Mains Test Series – 2018

Test – 06 (Paper – II), Answer Key

1(a) Union of two Subgroups is a Subgroup iff one of them is contained in the other.

Sol'n: Let H_1 & H_2 be two Subgroups of G.

Let $H_1 \subset H_2$ (or) $H_2 \subset H_1$.

To P.T $H_1 \cup H_2$ is a Subgroup of G.

Since $H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$ is a Subgroup

Since $H_2 \subset H_1 \Rightarrow H_2 \cup H_1 = H_1$ is a Subgroup

$\therefore H_1 \cup H_2$ is a Subgroup.

Conversely suppose that $H_1 \cup H_2$ is a Subgroup.

To P.T $H_1 \subset H_2$ or $H_2 \subset H_1$

If possible suppose that $H_1 \not\subset H_2$ (or) $H_2 \not\subset H_1$

Since $H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1$ and $a \notin H_2$ —— (1)

Again $H_2 \not\subset H_1 \Rightarrow \exists\, b \in H_2$ and $b \notin H_1$ —— (2)

from (1) & (2) we have

$a \in H_1$ and $b \in H_2 \Rightarrow a, b \in H_1 \cup H_2$.

Since $H_1 \cup H_2$ is a Subgroup of G.

$\therefore ab \in H_1 \cup H_2$

$\Rightarrow ab \in H_1$ (or) $ab \in H_2$

Let $ab \in H_1$

let $a \in H_1 \Rightarrow a^{-1} \in H_1$ ($\because H_1$ is Subgroup)

$\therefore a^{-1} \in H_1$, $ab \in H_1$

$\Rightarrow a^{-1}(ab) \in H_1$ (by closure axiom of H)

$\Rightarrow (a^{-1}a)b \in H_1$ (by associ)

$\Rightarrow eb \in H_1$ (by inverse)

$\Rightarrow b \in H_1$ (by identity)

which is contradiction to $b \notin H_1$.

Let $ab \in H_2$

let $b \in H_2 \Rightarrow b^{-1} \in H_2$

$\therefore b^{-1} \in H_2$, $ab \in H_2$

$\Rightarrow (ab)b^{-1} \in H_2$ by closure

$\Rightarrow a(bb^{-1}) \in H_2$

$\Rightarrow ae \in H_2$

$\Rightarrow a \in H_2$

which is contradiction to $a \notin H_2$

$\therefore$ our assumption that $H_1 \not\subset H_2$ (or) $H_2 \not\subset H_1$ is wrong

$\therefore$ Either $H_1 \subset H_2$ (or) $H_2 \subset H_1$

1(b) Let R be the ring of 3×3 matrices over reals. show that $S = \left\{ \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} / x \text{ real} \right\}$ is a subring of R and has unity different from unity of R.

Sol^n: Let $R = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} / a,b,c,d,e,f,g,h,i \in \mathbb{R} \right\}$

be the ring of all 3×3 matrices over reals w.r.t $+^n$ and $\times^n$.

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$

$\therefore S \neq \phi$

let $A = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$, $B = \begin{bmatrix} y & y & y \\ y & y & y \\ y & y & y \end{bmatrix} \in S$

then we have

$$A+(-B) = \begin{bmatrix} x+(-y) & x+(-y) & x+(-y) \\ x+(-y) & x+(-y) & x+(-y) \\ x+(-y) & x+(-y) & x+(-y) \end{bmatrix}$$

$$\in S \quad (\because x+(-y) \in \mathbb{R})$$

and we have

$$AB = \begin{bmatrix} 3xy & 3xy & 3xy \\ 3xy & 3xy & 3xy \\ 3xy & 3xy & 3xy \end{bmatrix} \in S$$

$\therefore A+(-B) \in S$ & $AB \in S$

$\therefore$ S is a subring of R

we have $AB = A \Rightarrow 3xy = x \Rightarrow \boxed{y = 1/3}$

$$\therefore \forall A \in S \ \exists \ B = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \in S \text{ such that}$$

$AB = A = BA$

Clearly the unity element of S different from the unity $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of R.

1(C) Prove that Every Infinite bounded subset of real numbers has a limit point.

Sol (i). Let S be an infinite bounded subset of $\mathbb{R}$.

(a) S is bounded $\Rightarrow \exists$ real numbers k & K s.t $k \le s \le K$ $\forall s \in S$.

(b) Let a set T be defined as follows:

$T = \{t \mid t >$ finitely many elements of $S\}$

(c) To prove that $T \ne \phi$

$k \le s \; \forall s \in S \Rightarrow k$ is greater than no element of S

$\Rightarrow k \in T \Rightarrow T \ne \phi$.

(d) To prove that T is bounded above:

for any $\epsilon > 0$, $K + \epsilon > K \ge s \; \forall s \in S$

$\Rightarrow K + \epsilon \notin T \Rightarrow K \notin T$

$\Rightarrow \forall t \in T$; $t < K \Rightarrow$ T is bounded above.

$\therefore$ T is non-empty bounded above subset of $\mathbb{R}$.

$\therefore$ T has the l.u.b. say u.

(e) To prove that u is a limit point of 'S'.

Let $(u-\epsilon, u+\epsilon)$ be any nbd of u

u is l.u.b of T $\Rightarrow \exists$ some $t \in T$ s.t $t > u-\epsilon$, $\epsilon > 0$

Now $t \in T \Rightarrow t >$ finitely many elements of 'S'.

$\Rightarrow u-\epsilon >$ finitely many elements of S

$\Rightarrow$ finitely many elements of 'S' lie to the left of $u-\epsilon$

$\Rightarrow$ infinitely many element of 'S' lie to the right of $u-\epsilon$. —— (1)

Also $u =$ l.u.b of $T \Rightarrow u+\epsilon \notin T$

$\Rightarrow u+\epsilon >$ infinitely many elements of S.

$\Rightarrow$ infinitely many elements of 'S' lie to the left of $u+\epsilon$. —— (2)

$u-\epsilon$ (u) $u+\epsilon$

combining (1) & (2),

$(u-\epsilon, u+\epsilon)$ has infinitely many elements of S. But $(u-\epsilon, u+\epsilon)$ is any nbd of 'u'

$\therefore$ every nbd of u has infinitely many elements of 'S'

Hence u is a limit point of 'S'.

1(d) Use Cauchy's theorem / Cauchy integral formula evaluate (i) $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ where $C: |z-i| = 2$

(ii) $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ where C is the circle $|z|=1$.

Sol^n: Let $\frac{z-1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$

$\therefore z-1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

Putting $z=-1$, we have $-2 = B(-1-2)$

$\Rightarrow \boxed{B = 2/3}$

Putting $z=2$, we have $1 = C(2+1)^2$

$\Rightarrow \boxed{C = 1/9}$

Putting $z=0$, we have $-1 = A(-2) + B(-2) + C$

$\Rightarrow -1 = -2A - 4/3 + 1/9$

$\Rightarrow 2A = -2/9 \Rightarrow \boxed{A = -1/9}$

$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = \frac{1}{9}\int_C \frac{1}{z+1}\,dz + \frac{2}{3}\int_C \frac{1}{(z+1)^2}\,dz + \frac{1}{9}\int_C \frac{1}{z-2}\,dz$

$|z-i|=2$ is a circle with centre i and radius 2 i.e. Centre is $(0,1)$ and radius is 2

Consider $\int_C \frac{1}{z+1}\,dz$

Here $z=-1$ is within the circle $|z-i|=2$ and $f(z)=1$

$\therefore \int_C \frac{1}{z+1}\,dz = 2\pi i(1) = 2\pi i$

Consider $\int_C \frac{1}{(z+1)^2}\,dz = \frac{2\pi i(0)}{2!} = 0$

$\therefore f'(z) = 0$ and $f'(-1) = 0$

consider $\int_C \frac{1}{z-2}\,dz$

Here $z=2$ lies outside the circle $|z-i|=2$ and $\frac{1}{z-2}$ is analytic in the $|z-i|=2$.

$\therefore$ By cauchy's integral theorem,

$$\int_C \frac{1}{z-2}\,dz = 0$$

$$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = -\frac{1}{9}(2\pi i) + 0 + 0 = \frac{-2\pi i}{9}$$

(iii) we have $f(z) = \sin^6 z$

Then $f(z)$ is analytic within and on C.

