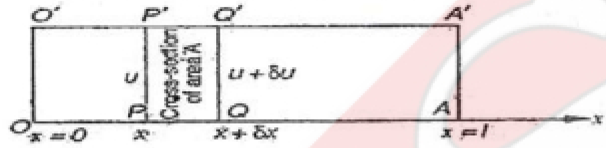
Applications of PDE continuation

Derivation of one-dimensional heat equation.

Consider the flow of heat by conduction in a bar OA. Let OA be taken as the x-axis. We consider an element $PQQ'P'$ of the bar as shown in the figure. Clearly the temperature $u(x, t)$ of the bar at any point P is function of x and the time t. Suppose that the bar is raised to an assigned temperature distribution at time $t = 0$ and then heat is allowed to flow by conduction. We wish to find $u(x, t)$ at any point x and at any time $t > 0$. We make the following assumptions. (19)



(*i*) The bar is homogenous, *i.e.*, the mass of the bar per unit volume is constant ρ (say).

(*ii*) The sides of the bar are insulated and the loss of heat from the sides by conduction or radiation can be neglected.

(*iii*) The amount of heat crossing any section of the bar is given by $kA\,(\partial u/\partial x)\delta t$, where

A = area of cross section of the bar, $\quad \partial u/\partial x$ = temperature gradient at the section PP'.

δt = time of flow of heat, $\quad K$ = themal conductivity of the material of the bar.

Now the quantity Q_1 of heat flowing into the element across the section PP' in time δt is given by

$$Q_1 = -KA(\partial u/\partial x)_x\,\delta t,$$

where the negative sign has been taken because heat flows in the direction of decreasing temperature.

Again the quantity Q_2 of heat flowing out of the element across the section QQ' in time δt is given by

$$Q_2 = -KA(\partial u/\partial x)_{x+\delta x}\,\delta t$$

Hence the quantity of heat retained by the element

$$= Q_1 - Q_2 = KA\,\delta t\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} \quad \dots (1)$$

Suppose the above heat raises the temperature of the element by a small quantity δu. Then the same quantity of heat is again given by

$$(\rho A\,\delta x)\sigma\,\delta u, \quad \dots (2)$$

where σ is specific heat of the bar. Since the expressions (1) and (2) are equal, we have

$$KA\,\delta t\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} = (\rho A\,\delta x)\sigma\delta u$$

or

$$K\frac{\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\}}{\delta x} = \rho\sigma\frac{\delta u}{\delta t} \quad \dots (3)$$

Now as $\delta x \to 0$ and $\delta t \to 0$, (3) reduces to

$$K\frac{\partial^2 u}{\partial x^2} = \rho\sigma\frac{\partial u}{\partial t} \quad \text{or} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \quad \dots (4)$$

where $k = K/\rho\sigma$ is called the diffusivity of the matrial of the bar.

Note: Equation (4) is also known as *one-dimensional diffusion equation.*

Method of indicating boundary conditions.:-

We begin with the case in which the end $x = l$ of the bar is kept at temperature u_0. Then, we write

$$u(l, t) = u_0, \quad \text{where } u_0 \text{ is a constant.} \quad \dots (i)$$

Condition (i) simply states that the boundary $x = l$ is held by some means at a constant temperature u_0 for all $t > 0$. (with temperature)

Next, we consider the case in which the end $x = l$ of the bar is insulated. Then, we write

$$(\partial u/\partial x)_{x=l} = 0 \quad (\dots (ii))$$

Condition (*ii*) indicates that the boundary $x = l$ is insulated.

From the emprical law of heat transfer, the flux of heat across a boundary (that is, the amount of heat per unit area per unit time conducted across the boundary) is proportional to the normal derivative of the temperatur u. Thus, when the boundary $x = l$ in thermally insulated, no heat flows into or out of the rod and so $(\partial u/\partial x)_{x=l} = 0$.

Boundary value problems in cartesian co-ordinates

We shall adopt a similar method of indicating boundary condition in two or three dimensional heat transfer problems.

An improtant note. In what follows, we have explained methods of solving boundary valve problems in some articles and indicated working rules for doing problems. You can solve a problem in one of the two following ways :

Method I. Prove the result of the relevant article completely. Then, compare the given problem with standard boundary value problem and use the relevant working rule as the case may be.

Method II. Proceed exactly as explained in the relevant article without using any result.

We have solved boundary value problem by both of the above two methods to make students familiar with the technique of solving boundary value problem. In examination, you can use any one of the above two methods.

PART I: PROBLEMS BASED ON ONE DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

Situation I. ***When ends of the rod are kept at zero temperature.***

2.2 A. *General solution of heat flow equation* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *by the method of separation of variables.* **[Delhi Maths (H) 1997; Meerut 1995; Ravishankar 2002]**

Sol. Given $$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

Let solution of (1) be of the form $$u(x, t) = X(x)\,T(t) \qquad \text{... (2)}$$

We then find, on substituting in (1), that

$$kX''T = XT' \quad \text{or} \quad X''/X = T'/kT, \qquad \text{... (3)}$$

where the dashes denote derivatives with respect to the relevant variable. Clearly the L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. Since x and t are independent variables, (3) can hold good if each side is equal to a constant, say μ. Then (3) leads to

$$X'' - \mu X = 0 \qquad \text{... (4)}$$

and $$T' = \mu k T \qquad \text{... (5)}$$

Three cases arise according as μ is zero, positive or negative.

Case I. Let $\mu = 0$. Then solutions of (4) and (5) are

$$X = a_1 x + a_2 \quad \text{and} \quad T = a_3 \qquad \text{... (6)}$$

Case II. Let μ *be + ve, say* λ^2, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' - \lambda^2 X = 0$ and $T' = \lambda^2 kT$. Solving these differential equations, we obtain

$$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x} \quad \text{and} \quad T = b_3 e^{\lambda^2 kt} \qquad \text{... (7)}$$

Case III. Let μ be – ve, say $-\lambda^2$, where $\lambda \neq 0$. Then (4) and (5) become respectively $X'' + \lambda^2 X = 0$ and $T' = -\lambda^2 kT$. Solving these, we get

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x \quad \text{and} \quad T = c_3 e^{-\lambda^2 kt} \qquad \text{... (8)}$$

Thus the various possible solutions are $$u(x, t) = A_1 x + A_2, \qquad \text{... (9)}$$

$$u(x,t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 kt} \qquad \text{... (10)}$$

and $$u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 kt} \qquad \text{... (11)}$$

where $A_1 = a_1 a_3$, $A_2 = a_2 a_3$, $B_1 = b_1 b_3$, $B_2 = b_2 b_3$, $C_1 = c_1 c_3$ and $C_2 = c_2 c_3$ are new arbitrary constants.

Now we have to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problem of heat conduction, temperature $u(x, t)$ must decrease with the increase of time. Accordingly the solution given by (11) is the only suitable solution.

2.2 B Solved examples based on Art. 2.2 A

Example 1. *Explain the method of separation of variables for finding solutions of second order linear partial differential equations. Hence, find the solution of one-dimensional diffusion equation* $\partial^2 z/\partial x^2 = (1/k)(\partial z/\partial t)$ *which tends to zero as* $t \to \infty$. **[Delhi, Maths (H). 1997]**

Sol. ***Method of separation of variables.*** A powerful method of solving the following second-order linear partial differential equation can be used in certain circumstances

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad \dots (i)$$

where $r = \partial^2 z/\partial x^2, s = \partial^2 z/\partial x\partial y, t = \partial^2 z/\partial y^2$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ and R, S, T, P, Q, Z and F are functions of x and y.

Let (*i*) possess a solution of the form $\qquad z = X(x)\, Y(y), \qquad \dots (ii)$
where X is a function of x alone and Y is a function of y alone.

Substituting the above value of z in (*i*), (*i*) reduces to

$$(1/X) f(D) X = (1/Y) g(D')\ Y, \qquad \dots (iii)$$

where $f(D)$, $g(D')$ are quadratic functions of $D(=\partial/\partial x)$ and $D'(=\partial/\partial y)$, respectively. When (*i*) reduces to (*iii*), we say that the equation (*i*) is separable in the variables x, y.

Now the L.H.S. of (*iii*) is a function of x alone whereas the R.H.S. is a function of y alone and the two can be equal only if each is equal to a constant, λ say. Then (*iii*) gives the following pair of second-order linear ordinary differential equations.

$$f(D)X = \lambda X \qquad \text{and} \qquad g(D')Y = \lambda Y. \qquad \dots (iv)$$

The problem of finding solutions of the form (ii) of (i) reduces to solving two equations given by (iv)

To find the required solution of the given diffusion equation.

Proceed exactly as in Art. 2.2 A upto equations (11). Then proceed as follows.

We have three possible solutions, namely, (9), (10) and (11). The given condition demands that $u \to 0$ as $t \to \infty$ we therefore reject the solutions given by (9) and (10). Hence, the desired solution is given by (11).

Ex. 2. *Use the method of separation of variables to solve the equation* $\partial^2 v/\partial x^2 = \partial v/\partial t$. *Given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$.

[Delhi Maths (Hons) 2002, Delhi Maths (Hons) Physics 2001, Lucknow UP (Tech) 2005]

Sol. Proceed as in Art. 2.2 A by noting that here $u = v$, $k = 1$ upto equation (11). Thus, we obtain the following three possible types of solutions of $\partial^2 v/\partial x^2 = \partial v/\partial t$.

$$v(x, t) = A_1 x + A_2 \qquad \dots (9)'$$

$$v(x, t) = (B_1 e^{\lambda x} + B_2 e^{-\lambda x}) e^{\lambda^2 t}, \lambda \neq 0 \qquad \dots (10)'$$

$$v(x, t) = (C_1 \cos\lambda x + C_2 \sin\lambda x) e^{-\lambda^2 t}, \lambda \neq 0 \qquad \dots (11)'$$

We are given that $v \to 0$ as $t \to \infty$. We, therefore, reject the solutions given by (9)′ and (10)′. Hence, the desired solution is given by (11)′, namely,

$$v(x,t) = (C_1 \cos\lambda x + C_2 \sin\lambda x) e^{-\lambda^2 t}, \lambda \neq 0 \qquad \dots (12)$$

The given boundary conditions are $\qquad v(0, t) = 0 \qquad \dots (13)$

and $\qquad v(l, t) = 0 \qquad \dots (14)$

Putting $x = 0$ in (12) and using B.C. (13), we obtain $C_1 = 0$ Then, (12) reduces to

$$v(x,t) = C_2 \sin \lambda x e^{-\lambda^2 t} \qquad \text{... (15)}$$

Putting $x = l$ in (15) and using B.C. (14), we obtain

$$0 = C_2 \sin \lambda l e^{-\lambda^2 t} \qquad \text{giving} \qquad C_2 \sin \lambda l = 0 \qquad \text{... (16)}$$

Since we are looking for a non-trivial solution, we take $C_2 \neq 0$. Hence (16) reduces to

$$\sin \lambda l = 0 \qquad \text{giving} \qquad \lambda l = n\pi \qquad \text{so that} \qquad \lambda = n\pi/l,\ n = 1, 2, 3, \ldots \qquad \text{... (17)}$$

Hence, from (11)′ a solution $v_n(x, t)$ of the given equation for some value of n is given by

$$v_n(x,t) = B_n \sin(n\pi x/l)\, e^{-(n^2\pi^2 t)/l^2}, \text{ by setting } C_2 = B_n. \qquad \text{... (18)}$$

Noting that the given equation $\partial^2 v/\partial x^2 = \partial v/\partial t$ is linear, its most general solution is obtained by applying the principle of superposition. Thus, we have

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-(n^2\pi^2 t)/l^2}$$

2.3 A. General solution of heat equation when both the ends of a bar are kept at temperature zero and the initial temperature is prescribed.

If both the ends of a bar of length a are at temperature zero and the initial temperature is to be prescribed function f(x) in the bar, then find the temperature at a subsequent time t.

$\boxed{\text{or}}$ *The faces x = 0 and x = a of an infinite slab are maintained at zero temperature. Given that the temperature u (x, t) = f(x) at t = 0. Determine the temperature at a subsequent time t.*

[Delhi Maths (H) 1997, 2005]

Sol. Here the temperature $u(x, t)$ in the given solid is governed by the one dimensional heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

Since the ends $x = 0$ and $x = a$ are kept at zero temperature, the boundary conditions are

$$u(0, t) = 0 \qquad \text{and} \qquad u(a, t) = 0, \quad \text{for all } t \qquad \text{... (2)}$$

The initial condition is given by $\qquad u(x, 0) = f(x),\ 0 < x < a \qquad$ ... (3)

Suppose that (1) has solutions of the form $\qquad u(x, t) = X(x)\,T(t) \qquad$ ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$kX''T = XT' \qquad \text{or} \qquad X''/X = T'/kT \qquad \text{... (5)}$$

Since x and t are independents variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (5) leads to the following equations:

$$X'' - \mu X = 0 \qquad \text{.... (6)}$$

and

$$T' = \mu k T. \qquad \text{...(7)}$$

Using (2), (4) gives $\qquad X(0)\,T(t) = 0 \qquad$ and $\qquad X(a)\,T(t) = 0 \qquad$ ... (8)

Since $T(t) = 0$ leads to $u = 0$, so suppose that $T(t) \neq 0$.

$\therefore$ From (8), $\qquad X(0) = 0 \qquad$ and $\qquad X(a) = 0 \qquad$ ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (10)

Using B.C. (9), (10) gives $0 = B$, $0 = Aa + B$ so that $A = B = 0$.

Hence $X(x) \equiv 0$ so that $u \equiv 0$, which does not satisfy (3). So we reject $\mu = 0$

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (11)

Using B.C. (9), (11) gives $0 = A + B$ and $0 = Ae^{a\lambda} + Be^{-a\lambda}$ ... (12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u = 0$,

which does not satisfy (3). So we also reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$, Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ... (13)

Using B.C. (9), (13) gives $0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$

So $\sin\lambda a = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3). Solving the trignometric equation $\sin\lambda a = 0$, we have

$\lambda a = n\pi$ so that $\lambda = n\pi/a$, where $n = 1, 2, ...$ ... (14)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/a)$ ... (15)

Using (14), (7) reduces to $\dfrac{dT}{T} = -\dfrac{n^2\pi^2 k}{a^2}dt$, as $\mu = -\lambda^2 = -\dfrac{n^2\pi^2}{a^2}$

or $(1/T)\,dT = -C_n^2 dt$, where $C_n^2 = (n^2\pi^2 k)/a^2$

whose general solution is $T_n(t) = D_n e^{-C_n^2 t}$ ... (16)

$$\therefore \quad u_n(x,t) = X_n(x)T_n(t) = E_n \sin(n\pi x/a)e^{-C_n^2 t} \quad \text{... (17)}$$

are solutions of (1), satisfying (2). Here $E_n\,(= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)e^{-C_n^2 t} \quad \text{... (18)}$$

Substituting $t = 0$ in (18) and using (3), we get $f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)$

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx, \qquad n = 1, 2, 3 \quad \text{... (19)}$$

Hence (18) is the required solution where E_n is given by (19).

2.3 B. Working rule for solving heat equation when both the ends of a bar of length a are kept at temperature zero and the initial temperature $f(x)$ is prescribed.

Step I. Proceed as in Art. 2.3 A and prove that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \text{... (1)}$$

subject to the boundary conditions $u(a, t) = u(a, t) = 0$, for all t ... (2)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (3)

is given by $u(x,t) = \sum_{n=0}^{\infty} E_n \sin(n\pi x/a)e^{-C_n^2 t}$ ... (4)

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n=1,2,3,\dots \quad \dots(5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \dots(6)$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III. Substitute the partiular values of k, a and $f(x)$ in (5) and (6) to get E_n and C_n^2.

