frac{d^2v}{dx^2} + \left(\frac{-2}{x(x^2-1)}\right)\frac{dv}{dx} = 0 \qquad$ Let $\frac{dv}{dx} = t$

$\frac{dt}{t} = \frac{2dx}{x(x^2-1)} = 2\left[-\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right]$

Integrate $\log t = -2\log x + \log (x-1) + \log(x+1) + \log c_1$

$t = c_1\left(\frac{x^2-1}{x^2}\right) = \frac{dv}{dx} \Rightarrow dv = c_1\left(1 - \frac{1}{x^2}\right)dx$

$v = c_1\left(x + \frac{1}{x}\right) + c_2$

$y = uv = vx = c_1(x^2+1) + c_2 x$

$y = Au + Bv \qquad u = x^2+1 \quad v = x \qquad W = \begin{vmatrix} x^2+1 & 2x \\ x & 1 \end{vmatrix}$

$= -x^2+1 \neq 0$

$A = \int \frac{-vR}{-x^2+1} = \int x\,dx = \frac{x^2}{2}$

$B = \int \frac{uR}{-x^2+1} = -\int (x^2+1)dx = -\left(\frac{x^3}{3} + x\right)$

$y_p = \frac{x^2}{2}(x^2+1) - \left(\frac{x^3}{3} + x\right)x = x^4\left(\frac{1}{2} - \frac{1}{3}\right) + x^2\left(\frac{1}{2} - 1\right)$

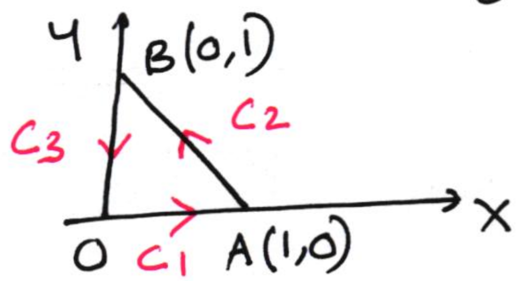
$= x^4\left(\frac{1}{6}\right) - \frac{x^2}{2} = \frac{x^4}{6} - \frac{x^2}{2}$

$y = c_1(x^2+1) + c_2 x + \frac{x^4}{6} - \frac{x^2}{2}$

6c) **Verify Green's theorem in the plane for** $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, **where** C **is the boundary curve of the region defined by** $x = 0, y = 0$, **where C is the boundary curve of the region defined by** $x = 0, y = 0, x + y = 1$

Green $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M dx + N dy = I$

$M = 3x^2 - 8y^2 \quad N = 4y - 6xy$



LHS

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$

$I = \iint_R 10y\,dx\,dy$

$= 10 \int_{x=0}^{x=1} \int_{y=0}^{1-x} y\,dx\,dy$

$= 10 \int_0^1 \frac{y^2}{2} \Big|_0^{1-x} dx$

$= 5 \int_0^1 (1-x)^2 dx$

$= 5 \int_0^1 (x-1)^2 dx$

$= \frac{5}{3} (x-1)^3 \Big|_0^1$

$= \frac{5}{3}$

RHS

C_1 : $y = 0 \quad dy = 0$

$I_1 = \int 3x^2 dx = x^3 \Big|_0^1 = 1$

C_3 : $x = 0 \quad dx = 0$

$I_2 = \int_1^0 4y\,dy = 2y^2 \Big|_1^0$
$= -2$

C_2 : $x = 1 - y \quad dx = -dy$

$I_2 = \int_0^1 \left(3(1-y)^2 - 8y^2\right)(-dy) + \left[4y - 6y(1-y)\right] dy$

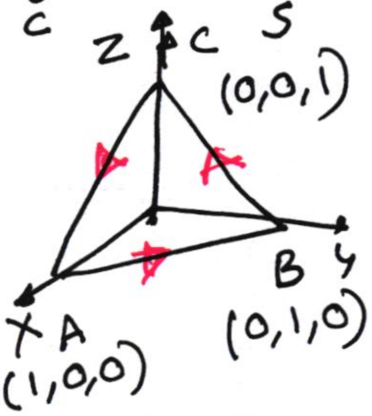
$= \int_0^1 (11y^2 + 4y - 3)\,dy$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

$I = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

7a) Verify Stokes' theorem for $\vec{F} = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$ over the plane surface : $x + y + z = 1$ lying in the first octant.