Using the n^{th} order derivative formula for the function $f(z)$, we have $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

Here $n=2$, $a=\pi/6$, $f(z) = \sin^6 z$.

$$\therefore \frac{2!}{2\pi i}\int_C \frac{\sin^6 z}{(z-\pi/6)^{2+1}}\,dz = f''\left(\frac{\pi}{6}\right) = \left[f''(z)\right]_{z=\pi/6}$$

$$= \left(30\sin^4 z\cos^2 z - 6\sin^6 z\right)_{z=\pi/6}$$

$$= 6\left\{\sin(\pi/6)^4\right\}\left[5\cos^2(\pi/6) - \sin^2(\pi/6)\right]$$

$$= 6\cdot\frac{1}{16}\left(5\cdot\frac{3}{4} - \frac{1}{4}\right) = \frac{21}{16}$$

Hence $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{21}{16}\pi i$.

1(c) Write the dual of the following problem:

Minimize $Z = x_1 + x_2 + x_3$

subject to the constraints

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_3 \geq 0$ and x_2 is unrestricted.

Sol'n: First convert the problem into standard primal form, as follows:

Change the objective function of minimization one, that is

$\text{Max } z' = -x_1 - x_2 - x_3$

where $z' = -z$.

The inequality $2x_2 - x_3 \geq 4$ can be written as

$-2x_2 + x_3 \leq -4$.

The equation $x_1 - 3x_2 + 4x_3 = 5$ can be expressed as pair of inequalities

$x_1 - 3x_2 + 4x_3 \leq 5$

$x_1 - 3x_2 + 4x_3 \geq 5$ (or) $-x_1 + 3x_2 - 4x_3 \leq -5$

Since the variable x_2 is unrestricted in sign, the given LP problem can be transformed into standard primal problem by substituting

$x_2 = x_2' - x_2''$ where $x_2' \geq 0,\ x_2'' \geq 0$.

Therefore, standard primal becomes:

$$Max\, z' = -x_1 - (x_2' - x_2'') - x_3$$

subject to the constraints

$$x_1 - 3(x_2' - x_2'') + 4x_3 \le 5$$
$$-x_1 + 3(x_2' - x_2'') - 4x_3 \le -5$$
$$x_1 - 2(x_2' - x_2'') \le 3$$
$$-2(x_2' - x_2'') - x_3 \le -4$$
$$x_1, x_2', x_2'', x_3 \geq 0.$$

The dual of the given standard primal is,

$$Min\, z_0 = 5w_1 - 5w_2 + 3w_3 - 4w_4$$

subject to the constraints

$$w_1 - w_2 + w_3 \geq -1$$
$$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$$
$$3w_1 - 3w_2 + 2w_3 + 2w_4 \geq 1$$
$$4w_1 - 4w_2 - w_4 \geq -1$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Again, we may write

$$Min\, w = 5w' + 3w_3 - 4w_4$$

Subject to

$$w' + w_3 \geq -1$$
$$-3w' - 2w_3 - 2w_4 \geq -1$$
$$3w' + 2w_3 + 2w_4 \geq 1$$
$$4w' - w_4 \geq -1$$

$w_3, w_4 \geq 0$ and w' is unrestricted.

(or) This dual can be written in more compact form as

$$Max\, z_0 = -5w' - 3w_3 + 4w_4$$

subject to

$$-w' - w_3 \le 1$$
$$3w' + 2w_3 + 2w_4 = 1$$
$$-4w' + w_4 \le 1$$

$w_3, w_4 \geq 0$ and w' is unrestricted.

2(a) → Let H be a subgroup of group G. Then $W = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G.

Sol'n: Let $a = ghg^{-1}$ and $b = gh_1g^{-1}$ be two elements of K. Then $\left[gHg^{-1} = \{ghg^{-1} / h \in H\}\right]$

$$ab^{-1} = ghg^{-1}(gh_1g^{-1})^{-1}$$
$$= ghg^{-1}\left((g^{-1})^{-1} h_1^{-1} g^{-1}\right)$$
$$= ghg^{-1}gh_1^{-1}g^{-1}$$
$$= ghh_1^{-1}g^{-1}$$

Now $h, h_1 \in H$ and H is a subgroup of G. Hence $hh_1^{-1} \in H$.

Then from (*) above

$$ab^{-1} = g(hh_1^{-1})g^{-1} \in gHg^{-1}$$

Hence gHg^{-1} is a subgroup of G for all $g \in G$.

Since the intersection of subgroups is a subgroup, W is a subgroup G. Let $x \in G, w \in W$. Then $w \in gHg^{-1}\ \forall g \in G$.

we have to show that $xwx^{-1} \in gHg^{-1}\ \forall g \in G$, which in turn will yield that $xwx^{-1} \in W$.

let $g \in G$ and let us suppose that $xwx^{-1} \in gHg^{-1}$.

Then $xwx^{-1} = ghg^{-1}$ for some $h \in H$.

thus $g^{-1}xwx^{-1}g = h \in H$.

$$\Rightarrow (g^{-1}x)w(g^{-1}x)^{-1} \in H$$

put $y = x^{-1}g$. Then $g = xy$. Hence in order to show that $xwx^{-1} \in gHg^{-1}$ for a given $g \in G$,

first we need to find $y \in G$ such that $g = xy$.

Since $g = x(x^{-1}g)$,
we can choose $y = x^{-1}g$.
So there exists $y \in G$ such that $g = xy$.
Since $y \in G$, we have $w \in yHy^{-1}$ and $w = yhy^{-1}$ for some $h \in H$.

Hence $xwx^{-1} = x(yhy^{-1})x^{-1}$

$$= xyhy^{-1}x^{-1}$$
$$= (xy)h(xy)^{-1}$$
$$= ghg^{-1} \in gHg^{-1}$$

Since $g \in G$ was arbitrary,

$$xwx^{-1} \in gHg^{-1} \text{ for all } g \in G.$$

Thus, W is a normal subgroup of G.

2(b). If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, Prove that f is uniformly continuous on $\mathbb{R}$.

Sol'n: Let f be continuous at a point $c \in \mathbb{R}$.

Let us choose $\epsilon > 0$. There exists a +ve δ such that

$|f(c+h) - f(c)| < \epsilon$ for all h satisfying $|h| < \delta$.

But $|f(c+h) - f(c)| = |f(c) + f(h) - f(c)| = |f(h)|$

Continuity of f at c implies $|f(h)| < \epsilon$ for all h satisfying $|h| < \delta$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that

$$|x_1 - x_2| < \delta$$

Then $|f(x_1 - x_2)| < \epsilon$

$f(x+y) = f(x) + f(y)$ gives $f(0+0) = f(0) + f(0)$

(or) $f(0) = 2f(0)$

$\Rightarrow f(0) = 0$

Also $0 = f(0) = f(x+(-x)) = f(x) + f(-x)$

$\therefore f(-x) = -f(x) \ \forall x \in \mathbb{R}$

$|f(x_1 - x_2)| = |f(x_1) + f(-x_2)| = |f(x_1) - f(x_2)|$

Thus $|f(x_1) - f(x_2)| < \epsilon$ for any two points x_1, x_2 in $\mathbb{R}$ satisfying $|x_1 - x_2| < \delta$. δ depends on ϵ only and not on the points x_1, x_2 in $\mathbb{R}$. This proves that f is uniformly continuous on $\mathbb{R}$.

2(c). The integral function $f(z)$ satisfies everywhere the inequality $|f(z)| \le A|z|^k$ where A and k are positive constants. Prove that $f(z)$ is a polynomial of degree not exceeding k.

Sol'n: Since $f(z)$ is analytic in the finite part of the plane, therefore by Taylor's theorem.

$f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $|z| < R$

Now, if $\max |f(z)| = M(r)$ on the circle $|z| = r \ (r < R)$, then by Cauchy's inequality, we have

$$|a_n| \le \frac{M(r)}{r^n} \text{ for all } n$$

$$= \frac{A(|z|^k)}{r^n} \quad \text{Since } M(r) = |f(z)| = A|z|^k \text{ when } |z| \to \infty.$$

$$= \frac{Ar^k}{r^n} = Ar^{k-n}$$

Hence as $r \to \infty$, the right handside tends to zero. Since $n > k$.

i.e. $a_n = 0$ for $n > k$.

i.e. all the coefficients a_n for which $n>k$ becomes zero.

$\therefore f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$

which is a polynomial of degree k.

2(d) Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\, d\theta = \pi\frac{1-p+p^2}{1-p}$, $0<p<1$.

