IFoS MATHEMATICS (OPT.)-2009

PAPER - II : SOLUTIONS

1(a) Prove that a non-empty subset H of a group G is normal subgroup of G $\Leftrightarrow$ for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$.

Sol) Let H be a normal subgroup of G.

Then we have to prove that for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$.

Now we have
$$(gx)(gy)^{-1} = (gx)(y^{-1}g^{-1}) \quad (\because \text{ by reversal of law of } G)$$
$$= g(xy^{-1})g^{-1} \quad (\text{by asso. prop. of } G)$$
$$\in H \quad (\because xy^{-1} \in H, g \in G, H \text{ is normal subgroup of } G).$$

Conversely suppose that for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$.

we ~~prove~~ that the nonempty subset H of G is a normal subgroup of G.

Now let $x, y \in H$ ~~then~~ we have
$$xy^{-1} = exy^{-1}e^{-1} \quad (\because e \in G).$$
$$= (ex)(ey)^{-1}$$
$$\in H \quad (\because x, y \in H \subseteq G, e \in G \Rightarrow (ex)(ey)^{-1} \in H).$$

$\therefore xy^{-1} \in H.$

$\Rightarrow$ H is a <u>subgroup</u> of G.

Again, let $h \in H \subseteq G$, $g \in G$ then we have
$$(gh)(ge)^{-1} \in H \Rightarrow (gh)(e^{-1}g^{-1}) \in H$$
$$\Rightarrow g(he)g^{-1} \in H \quad (\because e^{-1} = e)$$
$$\Rightarrow ghg^{-1} \in H \quad (\because he = h).$$
$\Rightarrow$ H is <u>normal</u> subgroup of G.

1.(b) Show that the function $f(x)=\frac{1}{x}$ is not uniformly continuous on $]0,1]$.

Sol. Now we show that the function $f(x)=\frac{1}{x}$ is not uniformly continuous on $]0,1]$.

Let $\epsilon>0$ be given. We show that for each $\delta>0$, $\exists\ x_1, x_2 \in]0,1]$ such that $|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|\geq\epsilon$.

For this,
let the sequence (a_n) be defined by

$$a_n=\frac{1}{n+2\epsilon}-\frac{1}{n} \quad \text{(1)}$$

Clearly which is convergent to zero. (i.e. $a_n\to 0$ as $n\to\infty$).

$\therefore$ Given $\delta>0$, $\exists$ +ve integer 'm' such that

$$|a_n-0|<\delta \quad \forall n\geq m. \quad \text{(2)}$$

$$\Rightarrow \left|\frac{1}{n+2\epsilon}-\frac{1}{n}\right|<\delta \quad \forall n\geq m$$

Let $x_1=\frac{1}{m+2\epsilon}\in(0,1]$ and $x_2=\frac{1}{m}\in(0,1]$ then $|x_1-x_2|<\delta$.

whereas $|f(x_1)-f(x_2)|=|m+2\epsilon-m|$
$=2\epsilon>\epsilon$.

$\therefore |x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|>\epsilon$.

Hence f is not uniformly continuous on $(0,1]$.

②

1(d) If G is a finite abelian group, then show that $o(ab)$ is a divisor of l.c.m. of $o(a)$, $o(b)$.

Sol^n: Let $o(a)=n$, $o(b)=m$, $o(ab)=k$.

Let $l = \text{l.c.m}(m,n)$

then m/l, n/l

$\Rightarrow l = mr_1$, $l = nr_2$

Now $(ab)^l = a^l b^l$ ($\because$ G is abelian)

$= a^{nr_2} b^{mr_1}$

$= (a^n)^{r_2} (b^m)^{r_1}$ ($\because$ $o(a)=n$, then by def n is the least +ve integer such that $a^n=e$ ‖ly $b^m=e$.)

$= e^{r_2} e^{r_1}$

$= e.e.$

$(ab)^l = e.$

$\Rightarrow ab/l$ ($\because$ W.K.T if $a \in G$ be a finite order n and also $a^m=e$. then $o(a)/n$).

$\Rightarrow k/l.$

i.e., $o(ab)$ is a divisor of l.c.m of $o(a)$, $o(b)$

1(e): Evaluate $\int_C \frac{2z+1}{z^2+z} dz$ by Cauchy's integral formula, where C is $|z| = \frac{1}{2}$.

Sol^n: Given that $\int_C \frac{2z+1}{z^2+z} dz$

$= \int_C \frac{2z+1}{z(z+1)} dz = \int_C \frac{2z+1/z+1}{(z-0)} dz$.

Comparing the given integral with $\int \frac{f(z)}{z-z_0} dz$

we get $f(z) = \frac{2z+1}{z+1}$, $z_0 = 0$.

Since $f(z)$ is analytic in $|z|=\frac{1}{2}$ and

$z_0 = 0$ is a point inside $|z|=\frac{1}{2}$

we apply Cauchy's integral formula

$$\int \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0).$$

Since $f(z_0) = f(0)$

$$= \frac{2(0)+1}{0+1} = 1$$

$$\therefore \int_{|z|=\frac{1}{2}} \frac{\frac{2z+1}{z+1}}{(z-0)}\,dz = 2\pi i\, f(0)$$

$$= 2\pi i\,(1)$$

$$= 2\pi i.$$

→ 2(b): Determine the analytic function $w = u + iv$, if $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$.

Sol^n: It is given that

$$u = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$

$$= \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} \quad \left(\because \cosh 2y = \frac{e^{2y} + e^{-2y}}{2}\right)$$

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Now $$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cdot \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = 0 - \frac{2\sinh 2y \cdot \sin 2x}{(\cosh 2y - \cos 2x)^2} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad \text{--- (2)}$$

Consider $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ (by C-R equs)

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad (\text{from (2)})$$

Integrating w.r.t. x, we get

$$v = \int \frac{2\sin 2x \cdot \sinh 2y}{(\cosh 2y - \cos 2x)^2}\,dx + \phi(y)$$

where $\phi(y)$ is a constant function of the integration.

$$= \sinh 2y \int \frac{2\sin 2x}{(\cosh 2y - \cos 2x)^2}\,dx + \phi(y).$$

④

$= \sinh 2y \int \frac{1}{t^2} dt + \phi(y)$ putting $\cosh 2y - \cos 2x = t$ $\Rightarrow 2\sin 2x \, dx = dt$

$= \sinh 2y \left(-\frac{1}{t}\right) + \phi(y)$

$$v = \frac{-\sinh 2y}{\cosh 2y - \cos 2x} + \phi(y) \quad \text{——— ③}$$

Differentiating ③ partially w.r.t y, we get

$$\frac{\partial v}{\partial y} = -\frac{\left[2\cosh 2y(\cosh 2y - \cos 2x) - \sinh 2y(\sinh 2y)(2)\right]}{(\cosh 2y - \cos 2x)^2} + \phi'(y)$$

$$= -\frac{\left[2(\cosh^2 2y - \sinh^2 2y) - 2\cosh 2y \cos 2x\right]}{(\cosh 2y - \cos 2x)^2} + \phi'(y)$$

$$= -\frac{\left[2(1) - 2\cosh 2y \cos 2x\right]}{(\cosh 2y - \cos 2x)^2} + \phi'(y) \quad (\because \cosh^2 2y - \sinh^2 2y = 1)$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + \phi'(y) \quad \text{——— ④}$$

By C-R condition

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + \phi'(y) = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} \quad (\text{from ① \& ④})$$

$\Rightarrow \phi'(y) = 0$

$\Rightarrow \phi(y) = C$ (constant)

$$\therefore ③ \equiv v = \frac{-\sinh 2y}{\cosh 2y - \cos 2x} + C$$

Hence the corresponding analytic function is

$f(z) = w = u + iv$

$$= \frac{\sin 2x - i(\sinh 2y)}{\cosh 2y - \cos 2x} + iC$$

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Q.(c) Find the multiplicative inverse of the element $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ of the ring M, of all matrices of order two over the integers.