Stoke thm : $\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n}\, ds = I$



LHS

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$F \cdot dr = x\,dx + z^2 dy + y^2 dz$

$\overrightarrow{AB}$: $x + y = 1 \quad z = 0 \quad dz = 0$

$dy = -dx$

$I_1 = \int_{x=1}^{x=0} x\,dx = -1/2$

$\overrightarrow{BC}$: $x = 0 \quad dx = 0 \quad y + z = 1$

$dz = -dy$

$z = 1 - y$

$I_2 = \int_{y=1}^{y=0} (1-y)^2 dy - y^2 dy$

$= \int_1^0 (1 - 2y)\,dy = y - y^2 \Big|_1^0 = 0$

$\overrightarrow{CA}$: $y = 0 \quad dy = 0 \quad x + z = 1$

$dz = -dx$

$I_3 = \int_{x=0}^{1} x\,dx = \frac{1}{2}$

$I_1 + I_2 + I_3 = -\frac{1}{2} + 0 + \frac{1}{2} = 0$

RHS

$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z^2 & y^2 \end{vmatrix}$

$= 2(y - z)\,\hat{i}$

$\phi = x + y + z - 1 \quad \nabla\phi = \hat{i} + \hat{j} + \hat{k}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

$(\nabla \times F) \cdot \hat{n} = \frac{2(y - z)}{\sqrt{3}}$

$ds = \frac{dx\,dy}{|n \cdot k|} = \frac{dx\,dy}{1/\sqrt{3}}$

$I = 2\iint (y - z)\,dx\,dy \qquad z = 1 - x - y$

$y - z = 2y + x - 1$

$= 2\iint (2y + x - 1)\,dx\,dy$

$= 2\int_{x=0}^{1}\int_{y=0}^{1-x} (2y + x - 1)\,dy$

$= 2\int^1 y^2 + xy - y \Big|_0^{1-x} dx$

$= 2\int_0^1 \left[\underbrace{(1-x)^2 + x(1-x) - (1-x)}_{=0}\right]$

$= 2\int 0\,dx$

7b) **Solve the following initial value problem by using Laplace's transformation**

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = h(t), \text{ where}$$

$$h(t) = \begin{cases} 2, & 0 < t < 4, \\ 0, & t > 4, \end{cases} \quad y(0) = 0, y'(0) = 0$$

$$y''(t) - 3y'(t) + 2y = h(t)$$

$$L[y''(t)] - 3L(y') + 2L(y) = L(h(t))$$

$$p^2 L(y(t)) - py(0) - y'(0) - 3[pL(y(t)) - y(0)] + 2L(y) = \int_0^4 e^{-pt} \cdot 2dt$$

$$(p^2 - 3p + 2) L(y(t)) = 2 \left. \frac{e^{-pt}}{-p} \right|_0^4$$

$$= 2\left(\frac{1-e^{-4p}}{p}\right)$$

$$L(y(t)) = \frac{2(1-e^{-4p})}{p(p-1)(p-2)} = (1-e^{-4p})\left[\frac{1}{p} - \frac{2}{p-1} + \frac{1}{p-2}\right]$$

$$= \frac{1}{p} - \frac{2}{p-1} + \frac{1}{p-2} - \frac{e^{-4p}}{p} - \frac{2e^{-4p}}{p-1} - \frac{e^{-4p}}{p-2}$$