Sol^n: Let $\displaystyle I = \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta}{1-2p\cos 2\theta+p^2}\, d\theta$$

$$= \frac{1}{2} \text{ real Part of } \int_0^{2\pi} \frac{1+e^{6i\theta}}{1-p\left(e^{2i\theta}+e^{-2i\theta}\right)+p^2}\, d\theta.$$

$$= \frac{1}{2} \text{ real part of } \int_C \frac{1+z^6}{1-p\left(z^2+\frac{1}{z^2}\right)+p^2}\, \frac{dz}{iz}$$

by putting $e^{i\theta} = z$

$\Rightarrow d\theta = \frac{dz}{iz}$

where C denotes the unit circle $|z|=1$.

$$= \frac{1}{2} \text{real part of } \frac{1}{i}\int_C \frac{z(1+z^6)}{(1-pz^2)(z^2-p)}\, dz$$

$$= \frac{1}{2} \text{real part of } \int_C f(z)\, dz \quad \text{where } f(z) = \frac{z(1+z^6)}{(1-pz^2)(z^2-p)}$$

poles of $f(z)$ are given by $(1-pz^2)(z^2-p)=0$

Thus $z=\pm\sqrt{p}$ and $z=\pm\frac{1}{\sqrt{p}}$ are the simple poles.

The only poles which lie with in C are $z=\pm\sqrt{p}$ as $p<1$.

Residue at $z=-\sqrt{p}$ is

$$\underset{z\to\sqrt{p}}{Lt}\ (z-\sqrt{p})\ \frac{z(1+z^6)}{(1-pz^2)(z^2-p)} = \frac{1}{2}\ \frac{1+p^3}{1-p^2}$$

and residue at $z=-\sqrt{p}$ is

$$\underset{z\to\sqrt{p}}{Lt}\ (z+\sqrt{p})\ \frac{z(1+z^6)}{(1-pz^2)(z^2-p)}$$

$$= \frac{1}{2}\ \frac{1+p^3}{1-p^2}$$

Sum of the residues $= \dfrac{1+p^3}{1+p^2}$

Hence by Cauchys Residue thedem, we have

$\int_C f(z)\,dz = 2\pi i \times$ Sum of residues within the Contour.

$$= 2\pi i\ \frac{1+p^3}{1-p^2}$$

$$I = \frac{1}{2}\text{ real part of } \frac{1}{i}\int_C f(z)\,dz$$

$$= \frac{1}{2}\text{ real part of } \frac{1}{i}\ 2\pi i\ \frac{1+p^3}{1-p^2}$$

$$= \frac{1}{2}\text{ real part of } 2\pi\ \frac{1+p^3}{1-p^2}.$$

$$= \pi\left(\frac{1-p+p^2}{1-p}\right)$$

3(a) (i) If in a ring R, with unity $(xy)^2 = x^2y^2$ for all $x, y \in R$ then show that R is commutative.

(ii) Show that the ring R of real valued continuous functions on $[0,1]$ has zero divisors.

Sol'n: (i) we have $(xy)^2 = x^2y^2\ \ \forall\ x, y \in R$ — ①

Replacing y by $y+1 \in R$ in ①, we get

$$[x(y+1)]^2 = x^2(y+1)^2$$

$$\Rightarrow (xy+x)^2 = x^2(y^2+2y+1)$$

$$\Rightarrow (xy+x)(xy+x) = x^2(y^2+2y+1)$$

$$\Rightarrow (xy)^2 + (xy)x + x(xy) + x^2 = x^2y^2 + 2x^2y + x^2 \quad \text{--- (2)}$$

$$\Rightarrow (xy)^2 + (xy)x + x(xy) + x^2 = (xy)^2 + 2x^2y + x^2$$

$$\Rightarrow (xy)x + x(xy) = 2x^2y \quad (\because \text{LCL \& RCL in } (R,+))$$

$$\Rightarrow xyx + x^2y = 2x^2y$$

$$\Rightarrow xyx = x^2y \;\forall\; x,y \in R \quad (\text{RCL}) \quad \text{--- (3)}$$

Replacing x by $x+1 \in R$ in (3),

$$(x+1)y(x+1) = (x+1)^2 y$$

$$\Rightarrow (x+1)(yx+y) = (x+1)(xy+y)$$

$$\Rightarrow xyx + xy + yx + y = x^2y + xy + xy + y$$

$$\Rightarrow yx = xy \;\forall\; x,y \in R \quad (\because \text{LCL \& RCL in } (R,+))$$

$\therefore$ R is a commutative ring.

(ii) Consider the functions f and g defined on $[0,1]$

by
$$f(x) = \tfrac{1}{2} - x, \quad 0 \le x \le \tfrac{1}{2}$$
$$= 0, \quad \tfrac{1}{2} \le x \le 1$$

and
$$g(x) = 0, \quad 0 \le x \le \tfrac{1}{2}$$
$$= x - \tfrac{1}{2}, \quad \tfrac{1}{2} \le x \le 1$$

then f and g are continuous functions and

$$f \neq 0,\ g \neq 0.$$

whereas $gf(x) = g(x)f(x) = 0.\left(\frac{1}{2}-x\right)$ if $0 \le x \le \frac{1}{2}$

$$= \left(x-\tfrac{1}{2}\right).0 = 0 \text{ if } \tfrac{1}{2} \le x \le 1$$

i.e. $gf(x) = 0$ for all x

i.e. $gf = 0$ but $f \neq 0,\ g \neq 0$.

3(b) For the series $\sum_1^\infty f_n(x)$ where $f_n(x) = n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2x^2}$, $x \in [0,1]$, show that $\sum_1^\infty \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\sum_{n=1}^\infty f_n(x)\right) dx$. Is the series $\sum_1^\infty f_n(x)$ uniformly convergent on $[0,1]$?

Sol^n: Let $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$. Then

$$S_n(x) = n^2 x e^{-n^2x^2}$$

for all $x \in (0,1]$, $e^{n^2x^2} > \frac{n^4x^4}{2} > 0$

therefore $0 < S_n(x) < \frac{2}{n^2x^3}$ for all $x \in (0,1]$.

By Sandwich theorem $\lim_{n\to\infty} S_n(x) = 0$, for all $x \in (0,1]$.

And for $x=0$, the sequence $\{S_n\}$ converges to 0.

Hence the series $\sum_1^\infty f_n(x)$ is convergent on $[0,1]$ and the sum function f is given by

$$f(x) = 0,\ x \in [0,1]$$

$$\therefore \int_0^1 \left(\sum_1^\infty f_n(x)\right) dx = 0$$

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}\left[-e^{-n^2x^2} + e^{-(n-1)^2x^2}\right]_0^1$$

$$= \frac{1}{2}\left[e^{-(n-1)^2} - e^{-n^2}\right].$$

Let $t_n = \int_0^1 f_1(x)\,dx + \int_0^1 f_2(x)\,dx + \dots + \int_0^1 f_n(x)\,dx$

Then $t_n = \frac{1}{2}\left[1 - e^{-n^2}\right]$ and

$$\lim_{n\to\infty} t_n = \frac{1}{2}$$

$$\therefore \sum_1^\infty \int_0^1 f_n(x)\,dx = \frac{1}{2} \neq \int_0^1 \left(\sum_1^\infty f_n(x)\right) dx$$

3(c) Using the simplex method solve the LPP problem:

minimize $Z = x_1 + x_2$

subject to

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$.

Sol: The objective function of the given LPP is of minimization type.
So, we convert it into maximization type

$\text{Max } Z' = \text{Min}(-Z)$

$= -x_1 - x_2$

Now we write the given LPP in the standard form $\text{Max } Z' = -x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

Subject to

$2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

$A_1, A_2, x_1, x_2, s_1, s_2 \geq 0$.

where s_1, s_2 are the surplus variables

A_1, A_2 are the artificial variables.

Now the IBFS is

$s_1 = s_2 = x_1 = x_2 = 0$ (Non-basic)

$A_1 = 4$, $A_2 = 7$ (Basic)

Thus the initial simplex table is

	C_j	-1	-1	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	2	1	-1	0	1	0	4	4
$-M$	A_2	1	(7)	0	-1	0	1	7	1 →
	$Z_j = \sum C_B a_{ij}$	$-3M$	$-8M$	M	M	$-M$	$-M$	$-11M$	
	$C_j = C_j - Z_j$	$-1+3M$	$-1+8M$ ↑	$-M$	$-M$	0	0		

From the above table,
the variable x_2 is entering variable, A_2 is the outgoing variable and omit column for this variable in the next simplex table. Here (7) is the key element and convert it into unity and all other elements in its column to zero.

