Mains Test Series - 2019

Test - 3 (Paper - I)

Answer Key

ODE, Statics & Dynamics, Vector Analysis

SECTION - A

Q.1.(a) Solve $(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$.

Solution:

Let $x^2+y^2=v$ and $x+y=u$ ——(1)

Differentiating (1), we have,

$2(x\,dx+y\,dy)=dv$ and $dx+dy=du$

$$\therefore \frac{dv}{du}=\frac{2(x\,dx+y\,dy)}{dx+dy}$$

$$=\frac{2\left\{x+y\left(\frac{dy}{dx}\right)\right\}}{1+\left(\frac{dy}{dx}\right)}$$

$$P=\frac{2(x+yp)}{1+p} \quad \text{——(2)}$$

where $P=dv/du$ and $p=dy/dx$

Re-writing the given equation, we get,

$$(x^2+y^2) - 2(x+y)\left(\frac{x+yp}{1+p}\right) + \left(\frac{x+yp}{1+p}\right)^2 = 0$$

$$\Rightarrow \quad v - 2u \times \frac{P}{2} + \left(\frac{P}{2}\right)^2 = 0 \quad \text{— using (1) \& (2)}$$

$$\Rightarrow \quad v = uP - \frac{P^2}{4}$$ which is in Clairaut's form.

Hence, it's general solution is given as

$$v = uC - \frac{C^2}{4}$$

$$\Rightarrow \boxed{(x^2+y^2) = C(x+y) - \frac{C^2}{4}} \quad \text{— by (1)}$$

is the required solution.

Hence, the result.

Q.1.(b)

(i) If $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, find $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$

where $a > 0$.

(ii) Find $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$.

Solution:

(i) Since $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$

$$\therefore L^{-1}\left\{\frac{e^{-1/pk}}{(pk)^{1/2}}\right\} = \frac{1}{k}\cdot\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)/k}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/pk}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)}}$$

Taking $k = 1/a$, we have,

$$\boxed{L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{(\pi t)}}} \quad \text{---} (1)$$

(ii) Let $f(p) = \log\left(1+\frac{1}{p^2}\right) = -\log\frac{p^2}{p^2+1}$

i.e. $f(p) = -2\log p + \log(p^2+1)$

$$\therefore\ f'(p) = -2\left(\frac{1}{p} - \frac{p}{p^2+1}\right)$$

$$\therefore\ L^{-1}\{f'(p)\} = -2(1-\cos t)$$

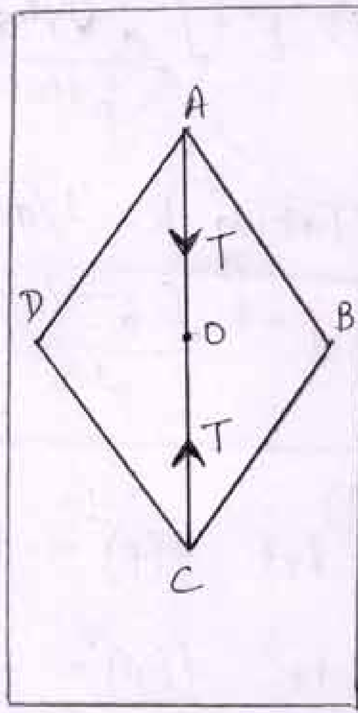
$$\Rightarrow -t\,L^{-1}\{f(p)\} = -2(1-\cos t)$$

$$\Rightarrow \boxed{L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\} = \frac{2}{t}(1-\cos t)} \quad \text{(ii)}$$

Hence, the result.

Q. 1. (c) Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.

Solution:



ABCD is a framework in the shape of parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC.

Let T be the tension in the string AC. The total weight W of all the four rods AB, BC, CD and DA can be taken as acting at O, the middle point of AC.

Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical.

Let $AC = 2x$

Give the system a small displacement in which AC remains vertical. x changes to $x + \delta x$ and The point A remains fixed, the point O changes and the length AC changes.

We have, $AO = x$.

By the principle of virtual work, we have,

$$-T\delta(AC) + W\delta(AO) = 0$$

$$\Rightarrow \quad -T\delta(2x) + W\delta(x) = 0$$

$$\Rightarrow \quad -2T\delta x + W\delta x = 0$$

$$\Rightarrow \quad [-2T + W]\,\delta x = 0$$

$$\Rightarrow \quad -2T + W = 0 \qquad (\because \delta x \neq 0)$$

$$\Rightarrow \quad T = +\frac{1}{2}W$$

$$\Rightarrow \quad T = \frac{1}{2}\left[\text{total weight of all the four rods}\right].$$

Hence, proved.

Q. 1. (d) A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are 'a' and 'b' respectively, and has velocity 'v' when half way between them. Show that the complete period is $\pi(b-a)/v$.

Solution:

In the figure alongside, A and B are the positions of instantaneous rest in S.H.M.

Let C be the middle point of AB. Then C is the centre of motion. Also, it is given that $OA = a$, $OB = b$.

The amplitude of the motion $= \frac{1}{2} AB = \frac{1}{2}(OB - OA)$

$$= \frac{1}{2}(b-a)$$

Now, in a S.H.M., the velocity at the centre = $(\sqrt{\mu}) \times$ amplitude

Since, in this case, the velocity at the centre is given to be v,

therefore,

$$v = \frac{1}{2}(b-a)\cdot\sqrt{\mu}$$

$$\Rightarrow \sqrt{\mu} = \frac{2v}{(b-a)}$$

Hence, the time period $T = \dfrac{2\pi}{\sqrt{\mu}}$

$$= 2\pi\left[\frac{b-a}{2v}\right]$$

$$\therefore \boxed{T = \frac{\pi(b-a)}{v}}$$

Hence, proved.

Q.1.(e) If $\nabla^2 f(\vec{r}) = 0$, Show that $f(\vec{r}) = c_1 \log \vec{r} + c_2$ where $r^2 = x^2+y^2$ and c_1, c_2 are arbitrary constants.

Solution:

Consider $\nabla^2 f(\vec{r}) = \nabla\cdot\left[\nabla(f(\vec{r}))\right]$

$$= \nabla\cdot\left[f'(\vec{r})\cdot\nabla\vec{r}\right]$$

$$= \nabla\cdot\left[f'(\vec{r})\cdot\frac{\vec{r}}{r}\right]$$

$$\left(\because \nabla\vec{r} = \frac{\vec{r}}{r}\right)$$

$$= \nabla\cdot\left[\frac{f'(\vec{r})}{r}\cdot\vec{r}\right]$$

$$= \left(\nabla\frac{f'(\vec{r})}{r}\right)\cdot\vec{r} + \frac{f'(\vec{r})}{r}\left(\nabla\cdot\vec{r}\right)$$

$$\left(\because \nabla\cdot(\phi\vec{r}) = \nabla\phi\cdot\vec{r} + \phi\,\nabla\cdot\vec{r}\right)$$

$$= \left[\frac{1}{r}f''(\vec{r})\cdot\nabla\vec{r} - \frac{f'(\vec{r})}{r^2}\cdot\nabla\vec{r}\right]\cdot\vec{r} +$$

$$\frac{f'(\vec{r})}{r}\cdot(2) \qquad \left(\because \nabla\cdot\vec{r} = 2 \text{ as } r^2 = x^2+y^2\right)$$

$$= f''(\vec{r}) - \frac{f'(\vec{r})}{r} + \frac{2f'(\vec{r})}{r}$$

$$\therefore \nabla^2 f(\vec{r}) = f''(\vec{r}) + \frac{f'(\vec{r})}{r}$$

Given $\nabla^2 f(\vec{r}) = 0 = f''(\vec{r}) + \dfrac{f'(\vec{r})}{r}$

$$\Rightarrow \frac{f''(\vec{r})}{f'(\vec{r})} = -\frac{1}{r}$$

On Integrating both the sides w.r.t. 'r', we get

$$\log f'(\vec{r}) = -\log r + \log c_1 \quad ; c_1 \text{ is constant} \qquad \text{———(1)}$$

$$\Rightarrow f'(\vec{r}) = c_1 \cdot r^{-1}$$

Once again integrating (1), we get,

$$\boxed{f(\vec{r}) = c_1 \log r + c_2} \quad ; c_2 \text{ is constant}$$

Hence, proved.

Q.2. (a). Use the method of variation of parameters to find the general solution of
$x^2y'' - 4xy' + 6y = -x^4\sin x$

Solution:

The given equation is $x^2y'' - 4xy' + 6y = -x^4\sin x$ —— (1)

Re-writing the given equation, we have

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = -x^2\sin x \quad \text{—— (2)}$$

Comparing (2) with $y'' + Py' + Qy = R$, we have

$P = -4/x$, $Q = 6/x^2$, $R = -x^2\sin x$

Consider, $y'' - (4/x)y' + (6/x^2)y = 0$

$\Rightarrow (x^2D^2 - 4xD + 6)y = 0$ where $D \equiv d/dx$ —— (3)

In order to apply the method of variation of parameters, we shall reduce (3) into linear differential equation with constant co-efficients.

