Number Systems

- Base r $\quad [a_1 a_2 . a_3 a_4] = a_1 \times r^1 + a_2 \times r^0 + a_3 \times r^{-1} + a_4 \times r^{-2}$

→ Conversions

① From Decimal system

A. Integers

41 to binary →

$$\begin{array}{c|cc} & \text{Divi} & \text{Rem} \\ 2 & 41 & 1 \\ 2 & 20 & 0 \\ 2 & 10 & 0 \\ 2 & 5 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \\ & 0 & \end{array}$$

$$\underset{32\ 16\ 8\ 4\ 2\ 1}{(101001)}$$

153 to octal →

$$\begin{array}{c|cc} 8 & 153 & 1 \\ 8 & 19 & 3 \\ 8 & 2 & 2 \\ & 0 & \end{array}$$

(231)

B. Fractions

0.6875 to binary →

$$\begin{aligned} 2 \times 0.6875 &= 1.375 \\ 2 \times 0.375 &= 0.750 \\ 2 \times 0.750 &= 1.5 \\ 2 \times 0.5 &= 1.0 \end{aligned}$$

(0.1011)

0.513 to octal →

$$\begin{aligned} 8 \times 0.513 &= 4.104 \\ 8 \times 0.104 &= 0.832 \\ 8 \times 0.832 &= 6.656 \\ 8 \times 0.656 &= 5.248 \\ 8 \times 0.248 &= 1.984 \ldots \end{aligned}$$

(0.40651)

Separately convert integer and fraction part.

→ Octal ≡ Base 8

· Break binary into triplets

Add extra zero

$$\underbrace{1}\,\underbrace{101}\,\underbrace{011}\,.\,\underbrace{011}\,\underbrace{001}\,\underbrace{110}$$

← →

$(153 \cdot 31\cancel{1})$

$\boxed{153 \cdot 316}$

→ Hexadecimal ≡ Base 16

· Break binary to quadruplets.

$$\underbrace{10}_{2}\ \underbrace{1011}_{B}\ .\ \underbrace{0110}_{6}\ \underbrace{0100}_{1}$$

$\cancel{(2B \cdot 61)}$

$\underline{2B \cdot 64}$

---

① Sign Magnitude $-(2^{n-1}-1)$ to $(2^{n-1}-1)$

② 2s complement (-2^{n-1}) to $(2^{n-1}-1)$

L From Signed Style ⇒ Flip bits and add 1 (Ensure no overflow

③ 1s complement

+ve ≡ Same as 2^s complement

−ve ≡ $2^s - 1$

* Complements.

1st Complement ≡ Flip all bits. $+7 = 0111$, $-7 = 1000$

└ +ve same as sign-magnitude

(Max $2^{n-1}-1$)
(Min -2^{n-1})

2s Complement ≡ $\boxed{\text{1st Complement} + 1}$

$$a_1 a_2 a_3 a_4 \equiv a_1 \times (-) 2^{4-1} + a_2 \times 2^2 + a_3 \times 2^1 + a_4 \times 2^0$$

- 2s Complement of 0101 is 1011
  - 0101 ⇓ (+5)
  - 1011 (−5)
  - 1010 (In 1s complement)

→ Operations

- Addition ≡ Trivial

- Subtraction
  └ 2s complement (unsigned case)
    1. A − B
    2. Take 2s complement of B = B'
    3. A + B'
    4. Overflow (discard bit, left over is answer)

    No overflow (Take 2st complement of output and a (−) sign)

erflow)

- Division (Normal)

```
1111 ) 11000101 ( 1101
        1111
       ------
        10011
         1111
       --------
        0010001.
            1111
          ------
              10
```

- Multiplication of 2 binaries – Same

Boolean Algebra

(*) $A + BC = (A+B)(A+C)$

- Demorgan's: $\overline{A+B} = \bar{A}\,\bar{B}$, $\overline{AB} = \bar{A} + \bar{B}$

- $\overline{A+B+C} = \bar{A}\bar{B}\bar{C}$

Example ⇒ Complement [Replace AND ⇄ OR, taking complements]

$\overline{A+BC+D\bar{E}} = \bar{A}(\bar{B}+\bar{C})(\bar{D}+\bar{\bar{E}})$

$\overline{A(B+C)(D+\bar{E})} = \bar{A} + (\bar{B}\bar{C}) + (\bar{D}E)$

- Order of operators ≡ $\boxed{(\,) \rightarrow \text{NOT} \rightarrow \text{AND} \rightarrow \text{OR}}$

Q $xy + x'z + yz = (xy+x')(xy+z) + \overline{yz}$
$= (x'+y)(\cancel{x+}z)(y+z) + yz$
$\neq (zx' + yx + yz)(y+z) + yz$
$= zx'y + yx + yz + zx' + xyz + yz + yz$
$= \cancel{yz} \cancel{+} yz + yx + zx'$

X

$= xy + x'z + (yz)(x+x') = xy \cancel{+} x'z + xyz + x'yz$
$= xy(1+z) + x'z(1+y) = \boxed{xy + x'z}$ Ans

Q $(x+y)(x'+z)(y+z) = (xz + yx' + yz)(y+z)$
$= \cancel{x}yz + yx' + yz + xz + yx'z + yz$
$= yz + yz + yx' + xz = yz + xz + yx'$

• Canonical forms.