Then the new simplex table is:

	C_j	-1	-1	0	0	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	A_1	b	θ
$-M$	A_1	$(13/7)$	0	-1	$1/7$	1	3	$\frac{21}{13}$ →
-1	x_2	$1/7$	1	0	$-1/7$	0	1	7
$z_j = \Sigma a_{ij} c_B$		$-\frac{13M}{7}-\frac{1}{7}$	-1	M	$-\frac{M}{7}+\frac{1}{7}$	$-M$	$-3M-1$	
$C_j = c_j - z_j$		$\frac{13M}{7}-\frac{6}{7}$ ↑	0	$-M$	$\frac{M}{7}-\frac{1}{7}$	0		

From the above table,
x_1 is the entering variable, A_1 is the outgoing variable and omit its column in the next simplex table. Here (13/7) is the key element and make it unity and all other elements in its column equal to zero. Then the revised simplex table is

	c_j	-1	-1	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	b	θ
-1	x_1	1	0	$-7/13$	$\frac{1}{13}$	$21/13$	
-1	x_2	0	1	$1/13$	$-2/13$	$10/13$	
$z_j = \Sigma a_{ij} c_B$		-1	-1	$6/13$	$1/13$	$-31/13$	
$C_j = c_j - z_j$		0	0	$-6/13$	$-1/13$		

From the above table, all C_j's ≤ 0.
there remains no artificial variable in the basis.
$\therefore$ The solution is an optimal BFS to the problem and is given by

$x_1 = 21/13$, $x_2 = 10/13$

$\therefore$ Max $z' = -31/13$

Hence the optimal value of the objective function is Min $z = -$Max z'
$= 31/13$

4(a). If R and S are two rings, then
$ch(R\times S)=0$ if $chR=0$ or $chS=0$
$=k$ where $k=l.c.m(chR, chS)$.

Sol^n: Let $chR=0$ and suppose $ch(R\times S)=t\neq 0$
Then $t(a,b)=(0,0)$, $\forall\, a\in R,\ b\in S$
$\Rightarrow (ta,tb)=(0,0)$
$\Rightarrow ta=0\ \forall\, a\in R$, a contradiction as $chR=0$
Thus $ch(R\times S)=0$
Similarly, if $chS=0$, then $ch(R\times S)=0$
Let now $chR=m$, $chS=n$ and let $k=l.c.m(m,n)$
Then $k(a,b)=(ka,kb)=(0,0)\ \forall\, a\in R,\ b\in S$
as m,n divide k.
Suppose $p(a,b)=(0,0)$, then $(pa,pb)=(0,0)$
$\Rightarrow pa=0=pb\Rightarrow m|p,\ n|p$
$\Rightarrow k|p\Rightarrow k\leq p\Rightarrow ch(R\times S)=k$.

4(b). A function f is defined on $[0,1]$ by $f(0)=0$ and
$f(x)=0$, if x be irrational
$=\frac{1}{q}$, if $x=\frac{p}{q}$ where p,q are +ve integers prime to each other
Show that f is integrable on $[0,1]$ and $\int_0^1 f=0$.

Sol^n: f is bounded on $[0,1]$. Let us choose a +ve ϵ such that $0<\epsilon<2$. Then there exists a natural number k such that $k<\frac{2}{\epsilon}<k+1$, by Archimedean property of $\mathbb{R}$.
Let the rational numbers in $(0,1]$ be arranged as

$\frac{1}{1}; \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \cdots; \frac{1}{k}, \cdots, \frac{k-1}{k};$

$\frac{1}{k+1}, \cdots, \frac{k}{k+1}; \cdots$

These are only a finite number of rational numbers of the form p/q in $[0,1]$ with denominator $\leq k$.

At every such point $f(x) \geq \frac{1}{k} > \frac{\epsilon}{2}$; and at all other rational points in $[0,1]$, $f(x) \leq \frac{\epsilon}{2}$.

Let the finite number of rational points for which $f(x) > \frac{\epsilon}{2}$ be $x_1, x_2, \cdots, x_m$ where $x_1 < x_2 < \cdots < x_m$.

Let us enclose the points by subinterval $\left[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}\right]$, $\left[x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}\right], \cdots, \left[x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}\right]$ such that

$\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\epsilon}{2}$.

Since each of these subintervals contain rational as well as irrational points, the oscillation of f in each of these subintervals is less than 1.

Let $P = (0, x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}, \cdots, x_m + \frac{\delta_m}{2}, 1)$.

then P is a partition of $[0,1]$ dividing $[0,1]$ into $2m+1$ subintervals, m of which enclose the points $x_1, x_2 \cdots x_m$.

In each of the remaining $m+1$ subintervals, the oscillation of f is less than $\frac{\epsilon}{2}$ and the sum of these $m+1$ subintervals is less than 1.

So $U(P,f) - L(P,f) < 1 \cdot \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot 1 = \epsilon$

$\therefore \exists$ a partition P of $[0,1]$ such that $U(P,f) - L(P,f) < \epsilon$.

this being a sufficient condition for integrability, f is integrable on $[0,1]$.

Let $P = (x_0, x_1, \ldots, x_n)$, where $0 = x_0 < x_1 < \cdots < x_n = 1$ be an arbitrary partition of $[0,1]$.

Let $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$; $r = 1, 2, \ldots n$.

Since every subinterval $[x_{r-1}, x_r]$ contains irrational points, $m_r = 0$ for $r = 1, 2 \ldots n$.

$\therefore L(P, f) = 0$.

Consequently, $\underline{\int_0^1} f = \sup\{L(P,f) : P \in P[a,b]\} = 0$

Since $f \in R[0,1]$, $\underline{\int_0^1} f = \int_0^1 f$ and

$\therefore \int_0^1 f = 0$.

4(c) If $w = u + iv$ represents the complex potential for an electric field and $v = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function u.

Sol'n: Here $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$\text{Now } \frac{\partial v}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\text{and } \frac{\partial v}{\partial x} = 2x - \frac{2x^2}{(x^2+y^2)^2}$$

By Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Integrating bothsides w.r.t x, we get

$$u = -2xy + \frac{y}{x^2+y^2} + \phi(y) \quad \text{——} (1)$$

where $\phi(y)$ stands for constant of integration

$$\text{Again } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore -2x + \frac{x^2-y^2}{(x^2+y^2)^2} + \phi'(y) = -2x + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c$; where c is constant of integration.

Substituting for $\phi(y)$ in (1), we get

$$u = -2xy + \frac{y}{x^2+y^2} + c$$

4(d) A methods engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production and they are given below. If only one method can be assigned to a work centre, determine the optimum assignment:

Increase in production (unit)

Work centres

Method	A	B	C
1	10	7	8
2	8	9	7
3	7	12	6
4	10	10	8

Sol:

The given problem is of maximization type. Since the elements of the given matrix relate to increase in production of units due to introduction of new methods. First of all, convert it into minimization problem by subtracting each element of the given matrix from maximum element 12. Since the problem is unbalanced one, introduce a dummy work centre

methods	A	B	C	Dummy
1	2	5	4	0
2	4	3	5	0
3	5	0	6	0
4	2	2	4	0

subtracting the smallest element of each column from all elements of that column we get

	A	B	C	D
1	0	5	0	0
2	2	3	1	0
3	3	0	2	0
4	0	2	0	0

Since the minimum no. of horizontal and vertical lines to cover up all zeros is 4, the reduced matrix will give the optimum solution.

0	5	$\cancel{0}$	$\cancel{0}$
2	3	1	0
3	0	2	$\cancel{0}$
$\cancel{0}$	2	0	$\cancel{0}$

The allocations as obtained from the above process are

1 ⟶ A

2 ⟶ Dummy

3 ⟶ B

4 ⟶ C.

The total production under the above assignment is

$10 + 12 + 8 = 30$ units

5(a) Find the general integral of the partial differential equation $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x=1, y=0$.

Sol'n: Given $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ —— ①

Given line is $x=1, y=0$. —— ②

Here the Lagrange's auxiliary equations of ① are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2x-2yz} \quad —— ③$$

Taking $z, 1, x$ as multipliers, each fraction of ③

$$= \frac{z\,dx + 1.dy + x\,dz}{0} \quad \text{so that } z\,dx + dy + x\,dz = 0$$

$\Rightarrow d(xz) + dy = 0$ and hence $xz + y = c_1$ — ④.