Let $x = e^z$ i.e. $\log x = z$ and let $D_1 \equiv d/dz$ —— (4)

Then, $xD = D_1$, $x^2D^2 = D_1(D_1-1)$ and so on,

Eq.ⁿ (3) reduces to

$\{D_1(D_1-1) - 4D_1 + 6\}y = 0$

$\Rightarrow (D_1^2 - 5D_1 + 6)y = 0$, whose auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$ giving $D_1 = 2, 3$.

$$\therefore \text{ C.F. of (1)} = c_1 e^{2z} + c_2 e^{3z}$$
$$= c_1 (e^z)^2 + c_2 (e^z)^3$$
$$= c_1 x^2 + c_2 x^3 \quad \text{—— (5)}$$

Let $u = x^2$ and $v = x^3$. Also, here $R = -x^2 \sin x$. —— (6)

Here, $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4$
$= x^4 \neq 0$

Hence, P.I. of (1) $= u f(x) + v g(x)$, —— (7)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^3 \times (-x^2 \sin x)}{x^4} dx$
$$= \int x \sin x \, dx$$
$$= x(-\cos x) - \int \{(-\cos x) \times 1\} dx$$
$$= -x \cos x + \sin x$$
—— (8)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{x^2 \times (-x^2 \sin x)}{x^4} dx$
$$= -\int \sin x \, dx = \cos x$$
—— (9)

Using (6), (8) and (9), (7) reduces to

P.I. of (1) $= x^2(-x\cos x + \sin x) + x^3 \cos x$
$= x^2 \sin x$ —— (10)

Hence, the required general solution is

$$y = C.F + P.I.$$

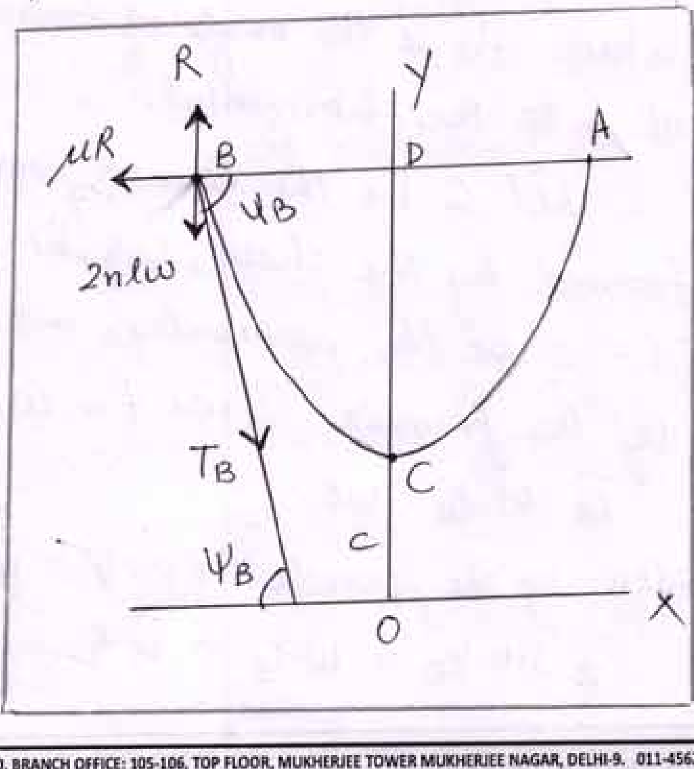
$$\therefore \boxed{y = c_1 x^2 + c_2 x^3 + x^2 \sin x}$$ where c_1, c_2 are arbitrary constants.

Hence, the result.

Q.2.(b). A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be 'n' times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\{\lambda + \sqrt{1+\lambda^2}\}$, $1/\lambda = \mu(2n+1)$ and μ is the coefficient of friction.

Solution :

Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A.



Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A. In this position of limiting equilibrium, the forces acting on the ring are:

(i) the weight $2nlw$ of the ring acting vertically downward.

(ii) the normal reaction R of the rod,

(iii) the force of limiting friction μR of the rod acting in the direction AB, and

(iv) the tension T_B in the string at B acting along the tangent to the string at B.

For the equilibrium of the ring at B, resolving the forces acting on it horizontally and vertically, we have,

$$\mu R = T_B \cos\psi_B \quad \text{——— (1)}$$

$$\text{and } R = 2nlw + T_B \sin\psi_B \quad \text{——— (2)}$$

where ψ_B is the angle of inclination of the tangent at B to the horizontal.

Let C be the lowest point of the catenary formed by the chain, OX be the directrix and $OC = c$ be the parameter. We have arc $CB = s_B = l$.

By the formula $T\cos\psi = wc$, we have

$$T_B \cos\psi_B = wc.$$

Also, by the formula $T\sin\psi = wx$, we have

$$T_B \sin\psi_B = w s_B = wl.$$

Putting these values in (1) and (2), we have

$$\mu R = wc \quad \text{and} \quad R = 2nlw + wl = (2n+1)wl$$

$$\therefore \mu(2n+1)wl = wc$$

$$\Rightarrow \mu(2n+1)l = c$$

But it is given that $\mu(2n-1) = 1/\lambda$

$$\Rightarrow 1/\lambda = c \quad \text{——— (3)}$$

Using the formula, $s = c\tan\psi$ for the point B, we have, $l = c\tan\psi_B$;

$$\Rightarrow \tan\psi_B = l/c = \lambda \quad \text{——— (4)}$$

Now, the required greatest possible distance of the ring from A $= AB$

$$= 2DB$$

$$= 2x_B$$

$$= 2c\log\left[\sec\psi_B + \tan\psi_B\right]$$

$$\left[\because x = c\log(\sec\psi + \tan\psi)\right]$$

$$= 2c\log\left[\tan\psi_B + \sqrt{(1+\tan^2\psi_B)}\right]$$

$$= 2\cdot\frac{l}{\lambda}\log\left[\lambda + \sqrt{(1+\lambda^2)}\right]$$

$$\left[\because c = \frac{l}{\lambda} \text{ from (3)}, \; \& \; \tan\psi_B = \lambda \text{ from (4)}\right]$$

i.e. $\boxed{\text{greatest possible distance of the ring from A} = \frac{2l}{\lambda}\log\left[\lambda + \sqrt{(1+\lambda^2)}\right]}$

Hence, proved.

Q.2. (c) Show that for the curve

$x = a(3u - u^3)$, $y = 3au^2$, $z = a(3u + u^3)$,

$$k = \tau = \frac{1}{3a(1+u^2)^2}.$$

Solution:

We know that,

$$\vec{r} = (x, y, z)$$

$$\text{i.e. } \vec{r} = (3au - au^3, 3au^2, 3au + au^3) \quad \text{--- (1)}$$

$$\frac{d\vec{r}}{du} = (3a - 3au^2, 6au, 3a + 3au^2) \quad \text{... on diff(1) w.r.t. } u \quad \text{--- (2)}$$

$$\left|\frac{d\vec{r}}{du}\right| = 3a\sqrt{2}(1+u^2) \quad \text{--- (3)}$$

Also, $$\frac{d^2\vec{r}}{du^2} = (-6au, 6a, 6au) \quad \text{... on diff. (2) w.r.t. } u \quad \text{--- (4)}$$

$$\frac{d^3\vec{r}}{du^3} = (-6a, 0, 6a) \quad \text{.... on diff. (4) w.r.t. } u \quad \text{--- (5)}$$

Now,

$$\frac{d\vec{r}}{du} \times \frac{d^2 r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3a-3au^2 & 6au & 3a+3au^2 \\ -6au & 6a & 6au \end{vmatrix}$$

$$= \hat{i}(36a^2u^2 - 18a^2 - 18a^2u^2) - \hat{j}(18a^2u - 18a^2u^3 + 18a^2u + 18a^2u^3) +$$

$$\hat{k}\,(18a^2 - 18a^2u^2 + 36a^2u^2)$$

$$= \hat{i}\,(18a^2u^2 - 18a^2) - \hat{j}\,(36a^2u) + \hat{k}\,(18a^2u^2 + 18a^2)$$

$$\therefore \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} = \left(18a^2u^2 - 18a^2, -36a^2u, 18a^2 + 18a^2u^2\right) \quad —(6)$$

$$\therefore \left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right| = 18a^2\sqrt{(u^2-1)^2 + (-2u)^2 + (1+u^2)^2}$$

$$= 18a^2 \cdot \sqrt{2}(u^2+1) \quad —(7)$$

Now,

$$\left(\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right) \cdot \frac{d^3\vec{r}}{du^3} = \left(\cancel{-108a^3u^2} + 108a^3 + 0 + 108a^3 + \cancel{108a^3u^2}\right)$$

$$= 216a^3. \quad —(8).$$

By using the formula,

$$\kappa = \frac{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|}{\left|\frac{d\vec{r}}{du}\right|^3}$$

$$= \frac{18a^2\sqrt{2}(u^2+1)}{\left[3a\sqrt{2}(1+u^2)\right]^3} = \frac{\cancel{18}a^{\cancel{2}}\cancel{\sqrt{2}}\cancel{(u^2+1)}}{\cancel{27}_{3}\,a^3\,2\cancel{\sqrt{2}}(u^2+1)^{\cancel{3}\,2}}$$

$\therefore \boxed{k = \dfrac{1}{3a(1+u^2)^2}}$ ——— (A)

Similarly, we obtain

$$\tau = \frac{\left[\frac{d\vec{r}}{du} \quad \frac{d^2\vec{r}}{du^2} \quad \frac{d^3\vec{r}}{du^3}\right]}{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|^2}$$

$$= \frac{(216)a^3}{\left[18a^2\sqrt{2}\,(u^2+1)\right]^2}$$

$$= \frac{\overset{18}{\cancel{216}}\,a^3}{\underset{3}{\cancel{18}} \times 18 \times a^4 \times \cancel{2} \times (1+u^2)^2}$$

$\therefore \boxed{\tau = \dfrac{1}{3a(1+u^2)}}$ ——— (B)

Hence, proved.

Q.2.(d) Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Solution:

Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $f_1 = x^2+y^2+z^2$ and $f_2 = x^2+y^2-z$

Then

$\text{grad } f_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and

$\text{grad } f_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n}_1$ = grad f_1 at the point $(2,-1,2)$ and

$\vec{n}_2$ = grad f_2 at the point $(2,-1,2)$.

