Krishna's

TEXT BOOK on

COMPLEX ANALYSIS

(For B.A. and B.Sc. V^{th} Semester students of Kumaun University)

Kumaun University Semester Syllabus *w.e.f.* 2018-19

By

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to

Lord

Krishna

Authors & Publishers

Preface

This book on **Complex Analysis** has been specially written according to the latest **Syllabus** to meet the requirements of **B.A. and B.Sc. Semester-V Students** of all colleges affiliated to Kumaun University.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall indeed be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to Mr. S.K. Rastogi, *M.D.*, Mr. Sugam Rastogi, *Executive Director*, Mrs. Kanupriya Rastogi, *Director* and entire team of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

— Authors

Syllabus

Complex Analysis

B.A./B.Sc. V Semester

Kumaun University

Fifth Semester – Second Paper

B.A./B.Sc. Paper-II M.M.-60

Complex Variables: Functions of a complex variable; Limit, continuity and differentiability.

Analytic functions: Analytic functions, Cauchy and Riemann equations, Harmonic functions.

Complex Integration: Complex integrals, Cauchy's theorem, Cauchy's integral formula, Morera's Theorem, Liouville's Theorem, Taylor's series, Laurent's series, Poles and singularities.

Residues: Residues, the Residue theorem, the principle part of a function, Evaluation of Improper real integrals.

Brief Contents

Krishna's

COMPLEX ANALYSIS

Chapters

Chapter

1

Functions of a Complex Variable

1 Complex Numbers

The equation $x^2 = -1$ has no solution in the set of real numbers because the square of every real number is either positive or zero. Therefore we feel the necessity to extend the system of real numbers. We all know that this defect is remedied by introducing complex numbers.

Complex Numbers: Definition: A number of the form $x + iy$, where $i = \sqrt{(-1)}$ and x, y are both real numbers, is called a *complex number*. A complex number is also defined as an ordered pair (x, y) of real numbers. A complex number $x + iy$ or (x, y) is usually denoted by the symbol z. If we write $z = x + iy$ or (x, y) then x is called the *real part* and y *the imaginary part* of the complex number z and these are denoted by $R(z)$ and $I(z)$ respectively. Thus in the complex number $z = \sqrt{3} + 5i$, we have $R(z)$ = the real part of $z = \sqrt{3}$, and $I(z)$ = the imaginary part of $z = 5$.

A complex number is said to be purely real if its imaginary part is zero, and purely imaginary if its real part is zero.

The complex number $a + 0i$ is simply written as a.

We shall denote the set of all complex numbers by **C**.

Equality of Two Complex Numbers:

Definition: *Two complex numbers*

$$z_1 = x_1 + iy_1 \quad \text{or} \quad (x_1, y_1)$$

and

$$z_2 = x_2 + iy_2 \quad \text{or} \quad (x_2, y_2)$$

are said to be equal if $x_1 = x_2$ *and* $y_1 = y_2$. *Thus two complex numbers are equal if and only if the real part of one is equal to the real part of the other and the imaginary part of one is equal to the imaginary part of the other.*

2 Addition of Complex Numbers

If $z_1 = x_1 + iy_1$ or (x_1, y_1) and $z_2 = x_2 + iy_2$ or (x_2, y_2) are any two complex numbers, then the sum of z_1 and z_2 written as $z_1 + z_2$ is defined by

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

or

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Thus $(3 + 5i) + (7 - 8i) = (3 + 7) + (5 - 8)\, i = 10 - 3i.$

Properties of the Addition of Complex Numbers:

The addition of complex numbers is commutative, associative, admits of identity element and every complex number possesses additive inverse.

Commutativity of addition in C: *To show that* $z_1 + z_2 = z_2 + z_1$ *where* z_1 *and* z_2 *are any complex numbers.*

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, where x_1, y_1, x_2, y_2 are real numbers.

We have

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

$= (x_1 + x_2, y_1 + y_2)$, by definition of addition in **C**

$= (x_2 + x_1, y_2 + y_1)$

[$\because$ Addition of real numbers is commutative]

$= (x_2, y_2) + (x_1, y_1) = z_2 + z_1.$

Hence $z_1 + z_2 = z_2 + z_1$, for all complex numbers z_1 and z_2.

Associativity of addition in C: *To show that* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, *for all complex numbers* z_1, z_2 *and* z_3.

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3)$,

where $x_1, y_1, x_2, y_2, x_3, y_3$ are real numbers.

We have

$$(z_1 + z_2) + z_3 = \{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3)$$

$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$, by def. of addition in **C**

$= (\{x_1 + x_2\} + x_3, \{y_1 + y_2\} + y_3)$, by def. of addition in **C**

$= (x_1 + \{x_2 + x_3\}, y_1 + \{y_2 + y_3\})$,

[$\because$ Addition of real numbers is associative]

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$, by def. of addition in **C**

$= (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\} = z_1 + (z_2 + z_3).$

Hence $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, for all complex numbers z_1, z_2 and z_3.

Additive Identity: The complex number (0, 0) or $0 + i0$ is the **additive identity**, since for every complex number (x, y), we have

$$(x, y) + (0,0) = (x + 0, y + 0) = (x, y) = (0,0) + (x, y).$$

The complex number (0, 0) is called the **zero complex number** and is simply written as 0.

A complex number $x + iy$ is said to be a non-zero complex number if at least one of x and y is not zero.

Additive Inverse: The complex number $(-x, -y)$ is the **additive inverse** of the complex number (x, y) since

$$(x, y) + (-x, -y) = (x - x, y - y) = (0,0)$$
$$= \text{the additive identity}$$

and also $\quad (-x, -y) + (x, y) = (0,0).$

The complex number $(-x, -y)$ is called the **negative** of the complex number (x, y) and we denote $(-x, -y)$ by $-(x, y)$.

Thus if $\quad z = (x, y)$, then $-z = -(x, y) = (-x, -y)$.

Cancellation law for addition in C. *If z_1, z_2, z_3 are any complex numbers, then*

$$z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2 .$$

3 Multiplication of Complex Numbers

If $z_1 = x_1 + iy_1$ or (x_1, y_1) and $z_2 = x_2 + iy_2$ or (x_2, y_2) are any two complex numbers, then the product of z_1 and z_2 written as $z_1 z_2$ is defined by

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

or $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$

Thus $$(3 + 5i)(7 + 6i) = (3 \times 7 - 5 \times 6) + (3 \times 6 + 5 \times 7) i = -9 + 53i,$$

or using the notation of ordered pairs, we have

$$(3,5)(7,6) = (3 \times 7 - 5 \times 6, 3 \times 6 + 5 \times 7) = (-9, 53).$$

Properties of the Multiplication of Complex Numbers:

The multiplication of complex numbers is commutative, associative, admits of identity element and every non-zero complex number possesses multiplicative inverse.

Commutativity of multiplication in C: *To show that $z_1 z_2 = z_2 z_1$, for all complex numbers z_1 and z_2.*

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, where x_1, y_1, x_2, y_2 are real numbers.

We have $$z_1 z_2 = (x_1, y_1)(x_2, y_2)$$
$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2), \text{ by def. of multiplication in } \mathbf{C}$$
$$= (x_2 x_1 - y_2 y_1, x_2 y_1 + y_2 x_1),$$
as real numbers are commutative for addition and multiplication

$= (x_2, y_2)(x_1, y_1)$, by def. of multiplication in **C**

$= z_2 \, z_1$.

Hence $z_1 z_2 = z_2 z_1$ for all complex numbers z_1 and z_2.

Associativity of multiplication in C : *To show that* $(z_1 z_2) z_3 = z_1 (z_2 z_3)$, *for all complex numbers* z_1, z_2 *and* z_3.

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3)$ where $x_1, y_1, x_2, y_2, x_3, y_3$ are real numbers.

We have $(z_1 z_2) z_3 = \{(x_1, y_1)(x_2, y_2)\}(x_3, y_3)$

$= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)(x_3, y_3)$, by def. of multiplication in C

$= (\{x_1 x_2 - y_1 y_2\} x_3 - \{x_1 y_2 + y_1 x_2\} y_3, \{x_1 x_2 - y_1 y_2\} y_3 + \{x_1 y_2 + y_1 x_2\} x_3)$

$= (x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3, x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$, ...(1)

by distributive law for real numbers.

Also $z_1 (z_2 z_3) = (x_1, y_1)\{(x_2, y_2)(x_3, y_3)\}$

$= (x_1, y_1)(x_2 x_3 - y_2 y_3, x_2 y_3 + y_2 x_3)$, by def. of multiplication in C

$= (x_1 \{x_2 x_3 - y_2 y_3\} - y_1 \{x_2 y_3 + y_2 x_3\}, x_1 \{x_2 y_3 + y_2 x_3\} + y_1 \{x_2 x_3 - y_2 y_3\})$

$= (x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3, x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 - y_1 y_2 y_3)$,

by distributive law for real numbers

$= (x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3, x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3)$, ...(2)

by laws of real numbers.

From (1) and (2), we have $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

Multiplicative identity: The complex number $(1, 0)$ or $1 + i0$ or simply 1 is the **multiplicative identity** since for every complex number (x, y), we have

$$(x, y)(1, 0) = (x \cdot 1 - y \cdot 0, \; x \cdot 0 + y \cdot 1) = (x, y) = (1, 0)(x, y).$$

Multiplicative inverse: The complex number (x, y) is called the **multiplicative inverse** of the complex number (a, b) if

$$(x, y)(a, b) = (1, 0) \text{ or simply } 1.$$

We have $(x, y)(a, b) = (1, 0)$

$\Rightarrow$ $(xa - yb, xb + ya) = (1, 0)$

$\Rightarrow$ $xa - yb = 1$ and $xb + ya = 0$.

The equations $xa - yb = 1$ and $xb + ya = 0$ give

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2},$$

provided $a^2 + b^2 \neq 0$ which implies that a and b are not both zero *i.e.*, (a, b) is a non-zero complex number.

Thus every non-zero complex number possesses multiplicative inverse and the multiplicative inverse of the complex number $(a, b) \neq (0, 0)$ is the complex number

$$\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

If z is a non-zero complex number, the multiplicative inverse of z is denoted by $1/z$ or z^{-1}.

Cancellation law for multiplication in C: *If z_1, z_2, z_3 are complex numbers and $z_3 \neq 0$, then* $z_1 z_3 = z_2 z_3 \Rightarrow z_1 = z_2$.

Multiplication distributes addition in C: *To show that*

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3, \text{ for all complex numbers } z_1, z_2 \text{ and } z_3.$$

Proof: Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3)$, where $x_1, y_1, x_2, y_2, x_3, y_3$ are real numbers.

We have $z_1 (z_2 + z_3) = (x_1, y_1) \{(x_2, y_2) + (x_3, y_3)\}$

$= (x_1, y_1)(x_2 + x_3, y_2 + y_3)$, by def. of addition in **C**

$= (x_1 \{x_2 + x_3\} - y_1 \{y_2 + y_3\}, x_1 \{y_2 + y_3\} + y_1 \{x_2 + x_3\})$, by def. of multiplication in **C**

$= (x_1 x_2 + x_1 x_3 - y_1 y_2 - y_1 y_3, x_1 y_2 + x_1 y_3 + y_1 x_2 + y_1 x_3)$, by distributive law for real numbers

$= (\{x_1 x_2 - y_1 y_2\} + \{x_1 x_3 - y_1 y_3\}, \{x_1 y_2 + y_1 x_2\} + \{x_1 y_3 + y_1 x_3\})$, by laws for real numbers

$= (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) + (x_1 x_3 - y_1 y_3, x_1 y_3 + y_1 x_3)$, by def. of addition in **C**

$= (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_3, y_3)$, by def. of multiplication in **C**

$= z_1 z_2 + z_1 z_3$.

Hence $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$, for all complex numbers z_1, z_2, z_3.

4 Difference of Two Complex Numbers

If z_1 and z_2 are two complex numbers, we define

$$z_1 - z_2 = z_1 + (-z_2).$$

Thus if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2)$$
$$= (x_1 - x_2, y_1 - y_2).$$

5 Division in C

Definition: *A complex number (a, b) is said to be divisible by a complex number (c, d) if there exists a complex number (x, y) such that*

$$(x, y)(c, d) = (a, b).$$

We have $(x, y)(c, d) = (a, b)$

$\Rightarrow$ $(xc - yd, xd + yc) = (a, b)$

$\Rightarrow$ $xc - yd = a$ and $xd + yc = b$.

The equations $xc - yd = a$ and $xd + yc = b$ give

$$x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{bc - ad}{c^2 + d^2}$$

provided $c^2 + d^2 \neq 0$ which implies that c and d are not both zero.

Thus division, except by $(0, 0)$, is always possible in the set of complex numbers. If z_1 and z_2 are two complex numbers such that $z_2 \neq 0$ then the quotient of the complex numbers z_1 and z_2 is defined by the relation

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 (z_2)^{-1}.$$

6 Modulus of a Complex Number

If $z = x + iy$ is any complex number then the non-negative real number $\sqrt{(x^2 + y^2)}$ is called the modulus of the complex number z and is denoted by $|z|$. Thus if $z = x + iy$, then $|z|$ *i.e.,* modulus of $z = \sqrt{(x^2 + y^2)}$. Obviously $|z| = 0$ if and only if $x = 0$ and $y = 0$ *i.e.,* if and only if $z = 0$. Also it can be easily seen that for any complex number z, $|z| \geq R(z)$ and $|z| \geq I(z)$.

Remember that for all real values of θ, we have

$$|\cos\theta + i\sin\theta| = \sqrt{(\cos^2\theta + \sin^2\theta)} = 1.$$

Thus the complex number $\cos\theta + i\sin\theta$ has always its modulus equal to 1 and is called uni-modular complex number.

If z_1 and z_2 are any two complex numbers, then

$$|z_1 z_2| = |z_1||z_2|.$$

If z_1 is any complex number and z_2 is any non-zero complex number, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

7 Conjugate of a Complex Number

If $z = x + iy$ is any complex number, then the complex number $x - iy$ is called the conjugate of the complex number z and is written as $\bar{z}$. Thus if

$$z = 3 + 4i, \text{ then } \bar{z} = 3 - 4i.$$

Obviously $|z| = |\bar{z}|$.

The following results are obvious and should be remembered :

(i) Two complex numbers are equal if and only if their conjugates are equal *i.e.,*

$$z_1 = z_2 \text{ if and only if } \overline{z_1} = \overline{z_2}.$$

(ii) $\overline{(\bar{z})} = z$.

(iii) We have $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2}$

$$= \bar{z}_1 - \bar{z}_2,\ \overline{z_1\ z_2} = \bar{z}_1\ \bar{z}_2$$

and $$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \text{ provided } z_2 \neq 0.$$

(iv) If $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2R(z).$$

(v) A complex number $z = x + iy$ is purely imaginary if and only if $z + \bar{z} = 0$.

(vi) If $z = x + iy$, then $z - \bar{z} = x + iy - (x - iy) = 2iy = 2iI(z)$.

(vii) A complex number z is purely real if and only if $z - \bar{z} = 0$.

(viii) If $z = x + iy$, then $z\,\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

$$= [\sqrt{(x^2 + y^2)}]^2$$

$$= |z|^2.$$

Thus the product of two conjugate complex numbers is a purely real number which is always ≥ 0 *i.e.*, which is never negative.

8 Modulus-Argument Form or Polar Standard Form or Trigonometric Form of a Complex Number

Every non-zero complex number $x + iy$ can always be put in the form $r(\cos\theta + i\sin\theta)$, where r and θ are both real numbers.

Let $x + iy = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$. Then equating real and imaginary parts on both sides, we get

$$x = r\cos\theta \qquad \ldots(1)$$

and $$y = r\sin\theta. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$x^2 + y^2 = r^2$$

or $r = +\sqrt{(x^2 + y^2)}$, taking the +ive sign before the radical sign

or $r = |z|$.

Thus r is known and is equal to the modulus of the complex number z.

Substituting this value of r in (1) and (2), we have

$$\cos\theta = \frac{x}{\sqrt{(x^2 + y^2)}}$$

and $$\sin\theta = \frac{y}{\sqrt{(x^2 + y^2)}}. \qquad \ldots(3)$$

If x and y are not both zero *i.e.,* if z is a non-zero complex number, then there exist values of θ which satisfy the equations (3) simultaneously. Any value of θ satisfying the equations (3) is called an **argument** or **amplitude** of the complex number z and we write

$$\theta = \arg z \qquad \text{or} \qquad \theta = \text{amp } z$$

Argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi + \theta$, where n is any integer.

The value of argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the **principal value** of the argument.

Usually by argument of a complex number we understand its principal value unless stated otherwise.

The zero complex number cannot be put in the form $r(\cos\theta + i\sin\theta)$ and thus the argument of zero complex number does not exist *i.e.,* is undefined.

If z is a non-zero complex number and r is a +ive real number, then the form $r(\cos\theta + i\sin\theta)$ in which z can always be put is called **modulus-argument form** or **polar form** or **trigonometric form** of z. Here r is modulus of z and θ is argument of z.

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we can write $z = r\, e^{i\theta}$. This is known as the **exponential form** of z.

To change the complex number $z = x + iy$ to modulus-argument form, we put $x = r\cos\theta$, $y = r\sin\theta$ and then we find the values of r and θ.

If x and y are both positive, the principal value of $\arg z$ lies between 0 and $\frac{1}{2}\pi$;

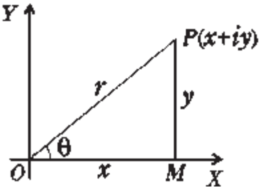
if x and y are both –ive, the principal value of $\arg z$ lies between $-\pi$ and $-\frac{1}{2}\pi$;

if x is +ive and y is –ive, it lies between $-\frac{1}{2}\pi$ and 0 ;

and if x is –ive and y is +ive, it lies between $\frac{1}{2}\pi$ and π.

9 The Geometrical Representation of Complex Numbers

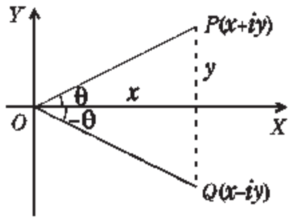
Argand Diagram: A complex number $z = x + iy$ can be represented by a point P in the cartesian plane whose coordinates are (x, y) referred to rectangular axes OX and OY, usually called the **real** and **imaginary** axes respectively.



The complex number $0 + i0$ corresponds to the origin, the real numbers $x = x + i0$ correspond to the points on the x-axis and the purely imaginary numbers $iy = 0 + iy$ correspond to the points on the y-axis.

Obviously the polar coordinates of the point P are (r, θ) where $r = OP = \sqrt{(x^2 + y^2)}$ is the modulus and $\theta = \angle POX = \tan^{-1}(y/x)$ is the argument of the complex number z.

Thus θ is the angle made by OP with positive direction of x-axis. This representation of complex numbers as points in the plane is due to Argand and is called the **Argand diagram** or **Argand plane** or **Complex plane**.



The complex number z is known as the **affix** of the point (x, y) which represents it.

If two complex numbers z_1 and z_2 are represented in the Argand diagram, then from the definitions of the difference of two complex numbers and the modulus of a complex number it is obvious that $|z_1 - z_2|$ is the distance between the points z_1 and z_2. It follows that for a fixed complex number z_0 and a given +ive real number r, the equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r.

If the complex number $z = x + iy$ is represented by the point $P(x, y)$ in the Argand plane, then its conjugate $\bar{z} = x - iy$ is represented by the point $Q(x, -y)$ which is the image of the point P in the **real axis** OX. If (r, θ) are the polar coordinates of P, then the polar coordinates of Q are $(r, -\theta)$ so that we have $|z| = |\bar{z}|$ and $\arg z = -\arg \bar{z}$.

Thus if the trigonometrical representation of a complex number z is $r(\cos\theta + i\sin\theta)$, then that of $\bar{z}$ is

$$r\{\cos(-\theta) + i\sin(-\theta)\} \text{ i.e., } r(\cos\theta - i\sin\theta).$$

Using exponential form, if $z = r e^{i\theta}$, then $\bar{z} = r e^{-i\theta}$.

Vector representation of a complex number: If we represent a complex number $z = x + iy$ by a point P in the Argand plane, then the length of the line segment OP is equal to the modulus of the complex number z and the direction of OP is represented by $\arg z$. Therefore the complex number z can be represented by the vector OP and we write $z = \overrightarrow{OP}$.

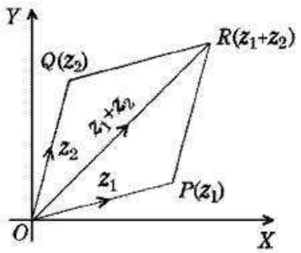
10 The Points on the Argand Plane Representing the Sum, Difference, Product and Division of Two Complex Numbers

(i) Representation of $z_1 + z_2$. Let the complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

be represented by the points P and Q on the Argand diagram. Then the coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. Complete the parallelogram $OPRQ$. Then the middle points of PQ and OR are the same. But the middle point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which is therefore also the middle point of OR and so the coordinates of R are $(x_1 + x_2, y_1 + y_2)$. Thus the point R corresponds to the complex number

$$(x_1 + x_2) + i(y_1 + y_2)$$



i.e., $(x_1 + iy_1) + (x_2 + iy_2)$ *i.e.,* $z_1 + z_2$.

Therefore the sum $z_1 + z_2$ of the complex numbers z_1, z_2 is geometrically represented by the vertex R of the parallelogram $OPRQ$ whose adjacent sides OP and OQ are represented by the complex numbers z_1 and z_2 .

The modulus and argument of $z_1 + z_2$ are given by

$$|z_1 + z_2| = OR \quad \text{and} \quad \arg(z_1 + z_2) = \angle ROX.$$

In vector notation, we have

$$z_1 + z_2 = \vec{OP} + \vec{OQ} = \vec{OP} + \vec{PR} = \vec{OR} .$$

To deduce that $|z_1 + z_2| \le |z_1| + |z_2|$

We know that in any triangle, the sum of any two sides is greater than the third side. Therefore from ΔOPR, we have

$OR \le OP + PR,$ the equality sign being also taken because the points O, P and R may be collinear

or $OR \le OP + OQ$

or $|z_1 + z_2| \le |z_1| + |z_2|$.

(ii) Representation of $z_1 - z_2$. Let the complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

be represented by the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively. Produce QO to Q' such that $OQ' = OQ$. Then the coordinates of the point Q' are $(-x_2, -y_2)$ and so the point Q' represents the complex number

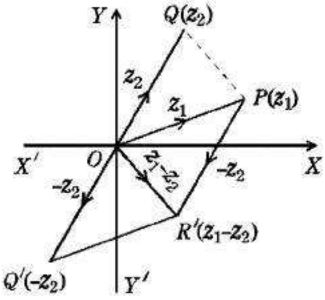
$$-x_2 - iy_2 \quad i.e., \quad -z_2 .$$

Complete the parallelogram $OQ'R'P$.

Then $-z_2 = \vec{OQ'}$

and $z_1 - z_2 = z_1 + (-z_2) = \vec{OP} + \vec{OQ'} = \vec{OP} + \vec{PR'} = \vec{OR'}$.

Thus the complex number $z_1 - z_2$ is geometrically represented by the vertex R' of the parallelogram $OQ'R'P$.



Since OQ is equal and parallel to $R'P$, therefore $OR'PQ$ is also a parallelogram and so $\vec{OR'} = \vec{QP}$.

Thus the complex number $z_1 - z_2$ is also represented by the vector $\vec{QP}$.

We have $|z_1 - z_2| = OR' = QP,$

and $\arg(z_1 - z_2) = \angle R'OX$

i.e., the angle through which OX has to rotate so as to be in the direction of QP.

To deduce that $|z_1 - z_2| \ge |z_1| - |z_2|$.

We know that in a triangle the difference of any two sides is less than the third side. Therefore from ΔOPQ, we have

$$OP - OQ \le QP$$

or $\quad |z_1| - |z_2| \le |z_1 - z_2|$

or $\quad |z_1 - z_2| \ge |z_1| - |z_2|.$

Remark: (a) Obviously $|z_1 - z_2| = QP$ and arg $(z_1 - z_2)$ is the angle through which OX has to rotate in anti-clockwise direction as to be parallel to line QP. It is often convenient to use the polar representation about some point z_0 other than the origin. The representation $z - z_0 = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$ means that ρ is the distance between z and z_0 *i.e.* $\rho = |z - z_0|$, and ϕ is the angle of inclination of vector $z - z_0$ with the real axis. Further if the vector $z - z_0$ is rotated about z_0 in the anti-clockwise direction through an angle θ and z' is the new position of z, then

$$z' - z_0 = \rho e^{i(\phi+\theta)} = \rho e^{i\phi} . e^{i\theta} = (z - z_0)\, e^{i\theta}. \qquad \textbf{(Note)}$$

(b) Let the lines AB and CD intersect at the point P_0 represented by the complex number z_0 and let P_1, P_2 be any two points on AB and CD represented by z_1 and z_2 respectively. Then the angle θ between the lines is given by

$$0 = \arg(z_2 - z_0) - \arg(z_1 - z_0) = \arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right).$$

[Note that here only principal values of the arguments are considered].

If AB coincides with CD, then $\arg\{(z_2 - z_0)/(z_1 - z_0)\} = 0$ or π so that $(z_2 - z_0)/(z_1 - z_0)$ is real. It follows that the points A, B, C, D are collinear.

If AB is perpendicular to CD, then

$$\arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \pm\frac{\pi}{2} \text{ and so } \frac{z_2 - z_0}{z_1 - z_0} \text{ is pure imaginary.}$$

(iii) Representation of $z_1 z_2$ and z_1 / z_2: Let P and Q be the points corresponding to the complex numbers z_1 and z_2, where

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$

Then $\quad OP = |z_1| = r_1, OQ = |z_2| = r_2,$

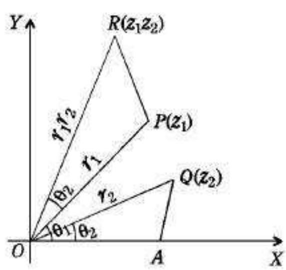
and $\quad \angle POX = \arg z_1 = \theta_1, \angle QOX = \arg z_2 = \theta_2.$

Representation of $z_1 z_2$ We have

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

From this representation of $z_1 z_2$ in standard polar form, we observe that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$



i.e., the modulus of the product of two complex numbers is equal to the product of their moduli ;

and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

i.e., the argument of the product of two complex numbers is equal to the sum of their arguments.

Now the point R in the Argand diagram representing the complex number $z_1 z_2$ can be obtained in the following manner.

Take the point A on OX such that $OA = 1$.

Draw the triangle OPR similar to the triangle OAQ such that the points R and Q lie on the opposite sides of OP,

$$\angle ROP = \angle QOA = \theta_2$$

and $\qquad \angle OPR = \angle OAQ.$

Then $\qquad \angle ROX = \angle POX + \angle ROP = \theta_1 + \theta_2 = \arg(z_1 z_2).$

Also from similar triangles OAQ and OPR, we have

$$\frac{OR}{OQ} = \frac{OP}{OA}$$

or $\qquad OR = \dfrac{OP \,.\, OQ}{OA} = OP \,.\, OQ \qquad [\because OA = 1]$

or $\qquad OR = r_1 r_2 = |z_1||z_2| = |z_1 z_2|.$

Thus OR is the modulus of the complex number $z_1 z_2$ and $\angle ROX$ is the argument of $z_1 z_2$. Hence the product $z_1 z_2$ is represented in the Argand diagram by the point R.

Remark : Multiplication by *i*.

Let $\qquad z = r(\cos\theta + i \sin\theta)$. Since $\quad i = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi,$

therefore $\qquad zi = r\left[\cos\left(\theta + \dfrac{\pi}{2}\right) + i\sin\left(\theta + \dfrac{\pi}{2}\right)\right].$

Hence multiplication of z with i rotates the vector for z through a right angle in the positive direction.

Representation of z_1/z_2: We have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

From the representation of z_1 / z_2 in standard polar form , we observe that

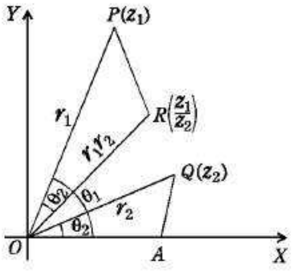
$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

i.e., the modulus of the quotient of two complex numbers is equal to the quotient of their moduli,

and $\qquad \arg(z_1 / z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

i.e., the argument of the quotient of two complex numbers is equal to the difference of their arguments.

Now the point R in the Argand diagram representing the complex number z_1 / z_2 can be obtained in the following manner :



Take the point A on OX such that $OA = 1$. Draw the triangle ORP similar to triangle OAQ such that the points R and Q are on the same side of OP,

$$\angle POR = \angle QOA = \theta_2$$

and $\qquad \angle OPR = \angle OQA.$

Then $\qquad \angle ROX = \angle POX - \angle POR$

$$= \theta_1 - \theta_2 = \arg (z_1 / z_2).$$

Also from similar triangles OAQ and ORP, we have

$$\frac{OR}{OA} = \frac{OP}{OQ} \text{ or } OR = \frac{OA \, . \, OP}{OQ} = \frac{OP}{OQ} \qquad [\because \; OA = 1]$$

or $\qquad OR = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|.$

Thus OR is the modulus of the complex number z_1 / z_2 and $\angle ROX$ is the argument of z_1 / z_2.

Hence the quotient z_1 / z_2 is represented in the argand diagram by the point R.

Remark: We have proved that

$$\arg (z_1 z_2) = \arg z_1 + \arg z_2 \quad \text{and} \quad \arg (z_1 / z_2) = \arg z_1 - \arg z_2.$$

But if $\arg z_1$ and $\arg z_2$ are the principal values of the arguments, then $\arg z_1 + \arg z_2$ need not represent the principal value of the argument of $z_1 z_2$. A similar remark applies to

$$\arg z_1 - \arg z_2.$$

For example, if $\quad z_1 = -1 + i, \; z_2 = 1 + i\sqrt{3},$

then $\qquad \arg z_1 = \frac{3}{4}\pi, \; \arg z_2 = \frac{1}{3}\pi,$

so that $\qquad \arg z_1 + \arg z_2 = \frac{3}{4}\pi + \frac{1}{3}\pi = \frac{13}{12}\pi > \pi.$

Therefore, $\arg z_1 + \arg z_2$ cannot be the principal value of $\arg (z_1 z_2)$.

11 Some Important Properties of Modulus and Arguments of Complex Numbers

Theorem 1: Modulus and Argument of the Conjugate of a Complex Number.

If z is any non-zero complex number, then $|\bar{z}| = |z|$ and $\arg \bar{z} = -\arg z$.

Proof: Let $|z| = r$ and $\arg z = \theta$.

Then from modulus-argument form of a complex number, we have

$$z = r(\cos\theta + i\sin\theta).$$

$\therefore \qquad \bar{z} = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)],$

which is modulus-argument form for $\bar{z}$.

Hence, $\qquad |\bar{z}| = r = |z|$ and $\arg \bar{z} = -\theta = -\arg z.$

Theorem 2: Modulus and Argument of the Product of two Complex Numbers.

If z_1 and z_2 are any two non-zero complex numbers, then

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $|z_1| = r_1, |z_2| = r_2$, $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$.

We have,

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2 [(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

From this representation of $z_1 z_2$ in the modulus-argument form, we observe that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

Theorem 3. Modulus and Argument of the Quotient of Two Complex Numbers:

If z_1 and z_2 are any two non-zero complex numbers, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then, $|z_1| = r_1, |z_2| = r_2$, $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$.

We have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \end{aligned}$$

From this representation of $\frac{z_1}{z_2}$ in the standard polar form, we observe that

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

Theorem 4: Triangle Inequality.

The modulus of the sum of two complex numbers can never exceed the sum of their moduli i.e., if z_1 and z_2 are any two complex numbers, then

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Proof: The inequality $|z_1 + z_2| \le |z_1| + |z_2|$ is obviously true if any of the two complex numbers z_1 and z_2 is zero. For, let $z_2 = 0$. Then

$$|z_1 + z_2| = |z_1 + 0| = |z_1| = |z_1| + 0 = |z_1| + |0| = |z_1| + |z_2|$$

and so the relation $|z_1 + z_2| \le |z_1| + |z_2|$ is true.

Now, let $z_1 \ne 0$ and $z_2 \ne 0$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

so that $|z_1| = r_1$ and $|z_2| = r_2$.

We have $z_1 + z_2 = (r_1 \cos\theta_1 + i r_1 \sin\theta_1) + (r_2 \cos\theta_2 + i r_2 \sin\theta_2)$

$= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + i (r_1 \sin\theta_1 + r_2 \sin\theta_2).$

$$\therefore \quad |z_1 + z_2| = \sqrt{(r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2} \qquad [\because \cos(\theta_1 - \theta_2) \leq 1]$$

$$= \sqrt{(r_1 + r_2)^2} = r_1 + r_2 = |z_1| + |z_2|.$$

Hence, $|z_1 + z_2| \leq |z_1| + |z_2|.$

Corollary : $|z_1 - z_2| \leq |z_1| + |z_2|$

We have $|z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2| = |z_1| + |z_2|$ $\quad [\because |z| = |-z|]$

$\therefore \quad |z_1 - z_2| \leq |z_1| + |z_2|$

Theorem 5: *The modulus of the difference of two complex numbers can never be less than the difference of their moduli.* **(Kumaun 2012, 13)**

Proof: Let z_1, z_2 be two complex numbers. We have to prove that

$$|z_1 - z_2| \geq |z_1| - |z_2|.$$

We have

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$= z_1 \overline{z_1} + z_2 \overline{z_2} - (z_1 \overline{z_2} + z_2 \overline{z_1})$$

$$= |z_1|^2 + |z_2|^2 - 2R(z_1 \overline{z_2}) \qquad [\text{See theorem 2}]$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1 \overline{z_2}| \qquad [\because R(z) \leq |z|]$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \qquad [\because |z| = |\bar{z}|]$$

$$= [|z_1| - |z_2|]^2.$$

Thus $|z_1 - z_2|^2 \geq [|z_1| - |z_2|]^2$

$\therefore \quad |z_1 - z_2| \geq |z_1| - |z_2|.$

Alternative Proof: Let

$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$$

so that $|z_1| = r_1$ and $|z_2| = r_2$.

We have $z_1 - z_2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i (r_1 \sin\theta_1 - r_2 \sin\theta_2).$

$$\therefore \quad |z_1 - z_2| = \sqrt{[(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2]}$$

$$= \sqrt{[r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)]}$$

$$\geq \sqrt{(r_1^2 + r_2^2 - 2r_1 r_2)} \qquad [\because \cos(\theta_1 - \theta_2) \leq 1]$$

$$= r_1 - r_2 = |z_1| - |z_2|.$$

Hence $|z_1 - z_2| \geq |z_1| - |z_2|.$

Corollary: $|z_1 + z_2| \geq |z_1| - |z_2|.$

We have $|z_1 + z_2| = |z_1 - (-z_2)| \geq |z_1| - |-z_2|$

$$= |z_1| - |z_2|. \qquad [\because |-z| = |z|]$$

Parallelogram Law

Theorem 6: *If z_1, z_2 are any complex numbers, then*

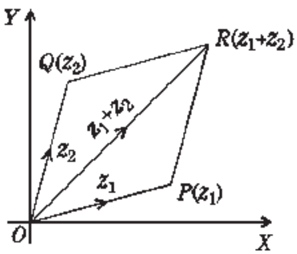
$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}.$$

Interpret the result geometrically.

Proof: We know that $|z|^2 = z\,\bar{z}$. Therefore, we have

$$\begin{aligned}
&|z_1 + z_2|^2 + |z_1 - z_2|^2\\
&= (z_1 + z_2)\,\overline{(z_1 + z_2)} + (z_1 - z_2)\,\overline{(z_1 - z_2)}\\
&= (z_1 + z_2)\,(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2)\\
&= 2\,z_1\,\bar{z}_1 + 2\,z_2\,\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2\\
&= 2\{|z_1|^2 + |z_2|^2\}. \qquad \ldots(1)
\end{aligned}$$

Geometrical interpretation: Let P and Q be the points in the Argand diagram represented by the complex numbers z_1 and z_2 respectively. Complete the parallelogram $OPRQ$. Then we have



$$z_1 = \vec{OP}, \; z_2 = \vec{OQ},$$

$$z_1 + z_2 = \vec{OP} + \vec{OQ} = \vec{OP} + \vec{PR} = \vec{OR},$$

$$z_1 - z_2 = \vec{OP} - \vec{OQ} = \vec{QP}.$$

$\therefore \qquad |z_1| = OP, |z_2| = OQ, |z_1 + z_2| = OR, |z_1 - z_2| = QP.$

Substituting these values in (1), we get

$$OR^2 + QP^2 = 2\,(OP^2 + OQ^2)$$

i.e., the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

12 The Order Relations Greater Than or Less Than do not apply to Complex Numbers

In the set of complex numbers the statements $z_1 > z_2$ or $z_1 < z_2$ are meaningless unless z_1 and z_2 are both real.

As $|z|, R(z), I(z)$ are all real numbers so the statements $|z_1| >$ or $< |z_2|$, $R(z_1) >$ or $< R(z_2)$ and $I(z_1) >$ or $< I(z_2)$ are meaningful.

Also $\qquad |z| = \sqrt{[\{R(z)\}^2 + \{I(z)\}^2]}$

or $\qquad |z|^2 = \{R(z)\}^2 + \{I(z)\}^2.$

From this, it is obvious that

$$|z| \geq |R(z)| \geq R(z)$$

and $\qquad |z| \geq |I(z)| \geq I(z).$

13 Some Important Results about Complex Numbers

(i) The separation of the complex number $\dfrac{a+ib}{c+id}$ into real and imaginary parts i.e., to put it in the form $A + iB$, where A and B are real numbers.

We have $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)}$,

multiplying the Nr and the Dr by the conjugate of the Dr

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$= A + iB, \quad \text{where} \quad A = \frac{ac+bd}{c^2+d^2} \text{ and } B = \frac{bc-ad}{c^2+d^2}.$$

Remember: To put the complex number $(a+ib)/(c+id)$ in the form $A+iB$, multiply its numerator and denominator by the conjugate of the denominator.

(ii) Effect of Multiplying a Complex number by Iota (i).

Let $z = r(\cos\theta + i\sin\theta)$.

Since $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$, therefore

$$z\, i = r(\cos\theta + i\sin\theta)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= r\left[\cos\left(\theta+\frac{\pi}{2}\right) + i\sin\left(\theta+\frac{\pi}{2}\right)\right].$$

Hence, multiplication of a complex number z by i results in rotating the vector joining the origin to the point representing z through a right angle in the positive direction i.e., anticlockwise direction.

(iii) If the complex numbers z_1 and z_2 are represented in the Argand plane by the points P and Q respectively, then $|z_2 - z_1| = PQ$ and $\arg(z_2 - z_1)$ is the angle through which OX has to rotate in anti-clockwise direction so as to be in the direction of the vector $\overrightarrow{PQ}$.

It is often convenient to use the polar representation of a complex number z about some point z_0 other than the origin. The representation

$$z - z_0 = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$$

means that $\rho = |z - z_0|$ *i.e.,* the distance between z and z_0, and ϕ is the angle of inclination of the vector $z - z_0$ with the real axis OX. If the vector $z - z_0$ is rotated about z_0 in the anti-clockwise direction through an angle θ and z_1 be the new position of z, then

$$z_1 - z_0 = \rho e^{i(\phi+\theta)} = \rho e^{i\phi}.e^{i\theta} = (z - z_0)e^{i\theta}.$$

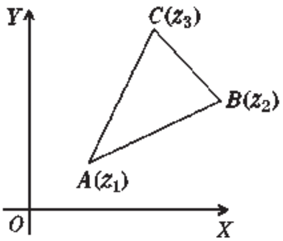
(iv) Angle between two intersecting lines in the Argand plane: Let the affixes of the points A, B, C in the Argand plane taken in the anticlockwise sense be the complex numbers z_1, z_2, z_3 respectively. Then

$$AB = |z_2 - z_1|, \; AC = |z_3 - z_1|, \quad \text{so that}$$

$$\frac{AC}{AB} = \frac{|z_3 - z_1|}{|z_2 - z_1|} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|.$$

Again $\arg(z_2 - z_1)$ is the angle which AB makes with the positive direction of x-axis and $\arg(z_3 - z_1)$ is the angle which AC makes with the positive direction of x-axis.

$\therefore$ angle between the lines AB and AC *i.e.*,



$$\angle BAC = \arg(z_3 - z_1) - \arg(z_2 - z_1)$$
$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right).$$

Hence $\left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \frac{AC}{AB}$ and $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \angle BAC.$

We can write $\frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB}(\cos\alpha + i\sin\alpha)$, where $\angle BAC = \alpha$.

(v) If the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4 respectively in the Argand plane, then

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$$

represents the angle through which DC is inclined to BA.

Also DC is perpendicular to BA

$$\Leftrightarrow \qquad \arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

$\Leftrightarrow$ $\quad \frac{z_1 - z_2}{z_3 - z_4}$ is a purely imaginary number.

14 Integral and Rational Powers of a Complex Number

Let z be a complex number and n a positive integer.

We define $\quad z^n = z \,.\, z \ldots\ldots$ upto n times.

Let z be a complex number. We define

$$z^0 = 1, \quad \text{and} \quad z^{-n} = (z^{-1})^n,$$

where n is a positive integer.

A complex number u is said to be an nth *root* of a complex number z, if $u^n = z$ and we write $u = z^{1/n}$.

If p/q is a rational number (p, q integers and $q \neq 0$), we define $z^{p/q}$ as the qth root of z^p

i.e., $\quad z^{p/q} = (z^p)^{1/q}$.

Binomial theorem: If z_1 and z_2 are complex numbers and n a positive integer, then

$$(z_1 + z_2)^n = z_1^{\,n} + {}^nC_1\, z_1^{\,n-1}\, z_2 + {}^nC_2\, z_1^{\,n-2}\, z_2^{\,2} + \ldots\ldots$$
$$+ {}^nC_r\, z_1^{\,n-r}\, z_2^{\,r} + \ldots\ldots + {}^nC_n\, z_2^{\,n}.$$

15 Geometrical Applications of Complex Numbers

Theorem 1: Formula for distance between two points.

If two complex numbers z_1 and z_2 be represented by the points A and B on the Argand Plane, then the distance $AB = |z_2 - z_1| = |z_1 - z_2|$.

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, these numbers on the Argand diagram are represented by the points $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

We have, $\quad z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1) = (x_2 - x_1) + i(y_2 - y_1)$.

$$\therefore \quad |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$= \text{distance } AB.$ [From Co-ordinate geometry]

Hence, $\quad AB = |z_2 - z_1| = |z_1 - z_2|$.

Theorem 2: *If two complex numbers z_1 and z_2 are represented by the points A and B on the Argand plane, then the affix of the point dividing AB internally in the ratio $m_1 : m_2$ is the complex number* $\left(\dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$.

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, these numbers on the Argand diagram are represented by the points $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

From co-ordinate geometry, the co-ordinates of the point P dividing AB internally in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right).$$

Hence, the affix of the point P

= the complex number representing the point P

$$= \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}\right) + i\left(\frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)$$

$$= \frac{m_1 (x_2 + iy_2) + m_2 (x_1 + iy_1)}{m_1 + m_2} = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

Corollary 1: *The affix of the middle point of z_1, z_2 is* $\frac{1}{2}(z_1 + z_2)$.

Corollary 2: *If z_1, z_2, z_3 be the affixes of the vertices of a triangle, then the centroid of the triangle has the affix* $\frac{1}{3}(z_1 + z_2 + z_3)$.

Corollary 3: Point dividing a line segment in the ratio $\lambda : 1$, $\lambda \neq -1$.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be the affixes of the points A and B respectively in the Argand plane.

If λ be a real number $\neq -1$, then there is a unique point C on AB such that

$$AC : CB = \lambda : 1.$$

The coordinates of C are given by $\left(\dfrac{x_1 + \lambda x_2}{1 + \lambda}, \dfrac{y_1 + \lambda y_2}{1 + \lambda}\right)$.

The affix of C is therefore $\dfrac{z_1 + \lambda z_2}{1 + \lambda}$.

16 Equation of a Straight Line in the Complex Plane

The *equation of a straight line* passing through a point of affix a in the complex plane can be put in the form

$$z = a + bt$$

where b is a non-zero complex number and the parameter t runs through all real values. It is easy to see that the two equations

$$z = a + bt \quad \text{and} \quad z = a' + b't$$

represent the same line if and only if $a' - a$ and b' are real multiples of b. The *lines are parallel* if b' is a real multiple of b and they are *equally directed* if b' is a positive multiple of b. We can identify the direction of a directed line by $\arg b$. If α is the *angle between the lines* $z = a + bt$ and $z = a' + b't$, then

$$\alpha = \arg\left(\frac{b'}{b}\right).$$

Note that it depends on the order in which the lines occur. It follows that the *lines are perpendicular if* $\dfrac{b'}{b}$ is purely imaginary.

The directed line $z = a + bt$ determines a *right half plane* consisting of all points z such that

$$I\left(\frac{z-a}{b}\right) < 0$$

and a *left half plane* consisting of all points z with

$$I\left(\frac{z-a}{b}\right) > 0.$$

Theorem 1: *The equation of any straight line passing through the origin and making an angle α with the real axis is $z = re^{i\alpha}$, where r is any real parameter.*

Proof: Let $z = x + iy$ be any point on the straight line passing through the origin and making an angle α with the real axis. Then

$$x = r\cos\alpha, \quad y = r\sin\alpha, \quad \text{where } r \text{ is any real number.}$$

$\therefore$ $$x + iy = r\cos\alpha + ir\sin\alpha = r(\cos\alpha + i\sin\alpha)$$

or $$z = re^{i\alpha}.$$

Hence, the equation of the required straight line is $z = re^{i\alpha}$, where r is a real parameter.

Theorem 2: *The equation of any straight line passing through the point z_1 and making an angle α with the real axis is $z = z_1 + re^{i\alpha}$, where r is any real parameter.*

Proof: Let $z_1 = x_1 + iy_1$.

Let $z = x + iy$ be any point on the straight line passing through the point z_1 and making an angle α with the real axis. Then

$$x - x_1 = r\cos\alpha, \quad y - y_1 = r\sin\alpha, \quad \text{where } r \text{ is any real number.}$$

$\therefore$ $$(x - x_1) + i(y - y_1) = r\cos\alpha + ir\sin\alpha$$

or $$(x + iy) - (x_1 + iy_1) = r(\cos\alpha + i\sin\alpha)$$

or $$z - z_1 = re^{i\alpha} \quad \text{or} \quad z = z_1 + re^{i\alpha}.$$

Hence, the equation of the required straight line is $z = z_1 + re^{i\alpha}$, where r is a real parameter.

Theorem 3: *The equation of the straight line joining the points z_1 and z_2 is*

$z = tz_1 + (1-t) z_2$, *where t is a real parameter.* **(Kumaun 2009)**

Proof: Let z be the affix of any point on the straight line joining the points z_1 and z_2.

Suppose the point z divides the join of z_1 and z_2 in the ratio $\lambda : 1$, where λ is any real number not equal to -1.

We have $$z = \frac{z_1 + \lambda z_2}{1+\lambda} \quad \text{or} \quad z = \left(\frac{1}{1+\lambda}\right) z_1 + \left(\frac{\lambda}{1+\lambda}\right) z_2. \qquad \ldots(1)$$

Put $\dfrac{1}{1+\lambda} = t$. Then $1 - t = 1 - \dfrac{1}{1+\lambda} = \dfrac{\lambda}{1+\lambda}$.

$\therefore$ the equation (1) becomes $z = tz_1 + (1-t) z_2$.

Hence, the equation of the straight line joining the points z_1 and z_2 is

$z = tz_1 + (1-t) z_2$, where t is a real parameter.

Remark: The above equation is the equation of the straight line joining the points z_1 and z_2 in **parametric form.**

Theorem 4: *The equation of the straight line joining the origin to the point z_1 is $z = tz_1$, where t is a real parameter.*

Proof: Let z be the affix of any point on the straight line joining the origin to the point z_1.

Suppose the point z divides the join of the point z_1 and the origin in the ratio $\lambda : 1$, where λ is any real number not equal to -1.

We have $$z = \frac{1 . z_1 + \lambda . 0}{1+\lambda} = \left(\frac{1}{1+\lambda}\right) z_1 . \qquad \ldots(1)$$

Put $$\frac{1}{1+\lambda} = t.$$

Then the equation (1) becomes $z = tz_1$.

Hence, the equation of the straight line joining the origin to the point z_1 is $z = tz_1$, where t is a real parameter.

Theorem 5: *The equation of the straight line joining the points z_1 and z_2 is*

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

Proof: Let z be the affix of any point on the straight line joining the points z_1 and z_2.

Then $$\arg \frac{z - z_1}{z_2 - z_1} = 0 \ \text{ or } \ \pi$$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} \text{ is purely real} \quad \Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \overline{\left(\frac{z - z_1}{z_2 - z_1}\right)}$$

$$\Rightarrow \qquad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

$$\Rightarrow \qquad (z - z_1)(\bar{z}_2 - \bar{z}_1) = (z_2 - z_1)(\bar{z} - \bar{z}_1)$$

$$\Rightarrow \qquad z(\bar{z}_2 - \bar{z}_1) - z_1(\bar{z}_2 - \bar{z}_1) = \bar{z}(z_2 - z_1) - \bar{z}_1(z_2 - z_1)$$

$$\Rightarrow \qquad z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + (z_1\bar{z}_2 - z_2\bar{z}_1) = 0$$

$$\Rightarrow \qquad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

Hence, the equation of the required straight line joining the points z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

Remark: The above equation is the equation of the straight line joining the points z_1 and z_2 in non-parametric form.

Corollary: *The necessary and sufficient condition for the points z_1, z_2 and z_3 to be collinear is*

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

Proof: The equation of the straight line joining the points z_2 and z_3 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0. \qquad \ldots(1)$$

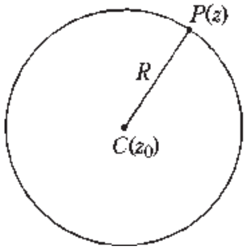
The points z_1, z_2 and z_3 are collinear if and only if the point z_1 lies on the straight line (1) *i.e.*, if and only if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

17 Equation of a Circle in the Complex Plane

Theorem 1: *The equation of a circle whose centre is at a point having affix z_0 and radius R is $|z - z_0| = R$.*

Proof: Let C be the centre of the circle and R be its radius. Then the affix of C is z_0. Let P be any point on the circle such that the affix of P is z. Then $CP = R \Rightarrow |z - z_0| = R$.



Since P is an arbitrary point on the circle, therefore, the equation of the circle is

$$|z - z_0| = R.$$

Corollary: *The equation of the circle whose centre is at the origin and radius R is* $|z| = R$.

Remark: The inequality $|z - z_0| < R$ represents the *inside* of the circle $|z - z_0| = R$ and the inequality $|z - z_0| > R$ represents the *outside* of the circle $|z - z_0| = R$. Similarly $|z| < R$ represents the *interior* of the circle $|z| = R$ and $|z| > R$ represents the *exterior* of the circle $|z| = R$.

Theorem 2: *The equation of the circle passing through the three given points in the Argand plane whose affixes are z_1, z_2 and z_3 is*

$$\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}.$$

Proof: Let A, B, C be three given points representing the complex numbers z_1, z_2, z_3 respectively. Let z be the complex coordinate of any point P on the circle. Then the angles $\angle ACB$ and $\angle APB$ are either equal as in figure (1) or have their sum equal to π as in figure (2).

Now from figure (1)

$$\angle ACB = \arg \frac{z_3 - z_2}{z_3 - z_1} \quad \text{and} \quad \angle APB = \arg \frac{z - z_2}{z - z_1}.$$

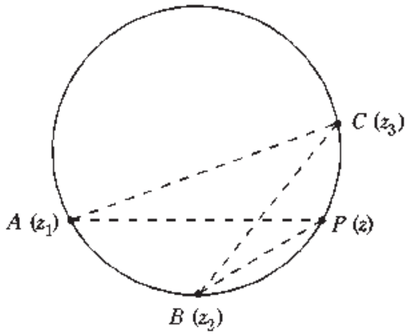


Fig. (1)

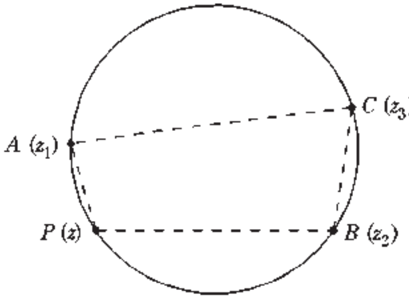


Fig. (2)

Hence in this case, we have

$$\arg \frac{z_3 - z_2}{z_3 - z_1} - \arg \frac{z - z_2}{z - z_1} = 0 \quad \text{or} \quad \arg \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)} = 0 \qquad \ldots(1)$$

Again from figure (2), we have

$$\angle ACB = \arg \frac{z_3 - z_2}{z_3 - z_1} \quad \text{and} \quad \angle APB = \arg \frac{z - z_1}{z - z_2}.$$

Hence in this case,

$$\arg \frac{z_3 - z_2}{z_3 - z_1} + \arg \frac{z - z_1}{z - z_2} = \pi \quad \text{or} \quad \arg \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)} = \pi \qquad \ldots(2)$$

It follows from (1) and (2) that $\dfrac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$

is purely real and so it must be equal to its conjugate. Hence we get

$$\frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}=\frac{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}$$

as the required equation of the circle.

Corollary: *The four points z_1, z_2, z_3, z_4 are concyclic if* $\dfrac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ *is purely real.*

Illustrative Examples

Example 1: *Express* $\dfrac{2+3i}{4+5i}$ *in the form $x+iy$.*

Solution: Multiplying the numerator and the denominator of the given fraction by the conjugate complex of the denominator, we have

$$\frac{2+3i}{4+5i}=\frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}=\frac{8-10i+12i-15i^2}{16-25i^2}$$

$$=\frac{23+2i}{16+25}=\frac{23+2i}{41}=\frac{23}{41}+\frac{2}{41}i. \qquad [\because\ i^2=-1]$$

$\therefore$ the real part $x=23/41$ and the imaginary part $y=2/41$.

Example 2: *Express $1-i$ in the modulus amplitude form.*

Solution: Let $1-i=r(\cos\theta+i\sin\theta)$.

Equating real and imaginary parts, we have

$$r\cos\theta=1 \qquad \ldots(1)$$

and $$r\sin\theta=-1. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$r^2=1+1=2.$$

$\therefore$ $$r=+\sqrt{2}.$$

Substituting the value of r in (1) and (2), we have

$$\cos\theta=1/\sqrt{2} \quad \text{and} \quad \sin\theta=-1/\sqrt{2}.$$

These give $\theta=-\pi/4$.

Hence $$1-i=\sqrt{2}\,\{\cos(-\pi/4)+i\sin(-\pi/4)\}.$$

Example 3: *Express* $\dfrac{1+7i}{(2-i)^2}$ *in the modulus amplitude form.*

Solution: Here $\dfrac{1+7i}{(2-i)^2}=\dfrac{1+7i}{4-4i+i^2}=\dfrac{1+7i}{3-4i}$, $\qquad [\because\ i^2=-1]$

$$=\frac{(1+7i)(3+4i)}{(3-4i)(3+4i)}=\frac{3+4i+21i+28i^2}{9-16i^2}=\frac{-25+25i}{25}=-1+i.$$

Now let $$-1+i=r(\cos\theta+i\sin\theta).$$

Equating real and imaginary parts, we have

$$-1 = r\cos\theta \qquad \ldots(1)$$

and $$1 = r\sin\theta. \qquad \ldots(2)$$

Squaring (1) and (2) and adding, we have

$$r^2 = 1 + 1 = 2.$$

$\therefore$ $$r = \sqrt{2}.$$

Now putting $r = \sqrt{2}$ in (1) and (2), we have

$$\cos\theta = -1/\sqrt{2} \quad \text{and} \quad \sin\theta = 1/\sqrt{2}, \quad \text{giving } \theta = 3\pi/4.$$

Hence $$\frac{1+7i}{(2-i)^2} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Example 4: *Put the following number in the trigonometrical form:*

$$(1 + i\tan\alpha), \textit{given } -\pi < \alpha < \pi, \alpha \neq \pm\frac{1}{2}\pi.$$

Solution: We have $1 + i\tan\alpha = 1 + i\dfrac{\sin\alpha}{\cos\alpha} = \dfrac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$

$$= \sec\alpha\,(\cos\alpha + i\sin\alpha).$$

Case I: If $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, then $\sec\alpha$ is +ive, *i.e.*, > 0 and so the desired trigonometrical form of $1 + i\tan\alpha$ is

$$1 + i\tan\alpha = \sec\alpha\,(\cos\alpha + i\sin\alpha).$$

Case II: If $-\pi < \alpha < -\frac{1}{2}\pi$, then $\sec\alpha$ is –ive and so $-\sec\alpha$ is +ive. Therefore in this case, we have

$$\begin{aligned} 1 + i\tan\alpha &= (-\sec\alpha)(-\cos\alpha - i\sin\alpha) \\ &= (-\sec\alpha)[\cos(\pi+\alpha) + i\sin(\pi+\alpha)], \end{aligned}$$

which is the desired form because in this case $-\pi < \pi + \alpha \leq \pi$.

Case III: If $\frac{1}{2}\pi < \alpha < \pi$, then again $\sec\alpha$ is –ive and so $-\sec\alpha$ is +ive. Therefore in this case, we have

$$\begin{aligned} 1 + i\tan\alpha &= (-\sec\alpha)(-\cos\alpha - i\sin\alpha) \\ &= (-\sec\alpha)[\cos(\pi-\alpha) - i\sin(\pi-\alpha)] \\ &= (-\sec\alpha)[\cos(\alpha-\pi) + i\sin(\alpha-\pi)], \end{aligned}$$

which is the desired form because in this case $-\pi < \alpha - \pi \leq \pi$.

Example 5: *Find the moduli and arguments of the following complex numbers :*

(i) $\left(\dfrac{2+i}{3-i}\right)^2$ *(ii)* $\dfrac{2+i}{4i + (1+i)^2}$.

Solution: (i) We have

$$\left(\frac{2+i}{3-i}\right)^2=\frac{3+4i}{8-6i}=\frac{(3+4i)(8+6i)}{(8-6i)(8+6i)}=\frac{50\,i}{100}=\frac{1}{2}\,i.$$

Let $\qquad \frac{1}{2}\,i=r\,(\cos\theta+i\sin\theta).$

Then $\qquad 0=r\cos\theta,\ \frac{1}{2}=r\sin\theta.$

Squaring and adding these relations, we get

$$r^2=\frac{1}{4}\text{ so that } r=\frac{1}{2}\cdot$$

Putting $\qquad r=\frac{1}{2}$, we have $\cos\theta=0,\sin\theta=1.$

The value of θ lying between $-\pi$ and π which satisfies both these equations is $\pi/2$.

$$\therefore \qquad \left|\left(\frac{2+i}{3-i}\right)^2\right|=r=\frac{1}{2}$$

and the principal value of $\arg\left(\frac{2+i}{3-i}\right)^2=\frac{\pi}{2}\cdot$

(ii) $$\frac{2+i}{4i+(1+i)^2}=\frac{2+i}{6i}=\frac{1}{6}-\frac{1}{3}\,i.$$

Let $\qquad \frac{1}{6}-\frac{1}{3}\,i=r\,(\cos\theta+i\sin\theta).$

Then $\qquad \frac{1}{6}=r\cos\theta,\ -\frac{1}{3}=r\sin\theta.$

Squaring and adding these relations, we get

$$r^2=\frac{1}{36}+\frac{1}{9}=\frac{5}{36}\text{ so that } r=\frac{\sqrt{5}}{6}\cdot$$

Dividing, we get $\tan\theta=-2 \Rightarrow \theta=-\tan^{-1}2.$

Hence required modulus $=r=\dfrac{\sqrt{5}}{6}$

and $\qquad$ argument $=\theta=-\tan^{-1}2.$

Example 6: *Show that the representative points of the complex numbers $i,-2-5i,1+4i$ and $3+10\,i$ are collinear.*

Solution: Let the representative points of the complex numbers $i,-2-5i,1+4i$ and $3+10i$ be A, B, C and D respectively. Then the cartesian coordinates of these points are

$$A\,(0,1),\ B\,(-2,-5),\ C\,(1,4)\text{ and } D\,(3,10).$$

The equation of the line AB is

$$y-1=\frac{-5-1}{-2-0}(x-0)\quad i.e.,\quad y-1=3x$$

i.e., $\qquad y-3x=1.$ $\qquad\qquad$ ...(1)

Substituting the coordinates of C *i.e.*, $(1, 4)$ in (1), we have $4 - 3 = 1$, which is satisfied. Therefore C (1, 4) lies on the line AB.

Again substituting the coordinates of D *i.e.*, $(3, 10)$ in (1), we have $10 - 9 = 1$, which is also satisfied.

Therefore D (3, 10) also lies on the line AB.

Hence the four points A, B, C and D are collinear.

Example 7: (i) *Show that* $\arg z + \arg \bar{z} = 2n\pi$, *where n is any integer.* **(Kumaun 2014)**

(ii) Show that amp $(z) - \text{amp}(-z) = \pm\pi$, *according as amp* (z) *is positive or negative.*

(iii) If $|z_1| = |z_2|$ *and amp* $z_1 + \text{amp}\, z_2 = 0$, *show that* z_1 *and* z_2 *are conjugate numbers.*

Solution: (i) Let $z = x + iy$, then $\bar{z} = x - iy$, where x, y are real.

We have $\arg z + \arg \bar{z} = \arg (z\bar{z}) = \arg \{(x + iy)(x - iy)\} = \arg (x^2 + y^2)$.

Now $x^2 + y^2$ is a positive real number, say a. Since a is a positive real number, therefore the representative point of a in the Argand plane will lie on the positive side of the real axis. So the principal value of arg a is 0 and the general value is $2n\pi$, where n is any integer.

Hence $\arg z + \arg \bar{z} = 2n\pi$.

(ii) **Case I:** When amp (z) is positive.

Let amp $(z) = \theta$, where $-\pi < \theta < \pi$

then amp $(-z) = -(\pi - \theta)$, where $0 < \theta < \pi$

Thus amp $(z) - \text{amp}(-z) = \theta + \pi - \theta = \pi$.

Case II: When amp (z) is negative amp $(-z) = \pi - (-\theta) = \pi + \theta$.

Therefore amp $(z) - \text{amp}(-z) = \theta - (\pi + \theta) = -\pi$.

(iii) Given amp $z_1 + \text{amp}\, z_2 = 0$ and $|z_1| = |z_2|$.

$\therefore$ amp $z_2 = -\text{amp}\, z_1$.

