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Date : ....

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

Minimize the more cuttings

MAINS TEST SERIES-2020

(JULY *to* DEC.-2020)

IAS/IFoS

MATHEMATICS

$\frac{202}{250}$

Under the guidance *of* K. Venkanna

ODE, VECTOR ANALYSIS AND DYNAMICS & STATICS

TEST CODE: TEST-3: IAS(M)/26-JULY-2020

Time: 3 Hours | **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: M.S. PRANAVI

Roll No.: 9849145602

Test Centre: HYDERABAD

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			06	
	(b)			08	
	(c)			07	
	(d)			07	36
	(e)			08	
2	(a)				
	(b)				
	(c)				
	(d)				
3	(a)			10	
	(b)			16	
	(c)			16	42
	(d)				
4	(a)			07	
	(b)			13	
	(c)			08	41
	(d)			13	
5	(a)			08	
	(b)			08	
	(c)			08	
	(d)			08	40
	(e)			08	
6	(a)				
	(b)				
	(c)				
	(d)				
7	(a)				
	(b)				
	(c)				
	(d)				
8	(a)			05	
	(b)			10	43
	(c)			12	
	(d)			16	
			Total Marks		

DO NOT WRITE ON THIS SPACE

SECTION – A

1. (a) Solve $(D^4 + D^2 + 1)\, y = ax^2 + be^x \sin 2x$. **[10]**

Given equation is $(D^4+D^2+1)y = ax^2+be^x\sin 2x$

Let us find the complimentary solution by Euler form ⇒

$m^4+m^2+1=0$

$m^2 = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{3}i}{2}$

$m^2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$m^2 = e^{i2\pi/3},\ e^{i4\pi/3}$

$m = \pm e^{i\pi/3},\ \pm e^{i2\pi/3}$

$m = \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$

$\therefore m = \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\therefore y_c = e^{x/2}\left[A\cos\frac{\sqrt{3}x}{2}+B\sin\frac{\sqrt{3}x}{2}\right]$

$+e^{-x/2}\left[C\cos\frac{\sqrt{3}x}{2}+D\sin\frac{\sqrt{3}x}{2}\right]$

$\boxed{\text{Now let's find } y_p}$

$y_p = \frac{1}{D^4+D^2+1}ax^2 + \frac{b}{D^4+D^2+1}e^x\sin 2x$

$y_p = \frac{1}{1+(D^2+D^4)}ax^2 + be^x\frac{1}{(D+1)^4+(D+1)^2+1}\sin 2x$

$= [1+(D^2+D^4)]^{-1}ax^2 + \frac{be^x}{D^4+4D^3+6D^2+4D+1+D^2+2D+1+1}\sin 2x$

$= [1-(D^2+D^4)+(D^2+D^4)^2+\dots]ax^2 + \frac{be^x}{D^4+4D^3+7D^2+6D+3}\sin 2x$

$4x^2(-4)$

$4(-4)D$

$7(-4)$

$= [ax^2-2a] + \frac{be^x}{16-16D-28+6D+3}\sin 2x$

$= ax^2-2a+\frac{be^x}{-10D-9}\sin 2x$

$= ax^2-2a-\frac{be^x(10D-9)}{100D^2-81}\sin 2x$

$= ax^2-2a-\frac{be^x}{64(-4)-9}[16\cos 2x+9\sin 2x]$

$y_p = ax^2-2a+\frac{be^x}{265}[16\cos 2x+9\sin 2x]$

$\therefore y = e^{x/2}\left[A\cos\frac{\sqrt{3}x}{2}+B\sin\frac{\sqrt{3}x}{2}\right]+e^{-x/2}\left[C\cos\frac{\sqrt{3}x}{2}+D\sin\frac{\sqrt{3}x}{2}\right]$

$+ax^2-2a+\frac{16}{265}be^x\cos 2x+\frac{9be^x\sin 2x}{265}$

06

1. (b) (i) Prove $L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\frac{s^2+4}{s^2}$

(ii) Evaluate $L^{-1}\{1/s(s+1)^3\}$ **[10]**

i) We know that $L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(p)\,dp.$

Let $f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$L\left[\frac{1-\cos 2t}{2}\right] = L\left[\frac{1}{2}\right] - \frac{1}{2}L[\cos 2t]$

$= \frac{1}{2p} - \frac{1}{2}\times\frac{p}{p^2+4}$

$F(p) = \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$

$\therefore L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(p)\,dp$

$= \int_p^\infty \frac{1}{2}\left(\frac{1}{p}\right) - \frac{1}{4}\times\frac{2p}{p^2+4}$

$= \left[\frac{1}{4}\log p^2 - \frac{1}{4}\log[p^2+4]\right]$

$= -\frac{1}{4}\left[\log\left(\frac{p^2+4}{p^2}\right)\right]_p^\infty$

$= -\frac{1}{4}\left[0 - \log\left(\frac{p^2+4}{p^2}\right)\right] = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$

Take $p = s$

$\boxed{= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)}$

Ok

(ii) $L^{-1}\left[\frac{1}{s(s+1)^3}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$ $\left(F(s) = 1/(s+1)^3\right)$ $\Rightarrow \int_0^t f(t)\,dt$

$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s+1)^3}\right]$

We know that $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

$L[t^2 e^{-t}] = (-1)^2 \times \frac{d^2}{ds^2}\left[\frac{1}{s+1}\right] = \frac{2}{(s+1)^3}$

$\therefore L^{-1}\left[\frac{1}{(s+1)^3}\right] = \boxed{\frac{t^2e^{-t}}{2} = f(t)}$

$-\frac{1}{2}\left[t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2\right]$

$\boxed{\frac{e^{-t}}{2}\left[t^2+2t+2\right] - 1}$

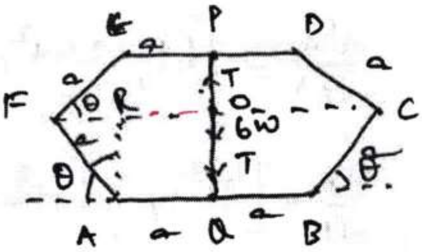
$\frac{1}{(s+1)^2}$ $\int te^{-t}$ te^{-t} t^2e^{-t} $(-1)^2\frac{d}{ds}\left[\frac{1}{s+1}\right]$ $(-1)^2\frac{1}{(s+1)^2}$

$L^{-1}\left[\frac{F(s)}{s}\right] = \frac{1}{2}\int_0^t t^2e^{-t}\,dt = \frac{1}{2}\left[t^2\int e^{-t}dt + \int 2te^{-t}\right] = \frac{1}{2}\left[-t^2e^{-t} + 2\left[-te^{-t} - e^{-t}\right]\right]$

$\boxed{= \frac{1}{2}\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right]_0^t}$

1. (c) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is 3W. **[10]**

Let ABCDEFG be a hexagon with BC, AF making angle θ with horizontal.



Now, let us apply Theory of virtual work.

Weight 6W acts at O, Let T be the Tension in string PQ.

AB is the equilibrium position → Let each side be of length a.