DNF $\equiv \sum$ Min $= \Sigma 1$

CNF $\equiv \prod$ Max $= \Pi 0$

x	y	z	f_1	
0	0	0	0	0
0	0	1	1	→ 1
0	1	0	0	2
0	1	1	0	3
1	0	0	1	→ 4
1	0	1	0	5
1	1	0	0	6
1	1	1	1	→ 7

① Sum of products / min terms: [NAND]

- Look where f_1 is 1 → keep it 1 (each term)

↳ $(\bar{x}\bar{y}z + x\bar{y}\bar{z} + xyz) = \Sigma 1, 4, 7$

② Product of sums / Max term [NOR] (Conjunctive NF)

- Look where f_1 is 0 → keep each term 0

↳ $(x+y+z)(x+\bar{y}+z)(x+\bar{y}+\bar{z})(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+z)$

$= \Pi(0, 2, 3, 5, 6)$.

③ Convert to sum of minterms.

- For each term lacking a variable v do $\boxed{t \to t(v+\bar{v})}$

$A + B'C = A(B+B')(C+C') + B'C(A+A')$

$= ABC + ABC' + AB'C + AB'C' + AB'C + A'B'C$

$= A'B'C + AB'C' + AB'C + ABC' + ABC$

$= \Sigma 1, 4, 5, 6, 7.$

⑨ Convert to product of maxterms.

- Use $x+yz = (x+y)(x+z)$
- If any missing variable $\Rightarrow$ OR with vv'

$$xy + x'z = (xy+x')(xy+z) = \cancel{(x+x')}(y+x')(x+z)(y+z)$$

$$= (y+x'+zz')(x+z+yy')(xx'+y+z)$$

$$= \cancel{(x'+y+z)}(x'+y+z')\cancel{(x+y+z)}(x+y'+z)(x+y+z)(x'+y+z)$$

$$= (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')$$

$$= \Pi(0,2,4,5)$$

* Some boolean functions

- Equivalence $\equiv xy + x'y'$ (XNOR)
- Implication $\equiv y' + x$

~~(x→y)~~ $\boxed{y \to x}$

~~y≠~~ $\boxed{p \Rightarrow q \text{ are same } \neg p \vee q}$

* Duality Principle

Replace 1 by 0, 0 by 1, + by · and · by +
value remains unchanged.

$$\boxed{A+BC \xleftrightarrow{\text{Dual}} A.(B+C)}$$

L Not A to $\bar{A}$

Now, for A, B, C.
If $A+BC = 0$
Then $A.(B+C) = 1$
for same A, B, C.

* Karnaugh Maps.

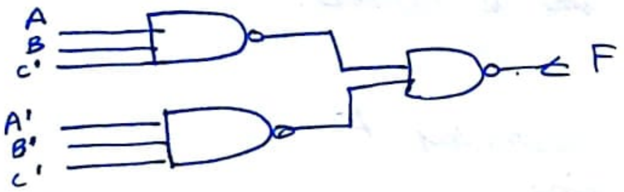
- Combine 1s to get sum of products form
- Combine 0s (+) Take complement to obtain product of sums.

$\boxed{Q}$ $F = \sum 0, 6.$

A \ BC	00	01	11	10
0	1	0	0	0
1	0	0	0	1

- Sum of products $= ABC' + A'B'C' =$

↳ NAND ⇒



A \ BC	00	01	11	10
0	1	0	0	0
1	0	0	0	1

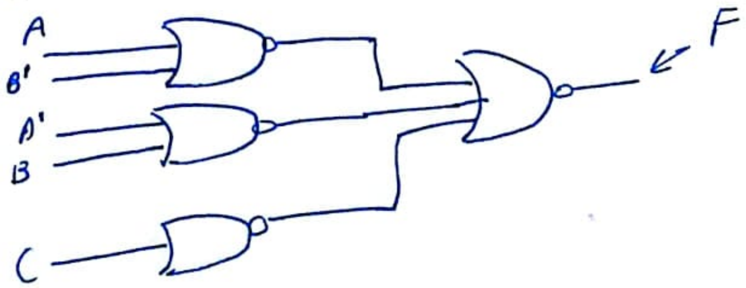
Product of Sums ≡

$$F' = \overline{[C] + \bar{A}B + A\bar{B}}$$

$$F = (A+\bar{B})(\bar{A}+B)(\bar{C}).$$

Use this

↳ NOR ⇒



① NAND Implementation

- Express as sum of products
- For each product a NAND Gate, 1 for NOT also
- Combine with a NAND

② NOR Implementation

- Express as product of sums.
- Same as NAND.

