A fixed wire is in the shape of the cardiod $r = a\,(1 + \cos\theta)$, the initial line being the downward vertical. A small ring of mass m can slide on the wire and is attached to the point $r = 0$ of the cardiod by an elastic string of natural length a and modulus of elasticity 4 mg. The string is released from rest when the string is horizontal. Show by using the laws of conservation of energy that

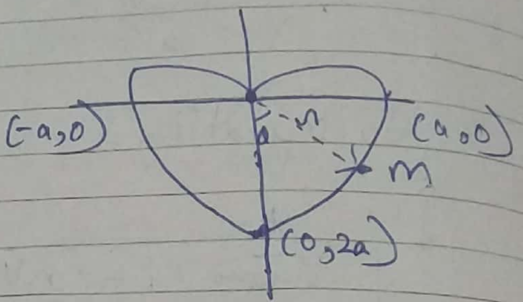
$a\dot{\theta}^2\,(1 + \cos\theta) - g\,\cos\theta\,(1 - \cos\theta) = 0$, g being the acceleration due to gravity. 10

2017

Ans) Given fixed wire is in the shape of a cardoid,

$r = a(1+\cos\theta)$

$r' = -a\sin\theta$



We have,

loss in PE of m = gain in KE of mass m + gain in PE of the string

$$mg(r\cos\theta) = \frac{1}{2}mv^2 + \int_0^{r-a} \frac{\lambda x}{a}\,dx$$

~~mg(r cos θ)~~

$x = r\cos\theta$
$y = r\sin\theta$

$v^2 = \dot{x}^2 + \dot{y}^2$

$\dot{x} = \left(\frac{dx}{dt}\right) = \cos\theta\,\dot{r} - \sin\theta\, r\,.\,\dot{\theta}$

$\dot{y} = \left(\frac{dy}{dt}\right) = \dot{r}\sin\theta + \cos\theta\,.\,r\,.\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 = v^2$

$v^2 = (a\,.\sin\theta\,\dot{\theta})^2 + (a(1+\cos\theta)\,.\,\dot{\theta})^2$

Now
mgr c
mg
gr

Now,

$$mgr\cos\theta = \frac{1}{2}m\left[a^2\sin^2\theta\cdot\dot{\theta}^2 + a^2\dot{\theta}^2 + a^2\cos^2\theta\cdot\dot{\theta}^2 + a^2\dot{\theta}^2\cdot 2\cos\theta\right] + \frac{\lambda}{2a}(r-a)^2$$

$$mgr\cos\theta = \frac{1}{2}m\left[\dot{\theta}^2a^2 + a^2\dot{\theta}^2(1+2\cos\theta)\right] + \frac{4mg}{2a}(r-a)^2$$

$$gr\cos\theta = \frac{1}{2}\left[a^2\dot{\theta}^2(2+2\cos\theta)\right] + \frac{2g}{a}(\cos\theta)^2a^2$$

$$ga(1+\cos\theta)\cdot\cos\theta = \cancel{\tfrac{1}{2}}\,a^2\dot{\theta}^2(1+\cos\theta) + \frac{2g}{a}(\cos\theta)^2\cdot a^2$$

$$g(1+\cos\theta)\cos\theta = a\dot{\theta}^2(1+\cos\theta) + 2g(\cos\theta)^2$$

$$a\dot{\theta}^2(1+\cos\theta) + 2g(\cos\theta)^2 - g\cos\theta - g(\cos\theta)^2 = 0$$

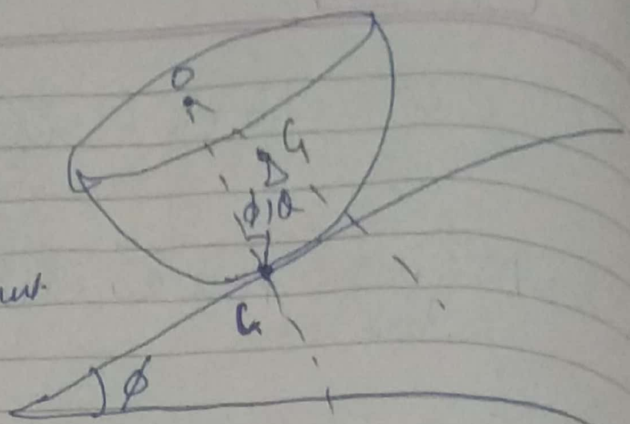
$$a\dot{\theta}^2(1+\cos\theta) + g(\cos\theta)^2 - g\cos\theta = 0$$

$$a\dot{\theta}^2(1+\cos\theta) + g\cos\theta(\cos\theta-1) = 0$$

//.

A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable ? 17

Ans 2



O is the centre of the hemisphere

G is the COG of the hemisphere

Let $OC = a$

$\therefore \; OG = \frac{3a}{8}$

OC is the common normal.