$-T\,\delta(PQ) + 6W\,\delta(OQ) = 0$

$\therefore OQ = a\sin\theta \qquad \therefore -T\,\delta(2a\sin\theta) + 6W\,\delta(a\sin\theta) = 0$

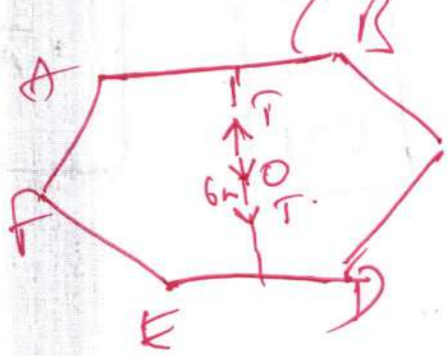
$PQ = 2a\sin\theta \qquad 6W(a\cos\theta)\,d\theta = T \times 2a\cos\theta\,d\theta$

$\boxed{\therefore T = 3W}$

For the string to be in ~~Tension~~ Equilibrium, it should be of 3W

$\boxed{T = 3W}$

07



Horizontal position — take it as top position

1. (d) If $\dfrac{d^2\mathbf{A}}{dt^2} = 6t\mathbf{i} - 24t^2\mathbf{j} + 4\sin t\mathbf{k}$, find **A** given that A = 2**i** + **j** and $\dfrac{d\mathbf{A}}{dt} = -\mathbf{i} - 3\mathbf{k}$ at t = 0. **[10]**

$$\frac{d^2A}{dt^2} = [6t, -24t^2, 4\sin t]$$

$$\frac{dA}{dt} = [3t^2+p, -8t^3+q, -4\cos t + r] \quad \left[\text{Integrating } \frac{d^2A}{dt^2}\right]$$

At $t=0$, $\frac{dA}{dt} = -i-3k = [-1, 0, -3]$

$$\left[\frac{dA}{dt}\right]_{t=0} = [p, q, r-4] = [-1, 0, -3]$$

$$\boxed{p=-1,\ q=0,\ r=1}$$

$$\therefore \frac{dA}{dt} = [3t^2-1, -8t^3, 1-4\cos t]$$

Integrating $\frac{dA}{dt}$, $A = [t^3-t+a, -2t^4+b, t-4\sin t+c]$

$$[A]_{t=0} = [2, 1, 0] = [a, b, c]$$

$$\boxed{\begin{matrix} a=2 \\ b=0 \\ c=1 \end{matrix}}$$

$$\therefore \boxed{\vec{A} = [t^3-t+2, -2t^4, t-4\sin t+1]}$$

07

1. (e) Find the curvature and the torsion of the space curve $x = a(3u - u^3), y = 3au^2, z = a(3u + u^3)$. **[10]**

Given

$$r(u) = \left[a(3u-u^3), 3au^2, a(3u+u^3)\right]$$

$$r' = \left[a(3-3u^2), 6au, a(3+3u^2)\right]$$

$$r'' = \left[-6au, 6a, 6au\right]$$

$$r''' = \left[-6a, 0, 6a\right]$$

Differentiate w.r.t. u for $r'(u)$, $r''(u)$, $r'''(u)$.

Curvature

$$k = \frac{|r' \times r''|}{|r'|^3}$$

$$\therefore r' \times r'' = \begin{vmatrix} i & j & k \\ a(3-3u^2) & 6au & a(3+3u^2) \\ -6au & 6a & 6au \end{vmatrix}$$

$$= \left[18a^2u^2 - 18a^2,\ \cancel{18a^2u^3 - 18}\ 36a^2u,\ 18a^2 + 18a^2u^2\right]$$

$$|r' \times r''| = \sqrt{(18)^2\left[a^4u^4 + a^4 - \cancel{2a^4u^2} + a^4u^4 + a^4 + \cancel{2a^4u^2} + 4a^4u^2\right]}$$

$$= \sqrt{(18)^2\left[2(a^4u^4 + 2a^4u^2 + a^4)\right]} = 18\sqrt{2}(a^2u^2 + a^2)$$

$$|r'| = \sqrt{36a^2u^4 + 36a^2u^2 + 36a^2} = \sqrt{18a^2(u^2+1)^2} = 3\sqrt{2}a(u^2+1)$$

$$k = \frac{18\sqrt{2} \times a^2 (u^2+1)}{(3\sqrt{2})^3 \times a^3 (u^2+1)^3} = \frac{18\sqrt{2}}{\frac{27 \times 2\sqrt{2}}{3}} \times \frac{1}{a(u^2+1)^2} = \boxed{\frac{1}{3a(u^2+1)^2}}$$

$$\tau = \frac{[r'\ r''\ r''']}{|r' \times r''|^2} = \frac{r' \cdot (r'' \times r''')}{(18\sqrt{2})^2 a^4 (u^2+1)^2} = \frac{6 \times 36 a^3}{\frac{18 \times 18 \times 2}{3} \times a^4 (u^2+1)^2} = \boxed{\frac{1}{3a(u^2+1)^2}}$$

08

2. (a) Find the orthogonal trajectories of $r = a(1 - \cos n\theta)$. **[10]**

2. (d) (i) Prove the identity

$$\nabla\left(\vec{A}\cdot\vec{B}\right)=\left(\vec{B}\cdot\nabla\right)\vec{A}+\left(\vec{A}\cdot\nabla\right)\vec{B}+\vec{B}\times\left(\nabla\times\vec{A}\right)+\vec{A}\times\left(\nabla\times\vec{B}\right)$$

(ii) Derive the identity

$$\iiint_V\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)dV=\iiint_S\left(\phi\nabla\psi-\psi\nabla\phi\right)\hat{n}\cdot dS$$

where V is the volume bounded by the closed surface S. **[15]**

Continuation of 3.(a)

$$[D'(D'-1)-2]v = e^{2z}$$

$$[D'^2-D'-2]v = e^{2z}$$

$\boxed{f(D)=0}$

m^2-m-2 $\quad \therefore V_C = C_1e^{2z} + C_2e^{-z}$ $\quad \frac{x^r}{r!}e^{az}$ $\quad \boxed{\begin{matrix} r=2 \\ n=1 \end{matrix}}$

$m = \frac{1 \pm \sqrt{9}}{2}$ $\quad V_p = \frac{e^{2z}}{D'^2-D'-2} = \frac{e^{2z}}{D'^2-D'-2} = \frac{z^2}{2}e^{2z}$ $\quad 4+2.$

$m = 2, -1$

$\therefore V_C = C_1x^2 + \frac{C_2}{x}$ $\quad \boxed{V = C_1x^2 + \frac{C_2}{x} + \frac{x^2(\log x)^2}{2}}$

$V_p = \frac{(\log x)^2 x^2}{2}$

$$\therefore \boxed{y = uV = (\log x)\left[C_1x^2 + \frac{C_2}{x} + \frac{x^2(\log x)^2}{2}\right]}$$

Which is the solution.

