Mains Test Series - 2020

COMMON TEST - [TEST-15 (Batch-I)] & [TEST-7 (Batch-II)]

Answer key

Paper-I, full Syllabus

1(a) Let $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(i) Find a vector w_1, different from u and v, so that $\langle \{u, v, w_1\} \rangle = \langle \{u, v\} \rangle$.

(ii) Find a vector w_2 so that $\langle \{u, v, w_2\} \rangle \neq \langle \{u, v\} \rangle$

Sol^n: (i) Let $S = \{u, v\}$; $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$; $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

(i) If we can find a vector w_1, that is linear combination of u and v, then $(\{u, v, w_1\})$ will be the same set as $(\{u, v\})$. Thus w_1 can be any linear combination of u and v.

for Example: $w_1 = 3u - v = \begin{bmatrix} 1 \\ 11 \\ -7 \end{bmatrix}$

(ii) Now we are looking for a vector w_2 that cannot be written as a linear combination of u and v. How can we find such a vector? Any vector that matches two components but not the third of any element of $(\{u, v\})$ will not be in the span.

for Example: $w_2 = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$; (which is nearly $2v$, but not quite.)

1(b). Let $T : \mathbb{C} \to M_{2,2}$ be given by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix}.$$ Find a basis of $R(T)$. Is T surjective?

Sol'n: The range of T is

$$R(T) = \left\{ \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix} \middle/ a,b,c,d \in \mathbb{C} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle/ a,b,c,d \in \mathbb{C} \right\}$$

$$= \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

These three matrices are linearly independent, so a basis of $R(T)$ is $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Thus T is not surjective, since the range has dimension 3 which is shy of $\dim(M_{2,2}) = 4$.

(Notice that the range is actually the subspace of symmetric 2×2 matrices in $M_{2,2}$).

1.(c)(ii) → If $z = (x+y) + (x+y)\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

Solution:

As z is a homogeneous function of degree 1, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad \text{--- (1)}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial x} = 1 + \phi(y/x) + (x+y)\phi'(y/x)(-y/x^2)$$

$$= 1 + \phi(y/x) - \frac{y(x+y)}{x^2}\phi'(y/x)$$

$$\frac{\partial z}{\partial y} = 1 + \phi(y/x) + \left(\frac{x+y}{x}\right)\phi'(y/x)$$

Substituting in (1) and (2), we have

$$x + y + \phi(y/x)(x+y) = z$$

$$\frac{\partial^2 z}{\partial x^2} = \phi'(y/x)(-y/x^2) + \frac{y(x+y)}{x^2}(y/x^2)\phi''(y/x)$$

$$+ \phi'(y/x)\left(y/x^2 - \frac{2y^2}{x^3}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \phi'(y/x)(1/x) + \frac{1}{x}\phi'(y/x) + \frac{(x+y)}{x^2}\phi'(y/x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -y/x^2\,\phi'(y/x) - y/x^2\,\phi'(y/x) - \frac{(x+y)}{x}\frac{y}{x^2}\cdot\phi''(y/x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\phi'(y/x) - \frac{(x+2y)}{x^2}\phi'(y/x) - \frac{y(x+y)}{x^3}\phi''(x)$$

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = \phi'(y/x)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + y/x\right) \quad \text{--- (3)}$$

$$y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right) = \phi'\left(\frac{y}{x}\right)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + \frac{y}{x}\right) \quad \text{--- (4).}$$

$\therefore$ from (3) & (4), we conclude that,

$$\boxed{x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)}$$

Hence Proved.

1(c)(i) Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

Sol^n: It is a (1^∞) form.

Therefore, let $K = \left(\frac{\tan x}{x}\right)^{1/x^2}$

So that $\log K = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

$$\therefore \lim_{x\to 0}\log K = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^2\tan x}$$

$$= \lim_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x + x\sec^2 x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\tan x}{\sin 2x + x}$$

$$= \lim_{x\to 0}\frac{\sec^2 x}{2\cos 2x + 1} = \frac{1}{3} \quad \left(\frac{0}{0}\text{ form}\right)$$

i.e. $\lim_{x\to 0}\log K = \frac{1}{3} \Rightarrow \lim_{x\to 0} K = e^{1/3}$

$$\therefore \lim_{x\to 0} K = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$$

$$= e^{1/3}$$

1(d) for the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Examine the continuity and differentiability

Sol let us approach $(0,0)$ along the path $y = x^4$

then $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{x\to 0}{Lt} \dfrac{x^2 - x(x^2)}{x^2 + x^4}$

$= \underset{x\to 0}{Lt} \dfrac{x^2(1-x)}{x^2(1+x^2)}$

$= \underset{x\to 0}{Lt} \dfrac{1-x}{1+x^2} = 1.$

$\neq f(0,0).$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

$\therefore$ f is not differentiable at $(0,0)$.

1(e) If the axes are rectangular, find the S.D between the lines $y = az + b$, $z = \alpha x + \beta$ and $y = a'z + b'$, $a = \alpha' x + \beta'$. Also deduce the condition for the lines to intersect.

Sol^n:

The equation of the given lines can be written in the symmetric form as

$$\frac{x + (\beta/\alpha)}{(1/\alpha)} = \frac{y - b}{a} = \frac{z}{1} \text{ and}$$

$$\frac{x + (\beta'/\alpha')}{(1/\alpha')} = \frac{y - b'}{a'} = \frac{z}{1}$$

If l, m, n the d.c's of required S.D, then we have $l(1/\alpha) + m.a + n.1 = 0$ and

$$l(1/\alpha') + m.a' + n.1 = 0$$

Solving these we get

$$\frac{l}{(a - a')} = \frac{m}{(1/\alpha') - (1/\alpha)} = \frac{n}{(1/\alpha)a' - (1/\alpha').a}$$

or

$$\frac{l}{\alpha\alpha'(a - a')} = \frac{m}{(\alpha - \alpha')} = \frac{n}{(a'\alpha' - a\alpha)} \quad \text{——(i)}$$

Also any point A on the first given line is $\left(\frac{-\beta}{\alpha}, b, 0\right)$ and a point on the second given line is $B\left(\frac{-\beta'}{\alpha'}, b', 0\right)$

$\therefore$ The required S.D = the projection of AB on the line whose d.c's are l, m, n given by (i)

$$= \left[l\left\{\left(\frac{-\beta}{\alpha}+\frac{\beta'}{\alpha'}\right)\right\} + m(b-b') + n(0-0)\right] \div \sqrt{(l^2+m^2+n^2)}$$

$$= \frac{\alpha\alpha'(a-a')(\alpha\beta'-\alpha'\beta)}{\alpha\alpha'\sqrt{(\Sigma l^2)}} + \frac{(\alpha-\alpha')(b-b')}{\sqrt{(\Sigma l^2)}}$$

$$= \frac{(a-a')(\alpha\beta'-\alpha'\beta)+(\alpha-\alpha')(b-b')}{\sqrt{[\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2]}}$$

If the given lines intersect, then this $SD=0$ and we have the required condition as

$$(a-a')(\alpha\beta'-\alpha'\beta)+(\alpha-\alpha')(b-b')=0.$$

2(a)(i) Suppose that $\{v_1, v_2, v_3 \ldots v_n\}$ is a set of vectors. Prove that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots v_n - v_1\}$ is a linearly dependent set.

Sol^n: consider the following linear combination

$$1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + \ldots + 1(v_n - v_1)$$

$$= v_1 - v_2 + v_2 - v_3 + v_3 - v_4 + \ldots + v_n - v_1$$

$$= v_1 + 0 + 0 + \ldots + 0 - v_1$$

$$= 0$$

This is a nontrivial relation of linear dependence, so the set is linearly dependent.

2(a)(ii) Suppose that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set in C^{35}. Prove that $\{v_1, v_1 + v_2, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.

Sol^n: set a linear combination of the vectors equal to the zero vector. using the unknown scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

i.e,

$$\alpha_1 v_1 + \alpha_2 (v_1 + v_2) + \alpha_3 (v_1 + v_2 + v_3) + \alpha_4 (v_1 + v_2 + v_3 + v_4) = 0$$

Using the distributive and commulative properties of vector addition and scalar multiplication, this equation can be writtet as $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) v_1 + (\alpha_2 + \alpha_3 + \alpha_4) v_2 + \alpha_4 (\alpha_3 + \alpha_4) + \alpha_4 v_4 = 0$

However, this is a relation of linear dependence on a linearly independent set, $\{v_1, v_2, v_3, v_4\}$. By definition of linear independence the scalars must all be zero.

This is the homogeneous system of equations, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$

$$\alpha_2 + \alpha_3 + \alpha_4 = 0$$
$$\alpha_3 + \alpha_4 = 0$$
$$\alpha_4 = 0.$$

solving these equations, we get

$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0.$

$\therefore$ The set

$\{v_1, v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4\}$ is linearly independent.

2(a)(iii) Find a basis for the subspace W of C^4,

$$W = \left\{ \begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} \middle|\ a, b, c, d \in C \right\}$$

Sol'n: we can rewrite an arbitrary vector of W as

$$\begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} = \begin{bmatrix} a \\ a \\ -2a \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ b \\ 2b \\ b \end{bmatrix} + \begin{bmatrix} -2c \\ -2c \\ 4c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ -d \\ d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} -2 \\ -2 \\ 4 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus we can write W as.

$$W = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

These four vectors span W, but we also need to determine if they are linearly independent.

