Mains Test Series - 2019

Test - 08

Section - A

Ques: 1) a) Give an example of two subgroups H, K which are not normal but HK is a subgroup.

Solution. Let $H = \{I, (12)\}$

$K = \{I, (123)(132)\}$

be two subgroups of S_4.

Here; $HK = \{I, (12), (123), (132), (12)(123), (12)(132)\}$

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$KH = \{I, (123), (132), (12), (123)(12), (132)(12)\}$

$KH = \{I, (12), (123), (132), (23), (13)\}$

Thus; $\boxed{KH = HK}$

$\Rightarrow$ HK is a subgroup.

Now; $H(123) = \{(123), (12)(123)\}$

$= \{(123), (23)\}$

$(123)H = \{(123), (13)\}$

or that $\boxed{H(123) \neq (123)H}$

i.e $\boxed{aH \neq Ha}$ for some $a \in S_4$

$\Rightarrow$ H is not normal in S_4.

Similarly;

$$K(12) = \{(123), (12)(123)\}$$
$$= \{(123), (23)\}$$

$$(12)K = \{(12)(123), (123)\}$$
$$= \{(123)(13)\}.$$

$\therefore$ $\boxed{aK \neq Ka}$ for some $a \in S_4$

$\Rightarrow$ K is not normal in S_4.

Ques: 1(b) Show by means of an example that we can find $A \subseteq B \subseteq R$, where A is an ideal of B, B is an ideal of R, but A is not an ideal of R.

Solution: given that

$$A \subseteq B \subseteq R$$

A is an ideal of B

B is an ideal of R

but T.p.t – A is not an ideal of R.

Let, R be the set containing matrices of 3×3 of the type $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix}$ over integers, then

R forms a ring under matrix addition & multiplication.

Take ; $A = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, x \text{ is an integer} \right\}$

$B = \left\{ \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \middle| \, u,v \text{ integers} \right\}$

To check whether, A is an ideal of B ;

$AB \in A$.

$\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ let $x=1$, $u=1$, $v=1$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in A.$

Hence, A is an ideal of 'B'

Similarly for B ,

$BR \in B$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in B$

Hence, B is an ideal of R

Now, to see , A is not an ideal of R.

$RA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \notin A$

Hence ; A is not an ideal of R.

Hence, verified

Q.1) c) Prove that the function 'f' defined on $\mathbb{R}$ by $f(x) = \frac{1}{x^2+1}$, $x \in \mathbb{R}$, is uniformly continuous of $\mathbb{R}$.

Solution:

Given; $f(x) = \frac{1}{x^2+1}$; $x \in \mathbb{R}$

$$f'(x) = -\frac{2x}{(x^2+1)^2}, \quad x \in \mathbb{R}$$

Therefore; $|f'(x)| < 2$ for all values of $x \in \mathbb{R}$

Let; x_1 and x_2 be any points in $\mathbb{R}$, such that

$$x_1 < x_2$$

f is continuous on $[x_1, x_2]$ and

f is differentiable on (x_1, x_2).

Now, by Mean Value Theorem,

there exists a point ξ_1 in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi_1)$$

Since; $|f'(x)| < 2$ for all $x \in \mathbb{R}$

$$|f(x_2) - f(x_1)| < 2\,|x_2 - x_1|$$

Let, us choose $\varepsilon > 0$.

There exists a positive $\delta = \frac{\varepsilon}{2}$, such that

$|f(x_1) - f(x_2)| < \varepsilon$ for all x_1, x_2 in $\mathbb{R}$

satisfying $|x_2 - x_1| < \delta$

This proves that 'f' is uniformly continuous on $\mathbb{R}$

Ques: 1) d) By using Riemann integral definition evaluate $\int_a^b x^{99}dx$; where $0<a<b$.

Solution: Let, $f(x) = x^{99}$, $x \in [a,b]$.

Then f is continuous on $[a,b]$ and therefore f is integrable on $[a,b]$.

Let; $P_n = (a, ah, ah^2, \ldots ah^{n-1}, b)$, where $h^n = b/a$

Then; $\|P_n\| = ah^{n-1}(h-1) = a\cdot\left(\frac{b}{a}\right)^{\frac{n-1}{n}}\left[\left(\frac{b}{a}\right)^{1/n}-1\right]$

$\{P_n\}$ is a sequence of partitions of $[a,b]$, such that $\lim \|P_n\| = 0$

Since, f is integrable on $[a,b]$,

$$\int_a^b f = \lim_{n\to\infty}\left[a(h-1)f(\xi_1) + ah(h-1)f(\xi_2) + \ldots + ah^{n-1}(h-1)f(\xi_n)\right]$$

for any particular choice of intermediate points $\xi_1, \xi_2, \ldots \xi_n$, where

$$\boxed{ah^{r-1} \leq \xi_r \leq ah^r}$$

Let, $\xi_r = ah^{r-1}$

Then $\int_a^b x^{99}dx = \lim\limits_{h\to 1} a^{100}(h-1)(1 + h^{100} + h^{200} + \ldots + h^{(n-1)100})$

$$\int_a^b x^{99}dx = \lim_{h\to 1} a^{100}(h-1)\left[\frac{h^{100n}-1}{h^{100}-1}\right]$$

$$\int_a^b x^{99}dx = \lim_{h\to 1} a^{100}(h-1)\left[\frac{h^{100n}-1}{h^{100}-1}\right] = \lim_{h\to 1}\frac{h-1}{h^{100}-1}\left[\left(\frac{b}{a}\right)^{100}-1\right]a^{100}$$

$$\int_a^b x^{99}dx = \left[\frac{b^{100}}{a^{100}}-1\right]a^{100}\cdot\frac{1}{100} = \left(\frac{b^{100}-a^{100}}{\cancel{a^{100}}}\times\cancel{a^{100}}\right)\frac{1}{100}$$

$$\Rightarrow \boxed{\int_a^b x^{99}dx = \frac{1}{100}\left[b^{100}-a^{100}\right]}$$ which is result.

Q.1) e) Prove that the function $u = e^x(x\cos y - y\sin y)$ satisfy Laplace's equation and find the corresponding analytic function $f(z) = u + iv$.

Solution:

Given, $u = e^x(x\cos y - y\sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - y\cos y - \sin y) = \phi_2(x,y)$$

$\therefore$ By Milne's method, we have.

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$f'(z) = e^z[z\cdot\cos 0 - 0\cdot\sin 0 + \cos 0]$$
$$- ie^z[-z\sin 0 - 0\cdot\cos 0 - \sin 0]$$

$$f'(z) = e^z[z\cdot 1 - 0 + 1] - ie^z[0 - 0 - 0]$$

$$\boxed{f'(z) = e^z(z+1)}$$

Integrating, we get

$$f'(z) = \int e^z(z+1)dz + c$$

$$f(z) = ze^z - \cancel{e^z} + \cancel{e^z} + c$$

$$\boxed{\therefore f(z) = ze^z + c}$$ which is the required solution.

Q.2) a) If G is finite group and H is a subgroup of G, then O(H) divides O(G). Is converse true? Justify your answer.

Sol:-

Given; G is finite group
and H is a subgroup of G.

Then, Let $O(G) = n$.

Since; corresponding to each element in G, we can define a right coset of H in G.

Hence, no. of distinct right cosets of H in G $\leq n$

using the property of equivalence classes, we know

$$G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_t$$

where t = no. of distinct right cosets of H in G

i.e $\boxed{t \leq n}$

$$\Rightarrow O(G) = O(Ha_1) + O(Ha_2) + \cdots + O(Ha_t)$$

$$\Rightarrow O(G) = O(H) + O(H) + \cdots + O(H) \quad \text{— t times}$$

$$\Rightarrow O(G) = t \cdot O(H)$$

Or that $\boxed{O(H) \mid O(G) = t}$

Hence proved O(H) divides O(G)

Now, to check, whether converse true or not.

Consider, the alternating group A_4.

$$O(A_4) = \frac{O(S_4)}{2} = \frac{\lfloor 4}{2} = 12$$

We show, although $6 \mid 12$, A_4 has no subgroup of order 6.

Suppose, H is a subgroup of A_4

and $o(H) = 6$.

The number of distinct 3-cycles in S_4 is

$$\frac{1}{3}\cdot\frac{4!}{(4-3)!} = \frac{4\times\cancel{3}\times 2\times 1}{\cancel{3}} = 8.$$

Again; each 3-cycle will be even permutation all these 3-cycles are in S_4.

Then, at least one 3-cycle, say $\sigma \notin H$ does not belong to H $(o(H)=6)$

Now, $\sigma \notin H \Rightarrow \sigma^2 \notin H$

because if $\sigma^2 \in H$

then $\Rightarrow \sigma^4 \in H$

$\Rightarrow \sigma \in H$

as $\sigma^3 = I$ as $o(\sigma)=3$

Let $K = \langle\sigma\rangle = \{I, \sigma, \sigma^2\}$

then $o(K) = 3 = o(\sigma)$

and $H \cap K = \{I\}$ $(\sigma, \sigma^2 \notin H)$.

$$\Rightarrow o(HK) = \frac{o(H)\cdot o(K)}{o(H\cap K)} = \frac{6\cdot 3}{1} = 18,$$

not possible as $HK \subseteq A_4$

and $\boxed{o(A_4) = 12 \neq 18}$

Hence, converse not true for Lagrange's theorem

Ques: 2(b)) Define Dedikinds property of real numbers, Prove that order - completeness property if and only if Dedikinds property.