Since the modulus of the one number is equal to the modulus of the other and amplitude of the one number is equal to the negative amplitude of the other. Hence the two numbers are conjugate to each other.

Example 8: *If P, Q, R are points of affix* $z_1, z_2, z_1 + z_2$ *respectively, show that OPRQ is a parallelogram.* **(Kumaun 2011)**

Solution: Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$,

so that $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

Thus the coordinates of O, P, Q, R are $(0, 0), (x_1, y_1), (x_2, y_2)$ and $(x_1 + x_2, y_1 + y_2)$ respectively.

Now the mid-point of PQ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

and the mid-point of OR is also

$$\left(\frac{0 + x_1 + x_2}{2}, \frac{0 + y_1 + y_2}{2}\right), \text{ i.e., } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Hence $OPRQ$ is a parallelogram.

Example 9: *If a and b are real numbers between 0 and 1 s.t. $z_1 = a + i, z_2 = 1 + ib$ and $z_3 = 0$ form an equilateral triangle then $a = \ldots\ldots\ldots\ldots, b = \ldots\ldots\ldots\ldots$.*

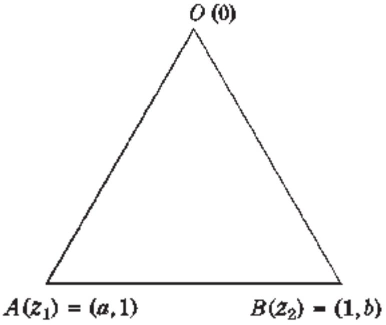
Solution: Take $O\,(0)$, $A\,(z_1 = a + i)$,

$$B\,(z_2 = 1 + ib).$$

Since ΔOAB is equilateral,

$$\therefore \quad OA = OB = AB$$

or $$(OA)^2 = (OB)^2 = (AB)^2.$$



Now $$(OA)^2 = (OB)^2$$

$$\Rightarrow \quad (a-0)^2 + (1-0)^2 = (b-0)^2 + (1-0)^2$$

$$\Rightarrow \quad a^2 = b^2 \Rightarrow a = \pm b \Rightarrow a = b \qquad \ldots(1)$$

as $a > 0, b > 0$.

Also $(OA)^2 = (AB)^2 \Rightarrow a^2 + 1 = (a-1)^2 + (1-b)^2$

$$\Rightarrow \quad a^2 + 1 = (a-1)^2 + (1-a)^2, \text{ by } (1)$$

$$\Rightarrow \quad 0 = a^2 + 1 - 4a$$

or $$a = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

or $$a = 2 - \sqrt{3} \text{ as } 0 < a < 1, 0 < b < 1.$$

Finally $$a = 2 - \sqrt{3},\ b = 2 - \sqrt{3}.$$

Example 10: *If the complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are complex conjugate to each other, then find the value of* $\boldsymbol{x}$. **(Kumaun 2008)**

Solution: We have $\overline{(\sin x + i\cos 2x)} = \cos x - i\sin 2x$

or $$\sin x - i\cos 2x = \cos x - i\sin 2x$$

or $$(\sin x - \cos x) + i(\sin 2x - \cos 2x) = 0$$

$$\Rightarrow \quad \sin x - \cos x = 0 \qquad \ldots(1)$$

and $$\cos 2x - \sin 2x = 0. \qquad \ldots(2)$$

$$(1) \Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4} \quad \text{or} \quad x = n\pi + \frac{\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots\ldots\ldots\ldots$$

$$(2) \Rightarrow \quad \tan 2x = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 2x = n\pi + \frac{\pi}{4} \quad \Rightarrow \quad x = \frac{n\pi}{2} + \frac{\pi}{8} = (4n+1)\frac{\pi}{8}$$

$$\therefore \quad x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \ldots\ldots\ldots\ldots$$

No value of x is common in both sets given by (1) and (2).

Hence there is no value of x for which both complex numbers are conjugate.

Example 11: If z_1 and z_2 are two non-zero complex numbers s.t.

$$|z_1 + z_2| = |z_1| + |z_2|,$$

then find arg $(z_1) -$ *arg* (z_2). **(Kumaun 2007)**

Solution: Here $|z_1 + z_2| = |z_1| + |z_2|$ gives

$$|(x_1 + x_2) + i(y_1 + y_2)| = (x_1^2 + y_1^2)^{1/2} + (x_2^2 + y_2^2)^{1/2}$$

where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$.

Squaring, we get

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2[(x_1^2 + y_1^2)(x_2^2 + y_2^2)]^{1/2}$$

or $x_1x_2 + y_1y_2 = [(x_1^2x_2^2 + y_1^2y_2^2) + (x_1^2y_2^2 + x_2^2y_1^2)]^{1/2}$.

Again squaring, we get $2\,x_1x_2 \,.\, y_1y_2 = x_1^2y_2^2 + x_2^2y_1^2$

or $(x_1y_2 - x_2y_1)^2 = 0$ or $x_1y_2 - x_2y_1 = 0$

or $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2} = k$, (say), which is real.

This gives $z_1 = x_1 + iy_1 = k(x_2 + iy_2) = kz_2$.

Now $\arg(z_1) - \arg(z_2) = \arg\left(\dfrac{z_1}{z_2}\right) = \arg\left(\dfrac{kz_2}{z_2}\right) = \arg(k)$

$$= 0, \text{ and } \pi, -\pi$$

according as $k > 0$ or $k < 0$.

Example 12: *Show that the equation of a straight line in the Argand plane can be put in the form* $z\bar{b} + b\,\bar{z} = c$, *where b is a non-zero complex constant and c is real.*

Solution: Let z_1, z_2 be two given points, say *A, B* on the Argand plane. Let z be any point, say *P*, on the line *AB*. Then we have

$$\arg\frac{z - z_1}{z_1 - z_2} = 0 \quad \text{or} \quad \pi$$

so that $\dfrac{z - z_1}{z_1 - z_2}$ is purely real *i.e.,* I.P. of $\dfrac{z - z_1}{z_1 - z_2} = 0$

or $$\left(\frac{z - z_1}{z_1 - z_2}\right) - \overline{\left(\frac{z - z_1}{z_1 - z_2}\right)} = 0$$

or $(z - z_1)(\overline{z_1} - \overline{z_2}) - (\bar{z} - \overline{z_1})(z_1 - z_2) = 0$

or $z(\overline{z_1} - \overline{z_2}) - \bar{z}(z_1 - z_2) + (z_1\overline{z_2} - \overline{z_1}z_2) = 0$. ...(1)

Since $\overline{z_1}z_2$ is conjugate of $z_1\overline{z_2}$ hence the number $z_1\overline{z_2} - \overline{z_1}z_2$ is purely imaginary.

Let $z_1\overline{z_2} - \overline{z_1}z_2 = ic$. ...(2)

Multiplying (1) by *i*, we get

$$zi(\overline{z_1} - \overline{z_2}) - \bar{z}\,i(z_1 - z_2) + i(z_1\overline{z_2} - \overline{z_1}z_2) = 0$$

or $zi(\overline{z_1} - \overline{z_2}) - \bar{z}\,i(z_1 - z_2) - c = 0$, using (2).

Now let $i(z_2 - z_1) = b$ then $i(\overline{z_1} - \overline{z_2}) = \bar{b}$.

Hence the equation takes the form

$$z\,\bar{b} + \bar{z}\,b = c.$$

Example 13: *Show that the equation of a circle in the Argand plane can be put in the form*

$$z\bar{z} + b\,\bar{z} + \bar{b}z + c = 0,$$

where c is a real and b a complex constant. **(Kumaun 2009)**

Solution: Let r be the radius and α the affix of the centre of the circle, say the point C. Let z be the affix of any point P on the circle. Then we have

$$|z - \alpha| = CP = r \quad \text{or} \quad |z - \alpha|^2 = r^2$$

or $$(z - \alpha)\,\overline{(z - \alpha)} = r^2 \quad \text{or} \quad (z - \alpha)(\bar{z} - \bar{\alpha}) = r^2$$

or $$z\,\bar{z} - \alpha\,\bar{z} - z\,\bar{\alpha} + \alpha\,\bar{\alpha} - r^2 = 0,$$

which is of the form $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$, where c is real, since $\alpha\,\bar{\alpha} - r^2$ is real, and b is complex.

Remark: The affix α of the centre of the circle $z\,\bar{z} + b\,\bar{z} + \bar{b}\,z + c = 0$ is given by $\alpha = -b$ and the radius r is given by $\alpha\,\bar{\alpha} - r^2 = c$.

We have $$\alpha\,\bar{\alpha} - r^2 = c \;\Rightarrow\; r^2 = \alpha\,\bar{\alpha} - c = (-b)\,\overline{(-b)} - c$$
$$= |-b|^2 - c = |b|^2 - c.$$

Hence, the centre of the circle $z\,\bar{z} + b\,\bar{z} + \bar{b}\,z + c = 0$ is the point $-b$ and its radius is $\sqrt{(b\,\bar{b} - c)}$ *i.e.,* $\sqrt{(|b|^2 - c)}$.

Example 14: *Show that the equation of a circle described on the line segment joining z_1 and z_2 as diameter is $(z - z_1)(\bar{z} - \overline{z_2}) + (z - z_2)(\bar{z} - \overline{z_1}) = 0$.*

Solution: Let z be the affix of any point P on the circle discribed on the line segment joining z_1 and z_2 as diameter. We then have

$$\arg \frac{z - z_1}{z - z_2} = \frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2}$$

so that $\dfrac{z - z_1}{z - z_2}$ is purely imaginary.

$\therefore$ $$\frac{z - z_1}{z - z_2} + \frac{\bar{z} - \overline{z_1}}{\bar{z} - \overline{z_2}} = 0$$

or $$(z - z_1)(\bar{z} - \overline{z_2}) + (z - z_2)(\bar{z} - \overline{z_1}) = 0,$$

which is the equation of the locus of the point z *i.e.,* the equation of the circle described on the line segment joining z_1 and z_2 as diameter.

Example 15: *Show that if the equation $z^2 + \alpha z + \beta = 0$ has a pair of conjugate complex roots, then α, β are both real and $\alpha^2 < 4\beta$.*

Solution: Let $x + iy$ and $x - iy$ be two conjugate complex roots of $z^2 + \alpha z + \beta = 0$.

Then sum of the roots $= 2x = \alpha$ ...(1)

and product of the roots $= x^2 + y^2 = \beta$. ...(2)

(1) and (2) show that α, β are both real. Also we have

$$4x^2 < 4x^2 + 4y^2 \qquad [\because y^2 > 0]$$

$$\Rightarrow \quad \alpha^2 < 4\beta.$$

Example 16: *Determine the regions of Argand diagram defined by*

(i) $|z^2 - z| < 1$, *(ii)* $|z - 1| + |z + 1| \le 4$.

Solution: (i) We have $|z^2 - z| < 1$

$$\Rightarrow \quad |r^2 (\cos 2\theta + i \sin 2\theta) - r(\cos\theta + i \sin\theta)| < 1$$

$$\Rightarrow \quad |(r^2 \cos 2\theta - r\cos\theta) + i(r^2 \sin 2\theta - r\sin\theta)|^2 < 1$$

$$\Rightarrow \quad (r^2 \cos 2\theta - r\cos\theta)^2 + (r^2 \sin 2\theta - r\sin\theta)^2 < 1$$

$$\Rightarrow \quad r^4 - 2r^3 (\cos 2\theta \cos\theta + \sin 2\theta \sin\theta) + r^2 < 1$$

$$\Rightarrow \quad r^4 - 2r^3 \cos\theta + r^2 - 1 < 0,$$

which represents the interior of the curve $r^4 - 2r^3 \cos\theta + r^2 - 1 = 0$.

(ii) We have $|z - 1| + |z + 1| \le 4$.

$$\Rightarrow \quad |z - 1|^2 + |z + 1|^2 + 2|z - 1||z + 1| \le 16$$

$$\Rightarrow \quad (z - 1)(\bar{z} - 1) + (z + 1)(\bar{z} + 1) + 2|(z - 1)(z + 1)| \le 16$$

$$\Rightarrow \quad z\bar{z} - z - \bar{z} + 1 + z\bar{z} + z + \bar{z} + 1 + 2|z^2 - 1| \le 16$$

$$\Rightarrow \quad 2|z|^2 + 2 + 2|z^2 - 1| \le 16$$

$$\Rightarrow \quad |z|^2 + |z^2 - 1| \le 7$$

$$\Rightarrow \quad |x + iy|^2 + |(x + iy)^2 - 1| \le 7$$

$$\Rightarrow \quad |x + iy|^2 + |(x^2 - y^2 - 1) + 2ixy| \le 7$$

$$\Rightarrow \quad (x^2 + y^2) + \sqrt{[(x^2 - y^2 - 1)^2 + 4x^2 y^2]} \le 7$$

$$\Rightarrow \quad \sqrt{[(x^2 - y^2 - 1)^2 + 4x^2 y^2]} \le 7 - (x^2 + y^2)$$

$$\Rightarrow \quad (x^2 - y^2 - 1)^2 + 4x^2 y^2 \le [7 - (x^2 + y^2)]^2$$

$$\Rightarrow \quad (x^2 - y^2)^2 + 1 - 2(x^2 - y^2) + 4x^2 y^2 \le 49 + (x^2 + y^2)^2 - 14(x^2 + y^2)$$

$$\Rightarrow \quad (x^2 + y^2)^2 + 1 - 2x^2 + 2y^2 \le 49 + (x^2 + y^2)^2 - 14x^2 - 14y^2$$

$$\Rightarrow \quad 12x^2 + 16y^2 \le 48 \quad \Rightarrow \quad 3x^2 + 4y^2 \le 12$$

$$\Rightarrow \quad (x^2/4) + (y^2/3) \le 1.$$

$\therefore$ The points z are on the boundary or in the interior of the ellipse

$$(x^2/4) + (y^2/3) = 1.$$

Example 17: *Prove that*

$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1 \text{ if } |z_1| < 1 \text{ and } |z_2| < 1.$$

(Kumaun 2010)

Solution: The inequality $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$ will hold good if

$$|z_1 - z_2| < |1 - \bar{z}_1 z_2| \quad \text{or} \quad |z_1 - z_2|^2 < |1 - \bar{z}_1 z_2|^2$$

or $$(z_1 - z_2)\,\overline{(z_1 - z_2)} < (1 - \bar{z}_1 z_2)\,\overline{(1 - \bar{z}_1 z_2)} \qquad [\because |z|^2 = z\,\bar{z}]$$

or $$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2)$$

or $$z_1 \bar{z}_1 - z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_2 \bar{z}_2 < 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z\,\bar{z}_1 z_2 \bar{z}_2$$

or $$|z_1|^2 + |z_2|^2 < 1 + |z_1|^2 |z_2|^2$$

or $$|z_1|^2 + |z_2|^2 - 1 - |z_1|^2 |z_2|^2 < 0$$

or $$(|z_1|^2 - 1)(1 - |z_2|^2) < 0. \qquad \ldots(1)$$

Now the inequality (1) will hold if $|z_1| < 1$ and $|z_2| < 1$.

Hence $$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1 \text{ if } |z_1| < 1 \text{ and } |z_2| < 1.$$

Example 18: *The vertices of a triangle are represented in Argand diagram by the complex numbers z_1, z_2, z_3. Interpret the modulus and argument of $\dfrac{z_2 - z_1}{z_3 - z_1}$ in terms of the sides and angles of the triangle.*

Solution: Let the points $z = z_1, z = z_2, z = z_3$ be A, B, C respectively in the Argand plane.

Then $$z_1 = \vec{OA},\ z_2 = \vec{OB},\ z_3 = \vec{OC}, \text{ where } O \text{ is the origin.}$$

We have $$z_2 - z_1 = \vec{OB} - \vec{OA} = \vec{AB} \quad \text{and} \quad z_3 - z_1 = \vec{OC} - \vec{OA} = \vec{AC}.$$

Now $$AB = |z_2 - z_1| \quad \text{and} \quad AC = |z_3 - z_1|.$$

$\therefore$ $$\frac{AB}{AC} = \frac{|z_2 - z_1|}{|z_3 - z_1|} = \left|\frac{z_2 - z_1}{z_3 - z_1}\right|.$$

Again $\arg(z_2 - z_1)$ is the angle which AB makes with the positive direction of x-axis and $\arg(z_3 - z_1)$ is the angle which AC makes with the positive direction of x-axis.

$\therefore$ angle between the lines AB and AC *i.e.,* $\angle BAC$

$$= \arg(z_2 - z_1) - \arg(z_3 - z_1) = \arg[(z_2 - z_1)/(z_3 - z_1)].$$

Hence $$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = \frac{AB}{AC} \quad \text{and} \quad \arg \frac{z_2 - z_1}{z_3 - z_1} = \angle BAC.$$

Example 19: *Prove that the area of the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argand diagram is*

$$\Sigma\,[(z_2 - z_3)\,|\,z_1\,|^2/4iz_1].$$

Solution: Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_3 = x_3 + iy_3$, so that the coordinates of the vertices of the given triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

Now the required area of the triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 & i\,y_1 & 1 \\ x_2 & i\,y_2 & 1 \\ x_3 & i\,y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\begin{vmatrix} x_1 & x_1 + i\,y_1 & 1 \\ x_2 & x_2 + i\,y_2 & 1 \\ x_3 & x_3 + i\,y_3 & 1 \end{vmatrix}, C_2 + C_1$$

$$= \frac{1}{2i}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\,\Sigma\,[x_1\,(z_2 - z_3)],$$

expanding the determinant along the first column

$$= \frac{1}{2i}\,\Sigma\left[\frac{1}{2}(z_1 + \overline{z_1})\,(z_2 - z_3)\right]$$

$$[\because z_1 + \overline{z_1} = (x_1 + i\,y_1) + (x_1 - i\,y_1) = 2x_1]$$

$$= \frac{1}{4i}\,\Sigma\,z_1\,(z_2 - z_3) + \frac{1}{4i}\,\overline{z_1}\,(z_2 - z_3)$$

$$= \frac{1}{4i}(0) + \frac{1}{4i}\,\Sigma\left[\frac{z_1\,\overline{z_1}}{z_1}(z_2 - z_3)\right] \qquad [\because \Sigma\,z_1\,(z_2 - z_3) = 0]$$

$$= \Sigma\left[\frac{(z_2 - z_3)\,|\,z_1\,|^2}{4iz_1}\right]. \qquad [\because z_1\,\overline{z_1} = |\,z_1\,|^2\,]$$

Example 20: *If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, prove that* $z_1^2 + 2z_2^2 + z_3^2 = 2z_2\,(z_1 + z_3)$.

Solution: Let the complex numbers z_1, z_2, z_3 represent the points A, B, C respectively in the Argand diagram.

Since $\angle\,ABC = 90°$, we have

$$\arg\frac{z_2 - z_1}{z_2 - z_3} = \frac{\pi}{2} \text{ or } -\frac{\pi}{2},$$

so that $\dfrac{z_2 - z_1}{z_2 - z_3}$ is purely imaginary.

Now if a complex number $z = x + iy$ is purely imaginary *i.e.*, $x = 0$ then $z + \bar{z} = 0$.

$$\therefore \quad \frac{z_2 - z_1}{z_2 - z_3} + \frac{\overline{z_2} - \overline{z_1}}{\overline{z_2} - \overline{z_3}} = 0 \quad \text{or} \quad \frac{z_2 - z_1}{z_2 - z_3} = -\frac{\overline{z_2} - \overline{z_1}}{\overline{z_2} - \overline{z_3}}. \qquad \ldots(1)$$

Again $\quad BA = BC$

so that $\quad |z_2 - z_1| = |z_2 - z_3| \quad$ or $\quad |z_2 - z_1|^2 = |z_2 - z_3|^2$

or $\quad (z_2 - z_1)(\overline{z_2} - \overline{z_1}) = (z_2 - z_3)(\overline{z_2} - \overline{z_3}). \qquad \ldots(2)$

Multiplying (1) and (2), we get

$$\frac{(z_2 - z_1)^2 (\overline{z_2} - \overline{z_1})}{(z_2 - z_3)} = -(\overline{z_2} - \overline{z_1})(z_2 - z_3)$$

or $\quad [(z_2 - z_1)^2 + (z_2 - z_3)^2](\overline{z_2} - \overline{z_1}) = 0$

or $\quad (z_2 - z_1)^2 + (z_2 - z_3)^2 = 0 \qquad [\because z_2 \neq z_1 \Rightarrow \overline{z_2} \neq \overline{z_1}]$

or $\quad z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3).$

Example 21: *Find the equations in complex variables of all the circles which are orthogonal to* $|z| = 1$ *and* $|z - 1| = 4$.

Solution: The given circles are $|z| = 1$ whose centre being the origin $(0, 0)$ and radius 1 and $|z - 1| = 4$ whose centre being the point $(1, 0)$ and radius 4. Let the equation of any circle intersecting the above two circles orthogonally be $|z - (\alpha + i\beta)| = r$ whose centre is the point (α, β) and radius r. Now two circles intersect orthogonally if the square of the distance between their centres is equal to the sum of the squares of their radii.

$\therefore \quad \alpha^2 + \beta^2 = 1 + r^2 \qquad \ldots(1)$

and $\quad (\alpha - 1)^2 + \beta^2 = 16 + r^2. \qquad \ldots(2)$

Subtracting (1) from (2), we get

$$-2\alpha + 1 = 15 \quad \text{or} \quad \alpha = -7.$$

Putting $\alpha = -7$ in (1), we get

$$\beta^2 = r^2 - 48 \quad \text{or} \quad r^2 = \beta^2 + 48.$$

Hence the required circles are

$$|z - (-7 + i\beta)| = \sqrt{(\beta^2 + 48)}$$

or $\quad |z + 7 - i\beta| = \sqrt{(\beta^2 + 48)}$, β is any real number

or $\quad |z + 7 + i\beta| = \sqrt{(\beta^2 + 48)}$, β is any real number.

Example 22: *If* $|z_1| = |z_2| = |z_3| = 1$ *and* $z_1 + z_2 + z_3 = 0$, *show that* z_1, z_2, z_3 *are the vertices of an equilateral triangle inscribed in a unit circle.*

Solution: Since $|z_1| = |z_2| = |z_3| = 1$, therefore the origin is the circumcentre of the triangle and its circum-radius is 1.

Now the affix of the centroid of the triangle whose vertices are z_1, z_2, z_3 is $\frac{1}{3}(z_1 + z_2 + z_3)$.

Since according to question $z_1 + z_2 + z_3 = 0$, therefore the origin is also the centroid of the triangle.

Thus for the given triangle, the centroid and the circumcentre coincide. Hence it is an equilateral triangle inscribed in a unit circle.

Comprehensive Exercise 1

1. Find real numbers A and B, if $A + iB = \dfrac{3 - 2i}{7 + 4i}$.

2. Find real numbers A and B, if $A + iB = \dfrac{1}{(1 - 2i)(2 + 3i)}$.

3. Find the value of the principal arguments of :

   (i) x (ii) $-x$

   (iii) iy (iv) $-iy$, where $x > 0$; $y > 0$.

4. Find the moduli and arguments of the following complex numbers :

   (i) $\dfrac{1 - i}{1 + i}$, (ii) $\dfrac{1 + 2i}{1 - (1 - i)^2}$, (iii) $\dfrac{3 - i}{2 + i} + \dfrac{3 + i}{2 - i}$.

5. Show that the origin and the points representing the roots of the equation $z^2 + pz + q = 0$ form an equilateral triangle if $p^2 = 3q$.

6. A, B, C, D, E are points on the complex plane representing complex numbers z_1, z_2, z_3, z_4, z_5 respectively. If $(z_3 - z_2) z_4 = (z_1 - z_2) z_5$, then prove that $\Delta\, ABC$ and ΔODE are similar, O being origin.

7. The roots z_1, z_2, z_3 of the equation $x^3 + 3ax^2 + 3bx + c = 0$,

   in which a, b, c c are complex numbers, correspond to the points A, B, C on the Argand plane. Find the centroid of the triangle ABC and show that it will be equilateral if $a^2 = b$.

8. If z_1 and z_2 are two complex numbers, prove that
$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2,$$
   if and only if, $z_1 \overline{z_2}$ is purely imaginary.

9. Prove that the centroid of the triangle whose vertices are z_1, z_2, z_3 is
$$\frac{z_1 + z_2 + z_3}{3}.$$

10. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

    Interpret the result geometrically and deduce that
$$|\alpha + \sqrt{(\alpha^2 - \beta^2)}| + |\alpha - \sqrt{(\alpha^2 - \beta^2)}| = |\alpha + \beta| + |\alpha - \beta|,$$
    all numbers involved being complex. **(Meerut 2000)**

11. A student writes the formula $\sqrt{(ab)} = \sqrt{a}\,\sqrt{b}$. Then substitutes $a = -1$ and $b = -1$ and finds $1 = -1$. Explain where he is wrong.

12. Let z_1, z_2 be complex numbers s.t. $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\dfrac{z_1 + z_2}{z_1 - z_2}$ will be purely imaginary.

13. Find the loci of the points z satisfying the conditions

(i) $arg \dfrac{z-1}{z+1} = \dfrac{\pi}{3}$. (ii) $z = a \cos t + b \sin t$

14. Prove that $|z_1| + |z_2| = \left|\frac{1}{2}(z_1 + z_2) + \sqrt{(z_1 z_2)}\right| + \left|\frac{1}{2}(z_1 + z_2) - \sqrt{(z_1 z_2)}\right|$.

15. If z_1, z_2 are the roots of $\alpha z^2 + 2\beta z + \gamma = 0$, prove that

$$|z_1| + |z_2| = \frac{1}{|\alpha|}[|-\beta + \sqrt{(\alpha\gamma)}| + |-\beta - \sqrt{(\alpha\gamma)}|].$$

16. Find the regions of the z-plane for which $\left|\dfrac{z-a}{z+\bar{a}}\right| < 1, = 1$ or > 1, where the real part of a is positive.

17. Prove that the sum and product of two complex numbers are real if and only if they are conjugate of each other.

18. Find the loci of the points z satisfying the conditions

(i) $\left|\dfrac{z-i}{z+i}\right| \geq 2$ (ii) $|z^2 - 1| < 1$.

19. Find the loci of the points z satisfying the conditions

(i) $|z-1| \geq 2$ (ii) $|z^2 - 1| < 2$.

20. Find the loci of the points z satisfying the conditions

(i) $\left|\dfrac{z-1}{z+1}\right| \leq 2$ (ii) $z = at + \dfrac{b}{t}$.

21. Show that the area of the triangle on the Argand plane formed by the complex numbers z, iz and $z + iz$ is $\frac{1}{2}|z|^2$.

22. Prove that $\left|\dfrac{z-1}{z+1}\right| =$ const. and amp $\left(\dfrac{z-1}{z+1}\right) =$ const. are orthogonal circles.

23. Show that locus of z such that $|z-a|.|z+a| = a^2, a > 0$ is a lemniscate.

(Kumaun 2008)

24. Let A and B be two complex numbers such that $\dfrac{A}{B} + \dfrac{B}{A} = 1$.

Prove that the origin and the points represented by A and B form vertices of an equilateral triangle.

Answers 1

1. $A = 1/5,\ B = 2/5$ **2.** $A = 8/65,\ B = 1/65$

3. (i) 0, (ii) π, (iii) $\frac{1}{2}\pi$, (iv) $-\frac{1}{2}\pi$

4. (i) $1; -\frac{\pi}{2}$, (ii) 1; 0, (iii) 2; 0 **7.** $-a$

13. (i) $x^2 + y^2 - (2/\sqrt{3})\,y - 1 = 0$,

(ii) $x^2(a_2^2 + b_2^2) + y^2(a_1^2 + b_1^2) - 2xy(a_1a_2 + b_1b_2) = (b_1a_2 - b_2a_1)^2$

16. $x >=< 0$

18. (i) $3x^2 + 3y^2 + 10y + 3 = 0$ (ii) $r^2 = 2\cos 2\theta$

19. (i) $(x-1)^2 + y^2 = 4$ (ii) $r^4 - 2r^2\cos 2\theta - 3 = 0$

20. (i) $3x^2 + 3y^2 + 10x + 3 = 0$

(ii) $x = a_1 t + \frac{b_1}{t}$, $y = a_2 t + \frac{b_2}{t}$

Eliminating 't' between these, we get the required locus.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Polar form of complex number $-5 + 5i$ is :

(a) $5\sqrt{2}\,e^{i\pi/4}$ (b) $5\sqrt{2}\,e^{-3i\pi/4}$

(c) $5\sqrt{2}\,e^{3i\pi/4}$ (d) none of these.

2. If the amplitude of the complex number z be θ, then amplitude of iz is :

(a) $-\theta$ (b) $\theta + \frac{\pi}{2}$

(c) $\theta + \pi$ (d) none of these.

(Kumaun 2008, 13)

3. If $z = x + iy$ then $z\bar{z} =$

(a) $x^2 + y^2$ (b) $x^2 - y^2$

(c) 0 (d) none of these.

4. If z_1, z_2 are any complex numbers, then $|z_1 + z_2|^2 + |z_1 - z_2|^2 =$

(a) $2\{|z_1|^2 - |z_2|^2\}$ (b) $2\{|z_1|^2 + |z_2|^2\}$

(c) $\{|z_1|^2 + |z_2|^2\}$ (d) none of these. **(Kumaun 2012)**

5. The multiplicative inverse of the complex number $(a, b) \neq (0, 0)$ is the complex number :

(a) $\left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right)$ (b) $\left(\frac{-a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

(c) $\left(\frac{-a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)$ (d) $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$.

6. The equation $|z-1|=|z+i|$ represents :
(a) a line through the origin whose slope is 1
(b) a line through the origin whose slope is -1
(c) an ellipse whose foci are at $z=1,-i$
(d) a circle through the origin.

7. Exp $(2 \pm 3i\pi) =$
(a) e^{-2} (b) $-e^2$
(c) $e^{\pm 3i}$ (d) none of these.

8. The points z_1, z_2, z_3, z_4 in order in the complex plane are the vertices of a parallelogram iff :
(a) $z_1+z_4=z_2+z_3$ (b) $z_1+z_3=z_2+z_4$
(c) $z_1+z_2=z_3+z_4$ (d) none of these.

9. If z_1 and z_2 are any two complex numbers, then
(a) $|z_1-z_2|=|z_1|-|z_2|$ (b) $|z_1-z_2|\le|z_1|+|z_2|$
(c) $|z_1-z_2|\le|z_1|-|z_2|$ (d) $|z_1-z_2|\ge|z_1|+|z_2|$.
(Kumaun 2014, 15)

10. In an Argand plane the centre of the circle $|4z-8+12i|=7$ has the affix
(a) $2-3i$ (b) $8-12i$
(c) $2+3i$ (d) $8+12i$.

11. In an Argand plane the radius of the circle $|5z+15-16i|=20$ is
(a) 20 (b) 2
(c) 4 (d) 10.

12. If $\sin(x+iy)=p+iq$, where p and q are real, then
(a) $q=\sin x \cos y$ (b) $q=\cos x \sin y$
(c) $q=\sin x \cosh y$ (d) $q=\cos x \sinh y$.

13. Equation $z\bar{z}+b\bar{z}+\bar{b}z+c=0$, where c is real and b is complex constant represents :
(a) a straight line (b) parabola
(c) ellipse (d) circle **(Kumaun 2009, 12)**

14. If z_1 and z_2 are two complex numbers than $|z_1-z_2|$ is :
(a) $\le|z_1|-|z_2|$ (b) $<|z_1|-|z_2|$
(c) $>|z_1|-|z_2|$ (d) $\ge|z_1|-|z_2|$ **(Kumaun 2010)**

15. $\arg z+\arg\bar{z}=$
(a) 0 (b) $n\pi$
(c) $\frac{n\pi}{2}$ (d) $2n\pi$
(Kumaun 2014)

Fill in the Blank(s)

Fill in the blanks "............" so that the following statements are complete and correct.

1. If $z = x + iy$ is any complex number then the non-negative real number .... is called the modulus of the complex number z.
2. Using exponential form if $z = re^{i\theta}$, then $\bar{z}$ = ...
3. A complex number $z = x + iy$ is purely imaginary if and only if $z + \bar{z}$ = ...
4. The value of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the .... value of the argument.
5. If two complex numbers z_1 and z_2 are represented in the Argand diagram, then the distance between the points z_1 and z_2 is ....
6. In an Argand plane the equation of the straight line joining the points z_1 and z_2 is $z = tz_1 +$ ......, where t is a real parameter.
7. In an Argand plane the affix of the point dividing the join of the points z_1 and z_2 internally in the ratio $m_1 : m_2$ is ....
8. In an Argand plane the equation of the circle whose centre is the point $2 + 3i$ and whose radius is 5 is ....
9. In an Argand plane the centre of the circle $|2z - 4 + 6i| = 7$ is the point whose affix is ....
10. In an Argand plane the radius of the circle $|3z - (6 + 4i)| = 9$ is ....
11. In an Argand plane the equation of the straight line joining the origin to the middle point of the points $2 + 4i$ and $6 + 8i$ is $z = t$ (......), where t is a real parameter.
12. In an Argand plane the equation of the circle described on the line segment joining the points z_1 and z_2 as diameter is $(z - z_1)(\bar{z} - \bar{z}_2) +$ ...... $= 0$.
13. If $e^z = a + ib$, where a and b are real, then b = ...
14. If $\cos(x + iy) = a + ib$, where a and b are real, then a = ...
15. If $\cos(x + iy) = a + ib$, where a and b are real, then b = ...

True or False

Write 'T' for true and 'F' for false statement.

1. The formula $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$ is valid only if at least one of the numbers a and b is non-negative.
2. The modulus of the sum of two complex numbers can exceed the sum of their moduli.
3. The modulus of the difference of two complex numbers can be less than the difference of their moduli.
4. Argument of negative real number is $\pm\pi$.
5. The order relations greater than or less than do not apply to complex numbers.
6. If z is any complex number, then $z\bar{z} = |z|$.

7. In an Argand plane $|z+i|=|z-i|$ represents a straight line.
8. In an Argand plane $|z+i|=2|z-i|$ represents a straight line.
9. In an Argand plane $|z+i|=2|z-i|$ represents a circle.
10. In an Argand plane the centre of the circle $|z-3-4i|=5$ has the affix $3+4i$.
11. In an Argand plane the radius of the circle $|5z-4+3i|=25$ is 25.
12. If $\sin(x-iy)=a+ib$, where a and b are real, then $b=\cos x \sinh y$.
13. If $\sin(\alpha+i\beta)=x+iy$, then $\dfrac{x^2}{\sin^2\alpha}+\dfrac{y^2}{\cos^2\alpha}=1$.

Answers

Multiple Choice Questions

1.	(c)	2.	(b)	3.	(a)	4.	(b)	5.	(d)
6.	(b)	7.	(b)	8.	(b)	9.	(b)	10.	(a)
11.	(c)	12.	(d)	13.	(d)	14.	(d)	15.	(d)

Fill in the Blank(s)

1. $\sqrt{(x^2+y^2)}$
2. $re^{-i\theta}$
3. 0
4. principal
5. $|z_1-z_2|$
6. $(1-t)z_2$
7. $\dfrac{m_1 z_2+m_2 z_1}{m_1+m_2}$
8. $|z-(2+3i)|=5$
9. $2-3i$
10. 3
11. $4+6i$
12. $(z-z_2)(\bar{z}-\bar{z}_1)$
13. $e^x \sin y$
14. $\cos x \cosh y$
15. $-\sin x \sinh y$

True or False

1.	*T*	2.	*F*	3.	*F*	4.	*T*	5.	*T*
6.	*F*	7.	*T*	8.	*F*	9.	*T*	10.	*T*
11.	*F*	12.	*F*	13.	*F*				

□

Chapter

2

Analytic Functions

1 Curves in the Argand Plane

We know that the equations of the type $x = x(t)$, $y = y(t)$, where t is the parameter, give the parametric representation of a curve in the plane. Using the complex variable z, these equations can be written as a single equation

$$z = z(t) = x(t) + i\,y(t) \text{ where } z = x + iy.$$

Definitions: (a) *In the Argand plane, a continuous complex valued function $z(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real valued continuous functions of a real variable t, defined in the range $\alpha \le t \le \beta$ where $\alpha < \beta$ is called a continuous **arc** or a **curve**.* We call $z(\alpha)$ and $z(\beta)$ the end points of the curve, $z(\alpha)$ is the initial point and $z(\beta)$ the terminal point of the curve. If $z(\alpha) = z(\beta)$ *i.e.,* if the initial and terminal points of a curve coincide, the curve is said to be a **closed curve.**

A point z_1 is a multiple point of the curve if the equation $z_1 = x(t) + iy(t)$ is satisfied by more than one value of t in the given range. In particular the multiple point is called a double point if the above equation is satisfied by two values of t in the given range.

(b) A curve Γ given by, $z(t) = x(t) + iy(t), \alpha \le t \le \beta$ is called a **Jordan arc** or a **simple curve** if $t_1 \ne t_2$ implies $z(t_1) \ne z(t_2)$ *i.e.*, $z(t)$ is one-one. A Jordan arc is a curve without multiple points.

(c) A closed curve Γ given by $z(t) = x(t) + iy(t), \alpha \le t \le \beta$, is called **simple** if $t_1 < t_2$ and $z(t_1) = z(t_2)$ imply $t_1 = \alpha$ and $t_2 = \beta$.

We usually refer to such curves as **simple closed Jordan curves.**

Illustration: The circle $z = \cos t + i \sin t, 0 \le t \le 2\pi$ is a simple closed Jordan curve since the values of $z(t)$ coincide only at the end points $t = 0$ and $t = 2\pi$.

The Jordan Curve Theorem: The theorem states that *a simple closed Jordan curve divides the Argand plane into two open domains which have the curve as common boundary.*

Of these two domains one is bounded and it is called the **interior domain** ; the other is unbounded and is called the **exterior domain.** For example, the circle $|z| = r$ divides the Argand plane into two open domains given by $|z| < r$ and $|z| > r$. Out of these $|z| < r$ is bounded and is the interior of the circle ; the other $|z| > r$ is unbounded and is the exterior of the circle, $|z| = r$. The circle is the common boundary of the two domains.

2 Functions of a Complex Variable

If certain rules be given by means of which it is possible to find one or more complex numbers w for every value of z in a certain domain D, w is said to be a function of z defined on the domain D and we write

$$w = f(z).$$

Since $z = x + iy$, $f(z)$ will be of the form $u + iv$ where u and v are functions of two real variables x and y. Thus we can write $w = u(x, y) + iv(x, y)$, x, y are real.

Single valued and multiple valued functions: If w takes only one value for each value of z in the region D, w is said to be a *uniform* or *single valued function* of z. If there correspond two or more values of w for some or all values of z in the region D, w is called *multiple valued function* of z.

3 Neighbourhood of a Point

A neighbourhood of a point z_0 in the Argand plane is the set of all points z such that $|z - z_0| < \delta$, where δ is an arbitrary small positive number. The number δ is called the radius of this neighbourhood.

Deleted neighbourhood: If from a neighbourhood of a point z_0, the point z_0 itself is deleted or excluded, we get a *deleted neighbourhood* of z_0.

4 Limit and Continuity

Let $f(z)$ be any function of the complex variable z defined in a bounded and closed domain D. Then l is said to be the limit of $f(z)$ as z approaches a along any path in D if for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number δ greater than zero such that

$$|f(z) - l| < \varepsilon,$$

for all values of z for which $0 < |z - a| < \delta$.

In symbols, we write $\lim_{z \to a} f(z) = l$.

Continuity: *A function $f(z)$ of a complex variable z defined in the closed and bounded domain D is said to be continuous at $a \in D$ if and only if for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number $\delta > 0$ such that*

$$|f(z) - f(a)| < \varepsilon \text{ whenever } |z - a| < \delta.$$

It follows from the definitions of limit and continuity that $f(z)$ is continuous at

$$z = a \text{ iff } \lim_{z \to a} f(z) = f(a).$$

We say that $f(z)$ is a continuous function in a domain D if it is continuous at every $z \in D$.

We can easily show that $f(z) = u(x, y) + i\,v(x, y)$ is a continuous function of z iff u and v are continuous functions of x and y.

Uniform continuity: *A function $f(z)$ defined in a domain D is said to be uniformly continuous in D if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$, where z_1 and z_2 are points in D.*

It should be noted carefully that uniform continuity is a property associated with a domain and not with a single point of it.

If $f(z)$ is uniformly continuous in a domain D, it is continuous in D.

A function which is continuous in a closed and bounded domain D is uniformly continuous in D, whereas a function continuous in an open domain D' may fail to be uniformly continuous in the domain D'.

5 Differentiability

Since the mode of definitions of continuity is the same both in case of the functions of the real and complex variables therefore definition of differentiability of a complex function is identical with that of the real function.

Let $w = f(z)$ be a function of a complex variable z defined in a domain D. Then $f(z)$ is said to be differentiable at a point z_0 of D iff

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ or } \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ *exists uniquely and finitely*

and this limit, if it exists finitely, *is called the* **differential coefficient** or **derivative** of f with respect to z at $z = z_0$.

It is denoted by $f'(z_0)$ or by $Df(z_0)$.

If the value of the above limit as $z \to z_0$ is not unique *i.e.*, if the **limit depends upon amp** Δz, we say that the derivative of $f(z)$ at $z = z_0$ does not exist or the function $f(z)$ is **non-differentiable** at $z = z_0$.

Hence if we have to show that $f(z)$ is non-differentiable, we should try different paths for Δz. Convenient paths for Δz are along real and imaginary axes *i.e.*, we can take Δz either **wholly real** or **wholly imaginary.**

Note: Since the derivative of a complex function has been defined in the same manner as the derivative of a function of a single real variable, therefore all the rules of differential calculus remain the same when applied to complex functions.

(Meerut 2002)

Theorem 1: *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.*

Proof: Let $f(z)$ be differentiable at z_0. Then,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and equals $f'(z_0)$.

Now we can write

$$f(z) - f(z_0) = (z - z_0) \frac{f(z) - f(z_0)}{z - z_0}, \text{ if } z \neq z_0.$$

Taking limit of both sides as $z \to z_0$, we get

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \left[(z - z_0) \frac{\{f(z) - f(z_0)\}}{z - z_0} \right]$$

$$= \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= f'(z_0) . 0 = 0,$$

so that $$\lim_{z \to z_0} f(z) = f(z_0).$$

Hence $f(z)$ is continuous at z_0. Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative.

The following example illustrates this fact :

Illustrative Examples

Example 1: *Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at origin.* **(Gorakhpur 2015)**

Solution: Let $f(z) = |z|^2$ where $z = x + iy$.

This function is continuous at every point because $x^2 + y^2$ is continuous at all points.

Now
$$f'(a) = \lim_{\Delta z \to 0} \frac{f(a + \Delta z) - f(a)}{\Delta z} \lim_{\Delta z \to 0} \frac{|a + \Delta z|^2 - |z|^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{(a + \Delta z)(\bar{a} + \Delta \bar{z}) - a\bar{a}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \left[\frac{a\Delta \bar{z} + \bar{a}\Delta z + \Delta z \Delta \bar{z}}{\Delta z}\right]$$
$$= \lim_{\Delta z \to 0} \left[a \frac{\Delta \bar{z}}{\Delta z} + \bar{a} + \Delta \bar{z}\right]$$
$$= \lim_{\Delta z \to 0} \left[a \frac{\Delta \bar{z}}{\Delta z} + \bar{a}\right] \qquad [\because \Delta \bar{z} \to 0 \text{ as } \Delta z \to 0]$$

$\because$ at $a = 0$, we have $\bar{a} = 0$, therefore $f'(a) = 0$.

Again at $a \neq 0$, let $\Delta z = r(\cos \phi + i \sin \phi)$.

Then $\qquad \Delta \bar{z} = r(\cos \phi - i \sin \phi)$.

$$\therefore \qquad \frac{\Delta \bar{z}}{\Delta z} = \frac{\cos \phi - i \sin \phi}{\cos \phi + i \sin \phi} = \cos 2\phi - i \sin 2\phi,$$

which does not tend to a unique limit as $\Delta z \to 0$ since this limit depends upon arg Δz.

Thus $f(z)$ is not differentiable for any non-zero value of z, though it is continuous everywhere.

Theorem 2. Rules of differentiation: *If $f(z)$ and $g(z)$ are analytic functions in a domain D, then their sum, product and quotient {provided $g(z) \neq 0$} are also analytic and we have*

(i) $\dfrac{d}{dz}[f(z) \pm g(z)] = \dfrac{d}{dz} f(z) \pm \dfrac{d}{dz} g(z)$

(ii) $\dfrac{d}{dz}[cf(z)] = c \dfrac{d}{dz} f(z)$

(iii) $\dfrac{d}{dz}[f(z)\, g(z)] = f(z) \dfrac{d}{dz} g(z) + g(z) \dfrac{d}{dz} f(z)$

(iv) $\dfrac{d}{dz}[f(z) / g(z)] = \dfrac{\left[\dfrac{d}{dz} f(z)\right] g(z) - \left[\dfrac{d}{dz} g(z)\right] f(z)}{[g(z)]^2}, [g(z) \neq 0]$

(v) *If $f(z) = F[g(z)]$, then* $\dfrac{d}{dz} f(z) = F'[g(z)]\, g'(z)$. [Chain rule]

6 Analytic, Holomorphic and Regular Functions

Let $f(z)$ be a single valued function defined in a domain D. Then $f(z)$ is said to be analytic at a point z_0 of D if it is differentiable not only at z_0 but also in some neighbourhood of z_0.

(Gorakhpur 2002, 03, 04, 07, 10, 16; Rohilkhand 10; Bundelkhand 11; Purvanchal 10)

A single valued function which is differentiable at each point of a domain D is said to be analytic in the domain D.

A function, which is analytic, is also called a **Holomorphic function.**

If a function $f(z)$ is analytic at some point in every neighbourhood of a point z_0 except at z_0 itself, then z_0 is called an **isolated singularity** of $f(z)$.

A function $f(z)$ defined in a domain D is said to have **removable singularity** at a point z_0 of D if $f(z)$ is not analytic at z_0 but can be made analytic by simply assigning a suitable value to the function $f(z)$ at the point z_0.

A function $f(z)$ is said to be **regular** at a point z_0 if it has a removable singularity at z_0 and if $f(z)$ is analytic in some deleted neighbourhood of z_0. Some authors use the term regular as a synonym for analytic.

7 Properties of Analytic Functions

If $f(z)$ and $g(z)$ are two analytic functions in a domain D, then

(i) $f(z) \pm g(z)$

(ii) $f(z) \,.\, g(z)$

(iii) $\dfrac{f(z)}{g(z)}$ *provided $g(z) \neq 0$ at any point of D*

and (iv) $k\, f(z)$ where k is any constant are also analytic in D.

8 The Necessary and Sufficient Conditions for $f(z)$ to be Analytic. Cauchy Riemann Equations. (Cartesian Form)

(Rohilkhand 2008, 11; Kumaun 09; Purvanchal 10)

(a) The necessary condition for $f(z)$ to be analytic:

Theorem 1: *If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at any point $z = x + iy$, the partial derivatives u_x, v_x, u_y, v_y should exist and satisfy the equations $u_x = v_y, u_y = -v_x$.*

(Meerut 2001; Garhwal 10; Gorakhpur 04, 10, 13)

Proof: Let $w = f(z) = u(x, y) + iv(x, y)$

We have $z = x + iy$, then $\Delta z = \Delta x + i\Delta y$. ...(1)

Since the function is differentiable at any point z, therefore the limit given by

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

must exist uniquely as $\Delta z \to 0$ along any path we choose.

Using relations (1) the above limit can be written as

$$\lim_{\Delta z \to 0} \left[\frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i \Delta y} + i \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i \Delta y} \right]. \quad \ldots(2)$$

Taking Δz to be wholly real, we get $\Delta y = 0$. In this case the limit given by (2) becomes

$$\lim_{\Delta x \to 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right]$$

$= \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$, since $f(z)$ is differentiable therefore the partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ must also exist

$$= u_x + i v_x . \quad \ldots(3)$$

Again taking Δz to be wholly imaginary, we get $\Delta x = 0$. In this case the limit given in (2) becomes

$$\lim_{\Delta y \to 0} \left[\frac{u(x, y + \Delta y) - u(x, y)}{i \Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i \Delta y} \right]$$

$= \dfrac{1}{i} \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}$, since $f(z)$ is differentiable, therefore $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ also exist

$$= - i u_y + v_y = v_y - i u_y . \quad \ldots(4)$$

Since the limit given by $\lim_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z}$ is unique, therefore equating real and imaginary parts of (3) and (4), we get

$$\boldsymbol{u_x = v_y, \; v_x = - u_y .}$$

These two equations are known as **Cauchy-Riemann partial differential equations.**

(b) Sufficient condition for $f(z)$ to be analytic:

Theorem 2: *The single valued continuous function $f(z)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy Cauchy-Riemann equations at each point of D.*

Proof: Let $w = f(z) = u(x, y) + i v(x, y)$.

We have $u = u(x, y)$, so that $u + \Delta u = u(x + \Delta x, y + \Delta y)$

$\therefore$ $\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$

$$= u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y) + u(x + \Delta x, y) - u(x, y)$$

$$= \Delta y \, u_y (x + \Delta x, y + \theta_1 \Delta y) + \Delta x \, u_x (x + \theta_2 \Delta x, y), \quad \ldots(1)$$

where $0 < \theta_1 < 1, \; 0 < \theta_2 < 1$, by the mean value theorem.

Since u_x and u_y are continuous in the given domain D, therefore by the definition of uniform continuity, we have

$$|u_y(x+\Delta x, y+\theta_1 \Delta y) - u_y(x, y)| < \varepsilon$$

and $\quad |u_x(x+\theta_2\, \Delta x, y) - u_x(x, y)| < \varepsilon, \qquad \ldots(2)$

provided $\quad |\Delta x| < \delta \quad \text{and} \quad |\Delta y| < \delta.$

Let $\quad u_y(x+\Delta x, y+\theta_1 \Delta y) - u_y(x, y) = \alpha_1$

and $\quad u_x(x+\theta_2\, \Delta x, y) - u_x(x, y) = \beta_1.$

Then from (2), we have $|\alpha_1| < \varepsilon_1, |\beta_1| < \varepsilon_1$.

Putting these values in (1), we get

$$\Delta u = \{\alpha_1 + u_y(x, y)\}\, \Delta y + \{\beta_1 + u_x(x, y)\}\, \Delta x.$$

Proceeding in the same way, we shall get

$$\Delta v = \{\alpha_2 + v_y(x, y)\}\, \Delta y + \{\beta_2 + v_x(x, y)\}\, \Delta x,$$

where $\quad |\alpha_2| < \varepsilon_2\,, |\beta_2| < \varepsilon_2$.

Now

$$\frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$= \frac{(u_y\, \Delta y + u_x\, \Delta x + \alpha_1\, \Delta y + \beta_1\, \Delta x) + i(v_y\, \Delta y + v_x\, \Delta x + \alpha_2\, \Delta y + \beta_2\, \Delta x)}{\Delta x + i\Delta y}$$

$$= \frac{-v_x\, \Delta y + u_x\, \Delta x + iu_x\, \Delta y + \alpha_1\, \Delta y + \beta_1\, \Delta x + i\alpha_2\, \Delta y + i\beta_2\, \Delta x}{\Delta x + i\Delta y}$$

$$[\because \ u_x = v_y\,,\ v_x = -u_y]$$

$$= \frac{(u_x + iv_x)(\Delta x + i\Delta y) + \alpha_1\, \Delta y + \beta_1\, \Delta x + i\alpha_2\, \Delta y + i\beta_2\, \Delta x}{\Delta x + i\Delta y}$$

$$= u_x + iv_x + \frac{(\alpha_1 + i\alpha_2)\, \Delta y}{\Delta x + i\Delta y} + \frac{(\beta_1 + i\beta_2)\, \Delta x}{\Delta x + i\Delta y}$$

or

$$\left|\frac{\Delta f}{\Delta z} - (u_x + iv_x)\right| = \left|\frac{(\alpha_1 + i\alpha_2)\, \Delta y}{\Delta x + i\Delta y} + \frac{(\beta_1 + i\beta_2)\, \Delta x}{\Delta x + i\Delta y}\right|$$

$$\leq \frac{|\alpha_1 + i\alpha_2|\, |\Delta y|}{|\Delta x + i\Delta y|} + \frac{|\beta_1 + i\beta_2|\, |\Delta x|}{|\Delta x + i\Delta y|}$$

$$\leq |\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2|$$

$$\{\because \ |\Delta x| \leq |\Delta x + i\Delta y| \text{ and } |\Delta y| \leq |\Delta x + i\Delta y|\}$$

$\therefore$

$$\left|\frac{\Delta f}{\Delta z} - (u_x + iv_x)\right| \leq 2\varepsilon_1 + 2\varepsilon_2.$$

Hence,

$$u_x + iv_x = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = f'(z).$$

Note: (i) $$|f'(z)|^2 = |u_x|^2 + |v_x|^2$$

$$= |u_x|^2 + |v_x|^2 = u_x v_y - v_x u_y, \text{ using Cauchy-Riemann equations}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}$$

= Jacobian of u and v with respect to x and y.

(ii) Lagrange's mean value theorem:

If $f(x)$ is defined and continuous in $a \le x \le a + h$, differentiable in $a < x < a + h$, then there exists a point $c = a + \theta h, (0 < \theta < 1)$ in $]\, a, a + h\, [$ such that

$$f(a + h) - f(a) = h\, f'(a + \theta h).$$

(iii) From theorem 2, we have

$$f'(z) = u_x + iv_x \quad \Rightarrow \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{dw}{dz} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial w}{\partial x}$$

Hence $$\frac{dw}{dz} = \frac{\partial w}{\partial x}.$$

An Important Observation:

Since $x = \frac{1}{2}(z + \bar{z}), y = \frac{1}{2i}(z - \bar{z}), u$ and v can be regarded as functions of two independent variables z and $\bar{z}$. If u and v have first order continuous derivatives, the condition that w shall be **independent of** $\bar{z}$ is

$$\frac{\partial w}{\partial \bar{z}} = 0 \quad \text{or} \quad \frac{\partial}{\partial \bar{z}}(u + iv) = 0$$

or $$\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) = 0$$

or $$\frac{1}{2}\cdot\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} = 0$$

or $$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

Whence equating real and imaginary parts to zero, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

which are the Cauchy-Riemann equations.

It follows that if $f(z)$ is an analytic function of z, then x and y can occur in $f(z)$ only in the combination of $x + iy$.

Illustrative Examples

Example 2: *Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous as well as analytic everywhere.*

Solution: Let $u(x, y) = \sin x \cosh y, v(x, y) = \cos x \sinh y$. Here u and v are both rational functions of x and y having non-zero denominators for all values of x and y, therefore u and v are both continuous everywhere.

Hence $f(z)$ is continuous everywhere.

We have $$\frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y$$

and $$\frac{\partial v}{\partial x} = -\sin x \sinh y, \frac{\partial v}{\partial y} = \cos x \cosh y.$$

These relations show $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

$\therefore$ u and v satisfy Cauchy-Riemann equations.

Thus $f(z)$ is analytic everywhere.

Example 3: *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin although the Cauchy-Riemann equations are satisfied at that point.*

(Meerut 2012; Kanpur 03; Rohilkhand 12; Purvanchal 10, 12; Agra 12)

Solution: Let $f(z) = u(x, y) + iv(x, y)$.

Then $$u(x, y) = \sqrt{|xy|}, \ v(x, y) = 0.$$

At the origin, we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0.$$

Similarly $$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0.$$

Hence Cauchy-Riemann equations are satisfied at origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}.$$

Suppose $z \to 0$ along $y = mx$, then we have

$$f'(0) = \lim_{z \to 0} \frac{\sqrt{|mx^2|}}{x + i\,mx} = \lim_{z \to 0} \frac{\sqrt{|m|}}{(1 + im)},$$

which is not unique, since it depends on m.

$\therefore$ $f'(0)$ does not exist.

Example 4: *Show that* $\dfrac{d}{dz}(\bar{z})$ *does not exist anywhere.*

(Kanpur 2000)

Solution: We have $z = x + iy$. Then $\bar{z} = x - iy$.

Now
$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{(z + \Delta z)} - \bar{z}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\overline{(x + iy + \Delta x + i\,\Delta y\,)} - \overline{(x + iy)}}{\Delta x + i\,\Delta y}$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x + \Delta x) - i\,(y + \Delta y) - (x - iy)}{\Delta x + i\,\Delta y}$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} \qquad \text{...(1)}$$

Taking $\Delta z \to 0$ along real axis, we get $\Delta y = 0$. In this case the limit given by (1) becomes 1.

Again taking $\Delta z \to 0$ along imaginary axis, we get $\Delta x = 0$. In this case the limit given by (1) becomes -1. Since the value of the limit given by (1) is not unique so $\dfrac{d}{dz}(\bar{z})$ does not exist. Hence $f(z) = \bar{z}$ is not analytic anywhere.

Example 5: *Find whether the following functions are analytic.*

(i) $f(z) = \bar{z}$ *(ii)* $f(z) = e^{z}$

(iii) $f(z) = \cos x \sin y + i \sin x \cos y$.

Solution: (i) We have $f(z) = u + iv = \bar{z} = \overline{x + iy} = x - iy$.

$\therefore \quad u = x, \quad v = -y.$

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial y} = -1.$$

We see that $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}\cdot$

Hence one of the Cauchy-Riemann equations is not satisfied.

$\therefore \quad f(z) = \bar{z}$ is not analytic.

(ii) We have $f(z) = u + iv = e^{z} = e^{x + iy} = e^{x} \,.\, e^{iy}$
$$= e^{x}(\cos y + i \sin y) = e^{x} \cos y + i\, e^{x} \sin y.$$

$\therefore \quad u = e^{x} \cos y, \quad v = e^{2x} \sin y.$

$$\therefore \quad \frac{\partial u}{\partial x} = e^{x} \cos y, \ \frac{\partial v}{\partial x} = e^{x} \sin y, \frac{\partial u}{\partial y} = -\,e^{x} \sin y, \ \frac{\partial v}{\partial y} = e^{x} \cos y.$$

We see that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}\cdot$

Hence Cauchy-Riemann equations are satisfied.

Also $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ exist and are continuous functions.

Hence $f(z) = e^z$ is an analytic function.

(iii) We have $f(z) = u + iv = \cos x \sin y + i \sin x \cos y$.

$\therefore$ $u = \cos x \sin y, \; v = \sin x \cos y.$

Now
$$\frac{\partial u}{\partial x} = -\sin x \sin y, \; \frac{\partial v}{\partial x} = \cos x \cos y,$$
$$\frac{\partial u}{\partial y} = \cos x \cos y, \; \frac{\partial v}{\partial y} = -\sin x \sin y.$$

We see that $\dfrac{\partial v}{\partial x} \neq -\dfrac{\partial u}{\partial y}$.

$\therefore$ One of the Cauchy-Riemann equations is not satisfied.

Hence $f(z) = \cos x \sin y + i \sin x \cos y$ is not analytic.

Example 6: *If $w = \log z$, find dw/dz and determine where w is non-analytic.*

Solution: Let
$$w = u(x, y) + iv(x, y)$$
$$= \log(x + iy)$$
$$= \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\frac{y}{x}$$

Then
$$u(x, y) = \frac{1}{2}\log(x^2 + y^2), \; v(x, y) = \tan^{-1}\frac{y}{x}$$

Now
$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \; \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}.$$
$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} \text{ and } \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}.$$

Since $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ except at origin therefore u and v satisfy Cauchy-Riemann equations and all the partial derivatives are continuous except at origin. Hence the function w is analytic everywhere except at origin.

Now
$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}$$
$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}, \text{ provided } z \neq 0.$$

Example 7: *If $w = f(z) = \dfrac{1 + z}{1 - z}$, find $\dfrac{dw}{dz}$ and determine where $f(z)$ is non-analytic.*

Solution: We have $w = f(z) = \dfrac{1 + z}{1 - z}$.

Now
$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{1+z+\Delta z}{1-(z+\Delta z)} - \dfrac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{[1-(z+\Delta z)](1-z)} = \frac{2}{(1-z)^2},$$

which exists for all finite values of z except $z = 1$.

Hence the function $f(z)$ is analytic for all finite values of z except $z = 1$.

Example 8: *Verify whether the real and imaginary parts of $w = \sin z$ satisfy Cauchy-Riemann equations.*

Solution: We have $w = u + iv = \sin z$

$$= \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

$\therefore \qquad u = \sin x \cosh y \quad$ and $\quad v = \cos x \sinh y.$

$$\therefore \qquad \frac{\partial u}{\partial x} = \cos x \cosh y, \; \frac{\partial u}{\partial y} = \sin x \sinh y,$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \; \frac{\partial v}{\partial y} = \cos x \cosh y.$$

We see that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

Hence, u and v satisfy Cauchy-Riemann equations.

9 Polar Form of Cauchy-Riemann Equations

(Purvanchal 2007, 09; Gorakhpur 07, 09, 11)

We have $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \qquad x^2 + y^2 = r^2 \qquad \ldots(1)$$

$$\text{and} \qquad \theta = \tan^{-1}(y/x). \qquad \ldots(2)$$

Differentiating (1) and (2) partially w.r.t. x and y, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \; \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\text{and} \qquad \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.$$

$$\text{Now} \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r},$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}$$

and $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}.$$

Cauchy-Riemann equations in cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Using the above relations, we get

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \quad \ldots(3)$$

and $$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}. \quad \ldots(4)$$

Multiplying (3) by $\cos\theta$ and (4) by $\sin\theta$ and adding, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}. \quad \ldots(5)$$

Again multiplying (3) by $\sin\theta$ and (4) by $\cos\theta$ and subtracting, we get

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \quad \ldots(6)$$

Equations (5) and (6) are the Cauchy-Riemann equations in polar form.

10 Derivative of $w = f(z)$ in Polar Form

We have $$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\frac{\partial r}{\partial x} + \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\partial \theta}{\partial x} \qquad [\because\ w = u + iv]$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta + \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\left(-\frac{\sin\theta}{r}\right), \qquad \text{[From article 9]}$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \left(-r\frac{\partial v}{\partial r} + i\,r\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r},$$

[From (5) and (6) of article 9]

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\theta - i\sin\theta)$$

$$= \frac{\partial w}{\partial r}e^{-i\theta}.$$

Illustrative Examples

Example 9: *Show that the function $f(z) = z^n$, where n is a positive integer is an analytic function.*

Solution: We have $f(z) = z^n$.

Now
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1}\Delta z + \frac{1}{2}n(n-1)z^{n-2}(\Delta z)^2 + \ldots + (\Delta z)^n - z^n}{\Delta z},$$

by binomial theorem

$$= \lim_{\Delta z \to 0} [nz^{n-1} + \frac{1}{2}n(n-1)z^{n-2}\Delta z + \ldots + (\Delta z)^{n-1}]$$

$$= nz^{n-1}.$$

$\therefore$ $f'(z)$ exists for all finite values of z.