Step IV. Substitute the values of coefficients E_n and C_n^2 obtained is step III in (4) to arrive at the desired solution of the given boundary value problem.

2.3 C. Solved example based on Art. 2.3 A and Art. 2.3 B

Ex. 1 (*a*) *A rod of length l with insulated sides, is initially at a uniform temperature* u_0. *Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperation u (x, t).* **[Andhra 1997, 2003]**

(b) *A thin rod of length* π *is first immersed in boiling water so that its temperature is 100°C thoughout. Then the rod is removed from water at t = 0 which is immediately put in ice so that the ends are kept at 0°C. Find w(x, t) if heat equation is* $a^2(\partial^2 w/\partial x^2) = \partial w/\partial t$. **[Nagpur 2002]**

Sol. (*a*) We can prove (actually prove in examination) that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \quad \dots(1)$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$, for all t. ... (2)

and the initial condition $u(x, 0) = f(x)$, $a < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)\, e^{-C_n^2 t},\ n=1,2,3,\dots \quad \dots(4)$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n=1,2,3 \quad \dots(5)$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \dots(6)$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = u_0$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx = \frac{2u_0}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l = \frac{2u_0}{n\pi}[1-(-1)^n], \quad \text{as} \quad \cos n\pi = (-1)^n$$

$$= \begin{cases} 0, & \text{if } n = 2m, \quad m=1,2,3,\dots \\ 4u_0/m\pi, & \text{if } n = 2m-1, \quad m = 1,2,3,\dots \end{cases}$$

Hence solution (4) reduces to

$$u(x,t) = \sum_{m=1}^{\infty} E_{2m-1}\sin\frac{(2m-1)\pi x}{l}e^{-C_{2m-1}^2 t} \quad \text{or} \quad u(x,t) = \frac{4u_0}{\pi}\sum_{m=1}^{\infty}\frac{l}{(2m-1)}\sin\frac{(2m-1)}{l}e^{-C_{2m-1}^2 t}$$

where $$C_{2m-1}^2 = \{(2m-1)^2\pi^2 k\}/l^2.$$

(*b*) Proceed as in part (*a*). Here $w = u$, $k = a^2$, $a = \pi$ and $u_0 = 100$.

Then, by (6), $C_n^2 = (2m-1)^2 a^2$ and so from (4) the required solution is

$$\therefore \quad w(x,t) = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(2m-1)x\, e^{-(2m-1)^2 a^2 t}$$

Ex. 2(*a*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = lx - x^2$. **[Delhi Maths (H) 2007]**

(*b*) *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ and $u(x, 0) = x$ when $0 \le x \le l/2$; $u(x, 0) = l - x$ when $l/2 \le x \le l$.

[Meerut 1998]

Sol. We can prove that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditions $\quad u(0, t) = u(a, t) = 0$ for all t ... (2)

and the initial condition $\quad u(x, 0) = f(x),\ 0 < x < a$ ... (3)

is given by
$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a) e^{-C_n^2 t} \qquad \text{... (4)}$$

where
$$E_n = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)dx,\ n = 1, 2, 3, \ldots \qquad \text{... (5)}$$

and
$$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (6)}$$

Part (*a*) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = lx - x^2$. Hence, (5) reduces to

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left\{\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right\} - (l - 2x)\left\{\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos(n\pi x)/l}{(n\pi)^3/l^3}\right\}\right]_0^l$$

[Using the chain rule of integration by parts]

$$= (2/l)\{(-2l^3/n^3\pi^3)\cos n\pi + (2l^3/n^3\pi^3)\} = (4l^2/n^3\pi^3)\{1 - (-1)^n\}$$

$$\therefore \quad E_n = \begin{cases} (8l^2)/(2m-1)^3\pi^3, \text{ if } n = 2m - 1 \text{ (odd) and } m = 1, 2, 3, \ldots \\ 0, \text{ if } n = 2m \text{ (even) where } m = 1, 2, 3, \ldots \end{cases}$$

Then, by (6), $C_n^2 = \{(2m-1)^2\pi^2 k\}/l^2$ and so from (4) the required solution is given by

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

Part (b) : Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and

$$f(x) = \begin{cases} x, & \text{when } 0 \le x \le l/2 \\ l - x, & \text{when } l/2 \le x \le l \end{cases} \qquad \text{... (6)}$$

$$\therefore (5) \Rightarrow E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^l f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$=\int_0^{l/2}\frac{2x}{l}\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}\frac{2}{l}(l-x)\sin\frac{n\pi x}{l}dx$$

$$=\left[\left(\frac{2x}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right)-\left(\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)}{(n\pi)^2/l^2}\right)\right]_0^{l/2}+\left[\left(\frac{2(l-x)}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right)-\left(-\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

[Using chain rule of integration by parts]

$$=-(l/n\pi)\cos(n\pi/2)+(2l/n^2\pi^2)\sin(n\pi/2)+(l/n\pi)\cos(n\pi/2)+(2l/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \qquad E_n=\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases}0, \text{ if } n=2m \text{ and } m=1,2,3,\ldots\\ 4l/(2m-1)^2\pi^2, \text{ if } n=2m-1 \text{ and } m=1,2,3,\ldots\end{cases}$$

Then, $\qquad (6) \Rightarrow C_n^2=\{(2m-1)^2\pi^2k\}/l^2$

$\therefore$ From (4), $\qquad u(x,t)=\dfrac{4l}{\pi^2}\sum_{m=1}^{\infty}\dfrac{1}{(2m-1)^2}\sin\dfrac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2kt\}/l^2}$

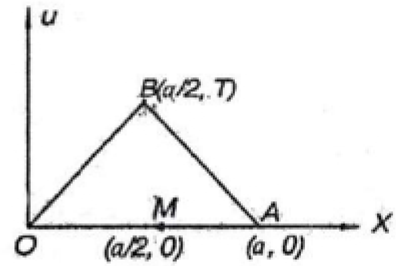
Ex. 2 (c) *Solve* $\partial u/\partial t=\partial^2u/\partial x^2$, $0<x<l$, $t>0$ *given that* $u(0,t)=u(l,t)=0$ *and* $u(x,0)=x(l-x), 0\le x\le l$. **[I.A.S. 2002]**

Sol. Refer solved Ex. 2(a). Here $k=1$ and hence the solution reduces to

$$u(x,t)=\frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}e^{-\{(2m-1)^2\pi^2 t\}/l^2}$$

Ex. 3. *A homogeneous rod of conducting material of length 'a' has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function u (x, t).*

Sol. We know that $u(x,t)$ is the solution of heat equation $\partial^2u/\partial x^2=(l/k)(\partial u/\partial t)$. Here the boundary conditions are $u(0,t)=u(a,t)=0$ for all $t\ge 0$. Let OA be the given rod and M be its middle point. Given that the temperature at the centre M is T and falls uniformly to zero at the two ends O and A of the rod. Hence, the temperature distribution at t = 0 is as given in the adjoining figure. The equations of straight lines OB and BA respectively are given by



$$u-0=\frac{T-0}{(a/2)-0}(x-0) \qquad \text{and} \qquad u-0=\frac{T-0}{(a/2)-a}(x-a) \qquad \ldots (i)$$

We can prove that the solution of heat equation $\qquad k(\partial^2u/\partial x^2)=\partial u/\partial t \qquad \ldots (1)$

subject to the boundary conditions $u(a,t)=u(a,t)=0$, $\quad$ for all $t \qquad \ldots (2)$

and the initial condition $\qquad u(x,0)=f(x), \quad 0<x<a \qquad \ldots (3)$

is given by $\qquad u(x,t)=\sum_{n=1}^{\infty}E_n\sin(n\pi x/a)^{-C_n^2 t} \qquad \ldots (4)$

where $\qquad E_n=\dfrac{2}{a}\displaystyle\int_0^a f(x)\sin(n\pi x/a)dx,\ n=1,2,3, \qquad \ldots (5)$

and $\qquad C_n^2=(n^2\pi^2k)/a^2 \qquad \ldots (6)$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k=k$, $a=a$. Also, from (i), we have

$$u(x,0)=f(x)=\begin{cases}(2Tx)/a, \text{ where } 0\le x\le a/2\\ \{2T(a-x)\}/a, \text{ where } a/2\le x\le a\end{cases} \quad \text{... (7)}$$

$$\therefore \quad (5)\Rightarrow E_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx=\frac{2}{a}\left[\int_0^{a/2} f(x)\sin\frac{n\pi x}{a}dx+\int_{a/2}^a f(x)\sin\frac{n\pi x}{a}dx\right]$$

$$=\frac{2}{a}\int_0^{a/2}\frac{2Tx}{a}\sin\frac{n\pi x}{a}dx+\frac{2}{a}\int_{a/2}^a\frac{2T(a-x)}{a}\sin\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$=\int_0^{a/2}\frac{4Tx}{a^2}\sin\frac{n\pi x}{a}dx+\int_{a/2}^a\frac{4T(a-x)}{a^2}\sin\frac{n\pi x}{a}dx$$

$$=\left[\left(\frac{4Tx}{a^2}\right)\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_0^{a/2}$$

$$+\left[\left\{\frac{4T(a-x)}{a^2}\right\}\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(-\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_{a/2}^a$$

$$=-(2T/n\pi)\cos(n\pi/2l)+(4T/n^2\pi^2)\sin(n\pi/2)+(2T/n\pi)\cos(n\pi/2)+(4T/n^2\pi^2)\sin(n\pi/2)$$

$$\therefore \quad E_n=\frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases}0, \text{ if } n=2m \text{ and } m=1,2,3,\ldots\\ \{8(-1)^{m+1}T\}/(2m-1)^2\pi^2, \text{ if } n=2m-1 \text{ and } m=1,2,3,\ldots\end{cases}$$

Then, by (6), $C_n^2=\{(2m-1)^2\pi^2k\}/a^2$ and so from (4), the required solution is given by

$$u(x,t)=\frac{8T}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{a}e^{-\{(2m-1)^2\pi^2kt\}/a^2}$$

Ex. 4. *Solve the one-dimensional diffusion equation* $\partial^2u/\partial x^2=(l/k)(\partial u/\partial t)$ *in the range* $0\le x\le 2\pi, t\ge 0$ *subject to the boundary conditions :* $u(x, 0)=\sin^3 x$ *for* $0\le x\le 2\pi$ *and* $u(0,t)=u(2\pi,t)=0$ *for* $t\ge 0$ **[Delhi Maths (H), 1998, 99, 2004, 06]**

Sol. Proceed upto equation (18) as in Art. 2.3 A by taking $a=2\pi$ and $f(x)=\sin^3 x$. Then, equation (18) for the present problem reduces to

$$u(x,t)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{nx}{2}\right)e^{-(n^2kt)/4}, \quad n=1,2,3,\ldots \quad \text{... } (i)$$

Putting $t=0$ in (i) and using given condition $u(x,0)=\sin^3 x$, we get

$$\sum_{n=1}^{\infty}E_n\sin(nx/2)=\sin^3 x=(3/4)\times\sin x-(1/4)\times\sin 3x$$

$$[\because \sin 3x=3\sin x-4\sin^3 x \Rightarrow \sin^3 x=(1/4)\times(3\sin x-\sin 3x)]$$

or
$$E_1\sin(x/2)+E_2\sin x+E_3\sin(3x/2)+E_4\sin 2x+E_5\sin(5x/2)$$
$$+E_6\sin 3x+E_7\sin(7x/2)+\ldots=(3/4)\times\sin x-(1/4)\times\sin 3x. \quad \text{... } (ii)$$

Equating the coefficients of the like terms on both sides of (ii), we get

$E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ when $n \neq 2$ or $n \neq 6$. Substituting these values in (*i*), the required solution is

$$u(x, t) = E_2 \sin x\, e^{-kt} + E_6 \sin 3x\, e^{-9kt} = (1/4) \times [3 \sin x\, e^{-kt} - \sin 3x\, e^{-9kt}].$$

Ex. 5. *Determine u such that* $\partial^2 u/\partial x^2 = (l/k)(\partial u/\partial t)$ *and satisfy the conditions (i)* $u \to 0$ *as* $t \to \infty$ *(ii)* $u = \sum_n c_n \cos nx$ *for* $t = 0$. **[Delhi Maths (H) 2004]**

Sol. Given $\qquad \partial^2 u/\partial x^2 = (l/k)(\partial u/\partial t)$. ... (1)

Also given that $\qquad u \to 0 \quad$ as $\quad t \to \infty$. ... (2)

and $\qquad u(x,0) = \sum_n c_n \cos nx$. ... (3)

Let a solution of (1) be $\qquad u(x, t) = X(x)\, T(t)$, ...(4)

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Substituting (4) in (1), we get $\qquad (1/X)X'' = (1/kT)T'$ ... (5)

Since the L.H.S of (5) is a function of x alone and the R.H.S. of (6) is a function of t alone, hence the two sides of (6) can be equal only if each side is equal to a constant, λ say. In view of condition (2), we choose $\lambda = -n^2$, where n is a non-zero constant. Then (6) gives

$(1/X)X'' = -n^2$ $\qquad$ so that $\qquad (D^2 + n^2)X = 0$, $\qquad$ where $\qquad D \equiv d/dx$.

and $\qquad (1/kT)T' = -n^2$ $\qquad$ so that $\qquad (1/t)dT = -n^2 k dt$.

Solving these $\qquad X_n(x) = a_n \cos nx + b_n \sin nx$ $\qquad$ and $\qquad T_n(t) = e_n e^{-n^2 kt}$

Keeping (3) and (4) in view, the most general solution of (1) may be/written as

$$u(x,t) = \sum_n u_n(x,t) = \sum_n X_n(x) T_n(t) \quad \text{or} \quad u(x,t) = \sum_n (c_n \cos nx + d_n \sin nx) e^{-n^2 kt}, \quad \text{... (1)}$$

where c_n $(= a_n e_n)$ and d_n $(= b_n e_n)$ are new arbitrary constants.

Putting $t = 0$ in (1) and using (3), we have $\qquad \sum_n c_n \cos nx = \sum_n (c_n \cos nx + d_n \sin nx)$,

showing that for the present problem $d_n = 0$. Then, from (1) the required solution is

$$u(x,t) = \sum_n c_n \cos nx\, e^{-n^2 kt}$$

Ex. 6. *A uniform rod 20 cm in length is insulated over its sides. Its ends are kept at 0°C. Its initial temperature is* $\sin(\pi x/20)$ *at a distance x from an end. Find temperature u(x, t) at time t. Given that* $\partial u/\partial t = a^2(\partial^2 u/\partial x^2)$. **[Nagpur 1996]**

Sol. Given $\qquad \partial u/\partial t = a^2(\partial^2 u/\partial x^2)$. ... (1)

Boundary Conditions (B.C.) : $\qquad u(0, t) = u(20, t) = 0$ for all t ... (2)

Initial Conditions (I.C.) : $\qquad u(x,0) = \sin(\pi x/20)$, $0 \le x \le 20$ ... (3)

Let a solution of (1) be $\qquad u(x,t) = X(x)\, T(t)$. ... (4)

Substituting this value of u in (1), we have

$XT' = a^2 X'' T$ $\qquad$ or $\qquad X''/X = T'/a^2 T$. ... (5)

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant say μ. Then, (5) gives $$X'' - \mu X = 0 \quad \ldots (6)$$

and $$T' = \mu a^2 T \quad \ldots (7)$$

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(20)\,T(t) = 0$ ... (8)

Since $T(t) \neq 0$, so (8) $\Rightarrow$ $X(0) = 0$ and $X(20) = 0$. ... (9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$. ... (10)

Using B.C (9), (10) gives $B = 0$ and $20A + B = 0$ so that $A = B = 0$.