Then,

$\vec{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{n}_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

The vectors $\vec{n}_1$ and $\vec{n}_2$ are along normals to the two surfaces at the point $(2,-1,2)$. If θ is the angle between these vectors, then

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| \cdot |\vec{n}_2| \cdot \cos\theta$$

$$\Rightarrow 16+4-4 = \sqrt{16+4+16} \cdot \sqrt{16+4+1} \cdot \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\cancel{16}\,8}{3\cancel{6}\sqrt{21}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

Hence, the result.

Q.3.(a) Solve $x^2\left(\frac{d^3y}{dx^3}\right)+2x\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{y}{x}\right)=10\left(1+\frac{1}{x^2}\right)$

Solution :

Multiplying both sides by x, the given equation becomes,

$$x^3\left(\frac{d^3y}{dx^3}\right)+2x^2\left(\frac{d^2y}{dx^2}\right)+2y=10\left(x+\frac{1}{x}\right)$$

i.e. $$(x^3D^3+2x^2D^2+2)y=10(x+x^{-1}) \quad \text{---(1)}$$

where $D\equiv d/dx$

Let $x=e^z$ so that $x=\log z$ and let $D_1\equiv d/dz$

Then (1) becomes,

$$[D_1(D_1-1)(D_1-2)+2D_1(D_1-1)+2]y=10[e^z+e^{-z}]$$

$$\Rightarrow (D_1^3-D_1^2+2)y=10e^z+10e^{-z} \quad \text{---(2)}$$

A.E. of (2) is $D_1^3-D_1^2+2=0$

$\Rightarrow (D_1+1)(D_1^2-2D_1+2)=0$ giving

$D_1=-1,\ 1\pm i.$

$\therefore$ C.F. $=c_1e^{-z}+e^z(c_2\cos z+c_3\sin z)$

i.e. $\quad\quad =c_1x^{-1}+x(c_2\cos\log x+c_3\sin\log x)$

Now,

P.I. corresponding to $10e^z = 10.\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z$

$$=10.\frac{1}{2(1-2+2)}.e^z$$

$$= 5x$$

P.I. corresponding to $10e^{-z} = 10 \cdot \frac{1}{(D_1+1)(D_1^2-2D_1+2)} \cdot e^{-z}$

$$= 10 \cdot \frac{1}{D_1+1} \cdot \frac{1}{1+2+2} e^{-z} = 2 \cdot \frac{1}{D_1+1} e^{-z} \cdot 1$$

$$= 2e^{-z} \cdot \frac{1}{D_1-1+1} \cdot 1 = 2 e^{-z} \frac{1}{D_1} \cdot 1 = 2e^{-z} z$$

$$= 2x^{-1} \log x.$$

$\therefore$ Required solution is $y = C.F + P.I.$

$$\therefore \boxed{y = c_1 x^{-1} + x\left(c_2 \cos \log x + c_3 \sin \log x\right) + 5x + 2x^{-1} \log x}$$

Q.3.(b) A body whose temperature is initially 100°C is allowed to cool in air, whose temperature remains at a constant temperature 20°C. It is given that after 10 minutes, the body has cooled to 40°C. Find the temperature of the body after half an hour.

Solution:

Let T be the temperature of the body in degree celsius and t be time in minutes. Then, by Newton's law of cooling, we get, $\frac{dT}{dt} = -\lambda(T-20)$

$$\Rightarrow \frac{dT}{T-20} = -\lambda\, dt \quad \text{—— (1)}$$

where λ is a positive constant of proportionality.

Integrating (1), $\log(T-20) - \log C = -\lambda t$

$$\Rightarrow T = 20 + Ce^{-\lambda t} \quad \text{—— (2)}$$

Initially, when $t=0$, $T=100$

so, (2) gives $C = 80$.

Then (2) reduces to $T = 20 + 80e^{-\lambda t}$ —— (3)

Given that $T=40$ when $t=10$, (3) gives

$$40 = 20 + 80e^{-10\lambda}$$

$$\Rightarrow 80e^{-10\lambda} = 20$$

$$\Rightarrow e^{-10\lambda} = 1/4$$

$$\Rightarrow e^{-\lambda} = (1/4)^{1/10}$$

$\therefore$ (3) reduces to $T = 20 + 80(e^{-\lambda})^t$

$$\Rightarrow T = 20 + 80\left(\tfrac{1}{4}\right)^{t/10}$$

$$\Rightarrow T = 20 + 80 \times \left(\frac{1}{4}\right)^3$$

$$\Rightarrow T = 20 + 80 \times \frac{1}{64}$$

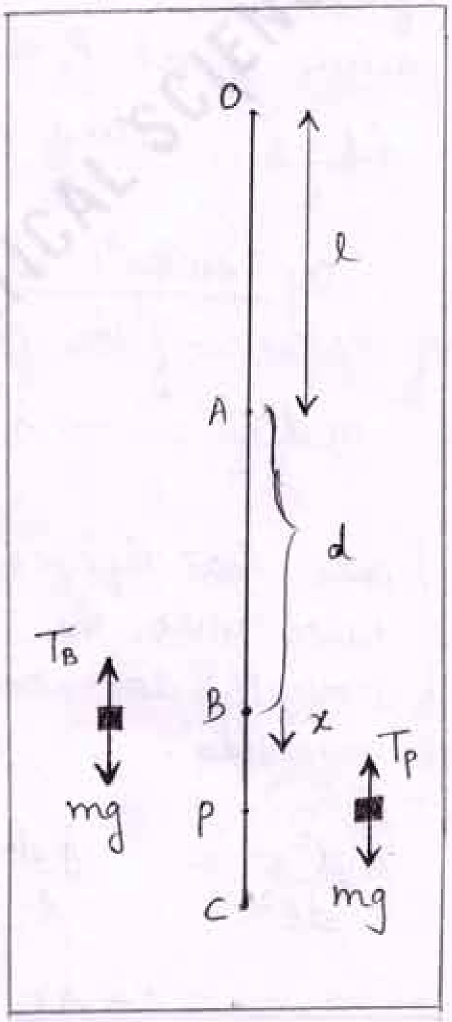
$$\Rightarrow T = 20 + 1.25$$

$$\Rightarrow \boxed{T = 21.25^\circ C}$$

Hence, the temperature of the body after half an hour is $21.25^\circ C$.

Q.3.(c) A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical expressions corresponding to these two weights, prove that $g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$.

Solution:



One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string.

Then AB is the statical expression in the string corresponding to this particle of mass m.

Let $AB = d$.

In the equilibrium position of the particle of mass m at B, the tension $T_B = \lambda(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l} = mg$$

$$\Rightarrow \frac{\lambda}{lm} = \frac{g}{d} \quad \text{——— (1)}$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP = x$. When the particle is at P, the tension T_P in the string P is $\lambda \frac{d+x}{l}$, acting vertically upwards.

By Newton's Second law of Motion, the equation of Motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{(d+x)}{l} + mg,$$

[Note that the weight mg of the particle has been taken with the +ve sign because it is acting vertically downwards, i.e., in the direction of x increasing.]

$$m\frac{d^2x}{dt^2} = -\frac{\lambda d}{l} - \frac{\lambda x}{l} + mg$$

$$= -\frac{\lambda x}{l} \qquad \left[\because \frac{\lambda d}{l} = mg\right]$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{lm}x = -\frac{g}{d}x \quad \cdots \text{from (1)}$$

Hence, the motion of the particle is Simple Harmonic about the centre B and its period is

$\dfrac{2\pi}{\sqrt{(g/d)}}$ i.e. $2\pi\sqrt{\dfrac{d}{g}}$

But according to the question, the period is t_1 when $d = c_1$ and the period is t_2 when $d = c_2$.

$t_1 = 2\pi\sqrt{(c_1/g)}$ and $t_2 = 2\pi\sqrt{(c_2/g)}$

so that, $t_1^2 - t_2^2 = \left(\dfrac{4\pi^2}{g}\right)(c_1 - c_2)$

$\Rightarrow \boxed{g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)}$

Hence, proved.

Q. 3.(d) Verify Stoke's theorem for $\vec{F} = -y^3\hat{i} + x^3\hat{j}$, where S is the circular disc $x^2+y^2 \le 1$, $z=0$.

Solution:

The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Stokes' Theorem :-

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot\hat{n}\, dS.$$

Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$ are parametric equations of C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left(-y^3\hat{i} + x^3\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \oint_C \left(-y^3 dx + x^3 dy\right)$$

$$= \int_{t=0}^{t=2\pi} \left\{-y^3\frac{dx}{dt} + x^3\frac{dy}{dt}\right\} dt$$

$$= \int_0^{2\pi} \left[-\sin^3 t(-\sin t) + \cos^3 t(\cos t)\right] dt$$

$$= \int_0^{2\pi} \left(\cos^4 t + \sin^4 t\right) dt$$

$$= 4\int_0^{\pi/2} \left(\cos^4 t + \sin^4 t\right) dt$$

$$= 4\times\left\{\frac{3.1.\pi}{4.2.2} + \frac{3.1.\pi}{4.2.2}\right\}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \frac{3\pi}{2} \quad \text{—— (I)}$$

Also, $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = \left(3x^2 + 3y^2\right)\hat{k}$

Here $\vec{n} = \hat{k}$ because the surface S is the xy-plane.

$$\therefore \left(\nabla\times\vec{F}\right)\cdot\hat{n} = \left(3x^2 + 3y^2\right)\hat{k}\cdot\hat{k} = 3\left(x^2 + y^2\right)$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = 3\iint_S (x^2+y^2)\, dS$$

$$= 3\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2 \cdot (r\,dr\,d\theta)$$

[Changing to polar coordinates]

$$= \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3}{4}(2\pi) = \frac{3\pi}{2} \quad \text{——(II)}$$

$\therefore$ from (I) and (II), we have,

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \frac{3\pi}{2}}$$

Hence, the Stoke's theorem is verified.