By using Row Reduced Echelon form (RREF),

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 1 \\ -2 & 2 & 4 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} \underline{1} & 0 & -2 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The basis of W,

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

2(b) By using Lagrange's multipliers method find the maximum value of the function $f(x,y,z)=x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.

Sol'n: we maximize the function

$f(x,y,z) = x+2y+3z$ subject to the

constraints $g(x,y,z) = x-y+z-1 = 0$ — (1)

and $h(x,y,z) = x^2+y^2-1 = 0$.

Let us consider a function F of independent variables x, y, z.

where $F = x+2y+3z + \lambda_1(x-y+z-1) + \lambda_2(x^2+y^2-1)$.

$dF = (1+\lambda_1+2\lambda_2 x)\,dx + (2-\lambda_1+2\lambda_2 y)\,dy + (3+\lambda_1)\,dz$

for stationary points, $dF=0$ $(\because F_x dx + F_y dy + F_z dz = dF)$

$\therefore F_x = 0 \Rightarrow 1+\lambda_1+2\lambda_2 x = 0$ — (i)

$F_y = 0 \Rightarrow 2-\lambda_1+2\lambda_2 y = 0$ — (ii)

$F_z = 0 \Rightarrow 3+\lambda_1 = 0 \Rightarrow \boxed{\lambda_1 = -3}$

$\therefore$ (i) $\equiv 1-3+2\lambda_2 x = 0 \Rightarrow \lambda_2 = \frac{1}{x}$

(ii) $\equiv 2-(-3)+2\lambda_2 y = 0 \Rightarrow \lambda_2 = \frac{5}{2}y$

$\Rightarrow \frac{1}{x} = -\frac{5}{2}y$

$\Rightarrow y = -\frac{5}{2}x$

Now, we have

$x^2+y^2=1 \Rightarrow x^2+\frac{25x^2}{4} = 1$

$\Rightarrow 29x^2 = 4$

$\Rightarrow x = \pm\frac{2}{\sqrt{29}}$

$\Rightarrow y = \mp\frac{5}{\sqrt{29}}$.

putting in ①; $x = \frac{2}{\sqrt{29}}$, $y = \frac{-5}{\sqrt{29}}$

$$z = 1 - x + y = 1 - \frac{2}{\sqrt{29}} - \frac{5}{\sqrt{29}} = 1 - \frac{7}{\sqrt{29}}$$

and putting in ①; $x = \frac{-2}{\sqrt{29}}$, $y = \frac{5}{\sqrt{29}}$,

we get $z = 1 + \frac{7}{\sqrt{29}}$.

$\therefore$ The corresponding values of f are

$$\pm\frac{2}{\sqrt{29}} + 2\left(\frac{-5}{\sqrt{29}}\right) + \left(1 \mp \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}.$$

$\therefore$ The maximum value of f on the given curve is $3+\sqrt{29}$.

2(c)(i) The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$. Find the equation of the plane in its new position.

Sol'n: Here we are to find the equation of the plane through the line of intersection of the planes $x-2y+3z=0$ and $2x+3y-4z-5=0$ and at right angles to the plane $x-2y+3z=0$

Now the equation of the plane through the line intersection of the planes

$$x-2y+3z=0 \text{ and } 2x+3y-4z-5=0 \text{ is}$$

$$(x-2y+3z)+\lambda(2x+3y-4z-5)=0$$

$$(1+2\lambda)x+(3\lambda-2)y+(3-4\lambda)z=5\lambda \quad \text{---} \text{ (1)}$$

If this plane is $\perp$ar to the plane $x-2y+3z=0$ then we have

$$a_1a_2+b_1b_2+c_1c_2=0$$

i.e. $1(1+2\lambda)-2(3\lambda-2)+3(3-4\lambda)=0$

$$\Rightarrow \lambda=7/8$$

Substituting the value of λ in (1), the required equation is

$$\left(1+\tfrac{7}{4}\right)x+\left(\tfrac{21}{8}-2\right)y+\left(3-\tfrac{7}{2}\right)z=\frac{35}{8}$$

$$\Rightarrow 22x+5y-4z=35$$

2(c)(ii) A variable plane is parallel to the given plane $x/a + y/b + z/c = 0$ and meets the axes in A, B, C respectively. prove that the circle ABC lies on the curve

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

Sol'n: the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ —— ①

It meets the axes at $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$.

the equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0. \quad \text{—— ②}$$

The required cone is generated by the lines drawn from O to meet the circle ABC [given by ① & ② together] and will be homogeneous. So making ② homogeneous with the help of ①, we get the required equation as

$$x^2+y^2+z^2-(ax+by+cz)(x/a+y/b+z/c)=0$$

$$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

3(a)(ii) Find a basis for $\langle S \rangle$, where

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Sol^n: We take these 5 vectors, put them into a matrix, and row reduce to discover the pivot columns, then the corresponding vectors in S will be linearly independent and span S, and thus will form a basis of S.

Using Row Reduced Echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 2 & 4 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & -2 \\ 0 & \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the independent vectors that span S are the first, second and fourth of the set, so a basis of S is

$$B = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

3.(a)(ii) Find the values of λ for which the equations $(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$

$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$

$$2x+(3\lambda+1)y+3(\lambda-1)z=0.$$

are consistent, and find the ratios of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greater of these values.

Sol: The given equations will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0$$

$R_2 \to R_2-R_1$

$$\Rightarrow \begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0$$

$C_3 \to C_3+C_2$.

$$\Rightarrow \begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix}=0$$

$$\Rightarrow (\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 6\lambda-2 \end{vmatrix}=0$$

$$\Rightarrow (\lambda-3)\left[(6\lambda^2-8\lambda+2)-10\lambda-2\right]=0$$

$\Rightarrow (\lambda-3)\left[6\lambda^2-18\lambda\right]=0$

$\Rightarrow 6(\lambda-3)\lambda(\lambda-3)=0$

$\Rightarrow \lambda(\lambda-3)^2=0$

$\Rightarrow \lambda=0,3.$

When $\lambda=0$, the equations become

$-x+y=0 \Rightarrow x=y.$

$\left.\begin{array}{l} -x-2y+3z=0 \\ 2x+y-3z=0 \end{array}\right\} \Rightarrow -3x+3z=0, \Rightarrow x=z.$

Hence $x=y=z.$

When $\lambda=3$, equations becomes identical

3.(b) (i) Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

Sol^n Let r be the radius of the circular base; h, the height; s the surface and v the volume of the open cylinder so that

$$S = \pi r^2 + 2\pi r h \rightarrow (1)$$
$$v = \pi r^2 h \rightarrow (2)$$

Here, as given s is constant and v a variable. Also h, r are variables. Substituting the value of h as obtained from (i) in (ii) we get

$$v = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) = \frac{Sr - \pi r^3}{2} \rightarrow (3)$$

which gives v in terms of one variable r.

As v must be necessarily non-negative, we have

$$Sr - \pi r^3 \geq 0 \Rightarrow \pi r^3 \leq Sr \Rightarrow r \leq \sqrt{S/\pi}$$

Also r is non-negative.

Thus 'r' varies in the interval $[0, \sqrt{S/\pi}]$.

Now $$\frac{dv}{dr} = \frac{S - 3\pi r^2}{2}.$$

So that $\frac{dv}{dr} = 0$ only when $r = \sqrt{S/3\pi}$, negative value of r being inadmissible. Thus v has only one stationary value.

Now $V=0$ for the end points $r=0$ and $\sqrt{S/\pi}$ and positive for every other admissible value of r. Hence V is greatest for

$$r = \sqrt{S/3\pi}$$

Substituting this value of r in (i) we get

$$h = \frac{S-\pi r^2}{2\pi r} = \frac{S-\pi\left(S/3\pi\right)}{2\pi\sqrt{(S/3\pi)}}$$

$$= \frac{2S}{3}\cdot\frac{1}{2\pi}\sqrt{\left(\frac{3\pi}{S}\right)} = \sqrt{\left(\frac{S}{3\pi}\right)}$$

Hence $h=r$ for the cylinder of greatest volume and given surface.

3(b)(ii) Show that $\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \pi \log 2$.

Sol^n: let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$.

$$\therefore I = \int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_0^{\pi/2} \log\left(\tan\theta + \frac{1}{\tan\theta}\right)\frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}.$$

$$= \int_0^{\pi/2} \log\left(\frac{\tan^2\theta+1}{\tan\theta}\right) d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \log\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta \Rightarrow \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta\, d\theta - \int_0^{\pi/2} \log \cos\theta\, d\theta.$$

$$= -\left(-\frac{\pi}{2}\log 2\right) - \left(-\frac{\pi}{2}\log 2\right)$$

$$= \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2 \quad \left(\because \int_0^{\pi/2} \log \sin x = \int_0^{\pi} \log \cos x = -\frac{\pi}{2}\log 2\right)$$

$$= \underline{\underline{\pi \log 2}} \text{ Ans.}$$

3(c) Normals at P and P', points of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, meet the xy-plane in G_3 and G_3' and make angles θ and θ' with PP'. Prove that $PG_3\cos\theta + P'G_3'\cos\theta' = 0$.

