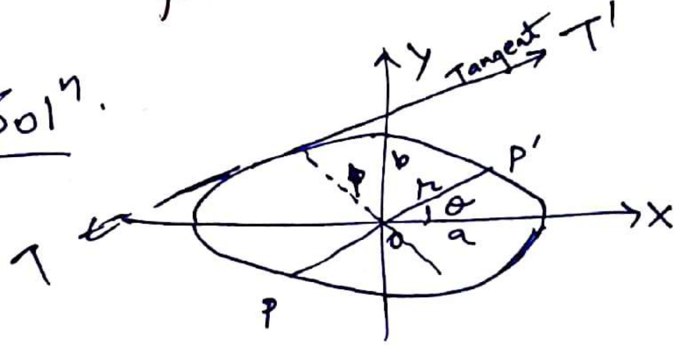
CSE 2017.

5 (e). Show that moment of inertia of an elliptical area of mass M and semi-axis a and b about a diameter of length r is $\frac{1}{4} M \frac{a^2 b^2}{r^2}$. Further moment of inertia about a tangent is $\frac{5M}{4} p^2$ where p is perpendicular distance ~~of~~ from centre of ellipse to tangent.

Sol^n.



$M_{pp'} = M_{ox} \cos^2\theta + M_{oy} \sin^2\theta$

M_{ox} = moment of inertia about ox

M_{oy} = MoI about O-Y

$M_{ox} = \frac{Mb^2}{4}$, $M_{oy} = \frac{Ma^2}{4}$

We know that

$$M_{pp'} = \frac{Mb^2}{4}\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta = \frac{M}{4}\left(b^2\cos^2\theta + a^2\sin^2\theta\right) \quad \text{--- (1)}$$

Let $p' = (r\cos\theta, r\sin\theta)$

$$\therefore \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \qquad \left[\because p' \text{ satisfies } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$$

Multiply by $a^2 b^2$

$$r^2\left[b^2\cos^2\theta + a^2\sin^2\theta\right] = a^2 b^2$$

$$b^2\cos^2\theta + a^2\sin^2\theta = a^2 b^2 / r^2 \quad \text{--- (2)}$$

Use (2) in (1),

$$M_{pp'} = \frac{M}{4}\frac{a^2 b^2}{r^2}$$

Proved

Part 2.

Let equation of tangent be

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$m = \tan\theta$ so we get [$TT' \parallel PP'$]

$$x\tan\theta - y + \sqrt{a^2\tan^2\theta + b^2} = 0 \quad \text{--- (3)}$$

Distance on (3) from $(0,0)$ is given by :-

$$p = \frac{\sqrt{a^2\tan^2\theta + b^2}}{\sqrt{1+\tan^2\theta}} = \cos\theta\sqrt{a^2\tan^2\theta + b^2}$$

$$p^2 = a^2\sin^2\theta + b^2\cos^2\theta \quad \text{--- (4)}$$

Use (4) in (1) we get

$$M_{PP'} = \frac{M}{4}(b^2\cos^2\theta + a^2\sin^2\theta) = \frac{Mp^2}{4}$$

By parallel axis theorem

$$M_{TT'} = M_{PP'} + M\,[\text{dist b/w } TT' \text{ and } PP']^2$$

$$= \frac{Mp^2}{4} + Mp^2 = \frac{5Mp^2}{4}$$

$$\therefore M_{TT'} = \frac{5Mp^2}{4}$$ Hence proved.

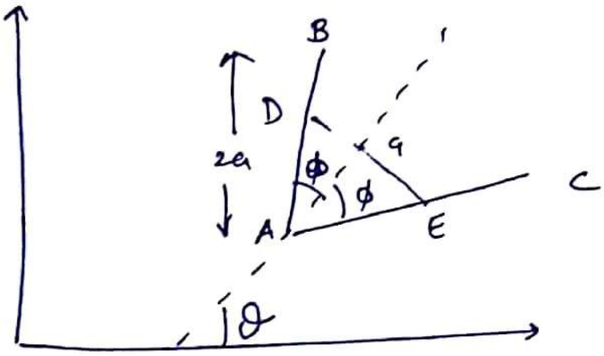
CSE 2017.

6(c). Two uniform rods AB and AC of mass m and length 2a, are smoothly hinged together at A and move on horizontal plane. At time t, mass centre of rod is at (ξ, η) refered to axes OX, OY and rods make angle $\theta \pm \phi$ with OX. Prove that kinetic energy of system is

$$m\left[\dot{\xi}^2+\dot{\eta}^2+\left(\frac{1}{3}+\sin^2\phi\right)a^2\dot{\theta}^2+\left(\frac{1}{3}+\cos^2\phi\right)a^2\dot{\phi}^2\right]$$

Also derive Lagrange's equations of motion for system if an external force with components [X, Y] along axes at A.