Q.4.(a) By using Laplace transform method, solve $(D^2+m^2)y = a\cos nt$, $t>0$, if $y>0$, $Dy = 0$ when $t=0$.

Solution:

$$(D^2+m^2)y = a\cos nt$$

Applying Laplace transform on both sides,

$$L\{D^2y + m^2y\} = L\{a\cos nt\}. \quad \text{--- (1)}$$

$$\because \left\{ \begin{array}{l} L\{D^2y\} = p^2 L(y(t)) - py(0) - y'(0) \\ L\{\cos at\} = \dfrac{p}{p^2+a^2} \end{array} \right\}$$

$\therefore$ (1) $\equiv$

$$p^2 L\{y\} - p(y(0)) - y'(0) + m^2 L(y) = a \cdot \frac{p}{p^2+n^2}$$

$$[\because y(0) = 0 \text{ and } y'(0) = 0]$$

$$\Rightarrow (p^2+m^2) L\{y\} = a \cdot \frac{p}{p^2+n^2}$$

$$\Rightarrow L\{y(t)\} = a \cdot \frac{p}{(p^2+n^2)(p^2+m^2)}$$

$$\Rightarrow L\{y(t)\} = \frac{ap}{m^2-n^2}\left[\frac{1}{p^2+n^2} - \frac{1}{p^2+m^2}\right]$$

Now, taking Inverse Laplace on both the sides, we have,

$$y(t) = \frac{a}{m^2-n^2} \cdot L^{-1}\left\{\frac{p}{p^2+n^2} - \frac{p}{p^2+m^2}\right\}$$

$$\therefore \boxed{y(t) = \frac{a}{m^2-n^2}\left\{\cos nt - \cos mt\right\}}$$

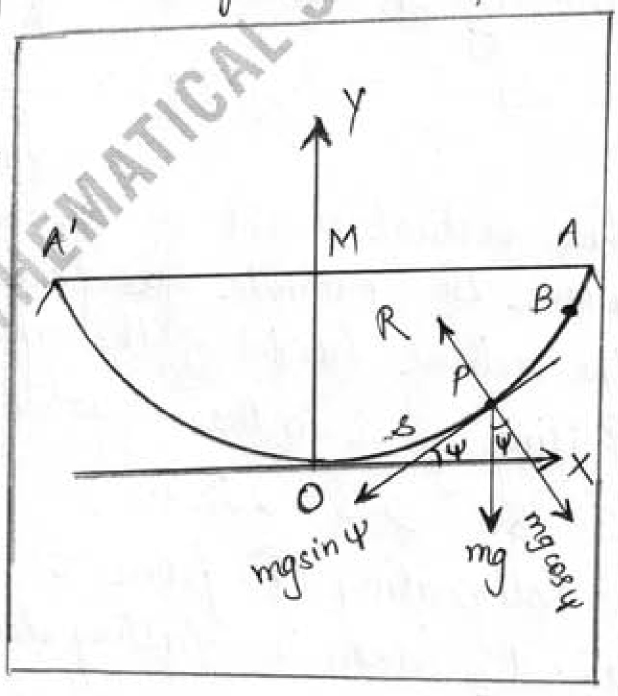
is the required equation.

Hence, the result.

Q.4.(b) → A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Solution:

Let a particle start from rest from the cusp A of the cycloid.



If P is the position of the particle after time t such that arc OP = s, the equations of motions along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{—(1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{—(2)}$$

for the cycloid, $s = 4a\sin\psi$. —(3)

from (1) and (3), we have, $\frac{d^2s}{dt^2} = -\frac{g}{4a}\cdot s$.

Multiplying both sides by $2(ds/dt)$ and then integrating, we have $\left(\frac{ds}{dt}\right)^2 = -(g/4a)s^2 + A$.

Initially at the cusp A, $s=4a$ and $\frac{ds}{dt}=0$.

$\therefore A = (g/4a)\cdot(4a)^2 = 4ag.$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{-g}{4a}s^2 + 4ag = \frac{g}{4a}(16a^2-s^2) \quad \text{--- (4)}$$

$$\Rightarrow \frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2-s^2)} \quad \text{--- (5)}$$

the $-$ve sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore dt = -2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{16a^2-s^2}} \quad \text{--- (6)}$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$. Putting $y=a$ in the equation $s^2=8ay$, we get,
$s^2=8a^2 \Rightarrow s=2\sqrt{2}\,a$.

$\therefore$ Integrating (5) from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{s=2\sqrt{2}a}\frac{ds}{\sqrt{16a^2-s^2}} = 2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$= 2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a} - \cos^{-1}(1)\right] = 2\sqrt{(a/g)}\cdot\left[\cos^{-1}\frac{1}{\sqrt{2}} - \cos^{-1}1\right] = 2\sqrt{(a/g)}\cdot\left[\frac{\pi}{4}-0\right]$$

$$\therefore \boxed{t_1 = \frac{\pi}{2}\sqrt{(a/g)}} \quad \text{—— (I)}$$

Again integrating (5) from $s = 2\sqrt{2}a$ to $s = 0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2 = -2\sqrt{(a/g)} \int_{s=2\sqrt{2}a}^{s=0} \frac{ds}{\sqrt{16a^2 - s^2}}$$

$$= 2\sqrt{(a/g)} \left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}$$

$$= 2\sqrt{(a/g)} \left[\cos^{-1} 0 - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\sqrt{(a/g)} \left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$\therefore \boxed{t_2 = \frac{\pi}{2}\sqrt{(a/g)}} \quad \text{—— (II).}$$

$\therefore$ from (I), (II), we have $\boxed{t_1 = t_2}$

i.e. The time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

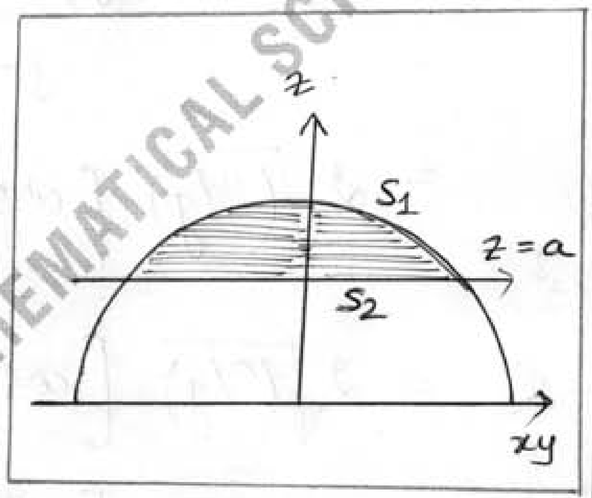
Hence, proved.

Q.4. (c) Let S be the spherical cap $x^2+y^2+z^2 = 2a^2$, $z \geq a$, together with its base $x^2+y^2 \leq a^2$, $z=a$. Find the flux of $\vec{F} = xz\hat{i} - yz\hat{j} - y^2\hat{k}$ outward through S

(i) by evaluating $\iint\limits_S \vec{F}\cdot\hat{n}\, d\sigma$ directly

(ii) by applying the divergence theorem.

Solution:



(i) $\iint\limits_{S_1} \vec{F}\cdot\hat{n}\, ds$ (curved spherical cap) $+$

$\iint\limits_{S_2} \vec{F}\cdot\hat{n}\, ds$ (disc $z=a$) region

$= \iint\limits_S \vec{F}\cdot\hat{n}\, d\sigma$ —— (I)

Consider,

$$\vec{n} = \nabla(x^2+y^2+z^2-2a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{2}\cdot a}$$

$$\therefore \vec{F}\cdot\hat{n} = \frac{x^2z - y^2z + zy^2}{\sqrt{2}\cdot a} \text{ and } \hat{n}\cdot\hat{k} = \frac{z}{\sqrt{2}a}$$

Let the curved surface be projected onto xy plane

then $$\iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \iint_{R: x^2+y^2=a^2} \frac{x^2 z - y^2 z + zy^2}{\sqrt{2}\,a} \cdot \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_{R: x^2+y^2=a^2} (x^2 - y^2 + y^2)\, dx\, dy$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r^2\cos^2\theta\,(r\,dr\,d\theta)$$

[Changing to polar co-ordinates]

$$= \int_0^{2\pi} \frac{a^4}{4}\cos^2\theta\, d\theta$$

$$= a^4 \frac{\Gamma 3/2 \cdot \Gamma 1/2}{2\Gamma 2} = \frac{\pi a^4}{4}$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \frac{\pi a^4}{4} \quad \text{——— (i)}$$

Now, consider $\iint_{S_2} \vec{F}\cdot\hat{n}\, dS$ = disc $(z=0)$ region

$\hat{n} = -\hat{k}$, $\vec{F}\cdot\hat{n} = -y^2$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, dS = \iint_{R: x^2+y^2=a^2} -y^2\, dx\, dy$$

$$= -\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} r^2\sin^2\theta \, r\,dr\,d\theta$$

[Changing to polar co-ordinates]

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = -\frac{\pi a^4}{4} \quad \text{——— (ii)}$$

$\therefore$ from (I), (i) & (ii), we have,

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \frac{\pi a^4}{4} - \frac{\pi a^4}{4} = 0$$

i.e. $\boxed{\iint_S \vec{F}\cdot\hat{n}\,d\sigma = 0}$

(ii) By Divergence theorem; we have,

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V \nabla\cdot\vec{F}\,dV.$$

Consider, $\nabla\cdot\vec{F} = (z - z + 0) = 0.$

$$\left[\because \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]$$

$$\therefore \boxed{\iiint_V \nabla\cdot\vec{F}\,dV = 0 \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,d\sigma = 0.}$$

Hence, the result.

SECTION - B

Q.5.(a) Solve $(D-1)^2(D^2+1)^2y = \sin^2(x/2)+e^x+x$.