Hence $f(z)$ is an analytic function.

Note: Applying the above formula for $z, z^2, z^3, \ldots$ and the rules of differentiation stated in theorem II we see that a **polynomial**

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

is an analytic function of z. More generally, **a rational function**

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m}$$

is an analytic function of z throughout any finite domain in the complex plane where the denominator does not vanish.

Example 10: *If* $f(z) = \dfrac{x^3 y(y - ix)}{x^6 + y^2}, z \neq 0$ *and* $f(0) = 0$, *show that* $\dfrac{f(z) - f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner.*

(Purvanchal 2008; Bundelkhand 11; Gorakhpur 11)

Solution: We have $\dfrac{f(z) - f(0)}{z} = \dfrac{f(z) - 0}{z} = \dfrac{f(z)}{z}$

$$= \frac{x^3 y(y - ix)}{(x^6 + y^2)(x + iy)} = \frac{-ix^3 y(x + iy)}{(x^6 + y^2)(x + iy)} = -i\frac{x^3 y}{x^6 + y^2}.$$

Let $z \to 0$ along the radius vector $y = mx$. Then we have

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-i x^3 . mx}{x^6 + (mx)^2} = \lim_{x \to 0} \frac{-i mx^2}{x^4 + m^2} = 0.$$

Now let $z \to 0$ along $y = x^3$. Then

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-i x^3 . x^3}{x^6 + (x^3)^2} = -\frac{i}{2} \neq 0.$$

Hence the result.

Example 11: *Show that the function* $f(z) = u + iv$,

where $$f(z) = \frac{x^3 (1+i) - y^3 (1-i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ *does not exist.* **(Meerut 2001; Rohilkhand 08; Garhwal 10; Gorakhpur 07, 11, 14)**

Solution: We have $u = \dfrac{x^3 - y^3}{x^2 + y^2}$, $v = \dfrac{x^3 + y^3}{x^2 + y^2}$, $z \neq 0$.

Here u and v are rational functions of x and y with non-zero denominators. Therefore u and v are continuous everywhere when $z \neq 0$.

To test the continuity at $z = 0$, changing u and v to polars by putting $x = r \cos \theta$, $y = r \sin \theta$, we get $u = r(\cos^3 \theta - \sin^3 \theta)$, $v = r(\cos^3 \theta + \sin^3 \theta)$, each of which tends to 0 as $r \to 0$ whatever may be the value of θ. Also we have $f(0) = 0$.

Since actual and limiting values of u and v are same at origin therefore $f(z)$ is continuous at origin. Hence $f(z)$ is continuous for all values of z.

Now at the origin, we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1,$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1,$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

Since $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, therefore u and v satisfy Cauchy-Riemann equations at origin.

We have $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)} \cdot \frac{1}{(x + iy)}.$$

Taking $z \to 0$ along $y = x$, we get

$$f'(0) = \lim_{x \to 0} \frac{i2x^3}{2x^2} \cdot \frac{1}{(x + ix)} = \frac{i}{1+i} = \frac{1}{2}(1+i).$$

Again taking $z \to 0$ along x-axis, we get

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = 1 + i.$$

Since $f'(0)$ has different values along different curves therefore $f'(0)$ is not unique. So $f'(0)$ does not exist.

Example 12: *Show that the function* $f(z) = e^{-z^{-4}}, z \neq 0$ *and* $f(0) = 0$ *is not analytic at* $z = 0$ *although the Cauchy-Riemann equations are satisfied at that point.*

(Meerut 2003; Rohilkhand 10; Purvanchal 07, 11; Kumaun 10; Gorakhpur 12, 14)

Solution: We have $f(z) = e^{-z^{-4}} = e^{-1/(x+iy)^4}$

$$= e^{-(x-iy)^4/(x^2+y^2)^4}$$

$$= e^{-(x^4+y^4-6x^2y^2-4ix^3y+4ixy^3)/(x^2+y^2)^4}$$

$$= e^{-(x^4+y^4-6x^2y^2)/r^8} \; e^{4ixy(x^2-y^2)/r^8}, \text{ where } x^2+y^2=r^2$$

$$= e^{-(x^4+y^4-6x^2y^2)/r^8}\left[\cos\frac{4xy(x^2-y^2)}{r^8} + i\sin\frac{4xy(x^2-y^2)}{r^8}\right].$$

We have at the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{e^{-x^{-4}} - 0}{x}$$

$$= \lim_{x \to 0} \frac{1}{x\, e^{1/x^4}} = \lim_{x \to 0} \frac{1}{x\left[1 + \frac{1}{x^4} + \frac{1}{2x^8} + \dots\right]}$$

$$= \lim_{x \to 0} \frac{1}{x + \frac{1}{x^3} + \frac{1}{2x^7} + \dots} = 0.$$

Similarly $$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{e^{-y^{-4}}}{y} = 0,$$

$$\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

$\therefore$ Cauchy-Riemann equations are satisfied at origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{e^{-z^{-4}}}{z}$$

$$= \lim_{r \to 0} \frac{e^{-r^{-4}} e^{-i\pi}}{re^{i\pi/4}}, \text{ taking } z \to 0 \text{ along } z = re^{i\pi/4}$$

$$= \lim_{r \to 0} \frac{e^{r^{-4}}}{r\, e^{i\pi/4}} = \lim_{r \to 0} \frac{e^{1/r^4}}{r\, e^{i\pi/4}} \to \infty.$$

$\therefore$ $f(z)$ is not analytic at $z = 0$.

Example 13: *Find the analytic function whose real part is* $\sin 2x / (\cosh 2y - \cos 2x)$.

Solution: Let $f(z) = u + iv$ be the required analytic function.

Then $$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

Now $$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \phi_1(x, y)$$

and $$\frac{\partial u}{\partial y} = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \phi_2(x, y)$$

The function $f(z)$ is given by

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c = \int \left[\frac{2\cos 2z - 2}{(1 - \cos 2z)^2} - i0\right] dz + c$$

$$= \int -\frac{2}{1 - \cos 2z}\, dz + c = \int -\operatorname{cosec}^2 z\, dz + c$$

$$= \cot z + c.$$

11 Orthogonal System

Two families of curves $u(x, y) = c_1$ *and* $v(x, y) = c_2$ *are said to form an orthogonal system if they intersect at right angles at each of their points of intersection.*

Differentiating $u(x, y) = c_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1, \text{ say}$$

Similarly from $v(x, y) = c_2$, we get

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2, \text{ say}$$

Now the two families of curves will intersect orthogonally if

$$m_1 m_2 = -1 \qquad \text{or} \qquad \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0.$$

Theorem: *If $f(z) = u + iv$ be an analytic function of $z = x + iy$, prove that the families of curves $u = c_1, v = c_2$ are orthogonal to each other, where c_1 and c_2 are parameters.*

(Rohilkhand 2008; Purvanchal 10)

Proof: It is given that $f(z) = u + iv$ is an analytic function therefore we have

$$u_x = v_y, v_x = -u_y. \qquad \text{...(1)}$$

The given systems are

$$u(x, y) = c_1, \qquad \text{...(2)}$$

$$v(x, y) = c_2. \qquad \text{...(3)}$$

Let m_1 and m_2 be the slopes of curves (2) and (3) respectively.

From (2) $\quad \dfrac{dy}{dx} = -\dfrac{\partial u / \partial x}{\partial u / \partial y} = m_1.$

From (3) $\quad \dfrac{dy}{dx} = -\dfrac{\partial v / \partial x}{\partial v / \partial y} = m_2.$

Now $\quad m_1 \, m_2 = \left(-\dfrac{\partial u / \partial x}{\partial u / \partial y} \right) \times \left(-\dfrac{\partial v / \partial x}{\partial v / \partial y} \right) = -1,$ from (1).

Hence the systems $u = c_1$ and $v = c_2$ are orthogonal to each other.

Illustrative Examples

Example 14: *If $f(z) = u + iv$ is an analytic function, regular in D, where $f(z) \neq 0$, prove that the curves $u = const., v = const.$, form two orthogonal families. Verify this in case of $f(z) = \sin z$.*

(Kumaun 2008)

Solution: (i) Suppose $f(z) = u + iv$ is an analytic function of z.

To prove that $u = \text{const.}, v = \text{const.}$ from two orthogonal families.

Prove this as in Theorem.

(ii) To verify the result (i) by taking $f(z) = \sin z$.

$$u + iv = f(z) = \sin z = \sin(x + iy)$$

$$= \sin x . \cos iy + \cos x . \sin iy$$

$$= \sin x . \cosh y + i \cos x . \sinh y$$

$$\therefore \qquad u = \sin x . \cosh y = c_1, \text{ say}$$

$$v = \cos x . \sinh y = c_2, \text{ say.}$$

Differentiating both w.r.t. x, we get

$$\cos x . \cosh y + \sin x . \sinh y \left(\frac{dy}{dx} \right)_1 = 0$$

and $\quad -\sin x \,.\, \sinh y + \cos x \,.\, \cosh y \left(\dfrac{dy}{dx}\right)_2 = 0$

or $\quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{\cos x \,.\, \cosh y}{-\sin x \,.\, \sinh y}$ and $\left(\dfrac{dy}{dx}\right)_2 = \dfrac{\sin x \,.\, \sinh y}{\cos x \cosh y}$.

Multiplying these two, we get

$$\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1.$$

Hence the verification follows.

12 Harmonic Function

Theorem 1: *Real and imaginary parts of an analytic function satisfy Laplace's equation.*

(Rohilkhand 2010)

Proof: Let $f(z) = u + iv$ be an analytic function. Then it satisfies Cauchy-Riemann equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots(1)$$

Since u and v are the real and imaginary parts of an analytic function therefore partial derivatives of u and v of all orders exist and are continuous functions of x and y.

From (1), we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly $\quad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0.$

Therefore the functions u and v satisfy the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Definition: *Any function of x and y possessing continuous partial derivatives of the first and second orders and satisfying Laplace's equation is called a harmonic function.*

Harmonic Conjugate of a function:

Definition: Let $u(x, y)$ be a harmonic function. Then a function $v(x, y)$ is said to be a harmonic conjugate of $u(x, y)$ if

(i) $v(x, y)$ is harmonic and

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ *i.e.,* u and v satisfy Cauchy-Riemann equations.

Theorem 2: *If $f(z) = u + iv$ is analytic in a domain D, then v is the harmonic conjugate of u. Conversely, if v is the harmonic conjugate of u in a domain D, then $f(z) = u + iv$ is analytic in D.*

Proof: Since $f(z) = u + iv$ is analytic in D, Cauchy-Riemann equations are satisfied *i.e.,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Differentiating partially with respect to x and y respectively and adding, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0.$$

Similarly differentiating partially with respect to y and x respectively and adding, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore$ u and v are harmonic functions in D and v is the harmonic conjugate of u because u and v satisfy Cauchy-Riemann equations.

Conversely, let v be the harmonic conjugate of u.

Then by the definition of the harmonic conjugate of u, v is harmonic and Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied. Also by the definition of harmonic functions u and v possess continuous partial derivatives of the first and second orders so that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous functions.

Hence, $f(z) = u + iv$ is analytic in D.

Remark: It is very important to note that if v is a harmonic conjugate of u in some domain D, then it is always not true that u is also the harmonic conjugate of v in D.

We illustrate this by the following example :

Let $\quad u = x^2 - y^2 \quad$ and $\quad v = 2xy.$

Then $f(z) = u + iv$ is analytic in D as shown below.

We have $\quad \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., Cauchy-Riemann equations are satisfied by u and v.

Also $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous functions.

$\therefore$ $f(z) = u + i\,v$ is analytic in D.

Hence, both u and v are harmonic functions and they satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\therefore$ v is the harmonic conjugate of u.

But if we define $\phi(z) = v + iu$, we see that

$$\frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x, \quad \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y.$$

We see that $\dfrac{\partial v}{\partial x} \neq \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y} \neq -\dfrac{\partial u}{\partial x}$.

Thus if $\phi(z) = v + iu$, then v and u do not satisfy Cauchy-Riemann equations.

$\therefore$ $\phi(z)$ is not analytic in D.

Hence, u is not the harmonic conjugate of v.

Theorem 3: *Two functions $u(x, y)$ and $v(x, y)$ are harmonic conjugates of each other if and only if they are constants.*

Proof: Let $u(x, y) = c_1$ and $v(x, y) = c_2$ $\forall$ $x, y \in D$, where c_1 and c_2 are constants.

Then
$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial^2 v}{\partial x^2} = 0,$$
$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus both u and v are harmonic functions.

Also
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., u and v satisfy Cauchy-Riemann equations.

Hence v is the harmonic conjugate of u.

Again
$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

i.e., v and u also satisfy Cauchy-Riemann equations.

Hence u is the harmonic conjugate of v.

Thus if both u and v are constants, they are harmonic conjugates of each other.

Conversely, let $u(x, y)$ and $v(x, y)$ be two harmonic functions such that they are harmonic conjugates of each other.

Then u and v satisfy Cauchy-Riemann equations *i.e.*,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(1)}$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Again v and u also satisfy Cauchy-Riemann equations *i.e.*,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \ldots(3)$$

and $$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \ldots(4)$$

From (1) and (4), we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x}$$

$\Rightarrow$ $$2\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u \text{ is independent of } x.$$

From (2) and (3), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \Rightarrow 2\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow u \text{ is independent of } y.$$

$\therefore$ $u(x, y)$ is independent of both x and y and consequently $u(x, y)$ is a constant function.

Similarly, we can show that $v(x, y)$ is also a constant function.

Hence if both u and v are harmonic conjugates of each other, they are constant functions.

Determination of the conjugate function:

If $f(z) = u + iv$ is an analytic function, u and v are called conjugate functions. Being given one of these say, $u(x, y)$, to determine the other $v(x, y)$.

We have $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$, since v is a function of x and y

or $$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy, \quad \ldots(1)$$

by Cauchy-Riemann equations.

The equation (1) is of the form $dv = M\,dx + N\,dy$,

where $$M = -\frac{\partial u}{\partial y}, \quad N = \frac{\partial u}{\partial x}.$$

Now $$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}.$$

Since $f(z)$ is an analytic function therefore u is a harmonic function *i.e.*, it satisfies Laplace's equation.

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

so that $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus equation (1) satisfies the condition of exact differential equation. Therefore v can be determined by integrating (1).

Illustrative Examples

Example 15: *Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.* **(Lucknow 2007, 13B, 14; Kumaun 14)**

Solution: We have $u = x^3 - 3xy^2$.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial^2 u}{\partial x^2} = 6x, \frac{\partial u}{\partial y} = -6xy \text{ and } \frac{\partial^2 u}{\partial y^2} = -6x.$$

Now $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, so that u satisfies Laplace's equation.

Also first and second order partial derivatives of u are continuous functions of x and y. Consequently u is a harmonic function.

Now $$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - i(-6xy)$$

$$= 3(x^2 - y^2 + 2ixy) = 3(x + iy)^2 = 3z^2.$$

Integrating $$f(z) = z^3 + c.$$

Example 16: *Show that the functions*

(i) $u = \frac{1}{2}\log(x^2 + y^2)$ **(Rohilkhand 2012; Garhwal 10)**

(ii) $u = \cos x \cosh y$ **(Kumaun 2012)**

are harmonic, find their harmonic conjugates.

Solution: (i) We have $u = \frac{1}{2}\log(x^2 + y^2)$.

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial^2 u}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

All the first and second order partial derivatives of u are continuous functions of x and y.

Also $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *i.e.*, u satisfies Laplace's equation.

$\therefore$ u is a harmonic function.

Let v be the harmonic conjugate of u.

We have $$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy, \text{ by Cauchy-Riemann equations}$$

$$= -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

or $$dv = \frac{x\,dy - y\,dx}{x^2+y^2}.$$

Integrating, we get $v = \tan^{-1}\frac{y}{x} + c$, where c is a real constant.

(ii) We have $u = \cos x \cosh y$.

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y \text{ and } \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y.$$

All the first and second order partial derivatives of u are continuous functions.

Also $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, *i.e.*, u satisfies Laplace's equation.

$\therefore$ u is a harmonic function.

Let v be the harmonic conjugate of u.

We have $$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy, \quad \text{by Cauchy-Riemann equations}$$

$$= -\cos x \sinh y\,dx - \sin x \cosh y\,dy$$

$$= -(\cos x \sinh y\,dx + \sin x \cosh y\,dy).$$

Integrating, $v = -(\sin x \sinh y) + c$, where c is a real constant.

Example 17: *Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$.* **(Bundelkhand 2011)**

Solution: Here $u = e^x \cos y$.

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

so $$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \text{ and } \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so u satisfies Laplace's equation.

Also first and second order partial derivatives of u are continuous therefore u is a harmonic function.

Since v is the harmonic conjugate of u, therefore

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy, v \text{ is a function of } x, y$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy, \quad \text{by Cauchy-Riemann equations}$$

$$= e^x \sin y \, dx + e^x \cos y \, dy.$$

Integrating $v = e^x \sin y + c$, where c is a real constant.

$\therefore$
$$f(z) = u + iv = e^x \cos y + i (e^x \sin y + c).$$

$$= e^x (\cos y + i \sin y) + i c = e^{x + iy} + d,$$

where d is a complex constant

$$= e^z + d.$$

13 Milne-Thomson's Method (Method of Constructing a Regular Function)

(Meerut 2002)

We have $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$.

Then $x = \frac{1}{2}(z + \bar{z}), \; y = \frac{1}{2i}(z - \bar{z})$.

We can write

$$f(z) = u\left[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right] + iv\left[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right]. \qquad \ldots(1)$$

This relation can be regarded a formal identity in two independent variables z and $\bar{z}$.

Putting $z = \bar{z}$ in (1), we get

$$f(z) = u(z, 0) + iv(z, 0).$$

We have $f'(z) = \frac{\partial f}{\partial x} = u_x + iv_x = u_x - iu_y$, by Cauchy-Riemann equations.

Let $u_x = \phi_1(x, y), u_y = \phi_2(x, y)$.

Then $f'(z) = \phi_1(x, y) - i\phi_2(x, y) = \phi_1(z, 0) - i\phi_2(z, 0)$.

Integrating, we get

$$f(z) = \int \phi_1(z, 0)\, dz - i \int \phi_2(z, 0)\, dz + c,$$

where c is constant of integration.

Similarly if v is given, we have

$$f(z) = \int \psi_1(z, 0)\, dz + i \int \psi_2(z, 0)\, dz + c',$$

where $v_y = \psi_1(x, y), \; v_x = \psi_2(x, y)$.

Theorem: *If the real part of an analytic function $f(z)$ is a given harmonic function $u(x, y)$,*

$$f(z) = 2u(z/2, z/2i) - u(0,0).$$

Proof: Let $f(z) = f(x+iy) = u(x, y) + iv(x, y)$.

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x, y) - iv(x, y)$.

Adding, we get

$$f(x+iy) + \overline{f(x+iy)} = 2u(x, y). \qquad \ldots(1)$$

Since $\overline{f(z)}$ is a function independent of z, therefore it can be regarded as a function of $\bar{z}$. So we can write

$$\overline{f(z)} = \bar{f}(\bar{z}).$$

We can rewrite relation (1) as

$$u(x, y) = \frac{1}{2}[f(x+iy) + \bar{f}(x-iy)]. \qquad \ldots(2)$$

We can regard (2) as a formal identity, therefore it holds even when x and y are complex. Putting $x = z/2$, $y = z/2i$, we get

$$u(z/2, z/2i) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right]$$

$$= \frac{1}{2}[f(z) + \bar{f}(0)].$$

$\therefore$ $$f(z) = 2u(z/2, z/2i) - \bar{f}(0).$$

Since $f(z)$ is only determined upto a purely imaginary constant, we may assume that $f(0)$ is real. So we have

$$\bar{f}(0) = u(0,0).$$

$\therefore$ $$f(z) = 2u(z/2, z/2i) - u(0,0).$$

Adding a purely imaginary constant, we have

$$f(z) = 2u(z/2, z/2i) - u(0,0) + ci, \text{ where } c \text{ is real.}$$

Illustrative Examples

Example 18: *If* $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$,

find the corresponding analytic function $f(z) = u + iv$.

Solution: Here $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

Then $$\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y + \cos 2x) - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2 + 2\cos 2x \cosh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_1(x, y).$$

$$\frac{\partial u}{\partial y} = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x, y).$$

By Milne-Thomson's method the function $f(z)$ is given by

$$f(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\,dz + c$$

$$= \int \left[\frac{2 + 2\cos 2z}{(1 + \cos 2z)^2} - i0\right] dz + c = \int \frac{2\,dz}{1 + \cos 2z} + c$$

$$= \int \sec^2 z\,dz + c$$

$$= \tan z + c.$$

Example 19: *Find the analytic function whose real part is given and hence find the imaginary part :*

(i) $e^x \sin y$ *(ii)* $\sin x \cosh y$ *(iii)* $x^2 - y^2$.

Solution: (i) Let $f(z) = u + iv$ be analytic.

Here, $u = e^x \sin y$.

$\therefore$ $\dfrac{\partial u}{\partial x} = e^x \sin y, \quad \dfrac{\partial u}{\partial y} = e^x \cos y.$

We apply the Milne-Thomson method to find $f(z)$.

Since $f(z) = u + iv$ is analytic, therefore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{[See Note (iii) at the end of article 8]}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left[\because \text{By Cauchy-Riemann equations, } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$= e^x \sin y - i e^x \cos y.$$

Putting $x = z$ and $y = 0$, we get

$$f'(z) = e^z \sin 0 - i e^z \cos 0 = -i.e^z.$$

Integrating with respect to z, we get

$$f(z) = -i e^z + \text{constant}$$

$$= -i e^{x + iy} + \text{constant}$$

$$= -i e^x e^{iy} + \text{constant}$$

$$= -i e^x (\cos y + i \sin y) + \text{constant}$$

$$= e^x \sin y - i e^x \cos y + ic$$

$$= u + i(c - e^x \cos y)$$

$\therefore$ $v = c - e^x \cos y.$

Hence, the required analytic function is

$$f(z) = u + iv = -ie^{z} + \text{constant}$$
$$= e^{x} \sin y + i(c - e^{x} \cos y)$$

and the imaginary part $v = c - e^{x} \cos y$.

(ii) Let $\quad f(z) = u + iv$ be analytic.

Here, $\quad u = \sin x \cosh y$.

$$\therefore \quad \frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y.$$

We apply the Milne-Thomson method to find $f(z)$.

Since $f(z) = u + iv$ is analytic, therefore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[Using Cauchy-Riemann equations]}$$
$$= \cos x \cosh y - i \sin x \sinh y.$$

Putting $x = z$ and $y = 0$, we get

$$f'(z) = \cos z \cosh 0 - i \sin z \sinh 0$$
$$= \cos z \qquad [\because \cosh 0 = 1, \sinh 0 = 0]$$

Integrating with respect to z, we get

$$f(z) = \sin z + \text{constant}$$
$$= \sin(x + iy) + \text{constant}$$
$$= \sin x \cos iy + \cos x \sin iy + \text{constant}$$
$$= \sin x \cosh y + i \cos x \sinh y + ic$$
$$= \sin x \cosh y + i(\cos x \sinh y + c)$$
$$= u + iv.$$

Hence, the required analytic function is

$$f(z) = u + iv = \sin z + \text{constant}$$
$$= \sin x \cosh y + i(\cos x \sinh y + c)$$

and the imaginary part $v = \cos x \sinh y + c$.

(iii) Let $f(z) = u + iv$ be analytic.

Here, $\quad u = x^2 - y^2$.

$$\therefore \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y.$$

We apply the Milne-Thomson method to find $f(z)$.

Since $f(z) = u + iv$ is analytic, therefore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \text{by Cauchy-Riemann equations}\right]$$

$$= 2x + 2iy.$$

Putting $x = z$ and $y = 0$, we get

$$f'(z) = 2z.$$

Integrating with respect to z, we get

$$f(z) = z^2 + \text{constant}$$

$$= (x + iy)^2 + \text{constant}$$

$$= x^2 - y^2 + 2ixy + ic$$

$$= (x^2 - y^2) + i(2xy + c) = u + iv.$$

Hence the required analytic function is $f(z) = z^2 + \text{constant} = (x^2 - y^2) + i(2xy + c)$ and the imaginary part $v = 2xy + c$.

Example 20: *Prove that the following functions are harmonic and find the harmonic conjugate.*

(i) $2x - x^3 + 3xy^2$ *(ii)* $e^{-x}(x \cos y + y \sin y)$. **(Agra 2012)**

Solution: (i) Let $u = 2x - x^3 + 3xy^2$.

Then $$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2, \frac{\partial u}{\partial y} = 6xy,$$

$$\frac{\partial^2 u}{\partial x^2} = -6x, \frac{\partial^2 u}{\partial y^2} = 6x.$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0.$$

$\therefore$ u is harmonic.

Let v be the harmonic conjugate of u.

Then $f(z) = u + iv$ is analytic.

$\therefore$ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left[\because \text{By Cauchy-Riemann equations } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$= 2 - 3x^2 + 3y^2 - i6xy.$$

To apply Milne-Thomson's method, putting $x = z$ and $y = 0$, we get

$$f'(z) = 2 - 3z^2.$$

Integrating with respect to z, we get

$$f(z) = 2z - z^3 + \text{a constant}$$

$$= 2(x+iy) - (x+iy)^3 + \text{a constant}$$

$$= 2x + i2y - x^3 + 3xy^2 - 3x^2 y i + iy^3 + \text{a constant}$$

$$= (2x - x^3 + 3xy^2) + i(2y - 3x^2 y + y^3) + ic$$

$$= u + i(2y - 3x^2 y + y^3 + c).$$

$\therefore$ $$v = 2y - 3x^2 y + y^3 + c.$$

Hence the harmonic conjugate of u is $v = 2y - 3x^2 y + y^3 + c$.

(ii) Let $u = e^{-x}(x\cos y + y\sin y)$.

Then $$\frac{\partial u}{\partial x} = e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x})$$

$$= e^{-x}(\cos y - x\cos y - y\sin y),$$

$$\frac{\partial u}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y),$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(-\cos y) + (\cos y - x\cos y - y\sin y)(-e^{-x})$$

$$= e^{-x}(-\cos y - \cos y + x\cos y + y\sin y)$$

$$= e^{-x}(x\cos y + y\sin y - 2\cos y),$$

and $$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\cos y + \cos y - y\sin y + \cos y)$$

$$= e^{-x}(-x\cos y - y\sin y + 2\cos y).$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(x\cos y + y\sin y - 2\cos y - x\cos y - y\sin y + 2\cos y) = 0.$$

$\therefore$ u is harmonic.

Let v be the harmonic conjugate of u.

Then $f(z) = u + iv$ is analytic.

$\therefore$ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left[\because \text{By Cauchy-Riemann equations, } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$= e^{-x}(\cos y - x\cos y - y\sin y) - ie^{-x}(-x\sin y + y\cos y + \sin y).$$

To apply Milne-Thomson's method putting $x = z$ and $y = 0$, we get

$$f'(z) = e^{-z}(1 - z - 0) - ie^{-z}(0) = e^{-z}(1 - z).$$

Integrating with respect to z, we get

$$f(z) = (1 - z)(-e^{-z}) - \int -e^{-z}(-1)\,dz + \text{constant}$$

$$= -(1 - z)e^{-z} - \int e^{-z}\,dz + \text{constant}$$

$$= -(1 - z)e^{-z} - (-e^{-z}) + \text{constant}$$

$$= e^{-z}(-1 + z + 1) + \text{constant}$$

$$= e^{-x - iy}(x + iy) + \text{constant}$$

$$= e^{-x}(\cos y - i\sin y)(x + iy) + \text{constant}$$

$$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y) + ic$$

$$= u + i[e^{-x}(y\cos y - x\sin y) + c].$$

$\therefore$ $$v = e^{-x}(y\cos y - x\sin y) + c.$$

Hence the harmonic conjugate of u is

$$v = e^{-x}(y\cos y - x\sin y) + c.$$

Example 21: *Find the orthogonal trajectory of the family of curves*

$$x^2 - y^2 + x = c.$$

Solution: Let $u = x^2 - y^2 + x$.

Then $$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y.$$

Let $f(z) = u + iv$ be analytic.

Then $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left[\because \text{ By Cauchy-Riemann equations, } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$= 2x + 1 - i(-2y).$$

To apply Milne-Thomson's method putting $x = z$ and $y = 0$, we get

$$f'(z) = 2z + 1.$$

Integrating with respect to z, we get

$$f(z) = z^2 + z + \text{constant}$$

$$= (x + iy)^2 + (x + iy) + \text{constant}$$

$$= (x^2 - y^2 + x) + 2ixy + iy + ic$$

$$= u + i(2xy + y + c)$$

$$= u + iv, \text{ where } v = 2xy + y + c.$$

We know that if $f(z) = u + iv$ is analytic, then the orthogonal trajectory of the family of curves u = constant is the family of curves v = constant.

Hence, $2xy + y = c$ where c is an arbitrary constant, is the orthogonal trajectory of the family of curves $x^2 - y^2 + x = c$.

Example 22: *If* $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, *find the corresponding analytic function* $f(z) = u + iv$.

Solution: We have $f(z) = u + iv$, $i f(z) = iu - v$.

$\therefore$ $$(1 + i) f(z) = u - v + i(u + v) = U + iV, \text{ say.}$$

Here $$V = u + v = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

Let $$\frac{\partial V}{\partial y} = \psi_1(x, y) \quad \text{and} \quad \frac{\partial V}{\partial x} = \psi_2(x, y).$$

Then $$\psi_1(x, y) = \frac{\partial V}{\partial y} = -\frac{\sin 2x\,(2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

and $$\psi_2(x, y) = \frac{\partial V}{\partial x} = \frac{2 \cos 2x\,(\cosh 2y - \cos 2x) - 2 \sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2 \cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}.$$

Now $$\psi_1(z, 0) = -\frac{\sin 2z\,(2 \sinh 0)}{(\cosh 0 - \cos 2z)^2} = 0$$

and $$\psi_2(z, 0) = \frac{2 \cos 2z \cosh 0 - 2}{(\cosh 0 - \cos 2z)^2} = \frac{2 \cos 2z - 2}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{1 - \cos 2z} = -\operatorname{cosec}^2 z.$$

By Milne's method, we have

$$(1 + i) f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\, dz + c$$

$$= \int (0 - i \operatorname{cosec}^2 z)\, dz + c = i \cot z + c$$

$\therefore$ $$f(z) = \frac{i}{1 + i} \cot z + \frac{c}{1 + i}$$

$$= \frac{1}{2}(1 + i) \cot z + d.$$

Example 23: *If* $f(z) = u + iv$ *is an analytic function of* z *and* $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2 \cos x - e^{y} - e^{-y}}$, *find* $f(z)$ *subject to the condition* $f\left(\dfrac{\pi}{2}\right) = 0$.

Solution: Here $$u-v=\frac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}}$$

$$=\frac{1}{2}\left[1+\frac{2\cos x+2\sin x-2e^{-y}}{2\cos x-e^{y}-e^{-y}}-1\right]$$

$$=\frac{1}{2}\left[1+\frac{2\sin x+e^{y}-e^{-y}}{2\cos x-e^{y}-e^{-y}}\right]=\frac{1}{2}\left[1+\frac{\sin x+\sinh y}{\cos x-\cosh y}\right].$$

Now $$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=\frac{1}{2}\left[\frac{\cos x(\cos x-\cosh y)+(\sin x+\sinh y)\sin x}{(\cos x-\cosh y)^2}\right]$$

$$=\frac{1}{2}\left[\frac{1-\cos x\cosh y+\sin x\sinh y}{(\cos x-\cosh y)^2}\right] \quad \ldots(1)$$

and $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=\frac{1}{2}\left[\frac{\cosh y(\cos x-\cosh y)+\sinh y(\sin x+\sinh y)}{(\cos x-\cosh y)^2}\right]$$

$$=\frac{1}{2}\left[\frac{\cosh y\cos x+\sinh y\sin x-1}{(\cos x-\cosh y)^2}\right]$$

or $$-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}=\frac{1}{2}\left[\frac{\cosh y\cos x+\sinh y\sin x-1}{(\cos x-\cosh y)^2}\right], \quad \ldots(2)$$

by Cauchy-Riemann equations.

Solving (1) and (2), we get

$$\frac{\partial u}{\partial x}=\frac{1}{2}\left[\frac{1-\cos x\cosh y}{(\cos x-\cosh y)^2}\right]=\phi_1(x,y)$$

and $$\frac{\partial v}{\partial x}=-\frac{\sin x\sinh y}{2(\cos x-\cosh y)^2}=\phi_2(x,y).$$

$\therefore$ $$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\phi_1(z,0)+i\,\phi_2(z,0)$$

$$=\frac{1}{2}\frac{1-\cos z}{(\cos z-1)^2}=\frac{1}{2(1-\cos z)}=\frac{1}{4}\operatorname{cosec}^2\frac{z}{2}.$$

$\therefore$ $$f(z)=\frac{1}{4}\int\operatorname{cosec}^2\frac{1}{2}z\,dz+c=-\frac{1}{2}\cot\frac{1}{2}z+c.$$

At $z=\dfrac{\pi}{2}$, $$f(z)=0.$$

$\therefore$ $$c=f\left(\frac{\pi}{2}\right)+\frac{1}{2}\cot\frac{\pi}{4}=\frac{1}{2}.$$

$\therefore$ $$f(z)=\frac{1}{2}\left(1-\cot\frac{1}{2}z\right).$$

Example 24: *If $f(z) = u + iv$ be an analytic function in a domain D, prove that $f(z)$ is constant in D if any one of the following conditions holds:*

(i) *$f'(z)$ vanishes identically in D.*

(ii) *$R(f(z)) = u =$ constant.* **(Kanpur 2003)**

(iii) *$I(f(z)) = v =$ constant.* **(Garhwal 2000)**

(iv) *$|f(z)| =$ constant.*

(v) *$\arg f(z) =$ constant.*

Solution: Since $f(z) = u + iv$ is analytic in D, therefore it satisfies Cauchy-Riemann equations,

$$\left.\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.\right\} \qquad \ldots(1)$$

(i) We have $\quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$, from (1)

$\therefore$ If $f'(z) = 0$, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0.$$

$$\therefore \qquad \frac{\partial u}{\partial x} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

Thus u and v are constants and consequently $f(z)$ is a constant function.

(ii) $\mathbf{R}(f(z)) = u =$ constant

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}.$$

Now $\quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$, from (1)

$= 0$

$\therefore$ $f(z)$ is a constant function.

(iii) $\mathbf{I}(f(z)) = v =$ constant

$$\Rightarrow \qquad \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0.$$

Now $\quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x}$, from (1)

$= 0$

$\therefore$ $f(z)$ is a constant function.

(iv) $|f(z)| =$ constant $\Rightarrow$ $u^2 + v^2 =$ constant

$$\Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \text{ and } u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \text{ and } -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial x} = 0, \text{ provided } u^2 + v^2 \neq 0.$$

Therefore if $u^2 + v^2 \neq 0$, u and v are constants and consequently $f(z)$ is constant. If $u^2 + v^2 = 0$ at a single point, it is constantly zero and so $f(z)$ is zero. Hence, in either case $f(z)$ is constant.

(v) Here $\quad \arg f(z) = \tan^{-1}\dfrac{v}{u}$.

$\arg f(z) = c$ (constant)

$$\Rightarrow \quad \tan^{-1}\frac{v}{u} = c \Rightarrow (v/u) = \tan c$$

$$\Rightarrow \quad u = v\cot c \Rightarrow u = kv, \text{ taking } \cot c = k.$$

We observe $u - kv = 0$ unless v is identically zero. But $u - kv$ is the real part of $(1 + ik) f$, therefore it follows from part (ii) that $(1 + ik) f$ is constant. But $(1 + ik)$ is a constant, therefore f is also a constant.

Example 25: *If $f(z)$ is an analytic function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\mathbf{R}\, f(z)|^2 = 2|f'(z)|^2.$$

Solution: Let $f(z) = u + iv$, where $z = x + iy$.

Then $\quad \mathbf{R}\, f(z) = u.$

Now $\quad \dfrac{\partial u^2}{\partial x} = 2u\dfrac{\partial u}{\partial x}$

and $\quad \dfrac{\partial^2 u^2}{\partial x^2} = 2\left(\dfrac{\partial u}{\partial x}\right)^2 + 2u\dfrac{\partial^2 u}{\partial x^2}$.

Similarly $\quad \dfrac{\partial^2 u^2}{\partial y^2} = 2\left(\dfrac{\partial u}{\partial y}\right)^2 + 2u\dfrac{\partial^2 u}{\partial y^2}$.

$$\therefore \quad \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]$$

or $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + 0,$$

since u is a harmonic function

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right],$$

using Cauchy-Riemann equations

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^2 = 2|f'(z)|^2, \text{ since } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Hence, $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\mathbf{R}\, f(z)|^2 = 2|f'(z)|^2.$$

Comprehensive Exercise 1

1. (i) Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic. **(Rohilkhand 2011; Purvanchal 11, 12)**

   (ii) If n is real, show that $r^n(\cos n\theta + i\sin n\theta)$ is analytic except possibly when $r=0$ and that its derivative is $nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$.

   (Kumaun 2015)

2. Show that the function $e^x(\cos y + i\sin y)$ is holomorphic and find its derivative. **(Lucknow 2006, 13)**

3. If $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$, $f(0)=0$, prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

   (Gorakhpur 2007; Purvanchal 12)

4. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$, $z \neq 0$, $f(0)=0$ in a region including the origin. **(Meerut 2002; Gorakhpur 09, 13)**

5. Show that an analytic function with constant argument is constant.

6. Show that an analytic function with constant modulus is constant.

   Or **(Kumaun 2009)**

   Show that an analytic function cannot have a constant modulus without reducing to a constant.

7. For what values of z the function w defined by $z = e^{-v}(\cos u + i\sin u)$, where $w = u + iv$ ceases to be analytic ?

8. Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function $f(z)$ in terms of z.

   (Purvanchal 2010, 12)

9. If $u = e^x (x \cos y - y \sin y)$, find the analytic function $u + iv$.

(Kanpur 2003, 11, 13; Gorakhpur 05; Rohilkhand 09, 11; Purvanchal 10; Meerut 12)

10. If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. **(Meerut 2001; Gorakhpur 06, 09, 13)**

11. Prove that if $u = x^2 - y^2, v = -y/(x^2 + y^2)$ both u and v satisfy Laplace's equation but $u + iv$ is not an analytic function of z. **(Agra 2012)**

12. Show that the function $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace's equation and find the corresponding analytic function $u + iv$.

13. Find the analytic function of which the real part is
$$e^{-x} \{(x^2 - y^2) \cos y + 2x \sin y\}.$$
(Purvanchal 2007; 09)

14. Construct the analytic function $f(z) = u + iv$, where
    (i) $u = x^3 - 3xy^2 + 3x + 1$, **(Gorakhpur 2007)**
    (ii) $u = y^3 - 3x^2 y$.

15. Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine the corresponding analytic function. **(Meerut 2000)**

16. If $f(z) = u + iv$ is analytic function and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z. **(Garhwal 2000; Purvanchal 09; Gorakhpur 11; Agra 12)**

17. If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.

(Rohilkhand 2008; Purvanchal 08; Gorakhpur 10)

18. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$.

(Gorakhpur 2008)

19. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and ψ any function of x and y with differential coefficients of the first and second orders, prove that

(i) $$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right\} |f'(z)|^2$$

(ii) $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right) |f'(z)|^2.$$

20. If $f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2$.

(Kanpur 2001)

21. If $w = f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f'(z)| = 0$.

(Kanpur 2002; Kumaun 07, 13)

If $|f'(z)|$ is the product of a function of x and a function of y, show that

$$f'(z) = \exp(\alpha z^2 + \beta z + \gamma),$$

where α is a real and β, γ are complex constants. **(Rohilkhand 2012)**

Answers 1

7. $z = 0$

9. $f(z) = ze^z + c$

10. $v = 3x^2 y - 6xy + 3y - y^3 + c$

13. $f(z) = (x + iy)^2 \cdot e^{-x} [\cos y - i \sin y] + c$

14. (i) $z^3 + 3z + 1 + ci$, (ii) $i(z^3 + c)$

16. $f(z) = e^z + c$

17. $f(z) = -iz^3 + d$

18. $f(z) = \cot \frac{1}{2} z + \frac{1}{2}(1 - i)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Cauchy-Riemann equations for $w = u + iv = f(z)$ are :

(a) $u_x = v_x, u_y = v_y$
(b) $u_x = v_y, u_y = v_x$
(c) $u_x = v_y, u_y = -v_x$
(d) none of these.

(Rohilkhand 2011; Kumaun 11)

2. Which of the following is not correct for analytic functions $f(z)$ and $g(z)$ in a region R?

(a) $f(z) + g(z)$ is analytic in R
(b) $f(z) - g(z)$ is analytic in R
(c) $f(z)\, g(z)$ is analytic in R
(d) $f(z)/g(z)$ is analytic in R. **(Kumaun 2013)**

3. Which of the following is correct for $w = f(z)$?

(a) $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x}$ (b) $\dfrac{dw}{dz} = -\dfrac{\partial w}{\partial x}$

(c) $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial y}$ (d) $\dfrac{dw}{dz} = -\dfrac{\partial w}{\partial y}$. **(Kumaun 2007)**

4. The derivative of a function $w = f(z)$ in polar form is given by :

(a) $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial r} e^{i\theta}$ (b) $\dfrac{dw}{dz} = -\dfrac{\partial w}{\partial \theta} e^{i\theta}$

(c) $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial r} e^{-i\theta}$ (d) $\dfrac{dw}{dz} = -\dfrac{\partial w}{\partial \theta} e^{-i\theta}$.

5. Any function of x and y possessing continuous partial derivatives of the first and second orders is called a harmonic function if it satisfies :

(a) Euler's equation (b) Laplace's equation

(c) Lagrange's equation (d) none of these.

6. An analytic function with constant modulus is :

(a) variable (b) constant

(c) may be variable or constant (d) none of these.

7. If $f(z) = u + iv$ is analytic function in a finite region and $u = x^3 - 3xy^2$, then v is:

(a) $3x^2 y - y^3 + c$ (b) $3x^2 y^2 - y^3 + c$

(c) $3x^2 y - y^2 + c$ (d) none of these.

8. The analytic function whose real part is $e^x \cos y$ is :

(a) $e^z + d$ (b) e^{2z}

(c) xe^z (d) none of these.

9. The analytic function whose real part is $e^x (x \cos y - y \sin y)$ is :

(a) $ze^z + c$ (b) $z^2 e^z$

(c) $ze^{x^2 + iy}$ (d) none of these.

10. The function w defined by $z = e^{-v} (\cos u + i \sin u)$ ceases to be analytic at z where z is :

(a) 1 (b) 0

(c) ∞ (d) none of these.

11. Which of the following functions $f(z)$ is analytic and bounded where $f(z) =$

(a) $\sin z$ (b) $\cos z$

(c) any polynomial of degree more than one

(d) none of these

Fill in the Blank(s)

Fill in the blanks "............" so that the following statements are complete and correct.

1. If $f(z) = u + iv$ be an analytic function of $z = x + iy$, then the families of curves $u =$ constant, $v =$ constant are .... to each other.
2. The function $f(z) = \bar{z}$ is not .... at any point.
3. If harmonic functions u and v satisfy Cauchy-Riemann equations, then $f(z) = u + iv$ is an .... function.
4. The function $w = |z|^2$ is continuous everywhere but nowhere differentiable except at the ....
5. If $f(z) = u + iv$ is analytic and $u = \frac{1}{2} \log (x^2 + y^2)$ then v is ....
6. An analytic function with constant real part is ....
7. The function $f(z) = \dfrac{1}{z(z-3)}$ is not analytic at $z =$ ....

True or False

Write 'T' for true and 'F' for false statement.

1. Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.
2. If a function $f(z) = u + iv$ is analytic at any point $z = x + iy$, then Cauchy-Riemann equations are satisfied at that point.
3. The function $f(z)$ defined by $f(z) = \dfrac{x^3 y (y - ix)}{x^6 + y^2}$, $z \neq 0$

   and $f(0) = 0$ is differentiable at $z = 0$.
4. Real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfy Laplace's equation.
5. The function $f(z) = |xy|^{1/2}$ is analytic at $z = 0$.
6. The function $f(z) = \dfrac{1}{(z-1)(z-2)}$ is not differentiable at $z = 1, z = 2$.
7. Cauchy-Riemann equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Answers

Multiple Choice Questions

1.	(c)	**2.**	(d)	**3.**	(a)	**4.**	(c)	**5.**	(b)
6.	(b)	**7.**	(a)	**8.**	(a)	**9.**	(a)	**10.**	(b)
11.	(a)								

Fill in the Blank(s)

1.	orthogonal	**2.**	analytic	**3.**	analytic
4.	origin	**5.**	$\tan^{-1}\left(\frac{y}{x}\right)$	**6.**	constant
7.	0, 3				

True or False

1.	*T*	**2.**	*T*	**3.**	*F.*	**4.**	*T*	**5.**	*F*
6.	*T*	**7.**	*T*						

□

Chapter 3

Complex Integration

1 Introduction

We are familiar with the theory of integration of a real variable. In the case of a real variable, the integration is considered from two points of view, namely, the indefinite integration as a process inverse to differentiation and definite integration as the limit of a sum. There is a similar distinction between definite and indefinite integrals of a complex variable. As in the case of real variables, the concept of indefinite integral as the process inverse to differentiation also extends to a function of a complex variable. The indefinite integral of a complex variable is a function whose derivative equals a given analytic function in a region. However the concept of definite integral of a real variable does not extend straightway to the domain of complex variables. For example, in the case of real variables, *the path of integration of* $\int_a^b f(x)\,dx$ *is always along the real axis (x-axis) from* $x = a$ *to* $x = b$. But *for a complex function* $f(z)$, *the path of the definite integral* $\int_a^b f(z)\,dz$ *may be along any curve joining the points* $z = a$ *and* $z = b$, *i.e., the value of the integral depends upon the path of integration.* However, this variation in the value of definite integral will disappear in some special circumstances. Definite integrals of a complex variable are usually known as **line integrals.**

The theory of line integrals, along with the theory of power series and residues forms a very useful and important part of the theory of functions of a complex variable. These theories contain some of the most powerful theorems which have application in both pure and applied mathematics.

2 Definitions

(i) Partition: Consider a closed interval $[a, b]$, where a and b are real numbers.

Divide $[a, b]$ into n sub-intervals

$$[a = t_0, t_1], [t_1, t_2], \ldots\ldots, [t_{n-1}, t_n = b]$$

by inserting $n - 1$ intermediate points $t_1, t_2, \ldots\ldots, t_{n-1}$ satisfying

$$a < t_1 < t_2 < \ldots\ldots < t_n.$$

Then we call the set $\mathbf{P} = \{t_0, t_1, t_2, \ldots\ldots, t_n\}$

a partition of $[a, b]$. The greatest number among $t_1 - t_0, t_2 - t_1, \ldots\ldots, t_n - t_{n-1}$ is called the **norm** of the partition **P** and is denoted by $|\mathbf{P}|$.

(ii) Arcs and closed curves: We know that the equation

$$z = z(t) = x(t) + i\, y(t),$$

where $a \le t \le b$ and $x(t), y(t)$ are continuous functions, represents an arc L in the Argand plane, *i.e., an arc is the set of all image points of a closed finite interval under a continuous mapping.*

The equations $x = x(t), y = y(t)$ give the parametric representation of the arc in the plane.

If $z'(t)$ exists and is continuous, the arc L is said to be **differentiable** or **continuously differentiable**. If in addition to the existence of $z'(t)$, we also have $z'(t) \ne 0$, the arc L is said to be **regular** or **smooth.** Geometrically, at every point of a smooth arc there exists a tangent whose direction is determined by $\arg z'(t)$. As a matter of fact, as t increases from a to b, z continuously traces out the arc L and at the same time $\arg z'(t)$ varies continuously since $z'(t)$ changes continuously without vanishing.

If among various representations of an arc L there exists at least one representation, such that the interval $[a, b]$ can be divided into a finite number of sub-intervals

$$[a, a_1], [a_1, a_2], \ldots\ldots, [a_{n-1}, b]$$

on each of which $z'(t)$ exists, then the arc L is said to be **piecewise differentiable**. If in addition to this we also have $z'(t) \ne 0$ on any of these sub-intervals, the arc L is said to be **piecewise smooth**.

If $t_1 \ne t_2 \Rightarrow z(t_1) \ne z(t_2)$, the arc L is called **simple** or **Jordan arc.**

If the points corresponding to the values a and b coincide, the arc L is said to be a **closed curve**.

If the arc L is defined by $z = z(t), a \le t \le b$, then the arc defined by $z = z(-t), -b \le t \le -a$ is called the **opposite arc** of L and is denoted by $-L$ or L^{-1}.

3 Rectifiable Arcs

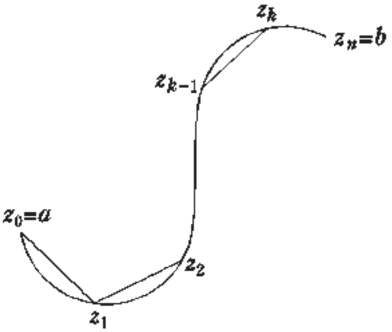
Consider the arc L defined by $z = z(t) = x(t) + i\,y(t), a \le t \le b$.

Let $\mathbf{P} = \{t_0, t_1, t_2, \ldots\ldots, t_n\}$ be any partition of $[a, b]$.

Corresponding to this partition, dividing the arc L into n sub-arcs

$$L_k = \text{arc } z_{k-1}\, z_k\,, (k = 1, 2, \ldots\ldots, n)$$

where $$z_k = z(t_k), (k = 0, 1, 2, \ldots\ldots, n).$$



Joining each of the points $z_0, z_1, z_2, \ldots\ldots, z_n$ to the next point by straight lines, we obtain a polygonal curve. The length of this polygonal curve is given by $\sum_{k=1}^{n} |z_k - z_{k-1}|$.

The arc L will be **rectifiable** if the least upper bound of the sum

$$|z_1 - z_0| + |z_2 - z_1| + \ldots\ldots + |z_n - z_{n-1}| \qquad \ldots(1)$$

taken over all partitions **P** is **finite**

i.e., if $$\sup_{\mathbf{P}} \sum_{k=1}^{n} |z_k - z_{k-1}| = l < \infty.$$

The non-negative real number l is called the **length** of the arc L. The arc L is said to be **non-rectifiable** if the sum (1) becomes arbitrarily large for suitably chosen partitions.

Contours: *A contour is a continuous chain of a finite number of regular arcs.*

If A is the starting point of the first arc and B the end point of the last arc, the integral of a function $f(z)$ along such a curve is written as $\int_{AB} f(z)\, dz$.

A contour is said to be ***closed*** *if it does not intersect itself and the starting point A of the first arc coincides with the end point B of the last arc.*

The integral along such closed contour C is written as $\int_C f(z)\, dz$. The boundaries of triangles and quadrilaterals are examples of closed contours.

Simply connected region and multiply connected region: A region in which every closed curve can be shrunk to a point without passing out of the region is called a **simply connected region** otherwise it is said to be **multiply connected.**

(Meerut 2002)

4 Functions of Bounded Variation

We can easily show that

$$|x(t_k) - x(t_{k-1})| \le |z(t_k) - z(t_{k-1})|,$$

$$|y(t_k) - y(t_{k-1})| \le |z(t_k) - z(t_{k-1})|,$$

$$|z(t_k) - z(t_{k-1})| = |x(t_k) - x(t_{k-1}) + i\{y(t_k) - y(t_{k-1})\}|$$

$$\le |x(t_k) - x(t_{k-1})| + |i||y(t_k) - y(t_{k-1})|$$

or $$|z(t_k) - z(t_{k-1})| \le |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})| \quad [\because |i| = 1]$$

From above inequalities we conclude that the sum

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \ldots\ldots + |z(t_n) - z(t_{n-1})|$$

and the sums

$$|x(t_1) - x(t_0)| + |x(t_2) - x(t_1)| + \ldots\ldots + |x(t_n) - x(t_{n-1})|,$$

$$|y(t_1) - y(t_0)| + |y(t_2) - y(t_1)| + \ldots\ldots + |y(t_n) - y(t_{n-1})|$$

are bounded at the same time.

The functions $x(t)$ and $y(t)$ are said to be of **bounded variation** if the latter two of the above three sums are bounded for all partitions **P** of $[a, b]$.

It can be easily proved that *an arc $z = z(t) = x(t) + i\,y(t)$, $a \le t \le b$ is rectifiable if and only if the functions $x(t)$ and $y(t)$ are of bounded variation in $[a, b]$.*

It is not hard to show that a smooth arc L is rectifiable and that its length l is given by the familiar formula

$$l = \int_a^b \sqrt{[x'(t)^2 + y'(t)^2]}\, dt \qquad \ldots(1)$$

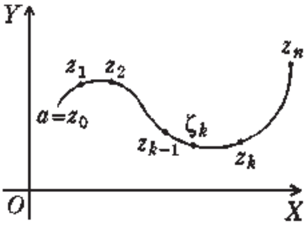
or equivalently by $l = \int_a^b |z'(t)|\, dt.$ ...(2)

5 Complex Integrals

(Meerut 2002)

Let $f(z)$ be a function of a complex variable z defined and continuous on an arc L, where L is a rectifiable arc defined by

$$z = z(t) = x(t) + i\,y(t), a \le t \le b.$$



Let there be any partition

$$\mathbf{P} = \{a = t_0, t_1, t_2, \ldots\ldots, t_n = b\} \text{ of } [a, b].$$

Form the sum

$$S_P = (z_1 - z_0) f(\zeta_1) + (z_2 - z_1) f(\zeta_2) + \ldots\ldots$$

$$+ (z_k - z_{k-1}) f(\zeta_k) + \ldots\ldots + (z_n - z_{n-1}) f(\zeta_n)$$

$$= \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k), \qquad \ldots(1)$$

where $z_k = z(t_k)$, $\zeta_k = z(\alpha_k)$, $t_{k-1} \le \alpha_k \le t_k$

and ζ_k is a point on each arc joining the points z_{k-1} to z_k.

Thus to form the sum S_P, we choose an arbitrary point ζ_k on each arc joining the points z_{k-1} to z_k and add the terms of the form $(z_k - z_{k-1}) f(\zeta_k)$, where k varies from 1 to n.
For convenience, we shall write $z_k - z_{k-1} = \Delta z_k$.
The function $f(z)$ is said to be integrable from a to b along the arc L if the sum S_P taken over all possible partitions **P** tends to a unique limit $\underline{\mathbf{I}}$ as $n \to \infty$ and $|\mathbf{P}| \to 0$.

$$\therefore \qquad \underline{\mathbf{I}} = \int_L f(z)\, dz = \lim_{\substack{n \to \infty \\ |\mathbf{P}| \to \infty}} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k).$$

$\int_L f(z)\, dz$ is called the **complex line integral** or simply line integral of $f(z)$ along L or the definite integral of $f(z)$ from a to b along L.

6 Evaluation of Some Integrals *ab-initio* (By Definition)

Illustrative Examples

Example 1: *Using the definition of an integral as the limit of a sum evaluate the following integrals*

(i) $\int_L dz$ *and* *(ii)* $\int_L z\, dz$,

where L *is any rectifiable arc joining the points* $z = \alpha$ *and* $z = \beta$.

Solution: Both the integrals exist since the integrand is a continuous function in each case.

(i) We have $\displaystyle\int_L dz = \lim_{n \to \infty} \sum_{k=1}^{n} (z_k - z_{k-1})$, by definition

$$= \lim_{n \to \infty} (z_1 - z_0 + z_2 - z_1 + \ldots\ldots + z_n - z_{n-1})$$

$$= \lim_{n \to \infty} (z_n - z_0) = \beta - \alpha. \qquad [\because z_0 = \alpha, z_n = \beta]$$

Note: If L is a closed curve, we have $\alpha = \beta$ and $\int_L dz = 0$.

(ii) We have $\displaystyle\int_L f(z)\, dz = \lim_{n \to \infty} \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$ by def.

$$\therefore \qquad \int_L z\, dz = \lim_{n \to \infty} \sum_{k=1}^{n} \zeta_k (z_k - z_{k-1}) \qquad \ldots(1)$$

where ζ_k is any point on the sub-arc joining z_{k-1} to z_k. Since ζ_k is arbitrary, therefore taking $\zeta_k = z_k$ and $\zeta_k = z_{k-1}$ successively in (1), we get

$$\int_L z\, dz = \lim_{n \to \infty} \sum_{k=1}^{n} z_k (z_k - z_{k-1})$$

and $$\int_L z\, dz = \lim_{n \to \infty} \sum_{k=1}^{n} z_{k-1} (z_k - z_{k-1}).$$

Adding these two integrals, we get

$$2 \int_L z\, dz = \lim_{n \to \infty} \sum_{k=1}^{n} (z_k + z_{k-1})(z_k - z_{k-1})$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} (z_k^2 - z_{k-1}^2)$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} (z_1^2 - z_0^2 + z_2^2 - z_1^2 + \ldots\ldots + z_n^2 - z_{n-1}^2)$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} (z_n^2 - z_0^2) = \beta^2 - \alpha^2, \text{ since } z_0 = \alpha,\ z_n = \beta.$$

$\therefore$ $$\int_L z\, dz = \frac{1}{2} (\beta^2 - \alpha^2).$$

If L is a closed curve, we have $\alpha = \beta$, so in this case $\int_L z\, dz = 0$.

Example 2: *Evaluate* $\int_L |dz|$ *(ab-initio)*

Solution: The above integral exists since the integrand is a continuous function. Here $f(z) = 1$ and we have $|dz|$ in place of dz.

We have $$\int_L |dz| = \lim_{n \to \infty} \sum_{k=1}^{n} |z_k - z_{k-1}|$$

$$= \lim_{n \to \infty} [|z_1 - z_0| + |z_2 - z_1| + \ldots\ldots + |z_n - z_{n-1}|]$$

$$= \lim_{n \to \infty} [\text{chord } z_1 z_0 + \text{chord } z_2 z_1 + \ldots\ldots + \text{chord } z_n z_{n-1}]$$

$$= \text{arc length of } L.$$

7 Reduction of Complex Integrals to Real Integrals

Theorem: *Let the arc L defined by*

$$z = z(t) = x(t) + i\, y(t),\ a \le t \le b$$

be continuously differentiable and let

$$f(z) = u(x, y) + i\, v(x, y)$$

be continuous over L. Then (i) L is rectifiable

(ii) $$\int_L f(z)\, dz = \int_a^b \{u(x(t), y(t))\, x'(t) - v(x(t), y(t))\, y'(t)\}\, dt$$

$$+ i \int_a^b \{v(x(t), y(t))\, x'(t) + u(x(t), y(t))\, y'(t)\}\, dt.$$

Proof: (i) Let $\mathbf{P} = \{a = t_0, t_1, t_2, \ldots\ldots, t_n = b\}$ be any partition of $[a, b]$.

We have $\displaystyle\sum_{k=1}^{n} |z_k - z_{k-1}| = \sum_{k=1}^{n} |z(t_k) - z(t_{k-1})|$, where $z_k = z(t_k)$

$$= \Sigma |x(t_k) - x(t_{k-1}) + i\{y(t_k) - y(t_{k-1})\}|,$$

since $z(t_k) = x(t_k) + i\, y(t_k)$

$$\leq \Sigma\, |x(t_k) - x(t_{k-1})| + \Sigma\, |y(t_k) - y(t_{k-1})|. \qquad \ldots(1)$$

Since $x(t)$ and $y(t)$ are continuously differentiable in $[a, b]$, therefore by Lagrange's mean value theorem there exist real numbers γ_k and δ_k in $]t_{k-1}, t_k[$ such that

$$x(t_k) - x(t_{k-1}) = (t_k - t_{k-1})\, x'(\gamma_k) \qquad \ldots(2)$$

$$y(t_k) - y(t_{k-1}) = (t_k - t_{k-1})\, y'(\delta_k). \qquad \ldots(3)$$

Again $z(t)$ is continuously differentiable in $[a, b]$, therefore the derivatives $x'(t)$and $y'(t)$ are continuous in $[a, b]$. Consequently $x'(t)$ and $y'(t)$ are bounded in $[a, b]$. Therefore there exists a real number M such that

$$|x'(t)| \leq M \text{ and}$$

$$|y'(t)| \leq M, \forall\, t \in [a, b]. \qquad \ldots(4)$$

From (1), (2), (3) and (4), we have

$$\Sigma |z_k - z_{k-1}| \leq \Sigma |t_k - t_{k-1}|\,|x'(\gamma_k)| + \Sigma |t_k - t_{k-1}|\,|y'(\delta_k)|$$

$$\leq \sum_{k=1}^{n} M |t_k - t_{k-1}| + \sum_{k=1}^{n} M |t_k - t_{k-1}| = 2M \sum_{k=1}^{n} |t_k - t_{k-1}|$$

$$= 2M(t_1 - t_0 + t_2 - t_1 + \ldots\ldots + t_n - t_{n-1})$$

$$= 2M(t_n - t_0) = 2M(b - a).$$

$$\therefore \quad \sum_{k=1}^{n} |z_k - z_{k-1}| \leq 2M(b - a). \qquad \ldots(5)$$

Thus we can say that $\displaystyle\sup_{\mathbf{P}} \sum_{k=1}^{n} |z_k - k_{k-1}| < \infty.$

Hence the arc L is rectifiable.

(ii) Consider the sum $S = \Sigma (z_k - z_{k-1})\, f(\zeta_k)$

where ζ_k is a point on the arc joining the points z_{k-1} and z_k. Let τ_k be the parameter of ζ_k. Then τ_k lies between z_{k-1} and z_k. We have

$$f[z(t)] = u[x(t), y(t)] + i\, v[x(t), y(t)]$$

$$= \phi(t) + i\,\psi(t).$$

Then $\quad \phi(t) = u[x(t), y(t)],\ \psi(t) = v[x(t), y(t)]. \qquad \ldots(6)$

Now $\quad S = \Sigma [x(t_k) + i\, y(t_k) - x(t_{k-1}) - iy(t_{k-1})][\phi(\tau_k) + i\,\psi(\tau_k)]$

$$= \Sigma\, [x(t_k) - x(t_{k-1}) + i\{y(t_k) - y(t_{k-1})\}][\phi(\tau_k) + i\,\psi(\tau_k)]$$

$$= \Sigma [(t_k - t_{k-1})\, x'(\gamma_k) + i(t_k - t_{k-1})\, y'(\delta_k)][\phi(\tau_k) + i\psi(\tau_k)],$$

from (2) and (3)

$$= \Sigma (t_k - t_{k-1})\, x'(\gamma_k)\, \phi(\tau_k) - (t_k - t_{k-1})\, y'(\delta_k)\, \psi(\tau_k)$$
$$+ i \Sigma (t_k - t_{k-1})\, x'(\gamma_k)\, \psi(\tau_k) + i \Sigma (t_k - t_{k-1})\, y'(\delta_k)\, \phi(\tau_k)$$
$$= S_1 + S_2 + i(S_3 + S_4), \qquad \ldots(7)$$

where $S_1 = \Sigma (t_k - t_{k-1})\, x'(\gamma_k)\, \phi(\tau_k)$,

$S_2 = - \Sigma (t_k - t_{k-1})\, y'(\delta_k)\, \psi(\tau_k)$,

$S_3 = \Sigma (t_k - t_{k-1})\, x'(\gamma_k)\, \psi(\tau_k)$,

and $S_4 = \Sigma (t_k - t_{k-1})\, y'(\delta_k)\, \phi(\tau_k)$.

