the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$ W, where W is the weight of the rod. 14

(c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{V}=\frac{D^2}{d^2}e^{(v^2-V^2)/2K}$$

where K is the pressure divided by the density and supposed constant. 13

8. (a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve :

$$\vec{r}=\vec{r}(u)=2\log_e u\hat{i}+4u\hat{j}+(2u^2+1)\hat{k}$$ 10

(b) Prove the vector identity :

curl $(\vec{f}\times\vec{g})=\vec{f}$ div $\vec{g}-\vec{g}$ div $\vec{f}+(\vec{g}\cdot\nabla)\vec{f}-(\vec{f}\cdot\nabla)\vec{g}$

and verify it for the vectors $\vec{f}=x\hat{i}+z\hat{j}+y\hat{k}$ and $\vec{g}=y\hat{i}+z\hat{k}$. 10

(c) Verify Green's theorem in the plane for

$$\oint_C[(3x^2-8y^2)dx+(4y-6xy)dy],$$

where C is the boundary of the region enclosed by the curves $y=\sqrt{x}$ and $y=x^2$. 10

(d) The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r}=(t^2-2t)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}.$$

At time $t=1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. 10

PAPER-II

I.F.S 2011

Instructions : *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. Answer any ***four*** parts from the following:

(a) Let G be a group, and x and y be any two elements of G. If $y^5=e$ and $yxy^{-1}=x^2$, then show that $O(x)=31$, where e is the identity element of G and $x\neq e$. 10

(b) Let Q be the set of all rational numbers.

Show that $Q(\sqrt{2})=\{a+b\sqrt{2}:a,b\in Q\}$ is a field under the usual addition and multiplication. 10

(c) Determine whether

$$f(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

is Riemann-integrable on [0, 1] and justify your answer. 10

(d) Expand the function

$$f(z)=\frac{2z^2+11z}{(z+1)(z+4)}$$

in a Laurent's series valid for $2<|z|<3$. 10

(e) Write the dual of the linear programming problem (LPP) :

Minimize $Z = 18x_1 + 9x_2 + 10x_3$

subject to

$$x_1 + x_2 + 2x_3 \geq 30$$
$$2x_1 + x_2 \geq 15$$
$$x_1, x_2, x_3 \geq 0$$

Solve the dual graphically. Hence obtain the minimum objective function value of the above LPP. 10

2. (a) Let G be the group of non-zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication. 13

(b) Let the function f be defined by

$$f(x) = \frac{1}{2^t}, \text{ when } \frac{1}{2^{t+1}} < x \leq \frac{1}{2^t}$$
$$(t = 0, 1, 2, 3, \ldots)$$
$$f(0) = 0$$

Is f integrable on $[0, 1]$? If f is integrable, then evaluate $\int_0^1 f\,dx$. 13

(c) Sketch the image of the infinite strip $1 < y < 2$ under the transformation $w = \frac{1}{z}$. 14

3. (a) Examine the convergence of

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$$

and evaluate, if possible. 13

(b) Let G be a group of order $2p$, p prime. Show that either G is cyclic or G is generated by $\{a, b\}$ with relations $a^p = e = b^2$ and $bab = a^{-1}$ 13

(c) Reduce the feasible solution $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ for the linear programming problem

Maximize $Z = x_1 + 2x_2 + 3x_3$

subject to

$$x_1 - x_2 + 3x_3 = 4$$
$$2x_1 + x_2 + x_3 = 6$$
$$x_1, x_2, x_3 \geq 0$$

to a basic feasible solution. 14

4. (a) Evaluate

$$\iint \sqrt{4x^2 - y^2}\, dx\, dy$$

over the triangle formed by the straight lines $y = 0$, $x = 1$, $y = x$. 13

(b) State Cauchy's residue theorem. Using it, evaluate the integral

$$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)}\, dz$$

counterclockwise around the circle $C : |z + 1| = 4$.

(c) A steel company has three open-hearth furnaces and four rolling mills. Transportation costs (rupees per quintal) for shipping steel from furnaces to rolling mills are given in the following table :

	M_1	M_2	M_3	M_4	*Supply (quintals)*
F_1	29	40	60	20	7
F_2	80	40	50	70	10
F_3	50	18	80	30	18
Demand (quintals)	4	8	8	15	

Find the optimal shipping schedule. 14

Section-B

5. Answer any *four* parts from the following:

(a) Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to its canonical form and solve. 10

(b) For the data

x	:	0	1	2	5
$f(x)$	:	2	3	12	147

find the cubic function of x.

(c) Solve by Gauss-Jacobi method of iteration the equations

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

(correct to two decimal places) 10

(d) Find the Lagrangian for a simple pendulum and obtain the equation describing its motion. 10

(e) With usual notations, show that ϕ and ψ for a uniform flow past a stationary cylinder are given by

$$\phi = U\cos\theta\left(r+\frac{a^2}{r}\right)$$

$$\psi = U\sin\theta\left(r-\frac{a^2}{r}\right)$$ 10

6. (a) A uniform string of length l is held fixed between the points $x = 0$ and $x = l$. The two points of trisection are pulled aside through a distance ε on opposite sides of the equilibrium position and is released from rest at time $t = 0$.
Find the displacement of the string at any latter time $t > 0$.
What is the displacement of the string at the midpoint? 16

(b) Draw a flow chart to declare the results for the following examination system :
60 candidates take the examination.
Each candidate writes one major and two minor papers.
A candidate is declared to have passed in the examination if he/she gets a minimum of 40 in all the three papers separately and an average of 50 in all the three papers put together.
Remaining candidates fail in the examination with an exemption in major if they obtain 60 and above and exemption in each minor if they obtain 50 and more in that minor.