Soln: Let the points P and P' be (α,β,γ) and (α',β',γ') respectively.

then the equations of the normal to the given ellipsoid at $P(\alpha,\beta,\gamma)$ & $P'(\alpha',\beta',\gamma')$ are

$$\frac{x-\alpha}{p\alpha/a^2}=\frac{y-\beta}{p\beta/b^2}=\frac{z-\gamma}{p\gamma/c^2}=r \text{ (say)} \quad \text{—(i)}.$$

[where $p=\frac{1}{\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}}}$ = length of the $\perp$lar from $(0,0,0)$ to the tangent plane $\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1$ of the ellipsoid at (α,β,γ)]

$\therefore$ The co-ordinates of any point Q on the normal (i) are $\left(\alpha+\frac{p\alpha r}{a^2},\ \beta+\frac{p\beta r}{b^2},\ \gamma+\frac{p\gamma r}{c^2}\right)$

where r is the distance of Q from P.

The normal at $P(\alpha,\beta,\gamma)$ meet the co-ordinate plane xy at G_3,

then putting $z=0$ in the above equation (i) of the normal we have,

$PG_3 = -c^2/p$.

Similarly, we can get $P'G_3' = -c^2/p'$

Now the direction cosines of the normals at P and P' are respectively

$p\alpha/a^2,\ p\beta/b^2,\ p\gamma/c^2$ and $p'\alpha'/a^2,\ p'\beta'/b^2,\ p'\gamma'/c^2$.

So the direction cosines of PP' are

$$\frac{\alpha'-\alpha}{PP'},\ \frac{\beta'-\beta}{PP'},\ \frac{\gamma'-\gamma}{PP'}$$

$$\therefore PG_3\cos\theta = -\frac{c^2}{p}\left[\frac{p\alpha}{a^2}\cdot\frac{\alpha'-\alpha}{PP'}+\frac{p\beta}{b^2}\cdot\frac{\beta'-\beta}{PP'}+\frac{p\gamma}{c^2}\cdot\frac{\gamma'-\gamma}{PP'}\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}-1\right]$$

($\because (\alpha,\beta,\gamma)$ lies on the given ellipsoid)

Similarly $$P'G_3'\cos\theta' = -\frac{c^2}{PP'}\left[1-\left(\frac{\alpha\alpha'}{a^2}+\frac{\beta\beta'}{b^2}+\frac{\gamma\gamma'}{c^2}\right)\right]$$

$$\therefore PG_3\cos\theta + P'G_3'\cos\theta' = 0$$

4(a)(i) Define $T: \mathbb{C}^2 \to \mathbb{C}^2$ by $T((z_1, z_2)) = (iz_1, (1+i)z_2 - z_1)$. Let $\mathbb{C}^2$ have the basis $S = \{(i,0), (0,1)\}$. Calculate M_T.

Sol^n:
$$T(i,0) = (-1,-i) = i(i,0) + (-i)(0,1)$$
$$T(0,1) = (0, 1+i) = 0(i,0) + (1+i)(0,1)$$

Therefore
$$M_T = \begin{pmatrix} i & 0 \\ -i & 1+i \end{pmatrix}$$

4(a)(ii) If A is a non-singular matrix, then show that $\text{adj adj } A = |A|^{n-2} A$.

Sol^n: we have $A(\text{adj} A) = |A| I_n$. ——— ①

If we take $\text{adj} A$ in place of A, then ① gives
$$(\text{adj}A)(\text{adj adj} A) = |\text{adj}A| I_n$$
$$\Rightarrow (\text{adj}A)(\text{adj adj}A) = |A|^{n-1} I_n \quad [\because |\text{adj} A| = |A|^{n-1}]$$

Pre-multiplying bothsides of this last relation by A, we get
$$A\{(\text{adj}A)(\text{adj adj } A)\} = A\{|A|^{n-1} I_n\}$$
$$(A \,\text{adj}A)(\text{adj adj } A) = |A|^{n-1}(A I_n)$$
($\because$ matrix multiplication is associative)

$$\Rightarrow (|A| I_n)(\text{adj adj } A) = |A|^{n-1} A \quad [\because A I_n = A]$$
$$\Rightarrow |A|(I_n \text{adj adj} A) = |A|^{n-1} A$$
$$\Rightarrow |A| \text{adj adj} A = |A|^{n-1} A \quad \text{——— ②}$$

Since A is non-singular, therefore $|A| \neq 0$.
so cancelling $|A|$ from bothsides of ②, we get
$$\text{adj adj } A = |A|^{n-2} A.$$

4.(a)(iii) Using Cayley-Hamilton theorem, find A^8, if $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Sol'n: The characteristic equation of the matrix A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow -(1-\lambda)(1+\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 5 = 0. \quad \text{--- (1)}$

By Cayley-Hamilton theorem, the matrix A must satisfy its characteristic equation (1).

So we must have

$A^2 - 5I = 0$

$\Rightarrow A^2 = 5I$

$\Rightarrow A^4 = A^2 \cdot A^2 = (5I)(5I)$
$= 25I.$

$\Rightarrow A^8 = (A^4)^2 = (25I)^2$

$= 625 I$

$= 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

$\therefore A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

4(b)(i) Let $z = f(t)$, $t = \frac{x+y}{xy}$. show that $x^2\frac{\partial z}{\partial x} = y^2\frac{\partial z}{\partial y}$.

Sol'n: Given $z = f(t)$, $t = \frac{x+y}{xy}$

$$= \frac{1}{x} + \frac{1}{y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial x} = f'(t)\left(-\frac{1}{x^2}\right) \Rightarrow x^2\frac{\partial z}{\partial x} = f'(t) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial y} = f'(t)\left(-\frac{1}{y^2}\right) \Rightarrow y^2\frac{\partial z}{\partial y} = f'(t) \quad \text{--- (2)}$$

$\therefore$ from (1) & (2) $\quad x^2\frac{\partial z}{\partial x} = y^2\frac{\partial z}{\partial y}$

4(b)(ii) Evaluate $\iiint z\,dx\,dy\,dz$ over the volume enclosed between the cone $x^2+y^2=z^2$ and the sphere $x^2+y^2+z^2=1$ on positive side of xy-plane.

Sol'n: $I = \iiint z\,dx\,dy\,dz$

$$= 4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{\frac{1}{2}-x^2}}\int_{z=\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dx\,dy\,dz$$

$$= 4\int_0^{1/\sqrt{2}}\int_0^{\sqrt{\frac{1}{2}-x^2}}\left[\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= 2\int_0^{1/\sqrt{2}}\int_0^{\sqrt{\frac{1}{2}-x^2}}\left[1-2(x^2+y^2)\right]dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$I = 2\int_0^{1/\sqrt{2}}\int_0^{\pi/2}(1-2r^2)\,r\,dr\,d\theta = 2\int_0^{1/\sqrt{2}}(r-2r^3)\,dr\int_0^{\pi/2}d\theta$$

$$= 2\left[\frac{r^2}{2}-\frac{r^4}{2}\right]_0^{1/\sqrt{2}}\left(\frac{\pi}{2}\right)$$

$$= \left[\frac{\frac{1}{2}}{2}-\frac{1}{2}\left(\frac{1}{4}\right)\right]\pi$$

$$= \left(\frac{1}{4}-\frac{1}{8}\right)(\pi)$$

$$= \frac{\pi}{8}$$

Q4(c) Sol'n The Coordinates of P, Q, P' and Q' are given by $P(a\cos\theta, b\sin\theta, c)$, $Q(-a\sin\theta, b\cos\theta, c)$, $P'(a\cos\theta, b\sin\theta, -c)$ and $Q'(-a\sin\theta, b\cos\theta, -c)$

$\therefore$ Equations to PQ' are

$$\frac{x-a\cos\theta}{-a\sin\theta-a\cos\theta} = \frac{y-b\sin\theta}{b\cos\theta-b\sin\theta} = \frac{z-c}{-c-c} = r$$

$\therefore$ $x-a\cos\theta = r[-a(\sin\theta+\cos\theta)]$,

$y-b\sin\theta = r[b(\cos\theta-\sin\theta)]$ and

$z-c=-2cr$

$\Rightarrow$ $x/a = \cos\theta - r(\sin\theta+\cos\theta)$,

$y/b = \sin\theta + r(\cos\theta-\sin\theta)$ and

$z = c(1-2r)$ ——— (i)

Eliminating r from these We get,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = [\cos\theta - r(\sin\theta-\cos\theta)]^2 + [\sin\theta + r(\cos\theta-\sin\theta)]^2$$

$$= 1+r^2\{(\sin\theta+\cos\theta)^2+(\cos\theta-\sin\theta)^2\}$$
$$-2r[\cos\theta(\sin\theta+\cos\theta)-\sin\theta(\cos\theta-\sin\theta)\}$$

$$= 1+r^2(2)-2r(1) = 2r^2-2r+1$$

or $2\left(\frac{x^2}{a^2}\right)+2\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1$ Which is a hyperboloid

4(c) CP, CQ are any two conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=c$, CP', CQ' are the conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$ drawn in the same directions as CP and CQ, Prove that the hyperboloid $2x^2/a^2+2y^2/b^2-z^2/c^2=1$ is generated by either PQ' or $P'Q'$.

