JFOS-2015

① Solve $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$.

Let $x=e^z$.

$x\frac{dy}{dx}=\frac{dz}{dx}$.

$((D)(D-1)+3D+1)y=\frac{1}{(1-e^z)^2}$.

$(D^2-D+3D+1)y=\frac{1}{(1-e^z)^2}$.

$(D^2+2D+1)y=\frac{1}{(1-e^z)^2}$.

$(D+1)^2=0$

$D=-1,-1$.

$y=\cancel{Ae^{-z}}\ (Az+B)e^{-z}$.

$y=\frac{1}{(D+1)^2}(1-e^z)^{-2}$.

We know that $\frac{1}{D+a}X=x^{-a}\int x^{a-1}X\,dx$.

$\Rightarrow y=\frac{1}{(D+1)}\frac{1}{(D+1)}(1-x)^{-2}$

$\Rightarrow \frac{1}{(D+1)}x^{-1}\int x^{1-1}(1-x)^{-2}\,dx$

$\Rightarrow \frac{1}{(D+1)}x^{-1}\int(1-x)^{-2}dx$

$=\frac{1}{(D+1)}x^{-1}(1-x)^{-1}$

$$\Rightarrow x^{-1} \int x^{1-1} x^{-1} (1-x)^{-1} \, dx$$

$$\Rightarrow y = x^{-1} \int x^{-1} (1-x)^{-1} \, dx$$

$$\Rightarrow y = x^{-1} \int \frac{1}{(x)(1-x)} \, dx$$

$$y = x^{-1} \int \left( \frac{1}{x} + \frac{1}{1-x} \right) dx$$

$$y = x^{-1} \left[ \ln x - \ln(1-x) \right].$$

$$\boxed{y = x^{-1} \ln \frac{x}{1-x}}$$

$$\boxed{\text{Solution } y = \frac{(A + B\log x)}{x} + x^{-1} \ln \frac{x}{1-x}.}$$

② Solve.

$$(D^4 + D^2 + 1)y = e^{\frac{-x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

$$D^4 + D^2 + D^2 + 1 - D^2$$

$$D^2(D^2+1) + D^2 + 1 - D^2$$

$$(D^2+1)(D^2+1) - D^2.$$

$$(D^2+1)^2 - D^2.$$

$$(D^2+1+D)(D^2+1-D)\, y = 0.$$

$$D^2 + D + 1 = 0 \qquad D^2 - D + 1 = 0.$$

$$\frac{-1 \pm \sqrt{1-4}}{2} \qquad \frac{1 \pm \sqrt{1-4}}{2}.$$

$$\frac{-1 \pm i\sqrt{3}}{2} \qquad \frac{1 \pm i\sqrt{3}}{2}.$$

$$c_1 e^{\frac{-x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + c_2\right) + c_3 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + c_4\right),$$

$$y = \frac{e^{\frac{-x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{D^4 + D^2 + 1}.$$

$$y = e^{\frac{-x}{2}} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{\left(D - \frac{1}{2}\right)^4 + \left(D - \frac{1}{2}\right)^2 + 1}.$$

$$y = e^{\frac{-x}{2}} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{\left(D - \frac{1}{2}\right)^2 \left(\left(D - \frac{1}{2}\right)^2 + 1\right) + 1}$$

$$y = e^{-x/2} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{\left(D^2 + \frac{1}{4} - D\right)\left(D^2 + \frac{1}{4} - D + 1\right) + 1}$$

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$$y = e^{-\frac{x}{2}} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{\left(D^2 - D + \frac{1}{4}\right)\left(D^2 - D + \frac{5}{4}\right) + 1}$$

$$D^2 = -\frac{3}{4}.$$

$$y = e^{-\frac{x}{2}} \cdot \frac{1}{\left(-\frac{3}{4} - D + \frac{1}{4}\right)\left(-\frac{3}{4} - D + \frac{5}{4}\right) + 1} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

$$y = e^{-\frac{x}{2}} \frac{1}{\left(-D - \frac{1}{2}\right)\left(-D + \frac{1}{2}\right) + 1} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