Hence $X(x) \equiv 0$ so that $u = 0$ which does not satisfy (3). Hence reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (11)

Using B.C. (9), (11) gives $A + B = 0$ and $Ae^{20\lambda} + Be^{-20\lambda} = 0$ ... (12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u \equiv 0$,

which does not satisfy (3), So we also reject $\mu = \lambda^2$

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad \ldots (13)$$

when A and B are arbitrary constants.

Using B.C. (9), (13) gives A = 0 and $A\cos 20\lambda + B\sin 20\lambda = 0$ giving $\sin 20\lambda = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Now $\sin 20\lambda = 0 \Rightarrow 20\lambda = n\pi \Rightarrow \lambda = n\pi/20$, where $n = 1, 2, 3,$ ... (14)

Hence non-zero solution $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin(n\pi x/20)$. ... (15)

Using (14), (7) reduces to $\dfrac{dT}{T} = -\dfrac{n^2\pi^2 a^2}{400}dt$, as $\mu = -\lambda^2 = -\dfrac{n^2\pi^2}{400}$

whose general solution is $$T_n(t) = D_n e^{-(n^2\pi^2 a^2/400)t}. \quad \ldots (16)$$

$$\therefore \quad u_n(x,t) = X_n(x)T_n(t) = E_n \sin(n\pi x/20)e^{-(n^2\pi^2 a^2/400)t},$$

are solutions of (1), satisfying (2). Here $E_n^1 (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider the most general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{20} e^{-(n^2\pi^2 a^2/400)t} \quad \ldots (17)$$

Putting time t = 0 in (17) and using (3), we get $\sin\dfrac{\pi x}{20} = \sum_{n=1}^{\infty} E_n \sin\dfrac{n\pi x}{20}$, ... (18)

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{20}\int_0^{20} \sin\frac{\pi x}{20}\sin\frac{n\pi x}{20}dx,\ n = 1,2,3,\ldots \quad \ldots (19)$$

Case I. If $n \neq 1$, then (19) gives

$$E_n = \frac{1}{20}\int_0^{20}\left[\cos\frac{(n-1)\pi x}{20} - \cos\frac{(n+1)\pi x}{20}\right]dx = \frac{1}{20}\left[\frac{20}{(n-1)\pi}\sin\frac{(n-1)\pi x}{20} - \frac{20}{(n+1)\pi}\sin\frac{(n+1)\pi x}{20}\right]_0^{20} = 0$$

Thus, $E_n = 0$ for $n = 2, 3, 4, \ldots$ ... (20)

Case II. If $n = 1$, then (19) gives

$$E_1 = \frac{1}{20}\int_0^{20}\left(2\sin^2\frac{\pi x}{20}\right) = \frac{1}{20}\int_0^{20}\left(1-\cos\frac{2\pi x}{20}\right)dx = \frac{1}{20}\left[x - \frac{10}{x}\sin\frac{\pi x}{10}\right]_0^{20} = 1. \quad \ldots (21)$$

Using (20) and (21) (17) reduces to

$$u(x,t) = E_1 \sin\frac{\pi x}{20} e^{-(\pi^2 a^2/400)t} = \sin\frac{\pi x}{20} e^{-(\pi^2 a^2/400)t}$$

Ex. 7. *Solve the one-dimensional diffusion equation $\partial^2 u/\partial x^2 = (l/k)(\partial u/\partial t)$ in the region $0 \le x \le \pi, t \ge 0$ when (i) u remains finite as $t \to \infty$, (ii) $u = 0$ if $x = 0$ or π, for all values of t; (iii) At $t = 0$, $u = x$ for $0 \le x \le \pi/2$, and $u = \pi - x$ for $\pi/2 < x \le \pi$.*

Sol. Given that $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ ... (1)

Also, $u(x, t) =$ finite quantity as $t \to \infty$. ... (2)

Boundary conditions are $u(0,t) = (\pi, t) = 0$ for each t. ... (3)

Initial condition is $u(x,0) = \begin{cases} x, \text{ when } 0 \le x \le \pi/2 \\ \pi - x, \text{ when } \pi/2 \le x \le \pi. \end{cases}$ ... (4)

Let a solution of (1) be $u(x, t) = X(x)\,T(t)$ ... (5)

Substituting (5) in (1), we get $(1/X)X'' = (1/kT)T'$ ... (6)

Since the L.H.S of (6) is a function of x alone whereas the R.H.S is a function of t alone, hence the two sides can be equal only if each side is equal to a constant, say μ. Hence (6) yields

$$X'' - \mu X = 0 \quad \ldots (7)$$

and $$dT/dt - \mu kT = 0 \quad \ldots (8)$$

In solving (7) and (8), three distinct cases arise :

Case I. When $\lambda = 0$, then the solutions of (7) and (8) will be of the forms

$X = a_1 x + a_2$ and $T = a_3$ ... (9)

Case II. When $\mu = \lambda^2, \lambda \ne 0$, then the solutions of (7) and (8) will be of the forms

$X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ and $T = b_3 e^{\lambda^2 kt}$ ... (10)

Case III. When $\mu = -\lambda^2, \lambda \ne 0$, then the solutions of (7) and (8) will be of the forms

$X = c_1 \cos\lambda x + c_2 \sin\lambda x$ and $T = c_3 e^{-\lambda^2 kt}$ ... (11)

Hence, in view of (5), the various solutions of (1) could be the following

$$u(x, t) = Ax + B \quad \ldots (12)$$

and $$u(x,t) = (Ce^{\lambda x} + De^{-\lambda x})e^{\lambda^2 kt} \quad \ldots (13)$$

$$u(x,t) = (E\cos\lambda x + F\sin\lambda x)e^{-\lambda^2 kt}, \qquad \text{... (14)}$$

where $A = a_1 a_3$, $B = a_2 a_3$, $C = b_1 b_3$, $D = b_2 b_3$, $E = c_1 c_3$ and $F = c_2 c_3$

Now, the condition (2) demands that u should remain finite as $t \to \infty$. We therefore reject solution (13).

Next, in view of B.C. (3), solution (12) gives $0 = A \cdot 0 + B$ and $0 = A \cdot \pi + B$. These give $A = B = 0$ and hence from (12), $u = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the solution (12) also. Thus, the only possible solution satisfying the condition (2) is given by (14).

Putting $x = 0$ in (4) and using B.C. $u(0, t) = 0$ given by (3), we obtain $E = 0$. Then, (14) is simplified as

$$u(x,t) = F\sin\lambda x\, e^{-\lambda^2 kt} \qquad \text{... (15)}$$

Putting $x = \pi$ in (15) and using the BC $u(\pi,t) = 0$ given by (3), we obtain

$$0 = F\sin\lambda\pi\, e^{-\lambda^2 kt} \quad \text{giving} \quad F\sin\lambda\pi = 0 \qquad \text{... (16)}$$

Now, in view of $E = 0$ we must take $F \neq 0$ in order to obtain a non-trivial solution of (1). Accordingly (16) yields

$\sin\lambda\pi = 0$ or $\lambda\pi = n\pi$ so that $\lambda = n$, $n = 1, 2, 3, \ldots$

Thus, from (15), we arrive at a solution of the form

$$u_n(x,t) = F_n \sin nx\, e^{-n^2 kt}, \; n = 1, 2, 3, \ldots$$

Noting that the diffusion equation (1) is linear, its most general solution is obtained by applying the principle of superposition. Thus, the general solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} F_n \sin nx\, e^{-n^2 kt} \qquad \text{... (17)}$$

Putting t = 0 in (17), we have $$u(x,0) = \sum_{n=1}^{\infty} F_n \sin nx, \qquad \text{... (18)}$$

which is a half-range Fourier-sine series and, hence

$$F_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} u(x,0)\sin nx\,dx + \int_{\pi/2}^{\pi} u(x,0)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx, \text{ using (5)}$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

[Using the chain rule of Integrating by parts]

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left(\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right)$$

$$\therefore \quad F_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4(-1)^{m+1}/\pi(2m-1)^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of F_n in (17), the required solution is

$$u(x,t)=\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin(2m-1)x\, e^{-(2m-1)^2kt}$$

Ex. 9. *Make use of the method of separating variables to solve*

$$\partial u/\partial t=c^2(\partial^2 u/\partial x^2),\ t>0,\ 0\le x\le 1 \qquad \text{... (1)}$$

$$u(0,t)=2, \qquad u(1,t)=3 \qquad \text{... (2)}$$

and $$u(x,0)=x(1-x) \qquad \text{... (3)}$$

Sol. Let $$g(x)=c_1x+c_2 \qquad \text{... (4)}$$

such that $g(0)=2$ and $g(1)=3$. ...(5)

Using (5), (4) reduces to $2=c_2$ and $3=c_1+c_2$, so that $c_1=1$ and $c_2=2$, Then (4) becomes

$$g(x)=x+2 \qquad \text{... (6)}$$

Let $$v(x,t)=\overset{u(x,t)}{v(x,t)}-g(x)=u(x,t)-x-2. \qquad \text{... (7)}$$

Using (7), (1), (2) and (3) respectively may be re-written as

$$\partial v/\partial t=c^2(\partial^2 v/\partial x^2) \qquad \text{... (8)}$$

$$v(0,t)=0, \qquad v(1,t)=0 \qquad \text{... (9)}$$

and $$v(x,0)=x(1-x)-x-2=-(x^2+2). \qquad \text{... (10)}$$

Proceed as Art. 2.3B taking $u=v$, $a=1$, $k=c^2$, $C_n^2=n^2\pi^2c^2$ and $f(x)=-(x^2+2)$

$$v(x,t)=\sum_{n=1}^{\infty}E_n\sin n\pi x\, e^{-n^2\pi^2c^2t} \qquad \text{... (11)}$$

and $$E_n=-2\int_0^1(x^2+2)\sin n\pi x\,dx=2\left[(x^2+2)\left(-\frac{\cos n\pi x}{n\pi}\right)-(2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

[Using the chain rule of integration by parts].

$$=-2\left[-\frac{3(-1)^n}{n\pi}+\frac{2(-1)^n}{n^3\pi^3}+\frac{2}{n\pi}-\frac{2}{n^3\pi^3}\right] \qquad \text{... (12)}$$

Using (11), (7) gives the required solution $\quad u(x,t)=v(x,t)+x+2$

or $$u(x,t)=x+2+\sum_{n=1}^{\infty}E_n\sin n\pi x\, e^{-n^2\pi^2c^2t},$$ where E_n is given by (12)

2.4 A. General solution of heat equation when both the ends of a bar are insulated and the initial temperature is prescribed.

If both the ends of a bar of length a are insulated and the initial temperature f(x) is prescribed, then to find the temperature at a subsequent time t.

Sol. Here the temperature $u(x,t)$ in the given bar is governed by one dimensional heat equation

$$k(\partial^2 u/\partial x^2)=\partial u/\partial t \qquad \text{... (1)}$$

Physical experiments show that the rate of heat flow is proportional to the gradient $\partial u/\partial x$ of the temperature $u(x,t)$. Hence if the ends $x=0$ and and $x=a$ of the bar are insulated, so that no heat can flow through the ends, we have

the boundary conditions: $\quad u_x(0,t)=u_x(a,t)=0$ for all t ... (2)

Also, the prescribed initial condition $\Rightarrow$ $u(x, 0) = f(x)$ for all x ... (3)

Suppose that (1) has solutions of the form $u(x, t) = X(x)\, T(t)$, ... (4)

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$kX''T = XT' \quad \text{or} \quad X''/X = T'/kT \qquad \text{... (5)}$$

Since x and t are independent variables, (5) can be true if each side is equal to the same constant, say μ. Thus, (5) yields. $X'' - \mu X = 0$ ... (6)

and $T' = \mu k T$ ... (7)

Differentiating (4) partially w.r.t. 'x', we get $u_x(x,t) = X'(x)T(t)$ ... (8)

Using (2), (8) gives $X'(0)T(t) = 0$ and $X'(a)T(t) = 0$ ... (9)

Since $T(t) = 0$ leads to $u = 0$, so assume that $T(t) \neq 0$. Hence, from (9), we get

$$X'(0) = 0 \quad \text{and} \quad X'(a) = 0 \qquad \text{... (10)}$$

Three cases arise :

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$, ... (11)

which yields $X'(x) = A$ ... (12)

Using B.C. (10), (12) gives $A = 0$. Then (11) reduces to $X(x) = B$.

Again, corresponding to $\mu = 0$, (7) yields

$dT/dt = 0$ so that $T = \text{constant} = E_0/2B$, say

$\therefore$ Corresponding to $\mu = 0$ a solution of the given boundary value problem from (4) is given by

$$u(x, t) = B \times (E_0/2B) = E_0/2 \qquad \text{... (13)}$$

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (14)

which yields $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ... (15)

Using B.C (10), (15) gives $0 = \lambda(A - B)$ and $0 = \lambda(Ae^{a\lambda} - Be^{-a\lambda})$.

These equations yield $A = B = 0$ so that $X(x) \equiv 0$ and hence $u = 0$,

which does not satisfy (3). So we reject $\mu = \lambda^2$

Case III. Let $\mu = -\lambda^2, \lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \qquad \text{... (16)}$$

which yields $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ... (16)′

Using B.C. (10), (16)′ gives $0 = B\lambda$ and $0 = -A\lambda\sin\lambda a + B\lambda\cos\lambda a$.

Thus, $B = 0$ and $A\lambda\sin\lambda = 0$. Since $\lambda \neq 0$ and $A \neq 0$ for a non-trivial solution, we must have

$\sin\lambda a = 0$ so that $\lambda a = n\pi$

Thus, $\lambda = n\pi/a$, $n = 1, 2, 3, \ldots$ ... (17)

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$X_n(x) = A_n\cos(n\pi x/a), \text{ by (16) and (17)} \qquad \text{... (18)}$$

Using (17), (7) reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 k}{a^2}dt, \qquad \text{as} \qquad \mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$$

or $$(1/T)dT = -C_n^2 dt \qquad \text{... (19)}$$

where $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (20)}$$

Solving (19), $$T_n(t) = D_n e^{-C_n^2 t} \qquad \text{... (21)}$$

$\therefore$ $$u_n(x,t) = X_n(X)T_n(t) = E_n \cos(n\pi x/a)e^{-C_n^2 t}, \text{ by (18) and (21)} \qquad \text{... (23)}$$

are solutions of (1). For $n = 1, 2, 3, \ldots$, each one these satisfy the boundary conditions (2). Here E_n $(= A_n D_n)$ is another arbitrary constant.

Thus, (13) and (23) constitute a set of infinite solutions of (1). To obtain a solution also satisfying the initial condition (3), we consider a linear combination of these solutions. Hence the complete solution of (1) may be taken in the following form.

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} u_n(x,t) \qquad \text{or} \qquad u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \text{... (24)}$$

Substituting $t = 0$ in (24) and using (3), we have $$f(x) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a}, \qquad \text{... (25)}$$

which is Fourier cosine series. So the constants E_0 and E_n are given by

$$E_0 = \frac{2}{a}\int_0^a f(x)dx \qquad \text{and} \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, \; n = 1, 2, 3. \ldots \qquad \text{... (26)}$$

Hence (24) is the required solution where E_0 and E_n are given by (26)

2.4 B. Working rule for solving heat equation when both the ends of a bar of length a are insulated and the initial temperature $f(x)$ is prescribed.

Step I. Proceed as in Art. 2.4 A and prove that the solution of the heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditions $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)
and the initial condition $u(x, 0) = f(x)$, for all x ... (3)

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \text{... (4)}$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx, n = 1, 2, 3\ldots \qquad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (6)}$$

Step II. Compare the given problem with (1), (2) and (3) and find particular values of k, a and $f(x)$.

Step III. Substitute the particular values of k a and $f(x)$ in (5) and (6) and calculate E_0, E_n and C_n^2.