$\downarrow$ Inverse laplace

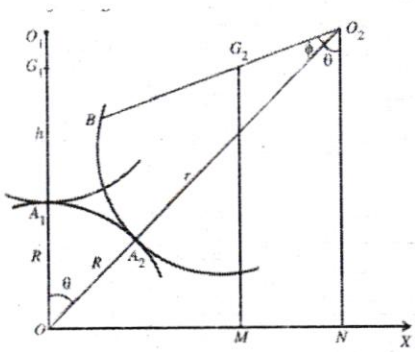
$$y(t) = \begin{cases} 1 - 2e^t + e^{2t} - 1 - 2e^{t-4} - e^{2t-4} & t > 4 \\ 1 - 2e^t + e^{2t} \end{cases}$$

$$y(t) = \begin{cases} e^{2t} - 2e^t + 2e^{t-4} - e^{2t-4} & t > 4 \\ 1 - 2e^t + e^{2t} & t < 4 \end{cases}$$

7c) Suppose a cylinder of any cross-section is balanced on another fixed cylinder, the contact of curved surfaces being rough and the common tangent line horizontal. Let ρ and ρ' be the radii of curvature of the two cylinders at the point of contact and h be the height of centre of gravity of the upper cylinder above the point of contact. Show that the upper cylinder is balanced in stable equilibrium if h $< \frac{\rho\rho'}{\rho+\rho'}$.

Let O be the centre of the spherical surface of the lower body which is fixed and O_1 that of the upper body which rests on the lower body, A_1 being their point of contact and the line OO_1 being vertical. If G_η is the centre of gravity of the upper body, then for the equilibrium of the upper body, the line A_1G_1 must be vertical ; let A_1G_η be h. The figure is a section of the bodies by a vertical plane through G_1.

Suppose the upper body is slightly displaced by pure rolling over the lower body. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of O_1A_1. Also G_2 is the new position of G_1 so that $BG_2 = A_1G_1 = h$.



Let $\angle A_1OA_2 = \theta$ and $\angle BO_2A_2 = \phi$; so that $\angle G_2O_2N = \theta + \phi$.
We have $O_1A_1 = r$ and $OA_1 = R$. Also $O_2A_2 = O_2B = r$ and $OA_2 = R$. Since the upper body rolls on the lower body without slipping, therefore

$$\text{arc } A_1A_2 = \text{arc } A_2B \text{ i.e., } R\theta = r\phi \text{ i.e., } \phi = (R/r)\theta.$$

Now in order to find the nature of equilibrium, we should find the height z of the centre of gravity G_2 in the new position above the fixed horizontal line OX. We have

$$\begin{aligned} z &= G_2M = O_2N - O_2G_2\cos(\theta + \phi) \\ &= O_2\cos\theta - (O_2B - BG_2)\cos(\theta + \phi) \\ &= (R + r)\cos\theta - (r - h)\cos(\theta + \phi) \end{aligned}$$

$$= (R + r)\cos\theta - (r - h)\cos\{\theta + (R/r)\theta\} \; [\because \phi = (R/r)\theta]$$

$$= (R + r)\cos\theta - (r - h)\cos\left\{\frac{\theta(r + R)}{r}\right\}.$$

For equilibrium we have $dz/d\theta = 0$
i.e., $-(R + r)\sin\theta + (r - h)\sin\left\{\frac{\theta(r+R)}{r}\right\}\frac{r+R}{r} = 0.$

This is satisfied by $\theta = 0$.

$$\frac{d^2z}{d\theta^2} = -(R \quad +r)\cos\theta + (r-h)\cos\left\{\frac{\theta(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^2$$

$$\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} = -(R+r) + (r-h)\left(\frac{r+R}{r}\right)^2$$

$$= \left(\frac{r+R}{r}\right)^2\left\{(r-h) - \frac{r^2}{R+r}\right\} = \left(\frac{r+R}{r}\right)^2\left\{r - \frac{r^2}{R+r} - h\right\}$$

$$= \left(\frac{r+R}{r}\right)^2\left\{\frac{rR}{R+r} - h\right\}.$$

This will be positive if

$$\frac{rR}{R+r} > h \text{ i.e., } \frac{1}{h} > \frac{R+r}{rR} \text{ i.e., } \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

$$\text{and negative, if } \frac{rR}{R+r} < h \text{ i.e., } \frac{1}{h} < \frac{1}{r} + \frac{1}{R}$$