We can write

$$S_1 = \sum_{k=1}^{n} (t_k - t_{k-1})\, x'(\tau_k)\, \phi(\tau_k) + \sum_{k=1}^{n} (t_k - t_{k-1})\, [x'(\gamma_k) - x'(\tau_k)]\, \phi(\tau_k). \qquad \ldots(8)$$

Since $x'(t)$ is continuous in the closed and bounded interval $[a, b]$, therefore it is uniformly continuous in $[a, b]$, so that for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x'(r) - x'(s)| < \varepsilon \text{ whenever } |r - s| < \delta$$

where r, s are in $[a, b]$.

Thus for any partition **P** of $[a, b]$ with norm $\leq \delta$, we have

$$|\Sigma (t_k - t_{k-1})\, [x'(\gamma_k) - x'(\tau_k)]\, \phi(\tau_k)| \leq \varepsilon (b - a)\, M_1 \qquad \ldots(9)$$

{Since $z(t)$ is continuous over L therefore $\phi(t)$ and $\psi(t)$ are continuous on $[a, b]$ and consequently they are bounded on $[a, b]$. Therefore there exists a number M_1 such that

$$|\phi(t)| \leq M_1, \forall\, t \in [a, b].\}$$

As $n \to \infty$ and $|\mathbf{P}| \to 0$, we conclude from (9) that the second term on the right side of (8) tends to zero and the first term tends to

$$\int_a^b \phi(t)\, x'(t)\, dt.$$

$\therefore$ $$\lim S_1 = \int_a^b \phi(t)\, x'(t)\, dt.$$

Similarly $$\lim S_2 = -\int_a^b \psi(t)\, y'(t)\, dt, \lim S_3 = \int_a^b \psi(t)\, x'(t)\, dt$$

and $$\lim S_4 = \int_a^b \phi(t)\, y'(t)\, dt.$$

Taking limit of both sides of (7) as $n \to \infty$ and $|\mathbf{P}| \to 0$ and using the above results, we get

$$\int_L f(z)\, dz = \int_a^b \{\phi(t)\, x'(t) - \psi(t)\, y'(t)\}\, dt + i \int_a^b \{\psi(t)\, x'(t) + \phi(t)\, y'(t)\}\, dt$$

$$= \int_a^b [u(x(t), y(t))\, x'(t) - v(x(t), y(t))\, y'(t)]\, dt + i \int_a^b [v(x(t), y(t))\, x'(t) + u(x(t), y(t))\, y'(t)]\, dt.$$

8 Some Elementary Properties of Complex Integrals

Prop. 1: $\int_L \{f(z) + \phi(z)\}\, dz = \int_L f(z)\, dz + \int_L \phi(z)\, dz.$

We can generalize this property for a finite number of functions.

Prop. 2: $\int_L f(z)\, dz = -\int_{-L} f(z)\, dz,$ *where* $-L$ *is the opposite arc of* L.

Prop. 3: $\int_{L_1 + L_2} f(z)\, dz = \int_{L_1} f(z)\, dz + \int_L f(z)\, dz$

where the end point of L_1 *coincides with the initial point of* L_2.

This property can be generalized for a finite number of arcs provided the end point of the preceding arc coincides with the initial point of the arc which follows it. Hence, if

$$L = L_1 + L_2 + \ldots + L_n$$

where the final point of L_k coincides with the initial point of

$$L_{k+1}\ (k = 1, 2, \ldots, n-1),$$

then $$\int_L f(z)\, dz = \int_L f(z)\, dz + \int_{L_2} f(z)\, dz + \ldots + \int_{L_n} f(z)\, dz.$$

Prop. 4: $\int_L c\, f(z)\, dz = c \int_L f(z)\, dz$, *where* c *is any complex constant.*

These properties can be easily proved by the definition of a complex integral as the limit of a sum.

Prop. 5: $$\int_L [c_1 f_1(z) + c_2 f_2(z) + \ldots + c_n f_n(z)]\, dz$$

$$= c_1 \int_L f_1(z)\, dz + c_2 \int_L f_2(z)\, dz + \ldots + c_n \int_L f_n(z)\, dz.$$

where $c_1, c_2, \ldots\ldots, c_n$ *are complex constants.*

This property follows directly from properties 1 and 4.

Prop. 6: $\left|\int_L f(z)\, dz\right| \le \int_L |f(z)|\,|dz|$

Proof: We have $\mathbf{R}\left[c \int_L f(z)\, dz\right] = \mathbf{R}\left[\int_L c\, f(z)\, dz\right]$, prop. 4

where c is a complex constant.

Since c is arbitrary therefore taking $c = e^{-i\theta}$ where θ is any real number.

$$\therefore \quad \mathbf{R}\left[e^{-i\theta} \int_L f(z)\, dz\right] = \mathbf{R}\left[\int_L e^{-i\theta} f(z)\, dz\right] = \int_L \mathbf{R}\,[e^{-i\theta} f(z)\, dz]$$

$$\le \int_L |e^{-i\theta} f(z)\, dz| \qquad [\because \mathbf{R}(z) \le |z|]$$

$$= \int_L |f(z)|\,|dz|. \qquad \ldots(1)$$

Again since θ is any real number, therefore taking $\theta = \arg \int_L f(z)\, dz$ so that we can write

$$\int_L f(z)\, dz = \left| \int_L f(z)\, dz \right| e^{i\theta}.$$

Also in this case, we have

$$\mathbf{R}\left[e^{-i\theta} \int_L f(z)\, dz\right] = \mathbf{R}\left[e^{-i\theta} \left| \int_L f(z)\, dz \right| e^{i\theta}\right]$$

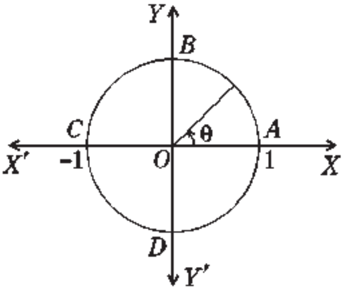
$$= \left| \int_L f(z)\, dz \right|. \qquad \ldots(2)$$

From (1) and (2), we have $\left| \int_L f(z)\, dz \right| \le \int_L |f(z)|\, |dz|.$

Illustrative Examples

Example 3: *Prove that the value of the integral of* $\frac{1}{z}$ *along a semi-circular arc* $|z| = 1$ *from* -1 *to* 1 *is* $-\pi i$ *or* πi *according as the arc lies above or below the real axis.*

Solution: The given circle is $|z| = 1$. Parametric equation of the circle is $z = e^{i\theta}$, where $0 \le \theta \le 2\pi$.



We have $\qquad dz = i e^{i\theta}\, d\theta.$

As z moves from -1 to 1 along the semi-circular arc above the real axis, θ varies from π to 0. In this case, we have

$$\int_{CBA} \frac{1}{z}\, dz = \int_{\pi}^{0} \frac{1}{e^{i\theta}}\, i e^{i\theta}\, d\theta$$

$$= i \int_{\pi}^{0} d\theta = -i\pi.$$

Again when z moves from -1 to 1 along the semi-circular arc below the real axis, θ varies from π to 2π.

$$\therefore \qquad \int_{CDA} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{i e^{i\theta}}{e^{i\theta}}\, d\theta = i \int_{\pi}^{2\pi} d\theta = \pi i.$$

Note: We have $\int_{CBA} \frac{dz}{z} = -i\pi$,

therefore $\qquad \int_{ABC} \frac{dz}{z} = i\pi.$

Also $\qquad \int_{CDA} \frac{dz}{z} = i\pi.$

Hence $\qquad \int_{ABCD} \frac{dz}{z} = \int_{ABC} \frac{dz}{z} + \int_{CDA} \frac{dz}{z} = 2\pi i.$

Example 4: *Find the value of the integral* $\int_0^{1+i} (x - y + i x^2)\, dz.$

(i) Along the straight line from $z = 0$ *to* $z = 1 + i.$

(ii) Along the real axis from $z = 0$ *to* $z = 1$ *and then along a line parallel to the imaginary axis from* $z = 1$ *to* $z = 1 + i.$ **(Meerut 2002; Rohilkhand 09; Gorakhpur 12, 14, 16)**

Solution: We have $z = x + i\, y$

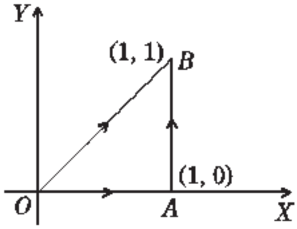
$\therefore \qquad dz = dx + i\, dy.$

Let A be the point of affix 1 and B be the point of affix $1 + i$ in the Argand plane. Join OB and AB.

(i) OB is the straight line joining $z = 0$ to $z = 1 + i.$

Obviously on OB, we have $y = x$

$\therefore \qquad dy = dx.$



Now $\int_{OB} (x - y + ix^2)\, dz$

$$= \int_0^1 (x - x + ix^2)(1 + i)\, dx$$

$$= \int_0^1 i(1 + i)\, x^2\, dx = (-1 + i)\left[\frac{1}{3} x^3\right]_0^1$$

$$= \frac{1}{3}(-1 + i).$$

(ii) OA is the line from $z = 0$ to $z = 1$ along the real axis and AB is the line from $z = 1$ to $z = 1 + i$ parallel to the imaginary axis. On the line OA, $y = 0$,

$\therefore \qquad z = x + iy = x$ and $dz = dx.$

$$\therefore \qquad \int_{OA} (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx = \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{i}{3}.$$

On the line AB, $x = 1$, therefore $z = 1 + iy$, $dz = i\, dy.$

$$\therefore \qquad \int_{AB} (x - y + ix^2)\, dz = \int_0^1 (1 + i - y)\, i\, dy$$

$$= i\left[(1 + i)\, y - \frac{y^2}{2}\right]_0^1 = -1 + \frac{i}{2}.$$

Hence $\int_0^{1+i} (x - y + i x^2)\, dz$ along the contour OAB

$$= \int_{OA} (x - y + i x^2)\, dz + \int_{AB} (x - y + i x^2)\, dz$$

$$= \frac{1}{2} + \frac{i}{3} - 1 + \frac{i}{2} = -\frac{1}{2} + \frac{5}{6} i.$$

Example 5: *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where* C *is the arc of the cycloid* $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$.

Solution: Here $f(z) = z^2 + 3z + 2$ is a polynomial so $f(z)$ is analytic in z-plane, therefore the integral of $f(z)$ between the points $(0, 0)$ and $(\pi a, 2a)$ is independent of the path joining these points.

The path of integration consists of :

(i) the part of real axis from the point $(0, 0)$ to the point $(\pi a, 0)$.

(ii) a line parallel to y-axis from the point $(\pi a, 0)$ to the point $(\pi a, 2a)$.

$$\therefore \quad \int_C (z^2 + 3z + 2)\, dz$$

$$= \int_0^{\pi a} (x^2 + 3x + 2)\, dx + \int_0^{2a} \{(\pi a + iy)^2 + 3(\pi a + iy) + 2\}\, i\, dy$$

$$= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right]_0^{\pi a} + \left[\frac{1}{3}(\pi a + iy)^3 + \frac{3}{2}(\pi a + iy)^2 + 2iy\right]_0^{2a}$$

$$= \frac{1}{3}(\pi a)^3 + \frac{3}{2}(\pi a)^2 + 2\pi a + \frac{1}{3}(\pi a + i2a)^3 + \frac{3}{2}(\pi a + i2a)^2 + 4ia - \frac{1}{3}(\pi a)^3 - \frac{3}{2}(\pi a)^2$$

$$= 2\pi a + \frac{1}{3}(\pi a + i2a)^3 + \frac{3}{2}(\pi a + i2a)^2 + 4ia.$$

Comprehensive Exercise 1

1. Evaluate $I = \int_{(0,1)}^{(2,5)} (3x + y)\, dx + (2y - x)\, dy$ along

   (i) the curve $y = x^2 + 1$ (ii) the line joining $(0, 1)$ and $(2, 5)$

   (iii) the line from $(0, 1)$ to $(0, 5)$ and then from $(0, 5)$ to $(2, 5)$.

2. Evaluate $\int (\bar{z})^2\, dz$ around the circle

   (i) $|z| = 1$, (ii) $|z - 1| = 1$.

3. Evaluate $\int_C (x^2 - iy^2)\, dz$ along the parabola $y = 2x^2$ from $(1, 2)$ to $(2, 8)$; and along the line joining $(1, 2)$ and $(2, 8)$.

4. Evaluate the integral $\int_0^{1+i} z^2\, dz$.

5. (i) Evaluate $\int_L \frac{dz}{z - a}$, where L represents the circle $|z - a| = r$.

   (Purvanchal 2011; Gorakhpur 15)

(ii) Evaluate $\int_C (z-a)^n \, dz$ when $n \neq -1$ and C is the circle $|z-a| = r$.

(Gorakhpur 2007)

(iii) Evaluate the integral $\int_C z^n \, dz, n \neq -1, C : |z| = 1.$

(Gorakhpur 2014)

Answers 1

1. (i) $\frac{88}{3}$; (ii) 32 ; (iii) 40

2. (i) 0, (ii) $4\pi i$

3. (i) $\frac{511}{3} - i\frac{49}{5}$; (ii) $\frac{65}{3} - 14i$

4. $\frac{1}{3}(1+i)^3$

5. (i) $2\pi i$ (ii) 0 (iii) 0

9 An Upper Bound for a Complex Integral

Theorem: *If a function $f(z)$ is continuous on a contour L of length l and if M be the upper bound of $|f(z)|$ on L i.e., $|f(z)| \leq M$ on L, then*

$$\left| \int_L f(z) \, dz \right| \leq M \, l.$$

(Gorakhpur 2008, 10)

Proof: Proceeding as in article 5, we have

$$S_P = \sum_{k=1}^{n} (z_k - z_{k-1}) \, f(\zeta_k).$$

Now
$$|S_P| = |\Sigma (z_k - z_{k-1}) \, f(\zeta_k)|$$
$$\leq \Sigma |(z_k - z_{k-1}) \, f(\zeta_k)| \qquad [\because |a+b| \leq |a| + |b|]$$
$$= \Sigma |z_k - z_{k-1}| |f(\zeta_k)|$$
$$\leq M \, \Sigma |z_k - z_{k-1}|, \text{ since } \zeta_k \text{ is a point on } L.$$

$\therefore$
$$\lim |\Sigma (z_k - z_{k-1}) \, f(\zeta_k)| \leq \lim M \, \Sigma |z_k - z_{k-1}|$$

or
$$\left| \int_L f(z) \, dz \right| \leq M \int_L |dz|.$$

Hence $\left| \int_L f(z) \, dz \right| \leq M \, l$, (see example 2 of article 6.)

10 Line Integrals as Functions of Arcs

We have seen that a line integral $\int_L f(z) \, dz$ over an arc L can be put in the form

$$\int_L (u + iv)(dx + i \, dy).$$

General line integrals of the form $\int_L p\,dx + q\,dy$ are often considered as **functions (or functionals)** of the arc L under the assumption that p and q are defined and continuous in a domain D and the arc L can vary freely in D. There exists a special class of integrals characterized by the property that the integral over an arc depends only on its end points. This means that if the two arcs L_1 and L_2 have the same initial point and the same final point, then

$$\int_{L_1} p\,dx + q\,dy = \int_{L_2} p\,dx + q\,dy.$$

Theorem 1: *The following statements are equivalent :*

A line integral of $f(z)$ over an arc L depends only on the end points of L .

The integral of $f(z)$ over any closed curve is zero.

Proof: First suppose that the line integral of $f(z)$ over any closed curve is zero. Let L_1 and L_2 be any two arcs having the same end points. Then $L_1 - L_2$ is a closed curve.

$\therefore$ $$\int_{L_1 - L_2} f(z)\,dz = 0$$

or $$\int_{L_1} f(z)\,dz - \int_{L_2} f(z)\,dz = 0, \text{ by a property of complex integration}$$

or $$\int_{L_1} f(z)\,dz = \int_{L_2} f(z)\,dz.$$

Hence the line integral of $f(z)$ over an arc depends only on its end points provided the integral of $f(z)$ over any closed curve is zero.

Conversely, suppose the line integral of $f(z)$ over any two arcs with same end points be same.

Consider a closed curve Γ. Then Γ and $-\Gamma$ have the same end points, so that

$$\int_{\Gamma} f(z)\,dz = \int_{-\Gamma} f(z)\,dz$$

We know $\int_{-\Gamma} f(z)\,dz = -\int_{\Gamma} f(z)\,dz$, by a property of complex integration.

$\therefore$ $$\int_{\Gamma} f(z)\,dz = -\int_{\Gamma} f(z)\,dz$$

or $$2\int_{\Gamma} f(z)\,dz = 0 \quad \text{or} \quad \int_{\Gamma} f(z)\,dz = 0.$$

Thus the integral of $f(z)$ over any closed curve is zero.

Theorem 2: *The line integral $\int p\,dx + q\,dy$, defined in a domain D, depends only on the end points of Γ if and only if there exists a function $U(x, y)$ in D such that*

$$\frac{\partial U}{\partial x} = p \quad \textit{and} \quad \frac{\partial U}{\partial y} = q.$$

Proof: **The 'if' part.** Let there exist a function $U(x, y)$ in D such that $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$. Also suppose a and b are the end points of Γ. Then we have

$$\int_\Gamma p\,dx + q\,dy = \int_a^b \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \right)$$

$$= \int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} \right) dt = \int_a^b \frac{d}{dt} U(x(t), y(t))\,dt$$

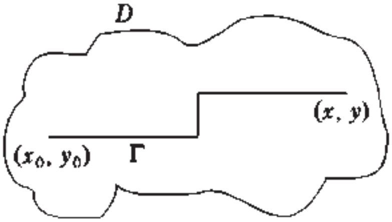
$$= [U(x(t), y(t))]_a^b = U(x(b), y(b)) - U(x(a), y(a)),$$

which shows that the line integral depends only on the end points of Γ.

The 'only if' part: Let us assume that the line integral $\int_\Gamma p\,dx + q\,dy$ depends only on the end points of Γ. Suppose (x_0, y_0) be a fixed point in D and (x, y) be any arbitrary point in D. Join (x_0, y_0) to (x, y) by a polygonal arc Γ contained in D having its sides parallel to coordinate axes.

Consider a function $U(x, y)$ given by

$$U(x, y) = \int_\Gamma p\,dx + q\,dy.$$



Then the function $U(x, y)$ is well defined since according to assumption the integral depends only on the end points of Γ. Also choose the last segment of Γ parallel to x-axis so that y becomes constant and $dy = 0$ and suppose that x varies without changing the other segments. Choosing x as a parameter on the last segment, we get

$$U(x, y) = \int^x p(x, y)\,dx + \text{constant}.$$

We have not specified the lower limit of x since it is insignificant for our purpose.

$$\therefore \qquad \frac{\partial U}{\partial x} = p.$$

Similarly choosing the last segment parallel to y-axis, we can show that $\frac{\partial U}{\partial y} = q$.

Remark: (i) It is customary to write $dU = \left(\frac{\partial U}{\partial x} dx\right) + \left(\frac{\partial U}{\partial y}\right) dy$ ...(1)

and we say that an expression $p\,dx + q\,dy$ is an **exact differential** if it can be written in the form (1). Using this terminology, the above theorem can be stated as :

An integral depends only on the end point if and only if the integrand is an exact differential.

(ii) We now determine the conditions under which

$$f(z)\,dz = f(z)\,dx + if(z)\,dy$$

is an exact differential. By definition of an exact differential, there must exist a function $F(z)$ in D such that

$$\frac{\partial F(z)}{\partial x} = f(z) \quad \text{and} \quad \frac{\partial F(z)}{\partial y} = if(z).$$

It follows that $\dfrac{\partial F}{\partial x} = -i\dfrac{\partial F}{\partial y}$, which is a Cauchy-Riemann equation.

Also $f(z)$ is by assumption continuous (otherwise $\int_L f(z)\,dz$ would not be defined).

Hence $F(z)$ is analytic.

(iii) From the discussion in (i) and (ii), we conclude :

The integral $\int_L f(z)\,dz$, with continuous f, depends only on the end points of L if and only if f is the derivative of an analytic function in D.

For example, for $n \geq 0$, the function $(z-a)^n$ is the derivative of $(z-a)^{n+1}/(n+1)$ which is an analytic function in the whole complex plane. If Γ is any closed curve, then it follows from theorem 1 and remark (iii) above that

$$\int_L (z-a)^n\,dz = 0.$$

If n is negative, but $\neq -1$, then also $\int_L (z-a)^n\,dz = 0$ for all closed curve Γ which do not pass through a, since in the complementary region of the point a the infinite integral is still analytic and single-valued. If $n=-1$, then (1) does not always hold. We have seen in problem 13 after article 8 that

$$\int_\Gamma \frac{dz}{z-a} = 2\pi i,$$

where Γ is any circle $|z-a| = r$.

11 Cauchy's Fundamental Theorem

In example 3, after article 8 the integral of $\dfrac{1}{z}$ round the circle $|z| = 1$ is $2i\pi$ whereas in the problem 8 after article 8 the integral of z^2 round the closed contour $OLMO$ is zero. Here we observe that the function $\dfrac{1}{z}$ is not analytic at $z=0$ which is an interior point of the circle $|z| = 1$ whereas the function z^2 is analytic throughout the interior and at the boundary of the triangle OLM. Now we shall prove a very important theorem known as **Cauchy's fundamental theorem** which states :

If $f(z)$ is analytic at all points within and on the closed contour C, then

$$\int_C f(z)\,dz = 0.$$

Theorem 1: *(Cauchy's Theorem). Let D be a simply connected region and let $f(z)$ be a single valued continuously differentiable function on D i.e., $f'(z)$ exists and is continuous at each point of D. Then*

$$\int_C f(z)\,dz = 0,$$

where C is any closed contour contained in D. **(Rohilkhand 2008; Gorakhpur 13)**

Proof: We have $\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy). \qquad \ldots(1)$$

To prove this theorem we shall use Green's theorem for a plane which states : if $P(x, y), Q(x, y), \dfrac{\partial Q}{\partial x}, \dfrac{\partial P}{\partial y}$ are all continuous functions of x and y in the region D, and C is any closed contour in D, then

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Since the function $f(z)$ is analytic in D therefore $f'(z)$ exists and $f'(z) = u_x + i v_x = v_y - i u_y$ (by Cauchy-Riemann equations). Also $f'(z)$ is given to be continuous at each point of D, therefore u, v, u_x, v_x, u_y and v_y are all continuous in D. Thus all the conditions of Green's theorem are satisfied. Hence from (1), we have

$$\int_C f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy,$$

by Cauchy–Riemann equations

$$= 0.$$

Note: This form of Cauchy's theorem is quite useful in applied mathematics as the continuity of the four partial derivatives u_x, v_x, u_y and v_y is generally assumed on physical grounds.

Cauchy-Goursat Theorem: In the statement of Cauchy's theorem the function $f'(z)$ is assumed to be a continuous function. It was Goursat who first proved the theorem without considering the continuity of $f'(z)$. The revised form of the theorem is known as **Cauchy-Goursat theorem.** We are giving here three independent proofs of this important theorem.

Theorem 2: *(Cauchy-Goursat theorem.) Let $f(z)$ be analytic in a simply connected domain D and let C be any closed continuous rectifiable curve in D. Then*

$$\int_C f(z)\,dz = 0.$$

(Meerut 2001; Gorakhpur 05, 08; Avadh 08; Purvanchal 08)

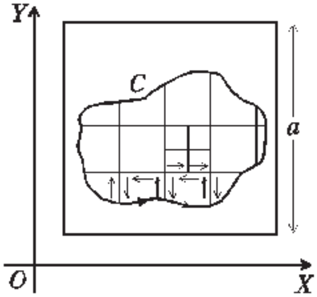
First we shall prove the following lemma known as **Goursat's lemma.**

Lemma: *Let $f(z)$ be analytic within and on a closed contour C. Then for every $\varepsilon > 0$, it is always possible to divide the region within C into a finite number of squares and partial squares whose boundaries are denoted by S_i ($i = 1, 2, \ldots\ldots, n$) such that there exists a point z_i within each S_i such that*

$$\left| \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i) \right| < \varepsilon \qquad \ldots(1)$$

for each point z ($\neq z_i$) within or on S_i ($i = 1, 2, \ldots\ldots, n$).

Proof of the lemma: Suppose the lemma is false. It means the lemma does not hold at least in one mesh *i.e.,* there exists $\varepsilon > 0$ such that in however small meshes (squares and partial squares) we subdivide the region within C there will be at least one mesh (square or a partial square) where the inequality (1) does not hold good.

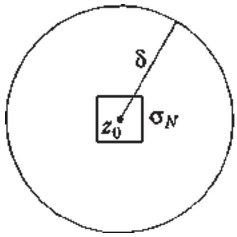


Let R denote the region within and on the closed contour C. Cover the region R by a network of finite number of meshes (squares and partial squares) by drawing lines parallel to the coordinate axes. Then as per assumption there is at least one mesh for which (1) does not hold. Let us denote it by σ_0. It may be a square or a partial square. Divide σ_0 into four equal squares. Then at least one of these squares contains the points of R for which (1) is not true. Suppose it is σ_1. Quadrisect σ_1 and repeat the above process. If this process comes to an end after a finite number of steps we arrive at a contradiction and the lemma is proved.

On the other hand if the process is continued indefinitely, we obtain a nested sequence of squares $\sigma_0, \sigma_1, \sigma_2, \ldots\ldots, \sigma_n \ldots$ each contained in the previous one, for which lemma is not true. Consequently there exists a point z_0 common to all the squares of the above sequence such that z_0 is the limit point of the set of points in R. Also $z_0 \in R$ because R is closed. Since $f(z)$ is analytic at every point which lies within and on the closed contour C therefore $f(z)$ is differentiable at z_0. So for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \qquad \ldots(2)$$

for all z for which $|z - z_0| < \delta$.



We can choose a positive integer N so large that the diagonal of the square σ_N is less than δ. Then all the squares σ_n ($n \geq N$) are contained in the circular neighbourhood

$$|z - z_0| < \delta \text{ of } z_0.$$

Also $z_0 \in \sigma_n$ for n.

Thus there exists a point z_i (here z_i is z_0) within each S_i for which inequality (1) is satisfied which contradicts the hypothesis. Thus the lemma is true.

Proof of the main theorem: The inequality (1) can be written as

$$f(z) = f(z_i) - z_i f'(z_i) + z f'(z_i) + (z - z_i)\,\eta_i(z) \qquad \ldots(3)$$

where $|\eta_i(z)| < \varepsilon$.

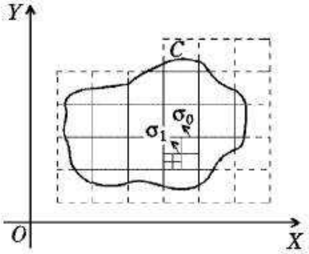
Since (3) also gives the value of $f(z)$ at any point on the boundary of S_i therefore integrating (3) around S_i, we get

$$\int_{S_i} f(z)\,dz = \{f(z_i) - z_i f'(z_i)\}\int_{S_i} dz + f'(z_i)\int_{S_i} z\,dz + \int_{S_i} (z - z_i)\,\eta_i(z)\,dz$$

$$= \int_{S_i} (z - z_i)\,\eta_i(z)\,dz, \qquad \ldots(4)$$

$$\left[\because \int_{S_i} dz = 0 = \int_{S_i} z\,dz\right]$$

It is clear from the adjoining diagram that the integral around the closed curve C is equal to the sum of the integrals around all the S_i because the line integrals along the common boundaries of every pair of adjacent meshes cancel each other. We are left only with the integrals along the arcs which form parts of C.



$$\therefore \qquad \int_C f(z)\,dz = \sum_{i=1}^{n} \int_{S_i} f(z)\,dz. \qquad \ldots(5)$$

From (4) and (5), we have

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{S_i} (z - z_i)\,\eta_i(z)\,dz$$

or

$$\left|\int_C f(z)\,dz\right| = \left|\sum_{i=1}^{n} \int_{S_i} (z - z_i)\,\eta_i(z)\,dz\right|$$

$$\leq \sum_{i=1}^{n} \left|\int_{S_i} (z - z_i)\,\eta_i(z)\,dz\right|$$

$$\leq \sum_{i=1}^{n} \int_{S_i} |z - z_i|\,|\eta_i(z)|\,|dz|$$

$$< \varepsilon \sum_{i=1}^{n} \int_{S_i} |z - z_i|\,|dz|. \qquad \ldots(6)$$

The boundary S_i of a mesh either completely or partially coincides with the boundary of a square. Let a_i be the length of a side of that square. The point z lies on S_i and z_i lies either on the boundary of S_i or inside S_i therefore the distance between the points z and z_i cannot be greater than the length $a_i\sqrt{2}$ of the diagonal of that square *i.e.*,

$$|z - z_i| \leq a_i\sqrt{2}$$

$$\therefore \qquad \int_{S_i} |z - z_i|\,|dz| \leq a_i\sqrt{2}\int_{S_i} |dz|. \qquad \ldots(7)$$

Now $\int_{S_i} |dz|$ represents the length of S_i. This length is $4ai$ if S_i is a complete square and it cannot exceed $(4a_i + l_i)$ if S_i is a partial square where l_i is the length of arc of C which forms a part of S_i.

Hence if S_i is a square, then inequality (7) gives

$$\int_{S_i} |z - z_i||dz| \le a_i \sqrt{2} \, . \, 4a_i = 4\sqrt{2} a_i^2 \qquad \text{...(8)}$$

and $$\int_{S_i} |z - z_i||dz| \le a_i \sqrt{2}\,(4\, a_i + l_i) \le 4\, a_i^2 \sqrt{2} + a\, l_i \sqrt{2} \qquad \text{...(9)}$$

if S_i is a partial square,

where a denotes the length of the side of the square which encloses the entire curve C together with the squares which are used in covering C. Obviously the sum of the areas a_i^2 of all these squares cannot exceed a^2.

If l denotes the arc length of C, we have from (6), (8) and (9)

$$\left| \int_C f(z)\, dz \right| < \varepsilon \sum_{i=1}^{n} (4\sqrt{2} a_i^2 + \sqrt{2} a\, l_i) \le \varepsilon\, (4\sqrt{2} a^2 + \sqrt{2} al)$$

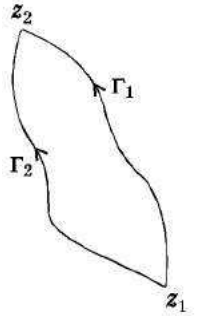
$$= \varepsilon K, \text{where } K \text{ is a constant.}$$

Since ε is arbitrary therefore we have $\int_C f(z)\, dz = 0$.

Corollary 1: *Let $f(z)$ be analytic in a simply connected region D. Then the integral along every rectifiable curve in D joining any two given points of D is the same i.e., it does not depend on the curve joining the two points.*

Proof: Let Γ_1 and Γ_2 be any two curves in the domain D joining two given points z_1 and z_2 of D. Let Γ be the closed curve consisting of Γ_1 and $-\Gamma_2$.

Then by Cauchy's theorem, we have



$$\int_\Gamma f(z)\, dz = 0 \quad \text{or} \quad \int_{\Gamma_1 + (-\Gamma_2)} f(z)\, dz = 0$$

or $$\int_{\Gamma_1} f(z)\, dz + \int_{-\Gamma_2} f(z)\, dz = 0$$

or $$\int_{\Gamma_1} f(z)\, dz - \int_{\Gamma_2} f(z)\, dz = 0$$

or $$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz.$$

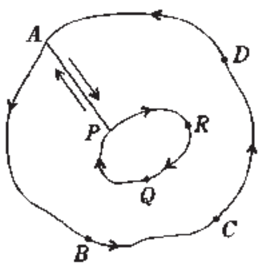
Note: In view of the above corollary, we may use the symbol $\int_a^b f(z)\, dz$ for the integral of $f(z)$ along any curve joining a and b.

Corollary 2: Extension of Cauchy-Goursat's theorem to multiply connected regions.

Let D be a doubly connected region bounded by two simple closed curves C_1 and C_2 such that C_2 is contained in C_1 and $f(z)$ is analytic in the region between these curves and continuous on C_1, then

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

where both C_1 and C_2 are traversed in the positive sense i.e., in anti-clockwise direction.



Proof: Connect the curve C_2 to C_1 by making a narrow cross-cut joining a point A of C_1 to a point P of C_2. Then $ABCDAPQRPA$ is the simply connected region in the interior of which $f(z)$ is analytic and on whose boundary $f(z)$ is continuous. Hence by Cauchy-Goursat's theorem,

we have $$\int_{ABCDAPQRPA} f(z)\,dz = 0$$

or $$\int_{ABCDA} f(z)\,dz + \int_{AP} f(z)\,dz + \int_{PQRP} f(z)\,dz + \int_{PA} f(z)\,dz = 0$$

or $$\int_{ABCDA} f(z)\,dz + \int_{PQRP} f(z)\,dz = 0,$$

since $$\int_{AP} f(z)\,dz = -\int_{PA} f(z)\,dz$$

or $$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

or $$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

or $$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz\,.$$

In general if C is a closed curve and $C_1, C_2, C_3, \ldots\ldots$ are the closed curves which lie inside C and if $f(z)$ is analytic function in the region between these curves and continuous on C, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \ldots\ldots$$

where integral along each curve is taken in positive sense.

Remark 1: Cauchy's fundamental theorem holds under less restrictive conditions. The following is one of the versions of this theorem.

Let $f(z)$ be continuous on and analytic within a rectifiable Jordan curve C. Then

$$\int_C f(z)\,dz = 0.$$

Note that in the above statement, it is not necessary for $f(z)$ to be analytic on C. Only the continuity of $f(z)$ is essential on C. However, we shall not try to prove the above assertion.

Remark 2: Cauchy-Goursat theorem gives only sufficient conditions for $\int_C f(z)\,dz$ to become zero. In certain cases $\int_C f(z)\,dz$ may vanish even if $f(z)$ is not an analytic function in C.

Illustration: *Evaluate* $\int_\gamma \frac{dz}{z^2}$, *where* γ *is defined by* $|z| = d, d > 0$.

Solution: Let $z = de^{i\theta}, 0 \le \theta \le 2\pi$. Then $dz = die^{i\theta}\, d\theta$.

Now
$$\int_\gamma \frac{1}{z^2}\, dz = \int_0^{2\pi} \frac{die^{i\theta}}{d^2 e^{i2\theta}}\, d\theta = \frac{i}{d}\int_0^{2\pi} e^{-i\theta}\, d\theta$$
$$= \frac{i}{d}\left[-\frac{e^{-i\theta}}{i}\right]_0^{2\pi} = -\frac{1}{d}[e^{-i2\pi} - 1] = -\frac{1}{d}(1-1) = 0.$$

$\therefore$ the integral of $1/z^2$ along the circle γ is zero but $1/z^2$ is not analytic at $z = 0$ which is the centre of γ.

If, however, the function $f(z)$ is assumed to be continuous within and on the boundary of C, vanishing of $\int_C f(z)\, dz$ will imply that $f(z)$ is an analytic function in C. This is **Morera's theorem** which will be proved later on.

12 Cauchy-Goursat Theorem (*Second proof*)

Lemma: *Let* $f(z)$ *be continuous on a domain* D *and let* C *be any continuous rectifiable curve in* D. *Then for every* $\varepsilon > 0$, *there exists a polygon* Δ *in* D *with vertices on* C *such that*
$$\left| \int_C f(z)\, dz - \int_\Delta f(z)\, dz \right| < \varepsilon.$$

Theorem: *Let* $f(z)$ *be analytic in a simply connected domain* D *and let* C *be any closed continuous rectifiable curve contained in* D. *Then*
$$\int_C f(z)\, dz = 0.$$

Illustration: *If* C *is the circle* $|z-2| = 5$, *determine whether* $\int_C \frac{dz}{z-3}$ *is zero.*

Solution: Putting $z - 2 = 5e^{i\theta}$, $dz = 5ie^{i\theta}\, d\theta$, we get
$$\int_C \frac{dz}{z-3} = \int_0^{2\pi} \frac{5ie^{i\theta}\, d\theta}{5e^{i\theta} - 1} = i\int_0^{2\pi} \left(1 - \frac{1}{5}e^{-i\theta}\right)^{-1} d\theta$$
$$= i\int_0^{2\pi} \left[1 + \frac{1}{5}e^{-i\theta} + \frac{1}{5^2}e^{-2i\theta} + \ldots\right] d\theta.$$

Now
$$\int_0^{2\pi} e^{-mi\theta}\, d\theta = -\frac{1}{mi}\left[e^{-mi\theta}\right]_0^{2\pi} = -\frac{1}{mi}\left[e^{-2\pi mi} - e^0\right]$$
$$= -\frac{1}{mi}[1-1] = 0, \text{ when } m \ne 0.$$

$\therefore$
$$\int_C \frac{dz}{z-3} = i\int_0^{2\pi} d\theta = 2\pi i \ne 0.$$

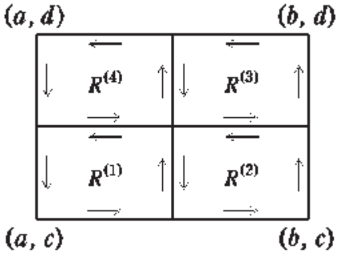
The reason that the integral is not zero is that $\dfrac{1}{z-3}$ is not analytic at $z=3$ which is an interior point of the circle $|z-2|=5$.

13 Cauchy-Goursat Theorem (Third proof)

Step 1. Cauchy's theorem for a rectangle:

By a rectangle R in the complex plane, we shall mean a set of points (x, y) such that

$$a \le x \le b, c \le y \le d.$$



We think of the perimeter of R as a simple closed curve consisting of four line segments whose direction is chosen in such a manner that the area R lies to the left of these directed segments. The vertices thus occur in the order $(a, c), (b, c), (b, d), (a, d)$. We shall refer to this closed curve as the **boundary curve or contour** of R and shall denote it by ∂R. We now prove Cauchy's theorem for a rectangle, namely :

If the function $f(z)$ is analytic on a rectangle R, then

$$\int_{\partial R} f(z)\, dz = 0,$$

where ∂R denotes the boundary curve of R.

Step 2: Cauchy's theorem for a circular disc:

If $f(z)$ is analytic in the open disc Δ defined by $|z-z_0|<r$, then

$$\int_C f(z)\, dz = 0$$

for every closed curve C in Δ.

Step 3: Cauchy's theorem for any closed curve:

If $f(z)$ is analytic in a simply connected domain D and C is any closed curve, then

$$\int_C f(z)\, dz = 0.$$

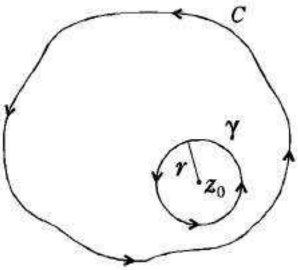
14 Cauchy's Integral Formula

Let $f(z)$ be an analytic function in a simply connected domain D enclosed by a rectifiable Jordan Curve C and let $f(z)$ be continuous on C. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz$$

where z_0 is any point of D. **(Meerut 2001; Bundelkhand 01; Purvanchal 07, 09; Kanpur 07; Gorakhpur 07, 08; Rohilkhand 12; Kumaun 12, 15)**

Proof: We describe a circle γ defined by the equation $|z - z_0| = \rho$ where $\rho < d$ (the distance of z_0 from C).



Then the function

$$\phi(z) = \frac{f(z)}{z - z_0}$$

is analytic in the doubly connected region bounded by C and γ.

$$\therefore \qquad \int_C \phi(z)\, dz = \int_\gamma \phi(z)\, dz$$

or
$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz \qquad \text{...(1)}$$

where C and γ are both traversed in the counter-clockwise direction (See the figure).

It is evident that the integral on the right-hand side of (1) is independent of ρ and so we may choose ρ as small as we please.

Now
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz + \frac{1}{2\pi i}\int_\gamma \frac{f(z_0)}{z - z_0}\, dz \qquad \text{...(2)}$$

Writing $z - z_0 = \rho e^{i\theta}$, $dz = \rho i e^{i\theta}\, d\theta$, we have

$$\int_\gamma \frac{f(z_0)}{z - z_0}\, dz = f(z_0)\int_0^{2\pi} \frac{\rho i e^{i\theta}}{\rho e^{i\theta}}\, d\theta = f(z_0)\int_0^{2\pi} i\, d\theta = 2\pi i\, f(z_0).$$

Hence (2) can be written as

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz + f(z_0)$$

or
$$\frac{1}{2\pi i}\int \frac{f(z)}{z - z_0}\, dz - f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz. \qquad \text{...(3)}$$

Since $f(z)$ is continuous at z_0, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \qquad \text{...(4)}$$

for all z satisfying the inequality $|z - z_0| < \delta$. Since ρ is at our choice, we can take $\rho < \delta$ so that the inequality (4) is satisfied for all points on γ. Hence

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z) - f(z_0)}{\rho e^{i\theta}} . \rho i e^{i\theta}\, d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z) - f(z_0)|\, d\theta < \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, d\theta \qquad \text{[By (4)]}$$

$$= \frac{1}{2\pi} . 2\pi\varepsilon = \varepsilon.$$

Thus
$$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0)\right| < \varepsilon. \qquad \text{...(5)}$$

Since ε is arbitrary and the left-hand side of (5) does not depend upon ρ, we conclude that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-z_0}\,dz - f(z_0) = 0$$

or $$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-z_0}\,dz = f(z_0). \qquad \text{...(6)}$$

Finally from (1) and (6), we obtain

$$f(z_0) = \frac{1}{2\pi i}\int_{C}\frac{f(z)}{z-z_0}\,dz. \qquad \text{...(7)}$$

Corollary 1: Extension of Cauchy's integral formula to multiply connected regions.

(We shall consider the case of doubly connected region)

If $f(z)$ is analytic in the region D bounded by two closed curves C_1 and C_2, then

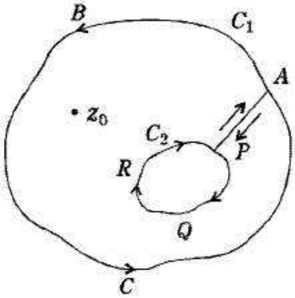
$$f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz$$

where z_0 is any point of D. **(Kumaun 2010)**

Make a cross–cut AP connecting the curves C_1 and C_2.

Then $f(z)$ is analytic in the region $ABCAPQRPA$.

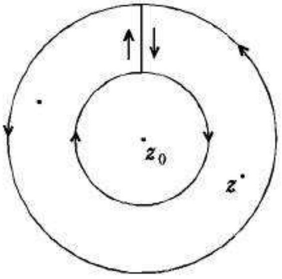
∴ By Cauchy's integral formula, we have



$$f(z_0) = \frac{1}{2\pi i}\int_{ABCAPQRPA}\frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\left[\int_{ABCA}\frac{f(z)}{z-z_0}\,dz + \int_{AP}\frac{f(z)}{z-z_0}\,dz + \int_{PQRP}\frac{f(z)}{z-z_0}\,dz + \int_{PA}\frac{f(z)}{z-z_0}\,dz\right]$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz,$$



the integrals along AP and PA cancel each other.

In particular, if C_1, C_2 are concentric circles with centre z_0 and radii ρ_1, ρ_2 $(\rho_1 > \rho_2)$, then for any point z in the annulus (ring shaped) region, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - \int_{C_2}\frac{f(z)}{z-z_0}\,dz\ .$$

Corollary 2: Gauss's mean value theorem:

If $f(z)$ is an analytic function on a domain D and if the circular region $|z-z_0| \le \rho$ is contained in D, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta.$$

In other words the value of $f(z)$ at the point z_0 equals the average of its values on the boundary of the circle $|z-z_0| = \rho$. **(Kumaun 2013)**

Proof: Let γ denote the circle $|z - z_0| = \rho$. The parametric equation of the circle is

$$z - z_0 = \rho e^{i\theta} \quad \text{or} \quad z = z_0 + \rho e^{i\theta}, 0 \le \theta \le 2\pi$$

and $$dz = \rho i e^{i\theta} d\theta.$$

By Cauchy's integral formula, we have

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.$$

15 Derivative of an Analytic Function

Theorem: *Let $f(z)$ be analytic function within and on the boundary C of a simply connected region D and let z_0 be any point within C. Then*

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz.$$

(Kanpur 2003; Purvanchal 08, 09, 10; Kumaun 09; Gorakhpur 09, 11)

Proof: Let $z_0 + h$ be any point within D in the neighbourhood of z_0. By Cauchy's integral formula, we have

$$f(z_0 + h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - (z_0 + h)} dz$$

and $$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

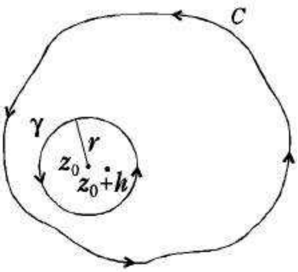
Now $$f(z_0 + h) - f(z_0) = \frac{1}{2\pi i} \int_C f(z) \left(\frac{1}{z - z_0 - h} - \frac{1}{z - z_0} \right) dz$$

$$= \frac{1}{2\pi i} \int_C \frac{h f(z)}{(z - z_0)(z - z_0 - h)} dz.$$

$\therefore$ $$\frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$= \frac{1}{2\pi i} \int_C f(z) \left[\frac{1}{(z - z_0 - h)(z - z_0)} - \frac{1}{(z - z_0)^2} \right] dz$$

$$= \frac{1}{2\pi i} \int_C \frac{h f(z)}{(z - z_0)^2 (z - z_0 - h)} dz. \qquad \ldots(1)$$



In order to prove the desired result we have to show that the right hand side of (1) approaches to zero as $h \to 0$. For this purpose draw a circle γ with centre z_0 and radius r lying entirely within C.

By Cauchy-Goursat theorem for multiply connected region, we have

$$\frac{1}{2\pi i}\int_C \frac{h\,f(z)}{(z-z_0)^2\,(z-z_0-h)}\,dz=\frac{1}{2\pi i}\int_\gamma \frac{h\,f(z)}{(z-z_0)^2\,(z-z_0-h)}\,dz\,. \quad \ldots(2)$$

Since h is arbitrary therefore choosing h such that the point z_0+h lies within γ and that $|h|<\frac{1}{2}r$.

Equation of the circle γ is $|z-z_0|=r$.

$\therefore$ For any point z on γ, we have

$$|z-(z_0+h)|=|z-z_0-h|\geq|z-z_0|-|h|$$
$$\geq r-\frac{1}{2}r=\frac{1}{2}r.$$

Again the function $f(z)$ is analytic in D therefore it is bounded in D so that there exists a positive constant M such that

$$|f(z)|\leq M.$$

Using these facts, we get from (1) and (2)

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}\,dz\right|$$

$$=\left|\frac{1}{2\pi i}\int_\gamma\frac{h\,f(z)\,dz}{(z-z_0)^2\,(z-z_0-h)}\right|$$

$$\leq\frac{|h|}{2\pi}\int_\gamma\frac{|f(z)|}{|z-z_0|^2\,|z-z_0-h|}\,|dz|$$

$$\leq\frac{|h|}{2\pi}\int_\gamma\frac{M}{r^2\,(\frac{1}{2}r)}\,|dz|=\frac{|h|\,M\,.2\pi r}{\pi r^3}=|h|\,.\text{ constant}$$

$\to 0$ as $h\to 0$.

$$\therefore \quad \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}\,dz.$$

Hence $f(z)$ is differentiable at z_0 and

$$f'(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}\,dz.$$

Note: The above formula for the derivative $f'(z_0)$ can be written formally by differentiating the integral in Cauchy's integral formula

$$f(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-z_0}\,dz$$

with respect to z_0 under the integral sign.

Thus $$f'(z_0)=\int_C\frac{d}{dz_0}\left(\frac{f(z)}{z-z_0}\right)dz=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}\,dz.$$

16 Higher Order Derivatives of an Analytic Function

Theorem: *Let $f(z)$ be analytic within and on the boundary C of a simply connected region D. If z_0 is any point within C, then $f(z)$ possesses derivatives of all orders at z_0 and all these derivatives are analytic at z_0. Also*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz.$$

(Garhwal 2000; Kanpur 07; Rohilkhand 12; Purvanchal 12)

Proof: Proceeding as in article 15 first prove that

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2}\,dz.$$

Now we shall show that $f'(z)$ is analytic at z_0.

We have
$$\frac{f'(z_0 + h) - f'(z_0)}{h} = \frac{1}{2\pi i}\int_C \left[\frac{1}{(z - z_0 - h)^2} - \frac{1}{(z - z_0)^2}\right]\frac{f(z)}{h}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2h(z - z_0) - h^2}{(z - z_0 - h)^2 (z - z_0)^2}\cdot\frac{f(z)}{h}\,dz$$

$$= \frac{2!}{2\pi i}\int_C \frac{(z - z_0) - \frac{1}{2}h}{(z - z_0 - h)^2 (z - z_0)^2}\, f(z)\,dz.$$

Now
$$\frac{f'(z_0 + h) - f'(z_0)}{h} - \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^3}\,dz$$

$$= \frac{2!}{2\pi i}\int_C f(z)\left[\frac{(z - z_0) - \frac{1}{2}h}{(z - z_0 - h)^2 (z - z_0)^2} - \frac{1}{(z - z_0)^3}\right]dz$$

$$= \frac{2!}{2\pi i}\int_C f(z)\left[\frac{\frac{3}{2}h(z - z_0) - h^2}{(z - z_0)^3 (z - z_0 - h)^2}\right]dz$$

$$= \frac{2!}{2\pi i}\int_\gamma \frac{h\left\{\frac{3}{2}(z - z_0) - h\right\}}{(z - z_0)^3 (z - z_0 - h)^2}\, f(z)\,dz,$$

by Cauchy-Goursat theorem for multi-connected region.

Here γ is the circle $|z - z_0| = r$ lying entirely within C.

$$\therefore \quad \left|\frac{f'(z_0 + h) - f'(z_0)}{h} - \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^3}\,dz\right|$$

$$\le \frac{2!}{2\pi}|h|\int_\gamma \frac{\frac{3}{2}|z - z_0| + |-h|}{|z - z_0|^3\,|z - z_0 - h|^2}\,|f(z)|\,|dz|$$

$$\leq \frac{2!}{2\pi}|h| \frac{\frac{3}{2}r + |h|}{r^3 \left(\frac{1}{2}r\right)^2} M.2\pi r \qquad \left[\because \int_\gamma |dz| = \text{perimeter of } \gamma\right]$$

The right hand side of above inequality tends to zero as $h \to 0$.

$$\therefore \quad \lim_{h \to 0} \frac{f'(z_0 + h) - f'(z_0)}{h} = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^3} dz.$$

Hence $f'(z)$ is differentiable at z_0 *i.e.*, **derivative of an analytic function is analytic**

and $$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^3} dz.$$

$\therefore$ the result is true for $n = 2$.

To complete the induction we have to show that the result is true for n if it is true for $n - 1$.

Now suppose that the result is true for $n - 1$ so that we assume that

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^n} dz$$

and $$f^{(n-1)}(z_0 + h) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0 - h)^n} dz$$

$\therefore$ $$f^{(n-1)}(z_0 + h) - f^{(n-1)}(z_0)$$

$$= \frac{(n-1)!}{2\pi i} \int_C f(z) \left[\frac{1}{(z - z_0 - h)^n} - \frac{1}{(z - z_0)^n}\right] dz$$

$$= \frac{(n-1)!}{2\pi i} \int_C f(z) \left[\frac{(z - z_0)^n - (z - z_0 - h)^n}{(z - z_0)^n (z - z_0 - h)^n}\right] dz. \qquad ...(3)$$

Now $$(z - z_0)^n - (z - z_0 - h)^n = [z - z_0 - (z - z_0 - h)]\,[(z - z_0)]^{n-1}$$

$$+ (z - z_0)^{n-2}(z - z_0 - h) + ... + \{(z - z_0) - h\}^{n-1}]$$

$$= h \sum_{r=1}^{n} (z - z_0)^{n-r} (z - z_0 - h)^{r-1}.$$

Hence we get from (3)

$$\frac{f^{(n-1)}(z_0 + h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$= \frac{(n-1)!}{2\pi i} \int_C f(z) \frac{\sum_{r=1}^{n} (z - z_0)^{n-r+1} (z - z_0 - h)^{r-1} - (z - z_0 - h)^n}{(z - z_0)^{n+1} (z - z_0 - h)^n} dz$$

$$= \frac{(n-1)!}{2\pi i} \int_\gamma f(z) \frac{\sum_{r=1}^{n} [(z - z_0)^{n-r+1} (z - z_0 - h)^{r-1} - (z - z_0 - h)^n]}{(z - z_0)^{n+1} (z - z_0 - h)^n} dz$$

where γ is the circle $|z - z_0| = \rho$ lying entirely within C

$$= \frac{(n-1)!}{2\pi i} \sum_{r=1}^{n} \int_\gamma f(z) \frac{(z-z_0)^{n-r+1} - (z-z_0-h)^{n-r+1}}{(z-z_0)^{n+1}(z-z_0-h)^{n-r+1}} dz$$

$$= \frac{(n-1)!}{2\pi i} \sum_{r=1}^{n} \int_\gamma f(z) \frac{h \sum_{s=0}^{n-r} (z-z_0)^{n-r-s}(z-z_0-h)^s}{(z-z_0)^{n+1}(z-z_0-h)^{n-r+1}} dz$$

$$= \frac{h(n-1)!}{2\pi i} \sum_{r=1}^{n} \int_\gamma \sum_{s=0}^{n} \frac{f(z)\,dz}{(z-z_0)^{r+s+1}(z-z_0-h)^{n-r-s+1}}$$

As before $|z-z_0-h| \geq \frac{1}{2}\rho$ and so

$$\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\leq \frac{|h|(n-1)!}{2\pi} \sum_{r=1}^{n} \sum_{s=0}^{n} \frac{M.2\pi\rho}{\rho^{r+s+1}(\frac{1}{2}\rho)^{n-r-s+1}}$$

$$\to 0 \text{ as } h \to 0.$$

$$\therefore \quad \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}} dz$$

or $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Hence the formula holds for all values of n. Thus $f(z)$ has derivatives of all orders and these are all analytic at z_0. The theorem is thus completely established.

Another method to show that the result is true for n if it is true for $n-1$. Suppose that the formula is true for $n-1$ *i.e.*,

$$f^{n-1}(z_0) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^n} dz.$$

Then $$f^{n-1}(z_0+h) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0-h)^n} dz.$$

Now $$\frac{f^{n-1}(z_0+h) - f^{n-1}(z_0)}{h}$$

$$= \frac{(n-1)!}{2\pi ih} \int_C f(z) \left\{ \frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \right\} dz$$

$$= \frac{(n-1)!}{2\pi ih} \int_C \frac{f(z)}{(z-z_0)^n} \left[\frac{1}{\left(1 - \frac{h}{z-z_0}\right)^n} - 1 \right] dz$$

$$= \frac{(n-1)!}{2\pi ih} \int_C \frac{f(z)}{(z-z_0)^n} \left[\left(1 - \frac{h}{z-z_0}\right)^{-n} - 1 \right] dz$$

$$= \frac{(n-1)!}{2\pi i h} \int_C \frac{f(z)}{(z-z_0)^n} \left[1 + \frac{nh}{z-z_0} \right.$$

$$\left. + \text{ (terms containing higher powers of } h) - 1 \right] dz,$$

by binomial theorem

$$= \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^n} \left[\frac{n}{z-z_0} + \text{terms containing } h \text{ in Nr.} \right] dz$$

$\therefore$ $$\lim_{h \to 0} \frac{f^{n-1}(z_0 + h) - f^{n-1}(z_0)}{h} = \frac{(n-1)!}{2\pi i} \int_C \frac{n\, f(z)}{(z-z_0)^{n+1}} dz$$

or $$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Hence the result holds for all values of n.

Therefore $f(z)$ possesses derivatives of all orders and these are themselves all analytic at z_0.

17 Morera's Theorem

This theorem is a sort of converse of Cauchy-Goursat theorem.

Theorem 1: *If $f(z)$ be continuous in a simply connected domain D and*

$$\int_\Gamma f(z)\, dz = 0$$

where Γ is any rectifiable closed Jordan curve in D, then $f(z)$ is analytic in D.

(Meerut 2001; Kanpur 03, 04; Gorakhpur 07, 09; Purvanchal 08, 11; Rohilkhand 12)

Proof: Suppose z is any variable point and z_0 is a fixed point in the region D. Also suppose Γ_1 and Γ_2 are any two continuous rectifiable curves in D joining z_0 to z and Γ is the closed continuous rectifiable curve consisting of Γ_1 and $-\Gamma_2$. Then we have

$$\int_\Gamma f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz$$

and $$\int_\Gamma f(z)\, dz = 0 \text{ (given)}$$

$\therefore$ $$\int_{\Gamma_1} f(z)\, dz = -\int_{\Gamma_2} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz,$$

i.e., the integral along every rectifiable curve in D joining z_0 to z is the same.

Now consider a function $F(z)$ defined by

$$F(z) = \int_{z_0}^{z} f(w)\, dw. \qquad \ldots(1)$$

As we have discussed above the integral (1) depends only on the end points z_0 and z. If $z + h$ is a point in the neighbourhood of z, then we have

$$F(z+h) = \int_{z_0}^{z+h} f(w)\, dw. \qquad \ldots(2)$$

From (1) and (2), we have

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw$$

$$= \int_{z_0}^{z+h} f(w)\,dw + \int_{z}^{z_0} f(w)\,dw$$

$$= \int_{z}^{z+h} f(w)\,dw. \qquad \ldots(3)$$

Since the integral on the right hand side of (3) is path independent therefore it may be taken along the straight line joining z to $z+h$, so that

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(w)\,dw - \frac{f(z)}{h}\,h$$

$$= \frac{1}{h}\left[\int_{z}^{z+h} f(w)\,dw - f(z)\int_{z}^{z+h} dw\right]$$

$$= \frac{1}{h}\int_{z}^{z+h} [f(w) - f(z)]\,dw. \qquad \ldots(4)$$

The function $f(w)$ is given to be continuous at z therefore for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(w) - f(z)| < \varepsilon \qquad \ldots(5)$$

where $\quad |w - z| < \delta.$

Since h is arbitrary therefore choosing $|h| < \delta$ so that every point w lying on the line joining z to $z+h$ satisfies (5).

From (4) and (5), we have

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{|h|}\int_{z}^{z+h} |f(w) - f(z)|\,|dw|$$

$$< \frac{1}{|h|}\,\varepsilon\int_{z}^{z+h} |dw|, \qquad [\text{From } (5)]$$

$$= \frac{1}{|h|}\,\varepsilon\,|h| = \varepsilon.$$

Since ε is small and positive, therefore we have

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = 0 \quad \text{or} \quad \lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

Hence $\quad F'(z) = f(z)$

i.e., $F(z)$ is differentiable for all values of z in D. Consequently $F(z)$ is analytic in D. Since the derivative of an analytic function is analytic therefore $f(z)$ is analytic in D.

In view of Cauchy-Goursat theorem and Morera's theorem, we may state the following theorem.

Theorem 1(a): *Let $f(z)$ be continuous in a simply connected domain D and let C be any rectifiable closed curve in D. Then necessary and sufficient condition for $f(z)$ to be analytic in D is that*

$$\int_C f(z)\,dz = 0.$$

18 Cauchy's Inequality

Let $f(z)$ be analytic in a domain D and let D contain the interior and the boundary of the circle γ defined by $|z - z_0| = \rho$ and if $|f(z)| \le M$ on γ, then

$$|f^n(z_0)| \le n! \frac{M}{r^n}.$$

(Kanpur 2008; Kumaun 09, 11, 13; Gorakhpur 09, 11; Purvanchal 10, 11; Rohilkhand 12)

Proof: We have $f^n(z_0) = \dfrac{n!}{2\pi i} \displaystyle\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$

or
$$|f^n(z_0)| = \left| \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \le \frac{n!}{2\pi} \int_\gamma \frac{|f(z)|}{|z - z_0|^{n+1}} |dz|$$

$$\le \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \int_\gamma |dz| \qquad [\because |f(z)| \le M]$$

$$= \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} 2\pi r.$$

Hence
$$|f^n(z_0)| \le n! \cdot \frac{M}{r^n}.$$

Illustrative Examples

Example 6: *If $f(z) = \dfrac{z^2 + 5z + 6}{z - 2}$, does Cauchy's theorem apply*

(i) when the path of integration **C** *is a circle of radius* 3 *with origin as centre.*

(Kumaun 2015)

(ii) when **C** *is a circle of radius* 1 *with origin as centre.*

Solution: Here $f(z) = \dfrac{z^2 + 5z + 6}{z - 2}$.

Obviously $f(z)$ is not analytic at $z = 2$.

(i) When the path of integration is the circle $|z| = 3$, the point $z = 2$ lies inside C so $f(z)$ is not analytic within C therefore Cauchy's theorem is not applicable *i.e.,*

$$\int_C \frac{z^2 + 5z + 6}{z - 2} dz \ne 0.$$

(ii) When C is the circle $|z| = 1$, the point $z = 2$ lies outside C as a result $f(z)$ is analytic within and on C. Hence Cauchy's theorem is applicable *i.e.,*

$$\int_C \frac{z^2 + 5z + 6}{z - 2} dz = 0.$$

Example 7: *Verify Cauchy's theorem for the function* $z^3 - iz^2 - 5z + 2i$ *if C is the circle* $|z-1| = 2$. **(Garhwal 2010)**

Solution: We have $f(z) = z^3 - iz^2 - 5z + 2i$.

Since $f(z)$ is a polynomial in z therefore it is analytic within C.

On C we can choose $z - 1 = 2e^{i\theta}$, $0 \le \theta \le 2\pi$

or $\quad z = 1 + 2e^{i\theta}$. $\quad \therefore \quad dz = 2ie^{i\theta}\, d\theta$.