Solution:

Dedikinds property of real numbers

let L and U are two subsets of R, such that

(i) $L \neq \phi$, $U \neq \phi$ (i.e. each set has atleast one element)

(ii) $L \cup U = R$ [i.e. each real number is either in L or in U]

and (iii) $x \in L, y \in U \Rightarrow x < y$ (i.e each member of L is smaller than each member of U);

then the subset 'L' has the greatest member or the subset 'U' has the smallest member.

i.e. there exists $\alpha \in R$, such that

$$x < a \Rightarrow x \in L$$
$$y > a \Rightarrow y \in U$$

In the following property / Theorem we shall establish the equivalence between the Order-completeness property and Dedikinds property.

Now To prove that

Order - completeness property $\Leftrightarrow$ Dedikind's Theorem

Proof:- First we shall prove that

$\boxed{\text{order completeness property} \Rightarrow \text{Dedikind's property}}$

Let, L and U be two subsets of R, such that

(i) $L \neq \phi$, $U \neq \phi$

(ii) $L \cup U = R$

and (iii) $x \in L, y \in U \Rightarrow x < y$

Then; we need to prove,

there exists $\alpha \in R$, such that

$$x < \alpha \Rightarrow x \in L, \quad y > \alpha \Rightarrow y \in U$$

from (i) $L \neq \phi$ $\quad x \in L$

Also; $U \neq \phi$, so let $y \in U$

Then (iii), we have

$$x < y \quad \forall \ x \in L$$

$\therefore$ y is an upper bound for L.

Thus, L is a non-empty subset of R which is bounded above. So by order-completeness property of real numbers, L must have its supremum in R.

Let; $\sup L = \alpha$

We shall show that $\alpha \in R$ is such that

$$x < \alpha \Rightarrow x \in L, \quad y > \alpha \Rightarrow y \in U$$

Let $y > \alpha$; Then $y \notin L$

because, $\alpha = \sup L$.

But from (ii), every real number is either in L or U.

So; $y \notin L \Rightarrow y \in U$. Thus, $y > \alpha \Rightarrow y \in U$

Again $x < \alpha$. Since $\alpha = \sup L$,

therefore 'x' cannot be an upper bound for L. So There exists $z \in L$, such that $z > x$ i.e $x < z$.

Now, from (iii), $z \in L$ and $x < z \Rightarrow x \notin U \Rightarrow x \in L$

Thus; $x < \alpha \Rightarrow x \in L$

$\therefore$ There exists $\alpha \in R$, such that

$x < a \Rightarrow x \in L$ and $y > a \Rightarrow y \in U$

Hence, $\boxed{\text{completeness property} \Rightarrow \text{Dedikind's Property}}$ — (A)

Now, conversely, we shall prove that

$\boxed{\text{Dedikind's property} \Rightarrow \text{Completeness property}}$ — A

Let S, be a non empty subset of R which is bounded above. So to prove – S has its supremum in $\mathbb{R}$.

Since, 'S' is bounded above,

$\therefore$ Let 'u' be an upper bound for 'S'; i.e.

$x \leq u, \forall x \in S$

Let; $U = \{s : s \in R$ and 's' is an upper bound for $S\}$

Since, 'u' is an upper bound for 'S',

therefore $u \in U$ and so $\boxed{U \neq \phi}$

Let $L = R - U$

Then; $S \subset L$ and $L \neq \emptyset$

Also, $L \cup U = (R-U) \cup U = R$

Now, let $x \in L$, $y \in U$

Then $x \neq y$, because by definition of L, L & U are disjoint.

Also, we cannot have $x > y$.

Because, If $x > y$, then

$y \in U \Rightarrow y$ is an upper bound of S

$\Rightarrow x$ is also an upper bound of S $[\because x > y]$

$\Rightarrow x \in U$, by definition of U

$\Rightarrow x \in L \cap U$ $[\because x \in L]$

$\Rightarrow L \cap U \neq \phi$, which is contradiction

$\therefore$ $x \in L, y \in U \Rightarrow x < y$

$\therefore$ By Dedikind's property, ~~there~~ exists $\alpha \in R$, such that

$x < \alpha \Rightarrow x \in L$, $y > \alpha \Rightarrow y \in U$

Now, we shall show that $\boxed{\alpha = \sup S}$

If $y > \alpha$, then $y \in U$ and so $y \notin L$. $[\because L \cap U = \phi]$ Disjoint.

Thus, if $y > \alpha$, then y cannot be in L and hence can also not be in S because $S \subset L$.

Thus, no real number $y > \alpha$ is in S.

α is an upper bound for S, i.e

$x \leq \alpha, \forall x \in S$

Now, let $\beta < \alpha$, Then,

$\beta < \alpha \Rightarrow \beta \in L$

$\Rightarrow \beta \notin U$

$\Rightarrow \beta$ is not an upper bound for S

Thus, α is an upper bound for S and no real number $\beta < \alpha$ can be upper bound for S

$\boxed{\therefore \alpha = \sup S}$

$\therefore$ Hence, $\boxed{\text{Dedikind's property} \Rightarrow \text{Order Completeness}}$ — (B)

Hence, from (A) & (B)

$\boxed{\text{Order-Completeness} \Leftrightarrow \text{Dedikind's Property}}$

Ques) 2(c)) Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young one lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs Re. 1 per week to feed. I have only 80 rupees to spend for hens. how many of each kind should I buy to give a profit more that Rs. 6 per week, assuming that I cannot house more than 20 hens.

Solution:-

Let number of old hens $= x$

number of young hens $= y$

Since, old can lay 3 eggs/week and young can lay 5 eggs/week

i.e $3x+5y$ = Total number of eggs obtained in a week.

Total cost of eggs per week, if each egg = ₹0.30

$\therefore$ Total gain = Rs. $0.30(3x+5y)$

Total expenditure for feeding $(x+y)$ hens at rate ₹.1 each will be = Rs. $1(x+y)$.

Thus total profit; z earned per week

z = Total gain − Total expenditure.

$$z = ₹\left[0.3(3x+5y) - 1(x+y)\right]$$

$$z = ₹\left[0.9x + 1.5y - x - y\right]$$

$$\boxed{z = 0.5y - 0.1x}$$

Since, old hens can be bought at ₹2 each and young one at ₹5 each; and there are only ₹80 available, then constraint is:

$$2x + 5y \leq 80$$

Since, not possible to house more than 20 hens

$$x + y \leq 20$$

Since, profit is restricted to be more than ₹6 this means $Z_{max} = 0.5y - 0.1x \geq 6$.

and also purchasing can't be negative

i.e $x \geq 0$, $y \geq 0$.

So, the problem becomes:

(Profit) $Z_{max} = 0.5y - 0.1x$

Subject to constraints:

$$2x + 5y \leq 80$$
$$x + y \leq 20$$
$$x, y \geq 0$$

Plot the graph.

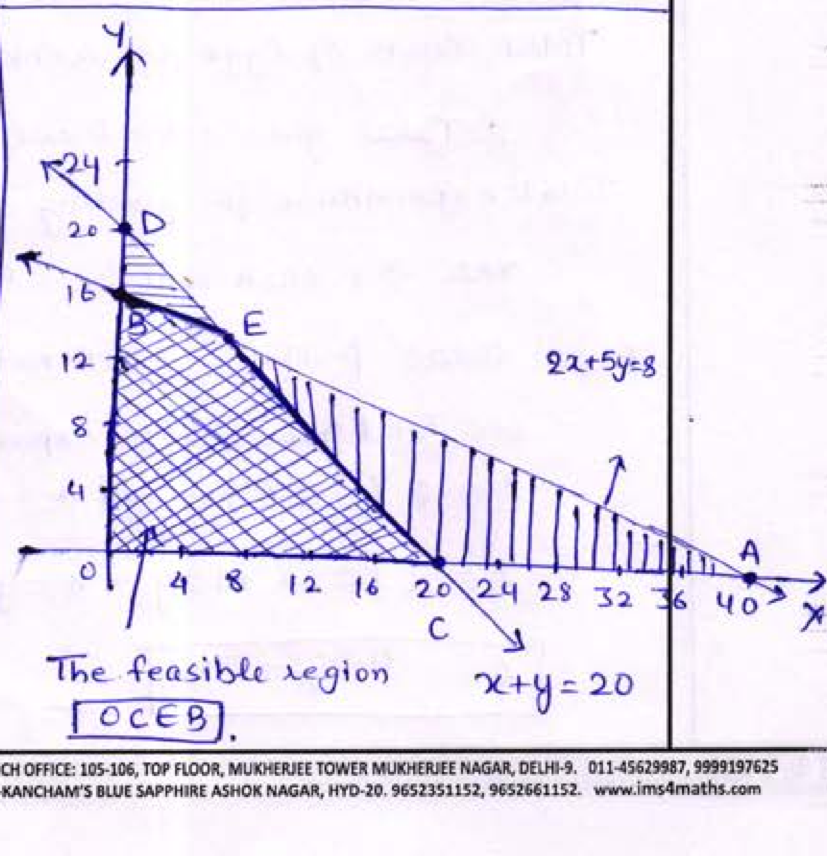
For ; $2x + 5y = 80$

x	0	40
y	16	0

for $x + y = 20$

x	0	20
y	20	0

Here, we conclude the feasible region OCEB.



The feasible region $\boxed{OCEB}$.