Step IV. Substitute the values of coefficients E_0, E_n and C_n^2 obtained is step III in (4) to arrive at the desired solution of the given boundary value problem.

2.4 C. Solved examples based on Art. 2.4 A and Art 2.4 B

Ex. 1 (*a*) *Solve* $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ *for* $0 < x < \pi, t > a$, if $u_x(0,t) = u_x(\pi,t) = 0$ and $u(x, 0) = \sin x$. **[I.A.S. 2002]**

(*b*) *Find the temperature in a laterally insulated bar of length a whose ends are insulated assuming that the initial temperature is* $f(x) = \begin{cases} x, \text{ if } a < x < a/2 \\ a - x, \text{ if } a/2 < x < a \end{cases}$

Sol. We can prove that (prove is examination for complete solution) that the solution of heat equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \ldots (1)$$

subject to the boundary conditions $u_x(0, t) = u_x(a, t) = 0$ for all t ... (2)

and the initial condition $u(x, t) = f(x),\ 0 < x < a$ ... (3)

is given by

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \ldots (4)$$

where

$$E_0 = \frac{2}{a}\int_0^a f(x)dx; \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx,\ n = 1, 2, 3, \ldots \qquad \ldots (5)$$

and

$$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \ldots (6)$$

(*a*) Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k,\ a = \pi$ and $f(x) = \sin x$. Hence, from (5), we get

$$E_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}\times 2 = \frac{4}{\pi} \qquad \ldots (7)$$

and

$$E_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} \{\sin(n+1) - \sin(n-1)x\}dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

Thus,

$$E_n = \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right] \qquad \ldots (8)$$

If $n = 2m - 1$ (odd) with $m = 1, 2, 3, \ldots$, then $E_n = 0$, by (8)

If $n = 2m$ (even) with $m = 1, 2, 3, \ldots$, then from (8), we get

$$E_{2m} = \frac{2}{m}\left(\frac{1}{2m+1} - \frac{1}{2m-1}\right) = -\frac{4}{\pi(4m^2-1)},\ m = 1, 2, 3 \qquad \ldots (9)$$

Also, from (6),

$$C_{2m}^2 = \left\{(2m)^2 \times \pi^2 \times k\right\}/\pi^2 = 4m^2\pi^2$$

Substituting the values of E_0 and E_{2m} given by (7) and (9) in (4), the required solution is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum\frac{\cos 2mx}{4m^2-1}e^{-4m^2kt}$$

(*b*) Comparing the given boundary value problem with boundary value problem given by (1), (2) and (3), we have $k = k,\ a = a$ and

$$u(x,0) = f(x) = \begin{cases} x, & \text{if } 0 < x < a/2 \\ a-x, & \text{if } a/2 < x < a \end{cases} \qquad \dots (7)$$

From (5) and (7), we have

$$E_0 = \frac{2}{a}\int_0^a f(x)dx = \frac{2}{a}\left[\int_0^{a/2} f(x)dx + \int_{a/2}^a f(x)\,dx\right]$$

$$= \frac{2}{a}\left[\int_0^{a/2} x\,dx + \int_{a/2}^a (a-x)dx\right] = \frac{2}{a}\left\{\left[\frac{x^2}{2}\right]_0^{a/2} + \left[ax - \frac{x^2}{2}\right]_{a/2}^a\right\}$$

$$= (2/a)\times\left\{a^2/8 + (a^2/2 - 3a^2/8\right\} = (2/a)\times(a^2/4) = a/2$$

$$E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx = \frac{2}{a}\left[\int_0^{a/2} f(x)\cos\frac{n\pi x}{a}dx + \int_{a/2}^a f(x)\cos\frac{n\pi x}{a}dx\right]$$

$$= \frac{2}{a}\int_0^{a/2} x\cos\frac{n\pi x}{a}dx + \frac{2}{a}\int_{a/2}^a (a-x)\cos\frac{n\pi x}{a}dx, \text{ using (7)}$$

$$= \frac{2}{a}\left[(x)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2)}\right)\right]_0^{a/2} + \frac{2}{a}\left[(a-x)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (-1)\left(-\frac{\cos(n\pi x/a)}{(n^2\pi^2/a^2}\right)\right]_{a/2}^a$$

[Using chain rule of integrating by parts]

$$= \frac{2}{a}\left[\frac{a}{2}\times\frac{a}{n\pi}\sin\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{a^2}{n^2\pi^2}\right] + \frac{2}{a}\left[-\frac{a^2}{n^2\pi^2}\cos n\pi - \frac{a}{2}\times\frac{a}{n\pi}\sin\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{a}\left[\frac{2a^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{a^2}{n^2\pi^2} - \frac{a^2}{n^2\pi^2}\cos n\pi\right] = \frac{2a}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

Also, from (6), we have $\qquad C_n^2 = (n^2\pi^2 k)/a^2$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t) = \frac{a}{4} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)\cos\frac{n\pi x}{a}e^{-(n^2\pi^2 kt)/a^2}$$

or

$$u(x,t) = \frac{a}{4} - \frac{8a}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{a}e^{-(4\pi^2 kt)/a^2} + \frac{1}{6^2}\cos\frac{6\pi x}{a}e^{-(36\pi kt/a^2} + \dots\right)$$

Ex. 2. *Find the solution of the one-dimensional diffusion equation $k(\partial^2 u/\partial x^2) = \partial u/\partial t$ satisfying the following boundary conditions : (i) u is bounded as $t \to \infty$ (ii) $u_x(0, t) = 0$, $u_x(a, t) = 0$ for all t (iii) $u(x, 0) = x(a-x)$, $0 < x < a$.*

Sol. We know that the bounded solution the diffusion equation

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \dots (1)$$

subject to the boundary conditions $\qquad u_x(0, t) = u_x(a, t) = 0$ for all $t \qquad \dots (2)$

and the initial condition $$u(x, 0) = f(x),\ 0 < x < a \quad \text{... (3)}$$

is given by $$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{a} e^{-C_n^2 t} \quad \text{... (4)}$$

where $$E_0 = \frac{2}{a}\int_0^a f(x)dx, \qquad E_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx,\ n = 1, 2, 3, \ldots \quad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \quad \text{... (6)}$$

Comparing the given boundary value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = a$ and $f(x) = ax - x^2$. So from (5), we have

$$E_0 = \frac{2}{a}\int_0^a (ax - x^2)dx = \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^2}{3}$$

$$E_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\left[(ax - x^2)\frac{\sin(n\pi x/a)}{(n\pi/a)} - (a-2x)\left(-\frac{\cos(n\pi x/a)}{n^2\pi^2/a^2}\right) + (-2)\left(-\frac{\sin(n\pi x/a)}{n^3\pi^3/a^3}\right)\right]_0^a$$

[Using chain rule of integrating by parts]

$$= \frac{2}{a}\left[-a\times\frac{a^2}{n^2\pi^2}(-1)^n - a\times\frac{a^2}{n^2\pi^2}\right] = -\frac{2a^2}{n^2\pi^2}\left\{1 + (-1)^n\right\}$$

Hence, if $n = 2m$ (even), then $E_n = E_{2m} = -(a^2/m^2\pi^2)$

and if $n = 2m-1$ (odd), then $E_n = E_{2m-1} = 0$.

Also, from (6), $C_n^2 = (n^2\pi^2 k)/a^2 = (4m^2\pi^2 k)/a^2$, if $n = 2m$

Substituting the above values of E_0, E_n and C_n^2 in (4), the required solution is given by

$$u(x,t) = \frac{a^2}{6} + \sum_{m=1}^{\infty}\left(-\frac{a^2}{m^2\pi^2}\right)\cos\frac{2m\pi x}{a} e^{-(4m^2\pi^2 kt)/a^2}$$

or $$u(x,t) = \frac{a^2}{6} - \frac{a^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2m\pi x}{a} e^{-(4m^2\pi^2 kt)/a^2}$$

Solution of heat equation when one end is insulated :-

While finding the solution of heat equation when one end is insulated while the other end is kept at a constant temperature, we proceed as explained in the following example.

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Example: *Obtain temperature distribution y (x, t) in a uniform bar of unit length whose one end is kept at 10° C and the other end is insulated. Further it is given that $y(x, 0) = 1 - x$, $0 < x < 1$.*

Sol. Suppose the bar be placed along the x-axis with its one end (which is at 10°C) at origin and the other end at $x = 1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity). Then we are to solve heat equation

$$\partial y/\partial t = k(\partial^2 y/\partial x^2) \quad \text{... (1)}$$

with boundary conditions $y_x(1, t) = 0, \quad y(0, t) = 10$ ... (2)

and initial conditions $y(x, 0) = 1 - x, \; 0 < x < 1.$ ... (3)

Let $y(x, t) = u(x, t) + 10$ ... (4)

i.e., $u(x, t) = y(x, t) - 10$ ... (5)

Using (4) or (5), (1), (2) and (3) reduce to $\partial u/\partial t = k(\partial^2 u/\partial x^2)$ ... (6)

$u_x(1, t) = 0, \qquad u(0, t) = 0$ ... (7)

and $u(x, 0) = y(x, 0) - 10 = -(x + 9).$ ... (8)

Suppose that (6) has solutions of the form $u(x, t) = X(x)\, T(t)$ ... (9)

Substituting this value of u in (6), we get

$XT' = kX''T$ or $X''/X = T'/kT.$ ... (10)

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (10) gives $X'' - \mu X = 0$ ... (11)

and $T' = \mu kT.$ ... (12)

Using (7), (9) gives $X'(1)T(t) = 0$ and $X(0)\, T(t) = 0$ ... (13)

Since T(t) = 0 leads to $u \equiv 0$, so we suppose that $T(t) \neq 0$. Then, from (13), we get

$X'(1) = 0$ and $X(0) = 0$ ... (14)

We now solve (11) under B.C. (14). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (11) is $X(x) = Ax + B$ ... (15)

From (15), $X'(x) = A.$ ... (15)′

Using B.C. (14), (15) and (15)′ give $0 = A$ and $0 = B.$

So from (15), $X(x) \equiv 0$, which leads to $u \equiv 0$. So reject $\mu = 0$.

Case II. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (11) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ ... (16)

so that $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ... (16)′

Using B.C. (14), (16) and (16)′ give $0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda}$ and $0 = A + B.$

These give $A = B = 0$ so that $X(x) \equiv 0$ and hence $u(x) = 0$ and hence $u(x) \equiv 0$.

So we reject $\mu = \lambda^2$.

Case III. Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (12) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{...(17)}$$

so that $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ...(17)′

Using B.C. (14), (17) and (17)′ give $0 = -A\lambda\sin\lambda + B\lambda\cos\lambda$ and $0 = A$

These give $A = 0$ and $\cos\lambda = 0,$ ... (18)

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$.

Now, $\cos\lambda = 0 \;\Rightarrow\; \lambda = (2n-1)\times(\pi/2) = (1/2)\times(2n-1)\pi \quad n = 1, 2, 3, \ldots$

So, $\mu = -\lambda^2 = -(1/4)\times(2n-1)^2\pi^2.$ ... (19)

Hence non-zero solutions $X_n(x)$ of (17) are given by $\quad X_n(x) = B_n \sin\{(2n-1)\pi x/2\}$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4}T \qquad \text{or} \qquad \frac{dT}{T} = -C_n^2\,dt \qquad \text{... (20)}$$

where
$$C_n^2 = (1/4)\times(2n-1)^2\pi^2 k \qquad \text{... (21)}$$

Solving (20),
$$T_n(t) = D_n e^{-C_n^2 t} \qquad \text{... (22)}$$

Thus,
$$u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$$

are solutions of (6), satisfying (7). Here E_n ($= B_n D_n$) is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \qquad \text{... (23)}$$

Putting $t = 0$ in (23) and using (8), we have $\quad -(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \qquad \text{... (24)}$

Multiplying both sides of (24) by $\sin\{(2m-1)\pi x/2\}$ and then integrating with respect to x from 0 to 1, we get

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx \qquad \text{... (25)}$$

But
$$\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \qquad \text{... (26)}$$

[Students are advised to prove result (26) in examination for complete solution]

Using (26), (25) gives
$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$$

$\therefore$
$$E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}\,dx$$

or
$$E_n = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{(2n-1)\pi/2}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right\}\right]_0^1$$

[on using chain rule of integration by parts]

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \qquad \left\{\begin{array}{l}\because \quad \cos\{(2n-1)\pi/2\} = 0 \\ \text{and } \sin\{(2n-1)\pi/2\} = (-1)^{n-1}\end{array}\right\} \qquad \text{... (27)}$$

Using (23) and (4), the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}.$$

where C_n and E_n are given by (21) and (27) respectively.

2.6. Miscellaneous examples on solution of heat or diffusion equation.

Ex. 1. (*a*) *An insulated rod of length l has its ends A and B kept at $a°$ celsius and $b°$ celsius respectively until steady state conditions prevail. The temperature at each end is suddenly reduced to zero degree celsius and kept so. Find the resulting temperature at any point of the rod taking the end A as origin.*

(*b*) *A rod 30 cm long has its ends A and B kept at 20° and 80° respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0° and kept so. Find the resulting function u(x, t) taking x = 0 at A.*

(*c*) *The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.*

Sol. (*a*) The temperature function $u(x, t)$ is a solution of heat equation

$$\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t). \qquad \text{... (i)}$$

When the steady state condition prevails, $\partial u/\partial t = 0$ and therefore (*i*) reduces to ordinary differential equation $d^2u/dx^2 = 0$ ... (*ii*)

Integrating (*ii*), $du/dx = c_1$, ... (*iii*)

Integrating (iii), $u(x) = c_1 x + c_2$, ... (*iv*)

where c_1 and c_2 and arbitrary constants. Putting $x = 0$ and $x = l$ in (*iv*) and using the fact that $u = a$ when $x = 0$ and $u = b$ when $x = l$, we have

$a = (c_1 \times 0) + c_2$ and $b = c_1 l + c_2$ so that $c_1 = (b - a)/l$ and $c_2 = a$.

So, by (*iv*), here $u(x, 0) = f(x) = \{(b - a)x\}/l + a$. ... (v)

Also, given boundary conditions are: $u(0, t) = u(l, t) = 0$ for all $t \geq 0$.

We can prove that (actually prove in examination as in Art. 2.4 B) that the solution of the

heat equation $$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (1)}$$

subject to the boundary conditons $u(0, t) = u(a, t) = 0$ for all $t > 0$ ... (2)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (3)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} e^{-C_n^2 t} \qquad \text{... (4)}$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx,\ n = 1, 2, 3, \qquad \text{... (5)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{...(6)}$$

Comparing the given boundry value problem with the boundary value problem given by (1), (2) and (3), we have $k = k$, $a = l$ and $f(x) = a + \{(b - a)x\}/l$. Hence, from (5), we have

$$E_n = \frac{2}{l}\int_0^l \left\{\frac{(b-a)x}{l} + a\right\}\sin\frac{n\pi x}{l}dx = \left[\frac{2}{l}\left\{\frac{(b-a)x}{l} + a\right\}\left(\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right) - \frac{2(b-a)}{l^2}\left(\frac{-\sin(n\pi x)/l}{(n\pi/l)^2}\right)\right]_0^l$$

[Using chain rule of integration by parts]

$$= -(2b/n\pi)\cos n\pi + (2a/n\pi) = (2/n\pi)\{a - b(-1)^n\}$$

Also, from (6), $$C_n^2 = (n^2\pi^2 k)/l^2$$

So, by (4), $$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{a-b(-1)^n}{n}\sin\frac{n\pi x}{l}e^{-(n^2\pi^2 kt)/l^2}$$

(*b*) Proceed as in part (*a*) taking $l = 30$, $a = 20$ and $b = 80$.