Now
$$\int_C f(z)\, dz = \int_0^{2\pi} [(1+2e^{i\theta})^3 - i(1+2e^{i\theta})^2 - 5(1+2e^{i\theta}) + 2i]\, 2ie^{i\theta}\, d\theta$$
$$= 2i \int_0^{2\pi} [8e^{4i\theta} + 4(3-i)e^{3i\theta} - 4(1+i)e^{2i\theta} + (-4+i)e^{i\theta}]\, d\theta$$
$$= 0, \text{ since } \int_0^{2\pi} e^{ik\theta} = 0 \text{ if } k \ne 0.$$

This verifies Cauchy's theorem for the function $f(z)$ and contour C.

Example 8: *Evaluate* $\int_C \frac{z-3}{z^2+2z+5}\, dz$ *where C is circle*

(a) $|z| = 1$ *and* *(b)* $|z+1-i| = 2$.

Solution: (a) We have
$$z^2 + 2z + 5 = z^2 + 2z + 1 + 4$$
$$= (z+1)^2 - (2i)^2 = (z+1+2i)(z+1-2i).$$

Let
$$\frac{z-3}{(z+1+2i)(z+1-2i)} = \frac{A}{z+1+2i} + \frac{B}{z+1-2i}.$$

$\therefore \quad z - 3 = A(z+1-2i) + B(z+1+2i)$

Putting $z = -1+2i$, we get
$$-1+2i-3 = A(0) + B(-1+2i+1+2i)$$

i.e., $\quad -4+2i = 4iB.$

$\therefore \quad B = \frac{2i-4}{4i} = \frac{1}{2} + i.$

Putting $z = -1-2i$, we get
$$-1-2i-3 = A(-1-2i+1-2i) \quad \text{or} \quad -4-2i = -4iA.$$

$\therefore \quad A = \frac{4+2i}{4i} = \frac{1}{2} - i.$

$$\therefore \quad \frac{z-3}{z^2+2z+5} = \frac{\frac{1}{2} - i}{z+1+2i} + \frac{\frac{1}{2} + i}{z+1-2i}.$$

$$\therefore \quad \int_C \frac{z-3}{z^2+2z+5}\, dz = \left(\frac{1}{2} - i\right) \int_C \frac{1}{z+1+2i}\, dz + \left(\frac{1}{2} + i\right) \int_C \frac{i}{z+1-2i}\, dz.$$

$f(z) = \frac{1}{z+1-2i}$ is analytic within and on the circle $|z| = 1$, as $z = -1-2i$ lies outside the circle $|z| = 1$.

$\therefore$ $$\int_C \frac{1}{z+1+2i}\, dz = 0, \quad \text{(by Cauchy's integral theorem).}$$

Similarly $f(z) = \dfrac{1}{z+1-2i}$ is analytic within and on the circle $|z| = 1$ as $z = -1 + 2i$ lies outside the circle $|z| = 1$.

$\therefore$ $$\int_C \frac{1}{z+1-2i}\, dz = 0 \quad \text{(by Cauchy's integral theorem).}$$

$\therefore$ $$\int_C \frac{z-3}{z^2+2z+5}\, dz = 0.$$

(b) $|z+1-i| = 2$ is the circle with centre $-1+i$ and radius 2.

The point $-1-2i$ lies outside the circle $|z+1-i| = 2$ and the point $-1+2i$ lies inside the circle $|z+1-i| = 2$.

$\therefore$ $$\int_C \frac{1}{z+1+2i}\, dz = 0, \text{ (by Cauchy's integral theorem)}$$

and $$\int_C \frac{1}{z+1-2i}\, dz = 2\pi i\,(1), \text{ since } f(z) = 1 \text{ and } f(-1+2i) = 1 = 2\pi i.$$

$\therefore$ $$\int_C \frac{z-3}{z^2+2z+5}\, dz = \left(\frac{1}{2} - i\right) 0 + \left(\frac{1}{2} + i\right) 2\pi i = \pi i - 2\pi = \pi(-2+i).$$

Comprehensive Exercise 2

1. Evaluate $\displaystyle\int_C \frac{z^2-4}{z(z^2+9)}\, dz$, where C is the circle $|z| = 1$.

2. Evaluate by Cauchy's integral formula $\displaystyle\int_C \frac{dz}{z(z+\pi i)}$, where C is $|z+3i| = 1$.

3. Find the value of $\displaystyle\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\, dz$ if C is the circle $|z| = 1$.

(Rohilkhand 2007)

4. Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, where the path of integration C is $|z| = 3$.

5. Evaluate $\displaystyle\int_C \frac{e^{3z}}{z+i}$ if C is the circle $|z+1+i| = 2$.

(Kumaun 2015)

6. Using Cauchy integral formula, calculate the following integrals :

(i) $\displaystyle\int_C \frac{z\, dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ described in positive sense.

(ii) $\displaystyle\int_C \frac{\cosh(\pi z)\, dz}{z(z^2+1)}$, where C is circle $|z| = 2$.

(iii) $\int_C \frac{e^{az}\, dz}{(z-\pi i)}$, where C is the ellipse $|z-2|+|z+2|=6$.

(iv) $\int_C \frac{dz}{z-2}$, where C is $|z|=3$.

(Kanpur 2003)

Answers 2

1. $-\frac{8\pi i}{9}$ 2. 0 3. $\frac{21}{16}\pi i$ 4. $\frac{8\pi i}{3e^2}$

5. $2\pi i e^{-3i}$ 6. (i) $\frac{\pi}{5}$, (ii) $4\pi i$, (iii) 0, (iv) $2\pi i$

19 Indefinite Integrals

Definition: *Suppose $f(z)$ is a single-valued analytic function in a simply connected region D, then a function $F(z)$ is called indefinite integral or primitive or anti-derivative of $f(z)$ if $F(z)$ is single-valued and analytic in D and $F'(z)=f(z)$, $z\in D$.*

Theorem 1: *A necessary and sufficient condition for the indefinite integral of a function $f(z)$ to exist in a simply connected domain D is that the function $f(z)$ is analytic in D.*

Also show that any two indefinite integrals of a function differ by a constant.

Proof: Condition is necessary: Let $F(z)$ be indefinite integral of $f(z)$. Then we have $F'(z)=f(z)$.

Therefore $F(z)$ is differentiable at every point $z\in D$. Consequently $F(z)$ is analytic in D. Since the derivative of an analytic function is analytic therefore $f(z)$ is analytic in D.

Condition is sufficient: Suppose $f(z)$ is analytic function in D. Take z_0 a fixed point and z any variable point in D. Then the integral of $f(z)$ along any path joining z_0 to z is the same.

Consider a function $F(z)$ defined by

$$F(z)=\int_{z_0}^{z} f(z)\, dz. \qquad \dots(1)$$

Now proceed as in Morera's theorem and prove that $F'(z)=f(z)$. Hence $F(z)$ given by (1) is the indefinite integral of $f(z)$. Thus every analytic function possesses indefinite integral in a simply connected domain.

Now we shall show that two indefinite integrals of a function differ by a constant.

Suppose $\phi(z)$ and $\psi(z)$ are two indefinite integrals of $f(z)$.

Then $\phi'(z)=f(z)$ and $\psi'(z)=f(z)$

so that $\phi'(z)=\psi'(z)$ or $\phi'(z)-\psi'(z)=0$

or $[\phi(z) - \psi(z)]' = 0$ or $\phi(z) - \psi(z) = c$, where c is any constant.

$\therefore$ $\phi(z) = \psi(z) + c$.

Hence the general indefinite integral of an analytic function $f(z)$ is given by

$$F(z) + c, \text{ where } F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta.$$

Fundamental Theorem of Integral Calculus For Complex Functions:

Theorem 2: *If $f(z)$ is a single valued analytic function in a simply connected domain D, then for $a, b \in D$ we have*

$$\int_a^b f(z)\, dz = F(b) - F(a), \text{ where } F(z) \text{ is any indefinite integral of } f(z).$$

Proof: We have $F(z)$ as an indefinite integral of $f(z)$, therefore

$$F'(z) = f(z)$$

and $F(z)$ is given by $F(z) = \int_{z_0}^{z} f(z)\, dz$, ...(1)

where z_0 is any fixed point and z is any variable point in D.

Now $F(b) = \int_{z_0}^{b} f(z)\, dz$ and $F(a) = \int_{z_0}^{a} f(z)\, dz$

$$\therefore \quad F(b) - F(a) = \int_{z_0}^{b} f(z)\, dz - \int_{z_0}^{a} f(z)\, dz$$

$$= \int_{z_0}^{b} f(z)\, dz + \int_{a}^{z_0} f(z)\, dz = \int_a^b f(z)\, dz.$$ **Proved.**

Since we have $f(z) = F'(z)$ therefore we can also write

$$F(b) - F(a) = \int_a^b F'(z)\, dz.$$

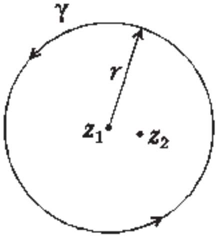
20 Integral Function: (Entire Function)

If a function $f(z)$ is analytic in every finite region of the z–plane, it is called an integral function or entire function.

Liouville's Theorem: *If $f(z)$ is an integral function satisfying the inequality $|f(z)| \le M$ for all values of z, where M is a positive constant, then $f(z)$ is constant.*

(Meerut 2000, 01; Bundelkhand 01; Avadh 07; Rohilkhand 07; Purvanchal 10)

Proof: Suppose z_1 and z_2 are any two points in the z–plane. With z_1 as centre and radius r draw a circle γ whose equation is $|z - z_1| = r$. Take r sufficiently large so that z_2 lies inside γ and $|z_2 - z_1| < \frac{1}{2} r$.



By Cauchy's integral formula, we have

$$f(z_1) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_1}\, dz$$

and $$f(z_2) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_2} dz.$$

Now $$f(z_2) - f(z_1) = \frac{1}{2\pi i} \int_\gamma f(z) \left[\frac{1}{z - z_2} - \frac{1}{z - z_1} \right] dz$$

$$= \frac{z_2 - z_1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_1)(z - z_2)} dz \cdot \qquad \ldots(1)$$

We have $|z - z_1| = r$,

$$|z - z_2| = |z - z_1 + z_1 - z_2| = |z - z_1 - (z_2 - z_1)|$$

$$\geq |z - z_1| - |z_2 - z_1| \geq r - \frac{1}{2} r = \frac{1}{2} r$$

and $|f(z)| \leq M$ (given).

Taking modulus of both sides of (1), we have

$$|f(z_2) - f(z_1)| = \left| \frac{(z_2 - z_1)}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_1)(z - z_2)} dz \right|$$

$$\leq \frac{|z_2 - z_1|}{2\pi} \int_\gamma \frac{|f(z)|}{|z - z_1||z - z_2|} |dz|$$

$$\leq \frac{|z_2 - z_1|}{2\pi} \cdot \frac{M}{r \cdot \frac{1}{2} r} \int_\gamma |dz|$$

$$\leq \frac{|z_2 - z_1|}{\pi} \cdot \frac{M}{r^2} 2\pi r = \frac{2|z_2 - z_1| M}{r}$$

$$\to 0 \text{ as } r \to \infty.$$

Consequently $f(z_2) - f(z_1) = 0$ or $f(z_2) = f(z_1)$.

Hence $f(z)$ is constant.

Alternative Proof: By Cauchy's inequality, we have

$|f^n(z_0)| \leq n! \dfrac{M}{r^n}$ where z_0 is any point in the z–plane and r is the radius of the circle γ defined by $|z - z_0| = r$. For $n = 1$, we have

$$|f'(z_0)| \leq \frac{M}{r} \cdot$$

As $r \to \infty$, $\qquad f'(z_0) = 0.$

Since the point z_0 is arbitrary therefore we conclude that $f'(z)$ vanishes at every point in the z-plane. Hence $f(z)$ is constant.

21 Expansion of Analytic Functions as Power Series

Theorem 1: Taylor's Theorem: *Let $f(z)$ be analytic at all points within a circle C_0 with centre z_0 and radius r_0. Then for every point z within C_0, we have*

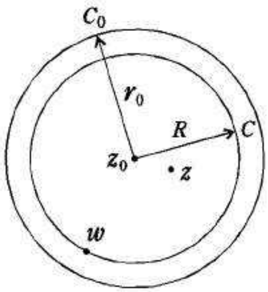
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots\ldots$$

$$+ \frac{f^n(z_0)}{n!}(z - z_0)^n + \ldots\ldots$$

$$= f(z_0) + \sum_{n=1}^{\infty} \frac{(z - z_0)^n}{n!} f^n(z_0).$$

(Kanpur 2001; Purvanchal 07; Gorakhpur 10; Rohilkhand 09, 12; Garhwal 10)

Proof: Consider a circle C_0 with centre z_0 and radius r_0.



Suppose z is any point inside the circle $|z - z_0| = r$. Draw a circle C with centre z_0 and radius R such that $r < R < r_0$ so that the point z lies inside C. If w is any point on C, equation of C is given by $|w - z_0| = R$.

Also by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw. \qquad \ldots(1)$$

Consider the identity

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{(w - z_0)\left(1 - \dfrac{z - z_0}{w - z_0}\right)}$$

$$= \frac{1}{w - z_0}\left[1 - \frac{z - z_0}{w - z_0}\right]^{-1}$$

$$= \frac{1}{w - z_0}\left[1 + \frac{z - z_0}{w - z_0} + \left(\frac{z - z_0}{w - z_0}\right)^2 + \ldots\ldots + \left(\frac{z - z_0}{w - z_0}\right)^{n-1} + \left(\frac{z - z_0}{w - z_0}\right)^n \frac{1}{1 - \dfrac{z - z_0}{w - z_0}}\right]$$

$$= \frac{1}{w - z_0} + \frac{z - z_0}{(w - z_0)^2} + \frac{(z - z_0)^2}{(w - z_0)^3} + \ldots\ldots + \frac{(z - z_0)^{n-1}}{(w - z_0)^n} + \frac{(z - z_0)^n}{(w - z_0)^n (w - z)}.$$

Multiplying both sides by $\dfrac{f(w)}{2\pi i}$ and integrating around C, we get

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} dw + \frac{(z - z_0)}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^2} dw$$

$$+ \ldots\ldots + \frac{(z - z_0)^{n-1}}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^n} dw + \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(w)}{(w - z)(w - z_0)^n} dw$$

or $$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2!} + \ldots\ldots$$
$$+ (z - z_0)^{n-1} \frac{f^{n-1}(z_0)}{(n-1)!} + S_n \qquad \ldots(2)$$

where $$S_n = \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(w)}{(w - z)(w - z_0)^n} dw.$$

In order to get the desired result we have to show that

$$S_n \to 0 \text{ as } n \to \infty.$$

We have $|z - z_0| = r$, $|w - z_0| = R$.

$\therefore$ $|w - z| = |w - z_0 - (z - z_0)| \geq |w - z_0| - |z - z_0| = R - r.$

If M denotes the greatest value of $f(w)$ on C, we have

$$|S_n| = \left| \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(w)}{(w - z)(w - z_0)^n} dw \right|$$
$$\leq \frac{|z - z_0|^n}{2\pi} \int_C \frac{|f(w)|}{|w - z||w - z_0|^n} |dw|$$
$$\leq \frac{r^n}{2\pi} \cdot \frac{M}{(R - r) R^n} \int_C |dw|$$
$$= \frac{M \cdot R}{(R - r)} \left(\frac{r}{R} \right)^n \qquad \left[\because \int_C |dw| = 2\pi R \right]$$

Since $r < R$ therefore $(r/R)^n \to 0$ as $n \to \infty$. Consequently $S_n \to 0$ as $n \to \infty$. Hence as $n \to \infty$, the limit of the sum of the first n terms on the right hand side of (2) is $f(z)$, so we can represent $f(z)$ by the infinite series

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2!} + \ldots\ldots$$
$$+ (z - z_0)^n \frac{f^n(z_0)}{n!} + \ldots\ldots$$
$$= f(z_0) + \sum_{n=1}^{\infty} (z - z_0)^n \frac{f^n(z_0)}{n!}.$$

It is known as **Taylor's series.**

If we put $z_0 = 0$ in the above series, we get

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} f^n(0),$$

which is known as **Maclaurin's series.**

Remark: For the validity of the expansion as a **Taylor's series**, it is essential that $f(y)$ be analytic at all points inside the circle C_0 for then the convergence of Taylor's series for $f(z)$ is assured.

Hence the greatest radius of C_0 is the distance from the point z_0 to the singularity of $f(z)$ which is nearest to z_0, since we require the function to be analytic at all points within C_0.

Theorem 2: Laurent's Theorem: *Let $f(z)$ be analytic in the ring shaped region D bounded by two concentric circles C_1 and C_2 with centre z_0 and radii r_1 and r_2 $(r_1 > r_2)$ and let z be any point of D. Then*

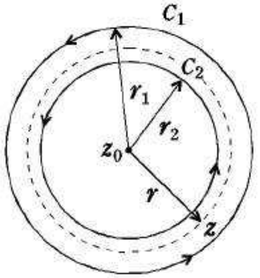
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where $$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^{n+1}} \, dw,$$

and $$b_n = \frac{1}{2\pi i} \int_{C_2} (w - z_0)^{n-1} f(w) \, dw, \; n = 1, 2, 3, \ldots\ldots$$

(Gorakhpur 2007, 13; Rohilkhand 11; Purvanchal 12)

Proof: The function $f(z)$ is given to be analytic in the ring shaped region D bounded by concentric circles C_1 and C_2 with centre z_0 and radii r_1 and r_2 $(r_1 > r_2)$. Let z be any point in the region D. Then by Cauchy's integral formula for doubly connected region, **we** have



$$f(z) = \frac{1}{2\pi i} \left[\int_{C_1} \frac{f(w)}{w - z} \, dw - \int_{C_2} \frac{f(w)}{w - z} \, dw \right]. \qquad \ldots(1)$$

For any point w on C_1, we have the identity

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{(w - z_0)\left(1 - \dfrac{z - z_0}{w - z_0}\right)}$$

$$= \frac{1}{w - z_0} \left[1 + \frac{z - z_0}{w - z_0} + \left(\frac{z - z_0}{w - z_0}\right)^2 + \ldots\ldots + \left(\frac{z - z_0}{w - z_0}\right)^{n-1} + \left(\frac{z - z_0}{w - z_0}\right)^n \frac{1}{1 - \dfrac{z - z_0}{w - z_0}} \right]$$

$$= \frac{1}{w - z_0} + \frac{z - z_0}{(w - z_0)^2} + \frac{(z - z_0)^2}{(w - z_0)^3} + \ldots + \frac{(z - z_0)^{n-1}}{(w - z_0)^n} + \frac{(z - z_0)^n}{(w - z_0)^n (w - z)}.$$

Multiplying both sides by $f(w)/2\pi i$ and integrating around C_1, we get

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z_0} \, dw + \frac{(z - z_0)}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^2} \, dw + \ldots$$

$$\ldots + \frac{(z - z_0)^{n-1}}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^n} \, dw + R_n$$

where $$R_n = \frac{(z - z_0)^n}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^n (w - z)} \, dw.$$

Putting $\qquad a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \dfrac{f(w)}{(w - z_0)^{n+1}}\, dw$ in the above relation, we get

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots\ldots$$

$$+ a_{n-1} (z - z_0)^{n-1} + R_n. \qquad \ldots(2)$$

Now we shall show that $R_n \to 0$ as $n \to \infty$.

Suppose $\qquad |z - z_0| = r$ so that $r_2 < r < r_1$.

We have $\qquad |w - z_0| = r_1$.

$\therefore \qquad |w - z| = |w - z_0 - (z - z_0)| \geq |w - z_0| - |z - z_0| = r_1 - r$.

Now
$$|R_n| = \left| \frac{(z - z_0)^n}{2\pi i} \int_{C_1} \frac{f(w)}{(w - z_0)^n (w - z)}\, dw \right|$$

$$\leq \frac{|z - z_0|^n}{2\pi} \int_{C_1} \frac{|f(w)|}{|w - z_0|^n\, |w - z|}\, |dw|$$

$$\leq \frac{r^n . M_1}{2\pi\, r_1^n (r_1 - r)} \int_{C_1} |dw|$$

where M_1 is the greatest value of $f(w)$ on C_1

$$= \frac{r^n . M_1}{2\pi\, r_1^n (r_1 - r)}\, 2\pi r_1 = \frac{M_1\, r_1}{r_1 - r} \left(\frac{r}{r_1} \right)^n .$$

Since $r < r_1$, therefore $(r / r_1)^n \to 0$ as $n \to \infty$. Consequently $R_n \to 0$ as $n \to \infty$.

$\therefore$ from (2), we have

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z}\, dw = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \qquad \ldots(3)$$

Again for the second integral in (1), consider the identity

$$-\frac{1}{w - z} = \frac{1}{(z - z_0) - (w - z_0)}, \quad \text{where } w \text{ is any point on } C_2$$

$$= \frac{1}{(z - z_0)\left(1 - \dfrac{w - z_0}{z - z_0}\right)}$$

$$= \frac{1}{z - z_0} \left[1 + \frac{w - z_0}{z - z_0} + \left(\frac{w - z_0}{z - z_0} \right)^2 + \ldots\ldots + \left(\frac{w - z_0}{z - z_0} \right)^{n-1} + \left(\frac{w - z_0}{z - z_0} \right)^n \frac{1}{1 - \dfrac{w - z_0}{z - z_0}} \right]$$

$$= \frac{1}{z - z_0} + \frac{w - z_0}{(z - z_0)^2} + \frac{(w - z_0)^2}{(z - z_0)^3} + \ldots\ldots$$

$$+ \frac{(w - z_0)^{n-1}}{(z - z_0)^n} + \frac{(w - z_0)^n}{(z - z_0)^n (z - w)} .$$

$$\therefore \qquad -\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i(z-z_0)}\int_{C_2} f(w)\,dw$$
$$+\frac{1}{2\pi i(z-z_0)^2}\int_{C_2}(w-z_0)\,f(w)\,dw + \ldots\ldots$$
$$+\frac{1}{2\pi i(z-z_0)^n}\int_{C_2}(w-z_0)^{n-1}\,f(w)\,dw + P_n,$$

where $$P_n = \frac{1}{2\pi i(z-z_0)^n}\int_{C_2}\frac{(w-z_0)^n\,f(w)}{z-w}\,dw.$$

Putting $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}(w-z_0)^{n-1}\,f(w)\,dw$ in the above relation, we get

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \ldots\ldots$$
$$+ b_n(z-z_0)^{-n} + P_n. \qquad \ldots(4)$$

Now we have to show that $P_n \to 0$ as $n \to \infty$. We have $|z-z_0| = r$, $|w-z_0| = r_2$ for C_2 where $r_2 < r$.

$$\therefore \qquad |z-w| = |(z-z_0)-(w-z_0)| \geq |z-z_0| - |w-z_0| = r - r_2.$$

Now $$|P_n| \leq \frac{1}{2\pi|z-z_0|^n}\int_{C_2}\frac{|w-z_0|^n\,|f(w)|}{|z-w|}\,|dw|$$
$$\leq \frac{1}{2\pi r^n}\cdot\frac{r_2{}^n M_2}{(r-r_2)}\int_{C_2}|dw|,$$

where M_2 is the greatest value of $f(w)$ on C_2

$$= \frac{M_2 r_2}{r-r_2}\left(\frac{r_2}{r}\right)^n.$$

$\therefore$ $P_n \to 0$ as $n \to \infty$, since $r_2 < r$.

Thus from (4), we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}. \qquad \ldots(5)$$

Substituting the values from (3) and (5) in (1), we get

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n},$$

where $$a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-z_0)^{n+1}}\,dw \quad\text{and}\quad b_n = \frac{1}{2\pi i}\int_{C_2}(w-z_0)^{n-1}\,f(w)\,dw.$$

Remark: We have $b_n = a_{-n}$. Therefore if C is any circle of radius r and centre z_0 such that $r_2 < r < r_1$ then since the integrand is analytic in $r_2 < |w-z_0| < r_1$, we can write

$$a_n = \frac{1}{2\pi i}\int_C\frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

and $$b_n = a_{-n} = \frac{1}{2\pi i}\int_C(w-z_0)^{n-1}\,f(w)\,dw.$$

In this case the resulting series becomes

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

Theorem 3: Uniqueness Theorem: *Suppose that we have obtained in any manner or as the definition of $f(z)$, the formula*

$$f(z)=\sum_{n=-\infty}^{\infty} P_n (z-z_0)^n, \ (r_2<|z-z_0|<r_1)$$

then the series is necessarily identical with Laurent's series of $f(z)$.

Proof: Suppose C is the circle defined by $|z-z_0|=r$, where $r_2<r<r_1$. Then the Laurent's series is given by

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where
$$a_n=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

$$=\frac{1}{2\pi i}\int_C \frac{1}{(w-z_0)^{n+1}}\sum_{m=-\infty}^{\infty} P_m (w-z_0)^m\,dw$$

$$=\frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m \int_C \frac{(w-z_0)^m}{(w-z_0)^{n+1}}\,dw$$

[Term by term integration is possible since the series is uniformly convergent on every closed subset of the annulus]

$$=\frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \frac{r^m e^{im\theta}}{r^{n+1} e^{i(n+1)\theta}}\, r i e^{i\theta}\, d\theta,$$

putting $w-z_0=re^{i\theta}$

$$=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} r^{m-n} e^{i(m-n)\theta}\, d\theta. \qquad \ldots(1)$$

When $m\neq n$, we have

$$\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta=\left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi}=\frac{1}{i(m-n)}[e^{i(m-n)2\pi}-e^0]$$

$$=\frac{1}{i(m-n)}(1-1)=0$$

and when $m=n$, we have

$$\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta=\int_0^{2\pi} d\theta=2\pi.$$

$\therefore$ We have $a_n=\dfrac{1}{2\pi}\cdot P_n\,.\,2\pi$, from (1)

$$=P_n.$$

Hence the given series is identical with the Laurent's series of $f(z)$.

Illustrative Examples

Example 9: *Expand* $f(z)=\dfrac{1}{(z+1)(z+3)}$ *in a Laurent's series valid for the regions*

(i) $|z|<1$ *(ii)* $1<|z|<3$

(iii) $|z|>3$ *(iv)* $0<|z+1|<2.$

(Purvanchal 2007, 09, 12; Gorakhpur 15)

Solution: We have $f(z)=\dfrac{1}{(z+1)(z+3)}\cdot$

Resolving into partial fractions, we get

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}\cdot$$

(i) $|z|<1$.

We have
$$f(z)=\frac{1}{2}(1+z)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$
$$=\frac{1}{2}[1-z+z^2-z^3+\ldots\ldots]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\ldots\right]$$
$$=\left(\frac{1}{2}-\frac{1}{6}\right)-\left(\frac{1}{2}-\frac{1}{18}\right)z+\left(\frac{1}{2}-\frac{1}{54}\right)z^2-\ldots\ldots$$
$$=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\ldots\ldots$$

(ii) $1<|z|<3$

Then we have $\dfrac{1}{|z|}<1$ and $\dfrac{|z|}{3}<1.$

Now
$$\frac{1}{2(z+1)}=\frac{1}{2z\left(1+\dfrac{1}{z}\right)}=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}$$
$$=\frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\ldots\right)=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\ldots\ldots$$

and
$$\frac{1}{2(z+3)}=\frac{1}{6\left(1+\dfrac{z}{3}\right)}=\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}=\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\ldots\ldots\right)$$
$$=\frac{1}{6}-\frac{1}{18}z+\frac{1}{54}z^2-\frac{1}{162}z^3+\ldots\ldots.$$

Thus the Laurent's series valid for the region $1<|z|<3$ is

$$f(z)=\ldots\ldots+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}+\frac{1}{6}-\frac{1}{18}z+\frac{1}{54}z^2-\frac{1}{162}z^3+\ldots\ldots$$

(iii) $|z| > 3$ .Then $(3/|z|) < 1$.

$$\therefore \quad f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\ldots\right) - \frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\ldots\ldots\right]$$

$$= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\ldots\ldots$$

(iv) $0 < |z+1| < 2$.

Let $z+1 = u$. Then we have $0 < |u| < 2$.

$$\therefore \quad f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\ldots\ldots\right] = \frac{1}{2u}-\frac{1}{4}+\frac{u}{8}-\frac{u^2}{16}+\ldots\ldots$$

$$= \frac{1}{2(z+1)}-\frac{1}{4}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+\ldots\ldots$$

Example 10: *Obtain the Taylor's and Laurent's series which represent the function*

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} \text{ in the regions}$$

(i) $|z| < 2$ **(Garhwal 2010)** *(ii)* $2 < |z| < 3$ **(Gorakhpur 2009, 11, 13)**

(iii) $|z| > 3$. **(Avadh 2008; Gorakhpur 09, 11, 13)**

Solution: Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = 1 - \dfrac{5z+7}{(z+2)(z+3)}$.

Resolving into partial fractions, we get

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

(i) $|z| < 2$. We have

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\ldots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\ldots\right)$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n.$$

(ii) $2 < |z| < 3$. Then $\dfrac{2}{|z|} < 1$ and $\dfrac{|z|}{3} < 1$.

$$\therefore \quad f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \ldots\ldots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \ldots\ldots\right)$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n.$$

(iii) $|z| > 3$. Then $\frac{3}{|z|} < 1$.

$$\therefore \qquad f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\ldots\right]$$

$$- \frac{8}{3}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \ldots\ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n.$$

Example 11: *Find different developments of* $\frac{1}{(z-1)(z-3)}$ *in powers of z according to the position of the point in the z-plane. Expand the function in Taylor's series about* $z = 2$ *and indicate the circle of convergence.*

Solution: Let $f(z) = \frac{1}{(z-1)(z-3)}$.

Resolving into partial fractions, we get

$$f(z) = -\frac{1}{2(z-1)} + \frac{1}{2(z-3)}.$$

Obviously $f(z)$ is regular everywhere except at $z = 1$ and 3.

(i) $0 < |z| < 1$.

$$f(z) = \frac{1}{2}(1-z)^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1} = \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}\left(1 - \frac{1}{3^{n+1}}\right)z^n$$

which is Taylor's expansion of $f(z)$ in $0 < |z| < 1$.

(ii) $1 < |z| < 3$.

$$f(z) = -\frac{1}{2z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1} = -\frac{1}{2z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

which is Laurent's series in the positive and negative powers of z in the region $1 < |z| < 3$.

(iii) $|z| > 3$.

$$f(z) = -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= -\frac{1}{2z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}(3^n - 1)\frac{1}{z^{n+1}}$$

which is a Laurent's series in the negative powers of z for $|z| > 3$.

Consider a circle with centre at $z = 2$. Then the distance of both the singularities $z = 1$ and $z = 3$ from the centre of the circle is 1. Hence if we draw the circle $|z-2| = 1$ then the function $f(z)$ is regular within this circle so that $f(z)$ can be expanded in a Taylor's series within this circle *i.e.*, in the region $|z-2| < 1$. Consequently $|z-2| = 1$ is the circle of convergence.

Now $$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{z^2 - 4z + 3} = \frac{1}{(z-2)^2 - 1}$$

$$= -[1-(z-2)^2]^{-1} = -\sum_{n=0}^{\infty}(z-2)^{2n},$$

which is a Taylor's expansion of $f(z)$ about $z = 2$.

Example 12: *Expand* $\log(1+z)$ *in a Taylor's series about* $z = 0$ *and determine the region of convergence for the resulting series.*

Solution: Let $f(z) = \log(1+z)$.

Taylor's expansion for $f(z)$ about $z = 0$ is given by

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!}f''(0) + \ldots\ldots \qquad \ldots(1)$$

We have $$f'(z) = \frac{1}{1+z},\ f''(z) = -\frac{1}{(1+z)^2},$$

$$f'''(z) = \frac{2}{(1+z^2)^3}, \ldots\ldots, f^n(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n}.$$

$\therefore$ $$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 2, \ldots\ldots,$$
$$f^n(0) = (-1)^{n-1}(n-1)!.$$

Substituting these values in (1), we get

$$\log(1+z) = z - \frac{1}{2!}z^2 + \frac{2}{3!}z^3 - \ldots\ldots + \frac{(-1)^{n-1}(n-1)!\,z^n}{n!} + \ldots\ldots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots + (-1)^{n-1}\frac{z^n}{n} + \ldots\ldots$$

Let u_n be the nth term of the series. Then we have

$$u_n = (-1)^{n-1}\frac{z^n}{n},\ u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}$$

$\therefore$ $$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{nz}\right| = \frac{1}{|z|}.$$

Hence by D'Alembert's ratio test the series converges for $|z| < 1$.

Example 13: *Find the Laurent expansion of* $\dfrac{z}{(z+1)(z+2)}$ *about the singularity* $z = -2$. *Specify the region of convergence.*

Solution: We have $f(z) = \dfrac{z}{(z+1)(z+2)} = \dfrac{2}{z+2} - \dfrac{1}{z+1}$

or $$f(z) = \frac{2}{(z+2)} - \frac{1}{z+1}. \qquad \text{...(1)}$$

To find Laurent expansion for $\phi(z) = \dfrac{1}{z+1}$ about $z = -2$, we write

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z+2)^n \qquad \text{...(2)}$$

where $a_n = \dfrac{\phi^{(n)}(-2)}{n!}$. But $\phi^{(n)}(z) = \dfrac{(-1)^n n!}{(z+1)^{n+1}}$.

$\therefore$ $$\frac{\phi^{(n)}(-2)}{n!} = \frac{(-1)^n}{(-2+1)^{n+1}} = \frac{(-1)^n}{(-1)^{n+1}} = -1 \quad \text{or} \quad a_n = \frac{\phi^{(n)}(-2)}{n!} = -1$$

Putting this in (2), we get

$$\frac{1}{z+1} = \phi(z) = \sum_{n=0}^{\infty} (-1)(z+2)^n \quad \text{or} \quad -\frac{1}{z+1} = \sum_{n=0}^{\infty} (z+2)^n.$$

Now (1) reduces to $f(z) = \dfrac{2}{2+z} + \sum_{n=0}^{\infty} (z+2)^n$. ...(3)

This is the required expansion.

Second Part: Let $\sum_{n=0}^{\infty} (z+2)^n = \Sigma u_n$.

Then $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(z+2)^{n+1}}{(z+2)^n}\right| = |z+2|.$$

Series will be convergent if $\left|\dfrac{u_{n+1}}{u_n}\right| < 1$ *i.e.,* if $|z+2| < 1$.

$\therefore$ Radius of convergence = 1.

Series is convergent $\forall$ z inside the circle whose centre is $z = -2$ and radius = 1.

Example 14: *Prove that* $\tan^{-1} z = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \ldots\ldots$ when $|z| < 1$.

Solution: Let $f(z) = \tan^{-1} z$.

By Taylor's theorem, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + \ldots\ldots$$

Taking $z_0 = 0$, we get

$$f(z) = f(0) + f'(0)\, z + \frac{z^2}{2!} f''(0) + \ldots\ldots \qquad \text{...(1)}$$

We have $f(z) = \tan^{-1} z$ so that $f(0) = \tan^{-1} 0 = 0$.

$$f'(z) = \frac{1}{1+z^2}, \quad f''(z) = -\frac{2z}{(1+z^2)^2},$$

$$f''' z = -\frac{2(1-3z^2)}{(1+z^2)^3}, \quad f^{iv}(z) = -\frac{24(-z+z^3)}{(1+z^2)^4},$$

$$f^{v}(z) = -\frac{24(-1+10z^2-5z^4)}{(1+z^2)^5} \text{ and so on.}$$

$\therefore$ $$f'(0) = 1, \; f''(0) = 0, \; f'''(0) = -2, \; f^{iv}(0) = 0, \; f^{v}(0) = 24 \text{ etc.}$$

Substituting all these values in relation (1), we get

$$\tan^{-1} z = z - \frac{1}{3} z^3 + \frac{1}{5} z^5 - \ldots\ldots$$

Example 15: *Prove that* $\cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n}\right)$

where $a_n = \frac{1}{2\pi} \int_0^{2\pi} \cosh n\theta \cosh(2\cos\theta)\, d\theta.$

(Gorakhpur 2007, 09; Rohilkhand 12; Purvanchal 08)

Solution: The function $\cosh\left(z + \frac{1}{z}\right)$ is analytic in every finite part of the z-plane except at $z = 0$. Thus the given function is analytic in the annulus $r \le |z| \le R$ where r is small and R is large so that we can expand $f(z)$ in a Laurent's series in the annulus $r < |z| < R$.

$\therefore$ $$\cosh\left(z + \frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n},$$

where $$a_n = \frac{1}{2\pi i} \int_C \cosh\left(z + \frac{1}{z}\right) \frac{dz}{z^{n+1}} \text{ and } b_n = \frac{1}{2\pi i} \int_C \cosh\left(z + \frac{1}{z}\right) z^{n-1}\, dz,$$

C is a circle with centre at origin.

Let C be the unit circle defined by $|z| = 1$. Then $z = e^{i\theta}, dz = ie^{i\theta}\, d\theta$.

$\therefore$ $$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \cosh(e^{i\theta} + e^{-i\theta}) \frac{ie^{i\theta}}{e^{i(n+1)\theta}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\theta)(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\theta) \cos n\theta\, d\theta,$$

{since the other integral becomes zero by the property $\int_0^{2a} f(\theta)\, d\theta = 0$ if $f(2a - \theta) = -f(\theta)$}

Now the function $\cosh\left(z+\dfrac{1}{z}\right)$ remains unchanged by replacing z by $1/z$, therefore we have

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\,d\theta = a_n.$$

Hence $$\cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

$$= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n} \qquad [\because b_n = a_n]$$

$$= a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n}),$$

where $$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\,d\theta.$$

Example 16: *Show that* $e^{\frac{1}{2}c(z-1/z)} = \sum_{n=-\infty}^{\infty} a_n z^n,$

where $$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\,d\theta.$$

(Meerut 2001, 02; Gorakhpur 2004, 06, 08, 11; Avadh 07)

Solution: The given function is analytic at every point in the z-plane except at $z=0$ so it is analytic in the annulus $r<|z|<R$ where r is small and R is large. Therefore it can be expanded in a Laurent's series in the region $r<|z|<R$.

$\therefore$ $$e^{\frac{1}{2}c(z-1/z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_C f(z)\frac{dz}{z^{n+1}},\quad b_n = \frac{1}{2\pi i}\int_C f(z)\,z^{n-1}\,dz$$

and C is any circle with centre at origin.

Let C be the unit circle defined by $|z|=1$.

Then $z = e^{i\theta}, dz = i e^{i\theta}\,d\theta.$

Now $$a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{1}{2}c(e^{i\theta}-e^{-i\theta})}\,\frac{i e^{i\theta}}{e^{i(n+1)\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ic\sin\theta}\,e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - c\sin\theta)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \{\cos(n\theta - c\sin\theta) - i\sin(n\theta - c\sin\theta)\}\,d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos (n\theta - c \sin \theta)\, d\theta, \qquad \ldots(1)$$

the second integral becomes zero by the property of definite integrals.

Since the given function remains unchanged if z is replaced by $(-1/z)$ therefore $b_n = (-1)^n a_n$.

Hence $$e^{\frac{1}{2} c (z - 1/z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

$$= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} (-1)^n a_n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_n z^n, \text{where } a_n \text{ is given by (1).}$$

Example 17: *If the function $f(z)$ is analytic when $|z| < R$ and has the Taylor's expansion $\sum_{n=0}^{\infty} a_n z^n$, show that for $r < R$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

(Avadh 2007)

Hence prove that if $|f(z)| \le M$ where $|z| < R$,

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2.$$

Solution: Here $f(z)$ is analytic within the circle $|z| = r, (r < R)$ therefore it can be expanded in a Taylor's series within the circle $|z| = r$.

$\therefore$ $$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{or} \quad f(r e^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \quad \text{where } z = r e^{i\theta}.$$

If $\overline{a_n}$ is the conjugate of a_n, we have

$$\overline{f(r e^{i\theta})} = \sum_{m=0}^{\infty} \overline{a_m}\, r^m e^{-im\theta}.$$

Now $$|f(r e^{i\theta})|^2 = f(r e^{i\theta})\, \overline{f(r e^{i\theta})}$$

$$= \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m}\, r^m e^{-im\theta}.$$

$\therefore$ $$\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left(\sum_{m=0}^{\infty} \overline{a_m}\, r^m e^{-im\theta} \right) d\theta. \qquad \ldots(1)$$

Since the two series for $f(r e^{i\theta})$ and $\overline{f(r e^{i\theta})}$ are absolutely convergent therefore their product is uniformly convergent for $0 \le \theta \le 2\pi$. Hence the term by term integration is justified.

Also we have

$$\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = 0 \text{ if } n \ne m$$

$$= 2\pi \text{ if } n = m.$$

Hence from (1), we have

$$\int_0^{2\pi} |f(r e^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} a_n \overline{a_n} \, r^{2n} \int_0^{2\pi} d\theta$$

$$= 2\pi \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n}$$

or $$\frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n}.$$

This proves the first result.

Again it is given that $|f(z)| \leq M$ when $|z| < R$.

$$\therefore \quad \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 \, d\theta \qquad [\because z = r e^{i\theta}]$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} M^2 \, d\theta = \frac{1}{2\pi} M^2 \, . 2\pi = M^2.$$

Hence $$\sum_{n=0}^{\infty} |a_n|^2 \, r^{2n} \leq M^2.$$ **Proved.**

Example 18: *If the function $f(z)$ is analytic and one valued in $|z - a| < R$, prove that when* $0 < r < R$,

$$f'(a) = \frac{1}{\pi r} \int_0^{2\pi} P(\theta) \, e^{-i\theta} \, d\theta$$

where $P(\theta)$ is the real part of $f(a + r e^{i\theta})$. **(Kumaun 2008)**

Solution: The function $f(z)$ is given to be analytic in $|z - a| < R$ and $r < R$ therefore $f(z)$ is also analytic inside the circle C defined by $|z - a| = r$ so that

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - a)^2} \, dz. \qquad \ldots(1)$$

Also we can expand $f(z)$ in a Taylor's series about $z = a$.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

or $$f(z) = f(a + re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \text{ putting } z - a = re^{i\theta}.$$

Then $$\overline{f(z)} = \sum_{n=0}^{\infty} \overline{a_n} \, r^n e^{-in\theta}.$$

Now $$\frac{1}{2\pi i} \int_C \frac{\overline{f(z)}}{(z - a)^2} \, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Sigma \, \overline{a_n} \, r^n e^{-in\theta}}{r^2 e^{i2\theta}} \, rie^{i\theta} \, d\theta$$

$$= \frac{1}{2\pi} \Sigma \, \overline{a_n} \, r^{n-1} \int_0^{2\pi} e^{-i(n+1)\theta} \, d\theta$$

$$= 0. \qquad \ldots(2)$$

From (1) and (2), we have

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\,dz$$

$$=\frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}\,dz$$

$$=\frac{1}{\pi i}\int_0^{2\pi} \frac{\text{real part of } f(a+re^{i\theta})\, ire^{i\theta}\, d\theta}{r^2 e^{i2\theta}} \qquad [\because z=a+re^{i\theta}]$$

$$=\frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta,$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

Comprehensive Exercise 3

1. Expand $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ for

   (i) $|z|<1$ (ii) $1<|z|<4$

   (iii) $|z|>4$. **(Garhwal 2000; Kanpur 04, 09, 12, 15)**

2. Express $f(z)=\dfrac{1}{z(z+1)^2(z+2)^3}$ in a Laurent's series in the region $\dfrac{5}{4}\le|z|\le\dfrac{7}{4}$.

3. (i) Find the Laurent series of the function $f(z)=\dfrac{1}{z^2(1-z)}$ about $z=0$. **(Kanpur 2004)**

   (ii) Find two Laurent's series expansions in power of z of the function

   $$f(z)=\frac{1}{z(1+z^2)}$$ **(Kumaun 2010)**

4. Obtain the Taylor's or Laurent's series which represents the function

   $$f(z)=\frac{1}{(1+z^2)(z+2)}$$

   when (i) $|z|<1$, **(Kumaun 2013)** (ii) $1<|z|<2$,

   (iii) $|z|>2$. **(Kanpur 2008)**

5. If $0<|z-1|<2$, then express $f(z)=\dfrac{z}{(z-1)(z-3)}$

   in a series of positive and negative powers of $(z-1)$. **(Rohilkhand 2010)**

6. Expand $f(z)=\dfrac{z-1}{z+1}$ as a Taylor's series about

   (i) $z=0$ (ii) $z=1$

   (iii) its Laurent's series for the domain $1<|z|<\infty$. **(Kanpur 2000)**

7. Find Laurent's series of the function $f(z) = \dfrac{1}{(z^2 - 4)(z + 1)}$ valid in the region $1 < |z| < 2.$ **(Kanpur 2001)**

8. Expand $\sin z$ in a Taylor's series about $z = \dfrac{\pi}{4}\cdot$ **(Kanpur 2002)**

9. (i) Expand $\dfrac{1}{z}$ as a Taylor's series about $z = 1.$

(ii) Determine Laurent's expansion of the function $f(z) = \dfrac{\sin z}{\left(z - \dfrac{\pi}{4}\right)^3}$ in the annulus $0 < \left| z - \dfrac{\pi}{4} \right| < 1.$

10. Represent the function $f(z) = \dfrac{4z + 3}{z(z - 3)(z + 2)}$ in Laurent's series

(i) within $|z| = 1$

(ii) in the angular region between $|z| = 2$ and $|z| = 3$ **(Kumaun 2014)**

(iii) exterior to $|z| = 3.$

11. Prove that $e^{u/z + vz} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n},$

$$\text{where } a_n = \frac{1}{2\pi} \int_0^{2\pi} \exp\{(u + v)\cos\theta\} \cos\{(v - u)\sin\theta - n\theta\}\, d\theta$$

$$\text{and} \quad b_n = \frac{1}{2\pi} \int_0^{2\pi} \exp\{(u + v)\cos\theta\} \cos\{(u - v)\sin\theta - n\theta\}\, d\theta.$$

(Rohilkhand 2011)

12. Show that $\sin\left\{c\left(z + \dfrac{1}{z}\right)\right\}$ can be expanded in a series of the type

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n},$$

where the coefficients of both z^n and z^{-n} are

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta.$$

(Garhwal 2000)

13. Show that if $c > 0$, then $e^{z + c^3/2z^2} = \sum_{n=-\infty}^{\infty} a_n z^n,$

$$\text{where } a_n = \frac{e^{-c/2}}{2\pi c^n} \int_0^{2\pi} e^{c(\cos\theta + \cos^2\theta)} \cos\{c\sin\theta(1 - \cos\theta) - n\theta\}\, d\theta.$$

14. By using the integral representation of $f^n(0)$, prove that

$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_C x^n \frac{e^{xz}}{n!\, z^{n+1}}\, dz,$$

where C is any closed contour surrounding the origin. Hence show that

$$\Sigma \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta}\, d\theta.$$

(Kanpur 2002; Kumaun 07, 08, 10; Gorakhpur 10)

Answers 3

1. (i) $f(z) = -1 - \frac{5}{4} z - \frac{17}{16} z^2 - \frac{65}{64} z^3 - \ldots\ldots$

(ii) $f(z) = \ldots\ldots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} - \ldots\ldots$

(iii) $f(z) = 1 - \frac{5}{z} + \frac{17}{z^2} - \frac{65}{z^3} + \ldots\ldots$

2. $$f(z) = \left(\frac{3}{z^2} + \frac{2}{z^3}\right) \sum_0^\infty (-1)^{n+1} (n+1) \frac{1}{z^n}$$
$$+ \frac{1}{16}\left(3z + \frac{17}{z} + 15\right) \sum_0^\infty (-1)^{n+1} (n+1)(n+2) \left(\frac{z}{2}\right)^n$$

3. (i) $f(z) = \frac{1}{z^2} + \frac{1}{z} + 1 + \sum_{n=1}^\infty z^n$

(ii) $f(z) = \frac{1}{z} - z + z^3 - z^5 + \ldots\ldots;\ f(z) = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \ldots\ldots$

4. (i) $f(z) = \frac{1}{10} \sum_{n=0}^\infty (-1)^n . \frac{z^n}{2^n} - \frac{z-2}{5} \sum_{n=0}^\infty (-1)^n z^{2n}$

(ii) $f(z) = \frac{1}{10} \sum_{n=0}^\infty (-1)^n \frac{z^n}{2^n} - \frac{z-2}{5z^2} \sum_{n=0}^\infty (-1)^n \frac{1}{z^{2n}}$

(iii) $f(z) = \frac{1}{5z} \sum_{n=0}^\infty (-1)^n \left(\frac{2}{z}\right)^n - \frac{1}{5}\left(\frac{1}{z} - \frac{2}{z^2}\right) \sum_{n=0}^\infty \frac{(-1)^n}{z^{2n}}$

5. $f(z) = -\frac{1}{2(z-1)} - \frac{3}{4} \sum_{n=0}^\infty \left(\frac{z-1}{2}\right)^n$

6. (i) $f(z) = 1 - 2 \sum_{n=0}^\infty (-1)^n z^n$ (ii) $f(z) = 1 - \sum_{n=0}^\infty \frac{(-1)^n (z-1)^n}{2^n}$

(iii) $f(z) = 1 - \frac{2}{z} \sum_{n=0}^\infty (-1)^n \frac{1}{z^n}$

7. $f(z) = -\frac{1}{24} \sum_{n=0}^\infty \left(\frac{z}{2}\right)^n + \frac{1}{8} \sum_{n=0}^\infty (-1)^n \left(\frac{z}{2}\right)^n - \frac{1}{3z} \sum_{n=0}^\infty \frac{(-1)^n}{z^n}$

8. $f(z) = \sum_{n=0}^\infty \sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) \frac{\left(z - \frac{\pi}{4}\right)^n}{n!}$

9. (i) $f(z) = \sum_{n=0}^\infty (-1)^n (z-1)^n$

(ii) $f(z) = \sum_{n=0}^{\infty} a_n \left(z - \frac{\pi}{4}\right)^n + \sum_{n=1}^{\infty} \frac{b_n}{\left(z - \frac{\pi}{4}\right)^n}$

where $a_n = \frac{1}{2\pi} \int_0^{2\pi} \left[\sin\phi \,.\, \cosh(\sin\theta) \,.\, \cos(m\theta) + \cos\phi \,.\, \sinh(\sin\theta) \,.\, \sin(m\theta)\right] d\theta$

$\phi = \frac{\pi}{4} + \cos\theta$, $m = n + 3$; and $b_n = a_{(-n)}$.

10. (i) $f(z) = -\frac{1}{2z} - \sum_{n=0}^{\infty} \left[\frac{1}{3^{n+1}} + (-1)^n \frac{1}{2^{n+2}}\right] z^n$

(ii) $f(z) = -\frac{1}{2z} - \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{n-1}}{z^{n+1}}$

(iii) $f(z) = -\frac{1}{2z} + \sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{n-1}}{z^{n+1}}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) and (d).

1. If C is closed contour $|z| = r$ and $n \neq -1$, then $\int_C z^n \, dz =$

(a) $2\pi i$ (b) 2π
(c) i (d) 0.

2. The value of $\int_C \frac{1}{z} dz$ where C is circle $z = e^{i\theta}, 0 \le \theta \le \pi$ is

(a) πi (b) $-\pi i$
(c) $2\pi i$ (d) 0

3. $\int_L dz$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to

(a) z (b) $b - a$
(c) $a - b - z$ (d) $z - a - b$

4. $\int_L |dz|$, where L is any rectifiable arc joining the points $z = a$ and $z = b$ is equal to

(a) $b - a$ (b) $|b - a|$
(c) arc length of L (d) 0

5. If C is circle $|z - a| = r$, then $\int_C \frac{dz}{z - a}$ is

(a) $2\pi i$ (b) $-2\pi i$

(c) πi (d) 0

6. If $f(z)$ is analytic in a simply connected domain D enclosed by a rectifiable Jordan curve C and $f(z)$ is continuous on C, then for any point z_0 in D, we have $f(z_0) =$

(a) $\frac{1}{2\pi} \int_C \frac{f(z)}{z - z_0} dz$ (b) $\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$

(c) $2\pi i \int_C \frac{f(z)}{z - z_0} dz$ (d) $2\pi \int_C \frac{f(z)}{z - z_0} dz$

7. Let $f(z)$ be continuous on a contour L of length l and let $|f(z)| \le M$ on L, then $\left| \int_L f(z)\, dz \right|$ is

(a) $\le Ml$ (b) $\ge Ml$

(c) $> Ml$ (d) $< Ml$ **(Kumaun 2007, 13)**

8. If a function $f(z)$ is analytic within a circle C with its centre $z = a$ and radius R, then for every point z inside C, $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$, where $a_n =$

(a) $\frac{f^{(n)}(a)}{n}$ (b) $\frac{f^{(n)}(a)}{n!}$

(c) $f^{(n)}(a)$ (d) none of these

9. Value of $\frac{2!}{2\pi i} \int_{|z|=3} \frac{z^2 + 3z + 4}{(z-1)^3} dz$ is

(a) 2 (b) 0

(c) πi (d) none of these

10. If C is a circle $|z| = 1$ then $\int_C \bar{z}\, dz$ is

(a) πi (b) $2\pi i$

(c) 0 (d) none of these

(Kumaun 2008. 15)

11. If L is a circle $|z| = r > 0$, then $\int_L \frac{dz}{z^2}$ is equal to :

(a) πi (b) 0

(c) $2\pi i$ (d) none of these

(Kumaun 2009. 11, 12)

12. If $f(z)$ is analytic within a circle C, given by $|z-a|=R$, and $|f(z)| \leq M$ on C, then

(a) $|f^n(a)| = \dfrac{M\,n!}{R^n}$ (b) $|f^n(a)| \geq \dfrac{M\,n!}{R^n}$

(c) $|f^n(a)| \leq \dfrac{M\,n!}{R^n}$ (d) $|f^n(a)| \neq \dfrac{M\,n!}{R^n}$

(Kumaun 2012, 15)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Let a function $f(z)$ be analytic in a simply connected domain D and let C be any closed continuous rectifiable curve in D. Then

$$\int_C f(z)\,dz = \ldots\ldots\,.$$

2. The path of the definite integral $\int_a^b f(z)\,dz$ is any ……… joining the points $z=a$ and $z=b$.

3. If C is straight line from $(1, 0)$ to $(1, 1)$, then the value of integral $\int_C \bar{z}\,dz$ is ……

4. Let D be a doubly connected region bounded by two simple closed curves C_1 and C_2 such that C_2 is contained in C_1 and $f(z)$ is analytic in the region between these curves and continuous on C_1, then

$$\int_{C_1} f(z)\,dz = \ldots\ldots\ldots,$$

where both C_1 and C_2 are traversed in the positive sense.

5. If L is any rectifiable arc joining the points $z=a$ and $z=b$, then $\int_L z\,dz$ is equal to …… .

6. If $f(z)$ be continuous in a simply connected domain D and $\int_\Gamma f(z)\,dz = 0$ where Γ is any rectifiable closed Jordan curve in D, then $f(z)$ is …… in D.

True or False

Write 'T' for true and 'F' for false statement.

1. A contour is said to be closed if it does not intersect itself and the starting point of the first arc in it coincides with the end point of the last arc.

2. If C is a closed curve with $z=a$ inside C, then $\int_C \dfrac{dz}{z-a} = 2\pi i.$

3. If $f(z)$ is an integral function satisfying the inequality $|f(z)||\,M$ for all values of z, where M is a positive constant, then $f(z)$ is constant.

4. For the indefinite integral of a function $f(z)$ to exist in a simply connected domain D, it is not necessary that $f(z)$ be analytic in D.

5. The series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}}$ represents Laurent's series.

Answers

Multiple Choice Questions

1.	(d)	2.	(a)	3.	(b)	4.	(c)	5.	(a)
6.	(b)	7.	(a)	8.	(b)	9.	(a)	10.	(b)
11.	(b)	12.	(c)						

Fill in the Blank(s)

1. 0.
2. curve
3. $\frac{1}{2} + i$
4. $\int_{C_2} f(z)\, dz$
5. $\frac{1}{2}(b^2 - a^2)$
6. analytic

True or False

1.	T	2.	T	3.	F	4.	F	5.	T

□

Chapter 4

Poles, Zeros and Singularities

1 The Zeros of an Analytic Function

Definition: *The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$.*

If $f(z)$ is analytic in a domain D and z_0 is any point of D, then we can expand $f(z)$ as Taylor's series about $z = z_0$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If $a_0 = a_1 = a_2 = \ldots\ldots = a_{m-1} = 0$ and $a_m \neq 0$, $f(z)$ is said to have a **zero of order** m at $z = z_0$.

In this case Taylor's expansion of $f(z)$ reduces to

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^{n+m}$$

$$= (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n.$$

A zero of order one ($m = 1$) is said to be a **simple zero.**

2 The Zeros are Isolated

(Gorakhpur 2004)

Theorem: *If $f(z)$ is an analytic function in a domain D, then unless $f(z)$ is identically zero, there exists a neighbourhood of each point in D throughout which the function has no zero except possibly at the point itself.*

Proof: Let $z = z_0$ be a zero of order m of the function $f(z)$.
Then we can write

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n. \qquad \ldots(1)$$

Let $$\phi(z) = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n.$$

Then $$\phi(z_0) = a_m \neq 0.$$

Now the series (1) is uniformly convergent and its each term is continuous at z_0 so that $\phi(z)$ is also continuous at z_0. Therefore for $\varepsilon > 0$ there will exist $\delta > 0$ such that

$$|\phi(z) - \phi(z_0)| < \varepsilon, \qquad \ldots(2)$$

where $$|z - z_0| < \delta.$$

Let $\varepsilon = \left|\dfrac{a_m}{2}\right|$ and δ_1 be the corresponding value of δ. Then we have from (2)

$$|\phi(z) - a_m| < \left|\frac{a_m}{2}\right|, \qquad \ldots(3)$$

where $$|z - z_0| < \delta_1.$$

Thus $\phi(z)$ is non-zero at any point in the neighbourhood of $|z - z_0| < \delta_1$. For if we have $\phi(z) = 0$, (3) will not hold. The argument also holds when $m = 0$ in which case $\phi = f$ and $f(z_0) \neq 0$.

Hence the zeros of an analytic function are isolated.

3 Singularities of an Analytic Function

(Gorakhpur 2006)

If a function is analytic at all points of a bounded domain except at a finite number of points then these exceptional points are called **singular points** or **singularities.** Thus *the singularity of a function is a point at which the function ceases to be analytic.*

4 Isolated and Non-isolated Singularities

(Purvanchal 2010)

If z_0 is a singularity of $f(z)$ and if $f(z)$ is analytic at each point in some neighbourhood of z_0, then z_0 is called an isolated singularity of $f(z)$ otherwise it is called non-isolated singularity.

For example the function $f(z) = \dfrac{z^2+5}{z(z-3)(z^2+1)}$ is analytic at every point except $z = 0, 3, \pm i$. These are the isolated singular points of $f(z)$.

Consider another function $f(z) = \dfrac{1}{\tan(\pi / z)}$. It has infinite number of isolated singularities which lie on the real axis from $z = -1$ to $z = 1$. These isolated singularities are given by $z = \pm \dfrac{1}{n}, n = 1, 2, 3, \ldots$. The origin $(z = 0)$ is also a singular point but it is not isolated since in every neighbourhood of 0 there are infinite number of other singularities.

The function Log z has a non-isolated singularity at origin since every neighbourhood of zero contains points on the negative real axis where Log z is not analytic.

5 Isolated Essential Singularities

Let $f(z)$ be an analytic function in a domain D except at the point $z = z_0$. Then there exists a deleted neighbourhood $0 < |z - z_0| < R$ in which $f(z)$ is analytic. In the annulus $0 < |z - z_0| < R$ the Laurent expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

The term $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ of the Laurent's series is called the principal part of $f(z)$ at $z = z_0$.

Now there arise three possibilities :

(i) The principal part contains infinite number of terms.

(ii) All the b_n are zero *i.e.*, there is no term in the principal part.

(iii) There are finite number of terms in the principal part. The above three possibilities give rise to three types of singularities :

Isolated essential singularity: *If there are infinite number of terms in the principal part of $f(z)$ at $z = z_0$, then z_0 is called an isolated essential singularity of $f(z)$.*

(Gorakhpur 2007)

For example $\sin \dfrac{1}{z}$ has an isolated essential singularity at $z = 0$ since $\sin \dfrac{1}{z} = \dfrac{1}{z} + \dfrac{1}{3!} \cdot \dfrac{1}{z^3} + \dfrac{1}{5!} \cdot \dfrac{1}{z^5} + \ldots\ldots$ has infinite number of terms in negative powers of z.

6 Removable Singularity

If all the coefficients b_n are zero i.e., if the principal part of $f(z)$ at $z = z_0$ consists of no terms, then z_0 is called removable singularity of $f(z)$.

We can remove this singularity by defining the function $f(z)$ at $z = z_0$ in such a way that it becomes analytic at z_0.

For example the function $f(z) = \dfrac{\sin(z-1)}{z-1}$ has removable singularity at $z = 1$ because

$$\frac{\sin(z-1)}{z-1} = \frac{1}{z-1}\left[(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \ldots\ldots\right]$$

$$= 1 - \frac{(z-1)^2}{2!} + \frac{(z-1)^4}{5!} - \ldots\ldots$$

contains no negative powers of $(z-1)$. This singularity can be made to disappear by defining $\dfrac{\sin(z-1)}{z-1} = 1$ at $z = 1$ so that the function becomes analytic at $z = 1$.

7 Pole

If the principal part of $f(z)$ at $z = z_0$ consists of a finite number of terms, say m, then $z = z_0$ is said to be a pole of order m of the function $f(z)$. **(Gorakhpur 2006; Purvanchal 10)**

For $m = 1$, the point $z = z_0$ is said to be a simple pole. If $z = z_0$ is a pole of order m, $f(z)$ has an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots\ldots + \frac{b_m}{(z-z_0)^m}.$$

8 Residue at Pole

Let z_0 be a pole of order m of the function $f(z)$. Then we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots\ldots + \frac{b_m}{(z-z_0)^m}, b_m \neq 0.$$

The coefficient $\boldsymbol{b}$, which may also be zero is called the **residue of $f(z)$** at $z = z_0$.

If $z = z_0$ is a simple pole, then we have $b_1 = \lim_{z \to z_0} (z - z_0) f(z)$.

9 Meromorphic Function

A function which has poles as its only singularities in the finite part of the plane is said to be a meromorphic function.

10 Entire Function

A function which has no singularity in the finite part of the plane is called an entire function.