The co-ordinates of the extreme points of the feasible regions are:

$O = (0,0)$ $\qquad$ $C = (20,0)$

$B = (0,16)$ $\qquad$ $E = [20/3, 40/3]$

Thus, the values of Z at these vertices are:

$Z_0 = 0\times 0.5 - 0\times 0.1 = 0$

$Z_C = 0\times 0.5 - 0.1\times 20 = -2$

$Z_B = 16\times 0.5 - 0\times 0.1 = 8$

$Z_E = 0.5\times \frac{40}{3} - 0.1\times\frac{20}{3} = \frac{20}{3} - \frac{2}{3} = \frac{18}{3} = 6$

Since, the max. value of $Z = ₹8$,

which occurs at point $B = (0,16)$

the solution to the given problem is

$\boxed{x = 0 \quad , y = 16}$

i.e $\boxed{\begin{array}{l}\text{Old hens} = 0 \\ \text{Young hens} = 16\end{array}}$

And hence:

$\boxed{\max . Z = \text{Rs. } 8}$

which is required optimal feasible solution.

Ques: 3(a):- Let G be defined as all formal symbols $x^i y^j$, $i=0,1$, $j=0,1,2,\dots,n-1$, where we assume.

$x^i y^j = x^{i'} y^{j'}$ if and only if $i=i'$, $j=j'$

$x^2 = y^n = e$, $n>2$

$xy = y^{-1}x$

(i) Find the form of the product $(x^i y^j)(x^k y^l)(x^\alpha y^\beta)$.

(ii) Using this, prove that G is a non-abelian group of order $2n$.

(iii) If 'n' is odd, prove that the center of G is (e), while if 'n' is even the center of G is larger than (e).

[This group G is known as a dihedral group.].

Solution:

(i) The form of the product $(x^i y^j)(x^k y^l)(x^\alpha y^\beta)$ — can solve yourself.

(ii) Here, consider G is a group [dihedral group] we can also write G as.

$$G = \left\{ \begin{matrix} y, y^2, \dots y^{n-1}, y^n = e \\ xy, xy^2, \dots xy^{n-1}, x \end{matrix} \middle| x^2 = e = y^n, xy = y^{-1}x \right\}$$

$\therefore O(G) = 2n$; We can write $G = D_{2n}$.

Now; $xy = y^{-1}x \Rightarrow xyx^{-1} = y^{-1}xx^{-1}$

$\Rightarrow xyx^{-1} = y^{-1}$

$\Rightarrow xyx \Rightarrow y^{-1}$ [as $x^2 = e \Rightarrow x = x^{-1}$]

$\therefore \quad (xy)(xy^2) = (xyx)y^2$
$= y^{-1}y^2 = y.$

$\Rightarrow \quad xy = y^{-1}x \Rightarrow yxy = x$
$yxy^2 = xy$

$\Rightarrow \quad \boxed{yx \neq xy}$

Hence, G is not an abelian group of order 2n.

iii) Now, we can compute the product of any two elements of G.
We first find $Z(G)$
Consider $\quad y^i \ (1 \le i \le n)$

Then; $\quad y^i(xy^i) = (y^ixy^i)y^{-i+j}$
$= xy^{-i+j}$

Note; $\quad xy = y^{-1}x \Rightarrow xyx^{-1} = y^{-1}$
$\Rightarrow (xyx^{-1})^i = y^{-i}$
$\Rightarrow xy^ix^{-1} = y^{-i}$
$\Rightarrow \boxed{y^ixy^{+i} = x}$

This is very useful relation in D_{2n}.
Also $\quad (xy^i)y^i = xy^{i+j}$
If $y^i \in Z(G)$, then y^i must commute with xy^j for all j, $1 \le j \le n$

$\therefore \quad xy^{-i+j} = xy^{i+j}$ for all j, $1 \le j \le n$.
$\Rightarrow y^{2i} = e$
$\Rightarrow o(y)$ divides $2i$
$\Rightarrow n$ divides $2i$.

Case I) $n = \text{odd}$.

Then; $n \mid 2i$ and $(n,2) = 1$.

$\Rightarrow n \mid i$

$\Rightarrow n \le i$; But $i \le n$

$\Rightarrow i = n$

So; $\boxed{y^i = y^n = e}$

Similarly, if $xy^i \in Z(G)$, then

$(xy^i)x = x(xy^i)$

$\Rightarrow xy^i x^{-1} = x^2 y^i$

$y^{-i} = x^2 y^i \Rightarrow y^{-i} = ey^i \Rightarrow y^{-i} = y^i$

$\Rightarrow y^{2i} = e$ as $x^2 = e$.

$\Rightarrow o(y) \mid 2i \Rightarrow n \mid 2i \Rightarrow n \mid i \Rightarrow n = i$

$\therefore \boxed{xy^i = xy^n = x.}$

But, $x \notin Z(G)$ as $xy = y^{-1}x$ (and $x \in Z(G)$ should imply $xy = yx$

i.e; $yx = y^{-1}x \Rightarrow y^2 = e \Rightarrow o(y) = n \mid 2$

$\Rightarrow n \le 2$, a contradiction as $n \le 3$).

So, $Z(G)$ does not contain any element of the type xy^{i}. Also if $y^i \in Z(G)$, then $\boxed{i = n}$

$\boxed{\therefore Z(G) = \{e\}}$

Case II) $n = \text{even}$, let $n = 2m$.

Then as above, $n \mid 2i$

$\Rightarrow 2m \mid 2i \Rightarrow m \mid i$

$\Rightarrow i = m$ or $\boxed{2m = n}$

i.e $y^i = y^m$ or $y^{2m} = y^n = e$

Clearly $y^m \in Z(G)$ as $(xy^k)y^m = xy^{k+m}$

and $y^m(xy^k) = (y^m x y^m) y^{k-m}$

$= xy^{k-m} \cdot y^{2m} = xy^{k-m}$ $[\because y^{2m} = e]$

$= xy^{k+m}$

as $\left[y^{2m} = e \Rightarrow y^m = y^{-m}\right]$

$\therefore$ only powers 'i' of y s.t., $y^i \in Z(G)$ are $i = m$ and $2m$.

Similarly, as in case 1, if $xy^i \in Z(G)$ then, $n|2i \Rightarrow 2m|2i \Rightarrow m|i \Rightarrow i = m$ or $2m$.

But $xy^{2m} = xy^n = xe = x \notin Z(G)$

Also; $(xy^m)y = xy^{m+1}$

and $y(xy^m) = yxy^m$

So, if $xy^m \in Z(G)$, then $xy^{m+1} = yxy^m$

$\Rightarrow xy = yx$

$\Rightarrow y^{-1}x = yx$.

$\Rightarrow y^2 = e$

$\Rightarrow o(y)$ divides 2

$\Rightarrow n$ divides 2

$\Rightarrow n \leq 2$, a contradiction.

$\boxed{\therefore Z(G) = \{e, y^m\}, \text{ where, } n = 2m}$

Thus, we proved that

$$\boxed{\begin{array}{l} \text{If } n = \text{odd then } Z(D_{2n}) = \{e\} \\ \quad n = \text{even then } Z(D_{2n}) = \{e, y^m\} \end{array}}$$

which is required result.

Ques: 3(b) Let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$; $x \in [0,1]$

Show that at $x=0$; $\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x)$.

Solution:- given;

$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \quad ; x \in [0,1]$$

Let, $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.

Then; $S_n(x) = \frac{nx}{1+n^2x^2}$. $\lim_{n\to\infty} S_n(x) = 0$

for all $x \in [0,1]$

The sequence $\{S_n\}$ converges to the function 'S', where $S(x) = 0$; $x \in [0,1]$.

Therefore, the series $\sum f_n(x)$ converges to $S(x)$ for all $x \in [0,1]$.

$$\frac{d}{dx}\left(\sum f_n(x)\right) = \frac{d}{dx} S(x) = 0 \text{, for all } x \in [0,1]$$

$$\frac{d}{dx} f_n(x) = \frac{n - n^2x^2}{(1+n^2x^2)} - \frac{(n-1)-(n-1)^2x^2}{1+(n-1)^2x^2}$$

At $x=0$

$$\left.\frac{d}{dx} f_n(x)\right|_{x=0} = \frac{n-0}{1+0} - \frac{(n-1)-0}{1+0} = n-(n-1) = 1.$$

At $x = 0.0$

$$\left.\sum \frac{d}{dx} f_n(x)\right|_{x=0} = 1+1+1+\cdots+1\ldots$$ This is divergent.

Hence, $\boxed{\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x) \text{ at } x=0}$

which is required result.

Ques: 3)c) Find the value of the integral

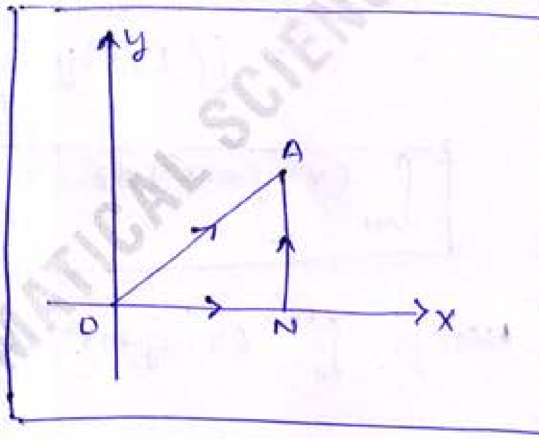
$$\int_0^{1+i} (x - y + ix)\, dz$$

(i) Along the straight line from $z=0$ to $z=1+i$

(ii) Along the real axis from $z=0$ to $z=1$ and then along a parallel line to imagenary axis from $z=1$ to $z=1+i$?

Solution:-

Let, A be the point of affix $1+i$ and N be the point of affix 1.