* For obtaining F' using

⇒ NAND ≡ Obtain product of sums for F and take NOT

⇒ NOR ≡ Obtain sum of products for F and take NOT.

* IEEE Double Representation (64-bit)

sign (1) | exponent (11) | mantissa (52-bit)

$\boxed{1, 8, 23}$

$$\text{Number} = (-1)^{\text{sign}} \times 2^{c-1023}(1+f).$$

32-bit $\boxed{2^{c-127}}$

(B) Number to IEEE and vice-versa

(1) 01000000011 10111001000 10000. ....

$$(-1)^0 \times 2^{1027-1023} \times \left(1 + 1.\tfrac{1}{2} + 1.\left(\tfrac{1}{2}\right)^3 + 1.\left(\tfrac{1}{2}\right)^4 + 1.\left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^8 + \left(\tfrac{1}{2}\right)^{12}\right)$$

$$2^4 \times \left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{256} + \frac{1}{4096}\right) = 27.56640625$$

(2) 18.9. $\boxed{\text{Sign} = 0}$

$$\log_2(18.9) = 4.24$$

Exponent $= \boxed{1023 + 4}$ (10000000011)

$$\frac{18.9}{2^4} = 1.18125 = \text{Mantissa.}$$

$0.18125 \times 2 = 0.3625$
$0.3625 \times 2 = 0.725$
$0.725 \times 2 = 1.45$
$0.45 \times 2 = 0.9$
$0.9 \times 2 = 1.8$
$0.8 \times 2 = 1.6$
$0.6 \times 2 = 1.2$

$\boxed{001011100. .....}$

Prev Year

2011

①

A \ BC	00	01	11	01
0			1	
1			1	

$\boxed{BC} = \boxed{ABC + \bar{A}BC.}$

② $x \wedge y = 0 \quad x \vee y = 1$

Prove $y = x'$

$x \vee y = 1$

1	0
0	1
1	1

$x \wedge y = 0$

0	0
0	1
1	0

x	y
0	1
1	0

2015

$$((p \wedge q) \rightarrow r) \vee ((p \wedge q) \rightarrow r')$$

Find DNF.
Is exp. contradiction/tautology?

$$(\neg(p \wedge q) \vee r) \vee (\neg(p \wedge q) \vee r')$$

$$\Rightarrow (p' \vee q' \vee r) \vee (p' \vee q' \vee r') \leftarrow \boxed{\text{Tautology}}$$

$$= (p' \vee q') \vee (1)$$

$$\text{DNF} \Rightarrow (\bar{p}\bar{q}\bar{r}) + (\bar{p}\bar{q}r) + (\bar{p}q\bar{r}) + (\bar{p}qr) + (p\bar{q}\bar{r}) + (p\bar{q}r) + (pq\bar{r}) + (pqr)$$

[Can create truth table and from there DNF is normal]

2010

$$A \oplus B = AB' + A'B$$

$$x \oplus y \oplus z = (xy' + x'y) \oplus z = (xy' + x'\cancel{z}y)z' + (xy' + x'\cancel{z}y)'z$$

$$= (x'+y) \cdot (x+y')z + (xy' + x'y)z'$$

$\cdot\ \underbrace{\underbrace{a + a'b}_{(a+a')(a+b) \Rightarrow (a+b)} + a'b'c}_{(a+b+c)} + a'b'c'd + \ldots$

$(a+b+c)$ - - - - - - - $\boxed{(a+b+c+\ldots)}$ Ans

$\cdot\ x'y'z + yz + xz = z(x+y+x')(x+y+y') = \underline{\underline{z}}$ Ans

$\cdot\ XY + \overline{XZ} + X\bar{Y}Z(XY+Z)$

$= XY + \bar{X} + \bar{Z} + X\bar{Y}Z = XY + \bar{Z} + (\bar{X}+X)(\bar{X}+\bar{Y}Z)$

$= \cancel{\bar{X}+Y+(\bar{Z}+X)(\bar{Z}+\bar{Y})} = XY + \bar{Z} + \bar{X} + \bar{Y}Z = \bar{X} + Y + \bar{Z} + \bar{Y}Z$

$= \cancel{\bar{X}+Y+\bar{Z}+\bar{Z}\bar{Y}+X\bar{Z}+X\bar{Y}}$

$= \bar{X} + \bar{Z} + Y + Z$

$= \underline{\underline{1}}$

Numerical Methods

A. Equations of one variable.

① Bisection Method.

- To find solution of $f(x)=0$ on $[a,b]$ where $f(a)$, $f(b)$ have opposite signs

ALGORITHM.