(c) Find the smallest positive root of the equation $x^3 - 6x + 4 = 0$ correct to four decimal places using Newton-Raphson method. From this root, determine the positive square root of 3 correct to four decimal places. 12

7. (a) For a steady Poiseuille flow through a tube of uniform circular cross-section, show that

$$w(R) = \frac{1}{4}\left(\frac{p}{\mu}\right)(a^2 - R^2)$$ 16

(b) Find the complementary function and particular integral of the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$ 12

(c) The velocity of a particle at time t is as follows :

t (seconds) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	36	60	94	136

Find its displacement at the 12th second and acceleration at the 2nd second.

8. (a) From a uniform sphere of radius a, a spherical sector of vertical angle 2α is removed. Find the moment of intertia of the remainder mass M about the axis of symmetry. 14

(b) Draw a flow chart to solve a quadratic equation with non-zero coefficients. The roots be classified as real distinct, real repeated and complex. 12

(c) Is

$$\vec{q} = \frac{k^2(x\hat{j} - y\hat{i})}{x^2 + y^2},$$

a possible velocity vector of an incompressible fluid motion? If so, find the stream function and velocity potential of the motion. 14

$$p = Ke$$

or, $$u\frac{\partial u}{\partial r} = -\frac{K}{e}\frac{\partial e}{\partial r}$$

By integrating, we have

$$\frac{1}{2}u^2 = -K\log e + K\log E$$

where E is an arbitrary constant

or, $$\log\frac{e}{E} = -\frac{u^2}{2K}$$

or, $$e = E\exp\left(-\frac{u^2}{2K}\right)$$

Again $e = e_1$ when $u = v$

then $$e_1 = E\exp\left(-\frac{v^2}{2K}\right)$$

and $e = e_2$ when $u = v$,

then $$e_2 = E\exp\left(\frac{-V^2}{2K}\right)$$

or, $$\frac{e_1}{e_2} = \frac{\exp\left(-v^2/2K\right)}{\exp\left(-V^2/2K\right)} \quad ...(2)$$

from (1) and (2), we have

$$\frac{v}{V} = \frac{D^2}{d^2}\exp.\left(\frac{v^2 - V^2}{2K}\right)$$ **Proved.**

PAPER-II

Section-A

1. (a): It is given that $(G, 0)$ is a group and $\exists\ x, y, \in G$, such that $y^5 = e$ and $yxy^{-1} = x^2$. Now, we have

$$(yxy^{-1})^2 = yxy^{-1} \,.\, yxy^{-1} = yx^2y^{-1}.$$
$$= y.yxy^{-1}y^{-1} = y^2xy^{-2}$$

$\therefore$ $$(yxy^{-1})^4 = (y^2xy^{-2})\ (y^2xy^{-2})$$
$$= y^2x^2y^{-2} = y^2.yxy^{-1}y^{-2}$$

i.e., $$(yxy^{-1})^4 = y^3xy^{-3}$$

Again $$(yxy^{-1})^8 = (y^3xy^{-3}) \,.\, (y^3xy^{-3})$$
$$= y^3x^2y^{-3}$$
$$= y^3.yxy^{-1}y^{-3} = y^4xy^{-4}$$

or $$(yxy^{-1})^{16} = (y^4xy^{-4}).(y^4xy^{-4}) = y^4x^2y^{-4}$$

or $$(yxy^{-1})^{16} = y^4.yxy^{-1}y^{-4}$$
$$= y^5xy^{-5} = x \qquad [\because y^5 = e]$$

or, $$(x^2)^{16} = x \quad \Rightarrow \quad x^{32} = x$$

$\Rightarrow$ $$x^{31} = e$$

$\Rightarrow$ $0(x) = 31$ **Proved.**

(b): First we show $Q(\sqrt{2}\,)$ is a ring under usual addition and multiplication.

i.e., We'll show that $(Q\sqrt{2}, +1)$ is a group under addition and closed under multiplication.

Let $x, y \in Q\sqrt{2}$ then $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$ where $a, b, c, d \in Q$.

Now $x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$.

Since $a + c$ and $b + d$ are elements of Q, therefore $(a + c) + (b + d)\sqrt{2} \in Q\sqrt{2}$.

Thus $x + y \in Q\sqrt{2}\ \ \forall\ x, y \in Q\sqrt{2}$.

Therefore $Q\sqrt{2}$ is closed with respect to addition.

Also, the elements of $Q\sqrt{2}$ are all real numbers and the addition of real number is associative.

Also $O + O\sqrt{2} \in Q\sqrt{2}$ as $O \in Q$.

If $a + b\sqrt{2} \in Q\sqrt{2}$ then $(O + O\sqrt{2}\,) + (a + b\sqrt{2}\,) = (a + b\sqrt{2}\,) + (O + O\sqrt{2}\,) = (a + b\sqrt{2}\,)$.

i.e., $O + O\sqrt{2}$ is additive identify.

Also, $\forall\ a + b\sqrt{2} \in Q\sqrt{2}$ there exist $-a - b\sqrt{2} \in Q\sqrt{2}$

Such that $(a + b\sqrt{2}) + (-a - b\sqrt{2})$

$= O + O\sqrt{2}$ (additive identity)

i.e., Every member of $Q\sqrt{2}$ has an additive inverse.

Hence, $(Q\sqrt{2} + 1)$ is a group.

Also $\forall\ a + b\sqrt{2},\ c + d\sqrt{2} \in Q\sqrt{2}$

$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + \sqrt{2}(bc + ad) \in Q\sqrt{2}$

as $ac + 2bd,\ bc + ad \in Q$.

i.e., $Q\sqrt{2}$ is closed under multiplication.