(a) Let, OA be line from $z=0$ to $z=1+j$

on OA, $y=x$, $z=x+ix$, $dz=(1+i)dx$

Hence, $\int_{OA}(x-y+ix)\,dz = \int_0^1 ix^2(1+i)\,dz$

$$= (-1+i)\left[\frac{x^3}{3}\right]_0^1 = \frac{-1+i}{3}.$$

(b) Natural real axis from $z=0$ to $z=1$ is the line ON and then from $z=1$ to $z=1+i$ a line parallel to imagenary axis is the line NA.

So, here the contour of integration consists of the lines ON and NA.

on ON, $y=0$, $z=x+iy=x$; $dz=dx$.

Hence, $\int_{ON}(x-y+ix^2)\,dz = \int_0^1 (x+ix)^2\,dx$

$$= \left\{ \frac{x^2}{2} + i\frac{x^3}{3} \right\}_0^1 = \frac{1}{2} + \frac{i}{3}$$

$$\therefore \boxed{\int_{ON} (x - y + ix^2)\, dz = \frac{1}{2} + \frac{i}{3}}$$

On line NA, $x = 1$ so it is $z = 1 + iy$, $dz = i\,dy$

Hence, $\displaystyle\int_{AN} (x - y + ix^2)\, dz = \int_0^1 ((1+i) - y)\, dy \cdot i$

$$= \left[(-1+i)y - \frac{iy^2}{2} \right]_0^1 = -1 + i - \frac{i}{2}$$

$$\boxed{\int_{AN} (x - y + ix^2)\, dz = -1 + \frac{i}{2}}$$

Hence, $\displaystyle\int_0^{1+i} (x - y + ix^2)\, dz$ along the contour ONA

= Integration along ON + Integration along NA

$$= \frac{1}{2} + \frac{i}{3} - 1 + \frac{i}{2} = -\frac{1}{2} + \frac{5i}{6}.$$

$$\boxed{\therefore \int_0^{1+i} (x - y + ix^2)\, dz = \frac{-1}{2} + \frac{5i}{6}}$$

which is required result.

Ques: 4(a) For any prime p, show that the polynomial $x^{p-1}+x^{p-2}+x^{p-3}+\cdots+x^2+x+1$ is irreducible over Q (rational number).

Solution:

Let; $f(x) = x^{p-1}+x^{p-2}+x^{p-3}+\cdots+x^2+x+1$

$$= \frac{x^p-1}{x-1} \quad [\text{sum of a G.P}).$$

Now,

$$f(x+1) = \frac{(x+1)^p-1}{(x+1)-1} = \frac{x^p+{}^pC_1x^{p-1}+\cdots+{}^pC_rx^{p-2}+\cdots+{}^pC_p(1)-1}{x+\cancel{1}-\cancel{1}}$$

$$f(x+1) = \frac{(x+1)^p-1}{(x+1)-1} = \frac{x^p+{}^pC_1x^{p-1}+\cdots+{}^pC_{p-1}x}{x}$$

$$f(x+1) = x^{p-1}+{}^pC_1x^{p-2}+{}^pC_2x^{p-3}+\cdots+{}^pC_{p-1}x\cdot x^{-1}$$

$$\boxed{f(x+1) = x^{p-1}+{}^pC_1x^{p-2}+{}^pC_2x^{p-3}+\cdots+{}^pC_{p-1}}$$

Since; p is a prime number. $p \mid {}^pC_r$ for all $1 \le r \le p-1$

Also; $\boxed{{}^pC_{p-1} = p \text{ or } p^2 \nmid {}^pC_{p-1}}$

Hence, by Eisenstein's criterion $f(x+1)$ and, therefore, $f(x)$ is irreducible over Q.

Q.4(b) Prove that the series $e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} - \frac{e^{-4x}}{4} + \cdots$

is uniformly convergent on $[0,1]$.

Solution:

given series

Let $v_n(x) = e^{-nx}$, $x \in [0,1]$

For each $x \in [0,1]$,

the sequence $\{v_n\}$ is monotonic

and for all $x \in [0,1]$.

$$|v_n(x)| = \frac{1}{e^{nx}} \leq 1 \text{ for all } n \in N$$

The series:

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is a convergent series of real numbers

$\therefore$ It is uniformly convergent on $[0,1]$

Hence, By Abel's test, the given series

$$\boxed{= e^{-x} - \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} - \frac{1}{4}e^{-4x} + \cdots \text{ is uniformly convergent on } [0,1]}$$

Hence, required result.

Ques: 4(c) A job shop has purchased 5 new machines of different type. There are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to and from these machines. Therefore, the objective is to assign the new machines to the available locations, in order to minimize the total cost of material handling. The estimated cost per unit time of material handling involving each of the machines is given below for the respective locations. Locations 1, 2, 3, 4 & 5 are not considered suitable for machine A, B, C, D and E, respectively. Find the optimal solution:

Location (Costs in Rs.)

	1	2	3	4	5
A	×	10	25	25	10
B	1	×	10	15	2
C	8	9	×	20	10
D	14	10	24	×	15
E	10	8	25	27	×

How would the optimal solution get modified if location 5 is also unsuitable for machine A?

Solution: Using Hungerian Method.

The given table can be transformed as below with unsuitable position cost = ∞

	1	2	3	4	5
A	∞	10	25	25	10
B	1	∞	10	15	2
C	8	9	∞	20	10
D	14	10	24	∞	15
E	10	8	25	27	∞

(i) choose, and substract the minimum element of each row.

	1	2	3	4	5
A	∞	0	15	15	0
B	0	∞	9	14	1
C	0	1	∞	12	2
D	4	0	14	∞	5
E	2	0	17	19	∞

In this transformation all the rows get at least one zero but not all the column.

(ii) Substract the minimum element of each column.

	1	2	3	4	5
A	∞	0	6	3	0
B	0	∞	0	2	1
C	0	1	∞	0	2
D	4	0	5	∞	5
E	2	0	8	7	∞

Here, all the columns and all the rows has atleast one zero

$n = 5$

Cover all zeros by minimum number of Horizontal and vertical lines, we observed that only 4 lines can cover all the zeros.

So, $r = 4 < n = 5$,

	1	2	3	4	5
A	∞	0	6	3	0
B	0	∞	0	2	1
C	0	1	∞	0	2
D	4	0	5	∞	5
E	2	0	8	7	∞

Min element among all non-covered elements = 2

So; Add 2 at crossing of Horizontal & vertical lines & substract two from non-covered elements

$\Downarrow$

	1	2	3	4	5
A	∞	2	6	3	0
B	0	∞	0	2	1
C	0	3	∞	0	2
D	2	0	3	∞	3
E	0	0	6	5	∞

Now, again cover all zeros with minimum number of horizontal and vertical lines.

$r = 5 = n = 5$

Hence, we obtain the optimal solution table

Now, to select the location for various machine

	1	2	3	4	5
A	∞	2	6	3	$\boxed{0}$
B	$\cancel{0}$	∞	$\boxed{0}$	2	1
C	$\cancel{0}$	3	∞	$\boxed{0}$	2
D	2	$\boxed{0}$	3	∞	3
E	$\boxed{0}$	$\cancel{0}$	6	5	$\cancel{\infty}$

Hence, the optimal solution is

$A \to 5$, $B \to 3$, $C \to 4$, $D \to 2$ & $E \to 1$.

Hence; Total Minimum cost $= 10 + 10 + 20 + 10 + 10$

$\boxed{\text{Total Min. Cost} = \text{Rs. } 60}$ Ans.

Now, if the location '5' is also not suitable for A (machine) – hence cost at $(1,5)=\infty$, Now the table becomes –

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & \infty & 10 & 25 & 25 & \infty \\ B & 1 & \infty & 10 & 15 & 2 \\ C & 8 & 9 & \infty & 20 & 10 \\ D & 14 & 10 & 24 & \infty & 15 \\ E & 10 & 8 & 25 & 27 & \infty \\ \hline \end{array}$$

Now applying the above procedure for this

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & \infty & 0 & 15 & 15 & \infty \\ B & 0 & \infty & 9 & 14 & 1 \\ C & 0 & 1 & \infty & 12 & 2 \\ D & 4 & 0 & 14 & \infty & 5 \\ E & 2 & 0 & 17 & 19 & \infty \\ \hline \end{array} \Rightarrow \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & \infty & 0 & 6 & 3 & \infty \\ B & 0 & \infty & 0 & 2 & 0 \\ C & 0 & 1 & \infty & 0 & 1 \\ D & 4 & 0 & 5 & \infty & 4 \\ E & 2 & 0 & 8 & 7 & \infty \\ \hline \end{array}$$

(row's) (column $\to$ 0)

Now, cover all zeros with minimum number of horizontal an vertical lines, we get $r=3<n=5$

$$\begin{bmatrix} \infty & 0 & 6 & 3 & \infty \\ 0 & \infty & 0 & 2 & 0 \\ 0 & 1 & \infty & 0 & 1 \\ 4 & 0 & 5 & \infty & 4 \\ 2 & 0 & 8 & 7 & \infty \end{bmatrix}$$

Hence, least element which is uncovered = 2

Add +2 = Junction of Horizontal & vertical lines

Substract 2 = uncovered elements

$$\Downarrow$$

$$\begin{bmatrix} \infty & 0 & 4 & 1 & \infty \\ 0 & \infty & 0 & 2 & 0 \\ 0 & 3 & \infty & 0 & 1 \\ 2 & 0 & 3 & \infty & 2 \\ 0 & 0 & 6 & 5 & \infty \end{bmatrix}.$$

By following the steps, the following assignment solution can be easily obtained.