Ans. $$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1-4(-1)^n}{n}\sin\frac{n\pi x}{30}e^{-(n^2\pi^2 kt)/900}$$

(*c*) Proceed as in part (*a*) taking $l = 50$, $a = 0$ and $b = 100$.

Ans. $$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{50}e^{-(n^2\pi^2 kt)/2500}$$

Ex. 2. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively untill steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

(*b*) *Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* **[Nagpur 1995]**

Sol. Part (*a*). The equation for conduction of heat is

$$\partial u/\partial t = k(\partial^2 u/\partial x^2) \qquad \text{... (1)}$$

Prior to temperature change at the end B when $t = 0$, the heat-flow was independent of time (steady state condition, for which $\partial u/\partial t = 0$). When u depends only on x, (1) reduces to

$d^2u/dx^2 = 0$ so that $u = c_1 x + c_2$ ... (2)

Given $u = 0$ for $x = 0$ and $u = 100$ for $x = l$,

$\therefore$ (2) gives, $0 = c_2$ and $100 = lc_1 + c_2$, giving $c_1 = 100/l$, $c_2 = 0$

$\therefore$ (2) becomes $u = (100/l)x$

Hence the boundary conditions for the subsequent flow are

$u(0, t) = 0$, for all t ... (3)

$u(l, t) = 0$, for all t ... (4)

and the initial condition is $u(x, 0) = (100/l)x$ ... (5)

We can prove that the solution of heat equation (refer Art. 2.4 B)

$$k(\partial^2 u/\partial x^2) = \partial u/\partial t \qquad \text{... (i)}$$

subject to the boundary conditions $u(0, t) = u(a, t) = 0$ for all t ... (*ii*)

and the initial condition $u(x, 0) = f(x)$, $0 < x < a$ ... (*iii*)

is given by $$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}e^{-C_n^2 t} \qquad \text{... (iv)}$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad \text{... (v)}$$

and $$C_n^2 = (n^2\pi^2 k)/a^2 \qquad \text{... (vi)}$$

Comparing the boundary value problem given by (1), (3), (4) and (5) with the boundary value problem given by (*i*), (*ii*) and (*iii*), we have $a = l$, $k = k$ and $f(x) = (100/l)x$. So, from (*v*) we get

$$E_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx = \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - \left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Also, from (*vi*) we have $$C_n^2 = (n^2x^2k)/l^2$$

$\therefore$ From (*iv*), the required solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-(kn^2\pi^2 t)/l^2}$$

Part (b). Here the initial condition is (5) as before. The new boundary conditions are

$$u(0, t) = 20, \quad \text{for all } t \qquad \text{... (7)}$$

and $$u(l, t) = 80, \quad \text{for all } t. \qquad \text{... (8)}$$

Note that in part (*a*), the two boundary values were both zero and the required solution was easily obtained. In the present part (*b*), the two boundary values are non-zero, so we modify the procedure as follows :

We split the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_1(x) + u_2(x, t) \qquad \text{... (9)}$$

where $u_1(x)$ is a solution of (1) involving x only and satisfying (7) and (8); $u_2(x, t)$ is then a function defined by (9). Hence $u_1(x)$ is a steady state solution of the form (2) and $u_2(x, t)$ may be treated as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 20$ for $x = 0$ and $u_1(x) = 80$ for $x = l$, (2) gives

$$u_1(x) = 20 + (60x/l) \qquad \text{... (10)}$$

Putting $x = 0$ in (9) and using (7), we get

$$u_2(0, t) = u(0, t) - u_1(0) = 20 - 20 = 0 \qquad \text{... (11)}$$

$\therefore$ Next, putting $x = l$ in (9) and using (8), we get

$$u_2(l, t) = u(l, t) - u_1(l) = 80 - 80 = 0 \qquad \text{... (12)}$$

Again, $$u_2(x, 0) = u(x, 0) - u_1(x) = (100x/l) - [(60x/l) + 20]$$

Thus, $$u_2(x, 0) = (40x/l) - 20 \qquad \text{... (13)}$$

Hence the boundary conditions and initial condition to the transient solution $u_2(x, t)$ are given by (11), (12) and (13).

So we now solve $$\partial u_2/\partial t = k(\partial^2 u_2/\partial x^2)$$

subject to boundary condition (11) and (12) and initial condition (13) as in Art. 2.3 B and obtain

$$u_2(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}e^{-(kn^2\pi^2 t/l^2)} \qquad \text{... (14)}$$

where $$E_n = \frac{2}{l}\int_0^l\left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}dx = -\frac{40}{n\pi}(1+\cos n\pi) = -\frac{40}{n\pi}[1+(-1)^n]$$

Thus, $$E_n = \begin{cases} 0, \text{ when } n \text{ is odd} \\ -80/n\pi, \text{ when } n \text{ is even}\end{cases}$$

Taking $n = 2m$, we have $$E_n = -(40/m\pi)$$

$\therefore$ From (14), we have $$u_2(x,t) = -\frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}e^{-(4km^2\pi^2 t/l^2)} \qquad \text{... (15)}$$

Combining (10) and (15), the required solution is

$$u(x,t) = 20 + \frac{40x}{l} - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}e^{-(4km^2\pi^2 t/l^2)}$$

EXERCISE 2(A)

1. Obtain a solution of one-dimensional diffusion equation $\partial^2\theta/\partial x^2 = (1/k)(\partial\theta/\partial t)$ in the form $$\theta(x,t) = \sum_{\lambda}(c_\lambda \cos\lambda x + d_\lambda \sin\lambda x)e^{-\lambda k t}$$

If the faces $x = 0$, $x = a$ of an infinite slab are maintained at zero temperature, and the initial distribution of temperature in the slab is described by the equation $\theta(x,0) = f(x)$, $0 \le x \le a$, using the above solution of the diffusion equation, show that the temperature at a subsequent time t is given by $$\theta(x,t) = \frac{2}{a}\sum_{n=1}^{\infty} e^{(-n^2\pi^2kt)/a^2} \sin\frac{n\pi x}{a}\int_0^a f(u)\sin\frac{n\pi u}{a}du.$$

2. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = l$ are kept at temperature zero and whose initial temperature is given by $$f(x) = \begin{cases} A, \text{ when } 0 < x < l/2 \\ 0, \text{ when } l/2 < x < l \end{cases}$$

Ans. $u(x,t) = \dfrac{4A}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin^2\dfrac{n\pi}{4}\sin\dfrac{n\pi x}{l}e^{-(n^2\pi^2kt)/l^2}$

3. Determine the solution of one-dimensional heat equation $\partial\theta/\partial t = a^2(\partial^2\theta/\partial x^2)$ under the boundary conditions $\theta(0,t) = 0, \theta(l,t) = 0$ when $t > 0$ and the initial condition $\theta(x,0) = x$ when $0 < x < l$, l being the length of the bar. **[Garhwal 2005, Meerut 1997]**

Hint. Refer Art. 2.3B. Here $k = a^2$, $u = \theta$, $a = l$ and $f(x) = x$. Then, as before
$$E_n = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}dx = \begin{cases} 2l/n\pi, n \text{ is odd} \\ -2l/n\pi, n \text{ is even} \end{cases}$$

4. Solve $\partial^2u/\partial x^2 = h^2(\partial u/\partial t)$ when $u(0, t) = u(l, t) = 0$ and $u(x, 0) = \sin(\pi x/l)$.

Ans. $u(x,t) = \sin(\pi x/l)e^{-\pi^2 t/h^2 l^2}$

5. Show that the solution of $\partial u/\partial t = k(\partial^2u/\partial x^2)$ subject to the conditions (*i*) u is not infinite for $t \to \infty$ (*ii*) $\partial u/dx = 0$ for $x = 0$ and $x = l$ (*iii*) $u = lx - x^2$ for $t = 0$, between $x = 0$ and $x = l$ is
$$u = \frac{1}{6}l^2 - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{l}{n^2}\cos\frac{n\pi x}{l}e^{-(4\pi^2n^2kt)/l l}$$

6. Solve $\partial u/dt = \alpha^2(\partial^2u/\partial x^2)$, given that (*i*) $u = 0$, when $x = 0$ and $x = l$ for all t, (*ii*) $u = 3\sin(\pi x/l)$ when $t = 0$ for all x, $0 < x < l$. **Ans.** $u(x,l) = 3\sin(\pi x/l)e^{-(\alpha^2\pi^2 t/l^2)}$

7. Find by the method of separation of variables the solution $u(x, t)$ of the boundary value problem $\partial U/\partial t = 3(\partial^2U/\partial x^2), t > 0, 0 < x < 2$, $U(0, t) = 0$, $U(2, t) = 0$, $t > 0$; $U(x, 0) = x$, $0 < x < 2$

Ans. $U(x,t) = \sum_{n=1}^{\infty}\dfrac{\sin(n\pi x/2)}{\sin(n\pi/2)}e^{-(3\pi^2n^2t)/4}$ **[Rohilkhand 1997]**

8. A bar of length unity, has its end at $x = 0$ insulated and its end at $x = 1$ is kept at temperature zero. Find an expression for the temperature $u(x, t)$ if
$$u(x,0) = \begin{cases} 1, \text{ where } 0 \le x \le 1/2 \\ 2(1-x) \quad , \text{when } 1/2 \le x \le 1 \end{cases}$$

Ans. $u = \dfrac{16}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos\{(2n-1)\pi x/2\}\cos\{(2n-1)\pi/4\}}{(2n-1)^2}e^{-\{(2n-1)^2\pi^2k^2t\}/4}$

9. The ends A and B of a rod 20 cm long have the temperatures at 30° and 80° until steady state prevails. The temperatures of the ends are changed to 40° and 60° respectively. Find the temperature distribution in the rod at time t. **[I.F.S 2003. Rajasthan 2006; I.A.S. 2005]**

$$\textbf{Ans. } u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n+1}{n}\sin\frac{n\pi x}{20}e^{-(c^2n^2\pi^2t)/400}$$

10. The temperature at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

$$\textbf{Ans. } u = 50 - \frac{400}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{50}e^{-(2n-1)^2\pi^2c^2t/2500}$$

11. Show that the solution of the equation $\partial^2\theta/\partial x^2 = \partial\theta/\partial t$ satisfying the conditions (i) $\theta \to 0$ as $t \to \infty$ (ii) $\theta = 0$ when $x = \pm a$ for all values of $t > 0$ (iii) $\theta = x$ when $t = 0$ and $-a < x < a$

is $$\theta = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{a}\exp\left(-\frac{n^2\pi^2 t}{a^2}\right), \text{ where } \exp a = e^a.$$

12. A rod of length l with insulated sides is initially at a uniform temperature U_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)e^{-(c^2\pi^2n^2t)/l^2}, \text{ where } b_n \text{ is given by } U_0 = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)$$

[Andhra 2003; Agra 2004; Kanpur 1997; Aligarh 2003]

Hint. For the required solution, take $\partial u/\partial t = c^2(\partial^2 u/\partial x^2)$ as heat equation

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1 A. Laplace's equation

We know that the two dimensional heat flow equation in steady state reduces to

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \qquad \text{... (1)}$$

which is known as *Laplace's equation in two dimensions.*

Likewise the three-dimensional heat flow equation in steady state reduces to

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0 \qquad \text{... (2)}$$

which is known as *Laplace's equation in three dimensions.*

Laplace's equation has wide applications in physics and engineering. The theory of its solution is known as the *potential theory* and its solutions are known as *potential functions* or *harmonic functions.* A Laplace equation is also known as *potential equation.*

If the problem involves rectangular boundaries, we prefer to take Laplace's equation in cartesian coordinates given by (1) and (2).

1 .B. Laplace's equation in plane polar coordinates

If the given boundary value problem involves circular boundaries, we would like to use Laplace's equation in polar coordinates (r, θ).

Transformation of Laplace's equation $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$ *into polar coordinates* (r, θ).

Sol. If (x, y) be the cartesian coordinates of the point P whose polar coordinates are (r, θ),

then $x = r\cos\theta$ and $y = r\sin\theta$. ... (1)

From (1), $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ ... (2)

From (2), $2r\,(\partial r/\partial x) = 2x$ and $2r\,(\partial r/\partial y) = 2y$

so that $\partial r/\partial x = x/r = \cos\theta$ and $\partial r/\partial y = y/r = \sin\theta$. ... (3)

Also, $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}. \qquad \text{... (4)}$$

and $$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y^2/x^2)}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad \text{... (5)}$$

Given that $$(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) = 0. \qquad \text{... (6)}$$

Now, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}, \text{ using (3) and (4)} \qquad \text{... (7)}$$

Then, (7) $\Rightarrow$ $$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \qquad \text{... (8)}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right), \text{ using (7) and (8)}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 v}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}\right)\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial v}{\partial r} + \cos\theta\frac{\partial^2 v}{\partial\theta\,\partial r} - \frac{1}{r}\left\{\cos\theta\frac{\partial v}{\partial\theta} + \sin\theta\frac{\partial^2 v}{\partial\theta^2}\right\}\right]$$

Thus, $$\frac{\partial^2 v}{\partial x^2} = \cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \qquad \text{... (9)}$$

Again, $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}, \text{ using (3) and (5)} \qquad \text{... (10)}$$

Then, (10) $\Rightarrow$ $$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}. \qquad \text{... (11)}$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$ using

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial v}{\partial r}+\frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial r}+\frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 v}{\partial r^2}+\cos\theta\left\{-\frac{1}{r^2}\frac{\partial v}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}\right\}\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial v}{\partial r}+\sin\theta\frac{\partial^2 v}{\partial\theta\,\partial r}+\frac{1}{r}\left\{-\sin\theta\frac{\partial v}{\partial\theta}+\cos\theta\frac{\partial^2 v}{\partial\theta^2}\right\}\right]$$

Thus, $$\frac{\partial^2 v}{\partial y^2}=\sin^2\theta\frac{\partial^2 v}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial v}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \quad \text{... (12)}$$

Adding (1) and (2), we obtain $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} \quad \text{... (13)}$$

Hence, using (6) the Laplace's equation in polar form is given by

$$\partial^2 v/\partial r^2+(1/r)\times(\partial v/\partial r)+(1/r^2)\times(\partial^2 u/\partial\theta^2)=0. \quad \text{... (14)}$$

1 C. Laplace's equation in cylindrical coordinates

If the given boundary value problem involves cylindrical boundaries, we would like to use Laplace's equation in cylindrical coordinates (r,θ,z)

Transformation of Laplace's equation $\partial^2 v/\partial x^2+\partial^2 v/\partial y^2+\partial^2 v/\partial z^2=0$ *in cylindrical coordinates* (r,θ,z).

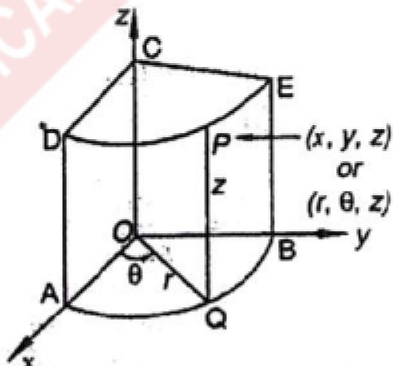

Fig. Cylindrical coordinates of $P(x, y, z)$ are $P(r, \theta, z)$

Sol. If (x, y, z) are the cartesian coordinates of the point P whose cylindrical coordinates are (r,θ,z), then we know that

$$x=r\cos\theta,\qquad y=r\sin\theta,\qquad z=z \quad \text{... (1)}$$

With $x=r\cos\theta$, $y=r\sin\theta$, proceed as in Art. 1.6B and prove that

$$\partial^2 v/\partial x^2+\partial^2 v/\partial y^2=\partial^2 v/\partial r^2+(1/r)\times(\partial v/\partial r)+(1/r^2)\times(\partial^2 v/\partial\theta^2) \quad \text{... (2)}$$

Adding $\partial^2 v/\partial z^2$ on both sides of (2), we have

$$\partial^2 v/\partial x^2+\partial^2 v/\partial y^2+\partial^2 v/\partial z^2=\partial^2 v/\partial r^2+(1/r)\times(\partial v/\partial r)+(1/r^2)\times(\partial^2 v/\partial\theta^2)+\partial^2 v/\partial z^2$$

Hence the Laplace's equation $\partial^2 v/\partial x^2+\partial^2 v/\partial y^2+\partial^2 v/\partial z^2=0$ reduces to

$$\partial^2 v/\partial r^2+(1/r)\times(\partial v/\partial r)+(1/r^2)\times(\partial^2 v/\partial\theta^2)+\partial^2 v/\partial z^2=0,$$

which is the required Laplace's equation is cylindrical coordinates.