Q5a(i) Solve $x\cos(y/x)(y\,dx + x\,dy) = y\sin(y/x)(x\,dy - y\,dx)$

Sol^n: $x\cos(y/x)(y\,dx + x\,dy) = y\sin(y/x)(x\,dy - y\,dx)$ — (1)

Rewriting (1)

$$\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0 \quad \text{— (2)}$$

from (2) $$\frac{dy}{dx} = \frac{\{x\cos(y/x) + y\sin(y/x)\}y}{\{y\sin(y/x) - x\cos(y/x)\}x}$$

$$\frac{dy}{dx} = \frac{[\cos(y/x) + (y/x)\sin(y/x)](y/x)}{[(y/x)\sin(y/x) - \cos(y/x)]} \quad \text{— (3)}$$

Take $y/x = v \Rightarrow y = vx$ so that

$$\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right) \quad \text{— (4)}$$

Using (3) & (4) $$v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v} - v = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$2\frac{dx}{x} = \frac{v\sin v - \cos v}{v\cos v}dv = \left[\frac{\sin v}{\cos v} - \frac{1}{v}\right]dv$$

Integrating, $2\log x = -\log\cos v - \log v + \log c$

$\Rightarrow \log x^2 = \log(c/v\cos v)$

$\Rightarrow x^2 v\cos v = c$

$\Rightarrow xy\cos(y/x) = c$

Q. 5(a)ii) solve $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0$

Sol^n: Comparing the given equation with $Mdx+Ndy=0$

Here $M = y(x^2y^2+2)$ and $N = x(2-2x^2y^2)$.

Showing that the given equation is of the form

$f_1(x,y)y\,dx + f_2(x,y)x\,dy = 0$

Again $Mx - Ny = xy(x^2y^2+2) - xy(2-x^2y^2)$

$= 3x^3y^3 \neq 0$

Showing that I.F of given equation

$= 1/(Mx-Ny) = 1/(3x^3y^3)$

Multiplying the given equation by $1/(3x^3y^3)$,

We get $\left(\frac{1}{3x} + \frac{2}{3x^3y^3}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$

Which is exact

its Solution is

$\left(\frac{1}{3}\right)\log x - (1/3x^2y^2) - (2/3)\log y = \left(\frac{1}{3}\right)\log C$

or $\log(x/cy^2) = (1/x^2y^2)$

or $x = cy^2e^{1/x^2y^2}$, C being an arbitary Constant

5(b) → Solve $(px^2+y^2)(px+y)=(p+1)^2$ by reducing it to Clairaut's form and find its singular solution.

Sol^n: Given differential equation is

$$(px^2+y^2)(px+y)=(p+1)^2.$$

put $u=xy$, $v=x+y$

$$\Rightarrow \frac{du}{dx}=y+x\frac{dy}{dx} \; ; \; \frac{dv}{dx}=1+\frac{dy}{dx}=1+p$$
$$=y+xp$$

$$\Rightarrow \frac{dv}{du}=\frac{1+p}{y+xp}$$

$$\Rightarrow P=\frac{1+p}{y+xp} \text{ where } P=\frac{dv}{du}\;;\; p=\frac{dy}{dx}$$

$$\Rightarrow P(y+xp)=1+p$$
$$\Rightarrow p(xP-1)=1-Py \Rightarrow p=\frac{1-Py}{xP-1} \quad \text{—} (1)$$

using (1), the given equation becomes

$$\left[\frac{1-Py}{(xP-1)}x^2+y^2\right]\left[\frac{(1-Py)x}{xP-1}+y\right]=\left[\frac{1-Py}{xP-1}+1\right]^2$$

$$\Rightarrow \left[x^2(1-Py)+y^2(xP-1)\right]\left[(1-Py)x+y(xP-1)\right]=(1-Py+xP-1)^2$$

$$\Rightarrow (x^2-Px^2y+y^2xP-y^2)(x-P\cancel{x}y+P\cancel{x}y-y)=P^2(x-y)^2$$

$$\Rightarrow \left[(x^2-y^2)-Pxy(x-y)\right](x-y)=P^2(x-y)^2$$

$$\Rightarrow (x-y)^2\left[(x+y)-Pxy\right]=P^2(x-y)^2$$

$$\Rightarrow x+y-Pxy=P^2$$

$$\Rightarrow v-Pu=P^2$$

$$\Rightarrow v=uP+P^2 \text{ which is a clairaut's equation.}$$

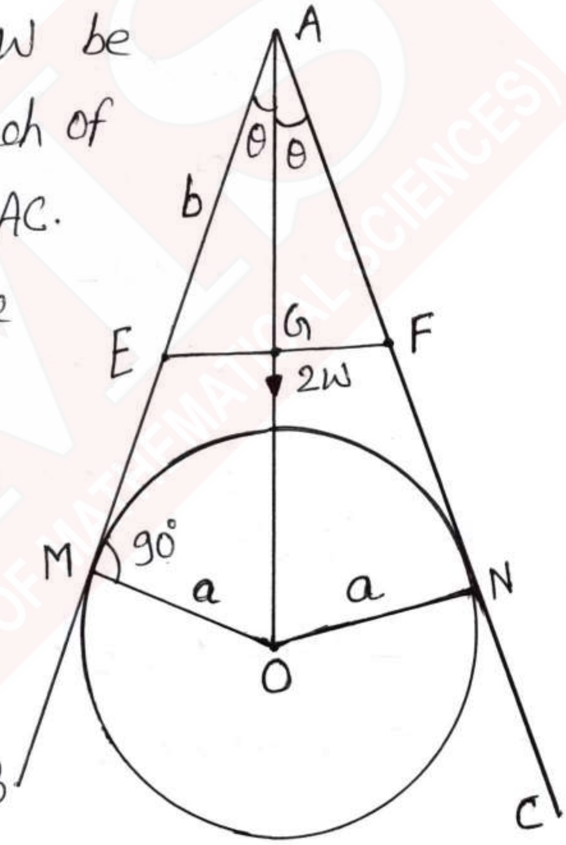
∴ The general solution is

$$v=cu+c^2$$

i.e, $x+y=cxy+c^2$

5(c) Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then $b\sin^3\theta = a\cos\theta$

Sol: Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC. If E and F are the middle points of AB and AC, then the total weight 2W of the two rods can be taken acting at G. the middle point of EF. The line AO is vertical. we have



$$\angle BAO = \angle CAO = \theta$$

Also $AB = 2b$, $AE = b$. If the rod AB

touches the circle at M, then $\angle OMA = 90^\circ$ and OM = the radius of the circle = a.

Give the rods a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90°. we have, the height of G above the fixed point O

$$= OG = OA - GA$$

$$= OM \operatorname{cosec}\theta - AE \cos\theta$$

$$= a \operatorname{cosec}\theta - b\cos\theta.$$

The equation of virtual work is

$$-2W\delta(OG) = 0 \quad \text{or} \quad \delta(OG) = 0$$

or $\delta(a \operatorname{cosec}\theta - b\cos\theta) = 0$

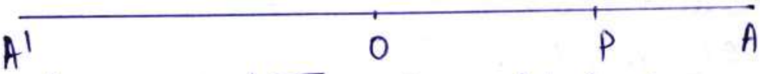
or $(-a \operatorname{cosec}\theta \cot\theta + b\sin\theta)\delta\theta = 0$

or $-a \operatorname{cosec}\theta \cot\theta + b\sin\theta = 0 \quad [\because \delta\theta \neq 0]$

or $a \operatorname{cosec}\theta \cot\theta = b\sin\theta$

or $a\cos\theta = b\sin^3\theta.$

5(d) Sol'n → Let the eq^n of the S.H.M with centre O as origin be $\dfrac{d^2x}{dt^2} = -\mu x$



The time period $T = 2\pi/\sqrt{\mu}$, let aplitude be a

Then $(dx/dt)^2 = \mu(a^2 - x^2)$ ——— (1)

When particle passes through P its velocity is given to be V in the direction OP. Also $OP = b$. So putting $x = b$ and $dx/dt = v$ in (1)

We get $v^2 = \mu(a^2 - b^2)$.

In S.H.M the time from P to A is equal to the time from A to P.

$\therefore$ the required time $= 2 \cdot$ time from A to P.

Now motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\,\sqrt{(a^2 - x^2)} \Rightarrow dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2 - x^2)}}$$

Let t_1 be the time from A to P. Then at A, $t = 0$, $x = a$ and at P, $t = t_1$ and $x = b$, Therefore integrating (3)

we get $\displaystyle\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{(a^2 - x^2)}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$

Hence required time $= 2t_1 = \dfrac{2}{\sqrt{\mu}}\cos^{-1}\left(\dfrac{b}{a}\right)$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{(a^2 - b^2)}}{b}\right\} = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{V}{b\sqrt{\mu}}\right) \quad [\text{from } (2)$$

$$= \frac{2}{2\pi/T}\tan^{-1}\left\{\frac{V}{b(2\pi/T)}\right\} \quad \left[\because T = 2\pi/\sqrt{\mu} \text{ so that } \sqrt{\mu} = 2\pi/T\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi b}\right)$$

Q5(d) → A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP; Prove that the time which elapses before it returns to P is $\dfrac{T}{\pi}\tan^{-1}\left(\dfrac{VT}{2\pi b}\right)$.