Sol) Let $M = \{[a_{ij}]_{2\times 2} / a_{ij} \in \mathbb{Z}\}$ be the given ring of all matrices of order 2×2 over the integers.

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \in M$ then $|A| = 6-5$
$= 1$
$\neq 0$.

$\therefore A$ is non-singular matrix.

$\therefore A^{-1}$ exists.

Now $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

$= \dfrac{1}{1} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \in M.$

$\therefore A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \in M \implies A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \in M.$

5

3(b): Evaluate by contour integration

$$\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}, \quad 0<a<1.$$

Sol^n: Let $I = \int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$ ——— (1)

Let the contour 'C' be the unit circle $|z|=1$. with centre at the origin.

Let $z=e^{i\theta}$ then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$= \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$\therefore \frac{1}{1-2a\sin\theta+a^2} = \frac{1}{1-2a\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]+a^2}$$

$$= \frac{i}{i-a\left(z-\frac{1}{z}\right)+a^2 i}$$

$$= \frac{zi}{(1+a^2)iz-az^2+a}$$

Since $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\therefore (1) \equiv \int_0^{2\pi} \frac{1}{1-2a\sin\theta+a^2}\,d\theta = \int_C \frac{iz \cdot \frac{dz}{iz}}{i(1+a^2)z-az^2+a}$$

where 'C' is unit circle of radius with centre at the origin.

$$= \int_C \frac{dz}{-az^2+i(1+a^2)z+a}$$

$$= \int_C \frac{dz}{ai^2z^2+iz+ia^2z+a} \quad (\because i^2=-1)$$

$$= \int_C \frac{dz}{iz(iaz+1)+a(iaz+1)}$$

$$= \int_C \frac{dz}{(iz+a)(iaz+1)}$$

$$= \int_C f(z)\,dz \quad \text{——— (2)}$$

where $f(z) = \dfrac{1}{(iz+a)(iaz+1)}$

$\therefore f(z)$ has poles at $z = -\frac{a}{i}$, $z = -\frac{1}{ia}$ of order 1.

But only the pole $z = -\frac{a}{i} = ai$ inside C.

$$\therefore \int_C f(z)\,dz = 2\pi i\left[\text{Residue at } z = ai\right] \quad \text{——— (3)}$$

Now residue at $z = ai$ of order 1 is

$$= \lim_{z\to ai} (z-ai)\frac{1}{(iz+a)(iaz+1)}$$

$$= \lim_{z\to ai} \cancel{(z-ia)}\frac{1}{i\cancel{(z-ai)}(iaz+1)}$$

$$= \lim_{z\to ai} \frac{1}{i(iaz+1)}$$

$$= \frac{1}{i(-a^2+1)} = \frac{1}{i(1-a^2)}.$$

$\therefore$ (3) $\equiv$

$$\int_C f(z)\,dz = 2\pi i \frac{1}{i(1-a^2)}$$

$$= \frac{2\pi}{1-a^2} \quad \text{where } 0<a<1$$

$$\therefore (2) \equiv \int_0^{2\pi} \frac{1}{1-2a\sin\theta + a^2}\,d\theta = \frac{2\pi}{1-a^2}; \text{ where } 0<a<1.$$

⑥

3(c) write the dual of the following LPP and hence solve it by graphical method.

Minimize $Z = 6x_1 + 4x_2$

constraints

$2x_1 + x_2 \geq 1$

$3x_1 + 4x_2 \geq 1.5$

$x_1, x_2 \geq 0$

Sol^n: Dual to the given LPP is

$\text{Max. } W = y_1 + 1.5y_2$ —— (1)

subject to the constraints

$2y_1 + 3y_2 \leq 6$ —— (2)

$y_1 + 4y_2 \leq 5$ —— (3)

$y_1, y_2 \geq 0$ —— (4)

Since every point which satisfies the condition $y_1 \geq 0$ & $y_2 \geq 0$ lies in the first quadrant only.

$\therefore$ The desired pair (y_1, y_2) is restricted the points of the first quadrant only.

Let us consider the constraint

$2y_1 + 3y_2 \leq 6$ as equality $2y_1 + 3y_2 = 6$ represents a line which passes through $(0,2)$ and $(3,0)$ in $y_1 y_2$ plane.

The origin $(0,0)$ gives $0 + 0 = 0 < 6$.

$\therefore$ All the points below and on this line satisfy the inequation $2y_1 + 3y_2 \leq 6$.

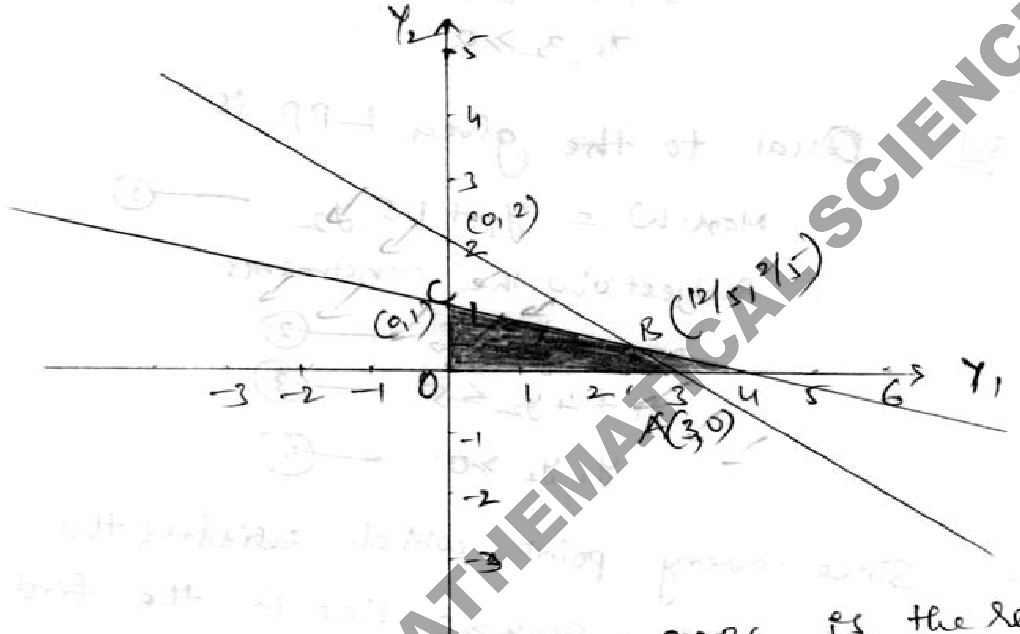
Let us consider the inequality $y_1+4y_2 \le 4$

as equality $y_1+4y_2=4$, represents a line

which passes through $(4,0)$, $(0,1)$ in y_1y_2 plane

$\therefore$ All the points below and on this line satisfy

the inequation $y_1+4y_2 \le 4$.