Hence the equilibrium is stable or unstable according as

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R} \text{ or } \frac{1}{h} < \frac{1}{r} + \frac{1}{R}.$$

Here R is the radius of the lower body and r that of the upper body and h is the height of the C.G. of the upper body above the point of contact.
Now it remains to discuss the case when $1/h = 1/r + 1/R$ i.e., $h = rR/(R+r)$.
In this case $d^2z/d\theta^2 = 0$. Hence we find $d^3z/d\theta^3$ and $d^4z/d\theta^4$. We have

$$\frac{d^3z}{d\theta^3} = (R+r)\sin\theta - (r-h)\sin\left\{\frac{\theta(r+R)}{r}\right)\cdot\left(\frac{r+R}{r}\right)^3$$

$$\frac{d^4z}{d\theta^4} = (R+r)\cos\theta - (r-h)\cos\left\{\frac{\theta(r+R)}{r}\right)\cdot\left(\frac{r+R}{r}\right)^4.$$

$$\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0} = 0$$

$$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R+r) - (r-h)\left(\frac{r+R}{r}\right)^4$$

$$= (R+r)\left\{1 - \frac{r-h}{r}\left(\frac{r+R}{r}\right)^3\right\}$$

$$= (R+r)\left\{1 - \frac{r-h}{r}\cdot\frac{R+r}{r}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(r - \frac{rR}{R+r}\right)\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \frac{r^2}{R+r}\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(1 + \frac{R}{r}\right)^2\right\}, \text{ which is negative.}$$

$$\left[\because h = \frac{rR}{r+R}\right]$$

This shows that z is maximum and so in this case the equilibrium is unstable.
Hence if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$, the equilibrium is stable
and if $\frac{1}{h} \le \frac{1}{r} + \frac{1}{R}$, the equilibrium is unstable.

8a-I) Find the general and singular solutions of the differential equation: $(x^2 - a^2)p^2 - 2xyp + y^2 + a^2 = 0$, where $p = \frac{dy}{dx}$. Also give the geometric relation between the general and singular solutions.

$$(x^2-a^2)p^2 - 2xyp + y^2 + a^2 = 0$$

$$x^2p^2 - 2xyp + y^2 - a^2p^2 + a^2 = 0$$

$$(xp-y)^2 + a^2(1-p^2) = 0 \Rightarrow (xp-y)^2 = a^2(p^2-1)$$

$$xp - y = \pm a\sqrt{p^2-1} \qquad y = px \mp a\sqrt{p^2-1}$$

$\downarrow$ Clairaut's form

Its general soln is

$$y = xc \mp a\sqrt{c^2-1}$$

$$\downarrow$$

$$(x^2-a^2)c^2 - 2xyc + y^2 + a^2 = 0$$

p-Disc & c-Disc are same as both in same Clairaut form

<u>p-Disc</u>: $B^2 - 4AC = 0$

$$(2xy)^2 - 4(x^2-a^2)(y^2+a^2) = 0$$

$$x^2y^2 - (x^2y^2 + x^2a^2 - a^2y^2 - a^4) = 0$$

$$(x^2 + y^2 + a^2)(a^2 = 0$$

$$x^2 - y^2 = a^2$$

$\hookrightarrow$ singular Solutn

Geometric relation between G.S and S.S is Hyperbola curve

8a-II) Solve the following differential equation :

$$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$$

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$3x+2=e^z$ $\quad$ $\therefore (3x+2)^2D^2 = 3^2 D_1(D_1-1)$

$D_1 \equiv \frac{d}{dz}$

$(3x+2)D = 3D_1$

Eqn. becomes $\left[9\left[D_1(D_1-1)\right]+5\times 3D_1-3\right]y = \left(\frac{e^z-2}{3}\right)^2+\left(\frac{e^z-2}{3}\right)+1$