1 .D. Laplace's equation in spherical coordinates

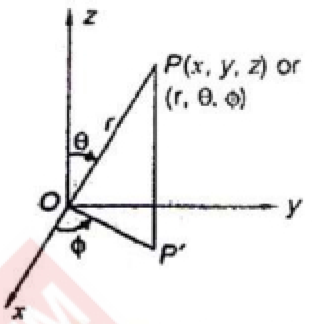

Fig: Shperical coordinates of $P(x, y, z)$ are $P(r, \theta, \phi)$

If the given boundary value problem involves spherical boundaries, we would like to use Laplace's equation in spherical coordinates (r, θ, ϕ)

Transformation of Laplaces equation

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2 = 0$$

in spherical coordinates (r, θ, ϕ)

Sol. If (x, y, z) are the carterial coordinates of the points P whose spherical polar coordinates are (r, θ, ϕ), then

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \quad \ldots (1)$$

$$\therefore \quad r^2 = x^2 + y^2 + z^2, \qquad \tan\theta = (x^2+y^2)^{1/2}/z, \qquad \tan\phi = y/x$$

$$\text{or} \quad r^2 = x^2 + y^2 + z^2, \qquad \theta = \tan^{-1}\left\{(x^2+y^2)^{1/2}/z\right\}, \qquad \phi = \tan^{-1}(y/x). \quad \ldots (2)$$

From (1) and (2), we have the following relations

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi, \qquad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta, \quad \ldots (3)$$

$$\frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}, \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}. \quad \ldots (4)$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \qquad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \qquad \frac{\partial \phi}{\partial z} = 0. \quad \ldots (5)$$

Now, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial x} + \dfrac{\partial v}{\partial \phi}\dfrac{\partial \phi}{\partial x} = \sin\theta\cos\phi\dfrac{\partial v}{\partial r} + \dfrac{\cos\theta\cos\phi}{r}\dfrac{\partial v}{\partial \theta} - \dfrac{\sin\phi}{r\sin\theta}\dfrac{\partial v}{\partial \phi}$, by (3), (4), (5)

$$\therefore \quad \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\sin\theta\cos\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial v}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial v}{\partial \phi}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 v}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial v}{\partial \theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\partial\phi}$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial \phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi}$$

$$+ \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial \phi} + \frac{\sin^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial v}{\partial \theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial \phi}. \quad \ldots (6)$$

Again, $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial y} + \dfrac{\partial v}{\partial \phi}\dfrac{\partial \phi}{\partial y} = \sin\theta\sin\phi\dfrac{\partial v}{\partial r} + \dfrac{\cos\theta\sin\phi}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{\cos\phi}{r\sin\theta}\dfrac{\partial v}{\partial \phi}$, by (3), (4) and (5)

$$\therefore \quad \frac{\partial}{\partial y}=\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$=\left(\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin\theta\sin\phi\frac{\partial v}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial v}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$=\sin^2\theta\sin^2\phi\frac{\partial^2 v}{\partial r^2}+\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 v}{\partial r\partial\theta}-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial v}{\partial\theta}+\frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\partial\phi}$$

$$-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial\phi}+\frac{\cos^2\sin^2\phi}{r}\frac{\partial v}{\partial r}+\frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\partial\phi}$$

$$-\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}+\frac{\cos^2\phi}{r}\frac{\partial v}{\partial r}+\frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}+\frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}. \quad \ldots (7)$$

Finally, $\dfrac{\partial v}{\partial z}=\dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial v}{\partial\theta}\dfrac{\partial\theta}{\partial z}+\dfrac{\partial v}{\partial\phi}\dfrac{\partial\phi}{\partial z}=\cos\theta\dfrac{\partial v}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial\theta}$, by (3), (4), (5)

Thus
$$\frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 v}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial v}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \quad \ldots (8)$$

Adding (6), (7) and (8), we see that $(\partial^2 v/\partial x^2)+(\partial^2 v/\partial y^2)+(\partial^2 v/\partial z^2)=0$ is transformed to

$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}=0,$$

which is the required Laplace's differential equation in spherical polar coordinates.

PART II: PROBLEMS BASED ON TWO-DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

2.5.8

2. A. General solution of the two dimensional heat (or diffusion) equation

(Delhi Maths (H) 1995; Nagpur 1995, 97; Pune 1998; Rajasthan 2001, 03)

Sol. Two-dimensional heat (or diffusion) equation is

$$\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=(1/k)\times(\partial u/\partial t) \quad \ldots (1)$$

Suppose that (1) has solutions of the form $u(x, y, t)=X(x)\,Y(y)\,T(t)$. ... (2)

Substituting this value of u in (1), we have

$$X''YT+XY''T=\frac{1}{k}XYT' \qquad \text{or} \qquad \frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{k}\frac{T'}{T} \quad \ldots (3)$$

Since x, y and t are independent variables, (3) is true if each term on each side is a constant, such that

$$\frac{X''}{X}=-n^2, \qquad \frac{Y''}{Y}=-m^2 \qquad \text{and} \qquad \frac{T'}{kT}=-p^2 \quad \ldots (4)$$

with
$$n^2+m^2=p^2 \quad \ldots (4)$$

We have chosen constants in (4) in such a way so that the solution u has the property that $u\to 0$ as $t\to\infty$. This is generally satisfied due to physical conditions of a real problem.

Solving differential equation given by (4), we have

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$$X_n(x) = A_n \cos nx + B_n \sin nx, \qquad Y_m(y) = C_m \cos my + D_m \sin my$$

and $\quad T_p(t) = E_p e^{-p^2 kt} = F_{nm} e^{-(n^2+m^2)kt}$, using (4)′

Hence a suitable solution of (1) is of the form

$$\therefore \quad u_{nm}(x, y, t) = F_{nm}(A_n \cos nx + B_n \sin nx)(C_m \cos my + D_m \sin my) e^{-(n^2+m^2)kt}$$

2.6

2. B. General solution of two-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditions

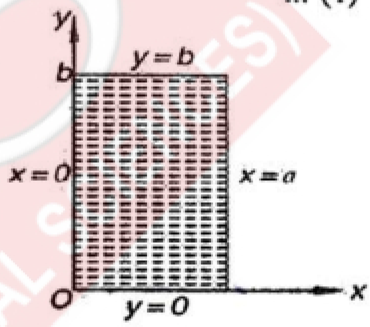
A thin rectangular plate whose surface is impervious to heat flow has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. Its four edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature. Determine the temperature at a point of the plate as t increases.

(Delhi Maths Hons. 2000, 01, Meerut 2001, 03, 06)

Sol. Here the temperature $u(x, y, t)$ in the plate is governed by the two-dimensional heat equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k) \times (\partial u/\partial t) \qquad \dots (1)$$

Since the edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature, so the boundary conditions are



$$u(0, y, t) = 0 \qquad \dots 2(a)$$
$$u(a, y, t) = 0 \qquad \dots 2(b)$$
$$u(x, 0, t) = 0 \qquad \dots 2(c)$$
$$u(x, b, t) = 0 \qquad \dots 2(d)$$

The initial condition is $\quad u(x, y, 0) = f(x, y) \qquad \dots (3)$

Suppose that (1) has solutions of the form

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \qquad \dots (4)$$

Substituting this in (1), we have

$$X''YT + XY''T = \frac{1}{k} XYT \qquad \text{or} \qquad \frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{kT} \qquad \dots (5)$$

where the dashes denote derivatives with respect to relevant variable. Since x, y and t are independent variables, (5) can only be true if each term on each side is equal to a constant.

Let $\quad X''/X = \mu_1 \quad$ so that $\quad X'' - \mu_1 X = 0 \qquad \dots (6)$

Using 2(*a*) and 2(*b*), (4) gives

$$X(0)\, Y(y)\, T(t) = 0 \qquad \text{and} \qquad X(a)\, Y(y)\, T(t) = 0 \qquad \dots (7)$$

Since $Y \equiv 0$ or $T \equiv 0$ leads to $u \equiv 0$, so we suppose that $Y \neq 0$ and $T \neq 0$. Then (7) gives

$$X(0) = 0 \qquad \text{and} \qquad X(a) = 0 \qquad \dots (8)$$

We now solve (6) satisfiying boundary conditions (8). Three cases *arise*:

Case I. Let $\mu_1 = 0$. Then solution of (6) is $\quad X(x) = Ax + B \qquad \dots (9)$

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ and hence $X(x) \equiv 0$. This lead to $u \equiv 0$, which does not satisfy (3). So we reject $\mu_1 = 0$.

Case II. Let $\mu_1 = \lambda_1^2, \lambda_1 \neq 0$. Then solution (6) is $\quad X(x) = Ae^{\lambda_1 x} + Be^{-\lambda_1 x} \qquad \dots (10)$

Using B.C. (8), (10) gives

$$0 = A + B \qquad \text{and} \qquad 0 = Ae^{a\lambda_1} + Be^{-a\lambda_1} \qquad \dots (11)$$

Solving (11), $A = B = 0$ so that $X(x) \equiv 0$. This lead to $u \equiv 0$, which does not satisfy (3). So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2, \lambda_1 \neq 0$. Then solution (6) is $X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x$. ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda_1 a + B\sin\lambda_1 a$.

Hence $A = 0$ and $\sin\lambda_1 a = 0$,

where we have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Now, $\sin\lambda_1 a = 0$ gives $\lambda_1 a = m\pi$, $m = 1, 2, 3, \ldots$

$$\therefore \quad \lambda_1 = m\pi/a, \quad m = 1, 2, 3 \ldots \qquad \ldots (13)$$

Hence non-zero solutions $X_m(x)$ of (6) are given by

$$X_m(x) = B_m \sin(m\pi x/a), \quad m = 1, 2, 3. \qquad \ldots (14)$$

Next, let $Y''/Y = \mu_2$ so that $Y'' - \mu_2 Y = 0$. ... (15)

Using 2(*c*) and 2(*d*), (4) gives as before $Y(0) = 0$ and $Y(b) = 0$. ... (16)

Solving (15) under B.C. (16) as before, we get

$$Y_n(y) = D_n \sin(n\pi y/b), \quad n = 1, 2, 3, \ldots \qquad \ldots (17)$$

where $\mu_2 = -\lambda_2^2$ and $\lambda_2 = n\pi/b$, $n = 1, 2, 3, \ldots$ ... (18)

In view of (6) and (15), (5) reduces to

$$T'/kT = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2(m^2/a^2 + n^2/b^2)$$

or $T' = -\lambda_{mn}^2 T$ or $(1/T)\,dT = -\lambda_{mn}^2\,dt$ ... (19)

where $\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2)$, $m = 1, 2, 3, \ldots;\ n = 1, 2, 3, \ldots$ ... (20)

Solving (19), $T_{mn}(t) = E_{mn} e^{-\lambda_{mn}^2 t}$, $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$,

$$\therefore \quad u_{mn}(x, y, t) = X_m(x) Y_n(y) T_{mn}(t) = F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}$$

are solutions, of (1), satisfying 2(*a*), 2(*b*), 2(*c*) and 2 (*d*). Here F_{mn} (= $B_m D_n$) is another arbitrary constant. In order to obtain a soluton also satisfying (3), we consider more general solution

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x, y, t) \quad i.e., \quad u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t} \qquad \ldots (21)$$

Putting $t = 0$ in (21) and using (3), we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \ldots (22)$$

which is double Fourier sine series. Accordingly, we get

$$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,dx\,dy \qquad \ldots (23)$$

Hence (21) is the required solution where F_{mn} is given by (23).

2.7 C. Working rule for solving two-dimesnional heat or diffusion equation

Step I : Proceed as in Art. 2.7 B and show that solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k) \times (\partial u/\partial t) \qquad \ldots (i)$$

subject to the boundary conditions

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \qquad (ii)$$

and the initial condition $$u(x, y, 0) = f(x, y) \qquad \text{... (iii)}$$

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t} \qquad \text{... (iv)}$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \qquad \text{... (v)}$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \quad m = 1, 2, 3, ..., \quad n = 1, 2, 3, \qquad \text{... (vi)}$$

Step II. Compare the given problem with (*i*) (*ii*) and (*iii*) and find particular values of k, a, b and $f(x, y)$.

Step III. Substitute the particular values of a, b and $f(x, y)$ in (*v*) and (*vi*) to compute F_{mn} and λ_{mn}^2.

Step IV. Substitute the values of F_{mn} and λ_{mn}^2 in (*iv*) to arrive at the desired solution of the given boundary value problem.

2.7 D Solved example based on Art. 2.7 A, Art. 2.7B and Art 2.7C

Ex. 1. *The four edges of a thin square plate of area π^2 are kept at temperature zero and the faces are perfectly insulated. The initial temperature is assumed to be $u(x, y, 0) = xy(\pi - x)(\pi - y)$. By applying the method of separating variables to the two diminsional heat equation $u_t = c^2\nabla^2 u$, determine the temperature $u(x, y, t)$ in the plate, where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$*

Sol. As is Art. 2.7 B, first prove that the solution of two dimensional heat equation

$$k\nabla^2 u = u_t, \quad i.e., \quad k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \partial u/\partial t \qquad \text{... (i)}$$

subject to the boundary conditions $$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \qquad \text{... (ii)}$$

and the initial condition $$u(x, y, 0) = f(x, y) \qquad \text{... (iii)}$$

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-\lambda_{mn}^2 t}, \qquad \text{... (iv)}$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \qquad \text{... (v)}$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \quad m = 1, 2, 3, ...; \quad n = 1, 2, 3 ... \qquad \text{... (vi)}$$

Comparing the given boundary value problem with the boundary value problem given by (*i*), (*ii*) and (*iii*), we have $k = c^2$, $a = b = \pi$ and $f(x, y) = xy(\pi - x)(\pi - y)$.