Hence, $(Q\sqrt{2}, + 1)$ is a commutative ring.

Now, we show it does not zero divisor.

Let $O \neq a + b\sqrt{2} \in Q\sqrt{2}$

such that $(a + b\sqrt{2})(c + d\sqrt{2}) = 0$.

then, $(ac + 2bd) + \sqrt{2}(bc + ad) = 0$,

$\Rightarrow \quad ac + 2bd = 0$

$\therefore \quad bc + ad = 0$

$\Rightarrow \quad d(a^2 - 2b^2) = 0$

However, $\quad a + b\sqrt{2} \neq 0$

$\Rightarrow \quad a^2 - 2b^2 \neq 0$

$\Rightarrow \quad d = 0$

Also, $\quad bc + ad = 0$

$\Rightarrow \quad c = 0$

i.e., $(a + b\sqrt{2})(c + d\sqrt{2}) = 0$

$\Rightarrow \quad (c + d\sqrt{2}) = 0$

i.e., It does not have a zero divisor.

or, $(Q\sqrt{2} + 1)$ is an integral domain.

To show it is a field, we have to show that every non-zero element of $Q\sqrt{2}$ is invertible.

let $O \neq a + b\sqrt{2} \in Q\sqrt{2}$

if possible there exist $x + y\sqrt{2} \in Q\sqrt{2}$ such that $(a + b\sqrt{2}).(x + y\sqrt{2}) = 1$.

$\Rightarrow (ax + 2by) + \sqrt{2}(bx + ay) = 1$.

Comparing the coefficients both the sides, we get

$ax + 2by = 1$ and $bx + ay = 0$.

$\Rightarrow$ Solving, we get

$$y = \frac{-b}{a^2 - 2b^2}$$

and $$x = \frac{a}{a^2 - 2b^2}$$

Clearly $x, y \in Q$.

i.e., $x + y\sqrt{2} \in Q\sqrt{2}$.

Hence, every non-zero element of $Q\sqrt{2}$ has a multiplicative inverse, hence, $(Q\sqrt{2} + 1)$ is a field.

(c): The given function

$$f(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x},\ x \in [0, 1].$$

Clearly, the function is not continuous on $[0, 1]$. (It is discontinuous at $x = 0$), but it is bounded and continuous on $[0, 1]$ and thus Riemann integrable on $[0, 1]$.

Consider the function

$$g(x) = x^2 \sin\frac{1}{x}, \qquad x \in [0, 1]$$
$$= 0, \qquad x = 0$$

is differentiable on $[0, 1]$ and satisfies. $g'(x) = f(x)\ \forall\ x \in [0, 1]$

$$\therefore \int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx = g(1) - g(0)$$

$$= \sin 1.$$

(d): The given function is $f(z) = \dfrac{2z^2 + 11z}{(z+1)(z+4)}$

or, $$f(z) = \frac{2(z^2 + 5z + 4) + z - 8}{(z+1)(z+4)}$$

$$= 2+\frac{z-8}{(z+1)(z+4)}$$

$$f(z) = 2+\frac{4}{z+4}-\frac{3}{z+1}$$

Now, for the given region $2 < |z| < 3$.

$$f(z) = 2+\frac{4}{z+4}-\frac{3}{z+1}$$

$$= 2+\frac{4}{4\left(1+\frac{z}{4}\right)}-\frac{3}{z\left(1+\frac{1}{z}\right)}$$

$2 < |z| < 3.$

$$= 2+\left(1+\frac{z}{4}\right)^{-1}-\frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= 2+\left(1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+....\right)$$

$$-\frac{3}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+...\right)$$

$$= 2+1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+...$$

$$-\frac{3}{z}+\frac{3}{z^2}-\frac{3}{z^3}+\frac{3}{z^4}...$$

$$f(z) = ...+\frac{3}{z^4}-\frac{3}{z^3}+\frac{3}{z^2}-\frac{3}{z}+3$$

$$-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+...$$

is the required Laurent's series expansion in the given region $2 < |z| < 3$.

(e): The given LPP is

minimize $z = 18x_1 + 9x_2 + 10x_3$

subject to

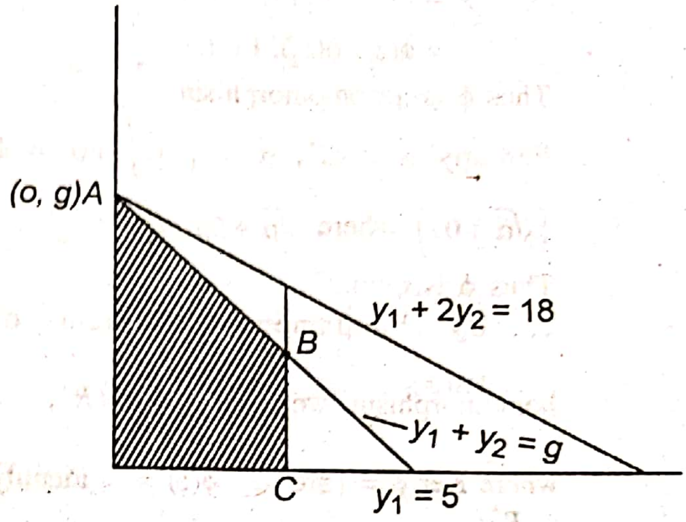
$$x_1 + x_2 + 2x_3 \geq 30$$
$$2x_1 + x_2 \geq 15$$
$$x_1, x_2, x_3 \geq 0$$

The dual of the above problem can be written as

$$y_1 + 2y_2 \leq 18$$
$$y_1 + y_2 \leq 9$$
$$2y_1 \leq 10 \qquad y_1 \leq 5$$

and the objective function

maximize $z' = 30y_1 + 15y_2$



Shaded area is the solution space.