$\frac{x^2 \log x}{3}$

3. (a) Reduce the equation $x^2 (\log x)^2 (d^2y / dx^2) - 2x \log x (dy/dx) + [2 + \log x - 2 (\log x)^2] y = x^2 (\log x)^3$ to normal form and hence solve it. **[14]**

Given equatn is $x^2(\log x)^2 \frac{d^2y}{dx^2} - 2x\log x \frac{dy}{dx} + [2+\log x - 2(\log x)^2]y = x^2(\log x)^3$

$\therefore \frac{d^2y}{dx^2} - \frac{2}{x\log x}\frac{dy}{dx} + \left[\frac{2}{x^2(\log x)^2} + \frac{1}{x^2\log x} - \frac{2}{x^2}\right]y = \log x$

It is of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

$P(x) = -\frac{2}{x\log x}$

$Q(x) = \frac{2}{x^2(\log x)^2} + \frac{1}{x^2(\log x)} - \frac{2}{x^2}$

$R(x) = \log x$

In normal form, $u = e^{-\frac{1}{2}\int P\,dx}$

$= e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{1}{t}dt} = t \quad (t = \log x)$

$= \log x$

$\boxed{\therefore u = \log x}$

$I = Q - \frac{1}{4}P^2 - \frac{1}{2}dP/dx$

$= \frac{2}{x^2(\log x)^2} + \frac{1}{x^2\log x} - \frac{2}{x^2} - \frac{1}{x^2(\log x)^2} - \frac{1}{2}\left[-\frac{2}{x}\cdot\left[\frac{-(1+\log x)}{(x\log x)^2}\right]\right]$

$-\frac{(\ldots)}{\ldots}$

$= \cancel{\frac{1}{x^2(\log x)^2}} + \cancel{\frac{1}{x^2\log x}} - \frac{2}{x^2} - \cancel{\frac{1}{x^2(\log x)^2}} - \cancel{\frac{1}{x^2\log x}}$

$= -2/x^2 \qquad \therefore I = -2/x^2$

$\frac{d^2v}{dx^2} + Iv = \frac{R(x)}{u}$

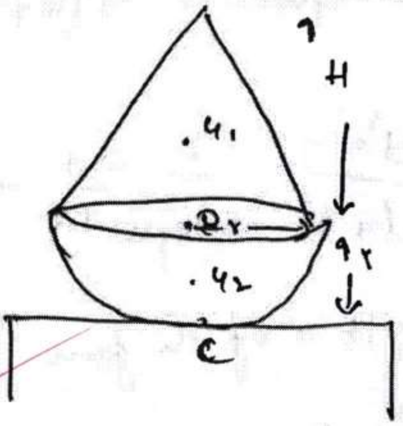
$\frac{d^2v}{dx^2} + \left(\frac{-2}{x^2}\right)v = 1$

$x^2\frac{d^2v}{dx^2} - 2v = x^2$ $\quad$ Let $x = e^z$

$[D'(D'-1)-2]v = e^{2z}$

$v = \frac{e^{2z}}{\ldots}$ $\quad$ Legendre Eqn

3. (b) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere. **[18]**



Let radius of hemisphere be "r"

Height of cone be "H"

Let G_1 be Centre of Gravity of Cone

G_2 be Centre of Gravity of Hemisphere.

$OG_1 = \frac{2}{3}H$ ∴ ~~Centre of Grav~~

$OG_2 = \frac{3}{8}r$

Height of Centre of Gravity of Body from Table is "h"

$$h = \frac{V_1(CG_1) + V_2(CG_2)}{V_1 + V_2}$$

$V_1 \Rightarrow$ Vol of Cone $\Rightarrow \frac{1}{3}\pi r^2 H$

$V_2 \Rightarrow$ vol of Hemp $\Rightarrow \frac{2}{3}\pi r^3$

$CG_1 = r + H/4$

$CG_2 = \frac{5}{8}r$

$$h = \frac{\frac{1}{3}\pi r^2 H\left(r + \frac{H}{4}\right) + \frac{2}{3}\pi r^3\left(\frac{5}{8}r\right)}{\frac{\pi r^2}{3}\left[H + 2r\right]}$$

For equilibrium to be stable

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$h = \frac{\frac{\pi r^2}{3}\left[Hr + \frac{H^2}{4} + \frac{5r^2}{4}\right]}{\frac{\pi r^2}{3}\left[H + 2r\right]}$$

$\rho_1 \rightarrow$ Curvature of Top surface

$\rho_2 \rightarrow$ Curvature of Table

$$\boxed{\begin{matrix}\rho_1 = r \\ \rho_2 = \infty\end{matrix}}$$

$$\therefore \frac{1}{h} > \frac{1}{r}$$

$$\frac{H+2r}{} > \frac{1}{r} = \cancel{H}r+2r^2 > \cancel{H}r + \frac{H^2}{4} + \frac{r^2}{4}$$

$$Hr + \frac{H^2}{4} + \frac{r^2}{4}$$

$$\frac{9r^2}{4} > \frac{3H^2}{4}$$

$$2r^2 - \frac{r^2}{4}$$

$$3r^2 > 3H^2$$

$$\boxed{H < \sqrt{3}r}$$

$$\frac{3r^2}{4} > \frac{H^2}{4}$$

$$\boxed{\sqrt{3}r > H} \Longrightarrow$$

$\boxed{\therefore \text{Greatest Height of cone for equi to be stable is } \sqrt{3}r}$

3. (c) (i) If **f** and **g** are irrotational then show that **f** × **g** is a solenoidal vector.

(ii) If **f** = (**a** × **r**) r^n,

show that div **f** = 0, curl **f** = (n + 2) r^n **a** – nr^{n-2} (**a** • **r**) **r**. **[5 + 13=18]**

~~f is~~ Given f and g are irrotational

$\therefore$ Curl f = 0 $\quad$ $\boxed{f \times g \text{ is Solenoidal}}$ — To prove.

Curl g = 0

$$\nabla \cdot (f \times g) = g.\text{curl } f - f \text{ curl } g$$

$$= g(0) - f(0) \qquad \therefore f \times g \text{ is Solenoidal}$$

$$= 0$$

$\therefore$ div $(f \times g) = 0$

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(ii) $f = r^n(a\times r)$

Let $\phi = r^n$ $\quad \mathrm{div}(\phi A) = \phi\, \mathrm{div} A + (\nabla\phi)\cdot A$

$A = a\times r$

$\mathrm{div}(r^n(a\times r)) = r^n \mathrm{div}(a\times r) + (\nabla r^n)\cdot(a\times r)$

$= r^n \mathrm{div}(a\times r) + n r^{n-1}\frac{\vec{r}}{r}\cdot(a\times r)$ $\quad \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$= r^n[r\,\mathrm{curl}\,a - a\,\mathrm{curl}\,r] + n r^{n-2}\,[\vec{r}\ \vec{a}\ \vec{r}]$

$= r^n[r\cdot 0 - a\cdot 0] + n r^{n-2}\times 0$

$= 0$ $\quad \left[\because \begin{array}{l} \mathrm{Curl}\,a = 0 \ (a \text{ is const.}) \\ \mathrm{Curl}(\vec{r}) = 0 \end{array}\right]$