INPUT endpoints a, b; error ERR; max iterations N_0

OUTPUT approx. solution p or message of failure

STEP 1 $i=1$; $FA = f(a)$.

STEP 2 While $i \le N_0$ do Steps 3-6.

STEP 3 Set $p = a + \frac{b-a}{2}$; $FP = f(p)$.

STEP 4 If $FP = 0$ or $\frac{(b-a)}{2} < ERR$ then
OUTPUT (p); // Procedure completed success
STOP.

STEP 5 Set $i = i+1$

STEP 6 If $FA \cdot FP > 0$ then set $a = p$;
$FA = FP$
else set $b = p$.

STEP 7 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
// unsuccessful
STOP

* Error : $|p_n - p| \le \frac{(b-a)}{2^n}$ $\qquad p_n = p + O\left(\frac{1}{2^n}\right)$

② Newton Raphson

$$f(p) = f(p_0) + (p-p_0) f'(p_0) + \underbrace{\frac{(p-p_0)^2}{2} f''(\xi(p))}_{}.$$

$\to 0$ as $(p-p_0)^2$ very small

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$$

Thus, $p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}$.

ALGORITHM:

given initial approx p_0, find solution to $f(x)=0$.

INPUT p_0; tolerance TOL; max iterations N_0

OUTPUT approximate solution p or message of failure

STEP 1 Set $i=1$.

STEP 2 While $i \leq N_0$ do steps 3-6.

STEP 3 Set $p = p_0 - \dfrac{f(p_0)}{f'(p_0)}$

STEP 4 If $|p-p_0| < TOL$ then OUTPUT (p).
STOP.

STEP 5 Set $i = i+1$.

STEP 6 Set $p_0 = p$.

STEP 7 OUTPUT ('The method failed after N_0 iterations, N_0=' N_0)

STOP.

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- Quadratic Convergence

Root be α, x_n be $\alpha+\varepsilon_n$, $x_{n+1} = \alpha+\varepsilon_{n+1}$

$$\alpha+\varepsilon_{n+1} = \alpha+\varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)} \Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha)+\varepsilon_n f'(\alpha)+\frac{\varepsilon_n^2}{2}f''(\alpha)+\dots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\dots}$$

$$f(\alpha)=0 \Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{\varepsilon_n f'(\alpha)+\frac{\varepsilon_n^2}{2}f''(\alpha)+\dots}{f'(\alpha)+\varepsilon_n f''(\alpha)+\dots}$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2 f''(\alpha)}{2(f'(\alpha)+\varepsilon_n f''(\alpha))} = \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

So, subsequent error is square of previous error

③ Regula Falsi

- Choose two initial approx p_0, p_1 such that $f(p_0).f(p_1) < 1$
- $P_2 = P_1 - f(P_1) \dfrac{P_1 - P_0}{f(P_1) - f(P_0)}$
- Check where solution lies — $f(P_1)f(P_2) < 0$ or $f(P_0)f(P_2) < 0$

ALGORITHM :

To find solution to $f(x) = 0$, where $f \in C[P_0, P_1]$ and $f(P_0) \cdot f(P_1) < 0$.

INPUT Initial approx p_0, p_1; Tolerance TOL; max iterations N_0.

OUTPUT Approx. solution or message of failure

STEP 1 Set $i = 2$.
$q_0 = f(p_0)$; $q_1 = f(p_1)$;

STEP 2 While $i \le N_0$ do steps 3-7.

STEP 3 Set $p = p_1 - q_1 \dfrac{(p_1 - p_0)}{q_1 - q_0}$.

STEP 4 If $|p - p_1| < TOL$ then OUTPUT (p); STOP.

STEP 5 Set $i = i+1$; $q = f(p)$.

STEP 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$:

STEP 7 Set $p_1 = p$.
$q_1 = q$.

B. System of Linear Equations

① Gaussian Elimination with Backward Substitution

To Solve : $E_1 : a_{11}x_1 + \dots \quad a_{1n}x_n = a_{1\,n+1}$

E_2

$\vdots$

$E_n : a_{n1}x_1 + \dots \quad a_{nn}x_n = a_{n,n+1}$

ALGORITHM

INPUT : No. of unknowns and equations $= n$, Augmented matrix $\tilde{A} = [a_{ij}]$ where $1 \le i \le n$, $1 \le j \le n+1$

OUTPUT : Solution $x_1, \dots x_n$ or message of no unique solution

STEP 1 : For $i = 1, 2 \dots n-1$ do 2-4

STEP 2 : Let p be smallest integer such that $i \le p \le n$, $a_{pi} \ne 0$.
If no such p then OUTPUT('No unique solution exists')
STOP.

STEP 3 : If $p \ne i$ then $(E_p) \leftrightarrow (E_i)$.