Solution:

The given equation on rewriting is given as,

$$(D-1)^2(D^2+1)^2y = \frac{(1-\cos x)}{2}+e^x+x \quad \text{---} (1)$$

The A.E. of (1) is given by $(D-1)^2(D^2+1)^2=0$

$\Rightarrow D = 1, 1, \pm i, \pm i$

So, C.F. $= (c_1+c_2x)e^x+(c_3+c_4x)\cos x + (c_5+c_6x)\sin x$,

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants

$\therefore$ P.I. corresponding to $(-1/2)\cos x =$

$$-\frac{1}{2}\cdot\frac{1}{(D^2+1)^2(D^2-2D+1)}\cdot\cos x$$

$$= -\frac{1}{2}\cdot\frac{1}{(D^2+1)^2(-1^2-2D+1)}\cos x = \frac{1}{4}\cdot\frac{1}{(D^2+1)^2}\cdot\frac{1}{D}\cos x$$

$$= \frac{1}{4}\cdot\frac{1}{(D^2+1)^2}\sin x = \text{Imaginary part of } \frac{1}{4}\times\frac{1}{(D^2+1)^2}\cdot e^{ix} \quad \text{---} (2)$$

Now, $\frac{1}{(D^2+1)^2}e^{ix}\cdot 1 = e^{ix}\cdot\frac{1}{[(D+i)^2+1]^2}\cdot 1 = e^{ix}\cdot\frac{1}{(D^2+2iD)^2}\cdot 1$

$$= e^{ix}\cdot\frac{1}{(2iD)^2\cdot(1+D/2i)^2}\cdot e^{0x} = -\frac{1}{4}\cdot e^{ix}\cdot\frac{1}{D^2}\times\frac{1}{(1+0)^2}\cdot e^{0x}$$

$$= -\frac{1}{4}e^{+ix}\cdot\frac{1}{D^2}\cdot 1 = -\frac{1}{4}e^{ix}\left(\frac{x^2}{2}\right) = -\frac{1}{8}x^2(\cos x + i\sin x) \quad \text{——— (3)}$$

Using ② and ③, P.I. corresponding to

$$\left(-\tfrac{1}{2}\right)\times\cos x = \frac{-x^2}{32}\times\sin x \quad \text{——— (4)}$$

P.I. corresponding to $1/2 = \frac{1}{2}\cdot\frac{1}{(D-1)^2(D^2+1)^2}\cdot e^{0x}$

$$= \frac{1}{2}\cdot\frac{1}{(0-1)^2(0+1)^2}\cdot e^{0x} = \frac{1}{2} \quad \text{——— (5)}$$

P.I. corresponding to $e^x = \frac{1}{(D-1)^2(D^2+1)^2}\cdot e^x = \frac{1}{(D-1)^2(1+1)^2}\cdot e^x$

$$= \frac{1}{4}\cdot\frac{1}{(D-1)^2}\cdot e^x = \frac{1}{4}\cdot\frac{x^2}{2!}e^x = \left(\frac{x^2}{8}\right)\cdot e^x.$$

P.I. corresponding to $x = \frac{1}{(D-1)^2(D^2+1)^2}\cdot x$

$$= (1-D)^{-2}(1+D^2)^{-2}x = (1+2D+\cdots)(1+\cdots)x$$

$$= (1+2D+\cdots)x = x+2 \quad \text{——— (6)}$$

$\therefore$ Required solution is $y = C.F + P.I.$

$$\boxed{\therefore y = (c_1+c_2x)e^x + (c_3+c_4x)\cos x + (c_5+c_6x)\sin x + \frac{1}{2} - \frac{x^2}{32}\sin x + \frac{x^2}{8}\cdot e^x + x + 2}$$

Hence, the result.

Q.5.(b) Find the orthogonal trajectories of cardioids $r = a(1-\cos\theta)$, 'a' being parameter.

Solution:

The given family of cardioids is $r = a(1-\cos\theta)$ —— (1)

Taking logarithm on both-sides of (1), we get,

$$\log r = \log a + \log(1-\cos\theta) \quad \text{—— (2)}$$

Differentiating (2) w.r.t. 'θ', we get,

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \quad \text{—— (3)}$$

Since (3) is free from parameter 'a', hence (3) is the differential equation of the given family (1).

Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\cdot\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\theta/2\cdot\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$$\Rightarrow \left(\frac{1}{r}\right)(dr) = -\tan(\theta/2)\,d\theta$$ on separating variables.

Integrating, we have,

$$\log r = 2\log\cos\theta/2 + \log c$$

$$\Rightarrow \log r = \log(c\cdot\cos^2\theta/2)$$

$$\Rightarrow r = c\cdot\cos^2\theta/2$$

$\Rightarrow \quad r = \frac{c}{2}(1+\cos\theta)$

$\Rightarrow \quad \boxed{r = b(1+\cos\theta)} \quad \text{——— (4)}$

where $b\left(=\frac{c}{2}\right)$ is arbitrary constant.

Hence, the result.

Q.5.(c) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, prove that

$$\tan\alpha = \tan A + \tan B.$$

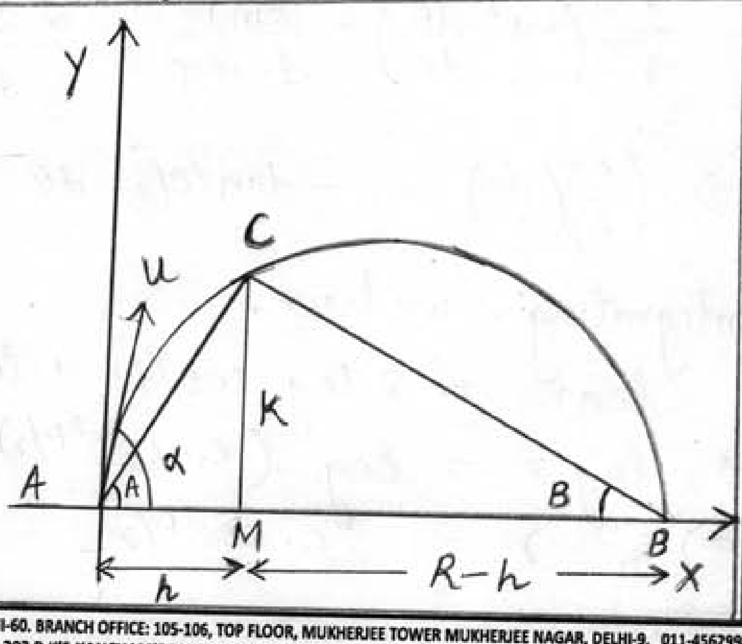
Solution:

Let A be the point of projection, u be the velocity of projection and α the angle of projection.

The particle while grazing over the vertex C falls at the point B.

If $AB = R$, then

$$R = \frac{2u^2 \sin\alpha\cos\alpha}{g} \quad \text{——— (1)}$$



Take the horizental line AB as the x-axis and the vertical line AY as the y-axis.
Let the co-ordinates of the vertex C be (h, k).
Then the point (h, k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore k = h\tan\alpha - \frac{1}{2}g\frac{h^2}{u^2\cos^2\alpha}$$

$$= h.\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$$= h.\tan\alpha\left[1 - \frac{h}{R}\right] \quad \text{— [by (1)]}$$

$$\therefore \frac{k}{h} = \tan\alpha\left(\frac{R-h}{R}\right)$$

$$\Rightarrow \tan A = \tan\alpha\left(\frac{R-h}{R}\right) \quad \left[\because \text{ from } \Delta CAM, \tan A = \frac{k}{h}\right]$$

$$\therefore \tan\alpha = \tan A\left(\frac{R}{R-h}\right)$$

$$= \tan A\left[\frac{(R-h)+h}{R-h}\right]$$

$$= \tan A\left[1 + \frac{h}{R-h}\right]$$

$$= \tan A + \tan A \cdot \frac{h}{R-h}$$

$$= \tan A + \frac{K}{h} \cdot \frac{h}{R-h} \qquad \left[\because \tan A = \frac{K}{h}\right]$$

$$\therefore \tan\alpha = (\tan A) + \frac{K}{(R-h)}$$

But from the ΔCMB,

$$\tan B = \frac{K}{(R-h)}$$

$$\therefore \boxed{\tan\alpha = \tan A + \tan B}$$

Hence, proved.

Q.5.(d) $\longrightarrow$ Find the value of $\vec{r}$ satisfying the equation

$$\frac{d^2\vec{r}}{dt^2} = 6t\,\hat{i} - 24t^2\,\hat{j} + 4\sin t\,\hat{k},$$

given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

Solution:

Given that $\frac{d^2\vec{r}}{dt^2} = (6t, -24t^2, 4\sin t)$ ——(1)

Integrate w.r.t. 't' on both sides,

$$\frac{d\vec{r}}{dt} = (3t^2, -8t^3, -4\cos t) + c \quad \text{——(2)}$$

Also, given that at $t=0$, $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$

$(-1,0,-3) = (0,0,-4) + c$

$\Rightarrow \quad c = (-1,0,1).$

i.e. $\dfrac{d\vec{r}}{dt} = (3t^2-1, -8t^3, -4\cos t + 1)$ ——— (3)

Now, once again, integrating (3), we have,

$$\vec{r} = (t^3-t, -2t^4, -4\sin t + t) + c' \quad \text{——— (4)}$$

Also, since it is given that,

at $t=0$, $\vec{r} = 2\hat{i}+\hat{j}$

$\Rightarrow (2,1,0) = (0,0,0) + c'$

$\Rightarrow \quad c' = (2,1,0)$

$\therefore \vec{r} = (t^3-t+2, -2t^4+1, -4\sin t + t)$

i.e. $\boxed{\vec{r} = (t^3-t+2)\hat{i} + (1-2t^4)\hat{j} + (t-4\sin t)\hat{k}}$

Hence, the result.