$\vec{r}\cdot(\vec{a}\times\vec{r}) = 0$ Since $\vec{a}\times\vec{r}$ is $\perp$ to $\vec{a}$

$\mathrm{Curl}(\phi A) = \phi\,\mathrm{Curl} A + \nabla\phi\times A$

$= r^n \mathrm{Curl}(a\times r) + \nabla r^n\times(a\times r)$ $\quad -a\frac{\partial}{\partial x}x$

$= r^n[(r\cdot\nabla)a - r\,\mathrm{div}a - (a\cdot\nabla)r + a\,\mathrm{div} r]$

$+ n r^{n-2}(\vec{r}\times(a\times\vec{r}))]$

$= r^n[-\vec{a} + 3\vec{a}] + n r^{n-2}[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}]$

$= 2r^n\vec{a} + n r^n\vec{a} - n r^{n-2}(\vec{a}\cdot\vec{r})\vec{r}$

$= (n+2)r^n\vec{a} - n r^{n-2}(a\cdot r)\vec{r}$ Proved.

$\boxed{y = (C_1 + C_2 \sin x) e^{\sin x} - \sin^2 x + 4\sin x - 5}$

4. (a) Solve the equation $d^2y/dx^2 + (2\cos x + \tan x) \times (dy/dx) + y\cos^2 x = \cos^4 x$. **[10]**

$$\frac{d^2y}{dx^2} + (2\cos x + \tan x)\frac{dy}{dx} + y\cos^2 x = \cos^4 x$$

~~Let~~ Let $P' = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $\quad Q' = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ $\quad R' = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Let $Q' = 1$ $\quad P = 2\cos x + \tan x \quad R = \cos^4 x$ $\quad \boxed{\text{Let } Q' = 1}$

$1 =$ $\quad Q = \cos^2 x$

$\boxed{\dfrac{dz}{dx} = \cos x}$ $\quad P' = \dfrac{-\sin x + (2\cos x + \tan x)(\cos x)}{\cos^2 x}$

$\boxed{z = \sin x}$ $\quad = \cancel{-\sin x - 4\sin x\cos x + 2\sec x - \frac{2\sin^2 x}{\cos x}}$

$\quad = \dfrac{-\sin x + 2\cos^2 x + \sin x}{\cos^2 x} = 2$

$\boxed{P' = 2}$ $\boxed{Q' = 1}$

$\boxed{y_c = (C_1 + C_2 z)e^{z}}$

$y_p = \dfrac{1}{(D+1)^2}(1-z^2) = [1 - 2D + 3D^2](1-z^2)$ $= \boxed{-z^2 + 4z - 5}$

$R' = \dfrac{\cos^4 x}{\cos^2 x} = \cos^2 x$

$\therefore$ Eq. changes to $\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz} + y = \cos^2 x$

$\boxed{(D^2 + 2D + 1)y = 1 - z^2}$ $\quad y_p = \dfrac{\cos^2 x}{D^2 + 2D + 1} = \dfrac{\cos^2 x}{2D}$

$m^2 + 2m + 1 = 0$ $\quad = \dfrac{1}{(D+1)^2}\cos^2 x = \dfrac{1}{2}\int \cos^2 x$

$m = -1, -1$ $\quad = \dfrac{1}{2}\int \cos^2 x \quad \dfrac{1}{2}\int \dfrac{1+\cos 2x}{2}$

$y_c = (C_1 + C_2 z)e^{z}$

$y_p = \dfrac{1}{D^2 + 2D + 1}(1 - z^2) = \dfrac{1}{(1+D)^2}(1-z^2) = (1 - z^2 + 4z - 6)$

$= (1 - 2D + 3D^2)(1-z^2) \quad = -z^2 + 4z - 5$

$\frac{1}{2}\int \frac{x}{2} + \frac{1}{4}$

$2 \quad -3$

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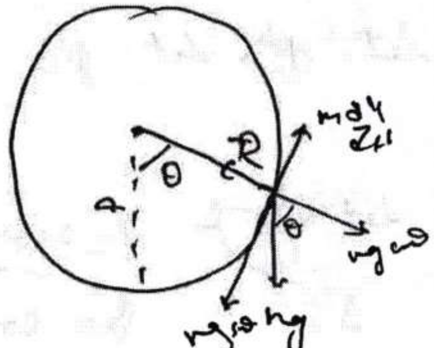
4. (b) A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time

$\sqrt{(a/g)} \cdot \log\left(\sqrt{5}+\sqrt{6}\right)$. **[15]**

$m\frac{d^2s}{dt^2} = mg - mg\sin\theta \rightarrow (1)$

$\frac{mv^2}{a} = R - mg\cos\theta \rightarrow (2)$

$\boxed{s = a\theta}$



$\frac{d^2s}{dt^2} = -g\sin\theta$

$\boxed{V = a\frac{d\theta}{dt}}$

$a\frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiply $2a\frac{d\theta}{dt}$

$\int 2a^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -\int 2ag\sin\theta\, d\theta$

$V^2 = 2ag \times 2\cos^2\theta/2$

$V^2 = 4ag\cos^2\theta/2$

$a\left(\frac{a\,d\theta}{dt}\right)^2 = 2ag\cos\theta + A$

$V = 2\sqrt{ag}\cos\theta/2$

$V^2 = 2ag\cos\theta + A$

It is projected with velocity, such that $v = 0$, $\theta = \pi$ (Highest point)

$0 = 2ag(-1) + A \quad \therefore V^2 = 2ag\cos\theta + 2ag$

$A = 2ag$

$\frac{mv^2}{a} = R - mg\cos\theta$ At $\theta_1 = \cos^{-1}(-1/3)$

$\frac{m}{a} \times 2ag(\cos\theta + 1) = -mg\cos\theta$ $\boxed{\cos\theta = -1/3}$ Stg being slow

$\cos\theta + 1 = -\frac{\cos\theta}{2}$

$\frac{3\cos\theta}{2} = -1$ $\quad 2\cos^2\theta/2 - 1 = -1/3$

$2\cos^2\theta/2 = 1/3$

$\boxed{\begin{array}{l}\cos\theta/2 = 1/\sqrt{6} \\ \sin\theta/2 = \sqrt{5}/\sqrt{6}\end{array}}$

$V = 2\sqrt{ag}\cos\theta/_2$ $\quad$ $\theta \Rightarrow 0^\circ$ to $\cos^{-1}(-1/3)$ $\quad$ $\cos\theta = -1/3$

$a\frac{d\theta}{dt} = 2\sqrt{ag}\cos\theta/_2$ $\quad$ $\sqrt{\frac{g}{a}}\, t = \left[\log\left[\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right]\right]_0^{\theta_1}$

$\frac{1}{2}\int \sec\theta/_2\, d\theta = \sqrt{\frac{g}{a}}\int dt$

$\sqrt{\frac{g}{a}}\, t = \log(\sec\theta_1/_2 + \tan\theta_1/_2)$

$\sqrt{\frac{g}{a}}\, t = \log(\sqrt{6} + \sqrt{5})$ $\quad$ $\left[\begin{array}{l}\text{since } \cos\theta_1/_2 = 1/\sqrt{6} \\ \sin\theta_1/_2 = \sqrt{5}/\sqrt{6}\end{array}\right]$