The shaded region OABC is the region

which satisfies the equations (2), (3) and (4)

$\therefore$ The shaded region OABC is called as

solution space or feasible region.

solving $2y_1+3y_2=6$ and $y_1+4y_2=4$

we get $y_1=12/5$, $y_2=2/5$.

The point B is given by $B(12/5, 2/5)$

$\therefore$ The four corners or extreme points

of the polygon are

$O(0,0)$, $A(3,0)$, $B(12/5, 2/5)$, $C(0,1)$.

The values of the objective function

$w = y_1+1.5y_2$ at these extreme points are

$w(O) = 0$

$w(A) = 3 + 1.5(0) = 3$

$w(B) = \frac{12}{5} + (1.5)\left(\frac{2}{5}\right) = \frac{12+3}{5} = 3$

$w(C) = 0 + 1.5(1) = 1.5$

$\therefore$ Maximum value ~~of~~ $w = 3$ at both the points $(3,0)$ and $(12/5, 2/5)$.

If we take any point on the line segment joining A and B, then the value of z at that point will also be the same.

→ 4(a): Show that $d(a) < d(ab)$, where a, b be two non-zero elements of a Euclidean domain R and b is not a unit in R.

Solⁿ: By the definition of the Euclidean domain, we have $d(ab) \geq d(a)$.

Let 'b' ~~be~~ not a unit in R.

Since a and b are non-zero elements of the Euclidean domain R.

$\therefore$ ab is also a non-zero element of R.

Now $a \in R$ and $0 \neq ab \in R$

$\therefore$ by definition of Euclidean domain $\exists$ elements q and r in R such that

$a = q(ab) + r$. ——— (1)

where either $r = 0$ or $d(r) < d(ab)$.

——— (2)

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If $r=0$ then

$a=qab$.

$\Rightarrow a-qab=0$

$\Rightarrow a(1-qb)=0$

$\Rightarrow 1-qb=0$ ($\because a\neq 0$ and R is an integral domain).

$\Rightarrow qb=1$.

$\Rightarrow b$ is invertible.

$\Rightarrow b$ is a unit in R.

Thus we get a contradiction.

Hence r cannot be zero.

$\therefore$ we must have

$d(r) < d(ab)$.

i.e, $d(ab) > d(r)$. —— ②.

Also from ①,

we have $r = a-qab$

$= a(1-qb)$.

$\therefore d(r) = d[a(1-qb)]$

But $d[a(1-qb)] \geq d(a)$ by the definition.

$\therefore d(r) \geq d(a)$. —— ③

$\therefore$ from ② & ③

we have

$d(a) \leq d(r) < d(ab)$

$\Rightarrow d(a) < d(ab)$.

→ **4(b):** Show that a field is an integral domain and a non-zero finite integral domain is a field.

Solⁿ: Let F be a field. then by definition F is a commutative ring with unity and every non-zero element is invertible w.r.t multiplication.

In order to prove that a field is an integral domain we have to prove that a field F has no zero divisors.

Let $a, b \in F$ and $a \neq 0$

Since F is a field.

For $a \neq 0 \in F \Rightarrow a^{-1}$ exists in F.

$\therefore aa^{-1} = a^{-1}a = 1.$

Now we have

$ab = 0$

$\Rightarrow a^{-1}(ab) = a^{-1}(0)$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1b = 0$

$\Rightarrow b = 0$

Similarly we can prove that $a, b \in F$

$b \neq 0$ and $ab = 0 \Rightarrow a = 0.$

$\therefore a, b \in F$ and $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

$\therefore$ F has no zero divisors

$\therefore$ A field is an integral domain.

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Now we show that a non-zero finite integral domain is a field.

Let F be the non-zero finite integral domain.

Let $F = \{a_1, a_2, \ldots, a_n\}$ and F contains 'n' distinct elements.

To prove that F is a field.

For this we enough to prove that the non-zero elements of F have multiplicative inverse.

Let $a \neq 0 \in F$.

$\therefore aa_1, aa_2, \ldots \ldots aa_n \in F$ (by closure property)

All these elements are distinct.

because: If possible

let $aa_i = aa_j$; $a_i, a_j \in F$

$\Rightarrow a(a_i - a_j) = 0$

$\Rightarrow a_i - a_j = 0$ ($\because a \neq 0$ & F is an ID i.e., F does not contain zero divisors)

$\Rightarrow a_i = a_j$

This is a contradiction to hypothesis that F contains 'n' distinct elements.

$\therefore$ Our assumption that $aa_i = aa_j$ is wrong.

$\therefore aa_1, aa_2, \ldots aa_n$ are all distinct elements in F which has exactly 'n' elements.

By the pigeon-hole principle, one of these products must be equal to one. ($\because$ F is an ID)

Let $aa_r = 1$ for some $a_r \in F$.

$\therefore a^{-1} = a_r$

$\therefore$ Every non-zero element of F has multiplicative inverse.

$\therefore$ F is a field.

4(c) Solve by simplex method, the following LPP.

Maximize $Z = 5x_1 + 3x_2$

constraints

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \geq 0.$$

Solⁿ: The objective function of the given LPP is of maximization type and R.H.S of all constraints are ≥ 0.

Now write the given LPP in the standard form.

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2.$$

subject to

$$\left.\begin{array}{l} 3x_1 + 5x_2 + s_1 + 0s_2 = 15 \\ 5x_1 + 2x_2 + 0s_1 + s_2 = 10 \end{array}\right\} \text{--- (1)}$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

where s_1, s_2 are slack variables.

Now the initial basic feasible solution is given by

setting $x_1 = x_2 = 0$ (non-basic)

$s_1 = 15;\ s_2 = 10.$ (basic)

$\therefore$ The initial basic feasible solution is $(0, 0, 15, 10)$

for which $Z = 0$.

Now we move from the current basic feasible solution to the next better basic feasible solution.

put the above information in tableau form:

	C_j	5	3	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	3	5	1	0	15	$\frac{15}{3}=5$
0	S_2	(5)	2	0	1	10	$\frac{10}{5}=2 \rightarrow$
$Z_j=\sum C_B a_{ij}$		0	0	0	0	0	
$C_j=c_j-z_j$		5 $\uparrow$	3	0	0		

from the above table,

x_1 is incoming variable as $C_j(=5)$ is maximum and the corresponding column is known as key column.

The minimum +ve ratio θ occurs in the second row.