(Purvanchal 2009)

11 Polynomials

An expression of the form $P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, *where* $a_0, a_1, \ldots\ldots, a_n$ *are complex numbers and* $a_n \neq 0$ *is said to be a polynomial of degree n.*

In particular, *every constant is a polynomial of degree* 0. The degree of the constant polynomial 0 remains undefined.

For example, z^n is a polynomial of degree n, $5 + 4z^2 + 2z^3$ is a polynomial of degree 3.

12 Behaviour of a Polynomial at Infinity

Let $P(z)$ be a polynomial of degree n defined by

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n, \ a_n \neq 0.$$

In order to discuss the behaviour of $P(z)$ at ∞, let us consider the function $P\left(\frac{1}{z}\right) = P_1(z)$. Then

$$P_1(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\ldots + \frac{a_n}{z^n}$$

$$= \frac{1}{z^n}(a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_n).$$

$P_1(z) \to \infty$ as $z \to 0$, because $a_n \neq 0$. Consequently $P(z)$ has a pole of order n at infinity. Hence every polynomial of degree n has a pole of order n at infinity.

13 Characterization of Polynomials

Theorem 1: *The order of a zero of a polynomial equals the order of its first non-vanishing derivative.*

Proof: Suppose $z = a$ is a zero of order m of a polynomial $P(z)$.

Then $\quad P(z) = (z - a)^m Q(z), Q(a) \neq 0.$

Differentiating both sides successively m times, we get

$$P'(z) = m(z - a)^{m-1} Q(z) + (z - a)^m Q'(z)$$

$$P''(z) = m(m-1)(z - a)^{m-2} Q(z) + 2m(z - a)^{m-1} Q'(z) + (z - a)^m Q''(z)$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$

$$P^m(z) = m!\,Q(z) + {}^mC_1\, m!\,(z - a)\,Q'(z) + \ldots\ldots + (z - a)^m Q^m(z).$$

Putting $z = a$ in above relations, we get

$$P(a) = P'(a) = P''(a) = \ldots\ldots = P^{m-1}(a) = 0$$

and $\qquad P^m(a) = m!\,Q(a) \neq 0.$

Hence the order of a zero of a polynomial equals the order of its non-vanishing derivative.

Theorem 2: (Luca's Theorem): *If all the zeros of a polynomial lie in a half plane, then all the zeros of its derivative also lie in the same half plane.*

Theorem 3: *Show that a function which has no singularity in the finite part of the plane and has a pole of order n at infinity is a polynomial of degree n.*

Theorem 4: *A polynomial of degree n has no singularities in the finite part of the plane but has a pole of order n at infinity.*

Theorem 5: *A function $f(z)$ is a polynomial of degree n if and only if $f(z)$ has no singularities in the finite part of the plane and has a pole of order n at infinity.*

Theorem 6: *If a function $f(z)$ is analytic for all finite values of z and as $|z| \to \infty, |f(z)| = a|z|^k$ then $f(z)$ is a polynomial of degree $\leq k$.*

14 Rational Function

A function $R(z)$ which is obtained by applying the rational operations of arithmetic (addition, subtraction, multiplication and division) finitely many times is called a rational function.

Thus a rational function $R(z)$ is of the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials given by

$$P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n, \;\; a_n \neq 0$$

$$Q(z) = b_0 + b_1 z + \ldots\ldots + b_m z^m, \;\; b_m \neq 0$$

having no factors in common.

$R(z)$ will tend to ∞ at the zeros of $Q(z)$. The zeros of $Q(z)$ are called **poles of $R(z)$** and the **order of a pole** of $R(z)$ is defined as the order of the corresponding zero of $Q(z)$.

We have $R'(z) = \dfrac{P'(z)\,Q(z) - Q'(z)\,P(z)}{Q^2(z)}$, provided $Q(z) \neq 0$.

Obviously numerator and denominator of $R'(z)$ are polynomials. Therefore the derivative of a rational function is also a rational function having the same poles as $R(z)$ and order of each pole is increased by one.

Poles and Zeros of a Rational Function at Infinity

Consider a rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

where $\qquad P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n, \;\; a_n \neq 0$

$$Q(z) = b_0 + b_1 z + \ldots\ldots + b_m z^m, \;\; b_m \neq 0.$$

Let $R(1/z) = R_1(z)$. Then the order of zero or pole at ∞ of $R(z)$ is defined as the order of the zero or pole of $R_1(z)$ at the origin. We have

$$R(z) = \frac{a_0 + a_1 z + \ldots\ldots + a_n z^n}{b_0 + b_1 z + \ldots\ldots + b_m z^m}$$

so that
$$R_1(z) = z^{m-n}\left(\frac{a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_n}{b_0 z^m + b_1 z^{m-1} + \ldots\ldots + b_m}\right).$$

Now there arise three cases :

Case 1: For $m > n$, $R_1(z)$ has a zero of order $m - n$ at the origin and consequently $R(z)$ has a zero of order $m - n$ at infinity.

Case 2: For $m < n$, $R_1(z)$ has a pole of order $n - m$ at the origin and therefore $R(z)$ has a pole of order $n - m$ at infinity.

Case 3: For $m = n$ we have $R(\infty) = R_1(0) = (a_n / b_m) \neq 0$ or ∞, therefore $R(z)$ has neither a zero nor a pole at infinity.

The rational function $R(z)$ has n zeros and m poles in the finite part of the plane. Therefore the total number of zeros and poles of $R(z)$ are as given below :

	Number of zeros			Number of poles		
	In the finite plane	**At ∞**	**Total**	**In the finite plane**	**At ∞**	**Total**
$m > n$	n	$m - n$	m	m	—	m
$m < n$	n	—	n	m	$n - m$	n
$m = n$	n	—	$m = n$	m	—	$n = m$

Hence the number of zeros of a rational function is equal to the number of its poles.

The total number of zeros or poles (the number of zeros and poles is equal) of a rational function is called its **order.**

15 Characterization of Rational Functions

Theorem 1: *If a single valued function $f(z)$ has no singularities other than poles in the finite part of the plane or at infinity, $f(z)$ is a rational function.*

Proof: Suppose $z_1, z_2, \ldots\ldots, z_k$ are poles of $f(z)$ of orders $m_1, m_2, \ldots\ldots, m_k$ in the finite part of the z-plane. Then we can write

$$f(z) = \frac{P(z)}{(z - z_1)^{m_1}(z - z_2)^{m_2} \ldots\ldots (z - z_k)^{m_k}}$$

where $P(z)$ is an analytic function for all finite values of z.

$$\therefore \quad P(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \ldots\ldots (z - z_k)^{m_k} f(z).$$

Since $P(z)$ is analytic for all finite values of z therefore the Taylor's expansion of $P(z)$ is of the form

$$P(z) = \sum_{n=0}^{\infty} a_n z^n. \quad \ldots(1)$$

Then $$P\left(\frac{1}{\zeta}\right) = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^n}, \text{ where } z = 1/\zeta.$$

The behavior of $P(z)$ at infinity is the same as the behavior of $P(1/\zeta)$ at $\zeta = 0$. Since the singularity of $P(z)$ at $z = \infty$ is a pole therefore the singularity of $P(1/\zeta)$ at $\zeta = 0$ is also a pole. As a result the expansion of $P(1/\zeta)$ will consist of a finite number of terms. Consequently the expansion (1) of $P(z)$ must contain a finite number of terms. Therefore $P(z)$ is a polynomial.

Since the numerator and denominator of $f(z)$ are polynomials therefore $f(z)$ is a rational function.

Theorem 2: *A rational function has no singularities other than poles.*

Proof: Let $f(z)$ be a rational function given by

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials having no factor in common.

The singularities of the function $f(z)$ in the finite part of the plane are given by $Q(z) = 0$. We know that zeros of $Q(z)$ are the poles of $\dfrac{1}{Q(z)}\cdot$ Hence the rational function $f(z) = \dfrac{P(z)}{Q(z)}$ has no singularities other than poles in the finite part of the plane.

Now we shall discuss the behavior of $f(z)$ near $z = \infty$.

Taking $$f(z) = \frac{a_0 + a_1 z + \ldots\ldots + a_n z^n}{b_0 + b_1 z + \ldots\ldots + b_m z^m},\ a_n \neq 0,\ b_m \neq 0,$$

$$= z^{n-m}\left[\frac{\dfrac{a_0}{z^n} + \dfrac{a_1}{z^{n-1}} + \ldots\ldots + a_n}{\dfrac{b_0}{z^m} + \dfrac{b_1}{z^{m-1}} + \ldots\ldots + b_m}\right].$$

The coefficient of z^{n-m} is regular for large values of $|z|$, therefore we can write

$$f(z) = z^{n-m}\left(c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots\ldots\right), c_0 \neq 0.$$

The power series in the bracket converges for $|z| > r$ if r is sufficiently large. As a result the behavior of $f(z)$ near $z = \infty$ depends on the value of $n - m$. For $n - m \leq 0$, the Laurent expansion of $f(z)$ near $z = \infty$ contains no positive powers of z therefore $f(z)$ is analytic near $z = \infty$. For $n - m > 0$, $f(z)$ has a pole of order $n - m$ at $z = \infty$. Hence all the singularities of a rational function are poles.

16 Theorems on Poles and Other Singularities

Theorem 1: *A function which has no singularity in the finite part of the plane or at infinity is constant.*

Proof: Let $f(z)$ be the function which has no singularity in finite part of the plane or at infinity.

Then $f(z)$ can be expanded as a Taylor's series in any circle $|z| = r$, where r is large.

$\therefore \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n. \qquad \ldots(1)$

Since $f(z)$ has no singularity at infinity therefore $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$. Also $f(1/z)$ has no singularity in the finite part of the plane because $f(z)$ has no singularity in the finite part of the plane. Consequently $f(1/z)$ can be expanded as a Taylor's series.

$\therefore \qquad f(1/z) = \sum_{n=0}^{\infty} b_n z^n. \qquad \ldots(2)$

Replacing z by $1/z$ in (1), we get

$$f(1/z) = \sum_{n=0}^{\infty} (a_n / z^n). \qquad \ldots(3)$$

From (2) and (3), we get

$$\sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n z^{-n}.$$

For the above relation to hold we must have

$$b_n = a_n = 0, \; n = 1, 2, 3, \ldots\ldots \text{ and } b_0 = a_0 .$$

Hence we have $f(z) = a_0 = b_0 =$ constant.

Theorem 2: *If z_0 is a pole of $f(z)$, there exists a neighbourhood of z_0 which contains no other pole of $f(z)$ i.e.,* **poles are isolated.**

Proof: If z_0 is a pole of order m of $f(z)$, there exists a deleted neighbourhood $0 < |z - z_0| < r$ of z_0 in which $f(z)$ is analytic and has a Laurent's expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

Thus $f(z)$ contains no other pole in the neighbourhood $0 < |z - z_0| < r$ *i.e.*, poles are isolated.

Theorem 3: *If $f(z)$ is a function such that for some positive integer m, a value $\phi(z_0)$ exists with $\phi(z_0) \neq 0$ such that the function $\phi(z) = (z - z_0)^m f(z)$ is analytic at z_0. Then $f(z)$ has a pole of order m at z_0.*

Proof: The function $\phi(z)$ is given to be analytic at z_0, so that it can be expanded in a Taylor's series about z_0.

$\therefore \qquad \phi(z) = (z - z_0)^m f(z)$

$$= \phi(z_0) + (z - z_0)\phi'(z_0) + \ldots\ldots + (z - z_0)^m \frac{\phi^m(z_0)}{m!} + \ldots\ldots$$

or
$$f(z) = \frac{\phi(z_0)}{(z - z_0)^m} + \frac{\phi'(z_0)}{(z - z_0)^{m-1}} + \ldots\ldots + \frac{\phi^{m-1}(z_0)}{(m-1)!} \cdot \frac{1}{(z - z_0)}$$
$$+ \sum_{n=m}^{\infty} \frac{\phi^n(z_0)}{n!}(z - z_0)^{n-m}.$$

Since we have $\phi(z_0) \neq 0$ therefore $f(z)$ has a pole of order m at z_0.

Also the residue at z_0 = coeff. of $\dfrac{1}{z - z_0} = \dfrac{\phi^{m-1}(z_0)}{(m-1)!}$.

Remark: It follows from the above theorem that if a function $f(z)$ can be put in the form $f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic at z_0 with $\phi(z_0) \neq 0$, then $f(z)$ has a pole of order m at z_0.

Theorem 4: *If $f(z)$ has a pole of order m at z_0, then the function ϕ defined by $\phi(z) = (z - z_0)^m f(z)$ has a removable singularity at z_0 and $\phi(z_0) \neq 0$.*

Also show that the residue at z_0 is given by $\dfrac{\phi^{m-1}(z_0)}{(m-1)!}$.

Proof: Since $z = z_0$ is a pole of order m of $f(z)$, therefore there exists a deleted neighbourhood of z_0 given by $0 < |z - z_0| < r$ $(r > 0)$ in which $f(z)$ has a Laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots\ldots + \frac{b_m}{(z - z_0)^m}, \qquad \ldots(1)$$

where $b_m \neq 0$.

Consider a function ϕ defined by

$$\phi(z) = (z - z_0)^m f(z). \qquad \ldots(2)$$

Then $\phi(z)$ is defined in the neighbourhood of z_0 except at z_0.

From (1) and (2), we have

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \ldots\ldots + b_m. \qquad \ldots(3)$$

Let us define $\phi(z_0) = b_m$. Then $\phi(z_0) \neq 0$, so that the expansion of $\phi(z)$ given by (3) is valid throughout a neighbourhood of z_0 including z_0. It can be easily shown that (3) is a convergent power series. Thus $\phi(z)$ is analytic at z_0. Therefore we have made $\phi(z)$ analytic at z_0 by setting $\phi(z_0) = b_m$. Hence $\phi(z)$ has a removable singularity at z_0.

Since $\phi(z)$ has become analytic at z_0, therefore (3) represents a Taylor's series for $\phi(z)$.

Consequently coefficient of

$$(z - z_0)^{m-1} = \frac{\phi^{m-1}(z_0)}{(m-1)!}.$$

But from (3) coefficient of $(z - z_0)^{m-1} = b_1$, which is the residue at z_0.

Hence the residue at $z_0 = \dfrac{\phi^{m-1}(z_0)}{(m-1)!}$.

In particular when z_0 is a simple pole, then the residue at z_0 is

$$\phi(z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Theorem 5: *Let a function $f(z)$ be analytic in an open domain D and let $\phi(z)$ be defined by $\phi(z) = \dfrac{1}{f(z)}$ where $f(z) \neq 0$. Then f has a zero of order m at a point z_0 in D if and only if ϕ has a pole of order m at z_0.*

Proof: The if part: Suppose the function $\phi(z)$ has a pole of order m at z_0. Then we have to show that $f(z)$ has a zero of order m at z_0.

Since $\phi(z)$ has a pole of order m at z_0 therefore we can write

$$\phi(z) = \frac{g(z)}{(z - z_0)^m}$$

where $g(z)$ is analytic function in a neighbourhood of z_0 and $g(z_0) \neq 0$.

It is given that $\phi(z) = \dfrac{1}{f(z)}$, so that $f(z) = \dfrac{1}{\phi(z)} = \dfrac{(z - z_0)^m}{g(z)}$.

Because $g(z)$ is an analytic function and $g(z_0) \neq 0$, therefore $f(z)$ has a zero of order m at z_0.

The only if part: Again suppose $f(z)$ has a zero of order m at z_0. Then we can write

$$f(z) = (z - z_0)^m h(z) \qquad \ldots(1)$$

where $h(z)$ is analytic and $h(z_0) \neq 0$.

$$\therefore \qquad \frac{1}{h(z)} = \frac{(z - z_0)^m}{f(z)}$$

is an analytic function in a neighbourhood of z_0 so it can be expanded in the Taylor's series about z_0.

Thus
$$\frac{1}{h(z)} = A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \ldots\ldots + A_m(z - z_0)^m + \ldots\ldots \qquad \ldots(2)$$

Now $\phi(z) = \dfrac{1}{f(z)}$ therefore from (1) and (2), we have

$$\phi(z) = \frac{1}{h(z)(z - z_0)^m} = \frac{A_0}{(z - z_0)^m} + \frac{A_1}{(z - z_0)^{m-1}} + \ldots\ldots$$
$$+ A_m + \sum_{n=1}^{\infty} A_{m+n}(z - z_0)^n.$$

Thus $\phi(z)$ has a pole of order m at z_0.

Theorem 6. (Riemann): *Let z_0 be an isolated singularity of $f(z)$ and if $|f(z)|$ is bounded on some deleted neighbourhood of z_0, z_0 is a removable singularity.* **(Kanpur 2008)**

Proof: Suppose $|f(z)|$ is bounded in a deleted neighbourhood $N(z_0)$ of z_0. Then there exists a positive number M such that $|f(z)| \leq M$ where M is the greatest value of $f(z)$ on a circle γ defined by $|z - z_0| = \rho$ where ρ is so small that γ lies entirely within $N(z_0)$. By Laurent's theorem, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Now
$$|a_n| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|f(z)||dz|}{|z - z_0|^{n+1}}$$

$$\leq \frac{M}{2\pi} \cdot \frac{1}{\rho^{n+1}} \int_\gamma |dz| = \frac{M}{2\pi\rho^{n+1}} 2\pi\rho = \frac{M}{\rho^n} \to 0 \quad \text{as} \quad n \to \infty$$

i.e., a_n becomes zero when n is negative so that the principal part of $f(z)$ contains no terms of negative powers of $z - z_0$ in the Laurent's expansion for $f(z)$. Hence $f(z)$ has removable singularity at z_0.

Behavior of a function in the neighbourhood of an essential singularity.
(Avadh 2008)

Theorem 7: (Weierstrass's Theorem): *Let z_0 be an essential singularity of a function $f(z)$ and let c be an arbitrary constant. Then for every $\varepsilon > 0$ and every neighbourhood $0 < |z - z_0| < \rho$ of z_0, there exists a point z of this neighbourhood such that $|f(z) - c| < \varepsilon$.*

OR

In every arbitrary neighbourhood of an essential singularity there exists a point (and therefore an infinite number of points) at which the function differs as little as we please from any pre-assigned number.

Proof: We shall prove the theorem by contradiction. Let the theorem be false. Then for a given $\varepsilon > 0$, there exists a constant c and a positive number ρ such that

$$|f(z) - c| > \varepsilon, \text{ where } z \text{ satisfies } 0 < |z - z_0| < \rho.$$

Thus
$$\frac{1}{|f(z) - c|} < \varepsilon \text{ for } 0 < |z - z_0| < \rho.$$

By Riemann's Theorem (Theorem 6) we see that the function $\dfrac{1}{f(z) - c}$ has a removable singularity at z_0 so that the principal part of Laurent's expansion for $\dfrac{1}{f(z) - c}$ contains no negative powers of $(z - z_0)$. Then we can write

$$\frac{1}{f(z) - c} = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If $a_0 \neq 0$, defining $\dfrac{1}{f(z_0) - c} = a_0$ or $f(z_0) = c + \dfrac{1}{a_0}$, then $\dfrac{1}{f(z) - c}$ becomes analytic and non-zero at z_0 so that $f(z)$ itself is analytic at z_0. This contradicts the initial assumption that z_0 is not an essential singularity of $f(z)$. Again if we take $a_0 = a_1 = a_2 = \ldots\ldots = a_{m-1} = 0$, we can write

$$\frac{1}{f(z)-c} = \sum_{n=m}^{\infty} a_n (z-z_0)^n$$

$$= a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \ldots\ldots$$

$$= (z-z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z-z_0)^n$$

which shows that z_0 is a zero of order m of $\dfrac{1}{f(z)-c}$ so that z_0 is a pole of order m of $f(z)-c$. Since c is merely a constant therefore $f(z)$ has a pole of order m at z_0 which again contradicts the hypothesis. Hence the theorem is true.

17 Limit Points of Zeros

Theorem 1: *Let $f(z)$ be analytic in a domain D and let E be a set of zeros of $f(z)$ having a limit point α in D. Then $f(z)$ vanishes for all $z \in D$.*

Proof: Since $f(z)$ is analytic in D therefore it is continuous in D. E is the set of zeros of $f(z)$ and α is the limit point of E in D therefore $f(z)$ vanishes at infinite number of points in every small neighbourhood of α. Since $f(z)$ is continuous at α therefore we must have $f(\alpha)=0$. But α cannot be a zero of $f(z)$ since zeros are isolated. Hence $f(z)$ vanishes identically in D *i.e.*, $f(z)$ vanishes for all $z \in D$.

Remark: If $f(z)$ is not analytic in D and $f(z)$ is not continuous at $z=\alpha$ then $f(z)$ must have a singularity at $z=\alpha$. In this case α is an isolated singularity but it is not a pole since $|f(z)|$ does not tend to ∞ as $z \to \alpha$ in any manner. Hence $z=\alpha$ which is the limit point of zeros is an isolated essential singularity of $f(z)$.

Theorem 2: (Identity Theorem): *If $f(z)$ and $g(z)$ are analytic in a domain D and if $f(z) = g(z)$ on a subset E of D which has a limit point α in D, then $f(z) = g(z)$ in the whole of D.*

Proof: Let $F(z) = f(z) - g(z)$. Then $F(z)$ is analytic in D. Since $f(z) = g(z)$ on E therefore $F(z)$ vanishes at an infinite number of points in every arbitrary small neighbourhood of α. The function $F(z)$ is continuous at $\alpha \in D$ therefore we have $F(\alpha)=0$. But zeros are isolated so that α cannot be a zero of $F(z)$ unless $F(z)$ vanishes identically in D *i.e.*, we must have $f(z) = g(z)$ in the whole of D.

18 The Limit Point of Poles

Theorem 1: *The limit point of a sequence of poles of a function $f(z)$ is a non-isolated essential singularity.*

Proof: Let z_0 be the limit point of a sequence of poles of $f(z)$. Then every neighbourhood of z_0 contains infinite number of points at which $f(z)$ becomes unbounded so that $f(z)$ cannot be analytic at z_0. Thus z_0 is a singularity of $f(z)$ which is not isolated. Hence z_0 is a non-isolated essential singularity of $f(z)$.

Illustration: The zeros of the function $\sin \frac{1}{z}$ are given by

$$z = \pm \frac{1}{n\pi}, n = 1, 2, 3, \ldots$$

The limit point of these zeros is the point $z = 0$. Thus 0 is an isolated singularity of $\sin \frac{1}{z}$.

Again the function $\tan \frac{1}{z}$ has poles at points given by

$$z = \frac{2}{n\pi}, n = \pm 1, \pm 3, \pm 5, \ldots$$

The limit point of this sequence of poles is $z = 0$ which is therefore a non-isolated essential singularity.

Theorem 2: (Picard's Theorem): *An integral function which is constant takes every finite value an infinite number of times with at most one possible exception.*

Proof: Recall that a function $f(z)$ is called an integral function if $f(z)$ has no singularities except at infinity. We shall not try to prove this theorem but only give an example.

The equation $e^z = A$ has an infinite number of roots if $A \neq 0$ as the reader can easily verify. But if $A = 0$, this equation has no finite root. Thus 0 is an exceptional value of e^z.

On the other hand, there exist integral functions with no exceptional values. The function $\sin z$ provides a simple example of such a case.

Illustrative Examples

Example 1: *Show that the function $e^{1/z}$ actually takes every value except zero an infinite number of times in the neighbourhood of $z = 0$.* **(Gorakhpur 2004)**

Solution: Let $f(z) = e^{1/z}$.

To prove the required result we have to show that $f(z)$ has an isolated essential singularity at $z = 0$.

We have $$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\ldots = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}.$$

The principal part of $f(z)$ contains infinite number of terms therefore $z = 0$ is an isolated essential singularity of $f(z)$.

Example 2: *Show that the function $(z^2 + 4) / e^z$ has an isolated essential singularity at* $\boldsymbol{z = \infty}$.

Solution: We have $f(z) = \dfrac{z^2 + 4}{e^z}$.

Putting $z = \frac{1}{y}$, we get $f\left(\frac{1}{y}\right) = \left(4 + \frac{1}{y^2}\right) e^{-1/y}$

$$= \left(4 + \frac{1}{y^2}\right)\left(1 - \frac{1}{y} + \frac{1}{2!}\cdot\frac{1}{y^2} - \frac{1}{3!}\frac{1}{y^3} + \ldots\right)$$

$$= 4 + \left[-\frac{4}{y} + (1+2)\frac{1}{y^2} + \left(-1 - \frac{2}{3}\right)\frac{1}{y^3} + \left(\frac{1}{2} + \frac{1}{6}\right)\frac{1}{y^4} + \ldots.\right]$$

$$= 4 + \left[-\frac{4}{y} + \frac{3}{y^2} - \frac{5}{3y^3} + \frac{2}{3y^4} - \ldots\ldots\right].$$

We have infinite number of terms in the negative powers of y in the principal part of the expansion of $f\left(\frac{1}{y}\right)$, therefore $f\left(\frac{1}{y}\right)$ has an isolated essential singularity at $y = 0$.

Hence $f(z)$ has an isolated essential singularity at $z = \infty$.

Example 3: *What kind of singularity have the following functions:*

(*i*) $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z = 0, z = \infty$ **(Kumaun 2009; Purvanchal 12)**

(*ii*) $\sin \dfrac{1}{1-z}$ *at* $z = 1$ **(Gorakhpur 2010, 13; Kumaun 14)**

(*iii*) $\sin z - \cos z$ *at* $z = \infty$

(*iv*) $\operatorname{cosec} \dfrac{1}{z}$ *at* $z = 0$ **(Gorakhpur 2008)**

(*v*) $\tan \dfrac{1}{z}$ *at* $z = 0$.

Solution: (i) Let $f(z) = \dfrac{\cot \pi z}{(z-a)^2} = \dfrac{\cos \pi z}{\sin \pi z\,(z-a)^2}$.

Poles of $f(z)$ are obtained by equating to zero the denominator of $f(z)$. Then we have

$$(z-a)^2 \sin \pi z = 0$$

$$\therefore \qquad \sin \pi z = 0 \quad \text{or} \quad (z-a)^2 = 0.$$

Now $\sin \pi z = 0$ gives $\pi z = n\pi$ or $z = n$, where n is any integer,

and $\qquad (z-a)^2 = 0$ gives $z = a$.

Hence $z = a$ is a double pole and $z = 0, \pm 1, \pm 2, \ldots\ldots$ are simple poles.

$z = \infty$ is a limit point of these simple poles therefore $z = \infty$ is non-isolated essential singularity.

(ii) Let $\qquad f(z) = \sin \dfrac{1}{1-z}$.

Zeros of $f(z)$ are given by

$$\sin \frac{1}{1-z} = 0 \quad \text{or} \quad \frac{1}{1-z} = n\pi \quad \text{or} \quad z = 1 - \frac{1}{n\pi},$$ where n is any integer.

$z = 1$ is a limit point of these zeros therefore $z = 1$ is an isolated essential singularity.

(iii) Let $\qquad f(z) = \sin z - \cos z$.

Zeros of $f(z)$ are given by

$$\sin z - \cos z = 0 \quad \text{or} \quad \sin z = \cos z$$

or $\qquad \tan z = 1 \quad$ or $\quad z = n\pi + \dfrac{\pi}{4}$, n is any integer.

$z = \infty$ is a limit point of these zeros which is therefore an isolated essential singularity.

(iv) Let $\qquad f(z) = \operatorname{cosec} \dfrac{1}{z} = \dfrac{1}{\sin(1/z)}$.

Poles of $f(z)$ are given by

$$\sin \frac{1}{z} = 0 \quad \text{or} \quad \frac{1}{z} = n\pi \quad \text{or} \quad z = (1/n\pi), \text{ where } n \text{ is any integer.}$$

Since $z = 0$ is a limit point of these poles therefore $z = 0$ is a non- isolated essential singularity.

(v) Let $\qquad f(z) = \tan(1/z) = \dfrac{\sin(1/z)}{\cos(1/z)}$.

Poles of $f(z)$ are given by

$$\cos(1/z) = 0 \quad \text{or} \quad \frac{1}{z} = 2n\pi \pm \frac{\pi}{2}$$

or $$z = \frac{1}{(2n \pm \frac{1}{2})\pi}, \text{ where } n \text{ is any integer.}$$

Since $z = 0$ is a limit point of these poles therefore $z = 0$ is a non-isolated essential singularity.

Example 4: *Show that $z = a$ is an isolated essential singularity of the function* $\dfrac{e^{c/(z-a)}}{e^{z/a} - 1}$.

Solution: We have

$$f(z) = \frac{e^{c/(z-a)}}{e^{z/a} - 1} = \frac{e^{c/(z-a)}}{e^{1 + (z-a)/a} - 1}$$

$$= \frac{1 + \dfrac{c}{z-a} + \dfrac{c^2}{2!(z-a)^2} + \dfrac{c^3}{3!(z-a)^3} + \ldots\ldots}{e\left[1 + \dfrac{z-a}{a} + \dfrac{(z-a)^2}{2!a^2} + \ldots\ldots\right] - 1}$$

$$= -\left[1 + \frac{c}{z-a} + \frac{c^2}{2!(z-a)^2} + \ldots\ldots\right] \times \left[1 - e\left\{1 + \frac{z-a}{a} + \frac{(z-a)^2}{2!a^2} + \ldots\ldots\right\}\right]^{-1}$$

$$= -\left\{1 + \frac{c}{z-a} + \frac{c^2}{2!(z-a)^2} + \ldots\ldots\right\} \left[1 + e\left\{1 + \frac{z-a}{a} + \frac{(z-a)^2}{2!a^2} + \ldots\ldots\right\} + e^2\left\{1 + \frac{z-a}{a} + \frac{(z-a)^2}{2!a^2} + \ldots\ldots\right\}^2 + \ldots\ldots\right]$$

Obviously in the expansion of $f(z)$ there are infinite number of terms containing negative powers of $(z-a)$. Hence $z = a$ is an isolated essential singularity of $f(z)$.

Example 5: *Discuss the nature of singularities of the following functions* :

(*i*) $\tan z$ (*ii*) $\dfrac{1}{z\,(1-z^2)}$ (*iii*) $\dfrac{z}{1+z^4}$ (*iv*) $\dfrac{\sin z}{(z-\pi)^2}$.

Solution: (i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

To obtain the singularities of $f(z)$ equating to zero the denominator of $f(z)$, we get

$$\cos z = 0 \quad \text{or} \quad z = 2n\pi \pm \frac{\pi}{2},\ n \in I$$

or $$z = (4n \pm 1)\frac{\pi}{2},\ n \in I \quad \text{or} \quad z = (2n+1)\frac{\pi}{2},\ n \in I.$$

Hence $z = (2n+1)\dfrac{\pi}{2}$, $(n \in I)$ give the simple poles of $f(z)$.

(ii) Let $f(z) = \dfrac{1}{z\,(1-z^2)}$.

Singularities of $f(z)$ are given by

$$z\,(1-z^2) = 0 \quad \text{or} \quad z = 0, -1, 1, \text{ which are the simple poles.}$$

(iii) Let $f(z) = \dfrac{z}{1+z^4}$.

Singularities of $f(z)$ are given by

$$1 + z^4 = 0 \quad \text{or} \quad z = (-1)^{1/4}$$

or $$z = (\cos \pi + i \sin \pi)^{1/4} = \{\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)\}^{1/4}$$

$$= \cos(2n+1)\frac{\pi}{4} + i \sin(2n+1)\frac{\pi}{4} = e^{i(2n+1)\frac{\pi}{4}}.$$

Putting $n = 0, 1, 2, 3,$ we get

$$z = e^{i\pi/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4} \text{ which are the simple poles of } f(z).$$

(iv) Let $f(z) = \dfrac{\sin z}{(z-\pi)^2}$.

Singularities of $f(z)$ are given by $(z-\pi)^2 = 0$.

Thus $z = \pi$ is a pole of order two of $f(z)$.

Comprehensive Exercise 1

1. Show that the function e^z has an isolated essential singularity at $z = \infty$. **(Gorakhpur 2016)**

2. What kind of singularity have the following functions :

   (i) $\cot z$ at $z = \infty$ **(Kumaun 2007)**

   (ii) $\sec \dfrac{1}{z}$ at $z = 0$. **(Kumaun 2007, 15)**

3. Specify the nature of singularity at $z = -2$ of

$$f(z) = (z-3)\sin\frac{1}{z+2}.$$

4. (i) Find zeros and poles of $\left(\frac{z+1}{z^2+1}\right)^2$.

(Kumaun 2008, 11)

(ii) What kind of singularity has the function $\frac{e^z}{z^2+4}$?

5. Find the kind of the singularities of the following functions :

(i) $\frac{1-e^z}{1+e^z}$ at $z = \infty$ **(Gorakhpur 2010, 13)**

(ii) $\frac{1}{\sin z - \cos z}$ at $z = \frac{\pi}{4}$ **(Kumaun 2013)**

(iii) $\sin\frac{1}{z}$ at $z = 0$

(iv) $z \operatorname{cosec} z$ at $z = \infty$. **(Purvanchal 2007)**

6. Find the zeros and discuss the nature of singularities of

$$f(z) = \frac{z-2}{z^2}\sin\frac{1}{z-1}.$$

(Kanpur 2007; Kumaun 12)

7. Show that the function e^{-1/z^2} has no singularities.

(Kumaun 2010; Gorakhpur 14)

Answers 1

2. (i) non-isolated essential singularity

(ii) non-isolated essential singularity

3. isolated essential singularity
4. $z = -1, -1$; $z = -i, -i, i, i$; $z = 2i, -2i$ are simple poles.
5. (i) non-isolated essential singularity ; (ii) simple pole.

(iii) isolated essential singularity

(iv) non-isolated essential singularity

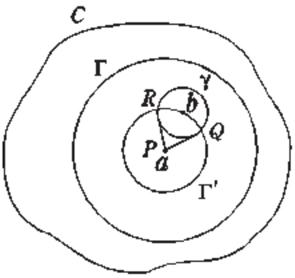
6. zeros of order one; isolated essential singularity

19 Maximum Modulus Principle

Theorem: *Let $f(z)$ be analytic within and on a simple closed contour C. Then $|f(z)|$ reaches its maximum value on C, unless $f(z)$ is a constant. In other words, if M is the maximum value of $|f(z)|$ on and within C, then, unless f is a constant, $|f(z)| < M$ for every point within C.*

(Rohilkhand 2009, 10)

(This is known as **Maximum modulus principle**.)



Since $f(z)$ is an analytic function, it is continuous within and on C so $|f(z)|$ must reach its maximum value M at some point on or within C. Let us assume that $f(z)$ is not constant. Then we wish to prove that $|f(z)|$ takes the value M at some point on C. If possible, suppose this value is not attained on the boundary of C but at a point a within C, so $|f(a)| = M$. Let Γ be a circle inside C with centre at a.

Since $|f(z)| = M$ is the maximum value of $|f(z)|$ and $f(z)$ is not a constant, there must exist a point, b, inside Γ such that $|f(b)| < M$.

Let $|f(b)| = M - \varepsilon$, where $\varepsilon > 0$. Since $|f(z)|$ is continuous at b, therefore for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||f(z)| - |f(b)|| < \frac{1}{2}\varepsilon, \qquad \ldots(1)$$

where $|z - b| < \delta$.

We have $|f(z)| - |f(b)| \le |f(z)| - |f(b)|$ therefore from (1), we have

$$|f(z)| - |f(b)| < \frac{1}{2}\varepsilon$$

or $$|f(z)| < |f(b)| + \frac{1}{2}\varepsilon = M - \varepsilon + \frac{1}{2}\varepsilon = M - \frac{1}{2}\varepsilon$$

$\therefore$ $$|f(z)| < M - \frac{1}{2}\varepsilon \text{ for all points } z \text{ satisfying } |z - b| < \delta.$$

Draw a circle Γ' with centre at a, which passes through b.

Since the arc QR of the circle Γ' lies inside γ therefore on this arc, we have

$$|f(z)| < M - \frac{1}{2}\varepsilon.$$

On the remaining arc of Γ', we have $|f(z)| \le M$.

Radius of the circle $\Gamma' = |b - a| = r$, say.

By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_{\Gamma'} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\,d\theta,$$

putting $z - a = re^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta.$$

Measuring θ from PQ in the anti-clockwise direction and taking $\angle QPR = \alpha$, we have

$$f(a) = \frac{1}{2\pi}\int_0^{\alpha} f(a + re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} f(a + re^{i\theta})\,d\theta.$$

Now $$|f(a)| \le \frac{1}{2\pi}\int_0^{\alpha} |f(a + re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} |f(a + re^{i\theta})|\,d\theta$$

$$< \frac{1}{2\pi}\int_0^{\alpha}\left(M - \frac{1}{2}\varepsilon\right)d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} M\,d\theta$$

$$= \frac{\alpha}{2\pi}\left(M - \frac{1}{2}\varepsilon\right) + \frac{M}{2\pi}(2\pi - \alpha) = M - \frac{\alpha\varepsilon}{4\pi}\cdot$$

$\therefore$ $$M = |f(a)| < M - \frac{\alpha\varepsilon}{2\pi}, \text{ which is absurd.}$$

Hence $|f(z)|$ cannot attain its maximum value at any point within C, so it must attain its maximum value on C.

20 Minimum Modulus Principle

Theorem 1: *Let $f(z)$ be analytic inside and on a closed contour C and let $f(z) \neq 0$ inside C. Then $|f(z)|$ must reach its minimum value on C.*

OR

If m is the minimum of $|f(z)|$ inside and on C, then unless f is constant, $|f(z)| > m$ for every point z inside C.

Proof: Since $f(z)$ is analytic inside and on C and $f(z) \neq 0$ inside C therefore $1/f(z)$ is analytic inside C. By the previous theorem $1/|f(z)|$ cannot reach its maximum value inside C so that $|f(z)|$ cannot attain its minimum value inside C. Since $f(z)$ is continuous on and within C therefore $|f(z)|$ must attain its minimum value at some point on C.

21 The Excess of the Number of Zeros over the Number of Poles of a Meromorphic Function. The Argument Principle

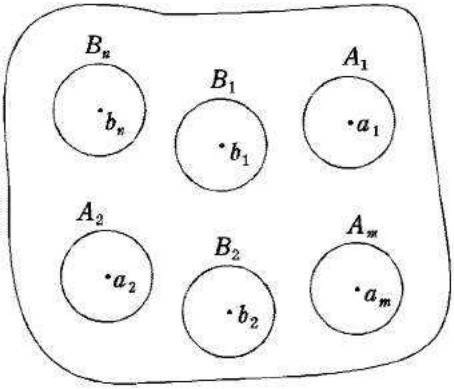
Theorem 1: *If $f(z)$ is meromorphic inside a closed contour C and has no zero on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros and P the number of poles inside C, (a pole or zero of order m must be counted m times). **(Avadh 2007; Kanpur 08; Gorakhpur 09, 11, 12)**

Proof: Let $z = ai\ (i = 1, 2, \ldots\ldots, m)$ be the zeros of $f(z)$ which lie inside C and r_i be the order of a_i. Again suppose $b_i\ (i = 1, 2, \ldots\ldots, n)$ be the poles of $f(z)$ inside C and s_i be the order of b_i. Then we have to show that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{m} r_i - \sum_{i=1}^{n} s_i.$$



Enclosing each zero and pole by non-overlapping circles $A_1, A_2, \ldots\ldots, A_m$

and $B_1, B_2, \ldots\ldots, B_n$ respectively each of radii ρ. Since poles and zeros are isolated, we can always find such ρ. Therefore

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{m}\int_{A_i}\frac{1}{2\pi i}\frac{f'(z)}{f(z)}\,dz + \sum_{i=1}^{n}\frac{1}{2\pi i}\int_{B_i}\frac{f'(z)}{f(z)}\,dz. \qquad \ldots(1)$$

Since a_i is a zero of order r_i of $f(z)$, we may write

$$f(z) = (z - a_i)^{r_i}\,\phi_i(z), \text{ where } \phi_i \text{ is analytic and non-zero at } a_i.$$

Taking log of both sides, we get

$$\log f(z) = r_i \log(z - a_i) + \log \phi_i(z).$$

Differentiating both sides w.r.t. z, we get

$$\frac{f'(z)}{f(z)} = \frac{r_i}{z - a_i} + \frac{\phi_i'(z)}{\phi_i(z)}.$$

We have $$\int_{A_i}\frac{\phi_i'(z)}{\phi_i(z)}\,dz = 0,$$

since $\dfrac{\phi_i'(z)}{\phi_i(z)}$ is analytic at $z = a_i$,

and $$\int_{A_i}\frac{r_i}{z - a_i}\,dz = r_i\int_0^{2\pi}\frac{\rho\, i e^{i\theta}}{\rho e^{i\theta}}\,d\theta = 2\pi i r_i.$$

$$\therefore \quad \sum_{i=1}^{m}\frac{1}{2\pi i}\int_{A_i}\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{m}\frac{1}{2\pi i}2\pi i r_i = \sum_{i=1}^{m} r_i. \qquad \ldots(2)$$

Since b_i is a pole of order s_i of $f(z)$, we may write

$$f(z) = \frac{\psi_i(z)}{(z - b_i)^{s_i}}, \text{ where } \psi_i \text{ is analytic and non-zero at } b_i.$$

Taking log of both sides and differentiating w.r.t. z, we get

$$\frac{f'(z)}{f(z)} = \frac{\psi_i'(z)}{\psi_i(z)} - \frac{s_i}{z - b_i}.$$

Proceeding as above, we have $\displaystyle\int_{B_i}\frac{f'(z)}{f(z)}\,dz = -2\pi i s_i$.

$$\therefore \quad \sum_{i=1}^{n}\frac{1}{2\pi i}\int_{B_i}\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{n}\frac{1}{2\pi i}(-2\pi i s_i) = -\sum_{i=1}^{n} s_i. \qquad \ldots(3)$$

Hence from (1), (2), (3), we have

$$\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{m} r_i - \sum_{i=1}^{n} s_i = N - P. \qquad \ldots(4)$$

Corollary 1: *$N - P = \frac{1}{2\pi} \Delta C \arg f(z)$ where ΔC denotes the variation in $\arg f(z)$ as z moves once round C.*

Proof: By the above theorem, we have

$$N - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

Let $f(z) = R e^{i\phi}$.

Then $R = | f(z) |, \phi = \arg f(z)$

and $f(z) dz = d f(z) = d(R e^{i\phi}) = e^{i\phi} (dR + iR d\phi)$.

We have
$$N - P = \frac{1}{2\pi i} \int_C \left(\frac{dR}{R} + i\, d\phi \right)$$
$$= \frac{1}{2\pi i} \int_C \frac{dR}{R} + \frac{1}{2\pi} \int_C d\phi.$$

Now
$$\int_C \frac{dR}{R} = [\log R]_C = 0,$$

since $\log R$ retains its original value if z moves once round C.

Also
$$\frac{1}{2\pi} \int_C d\phi = \frac{1}{2\pi} [\phi]_C = \frac{1}{2\pi} \Delta C \arg f(z).$$

Hence we have

$$N - P = \frac{1}{2\pi} \Delta C \arg f(z). \qquad \ldots(5)$$

Thus the excess of the number of zeros over the number of poles of a meromorphic function equals $(1/2\pi)$ times the increase in $\arg f(z)$ as z goes once round C.
This is known as the **argument principle.**

Corollary 2: When $f(z)$ is analytic, we have $P = 0$ and in this case

$$N = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi} \Delta C \arg f(z)$$

(Kanpur 2007)

i.e., the number of zeros of an analytic function $f(z)$ within C is $(1/2\pi)$ times the increase in $\arg f(z)$ as z goes once round C.
This is known as the **argument principle for an analytic function.**

Remark: (i) We observe that the variation in $\arg f(z)$ as z moves round C is always equal to an integer. If z_0 is any point on C, we have

$$\Delta C \arg f(z) = [\arg f(z_0)]^* - \arg f(z_0)$$

where $[\arg f(z_0)]^*$ is the value of the argument after the contour C has been traversed. Since any two values of an argument differ by an integral multiple of 2π therefore we have

$$\frac{1}{2\pi} \Delta C \arg f(z) = \frac{1}{2\pi} \cdot 2\pi m = m, \text{ where } m \text{ is an integer.}$$

(ii) We can use the formulae (4) and (5) to count the number of times, N_α, a function $f(z)$ takes the values α. $f(z)-\alpha=0$ iff $f(z)=\alpha$ and then (4) gives

$$N_\alpha - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-\alpha}\,dz.$$

Similarly from (5), we get

$$N_\alpha - P = \frac{1}{2\pi}\,\Delta C \arg[f(z)-\alpha].$$

22 Rouche's Theorem

(Gorakhpur 2007, 10, 14; Avadh 07; Purvanchal 09; Kanpur 07; Rohilkhand 08, 09)

Theorem 1: *Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)|<|f(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.*

Proof: We observe that both $f(z)$ and $f(z)+g(z)$ are non-zero on the boundary C. If at some point a on C, we have $f(a)=0$, then $|g(a)|<|f(a)| \Rightarrow g(a)=0$, which contradicts the hypothesis that $|g(z)|<|f(z)|$ on C. Similarly if we take $f(a)+g(a)=0$, then $|g(a)|=|f(a)|$ which is again a contradiction.

Hence neither $f(z)$ nor $f(z)+g(z)$ has a zero on C.

Let $\quad F(z)=g(z)/f(z)$.

Then $\quad g(z)=f(z)\,F(z)$ so that

$$g'(z)=f'(z)\,F(z)+f(z)\,F'(z).$$

Suppose M and N are the number of zeros of $f(z)$ and $f(z)+g(z)$ inside and on C.

Since $f(z)$ and $f(z)+g(z)$ are analytic within and on C, we have

$$M=\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz \quad\text{and}\quad N=\frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}\,dz,$$

$$\left[\text{Using the formula } N-P=\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz\right].$$

Now

$$N-M=\frac{1}{2\pi i}\int_C \frac{f'+f'F+fF'}{f+fF}\,dz-\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f'(1+F)+fF'}{f(1+F)}\,dz-\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz+\frac{1}{2\pi i}\int_C \frac{F'}{1+F}\,dz-\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$$

$$=\frac{1}{2\pi i}\int_C F'(1+F)^{-1}\,dz$$

$$= \frac{1}{2\pi i} \int_C F' (1 - F + F^2 - F^3 + \ldots\ldots) \, dz \qquad [\because |F(z)| < 1]$$

$$= \frac{1}{2\pi i} \int_C F' \, dz - \frac{1}{2\pi i} \int_C F' F \, dz + \frac{1}{2\pi i} \int_C F' F^2 \, dz - \ldots\ldots$$

$$= 0,$$

since $f(z)$ and $g(z)$ are analytic and $g(z) \neq 0$ at any point on C, so F and F' are also analytic within and on C, consequently each integral is separately zero.

Hence $N = M$ *i.e.*, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Alternative Proof: First show that neither $f(z)$ nor $f(z) + g(z)$ has zero on C. (Proceed as above)

Suppose M and N are the number of zeros of $f(z)$ and $f(z) + g(z)$ inside and on C.

Since $f(z)$ and $f(z) + g(z)$ are analytic within and on C, by the argument principle for analytic functions, we have

$$M = \frac{1}{2\pi} \Delta C \arg f$$

and

$$N = \frac{1}{2\pi} \Delta C \arg (f + g).$$

Now

$$N - M = \frac{1}{2\pi} \{\Delta C \arg (f + g) - \Delta C \arg f\}$$

$$= \frac{1}{2\pi} \left\{\Delta C \arg f \left(1 + \frac{g}{f}\right) - \Delta C \arg f\right\}$$

$$= \frac{1}{2\pi} \left[\Delta C \left\{\arg f + \arg \left(1 + \frac{g}{f}\right) - \arg f\right\}\right]$$

$$= \frac{1}{2\pi} \Delta C \arg \left(1 + \frac{g}{f}\right)$$

$$= \frac{1}{2\pi} \Delta C \arg w, \text{ where } w = 1 + (g/f).$$

Since $|g| < |f|$, we have $|w - 1| = \left|\frac{g}{f}\right| < 1$ so that the point w always lies inside the circle with centre $w = 1$ and radius unity. Thus the point w always lies to the right of the imaginary axis consequently $\arg w = \arg \left(1 + \frac{g}{f}\right)$ always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It follows that $\arg \left(1 + \frac{g}{f}\right)$ returns to its original value when z describes C. Since $\arg \left(1 + \frac{g}{f}\right)$ cannot increase or decrease by a multiple of 2π, we have $\Delta C \arg \left(1 + \frac{g}{f}\right) = 0$.

Hence $N - M = 0$, which gives $N = M$.

23 Fundamental Theorem of Algebra

Theorem 1: (Fundamental Theorem of Algebra):

(Gorakhpur 2004; Purvanchal 08; Rohilkhand 08, 09)

Let $P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$, *where* $n \geq 1$ *and* $a_n \neq 0$ *so that* $P(z)$ *is a polynomial of degree one or greater. Then the equation* $P(z) = 0$ *has at least one root.*

(We shall prove it with the help of Liouville's theorem since its proof by purely algebraic method is extremely difficult).

Proof: We shall prove it by contradiction. Suppose $P(z)$ is not zero for any value of z. Then

$$f(z) = \frac{1}{P(z)}, \text{ is analytic everywhere.}$$

We have
$$f(z) = \frac{1}{P(z)} = \frac{1}{a_0 + a_1 z + \ldots\ldots + a_n z^n}$$

$$= \frac{1}{z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots\ldots + a_n \right)} \to 0 \text{ as } z \to \infty.$$

$\therefore$ For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z)| < \varepsilon$ for $|z| > \delta$.

Since $f(z)$ is continuous in the bounded closed domain $|z| \leq \delta$ therefore $f(z)$ is bounded in the closed domain $|z| \leq \delta$ so there exists a positive number K such that

$$|f(z)| < K \quad \text{for} \quad |z| \leq \delta.$$

If $M = \max(\varepsilon, K)$, then we have $|f(z)| = \left| \frac{1}{P(z)} \right| < M$, for every z.

Hence by Liouville's theorem $f(z)$ is constant. This gives a contradiction since $P(z)$ is not constant for $n = 1, 2, 3, \ldots\ldots$ and $a_n \neq 0$. Thus $P(z)$ must be zero for at least one value of z *i.e.,* $P(z) = 0$ must have at least one root.

Corollary: *Every polynomial equation*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n = 0,$$

where $n \geq 1, a_n \neq 0$ *has exactly* n *roots.* **(Gorakhpur 2006, 11, 13)**

Proof: By the fundamental theorem of Algebra $P(z) = 0$ has at least one root, say α_1. Then we have $P(\alpha_1) = 0$.

Now
$$P(z) = P(z) - P(\alpha_1)$$
$$= (a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n) - (a_0 + a_1 \alpha_1 + a_2 \alpha_1^2 + \ldots\ldots + a_n \alpha_1^n)$$
$$= a_1 (z - \alpha_1) + a_2 (z^2 - \alpha_1^2) + \ldots\ldots + a_n (z^n - \alpha_1^n)$$
$$= (z - \alpha_1) P_1(z), \text{ where } P_1(z) \text{ is a polynomial of degree } n - 1.$$

Again by fundamental theorem of Algebra $P_1(z) = 0$ must have at least one root, say α_2, (α_2 may be equal to α_1).

Proceeding as above, we have $\quad P(z) = (z - \alpha_1)(z - \alpha_2) P_2(z)$,

where $P_2(z)$ is a polynomial of degree $n - 2$.

Continuing in this way we see that $P(z) = 0$ has exactly n roots.

Alternative Proof: We shall use Rouche's theorem to show that $P(z) = 0$ has exactly n roots.

Let $\qquad f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + \ldots\ldots + a_{n-1} z^{n-1}$.

Then $$\frac{g(z)}{f(z)} = \frac{1}{a_n}\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots\ldots + \frac{a_{n-1}}{z}\right) \to 0 \text{ as } z \to \infty.$$

$\therefore$ there exists a $\delta > 0$ such that

$$\left|\frac{g(z)}{f(z)}\right| < 1 \quad \text{for} \quad |z| > \delta. \qquad \ldots(1)$$

We now take the closed curve C as the circle $|z| = \delta + 1$. Then from (1), $|g(z)| < |f(z)|$ on C. Also $f(z)$ and $g(z)$ are analytic on and inside C. Therefore by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. Since $f(z) = a_n z^n$ has n zeros at the origin therefore $f(z) + g(z)$ must also have n zeros. Hence the polynomial equation $P(z) = 0$ has exactly n roots.

Illustrative Examples

Example 6: *Prove that all the roots of* $z^7 - 5z^3 + 12 = 0$ *lie between the circles* $|z| = 1$ *and* $|z| = 2$. **(Kanpur 2008; Gorakhpur 14)**

Solution: Let C_1 represent the circle $|z| = 1$ and C_2 represent the circle $|z| = 2$. Suppose $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

We observe that both $f(z)$ and $g(z)$ are analytic within and on C_1.

Now $$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7 - 5z^3}{12}\right| \le \frac{|z|^7 + |-5z^3|}{12} = \frac{|z|^7 + 5|z|^3}{12} = \frac{1+5}{12} = \frac{1}{2},$$

since $|z| = 1$ on C.

$$\therefore \quad \left|\frac{g(z)}{f(z)}\right| < 1 \quad \text{or} \quad |g(z)| < |f(z)| \text{ on } C_1.$$

Hence by Rouche's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside C_1 as $f(z) = 12$. Since $f(z) = 12$ has no zeros inside C_1 therefore $f(z) + g(z) = z^7 - 5z^3 + 12$ has no zeros inside C_1.

Now consider the circle C_2. Take $F(z) = z^7, \phi(z) = 12 - 5z^3$. We observe that both $F(z)$ and $\phi(z)$ are analytic within and on C_2. We have

$$\left|\frac{\phi(z)}{F(z)}\right| = \frac{|12 - 5z^3|}{|z|^7} \leq \frac{|12| + 5|z|^3}{|z|^7} = \frac{12 + 5.2^3}{2^7} = \frac{52}{128} < 1,$$

since $|z| = 2$ on C_2.

Thus on C_2, $|\phi(z)| < |F(z)|$. Hence by Rouche's theorem $F(z) + \phi(z) = z^7 - 5z^3 + 12$ has the same number of zeros as $F(z) = z^7$ inside C_2. Since $F(z) = z^7$ has all the seven zeros inside the circle $|z| = 2$ as they are all located at the origin therefore all the seven zeros of $z^7 - 5z^3 + 12$ lie inside the circle C_2.

Hence all the roots of the equation $z^7 - 5z^3 - 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Example 7: *Use Rouche's theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.*

(Kanpur 2007)

Solution: Let $f(z) = z^5$ and $g(z) = 15z + 1$.

Now on the circle $|z| = 2$, we have

$$|f(z)| = |z|^5 = 2^5 = 32$$

and $\quad |g(z)| = |15z + 1| \leq 15|z| + 1 = 31.$

$\therefore \quad |g(z)| < |f(z)|$ on the circle $|z| = 2$.

Hence by Rouche's theorem the function

$$f(z) + g(z) = z^5 + 15z + 1$$

has as many zeros in $|z| < 2$ as the function $f(z) = z^5$. Since the function $f(z)$ has a zero of order 5 at $z = 0$ therefore all the five roots of $z^5 + 15z + 1 = 0$ must lie inside the disc $|z| < 2$.

Again for $|z| = \frac{3}{2}$, we have

$$|z^5 + 1| \leq |z|^5 + 1 = \frac{243}{32} + 1 < \frac{45}{2} = |15z|.$$

The function $z^5 + 15z + 1$ has as many zeros in $|z| < \frac{3}{2}$ as the function $15z$. Since $15z$ has exactly one zero in this region, so does $z^5 + 15z + 1$. Hence four of the zeros of $z^5 + 15z + 1$ must lie in the ring $\frac{3}{2} < |z| < 2$.

Comprehensive Exercise 2

1. If $a > e$, use Rouche's theorem to prove that the equation
$$e^z = az^n$$
has n roots inside the circle $|z| = 1$. **(Kanpur 2008)**

2. Using Rouche's theorem determine the number of zeros of the polynomial
$$P(z) = z^{10} - 6z^7 + 3z^3 + 1 \text{ in } |z| < 1.$$

3. Apply Rouche's theorem to determine the number of roots of the equation
$$z^8 - 4z^5 + z^2 - 1 = 0$$
that lie inside the circle $|z| = 1$.

4. Show that the polynomial $z^5 + z^3 + 2z + 3$ has just one zero in the first quadrant of the complex plane.

Answers 2

2. seven zeros
3. five roots

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) and (d).

1. If a function is analytic at all points of a bounded domain except at finitely many points, then these exceptional points are called
   (a) zeros (b) singularities
   (c) poles (d) simple points.

2. A function which has poles as its only singularities in the finite part of the plane is said to be
   (a) an analytic function (b) an entire function
   (c) a meromorphic function (d) none of these.

3. For the function $f(z) = e^z$, $z = \infty$ is

(a) isolated essential singularity (b) pole

(c) ordinary point (d) none of these.

4. Number of poles of the function $f(z) = \tan \frac{1}{z}$ is

(a) 2 (b) 4

(c) infinite (d) none of these.

5. Number of zeros of the function $f(z) = \sin \frac{1}{z}$ is

(a) 3 (b) 4

(c) infinite (d) none of these.

6. The number of isolated singular points of $f(z) = \dfrac{z+3}{z^2(z^2+2)}$ is

(a) 1 (b) 2

(c) 3 (d) 4. **(Kumaun 2007, 11)**

7. If $f(z) = z^5 - 3iz^2 + 2z + i - 1$ and C encloses zeros of $f(z)$, then $\int_C \dfrac{f'(z)}{f(z)}\, dz$ is

(a) $5\pi i$ (b) 0

(c) $10\pi i$ (d) none of these.

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The function $f(z) = \dfrac{e^z}{(z-1)^3}$ has a pole of order ……… at $z = 1$.

2. The function $f(z) = \dfrac{\sin(z-1)}{z-1}$ has removable singularity at $z =$ ……… .

3. The function $f(z) = e^{1/z}$ has an isolated essential singularity at $z =$ …… .

True or False

Write 'T' for true and 'F' for false statement.

1. The zeros of an analytic function are isolated.
2. A rational function has singularities other than poles.
3. Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $|g(z)| < |f(z)|$ on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.
4. Every polynomial of degree n has exactly n zeros.

Answers

Multiple Choice Questions

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (c)
6. (c)	**7.** (c)			

Fill in the Blank(s)

1. 3 **2.** 1 **3.** 0

True or False

1. *T* **2.** *F* **3.** *T* **4.** *T*

□

Chapter

5

The Calculus of Residues

1 Residue at a Pole

Definition: Let $z = a$ be a pole of order m of a single-valued function $f(z)$ and γ be any circle of radius r and centre $z = a$ containing no other singularities except $z = a$. Then the function $f(z)$ is regular within the region $0 < |z - a| < r$ so we can expand $f(z)$ in a Laurent's series in the region $0 < |z - a| < r$.

$$\therefore \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where $$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz$$

and $$b_n = \frac{1}{2\pi i} \int_\gamma (z-a)^{n-1} f(z) \, dz.$$

In particular, $$b_1 = \frac{1}{2\pi i} \int_\gamma f(z) \, dz.$$

The coefficient b_1 is called the residue of $f(z)$ at $z = a$.

2 Computation of Residue at a Finite Pole

(i) Residue at a simple pole: If $z = a$ is a simple pole of $f(z)$ then the Laurent's expansion of $f(z)$ is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}.$$

$$\therefore \qquad b_1 = \lim_{z \to a} (z-a) f(z).$$

Hence the residue at the simple pole $z = a$ is given by

$$\lim_{z \to a} (z-a) f(z)$$

Another form is obtained as follows :

If $\quad f(z) = \dfrac{\phi(z)}{\psi(z)}$ where $\psi(z) = (z-a) F(z), F(a) \neq 0$, then

$$\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z \to a} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots\ldots]}{\psi(a) + (z-a)\psi'(a) + \ldots\ldots}, \text{ by Taylor's theorem}$$

$$= \lim_{z \to a} \frac{\phi(a) + (z-a)\phi'(a) + \ldots\ldots}{\psi'(a) + \ldots\ldots}, \text{ since } \psi(a) = 0$$

$$= \frac{\phi(a)}{\psi'(a)}.$$

Hence the residue at the simple pole $z = a$ is $\dfrac{\phi(a)}{\psi'(a)}$.

(ii) Residue at a pole of order greater than unity :
(Residue at a pole of order m) **(Gorakhpur 2014)**

Let $z = a$ be a pole of order m of $f(z)$ and suppose

$$f(z) = \frac{\phi(z)}{(z-a)^m}.$$

We then have, by the definition of a pole,

$$f(z) = \frac{\phi(z)}{(z-a)^m} = \psi(z) + \sum_{r=1}^{m} \frac{M_r}{(z-a)^r},$$

where $\psi(z)$ is regular at $z = a$,

or $$\frac{\phi(z)}{(z-a)^m} = \psi(z) + \frac{M_1}{(z-a)} + \frac{M_2}{(z-a)^2} + \ldots + \frac{M_m}{(z-a)^m}.$$

$$\therefore \qquad \phi(z) = \psi(z)(z-a)^m + M_1 (z-a)^{m-1} + M_2 (z-a)^{m-2} + \ldots. M_m$$

Differentiating both the sides with respect to z, $(m-1)$ times, we have

$$\phi^{(m-1)}(z) = \psi^{(m-1)}(z)(z-a)^m + (m-1)\,\psi^{m-2}(z)\,.\,m\,(z-a)^{m-1}$$

$$+ \frac{(m-1)(m-2)}{2!}\,.\,\psi^{(m-3)}\,.\,(z)\,m\,(m-1)\,.\,(z-a)^{m-2} + \ldots$$

$$+ \psi(z)\frac{m!}{1!}(z-a) + M_1\,(m-1)!$$

$\therefore$ $$\phi^{(m-1)}(a) = M_1\,(m-1)!.$$

Hence $M_1 = \dfrac{\phi^{(m-1)}(a)}{(m-1)!}$ which is the required residue at $z = a$.

In particular if $f(z) = \dfrac{\phi(z)}{(z-a)^2}$, the residue at $z = a$ is $\phi'(a)$ and if $f(z) = \dfrac{\phi(z)}{(z-a)^3}$, the residue at $z = a$ is $\dfrac{\phi''(a)}{2!}$ etc.

Alternative Proof (a): Suppose $z = a$ is a pole of order m. Then $f(z)$ is of the form $\dfrac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic.

Residue of $f(z)$ at $z = a$ is given by

$$b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-a)^m}\,dz$$

$$= \frac{\phi^{m-1}(a)}{(m-1)!}, \text{ by Cauchy's integral formula.}$$

Hence *the residue of $f(z)$ at the pole of order m is* $\dfrac{\boldsymbol{\phi^{m-1}(a)}}{\boldsymbol{(m-1)!}}$, *where $z = a$ is the pole of order m.*

Alternative Proof (b): If $z = a$ is the pole of order m of $f(z)$ then we have $f(z) = \dfrac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic at $z = a$.