$\rightarrow \left(9D_1^2+6D_1-3\right)y = e^{2z}-e^z+7$

$\rightarrow$ AE : $9m^2+6m-3=0 \Rightarrow m=-1, 1/3$

$y_{CF} = c_1e^{-z}+c_2e^{z/3} = \frac{c_1}{e^z}+c_2(e^z)^{1/3} = \frac{c_1}{3x+2}+c_2(3x+2)^{1/3}$

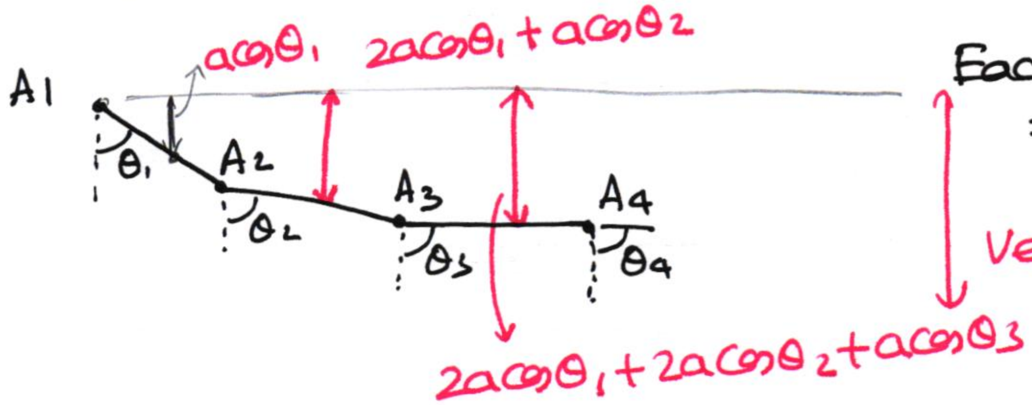
$\rightarrow y_P = \frac{e^{2z}-e^z+7}{9D^2+6D-3} = \frac{e^{2z}}{9\times4+6\times2-3} - \frac{e^z}{9+6-3}+\frac{7}{-3}$

$= \frac{e^{2z}}{45}-\frac{e^z}{12}-\frac{7}{3} =$

$= \frac{(3x+2)^2}{45}-\frac{(3x+2)}{12}-\frac{7}{3}$

$y = y_{CF}+y_P$

8b) A chain of n equal uniform rods is smoothly jointed together and suspended from its one end A_1. A horizontal force $\vec{P}$ is applied to the other end A_{n+1} of the chain. Find the inclinations of the rods to the downward vertical line in the equilibrium configuration.



Each rod length $= 2a$

Vertical distance

$V =$ Potential $= -Wa\cos\theta_1$
$-W(2a\cos\theta_1 + a\cos\theta_2)$
$-W(2a\cos\theta_1 + 2a\cos\theta_2 + a\cos\theta_3)$
$-W(2a\cos\theta_1 + 2a\cos\theta_2 + \cdots$
$\vdots$
$-W(2a\cos\theta_1 + 2a\cos\theta_2 + \cdots + 2a\cos\theta_{n-1} + \cos\theta_n)$

$= (-Wa)\left[\cos\theta_1(1+2(n-1)) + \cos\theta_2(1+2(n-2)) + \cos\theta_3(1+2(n-3)) + \cdots \cos\theta_n\right)$

first   next

$= (-Wa)\left[(2n-1)\cos\theta_1 + (2n-3)\cos\theta_2 + (2n-5)\cos\theta_3 + \cdots + 3\cos\theta_{n-1} + \cos\theta_n\right]$

Virtual displacement

work done by gravity is $\delta W_1 = -\delta V$

$\delta W_1 = Wa\left[(2n-1)\sin\theta_1\,\delta\theta_1 + (2n-3)\sin\theta_2\,\delta\theta_2 + (2n-5)\sin\theta_3\,\delta\theta_3 + 3\sin\theta_{n-1}\,\delta\theta_{n-1} + \sin\theta_n\,\delta\theta_n\right)$