5.(e) Verify Green's theorem in the plane for

$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region defined by: $y=\sqrt{x}$, $y=x^2$.

Sol'n: By Green's theorem in plane, we have

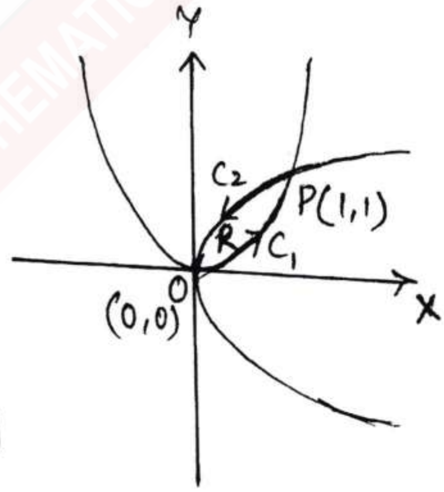
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = 3x^2-8y^2$, $N = 4y-6xy$

The parabola $y=\sqrt{x}$ i.e. $y^2=x$ and the parabola $y=x^2$ intersect at the points $(0,0)$ and $(1,1)$. The closed curve C consists of the arc C_1 of the parabola $y=x^2$ and the arc C_2 of the parabola $y=\sqrt{x}$. Also R is the region bounded by the closed curve C.

We have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$$= \iint_R \left[\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right] dx\,dy$$

$$= \iint_R (-6y+16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dx\,dy \quad [\because \text{for the region R, x varies from 0 to 1 \& y varies from } x^2 \text{ to } \sqrt{x}]$$

$$= \int_0^1 5\left[y^2\right]_{y=x^2}^{\sqrt{x}} dx = 5\int_0^1 [x-x^4]\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2}-\frac{1}{5}\right] = \frac{15}{10} = 3/2 \quad \text{——— (1)}$$

Now the line integral along the closed curve C.

$$= \oint_C (M\,dx + N\,dy) = \int_{C_1} (M\,dx + N\,dy) + \int_{C_2} (M\,dx + N\,dy)$$

Along C_1, $x^2 = y$, $dy = 2x\,dx$ and x varies from 0 to 1.

$\therefore$ line integral along $C_1 = \int_0^1 [(3x^2 - 8x^4)dx + (4x^2 - 6x^3)2x\,dx]$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)dx = [x^3 + 2x^4 - 4x^5]_0^1 = 1 + 2 - 4 = -1.$$

Along C_2, $y^2 = x$.

$\therefore dx = 2y\,dy$ and limits for y are 1 to 0.

$\therefore$ line integral along C_2

$$= \int_1^0 [(3y^4 - 8y^2)2y\,dy + (4y - 6y^3)dy]$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0$$

$$= -1 + 11/2 - 2 = 5/2$$

$\therefore$ Total line integral along the closed curve C

$$= -1 + 5/2 = 3/2. \quad \text{——— } (2)$$

from (1) and (2), we see that Green's theorem is verified.

6(a)(i) Evaluate $L^{-1}\left\{e^{4-3s}/(s+4)^{5/2}\right\}$

Sol'n: Let $f(s) = \dfrac{e^4}{(s+4)^{5/2}}$

and $F(t) = L^{-1}\{f(s)\}$ —— (1).

$\therefore F(t) = L^{-1}\left\{\dfrac{e^4}{(s+4)^{5/2}}\right\}$

$= e^4 L^{-1}\left\{\dfrac{1}{(s+4)^{5/2}}\right\} = e^4 e^{-4t} L^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

$\Rightarrow F(t) = e^{4(1-t)} \dfrac{t^{3/2}}{\Gamma(5/2)} = e^{4(1-t)} \dfrac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \dfrac{4t^{3/2}e^{4(1-t)}}{3\sqrt{\pi}}$ —— (2)

Hence by second shifting theorem, we have

$L^{-1}\left\{e^{-3s} f(s)\right\} = \begin{cases} F(t-3), & t>3 \\ 0, & t<3 \end{cases}$

$\Rightarrow L^{-1}\left\{e^{-3s}\dfrac{e^4}{(s+4)^{5/2}}\right\} = \begin{cases} (4/3\sqrt{\pi})(t-3)^{3/2} e^{4[1-(t-3)]}, & t>3 \\ 0, & t<3 \end{cases}$

(using (1) & (2))

$\Rightarrow L^{-1}\left\{\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right\} = \begin{cases} (4/3\sqrt{\pi})(t-3)^{3/2} e^{-4(t-4)}, & t>3 \\ 0, & t<3 \end{cases}$

$= (4/3\sqrt{\pi})(t-3)^{3/2} e^{-4(t-4)} H(t-3),$

interms of Heaviside unit step function.

6(a)(ii) By using Laplace transform solve $(D^2+m^2)x = a\sin nt$, $t>0$ where x, Dx equal to x_0 and x_1, when $t=0$, $n\neq m$.

Sol^n: Re-writing the given equation and conditions, we get

$$x'' + m^2 x = a\sin nt \quad \text{——} (1)$$

with initial conditions: $x(0) = x_0$ and $x'(0) = x_1$ —— (2)

Taking Laplace transform of bothsides of (1), we get

$$L\{x''\} + m^2 L\{x\} = aL\{\sin nt\}$$

$$\Rightarrow s^2 L\{x\} - sx(0) - x'(0) + m^2 L\{x\} = \frac{an}{(s^2+n^2)}$$

$$\Rightarrow (s^2+m^2)L\{x\} - sx_0 - x_1 = \frac{an}{(s^2+n^2)}, \text{ using } (2)$$

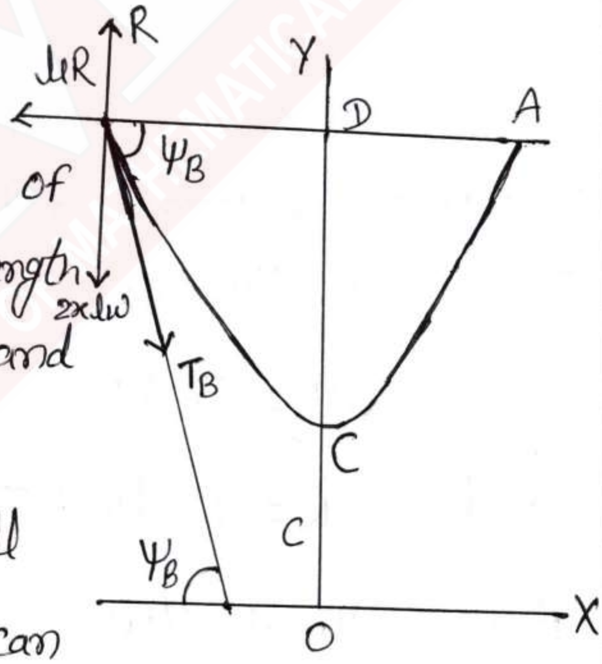
$$\Rightarrow L\{x\} = \frac{x_0 s}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{an}{(s^2+m^2)(s^2+n^2)}$$

$$\therefore L\{x\} = \frac{x_0 s}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{an}{(s^2+m^2)(s^2+n^2)}$$

Taking inverse Laplace transform of bothsides, we get

$$x = x_0 \cos mt + \frac{x_1}{m}\sin mt + \frac{an}{m^2-n^2}\left[\frac{1}{n}\sin nt - \frac{1}{m}\sin mt\right]$$

6(b). A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\{\lambda+\sqrt{(1+\lambda^2)}\}$, where $1/\lambda = \mu(2n+1)$ and μ is the coefficient of friction.



Sol: Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A. Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A.

In this position of limiting equilibrium the forces acting on the ring are:

(i) the weight $2nlw$ of the ring acting vertically downwards,

(ii) the normal reaction R of the rod,

(iii) the force of limiting friction μR of the rod acting in the direction AB, and

(iv) the tension T_B in the string at B acting along the tangent to the string at B.

For the equilibrium of the ring at B. resolving the forces acting on it horizontally and vertically, we have

$$\mu R = T_B \cos\psi_B \quad \text{——— (1)}$$

and $$R = 2nlw + T_B \sin\psi_B, \quad \text{——— (2)}$$

where ψ_B is the angle of inclination of the tangent at B to the horizontal.

Let C be the lowest point of the catenary formed by the chain, OX be the

directrix and $OC = c$ be the parameter.
We have arc $CB = s_B = l$. By the formula $T\cos\psi = wc$, we have $T_B \cos\psi_B = wc$. Also by the formula $T\sin\psi = ws$, we have

$$T_B \sin\psi_B = ws_B = wl.$$

Putting these values in ① and ②, we have

$$\mu R = wc \quad \text{and} \quad R = 2nlw + wl$$
$$= (2n+1)wl.$$

$$\therefore \mu(2n+1)wl = wc$$

or $\mu(2n+1)l = c$.