Again, taking $x, y, \frac{1}{2}$ as multipliers, each fraction of ③

$$= \frac{x\,dx + y\,dy + \frac{1}{2}dz}{0} \quad \text{so that } x\,dx + y\,dy + \frac{1}{2}dz = 0$$

$\Rightarrow 2x\,dx + 2y\,dy + dz = 0$ and so $x^2+y^2+z = c_2$.

Since the required curve given by ④ and ⑤ passes through the line ②, so putting $x=1$ and $y=0$ in ④ and ⑤, we get

$z = c_1$ and $1+z = c_2$ so that $1+c_1 = c_2$ —— ⑥

Substituting the values of c_1 and c_2 from ④ and ⑤ in ⑥, the equation of the required surface is given by

$$1 + xz + y = x^2 + y^2 + z$$

$$\Rightarrow x^2 + y^2 + z - xz - y = 1$$

5(b) Find complete integral of $(x^2-y^2)pq - xy(p^2-q^2) = 1$.

Sol^n: Here $f(x,y,z,p,q) = (x^2-y^2)pq - xy(p^2-q^2) - 1 = 0$ —— ①

Then, Charpit's auxiliary equation are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2pqx - z(p^2-q^2)} = \frac{dq}{-2pqy - x(p^2-q^2)} = \frac{dx}{-(x^2-y^2)y + 2pxy}$$

$$= \frac{dy}{-(x^2-y^2)p - 2pxy}$$

Using x, y, p, q as multipliers, each fraction

$$= \frac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \frac{d(xp) + d(yq)}{0}$$

$$\Rightarrow d(xp + qy) = 0 \Rightarrow xp + yq = a \Rightarrow p = (a - qy)/x \quad —— ②$$

Using ②, ① $\Rightarrow (x^2-y^2)\left(\frac{a-qx}{x}\right)q - xy\left[\left(\frac{a-qy}{x}\right)^2 - q^2\right] - 1 = 0$

$$\Rightarrow \frac{a-qy}{x}\left\{(x^2-y^2)q - (a-qy)y\right\} + xyq^2 - 1 = 0$$

$$\Rightarrow \{(a-qy)/x\}(x^2q - ay) + xyq^2 - 1 = 0$$

$$\Rightarrow (a-qy)(x^2q - ay) + x^2yq^2 - x = 0$$

$$\Rightarrow \quad aq(x^2+y^2) = a^2y + x$$

$$\therefore q = \frac{a^2y + x}{a(x^2+y^2)} \quad \text{and} \quad p = \frac{1}{x}\left[a - \frac{(a^2y+x)y}{a(x^2+y^2)}\right] = \frac{a^2x - y}{a(x^2+y^2)}$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)\,dx + (a^2y + x)\,dy}{a(x^2+y^2)} = a\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{x\,dy - y\,dx}{a(x^2+y^2)}$$

Integrating $z = (a/2)\log(x^2+y^2) + \frac{1}{a}\tan^{-1}(y/x) + b$.

5(c) Given that $f(0)=1$, $f(1)=3$, $f(3)=55$, find the unique polynomial of degree 2 or less which fits the given data. Find the bound on the error.

Sol^n: We have $x_0=0$, $x_1=1$, $x_2=3$,

$f_0=1$; $f_1=3$, $f_2=55$.

The Lagrange's fundamental polynomials are given by

$$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-1)(x-3)}{(-1)(-3)}=\frac{1}{3}(x^2-4x+3)$$

$$l_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{x(x-3)}{(1)(-2)}=\frac{1}{2}(3x-x^2)$$

$$l_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{x(x-1)}{3(2)}=\frac{1}{6}(x^2-x)$$

Hence, the Lagrange's quadratic interpolating polynomial is given by

$$P_2(x)=l_0(x)f_0+l_1(x)f_1+l_2(x)f_2$$
$$=\frac{1}{3}(x^2-4x+3)+\frac{3}{2}(3x-x^2)+\frac{55}{6}(x^2-x)$$
$$=8x^2-6x+1.$$

We have

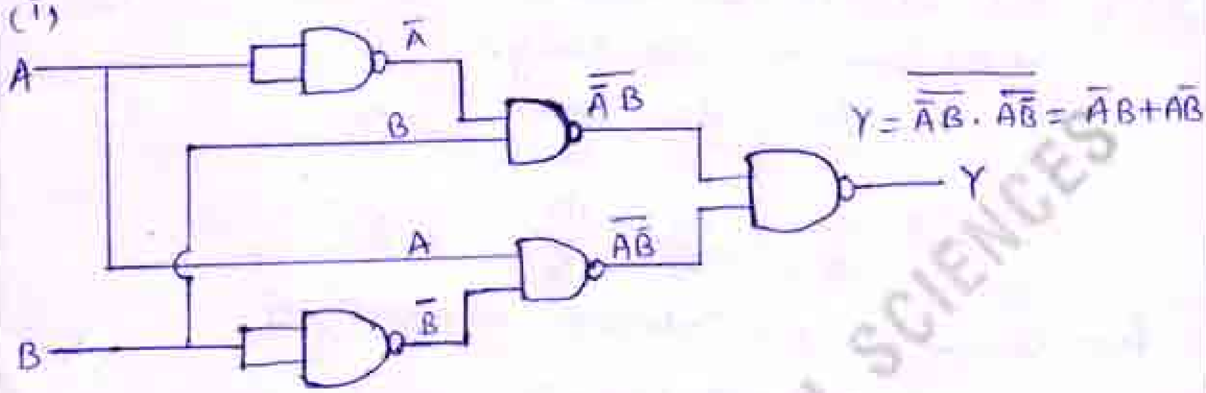
$$|E_2(f;x)|\le\frac{1}{6}M_3\left[\max_{0\le x\le 3}|x(x-1)(x-3)|\right]$$
$$=\frac{1}{6}(2.1126)M_3=0.3521M_3$$

where $M_3=\max_{0\le x\le 3}|f'''(x)|$ and since the maximum of $|x(x-1)(x-3)|$ occurs at $x=2.2152$.

5(d). (i) Implement $Y = \bar{A}B + A\bar{B}$ using NAND gates only.
(ii) Find the hexadecimal equivalent of the decimal number $(587632)_{10}$.

Sol'n:

(i)



(ii)

16	587632	
16	36727	0
16	2295	7
16	143	7
16	8	15
	0	8

Here, 15 = F

$\therefore (587632)_{10} = (8F770)_{16}$

5(e) Prove that the necessary and sufficient condition that vortex lines may be at right angles to the streamlines are $u, v, w = \mu\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$, where μ and ϕ are functions of x, y, z, t.

Sol^n: The differential equations of streamlines and vortex lines are respectively.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{--- (1)}$$

and $$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad \text{--- (2)}$$

(1) and (2) will intersect orthogonally iff

$$u\xi + v\eta + w\zeta = 0$$

$$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

But this is the condition that

$u\,dx + v\,dy + w\,dz$ is perfect differential

$$\Rightarrow u\,dx + v\,dy + w\,dz = \mu\, d\varphi$$

$$= \mu\left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right)$$

this $\Rightarrow u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$.

6(a) → solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$.

Sol^n: Here A.E is $m^2 - m - 2 = 0$ so that $m = 2, -1$.

So C.F $= \phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{(D-2D')}\frac{1}{D+D'}\left\{(2x^2+xy-y^2)\sin xy - \cos xy\right\}$$

$$= \frac{1}{D-2D'}\frac{1}{D+D'}\left\{(2x-y)(x+y)\sin xy - \cos xy\right\}$$

$$= \frac{1}{D-2D'}\int\left\{(x-c)(2x+c)\sin x(c+x) - \cos x(c+x)\right\}dx$$

[Taking $c = y-x$]

$$= \frac{1}{D-2D'}\int\left\{(x-c)(2x+c)\sin(cx+x^2) - \cos(cx+x^2)\right\}dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos(cx+x^2) + \int\cos(cx+x^2)dx - \int\cos(cx+x^2)dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy \quad \text{as } c = y-x$$

$= \int(c'-4x)\cos(c'x-2x^2)dx$, where $c' = y+2x$

$= \int\cos t\,dt = \sin t$,

putting $c'x - 2x^2 = t$
$(c'-4x)dx = dt$

$= \sin(c'x - 2x^2)$

$= \sin xy$, as $c' = y+2x$

So solution is $z = \phi_1(y+2x) + \phi_2(y-x) + \sin xy$

6(b) Find a partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol: Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——— (1)

Differentiating (1) w.r.t x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \Rightarrow c^2x+a^2z\frac{\partial z}{\partial x}=0 \quad \text{——— (2)}$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \Rightarrow c^2y+b^2z\frac{\partial z}{\partial y}=0 \quad \text{——— (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we have

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{——— (4)}$$

& $$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \quad \text{——— (5)}$$

From (2), $c^2=-\frac{a^2z}{x}\left(\frac{\partial z}{\partial x}\right)$

putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2} \quad \text{——— (6)}$$

Similarly, from (3) & (5),

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \quad \text{——— (7)}$$

Differentiating (2) partially w.r.t y, we get

$$a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)+z\frac{\partial^2 z}{\partial x\partial y}\right\}=0$$

i.e., $$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+z\frac{\partial^2 z}{\partial x\partial y}=0 \quad \text{——— (8)}$$

$\therefore$ (6), (7) and (8) are three possible forms of the required partial differential equations.