$\tan\theta_1/_2 = \sqrt{5}$

$\sec\theta_1/_2 = \sqrt{6}$

$$\boxed{\therefore t = \sqrt{\frac{a}{g}}\log(\sqrt{5} + \sqrt{6})}$$

4. (c) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$ and the work done in moving an object in this field from (1, -2, 1) to (3, 1, 4). **[10]**

Scalar potential of F $\Rightarrow$ $\nabla\phi = F$, ϕ is potential

$\frac{\partial\phi}{\partial x} = 2xy + z^3 = F_x \Rightarrow F = x^2y + z^3x + f(y,z)$

$\frac{\partial\phi}{\partial y} = x^2 = F_y \longrightarrow F = x^2y + f(x,z)$

$\frac{\partial\phi}{\partial z} = 3xz^2 = F_z$ $\quad$ $F = xz^3 + f(x,y)$

By compared all equation, we get

$f(x,z) = xz^3$ $\quad$ $f(x,y) = x^2y$.

$f(y,z) = 0$

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$$\boxed{\therefore \phi = x^2y + xz^3}$$

Work done $= \left[\phi\right]_{(1,-2,1)}^{(3,1,4)}$

$= \left[(3)^2(1) + 3(4)^3\right] - \left[(1)(-2) + (1)(1)\right]$

$= \left[9 + 192\right] - \left[-2 + 1\right]$

$= 201 - [-1]$

$= \boxed{202}$

$\boxed{\therefore \text{Work done} = 202\text{ J}}$

4. (d) Verify Stoke's theorem for the vector
$\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ taken over the half of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane. **[15]**

Stoke's thm $\Rightarrow \iint (\text{Curl } F)\cdot\hat{n}\,ds = \int \vec{F}\cdot d\vec{r}$

Given surface $= x^2+y^2+z^2 = a^2$, above xy plane ✓

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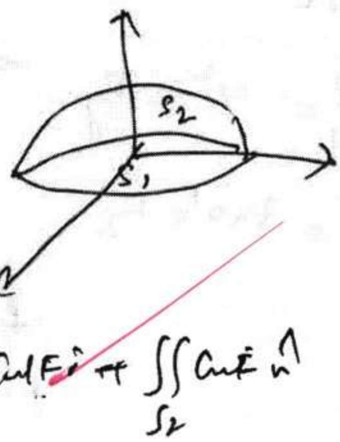
$F = [z, x, y]$

$\cancel{\hat{n} = \hat{k}}$



Let S_1 be xy plane $(x^2+y^2=a^2)$

S_2 be above ple.

$$\iiint \text{div}(\text{Curl } F) = \iint_S \text{Curl } F\cdot\hat{n}\,ds = \iint_{S_1} \text{Curl } F\cdot\hat{n} + \iint_{S_2} \text{Curl } F\cdot\hat{n}$$

$\operatorname{div}(\operatorname{Curl} F) = 0 \quad \therefore \iint_{S_2} (\operatorname{Curl} F)\cdot \hat{n} = -\iint_{S_1} (\operatorname{Curl} F)\cdot \hat{n}$

(Above xy plane) = − (on xy plane)

∴ On xy plane

$F = [z, x, y]$ $\quad \operatorname{Curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \hat{i}[1] - \hat{j}[-1] + \hat{k}[1]$

$\hat{n} = \cancel{0} - \hat{k}$

$\operatorname{Curl} F = [1, -1, 1]$

$$\iint_{S_2} (\operatorname{Curl} F)\,\hat{n} = -\iint_{S_1} [1,-1,1]\cdot[0,0,-1]\, dS$$

$$= \iint dS = \boxed{\pi a^2} \left[\begin{array}{c} \text{Circle of radius } a \\ x^2+y^2=a^2 \end{array}\right]$$

Now $\int \vec{F}\cdot dr \Rightarrow$ with $x^2+y^2=a^2$ as Boundary $z=0$

$x = a\cos t \quad 0 \le t \le 2\pi$

$y = a\sin t$

$z = 0 \qquad F = [0, a\cos t, a\sin t]$

$$\int F(t)\cdot \frac{dr}{dt}\, dt = \int_0^{2\pi} [0, a\cos t, a\sin t]\cdot[-a\sin t, a\cos t, 0]\, dt$$

$$= \int_0^{2\pi} a^2\cos^2 t\, dt = a^2 \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= a^2 \cdot \frac{1}{2}\left(t + \sin 2t\right)_0^{2\pi}$$

$$= \boxed{\pi a^2}$$

$\therefore \iint_{S_2} (\operatorname{Curl} F)\cdot \hat{n} = \int F\cdot dr$

Stoke's is verified.

Please check (Q) 29 for whole Construction

SECTION – B

5. (a) Solve dy/dx + (x – y – 2)/(x – 2y – 3) = 0. **[10]**

Given $\frac{dy}{dx} = -\left(\frac{x-y-2}{x-2y-3}\right)$

Let $x = X + h$, $y = Y + k$ such that $h - k - 2 = 0$, $h - 2k - 3 = 0$

$\therefore h = 1, k = -1$

$\boxed{x = X + 1,\ y = Y - 1}$

$\therefore dy = dY$, $dx = dX$

$\frac{dY}{dX} = -\left(\frac{X-Y}{X-2Y}\right)$

Let $Y = vX$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$v + X\frac{dv}{dX} = -\left[\frac{X(1-v)}{X(1-2v)}\right]$

$X\frac{dv}{dX} = \frac{v-1}{1-2v} - v$

$X\frac{dv}{dX} = \frac{v-1-v+2v^2}{1-2v}$

$X\frac{dv}{dX} = \frac{2v^2-1}{1-2v}$

$\int \frac{1-2v}{2v^2-1} = \int \frac{dX}{X}$

$-\frac{1}{2}\int \frac{4v}{2v^2-1}dv$

$-\frac{1}{2}\log|2v^2-1|$

$\log X + \frac{1}{2}\log|2v^2-1| = \log|cX|$

$\log(cX\sqrt{2v^2-1})$

$-\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|$

$\int \frac{1}{1-2v^2}dv$

$\frac{1}{2}\left[\frac{1}{\sqrt{2}v+1} - \frac{1}{\sqrt{2}v-1}\right]$

$\frac{(\sqrt{2}v+1)-(\sqrt{2}v-1)}{(\sqrt{2}v+1)(\sqrt{2}v-1)}$

$-\frac{1}{2\sqrt{2}}\int\frac{(1+\sqrt{2}v)+(1-\sqrt{2}v)}{(1+\sqrt{2}v)(1-\sqrt{2}v)}$

(CIOMLA)

$-\frac{1}{2\sqrt{2}}\int\left(\frac{1}{1+\sqrt{2}v} + \frac{1}{1-\sqrt{2}v}\right)$