Q. 5. (c)

(i) If $\vec{u} = (1/r)\vec{r}$, show that $\nabla \times \vec{u} = 0$.

(ii) If $\vec{u} = (1/r)\vec{r}$, find grad. (div $\vec{u}$).

Solution:

(i) Given $u = \dfrac{\vec{r}}{r}$,

Consider, $\nabla \times \vec{u} = \nabla \times \frac{1}{r}\vec{r}$

$$= \left(\nabla\left(\frac{1}{r}\right) \times \vec{r}\right) + (\nabla \times \vec{r})\frac{1}{r}$$

$$\left[\because (\nabla \times \phi A) = \nabla\phi \times A + \phi(\nabla \times A)\right]$$

$$= \frac{-1}{r^2}\cdot\left(\frac{\vec{r}}{r} \times \vec{r}\right) + 0 \times \frac{1}{r} \quad \left[\because \nabla \times \vec{r} = 0\right]$$

$$= 0 + 0 \qquad \left[\because \vec{r} \times \vec{r} = 0\right]$$

$$= 0$$

$$\therefore \boxed{\nabla \times \vec{u} = 0 \text{ for } \vec{u} = \left(\frac{1}{r}\right)\vec{r}.} \quad \text{--- (I)}$$

(ii) $\vec{u} = \left(\frac{1}{r}\right)\vec{r}$,

Consider, $\nabla \cdot \vec{u} = \nabla\cdot\left(\frac{1}{r}\right)\cdot\vec{r} = \frac{1}{r}(\nabla\cdot\vec{r}) + \nabla\left(\frac{1}{r}\right)\cdot\vec{r}$

$$\left[\because \nabla\cdot(\phi A) = \nabla\phi \cdot A + \phi(\nabla\cdot A)\right]$$

$$= \frac{3}{r} - \frac{1}{r^2}\cdot\frac{\vec{r}\cdot\vec{r}}{r} \quad \left[\because \nabla\cdot\vec{r} = 3\right]$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

$$\therefore \text{grad}(\nabla\cdot\vec{u}) = \nabla\left(\frac{2}{r}\right) = 2\cdot\frac{-1}{r^2}\cdot\frac{\vec{r}}{r} = \frac{-2\vec{r}}{r^3}$$

$$\therefore \boxed{\text{grad}(\text{div } \vec{u}) = \frac{-2\vec{r}}{r^3}} \quad \text{--- (II)}$$

Hence the result.

Q.6.(a)

(i) Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

(ii) Solve $(1+y^2)\,dx + (x - e^{-\tan^{-1}y})\,dy = 0$, $y(1) = 0$.

Solution:

Given,

$$(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0 \quad \text{——(1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$, we get,

$$M = 2xy^4e^y + 2xy^3 + y \quad \text{and} \quad N = x^2y^4e^y - x^2y^2 - 3x \quad \text{——(2)}$$

Here,

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1 \quad \text{and}$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3.$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y}(2xy^4e^y + 2xy^3 + y)$$

$$= -\frac{4M}{y}$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}$$, which is a function of y alone.

$$\Rightarrow \text{I.F. of (1)} = e^{\int(-4/y)\,dy} = e^{-4\log y} = \left(1/y^4\right).$$

Multiplying (1) by $1/y^4$, we have,

$\left\{2xe^y + (2x/y) + (1/y^3)\right\} dx + \left\{x^2 e^y - (x^2/y^2) - 3(x/y^4)\right\} dy = 0$ whose solution is

$$\int \left\{2xe^y + (2x/y) + (1/y^3)\right\} dx = c$$

Treating y as constant

$$\Rightarrow \boxed{x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c} \quad \text{——— (I)}$$

is the required solution.

(ii) Given equation is

$$(1+y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0 \quad \text{—— (1)}$$

Rewriting (1), we have,

$$\frac{dx}{dy} + \frac{x - e^{-\tan^{-1} y}}{1+y^2} = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}\, x = \frac{e^{-\tan^{-1} y}}{1+y^2} \quad \text{——— (2)}$$

which is clearly linear.

It's I.F. $= e^{\int (1/1+y^2)\, dy} = e^{\tan^{-1} y}$ and so its solution is given as,

$$x \cdot e^{\tan^{-1} y} = \int \left(e^{\tan^{-1} y} \times \frac{e^{-\tan^{-1} y}}{1+y^2} \right) dy + c$$

$$\Rightarrow x \cdot e^{\tan^{-1} y} = \tan^{-1} y + c \qquad \text{—— (3)}$$

Now, putting $x = 1, y = 0$ in (2), we get

$$\boxed{c = 1}$$

Hence, the required solution is

$$\boxed{x \cdot e^{\tan^{-1} y} = \tan^{-1} y + 1} \qquad \text{—— (II)}$$

Hence, the result.

Q. 6. (b) Find the general and singular solution of $y^2(y - xp) = x^4 p^2$.

Solution:

The given equation is $y^2(y - xp) = x^4 p^2$ —— (1)

Putting $x = 1/u$, $y = 1/v$ so that

$dx = -(1/u^2)\, du$, $dy = -(1/v^2)\, dv$, we get.

$$\frac{dy}{dx} = \frac{u^2}{v^2} \frac{dv}{du}$$

$$\Rightarrow p = \frac{u^2}{v^2} P, \text{ where } P = \frac{dv}{du} \text{ and } p = \frac{dy}{dx}$$

$\therefore$ Putting $x = 1/u$, $y = 1/v$, $p = (u^2 P)/v^2$ in (1),

we have,

$$\left(\frac{1}{v^2}\right)\left\{\frac{1}{v} - \frac{1}{u}\frac{u^2 P}{v^2}\right\} = \left(\frac{1}{u^4}\right)\left(\frac{u^4 P^2}{v^4}\right)$$

$\Rightarrow$ $\underline{v = uP + P^2}$, which is in <u>Clairaut's form</u>.

$\therefore$ The required general solution is

$$v = uC + C^2$$

$$\Rightarrow 1/y = c/x + c^2$$

$$\Rightarrow x = cy + c^2 xy.$$

i.e. $\boxed{xyc^2 + yc - x = 0}$ ——— (2)

which is a quadratic equation in c and so it's c-discriminant relation is

$$y^2 - 4(xy)(-x) = 0$$

$$\Rightarrow y(y + 4x^2) = 0.$$

Now, $y = 0$ gives $p = \frac{dy}{dx} = 0$.

These values satisfy (1). So, $\boxed{y = 0 \text{ is a singular solution}}$. Again, $y = -4x^2$ gives $p = \frac{dy}{dx} = -8x$.

These values satisfy (1). Hence, $\boxed{y + 4x^2 = 0 \text{ is also singular solution}}$.

Q.6.(c) Reduce the equation $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$, $(x>0)$ into the normal form and hence solve it.

Solution:

The given equation is $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$.

Rewriting it, we get,

$$y'' - \frac{2}{x}(1+x)y' + \frac{2(1+x)}{x^2}y = x \quad \text{—— (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2(1+x)}{x^2}, \quad R = x. \quad \text{—— (2)}$$

We choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x}(1+x)dx} = e^{\int\left(\frac{1}{x}+1\right)dx}$

$$= e^{\ln x + x} = x\cdot e^x. \quad \text{—— (3)}$$

Let the required general solution be $y = uv$ —— (4)

Then v is given by the normal form,

$$\frac{d^2v}{dx^2} + Iv = S, \quad \text{—— (5)}$$

where,

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{2(1+x)}{x^2} - \frac{1}{4}\times\frac{4}{x^2}(1+x)^2 - \frac{1}{2}\times\frac{2}{x^2}$$

$$= \cancel{\frac{2}{x^2}} + \cancel{\frac{2}{x}} - 1 - \cancel{\frac{1}{x^2}} - \cancel{\frac{2}{x}} - \cancel{\frac{1}{x^2}}$$

$$= -1$$

$$S = \frac{R}{u} = \frac{x}{x \cdot e^{x}} = e^{-x}$$

Then (5) becomes, $\frac{d^2V}{dx^2} - V = e^{-x}$

$$\Rightarrow (D^2 - 1)v = e^{-x} \text{ where } D \equiv d/dx \quad \text{——— (6)}$$

The A.E. of (6) is given by $D^2 - 1 = 0$
$\Rightarrow D = \pm 1$

$$\therefore \; C.F. = c_1 e^{x} + c_2 e^{-x} \quad \text{——— (7).}$$

$$P.I. = \frac{1}{(D+1)(D-1)} \cdot e^{-x} = \frac{x}{1!} x - \frac{1}{2} e^{-x} = -\frac{x \cdot e^{-x}}{2}$$

$$\therefore \; v = c_1 e^{x} + c_2 e^{-x} - \frac{x \cdot e^{-x}}{2} \quad \text{——— (8)}$$

$\therefore$ from (4), (3) and (8), we have

$$\boxed{y = c_1 x e^{2x} + c_2 x - \frac{x^2}{2}}$$ which is the required solution.

Hence, the result.

Q.6.(d) $\rightarrow$ Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Solution: Let $F(t) = \sin t$

Now $\lim\limits_{t\to 0} \frac{F(t)}{t} = \lim\limits_{t\to 0} \frac{\sin t}{t} = 1.$

Since $L\{\sin t\} = \frac{1}{p^2+1} = f(p)$, say,

therefore, from

we have,

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)\,dx = \int_p^\infty \frac{dx}{x^2+1}$$

$$= \left(\tan^{-1} x\right)_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p$$

$$= \tan^{-1} 1/p$$

i.e. $\boxed{L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}}$ ——— (I)

Hence, Proved.