6(c) The equation $x^2+ax+b=0$ has two real roots α and β show that the iteration method $x_{k+1}=-\frac{(ax_k+b)}{x_k}$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$ and that $x_{k+1}=\frac{-b}{x_k+a}$ is convergent near $x=\alpha$ if $|\alpha|<|\beta|$. Show also that iteration method $x_{k+1}=-\frac{(x_k^2+b)}{a}$ is convergent near $x=\alpha$ if $2|\alpha|<|\alpha+\beta|$.

Sol^n: The iterations are given by

$$x_{k+1}=-\frac{(ax_k+b)}{x_k}=g(x_k)\ (\text{say})$$
$$k=0,1,2---$$

By the known theorem

If $g(x)$ and $g'(x)$ are continuous in an interval about a root α of the equation $x=g(x)$ and if $|g'(x)|<1$ for all x in the interval, then the successive approximations $x_1, x_2, ---$ given by

$$x_k=g(x_{k-1}),\quad k=1,2,3----$$

converges to the root α provided that the initial approximation x_0 is chosen in the in the interval.

$\therefore$ These iterations converge to α if

$$|g'(x)|<1 \text{ near } \alpha.$$

i.e. $|g'(x)|=\left|\frac{-b}{x^2}\right|<1$

Note that $g'(x)$ is continuous near α.

If the iterations converge to $x=\alpha$, then we require $|g'(\alpha)|=\left|\frac{-b}{\alpha^2}\right|<1$

$$\text{Thus } |b|<|\alpha|^2$$
$$\text{i.e. } |\alpha|^2>|b| \quad \text{—— (1)}$$

Given that α and β are roots of the equation $x^2+ax+b=0$

then $\alpha+\beta = -a$ and $\alpha\beta = b \Rightarrow |b| = |\alpha||\beta|$ —③

Substituting ③ in ①, we get

$$|\alpha|^2 > |b| = |\alpha||\beta|$$

$$\Rightarrow |\alpha|^2 > |\alpha||\beta|$$

$$\Rightarrow |\alpha| > |\beta|$$

Now, if $x = \frac{-b}{x+a}$

The iteration $x_{k+1} = \frac{-b}{x_k + a} = g(x_k)$ (Say)

Converges to α if

$$|g'(x)| = \left|\frac{b}{(x+a)^2}\right| < 1 \text{ in an interval containing } \alpha.$$

Inparticular we require

$$|g'(\alpha)| = \left|\frac{b}{(\alpha+a)^2}\right| < 1$$

$$\Rightarrow (\alpha+a)^2 > |b|.$$

But we have $\alpha+\beta = -a$ & $\alpha\beta = b$

$$\Rightarrow \beta^2 > |b| = |\alpha||\beta|$$

$$\Rightarrow |\beta|^2 > |\alpha||\beta|$$

$$\Rightarrow |\beta| > |\alpha|$$

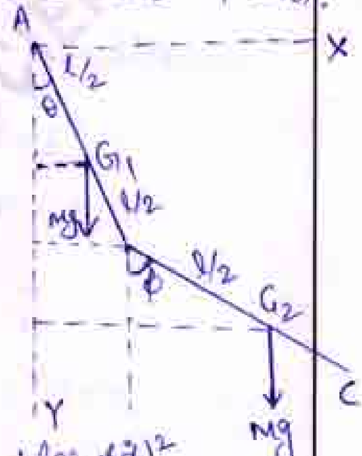
$\therefore x_{k+1} = \frac{-b}{x_k + a}$ is Convergent near

$x = \alpha$ if $|\beta| > |\alpha|$.

6(d) Two equal rods AB and BC each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $2\pi/n$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

Sol^n: Let AB and BC be the rods of equal length l and mass M. At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by



$x_{G_1} = \frac{1}{2} l\sin\theta$; $y_{G_1} = \frac{1}{2} l\cos\theta$

$x_{G_2} = l\sin\theta = \frac{1}{2} l\sin\phi$, $y_{G_2} = l\cos\theta + \frac{1}{2} l\cos\phi$

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2, then

$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\frac{1}{2} l\cos\theta\dot{\theta}\right)^2 + \left(-\frac{1}{2} l\sin\theta\dot{\theta}\right)^2$

$= -\frac{1}{4} l^2\dot{\theta}^2$

$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot{\theta} + \frac{1}{2} l\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \frac{1}{2} l\sin\phi\dot{\phi}\right)^2$

$= l^2\left[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$

$= l^2\left[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right]$, ($\because \theta, \phi$ are small)

If T be the total K.E. and W the work function of the system, then

T = K.E of rod AB + K.E of rod BC

$= \left[\frac{1}{2} M\cdot\frac{1}{3}\left(\frac{1}{2} l\right)^2\dot{\theta}^2 + \frac{1}{2} M\cdot v_{G_1}^2\right] + \left[\frac{1}{2} m\cdot\frac{1}{3}\left(\frac{1}{2} l\right)\dot{\phi}^2 + \frac{1}{2} M\cdot v_{G_2}^2\right]$

$= \frac{1}{2} Ml^2\left(\frac{4}{3}\dot{\theta}^2 + \frac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$

and $W = Mg\,y_{G_1} + Mg\,y_{G_2} + C = Mg\left[\frac{1}{2} l\cos\theta + l\cos\theta + \frac{1}{2}\cos\phi\right] + C$

$= \frac{1}{2} Mgl(3\cos\theta + \cos\phi)$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2} Ml^2\left(\frac{8}{3}\dot{\theta} + \dot{\phi}\right)\right] - 0 = \frac{1}{2} Mgl(-3\sin\theta) = -\frac{3}{2} Mgl\theta$ ($\because \theta$ is small)

$\Rightarrow 8\ddot{\theta} + 3\ddot{\phi} = -9c\theta$, (where $c = g/l$) —— ①.

equations ① and ② can be written as

$(8D^2+9c)\theta + 3D^2\phi = 0$ and $3D^2\theta + (2D^2+3c)\phi = 0$

Eliminating ϕ b/w these two equations, we get

$[(2D^2+3c)(8D^2+9c) - 9D^4] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$

If the periods of normal oscillations are $2\pi/n$, then the solution of ③, must be

$\theta = A\cos(nt+B)$ $\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$

Substituting in ③ we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0$, $\because \theta \neq 0$.

$\therefore n^2 = \dfrac{42c \pm \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$

$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$ $(\because c = g/l)$.

7(a) Reduce the equation $yr+(x+y)s+xt=0$ to canonical form and hence find its general solution.