Sol^n.



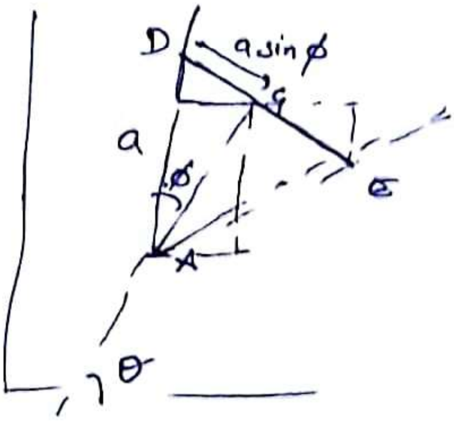
For the 2 rods,

$$T=\frac{1}{2}m\left\{\dot{x}_D^2+\dot{y}_D^2\right\}+\frac{1}{2}\left(\frac{m4a^2}{12}\right)\left(\dot{\theta}+\dot{\phi}\right)^2+$$

$$\frac{1}{2}m\left\{\dot{x}_E^2+\dot{y}_E^2\right\}+\frac{1}{2}\left(\frac{m4a^2}{12}\right)\left(\dot{\theta}-\dot{\phi}\right)^2 \quad \text{—(1)}$$

$G = \{\xi, \eta\}$ = Centre of mass

From the diagram



$\therefore\ x_D = \xi - a\sin\phi\sin\theta$

$x_E = \xi + a\sin\phi\sin\theta$

$y_D = \cancel{\xi}\ \eta + a\sin\phi\cos\theta$

$y_E = \eta\ \cancel{+} - a\sin\phi\cos\theta$

$$\therefore\ \dot{x}_D = \dot{\xi} - a\cos\phi\sin\theta\,\dot{\phi} - a\sin\phi\cos\theta\,\dot{\theta}$$

$$\dot{x}_E = \dot{\xi} + \cancel{a\sin\phi\, c}\ a\cos\phi\sin\theta\,\dot{\phi} + a\sin\phi\cos\theta\,\dot{\theta}$$

$$\dot{y}_D = \dot{\eta} + a\cos\phi\cos\theta\,\dot{\phi} - a\sin\phi\sin\theta\,\dot{\theta}$$

$$\dot{y}_E = \dot{\eta}\ \cancel{+} - a\cos\phi\cos\theta\,\dot{\phi} + a\sin\phi\sin\theta\,\dot{\theta}$$

So
$$\dot{x}_D^2 + \dot{x}_E^2 = 2\dot{\xi}^2 + 2(a\cos\phi\sin\theta\,\dot{\phi} + a\sin\phi\cos\theta\,\dot{\theta})^2$$
$$= 2\dot{\xi}^2 + 2a^2\cos^2\phi\sin^2\theta\,\dot{\phi}^2 + 2a^2\sin^2\phi\cos^2\theta\,\dot{\theta}^2 + 4a^2\sin\phi\cos\phi\sin\theta\cos\theta\,\dot{\phi}\dot{\theta}$$

Similarly
$$\dot{y}_D^2 + \dot{y}_E^2 = 2\dot{\eta}^2 + 2a^2\cos^2\phi\cos^2\theta\,\dot{\phi}^2 + 2a^2\sin^2\phi\sin^2\theta\,\dot{\theta}^2 - 4a^2\sin\phi\cos\phi\sin\theta\cos\theta\,\dot{\phi}\dot{\theta}$$

$$\therefore\ \dot{x}_D^2 + \dot{x}_E^2 + \dot{y}_D^2 + \dot{y}_E^2 = 2\left[\dot{\xi}^2 + \dot{\eta}^2 + a^2\cos^2\phi\,\dot{\phi}^2 + a^2\sin^2\phi\,\dot{\theta}^2\right] \quad (2)$$

Rearranging (1),
$$T = \frac{1}{2}m\{\dot{x}_D^2 + \dot{y}_D^2 + \dot{x}_E^2 + \dot{y}_E^2\} + \frac{1}{3}\frac{ma^2}{2}\left[2\dot{\phi}^2 + 2\dot{\theta}^2\right]$$

Using (2) in above equation.

$$T = m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right]$$

— Proved

Now constant force $[X, Y]$ acts at A.