STEP 4 : for $j = i+1, \dots n$ do Steps 5 and 6

STEP 5 : Set $m_{ij} = a_{ji}/a_{ii}$

STEP 6 : Perform $(E_j - m_{ji}E_i) \rightarrow E_j$

STEP 7 : If $a_{nn} = 0$ then OUTPUT ('no unique solution exists')
STOP.

STEP 8 : Set $x_n = a_{n,n+1}/a_{nn}$

STEP 9 : For $i = n-1, \dots 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] / a_{ii}$

STEP 10 : OUTPUT $(x_1 \dots x_n)$
STOP.

② Gauss-Jordan Elimination (Only row transforms)

- Extend Gaussian elimination to remove upper Δ elements as well.

Example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \quad \text{[Gauss-Jordan]}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3/(-13) \\ R_2 \to R_2 - 5R_3 \\ R_1 \to R_1 + 4R_3 \end{array}$$

③ Gauss-Siedel Iterative

- Use k^{th} equation for k^{th} variable
- Use updated values obtained even in ongoing iteration.

ALGORITHM:

To Solve $Ax=b$ given initial approximation $x^{(0)}$.

INPUT : number of equations and unknowns n; entries a_{ij} $i,j \in [1,n]$
entries b_i $1 \le i \le n$ of b; Initial approx xo_i $1 \le i \le n$ of $x^{(0)}$;
tolerance TOL; max iterations N_o.

OUTPUT : Approximate solution $x_1, \dots x_n$ or number of iters exceeded.

STEP 1: Set $k=1$.

STEP 2: While $k \le N$ do Steps 3-6.

STEP 3: For $i=1, \dots n$.

$$\text{Set } x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} xo_j + b_i\right]$$

STEP 4: If $||x - Xo|| < TOL$ then OUTPUT $(x_1, \dots x_n)$;
STOP.

STEP 5: Set $k = k+1$

STEP 6: For $i=1, \dots n$ Set $Xo_i = x_i$

STEP 7 OUTPUT ('Max iterations exceeded'),
STOP.

3] Converges if in each equation absolute value of largest coefficient is greater than sum of absolute values of remaining coefficients.

C. Interpolation

① Lagrange Interpolation

. Given $n+1$ points (x_i, y_i) $1 \le i \le n+1$ $\boxed{y_i = f(x_i)}$

We wish to find a polynomial $P(x)$ to approximate

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x).$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})(\dots)(x_k-x_n)}$$

ALGORITHM :

INPUT Pair of values $(x_i, f(x_i))$ for $1 \le i \le n+1$; n

OUTPUT Polynomial p.

STEP 1 Set $p = 0$.

STEP 2 For $i = 1$ to $n+1$ do STEPS 3-5.

STEP 3 : Set $func = 1$.

STEP 4 : For $j = 1$ to $n+1$

If $(j \ne i)$ then $func = func \times \frac{(x-x_j)}{(x_i-x_j)}$.

STEP 5 : $p = p + f(x_i)\, func$.

STEP 6 OUTPUT (p).

STOP.

* Error : $\frac{f^{(n+1)}(\xi)}{(n+1)!}\{(x-x_0)\dots(x-x_n)\}$

$\xi \in (x_0, x_n)$.

For $E(0.5)$, Put $x = \underline{\underline{0.5}}$

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(1) Newton's Divided Differences

Divided difference notation [Equi-spaced points]

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_{i+1}, x_i]}{x_{i+2} - x_i}$$

- Taking $a_k = f[x_0, x_1 \dots x_k]$

$$P_n(x) = f[x_0] + f[x_0, x_1](x_1 - x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \dots a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$\boxed{P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots x_k](x-x_0)\dots(x-x_{k-1})}$$

→ Evaluation Table.

x_0	$f[x_0]$	A $f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$	$f[x_0, x_1, x_2] = \frac{B - A}{x_2 - x_0}$
x_1	$f[x_1]$	B $f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_1, x_2, x_3] = \frac{C - B}{x_3 - x_1}$
x_2	$f[x_2]$	C $f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$	$f[x_2, x_3, x_4] = \frac{D - C}{x_4 - x_2}$
x_3	$f[x_3]$	D $f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3}$	$f[x_3, x_4, x_5] = \frac{E - D}{x_5 - x_3}$
x_4	$f[x_4]$	E $f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4}$	
x_5	$f[x_5]$		

ALGORITHM:

INPUT : $x_0, x_1, \ldots x_n$; values $f(x_0), f(x_1) \ldots f(x_n)$ as $F_{0,0}, F_{1,0} \ldots F_{n,0}$

OUTPUT : $F_{0,0}$ F_{11}, F_{22}, ... F_{nn} such that.

$$P_n(x) = F_{0,0} + \sum_{1}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j) \qquad F_{ii} \text{ is } f[x_0, \ldots x_i]$$

STEP 1 : For $i = 1, 2, \ldots n$

For $j = 1, 2, \ldots i$.