$\therefore$ S_2 is outgoing variable and the common intersection element (5) is the key element.

Now convert the key element to unity and all other elements in its column to zero.

Then we obtain a new iterated simplex tableau as

	C_j	5	3	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	0	(19)	5	-3	45	$\frac{45}{19}=2.3 \rightarrow$
5	x_1	1	2/5	0	1/5	2	$\frac{2\times5}{2}=5$
$Z_j=\sum C_B a_{ij}$		5	2	0	1	10	
$C_j=$		0	1 $\uparrow$	0	-1		

$5R_1-3R_2$

$R_2 \rightarrow \frac{R_2}{5}$

from the above tableau,

x_2 is incoming variable as $C_j(=1)$, S_1 is the outgoing variable and (19) is the key element.

Now convert the key element to unity and all other elements in its column to zero.

Then we get the new iterated simplex tableau as

	C_j	5	3	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	b
3	x_2	0	1	5/19	-3/19	45/19
5	x_1	1	0	-2/19	5/19	20/19
$Z_j = \Sigma C_B a_{ij}$		5	3	5/19	16/19	235/19
$C_j =$		0	0	-5/19	-16/19	

As C_j is either zero or negative (i.e, $C_j \leq 0$) under all columns, the above tableau gives the optimal basic feasible solution.

$\therefore$ The optimal solution is

$$x_1 = \frac{20}{19} \qquad x_2 = \frac{45}{19}$$

and maximum $Z = 235/19$.

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section - B

5(a) Find complete and singular integrals of $(p^2+q^2)y = qz$.

Sol^n: The given equation is

$$f(x,y,z,p,q) = (p^2+q^2)y - qz = 0 \quad \text{---} \text{①}$$

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y+qz-2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

Taking the first two fractions we get

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$\Rightarrow pdp + qdq = 0$$

integrating, we get

$$p^2+q^2 = a^2 \quad \text{---} \text{②}$$

Using ②, ① gives

$$a^2y - qz = 0$$

$$\Rightarrow a^2y = qz$$

$$\Rightarrow \boxed{q = \frac{a^2y}{z}}$$

putting this value of q in ②,

we get

$$p = \sqrt{a^2-q^2}$$

$$= \sqrt{a^2 - \frac{a^4y^2}{z^2}}$$

$$= \frac{a}{z}\sqrt{z^2-a^2y^2}.$$

(11)

Now putting these values of p and q in

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

Integrating

$$(z^2-a^2y^2)^{1/2} = ax+b$$

$$\Rightarrow z^2-a^2y^2 = (ax+b)^2 \quad \text{--- (3)}$$

which is a required complete integral.

Singular integral:

Differentiating (3) partially w.r.t a and b, we have

$$0 = -2ay^2 + 2(ax+b)x \quad \text{--- (4)}$$

and $0 = 2(ax+b)$ --- (5)

Eliminating a and b between (3), (4) and (5),

we get $\boxed{z=0}$

which is clearly satisfies (1) and hence it is the singular integral.

→ 5(b): Obtain the iterative scheme for finding p^{th} root of a function of single variable using Newton-Raphson method. Hence, find $\sqrt[7]{277234}$ correct to four decimal places.

Sol^n Let $x = \sqrt[p]{N}$.

then $x^p - N = 0$.

Let $f(x) = x^p - N$ ——— (1)

then $f'(x) = p x^{p-1}$.

By Newton–Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^p - N}{p x_n^{p-1}}$$

$$= \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$$

$$= \frac{(p-1) x_n^p + N}{p x_n^{p-1}}$$

$$\boxed{x_{n+1} = \frac{1}{p}\left[(p-1)x_n + \frac{N}{x_n^{p-1}}\right]} \quad \text{——— (2)}$$

which is the required iterative formula for finding $\sqrt[p]{N}$. (i.e p^{th} root of single variable).

To find the value $\sqrt[7]{277234}$ (correct to four decimal places) :–

Here putting $p = 7$ and $N = 277234$ in (2), we get

$$x_{n+1} = \frac{1}{7}\left[6x_n + \frac{277234}{x_n^6}\right] \quad \text{——— (3)}$$

(12)

Since an approximate value of

$$\sqrt[7]{277234} = 6.$$

we take $x_0 = 6.$

then $x_{n+1} = \frac{1}{7}\left[6x_0 + \frac{277234}{(x_0)^6}\right]$

$$x_1 = \frac{1}{7}\left[36 + \frac{277234}{(6)^6}\right].$$

$$\boxed{x_1 = 5.99172668}.$$

Now

$$x_2 = \frac{1}{7}\left[6x_1 + \frac{277234}{(5.99172668)^6}\right]$$

$$\boxed{x_2 = 5.99169225|}.$$

Now

$$x_3 = \frac{1}{7}\left[6x_2 + \frac{277234}{(x_2)^6}\right]$$

$$\boxed{x_3 = 5.99169225}.$$

Since $x_2 \approx x_3$ upto four decimal places

$\therefore$ we have

$$\sqrt[7]{277234} = 5.9916$$

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5.(c) convert the following binary numbers to the base indicated:

(i) $(10111011001.101110)_2$ to octal.

(ii) $(10111011001.10111000)_2$ to hexadecimal.

(iii) $(0.101)_2$ to decimal.

Sol: (i) Binary number can be converted into an equivalent octal number by making groups of three bits starting from LSB (Least significant Bit) and moving towards MSB (Most significant Bit) for integer part of the number and then replacing each group of three bits by its octal representation.

For fractional part, the groupings of three bits are made starting from the binary point.

Also forming the 3-bit groupings, 0's may be required to complete the first group in the integer part and the last group in the fractional part.

$$(10111011001.101110)_2 = \underbrace{010}_{2}\,\underbrace{111}_{7}\,\underbrace{011}_{3}\,\underbrace{001}_{1}.\underbrace{101}_{5}\,\underbrace{110}_{6}$$

$$= (2731.56)_8.$$

(ii) Binary number can be converted into an equivalent hexadecimal ~~number~~ by making groups of four bits and following the above problem procedure.

$(10111011001.10111000)_2 = \underbrace{0101}_{5}\ \underbrace{1101}_{D}\ \underbrace{1001}_{9}.\underbrace{1011}_{B}\ \underbrace{1000}_{8}$

$= (5D9.B8)_{16}$

(iii) $0.101 = 1\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3}$

$= \frac{1}{2} + 0 + \frac{1}{8}$

$= 0.5 + 0.125$

$= (0.625)_{10}$

5(d): A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity of the ball and cannon at the moment when it leaves the cannon is 'u'. Show that the cannon will recoil a distance

$$\left(\frac{mu}{M+m}\right)^2 \frac{1}{2\mu g}.$$

along the plane, m being the mass of the ball.

Solⁿ: Let 'I' be the impulse between the cannon and the ball. If 'v' is the velocity of the ball and 'V' be the velocity of the cannon in opposite direction, then the relative velocity of the ball and cannon at the moment the ball leaves the cannon is

$$v + V = u \text{ (given)} \quad \text{——} (1)$$

Also since, impulse = Change in momentum.

