[IFoS 2021 P-1 MATHS] 16/04/22

1(a) $q(x,y,z) = 2x^2+2y^2+6z^2+2xy-6yz-6zx$

standard basis,

$SB = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$B^* = \left\{\left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}\right), \left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}, 0\right), \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)\right\}$$

To find expression of $q(x,y,z)$ with respect to the basis B

(Here, (x,y,z) are co-ordinates of vector x).

Here, we notice that B is an orthonormal basis. Hence, setting

$$\varphi = \begin{pmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{-2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \end{pmatrix}$$

φ is a change of coordinate matrix.

Also, we find matrix of quadratic form

$$\begin{aligned} q(x,y,z) = {} & 2x^2 + xy - 3xz \\ & xy + 2y^2 - 3yz \\ & -3zx - 3yz + 6z^2 \end{aligned}$$

$$\therefore \quad A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{pmatrix}$$

$$\phi^t A \phi = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(use Calci)}$$

Hence, the given quadratic form gets transformed as

$$q'(s_1, s_2, s_3) = 9s_1^2 + s_2^2$$

Where (s_1, s_2, s_3) are co-ordinates of vector X with respect to new orthonormal basis B.

Since, $q'(s_1, s_2, s_3) = 9s_1^2 + s_2^2 \geq 0$ for all real values of variables s_1, s_2 and s_3 and $q'(s_1, s_2, s_3) = 0$ only if $(s_1, s_2, s_3) = (0, 0, 0)$

Hence, given quadratic form is positive-definite.

1(b) Let A and B are hermitian matrices

i.e. $A^\theta = A$, $B^\theta = B$ — ①

where A^θ is conjugate-transpose of A.

Now, let their product AB is hermitian

i.e. $(AB)^\theta = AB$ — (2)

To prove; A and B commute

i.e. $AB = BA$

Consider, $(AB)^\theta = B^\theta A^\theta$

$= BA$ using (1)

using (2), we get that $AB = BA$.

~~Again~~ let A and B commute ~~i.e AB~~

~~To prove~~ i.e $AB = BA$ — (3)

To prove, AB is hermitian

i.e. $(AB)^\theta = AB$

Consider, $(AB)^\theta = B^\theta A^\theta$

$= BA$ using (1)

$= AB$ using (3)

Hence proved.

non-diagonal

Example of a pair of 3×3 symmetric matrices such that their product is again symmetric.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, A and B are symmetric matrices as $A^t = A$, $B^t = B$.

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, AB is again symmetric matrix.

~~Also~~,

$$BA = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, we notice that $AB = BA$

i.e. A and B commute.

1(c)(i) $I = \int_0^2 x(8-x^3)^{1/3}dx$

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Put $x^3 = 8t \Rightarrow 3x^2 dx = 8dt$

limits : x from 0 to 2 $\Rightarrow t$ from 0 to 1.

$$\therefore I = \int_0^1 (8t)^{1/3} (8-8t)^{1/3} \cdot \frac{8dt}{3(8t)^{2/3}}$$

$$= \frac{8}{3}\int_0^1 t^{-1/3} (1-t)^{1/3} dt$$

$$= \frac{8}{3}\beta\left(-\frac{1}{3}+1, \frac{1}{3}+1\right) = \frac{8}{3}\beta\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$= \frac{8}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} = \frac{8}{3}\times\frac{1}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}$$

$$= \frac{8}{9} \cdot \frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{16\sqrt{3}\,\pi}{27} \quad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}\right]$$

$\left(\Gamma(n+1) = n\Gamma n, \Gamma(2) = 1\right)$

(ii) $I = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$

$$= \int_0^1 x^2(1-x^5)^{-1/2} dx$$

Put, $x^5 = t \Rightarrow 5x^4 dx = dt$

limits : x from 0 to 1 $\Rightarrow t$ from 0 to 1.

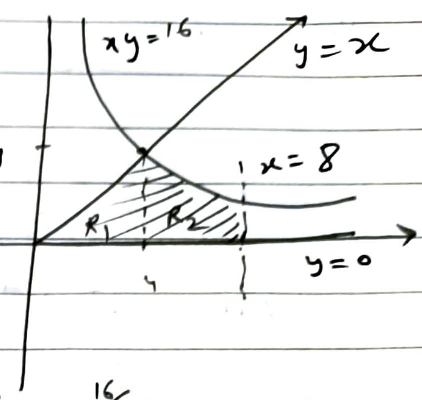
$$I = \int_0^1 t^{2/5} (1-t)^{-1/2} \frac{dt}{5t^{4/5}}$$