$\therefore$ From (*vi*), $$\lambda_{mn}^2 = \pi^2 c^2 (m^2/\pi^2 + n^2/\pi^2) = c^2(m^2 + n^2) \qquad \text{... (vii)}$$

$\therefore$ (*iv*) gives $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin mx \sin xy\, e^{-c^2(m^2+n^2)t} \qquad \text{... (viii)}$$

From (*v*), $$F_{mn} = \frac{4}{\pi^2}\int_{x=0}^{\pi}\int_{y=0}^{\pi} xy(\pi - x)(\pi - y) \sin mx \sin ny\, dx\, dy$$

$$= \frac{4}{\pi^2}\left[\int_0^{\pi}(\pi x - x^2)\sin mx\, dx\right]\times\left[\int_0^{\pi}(\pi y - y^2)\sin ny\, dy\right] \qquad \text{... (ix)}$$

Using the chain rule of integrating by parts, we get

$$\int_0^{\pi}(\pi x - x^2)\sin mx\,dx = \left[(\pi x - x^2)\left(-\frac{\cos mx}{m}\right) - (\pi - 2x)\left(-\frac{\sin mx}{m^2}\right) + (-2)\left(\frac{\cos mx}{m^3}\right)\right]_0^{\pi} = \frac{2}{m^3}[1-(-1)^m].$$

Similarly, $$\int_0^{\pi}(\pi y - y^2)\sin ny\,dy = \frac{2}{n^3}[1-(-1)^n].$$

$\therefore$ $$F_{mn} = \frac{16}{\pi^2 m^3 n^3}[1-(-1)^m][1-(-1)^n], \text{ using } (ix)$$

or $$F_{mn} = \begin{cases} 0, \text{ when } & m = 2p \text{ or } n = 2q \\ \dfrac{64}{\pi^2(2p-1)^3(2q-1)^3}, & \text{when } m = 2p-1, n = 2q-1 \end{cases} \qquad \ldots (x)$$

Using (*viii*) and (*x*), the desired solution is given by

$$u(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} F_{pq}\sin[(2p-1)x]\sin[(2q-1)y]e^{-c^2\{(2p-1)^2+(2q-1)^2\}t}$$

where $$F_{pq} = 64/\pi^2(2p-1)^3(2q-1)^3.$$

Ex. 2. *Find temperature distribution inside a square plate of side a having boundary condition $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, a, t) = 0$ and initial condition*

$$u(x,y,0) = \cos\{\pi(x-y)/a\} - \cos\{\pi(x+y)/a\}$$ **[Delhi B.Sc. (Hons.) Physics 1998]**

Sol. As in Art. 2.7 B, first prove that the solution of two-dimensional heat equation

$$k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \partial u/\partial t \qquad \ldots (i)$$

subject to the boundary conditions $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ ... (*ii*)

and the initial condition $u(x, y, a) = f(x, y)$ ... (*iii*)

is given by $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-\lambda_{mn}^2 t} \qquad \ldots (iv)$$

where $$F_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \qquad \ldots (v)$$

and $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/b^2), \quad m = 1, 2, 3, \ldots, \quad n = 1, 2, 3, \ldots \qquad \ldots (vi)$$

Comparing the given boundary value problem with the boundary value problm given by (*i*), (*ii*) and (*iii*), we have $b = a$ and $f(x,y) = \cos\{\pi(x-y)a\} - \cos\{\pi(x+y)/a\}$

$\therefore$ From (*vi*), $$\lambda_{mn}^2 = \pi^2 k(m^2/a^2 + n^2/a^2) = \{\pi^2 k(m^2+n^2)\}/a^2 \qquad \ldots (vii)$$

and from (*iv*), $$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}e^{-\lambda_{mn}^2 t} \qquad \ldots (viii)$$

From (*v*), $$F_{mn} = \frac{4}{a^2}\int_{x=a}^{a}\int_{y=0}^{a}\left\{\cos\frac{\pi(x-y)}{a} - \cos\frac{\pi(x+y)}{a}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}dxdy$$

$$= \frac{4}{a^2}\int_{x=0}^{a}\int_{y=0}^{a}\left(2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy \qquad \ldots (ix)$$

or $$F_{mn} = \frac{2}{a^2}\left(\int_{x=0}^{a} 2\sin\frac{\pi x}{a}\sin\frac{m\pi x}{a}\,dx\right)\times\left(\int_{y=0}^{a} 2\sin\frac{\pi y}{a}\sin\frac{n\pi y}{a}\,dy\right)$$

$$= \frac{2}{a^2}\left(\int_{x=0}^{a}\left\{\cos\frac{(m-1)\pi x}{a} - \cos\frac{(m+1)\pi x}{a}\right\}dx\right)\times\left(\int_{y=0}^{a}\left\{\cos\frac{(n-1)\pi y}{a} - \frac{\cos(n+1)\pi y}{a}\right\}dy\right)$$

$$F_{mn} = \frac{2}{a^2}\times\left[\frac{\sin\{(m-1)\pi x/a\}}{(m-1)\pi/a} - \frac{\sin\{(m+1)\pi x/a\}}{(m+1)\pi/a}\right]_0^a \times\left[\frac{\sin\{(n-1)y\pi/a\}}{(n-1)\pi/a} - \frac{\sin\{(n+1)y\pi/a\}}{(n+1)\pi/a}\right]_0^a \quad ...(x)$$

When $m \neq 1$ and $n \neq 1$, then $\qquad (x) \Rightarrow F_{mn} = 0$

When $m = 1$ and $n = 1$, (*ix*) reduces to

$$F_{11} = \frac{8}{a^2}\int_{x=0}^{a}\int_{y=0}^{a}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a} = \frac{2}{a^2}\left(\int_{x=0}^{a} 2\sin^2\frac{\pi x}{a}\,dx\right)\times\left(\int_{y=0}^{a} 2\sin^2\frac{\pi y}{a}\,dy\right)$$

$$= \frac{2}{a^2}\left(\int_{x=0}^{a}\left(1-\cos\frac{2\pi x}{a}\right)dx\right)\times\left(\int_{y=0}^{a}\left(1-\cos\frac{2\pi y}{a}\right)dy\right) = \frac{2}{a^2}\left[x - \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_0^a \times\left[y - \frac{a}{2\pi}\sin\frac{2\pi y}{a}\right]_0^a$$

Hence, $\qquad F_{11} = (2/a^2) \times a \times a = 2$

Thus, $$F_{mn} = \begin{cases} 2, \text{ if } m = 1, n = 1 \\ 0, \text{ otherwise}\end{cases} \quad ...(xi)$$

from (*vii*), $$\lambda_{11}^2 = \pi^2 k(1/a^2 + 1/a^2) = (2\pi^2 k)/a^2 \quad ...(xii)$$

Using (*xi*) and (*xii*), (*viii*) yields the required solution

$$u(x,y,t) = F_{11}\sin(\pi x/a)\sin(\pi y/a)\,e^{-\lambda_{11}^2 t} \qquad i.e., \qquad u(x,y,t) = 2\sin(\pi x/a)\sin(\pi y/a)e^{-(2\pi^2 kt)/a^2}$$

Ex. 3. *Show that the two dimensional diffusion equation* $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = (1/k)(\partial\theta/\partial t)$

can be put in the form, $$\theta(x,y,t) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu}\cos(\lambda x + \epsilon_\lambda)\cos(\mu y + \epsilon_\mu)e^{-(\lambda^2+\mu^2)kt}$$

[Delhi Maths (Hons) 1994]

Sol. Suppose the given equation has solutions of the form

$$\theta(x,y,t) = X(x)Y(y)T(t) \quad ...(1)$$

Substitution this value of θ in the given equation, we get

$$X''YT + XY''T = (1/k)\times XYT' \quad \text{or} \quad X''/X + Y''/Y = (1/k)\times(T'/T) \quad ...(2)$$

Since x, y and t are independent variables, (2) is true if each term on each side is a constant, such that

$$X''/X = -\lambda^2, \qquad Y''/Y = -\mu^2 \qquad \text{and} \qquad T'/kT = -v^2, \quad ...(3)$$

where $$v^2 = \lambda^2 + \mu^2 \quad ...(4)$$

Solutions of equations in (3) can be put in the form

$$X(x) = \cos(\lambda x + \epsilon_\lambda), \qquad Y(y) = \cos(\mu y + \epsilon_\mu), \qquad T(t) = A_{\lambda\mu}e^{-(\lambda^2+\mu^2)kt}$$

$\therefore$ By the principle of superposition, the required general solution is

$$\theta(x,y,t) = \sum_{\lambda}\sum_{\mu} A_{\lambda\mu}\cos(\lambda x + \epsilon_\lambda)\cos(\mu y + \epsilon_\mu)e^{-(\lambda^2+\mu^2)kt}$$

EXERCISE 2(B)

Ex. 1. Solve the two dimensional diffusion equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k)(\partial u/\partial t)$, in the region $0 < x < a$, $0 < y < b$, $t > 0$ with the following boundary and initial conditions: $u(x, y, t) = 0$ and where C is the boundary of the rectangle defined by $0 \le x \le a$ and $0 \le y \le b$ and $u(x, y, 0) = f(x, y)$, $0 < x < a$, $0 < y < b$.

Sol. This is exactly the same as Art. 2.7B

Ex. 2. A rectangular plate bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$ has an initial distribution given by $V = A\sin(\pi x/a)\sin(\pi y/b)$. The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature at any point.

[Meerut 2000, Kurukshetra 2004]

Ans. $u(x,y,t) = A\sin(\pi x/a)\sin(\pi y/a)e^{-\pi^2 kt(1/a^2+1/b^2)}$

Ex. 3. Solve $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = \partial V/\partial t$, if $V = 0$ when $t = \infty$, $x = 0$ or l and $y = 0$ or l.

Ex. 4. A square plate with sides of unit length has its faces insulated and its sides kept at 0°C. If the initial temperature is specified, determine the subsequent temperature at any point of the plate.

PART III : PROBLEMS BASED ON THREE-DIMENSIONAL HEAT (OR DIFFUSION) EQUATION

2.8 A. Solution of three-dimensional heat (diffusion) equation. [Kanpur 2000]

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)(\partial u/\partial t) \quad \ldots (1)$$

Suppose that (1) has solutions of the form

$$u(x, y, z, t) = X(x)\, y(y)\, Z(z)\, T(t) \quad \ldots (2)$$

where X, Y, Z and T are respectively the functions of x, y, z and t alone. Substituting this value of u in (1), we have
$$X''YZT + XY''ZT + XYZ''T = (1/k)XYZT'$$

or
$$X''/X + Y''/Y + Z''/Z = (1/k)(T'/T) \quad \ldots (3)$$

Since x, y, z and t are independent variables, (3) is true only if each term on each side is a constant such that

$$X''/X = -n^2, \quad Y''/Y = -m^2, \quad Z''/Z = -l^2 \quad \text{and} \quad T'/kT = -p^2 \quad \ldots (4)$$

where
$$n^2 + m^2 + l^2 = p^2 \quad \ldots (5)$$

We have chosen constants in (4) in such a manner so that the solution $u(x, y, z, t)$ has the property that $u \to 0$ and $t \to \infty$. This is generally satisfied due to physical conditions of actual physical problem Solving equations in (4), we have

$$X_n(x) = A_n \cos nx + B_n \sin nx, \qquad Y_m(y) = C_m \cos my + D_m \sin my.$$

$$Z_l(z) = E_l \cos lz + F_l \sin lz, \quad \text{and} \quad T_p(t) = G_p e^{-p^2 kt} = H_{nml} e^{-(n^2+m^2+l^2)kt}$$

$$\therefore u_{nml}(x,y,z,t) = H_{nml}(A_n \cos nx + B_n \sin nx)(C_m \cos my + D_m \sin my)(E_l \cos lz + F_l \sin lz)e^{-(n^2+m^2+l^2)kt}$$

are solutions of (1). Hence the general solution of (1) is given by

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} u_{nml}(x,y,z,t).$$

2.8 B. Solution of three-dimensional heat (or diffusion) equation satisfying the given boundary and initial conditions.

Find the solution of the three dimensional diffusion equation in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$, $\partial^2 u/\partial u^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)(\partial u/\partial t)$ *with the boundary and initial conditions:* $u(0, y, z, t) = 0 = u(a, y, z, t)$; $u(x, 0, z, t) = 0 = u(x, b, z, t)$, $u(x, y, 0, t) = 0 = u(x, y, c, t)$ and $u(x, y, z, 0) = f(x, y, z)$.

Sol. Given $$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/k)(\partial u/\partial t) \quad \dots (1)$$

$$u(0, y, z, t) = 0 = u(a, y, z, t), \quad \text{for all } t > 0 \quad \dots (2)$$

$$u(x, 0, z, t) = 0 = u(x, b, z, t), \quad \text{for all } t > 0 \quad \dots (3)$$

$$u(x, y, 0, t) = 0 = u(x, y, c, t), \quad \text{for all } t > 0 \quad \dots (4)$$

and $$u(x, y, z, 0) = f(x, y, z), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c \quad \dots (5)$$

Let a solution of (1) be of the form $$u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t) \quad \dots (6)$$

Substituting this in (1), we have $$X''/X + Y''/Y + Z''/Z = (1/k)(T'/T) \quad \dots (7)$$

Since x, y, z and t are independent variables, (7) is true if each term on each side is a negative constant. [We have chosen negative constant because non-negative constant will give rise to trivial solution of (1)]. Thus, we have

$$X''/X = -\lambda_1^2, \quad Y''/Y = -\lambda_2^2, \quad Z''/Z = -\lambda_3^2 \quad \text{and} \quad T'/kT = -\lambda^2, \quad \dots (8)$$

where $$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \dots (9)$$

Solutions of (8) are given by $$X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x \quad \dots (10)$$

$$Y(y) = C\cos\lambda_2 y + D\sin\lambda_2 y \quad \dots (11)$$

$$Z(z) = E\cos\lambda_3 z + \sin\lambda_3 z \quad \dots (12)$$

and $$T(t) = Ge^{-\lambda^2 kt} \quad \dots (13)$$

Put $x = 0$ and $x = a$ by turn in (6). Using (2) and the fact that $Y(y) \neq 0, X(z) \neq 0, T(t) \neq 0$ for non-trivial solution of (1), we get

$$X(0) = 0 \quad \text{and} \quad X(a) = 0. \quad \dots (14)$$

Putting $x = 0$ and $x = a$ by turn in (10) and using (14), we get

$$0 = A \quad \text{and} \quad 0 = A\cos\lambda_1 a + B\sin\lambda_1 a$$

$$\Rightarrow \quad A = 0 \quad \text{and} \quad \sin\lambda_1 a = 0,$$

where we have taken $B \neq 0$, since otherwise $X(x) = 0$ and hence we have trivial solution $u = 0$ for (1)

Now $$\sin\lambda_1 a = 0 \quad \Rightarrow \quad \lambda_1 a = l\pi \quad \Rightarrow \quad \lambda_1 = l\pi/a, \quad l = 1, 2, 3, \dots \quad \dots (15)$$

and then from (10), $$X_l(x) = B_l \sin(l\pi x/a), \quad n = 1, 2, 3, \dots \quad \dots (16)$$

Similarly, boundary condition (3) and (11) give

$$\lambda_2 = m\pi/b \quad Y_m(y) = D_m \sin(m\pi y/b), \quad m = 1, 2, 3, \dots \quad \dots (17)$$

and finally, boundary condition (4) and (12) give

$$\lambda_3 = n\pi/c, \quad Z_n(z) = F_n \sin(n\pi z/c), \quad n = 1, 2, 3, \dots \quad \dots (18)$$

Using (15), (17) and (18), (9) reduces to

$$\lambda^2 = \pi^2(l^2/a^2 + m^2/b^2 + n^2/c^2)$$

Using (6), (13), (16), (17) and (18), a solution of (1) is given by

$$H_{lmn} \sin(l\pi x/a)\sin(m\pi y/b)\sin(n\pi z/c)e^{-\lambda^2 kt},$$

where H_{lmn} (= $B_l\, D_m\, F_n\, G$) is another arbitrary constant Now, using the principle of superposition, the general solution of (1) is

$$u(x,y,z,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}e^{-\lambda^2 kt} \qquad \text{... (20)}$$

where λ is given by (19),

Putting $t = 0$ in (20) and using (5), we have

$$f(x,y,z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c} \qquad \text{... (21)}$$

which is Fourier sine series for three variables x, y, z and so

$$H_{lmn} = \frac{8}{abc}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} f(x,y,z)\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c} \qquad \text{... (22)}$$

The required solution of (1) is given by (21) and (22).