But it is given that

$$\mu(2n+1) = 1/\lambda$$

$$\therefore l/\lambda = c \quad \text{——— } ③$$

Using the formula $s = c\tan\psi$ for the point B, we have

$$l = c\tan\psi_B$$

$$\therefore \tan\psi_B = l/c = \lambda \quad \text{——— } ④$$

Now the required greatest possible distance of the ring from A

$$= AB = 2DB = 2x_B$$

$$= 2c\log(\sec\psi_B + \tan\psi_B)$$

$$\left[\because x = c\log(\sec\psi + \tan\psi)\right]$$

$$= 2c\log\left[\tan\psi_B + \sqrt{(1+\tan^2\psi_B)}\right]$$

$$= \frac{2l}{\lambda}\log\left[\lambda + \sqrt{(1+\lambda)^2}\right]$$

[$\because$ from ③, $c = l/\lambda$ and

from ④, $\tan\psi_B = \lambda$]

6(c)(i) Find (i) the curvature k (ii) the torsion τ for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$

Solution :

Given that $\vec{r} = (t - t^3/3)\hat{i} + t^2\hat{j} + (t + t^3/3)\hat{k}$

$$\frac{d\vec{r}}{dt} = (1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}.$$

$$\frac{d^2\vec{r}}{dt^2} \Rightarrow -2t\hat{i} + 2\hat{j} + 2t\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2}$$

$$= \sqrt{1 + t^4 - \cancel{2t^2} + 4t^2 + 1 + t^4 + \cancel{2t^2}}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{2 + 2t^4 + 4t^2}$$

Now; $$\left[\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} i & j & k \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \end{vmatrix}$$

$$= i(4t^2 - 2 - 2t^2) - j(2t - \cancel{2t^3} + 2t + \cancel{2t^3}) + \hat{k}(2 - 2t^2 + 4t^2)$$

$$= 2(t^2 - 1)\hat{i} - \hat{j}\,4t + \hat{k}\,2(t^2+1)$$

$$= 2(t^2-1)\hat{i} - 4t\hat{j} + 2(t^2+1)\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{4(t^2-1)^2 + 16t^2 + 4(t^2+1)^2}$$

$$= 2\sqrt{(t^2-1)^2 + 4t^2 + (t^2+1)^2}$$

$$= 2\sqrt{2 + 2t^4 + 4t^2}$$

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = 2\left|\frac{d\vec{r}}{dt}\right|$$

$$\frac{d^3\vec{r}}{dt^3} = -2\hat{i}+0\hat{j}+2\hat{k}$$

$$\left[\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 4\left[(1-t^2)(1) - 2t(-t+t) + (1+t^2)(1)\right]$$

$$= 4\left[1-\cancel{t^2}+1+\cancel{t^2}\right] = 8$$

$$\text{Curvature} = K = \frac{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{2\cancel{\left|\frac{d\vec{r}}{dt}\right|}}{\left|\frac{d\vec{r}}{dt}\right|^{\cancel{3}\,2}}$$

$$K = \frac{2}{\left(\sqrt{2+2t^4+4t^2}\right)^2} = \frac{2}{2t^4+4t^2+2}$$

$$\boxed{K = \frac{1}{t^4+2t^2+1} = \frac{1}{(t^2+1)^2}}$$

$$\text{Torsion }(\tau) = \frac{\left[\frac{dr}{dt}\cdot\frac{d^2r}{dt^2}\cdot\frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|^2} = \frac{8}{4(2t^4+4t^2+2)}$$

$$\tau = \frac{\cancel{8}}{\cancel{8}(t^4+2t^2+1)} \Rightarrow \boxed{\tau = \frac{1}{(t^2+1)^2}}$$

Ans.

Hence; $\boxed{K = \tau = \frac{1}{(t^2+1)^2}}$

is the required solution

6.(c)(ii) If $A = 5t^2 i + t j - t^3 k$ and $B = \sin t\, i - \cos t\, j$, find

(i) $\frac{d}{dt}(A \cdot B)$ (ii) $\frac{d}{dt}(A \times B)$ (iii) $\frac{d}{dt}(A \cdot A)$.

Sol: we have

$$\frac{dA}{dt} = 10t\, i + j - 3t^2 k \quad \text{and} \quad \frac{dB}{dt} = \cos t\, i + \sin t\, j.$$

(i) $\frac{d}{dt}(A \cdot B) = A \cdot \frac{dB}{dt} + \frac{dA}{dt} \cdot B$

$$= (5t^2 i + t j - t^3 k) \cdot (\cos t\, i + \sin t\, j) + (10t\, i + j - 3t^2 k) \cdot (\sin t\, i - \cos t\, j)$$

$$= 5t^2 \cos t + t \sin t + 10t \sin t - \cos t$$

$$= (5t^2 - 1)\cos t + 11t \sin t$$

(ii) $A \times B = (5t^2 i + t j - t^3 k) \times (\sin t\, i - \cos t\, j)$

$$= \begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= -t^3 \cos t\, i - (0 + t^3 \sin t)\, j + (-5t^2 \cos t - t \sin t)\, k$$

$$= -t^3 \cos t\, i - t^3 \sin t\, j - (5t^2 \cos t + t \sin t)\, k$$

$\therefore \frac{d}{dt}(A \times B) = (t^3 \sin t - 3t^2 \cos t)\, i + (t^3 \cos t + 3t^2 \sin t)\, j$
$\quad - (10t \cos t - 5t^2 \sin t + \sin t + t \cos t)\, k$

$$= t^2(t \sin t - 3\cos t)\, i - t^2(t \cos t + 3\sin t)\, j - (10t \cos t - 5t^2 \sin t + \sin t + t\cos t)\, k$$

(iii) $\frac{d}{dt}(A \cdot A) = \frac{dA}{dt} \cdot A + A \cdot \frac{dA}{dt} = 2A \cdot \frac{dA}{dt}$

$$= 2(5t^2 i + t j - t^3 k) \cdot (10t\, i + j - 3t^2 k)$$

$$= 2[50t^3 + t + 3t^5] = 100t^3 + 2t + 6t^5.$$

7(a) Solve $(x^2D^2 - xD + 1)y = (\log x \sin\log x + 1)/x$.

Sol^n: Given $(x^2D^2 - xD + 1)y = x^{-1}[1 + \log x \sin\log x]$ — ①

Let $x = e^z \Rightarrow z = \log x$ and let $D_1 = d/dz$.

Then ① becomes

$$(D_1(D_1 - 1) - D_1 + 1)y = e^{-z}(z\sin z + 1)$$

$$\Rightarrow (D_1^2 - 2D_1 + 1)y = e^{-z}(1 + z\sin z) \quad \text{—②}$$

$$= e^{-z} + e^{-z} z\sin z$$

Auxiliary equation of ② is

$$D_1^2 - 2D_1 + 1 = 0 \Rightarrow (D_1 - 1)^2 = 0 \Rightarrow D_1 = 1, 1$$

$$\therefore C.F = y_c = (c_1 + c_2 z)e^{z}$$

$$= (c_1 + c_2 \log x)x.$$

where c_1 and c_2 are arbitrary constants

P.I. corresponding to $e^{-z} z\sin z$.

$$= \frac{1}{(D_1 - 1)^2} e^{-z} z\sin z$$

$$= e^{-z} \frac{1}{[(D_1 - 1) - 1]^2} z\sin z$$

$$= e^{-z} \frac{1}{(D_1 - 2)^2} z\sin z$$

$$= e^{-z}\left[z \frac{1}{(D_1 - 2)^2}\sin z - 2(D_1 - 2)\frac{1}{(D_1 - 2)^4}\sin z\right]$$

$$= e^{-z}\left[z \frac{1}{D_1^2 - 4D_1 + 4}\sin z - \frac{2}{(D_1 - 2)^3}\sin z\right]$$

$$= e^{-z}\left[z \frac{1}{-1 - 4D_1 + 4}\sin z - \frac{2}{D_1^3 - 6D_1^2 + 12D_1 - 8}\sin z\right]$$

$$= e^{-z}\left[z \frac{1}{3 - 4D_1}\sin z - \frac{2}{11D_1 - 2}\sin z\right]$$

$$= e^{-z}\left[z\frac{(3+4D_1)}{9-16D_1^2}\sin z - \frac{2(11D_1+2)}{121D_1^2-4}\sin z\right]$$

$$= e^{-z}\left[z\frac{(3+4D_1)}{25}\sin z - \frac{2(11D_1+2)}{121(-1)-4}\sin z\right]$$

$$= e^{-z}\left[z\frac{(3\sin z+4\cos z)}{25} + \frac{2(11\cos z+2\sin z)}{125}\right]$$

$$= \frac{e^{-z}}{125}\left[z(15\sin z+20\cos z) + 22\cos z+4\sin z\right]$$

$$= \frac{e^{-z}}{125}\left[(15z+4)\sin z + (20z+22)\cos z\right]$$

P.I. corresponding to $e^{-z} = \frac{1}{D_1^2-2D_1+1}e^{-z} = \frac{e^{-z}}{4}$.

$$\therefore y_p = \frac{e^{-z}}{4} + \frac{e^{-z}}{125}\left[(15z+4)\sin z + 2(10z+11)\cos z\right]$$

$$= \frac{1}{4x} + \frac{1}{125x}\left[(15\log x+4)\sin(\log x) + 2(10\log x+11)\cos(\log x)\right]$$

$\therefore y = y_c + y_p$

$$= (c_1+c_2\log x)x + \frac{1}{4x} + \frac{1}{125x}\left[(15\log x+4)\sin(\log x) + 2(10\log x+11)\cos(\log x)\right]$$

which the required solution

7(b) Sol^n Let the particle of mass m start from rest from Cusp A of the Cycloid. If P is the position of particle after time t such that arc OP = s the eq^n of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad —(1)$$

$$m\frac{v^2}{\rho} = R - mg\cos\psi \quad —(2)$$

and from the Cycloid, $s = 4a\sin\psi$ —(3)

from (1) & (3) We have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$.