$A \to 4, B \to 3, C \to 5, D \to 2$ and $E \to 1$

Total minimum cost $= 25 + 10 + 10 + 10 + 10$

$\boxed{\text{Total Min. Cost} = \text{Rs. } 65}$

OR

$A \to 2, B \to 3, C \to 4, D \to 5$ and $E \to 1$

Total minimum cost $= 10 + 10 + 20 + 15 + 10$

$\boxed{\text{Total Min. Cost} = \text{Rs. } 65}$

which is required result.

Section - B

Ques: 5) a) Solve

$$\left(\partial^2 z/\partial x^2\right) - \left(\partial^2 z/\partial y^2\right) + \left(\partial z/\partial x\right) + 3\left(\partial z/\partial y\right) - 2 = e^{x-y} - x^2 y$$

Solution:-

Given equation:

$$\left(\frac{\partial^2 z}{\partial x^2}\right) - \left(\frac{\partial^2 z}{\partial y^2}\right) + \left(\frac{\partial z}{\partial x}\right) + 3\left(\frac{\partial z}{\partial y}\right) - 2 = e^{x-y} - x^2 y$$

which can be written as

$$(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2 y$$

$$\{(D-D')(D+D') + 2(D+D') - (D-D'+2)\}z = e^{x-y} - x^2 y$$

$$\{(D+D')(D-D'+2) - (D-D'+2)\}z = e^{x-y} - x^2 y$$

$$\{(D-D'+2)(D+D'-1)\}z = e^{x-y} - x^2 y$$

$$\therefore \boxed{C.F = e^{-2x}\phi_1(y+x) + e^{x}\phi_2(y-x)} \quad \text{——} \ (1)$$

where, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{x-y}

$$= \frac{1}{(D-D'+2)(D+D'-1)} e^{x-y}$$

[from $e^{x-y} \Rightarrow D = 1, D' = -1$] → co-efficient of x & y.

$$= \frac{1}{(1+1+2)(1-1-1)} e^{x-y} = \boxed{\frac{-1}{4} e^{x-y}.} \quad \text{——} \ (2)$$

Now P.I corresponding to $(-x^2 y)$.

$$= \frac{1}{(D-D'+2)(D+D'-1)}(-x^2 y)$$

$$= \frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\left\{1-(D+D')\right\}^{-1}.x^2y$$

$$= \frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\cdots\right\}\left\{1+(D+D')+(D+D')^2+\cdots\right\}x^2y$$

$$= \frac{1}{2}\left[1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+\cdots\right]\times$$

$$\left[1+D+D'+D^2+2DD'+3D^2D'+\cdots\right]x^2y$$

$$= \frac{1}{2}\left[1+\frac{1}{2}D+\frac{3}{2}D'+\frac{3}{4}D^2+\frac{3}{2}DD'+\cdots\right]x^2y$$

$$\Rightarrow \boxed{\frac{1}{2}\left\{x^2y+xy+\frac{3}{2}x^2+3x+\frac{21}{4}\underbrace{\cdots}_{\to 0}\right\}} \quad \text{——(3)}$$

From (1), (2) and (3),

General solution in z = C.F + P.I.

$$\boxed{z = e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)-\frac{1}{4}e^{x-y}+\frac{1}{2}x^2y+\frac{xy}{2}+\frac{3}{4}x^2+\frac{3}{2}x+\frac{21}{8}.}$$

which is required solution.

Ques: 5.b) Reduce the following equation to a canonical form and hence solve it:

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$$

Solution:

Given, P.D.E

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$$

Now, Discriminant

$$D = B^2 - 4AC = (x+y)^2 - 4xy$$

$$D = x^2 + y^2 + 2xy - 2xy$$

$$\boxed{D = (x-y)^2 > 0}$$

Hence, the given PDE is hyperbolic everywhere except along $x=y$; whereas on the line $y=x$ it is parabolic. Hence; for $y \neq x$ characteristic equations are –

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2-4AC}}{2A} = \frac{(x+y) \mp (x-y)}{2y}.$$

Therefore; $\frac{dy}{dx} = 1$ and $\frac{dy}{dx} = \frac{x}{y}$

Hence, on integration, we obtained

$$y = x + c_1 \qquad y^2 = x^2 + c_2$$

Hence, the characteristic equations are

$$\boxed{\xi = y - x \quad , \quad \eta = y^2 - x^2}$$

These are straight lines and rectangular hyperbolas.

The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = y-x-y+x = 0$$

$$\bar{B} = -2(x-y)^2 \quad ; \quad \bar{C} = 0 \quad ; \bar{D} = 0$$

$$\bar{E} = 2(x-y) \qquad \bar{F} = \bar{G} = 0$$

Thus, the canonical equation for given PDE is

$$-2(x-y)^2 u_{\xi\eta} + 2(x-y)u_\eta = 0$$

or
$$-2\xi^2 u_{\xi\eta} + 2(-\xi)u_\eta = 0$$

or
$$\xi u_{\xi\eta} + u_\eta = \frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\eta}\right) = 0$$

Integrating yields

$$\xi\frac{\partial u}{\partial\eta} = f(\eta)$$

Again, integrating with respect to η, we obtain

$$u = \frac{1}{\xi}\int f(\eta)d\eta + g(\xi)$$

Hence,
$$u = \frac{1}{y-x}\int f(\eta)d\eta + g(\xi)$$

Hence,

$$\boxed{u = \frac{1}{y-x}\int f(y^2-x^2)\,d(y^2-x^2) + g(y-x)}$$

is the general solution.

Q.5(c): Using Newton's forward formula find the number of men getting wages between Rs.10 and Rs.15 from the following data:

Wages in Rs. :	0-10	10-20	20-30	30-40
Frequency :	9	30	35	42

Solution:

The above wage-frequency table can be redrawn as

wages less than	10	20	30	40
Frequency	9	39	74	116

Now, make the difference table:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

Here; $x_0 = 10$, $x = 15$, $h = 10$

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

By Newton's forward formula.

$$y_{15} = y_{10} + p\Delta y_{10} + \frac{p(p-1)}{2!}\Delta^2 y_{10} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{10}$$

$$y_{15} = 9 + 0.5\times 30 + \frac{0.5(-0.5)}{2}\times 5 + \frac{0.5(-0.5)(-1.5)\cancel{2}}{3\times\cancel{2}}$$

$$y_{15} = 9 + 15 + (-0.625) + 0.125$$

$$y_{15} = 24 - 0.5 = \underline{23.5}$$

The number of men getting wages less than $15 = 23.5$

$< 15 = 24$ i.e

But, the no. of men with wages less than $10 = 9$

Hence,

$$\boxed{\text{Number of men with wages b/w Rs.10 to Rs.15} = 24 - 9 = 15}$$

which is required result.

Ques: 5(d): The velocity 'v' is a particle at distance 's' from a point on its path is given by the table.

s (ft):	0	10	20	30	40	50	60
v (ft/sec):	47	58	64	65	61	52	38

Estimate the time taken to travel 60ft by using Simpson's $1/3^{rd}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

Solution: We know that

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$$

$$t = \int dt = \int \frac{ds}{v}$$

$$\Rightarrow \boxed{t = \int \frac{ds}{v}} \quad \text{——} \ (1)$$

So, the above table can be re-drawn as given below –

s	0	10	20	30	40	50	60
$y=1/v$	0.02127	0.01724	0.015625	0.015385	0.01639	0.01923	0.02631
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = \frac{b-a}{n} = \frac{60-0}{6} = 10$$

Using Simpson's 1/3 rule:

$$T = \frac{h}{3}\left[y_0+y_6+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$T = \frac{10}{3}\left[0.02127+0.02631+2(0.015625+0.01639)+4(0.01724+0.015385+0.01923)\right]$$

$$T = \frac{10}{3}\left[0.04758+2\times 0.032015+4\times 0.051855\right]$$

$$\boxed{T_{1/3} = 1.06538 \text{ seconds}} \quad \text{——— (2)}$$

Now, Using Simpson's 3/8 rule:-

$$T = \int y\,ds = \frac{3h}{8}\left[y_0+y_6+2y_3+3(y_1+y_2+y_4+y_5)\right]$$

$$T = \frac{3\times 10}{8}\left[0.02127+0.02631+2\times 0.01538+3(0.01724+0.01639+0.015625+0.01923)\right]$$

$$T = \frac{15}{4}\left[0.04758+0.03076+3(\qquad)\right]$$

$$T = \frac{15}{4}\times 0.283795 = 1.06423$$

$$\boxed{T_{3/8} = 1.06423 \text{ seconds}} \quad \text{——— (3)}$$

From (2) and (3)

Difference b/w $T_{1/3}$ & $T_{3/8}$ $= 1.06538 - 1.06423$

$= 0.00115$ seconds.

Hence, $\boxed{\text{Difference} = 0.00115 \text{ seconds}}$ is the required result.

Ques: 5(e): In an incompressible fluid the vorticity at every point is constant in magnitude and direction; prove that the components of velocity of u, v, w are the solutions of Laplace equation.

Solution:-

Let, $\underline{\Omega}$ be the vorticity at any point in an incompressible fluid, then

$$\underline{\Omega} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$$

where; $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$; $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The magnitude and direction cosines of its direction are given by

$$\boxed{\Omega = \sqrt{\xi^2 + \eta^2 + \zeta^2}} \text{ and } \boxed{\frac{\xi}{\Omega}, \frac{\eta}{\Omega}, \frac{\zeta}{\Omega}}$$

Differentiating η partially with regard to z and ζ with regard to y and subtracting, we have.

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y}\right) = 0 \quad \text{——} (1)$$

By equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or $$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}. \quad \rightarrow (2)$$

put (2) in (1), we get

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0} \Rightarrow \boxed{\nabla^2 u = 0}$$

Hence, velocity component satisfy Laplace's Equation.