The point A, B, C are (0, 9) (5, 4) and (5, 0)

$Z^1_A = 135$, $Z^1_B = 210$ and $Z^1_C = 150$.

Hence, by dual principle minimum value of $z = 210$.

2. (a): From question, we have

$$G = \{z : z \in C \text{ and } z \neq 0\}$$

and $\quad N = \{z \in C : |z| = 1\}$

let R^+ denote the group of all positive real numbers under multiplication.

Define a mapping $\phi : G \to R^+$ as

$$\phi(a + bi) = a^2 + b^2$$

for all $\quad z = a + bi \neq 0 \in G. \quad ...(i)$

As both a and b cannot be zero, $a^2 + b^2 > 0$.

We show that ϕ is a homomorphism.

let $\quad z_1 = a_1 + ib_1,$

$\quad z_2 = a_2 + b_2 i$

be any two elements of G. Then

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$$

Using (1), we have

$$\phi(z_1z_2) = (a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2$$
$$= a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2$$
$$= (a_1^2 + b_1^2)(a_2^2 + b_2^2)$$
$$= \phi(z_1)\,\phi(z_2), \text{ by } (i).$$

Thus ϕ is a homomorphism.

For any $\alpha \in R^+$; $\alpha = \left(\sqrt{\alpha}\right)^2 + 0 = \phi\left(\sqrt{\alpha} + 0.i\right)$ where $\sqrt{\alpha} + 0.i \in G$.

Thus ϕ is onto.

$\therefore$ By Fundamental theorem of homomorphism, we have $\dfrac{G}{\text{Ker}\phi} \approx R^+$,

where Ker $\phi = \{z \in G : \phi(z) = 1$, identity of $R^+\}$

Now, $z = a + bi$ Ker $\phi \Leftrightarrow a^2 + b^2 = 1 \Leftrightarrow \sqrt{a^2+b^2} = 1$.

$\therefore z \in$ Ker $\phi \Leftrightarrow |z| = 1 \Leftrightarrow z \in N$.

Thus, ker ϕ = N, putting in (2), we have

$$\frac{G}{N} \approx R^+$$

Hence the result.

(b): From question,

$$f(x) = \frac{1}{2^t}, \text{ when } \frac{1}{2^{t+1}} < x \le \frac{1}{2^t}$$
$$f(0) = 0$$

i.e.
$$f(x) = 1, \text{ when } \frac{1}{2} < x \le \frac{1}{1}$$
$$= \frac{1}{2} \text{ when } \frac{1}{2^2} < x \le \frac{1}{2}$$
$$= \frac{1}{2^2} \text{ when } \frac{1}{2^3} < x \le \frac{1}{2^2}$$
$$\frac{1}{2^{t-1}} \text{ when } \frac{1}{2^t} < x \le \frac{1}{2^{t-1}}$$
$$= 0 \text{ where } x = 0.$$

i.e., we notice that f is bounded and monotonic increasing on [0, 1]. Hence f is integrable. In other words, f is continuous on [0, 1] except at the set of points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$ which have only one limit point 0, hence f is integrable.

$$\therefore \int_{1/2^t}^{1} f dx = \int_{1/2}^{1} f dx + \int_{1/2^2}^{1/2} f dx + \int_{1/2^3}^{1/2^2} f dx + \ldots\ldots \int_{1/2^t}^{1/2^{t-1}} f dx$$

$$= \int_{1/2}^{1} 1.dx + \frac{1}{2}\int_{1/2^2}^{1/2} dx + \frac{1}{2^2}\int_{1/2^3}^{1/2^2} dx + \ldots\ldots + \frac{1}{2^{t-1}}\int_{1/2^t}^{1/2^{t-1}} dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots\ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \ldots\ldots + \left(\frac{1}{2^2}\right)^{t-1}\right]$$

$$= \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{4}\right)^t}{1 - \frac{1}{4}} = \frac{2}{3}\left(1 - \frac{1}{4^t}\right)$$

Proceeding to the limit when $t \to \infty$, we get

$$\int_0^1 f dx = \frac{2}{3}.$$

(c): The given transformation is $\omega = \dfrac{1}{z}$ in the infinite strip $1 < y < 2$.

We have $\omega = \dfrac{1}{z}$

or, $$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow \quad u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

which given

$$\frac{u}{v} = -\frac{x}{y} \quad \text{or} \quad x = -\frac{uy}{v}$$

Then, $$v = -\frac{y}{\frac{u^2y^2}{v^2}+y^2} = \frac{v^2}{(u^2+v^2)y}$$

or, $$y = -\frac{v}{(u^2+v^2)}$$

If $y = 1$

then, $$-\frac{v}{u^2+v^2} = 1$$

or, $$u^2 + v^2 + v = 0$$

or, $$u^2+\left(v+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Thus, $y = 1$ is transformed to circle

$$u^2+\left(v+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

If $y = 2,$

then, $$-\frac{v}{u^2+v^2} = 2$$

or, $$u^2 + v^2 + \frac{2v}{2} = 0$$

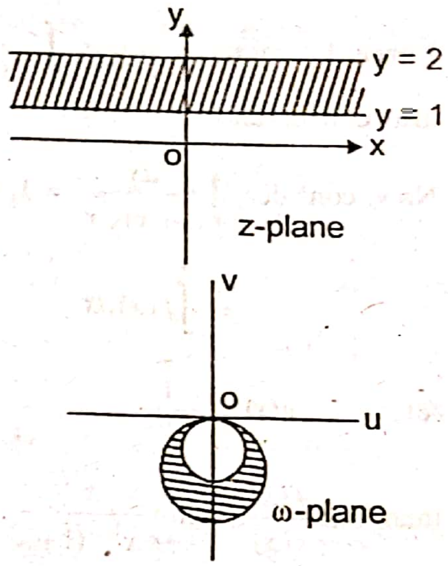
$$\Rightarrow \quad u^2+\left(u+\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^2$$

then, $y = 2$ is transformed to circle

$$u^2+\left(v+\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^2$$

Hence, the infinite strip $1 < y < 2$ in the z-plane is transformed into the region common to circle $u^2+\left(v+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ and $u^2+\left(v+\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^2$ is the ω-plane.