$\therefore I = m(v-0)$ (for the ball)

and $I = M(V-0)$. (for the cannon)

$\therefore mv = MV$

$\Rightarrow \boxed{v = \frac{MV}{m}}$. ——— (2)

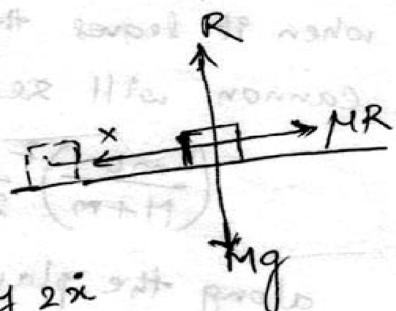
substituting (2) in (1), we get

$$\frac{MV}{m} + V = u$$

$$\Rightarrow V(M+m) = mu.$$

$$\Rightarrow \boxed{V = \frac{mu}{M+m}} \text{ ——— (3)}$$

If the cannon moves through a distance x in the direction opposite to the direction of motion of the ball in time 't', then on the rough plane, for the cannon the equation of motion is



$$M\ddot{x} = -\mu R = -\mu Mg$$

$$\Rightarrow \ddot{x} = -\mu g$$

Multiplying both sides by $2\dot{x}$ and intergrating, we get

$$\dot{x}^2 = -2\mu g x + C \text{ — (4)}$$

But initially when $x=0$, $\dot{x} = V$ (starting velocity of the cannon)

$$\therefore \boxed{C = V^2}$$

$$\therefore (4) \equiv \dot{x}^2 = -2\mu g x + V^2$$

$$\dot{x}^2 = V^2 - 2\mu g x$$

when the cannon comes to rest $\dot{x} = 0$

$$\therefore 0 = V^2 - 2\mu g x.$$

14

$$\Rightarrow x = \frac{v^2}{2\mu g}$$

$$x = \left(\frac{mu}{M+m}\right)\cdot\frac{1}{2\mu g} \quad (\text{from } ③)$$

which is the required distance.

5(e): If the velocity of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$ where $r^2 = x^2+y^2+z^2$, prove that the velocity potential is $\cos\theta/r^2$.

Sol^n: Given $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$, $w = \frac{3z^2-r^2}{r^5}$

If ϕ is the velocity potential, then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\Rightarrow d\phi = -(u\,dx + v\,dy + w\,dz)$$

$$= -\frac{1}{r^5}\left(3xz\,dx + 3yz\,dy + (3z^2-r^2)\,dz\right)$$

$$= -\frac{1}{r^5}\left\{3z(x\,dx + y\,dy + z\,dz) - r^2 dz\right\}$$

$$= -\frac{3z}{2}\frac{d(x^2+y^2+z^2)}{r^5} + \frac{dz}{r^3}$$

$$= -\frac{3z}{2}\frac{d(r^2)}{r^5} + \frac{dz}{r^3}$$

$$= -\frac{3z}{2}\frac{2r\,dr}{r^5} + \frac{dz}{r^3}$$

$$= \frac{-3}{r^4}dr + \frac{dz}{r^3}$$

$$d\phi = d\left(\frac{z}{r^3}\right)$$

Integrating, we get

$$\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3}$$

$$= \frac{\cos\theta}{r^2},$$

neglecting constant of integration.

velocity potential: Suppose $q = ui + vj + wk$ is velocity at any point $P(x, y, z)$. Also suppose the expression $u\,dx + v\,dy + w\,dz$ is an exact diff. equation, say $-d\phi$. Then
$-d\phi = u\,dx + v\,dy + w\,dz$
i.e., $-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz$.

6(a) : A rod of length l with insulated sides, is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°C$ and are kept at that temperature. Find the temperature distribution in the rod at any time t.

Sol^n: Here the temperature $u(x,t)$ in the given solid is governed by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Since the ends $x=0$ and $x=l$ are kept at zero temperature,

$\therefore$ the boundary conditions are

$$u(0,t) = 0, \; u(l,t) = 0 \quad \forall t \quad \text{--- (2)}$$

The initial condition is given by

$$u(x,0) = u_0. \quad \text{--- (3)}$$

Suppose that (1) has solution of the form $u(x,t) = X(x)\,T(t)$ --- (4)

where X is a function of x alone and T is a function of t only.

Substituting the value of u in (1), we get

$$kX''T = XT'$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = \mu \text{ (say)}$$

$$\therefore X'' - \mu X = 0 \quad \text{--- (5)}$$

$$\text{and } T' = \mu k T \quad \text{--- (6)}$$

(15)

using (2), (4) gives

$X(0)\,T(t)=0$ and $X(l)\,T(t)=0$. ——— (7)

since $T(t)=0$ leads to $u=0$ (which is trivial)

so suppose that $T(t)\neq 0$.

$\therefore$ from (7)

$X(0)=0,\ X(l)=0$ ——— (8).

we now solve (5) under the boundary condition (8).

Three cases arise:

Case (1): Let $\mu=0$. Then the solution of (5) is $X(x)=Ax+B$ ——— (9)

Using (8), (9) gives

$B=0,\ 0=Al+B$

$\Rightarrow A=0,\ B=0$

Hence $X(x)=0$

so that $u=0$.

which doesnot satisfy (3).

Case (2): Let $\mu=\lambda^2,\ \lambda\neq 0$.

Then the solution of (5) is

$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$ ——— (10)

Using (8), (10) gives $\left.\begin{aligned}0&=A+B\\ \text{and } 0&=Ae^{l\lambda}+Be^{-l\lambda}.\end{aligned}\right\}$ ——— (11)

solving (11), $A=B=0$.

so that $X(x)=0$ and

hence $u\equiv 0$.

which doesnot satisfy (3).

Case(3): Let $\mu = -\lambda^2, \lambda \neq 0$. Then the solution of (5) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad \text{--- (12)}$$

Using (8), (12) gives

$$0 = A \quad \text{and} \quad 0 = A\cos\lambda l + B\sin\lambda l.$$

$$\Rightarrow B\sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = 0 \quad (\because B \neq 0. \text{ otherwise } X = 0 \text{ and so } u = 0 \text{ which doesnot satisfy (3)}).$$

$$\Rightarrow \lambda l = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l} \quad \text{--- (13)}$$

where $n = 1, 2, 3, \ldots$

Hence non-zero solutions $X_n(x)$ of (5) are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (14)}$$

Using (13), (6) reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 k}{l^2}dt \quad \left(\because \text{from (6) } T' = \mu kT \Rightarrow \frac{dT}{dt} = \mu kT \Rightarrow \frac{dT}{T} = \mu k\,dt = -\lambda^2 k\,dt = -\frac{n^2\pi^2}{l^2}k\,dt \text{ from (13)}\right)$$

$$\Rightarrow \frac{dT}{T} = -C_n^2\,dt$$

where $C_n^2 = \dfrac{n^2\pi^2 k}{l^2}$.

whose general solution is

$$T_n(t) = D_n e^{-C_n^2 t}. \quad \text{--- (15)}$$

$$\therefore u_n(x,t) = X_n(x)\,T_n(t)$$

$$= B_n \sin\left(\frac{n\pi x}{l}\right) D_n e^{-C_n^2 t}$$

$$= B_n D_n \sin\left(\frac{n\pi x}{l}\right) e^{-C_n^2 t}$$

$$u_n(x,t) = E_n \sin\left(\frac{n\pi x}{l}\right) e^{-C_n^2 t} \quad \text{--- (16)}$$

where $E_n = B_n D_n$ are solutions of (1), satisfying (2).