Set. $F_{i,j} = \dfrac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$

STEP 2 : OUTPUT $(F_{00}, F_{11} \ldots F_{nn})$

STOP.

TWO VERSIONS : ~~$\frac{p(p+1)}{2}$, Not $^pC_2$~~

- Forward ⇒ Upper diagonal [Used for values at beginning]
- Backward ⇒ Lower diagonal [For values at end].

Newton's Forward and backward interpolation

① x y

Find

100

150

200

250

300

350

400

Ta

A

S

(ii).

①

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Find y at $x = 218, 410$.

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.4			
150	13.03		-0.39		
		2.01		0.15	
200	15.04		-0.24		-0.07
		1.77		0.08	
			-0.16		-0.05
250	16.81			0.03	
		1.61			-0.01
			-0.13		
300	18.42			0.02	
		1.48			
350	19.9		-0.11		
		1.37			
400	21.27				

→ Difference table.

i) Take $x_0 = 200$, $h = 50$.

As $x = 218 \Rightarrow \boxed{p = \frac{x - x_0}{h}} = \frac{18}{50} = 0.36$

So, $\boxed{y(218) = y_0 + p\Delta y_0 + {}^pC_2 \Delta^2 y_0 + {}^pC_3 \Delta^3 y_0}$

$= 15.04 + 0.36 \times 1.77 + \frac{0.36 \times (-0.64)}{2} \times (-0.16)$

$+ \frac{0.36 \times (-0.64) \times (-1.64)}{3.2.} \times (0.03)$

$= 15.696$

(ii). Use backward $\Rightarrow x = 410$

Take $x_n = 400 \Rightarrow p = \frac{10}{50}$

$y(410) = y(400) + p \times 1.37 + \underbrace{{}^pC_2}_{\boxed{\frac{p(p+1)}{2}}} \times (-0.11) + \underbrace{{}^pC_3}_{\boxed{\frac{p(p+1)(p+2)}{3.2.}}} \times (0.02)$

$= 21.53$

Q3. Estimate number of students with marks b/w 40 and 45.

30-40	40-50	50-60	60-70	70-80
31	42	51	35	31

⇒ Cumulative frequency table

Marks less than	40	50	60	70	80
	31	73	124	159	190

x	y	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

→ To find value at 45°

$x = 45$, $x_0 = 40$, $h = 10 \Rightarrow p = \frac{5}{10} = \frac{1}{2}$.

$y(45) = y(40) + p\,42 + {}^{p}C_2 \cdot 9 + {}^{p}C_3(-25) + {}^{p}C_4(37)$.

$= 31 + 21 + (-1.125) + (-1.5625) + (-1.4453)$

$= 47.87 \sim 48$

So, between 45 and 40 ⇒ $48 - 31 = \boxed{17}$ Ans

③ When variable differences, Use Newton's divided differences

Find $f(x)$

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335.

$\Rightarrow$ Table.

		Δ	Δ^2	Δ^3	Δ^4
-4	1245				
		-404			
-1	33		94.		
		-28		-14.	
0	5		10		3.
		2.		13.	
2	9		88		
		442.			
5	1335.				

$$f(x) = f(x_0) + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \ldots$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)x(-14)$$
$$+ (x+4)(x+1)x(x-2)3.$$

$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5$$ An

D. Numerical Integration

① Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] - \boxed{\frac{h^3}{12} f''(\xi).}$$

• Composite Trapezoidal

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$$

② Simpson's Rules (1/3 is better)

$\frac{1}{3}$ rd $\Rightarrow n=2$ $$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] - \frac{h^5}{90} f^{(4)}(\xi).$$

$\frac{3}{8}$ th $\Rightarrow n=3$ $$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right] - \frac{3h^5}{80} f^{(4)}(\xi)$$

• Composite Simpson

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

$$h = \frac{b-a}{n}$$

ALGORITHM FOR COMPOSITE SIMPSON'S.

- To approximate ${}_a\int^b f(x)\,dx$

INPUT ends a, b; even +ve integer n.

OUTPUT approximation P.

STEP 1 Set $h = \frac{b-a}{n}$; $P0 = f(a) + f(b)$; $P1 = 0$; $P2 = 0$;
odd terms even terms

STEP 2 For $i = 1, \dots n-1$ do Steps 4-5

STEP 4 Set $X = a + ih$

STEP 5 If i is even then $P2 = P2 + f(X)$
else set $P1 = P1 + f(X)$

STEP 6 Set $P = \frac{h}{3} \times (P0 + 2P2 + 4P1)$

STEP 7 Output (P).
STOP.