The two regions are shown in the following figure.



3. (a): The given integral is

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \text{I (say)}$$

clearly, 0 and ∞ are the point of infinite discontinuity

or, $$\text{I} \int_0^1 \frac{dx}{(1+x)\sqrt{x}} + \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$$

Clearly, I will be convergent iff both the integral on the right hand sides are seperately convergent

let $$\text{I}_1 = \int_0^1 \frac{dx}{(1+x)\sqrt{x}} = \int_0^1 f(x)\,dx$$

where $$f(x) = \frac{1}{(1+x)\sqrt{x}}$$

let $$g(x) = \frac{1}{\sqrt{x}}$$

then, $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1}{(1+x)} = 1.$$

<a finite number>

Also $$\int_0^1 g(x)dx = 2x^{1/2}\Big|_0^1 = 2$$

<converges>

Hence. by comparsion test $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ is also convergent.

Now, consider $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = I_2$ (say)

$$= \int_1^\infty f(x)\,dx$$

let $g(x) = \frac{1}{x^{3/2}}$

then, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{3/2}}{x^{1/2}(1+x)}$

$$= \lim_{x\to\infty} \frac{x^{3/2}}{x^{3/2}\left(1+\frac{1}{x}\right)}$$

$= 1$ <a finite number>

also $\int_1^\infty g(x)\,dx = 2$ <convergent>

Hence, by comparison test, $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ is also convergent

i.e., $I = \int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent.

Now consider $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

put $y = \frac{1}{1+x}$

then, $x = \frac{1}{y} - 1$

or, $dx = -\frac{1}{y^2}dy$

At $x = 0$, $y = 1$ and at $x = \infty$, $y = 0$.
Hence, the above integral can be rewritten as

$$\int_1^0 \frac{y}{\left(\frac{1}{y}-1\right)^{1/2}}\left(-\frac{1}{y^2}dy\right) = \int_1^0 y^{\frac{-1}{2}}(1-y)^{\frac{-1}{2}}dy$$

$$= \int_1^0 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{2}-1}\,dy = B\left(\frac{1}{2},\frac{1}{2}\right)$$

<Beta function>

$$= \frac{\boxed{\left(\frac{1}{2}\right)}\boxed{\left(\frac{1}{2}\right)}}{\boxed{\left(\frac{1}{2}+\frac{1}{2}\right)}} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{1} = \pi.$$

(b): From question,

$O(G) = 2p$. (where p is prime)

Then, by the Sylow's 1st theorem it must have p-sylow subgroup of order p.

And the number of such sylow subgroups are given by $1 + Kp$ such that $1 + Kp/O(G)$, where $K = 0, 1, 2......$

Clearly, only $K = 0$ satisfy the above equation.

i.e., there exists exactly one p-sylow subgroup of G. Say H, $O(H) = p$.

Hence, H is normal subgroup.

Also by Cauchy's theorem, G has subgroups of order 2 and p.

i.e., G has an element of order 2 and an element of order p. Clearly, G can be generated by a and b.

Also $\langle a \rangle$ has index 2, it is normal and therefore.

$bab^{-1} = a^k$ for some positive integer k less than p.

Thus $a^{k^2} = \left(a^k\right)^k = \left(bab^{-1}\right)^k = ba^kb^{-1}$

$= b.bab^{-1}b^{-1} = b^2ab^{-2}$

Also $O(b) = 2 \Rightarrow a^{k^2} = a \Rightarrow a^{k^2-1} = e.$

Also $O(a) = p.$

It implies that p divides $K^2 - 1 = (K-1)(K+1)$.

Since $1 \le K < p$, this implies that $K - 1 = 0$ or $K + 1 = p$.

In the first case, we have $K = 1$.

$\Rightarrow \quad bab^{-1} = a \quad$ or $\quad ba = ab$

Also, $\quad O(a) = 2, \quad O(b) = p$

i.e., $\quad (2, p) = 1 \quad$ and $\quad ab \in G$

$O(ab) = 2p.$

Also, $\quad O(G) = 2p \quad$ and $ab \in G$.

Hence, G is a cyclic group.

In the second case, we have $K + 1 = p$.

i.e., $2p$ elements of G can be written in the form of $a^i b^j$ where $O \le i \le 1$ and $O \le j \le p - 1$.

Again, $bab^{-1} = a^K$

or $\quad bab^{-1}.a = a^{K+1} = a^p = e$

or $\quad bab^{-1} = a^{-1}$

Also, $\quad O(b) = 2$

$\Rightarrow \quad bab^{-1} = a^{-1}$

$\Rightarrow \quad bab = a^{-1} \qquad [\because b = b^{-1}]$

Hence the result.