Ques) 6) a.

(i) Form a partial differential Equation by eliminating the arbitrary function 'f' and 'g' from

$$z = yf(x) + xg(y).$$

Solution:-

Given; $z = yf(x) + xg(y)$ —— (1)

Differentiating (1), w.r.t 'x' and 'y', we get.

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) \quad \text{—— (2)}$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y) \quad \text{—— (3)}$$

Differentiating (3), w.r.t 'x', we have.

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \quad \text{—— (4)}$$

from (2) & (3)

$$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] \quad \& \quad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

substituting the values of $f'(x)$ & $g'(y)$ in (4)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

$$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{xg(y) + yf(x)\}$$

$$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z \quad \text{— from (1)}$$

Hence; $\boxed{xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z}$

is required PDE without function f & g.

6(a) ii) Find the complete and the singular integral of
$$4xyz = pq + 2px^2y + 2qxy^2.$$

Solution: given; $4xyz = pq + 2px^2y + 2qxy^2$ — (1)

Which can be written as

$$4xyz = \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + 2\cdot\frac{\partial z}{\partial x}\cdot x^2y + 2\frac{\partial z}{\partial y}\cdot xy^2$$

or $z = \frac{1}{2x}\frac{\partial z}{\partial x}\cdot\frac{1}{2y}\frac{\partial z}{\partial y} + x^2\left(\frac{1}{2x}\cdot\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)$ — (2)

Put $2x\,dx = dX$ and $2y\,dy = dY$ — (3)

so that $x^2 = X$ and $y^2 = Y$ — (4)

using (3) in (2), (2) becomes

$$z = \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y} + X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y}$$

$\boxed{z = XP + YQ + PQ}$ — (5)

where; $P = \frac{\partial z}{\partial X}$, $Q = \frac{\partial z}{\partial Y}$

(5) is in the form of $z = XP + YQ + f(P,Q)$

$\therefore$ Solution of (5) is

$\boxed{z = aX + bY + ab}$ a, b - arbitrary constants.

which is complete Integral — (6)

Differentiating (6), w.r.t a & b, we have

$0 = x^2 + b$ and $0 = y^2 + a$

So, that $b = -x^2$, $a = -y^2$ — (7)

Eliminating 'a' & 'b', the required singular integral is —

$$z = -x^2y^2 - \cancel{x^2y^2} + \cancel{x^2y^2} = -x^2y^2$$

$\boxed{z = -x^2y^2}$ which is required singular integral.

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Q.6 b) Apply Gauss-Seidal iteration method to solve the equations:

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Let, given equations can be re-written as

$$\left.\begin{aligned} x &= \frac{17 - y + 2z}{20} = \frac{17}{20} - 0.05y + 0.1z \\ y &= \frac{-18 - 3x + z}{20} = 0.9 - 0.15x + 0.05z \\ z &= \frac{25 - 2x + 3y}{20} = 1.25 - 0.1x + .15y \end{aligned}\right\} -(1)$$

$\boxed{\text{I}^{st} \text{ Iteration}}$

We start from the approximation $x_0 = y_0 = z_0 = 0$

Substituting $y = y_0$, $z = z_0$ in right side of eq^n (1)

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Put

$$x_1 = 0.8500,\ z_0 = 0 \Rightarrow y_1 = \frac{1}{20}(-18 - 3\times 0.8500 + 0)$$

$$y_1 = -1.0275.$$

Put ; $x_1 = 0.8500$, $y_1 = -1.0275$

$$z_1 = \frac{1}{20}[25 - 2\times 0.8500 + 3\times -1.0275] = 1.0109.$$

$$\boxed{\therefore\ x_1 = 0.8500\ ;\ y_1 = -1.0275\ ;\ z_1 = 1.0109}$$

II^nd Iteration :-

$$y_1 = -1.0275;\ z_1 = 1.0109 \Rightarrow x_2 = \frac{1}{20}(17 + 1.0275 + 2\times 1.0109)$$

$$x_2 = 1.0025.$$

$$x_2 = 1.0025,\ z_1 = 1.0109 \Rightarrow y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$x_2 = 1.0025,\ y_2 = -0.9998 \Rightarrow z_2 = \frac{1}{20}[25 - 2x_2 + 3y_2] = 0.9998.$$

$$\boxed{x_2 = 1.0025\ ;\ y_2 = -0.9998\ ;\ z_2 = 0.9998,}$$

Third Iteration

Put $y_2 = -0.9998$; $z_2 = 0.9998$

$$x_3 = \frac{1}{20}[17 - y_2 + 2z_2] = 1.0000$$

$$y_3 = \frac{1}{20}[-18 - 3x_3 + z_2] = -1.0000$$

$$z_3 = \frac{1}{20}[25 - 3x_3 + 2y_3] = 1.0000.$$

From 2nd and 3rd iteration being practically the same; we can stop. Hence, solution is

$\boxed{x = 1 \ , \ y = -1 \ , \ z = 1}$ required solution.

Q.6) c) Obtain the principal disjunctive and conjunctive normal forms of

$$p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)].$$

Solution:

(1) For principal Disjunctive normal form. (PDnf)

$$p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$$

$$\Leftrightarrow \sim p \wedge [(\sim p \vee q) \wedge (q \wedge p)]$$

[using De Morgan's law and equivalence

$$p \to q \Leftrightarrow \sim p \vee q$$

$$\Leftrightarrow \sim p \vee [\sim p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)]$$

$$\Leftrightarrow \sim p \vee (q \wedge p)$$

$$\Leftrightarrow [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p)$$

$$\Leftrightarrow \boxed{(\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)}$$

This is the desired PDnf.

(ii) For the principal conjunctive normal function (PCNF)

$$p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$$

$$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$$

$$\Leftrightarrow [\sim p \vee (\sim p \vee q)] \wedge [\sim p \vee (q \wedge p)]$$

$$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee p)$$

$$\Leftrightarrow \boxed{\sim p \vee q}$$

Which is required PCnf).

Q.6) d) Write Hamilton's equations for a particle of mass 'm' moving in a plane under a force which is some function of distance from the origin.

Solution:-

Let, $P(r, \theta)$ be the co-ordinates of a particle of mass 'm' at time 't', referred to the pole at 'O' and OX as initial line.

The components of velocity at P along OP and $\perp^r$ to it are $\dot{r}$ and $r\dot{\theta}$ respectively.

$\therefore$ (vel)^2 of m at P ; $\boxed{v^2 = \dot{r}^2 + (r\dot{\theta})^2}$

$\therefore$ Kinetic energy ; $\boxed{T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)}$

Since, the force at 'P' is some function of r, therefore the potential 'V' is function of 'r' alone and will be independent of θ.

i.e $\boxed{V = V(r)}$

Thus, $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

Here, 'r' and 'θ' are the generalised coordinates,

$\therefore P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ — ①

since, L does not contain 't' explicitly.

$\therefore H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

$= \frac{1}{2}m\left[(P_r/m)^2 + r^2(P_\theta/mr^2)^2\right] + V(r)$ — from ①

$$\boxed{H = \frac{1}{2m}\left[P_r^2 + P_\theta^2/r^2\right] + V(r)}$$

Hence, four hamilton's equations are.

$$\dot{P}_r = -\frac{\partial H}{\partial r} = \frac{P_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad \text{— } H_1$$

$$\dot{r} = -\frac{\partial H}{\partial P_r} = \frac{P_r}{m} \quad \text{— } H_2$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{— } H_3$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta^2}{mr^2} \quad \text{— } H_4$$

are required Hamilton's equations.

Q.7(a) A thin annulus occupies the region $0 < a \leq r \leq b$, $0 \leq \theta \leq 2\pi$, where $b > a$. The faces are insulated, and along the inner edge, the temperature is held at $0°$, while along the outer edge, the temperature is held at $100°$. Find the temperature distribution in the annulus.

Solution:-

Mathematically, the problem is to solve.

PDE : $\nabla^2 T = 0$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

Subject to constraints (BCs)

$$T(a,\theta) = 0°$$

$$T(b,\theta) = 100°, \quad 0 \leq \theta \leq 2\pi.$$

Hence, the possible general solution is given by equation

$$\boxed{T(r) = c_1 \log r + c_2} \quad \text{——} (1)$$

Now, using the first BCs:

we obtain

$$T(a,\theta) = c_1 \log r + c_2.$$

$$0° = c_1 \log r + c_2$$

$$\boxed{c_2 = -c_1 \log r} \quad \text{—} (2)$$

Here, $r = a$

$$\therefore \boxed{c_2 = -c_1 \log a} \quad \text{—} (2A)$$

Now, the other end BCs.

$$T(b,\theta) = c_1 \log b + c_2$$

$$100^\circ = c_1 \log b + c_2$$

$$100^\circ = c_1 \log b - c_1 \log a$$

$$100^\circ = c_1 \log(b/a)$$

$$\boxed{c_1 = \frac{100}{\log(b/a)}} \quad \text{—— (3)}$$

Put (3) in (2A), we get

$$\boxed{c_2 = -\frac{100 \log a}{\log(b/a)}} \quad \text{—— (4)}$$

using (3) and (4) in equation (1), we get

$$T(r) = \frac{100}{\log(b/a)} \cdot \log r - \frac{100 \log a}{\log(b/a)}$$

$$T(r) = \frac{100}{\log(b/a)} [\log r - \log a]$$

$$T(r) = \frac{100 \log(r/a)}{\log(b/a)}$$

$$\Rightarrow \boxed{T(r) = \frac{100 \log(r/a)}{\log(b/a)}}$$

is the required temperature distribution in the annulus.