Multiplying both side $2(ds/dt)$ and then integrating.

$$\left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

Intialy at Cusp A, $s = 4a$ and $\frac{ds}{dt} = 0$

$$\therefore A = \frac{g}{4a}(4a)^2 = 4ag$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + 4ag = \frac{g}{4a}(16a^2 - s^2)$$

$$ds/dt = \frac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2 - s^2)} \quad —(4)$$

The -ve sign is taken because the particle is moving in the direction of s decreasing

Seperating the Variable, we have.

$$dt = -2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2 - s^2)}} \quad —(5)$$

If t_1 is the time from cusp A to the vertex, then integrating (5)

$$t_1 = -2\sqrt{(a/g)} \cdot \int_{4a}^{0} \frac{ds}{\sqrt{(16a^2 - s^2)}}$$

$$= 2\sqrt{(a/g)} \left[\cos^{-1}\frac{s}{4a} \right]_{4a}^{0}$$

$$= 2\sqrt{(a/g)} \cdot \frac{\pi}{2} = \pi\sqrt{(a/g)}$$

Again if t_2 is time taken to move from the cusp A to half the distance along the arc to the vertex $s = 2a$
then integrating (5).

$$t_2 = -2\sqrt{(a/g)} \int_{s=4a}^{2a} \frac{ds}{\sqrt{(16a^2 - s^2)}}$$

$$= 2\sqrt{(a/g)} \left[\cos^{-1}\frac{s}{4a} \right]_{4a}^{2a}$$

$$= 2\sqrt{(a/g)} \left[\cos^{-1}\frac{1}{2} - \cos^{-1} 1 \right]$$

$$= 2\sqrt{(a/g)} \cdot \left(\frac{\pi}{3}\right) = \left(\frac{2}{3}\right) t_1.$$

Q.7(b) A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc to the vertex, two thirds of the time of descent will have elapsed.

7.(c)(i) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1,1,2)$ has maximum magnitude 4 in the direction parallel to y-axis.

Sol^n: we have $\text{grad}\,\phi = \left(\frac{\partial\phi}{\partial x}\right)\vec{i} + \left(\frac{\partial\phi}{\partial y}\right)\vec{j} + \left(\frac{\partial\phi}{\partial z}\right)\vec{k}$

$$= 2ax\vec{i} + 2by\vec{j} + 2cz\vec{k}$$

$$= 2a\vec{i} + 2b\vec{j} + 2c\vec{k} \text{ at the point } (1,1,2)$$

Now the directional derivative of ϕ at the point $(1,1,2)$ is maximum in the direction vector grad ϕ at this point. According to the question this directional derivative is maximum in the direction parallel to y-axis. ie, in the direction parallel to the vector j so if the direction of the vector $2a\vec{i} + 2b\vec{j} + 4c\vec{k}$. is parallel to the vector j, we must have $2a = 0$, $4c = 0$, i.e, $a = 0$ and $c = 0$.

Then grad ϕ at $(1,1,2) = 2bj$

Also the maximum value of directional derivative

$$= |\text{grad}\,\phi|$$

$\therefore 4 = |2bj|$, since according to the question the maximum value of directional derivative is 4.

$$\therefore 2b = 4 \Rightarrow b = 2$$

Hence $a = 0$, $b = 2$, $c = 0$

7(c)ii find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$.

Solⁿ:- Angle between two surfaces at a point is the angle between the normals to the surfaces at the point

Let $f_1 = x^2+y^2+z^2$

and $f_2 = x^2+y^2-z$

Then $\text{grad } f_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\text{grad } f_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n_1} = \text{grad } f_1$ at the point $(2,-1,2)$

and $\vec{n_2} = \text{grad } f_2$ at the point $(2,-1,2)$. Then,

$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

and $\vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point $(2,-1,2)$. If θ is the angle between these vectors then,

$$\vec{n_1}\cdot\vec{n_2} = |\vec{n_1}||\vec{n_2}|\cos\theta$$

$$16+4-4 = \sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$\therefore \cos\theta = \frac{16}{6\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

7(c)iii Evaluate $\iint_S (\nabla \times F)\cdot \hat{n}\, ds$, where $F = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ and S is the surface of the paraboloid $z = 4-(x^2+y^2)$ above the xy-plane.

Sol^n:- The surface $z = 4-(x^2+y^2)$ meets the plane $z=0$ in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane S_1, we have $n = -\hat{k}$.

Note that $\hat{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl}\, F)\cdot n\, ds = \iiint_V \text{div}(\text{curl}\, F)\, dv$$

$$= 0 \quad (\because \text{div curl} F = 0)$$

$$\therefore \iint_S \text{curl} F\cdot n\, ds + \iint_{S_1} \text{curl} F\cdot \hat{n}\, ds = 0$$

$(\because S'$ consists of S and $S_1)$

$\Rightarrow \iint_S (\text{curl} F)\cdot \hat{n}\, ds - \iint_{S_1} (\text{curl} F)\cdot \hat{k}\, ds = 0$

($\because$ on S_1, $\vec{n} = -\vec{k}$)

$\Rightarrow \iint_S (\text{curl} F)\cdot n\, ds = \iint_{S_1} (\text{curl} F)\cdot k\, ds$

Now $\text{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$= 0\hat{i} - 2z\hat{j} + (3y-1)k$

$\therefore (\text{curl} F)\cdot k = \left[-2z\hat{j} + (3y-1)\hat{k}\right]\cdot \hat{k}$

$= 3y-1$ over the surface S_1 bounded by the circle $x^2+y^2=4,\ z=0$

Hence $\iint_S (\text{curl} F)\cdot \hat{n}\, ds = \iint_{S_1} (3y-1)\, ds$

$= \int_{-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3y-1)\, dx\, dy$

$= 2\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} (-1)\, dx\, dy$ ($\because 3y$ is an odd function of y)

$= -2\int_{-2}^{2} [y]_0^{\sqrt{4-x^2}}$

$= -2\int_{-2}^{2} \sqrt{4-x^2}\, dx = -4\int_0^2 \sqrt{4-x^2}\, dx$

$= -4\left[\frac{x}{2}\left(\sqrt{4-x^2}\right) + 2\sin^{-1}\frac{x}{2}\right]$

$= -4\left[2\cdot\frac{\pi}{2}\right] = -4\pi$ Ans.

8(a)ii) Find the orthogonal trajectories of $r = a(1+\cos n\theta)$.

Sol^n: Given family is $r = a(1+\cos n\theta)$, where a is parameter. —— (1)

Taking logarithm of bothsides,

$$\log r = \log a + \log(1+\cos n\theta) \quad \text{—— (2)}$$

Differentiating (2) w.r.t θ

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = -(n\sin n\theta)/(1+\cos n\theta) \quad \text{—— (3)}$$

which is differential equation of the family of curves (1). Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3) the differential equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta}$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}\,d\theta$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{2\cos^2\left(\frac{n\theta}{2}\right)d\theta}{2\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)}$$

$$\Rightarrow n\frac{dr}{r} = \cot\left(\frac{n\theta}{2}\right)d\theta.$$

Integrating, $n\log r = \frac{2}{n}\times\log\sin\left(\frac{n\theta}{2}\right) + \frac{1}{n}\log c$, c being arbitrary constant.

$$n^2\log r = \log\sin^2\left(\frac{n\theta}{2}\right) + \log c$$

$$\Rightarrow r^{n^2} = c\sin^2\left(\frac{n\theta}{2}\right)$$

$$\Rightarrow r^{n^2} = \frac{c}{2}(1-\cos n\theta)$$

$$\Rightarrow r^{n^2} = b(1-\cos n\theta), \quad \text{taking } b = c/2$$

which is the equation of required orthogonal trajectories with b as parameter.

8(a)(ii) Use the method of variation of parameters to find the general solution of $x^2y''-4xy'+6y=-x^4\sin x$.