(c): We know that any solution to a L.P.P. which satisfies the non-negativity restrictions of the problem is called its **feasible solution** on the other hand, when there are m constraints and $(m + n)$ variables (m being $\le n$), the starting solution is found by setting n variables equal to zero and then solving the remaining m equations, provided the solution exists and is unique. The n zero variables are known as **non-basic variables** while the remaining m variables are called **basic variables** and they form a **basic solution.**

Now from question,

The given constraints are

$$\left.\begin{aligned} x_1 - x_2 + 3x_3 &= 4 \\ 2x_1 + x_2 + x_3 &= 6 \end{aligned}\right] \qquad \text{...(1)}$$

clearly, it has 2 constraints and 3 variables. Hence, the basic solutions are founded first by making any one of x_1, x_2, x_3 equal to zero.

Choose $x_1 = 0$ <as the non-basic variables> then (1) reduces to

$$-x_2 + 3x_3 = 4$$
$$x_2 + x_3 = 6$$

$\Rightarrow \quad 4x_3 = 10$

$\Rightarrow \quad x_3 = \frac{5}{2}$

$\Rightarrow \quad x_2 = 6 - \frac{5}{2} = \frac{7}{2}$

Clearly $x_2, x_3 \ge 0$

Also the **objective function**

i.e., This form a basic solution, where $x_1 = 0, x_2 = \frac{7}{2}, x_3 = \frac{5}{2}$.

And $z = x_1 + 2x_2 + 3x_3 = \frac{2.7}{2} + \frac{3.5}{2} = \frac{29}{2}$.

Choosing $x_2 = 0$ <as non-basic variables> then (1) reduces to

$$x_1 + 3x_3 = 4$$
$$2x_1 + x_3 = 6$$

Solving, we get

$$x_3 = \frac{2}{5}, \quad x_1 = \frac{14}{5}$$

Clearly, it also form basic solution with $x_1 = \frac{14}{5}, x_2 = 0, x_3 = \frac{2}{5}$.

And $\quad z = x_1 + 2x_2 + 3x_3$

$$= \frac{14}{5} + 0 + \frac{6}{5} = 4.$$

Choosing $x_3 = 0$,

(1) reduces to

$$x_1 - x_2 = 4$$
$$2x_1 + x_2 = 6$$

$\Rightarrow \quad 3x_1 = 10 \qquad x_1 = \frac{10}{3}$

$\therefore \quad \frac{10}{3} - x_2 = 4 \quad \Rightarrow x_2 = -\frac{2}{3}$

As $x_2 < 0$ it does not form a basic feasible solution.

i.e., Basic feasible solutions of (1) are

$x_1 = 0, \quad x_2 = \frac{7}{2}, \quad x_3 = \frac{5}{2}$

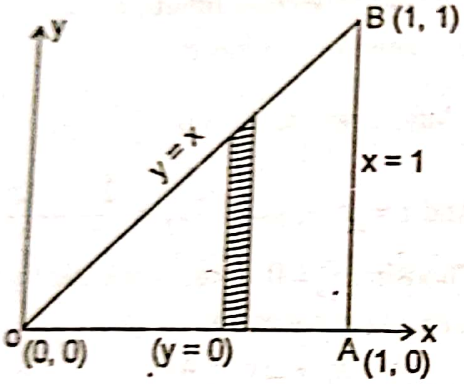
$$x_1 = \frac{14}{5}, \quad x_2 = 0, \quad x_3 = \frac{2}{5}$$

Out of these two first one also form the optimal solution of (1) for which z has optimal value.

4. (a): The given integral is

$$I = \iint \sqrt{4x^2 - y^2}\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{4x^2 - y^2}\, dx\, dy$$



$$= \int_0^1 \left[\frac{y\sqrt{4x^2 - y^2}}{2} + \frac{4x^2}{2} \sin^{-1}\left(\frac{y}{2x}\right) \right]_0^x dx$$

$$= \int_0^1 \left[\frac{x\sqrt{4x^2 - x^2}}{2} + \frac{4x^2}{2} \sin^{-1}\left(\frac{x}{2x}\right) \right] dx$$

$$= \int_0^1 \left(\frac{\sqrt{3}}{2} x^2 + \frac{\pi}{3} x^2 \right) dx$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \int_0^1 x^2\, dx = \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \cdot \frac{1}{3}$$

$$= \frac{(2\pi + 3\sqrt{3})}{18}.$$

(b): Cauchy's Residue Theorem

If $f(z)$ is regular, except at a finite number of poles within a closed contour C and continuous on the boundary of C, then

$$\int_C f(z)\, dz = 2\pi i \Sigma R,$$

where ΣR is the sum of the residues of $f(z)$ at its poles within C.

Now, the given integral

$$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)}\, dz$$

The poles of integrand we given by roots of $(z + 2)(z^2 - 4) = 0 \Rightarrow z = 2, -2, -2$.

Also the given the circle $|z + 1| = 4$

Clearly all the poles within the closed contour C.

Now residue at $z = 2$

$$\lim_{z \to 2} \frac{(z-2)e^{z/2}}{(z+2)(z^2-4)} = \lim_{z \to 2} \frac{e^{z/2}}{(z+2)^2}$$

$$= \frac{e^{2/2}}{(2+2)^2} = \frac{e}{4}.$$

Also residue at $z = -2$

$$\lim_{z \to -2} \frac{d}{dz} \left(\frac{e^{z/2}}{z-2} \right)$$

$$= \lim_{z \to -2} \frac{(z-2)e^{z/2} \cdot \frac{1}{2} - e^{z/2} \cdot 1}{(z-2)^2}$$

$$= \frac{\left(\frac{-4}{2}\right) e^{-1} - e^{-1}}{16} = \frac{-3}{16e}$$

Hence the required integral is

$$I = \int \frac{e^{z/2}}{(z+2)(z^2-4)}\, dz \; 2\pi i \left[\frac{e}{4} - \frac{3}{16e} \right]$$

$$= \frac{\pi i}{2} \left[e - \frac{3}{4e} \right].$$

(c):

	M_1	M_2	M_3	M_4	Supply (quintals)
F_1	(4) 29	40	60	(3) 20	7
F_2	80	40	(8) 50	(2) 70	10
F_3	50	(8) 18	80	(10) 30	18
Demand (quintals)	4	8	8	15	35

Here, the total demand and the total supply being the same *i.e.* 35, the problem is balanced.