GC is the vertical.

In ΔOGC, applying sine rule

$$\frac{OG}{\sin\phi} = \frac{OC}{\sin(\pi-\theta)} = \frac{GC}{\sin(\pi-(\phi+\pi-\theta))}$$

Consider

$$\frac{OG}{\sin\phi} = \frac{OC}{\sin\theta}$$

$$\frac{3a}{8\sin\phi} = \frac{a}{\sin\theta}$$

$\cancel{\sin\theta = \frac{3}{8}\sin\phi}$ $\sin\theta = \frac{8}{3}\sin\phi$

$\therefore$, $\sin\theta < 1$

$\frac{8}{3}\sin\phi < 1$

$\cancel{\sin\phi < \frac{8}{3}}$ $\sin\phi < \frac{3}{8}$ —(1)

Also,

$$\frac{CG}{\sin(\theta-\phi)} = \frac{OG}{\sin\phi}$$

$$CG = \left(\frac{3a}{8}\right) \frac{\sin(\theta-\phi)}{\sin\phi}$$

Now, for equilibrium to be stable,

$$\frac{1}{h} > \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \sec\phi$$

$h = CG$
$\rho_1 = a$
$\rho_2 = \infty$

$$\frac{8}{3a} \frac{\sin\phi}{\sin(\theta-\phi)} > \left(\frac{1}{a}\right) \sec\phi$$

$$\frac{8}{3} \frac{\sin\phi}{\sin(\theta-\phi)} > \sec\phi$$

$$8\sin\phi\cos\phi > 3\sin(\theta-\phi)$$

$$8\sin\phi\cos\phi > 3\sin\theta\cos\phi - 3\cos\theta\sin\phi$$

$$8\sin\phi\cos\phi > 3 \cdot \frac{8}{3}\sin\phi\cos\phi - 3\sqrt{1-\frac{64}{9}\sin^2\phi} \cdot \sin\phi$$

$$\cancel{-\sqrt{9-64\sin^2\phi} \cdot \sin\phi}$$

$$-\sin\phi\sqrt{9-64\sin^2\phi} < 0$$

$$\sin\phi\sqrt{9-64\sin^2\phi} > 0$$

$$9-64\sin^2\phi > 0$$

$$\sin\phi < \frac{3}{8}$$

Hence, the equilibrium is stable.

Ans 3

The

1)

2)

A particle is free to move on a smooth vertical circular wire of radius a. At time $t = 0$ it is projected along the circle from its lowest point A with velocity just sufficient to carry it to the highest point B. Find the time T at which the reaction between the particle and the wire is zero. 17

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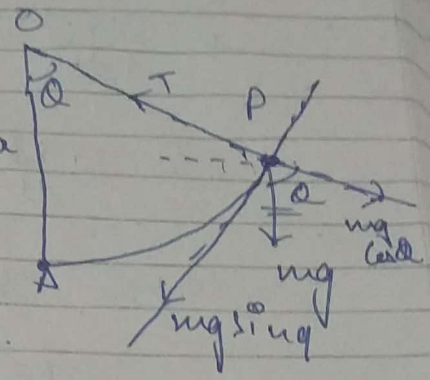
Ans 3

There are 2 acceleration

1) $\left(\frac{v^2}{a}\right)$ acting towards the centre

2) $\left(\frac{d^2s}{dt^2}\right)$ tangential acceleration.



At P,

$$m\left(\frac{d^2s}{dt^2}\right) = -mg\sin\theta \quad -①$$

$$T - mg\cos\theta = \frac{mv^2}{a} \quad -②$$

Also,

arc (AP) = s

$$s = a\theta$$

$$\frac{ds}{dt} = a\frac{d\theta}{dt} \Rightarrow v = a\frac{d\theta}{dt} \quad -③$$

$$\frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2} \quad -④$$

Putting ④ in ①,

$$ma\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

Multiplying by $2\left(\frac{d\theta}{dt}\right)$

$$a\,2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = -2\left(\frac{d\theta}{dt}\right)g\sin\theta$$

Integrating,

$$a \cdot \left(\frac{d\theta}{dt}\right)^2 = +2g\cos\theta + A$$

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A \quad \text{(from ③)}$$

When ~~$\theta = 0$,~~

$$\theta = \pi, \quad v = 0$$

$$0 = -2ag + A$$
$$A = 2ag$$

$$v^2 = (2ag\cos\theta) + 2ag$$

$$v^2 = 2ag(1 + \cos\theta)$$

Putting this in ②,

$$T - mg\cos\theta = \frac{m}{a}\left(2ag(1+\cos\theta)\right)$$

$$T - mg\cos\theta = 2mg(1+\cos\theta)$$

When $T = 0$

$$-g\cos\theta = 2g + 2g\cos\theta$$

$$-3g\cos\theta = 2g$$

$$\cos\theta = -\left(\frac{2}{3}\right)$$

Now,

$$v^2 = 2ag(1+\cos\theta)$$

$$a^2\left(\frac{d\theta}{dt}\right)^2 = 2ag(1+\cos\theta)$$

$$a\left(\frac{d\theta}{dt}\right) = \sqrt{2ag}\,\sqrt{1+\cos\theta}$$

$$\frac{\sqrt{a}}{\sqrt{2}\cdot\sqrt{g}}\quad\frac{d\theta}{\sqrt{1+\cos\theta}} = dt$$