Q.7(b)> Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\frac{dy}{dx} = x^2+y^2$, $y(0)=1$, taking $h=0.1$.

Solution :-

Given : $f(x,y) = \frac{dy}{dx} = \frac{x^2+y^2}{10}$

$y(0)=1$, $h=0.1$. ; we need to find
$y(0.2) = ?$
$y(0.4) = ?$

Ist Iteration

$x_0=0$, $y_0=1$, $h=0.1$.

Using Runge kutta Method

$$\boxed{y(0.1) = y(0)+K}$$

where $\boxed{K = \frac{1}{6}(K_1+K_4+2(K_2+K_3))}$

Now to find; $K \to$

$K_1 = hf(x_0,y_0) = 0.1f(0,1) = 0.01$

$K_2 = hf(x_0+\frac{1}{2}h, y_0+\frac{1}{2}K_1) = 0.1f(0.05, 1.005) = 0.010125$

$K_3 = hf(x_0+\frac{1}{2}h, y_0+\frac{1}{2}K_2) = 0.1f(0.05, 1.00505) = 0.010126$

$K_4 = hf(x_0+h, y_0+K_3) = 0.1f(0.1, 1.010126) = 0.0103035$

$K = \frac{1}{6}[0.01 + 0.0103035 + 2(0.010125 + 0.010126)]$

$K = \frac{1}{6} \times 0.06083 \Rightarrow \boxed{K = 0.01014}$

$\therefore y(0.1) = y(0)+K = 1+0.01014$

$$\boxed{y(0.1) = 1.01014}$$

II nd Iteration

$x = 0.1,\ h = 0.1,\ y(0.1) = 1.01014$

$k_1 = hf(x_{0.1}, y_{0.1}) = 0.1 f(0.1, 1.01014) = 0.0103038$

$k_2 = hf(x_{0.1}+\frac{h}{2}, y_{0.1}+\frac{k_1}{2}) = 0.1 f(0.15, 1.01529) = 0.010533$

$k_3 = hf(x_{0.1}+\frac{h}{2}, y_{0.1}+\frac{k_2}{2}) = 0.1 f(0.15, 1.0154065) = 0.0105355$

$k_4 = hf(x_{0.1}+h, y_{0.1}+k_3) = 0.1 f(0.2, 1.0206755) = 0.0108178$

$k = \frac{1}{6}(0.010308 + 0.0108178 + 2(0.010533 + 0.0105355)]$

$k = 0.0105431$

$y(0.2) = y(0.1) + k = 1.01014 + 0.01054$

$\boxed{y(0.2) = 1.02068}$ ↓ required solution.

III rd Iteration

$x = 0.2,\ h = 0.1,\ y(0.2) = 1.02068$

$k_1 = 0.010818$

$k_2 = 0.0115358$

$k_3 = 0.01115703$

$k_4 = 0.01142945$

$k = \frac{1}{6}[0.010818 + 0.01142945 + 2(0.01115358 + 0.01115703)]$

$k = 0.011432$

$\Rightarrow \boxed{y(0.3) = 1.02068 + 0.0111432 = 1.0318}$

IV th Iteration

$x = 0.3,\ h = 0.1,\ y(0.3) = 1.0318.$

$k_1 = 0.011546$

$k_2 = 0.0119906$

$k_3 = 0.0119952$

$k_4 = 0.0124951$

$k = \frac{1}{6}(0.011546 + 0.0124951 + 2(0.0119906 + 0.0119952)]$

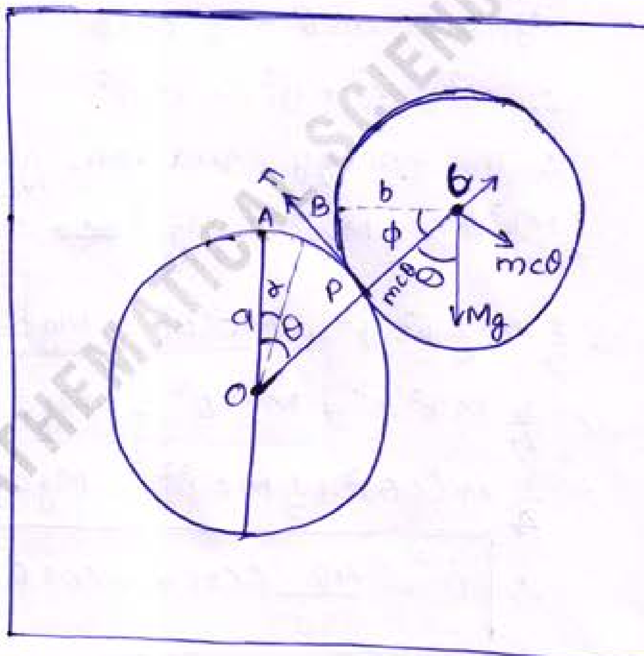
$k = 0.0120021$

$y(0.4) = y(0.3) + k = 1.0318 + 0.0120021 = 1.0438$

$\boxed{y(0.4) = 1.0438}$ ↓ required solution

Q.7(C)) A rough solid cylinder rolls down a second rough cylinder which is fixed with its axis horizontal. If the plane through their axis makes an angle 'α' with the vertical when first cylinder is at rest, show that the cylinders will separate when this angle of inclination is $\cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$.

Solution:-



Let, O be the centre 'a' be the radius of fixed cylinder. Let, 'C' be the centre of and 'b' be the radius of the cylinder resting on the fixed cylinder. with its point 'B' in contact of 'A' of the fixed cylinder such that OA makes an angle 'α' to the vertical.

Upper cylinder rolls at time 't', Let 'P' be the point of contact of the two cylinder such that the line OC joining the centres of makes an angle 'θ' to the vertical.

Let, CB make an angle ϕ to the vertical at time 't'.

Since, there is pure rolling.

$\therefore$ Arc AP = Arc BP or $a(\theta-\alpha) = b(\phi-\theta)$

i.e $b\phi = (a+b)\theta - a\alpha$ i.e $\boxed{b\phi = c\theta}$ — (1)

where, $\alpha \to$ is very small

$a+b = c$.

Let, R be the normal reaction,
F the friction acting on the upper sphere.
Therefore, the equation of motion of cylinder along CO is $\Rightarrow$ $\boxed{Mc\dot{\theta}^2 = Mg\cos\theta - R}$ ——— (2)

The coordinates (x_c, y_c) of the centre 'C' referred to the horizontal and vertical lines through 'O' as axes are given by –

$$x_c = OC\sin\theta = c\sin\theta$$
$$y_c = OC\cos\theta = c\cos\theta$$
$$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$$

$\therefore$ The energy equation, gives

$$\frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}Mv_c^2 = Mg(c\cos\alpha - c\cos\theta)$$
$$\Rightarrow \frac{1}{2}M\cdot\frac{1}{2}b^2\cdot\dot{\phi}^2 + Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta)$$
$$\Rightarrow \frac{1}{4}Mb^2\dot{\phi}^2 + Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta)$$
$$\Rightarrow \frac{1}{4}M(c\dot{\theta})^2 + \frac{1}{2}Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta) \text{ - from (1)}$$
$$\therefore \boxed{\dot{\theta}^2 = \frac{4g}{3c}(\cos\alpha - \cos\theta)}$$

Substituting in (2), we get

$$R = Mg\cos\theta - Mc\dot{\theta}^2 = Mg\cos\theta - M\cancel{c}\cdot\frac{4g}{3\cancel{c}}(\cos\alpha - \cos\theta)$$

$$\boxed{R = \frac{1}{3}Mg(7\cos\theta - 4\cos\alpha)}$$

The cylinders will separate when $R = 0$

i.e. $\frac{1}{3}Mg(7\cos\theta - 4\cos\alpha) = 0$ $\qquad M \neq 0$, $g \neq 0$

$$\therefore 7\cos\theta - 4\cos\alpha = 0$$
$$\cos\theta = \frac{4\cos\alpha}{7} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{4}{7}\cos\alpha\right)}$$

Which is required result.

Q. 8(a) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l-x)$, find its displacement.

Solution :-

- string has fixed end points $- x=0$ & $x=l$ initially at rest in its equilibrium position.

$\therefore$ The required displacement $y(x,t)$ of the string is the solution of the wave equation.