(16)

In order to obtain a solution also satisfying (3), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) e^{-c_n^2 t} \quad \text{——— (17)}$$

Substituting $t=0$ in (17), and using (3)

we get

$$u_0 = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right). \quad \left(\because \text{from (3)}\ u(x,0) = u_0\right)$$

which is Fourier sine series.

So the constants E_n are given by

$$E_n = \frac{2}{l}\int_0^l u_0 \sin\left(\frac{n\pi x}{l}\right) dx.$$

$$= \frac{2}{l}u_0 \left(-\frac{l}{n\pi}\right)\left[\cos\frac{n\pi x}{l}\right]_0^l$$

$$= -\frac{2u_0}{n\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{2u_0}{n\pi}\left[1 - \cos n\pi\right]$$

$$= \frac{2u_0}{n\pi}\left[1 - (-1)^n\right] \quad \left(\because \cos(n\pi) = (-1)^n\ \forall n \in N\right)$$

$$= \begin{cases} 0 & \text{if } n = 2m \ ;\ m = 1,2,3,\ldots \\ \dfrac{4u_0}{n\pi} & \text{if } n = 2m-1,\ m = 1,2,3,\ldots \end{cases}$$

Hence solution (17) reduces to

$$u(x,t) = \sum_{m=1}^{\infty} E_{2m-1} \sin\left[\frac{(2m-1)\pi x}{l}\right] e^{-c_{2m-1}^2 t}$$

i.e., $$u(x,t) = \sum_{m=1}^{\infty} \frac{4u_0}{\pi(2m-1)} \sin\left\{\frac{(2m-1)\pi x}{l}\right\} e^{-c_{2m-1}^2 t}$$

$$= \frac{4u_0}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m-1)} \sin\left\{\frac{(2m-1)\pi x}{l}\right\} e^{-c_{2m-1}^2 t}$$

where $c_{2m-1}^2 = \dfrac{(2m-1)^2\pi^2 k}{l}$.

6.(b) convert the following to the base indicated against each:

(i) $(266.375)_{10}$ to base 8

(ii) $(341.24)_5$ to base 10

(iii) $(43.3125)_{10}$ to base 2.

Sol.

(i) $(266.375)_{10}$

Integer part:

8	266
8	33 remainder 2
8	4 remainder 1
	0 remainder 4

$\therefore (266)_{10} = (412)_8$

fractional part:

Fraction	Fraction × 8	Remainder new fraction	Integer.
0.375	3.000	0.000	3
0.000	0.000	0.000	0.

$$(0.375)_{10} = (0.3)_8$$

$$\therefore (266.375)_{10} = (412.3)_8$$

(ii) $(341.24)_5 = 3\times5^2 + 4\times5^1 + 1\times5^0 + 2\times5^{-1} + 5\times5^{-2}$

$= 3\times25 + 20 + 1 + \frac{2}{5} + \frac{5}{25}$

$= 75 + 20 + 1 + 0.4 + 0.2$

$= 96.6$

(iii) $(43.3125)_{10}$

Taking the integer part first:

$$\begin{array}{r|l}
2 & 43 \\ \hline
2 & 21 \text{ remainder } 1 \\ \hline
2 & 10 \text{ remainder } 1 \\ \hline
2 & 5 \text{ remainder } 0 \\ \hline
2 & 2 \text{ remainder } 1 \\ \hline
2 & 1 \text{ remainder } 0 \\ \hline
 & 0 \text{ remainder } 1
\end{array} \uparrow$$

$$(43)_{10} = (101011)_2$$

Taking the fractional part:

Fraction	Fraction × 2	Remainder new fraction	Integer
0.3125	0.625	0.625	0
0.625	1.25	0.25	1
0.25	0.5	0.5	0
0.5	1.0	0.0	1

$\therefore (0.3125)_{10} = 0101$

Adding the binary equivalent of 43 and 0.3125.

$$\begin{array}{r}
101011.0000 \\
0.0101 \\ \hline
101011.0101
\end{array}$$

$$\therefore (43.3125)_{10} = (101011.0101)_2$$

(6).(d) Using Runge-kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$. Initial conditions are at $x=0$, $y=1$ and $y'=0$. Use four decimal places for computations.

Sol. The given equation is

$y'' = xy'^2 - y^2$ which is a second order differential equation

By writing $\frac{dy}{dx} = p$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = p = f(x, y, p)$$

$$\frac{dp}{dx} = xp^2 - y^2 = \phi(x, y, p)$$

we have $x_0 = 0$, $y_0 = 1$, $p_0 = y_0' = 0$.

Taking $h = 0.2$.

Using $k_1, k_2, \ldots$ for $f(x, y, p)$ and $l_1, l_2, \ldots$ for $\phi(x, y, p)$.

Runge kutta formulae become.

$$k_1 = hf(x_0, y_0, p_0) = (0.2)\, p_0 = (0.2)(0) = 0$$

$$l_1 = h\phi(x_0, y_0, p_0) = (0.2)[x_0 p_0^2 - y_0^2]$$
$$= (0.2)[0-1] = -0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right)$$
$$= h\left(p_0 + \frac{l_1}{2}\right) = (0.2)\left(0 - \frac{0.2}{2}\right)$$
$$= (0.2)(0 - 0.1)$$
$$= -0.02$$

$l_2 = h\,\phi\left(x_0+\frac{h}{2},\, y_0+\frac{k_1}{2},\, p_0+\frac{l_1}{2}\right)$
$= (0.2)(0.1, 1, -0.1) = (0.2)\left[(0.1)[-0.1]^2-1\right]$
$= (0.2)(-0.999) = -0.1998$

$k_3 = h\,f\left(x_0+\frac{h}{2},\, y_0+\frac{k_2}{2},\, p_0+\frac{l_2}{2}\right)$
$= (0.2)\,f(0.1, -0.99, -0.0999) = (0.2)(-0.0999)$
$= -0.01998$

$l_3 = h\,\phi\left(x_0+\frac{h}{2},\, y_0+\frac{k_2}{2},\, p_0+\frac{l_2}{2}\right)$
$= (0.2)\,\phi(0.1, -0.99, -0.0999)$
$= (0.2)\left[(0.1)(-0.0999)^2-(-0.99)^2\right] = (0.2)(-0.9791)$
$= -0.1958$

$k_4 = h\,f(x_0+h,\, y_0+k_3,\, p_0+l_3)$
$= h\,f(0.1, 0.98, -0.1958) = (0.2)(-0.1958)$
$= -0.0392$