The residue at $z = a$ is the coefficient of $(z-a)^{-1}$ in $f(z)$

$$= \text{coeff. of } (z-a)^{m-1} \text{ in } \phi(z)$$

$$= \text{coeff. of } (z-a)^{m-1} \text{ in } [\,\phi(a) + (z-a)\,\phi'(a) + \ldots\ldots$$

$$+ \frac{(z-a)^{m-1}}{(m-1)!}\phi^{m-1}(a) + \ldots\ldots]$$

$$= \frac{\phi^{m-1}(a)}{(m-1)!}.$$

Remark: We have seen that the residue of $f(z)$ at the pole $z = a$ is the coefficient of $\dfrac{1}{(z-a)}$ in the Laurent's expansion of $f(z)$. If we put $z - a = t$ or $z = a + t$, where t is small then the Laurent's expansion of $f(z)$ becomes

$$f(a+t) = \sum_{n=0}^{\infty} a_n t^n + \frac{b_1}{t} + \frac{b_2}{t^2} + \ldots\ldots + \frac{b_m}{t^m}.$$

We see that b_1 is the coefficient of $1/t$ in the above expansion.

Thus *to find the residue of* $f(z)$ *at* $z = a$, *put* $z = a + t$ *in* $f(z)$ *and expand in powers of* t, *where* t *is small, the coefficient of* $1/t$ *is the residue at* $z = a$.

3 Residue at Infinity

Definition: *If the function* $f(z)$ *has an isolated singularity at infinity or is analytic there then the residue of* $f(z)$ *at* $z = \infty$ *is given by*

$$\frac{1}{2\pi i}\int_C f(z)\, dz$$

where C *is a large circle containing all the finite singularities of* $f(z)$ *and integral along* C *is performed in a clockwise direction provided that this integral has a definite value.*

If the integral along C is taken in anti-clockwise direction the residue at infinity is

$$-\frac{1}{2\pi i}\int_C f(z)\, dz.$$

4 Computation of Residue at Infinity

Method I: By definition the residue of $f(z)$ at $z = \infty$ is $\frac{1}{2\pi i}\int_C f(z)\, dz$ taken in clockwise direction round a large circle C which encloses in its interior all other singularities.

Therefore the residue of $f\left(\frac{1}{w}\right)$ at $w = 0$ is given by

$$\frac{1}{2\pi i}\int_\gamma f\left(\frac{1}{w}\right)\left(-\frac{dw}{w^2}\right) = \frac{1}{2\pi i}\int_\gamma -\frac{1}{w^2} f\left(\frac{1}{w}\right) dw,$$

taken in anti-clockwise direction around a small circle γ with centre at origin.

Hence the residue of $f(z)$ at infinity

$$= \lim_{w \to 0}\left[\frac{-w\, f(1/w)}{w^2}\right] = \lim_{z \to \infty}\,[-zf(c)],$$

provided the limit exists.

Method II: Suppose $f(z)$ has a pole of order m at infinity. Then $f(1/w)$ has a pole of order m at $w = 0$.

By Laurent's theorem the expansion of $f(1/w)$ at $w = 0$ is given by

$$f\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} a_n w^n + \sum_{n=1}^{m} \frac{b_n}{w^n}.$$

Therefore the expansion of $f(z)$ in the neighbourhood of $z = \infty$ is given by

$$f(z) = \sum_{n=1}^{m} b_n z^n + \sum_{n=0}^{\infty} a_n z^{-n}.$$

Now $$\int_C f(z)\,dz = \int_C \sum_{n=1}^{m} b_n z^n\,dz + \int_C \sum_{n=0}^{\infty} a_n z^{-n}\,dz$$

$$= \sum_{n=1}^{m} \int_C b_n z^n\,dz + \sum_{n=0}^{\infty} \int_C a_n z^{-n}\,dz$$

$$= \int_C \frac{a_1}{z}\,dz,$$ all other integrals vanish since each of them is of the form $\int_C \frac{dz}{z^k}$, $k \neq 1$

$$= a_1 \cdot 2\pi i \qquad \left[\because \int_C \frac{dz}{z} = 2\pi i\right]$$

Thus the residue at infinity $= -\dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz = -a_1$, which is the coefficient of $1/z$ with sign changed in the expansion of $f(z)$ in the neighbourhood of $z = \infty$.

Hence the residue of $f(z)$ at infinity is the negative of the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ in the neighbourhood of $z = \infty$.

An Important Observation: We can easily show that the residue of a function at a finite point is zero if the function is analytic there. On the other hand a function may be analytic at $z = \infty$ but yet has a residue there.

Consider $f(z) = \dfrac{1}{z-a}\cdot$ The only singularity of $f(z)$ is a simple pole at $z = a$. The function is analytic at $z = \infty$.

Now the residue at infinity $$= -\frac{1}{2\pi i}\int_C \frac{dz}{z-a}$$

$$= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}\,d\theta, \text{ putting } z - a = re^{i\theta}$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} d\theta = -1.$$

5 Cauchy's Residue Theorem

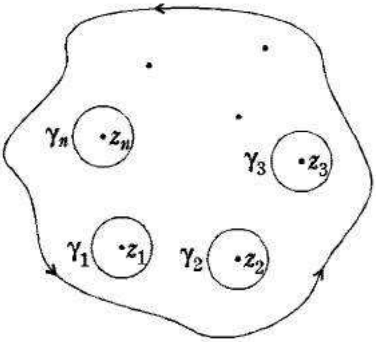
If $f(z)$ is regular, except at a finite number of poles within a closed contour C and continuous on the boundary of C, then

$$\int_C f(z)\,dz = 2\pi i\,\Sigma R,$$

where ΣR is the sum of the residues of $f(z)$ at its poles within C.

(Meerut 2001; Kanpur 01; Gorakhpur 06, 11; Rohilkhand 08, 12)

Proof: Let $z_1, z_2, \ldots\ldots, z_n$ be the n poles within the closed contour C. Let $\gamma_1, \gamma_2, \ldots\ldots, \gamma_n$ be the circles with centres $z_1, z_2, \ldots\ldots, z_n$ respectively and each of radius r so small that all the circles lie entirely within C and do not overlap. Then $f(z)$ is analytic in the region lying between C and the circles. Then by Cauchy's theorem



$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz = 0$$

or

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

$$+ \int_{\gamma_2} f(z)\,dz + \ldots\ldots + \int_{\gamma_n} f(z)\,dz. \qquad \ldots(1)$$

Suppose $f(z)$ has a pole of order m_1 at $z = z_1$ then we have

$$f(z) = \phi_1(z) + \sum_{k=1}^{m_1} \frac{b_r}{(z - z_1)^r}, \text{where } \phi_1 \text{ is analytic within and on } \gamma_1.$$

Now

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} \phi_1(z)\,dz + \int_{\gamma_1} \frac{b_1}{z - z_1}\,dz + \ldots\ldots$$

$$+ \int_{\gamma_1} \frac{b_{m_1}}{(z - z_1)^{m_1}}\,dz. \qquad \ldots(2)$$

We have $\int_{\gamma_1} \phi_1(z)\,dz = 0$ since ϕ_1 is analytic within and on γ_1.

Also

$$\int_{\gamma_1} \frac{b_1}{z - z_1}\,dz = \int_0^{2\pi} \frac{b_1\, rie^{i\theta}}{r\,e^{i\theta}}\,d\theta, \text{ putting } z - z_1 = r\,e^{i\theta}$$

$$= \int_0^{2\pi} b_1\, i\,d\theta = 2\pi\, i\, b_1$$

and

$$\int_{\gamma_1} \frac{b_{m_1}}{(z - z_1)^{m_1}}\,dz = \int_0^{2\pi} \frac{b_{m_1}\, r\, i\, e^{i\theta}}{r^{m_1}\, e^{i m_1 \theta}}\,d\theta, \text{ putting } z - z_1 = re^{i\theta}$$

$$= \frac{i\, b_{m_1}}{r^{m_1 - 1}} \int_0^{2\pi} e^{-i(m_1 - 1)\theta}\,d\theta = 0,\ m_1 \neq 1.$$

Substituting these values in (2), we get

$$\int_{\gamma_1} f(z)\,dz = 2\pi\, i\, b_1 = 2\pi\, i \times \text{residue of } f(z) \text{ at } z = z_1.$$

Proceeding as above, we have

$$\int_{\gamma_2} f(z)\,dz = 2\pi\, i \times \text{residue of } f(z) \text{ at } z = z_2 \text{ and so on.}$$

Hence from (1), we have

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of the residues at } z_1, z_2, \ldots\ldots, z_n)$$

$$= 2\pi\, i\, \Sigma\, R.$$

Corollary: *If an analytic function has singularities at a finite number of points (including that at infinity), then the sum of the residues at these points along with infinity is zero.*

Let C be the circle enclosing within it all the singularities excluding infinity. Then by the previous theorem, we have

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \text{sum of the residues at all the finite singular points within } C,$$

also the residue at infinity is $-\dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz$.

Hence the sum of the residues at all the finite poles along with infinity is zero.

Illustrative Examples

Example 1: *Find the residue of* $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ *at* $z=1$.

(Kumaun 2007, Rohilkhand 09)

Solution: Let $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$.

$z=1$ is the pole of order 4 of $f(z)$.

To find the residue at $z=1$ we shall put $z=1+t$ in $f(z)$ then the coefficient of $1/t$ will be the residue at $z=1$.

Now $$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} = \frac{(1+t)^3}{t^4(t-1)(t-2)}, \text{ putting } z-1=t$$

$$= \frac{1}{2t^4}(1+t)^3(1-t)^{-1}\left(1-\frac{1}{2}t\right)^{-1}$$

$$= \frac{1}{2t^4}(1+3t+3t^2+t^3)(1+t+t^2+t^3+\ldots\ldots)\left(1+\frac{1}{2}t+\frac{1}{4}t^2+\frac{1}{8}t^3+\ldots\ldots\right)$$

$$= \frac{1}{2t^4}(1+3t+3t^2+t^3)\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\frac{15}{8}t^3+\ldots\ldots\right).$$

The coefficient of $1/t$ in the above expansion

$$= \frac{1}{2}\left(\frac{15}{8}+\frac{21}{4}+\frac{9}{2}+1\right) = \frac{101}{16}$$

which is the residue at $z=1$.

Example 2: *Determine the poles of the function* $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *and the residue at each point.*

Hence evaluate $\displaystyle\int_C f(z)\,dz$ *where* C *is the circle* $|z| = 2\cdot 5$.

Solution: We have $f(z) = \dfrac{z^2}{(z-1)^2(z+2)} = \dfrac{1}{(z-1)^2}\phi(z)$, where $\phi(z) = \dfrac{z^2}{z+2}$.

Here $z=1$ is a pole of order 2 of $f(z)$ and $z=-2$ is a simple pole.

Now residue at $z = 1$ is

$$\frac{1}{1!}[\phi'(z)]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

Residue at $z = -2$ is $\lim_{z \to -2} (z+2) f(z) = \lim_{z \to -2} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}.$

The function $f(z)$ is analytic on $|z| = 2 \cdot 5$ and at all points inside it except at $z = 1, -2$ therefore by residue theorem, we have

$$\int_C f(z)\, dz = 2\pi i\,[\text{residue at } (z=-2) + \text{residue at } (z=1)]$$

$$= 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i.$$

Example 3: *Find the residues of* $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ *at all its poles in the finite plane.*

Solution: Here $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)} = \dfrac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}.$

Poles of $f(z)$ are given by $(z+1)^2(z+2i)(z-2i) = 0.$

$f(z)$ has a double pole at $z = -1$ and simple poles at $z = 2i, -2i.$

Residue at $z = 2i$ is

$$\lim_{z \to 2i} (z-2i) f(z) = \lim_{z \to 2i} \frac{z^2-2z}{(z+1)^2(z+2i)}$$

$$= \frac{(2i)^2 - 2 \cdot 2i}{(2i+1)^2(2i+2i)} = \frac{7+i}{25}.$$

Residue at $z = -2i$ is

$$\lim_{z \to -2i} (z+2i) f(z) = \lim_{z \to -2i} \frac{z^2-2z}{(z+1)^2(z-2i)}$$

$$= \frac{(-2i)^2 - 2\cdot(-2i)}{(-2i+1)^2(-2i-2i)} = \frac{7-i}{25}.$$

Residue at $z = -1$ is $\dfrac{1}{1!}\left[\dfrac{d}{dz}\phi(z)\right]_{z=-1}$, where $\phi(z) = \dfrac{z^2-2z}{z^2+4}$ is

$$= \left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)\right]_{z=-1} = \left[\frac{2z^2+8z-8}{(z^2+4)^2}\right]_{z=-1} = -\frac{14}{25}.$$

Example 4: *Find the residues of* $e^z \operatorname{cosec}^2 z$ *at all its poles in the finite plane.*

Solution: Let $f(z) = e^z \operatorname{cosec}^2 z = \dfrac{e^z}{\sin^2 z}.$

The poles of $f(z)$ are given by $\sin^2 z = 0$

or $\quad z = m\pi,\ m \in I$ are the poles of $f(z)$ of order 2.

The limit point of these poles is $z = \infty$ which is therefore a non-isolated essential singularity.

Putting $z = m\pi + t$ in $f(z)$, we get

$$f(m\pi + t) = \frac{e^{m\pi + t}}{\sin^2(m\pi + t)} = e^{m\pi} e^{t} \cdot \frac{1}{\sin^2(m\pi + t)}$$

$$= \frac{e^{m\pi}\left(1 + t + \frac{1}{2!}t^2 + \ldots\right)}{\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \ldots\right)^2}$$

$$= e^{m\pi} \frac{1}{t^2}\left(1 + t + \frac{t^2}{2!} + \ldots\right)\left[1 - \left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots\right)\right]^{-2}$$

$$= e^{m\pi} \frac{1}{t^2}\left(1 + t + \frac{t^2}{2!} + \ldots\right)\left[1 + 2\left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots\right) + 3\left(\frac{t^2}{3!} - \frac{t^4}{5!} + \ldots\right)^2 + \ldots\right]$$

Now residue at $z = m\pi$ is the coeff. of $\frac{1}{t}$ in the above expansion $= e^{m\pi}$.

Example 5: *Evaluate* $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.

Solution: We have $\quad f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$.

The function $f(z)$ is analytic at every point within C except at the poles $z = 1, 2$.

Residue at $z = 2$ is $\lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1.$

Residue at $z = 1$ is

$$\frac{1}{1!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)\right]_{z=1} = 2\pi + 1.$$

$\therefore$ By residue theorem, we have

$$\int_C f(z)\, dz = 2\pi i\,[\text{sum of residues at } z = 1 \text{ and } z = 2]$$

$$= 2\pi i\,[1 + (2\pi + 1)]$$

$$= 4\pi i\,(\pi + 1).$$

Comprehensive Exercise 1

1. Find the residue of $\dfrac{1}{(z^2+1)^3}$ at $z = i$. **(Kumaun 2009)**

2. Find the residue of $\dfrac{1}{(z^2+a^2)^2}$ at $z = ia$. **(Kumaun 2010, 12, 13)**

3. Find the residue of $\dfrac{z^2}{z^2+a^2}$ at $z = ia$. **(Kumaun 2015)**

4. Find the residues of the function $\dfrac{\cot \pi z}{(z-a)^2}$.

5. Find the residues of the function $\dfrac{z^4}{(c^2+z^2)^4}$. **(Kumaun 2011)**

6. Find the residues of $\dfrac{z^2}{(z-a)(z-b)(z-c)}$ at infinity.

Answers 1

1.	$-\dfrac{3i}{16}$	2.	$\dfrac{-i}{4a^3}$	3.	$\dfrac{ia}{2}$
4.	$\dfrac{1}{\pi(n-a)^2}$	5.	$-\dfrac{i}{32c^3}$	6.	-1

6 Evaluation of Real Definite Integrals by Contour Integration

Now we shall evaluate the real definite integrals with the help of contour integration and Cauchy's residue theorem by properly choosing the integrand and the contour. It should be noted that a large number of real definite integrals whose evaluation by usual methods is difficult can be easily evaluated by using Cauchy's residue theorem, yet there are many integrals which cannot be evaluated by contour integration. Before discussing the procedure of the evaluation of definite integrals we are going to prove two useful theorems :

Theorem 1: *If* $\lim_{z \to a} (z-a) f(z) = A$ *and if* C *is the arc* $\theta_1 \le \theta \le \theta_2$ *of the circle* $|z-a| = r$, *then* $\lim_{r \to 0} \int_C f(z)\, dz = i A (\theta_2 - \theta_1)$.

In particular, if $(z-a)\, f(z) \to 0$ as $z \to 0$, then we have

$$\int_C f(z)\, dz \to 0 \text{ as } z \to 0.$$

Theorem 2: *If C is an arc* $\theta_1 \le \theta \le \theta_2$ *of the circle* $|z| = R$ *and if* $\lim_{R \to \infty} z\, f(z) = A$ *then* $\lim_{R \to \infty} \int_C f(z)\, dz = i(\theta_2 - \theta_1)\, A.$

7 Integration Round the Unit Circle

(Gorakhpur 2016)

We shall discuss here the method of evaluation, by contour integration, of the integrals which are of the type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

where integrand is a rational function of $\cos\theta$ and $\sin\theta$.

Putting $z = e^{i\theta}$, we have $dz = i e^{i\theta}\, d\theta$

and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$.

Now $$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \frac{1}{i}\int_C f\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z}$$

$$= \int_C \phi(z)\, dz, \text{ say,}$$

where C is the circle $|z| = 1$.

It is obvious that $F(z)$ is a rational function of z.

Thus by residue theorem we have

$$\int_C \phi(z)\, dz = 2\pi i\, \Sigma\, R_C,$$

where $\Sigma\, R_C$ is the sum of the residues of $\phi(z)$ at its poles inside C.

Illustrative Examples

Example 6: *Show that*

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{(a^2 - b^2)}},\ a > b > 0.$$

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$

$$= \int_0^{2\pi} \frac{d\theta}{a + \frac{1}{2} b\,(e^{i\theta} + e^{-i\theta})} = \frac{1}{i} \int_C \frac{dz}{z\left\{a + \frac{1}{2} b\left(z + \frac{1}{z}\right)\right\}},$$

putting $e^{i\theta} = z, i e^{i\theta}\, d\theta = dz$

$$= \frac{2}{i} \int_C \frac{dz}{bz^2 + 2az + b}, \text{ where } C \text{ is the unit circle } |z| = 1.$$

The poles of the integrand $f(z) = \dfrac{2}{i\,(bz^2 + 2az + b)}$ are given by

$$bz^2 + 2az + b = 0$$

or $$z = \frac{-2a \pm \sqrt{(4a^2 - 4b^2)}}{2b} = \frac{-a \pm \sqrt{(a^2 - b^2)}}{b}.$$

Let $$\alpha = \frac{-a + \sqrt{(a^2 - b^2)}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{(a^2 - b^2)}}{b}.$$

Since $a > b > 0$ therefore $|\beta| > 1$. Also $|\alpha\beta| = 1$ so we have $|\alpha| < 1$.

Thus $z = \alpha$ is the simple pole lying inside C.

Since α, β are the roots of $bz^2 + 2az + b = 0$, therefore we have

$$f(z) = \frac{2}{i\,b\,(z - \alpha)(z - \beta)}.$$

Residue of $f(z)$ at the simple pole $z = \alpha$ is

$$= \lim_{z \to \alpha} (z - \alpha) \frac{2}{i\,b\,(z - \alpha)(z - \beta)} = \lim_{z \to \alpha} \frac{2}{b\,i\,(z - \beta)}$$

$$= \frac{2}{b\,i\,(\alpha - \beta)} = \frac{2}{b\,i\,\dfrac{2\sqrt{(a^2 - b^2)}}{b}} = \frac{1}{i\sqrt{(a^2 - b^2)}}.$$

Hence, $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = 2\pi i$. sum of the residues of $f(z)$ at the poles inside C

$$= 2\pi i \cdot \frac{1}{i\sqrt{(a^2 - b^2)}} = \frac{2\pi}{\sqrt{(a^2 - b^2)}}.$$

Similarly, we can evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$.

Example 7:*Evaluate* $\displaystyle\int_{-\pi}^{2\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta, a > 1$

by putting $e^{i\theta} = z$ *and using the theory of residues.*

Solution: Let $I = \displaystyle\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta = 2\int_0^{\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta$

$$= \int_0^{2\pi} \frac{2a\cos\theta}{2a + 2\cos\theta}\, d\theta$$

$$= \text{real part of} \int_0^{2\pi} \frac{2a\, e^{i\theta}}{2a + 2\cos\theta}\, d\theta$$

$$= \text{real part of} \int_C \frac{2az}{2a + z + \frac{1}{z}} \cdot \frac{dz}{iz}$$

$$= \text{real part of} \int_C \frac{2az}{i(z^2 + 2az + 1)}\, dz$$

$$= \text{real part of} \int_C f(z)\, dz, \text{ say,} \qquad \ldots(1)$$

where C is the unit circle $|z| = 1$.

Solving $z^2 + 2az + 1 = 0$, we get

$$z = -a + \sqrt{(a^2 - 1)} = \alpha, \; z = -a - \sqrt{(a^2 - 1)} = \beta,$$

which are the simple poles of $f(z)$.

Since $|\beta| > 1$ and $|\alpha\beta| = 1$ therefore $|\alpha| < 1$. Thus $z = \alpha$ is the simple pole inside C.

Residue at $z = \alpha$ is $\lim_{z \to \alpha} (z - \alpha) f(z)$

$$= \lim_{z \to \alpha} \frac{(z - \alpha)\, 2az}{i(z - \alpha)(z - \beta)} = \lim_{z \to \alpha} \frac{2az}{i(z - \beta)} = \frac{2a\alpha}{i(\alpha - \beta)}$$

$$= \frac{2a\{-a + \sqrt{(a^2 - 1)}\}}{2i\sqrt{(a^2 - 1)}} = ai\left\{\frac{a}{\sqrt{(a^2 - 1)}} - 1\right\}.$$

$\therefore$ by Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = 2\pi i \,.\, ai\left\{\frac{a}{\sqrt{(a^2 - 1)}} - 1\right\} = 2a\pi\left\{1 - \frac{a}{\sqrt{(a^2 - 1)}}\right\}.$$

Hence, $\displaystyle\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta = \text{real part of} \int_C f(z)\, dz$, from (1)

$$= 2a\pi\left\{1 - \frac{a}{\sqrt{(a^2 - 1)}}\right\}.$$

Example 8: *Prove that* $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta = \pi\, \frac{1 - p + p^2}{1 - p}, 0 < p < 1.$

Solution: We have $\displaystyle I = \int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta$$

$$= \frac{1}{2}\, \text{real part of} \int_0^{2\pi} \frac{1 + e^{i6\theta}}{1 - 2p\cos 2\theta + p^2}\, d\theta$$

$$= \frac{1}{2} \text{ real part of } \int_C \frac{1 + z^6}{1 - p\left(z^2 + \frac{1}{z^2}\right) + p^2} \cdot \frac{dz}{iz}, \text{ putting } z = e^{i\theta}$$

$$= \frac{1}{2} \text{ real part of } \int_C \frac{z(1 + z^6)}{i(1 - pz^2)(z^2 - p)} dz$$

$$= \frac{1}{2} \text{ real part of } \int_C f(z)\, dz, \text{ say,}$$

where C is the unit circle.

$z = \pm\sqrt{p}, \pm 1/\sqrt{p}$ are the simple poles of $f(z)$.

Since $0 < p < 1$ therefore $z = \sqrt{p}, -\sqrt{p}$ are the only simple poles which lie within C.

Residue at $z = \sqrt{p}$ is

$$\lim_{z \to \sqrt{p}} (z - \sqrt{p}) f(z) = \lim_{z \to \sqrt{p}} \frac{(z - \sqrt{p})\, z(1 + z^6)}{i(1 - pz^2)(z^2 - p)} = \frac{1}{2i}\left(\frac{1 + p^3}{1 - p^2}\right)$$

and residue at $z = -\sqrt{p}$ is

$$\lim_{z \to -\sqrt{p}} (z + \sqrt{p}) f(z) = \lim_{z \to -\sqrt{p}} (z + \sqrt{p}) \frac{z(1 + z^6)}{i(1 - pz^2)(z^2 - p)}$$

$$= \frac{1}{2i} \cdot \frac{1 + p^3}{(1 - p^2)}.$$

$\therefore$ Sum of the residues at the poles inside C

$$= \frac{1}{i}\left(\frac{1 + p^3}{1 - p^2}\right).$$

Hence,
$$\int_0^{2\pi} \frac{1 + e^{i6\theta}}{1 - 2p\cos 2\theta + p^2} = 2\pi i \cdot \frac{1}{i}\left(\frac{1 + p^3}{1 - p^2}\right) = 2\pi\left(\frac{1 + p^3}{1 - p^2}\right).$$

$\therefore$
$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2} d\theta = \frac{1}{2} \text{ real part of } \int_0^{2\pi} \frac{1 + e^{i6\theta}}{1 - 2p\cos 2\theta + p^2} d\theta$$

$$= \pi\left(\frac{1 + p^3}{1 - p^2}\right) = \pi \frac{(1 + p + p^2)}{1 - p}.$$

Example 9: *(i) Prove that* $\displaystyle\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{(1 + a^2)}}, a > 0$

(Avadh 2007)

(ii) $\displaystyle\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{2\sqrt{1 + a^2}}, a > 0.$

Solution: (i) We have $\displaystyle I = \int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 2\sin^2\theta}$

$$= \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 1 - \cos 2\theta} = \int_0^{2\pi} \frac{a\, dt}{2a^2 + 1 - \cos t}, \text{ putting } 2\theta = t$$

$$= \int_C \frac{a}{2a^2 + 1 - \frac{1}{2}(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= \frac{2a}{i} \int_C \frac{dz}{2z(2a^2 + 1) - z^2 - 1}$$

$$= 2ai \int_C \frac{dz}{z^2 - 2(2a^2 + 1) z + 1} = \int_C f(z)\, dz, \text{ say,}$$

where C is the unit circle.

Solving $\quad z^2 - 2(2a^2 + 1) z + 1 = 0$, we get

$$z = (2a^2 + 1) \pm 2a\sqrt{(a^2 + 1)}$$

or

$$z = (2a^2 + 1) + 2a\sqrt{(a^2 + 1)} = \alpha,$$

$$z = (2a^2 + 1) - 2a\sqrt{(a^2 + 1)} = \beta.$$

Thus $z = \alpha, \beta$ are the simple poles of $f(z)$.

Since $a > 0$ therefore $|\alpha| > 1$. Also we have $|\alpha\beta| = 1$ therefore $|\beta| < 1$.

Thus $z = \beta$ is the only simple pole inside C.

Residue at $z = \beta$ is

$$\lim_{z \to \beta} (z - \beta) f(z) = \lim_{z \to \beta} (z - \beta) \frac{2ai}{(z - \alpha)(z - \beta)} = \frac{2ai}{\beta - \alpha}$$

$$= -\frac{2ai}{4a\sqrt{(a^2 + 1)}} = -\frac{i}{2\sqrt{(a^2 + 1)}}.$$

Hence, $\quad \displaystyle\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2 \theta} = 2\pi i.$ sum of the residues inside C

$$= 2\pi i \left[-\frac{i}{2\sqrt{(a^2 + 1)}} \right] = \frac{\pi}{\sqrt{(a^2 + 1)}}.$$

(ii) $\quad \displaystyle\int_0^{2\pi} \frac{a\, d\theta}{a^2 + \sin^2 \theta} = 2\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2 \theta} = \frac{2\pi}{2\sqrt{(1 + a^2)}}.$

Example 10: *By the method of contour integration, prove that*

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!},$$

where n is a positive integer. **(Gorakhpur 2005; Kanpur 07, 08)**

Solution: We have

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta.$$

$[\because \cos(-\theta) = \cos\theta]$

Now consider $I = \displaystyle\int_0^{2\pi} e^{\cos\theta}\, e^{i(\sin\theta - n\theta)}\, d\theta$

$$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\, e^{-in\theta}\, d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\, d\theta$$

$$= \int_C e^z z^{-n} \cdot \frac{dz}{iz}, \text{ putting } z = e^{i\theta}$$

$$= \int_C \frac{e^z}{i z^{n+1}}\, dz$$

$$= \int_C f(z)\, dz, \text{say,}$$

where C is the unit circle.

The function $f(z)$ has a pole of order $(n+1)$ at $z = 0$.

The residue at $z = 0$ is $\frac{1}{n!}\left[D^n \frac{e^z}{i}\right]_{z=0} = \frac{1}{in!}\cdot$

Hence $\qquad I = 2\pi i \cdot \frac{1}{in!} = \frac{2\pi}{n!}\cdot$

Equating real parts on both sides, we get

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}\cdot$$

Comprehensive Exercise 2

1. Prove that $\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0.$

(Kanpur 2007, Rohilkhand 12)

2. (i) Show that $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}, \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{2}\cdot$

(Kumaun 2012)

(ii) Show that $\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{(1 - a^2)}}, \; a^2 < 1.$

(iii) Prove that $\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{(a^2 - b^2)}}\cdot$

3. Use the method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2}, \quad 0 < a < 1.$$

4. Apply the method of contour integration to prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = \frac{\pi}{6}\cdot$$

5. By the method of contour integration prove that

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta = \frac{\pi a^2}{1 - a^2}, \quad (-1 < a < 1).$$

(Gorakhpur 2006)

6. Evaluate $\int_0^{2\pi} \dfrac{\sin n\theta}{1 + 2a\cos\theta + a^2}\, d\theta$ (Rohilkhand 2010)

and $\int_0^{2\pi} \dfrac{\cos n\theta}{1 + 2a\cos\theta + a^2}\, d\theta,$

$a^2 < 1$ and n is a positive integer.

7. Show that $\int_0^{\pi} \tan(\theta + ia)\, d\theta = i\pi$, where $R(a) > 0$.

8. Prove that $\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + \sin\theta)\, d\theta = (-1)^n \dfrac{2\pi}{n!}$,

where n is a positive integer.

Answers 2

6. $\dfrac{2\pi(-1)^n a^n}{1 - a^2}$; 0

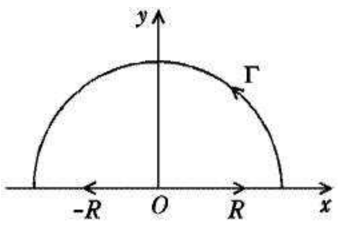
8 Evaluation of the Integral $\int_{-\infty}^{\infty} f(x)\, dx$

Theorem: *If the function $f(z)$ is analytic in the upper half of the z-plane except at a finite number of poles in it, having no poles on the real axis and if further $z\, f(z)$ tends to zero as $|z|$ tends to infinity then by contour integration*

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i\, \Sigma R^{+}$$

where ΣR^{+} represents the sum of the residues at the poles in the upper half plane.

Proof: Under the given conditions the integral $\int_{-\infty}^{\infty} f(x)\, dx$ is convergent. To evaluate such integrals we shall integrate $f(z)$ round a contour C consisting of a semi-circle Γ of radius R large enough to include all the poles of $f(z)$ and the part of the real axis from $x = -R$ to $x = R$. The only singularities of $f(z)$ in the upper half plane are poles.



$\therefore$ By Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = 2\pi i\, \Sigma R^{+}$$

or
$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 2\pi i \Sigma R^{+} \qquad \ldots(1)$$

where ΣR^+ represents the sum of the residues of $f(z)$ at the poles in the upper half plane.

Since $z f(z) \to 0$ as $|z| \to \infty$, therefore we have

$$\int_{\Gamma} f(z)\, dz = 0. \qquad \text{[By theorem 2 of article 6]}$$

Also
$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx$$

$$= P \int_{-\infty}^{\infty} f(x)\, dx, \text{ where } P \text{ stands for principal value of the integral}$$

$$= \int_{-\infty}^{\infty} f(x)\, dx, \text{ since the integral is convergent.}$$

Now taking limit of both sides of (1) when $R \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i\, \Sigma R^+.$$

Corollary: If the function $f(z)$ is of the form $P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are both polynomials such that (i) $Q(z) = 0$ has no real roots (ii) degree of $P(z)$ is at least two less than that of $Q(z)$ so that $z f(z) \to 0$ as $|z| \to \infty$ then

$$\int_{-\infty}^{\infty} f(z)\, dz = 2\pi i\, \Sigma R^+.$$

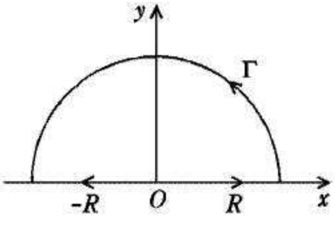
Illustrative Examples

Example 11: *(i) If $a > 0$, prove that* $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$.

(ii) Prove that $\int_0^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{\pi}{4}$. **(Kumaun 2014)**

Solution: (i) Consider

$$\int_C f(z)\, dz = \int_C \frac{dz}{(a^2 + z^2)^2},$$



where C is the contour consisting of a large semi-circle Γ of radius R together with real axis from $-R$ to R.

$z = ai, -ai$ are the poles of $f(z)$ of second order. Out of these only $z = ai$ lies inside C.

Residue at $z = ai$ is

$$\phi'(ai) = [(d/dz)(z + ai)^{-2}]_{z = ai} = [-2(z + ai)^{-3}]_{z = ai} = \frac{1}{4a^3 i}.$$

By Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} \frac{dx}{(a^2 + x^2)^2} + \int_{\Gamma} \frac{dz}{(a^2 + z^2)^2} = 2\pi i\, \Sigma R^+. \qquad \ldots(1)$$

Now $$\lim_{R \to \infty} z\, f(z) = \lim_{z \to \infty} \frac{z}{(a^2 + z^2)^2} = 0,$$

$\therefore$ $$\lim_{R \to \infty} \int_\Gamma \frac{dz}{(a^2 + z^2)^2} = 0$$

and $$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(a^2 + x^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)^2}.$$

Hence we have from relation (1)

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)^2} = 2\int_0^{\infty} \frac{dx}{(a^2 + x^2)^2} = 2\pi i \cdot \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

or $$\int_0^{\infty} \frac{dx}{(a^2 + x^2)^2} = \frac{\pi}{4a^3}.$$

(ii) Proceed as in part (i) taking $a = 1$.

Example 12: *Prove that* $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3}\, dx = \frac{\pi}{8a^3}$, *provided that* $R(a)$ *is +ve. What is the value of this integral when* $R(a)$ *is negative ?*

Solution: Consider the integral $\int_C f(z)\, dz = \int_C \frac{z^2}{(z^2 + a^2)^3}\, dz,$

where C is the contour as described in Ex. 11.

By Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} \frac{x^2}{(x^2 + a^2)^3}\, dx + \int_\Gamma \frac{z^2}{(z^2 + a^2)^3}\, dz = 2\pi i\, \Sigma\, R^+ \quad \ldots(1)$$

Since $$\lim_{R \to \infty} z\, f(z) = \lim_{z \to \infty} z \cdot \frac{z^2}{(z^2 + a^2)^3} = 0,$$ therefore we have

$$\lim_{R \to \infty} \int_\Gamma \frac{z^2}{(z^2 + a^2)^3}\, dz = 0.$$

Also $z = \pm ai$ are the poles of $f(z)$ of order three. Out of these only $z = ai$ lies inside C.

Residue at $z = ai$ is $$\frac{1}{2!}\left[(d^2/dz^2) \frac{z^2}{(z + ai)^3}\right]_{z = ai} = \frac{1}{16\, a^3 i}.$$

Now from (1), we have

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2 + a^2)^3}\, dz = 2\pi i \cdot \frac{1}{16\, a^3\, i} = \frac{\pi}{8a^3}.$$

If $R(a) < 0$ then $z = -ai$ is the pole inside C.

In this case, $$\int_{-\infty}^{\infty} \frac{z^2}{(z^2 + a^2)^3}\, dz = -\frac{\pi}{8a^3}.$$

Aliter: Consider $\int_C \frac{z^2\, dz}{(z^2+a^2)^3} = \int_C f(z)\, dz$, where C is the same contour as in Ex. 11.

By residue theorem,

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i\, \Sigma R^+.$$

Since $\lim_{z \to \infty} z\, f(z) = 0$, we have $\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$.

$\therefore$ Proceeding to the limits, when $R \to \infty$, we get from (1),

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i\, \Sigma R^+. \qquad \text{...(2)}$$

Now $f(z)$ has poles at $z = \pm ai$ of order three, of which $z = ai$ lies inside C provided $R(a)$ is positive.

Residue of $f(z)$ (at $z = ai$) = coeff. of $\frac{1}{t}$ in $f(t + ai)$

$$= \text{coeff. of } \frac{1}{t} \text{ in } \frac{(t+ai)^2}{[(t+ai)^2 + a^2]^3}.$$

Now

$$\frac{(t+ai)^2}{(t^2+2ait)^3} = -\frac{1}{8a^3 it^3}(t^2+2ait-a^2)\left(1+\frac{t}{2ai}\right)^{-3}$$

$$= -\frac{1}{8a^3\, it^3}[t^2+2ait-a^2]\left[1-\frac{3t}{2ai}-\frac{6t^2}{4a^2}+\ldots\ldots\right]$$

$\therefore$ coeff. of $\frac{1}{t}$ in $f(t+ai) = -\frac{1}{8a^3 i}\left[1-\frac{6ai}{2ai}+\frac{6a^2}{4a^2}\right] = \frac{1}{16a^3 i}$.

$\therefore$ Residue of $f(z)$ (at $z = ai$) $= \frac{1}{16a^3 i}$.

$\therefore$ From (2), $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \cdot \frac{1}{16a^3 i} = \frac{\pi}{8a^3}$. ...(3)

When $R(a)$ is $-$ ve, pole within C is at $z = -ai$.

$\therefore$ In this case $\int_{-\infty}^{\infty} f(x)\, dx = -\frac{\pi}{8a^3}$. [Replacing $-a$ by a in (3)]

Example 13: *Evaluate* $\int_0^\infty \frac{dx}{x^4+a^4}$, $a > 0$.

(Meerut 2002; Purvanchal 09; Gorakhpur 14, 16)

Solution: Consider the integral $\int_C f(z)\, dz = \int_C \frac{dz}{z^4+a^4}$,

where C is the contour as described in Ex. 11.

By Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_\Gamma \frac{dz}{z^4 + a^4} = 2\pi i \,\Sigma\, R^+.$$

Since $\lim_{R \to \infty} z\, f(z) = 0$ therefore $\lim_{R \to \infty} \int_\Gamma \frac{dz}{z^4 + a^4} = 0$

so that $$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^4 + a^4} = 2\pi i \,\Sigma\, R^+$$

or $$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i \,\Sigma\, R^+. \qquad \ldots(1)$$

Poles of $f(z)$ are given by $z^4 + a^4 = 0$

or $$z = a\, e^{(2n+1)\pi i/4}, \text{ where } n = 0, 1, 2, 3.$$

$\therefore$ $z = a\, e^{i\pi/4}, a\, e^{3i\pi/4}, a\, e^{5i\pi/4}, a\, e^{7i\pi/4}$ are the simple poles of $f(z)$.

Out of these only $z = a\, e^{i\pi/4}, a\, e^{i3\pi/4}$ lie inside C.

If α denotes any of these poles then residue at $z = \alpha$ is

$$\lim_{z \to \alpha} (z - \alpha) f(z) = \lim_{z \to \alpha} \frac{z - \alpha}{z^4 + a^4} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to \alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4} = -\frac{\alpha}{4a^4}. \qquad [\because \alpha^4 = -a^4]$$

Now sum of the residues at poles inside C

$$= -\frac{1}{4a^4}\{a\, e^{i\pi/4} + a\, e^{i3\pi/4}\} = -\frac{1}{4a^3} \cdot \frac{2i}{\sqrt{2}} = -\frac{i\sqrt{2}}{4a^3}.$$

Hence from (1), we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i\left(-\frac{i\sqrt{2}}{4a^3}\right) = \frac{\pi\sqrt{2}}{2a^3}$$

or $$\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}.$$

Example 14: *Prove by contour integration that*

$$\int_0^{\infty} \frac{dx}{(a + bx^2)^n} = \frac{1}{2^n b^{1/2}} \cdot \frac{1.3.5 \ldots\ldots (2n-3)}{1.2.3 \ldots\ldots (2n-1)} \cdot \frac{1}{a^{(n-1)/2}}.$$

(Rohilkhand 2011)

Solution: Consider the integral $\int_C \frac{dz}{(a + bz^2)^n} = \int_C f(z)\,dz$, where C is the same contour as described in Ex. 11.

By Cauchy's residue theorem , we have

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{dx}{(a + bx^2)^n} + \int_\Gamma \frac{dz}{(a + bz^2)^n} = 2\pi i \,\Sigma\, R^+.$$

Since $\lim_{R \to \infty} z\, f(z) = 0$ therefore $\lim_{R \to \infty} \int_\Gamma \frac{dz}{(a + bz^2)^n} = 0$

so we have

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(a + bx^2)^n} = 2\pi i \,\Sigma\, R^+ \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{dx}{(a + bx^2)^n} = 2\pi i \,\Sigma\, R^+. \qquad \ldots(1)$$

$z = \pm i\sqrt{(a/b)}$ are the poles of $f(z)$ of order n. The pole $z = \sqrt{(a/b)}\, i$ lies inside C.

Residue at $z = i\sqrt{(a/b)}$ is

$$\frac{1}{(n-1)!}\left[D^{n-1} \frac{1}{b^n \left\{ z + i\sqrt{\left(\frac{a}{b}\right)} \right\}^n} \right]_{z = \sqrt{(a/b)}\, i}$$

$$= \frac{1}{(n-1)!} \cdot \frac{1}{b^n} \left[\frac{(-n)(-n-1)\ldots\ldots\{-n-(n-1)+1\}}{\left\{ z + i\sqrt{\left(\frac{a}{b}\right)} \right\}^{2n-1}} \right]_{z = i\sqrt{(a/b)}}$$

$$= \frac{(-1)^{n-1}}{(n-1)!\, b^n} \cdot \frac{n(n+1)\ldots\ldots(2n-2)}{\{2i\sqrt{(a/b)}\}^{2n-1}}$$

$$= \frac{(-1)^{n-1}\, 1.2.3 \ldots\ldots (n-1).\, n(n+1) \ldots\ldots (2n-2)}{2^{2n-1} \{\sqrt{(a/b)}\}^{2n-1}\, i^{2n-1}\, b^n\, (n-1)!.(n-1)!}$$

$$= -\frac{1.3.5\ldots\ldots(2n-3).\, 2^{n-1}\, 1.2.3\ldots\ldots(n-1)}{2^{2n-1}\, a^{n-1/2}\, b^{1/2}\, (n-1)!.(n-1)!}\, i$$

$$= -\frac{1.3.5\ldots\ldots(2n-3)\, i}{2^n\, a^{n-1/2}\, b^{1/2}\, 1.2.3\ldots\ldots(n-1)}.$$

Hence from (1), we have

$$\int_{-\infty}^{\infty} \frac{dx}{(a + bx^2)^n} = 2\pi i \left\{ -\frac{1.3.5\ldots\ldots(2n-3)\, i}{2^n\, a^{n-1/2}\, b^{1/2}\, 1.2.3\ldots\ldots(n-1)} \right\}$$

or
$$\int_0^{\infty} \frac{dx}{(a + bx^2)^n} = \frac{\pi}{2^n\, a^{n-1/2}\, b^{1/2}} \cdot \frac{1.3.5\ldots\ldots(2n-3)}{1.2.3\ldots\ldots(n-1)}.$$

Comprehensive Exercise 3

1. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$.

2. Evaluate $\int_0^{\infty} \frac{x^2}{(1+x^2)^3}\, dx$.

3. Evaluate $\int_0^\infty \frac{dx}{1+x^2}$.

(Kanpur 2007; Rohilkhand 10; Kumaun 10, 11; Gorakhpur 05, 11, 13)

4. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^3(b+c)^3}, b>0, c>0.$

5. Use the method of contour integration to prove that

$$\int_0^\infty \frac{x^6}{(x^4+a^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}, a>0.$$

Answers 3

2. $\frac{\pi}{16}$

3. $\frac{\pi}{2}$

9 Jordan's Inequality. $\frac{2\theta}{\pi} \le \sin\theta \le \theta$, where $0 \le \theta \le \frac{\pi}{2}$.

Let $y = \cos x$. We know that as θ increases from 0 to $\pi/2$, $\cos\theta$ decreases from 1 to 0. Consequently the mean ordinate of the graph $y = \cos x$ also decreases steadily over the range $0 \le x \le \theta$.

The mean ordinate is $\frac{1}{\theta}\int_0^\theta \cos\theta\, d\theta = \frac{\sin\theta}{\theta}$.

Hence when $0 \le \theta \le \frac{\pi}{2}$, $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ or $\frac{2\theta}{\pi} \le \sin\theta \le \theta$.

This is known as Jordan's inequality.

10 Jordan's Lemma

If $f(z)$ tends to zero uniformly as $z \to \infty$ and $f(z)$ is meromorphic in the upper half plane then

$$\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)\,dz = 0, \ (m>0)$$

where Γ denotes the semi-circle $|z| = R$, $I(z) > 0$.

Proof: Here we assume that R is large enough so as to include within it all the singularities of $f(z)$ and $f(z)$ has no singularity on Γ.

Since $\lim_{R\to\infty} f(z) = 0$ therefore for $\varepsilon > 0$ there exists $R_0 > 0$ such that $|f(z)| < \varepsilon$ when $|z| = R \le R_0$.

Now let Γ denote any semi-circle with radius $R \geq R_0$. Putting $z = Re^{i\theta}$, we get

$$\int_\Gamma e^{imz} f(z)\, dz = \int_0^\pi e^{imRe^{i\theta}} f(Re^{i\theta})\, R i e^{i\theta}\, d\theta$$

$$= \int_0^\pi e^{imR\cos\theta} e^{-mR\sin\theta} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta.$$

$$\left|\int_\Gamma e^{imz} f(z)\, dz\right| \leq \int_0^\pi |e^{imR\cos\theta}|\, |e^{-mR\sin\theta}|\, |f(Re^{i\theta})|\, |Rie^{i\theta}|\, d\theta$$

$$< \int_0^\pi e^{-mR\sin\theta}\, \varepsilon R\, d\theta \quad [\because |f(z)| = |f(Re^{i\theta})| < \varepsilon]$$

$$= 2\varepsilon R \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

$$\leq 2\varepsilon R \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta, \text{ by Jordan's inequality}$$

$$= \frac{2\varepsilon R(1 - e^{-mR})}{2mR/\pi} = \frac{\varepsilon\pi}{m}(1 - e^{-mR}) < \frac{\varepsilon\pi}{m}.$$

Hence $$\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)\, dz = 0.$$

11 Evaluation of the Integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx\, dx\,;\ \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx\, dx,\ m > 0, \text{ where}$$

(i) $P(x)$, $Q(x)$ are polynomials, (ii) $\deg Q(x) > \deg P(x)$

(iii) $Q(x) = 0$ has no real roots.

Under the above mentioned conditions the given integrals are convergent. Consider

$$\int_C e^{imz} f(z)\, dz = \int_C e^{imz} \frac{P(z)}{Q(z)}\, dz,$$

where C is the contour consisting of a semi-circle Γ of radius R so large as to include all the poles of the integrand in the upper half plane and part of the real axis from $-R$ to R.

By Cauchy's residue theorem, we have

$$\int_C e^{imz} f(z)\, dz = \int_{-R}^{R} e^{imx} f(x)\, dx + \int_\Gamma e^{imz} f(z)\, dz = 2\pi i \Sigma R^+.$$

We have $$\lim_{R\to\infty} \int_\Gamma e^{iz} f(z)\, dz = 0, \text{ by Jordan's lemma.}$$

$$\therefore \quad \lim_{R\to\infty} \int_{-R}^{R} e^{ix} f(x)\, dx = 2\pi i\, \Sigma R^+$$

or $$\int_{-\infty}^{\infty} f(x) \cos mx\, dx + i \int_{-\infty}^{\infty} f(x) \sin mx\, dx = 2\pi i\, \Sigma R^+.$$

Equating real and imaginary parts on both sides, we shall get the values of the given integrals.

Illustrative Examples

Example 15: *Prove that* $\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{2a} e^{-ma}, m \ge 0.$

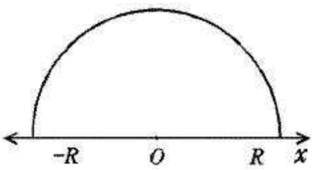
(Kumaun 2012, 13)

Deduce that $\int_0^\infty \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}.$

(Gorakhpur 2007, 09, 11, 15)

Solution: Consider the integral

$$\int_C \frac{e^{imz}}{a^2+z^2}\,dz = \int_C f(z)\,dz,$$



where C is the contour consisting of a large semi-circle Γ of radius R containing all the poles of the integrand in the upper half plane and the part of real axis from $-R$ to R.

By Cauchy's residue theorem , we have

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{imx}}{a^2+x^2}\,dx + \int_\Gamma \frac{e^{imz}}{a^2+z^2}\,dz = 2\pi i\,\Sigma R^+.$$

Since $\lim_{z\to\infty} \frac{1}{z^2+a^2} = 0$ therefore we have

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0, \text{ by Jordan's lemma.}$$

$$\therefore \quad \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{imx}}{a^2+x^2}\,dx = 2\pi i\,\Sigma R^+$$

or
$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2}\,dx = 2\pi i\,\Sigma R^+ \qquad \dots(1)$$

$z = \pm ai$ are the simple poles of $f(z)$. The pole $z = ai$ lies inside C.

Residue at $z = ai$ is

$$\lim_{z\to ai} (z-ai)\,f(z) = \lim_{z\to ai} \frac{e^{imz}}{z^2+a^2}(z-ai) = \frac{e^{-ma}}{2ia}.$$

From (1), we have

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = \frac{2\pi i\,e^{-ma}}{2\,i\,a} = \frac{\pi}{a} e^{-ma}.$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-ma} \quad \text{or} \quad \int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-ma}.$$

Differentiating both sides w.r.t. m, we get

$$\int_0^\infty -\frac{x \sin mx}{a^2+x^2}\,dx = \frac{\pi}{2a}(-a)\,e^{-ma} \quad \text{or} \quad \int_0^\infty \frac{x \sin mx}{a^2+x^2}\,dx = \frac{\pi}{2} e^{-ma}.$$

Example 16: *Evaluate* $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx,\ a>0, b>0.$

Solution: Consider $\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = \int_C f(z)\,dz,$

where C is the same contour as described in Ex. 15.

By Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}\,dx + \int_\Gamma \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = 2\pi i\,\Sigma R^+.$$

By Jordan's lemma, we have

$$\lim_{R\to\infty} \int_\Gamma \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = 0.$$

$$\therefore \qquad \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = 2\pi i \Sigma R^+$$

or
$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = 2\pi i\,\Sigma R^+. \qquad \ldots(1)$$

$z = \pm ib$ are the double poles of $f(z)$. Out of these only $z = ib$ lies in the upper half plane.

Residue at $z = ib$ is

$$\phi'(ib) = \left[D\,\frac{e^{iaz}}{(z+ib)^2}\right]_{z=ib}$$

$$= \left[\frac{iae^{iaz}(z+ib)^2 - 2e^{iaz}(z+ib)}{(z+ib)^4}\right]_{z=ib}$$

$$= \frac{e^{-ab}\,[ia(2ib)-2]}{(2ib)^3} = \frac{e^{-ab}}{4b^3 i}(ab+1).$$

Hence from (1), we have

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = 2\pi i\cdot\frac{e^{-ab}}{4ib^3}(ab+1)$$

or
$$\int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{(x^2+b^2)^2}\,dx = \frac{\pi e^{-ab}}{2b^3}(ab+1).$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi(ab+1)}{2b^3 e^{ab}}$$

or
$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi(1+ab)}{4b^3 e^{ab}}.$$

Example 17: *Prove that*

$$\int_{-\infty}^{\infty} \frac{\sin x}{(1-x+x^2)^2}\, dx = \frac{2\pi(\sqrt{3}+2)}{3\sqrt{3}}\, e^{-\sqrt{3}/2} \sin\frac{1}{2}.$$

Solution: Let $\int_C f(z)\, dz = \int_C \frac{e^{iz}}{(1-z+z^2)^2}\, dz$, where C is the same contour as in Ex. 15.

By residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i\, \Sigma R^+.$$

By Jordan's lemma, we have

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0.$$

$$\therefore \qquad \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = 2\pi i\, \Sigma\, R^+$$

or $$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i\, \Sigma R^+. \qquad \ldots(1)$$

$z=(1\pm i\sqrt{3})/2$ are the poles of $f(z)$ of second order. The only pole which lies within C is $(1+i\sqrt{3})/2=\alpha$, say.

Putting $z=\alpha+t$ in $f(z)$, we get

$$f(\alpha+t) = \frac{e^{i(\alpha+t)}}{\{t^2+(2\alpha-1)\,t+\alpha^2-\alpha+1\}^2}$$

$$= \frac{e^{i(\alpha+t)}}{\{t^2+(2\alpha-1)\,t\}^2}, \text{ since } \alpha^2-\alpha+1=0$$

$$= \frac{e^{i(\alpha+t)}}{(2\alpha-1)^2\, t^2}\left[1+\frac{t}{2\alpha-1}\right]^{-2}$$

$$= \frac{e^{i\alpha}}{(2\alpha-1)^2\, t^2}(1+it+\ldots\ldots)\left(1-\frac{2t}{2\alpha-1}+\ldots\ldots\right).$$

Residue at $z=\alpha$ is the coefficient of $(1/t)$ in the expansion of

$$f(\alpha+t) = e^{i\alpha}\left[\frac{i}{(2\alpha-1)^2}-\frac{2}{(2\alpha-1)^3}\right]$$

$$= e^{i\alpha}\left[\frac{i(2\alpha-1)-2}{(2\alpha-1)^3}\right]$$

$$= e^{i(1+i\sqrt{3})/2}\, \frac{[i(1+i\sqrt{3}-1)-2]}{(i\sqrt{3})^3}$$

$$= \frac{e^{-\sqrt{3}/2}\, e^{i/2}\, (\sqrt{3}+2)}{i 3\sqrt{3}}.$$

Hence from (1), we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 - x + 1)^2}\, dx = \frac{2\pi e^{-\sqrt{3}/2}}{3\sqrt{3}} (\sqrt{3}+2)\, e^{i/2}.$$

Equating imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2 - x + 1)^2}\, dx = \frac{2\pi\, e^{-\sqrt{3}/2}}{3\sqrt{3}} (\sqrt{3}+2) \sin\frac{1}{2}.$$

Example 18: *Prove that* $\int_0^{\infty} \frac{\cos x^2 + \sin x^2 - 1}{x^2}\, dx = 0.$

Solution: Let $\int_C f(z)\, dz = \int_C \frac{e^{iz^2} - 1}{z^2}\, dz$, where C is the same contour as in Example 15.

Since $f(z)$ has no poles in the upper half plane therefore by Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 0. \qquad \ldots(1)$$

We have $\int_{\Gamma} f(z)\, dz = \int_0^{\pi} \frac{\exp(i R^2 e^{i2\theta}) - 1}{R^2 e^{i2\theta}} R i e^{i\theta}\, d\theta,$ putting $z = R e^{i\theta}$

$$= \frac{i}{R} \int_0^{\pi} e^{-i\theta} [\exp\{i R^2 (\cos 2\theta + i \sin 2\theta)\} - 1]\, d\theta$$

$$= \int_0^{\pi} \frac{i}{R} e^{-i\theta} [\exp(-R^2 \sin 2\theta) \exp(i R^2 \cos 2\theta) - 1]\, d\theta.$$

Now $\left| \int_{\Gamma} f(z)\, dz \right|$

$$\leq \frac{1}{R} \int_0^{\infty} | i e^{-i\theta} | [\exp(-R^2 \sin 2\theta) | \exp(iR^2 \cos 2\theta) | + | -1 |]\, d\theta$$

$$\leq \frac{1}{R} \int_0^{\infty} [\exp(-R^2 \sin 2\theta) + 1]\, d\theta,$$ which tends to zero as $R \to \infty$.

$\therefore$ $\int_{\Gamma} f(z)\, dz = 0$ when $R \to \infty$.

Thus when $R \to \infty$, we have from (1)

$$\int_{-\infty}^{\infty} f(x)\, dx = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{ix^2} - 1}{x^2}\, dx = 0$$

or $$2 \int_0^{\infty} \frac{e^{ix^2} - 1}{x^2}\, dx = 0.$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty \frac{\cos x^2 - 1}{x^2}\, dx = 0 \quad \text{and} \quad \int_0^\infty \frac{\sin x^2}{x^2}\, dx = 0.$$

Adding these relations, we get

$$\int_0^\infty \frac{\cos x^2 + \sin x^2 - 1}{x^2}\, dx = 0.$$

Example 19: *Prove by contour integration* $\int_0^\infty \frac{\log (1 + x^2)}{1 + x^2}\, dx = \pi \log 2.$

(Kanpur 2008; Gorakhpur 09, 13)

Solution: Consider

$$\int_C \frac{\log (i + z)}{1 + z^2}\, dz = \int_C f(z)\, dz, \text{ where } C \text{ is the contour of Ex. 15.}$$

By residue theorem, we have

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i\, \Sigma R^+. \qquad \dots(1)$$

Now
$$\lim_{z \to \infty} z\, f(z) = \lim_{z \to \infty} \frac{z \log (i + z)}{(i + z)(z - i)}$$

$$= \lim_{z \to \infty} \frac{z}{z - i} \cdot \lim_{z \to \infty} \frac{\log (i + z)}{i + z} = 0.$$

$\therefore$ when $R \to \infty$, we have from (1)

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i\, \Sigma R^+ \qquad \dots(2)$$

$z = \pm i$ are the simple poles of $f(z)$ and only $z = i$ lies inside C.

Residue at $z = i$ is

$$\lim_{z \to i} (z - i)\, f(z) = \lim_{z \to i} \frac{\log (i + z)}{(z + i)} = \frac{\log 2i}{2i} = \frac{\log 2 + i(\pi / 2)}{2i}.$$

Hence from (2), we have

$$\int_{-\infty}^{\infty} \frac{\log (i + x)}{1 + x^2}\, dx = \pi \{\log 2 + i(\pi / 2)\}.$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2} \log (x^2 + 1)}{x^2 + 1}\, dx = \pi \log 2 \quad \text{or} \quad \int_0^\infty \frac{\log (1 + x^2)}{1 + x^2}\, dx = \pi \log 2.$$

Comprehensive Exercise 4

1. Prove that $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5}\, dx = -\frac{\pi}{e} \sin 2.$

2. (i) Apply the calculus of residues to evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx, \ a > 0.$

(Avadh 2008; Gorakhpur 2011)

(ii) Prove that $\int_0^{\infty} \frac{\cos x}{a^2 + x^2} dx = \frac{\pi e^{-a}}{a}, a > 0.$

3. Show that $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 2x + 5} dx = \frac{\pi}{2e^2} \sin 1.$

4. Prove that $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right), \ a > 0, b > 0.$

5. Prove that when $m > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4 + x^2 + 1} dx = \frac{2\pi}{\sqrt{3}} \sin \left(\frac{m}{2} + \frac{\pi}{6} \right) e^{-(1/2) m \sqrt{3}}.$$

6. Prove that $\int_0^{\infty} \frac{\cos mx}{x^4 + a^4} dx = \frac{\pi}{2a^3} e^{-ma/\sqrt{2}} \sin \left(\frac{ma}{\sqrt{2}} + \frac{\pi}{2} \right).$

(Gorakhpur 2008)

Deduce that $\int_0^{\infty} \frac{x \sin mx}{x^4 + a^4} dx = \frac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}.$

7. Prove by contour integration that $\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx = \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}.$

8. If $a \geq 4$, prove that

(i) $\int_0^{\infty} \frac{(1 + x^2) \cos ax}{1 + x^2 + x^4} dx = \frac{\pi}{\sqrt{3}} e^{-a(\sqrt{3}/2)} \cos \frac{a}{2}$

(ii) $\int_0^{\infty} \frac{x \sin ax}{1 + x^2 + x^4} dx = \frac{\pi}{\sqrt{3}} e^{-a(\sqrt{3}/2)} \sin \frac{a}{2}.$

Answers 4

2. (i) $\frac{\pi}{2} e^{-a}$

12 Poles Lie on the Real Axis

We shall now discuss the case when the integrand has poles on the real axis as well as within the semi-circle.

Theorem: *Let $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials such that $Q(z)$ has only non-repeated real roots, that is $f(z)$ has only simple poles on the real axis. Let $m > 0$ and let the degree of $Q(z)$ exceeds that of $P(z)$, then*

$$P \int_{-\infty}^{\infty} e^{imx} f(x) \, dx = 2\pi i \sum_{k=1}^{p} Res(a_k) + \pi i \sum_{k=1}^{q} Res(b_k),$$

where $a_1, a_2 \ldots, a_p$ are the zeros of $Q(z)$ in the region $Im \ z > 0$ and $b_1, b_2, \ldots, b_q$ are its zeros in the real axis, and $Res(\alpha)$ denotes the residue of $e^{imz} f(z)$ at α.

Illustrative Examples

Example 20: *If $m > 0$, show that* $P\int_{-\infty}^{\infty} \frac{\cos mx}{x-b}\,dx = -\sin mb$.

Solution: Referring to the above theorem, let $f(z) = \frac{1}{z-b}\cdot$ Here $f(z)$ has simple real pole at $z = b$.

$$\therefore \qquad \text{Res}_{z=b} \frac{e^{imz}}{z-b} = \lim_{z \to b} (z-b)\cdot\frac{e^{imz}}{(z-b)} = e^{imz}.$$

Hence
$$P\int_{-\infty}^{\infty} \frac{e^{imz}}{z-b}\,dz = \pi i e^{imb}.$$

Equating real parts on both sides, we get

$$P\int_{-\infty}^{\infty} \frac{\cos mx}{x-b}\,dx = -\pi \sin mb.$$

Example 21: *If $m > 0$, prove that* $P\int_{0}^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}\cdot$

Solution: Referring to the above theorem, let $f(z) = \frac{1}{z}\cdot$ Here $f(z)$ has simple real pole at $z = 0$.

$$\therefore \qquad \text{Res}_{z=0} \frac{e^{imz}}{z} = \lim_{z \to 0} (z-0)\frac{e^{imz}}{z} = 1.$$

Thus
$$P\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\,dx = \pi i \times \text{Residue of } \frac{e^{imz}}{z} \text{ at } z = 0$$
$$= \pi i \times 1 = \pi i.$$

Equating real and imaginary parts on both sides, we get

$$P\int_{-\infty}^{\infty} \frac{\cos mx}{x}\,dx = 0 \qquad \ldots(1)$$

and
$$P\int_{-\infty}^{\infty} \frac{\sin mx}{x}\,dx = \pi. \qquad \ldots(2)$$

Note: The principal part 'P' has been dropped in (2), since $\lim_{x \to 0} \frac{\sin mx}{x} = m$, whereas in the first integral the integrand becomes unbounded at the origin. Hence from (2), we get the required result.