$$y = e^{-\frac{x}{2}} \frac{1}{\left(D^2 - \frac{1}{4}\right) + 1} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

$$y = e^{-\frac{x}{2}} \frac{1}{D^2 + \frac{3}{4}} \cos\left(\frac{x\sqrt{3}}{2}\right).$$

$$y = e^{-\frac{x}{2}} \frac{x \cdot \cos\left(\frac{x\sqrt{3}}{2}\right)}{2D}.$$

$$y = e^{-\frac{x}{2}} \frac{x\, D\left(\cos\left(\frac{x\sqrt{3}}{2}\right)\right)}{-2\frac{3}{4}}.$$

$$y = e^{-\frac{x}{2}} \cdot \frac{x\left(-\sin\left(\frac{x\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}\right)}{-\sqrt{3}} \cancel{2}.$$

$$\boxed{y = \frac{x e^{-x/2}}{\sqrt{3}} \sin\left(\frac{x\sqrt{3}}{2}\right) + c_1 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + c_2\right) + c_3 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x + c_4\right)}$$

③ $x^2p^2 + yp(2x+y) + y^2 = 0$

Substitution is $y = u$ and $xy = v$.

$dy = du$ and $xdy + ydx = dv$.

$$\frac{dv}{du} = \frac{xdy + ydx}{dy}$$

$$\frac{dv}{du} = x + y\frac{dx}{dy}$$

$$\boxed{x + \frac{y}{P} = \frac{dv}{du}}$$

$$\frac{y}{p} = P - x$$

$$\boxed{p = \frac{y}{P-x}}$$

$$\frac{x^2y^2}{(P-x)^2} + \frac{y^2(2x+y)}{(P-x)} + y^2 = 0$$

$$(P-x)^2 = 0$$

$$x^2 + (2x+y)(P-x) + \cancel{y}(P-x)^2 = 0$$

$$x^2 + [2Px - 2x^2 + Py - xy] + \cancel{y}[P^2 + x^2 2Px] = 0$$

$$x^2 + [2Px - 2x^2 + Py - xy + P^2\cancel{y} + x^2\cancel{y} - 2Px\cancel{y} = 0$$

$$\cancel{x^2} + \cancel{2}Px - \cancel{2x^2} + Py - xy$$

$$P^2 + Py - xy = 0$$

$$P^2 + Pu - v = 0$$

$$\boxed{v = Pu + P^2}$$ Clairaut form.

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general solution is $\Rightarrow$

$$\boxed{v = cu + c^2}$$

differentiating w.r.t to c.

$$0 = u + 2c$$

$$c^2 + cu - v = 0$$

c discriminant $\Rightarrow$ $u^2 + 4v = 0$

Singular solution is $\boxed{u^2 + 4v = 0}$

(4) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$ by changing independent variable.

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2 y = 8x^2 \sin x^2$$

Independent variable $= x$.

Dependent variable $= y$.

$R = 8x^2 \sin x^2$

$Q = -4x^2$

$P = -\frac{1}{x}$

choosing $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\Rightarrow \frac{dz}{dx} = 2x$$

$$z = x^2$$

then $P_1 = \dfrac{\frac{d^2z}{dx^2} + P(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$. $\quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$P_1 = 2 - \frac{1}{x} \cdot 2x = 0$

$Q_1 = \frac{-4x^2}{4x^2} \Rightarrow -1$

$R_1 = \frac{\overset{2}{\cancel{8}}x^2 \sin x^2}{\cancel{4}x^2}$

$$\boxed{R_1 = 2\sin x^2}$$

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$$\frac{d^2y}{dz^2} - y = 2\sin z$$

$$(D_1'^2 - 1)y = 2\sin z$$

C. Functi $\Rightarrow$ $c_1e^z + c_2e^{-z}$

P. I. $\Rightarrow$ $\dfrac{2\sin z}{D_1^2 - 1} \Rightarrow \dfrac{2\sin z}{-1-1} \Rightarrow -\sin z$

$$\boxed{y = c_1e^z + c_2e^{-z} - \sin z}$$