Sol^n: Given $yr+(x+y)s+xt=0 \rightarrow (1)$

comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=y, S=x+y$ and $T=x$ So that

$$S^2-4RT=(x+y)^2-4xy=(x-y)^2>0 \text{ for } x\neq y.$$

ans so (1) is hyperbolic. Its λ-quadratic equation

$R\lambda^2+S\lambda+T=0$ reduces to $y\lambda^2+(x+y)\lambda+x=0$

(or) $(y\lambda+x)(\lambda+1)=0$

So that $\lambda=-1, -x/y$. Then the corresponding characteristic equations are given by

$$\frac{dy}{dx}-1=0 \text{ and } \frac{dy}{dx}-(x/y)=0$$

Integrating these $y-x=c_1$ and $\frac{y^2}{2}-x^2/2=c_2$

In order to reduce one (1) to its canonical form. we choose.

$$u=y-x \text{ and } v=y^2/2-x^2/2 \longrightarrow (2)$$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right), \text{ using (2)} \rightarrow (3)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}), \text{ using (2)} \rightarrow (4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right) \text{ using } (3)$$

$$=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

$$\Rightarrow -\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right)-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]-\frac{\partial z}{\partial v}$$

$$=-\left(-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}\right)-x\left(-\frac{\partial^2 z}{\partial u\partial v}-x\frac{\partial^2 z}{\partial v^2}\right)-\frac{\partial z}{\partial v} \text{ using } (2)$$

$$r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}$$

Now, $t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial v}\right) \text{ by } (4)$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}.$$

$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}$$

Also $s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$ using (4)

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial}\left(y\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$\Rightarrow -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial u\partial u} - y\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2}.\ \text{using (2)}$$

$$\therefore\ s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2} \quad \to (7)$$

using (5), (6) and (7) in (1) we get

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right) + (x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\}$$
$$+ x\left\{\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right\} = 0$$

(or) $\left\{4xy - (x+y)^2\right\}\frac{\partial^2 z}{\partial u\partial v} - y\frac{\partial z}{\partial v} + x\frac{\partial z}{\partial v} = 0$

(or) $(y-x)^2\frac{\partial^2 z}{\partial u\partial v} + (y-x)\frac{\partial z}{\partial v} = 0$

(or) $u^2 \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial v} = 0$ (or) $u \frac{\partial^2 z}{\partial v \partial v} + \frac{\partial z}{\partial v} = 0 \rightarrow (8)$

[$\because u \neq 0$ and $y - x = u$, by (2)]

(8) is the required canonical form of (1).

Solution of (8) multiplying both sides of (8) by v we get

$uv\left(\frac{\partial^2 z}{\partial u \partial v}\right) + v\left(\partial z/\partial v\right) = 0$ (or) $(uvDD' + vD')z = 0 \rightarrow (9)$

where $D \equiv \frac{\partial}{\partial u}$ and $D' \equiv \partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables x and y as follows

let $u = e^x$ and $v = e^y$ so that $x = \log u$, $y = \log v \rightarrow (10)$

let $D_1 \equiv \partial/\partial x$ and $D_1' \equiv \partial/\partial y$ then (9) reduces to

$(D_1 D_1' + D_1')z = 0$ (or) $D_1'(D+1)z = 0$

its general solution is

$z = e^{-x}\phi_1(y) + \phi_2(x) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$

(or) $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$,

where ψ_1 and ψ_2 are arbitrary functions.

7(b). Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ by Gauss-Jordan method.

Sol^n: we place an identity matrix adjacent to the given matrix as a first step and the resulting augmented matrix is given by

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \longleftarrow (1)$$

$R_2 \to R_2 - 4R_1$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \longleftarrow (2)$$

$R_3 \to R_3 - 3R_1$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \longleftarrow (3)$$

$R_1 \to R_1 + R_2$
$R_3 \to R_3 + 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \longleftarrow (4)$$

$R_3 \to R_3/-10$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \longleftarrow (5)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & -1 & 0 & 3/2 & 0 & -1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 4R_3 \\ R_2 \to R_2 + 5R_3 \end{array} \quad \text{——— (6)}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] R_2 \to -1R_2 \quad \text{——— (7)}$$

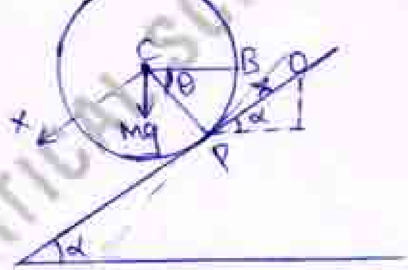
Hence we have

$$A^{-1} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \longleftarrow (8)$$

It can easily verified that $[A][A^{-1}] = [I]$

7(d) A Sphere of radius a and mass M rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamiltons equations.

Sol^n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting intially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during this time let it turn through an angle θ.



Since there is no slipping

$\therefore x = OA = \text{arc } AB = a\theta,$

so that $\dot{x} = a\dot{\theta}$

If T and V are the Kinetic and potential energies of the sphere, then

$$T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$$

$$\Rightarrow T = \frac{7}{10}M\dot{x}^2$$

and $V = -MgOL = -Mgx\sin\alpha$ (since the sphere move down the plane)

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised coordinate

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x}$$

Since L does not contain t explicitly

$$\therefore H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha$$

$$\Rightarrow H = \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx\sin\alpha = \frac{5}{14M}p_x^2 - Mgx\sin\alpha \text{ from } (1)$$

Hence the two Hamilton's equations are $\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \quad -(H_1)$.

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M} p_x \quad — (H_2)$$

Differentiating (H_2) and using (H_1), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M} Mg\sin\alpha$$

$\Rightarrow \ddot{x} = 5/7\, g\sin\alpha$

which gives the required acceleration.

8(a) The ends A and B of a rod 20cm long have the temperatures at 30° and 80° until steady state prevails. The temperatures of the ends are changed to 40° and 60° respectively. Find the temperature distribution in the rod at time t.

sol:- Let the equation for conduction of heat is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad —①$$

prior to temperature change at the end B when $t=0$, the heat flow was independent of time (steady state condition, for which $\partial u/\partial t = 0$).

When u depends only on x, ① reduces to

$$\frac{d^2u}{dx^2} = 0 \Rightarrow u = c_1 x + c_2 \quad —②$$

A ← 20cm → B

Given that $u = 30$ for $x = 0$
and $u = 80$ for $x = 20$ cm $\Big\}$ —③

by using ③, ② becomes $u = \frac{5}{2}x + 30$

∴ The initial condition is given by

$$u(x, 0) = \frac{5}{2}x + 30 \quad —④$$

Given that the boundary condition are

$u(0,t) = 40 \quad \forall t$ —— (5)

$u(20,t) = 60 \quad \forall t$ —— (6)

Now the boundary values are non-zero, so we modify the procedure as follows.

we split the temperature function $u(x,t)$ into two parts as

$u(x,t) = u_1(x) + u_2(x,t)$ —— (7)

where $u_1(x)$ is a solution of (1) involving x only and satisfying the boundary conditions (5) and (6).

$u_2(x,t)$ is then a function defined by (7).

Hence $u_1(x)$ is a steady state solution of the form (2) and $u_2(x,t)$ may be treated as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 40$ for $x=0$ & $u_1(x) = 60$ for $x=20$

using (2), we get $\boxed{u_1(x) = x+40}$ —— (8)

[$u = c_1x + c_2$
$\therefore 40 = c_1(0) + c_2$
$\Rightarrow c_2 = 40$
& $60 = c_1(20) + 40$
$\Rightarrow c_1 = 1$
$u_1(x) = 1(x) + 40$
$= x + 40$]

putting $x=0$ in (7) and using (5),

we get

$\underline{u_2(0,t)} = u(0,t) - u_1(0) = 40 - 40 = 0$ —— (9)

putting $x=20$ in (7) and using (6), we get

$\underline{u_2(20,t)} = u(20,t) - u_1(20) = 60 - 60 = 0$ —— (10)

Also, $u_2(x,0) = u_1(x,0) - u_1(x)$

$= \frac{5}{2}x + 30 - (x+40)$

$\underline{u_2(x,0)} = \frac{3}{2}x - 10$ —— (11)

Hence the boundary conditions and initial condition to the transient solution $u_2(x,t)$ are given by (9), (10) and (11).

So we now solve $\frac{\partial u_2}{\partial t} = k \frac{\partial^2 u_2}{\partial x^2}$ —— (12)

subject to boundary conditions (9) & (10)
and initial condition (11).

Now taking $u_2(x,t) = X(x)\,T(t)$

$\therefore$ From (12), we have $XT' = k X'' T$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} = \mu \text{ (say)}$$

$$\Rightarrow X'' - \mu X = 0 \text{ —— (13)} \quad \& \quad T' = \mu k T \text{ —— (14)}.$$

Using (9) & (10), (13) gives

$X(0)T(t) = 0$ & $X(20)T(t) = 0$ —— (15)

$\Rightarrow X(0) = 0$ & $X(20) = 0$ ($\because T(t) \neq 0$ otherwise leads to $u=0$)

we now solve (13) under B.C. (15).

Three cases arise.

Case(1) Let $\mu = 0$. The solution of (13) is $X(x) = Ax + B$

Using B.C. (15), we get $A = B = 0$

$\therefore X(x) = 0$ so that $u = 0$
which doesn't satisfy (11)
so we reject $\mu = 0$.

Case (2) : Let $\mu = \lambda^2$, $\lambda \neq 0$. Then $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$.

by using B.C. (15), we get $A = B = 0$ so that
$X(x) = 0$ and hence $u = 0$
which doesnot satisfy (11).
so we reject $\mu = \lambda^2$.