$-\frac{1}{2\sqrt{2}}\left[\frac{1}{\sqrt{2}}\log|1+\sqrt{2}v| - \frac{1}{\sqrt{2}}\log|1-\sqrt{2}v|\right]$

$\frac{1}{2\sqrt{2}}\log\left|\frac{1-\sqrt{2}v}{1+\sqrt{2}v}\right| - \frac{1}{2}\log|2v^2-1| = \log|cX|$

$\left(\frac{\sqrt{2}v-1}{\sqrt{2}v+1}\right)^{1/\sqrt{2}} \cdot \frac{1}{\sqrt{2v^2-1}} = c'X^2$

$\left(\frac{\sqrt{2}(y+1)-(x-1)}{\sqrt{2}(y+1)+(x-1)}\right)^{1/\sqrt{2}} \cdot \frac{1}{\sqrt{2(y+1)^2-(x-1)^2}} = C(x-1)$

~~$X\sqrt{2Y^2-X^2} = Ce^{-\frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}Y/X)}$~~

$\boxed{\sqrt{2(y+1)^2-(x-1)^2} = Ce^{-\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}(y+1)}{x-1}\right)}}$

5. (b) (i) Prove that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\left(\frac{2}{3}\right)$.

(ii) If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$, *find* $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$ **[10]**

$\int_0^\infty \frac{f(bx)-f(ax)}{bx-ax} = \log(a/b)$

(i) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$

We know that $\boxed{\int_0^\infty \frac{\cos bx - \cos ax}{bx-ax}\,dx = \log\left(\frac{a}{b}\right)} \rightarrow (1)$

$\lim_{t\to 0} \frac{\cos bx - \cos ax}{x}$

Let $2t = p$

$\int_0^\infty \frac{\cos 3p - \cos 2p}{p/2} \times \frac{dp}{2} = \int_0^\infty \frac{\cos 3p - \cos 2p}{p}\,dp$

$\int_0^\infty \frac{\cos 3p - \cos 2p}{3p-2p}\,dp = \int_0^\infty \frac{\cos 3p - \cos 2p}{(3-2)p}\,dp = \log\left(\frac{2}{3}\right)$

[By (1)]

$\boxed{\therefore \log(2/3)}$

(ii) Let $F(s) = \frac{s}{(s^2+1)^2}$ $\quad L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$f(t) = \frac{1}{2}t\sin t.$

$t\sin t$

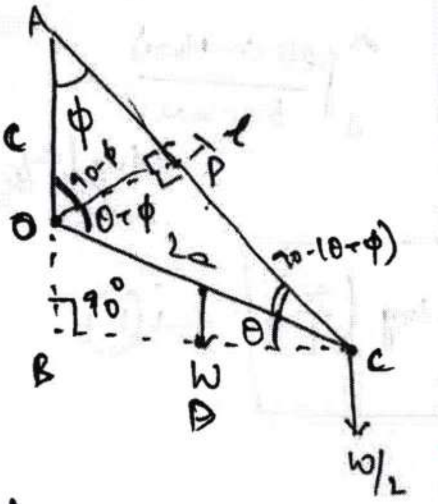
$\therefore L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{\frac{s}{(s^2+1)^2}}{s}\right] = \int_0^t \frac{1}{2}t\sin t$

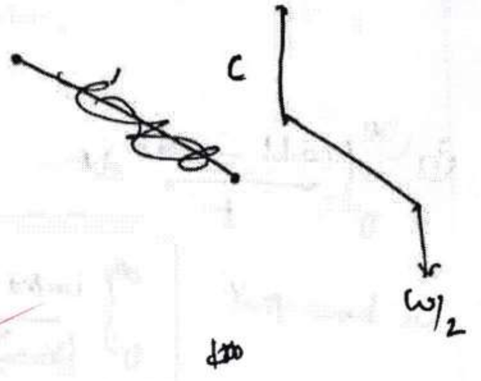
$= \frac{1}{2}\left[t\int \sin t - \int \int \sin t\right]$ $\quad = \frac{1}{2}\left[-t\cos t + \sin t\right]_0^t$

$= \frac{1}{2}\left[-t\cos t - \int -\cos t\right]$ $\quad = \frac{1}{2}\left[-t\cos t + \sin t\right] = \boxed{\frac{1}{2}(t\cos t - \sin t)}$

08

5. (c) A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight W/2, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is $\ell W/c$. **[10]**





$90+x+\theta+\phi=180°$

$x=90-(\theta+\phi)$

$\phi+90-\theta$

$+90-\theta-\phi$

$\phi+90+\theta$

$90-\theta-\phi+\theta$

$90-\phi$

By moments principle $\Rightarrow$

$$\cancel{-T(OP)}\; T(OP)-W(BD)-\frac{W}{2}(BC)=0$$

$$\boxed{OP=c\sin\phi} \qquad BD=a\cos\theta$$
$$BC=2a\cos\theta.$$

$$T(c\sin\phi)=W(a\cos\theta)+\frac{W}{2}(2a\cos\theta)$$

$$\boxed{T(c\sin\phi)=2Wa\cos\theta}$$

Also $BC = l\sin\phi = 2a\cos\theta$ [in ΔABC]

$$T=\frac{2Wa\cos\theta}{c\sin\phi}=\frac{\cancel{2}W\times\frac{l\sin\phi}{\cancel{2}}}{c\sin\phi}=\frac{Wl}{c}$$

$\therefore T=\frac{lW}{c}$

Hence proved.

08

5. (d) A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is $2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$. **[10]**

In a S.H.M., we know that $v^2 = \mu(a^2 - x^2)$

where v is velocity at distance x from centre

$\sqrt{\mu}$ is ~~Accele~~ Angular velocity

$T = \frac{2\pi}{\sqrt{\mu}}$ [T is Time period]

At x_1 $v_1^2 = \mu(a^2 - x_1^2)$

$v_2^2 = \mu(a^2 - x_2^2)$

$\frac{v_1^2}{\mu} + x_1^2 = \frac{v_2^2}{\mu} + x_2^2$

$x_1^2 - x_2^2 = \frac{1}{\mu}(v_2^2 - v_1^2)$

$\therefore \mu = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$ $\therefore T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}}$

$$\boxed{\therefore T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}}$$

$\boxed{\therefore \text{Proved}}$

8

x_1, and x_2 are distance from centre, v_1 and v_2 are velocities.