$l_4 = h\,\phi(x_0+h,\, y_0+k_3,\, p_0+l_3)$
$= (0.2)(0.2, 0.98, -0.1958)$
$= (0.2)\left[(0.2)(-0.1958)^2-(0.98)^2\right]$
$= -0.1905$

$\therefore k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{-0.11916}{6}$
$= -0.01986$
$= -0.0199$

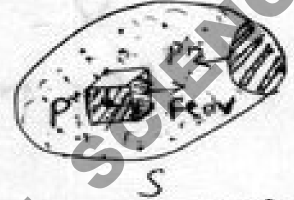
$l = \frac{1}{6}(l_1+2l_2+2l_3+2l_4) = \frac{-1.1817}{6} = -0.19695$
$= -0.1970$

Hence at $x=0.2$
$y = y_0+k = 1-0.0199 = 0.9801$
and $y' = p = p_0+l = 0-0.1970 = -0.1970$
$\therefore$ The required solution is
$y(0.2) = 0.9801 \;;\; y'(0.2) = -0.1970$

7(a). prove that the equation of motion of a homogeneous inviscid liquid moving under conservative forces may be written as

$$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times \text{curl}\,\vec{q} = -\text{grad}\left[\frac{p}{\rho} + \frac{1}{2}q^2 + \vec{\Omega}\right]$$

Sol: Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid at an instant t.



Let ρ be the density of the fluid particle 'p' with in the closed surface and dv be the volume enclosing p. The mass of the element $\rho\,dv$ will always remain constant.

Consider 'q' be the velocity of the fluid particle p then the momentum of the volume is $M = \int q\,\rho\,dv$ ——— ①

The rate of change of momentum is given by differentiating ① w.r.t t,

thus we have

$$\frac{dM}{dt} = \int \frac{dq}{dt}(\rho\,dv) + \int q\,\frac{d}{dt}(\rho\,dv)$$

$$= \int \frac{dq}{dt}(\rho\,dv) + 0$$

($\because$ the mass $\rho\,dv$ remains unchanged throughout the motion.

i.e, $\rho\,dv$ remains constant for all time. $\therefore \frac{d}{dt}(\rho\,dv) = 0$)

$$= \int \frac{dq}{dt}\,\rho\,dv$$ ——— ①

Let $\hat{n}$ be the unit outward normal vector on the surface element dS.

Suppose F is the external force per unit mass acting on the fluid and p the pressure at any point on the element dS.

Total surface force is

$$\int_V F\rho\, dv + \int_S p(-\hat{n})\, dS$$

(for pressure acts along inward normal)

$$= \int_V F\rho\, dv + \int_V \nabla p\, dv \quad \text{(by Gauss theorem)}$$

$$= \int_V (F\rho - \nabla p)\, dv \quad \text{——— (2)}$$

By Newton's second law of motion,

rate of change of momentum = total applied force.

i.e, $\int \frac{dq}{dt} \rho\, dv = \int (F\rho - \nabla p)\, dv$ (from (1) & (2))

$$\Rightarrow \int \left(\frac{dq}{dt}\rho - F\rho + \nabla p\right) dv = 0$$

Since S is arbitrary and so V is arbitrary so that the integrand of the last integral vanishes;

i.e, $\frac{dq}{dt}\rho - F\rho + \nabla p = 0.$

$$\Rightarrow \frac{dq}{dt} = F - \frac{1}{\rho}\nabla p \quad \text{——— (3)}$$

which is known as Euler's equation of motion.

(20)

The equation (3) may be expressed as

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + q\cdot\nabla\right)q = F - \frac{1}{\rho}\nabla p \quad \left(\because \frac{d}{dt} = \frac{\partial}{\partial t} + q\cdot\nabla\right)$$

$$\Rightarrow \frac{\partial q}{\partial t} + (q\cdot\nabla)q = F - \frac{1}{\rho}\nabla p \quad \text{——— (4)}$$

Since $\nabla(q\cdot q) = 2\left[q\times \text{curl}\, q + (q\cdot\nabla)q\right]$

$$\Rightarrow (q\cdot\nabla)q = \nabla\left(\frac{1}{2}q^2\right) - q\times \text{curl}\, q.$$

$$\therefore (4) \equiv \frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times\text{curl}\, q = F - \frac{1}{\rho}\nabla p.$$

$$\text{——— (5)}$$

Since the forces are conservative.

$\Rightarrow F = -\nabla\vec{\Omega}$.

where $\vec{\Omega}$ is known as force potential which measures the potential energy of the field.

$\therefore (5) \equiv$

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times\text{curl}\, q = -\nabla\vec{\Omega} - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \frac{\partial q}{\partial t} - q\times\text{curl}\, q = -\nabla\left(\frac{1}{2}q^2\right) - \nabla\vec{\Omega} - \frac{1}{\rho}\nabla p$$

$$= -\nabla\left[\frac{1}{2}q^2 + \vec{\Omega} + \frac{p}{\rho}\right]$$

$$= -\text{grad}\left[\frac{p}{\rho} + \frac{1}{2}q^2 + \vec{\Omega}\right]$$

$$\therefore \frac{\partial \vec{q}}{\partial t} - \vec{q}\times\text{curl}\,\vec{q} = -\text{grad}\left[\frac{p}{\rho} + \frac{1}{2}q^2 + \vec{\Omega}\right]$$

→ 1(b) find the general solution of

$$\{D^2 - DD' - 2D'^2 + 2D + 2D'\}z = e^{2x+3y} + xy + \sin(2x+y)$$

Sol^n: The given equation can be rewritten as

$$(D^2 - 2DD' + 2D + DD' - 2D'^2 + 2D')z = e^{2x+3y} + xy + \sin(2x+y)$$

$$(D+D')(D-2D'+2)z = e^{2x+3y} + xy + \sin(2x+y) \quad \text{——} (1)$$

$\therefore$ C.F $= \phi_1(y-x) + e^{-2x}\phi_2(y+2x)$

where ϕ_1, ϕ_2 are arbitrary functions

P.I corresponding to e^{2x+3y} is

$$= \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y}$$

$$= \frac{1}{(2+3)(2-6+2)} e^{2x+3y} = \frac{1}{-10} e^{2x+3y}$$

P.I corresponding to xy:

$$= \frac{1}{(D+D')(D-2D'+2)} xy$$

$$= \frac{1}{D\left[1+\frac{D'}{D}\right] \cdot 2\left[1+\left(\frac{D}{2}-D'\right)\right]} xy$$

$$= \frac{1}{2D}\left[1+\frac{D'}{D}\right]^{-1}\left\{1+\left(\frac{D}{2}-D'\right)\right\}^{-1} xy$$

$$= \frac{1}{2D}\left\{1-\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right\}\left\{1-\left(\frac{D}{2}-D'\right)+\left(\frac{D}{2}-D'\right)^2+\cdots\right\}(xy)$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

(21)

$$= \frac{1}{2}\left[\frac{x^2y}{2} - \frac{xy}{2} + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{4}\right]$$

$$= \frac{x^2y}{4} + \frac{3x^2}{8} - \frac{xy}{4} - \frac{x}{2} - \frac{x^3}{12}$$