Case(3) : Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then $X(x) = A\cos\lambda x + B\sin\lambda x$.

Using B-C (15), we get

$A = 0$ & $A\cos 20\lambda + B\sin 20\lambda = 0$

$\Rightarrow B\sin 20\lambda = 0$

$\Rightarrow \sin 20\lambda = 0$ ($B \neq 0$)

$\Rightarrow 20\lambda = n\pi$, $n = 1, 2, \ldots$

$\Rightarrow \lambda = \frac{n\pi}{20}$, $n = 1, 2, \cdots$ —— (16)

Hence non-zero solutions $X_n(x)$ of (13) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{20}\right)$.

Using (16), (14) gives $\frac{dT}{T} = -\frac{n^2\pi^2 k\,dt}{400}$

$$\Rightarrow T_n(t) = C_n e^{-\frac{n^2\pi^2 kt}{400}} \quad \left(\because \mu = -\lambda^2 = -\frac{n^2\pi^2}{(20)^2}\right)$$

$$\therefore u_2(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{20}\right) e^{-\frac{n^2\pi^2 kt}{400}}$$

where $D_n = B_n C_n$.

where $D_n = \frac{2}{20}\int_0^{20} \left(\frac{3x}{2} - 10\right) \sin\frac{n\pi x}{20}\,dx$

$$= \frac{1}{10}\left[\left(\frac{3x}{2} - 10\right)\left(-\cos\frac{n\pi x}{20}\right)\frac{20}{n\pi} + \frac{3}{2}\sin\frac{n\pi x}{20}\cdot\left(\frac{20}{n\pi}\right)^2\right]$$

$$= \frac{1}{10}\left[\frac{20}{n\pi}\left[20\left(-\cos n\pi\right) - 10\right] + 0\right]$$

$$= -\frac{20}{n\pi}\left[2\cos n\pi + 1\right]$$

$$= -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

$$\therefore u_2(x,t) = -\frac{20}{\pi}\sum_{n=1}^{\infty} \frac{\left[2(-1)^n + 1\right]}{n} \sin\left(\frac{n\pi x}{20}\right) e^{-\frac{n^2\pi^2 kt}{400}} \quad \text{—— (17)}$$

$\therefore$ From (8) & (17), (7) gives

$$u(x,t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{\left[2(-1)^n + 1\right]}{n} \sin\frac{n\pi x}{20}\, e^{-\frac{n^2\pi^2 kt}{400}}$$

which is the required solution

8(b) Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$. Use the fourth order classical Runge-Kutta method. Compare with the exact solution.

Sol: Given that $\frac{du}{dt} = -2tu^2 = f(t,u)$

$h = 0.2$

$t_0 = 0,\ u_0 = 1$

Now $K_1 = hf(t_0, u_0) = -2(0.2)(1)^2 = 0$

$K_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{K_1}{2}\right) = -2(0.2)\left(\frac{0.2}{2}\right)(1)^2 = -0.04$

$K_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_2\right) = -2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2$

$= -0.038416$

$K_4 = hf(t_0 + h, u_0 + K_3) = -2(0.2)(0.2)(0.961584)^2$

$= -0.0739715$

By Runge Kutta fourth order method

$u_1 = u(0.2) = u_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0 - 0.08 - 0.076832 - 0.0739715]$

$= 0.9615328$

$\therefore u(0.2) = 0.9615328$

For second step, we have

$t_1 = 0.2,\ u_1 = 0.9615328$

$K_1 = hf(t_1, u_1) = -2(0.2)(0.2)(0.9615328)^2$

$= -0.0739636$

$K_2 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{K_1}{2}\right) = -2(0.2)(0.3)(0.924551)^2$

$= -0.1025753$

$K_3 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{K_2}{2}\right) = -2(0.2)(0.3)(0.9102451)^2$

$= -0.0994255$

$k_4 = hf(t_1+h,\ u_1+k_3) = -2(0.2)(0.4)(0.8621073)^2$
$= -0.1189166$

∴ By Runge kutta fourth order method

$u(0.4) = u_2 = u_1 + \frac{1}{6}[-0.0739636 - 0.2051506$
$- 0.1988510 - 0.1189166]$

$u(0.4) = 0.8620525$

The exact solution of $u' = -2tu^2$

$\frac{du}{dt} = -2tu^2$

$\Rightarrow \frac{du}{+u^2} = -2t\,dt$

$\Rightarrow -\frac{1}{u} = -t^2 + c$ — ①

$\Rightarrow -1 = -0 + c$ $\left(\because \begin{matrix} u=1 \\ t=0 \end{matrix}\right)$

$\Rightarrow c = -1$

∴ from ① $-\frac{1}{u} = -t^2 - 1$

$\Rightarrow \frac{1}{u} = 1+t^2$

$\Rightarrow \boxed{u = \frac{1}{1+t^2}}$

The exact solution is

$u(0.2) = 0.961538$

$u(0.4) = 0.862069$

∴ The absolute errors in the numerical solutions

are $\epsilon(0.2) = |0.961539 - 0.961533| = 0.000006$

$\epsilon(0.4) = |0.862069 - 0.862053| = 0.000016$

8(c) → Prove that liquid motion is possible when velocity at (x, y, z) is given by

$$u = \frac{3x^2 - r^2}{r^5}, \quad v = \frac{3xy}{r^5}, \quad w = \frac{3xz}{r^5} \quad \text{where } r^2 = x^2+y^2+z^2$$

and the streamlines are the intersection of the surfaces, $(x^2+y^2+z^2)^3 = c(y^2+z^2)^2$, by the planes passing through OX. Is this irrotational?

Solⁿ: To P.T the liquid motion is possible. For this we have to show that the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{—— ①}$$

is satisfied.

$$r^2 = x^2+y^2+z^2 \quad \text{—— ②}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{—— ③}$$

$$\frac{\partial u}{\partial x} = \frac{(6x-2x)r^5 - 5r^3x(3x^2-r^2)}{r^{10}}$$

$$\frac{\partial v}{\partial y} = \frac{3x}{r^{10}}(r^5 - 5r^3y^2) \text{ and } \frac{\partial w}{\partial z} = \frac{3x}{r^{10}}(r^5 - 5r^3z^2)$$

This implies

$$\frac{\partial u}{\partial x} = \frac{3x}{r^7}(3r^2 - 5x^2)$$

$$\frac{\partial v}{\partial y} = \frac{3x}{r^7}(r^2 - 5y^2) \text{ and } \frac{\partial w}{\partial z} = \frac{3x}{r^5}(r^2 - 5z^2)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Hence the result.

To determine streamlines:

Streamlines are the solution of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{--- (4)}$$

putting the values

$$\frac{dx}{3x^2-r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} = \frac{x\,dx + y\,dy + z\,dz}{x(3r^2-r^2)}$$

$$= \frac{y\,dy + z\,dz}{3x(y^2+z^2)}$$

$$\Rightarrow \frac{dy}{3xy} = \frac{dz}{3xz}.$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y - \log z = \log a$$

$$\Rightarrow y = az \quad \text{--- (5)}$$

This is a plane through ox.

and $$\frac{x\,dx + y\,dy + z\,dz}{2(x^2+y^2+z^2)} = \frac{-y\,dy + z\,dz}{3(y^2+z^2)}$$

$$\Rightarrow \tfrac{1}{2}\log(x^2+y^2+z^2) = \tfrac{1}{3}\log(y^2+z^2) + \tfrac{1}{6}\log b$$

$$\Rightarrow (x^2+y^2+z^2)^3 = b(y^2+z^2)^2 \quad \text{--- (6)}$$

The required streamlines are given by the intersection of surfaces (6) by the planes (5) passing through ox.

finally, to show that the motion is irrotational, we should verify the conditions: $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0,\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ --- (7)

from $\frac{\partial u}{\partial y} = \frac{-3y(5x^2-r^2)}{r^7}$, $\frac{\partial u}{\partial z} = \frac{-3z(5x^2-r^2)}{r^7}$, $\frac{\partial v}{\partial x} = \frac{3y(r^2-5x^2)}{r^7}$

$\frac{\partial v}{\partial z} = \frac{-15xyz}{r^7}$, $\frac{\partial w}{\partial x} = \frac{3z(r^2-5x^2)}{r^7}$, $\frac{\partial w}{\partial y} = \frac{-15xyz}{r^7}$

with these values (7), are all satisfied. Hence the motion is irrotational.