Now, $L\left\{\frac{\sin at}{t}\right\} = a\,L\left\{\frac{\sin at}{at}\right\}$

$$= a\cdot\frac{1}{a}\cdot\tan^{-1}\frac{1}{(p/a)}, \left[\text{since } L\{F(at)\} = \frac{1}{a}\cdot f(p/a)\right]$$

$\therefore \boxed{L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{p}}$ ——— (II)

Again, since $L\{\cos at\} = \dfrac{p}{p^2+a^2} = f(p)$,

we have,

$$L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{x}{x^2+a^2}\,dx$$

$$= \left[\frac{1}{2}\log(x^2+a^2)\right]_p^\infty$$

$$= \frac{1}{2}\lim_{x\to\infty}\log(x^2+a^2) - \frac{1}{2}\log(p^2+a^2)$$

which does not exist since $\lim\limits_{x\to\infty}\log(x^2+a^2)$ is infinite.

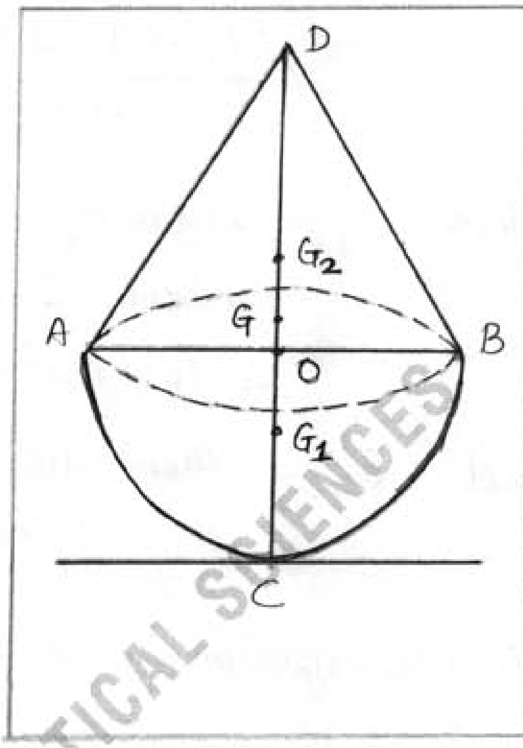
$$\boxed{\text{Hence, } L\left\{\frac{\cos at}{t}\right\} \text{ does not exist.}} \quad \text{—(III)}$$

Hence, the result

Q.7. (a) A body consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table, show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

Solution :

AB is the common base of the hemisphere and the



cone and COD is their common axis which must be vertical for equilibrium.
The hemisphere touches the table at C.

Let H be the height OD of the cone and r be the radius OA or OC of the hemisphere. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then

$$OG_1 = 3r/8 \text{ and } OG_2 = H/4.$$

If h be the height of the centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact C, then using the formula $x = \dfrac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$, we have

$$h = \frac{1/3\,\pi r^2 H \cdot CG_2 + \frac{2}{3}\pi r^3 \cdot CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{\frac{1}{3}\pi r^2 H\left(r + \frac{1}{4}H\right) + \frac{2}{3}\pi r^3 \cdot \frac{5}{8} r}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r}$$

Here, ρ_1 = radius of curvature at the point of contact C of the upper body which is spherical $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$.

$\therefore$ the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

i.e., $\frac{1}{h} > \frac{1}{r} + \frac{1}{\infty}$

i.e., $\frac{1}{h} > \frac{1}{r}$

i.e., $h < r$

i.e., $\frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r} < r$

i.e., $\cancel{Hr} + \frac{1}{4}H^2 + \frac{5}{4}r^2 < \cancel{Hr} + 2r^2$

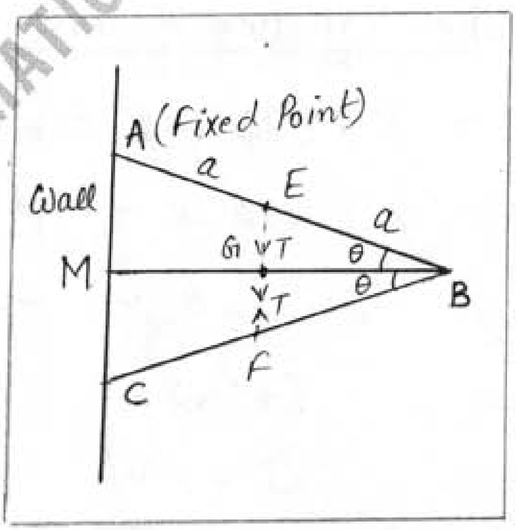
$\Rightarrow \frac{1}{\cancel{4}}H^2 < \frac{3}{\cancel{4}}r^2$

$\Rightarrow H^2 < 3r^2$

$\Rightarrow \boxed{H < r\sqrt{3}.}$ Hence, the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

Q. 7. (b) → One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are jointed by an elastic string, of natural length 'a' and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2\sin^{-1}(3/4)$.

Solution:



AB and BC are two rods each of length $2a$ and weight W smoothly joined together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a.

Let T be the tension in the string EF.

The total weight $2W$ of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M.

Let $\angle ABM = \theta = \angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF = 2EG = 2EB \sin\theta = 2a \sin\theta$

Also, the depth of G below the fixed point A.

$= AM = AB \sin\theta = 2a \sin\theta$.

The equation of virtual work is

$$-T\delta(2a\sin\theta) + 2W\delta(2a\sin\theta) = 0$$

$$\Rightarrow (-2aT\cos\theta + 4aW\cos\theta)\,\delta\theta = 0$$

$$\Rightarrow \quad 2a\cos\theta(-T + 2W)\,\delta\theta = 0$$

$$\Rightarrow \quad -T + 2W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$\Rightarrow \quad T = 2W$$

Also, by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda \frac{2a\sin\theta - a}{a},$$

where λ is the modulus of elasticity of the string

$$= 4W(2\sin\theta - 1) \qquad (\because \lambda = 4W).$$

Equating the two values of T, we have

$$2W = 4W(2\sin\theta - 1)$$

$$\Rightarrow \quad 1 = 2(2\sin\theta - 1)$$

$$\Rightarrow \quad 1 = 4\sin\theta - 2$$

$$\Rightarrow \quad 4\sin\theta = 3$$

$$\Rightarrow \quad \sin\theta = 3/4$$

$$\Rightarrow \quad \theta = \sin^{-1}(3/4)$$

$\therefore$ In Equilibrium, the whole angle between AB and BC $= \boxed{2\theta = 2\sin^{-1}(3/4)}$.

Hence, proved.

Q.7.(c) A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$, $a>b$ and is projected from an apse at distance $(a+b)$ with velocity $\sqrt{\frac{\mu}{(a+b)}}$. Show that the equation of its path is $r = a+b\cos\theta$.

Solution:

Here, the central acceleration

$$P = \mu\{3au^4 - 2(a^2-b^2)u^5\}.$$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left\{3au^4 - 2(a^2-b^2)u^5\right\}$$

$$\Rightarrow h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \mu\left\{3au^2 - 2(a^2-b^2)u^3\right\}.$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left\{au^3 - 2(a^2-b^2)\frac{u^4}{4}\right\} + A$$

$$\Rightarrow v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left\{2au^3 - (a^2-b^2)u^4\right\} + A$$

where A is constant. ——— (1)

But initially at an apse, $r = a+b$, $u = 1/(a+b)$, $du/d\theta = 0$ and $v = \sqrt{\mu}/(a+b)$.

$\therefore$ from (1), we have,

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right] + A$$

$\therefore h^2 = \mu$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left\{2au^3 - (a^2-b^2)u^4\right\}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = -u^2 + 2au^3 - (a^2-b^2)u^4$$

——— (2)

But $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$.

Substituting in (2), we have,

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2 + 2ar - (a^2-b^2)\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2$$

$$= b^2 - (r^2 - 2ar + a^2)$$

$$= b^2 - (r-a)^2$$

$$\therefore \frac{dr}{d\theta} = \sqrt{\{b^2 - (r-a)^2\}}$$

$$\Rightarrow d\theta = \frac{dr}{\sqrt{\{b^2-(r-a)^2\}}}$$

Integrating, $\theta = \sin^{-1}\left(\frac{r-a}{b}\right) + B'$ ——— (3)

But initially, when $r = a+b$, let us take $\theta = 0$.

Then from (3), $B' = -\sin^{-1}(1) = -\pi/2$.

Substituting in (3), we have

$$\theta + \pi/2 = \sin^{-1}\left(\frac{r-a}{b}\right) \Rightarrow r - a = b\sin\left(\frac{\pi}{2}+\theta\right)$$

$\Rightarrow \boxed{r = a + b\cos\theta}$, which is the required equation of the path.

Hence, proved.

Q. 8. (a) If $\vec{R}$ be a unit vector in the direction of $\vec{r}$, prove that $\vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{r^2} \cdot \vec{r} \times \frac{d\vec{r}}{dt}$,

where $r = |\vec{r}|$.

Solution:

We have $\vec{r} = r\vec{R}$; so that $\vec{R} = \frac{1}{r} \cdot \vec{r}$

$$\therefore \frac{d\vec{R}}{dt} = \frac{1}{r} \cdot \frac{d\vec{r}}{dt} - \frac{1}{r^2} \frac{dr}{dt} \cdot \vec{r}.$$

Hence, $\vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{r} \cdot \vec{r} \times \left(\frac{1}{r} \cdot \frac{d\vec{r}}{dt} - \frac{1}{r^2} \frac{dr}{dt} \vec{r} \right)$

$$= \frac{1}{r^2} \cdot \vec{r} \times \frac{d\vec{r}}{dt} - \frac{1}{r^3} \frac{dr}{dt} \vec{r} \times \vec{r}$$

i.e. $\boxed{\vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{r^2} \cdot \vec{r} \times \frac{d\vec{r}}{dt}}$ $\quad [\because \vec{r} \times \vec{r} = 0]$

Hence, proved.