$$= \frac{1}{5}\int_0^1 t^{-2/5} (1-t)^{-1/2} dt$$

$$= \frac{1}{5}\beta\left(\frac{-2}{5}+1, \frac{-1}{2}+1\right)$$

$$= \frac{1}{5}\beta\left(\frac{3}{5}, \frac{1}{2}\right)$$

1(d) $I = \iint_R x^2 \, dx \, dy$



$$= \iint_{R_1} x^2 \, dx \, dy + \iint_{R_2} x^2 \, dx \, dy$$

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 \, dy \, dx + \int_{x=4}^{8} \int_{y=0}^{16/x} x^2 \, dy \, dx$$

$$= \int_{x=0}^{4} [y]_{y=0}^{y=x} \cdot x^2 \, dx + \int_{x=4}^{8} [y]_{y=0}^{16/x} x^2 \, dx$$

$$= \int_0^4 x \cdot x^2 \, dx + \int_4^8 \frac{16}{x} \cdot x^2 \, dx$$

$$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx$$

$$= \left. \frac{x^4}{4} \right|_0^4 + \left. \frac{16x^2}{2} \right|_4^8$$

$$= 64 + 8(64 - 16) = 64 + 8 \times 48$$

$$= 64 + 384$$

$$= \underline{448}$$

1(e) Equation of a plane passing through the point $(1,-1,1)$ is

$$a(x-1)+b(y+1)+c(z-1)=0 \quad \text{---(1)}$$

Again, it passes through point $(-2,1,-1)$

$\therefore \quad a(-2-1)+b(1+1)+c(-1-1)=0$

i.e. $\quad -3a+2b-2c=0$

or $\quad 3a-2b+2c=0 \quad \text{---(2)}$

Given plane is perpendicular to plane
$2x+y+z+5=0$

$\therefore \quad 2a+b+c=0 \quad \text{---(3)}$

solving equations (2) and (3) for a, b, c

$$\frac{a}{-2-2}=\frac{b}{4-3}=\frac{c}{3-(-4)}$$

$$\frac{a}{-4}=\frac{b}{1}=\frac{c}{7}$$

using ~~in~~ in Eqn (1)

$$-4(x-1)+(y+1)+7(z-1)=0$$

i.e. $\quad -4x+y+7z+4+1-7=0$

$$-4x+y+7z-2=0$$

or $\quad 4x-y-7z+2=0.$

Which is the required eqn of plane.

2(a).

$$f(x) = x^2+4x-3$$
$$e_1 = x^2-2x+5$$
$$e_2 = 2x^2-3x$$
$$e_3 = x+3$$

Let, $f(x) = ae_1 + be_2 + ce_3$, $a, b, c \in \mathbb{R}$

$$x^2+4x-3 = a(x^2-2x+5)+b(2x^2-3x)+c(x+3)$$
$$= (a+2b)x^2+(-2a-3b+c)x+(5a+3c) \quad -(\star)$$

comparing the coefficients.

$$a+2b = 1$$
$$-2a-3b+c = 4$$
$$5a + 3c = -3$$

Solving for a, b, c, we get

$a = -3$, $b = 2$, $c = 4$ [use calci]

$$\therefore f(x) = -3(x^2-2x+5)+2(2x^2-3x)+4(x+3)$$

Now, we have to show that set $\{e_1, e_2, e_3\}$ forms a basis of all quadratic polynomials over $\mathbb{R}$.

First we show that set $\{e_1, e_2, e_3\}$ is linearly independent. consider

$$ae_1 + be_2 + ce_3 = 0$$

i.e. $(a+2b)x^2 + (-2a-3b+c)x + (5a+3c) = 0$ [using $(\star)$]

$$\Rightarrow \quad a+2b = 0$$
$$-2a-3b+c = 0$$
$$5a + 3c = 0$$

Coefficient matrix of this homogeneous system of linear equations is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix}, \quad \therefore \quad |A| = 1(-9-0)-2(-6-5)$$
$$= 13 \neq 0$$

As, $|A| \neq 0$, the given system has only trivial solution i.e.

$$a = 0, \quad b = 0, \quad c = 0$$

Hence, set $\{e_1, e_2, e_3\}$ is linearly independent.

Also, this set has dimension 3 which is equal to the dimension of set of all quadratic polynomials over $\mathbb{R}$

$$V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

Hence, set $\{e_1, e_2, e_3\}$ from a basis of vector space V.

2(b). Let a point $P(\alpha, \beta)$ lies on given ellipse.

$$\therefore \quad \frac{\alpha^2}{4} + \frac{\beta^2}{9} = 1 \qquad \text{——(1)}$$

Distance of point $P(\alpha, \beta)$ from given line

$$D = \frac{2\alpha + \beta - 10}{\sqrt{5}} \qquad \text{——(2)}$$

Let us, minimize 'D' along with constaint (1), using Lagrange's multiplies method.