Sol'n: Given that $x^2y''-4xy'+6y=-x^4\sin x$

The given equation in standard form $y_2+Py_1+Qy=R$

is $y''-\frac{4y'}{x}+\frac{6}{x^2}y=-x^2\sin x$

Consider $y''-\frac{4}{x}y'+\frac{6}{x^2}y=0$

$\Rightarrow x^2y''-4xy'+6y=0$

$\Rightarrow (x^2D^2-4xD+6)y=0$, $D\equiv\frac{d}{dx}$ ——— (2)

which is a homogeneous equation.

putting $x=e^z$ and $D_1=\frac{d}{dz}$

$\Rightarrow \log x=z$

Then from (2), we have

$[D_1(D_1-1)-4D_1+6]y=0$

$(D_1^2-D_1-4D_1+6)y=0$

$(D_1^2-5D_1+6)y=0$ ——— (3)

Auxiliary equation of (3) is $D_1^2-5D_1+6=0$

$\Rightarrow (D_1-2)(D_1-3)=0$

$\Rightarrow D_1=2,3$

$\therefore$ The complementary function of (3)

is $y_c=c_1e^{2z}+c_2e^{3z}$

$\Rightarrow y_c=c_1x^2+c_2x^3$ $(\because x=e^z)$

Now let $u=x^2$, $v=x^3$ & $-x^2\sin x$

let $\boxed{y_p=Au+Bv}$ be a particular integral of (1) where A and B are functions of x.

and $u = x^2$, $v = x^3$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - u'v$

$= x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4 \neq 0.$

Now $A = \int \frac{-VR}{uv' - u'v} dx = \int \frac{-x^3(-x^2 \sin x)}{x^4} dx$

$= \int x \sin x \, dx$

$= \left[x(-\cos x) - \int 1(-\cos x) dx\right]$

$= -x\cos x + \sin x$

and $B = \int \frac{uR}{uv' - u'v} dx$

$= \int \frac{x^2(-x^2 \sin x)}{x^4} dx$

$= -\int \sin x \, dx = \cos x$

$\therefore y_p = x^2[-x\cos x + \sin x] + x^3 \cos x = x^2 \sin x$

$\therefore$ The general solution of (1) is

$$\boxed{y = c_1 x^2 + c_2 x^3 + x^2 \sin x}$$

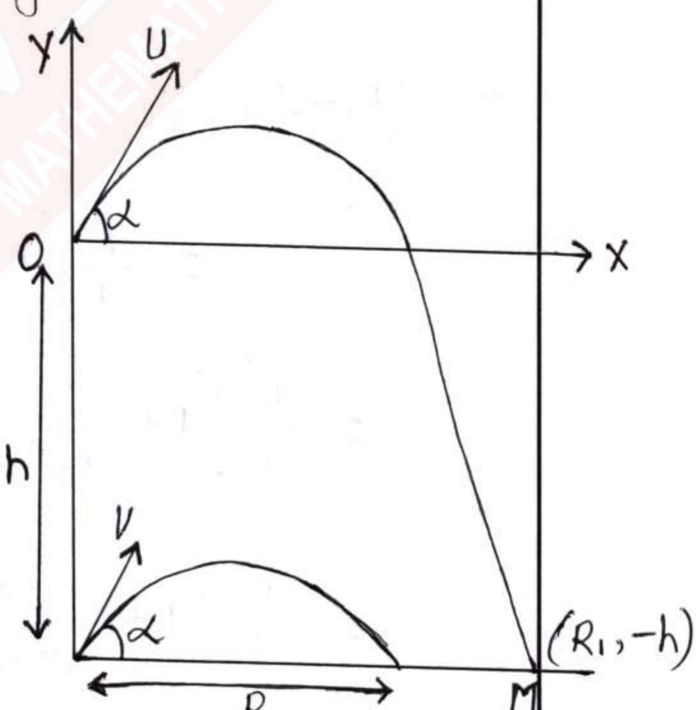
8(b) A gun is firing from the sea level out to sea. It is then mounted in a battery h feet higher up and fired at the same elevation α. Show that the range is increased by

$$\frac{1}{2}\left\{\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2}-1\right\}$$

of itself, v being the velocity of projection.

Sol: Let R be the original range. Then

$$R=\frac{2v^2\sin\alpha\cos\alpha}{g}. \quad \text{——} (1)$$



Let O be a point at a height h above the water level. Let R_1 be the range on the sea when the shot is fired from O.

Referred to the horizontal and vertical lines OX and OY in the plane of projection as the coordinate axes, the coordinates of the point M where the

Shot strikes the water are $(R_1, -h)$.

The point $(R_1, -h)$ lies on the curve

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v^2\cos^2\alpha}.$$

$$\therefore -h = R_1\tan\alpha - \frac{1}{2}\frac{gR_1^2}{v^2\cos^2\alpha}$$

or $$R_1^2 - \frac{2}{g}v^2\sin\alpha\cos\alpha R_1 - \frac{2}{g}v^2h\cos^2\alpha = 0$$

or $$R_1^2 - RR_1 - \frac{2}{g}v^2h\cos^2\alpha = 0$$

or $$R_1^2 - RR_1 = \frac{2}{g}v^2h\cos^2\alpha$$

or $$\left(R_1 - \frac{1}{2}R\right)^2 = \frac{1}{4}R^2 + \frac{2}{g}v^2h\cos^2\alpha$$

$$= \frac{R^2}{4}\left[1 + \frac{1}{R^2}\cdot\frac{8}{g}v^2h\cos^2\alpha\right]$$

$$= \frac{R^2}{4}\left[1 + \frac{g^2}{4v^4\sin^2\alpha\cos^2\alpha}\cdot\frac{8}{g}v^2h\cos^2\alpha\right].$$

[by ①]

$$= \frac{R^2}{4}\left[1 + \frac{2gh}{v^2\sin^2\alpha}\right]$$

$$\therefore R_1 - \frac{1}{2}R = \frac{1}{2}R\left(1 + \frac{2gh}{v^2\sin^2\alpha}\right)^{1/2}$$

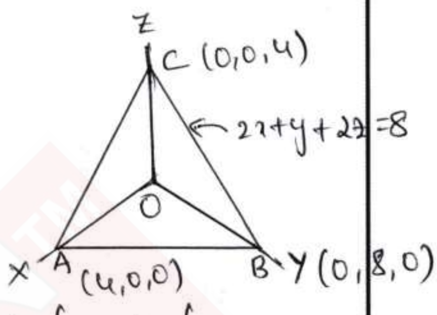
So that

$$R_1 - R = \frac{1}{2}R\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - \frac{1}{2}R$$

$$= \frac{1}{2}\left\{\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - 1\right\}R.$$

Hence the range is increased by $\frac{1}{2}\left\{\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - 1\right\}$ of its former value.

8(c) Verify Stokes theorem for $F = xz\,i - y\,j + x^2 y\,k$, where S is the surface of the region bounded by $x=0, y=0$, $z=0$, $2x+y+2z=8$ which is not included in the xz-plane.



Sol'n: By Stoke's theorem, we have

$$\oint_C F\cdot dr = \int_S (\nabla\times F)\cdot \hat{n}\, ds$$

$$\int_C F\cdot dr = \int_C xz\,dx - y\,dy + x^2 y\,dz = \int_{OA} + \int_{OC} + \int_{AC}$$

xz-plane
$[\because y=0 \Rightarrow dy=0]$

Along OA $\to$ $z=4 \Rightarrow dz=0$, $y=0, z=0$

Along OC $\to$ $z=4 \Rightarrow dz=0$, $x=0, y=0$

$2x+2z=8$
$z=4-x$

$$= \int_{AC} xz\,dx$$

$$= \int_0^4 x(4-x)\,dx$$

$$= \int_0^4 (4x-x^2)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^4$$

$$= 32 - \frac{64}{3} = \frac{32}{3}$$

Now
$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -y & x^2 y \end{vmatrix}$$

$$= i(x^2) + j\,(x-2xy) + k(0) = x^2\,i + (x-2xy)\,j$$

Now surface S consists of three surfaces.
(i) S_1 : plane AOB (xy-plane) (ii) S_2 = plane OBC (yz-plane)
(iii) S_3 : plane ABC

On surface S_1: $\hat{n} = -\hat{k}$

$$\therefore (\nabla\times F)\cdot\hat{n} = 0$$

$$\therefore \iint_S (\nabla\times F)\cdot\hat{n}\,ds = 0$$

On surface S_2: $\hat{n} = -\hat{i}$

$$(\nabla\times F)\cdot\hat{n} = -x^2$$

$$\iint_{S_2} (\nabla\times F)\cdot\hat{n}\,ds = \int -x^2\,ds = 0 \quad (\because x=0)$$

On surface S_3: $\hat{n} = \dfrac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

$$\therefore \iint_{S_3} (\nabla\times F)\cdot\hat{n}\,ds = \iint_{S_3} \frac{2x^2+x-2xy}{3}\,\frac{dx\,dy}{|n\cdot\hat{k}|}$$

$$= \iint_{S_3} \frac{(2x^2+x-2xy)}{2}\,dx\,dy$$

$$= \int_{x=0}^{4}\int_{y=0}^{8-2x} \left(\frac{2x^2+x-2xy}{2}\right)dx\,dy$$

$$= \frac{1}{2}\int_0^4 \left(2x^2y+xy-xy^2\right)_0^{8-2x} dx$$

$$= \frac{1}{2}\int_0^4 2x^2(8-2x)+x(8-2x)-x(8-2x)^2\,dx$$

$$= \frac{1}{2}\int_0^4 (46x^2-8x^3-56x)\,dx$$

$$= \frac{1}{2}\left[46\left(\frac{x^3}{3}\right)-2x^4-28x^2\right]_0^4$$

$$= \frac{1}{2}\left[\frac{46}{3}(64)-2(256)-28(16)\right]$$

$$= \frac{1}{2}\left[\frac{2944}{3}-960\right] = \frac{1}{2}\left(\frac{64}{3}\right) = \frac{32}{3}$$

$$\therefore \oint_C F\cdot dr = \iint_S (\nabla\times F)\cdot\hat{n}\,ds.$$