$$\therefore V = -\int_A F \cdot dr$$

Coordinates of A = $[\xi - a\cos\phi\cos\theta,\ \eta - a\cos\phi\sin\theta]$

$$\therefore V = X(a\cos\phi\cos\theta - \xi) + Y(a\cos\phi\sin\theta - \eta) + \text{const}$$

$$L = T - V.$$

$$L = m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right] + Y[\eta - a\cos\phi\sin\theta] + \text{const.}$$
$$+ X[\xi - a\cos\phi\cos\theta]$$

Eq. of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

For ξ.

$$\frac{d}{dt}\left[2m\dot{\xi}\right] - X = 0$$

$$\Rightarrow \ddot{\xi} - \frac{X}{2m} = 0 \quad \text{— (I)}$$

For η.

$$\ddot{\eta} - \frac{Y}{2m} = 0 \quad \text{— (II)}$$

For ϕ.

$$\frac{d}{dt}\left[\left(\frac{1}{3}+\cos^2\phi\right)2a^2\dot{\phi}\right]-\left(2\sin\phi\cos\phi\, a^2\dot{\theta}^2 - 2\sin\phi\cos\phi\, a^2\dot{\phi}^2 + Xa\sin\phi\cos\theta + Ya\sin\phi\sin\theta\right) = 0$$

$$2a^2\left(\frac{1}{3}+\cos^2\phi\right)\ddot{\phi} - \cancel{2a^2} - a^2\left(\sin 2\phi\,\dot{\theta}^2 \mp \sin 2\phi\,\dot{\phi}^2\right) - a\sin\phi\left(X\cos\theta + Y\sin\theta\right) = 0$$

$$2a^2\left(\frac{1}{3}+\cos^2\phi\right)\ddot{\phi} - a^2\sin 2\phi\left(\dot{\theta}^2-\dot{\phi}^2\right) - a\sin\phi\left(X\cos\theta + Y\sin\theta\right) = 0$$

—— III

For θ.

$$\frac{d}{dt}\left[\left(\frac{1}{3}+a\sin^2\phi\right)2a^2\dot{\theta}\right]-\left(\cancel{2\sin}\; Xa\cos\phi\sin\theta - Ya\cos\phi\cos\theta\right) = 0$$

$$2a^2\left(\frac{1}{3}+\sin^2\phi\right)\ddot{\theta} - a\cos\theta\left(X\sin\theta \mp Y\cos\theta\right) = 0$$

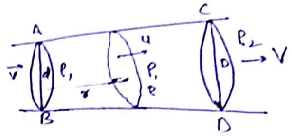
—— IV

7(c) A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. If V and v be the corresponding velocities of the streams and if the motion is assumed to be steady and diverging from the vertex of the cone, then prove that

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2-V^2)/2K}.$$

where K is the pressure divided by the density and is constant (15)

Sol: Let p be the pressure, ρ the density and u the velocity at distance r from AB.



Then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$ (since the motion is steady)

or, $$u\frac{\partial u}{\partial r} = -\frac{K}{\rho}\frac{\partial \rho}{\partial r}$$ [since as given, $K = \frac{p}{\rho} \Rightarrow p = K\rho$]

By integrating w.r.t 'r', we get

$$\frac{u^2}{2} = -K\log\rho + c \quad \text{—(1)}$$

Boundary conditions are

(i) $\rho = \rho_1$ when $u = v$

(ii) $\rho = \rho_2$ when $u = V$

subjecting (1) to (i) and (ii) we get

$$\frac{v^2}{2} = -K\log\rho_1 + c \quad \text{— (2)}$$

and $$\frac{V^2}{2} = -K\log\rho_2 + c \quad \text{—(3)}$$

subtracting (2) from (3), we get

$$\frac{V^2-v^2}{2} = K\log\frac{\rho_1}{\rho_2} \quad \text{—— (4)}$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = e^{\frac{(V^2-v^2)}{2K}}$$

By the eqn of continuity

flux at A = flux at B

$$\pi\left(\frac{d}{2}\right)^2 v \cdot \rho_1 = \pi\left(\frac{D}{2}\right)^2 \cdot V \cdot \rho_2$$

or, $$\frac{\rho_1}{\rho_2} = \frac{V}{v}\,\frac{D^2}{d^2} \quad \cdots (5)$$

Now, from (4) and (5), we have

$$\frac{V}{v}\,\frac{D^2}{d^2} = e^{\left(\frac{V^2-v^2}{2K}\right)}$$

or, $$\frac{V}{v} = \frac{d^2}{D^2}\, e^{\left(\frac{v^2-V^2}{2K}\right)}$$

or $$\boxed{\frac{v}{V} = \frac{D^2}{d^2}\, e^{\frac{(v^2-V^2)}{2K}}}$$

8(c) If the velocity of an incompressible fluid at the point (x,y,z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$, $r^2 = x^2+y^2+z^2$, then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. further, determine the streamlines. (15)

Soln.