Consider,

$$f(\alpha, \beta) = \frac{1}{\sqrt{5}}(2\alpha + \beta - 10) + \lambda\left(\frac{\alpha^2}{4} + \frac{\beta^2}{9} - 1\right)$$

for max or mini values, $df = 0$

i.e. $f_\alpha = 0$, $f_\beta = 0$

$$\frac{1}{\sqrt{5}} \cdot 2 + \lambda \frac{2\alpha}{4} = 0 \qquad \text{——(3)}$$

$$\frac{1}{\sqrt{5}} \cdot 1 + \lambda \cdot \frac{2\beta}{9} = 0 \qquad \text{——(4)}$$

Multiplying (3) with α and (4) with β and adding,

$$\frac{1}{\sqrt{5}}(2\alpha + \beta) + \lambda \cdot 2\left(\frac{\alpha^2}{4} + \frac{\beta^2}{9}\right) = 0$$

$$\frac{1}{\sqrt{5}}(D\sqrt{5} + 10) + 2\lambda \cdot (1) = 0.$$

$$D = 2\lambda - 2\sqrt{5} \qquad \text{——(5)}$$

Also, from (3) and (4)

$$\alpha = \frac{-4}{\sqrt{5}} \cdot \frac{1}{\lambda} \quad ; \quad \beta = \frac{-9}{2\sqrt{5}} \cdot \frac{1}{\lambda}$$

using in (1)

$$\frac{1}{4}\left(\frac{16}{5}\right) \cdot \frac{1}{\lambda^2} + \frac{1}{9}\left(\frac{81}{20}\right)\frac{1}{\lambda^2} = 1$$

$$\left(\frac{16}{20} + \frac{9}{20}\right) = \lambda^2$$

i.e. $\lambda^2 = \frac{25}{20} = \frac{5}{4}$

i.e. $\lambda = \pm \frac{\sqrt{5}}{2}$

Hence, from (5),

$$D = 2(\lambda) - 2\sqrt{5}$$

If $\lambda = \frac{\sqrt{5}}{2} \Rightarrow D = \sqrt{5} - 2\sqrt{5} = -\sqrt{5}$

$\lambda = -\frac{\sqrt{5}}{2} \Rightarrow D = -\sqrt{5} - 2\sqrt{5} = -3\sqrt{5}$

$|D| = \sqrt{5}$ or $3\sqrt{5}$

Hence, the shortest distance is $\sqrt{5}$ unit.

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Verification: Any point on ellipse is
$P(2\cos\theta, 3\sin\theta)$

$\therefore\ D = \frac{1}{\sqrt{5}}(4\cos\theta + 3\sin\theta - 10)$

Max/ min value of D is

$$\frac{1}{\sqrt{5}}\left(\pm\sqrt{16+9} - 10\right)$$

i.e. $\frac{-5}{\sqrt{5}}$ ~~i.e. $\pm\sqrt{5}$~~ or $\frac{-15}{\sqrt{5}}$

ie $-\sqrt{5}$ or $-3\sqrt{5}$

$\therefore\ |D|$ value is $\sqrt{5}$ or $3\sqrt{5}$

2(c) Vertex is given as $A(1,2,1)$

Guiding circle : $x^2+y^2+z^2=5, \quad x+y-z=1.$

$$\text{—(1)} \qquad\qquad \text{—(2)}$$

Eqns of any generating line through point $(1,2,1)$

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-1}{n} = r \text{ (say)} \qquad \text{—(3)}$$

Any point on (3) is

$$(lr+1, \; mr+2, \; nr+1)$$

Since, generator (3) must meet the curve given by (1) and (2), thus this point must satisfy (1) and (2) for some value of r.

$$\therefore \; (lr+1)^2+(mr+2)^2+(nr+1)^2 = 5$$

i.e. $r^2(l^2+m^2+n^2) + 2r(l+2m+n) + 1 = 0 \qquad \text{—(4)}$

and $\quad (lr+1)+(mr+2)-(nr+1) = 1$

$$r(l+m-n) = -1 \Rightarrow r = \frac{-1}{l+m-n}$$

using in (4)

$$\frac{1}{(l+m-n)^2}(l^2+m^2+n^2) - \frac{2}{(l+m-n)}(l+2m+n)+1 = 0$$

$$(l^2+m^2+n^2) - 2(l+2m+n)(\cancel{l}+m-n) + (l+m-n)^2 = 0.$$

$$(\cancel{l^2}+\cancel{m^2}+n^2) - 2(\cancel{l^2}+2lm+\cancel{nl}+lm+2m^2+mn-\cancel{ln}-2mn-n^2$$
$$+(\cancel{l^2}+\cancel{m^2}+n^2+2lm-2mn-2ln) = 0.$$

$$-2m^2+4n^2-4lm-2ln = 0.$$

Eliminating, l, m, n with the ~~sto~~ help of (3)

$$2(y-2)^2 - 4(z-1)^2 + 4(x-1)(y-2) + 2(x-1)(z-1) = 0$$

$$\Rightarrow \; y^2-2z^2+2xy+xz-5x-4y+3z+7 = 0$$

Which is the required eqn of cone.