$$\frac{\sqrt{a}}{\sqrt{2g}}\cdot\frac{d\theta}{\sqrt{2}\cdot\cos(\theta/2)} = dt$$

$$\frac{\sqrt{a}}{2\sqrt{g}}\cdot\sec(\theta/2)\,d\theta = dt$$

$$\frac{\sqrt{a}}{2\sqrt{g}}\int_{\theta=0}^{\cos\theta=(-2/3)}\sec(\theta/2)\,d\theta = \int_{t=0}^{t_1} dt$$

$$\frac{\sqrt{a}}{2\sqrt{g}}\left[\log\left(\sec\left(\frac{\theta}{2}\right)+\tan\left(\frac{\theta}{2}\right)\right)\cdot 2\right]_{\theta=0}^{\cos\theta=(-2/3)} = t_1$$

$\cancel{\frac{\sqrt{a}}{\sqrt{g}}\left[\log\right]}$

$$\cos\theta = -2/3$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{6}}$$

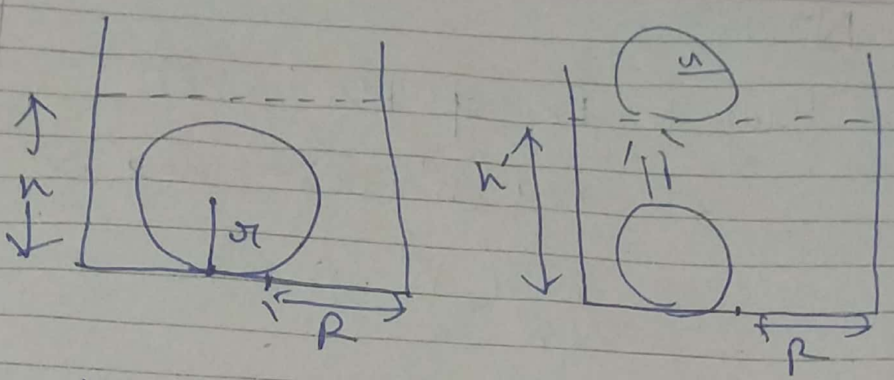
$$\sec\left(\frac{\theta}{2}\right) = \sqrt{6}$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{5}$$

$\frac{\sqrt{a}}{\sqrt{g}} \log\left(\sqrt{6}+\sqrt{5}\right) = t_1$ //

A spherical shot of W gm weight and radius r cm, lies at the bottom of cylindrical bucket of radius R cm. The bucket is filled with water up to a depth of h cm $(h > 2r)$. Show that the minimum amount of work done in lifting the shot just clear of the water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm gm. W' gm is the weight of water displaced by the shot. 16

Ans 4



Let

r = radius of the shot.
R = radius of the cylindrical ~~set~~ bucket
h = height of the bucket.

When the shot is lifted out, depth of water will decrease.

Volume of shot $= \frac{4}{3}\pi r^3$

Volume of ~~cyli~~ cylindrical bucket when shot is immersed $= \pi R^2 h$

Volume of cylindrical bucket when shot is lifted out $= \pi R^2 h'$

$$\frac{4}{3}\cancel{\pi} r^3 = \cancel{\pi} R^2 h - \cancel{\pi} R^2 h'$$

$$\frac{4}{3} r^3 = R^2 (h - h')$$

$$h' = h - \frac{4}{3}\frac{r^3}{R^2}$$

Scanned with CamScanner

Now,

Change in PE of the shot

$= mg(h'+r) - mg\, r$

$= W(h'+r-r)$

$= Wh'$

$= W\left(h - \frac{4}{3}\frac{r^3}{R^3}\right)$

W' is the weight of the water displaced by the shot.

Change in PE of the displaced water=

$= W'(r) - W'\frac{(h+h')}{2}$

$= W'\left(r - \frac{h}{2} - \frac{h'}{2}\right)$

$= W'\left(r - \frac{h}{2} - \frac{h}{2} + \frac{2}{3}\frac{r^3}{R^2}\right)$

$= W'\left(r - h + \frac{2}{3}\frac{r^3}{R^2}\right)$

Hence

Work done $= W\left(h - \frac{4}{3}\frac{r^3}{R^3}\right) + W'\left(r - h + \frac{2}{3}\frac{r^3}{R^2}\right)$

//.