The initial basic feasible solution is obtained by Vogel Approximation Method <VAM>.

i.e. the difference between the smallest and next to the smallest in each row and each computed are computed and displayed within brackets against the respective rows and column. Clearly, largest of these difference is (22), which is associated with the second column.

Table 1

29	40	60	20	7	(9)
80	40	50	70	10	(10)
50	(8) 18	80	30	18	(12)
4	8	8	15		
(21)	(22)	(10)	(10)		
	↑				

Clearly, C_{32} (=18) is the minimum cost, we allocate $x_{32} = \min(8, 18) = 8$. This exhaust the demand at second column and therefore we cross it.

Table 2

(4) 29	60	20	7	(9)
80	50	70	10	(20)
50	80	30	10	(20)
4	8	15		
(21)	(10)	(10)		

The row and column differences are computed for reduced table and displayed within brackets.

60	20	3	(40)
50	70	10	(20)
80	(10) 30	10	(50)
8	15.		
(10)	(10)		

60	20	3	(40)
50	70	10	(20)
8	5		

(8) 50	(2) 70	10
8	2	

Hence, the initial optimal solution.

$29 \times 4 + 20 \times 3 + 8 \times 50 + 2 \times 70 + 8 \times 18 + 10 \times 30 = 1160.$

Section-B

5. (a): The given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

The characteristic equation is given by,

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1, -1$$

Hence, $\frac{dy}{dx} - 1 = 0 \quad \Rightarrow \quad y - x = \text{constant}$

Choose $\quad u = y + x$

and $\quad v = y - x$

as the parameter then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} = \left(\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial v}\right)z$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)z$$

Now, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)z$$

$$= \left(\frac{\partial^2}{\partial u^2} - \frac{2\partial^2 u}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}\right)z$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Similarly, $$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)z$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

and, $$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)z$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}$$

putting these values in the parent equation, we get

$$\frac{\partial^2 u}{\partial v^2} - \frac{2\partial^2 u}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{2\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial v^2} + \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u^2} = 0$$

integrating,

Hence, $$\frac{\partial z}{\partial u} = \phi(v)$$

$$\Rightarrow \quad z = u(\phi(v)) + \psi(v)$$

or, $$z = (y + x)\,\phi(y - x) + \psi(y - x)$$

is the required solution.

(b): The given data are

x	0	1	2	5
$f(x)$	2	3	12	147

Clearly, from Lagrange interpolation, we will get a cubic polynomial in x.

Using Lagrange interpolation, we have

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\cdot 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\cdot 3 + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

$$= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2(x^3 - 6x^2 + 5x) + \frac{49}{20}(x^3 - 3x^2 + 2x)$$

$$= x^3\left(\frac{49}{20} - 2 + \frac{3}{4} - \frac{1}{5}\right) + x^2\left(\frac{8}{5} - \frac{21}{4} + 12 - \frac{147}{20}\right) + \left(\frac{15}{2} - 10 + \frac{49}{10} - \frac{17}{5}\right)x + 2$$

$$f(x) = x^3 + x^2 - x + 2$$

i.e. $f(x) = x^3 + x^2 - x + 2$ is the required polynomial.

(c): The given system of equations are

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{27}(85 + z - 6y) \\ y &= \frac{1}{15}(72 - 6x - 2z) \\ z &= \frac{1}{54}(110 - x - y) \end{aligned}\right\} \quad \ldots(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting on the right sides of the equations (i), we get

$x_1 = 3.148$, $y_1 = 4.8$ $z_1 = 2.037$.

Putting these values in the right sides of equation (i), we get

$$x_2 = \frac{1}{27}(85 + 2.037 - 6 \times 4.8) = 2.156$$

$$y_2 = \frac{1}{15}(72 - 6 \times 3.148 - 2 \times 2.037) = 3.2692$$

$$z_2 = \frac{1}{54}(110 - 3.148 - 4.8) = 1.8898$$

Putting these value of the right sides of equation (i), we get

$$x_3 = \frac{1}{27}(85 + 1.8898 - 6 \times 3.2692) = 2.491$$

$$y_3 = \frac{1}{15}(72 - 6 \times 2.156 - 2 \times 1.8898) = 3.6856$$

$$z_3 = \frac{1}{54}(110 - 2.156 - 3.2692)$$
$$= 1.93657$$

Similarly,

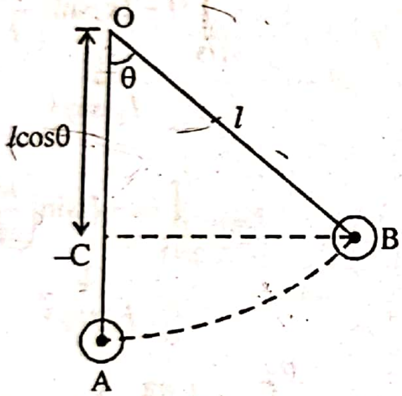
$x_4 = 2.40086$, $y_4 = 3.67933$, $z_4 = 1.92265$
$x_5 = 2.40172$, $y_5 = 3.583$, $z_5 = 1.9244$
$x_6 = 2.4232$, $y_6 = 3.582$, $z_6 = 1.926$

is the required answer.