Given

$$u = \frac{3xz}{r^5}$$

$$v = \frac{3yz}{r^5}, \quad — (1)$$

$$w = \frac{3z^2-r^2}{r^5} = \frac{3z^2}{r^5} - \frac{1}{r^3} \quad \cdots (2)$$

since, $r^2 = x^2+y^2+z^2$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad (3)$$

and $$\frac{\partial r}{\partial z} = \frac{z}{r}$$

I. To prove that liquid motion is possible

from (1), (2) and (3), we have

$$\frac{\partial u}{\partial x} = 3z\left[\frac{1}{r^5} + \frac{(-5x)}{r^6}\frac{\partial r}{\partial x}\right] = \frac{3z}{r^5} - \frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y} = 3z\left[\frac{1}{r^5} + \frac{(-5y)}{r^6}\frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\frac{z}{r}$$

$$= \frac{9z}{r^5} - \frac{15z^3}{r^7}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = \frac{15z}{r^5} - \frac{15z}{r^7}\times r^2 = 0$$

(using (3))

since the eqn of continuity is satisfied by the given values of u, v and w, so the motion is possible.

II. To show that $\phi = \frac{\cos\theta}{r^2} = \frac{z}{r^3}$

Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\text{or, } d\phi = -(u\,dx + v\,dy + w\,dz)$$

$$\text{or, } d\phi = -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right] = \frac{r^2dz - 3z(x\,dx + y\,dy + z\,dz)}{r^5}$$

$$d\phi = \frac{r^2dz - \frac{3z}{2}d(x^2+y^2+z^2)}{r^5} = \frac{r^2dz - \frac{3z}{2}d(r^2)}{r^5}$$

$$= \frac{r^2dz - 3r^{\cancel{2}}z\,dr}{r^5} = \boxed{\frac{dz}{r^3} - \frac{3z}{r^4}dr = d\left(\frac{z}{r^3}\right)}$$

$$= \frac{r^3dz - 3r^2z\,dr}{(r^3)^2}$$

$$\Rightarrow d\phi = d\left(\frac{z}{r^3}\right)$$

Integrating $\boxed{\phi = \frac{z}{r^3}}$ (neglecting constant of integration, as it has no significance in ϕ)

[as $z = r\cos\theta$ in spherical coordinates (r,θ,ϕ)]

$$\text{So, } \phi = \frac{r\cos\theta}{r^3}$$

$$\text{i.e. } \boxed{\phi = \frac{\cos\theta}{r^2}}$$

Aliter:

$$\frac{\partial\phi}{\partial x} = -u = -\frac{3xz}{r^5}$$

Integrating, w.r.t. x,

$$\phi = -\frac{3z}{2}\int (2x)(x^2+y^2+z^2)^{-5/2}\,dx = \left(-\frac{3z}{2}\right)\left(\frac{-2}{3}\right)(x^2+y^2+z^2)^{-3/2}$$

$$\phi = \frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}.$$

(on neglecting constant of integration)

Step III: Streamlines

Stream lines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w},$$

putting the values of respective terms, we obtain

$$\underset{(i)}{\frac{dx}{3xz}} = \underset{(ii)}{\frac{dy}{3yz}} = \underset{(iii)}{\frac{dz}{3z^2-r^2}} = \underset{(iv)}{\frac{x\,dx+y\,dy+z\,dz}{3z(x^2+y^2+z^2)-r^2z}} = \underset{(v)}{\frac{x\,dx+y\,dy+z\,dz}{2r^2z}}$$

(as $x^2+y^2+z^2=r^2$)

from (i) and (ii), we have

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, $\log x = \log y + \log a$ or $\boxed{x = ay}$ — (5)

from (i) and (v), we get

$$\frac{dx}{3x} = \frac{x\,dx+y\,dy+z\,dz}{2(x^2+y^2+z^2)}$$

or, $$\frac{4\,dx}{x} = 3\left(\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}\right)$$

Integrating, $4\log x = 3\log(x^2+y^2+z^2) + \log b$

or, $$\boxed{x^4 = b(x^2+y^2+z^2)^3} \quad — (6)$$

The required streamlines are the curves of intersection of (5) and (6)