Composite $R = -\frac{h^5}{90}\left[f^4(\xi_1) + f^4(\xi_2) + \dots f^4(\xi_n)\right]$

Take $f^4(\eta) = \max_i |f^4(\xi_i)| \Rightarrow |R| \le N f^{(4)}(\eta)\frac{h^5}{90} = \frac{f^4(\eta)(b-a)^5 \cdot \cancel{N}}{2^5 \cdot N^{\cancel{5}4}\, 90}$

$$\le \frac{f^{(4)}(\eta)(b-a)^5}{2880\, N^4} = \frac{(b-a)}{180} h^4 f^{(4)}(\eta)$$

* Error finding

① Simpson's.

- Order $f^{4}(\varepsilon)$.

So error is 0 for degree 3 polynomials

- Error is $\frac{C}{4!} f^{(4)}(\eta)$

To find C $\Rightarrow$

$$C = \int_a^b x^4 - \frac{\cancel{3}h}{3}\left[a^4 + 4.\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$= \frac{a^5 - b^5}{5} - \frac{h}{3}\left(a^4 + 4.(a+h)^4 + (a+2h)^4\right)$$

$$= \frac{(a+2h)^5}{5} - \frac{\cancel{a^5}}{\cancel{5}} - \frac{a^4\cancel{h}}{\cancel{3}} - \frac{h}{3}\left(\cancel{4a^4} + 16\cancel{a^3}h + 24\cancel{a^2}h^2 + 16\cancel{a}h^3 + 4h^4\right)$$

$$- \frac{h}{3}\left(\cancel{a^4} + 8\cancel{a^3}h + 24\cancel{a^2}h^2 + 32\cancel{a}h^3 + 16h^4\right)$$

$$\frac{\cancel{a^5} + 10a^4h + 40a^3h^2 + 80a^2h^3 + 80ah^4 + 32h^5}{5}$$

$$a^4h\left(2 - \frac{1}{3} - \frac{4}{3} - \frac{1}{3}\right)^{\to 0} + a^3h^2\left(8 - \frac{16}{3} - \frac{8}{3}\right)^{\to 0} + a^2h^3\left(16 - 8 - 8\right)^{\to 0}$$

$$+ ah^4\left(16 - \frac{16}{3} - \frac{32}{3}\right)^{\to 0} + h^5\left(-\frac{4}{3} - \frac{16}{3}\right) + \frac{32}{5}$$

$$\frac{-100 + 96}{15} = \left(-\frac{4}{15}h^5\right)$$

$$-\frac{\cancel{4}\,h^5}{15.\cancel{4}\times 3\times 2} = \boxed{-\frac{h^5}{90}}$$ Error

③ Gaussian Quadrature Formula

• n point Quadrature formula.

$$\int_a^b f(x)\,dx = C_1 f(x_1) + C_2 f(x_2) + \dots C_n f(x_n)$$

To find : $C_1, C_2 \dots C_n$ and $x_1, \dots x_n$

→ For $n=2$ and $\int_{-1}^{1}$ — $C_1, C_2 = 1$ $x_1, x_2 = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

↳ $C_1 + C_2 = \int_{-1}^{1} dx$ $\quad C_1 x_1 + C_2 x_2 = \int_{-1}^{1} x\,dx$

$C_1 x_1^2 + C_2 x_2^2 = \int_{-1}^{1} x^2\,dx$ $\quad C_1 x_1^3 + C_2 x_2^3 = \int_{-1}^{1} x^3\,dx$

Values :

• 2-points. $C_1 = C_2 = 1$

$C_1 f(x_1) + C_2 f(x_2)$ $\quad x_1 = \frac{1}{\sqrt{3}}, \; x_2 = -\frac{1}{\sqrt{3}}.$

• 3-points.

$C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$

$C_1 = 8/9 \quad x_1 = 0.$

$C_2 = 5/9 \quad x_2 = \sqrt{\frac{3}{5}}$

$C_3 = 5/9 \quad x_3 = -\sqrt{\frac{3}{5}}.$

To convert $\int_a^b f(x)\,dx$ to $\int_{-1}^{1}$

Put $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$

For $n=2$ we have polynomial of degree 3 on RHS $(2n-1)$

2-pt · Consider $f(x)$ as 3 degree polynomial

$$\int_a^b f(x)\,dx = \int_a^b a_0 + a_1 x + a_2 x^2 + a_3 x^3 \, dx$$

$$a_0(b-a) + a_1\frac{(b^2-a^2)}{2} + a_2\frac{(b^3-a^3)}{3} + a_3\frac{(b^4-a^4)}{4} \quad -(1)$$

$$= C_1 f(x_1) + C_2 f(x_2)$$

$$(2) - = C_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + C_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

Equate coeff. of a_0, a_1, a_2, a_3 in (1) and (2)

$$b-a = C_1 + C_2 \qquad \frac{b^2-a^2}{2} = C_1 x_1 + C_2 x_2$$

$$\frac{b^3-a^3}{3} = C_1 x_1^2 + C_2 x_2^2 \qquad \frac{b^4-a^4}{4} = C_1 x_1^3 + C_2 x_2^3$$