5. (e) Find the work done in moving the particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1, z=0$ under the field of force given by $\vec{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$

[10]

$\vec{F} = (2x-y+z)\vec{i} + (x+y-z^2)\vec{j} + (3x-2y+4z)\vec{k}$

Given boundary $= \frac{x^2}{25}+\frac{y^2}{16} = 1$, $z=0$

Let $\boxed{x = 5\cos t \quad y = 4\sin t \quad z=0}$ t is parameter $\quad r(t) = [5\cos t, 4\sin t, 0] \quad$ [Parametric equation in t]

$0 \le t \le 2\pi$

Work done $= \int \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(r(t))\cdot \frac{d\vec{r}}{dt}\, dt$

$= \int_0^{2\pi} [(10\cos t - 4\sin t), (5\cos t + 4\sin t), (15\cos t - 8\sin t)] \cdot [-5\sin t, 4\cos t, 0]\, dt$

$= \int_0^{2\pi} (-50\sin t\cos t + \cancel{20\cos t} + 20\cos^2 t + 16\sin t\cos t)\, dt$

$= \int_0^{2\pi} (20 - 17\sin 2t)\, dt \qquad \left[\begin{array}{l}\sin^2 t + \cos^2 t = 0 \\ \int_0^{2\pi} \cos t\, dt = 0\end{array}\right]$

$= 20[\int dt] + \frac{17}{2}[\cos 2t]_0^{2\pi}$

$= \boxed{40\pi}$ $\qquad \therefore$ Work done $= 40\pi$

$\therefore$ Work done in move particle around the ellipse is 40π

8

6. (a) ~~Justify that a differential equation of the form :~~

~~$[y + x f(x^2 + y^2)]\, dx+[y f(x^2 + y^2) - x]\, dy = 0$,~~

~~where $f(x^2 + y^2)$ is an arbitrary function of $(x^2 + y^2)$, is not an exact differential equation and $\frac{1}{x^2+y^2}$ is an integrating factor for it. Hence solve this differential equation for $f(x^2 + y^2) = (x^2 + y^2)^2$.~~ **[12]**

Contd. of 5 (c)

$$\int \frac{1-2V}{2V^2-1} = \int \frac{dX}{X} \quad \left(\text{Contd. of } 5 \cdot c)\right)$$

$$\frac{1}{2}\int \frac{(\sqrt{2}V+1)+(\sqrt{2}V-1)}{(\sqrt{2}V+1)(\sqrt{2}V-1)} + \frac{1}{2}\int \frac{4V}{2V^2-1} = \log|cX|$$

$$\frac{1}{2}\log\left|\frac{\sqrt{2}V-1}{\sqrt{2}V+1}\right| - \frac{1}{2}\log|2V^2-1| = \log|cX|$$

$$\log\left|\frac{\cancel{\sqrt{2}V-1}}{\sqrt{2}V+1\,(\cancel{\sqrt{2}V-1})(\sqrt{2}V+1)}\right| = \log|cX^2|$$

$$\frac{1}{(\sqrt{2}V+1)^2} = cX^2 \qquad \boxed{V = Y/X}$$

$$c(X+\sqrt{2}Y)^2 = 1$$

$$\frac{\cancel{X^2}}{(\sqrt{2}Y+X)^2} = c\cancel{X} \qquad c((x-1)+\sqrt{2}(y+1))^2 = 1$$

$$\boxed{\begin{aligned} X &= x-1 \\ Y &= y+1 \end{aligned}}$$

$$\boxed{(x+\sqrt{2}y+\sqrt{2}-1)^2 = k \quad (k \text{ is const})}$$

$$(x+\sqrt{2}y+\sqrt{2}-1)^2 = c$$

6. (b) Solve by the method of variation of parameters $d^2y/dx^2 + (1 - \cot x)(dy/dx) - y \cot x = \sin^2 x$. **[14]**

8. (a) If $\mathbf{F}\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\mathbf{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\mathbf{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\mathbf{k}$

prove that (i) $\mathbf{F} = \mathbf{r} \times \nabla f$, (ii) $\mathbf{F} \cdot \mathbf{r} = 0$, (iii) $\mathbf{F} \cdot \nabla f = 0$.

[06]

Given $F = \left[y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y},\ z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z},\ x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right]$

$r = (x, y, z)$

$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

$$r \times \nabla f = \begin{vmatrix} i & j & k \\ x & y & z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right] - \vec{j}\left[x\frac{\partial f}{\partial z} - y\frac{\partial f}{\partial x}\right] + \vec{k}\left[x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right]$$

$$= \vec{F}$$

$\boxed{\text{Proved}}$

(ii) $\vec{F}\cdot\vec{r} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)x + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)y + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)z$

$= \sum xy\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial z}\right) + xz\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y}\right) + yz\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}\right)$

$= 0$ (Proved)

(iii) $F\cdot\nabla f = \left[y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right]\frac{\partial f}{\partial x} + \left[z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right]\frac{\partial f}{\partial y} + \left[x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right]\frac{\partial f}{\partial z}$

$= \sum y\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)$

$= 0$ (Proved)

05

05

8. (b) Find div grad r^m and verify that $\nabla \times \nabla r^m = 0$. **[12]**

$\nabla r^m = m r^{m-1} \nabla r$ $\left[\begin{array}{l} \nabla f(r) = f'(r) \nabla r \\ \nabla r = \frac{\vec{r}}{r} \end{array}\right]$

$= m r^{m-1} \cdot \frac{\vec{r}}{r}$

$= m r^{m-2} \vec{r}$

$\text{div}\left(m r^{m-2} \vec{r}\right) \Rightarrow$ Let $\phi = m r^{m-2}$

$A = \vec{r}$

$\text{div}(\phi A) = \phi \, \text{div} A + \nabla\phi \cdot A$

$= m r^{m-2} \text{div} \vec{r} + \nabla m r^{m-2} \cdot \vec{r}$ $\left(\text{div} \vec{r} = 3\right)$

$= 3 m r^{m-2} + m(m-2) r^{m-3} \cdot \frac{\vec{r}}{r} \cdot \vec{r}$

$= 3m r^{m-2} + m(m-2) r^{m-2}$

$= \boxed{r^{m-2}(m^2 + m)}$

$\text{Curl}\left(\nabla r^m\right) = \text{Curl}\left(m r^{m-2} \vec{r}\right)$ $\phi = m r^{m-2}$

$A = \vec{r}$

$= m r^{m-2} \text{Curl}(\vec{r}) + \left(\nabla m r^{m-2}\right) \times \vec{r}$ $\left[\text{Curl}\, \vec{r} = 0\right)$

$= 0 + m(m-2) r^{m-3} \cdot \frac{\vec{r}}{r} \times \vec{r}$

$\left(\vec{r} \times \vec{r}\right) = 0$ ✓

$= 0$

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8. (c) Verify Green's theorem in a plane for

$$\oint_c \left[\left(x^2 - 2xy\right) dx + \left(x^2 y + 3\right) dy \right]$$

where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$. **[14]**

$\underline{C_1} \to (2,4) \to (0,0)$ $0 \le t \le 4$

$y^2 = 8x$ $\boxed{y = t \quad x = t^2/8}$

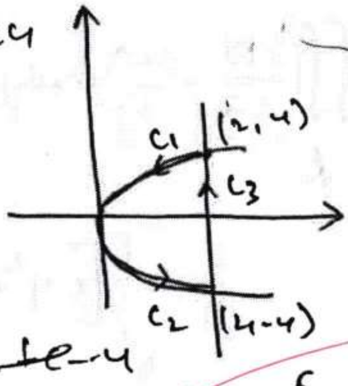
$\underline{C_2} \to (0,0) \to (2,-4)$

$y^2 = 8x$ $\boxed{y = t \quad x = t^2/8}$ $0 \ge t \ge -4$

$-4 \le t \le 0$

$\underline{C_3} \to (2,-4) \to (2,4)$

$\boxed{x = 2}$ $y = t$ $\boxed{-4 \le t \le 4}$



$\frac{t^5}{256}$ $2\frac{t^2}{8}t$

$$\int_{C_1} (x^2 - 2xy)dx + (x^2y + 3)dy = \int_4^0 \left(\frac{t^4}{64} - \frac{t^3}{4}\right)\frac{t}{4}dt + \int \left(\frac{t^5}{64} + 3\right)dt$$

$$= \int_4^0 \frac{5t^5}{256}dt - \int_4^0 \frac{t^4}{16}dt + 3\int_4^0 dt = -\frac{4^6}{256 \times 6} + \frac{4^5}{80} + 12 \quad \frac{t^5}{64 \times 4} + \frac{4t^5}{64 \times 4}$$