P.I. corresponding to $\sin(2x+y)$:

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{-2^2 - (-2\cdot 1) - 2(-1^2) + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{-4+2+2+2D+2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \sin(2x+y)$$

$$= \frac{D-D'}{2(D^2-D'^2)} \sin(2x+y)$$

$$= \frac{1}{2}\frac{1}{-2^2-(-1^2)}(D-D')\sin(2x+y)$$

$$= \frac{1}{2(-3)}(D-D')\sin(2x+y)$$

$$= -\frac{1}{6}\left[\cos(2x+y)\cdot 2 - \cos(2x+y)\cdot 1\right]$$

$$= -\frac{1}{6}\left[2\cos(2x+y) - \cos(2x+y)\right]$$

$$= -\frac{1}{6}\cos(2x+y).$$

$\therefore$ The required solution is

$$z = C.F + P.I.$$

i.e, $z = \phi_1(y-x) + e^{-2x}(y+2x) - \frac{1}{10}e^{2x+3y}$

$$+ \frac{x^2y}{4} + \frac{3x^2}{8} - \frac{xy}{4} - \frac{x}{2} - \frac{1}{12}x^3 - \frac{1}{6}\cos(2x+y)$$

7(c) From the following data

x	1	8	27	64
y	1	2	3	4

Calculate $y(20)$, using Lagrangian interpolation technique. Use four decimal points for computations.

Sol^n: we have

$x_0 = 1$, $x_1 = 8$, $x_2 = 27$, $x_3 = 64$.

and $y_0 = f(x_0) = 1$, $y_1 = f(x_1) = 2$, $y_2 = f(x_2) = 3$

$y_3 = f(x_3) = 4$

Now the Lagrange's interpolation formula with four points is

$$f(x) = p_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore f(20) = \frac{(20-8)(20-27)(20-64)}{(1-8)(1-27)(1-64)}(1) + \frac{(20-1)(20-27)(20-64)}{(8-1)(8-27)(8-64)}(2)$$
$$+ \frac{(20-1)(20-8)(20-64)}{(27-1)(27-8)(27-64)}(3) + \frac{(20-1)(20-8)(20-27)}{(64-1)(64-8)(64-27)}(4)$$

$$= \frac{(12)(\cancel{-7})(-44)}{(\cancel{-7})(-26)(-63)}(1) + \frac{(19)(\cancel{-7})(-44)}{(\cancel{7})(-19)(-56)}(2)$$
$$+ \frac{(\cancel{19})(\cancel{12})^{6}(-44)}{(\cancel{26})_{13}(\cancel{19})(-37)}(3) + \frac{(19)(\cancel{12})^{3}(\cancel{-7})}{(\cancel{7})(\cancel{56})_{14\,7}(37)}(4)^{2}$$

$$= \frac{-88}{273} + \frac{11}{7} + \frac{792}{481} - \frac{114}{259}$$

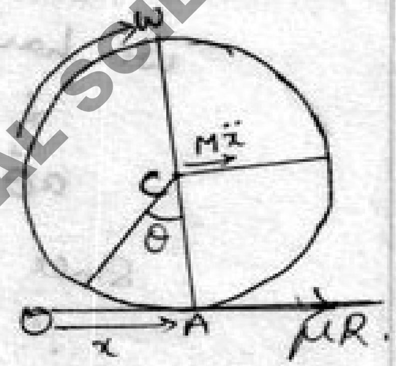
$$= -0.3223 + 1.5714 + 1.6466 - 0.4402$$

$$= 3.218 - 0.7625 = 2.4555.$$

→8(a): A homogeneous sphere of radius a, rotating with angular velocity ω about horizontal diameter, is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\frac{2\omega a}{7\mu g}$ and that then the sphere will roll with angular velocity $\frac{2\omega}{7}$.

Sol^n:

As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.



Initial velocity of the point of contact

= Initial velocity of the centre C + initial velocity of the point of contact w.r.t the centre C.

= $0 + a\omega$ in the direction from right to left,

i.e, the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t

the equations of motion are

$M\ddot{x} = \mu R$, where $R = Mg$ —— ①

and $Mk^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu Ra$ —— ②

from ①, we have $\ddot{x} = \mu g$ —— ③

and from ②, we have

$a\ddot{\theta} = -\frac{5}{2}\mu g$ —— ④

Integrating ③ & ④,

we have

$\dot{x} = \mu g t + c_1$ and

$a\dot{\theta} = -\frac{5}{2}\mu g t + c_2$

Since initially when $t=0$, $\dot{x}=0$, $\dot{\theta}=\omega$

$\therefore c_1 = 0$ and $c_2 = a\omega$.

$\therefore \dot{x} = \mu g t$ —— ⑤

and $a\dot{\theta} = -\frac{5}{2}\mu g t + a\omega$. —— ⑥

velocity of the point of contact $= \dot{x} - a\dot{\theta}$

$\therefore$ The point of contact will come to rest

when $\dot{x} - a\dot{\theta} = 0$.

i.e, when $\mu g t - \left(-\frac{5}{2}\mu g t + a\omega\right) = 0$

i.e, when $t = \frac{2a\omega}{7\mu g}$

Therefore after time $\frac{2a\omega}{7\mu g}$ the slipping will stop and pure rolling will commence.

putting this value of t in ⑥, we get

$\dot{\theta} = \frac{2\omega}{7}$

∴ when rolling commences, let F be the frictional force. Therefore the equations of motion are

$$M\ddot{x} = F, \quad \text{—— (7)}$$

$$M\cdot\frac{2}{5}a^2\ddot{\theta} = -Fa \quad \text{—— (8)}$$

and $\dot{x} - a\dot{\theta} = 0$ —— (9)

From (9) $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

now from (7) & (8), we get

$$M\ddot{x} = F = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow M\ddot{x} = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow \ddot{x} = -\frac{2}{5}a\ddot{\theta}$$

$$\Rightarrow a\ddot{\theta} = -\frac{2}{5}a\ddot{\theta} \quad (\because \ddot{x} = a\ddot{\theta})$$

$$\Rightarrow \frac{7}{5}a\ddot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} = 0. \quad (\because \tfrac{7}{5}a \neq 0)$$

Integrating

$$\dot{\theta} = \text{constant}$$

$$\dot{\theta} = \frac{2\omega}{7}.$$

8(b): Derive composite $\frac{1}{3}$rd Simpson's rule. Hence evaluate $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates. Tabulate the integrand for these ordinates to four decimal places.

Solⁿ: we have

$f(x) = e^{-x^2}$

$a=0,\ b=0.6\ ;\ n=6$

$\therefore h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1.$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
$y=f(x)$	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^{0.6} e^{-x^2}dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_2+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6977)+4(0.99+0.9139+0.7788)+2(0.9608+0.8521)\right]$$

$$= \frac{0.1}{3}\left[1.6977+10.7308+3.6258\right]$$

$$= \frac{0.1}{3}(16.0543)$$

$$= 0.5351$$

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