ALITER ≡

$$\int_a^b 1\,dx = C_1 + C_2$$

$$\int_a^b x\,dx = C_1 x_1 + C_2 x_2$$

$$\int_a^b x^2 dx = C_1 x_1^2 + C_2 x_2^2$$

$$\int_a^b x^3 dx = C_1 x_1^3 + C_2 x_2^3$$

For 2pt $\Rightarrow$

$$C_1 = \frac{b-a}{2} \qquad C_2 = \frac{b-a}{2}.$$

$$x_1 = \frac{(b-a)}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}.$$

E. Solution of ODEs

① Euler's method

To Solve $\frac{dy}{dt} = f(t,y)$ $\quad a \le t \le b \quad y(a) = \alpha.$

$\rightarrow w_0 = \alpha$

$w_{i+1} = w_i + hf(t_i, w_i)$

$t_i = a + ih$

$h = \frac{b-a}{n}$

ALGORITHM

INPUT ends a, b; integer n; initial condition α.

OUTPUT approx w to y at all $(N+1)$ values of t.

STEP 1 Set $h = \frac{(b-a)}{n}$; $t = a$; $w = \alpha$

OUTPUT (t, w).

STEP 2 For $i = 1, 2, \ldots n$ do STEPS 3-4.

STEP 3 Set $w = w + hf(t, w)$

$t = a + ih$.

STEP 4 OUTPUT (t, w)

STEP 5 STOP.

② Runge- Kutta methods

· MOST IMPORTANT

ORDER 4.

$$\omega_0 = \alpha.$$
$$k_1 = hf(t_i, \omega_i)$$
$$k_2 = hf\left(t_i + \frac{h}{2}, \omega_i + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(t_i + \frac{h}{2}, \omega_i + \frac{k_2}{2}\right)$$
$$k_4 = hf(t_{i+1}, \omega_i + k_3)$$
$$\omega_{i+1} = \omega_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

ALGORITHM

To approx solutions of $y' = f(t,y)$, $y(a) = \alpha$ at $(N+1)$ equally spaced numbers in $[a,b]$

INPUT a, b; integer N; α.

OUTPUT approx ω to y at $N+1$ points

STEP 1 Set $h = \frac{b-a}{N}$; $t = a$; $\omega = \alpha$.
OUTPUT (t, ω).

STEP 2 For $i = 1, 2, \ldots n$ do STEP 3-5.

STEP 3 Set $K_1 = hf(t, \omega)$
$K_2 = hf(t + \frac{h}{2}, \omega + \frac{K_1}{2})$
$K_3 = hf(t + \frac{h}{2}, \omega + \frac{K_2}{2})$
$K_4 = hf(t + h, \omega + K_3)$

STEP 4 Set $\omega = \omega + (K_1 + 2K_2 + 2K_3 + K_4)/6$
$t = a + ih$

STEP 5 OUTPUT (t, ω)

STEP 6 STOP

ORDER 2 (Mid-point method)

$$\boxed{\begin{aligned} w_0 &= \alpha. \\ w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right) \end{aligned}}$$

Order 2. (Heun's)

$$\boxed{\begin{aligned} k_1 &= h f(x_0, y_0) \\ k_2 &= h f(x_0 + h, y_0 + k_1) \\ y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \end{aligned}}$$

$\Rightarrow$ Also known as modified Euler.

ORDER 3

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f(x_0 + h, y_0 + k_1')$$

where $k' = hf(x_0 + h, y_0 + k_1)$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Scanned by CamScanner

* Solving simultaneous ODE's.

$$\frac{dy}{dx} = z \quad \{f(x,y,z)\} \qquad y(0) = 3$$

$$\frac{dz}{dx} = 6y - z \quad \{g(x,y,z)\} \qquad z(0) = 1.$$

$$y_{i+1} = y_i + \frac{1}{6}\{k_0 + 2k_1 + 2k_2 + k_3\}$$

$hf(x_i+h, y_i+k_2, z_i+l_2)$

$hf(x_i, y_i, z_i)$ $\quad$ $hf\left(x_i+\frac{h}{2}, y_i+\frac{k_0}{2}, z_i+\frac{l_0}{2}\right)$

$$z_{i+1} = z_i + \frac{1}{6}\{l_0 + 2l_1 + 2l_2 + l_3\}$$

$hg(x_i, y_i, z_i)$ $\quad$ $hg\left(x_i+\frac{h}{2}, y_i+\frac{k_1}{2}, z_i+\frac{l_1}{2}\right)$

Calculations:

$k_0 \rightarrow l_0 \rightarrow k_1 \rightarrow l_1 \rightarrow k_2, l_2 \rightarrow k_3, l_3$

$y_{i+1} \rightarrow z_{i+1}$

Regula Falsi

Assume $f(a).F(b) < 0$
$a > b$