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right)} \quad \text{—— (1)}$$

Subject to boundary conditions:

$$y(0,t) = y(l,t) = 0 \text{; for all } t \geq 0 \quad \text{—— (2)}$$

and given initial conditions, namely:

initial displacement $= y(x,0) = f(x) = 0$ —— (3A)

and initial velocity $= \left(\partial y/\partial t\right)_{t=0} = g(x) = k(lx - x^2)$ —— (3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi ct}{l}} \quad \text{—— (4)}$$

where; $F_n = \dfrac{2}{n\pi c}\displaystyle\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$

$$F_n = \frac{2k}{n\pi c}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx \quad \text{—— 5}$$

$$\therefore F_n = \frac{2k}{n\pi c}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$F_n = \frac{2k}{n\pi c}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^3}{cn^4\pi^4}\left[1-(-1)^n\right]$$

$$\therefore F_n = \begin{cases} 0, & \text{if } n = 2m, \text{ and } m = 1,2,3\ldots \text{ (even)} \\ 8kl^3/cn^4\pi^4 = \dfrac{8kl^3}{c\pi^4(2m-1)^4}; & \text{if } n = 2m-1 \text{ (odd)} \end{cases}$$

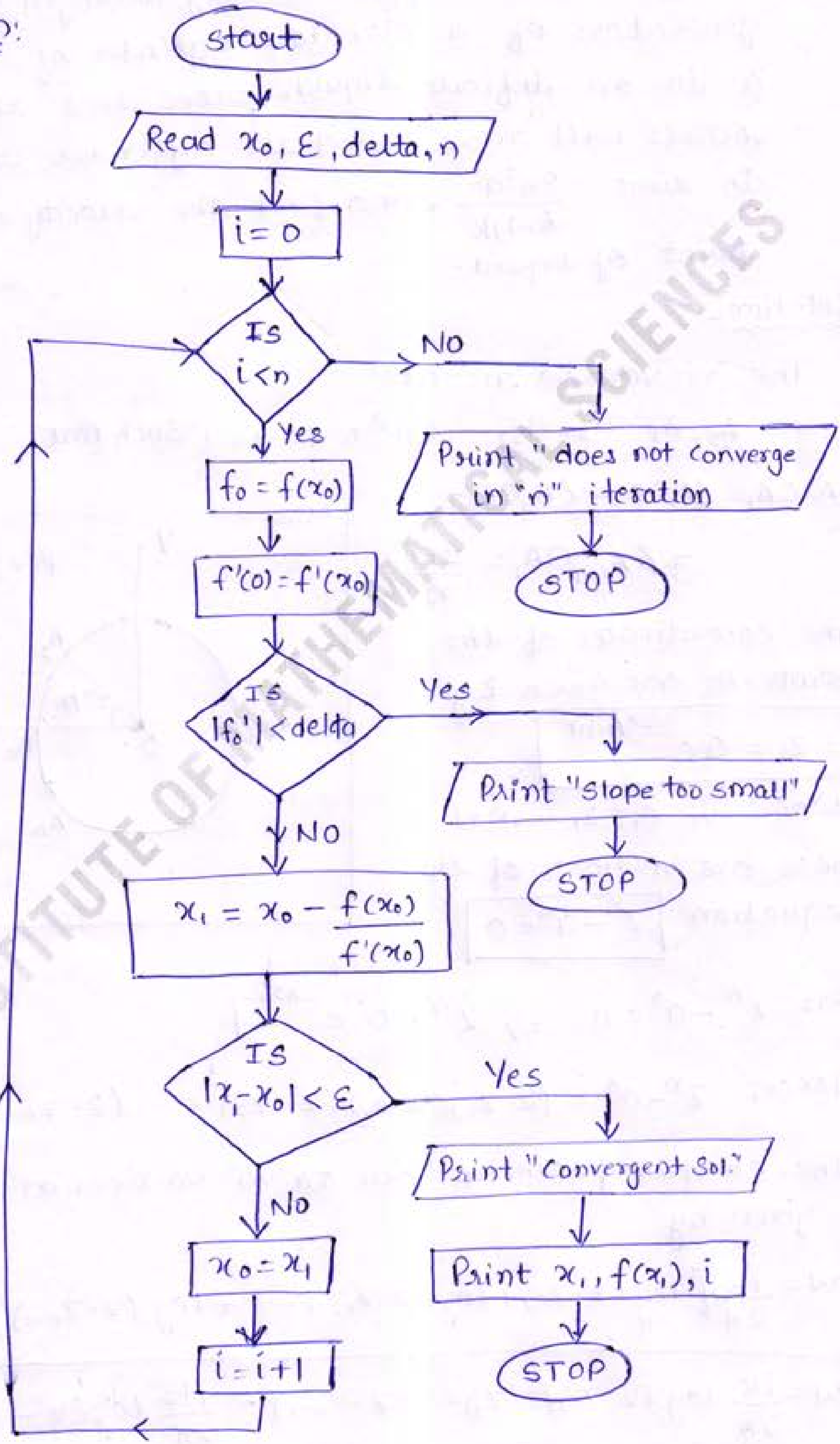
Substituting these values in (4),

$$\boxed{y(x,t) = \frac{8kl^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\cdot\sin\frac{(2m-1)\pi ct}{l}}$$

which is the required displacement.

Q.8. b) Draw a flow chart for Newton - Raphson method.

Solution:



Ques: 8(c)) If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius 'a' in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of liquid.

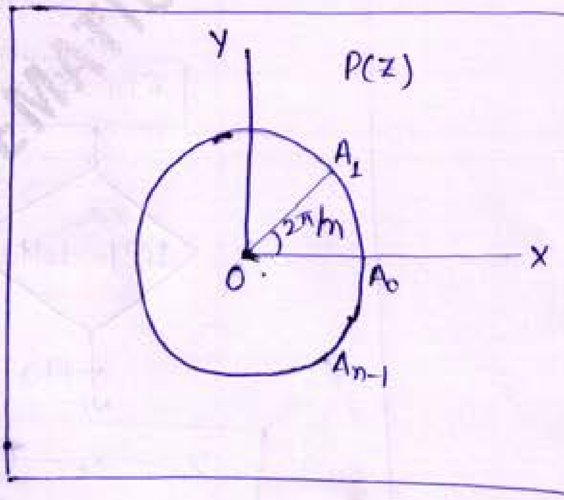
Solution:-

The 'n' vortices are at

$A_0, A_1, A_2, A_3, \ldots, A_n$ (0-1), Such that

$\angle A_0OA_1 = \angle A_1OA_2 = \angle A_2OA_3 = \ldots$

$\ldots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$



The co-ordinates of the points A_r are given by

$$\boxed{z = z_r = a e^{(2\pi/n)ir}}$$

where $r = 0, 1, 2, \ldots, n-1$

These are 'n' roots of the equation $\boxed{z^n - a^n = 0}$

$\left[\text{For } z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi r i}\right]$

Hence; $z^n - a^n = (z-z_0)(z-z_1)(z-z_2)\cdots\cdots(z-z_{n-1})$

The complex potential due to 'n' vortices at P is given by

$$W = \frac{ik}{2\pi}\left[\log(z-z_0) + \log(z-z_1) + \cdots + \log(z-z_{n-1})\right]$$

$$\boxed{W = \frac{ik}{2\pi}\log(z-z_0)(z-z_1)\cdots(z-z_{n-1}) = \frac{ik}{2\pi}\log(z^n - a^n)} \quad -(1)$$

For the point A_0, $z=a$ so that

$r=a$, $\theta=0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\left[\log(z^n-a^n) - \log(z-a)\right]$$

$$\phi'+\psi' = \frac{ik}{2\pi}\left[\log(r^n e^{in\theta}-a^n) - \log(re^{i\theta}-a)\right]$$

$$\therefore \psi' = \frac{k}{4\pi}\left[\log(r^{2n}+a^{2n}-2r^n a^n\cos n\theta) - \log(r^2+a^2-2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1}-2nr^{n-1}a^n\cos n\theta}{r^{2n}+a^{2n}-2r^na^n\cos n\theta} - \frac{2r-2a\cos\theta}{r^2+a^2-2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi}\left[\frac{2nr^na^n\sin n\theta}{r^{2n}-2r^na^n\cos n\theta+a^{2n}} - \frac{2ra\sin\theta}{r^2+a^2-2ra\cos\theta}\right]$$

$$\left.\frac{\partial\psi'}{\partial r}\right|_{r=a} = \frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right)-\left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right]$$

$$\boxed{\left.\frac{\partial\psi'}{\partial r}\right|_{r=a} = \frac{k}{4\pi a}[n-1]}$$

$$\boxed{\left.\frac{\partial\psi'}{\partial\theta}\right|_{r=a} = \frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta} - \frac{\sin\theta}{1-\cos\theta}\right]}$$

Since, $\lim\limits_{x\to0}\frac{F(x)}{G(x)} = \lim\limits_{x\to0}\frac{F'(x)}{G'(x)} = \lim\limits_{x\to0}\frac{F''(x)}{G''(x)}$, [form $\frac{0}{0}$]

$$\left.\frac{\partial\psi'}{\partial\theta}\right|_{r=a} = \frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta} - \frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta\to0$$

$$\left.\frac{\partial\psi'}{\partial\theta}\right|_{r=a} = \frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta} - \frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta\to0$$

$$\boxed{\left.\frac{\partial\psi'}{\partial\theta}\right|_{r=a} = \frac{k}{4\pi}[0+0] = 0}$$

Finally;

$$\frac{\partial \psi'}{\partial \theta_1} = \frac{K}{4\pi a}(n-1), \quad \frac{\partial \psi'}{\partial \theta} = 0 \quad \text{as } r \to a, \theta \to 0$$

consequently, the velocity q_0 of the vortex A_0 is given by

$$q_0 = \left[\left(\frac{\partial \psi'}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \psi'}{\partial \theta}\right)^2\right]^{+1/2}$$

$$q_0 = \left[\left(\frac{K}{4\pi a}(n-1)\right)^2 + \left(\frac{1}{r}\cdot 0\right)^2\right]^{1/2} = \left[\left(\frac{K(n-1)}{4\pi a}\right)^{2}\right]^{1/2}$$

$$\boxed{q_0 = \frac{K(n-1)}{4\pi a}}_A$$

This proves that the whole of velocity is along tangent and there is no velocity along the normal to the circle.

Hence, the vortices will move round the cylinder with uniform velocity $\frac{K(n-1)}{4\pi a}$.

The time of one complete revolution

$$\text{Time} = \frac{\text{Distance}}{\text{Velocity}}$$

$$= \frac{2\pi a}{\frac{K(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{K(n-1)}$$

Hence, time of one complete revolution

$$\boxed{T = \frac{8\pi^2 a^2}{K(n-1)}}$$ which is the required result