Work done by force P

$\delta W_2 = P \times$ Horizontal displacement

$$\delta W_2 = P\delta\left(2a\sin\theta_1 + 2a\sin\theta_2 + \cdots 2a\sin\theta_n\right)$$
$$= 2Pa\left(\cos\theta_1\delta\theta_1 + \cos\theta_2\delta\theta_2 + \cdots \cos\theta_n\delta\theta_n\right)$$

Total workdone $\delta W = \delta w_1 + \delta w_2$

$$\delta W = \left[2P\cos\theta_1 - (2n-1)w\sin\theta_1\right]a\,\delta\theta_1$$
$$+\left[2P\cos\theta_2 - (2n-3)w\sin\theta_2\right]a\,\delta\theta_2$$
$$+ \cdots \left[2P\cos\theta_n - w\sin\theta_n\right]a\,\delta\theta_n$$
$$= 0$$

Equilibrium $\delta W = 0$

$$\left.\begin{aligned} &2P\cos\theta_1 - (2n-1)w\sin\theta_1 = 0\\ &2P\cos\theta_2 - (2n-3)w\sin\theta_2 = 0\\ &\vdots\\ &2P\cos\theta_n - w\sin\theta_n = 0\end{aligned}\right]$$

Hence $\tan\theta_1 = \dfrac{2P}{w(2n-1)}$ $\quad \tan\theta_2 = \dfrac{2P}{w(2n-3)} \cdots \tan\theta_n = \dfrac{2P}{w}$

observation $\dfrac{1}{\tan\theta_1} = \dfrac{w(2n-1)}{2P}$ $\quad \dfrac{1}{\tan\theta_2} = \dfrac{w(2n-3)}{2P}$

$\dfrac{1}{\tan\theta_3} = \dfrac{w(2n-5)}{2P} \cdots \dfrac{1}{\tan\theta_n} = \dfrac{w}{2P}$

$\dfrac{1}{\tan\theta_1}, \dfrac{1}{\tan\theta_2}, \dfrac{1}{\tan\theta_3}, \cdots \dfrac{1}{\tan\theta_n}$ are Arithmetic progression with common difference -2

8c) Using Gauss' divergence theorem, evaluate

$\iint_S \vec{F} \cdot \vec{n} dS$, **where** $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ **and S is the cylinder formed by the surfaces** $z_0 = 0, z = 1, x^2 + y^2 = 4$

$$\int_S F.n\,ds = \int_V \nabla.F\,dV = I$$

$$\nabla.F = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2-1) = 1-1+2z = 2z$$

$$I = \int_V 2z\,dV \qquad V: \text{Cylinder}$$

Method 1 : Use Cylinder Coord System to Solve Fast

$$dV = \rho\,d\rho\,d\theta\,dz \qquad I = \int_{\rho=0}^{\rho=2}\int_{\theta=0}^{2\pi}\int_{z=0}^{z=1}(2z)(\rho\,d\rho\,d\theta\,dz)$$

$$I = 2\left.\frac{z^2}{2}\right|_0^1 \left.\frac{\rho^2}{2}\right|_0^2 (2\pi) = 2\times\frac{1}{2}\times 2\times 2\pi = \underline{4\pi}$$

Method 2 : Cartesian system

$$I = \int_{z=0}^{z=1}\int_{y=-2}^{y=2}\int_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} 2z\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=1}\int_{y=-2}^{y=2} 2zx\Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 4z\sqrt{4-y^2}\,dy\,dz = \int_{y=-2}^{2} 4.\frac{z^2}{2}\sqrt{4-y^2}\Big|_0^1 dy$$

$$= 2\int_{y=-2}^{2}\sqrt{4-y^2}\,dy = 4\int_0^2\sqrt{4-y^2}\,dy$$

$$= 4\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right|_0^2 = 4\times 2\times \sin^{-1}1 =$$

$$= 4\times 2\times\frac{\pi}{2} = 4\pi$$