Indenting Method: We can avoid the poles which lie on the real axis by drawing semi-circles of small radii about these poles as centres. This method is known as 'indenting at a point'.

Example 22: *(i) Evaluate* $\int_0^{\infty} \frac{\sin mx}{x}\, dx, m > 0.$ **(Gorakhpur 2007, 10, 13; Kumaun 07)**

(ii) By integrating e^{iz} / z *around a suitable contour, prove that*

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}\cdot$$

(Rohilkhand 2011; Kumaun 14)

Solution: (*i*) Consider the integral $\int_C f(z)\, dz = \int_C \frac{e^{imz}}{z}\, dz$

where C is the contour consisting of (1) the upper half of the circle $|z| = R$

(2) real axis from r to R where r is small and R is large

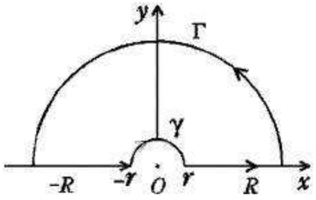
(3) real axis from $-R$ to $-r$

(4) upper half of the circle $\gamma, |z| = r$.

Obviously the function $f(z)$ has no singularity inside C, therefore by Cauchy's residue theorem, we have

$$\int_C f(z)\, dz = \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz + \int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)\, dz$$
$$= 0. \qquad \ldots(1)$$

By Jordan's lemma, we have



$$\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0.$$

Again $$\lim_{z \to 0} z\, f(z) = 1 \text{ therefore}$$

$$\lim_{r \to 0} \int_\gamma f(z)\, dz = i(0 - \pi) = -i\pi.$$

Thus when $r \to 0, R \to \infty$, we get from (1)

$$\int_0^{\infty} f(x)\, dx + \int_{-\infty}^0 f(x)\, dx - i\pi = 0$$

or $$\int_{-\infty}^{\infty} f(x)\, dx = i\pi \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{imx}}{x}\, dx = i\pi.$$

Equating imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x}\, dx = \pi \qquad \text{or} \qquad \int_0^{\infty} \frac{\sin mx}{x}\, dx = \frac{\pi}{2}\cdot$$

(ii) Proceed as in part (i) taking $m = 1$.

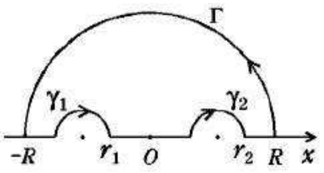
Example 23: *Prove that if* $a > 0$

(i) $P \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \frac{\pi \sin a}{a}$ *(ii)* $P \int_{-\infty}^{\infty} \frac{\sin x}{a^2 - x^2}\, dx = 0.$

(Gorakhpur 2008, 09)

Solution: Consider the integral

$$\int_C f(z)\,dz = \int_C \frac{e^{iz}}{a^2 - z^2}\,dz,$$



where C is the contour consisting of the large semi-circle Γ of radius R, indented at $z = a$ and $z = -a$. γ_1 and γ_2 are the small semi-circles with radii r_1 and r_2 and centres at $z = -a$ and $z = a$ respectively.

The function $f(z)$ is regular within and on the contour C therefore by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{-(a+r_1)} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-(a-r_1)}^{a-r_2} f(x)\,dx + \int_{\gamma_2} f(z)\,dz + \int_{a+r_2}^{R} f(x)\,dx = 0. \quad \ldots(1)$$

Since $1/(a^2 - z^2) \to 0$ as $z \to \infty$, therefore by Jordan's lemma, we have

$$\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0.$$

Also $$\lim_{z \to -a} (a+z)\,f(z) = \lim_{z \to -a} \frac{e^{iz}}{a - z} = \frac{e^{-ia}}{2a},$$

therefore $$\lim_{r_1 \to 0} \int_{\gamma_1} f(z)\,dz = i\,\frac{e^{-ia}}{2a}(0 - \pi) = -i\,\frac{\pi}{2a}\,e^{-ia}.$$

Similarly $$\lim_{r_2 \to 0} \int_{\gamma_2} f(z)\,dz = i\,\frac{\pi}{2a}\,e^{ia}.$$

Hence when $R \to \infty, r_1 \to 0, r_2 \to 0$, we have from (1)

$$\int_{-\infty}^{-a} f(x)\,dx + \left(-i\,\frac{\pi}{2a}\,e^{-ia}\right) + \int_{-a}^{a} f(x)\,dx + \frac{i\pi}{2a}\,e^{ia} + \int_a^\infty f(x)\,dx = 0$$

or $$P\int_{-\infty}^{\infty} f(x)\,dx = -i\,\frac{\pi}{2a}\,(e^{ia} - e^{-ia})$$

or $$P\int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2}\,dx = \frac{\pi}{a}\sin a.$$

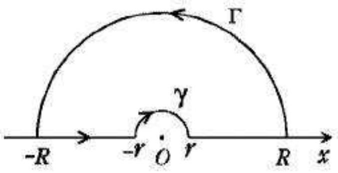
Equating real and imaginary parts on both sides, we get

$$P\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi}{a}\sin a, \quad P\int_{-\infty}^{\infty} \frac{\sin x}{a^2 - x^2}\,dx = 0.$$

Example 24: *Prove that* $\displaystyle\int_0^\infty \frac{\sin x}{x(x^2 + a^2)}\,dx = \frac{\pi}{2a^2}(1 - e^{-a}), a > 0.$

Solution: Consider the integral $\displaystyle\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z(z^2 + a^2)}\,dz,$

where C is the contour consisting of



(i) a large semi-circle $\Gamma, |z| = R$ in the upper half plane

(ii) real axis from r to R

(iii) real axis from $-R$ to $-r$

(iv) a small semi-circle $\gamma, |z| = r$.

$z = 0, \pm ia$ are the simple poles of $f(z)$. Out of these only $z = ia$ lies within C.

Residue at $z = ai$ is

$$\lim_{z \to ai} (z - ai) f(z) = \lim_{z \to ai} \frac{e^{iz}}{z(z + ai)} = \frac{e^{-a}}{-2a^2}.$$

By residue theorem, we have

$$\int_C f(z)\,dz = \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_\gamma f(z)\,dz$$

$$= 2\pi i\, \Sigma R^+. \qquad \ldots(1)$$

By Jordan's lemma, we have $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0.$

Also $$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^{iz}}{z^2 + a^2} = \frac{1}{a^2},$$

$\therefore$ $$\lim_{r \to 0} \int_\gamma f(z)\,dz = i \cdot \frac{1}{a^2}(0 - \pi) = -\frac{\pi i}{a^2}.$$

$\therefore$ when $r \to 0$ and $R \to \infty$, we have from (1)

$$\int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx - \frac{i\pi}{a^2} = 2\pi i\left(-\frac{e^{-a}}{2a^2}\right)$$

or $$\int_{-\infty}^\infty f(x)\,dx = \frac{i\pi}{a^2}(1 - e^{-a}).$$

Equating imaginary parts on both sides, we get

$$\int_{-\infty}^\infty \frac{\sin x}{x(x^2 + a^2)}\,dx = \frac{\pi}{a^2}(1 - e^{-a}).$$

Example 25: *Show that if a and m are positive,*

$$\int_0^\infty \frac{\sin^2 mx}{x^2 (a^2 + x^2)^2}\,dx = \frac{\pi}{8a^5}\{e^{-2am}(2am + 3) + 4am - 3\}.$$

Solution: We have $$\int_0^\infty \frac{\sin^2 mx}{x^2 (a^2 + x^2)^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1 - \cos 2mx}{x^2 (a^2 + x^2)^2}\,dx.$$

Consider $$\int_C f(z)\,dz = \frac{1}{2}\int_C \frac{1 - e^{i2mz}}{z^2 (a^2 + z^2)^2}\,dz,$$

where C is the same contour as in Ex. 24.

$z = 0, \pm ia$ are the poles of $f(z)$ of order two. Only $z = ai$ lies within C.

Putting $z = ai + t$ in $f(z)$, we get

$$f(ai+t) = \frac{1}{2}\frac{\{1-e^{i2m(ai+t)}\}}{(ai+t)^2\{a^2+(ai+t)^2\}^2} = \frac{1}{2}\frac{(1-e^{-2am}e^{i2mt})}{(ai+t)^2(2ait+t^2)^2}$$

$$= \frac{1}{8a^4t^2}(1-e^{-2am}e^{i2mt})\left(1+\frac{t}{ai}\right)^{-2}\left(1+\frac{t}{2ai}\right)^{-2}$$

$$= \frac{1}{8a^4t^2}\{1-e^{-2am}(1+i2mt+\ldots\ldots)\}\left(1-\frac{2t}{ai}+\ldots\ldots\right)\left(1-\frac{t}{ai}+\ldots\ldots\right)$$

$$= \frac{1}{8a^4t^2}(1-e^{-2am}-i2me^{-2am}t)\left(1-\frac{2t}{ai}\right)\left(1-\frac{t}{ai}\right)$$

neglecting higher powers of t since t is small

$$= \frac{1}{8a^4t^2}\{(1-e^{-2am})-i2me^{-2am}t\}\left(1-\frac{3}{ai}t\right).$$

$\therefore$ Residue at $z\ (=ai)$ is = coefficient of $(1/t)$ in $f(ai+t)$

$$= \frac{1}{8a^4}\left\{-\frac{3}{ai}(1-e^{-2am})-2im\,e^{-2am}\right\}$$

$$= \frac{1}{8a^5}\{3i(1-e^{-2am})-2iame^{-2am}\}.$$

By residue theorem, we have

$$\int_C f(z)\,dz = \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_{-R}^{-r} f(x)\,dx + \int_\gamma f(z)\,dz$$

$$= 2\pi i\,\Sigma R^+. \qquad \ldots(1)$$

By Jordan's lemma, $\int_\Gamma f(z)\,dz = 0$, when $R\to\infty$.

Since $\lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{1}{2}\frac{(1-e^{i2mz})}{z(a^2+z^2)^2}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{z\to 0}\frac{-i2me^{i2mz}}{2\{(a^2+z^2)^2+z\,2(a^2+z^2).2z\}} = \frac{-im}{a^4}.$$

$$\therefore \quad \lim_{r\to 0}\int_\gamma f(z)\,dz = i\left(-\frac{im}{a^4}\right)(0-\pi) = \frac{-m\pi}{a^4}.$$

Hence when $r\to 0, R\to\infty$, we have from (1)

$$\int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx - \frac{m\pi}{a^4} = 2\pi i\cdot\frac{i}{8a^5}\{3(1-e^{-2am})-2ame^{-2am}\}$$

or
$$\int_{-\infty}^\infty f(x)\,dx = \frac{\pi}{4a^5}[4am-3+e^{-2am}(3+2am)].$$

Equating real parts on both sides, we get

$$\frac{1}{2}\int_{-\infty}^\infty \frac{1-\cos 2mx}{x^2(a^2+x^2)^2}\,dx = \frac{\pi}{4a^5}[4am-3+e^{-2am}(3+2am)]$$

or $$\int_{-\infty}^{\infty} \frac{\sin^2 mx}{x^2 (a^2 + x^2)^2} dx = \frac{\pi}{4a^5} [4am - 3 + e^{-2am} (3 + 2am)]$$

or $$\int_{0}^{\infty} \frac{\sin^2 mx}{x^2 (a^2 + x^2)^2} dx = \frac{\pi}{8a^5} [4am - 3 + e^{-2am} (3 + 2am)].$$

Example 26: *Evaluate* $\int_0^\infty \frac{x - \sin x}{x^3 (a^2 + x^2)} dx, a > 0.$

Solution: Consider $\int_C f(z)\, dz = \int_C \frac{z - i + ie^{iz}}{z^3 (a^2 + z^2)} dz$, where C is the same contour as in Ex. 24.

$z = 0$ is the pole of $f(z)$ of order two and $z = \pm ia$ are the simple poles of $f(z)$. Out of these only $z = ai$ lies within C.

Residue at $z = ai$ is

$$\lim_{z \to ai} (z - ai) f(z) = \lim_{z \to ai} \frac{z - i + ie^{iz}}{z^3 (ai + z)} = \frac{(a - 1 + e^{-a}) i}{2a^4}.$$

By residue theorem , we have

$$\int_C f(z)\, dz = \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz + \int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)\, dz$$

$$= 2\pi i\, \Sigma R^+. \qquad \ldots(1)$$

By Jordan's lemma, when $R \to \infty$, $\int_\Gamma f(z)\, dz \to 0$.

Since $$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z - i + ie^{iz}}{z^2 (a^2 + z^2)} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to 0} \frac{z - i + i\left(1 + iz + \frac{1}{2} i^2 z^2 + \ldots\ldots\right)}{z^2 (a^2 + z^2)}$$

$$= \lim_{z \to 0} \frac{-\frac{1}{2} iz^2 + \ldots\ldots}{z^2 (a^2 + z^2)}$$

$$= \lim_{z \to 0} \frac{-\frac{1}{2} i + \text{terms containing } z \text{ in Nr.}}{a^2 + z^2} = -\frac{i}{2a^2}.$$

$\therefore$ $$\lim_{r \to 0} \int_\gamma f(z)\, dz = i\left(-\frac{i}{2a^2}\right)(0 - \pi) = -\frac{\pi}{2a^2}.$$

Hence when $r \to 0, R \to \infty$, we have

$$\int_0^\infty f(x)\, dx + \int_{-\infty}^0 f(x)\, dx - \frac{\pi}{2a^2} = 2\pi i \frac{(a - 1 + e^{-a}) i}{2a^4}$$

or $$\int_{-\infty}^\infty f(x)\, dx = \frac{\pi}{2a^2} - \frac{\pi}{a^4} (a - 1 + e^{-a}).$$

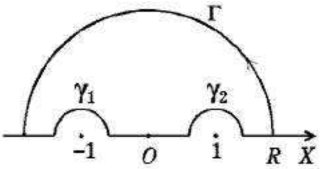
Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3 (a^2 + x^2)} dx = \frac{\pi}{a^4} \left(\frac{1}{2} a^2 - a + 1 - e^{-a}\right)$$

or
$$\int_{0}^{\infty} \frac{x - \sin x}{x^3 (a^2 + x^2)} dx = \frac{\pi}{2a^4} \left(\frac{1}{2} a^2 - a + 1 - e^{-a}\right).$$

Example 27: *Prove that* $P \int_0^{\infty} \frac{x^4}{x^6 - 1} dx = \frac{\pi \sqrt{3}}{6}$.

Solution: Consider $\int_C f(z)\, dz = \int_C \frac{z^4}{z^6 - 1} dz$, where C is the contour consisting of semi-circle Γ of radius R in the upper half plane indented at $z = -1, 1,$ r_1 and r_2 are the radii of the small semi-circles γ_1 and γ_2 with centres at $z = -1$ and $z = 1$ respectively.



$$z = e^{2n\pi i/6}, n = 0, 1, 2, 3, 4, 5$$

are the simple poles of $f(z)$ of which only $z = e^{i\pi/3}, e^{i2\pi/3}$ lie within C. Let α denote any of these poles.

Residue at $z = \alpha$ is $\lim_{z \to \alpha} \frac{z^4}{D(z^6 - 1)} = \lim_{z \to \alpha} \frac{z^4}{6z^5} = \frac{1}{6\alpha}$.

Sum of the residues at poles inside C is

$$= \frac{1}{6} (e^{-i\pi/3} + e^{-i2\pi/3}) = \frac{1}{6} (e^{-i\pi/3} - e^{i\pi/3})$$

$$= -\frac{2i}{6} \sin \frac{\pi}{3} = -\frac{\sqrt{3}\, i}{6}.$$

By residue theorem, we have

$$\int_C f(z)\, dz = \int_{\Gamma} f(z)\, dz + \int_{-R}^{-(1+r_1)} f(x)\, dx + \int_{\gamma_1} f(z)\, dz$$

$$+ \int_{-(1-r_1)}^{1-r_2} f(x)\, dx + \int_{\gamma_2} f(z)\, dz + \int_{1+r_2}^{R} f(x)\, dx$$

$$= 2\pi i \Sigma R^+. \qquad \ldots(1)$$

Now
$$\left| \int_{\Gamma} f(z)\, dz \right| \le \int_0^{\pi} \left| \frac{R^4 e^{i4\theta} R i e^{i\theta}}{R^6 e^{i6\theta} - 1} \right| d\theta \le \int_0^{\pi} \frac{R^5}{R^6 - 1} d\theta$$

$$= \frac{R^5 \pi}{R^6 - 1} \to 0 \text{ as } R \to \infty.$$

$\therefore$
$$\lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0.$$

Since
$$\lim_{z \to -1} (z + 1) f(z) = \lim_{z \to -1} \frac{(z + 1) z^4}{z^6 - 1} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to -1} \frac{z^4 + 4z^3(z+1)}{6z^5} = -\frac{1}{6},$$

$$\therefore \quad \lim_{r_1 \to 0} \int_{\gamma_1} f(z)\, dz = -\frac{1}{6} i(0 - \pi) = \frac{i\pi}{6}.$$

Since $\displaystyle\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{z^4}{z^6 - 1}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{z \to 1} \frac{z^4 + 4z^3(z-1)}{6z^5} = \frac{1}{6},$$

$$\therefore \quad \lim_{r_2 \to 0} \int_{\gamma_2} f(z)\, dz = \frac{-i\pi}{6}.$$

Hence as $r_1 \to 0, r_2 \to 0, R \to \infty$, we have from (1)

$$\int_{-\infty}^{-1} f(x)\, dx + \frac{i\pi}{6} + \int_{-1}^{1} f(x)\, dx + \left(-\frac{i\pi}{6}\right) + \int_{1}^{\infty} f(x)\, dx = 2\pi i \left(-\frac{\sqrt{3}}{6} i\right)$$

or $\displaystyle P \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi\sqrt{3}}{3}$ or $\displaystyle P \int_{0}^{\infty} f(x)\, dx = \frac{\pi\sqrt{3}}{6}.$

13 Integrals of Many Valued Functions

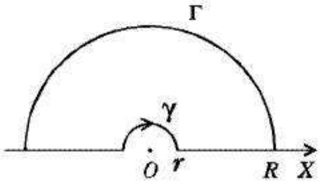
Such integrals involve functions of the type $\log z, z^a$ where a is not an integer.

These integrals are not single valued. To evaluate such integrals we consider only those contours whose interiors do not contain any branch point. For these integrals we generally use **double circle** contour indented at the centre.

Illustrative Examples

Example 28: *Prove that* $\displaystyle\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\, dx = -\frac{\pi}{4}$, *using as a contour a large semi-circle in the upper half plane indented at the origin.*

Solution: Consider $\displaystyle\int_C \frac{\log z}{(1+z^2)^2}\, dz = \int_C f(z)\, dz,$



where C is the contour as given in the question. (See figure).

$z = \pm i$ are the double poles of $f(z)$. Only $z = i$ lies inside C.

Residue at $z = i$ is $\phi'(i)$, where

$$\phi(z) = \frac{\log z}{(z+i)^2} \quad \text{or} \quad \log \phi(z) = \log \log z - 2 \log (z+i).$$

Differentiating,

$$\frac{\phi'(z)}{\phi(z)} = \frac{1}{z \log z} - \frac{2}{z+i} \quad \text{or} \quad \phi'(z) = \phi(z)\left[\frac{1}{z \log z} - \frac{2}{z+i}\right]$$

$$\therefore \quad \phi'(i) = \frac{\log i}{-4}\left[\frac{1}{i \log i} - \frac{1}{i}\right] = -\frac{(1-\log i)}{4i}$$

$$= -\frac{1}{4i}(1 - \log e^{i\pi/2}) = -\frac{1}{4i}\{1 - i(\pi/2)\}.$$

By residue theorem, we have

$$\int_C f(z)\,dz = \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_R^r f(x e^{i\pi})\, e^{i\pi}\,dx$$

$$+ \int_\gamma f(z)\,dz = 2\pi i\, \Sigma R^+. \qquad \ldots(1)$$

Now
$$\left|\int_\Gamma f(z)\,dz\right| \le \int_0^\pi \left|\frac{\log(R e^{i\theta})}{(1+R^2 e^{i2\theta})^2} R i e^{i\theta}\right| d\theta \le \int_0^\pi \frac{\log R + \theta}{(R^2-1)^2} R\,d\theta$$

$$= \frac{\left(\pi \log R + \frac{1}{2}\pi^2\right)R}{(R^2-1)^2} = \frac{R^2}{(R^2-1)^2}\left(\frac{\pi \log R}{R} + \frac{1}{2}\pi^2 \frac{1}{R}\right)$$

$$\to 0 \text{ as } R \to \infty, \text{ since } \lim_{R\to\infty} \frac{\log R}{R} = 0.$$

Similarly
$$\left|\int_\gamma f(z)\,dz\right| \le \int_0^\pi |f(re^{i\theta})\, rie^{i\theta}|\,d\theta \to 0 \text{ as } r \to 0,$$

$$\text{since } \lim_{r\to 0} r \log r = \lim_{r \to 0} \frac{\log r}{1/r} = 0.$$

Hence when $r \to 0, R \to \infty$, we have from (1)

$$\int_0^\infty f(x)\,dx - \int_\infty^0 f(x e^{i\pi})\,dx = 2\pi i\, \Sigma R^+$$

or
$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + \int_0^\infty \frac{\log x e^{i\pi}}{(1+x^2 e^{i2\pi})^2}\,dx$$

$$= 2\pi i\left[-\frac{1}{4i}\left(1 - \frac{i\pi}{2}\right)\right] = -\frac{\pi}{2}\left(1 - \frac{i\pi}{2}\right).$$

Equating real parts on both sides, we get

$$2\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{2} \quad \text{or} \quad \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

Example 29: *Prove that* $\int_0^\infty \frac{x^b}{1+x^2}\,dx = \frac{\pi}{2} \sec \frac{\pi b}{2}, -1 < b < 1.$

Solution: Consider $\int_C f(z)\,dz = \int_C \frac{z^b}{1+z^2}\,dz$, where C is the same contour as in Example 28.

Here we have avoided the branch point 0 of z^b by indenting at origin.

$z = \pm i$ are the simple poles of $f(z)$. Only $z = i$ lies inside C.

Residue at $z = i$ is

$$\lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{z^b}{z + i} = \frac{i^b}{2i}$$

$$= \frac{\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^b}{2i} = \frac{1}{2i}\left(\cos \frac{b\pi}{2} + i \sin \frac{b\pi}{2}\right).$$

By residue theorem, we have

$$\int_C f(z)\, dz = \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz + \int_R^r f(x e^{i\pi}) e^{i\pi}\, dx$$

$$+ \int_\gamma f(z)\, dz = 2\pi i \Sigma R^+. \quad \ldots(1)$$

Now $$\left|\int_\Gamma f(z)\, dz\right| \le \int_0^\pi \left|\frac{R^b e^{ib\theta}\, i R e^{i\theta}}{1 + R^2 e^{i2\theta}}\right| d\theta \le \int_0^\pi \frac{R^{b+1}}{R^2 - 1}\, d\theta$$

$$= \frac{R^{b+1}}{R^2 - 1}\pi \to 0 \text{ as } R \to \infty, \text{ since } -1 < b < 1.$$

Similarly $$\left|\int_\gamma f(z)\, dz\right| \le \int_\pi^0 \frac{r^{b+1}}{1 - r^2}\, d\theta \to 0 \text{ as } r \to 0. \text{ since } b + 1 > 0.$$

Hence from (1), we have

$$\int_0^\infty f(x)\, dx - \int_\infty^0 f(x e^{i\pi})\, dx = 2\pi i \cdot \frac{1}{2i}\left(\cos \frac{\pi b}{2} + i \sin \frac{\pi b}{2}\right)$$

or $$\int_0^\infty \frac{x^b}{1 + x^2}\, dx + \int_0^\infty \frac{e^{ib\pi} x^b}{1 + x^2 e^{i2\pi}}\, dx = \pi\left(\cos \frac{\pi b}{2} + i \sin \frac{\pi b}{2}\right).$$

Equating real parts on both sides, we get

$$\int_0^\infty \frac{x^b (1 + \cos \pi b)}{1 + x^2}\, dx = \pi \cos \frac{\pi b}{2}$$

or $$\int_0^\infty \frac{x^b}{1 + x^2}\, dx = \frac{\pi \cos (\pi b / 2)}{1 + \cos \pi b} = \frac{\pi}{2} \sec\left(\frac{\pi b}{2}\right).$$

Example 30: *Prove that*

$$\int_0^\infty \frac{x^{a-1}}{x^2 + x + 1}\, dx = \frac{2\pi}{\sqrt{3}} \cos\left(\frac{2a\pi + \pi}{6}\right) \operatorname{cosec} a\pi, (0 < a < 2).$$

Solution: Consider $\int_C f(z)\, dz = \int_C \frac{z^{a-1}}{z^2 + z + 1}\, dz,$

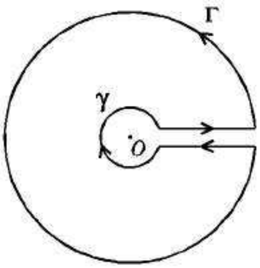
where C is the contour consisting of

(1) a large circle $\Gamma, |z| = R$

(2) radius vector $\theta = 2\pi$

(3) a small circle $\gamma, |z| = \rho$

(4) radius vector $\theta = 0$.



By residue theorem, we have

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_R^\rho f(r e^{i2\pi})\, e^{i2\pi}\,dr$$
$$+ \int_\gamma f(z)\,dz + \int_\rho^R f(x)\,dx = 2\pi i \Sigma R^+. \qquad \ldots(1)$$

The poles of $f(z)$ are given by

$$z^2 + z + 1 = 0 \quad \text{or} \quad z = \frac{-1 \pm i\sqrt{3}}{2}.$$

Thus $z = \dfrac{-1}{2} + i\dfrac{\sqrt{3}}{2} = e^{2\pi i/3} = \alpha$

and $z = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2} = e^{4\pi i/3} = \beta$ are the simple poles of $f(z)$ and both lie in C.

Residue at $z = \alpha$ is $\displaystyle\lim_{z \to \alpha} (z - \alpha) f(z) = \lim_{z \to \alpha} \frac{z^{a-1}}{z - \beta} = \frac{\alpha^{a-1}}{\alpha - \beta}$.

Sum of the residues at poles inside C is

$$= \frac{\alpha^{a-1}}{\alpha - \beta} + \frac{\beta^{a-1}}{\beta - \alpha} = \frac{1}{\alpha - \beta}(\alpha^{a-1} - \beta^{a-1})$$

$$= \frac{1}{i\sqrt{3}}\left[e^{i2\pi(a-1)/3} - e^{i4\pi(a-1)/3}\right]$$

$$= \frac{e^{i\pi(a-1)}}{i\sqrt{3}}\left[e^{-i\pi(a-1)/3} - e^{i\pi(a-1)/3}\right]$$

$$= \frac{2e^{i\pi a}}{\sqrt{3}} \sin\frac{(a-1)\pi}{3} = -\frac{2}{\sqrt{3}} e^{i\pi a} \cos\left\{\frac{\pi}{2} + (a-1)\frac{\pi}{3}\right\}$$

$$= -\frac{2}{\sqrt{3}} e^{ia\pi} \cos\left(\frac{\pi + 2a\pi}{6}\right).$$

Now $$\left|\int_\Gamma f(z)\,dz\right| \le \int_0^{2\pi} \left|\frac{R^{a-1}\exp\{i\theta(a-1)\}}{1 + Re^{i\theta} + R^2 e^{i2\theta}}\, iRe^{i\theta}\right| d\theta$$

$$\le \int_0^{2\pi} \frac{R^a}{R^2 - R - 1}\,d\theta = \frac{2\pi R^a}{R^2 - R - 1} \to 0$$

as $R \to \infty$, since $0 < a < 2$.

Similarly $$\left|\int_\gamma f(z)\,dz\right| \le \int_{2\pi}^0 \frac{\rho^a}{1 - \rho - \rho^2}\,d\theta \to 0 \text{ as } \rho \to 0.$$

Hence when $R \to \infty, \rho \to 0$, we have from (1)

$$\int_0^\infty f(x)\,dx + \int_\infty^0 f(r e^{i2\pi})\, e^{i2\pi}\,dr = 2\pi i \Sigma R^+$$

or $$\int_0^{\infty} \frac{x^{a-1}}{1+x+x^2}\,dx - \int_0^{\infty} \frac{x^{a-1}\, e^{i(a-1)2\pi}}{1+xe^{i2\pi}+x^2 e^{i4\pi}}\,dx$$

$$= 2\pi i \left\{-\frac{2}{\sqrt{3}}\, e^{i\pi a} \cos\left(\frac{\pi+2a\pi}{6}\right)\right\}$$

or $$\int_0^{\infty} \frac{x^{a-1}}{1+x+x^2}\,(1-e^{2ia\pi})\,dx = -\frac{4\pi i}{\sqrt{3}}\, e^{i\pi a} \cos\left(\frac{\pi+2a\pi}{6}\right)$$

or $$\frac{(e^{ia\pi}-e^{-ia\pi})}{2i} \int_0^{\infty} \frac{x^{a-1}}{1+x+x^2}\,dx = \frac{2\pi}{\sqrt{3}} \cos\left(\frac{\pi+2a\pi}{6}\right)$$

or $$\int_0^{\infty} \frac{x^{a-1}}{1+x+x^2}\,dx = \frac{2\pi}{\sqrt{3}} \cos\left(\frac{\pi+2a\pi}{6}\right) \operatorname{cosec} a\pi.$$

Example 31: *By integrating* $\dfrac{e^{iz}\log(-iz)}{z^2+4}$ *round the contour consisting of a large semi-circle in the upper half plane indented at the origin or otherwise prove that*

$$\int_0^{\infty} \frac{2\cos x \log x + \pi \sin x}{x^2+4}\,dx = \frac{1}{2}\pi e^{-2}\log 2.$$

Solution: Consider $\int_C f(z)\,dz = \int_C \dfrac{e^{iz}\log(-iz)}{z^2+4}\,dz$, where C is the contour as given above. (See fig. of Ex. 28).

$z = \pm 2i$ are the simple poles of $f(z)$. Only $z = 2i$ lies within C.

Residue at $z = 2i$ is

$$\lim_{z\to 2i} (z-2i)\,f(z) = \lim_{z\to 2i} \frac{e^{iz}\log(-iz)}{z+2i} = \frac{e^{-2}\log 2}{4i}.$$

By residue theorem, we have

$$\int_C f(z)\,dz = \int_r^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_R^r f(x e^{i\pi})\, e^{i\pi}\,dx$$

$$+ \int_\gamma f(z)\,dz = 2\pi i\,\Sigma R^+ \qquad \ldots(1)$$

Now $$\left|\int_\Gamma f(z)\,dz\right| \le \int_0^{\pi} \left|\frac{e^{iRe^{i\theta}}\log(-iRe^{i\theta})}{4+R^2 e^{i2\theta}}\, iRe^{i\theta}\right| d\theta$$

$$\le \int_0^{\pi} \frac{\log R + \theta + (\pi/2)}{R^2-4}\, R\,d\theta$$

$$\to 0 \text{ as } R \to \infty, \text{ since } \lim_{R\to\infty} \frac{\log R}{R} = 0.$$

Similarly $$\left|\int_\gamma f(z)\,dz\right| \to 0 \text{ as } r \to 0.$$

Hence when $r \to 0$, $R \to \infty$, we have from (1)

$$\int_0^{\infty} f(x)\,dx - \int_{\infty}^{0} f(x\,e^{i\pi})\,dx = 2\pi i \frac{e^{-2}\log 2}{4i}$$

or $$\int_0^{\infty} \frac{e^{ix}\log(-ix)}{x^2+4}\,dx + \int_0^{\infty} \frac{e^{-ix}\log(ix)}{x^2+4}\,dx = \frac{\pi}{2} e^{-2}\log 2$$

or $$\int_0^{\infty} \frac{(\cos x + i\sin x)\left(\log x - i\frac{\pi}{2}\right)}{x^2+4}\,dx$$

$$+ \int_0^{\infty} \frac{(\cos x - i\sin x)\left(\log x + i\frac{\pi}{2}\right)}{x^2+4}\,dx = \frac{\pi}{2} e^{-2}\log 2.$$

Equating real parts on both sides, we get

$$2\int_0^{\infty} \frac{\left(\cos x \log x + \frac{\pi}{2}\sin x\right)\,dx}{x^2+4} = \frac{\pi}{2} e^{-2}\log 2$$

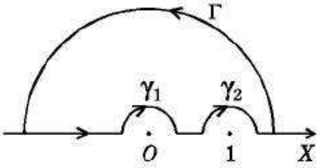
or $$\int_0^{\infty} \frac{2\cos x\log x + \pi\sin x}{x^2+4}\,dx = \frac{\pi}{2} e^{-2}\log 2.$$

Example 32: *Prove that*

(i) $\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}$, $0 < a < 1$

(ii) $\int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \pi \cot a\pi$, $0 < a < 1$.

Solution: Let $\int_C f(z)\,dz = \int_C \frac{z^{a-1}}{1-z}\,dz$,



where C is the contour consisting of a large semi-circle $\Gamma, |z| = R$ in the upper half plane indented at $z = 0, z = 1$. γ_1 and γ_2 are the semi-circles in the upper half plane with radii ρ_1 and ρ_2 and centres $z = 0, z = 1$ respectively.

By Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = \int_{\Gamma} f(z)\,dz + \int_{-R}^{-\rho_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz$$

$$+ \int_{\rho_1}^{1-\rho_2} f(x)\,dx + \int_{\gamma_2} f(z)\,dz + \int_{1+\rho_2}^{R} f(x)\,dx = 0, \quad \ldots(1)$$

since $f(z)$ has no pole inside C.

Now $$\left| \int_{\Gamma} f(z)\,dz \right| \le \int_0^{\pi} \left| \frac{R^{a-1}\,e^{i(a-1)\theta}}{1 - R\,e^{i\theta}}\, i\,R\,e^{i\theta} \right| d\theta$$

$$\leq \int_0^{\infty} \frac{R^a}{R-1}\, d\theta = \frac{R^a \pi}{R-1} \to 0 \text{ as } R \to \infty, \text{ since } a < 1.$$

Since $$\lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \frac{z^a}{1-z} = 0, a > 0,$$

$\therefore$ $$\lim_{\rho_1 \to 0} \int_{\gamma_1} f(z)\, dz = -i(\pi - 0)\, 0 = 0,$$

$$\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{z^{a-1}}{1-z} = -1,$$

$\therefore$ $$\lim_{\rho_2 \to 0} \int_{\gamma_2} f(z)\, dz = i\pi.$$

Hence as $\rho_1 \to 0, \rho_2 \to 0, R \to \infty$, we have from (1)

$$\int_{-\infty}^{0} f(x)\, dx + \int_0^1 f(x)\, dx + i\pi + \int_1^{\infty} f(x)\, dx = 0$$

or $$\int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx = -i\pi$$

or $$\int_0^{\infty} f(-x)\, dx + \int_0^{\infty} f(x)\, dx = -i\pi,$$

putting $-x$ for x in the first integral

or $$\int_0^{\infty} \frac{(-x)^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = -i\pi$$

or $$\int_0^{\infty} \frac{(-1)^{a-1} x^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = -i\pi$$

or $$\int_0^{\infty} \frac{(e^{i\pi})^{a-1} x^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = -i\pi$$

or $$\int_0^{\infty} \frac{e^{-i\pi} e^{ia\pi} x^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = -i\pi$$

or $$-\int_0^{\infty} \frac{e^{ia\pi} x^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = -i\pi.$$

Equating real and imaginary parts on both sides, we get

$$-\int_0^{\infty} \frac{\cos a\pi\, x^{a-1}}{1+x}\, dx + \int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = 0$$

or $$\int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = \cos a\pi \int_0^{\infty} \frac{x^{a-1}}{1+x}\, dx \quad \ldots(2)$$

and $$-\int \frac{\sin a\pi\, x^{a-1}}{1+x}\, dx = -\pi \quad \text{or} \quad \int \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin a\pi}.$$ **Proved.**

Substituting the above value in (2), we get

$$\int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = \pi \cot a\pi.$$

Comprehensive Exercise 5

1. Apply the calculus of residues to evaluate $\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx, \ a > b > 0.$

2. Prove that $\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx = -\pi (a - b), \ a > 0, \ b > 0.$

3. Prove that $\int_0^\infty \frac{\sin mx}{x (x^2 + a^2)^2} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3} \left(m + \frac{2}{a} \right), m > 0, a > 0.$

4. Prove that $\int_0^\infty \frac{\sin^2 mx}{x^2 (a^2 + x^2)} dx = \frac{\pi}{4a^3} (e^{-2ma} - 1 + 2ma), \ m > 0, a > 0.$

5. Prove that $\int_0^\infty \frac{\sin \pi x}{x (1 - x^2)} dx = \pi.$

6. Prove by contour integration $\int_0^\infty \frac{\log (1 + x^2)}{1 + x^2} dx = \pi \log 2$ **(Gorakhpur 2009, 13)**

 and deduce that $\int_0^1 \frac{\log (x + 1/x)}{1 + x^2} dx = \frac{\pi}{2} \log 2.$

7. By integrating $\frac{(\log z)^2}{1 + z^2}$ round a suitable contour prove that

$$\int_0^\infty \frac{(\log x)^2}{1 + x^2} dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\log x}{1 + x^2} dx = 0.$$

(Gorakhpur 2003, 16)

8. Evaluate $\int_0^\infty \frac{\log x}{(1 + x)^3} dx.$

Answers 5

1. $-\frac{\pi}{2} (a - b)$

8. $-\frac{1}{2}$

14 Rectangular and other Contours

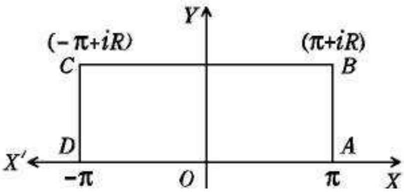
Example 33: *Integrating* $\frac{z}{1 - ae^{-iz}}$ *round the rectangle with vertices at* $z = \pm \pi, \pm \pi i R$, *prove that if* $a \geq 1$,

$$\int_0^\pi \frac{ax \sin x}{1 - 2a \cos x + a^2} dx = \pi \log \left(1 + \frac{1}{a} \right).$$

What is the value of this integral when $0 < a < 1$?

Solution: Consider the integral

$$\int_C \frac{z}{1-ae^{-iz}}\,dz = \int_C f(z)\,dz,$$



where C is the rectangle with sides $x = \pm\pi$, $y = 0$, $y = R$. $f(z)$ has simple poles given by

$$e^{iz} = a = \exp(\log a + 2n\pi i)$$

or $$z = -i(\log a + 2n\pi i), n = 0, \pm 1, \pm 2$$

and so on.

If $0 < a < 1$, there is only one pole $z = -i \log a$ which lies inside the rectangle C. If $a > 1$ there is no pole inside C.

Residue at $z = -i\log a$ is

$$\left[\frac{z}{D(1-a\,e^{-iz})}\right]_{z=-i\log a} = \frac{-i\log a}{ai\exp(-\log a)} = -\log a.$$

By Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} f(x)\,dx + \int_0^R f(\pi+iy)\,i\,dy + \int_{\pi}^{-\pi} f(x+iR)\,dx$$

$$+ \int_R^0 f(-\pi+iy)\,i\,dy = 2\pi i(-\log a). \quad \ldots(1)$$

Now $$\left|\int_{\pi}^{-\pi} f(x+iR)\,dx\right| \le \int_{\pi}^{-\pi} \frac{|x+iR|}{|1-a\exp(-ix+R)|}\,dx$$

$$\le \int_{\pi}^{-\pi} \frac{|x|+R}{a\,e^R - 1}\,dx \to 0, \text{ as } R \to \infty.$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{x\,dx}{1-a\,e^{-ix}} = \int_{-\pi}^{0} \frac{x\,dx}{1-a\,e^{-ix}} + \int_0^{\pi} \frac{x\,dx}{1-a\,e^{-ix}}.$$

Putting $-x$ for x in the first integral, we get

$$I = \int_{\pi}^{0} \frac{x\,dx}{1-ae^{ix}} + \int_0^{\pi} \frac{x\,dx}{1-ae^{-ix}} = \int_0^{\pi} x\left(\frac{1}{1-ae^{-ix}} - \frac{1}{1-ae^{ix}}\right)dx$$

$$= \int_0^{\pi} \frac{-x\,a\,(e^{ix}-e^{-ix})}{1-a\,(e^{ix}+e^{-ix})+a^2}\,dx = \int_0^{\pi} \frac{-2ixa\sin x}{1-2a\cos x+a^2}\,dx.$$

Also $$\lim_{R\to\infty}\left[\int_0^R f(\pi+iy)\,i\,dy + \int_R^0 f(-\pi+iy)\,i\,dy\right]$$

$$= \int_0^{\infty}\left[\frac{\pi+iy}{1-a\exp(-i\pi+y)} - \frac{(-\pi+iy)}{1-a\exp(\pi i+y)}\right] i\,dy$$

$$= \int_0^{\infty} \frac{2\pi}{1+ae^{y}}\,i\,dy = \int_0^{\infty} \frac{2\pi i\,e^{-y}}{e^{-y}+a}\,dy$$

$$= -2\pi i\,[\log(e^{-y}+a)]_0^{\infty}$$

$$= 2\pi i\,[\log(1+a) - \log a].$$

Hence as $R \to \infty$, we have from (1)

$$\int_0^{\pi} \frac{-2ixa \sin x}{1-2a\cos x + a^2}\,dx + 2\pi i\,\{\log(1+a) - \log a\} = 2\pi i\,(-\log a),$$

$$0 < a < 1$$

or $$\int_0^{\pi} \frac{xa \sin x}{1-2a\cos x + a^2}\,dx = \pi \log(1+a).$$

When $a > 1$, we have from (1)

$$\int_0^{\pi} \frac{-2ixa \sin x}{1-2a\cos x + a^2}\,dx + 2\pi i\,\{\log(1+a) - \log a\} = 0$$

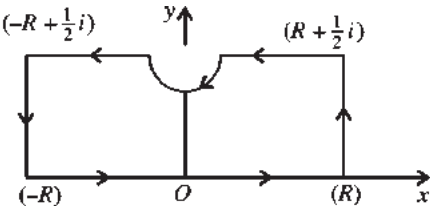
or $$\int_0^{\pi} \frac{ax \sin x}{1-2a\cos x + a^2}\,dx = \pi \log\left(1+\frac{1}{a}\right).$$

Example 34: *Apply the calculus of residues to prove that*

$$\int_0^{\infty} \frac{\cosh ax}{\cosh \pi x}\,dx = \frac{1}{2}\sec\frac{a}{2},\ -\pi < a < \pi.$$

Solution: Consider

$$\int_C \frac{e^{az}}{\cosh \pi z}\,dz = \int_C f(z)\,dz,$$



where C is the rectangle with corners at $R, R+\frac{1}{2}i, -R+\frac{1}{2}i$ and $-R$ indented at $\frac{1}{2}i$. Within and on this contour, $f(z)$ is regular and so by Cauchy's theorem, we get

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^{1/2} f(R+iy)\,i\,dy$$

$$+\int_R^{\rho} f\left(x+\frac{1}{2}i\right)dx + \int_{\gamma} f(z)\,dz + \int_{-\rho}^{-R} f\left(x+\frac{1}{2}i\right)dx$$

$$+\int_{1/2}^{0} f(-R+iy)\,i\,dy = 0. \qquad ...(1)$$

Since $$\lim_{z \to \frac{1}{2}i} \left(z-\frac{1}{2}i\right) f(z) = \lim_{z \to \frac{1}{2}i} \frac{\left(z-\frac{1}{2}i\right)e^{az}}{\cosh \pi z} = \frac{\exp(ai/2)}{\pi \sinh \frac{1}{2}\pi i} = \frac{\exp(ai/2)}{\pi i},$$

we have $$\lim_{\rho \to 0} \int_{\gamma} f(z)\,dz = (-\pi i)\cdot\frac{1}{\pi i}\cdot\exp(ai/2) = -\exp(ai/2),$$

$$\left|\int_0^{1/2} f(R+iy)\, i\, dy\right| \le \int_0^{1/2} \left|\frac{\exp\{aR+aiy\}}{\cosh \pi(R+iy)}\, i\, dy\right|$$

$$\le \int_0^{1/2} \frac{e^{aR}\, dy}{\cosh \pi R \cos \pi y - \sinh \pi R \sin \pi y}$$

$$[\because |\cosh \pi R \cos \pi y + i \sinh \pi R \sin \pi y|$$
$$\le |\cosh \pi R \cos \pi y| - |i \sinh \pi R \sin \pi y|]$$

$$\int_0^{1/2} \frac{e^{aR}\, dy}{e^{\pi R}(\cos \pi y - \sin \pi y)} \to 0 \text{ as } R \to \infty,\ a < \pi.$$

$[\because$ R is large we can write $e^{\pi R}$ for $\cosh \pi R$ and $\sinh \pi R]$

Similarly $\quad \int_{1/2}^{0} f(-R+iy)\, i\, dy \to 0$ as $R \to \infty$.

Hence when $R \to \infty$, and $\rho \to 0$, we get from (1),

$$\int_{-\infty}^{\infty} f(x)\, dx + \int_{\infty}^{0} f\left(x + \frac{1}{2} i\right) dx - e^{ai/2} + \int_{0}^{-\infty} f\left(x + \frac{1}{2} i\right) = 0$$

or $$\int_{-\infty}^{\infty} f(x)\, dx - \int_{-\infty}^{\infty} f\left(x + \frac{1}{2} i\right) dx = e^{ai/2}$$

or $$\int_{-\infty}^{\infty} \left[\frac{e^{ax}}{\cosh \pi x} - \frac{\exp\{a(x + \frac{1}{2} i)\}}{\cosh \pi (x + \frac{1}{2} i)}\right] dx = e^{ai/2}$$

or $$\int_{-\infty}^{\infty} \left[\frac{e^{ax}}{\cosh \pi x} - e^{ax} \cdot \frac{e^{a i/2}}{i \sinh \pi x}\right] dx = \left(\cos \frac{1}{2} a + i \sin \frac{1}{2} a\right)$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx - \int_{-\infty}^{\infty} \frac{\sin \frac{1}{2} a \,.\, e^{ax}}{\sinh \pi x}\, dx = \cos \frac{1}{2} a \qquad ...(2)$$

and $$\int_{-\infty}^{\infty} \frac{e^{ax} \cos \frac{1}{2} a}{\sinh \pi x}\, dx = \sin \frac{1}{2} a \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{ax}}{\sinh \pi x}\, dx = \tan \frac{1}{2} a. \qquad ...(3)$$

From (2) and (3), we get

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx - \sin \frac{1}{2} a \tan \frac{1}{2} a = \cos \frac{1}{2} a$$

or $$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \sec \frac{1}{2} a$$

or $$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\, dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \sec \frac{1}{2} a.$$

Putting $x = -y$ in the first integral, we get

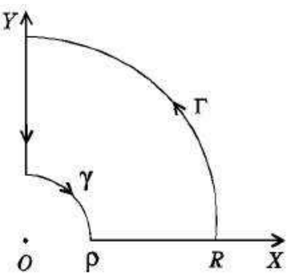
$$-\int_{\infty}^{0} \frac{e^{-ay}}{\cosh \pi y}\, dy + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \sec \frac{1}{2} a$$

or
$$\int_{0}^{\infty} \frac{e^{ax} + e^{-ax}}{\cosh \pi x}\, dx = \sec \frac{1}{2} a$$

or
$$\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2} \sec \frac{1}{2} a.$$

Example 35: *By integrating $e^{iz} / \sqrt{z}$ along a suitable path prove that*

$$\int_{0}^{\infty} \frac{\cos x}{\sqrt{x}}\, dx = \int_{0}^{\infty} \frac{\sin x}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2}}\,.$$



Solution: Consider $\int_C \frac{e^{iz}}{\sqrt{z}}\, dz = \int_C f(z)\, dz,$

where C is a quadrant of a circle indented at the origin which is the centre of the circle. Since $f(z)$ is regular within and on C, therefore by Cauchy's theorem, we have

$$\int_C f(z)\, dz = \int_{\rho}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz + \int_{R}^{\rho} f(r e^{i\pi/2})\, e^{i\pi/2}\, dr + \int_{\gamma} f(z)\, dz = 0. \qquad \ldots(1)$$

Now
$$\left| \int_{\Gamma} f(z)\, dz \right| \le \int_{0}^{\pi/2} \left| \frac{\exp(i R e^{i\theta})}{R^{1/2}\, e^{i\theta/2}}\, i R e^{i\theta}\, d\theta \right|$$
$$\le \int_{0}^{\pi/2} e^{-R \sin\theta}\, R^{1/2}\, d\theta$$
$$\le \int_{0}^{\pi/2} e^{-(2\theta/\pi)R}\, R^{1/2}\, d\theta, \text{ by Jordan's inequality}$$
$$= \frac{\pi}{2\sqrt{R}} (1 - e^{-R}) \to 0 \text{ as } R \to \infty.$$

Similarly,
$$\left| \int_{\gamma} f(z)\, dz \right| \le \int_{\pi/2}^{0} e^{-\rho \sin\theta}\, \rho^{1/2}\, d\theta \le \int_{\pi/2}^{0} \rho^{1/2}\, d\theta,$$

since $e^{-\rho \sin\theta} \le 1$ for small values of ρ

$$\to 0 \text{ as } \rho \to 0.$$

Hence when $R \to \infty$, $\rho \to 0$, we have from (1)

$$\int_{0}^{\infty} \frac{e^{ix}}{\sqrt{x}}\, dx - i \int_{0}^{\infty} \frac{e^{ir e^{i\pi/2}}}{\sqrt{r\, e^{i\pi/4}}}\, dr = 0$$

or $$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx = i\,e^{-i\pi/4}\int_0^\infty r^{-1/2}\,e^{-r}\,dr$$

$$= i\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\Gamma\,\frac{1}{2} = (1+i)\sqrt{\frac{\pi}{2}}\,.$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}\,,\ \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}\,.$$

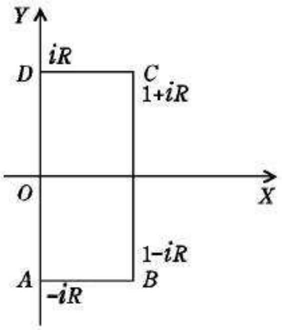
15 To find the Residue by Knowing the Integral First

So far we evaluated the integrals with the help of residues of the function $f(z)$ and Cauchy's residue theorem. Now we shall illustrate how can we find the residues of $f(z)$ by integrating $f(z)$ and then using the residue theorem.

Illustrative Examples

Example 36: *Find the residue of* $\tan^{n-1}\pi z$ *at* $z = \frac{1}{2}$, *where n is an even positive integer.*

Solution: Let $f(z) = \tan^{n-1}\pi z = \left(\frac{\sin \pi z}{\cos \pi z}\right)^{n-1}$.



Poles of $f(z)$ are given by

$$\cos \pi z = 0$$

or $$e^{i\pi z} + e^{-i\pi z} = 0$$

or $$e^{i2\pi z} = -1 = e^{(2r+1)\pi i}$$

or $$z = \frac{1}{2}(2r+1), r = 0, \pm 1, \pm 2, \ldots\ldots$$

Consider a rectangular contour C with sides $x = 0, 1$, $y = \pm R$. The only pole which lies inside C is $z = \frac{1}{2}$.

By Cauchy's residue theorem, we have the residue at $(z = \frac{1}{2})$

$$= \frac{1}{2\pi i}\left[\int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz\right], \quad \ldots(1)$$

where $f(z) = \tan^{n-1}\pi z$.

On BC, we have $z = 1 + iy, dz = i\,dy$,

$$\therefore \qquad \int_{BC} \tan^{n-1} \pi z \, dz = \int_{-R}^{R} \tan^{n-1} \{\pi (1 + iy)\} \, i \, dy$$

$$= \int_{-R}^{R} i \tan^{n-1} i\pi y \, dy$$

and on DA, we have $z = iy, dz = i \, dy$,

$$\therefore \qquad \int_{DA} \tan^{n-1} \pi z \, dz = \int_{R}^{-R} \tan^{n-1} (\pi i y) \, i \, dy$$

$$= - \int_{-R}^{R} i \tan^{n-1} (\pi i y) \, dy.$$

Thus $$\int_{BC} \tan^{n-1} \pi z \, dz + \int_{DA} \tan^{n-1} \pi z \, dz = 0.$$

Hence from (1), we have the residue at $(z = \frac{1}{2})$

$$= \frac{1}{2\pi i} \left[\int_{AB} \tan^{n-1} \pi z \, dz + \int_{CD} \tan^{n-1} \pi z \, dz \right]$$

$$= \frac{1}{2\pi i} \left[\int_{0}^{1} \tan^{n-1} \pi (x - iR) \, dx + \int_{1}^{0} \tan^{n-1} \pi (x + iR) \, dx \right].$$

Now $$\tan \pi (x - iR) = \frac{1}{i} \frac{\exp i\pi (x - iR) - \exp \{- i\pi (x - iR)\}}{\exp i\pi (x - iR) + \exp \{- i\pi (x - iR)\}}$$

$$= \frac{1}{i} \frac{1 - \exp \{- 2i\pi (x - iR)\}}{1 + \exp \{- 2i\pi (x - iR)\}}$$

$$= \frac{1}{i} \frac{1 - \exp (- 2i\pi x) \exp (- 2\pi R)}{1 + \exp (- 2i\pi x) \exp (- 2\pi R)} \to \frac{1}{i} \text{ as } R \to \infty.$$

Similarly $\tan (\pi x + i\pi R) \to - \frac{1}{i}$ as $R \to \infty$.

$\therefore$ the residue at $z = \frac{1}{2}$ is

$$= \frac{1}{2\pi i} \left[\int_{0}^{1} \left(\frac{1}{i}\right)^{n-1} dx + \int_{1}^{0} \left(- \frac{1}{i}\right)^{n-1} dx \right]$$

$$= \frac{1}{2\pi i} \left[\frac{1}{i^{n-1}} - \frac{(-1)^{n-1}}{i^{n-1}} \right]$$

$$= \frac{1}{2\pi i} \left[\frac{1}{i^{n-1}} + \frac{1}{i^{n-1}} \right], n \text{ is an even positive integer}$$

$$= \frac{1}{\pi i^{n}} = \frac{1}{\pi} (-1)^{n/2}.$$

Comprehensive Exercise 6

1. Integrating $\dfrac{e^{iaz}}{e^{2\pi z}-1}$, (a is real) round the rectangle of sides $x=0$, $x=R$, $y=0$, $y=1$, indented at 0 and i, prove that
$$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{1}{2}a - \frac{1}{2a}.$$

2. Integrating $\log(1-e^{i2z})$ round a suitable contour, prove that
$$\int_0^\pi \log \sin x\,dx = -\pi \log 2.$$

3. By integrating e^{-z^2} round the rectangle whose vertices are $0, R, R+ia, ia$, show that
$$\int_0^\infty e^{-x^2}\cos(2ax)\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-a^2}$$
and $$\int_0^\infty e^{-x^2}\sin(2ax)\,dx = e^{-a^2}\int_0^a e^{x^2}\,dx.$$

4. By integrating $\dfrac{z}{a-e^{-iz}}$ round the rectangle with vertices at $\pm\pi, \pm\pi+iR$,
prove that $$\int_0^\pi \frac{x\sin x\,dx}{1+a^2-2a\cos x} = \begin{cases} \dfrac{\pi}{a}\log(1+a), & \text{if } 0<a<1 \\ \dfrac{\pi}{a}\log\left(\dfrac{1+a}{a}\right), & \text{if } a>1. \end{cases}$$

5. By integrating $\dfrac{\exp(iaz^2)}{\sinh \pi z}$ round the rectangle with vertices $\pm R \pm \frac{1}{2}i$, show that if $0<a\le\pi$,
$$\int_0^\infty \cos(ax^2)\frac{\cosh ax}{\cosh \pi x}\,dx = \frac{1}{2}\cos\frac{a}{4} \text{ and } \int_0^\infty \sin(ax^2)\frac{\cosh ax}{\cosh \pi x}\,dx = \frac{1}{2}\sin\frac{a}{4}.$$

6. By integrating e^{iz^2}/z round a suitable contour, prove that
$$\int_0^\infty \frac{\sin x^2}{x}\,dx = \frac{\pi}{4}.$$
Deduce that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

(Gorakhpur 2004, 07, 10)

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) and (d).

1. The number of poles of $f(z) = \dfrac{1}{z(z^2+3)(z^2+2)^3}$ inside the circle $|z| = 1$ is

(a) 1 (b) 9

(c) 5 (d) 2.

2. If C is circle $|z| = 3$, then the value of $\int_C \dfrac{dz}{(z-1)(z+1)}$ is

(a) 1 (b) 0

(c) 2 (d) none of these.

3. The residue of the function $f(z) = \dfrac{z+1}{(z-1)(z-2)}$ at $z = 1$ is

(a) -2 (b) 2

(c) 1 (d) -1. **(Kumaun 2014)**

4. The residue of $\dfrac{1}{z^3 - z^5}$ at $z = \infty$ is

(a) 1 (b) 0

(c) -1 (d) 2.

5. The residue of $z^3 \cos\left(\dfrac{1}{z-2}\right)$ at $z = 2$ is

(a) $\dfrac{143}{124}$ (b) $-\dfrac{22}{123}$

(c) $-\dfrac{143}{24}$ (d) none of these.

6. The residue of the function $\dfrac{1}{(z^2+1)^3}$ at $z = i$ is

(a) $-\dfrac{3i}{16}$ (b) $\dfrac{3i}{16}$

(c) $-\dfrac{3}{16i}$ (d) none of these.

7. The value of $\int_0^{2\pi} \dfrac{d\theta}{a + b\cos\theta}$, $a > b > 0$ is

(a) $\dfrac{\pi}{\sqrt{(a^2 - b^2)}}$ (b) $\dfrac{2\pi}{\sqrt{(a^2 - b^2)}}$

(c) $\dfrac{3\pi}{\sqrt{(a^2 - b^2)}}$ (d) none of these.

8. The value of $\int_0^{\infty} \dfrac{dx}{1 + x^2}$ is

(a) 0 (b) 1

(c) $\dfrac{\pi}{2}$ (d) none of these.

9. If $f(z) = \dfrac{e^{miz}}{z^2 + a^2}$, then residue of $f(z)$ at $z = ai$ is

(a) $\dfrac{e^{-ma}}{2ia}$ (b) $\dfrac{e^{ma}}{2ia}$

(c) $\dfrac{e^{-ma}}{2a}$ (d) none of these.

(Kumaun 2007)

10. If $f(z) = \dfrac{\sin 2z}{(z+1)^3}$, the residue of $f(z)$ at $z = -1$ is

(a) $2\sin 2$ (b) $-2\sin 2$

(c) 0 (d) none of these.

11. The residue of $\dfrac{z^2}{(z-a)(z-b)(z-c)}$ at infinity is :

(a) 1 (b) -1

(c) a (d) b

(Kumaun 2010, 13)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The residue of a function $f(z)$ at ∞ is given by $\lim\limits_{z \to \infty}$ ....
2. If $f(z)$ is regular, except at a finite number of poles within a closed contour C and continuous on the boundary of C, then $\int_C f(z)\,dz$ is .... , where ΣR is the sum of residues of $f(z)$ at its poles within C.
3. Residue of $\dfrac{1}{z^3 - z^5}$ at $z = \pm 1$ is ....
4. Residue of $f(z) = \dfrac{e^{miz}}{z^2 + a^2}$ at $z = -ai$ is ....

5. Residue of $\cos\left(\dfrac{1}{z-2}\right)$ at $z = 2$ is ....

6. If $\lim\limits_{z \to a} (z - a) f(z) = A$ and if C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z - a| = r$, then $\lim\limits_{r \to 0} \int_C f(z)\, dz =$ ....

7. When $0 \leq \theta \leq \dfrac{\pi}{2}, \dfrac{2}{\pi} \leq \dfrac{\sin\theta}{\theta} \leq 1$. This is known as ...... inequality.

8. If $f(z) = \dfrac{z^2}{(z-1)(z-2)(z-3)}$, residue of $f(z)$ at $z = 1$ is ....

True or False

Write 'T' for true and 'F' for false statement.

1. If $f(z) = \dfrac{\phi(z)}{\psi(z)}$ where $\psi(z) = (z - a) F(z)$, $F(a) \neq 0$, then residue at the simple pole $z = a$ is $\dfrac{\phi(a)}{\psi'(a)}$.

2. The residue of $f(z)$ at infinity is the coefficient of $\dfrac{1}{z}$ in the expansion of $f(z)$ in the neighbourhood of $z = \infty$.

3. If a function $f(z)$ is analytic except at finite number of singularities (including that at infinity), then the sum of residues of these singularities is zero.

4. If $f(z)$ is of the form $\dfrac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic, the residue of $f(z)$ at $z = a$ is $\dfrac{\phi^m(a)}{(m-1)!}$.

5. If C is an arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and if $\lim\limits_{R \to \infty} z f(z) = 0$, then $\lim\limits_{R \to \infty} \int_C f(z)\, dz = 0$.

6. Residue of $\dfrac{1}{z(1 - z^2)}$ at $z = 0$ is 1.

7. The value of $\int_0^{\pi} \dfrac{1 + 2\cos\theta}{5 + 3\cos\theta}\, d\theta$ is $\dfrac{\pi}{2}$.

8. Residue of $\dfrac{z^2}{(z+1)^2}$ at $z = i$ is $\dfrac{i}{4}$.

Answers

Multiple Choice Questions

1.	(a)	**2.**	(b)	**3.**	(a)	**4.**	(b)	**5.**	(c)
6.	(a)	**7.**	(b)	**8.**	(c)	**9.**	(a)	**10.**	(a)
11.	(b)								

Fill in the Blank(s)

1. $-z\ f(z)$ **2.** $2\pi i\,\Sigma\, R$ **3.** $-\dfrac{1}{2}$

4. $\dfrac{i}{2a}\, e^{m a}$ **5.** 0

6. $iA\,(\theta_2-\theta_1)$ **7.** Jordan's **8.** $\dfrac{1}{2}$

True or False

1.	*T*	**2.**	*F*	**3.**	*T*	**4.**	*F*	**5.**	*T*
6.	*T*	**7.**	*F*	**8.**	*F*				

□