$= -\frac{28}{15}$ $\underline{5t^5}$

$= \frac{1}{32} \times 16 \times 16 \times 4 - \frac{1}{16} \times 16 \times 16 + 12$

$\frac{5}{256} \times 16 \times 16 \times 16$

$= 32 - 16 + 12$; $\frac{5}{256} \times \frac{1}{6_3} \times 16 \times 16 \times 16 - \frac{1}{16} \times \frac{1}{5} \times 16 \times 16 \times 4 - 12$

$= 28$

$= \frac{40}{3} - \frac{64}{5} - 12 = \frac{40}{3} - \frac{124}{5}$

12

$$\underline{C_2} \quad \int_0^{-4} \frac{5t^5}{256}dt - \int_0^{-4} \frac{t^4}{16}dt + 3\int_0^{-4} dt = \left[\frac{t^6}{256 \times 6}\right]_0^{-4} - \left[\frac{t^5}{80}\right]_0^{-4} - 12$$

$= \frac{5}{256} \times \frac{1}{6}(64)^6 - \frac{1}{80}(-1)(4)^5 - 12 = \frac{-40}{3} + \frac{124}{5} = -$

$\frac{52}{15}$

$C_3 = \int_{-4}^{4} (4t+3)\,dt = \left(2t^2+3t\right)_{-4}^{4} = 3[8] = 24.$

$\therefore C_1 + C_2 + C_3 = 24 + \frac{12}{15} + \frac{24}{15} = \boxed{\frac{128}{5}}$

$\iint_R \left(\frac{\partial xy}{\partial x} - \frac{\partial xy}{\partial y}\right) dx\,dy = \int_0^2 \int_{-2\sqrt{2x}}^{2\sqrt{2x}} (2xy + 2x)\,dx\,dy$

$= \int_0^2 \left[x y^2 + 2xy\right]_{-2\sqrt{2x}}^{2\sqrt{2x}} = \int_0^2 2x \times 4\sqrt{2}\sqrt{x}$

$= \int 2x \times 2\sqrt{8x}$ $= \int_0^2 \cancel{8\sqrt{2}\, x^{3/2}}\,dx$ $= 8\sqrt{2}\int_0^2 x^{3/2}\,dx = 8\sqrt{2} \times \frac{2}{5} \times 4\sqrt{2}$

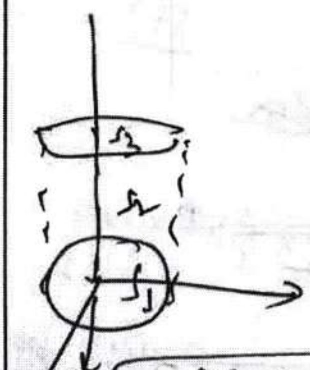
$= \boxed{128/5}$

$\int x^{3/2}\,dx$ $x^{5/2} \times \frac{2}{5}$ $2^{5/2}$ $(2^5)^{1/2}$ $\sqrt{32}$

8. (d) Verify the divergence theorem for

$\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$

taken over the region bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **[18]**



S_1, $z = 0$, $x^2+y^2=4$

$F = [4x, -2y^2, 0]$ $\hat{n} = -\hat{k}$

~~Curl F = $\begin{vmatrix} \vec{i} & \vec{j} \\ \partial/\partial x & \partial/\partial y \end{vmatrix}$~~

$\boxed{\vec{F}\cdot\hat{n} = 0}$

$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,dS = 0}$

$\iint_{S_2} \vec{F}\cdot\hat{n}\,dS$

S_3, $z = 3$, $x^2+y^2=4$ $\hat{n} = \hat{k}$

$F = [4x, -2y^2, 9]$

16

$$= \iint 9\, ds = 9 \times \pi (4) = 36\pi \left[\text{Radius} = 4\right]$$

$\underline{S_2} \rightarrow x^2+y^2=4 \qquad x = 2\cos\theta \qquad r=2$

$z = 0 \rightarrow 3 \qquad y = 2\sin\theta \qquad 2$

$$\hat{n} = \frac{2x, 2y, 0}{\sqrt{4x^2+4y^2}} = \left[\frac{x}{2}, \frac{y}{2}, 0\right] \qquad dxdy = 2\,d\theta\, dz$$

$$\vec{F}\cdot\hat{n} = (4x, -4y^2, 0)\cdot\left(\frac{x}{2}, \frac{y}{2}, 0\right) = \left(2x^2 - y^3\right)$$

$$\iint (2x^2 - y^3)\, ds \qquad |J|\, d\theta\, dz \qquad \boxed{|J| = 2}$$

$$\iint (8\cos^2\theta - 8\sin^3\theta) \times 2\, d\theta\, dz$$

$$16\int_0^{2\pi}\int_0^{3}(\cos^2\theta - \sin^3\theta)\, d\theta\, dz$$

$$16\int(\cos^2\theta - \sin^3\theta)\times 3 = 48\int_0^{2\pi}(\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - 48\int \sin^3\theta\, d\theta$$

$$\boxed{\int_0^{2\pi}\sin^3\theta\, d\theta = 0}$$

$$= 48 \times \frac{1}{2} \times 2\pi = 48\pi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \sum \iint_{S_1} \vec{F}\cdot\hat{n} = (48+36)\pi = 84\pi.$$

$\underline{\text{Now}}$ $\quad \text{div} F = 4 - 4y + 2z$

$$\iiint \text{div} F\, dV = \iiint (4 - 4y + 2z)\, dx\, dy\, dz$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3}(4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z-4zy+z^2\right]_0^3 = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-12y+9)\,dy\,dx$$

$$= \int_{-2}^{2}\left[21y-6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = \int_{-2}^{2} 42\sqrt{4-x^2}\,dx$$

$$= 42\times\int_{-2}^{2}\sqrt{4-x^2}\,dx \qquad x = 2\sin\theta$$

$$= 42\times\int_{-\pi/2}^{\pi/2} 2\cos\theta\times 2\cos\theta\,d\theta = 42\times 4\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 42\times 4\times\frac{1}{2}\times\pi$$

$$= \boxed{84\pi}$$

$\therefore$ Div Thm is verified