Exercise. *The faces of a solid parallelopiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ *are kept at zero temperature. If, initially the temperature of the solid is given by* $u(x, y, z, 0) = f(x, y, z)$

Show that $u(x,y,z,t) = \dfrac{8}{abc}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} F(m,n,q)\; e^{-\lambda^2 kt} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi z}{c}$, *where*

$$F(m,n,q) = \int_0^a\int_0^b\int_0^c f(x,y,z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{q\pi z}{c}, \quad \textit{where} \quad \lambda^2 = \pi^2(m^2/a^2 + n^2/b^2 + q^2/c^2),$$

[Meerut 2005]

Hint. This is same as Art. 2.8B. Here $l = m$, $m = n$, $n = q$ and $H_{lmn} = F(l,m,n)$. With these changes, we get the desired solution.

(38)

Derivation of two-diamensional wave equation. :

Consider a tightly stretched membrane (*e.g.* the membrane of a drum). Let the membrane be distored and further let at time $t = 0$, it be released and allowed to vibrate. Let $A' B' C' D'$ be the shape of an element of the membrane at any time t. Let $ABCD$ be the projection of $A' B' C' D'$ on the xy-plane. We wish to obtain deflection $u(x, y, t)$ at any point (x, y) and at any time $t > 0$. We make the following assumptions:

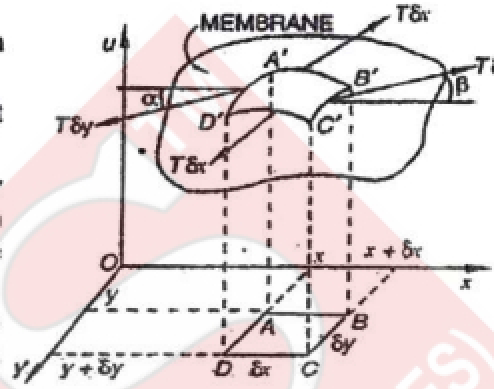
(*i*) The membrane is homogeneous, *i.e.*, the mass of the membrane per unit area is constant ρ (say).

(*ii*) The entire motion takes place in a direction perpendicular to xy-plane.

(*iii*) The string is perfectly elastic and it does not produce resistance to bending.

(*iv*) The tension T per unit length developed by stretching the membrane is the same at all points and in all directions. We further assume that the absolute value of T does not change during the motion.

(*v*) The deflection $u(x, y, t)$ is small as compared to the size of the membrane. All angles of inclination are small.



(*vi*) The slopes $\partial u/\partial x$ and $\partial u/\partial y$ are small so that their higher powers can be neglected.

We now consider motion of the element $A'B'C'D'$. The forces $T\delta y$ on its opposite edges $B'C'$ and $A'D'$ of length δy act at angles α and β to the horizontal. These have a vertical component equal to

$$= (T\delta y)\sin\beta - (T\delta y)\sin\alpha = T\delta y(\sin\beta - \sin\alpha)$$

$= T\delta y(\tan\beta - \tan\alpha)$, since α and β are small, so $\sin\alpha = \alpha = \tan\alpha$, $\sin\beta = \beta = \tan\beta$.

$$= T\delta y\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\}$$

Similarly the forces $T\delta x$ acting on the edges $A'B'$ and $D'C'$ of length δx have a vertical component equal to

$$= T\delta x\{(\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y\}.$$

Now the area of the element $A'B'C'D'$ is $\delta x\,\delta y$ so that its mass is $\rho\,\delta x\,\delta y$. Again the acceleration of the element in vertical direction is $\partial^2 u/\partial t^2$. Hence using $P = mf$, the equation of the motion of the element is

$$T\delta y\{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x\} + T\delta x\{\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y\} = (\rho\,\delta x\,\delta y)\,(\partial^2 u/\partial t^2)$$

or
$$\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial u)_x}{\delta x} + \frac{(\partial u/\partial y)_{y+\delta y} - (\partial u/\partial y)_y}{\delta y} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Now as $\delta x \to 0$ and $\delta y \to 0$, the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = T/\rho$. This equation is called *two-dimensional wave equation*.

PART V: PROBLEMS BASED ON TWO-DIMENSIONAL WAVE EQUATION

2.11A. General soution of two-diniensional wave equation

To solve $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$ *subject to the boundary conditions* $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ *and initial conditions* $u(x, y, 0) = f(x, y)$ *and* $(\partial u/\partial t)_{t=0} = g(x, y)$

Sol. We are to solve $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)\times(\partial^2 u/\partial t^2)$ ... (1)

subject the boundary conditions $u(0, y, t) = 0$ ... 2(*a*)

$u(a, y, t) = 0$ ... 2(*b*)

$u(x, 0, t) = 0$ ... 2(*c*)

and $u(x, b, t) = 0$ ... 2(*d*)

and the two initial conditions

$u(x, y, 0) = f(x, y)$ [$\because$ given initial displacement $= f(x, y)$] ... 3(*a*)

and $u_t(x, y, 0) = g(x, y)$ [$\because$ given initial velocity $= g(x, y)$] ... 3(*b*)

where $u(x, y, t)$ gives the displacement of the point (x, y) of the vibrating membrane from rest $(u = 0)$ at time t.

Suppose that (1) has solutions of the form $u(x, y, t) = X(x)\, Y(y)\, T(t)$ ... (4)

Substituting this in (1), we have $X''YT + XY''T = (1/c^2)XYT''$

or $X''/X + Y''/Y = T''/(c^2T)$ ... (5)

Since x, y and t are independent variables, (5) can only be true if each term on each side is equal to a constant.

Let $X''/X = \mu_1$ so that $X'' - \mu_1 X = 0$ ... (6)

Using 2(*a*) and 2(*b*), (4) gives

$X(0)\, Y(y)\, T(t) = 0$ and $X(a)\, Y(y)\, T(t) = 0$ ... (7)

We assume that $Y(y) \neq 0$ and $T(t) \neq 0$. So by (7), we have

$X(0) = 0$ and $X(a) = 0$ ... (8)

We now solve (6) under B.C. (8). Three cases arise :

Case I. Let $\mu_1 = 0$. Then solution of (6) is $X(x) = Ax + B$ ... (9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ and hence $X(x) \neq 0$. This leads to $u = 0$, which does not satisfy 3(*a*) and 3(*b*). So we reject $\mu_1 = 0$.

Case II. Let $\mu_1 = \lambda_1^2$, $\lambda_1 \neq 0$. Then solution of (6) is $X(x) = Ae^{x\lambda_1} + Be^{-x\lambda_1}$ ...(10)

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{a\lambda_1} + Be^{-a\lambda_1}$ ... (11)

Solving (11), $A = B = 0$ so that $X(x) \equiv 0$. This leads to $u \equiv 0$. So we reject $\mu_1 = \lambda_1^2$.

Case III. Let $\mu_1 = -\lambda_1^2$, $\lambda_1 \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x$ ... (12)

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda_1 a + B\sin\lambda_1 a$

so that $A = 0$ and $\sin\lambda_1 a = 0$,

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ so that $u \equiv 0$ which does not satisfy 3(*a*) and 3(*b*).

Now, $\sin\lambda_1 a = 0$ gives $\lambda_1 a = m\pi$, $m = 1, 2, 3,$

Hence $\lambda_1 = m\pi/a$, $m = 1, 2, 3, \ldots$ ... (13)

Hence non-zero solutions $X_m(x)$ of (6) are given by

$$X_m(x) = B_m \sin(m\pi x/a), \qquad m = 1, 2, 3, \ldots \quad \text{... (14)}$$

Next, let $Y''/Y = \mu_2$ so that $Y'' - \mu_2 y = 0$ ... (15)

Using 2(*c*) and 2(*d*), (4) gives as before

$$Y(0) = 0 \qquad \text{and} \qquad Y(b) = 0 \qquad \text{... (16)}$$

Solving (15) under B.C. (16) as before, we get

$$Y_n(y) = D_n \sin(n\pi y / b), \qquad n = 1, 2, 3, ... \qquad \text{... (17)}$$

where $\quad \mu_2 = -\lambda_2^2 \quad$ and $\quad \lambda_2 = n\pi / b, \quad n = 1, 2, 3, ... \qquad$ ... (18)

In view of (6) and (15), (5) reduces to

$$T''/c^2T = \mu_1 + \mu_2 = -\lambda_1^2 - \lambda_2^2 = -\pi^2(m^2/a^2 + n^2/b^2), \text{ using (13) and (18)}$$

or
$$T'' + \lambda_{mn}^2 T = 0 \qquad \text{... (19)}$$

where
$$\lambda_{mn}^2 = c^2\pi^2(m^2/a^2 + n^2/b^2) \qquad \text{... (20)}$$

Solving (19), $\quad T_{mn}(t) = E_{mn} \cos \lambda_{mn} t + F_{mn} \sin \lambda_{mn} t \qquad$ ... (21)

$\therefore \quad$ Functions $\quad u_{mn}(x, y, t) = X_m(x)\, Y_n(y)\, T_{mn}(t)$

i.e., $\quad u_{mn}(x, y, t) = (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t) \sin(m\pi x/a) \sin(n\pi y/b) \qquad$ ... (22)

are solutions of (11) satisfying 2(*a*), 2(*b*), 2(*c*) and 2(*d*). Here $A_{mn} = (B_m D_n E_{mn})$ and B_{mn} (= $B_m D_n F_{mn}$) are new arbitrary constants. In order to obtain a solution also satisfying 3(*a*) and 3(*b*), weconsider more general solution, that is,

$$u(x, y, t) = \sum_{m=1} \sum_{n=1} (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t) \sin(m\pi x/a) \sin(n\pi y/b) \qquad \text{... (23)}$$

Differentiating (23) pertially w.r.t. '*t*' we get

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-A_{mn}\lambda_{mn} \sin \lambda_{mn} t + B_{mn}\lambda_{mn} \cos \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{... (24)}$$

Putting $t = 0$ in (23) and (24) and using 3 (*a*) and 3 (*b*), we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{... (25)}$$

and
$$g(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn}\lambda_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{... (26)}$$

which are double Fourier sine series. Accordingly, we get

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \qquad \text{... (27)}$$

and
$$B_{mn}\lambda_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$$

so that $\quad B_{mn} = \dfrac{4}{ab\lambda_{mn}} \displaystyle\int_{x=0}^{a} \int_{y=0}^{b} g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$, $m = 1, 2, 3 ..., n = 1, 2, 3...$ $\qquad$ ... (28)

Thus the desired deflection $u(x, y, t)$ of the given membrane is given by (23) wherein A_{mn} and B_{mn} are given by (27) and (28) respectively.

Particular Case I. If the initial velocity $u_t(x, y, 0) = g(x, y) = 0$, then $B_{nm} = 0$ by (28). Thus, in this case the solution (23) reduces to

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad \text{... (29)}$$

Particular Case II. If the initial displacement $= u(x, y, 0) = f(x, y) = 0$, then $A_{mn} = 0$ by (27). Thus, in this case the solution (23) reduces to

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\sin(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \text{... (30)}$$

Eigenfunctions (or characteristic functions) and eigenvalues (or characteristic values) of the vibrating membrane. Definitions

The numbers λ_{mn} $(m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots)$ given by (20) are called the *eigenvalues* and the functions $u_{mn}(x, y, t)$ $(m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots)$ are called the corresponding *eigenfunctions* of the vibrating membrane. The *frequency* of u_{mn} is given by $\lambda_{mn}/2\pi$.

2.11 B. Solved examples based on Art. 2.11 A

Ex. 1. *Solve the two-dimensional wave equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)(\partial^2 u/\partial t^2)$ *by the method of separation of variables under the following initial and boundary conditions:*

$u(x, y, 0) = f(x, y)$, $(\partial u/\partial t)_{t=0} = 0, 0 \le x \le a, 0 \le y \le b$ $u(0, y, t) = 0 = u(a, y, t)$ for $t \ge 0$

$u(x, 0, t) = 0 = u(x, b, t)$, for $t \ge 0$ **[Delhi Maths (Hons) 1996]**

Sol. Proceed as in Art. 2.11A upto equation (29).

Ex. 2. *Find the deflection* $u(x, y, t)$ *of the square membrane of each side unity and* $c = 1$, *if the initial velocity is zero and the initial deflection is* $f(x,y) = A\sin\pi x\sin 2\pi y$.

Sol. The deflection $u(x, y, t)$ of the given membrane is governed by the two dimensional wave equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial^2 u/\partial t^2 \quad \text{... (1)}$$

subject to the boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ... (2)

and the initial deflection $= u(x, y, 0) = f(x, y) = A\sin\pi x\sin 2\pi y$ ... 3(*a*)

and the initial velocity $= u_t(x, y, 0) = g(x, y) = 0$ ... 3(*b*)

Proceed as in Art. 2.11A upto equation (29) and prove that the solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)\times(\partial^2 u/\partial t^2) \quad \text{... (4)}$$

subject to the boundary conditions $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$ ... (5)

and initial condition: $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y) = 0$ ... (6)

is given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \text{... (7)}$$

where

$$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \quad \text{... (8)}$$

and

$$\lambda_{mn}^2 = c^2\pi^2(m^2/a^2 + x^2/b^2) \quad \text{... (9)}$$

Comparing the given boundary value problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem given by (4), (5) and (6), we have $c = 1$, $a = b = 1$ and $f(x, y) = A\sin\pi x\sin 2\pi y$.

$\therefore$ From (8), $$A_{mn} = 4\int_{x=0}^{1}\int_{y=0}^{1} A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dx\, dy$$

or $$A_{mn} = A\left(\int_0^1 2\sin \pi x \sin m\pi x \, dx\right)\times\left(\int_0^1 2\sin 2\pi y \sin n\pi y \, dy\right) \qquad \text{... (10)}$$

But $$\left.\begin{aligned}\int_0^1 2\sin \pi x \sin m\pi x \, dx &= 0 \text{ if } m \neq 1\\ &= 1 \text{ if } m = 1\end{aligned}\right\} \qquad \text{... (11)}$$

and $$\left.\begin{aligned}\int_0^1 2\sin 2\pi y \sin n\pi y \, dy &= 0 \text{ if } n \neq 2\\ &= 1 \text{ if } n = 2\end{aligned}\right\} \qquad \text{... (12)}$$

For complete solution results (11) and (12) should be proved in examination.

Using (11) and (12), (10) gives $\quad A_{12} = 1 \quad$ and $\quad A_{mn} = 0$, otherwise.

Also, $\quad$ (9) $\Rightarrow \quad \lambda_{12} = \pi(1^2 + 2^2)^{1/2} = \pi\sqrt{5}$

Now, from (7) the required solution is given by

$$u(x, y, t) = A_{12} \cos \lambda_{12} t \sin \pi x \sin 2\pi y = \cos(\pi t\sqrt{5}) \sin \pi x \sin 2\pi y.$$

EXERCISE 2(D)

1. Discuss a solution of the wave equation $\nabla^2 z = (1/c^2)(\partial^2 z/\partial t^2)$ satisfied by a thin membrane bounded by a rectangle $x = 0$, $x = a$, $y = 0$, $y = b$ and $z = f(x, y)$, $\partial z/\partial t = 0$ at $t = 0$.

2. Obtain the solution of the two-dimensional wave equation $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$ subject to the boundary condition that $u = 0$ on the curve Γ (which is a rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$) for all t, and the initial onditions $u = f(x, y)$ and $\partial u/\partial t = 0$ at $t = 0$ for all (x, y) in the region S bounded by Γ. **[Delhi Maths (H) 1996, 2006]**

Hint: For complete solution proceed as in Art. 2.11A upto equation (29).

3. Solve the partial differential equation for the vibrations of a square elastic membrane fixed at edges, $\partial^1 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)(\partial^2 u/\partial t^2)$. The length of each side of the square is L, c is the velocity of elastic waves and $u(x, y, t)$ is the displacement of the membrane. Obtain the lowest frequency of vibrations, if $c = 10{,}000$ metres/sec. and $L = \sqrt{2}$ metres.