Q. 8. (b) If $\vec{V}(x, y, z)$ is a vector function invariant under a rotation of axes, then prove that curl $\vec{V}$ is a vector invariant under this rotation.

Solution:

Let O be the fixed origin. Let Ox, Oy, Oz be one system of rectangular axes and Ox', Oy', Oz'

be the other system of rectangular axes. Take $\hat{i}, \hat{j}, \hat{k}$ as unit vectors along Ox, Oy, Oz and $\hat{i}', \hat{j}', \hat{k}'$ as unit vectors along Ox', Oy', Oz'.

Let P be any point in space whose co-ordinates are (x, y, z) or (x', y', z') with respect to the two systems of axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the lines Ox', Oy', Oz' with respect to the co-ordinate axes Ox, Oy, Oz.

The scheme of transformation will be as follows:

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \quad \text{——— (1)}$$

Also, we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $l\hat{i} + m\hat{j} + n\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along co-ordinate axes. Therefore,

$$\left.\begin{aligned} \hat{i}' &= l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k} \\ \hat{j}' &= l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k} \\ \hat{k}' &= l_3 \hat{i} + m_3 \hat{j} + n_3 \hat{k} \end{aligned}\right\} \quad \text{——— (2)}$$

Now, suppose the function $\vec{V}(x,y,z)$ becomes $\vec{V}'(x',y',z')$ after rotation of axes, then by hypothesis,

$$\vec{V}(x,y,z) = \vec{V}'(x',y',z')$$

By chain rule of differentiation, we have

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial \vec{V}'}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial \vec{V}'}{\partial y'} \cdot \frac{\partial y'}{\partial x} + \frac{\partial \vec{V}'}{\partial z'} \cdot \frac{\partial z'}{\partial x}$$

But from (1),

$$\frac{\partial x'}{\partial x} = l_1, \quad \frac{\partial y'}{\partial x} = l_2, \quad \frac{\partial z'}{\partial x} = l_3.$$

$$\left.\begin{aligned} \therefore \frac{\partial \vec{V}}{\partial x} &= l_1 \frac{\partial \vec{V}'}{\partial x'} + l_2 \frac{\partial \vec{V}'}{\partial y'} + l_3 \frac{\partial \vec{V}'}{\partial z'} \\ \text{Similarly,} \quad \frac{\partial \vec{V}}{\partial y} &= m_1 \frac{\partial \vec{V}'}{\partial x'} + m_2 \frac{\partial \vec{V}'}{\partial y'} + m_3 \frac{\partial \vec{V}'}{\partial z'} \\ \text{and} \quad \frac{\partial \vec{V}}{\partial z} &= n_1 \frac{\partial \vec{V}'}{\partial x'} + n_2 \frac{\partial \vec{V}'}{\partial y'} + n_3 \frac{\partial \vec{V}'}{\partial z'} \end{aligned}\right\} \quad \text{— (3)}$$

Taking cross product of these three equations by $\hat{i}, \hat{j}, \hat{k}$ respectively, adding and using the results (2), we get,

$$\hat{i}\times\frac{\partial\vec{V}}{\partial x}+\hat{j}\times\frac{\partial\vec{V}}{\partial y}+\hat{k}\times\frac{\partial\vec{V}}{\partial z}=$$

$$\hat{i}'\times\frac{\partial\vec{V}'}{\partial x'}+\hat{j}'\times\frac{\partial\vec{V}'}{\partial y'}+\hat{k}'\times\frac{\partial\vec{V}'}{\partial z'}$$

$$\Rightarrow \boxed{\text{curl}\,\vec{V}=\text{curl}\,\vec{V}'}$$

$\Rightarrow$ curl $\vec{V}$ is a vector invariant function under this rotation.

Hence, proved.

Q.8.(c) Let $\rho=(x^2+y^2+z^2)^{1/2}$.
Show that $\nabla(\rho^n)=n\rho^{n-2}\vec{R}$, where $\vec{R}=\hat{i}x+\hat{j}y+\hat{k}z$. Is there a value of n for which $\vec{F}=\nabla(\rho^n)$ represents the "inverse-square law" field? If so, what is this value of n?

Solution:

Given $\rho=(x^2+y^2+z^2)^{1/2}$

$\Rightarrow \rho=(r^2)^{1/2}=r$. (by assuming $r^2=x^2+y^2+z^2$)

$$\Rightarrow \nabla(\rho^n) = n\rho^{n-1}\cdot\frac{\vec{\rho}}{\rho} \quad \left[\because \text{grad}(f(\vec{r})) = f'(r)\cdot\frac{\vec{r}}{r} \text{ by definition}\right]$$

$$= n\rho^{n-2}\cdot\vec{\rho}$$

$$= n\rho^{n-2}\cdot\vec{R}$$

And, $\left(\because \rho = (x^2+y^2+z^2)^{1/2},\ R = x\hat{i}+y\hat{j}+z\hat{k}\right)$

i.e. $\boxed{\nabla(\rho^n) = n\rho^{n-2}\vec{R}}$ ——— (I)

Hence, proved.

For inverse square law field, $\vec{F} = \nabla(\rho^n)$ should be proportional to ρ^{-2}.

$$\Rightarrow \vec{F} \propto \frac{1}{\rho^2}$$

$$\Rightarrow \vec{F} = \frac{k}{\rho^2} \text{ (where k is a constant)}$$

Now, $\nabla(\rho^n) = n\rho^{n-2}\vec{R} = \vec{F}$

$$\Rightarrow n\rho^{n-2}\vec{R} = \frac{k}{\rho^2} \Rightarrow k = n\rho^n\vec{R}$$

$$\Rightarrow nR = \frac{k}{\rho^n} \Longrightarrow nR \propto \frac{1}{\rho^n}$$

$$\Rightarrow n(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k}) = k$$

For this relation to be a constant $\rho^n R$ should be a unit vector in the direction of R.

$\Rightarrow (x^2+y^2+z^2)^{n/2} \cdot (x\hat{i}+y\hat{j}+z\hat{k})$ is a unit vector

$\Rightarrow \boxed{n=-1}$ $\left[\because \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} \text{ is a unit vector}\right]$

So, for $n=-1$, $\vec{F} = \nabla(e^n)$ represents an "inverse square law" field.

Hence, the result.

Q. 8. (d)

If $\vec{F} = (y^2+z^2-x^2)\hat{i} + (z^2+x^2-y^2)\hat{j} + (x^2+y^2-z^2)\hat{k}$, evaluate $\iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds$ taken over the portion of the surface $x^2+y^2+z^2-2ax+az=0$ above the plane $z=0$, and verify Stoke's theorem.

Solution:

The surface meets the plane $z=0$ in the circle C given by $x^2+y^2-2ax=0$, $z=0$.
[where the given surface is $x^2+y^2+z^2-2ax+az=0$]

The polar equation of the circle C lying in the xy-plane is $r = 2a\cos\theta$, $0 \le \theta \le \pi$.

Also, the equation $x^2+y^2-2ax=0$ can be written as $(x-a)^2+y^2=a^2$.

Therefore the parametric equations of the circle C can be taken as

$$x = a + a\cos t,\ y = a\sin t,\ z=0,\ 0 \le t \le 2\pi.$$

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C.

By an application of divergence theorem, we have,

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_{S_1} \text{curl}\,\vec{F}\cdot\hat{k}\,dS$$

Now, $\text{curl}\,\vec{F}\cdot\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}\cdot\hat{k}$

$$= \left[\frac{\partial}{\partial x}(z^2+x^2-y^2) - \frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\hat{k}\cdot\hat{k} \quad [\because \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0]$$

$\therefore \text{curl}\,\vec{F}\cdot\hat{k} = 2(x-y)$.

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_{S_1} \text{curl}\,\vec{F}\cdot\hat{k}\,dS$$

$$= \iint_{S_1} 2(x-y)\,dS$$

$$= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2a\cos\theta} (r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

[changing to polar co-ordinates]

$$= 2\int_{\theta=0}^{\theta=\pi} (\cos\theta-\sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= 2\times\frac{8a^3}{3}\int_0^{\pi}(\cos\theta-\sin\theta)\cos^3\theta\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi}\cos^4\theta\,d\theta \quad \left[\because \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta = 0\right]$$

$$= 2\times\frac{16a^3}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= 2\times\frac{16a^3}{3}\times\frac{3\times1}{4\times2}\times\frac{\pi}{2} = \boxed{2\pi a^3} \quad \text{—(I)}$$

Also, $\int_C \vec{F}\cdot d\vec{r} = \int_C (y^2+z^2-x^2)\,dx + (z^2+x^2-y^2)\,dy + (x^2+y^2-z^2)\,dz$

$$= \int_C (y^2-x^2)\,dx + (x^2-y^2)\,dy \quad [\because \text{on } C,\ z=0 \text{ and } dz=0]$$

$$= \int_0^{2\pi}(x^2-y^2)\left(\frac{dy}{dt}-\frac{dx}{dt}\right)dt$$

$$= \int_0^{2\pi} \left[(a+a\cos t)^2 - a^2\sin^2 t\right](a\cos t + a\sin t)\,dt$$

$$= a^3 \int_0^{2\pi} (1+\cos^2 t + 2\cos t - \sin^2 t)(\cos t + \sin t)\,dt$$

$$= a^3 \int_0^{2\pi} 2\cos^2 t\,dt, \text{ (the other integrals vanish)}$$

$$= 2a^3 \times 4 \int_0^{\pi/2} \cos^2 t\,dt$$

$$= 8a^3 \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \boxed{2\pi a^3}. \quad \text{—— (II)}$$

$\therefore$ from (I) and (II),

we have,

$$\boxed{\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \int_C \vec{F}\cdot d\vec{r}.}$$

Hence, the Stoke's theorem is verified.