(d): For simple pendulum,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

let at A, the potential energy of the bob be zero.



So, from $A \to B$, there is a rise in the height of the bob by $l(1 - \cos\theta)$

$\therefore \quad V = mgl(1 - \cos\theta)$

Thus $\quad L = T - V$

$$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Now by Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\theta}}\left\{\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right\}\right]$$

$$-\frac{\partial}{\partial \theta}\left\{\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right\} = 0$$

$$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

If θ is very small, $\sin\theta \approx \theta$

$$\therefore \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

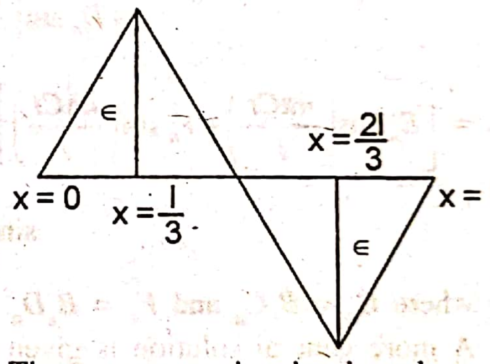
which is the equation for SHM with the time period $T = 2\pi\sqrt{\frac{l}{g}}$.

6. (a): The initial deflection is given by

$$f(x) = \frac{3\epsilon x}{l} \quad \text{when} \quad 0 \le x \le \frac{l}{3}$$

$$= \frac{3\epsilon(l-2x)}{l} \quad \text{when} \quad \frac{l}{3} \le x \le \frac{2l}{3}$$

$$= \frac{3\epsilon(x-l)}{l} \quad \text{when} \quad \frac{2l}{3} \le x \le l$$



The wave equation is given by,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

By separation of variables

$$u(x, t) = X(x)\,T(t)$$

Putting these in equation (1), we get

$$X''T = \frac{1}{c^2}T''X \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{C^2T}$$

As L.H.S. of the above equation is a function of x only whereas on the R.H.S. of the above equation is a function of t only. Hence, both must be equated to a constant. It can be further shown that for non-trivial, it must be equal to a negative number

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{C^2T} = -\lambda^2 \text{ (say)}$$

Now, $\quad \dfrac{X''}{X} = -\lambda^2 \Rightarrow X'' = -\lambda^2 X$

or, $\quad X(x) = A\cos\lambda x + B\sin\lambda x$

Now from equation, $u(o, t) = 0$ and $u(l, t) = 0$ and $u_t(x, o) = 0$

Now, $x(o) = 0 \Rightarrow A = 0$

and $X(l) = 0 \Rightarrow \lambda = \frac{n\pi}{l}$

$$\Rightarrow X(x) = \mathrm{B}\sin\left(\frac{n\pi}{l}x\right)$$

Similarly,

$$T(t) = C\cos\frac{n\pi Ct}{l} + D\sin\frac{n\pi Ct}{l}$$

$$\Rightarrow u_n(x, t) = X_n(x)$$

$$T_n(t) = B_n \sin\left(\frac{n\pi x}{l}\right)\left[C_n \cos\left(\frac{n\pi Ct}{l}\right) + D_n \sin\left(\frac{n\pi C_t}{l}\right)\right]$$

$$= \left[E_n \cos\left(\frac{n\pi Ct}{l}\right) + F_n \sin\left(\frac{n\pi Ct}{l}\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

where $E_n = B_nC_n$ and $F_n = B_nD_n$.

A more general solution is given by

$$u(x, t) = \sum_{n=1}^{\infty}\left[E_n \cos\left(\frac{n\pi Ct}{l}\right) + F_n \sin\left(\frac{n\pi Ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

Also, $u_t(x, o) = 0 \Rightarrow F_n = 0$

Thus, the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi Ct}{l}\right)\sin\left(\frac{n\pi}{l}x\right)$$

$$\Rightarrow u(x, o) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\therefore \quad E_n = \frac{2}{l}\int_0^l u(x,o)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/3}\frac{3\in x}{l}\sin\left(\frac{n\pi}{l}x\right)dx + \int_{l/3}^{2l/3}\frac{3\in}{l}(l-2x)\sin\left(\frac{n\pi}{l}x\right)dx + \int_{2l/3}^{l}\frac{3\in}{l}(x-l)\sin\left(\frac{n\pi}{l}x\right)dx\right]$$

$$E_n = \frac{2}{l}\cdot\frac{3\in}{l}\left[\underbrace{\int_0^{l/3} x\sin\left(\frac{n\pi}{l}x\right)dx}_{I_1\text{(say)}} + \underbrace{\int_{l/3}^{2l/3}(l-2x)\sin\left(\frac{n\pi}{l}x\right)dx}_{I_2\text{(say)}} + \underbrace{\int_{2l/3}^{l}(x-l)\sin\left(\frac{n\pi}{l}x\right)dx}_{I_3\text{(say)}}\right]$$

then, $$I_1 = \int_0^{l/3} x\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \left.\frac{-x\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^2}\right|_0^{l/3}$$

$$= \frac{-\frac{l}{3}\cos\left(\frac{n\pi}{3}\right)}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2}$$

$$I_2 = \int_{l/3}^{2l/3}(l-2x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \left.\frac{-(l-2x)\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}} + \frac{2\sin\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^2}\right|_{l/3}^{2l/3}$$

$$= \frac{\frac{l}{3}\cos\left(\frac{2n\pi}{3}\right)}{\frac{n\pi}{l}} + \frac{2\sin\left(\frac{2n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2} + \frac{\frac{l}{3}\cos\left(\frac{n\pi}{3}\right)}{\frac{n\pi}{l}} - \frac{2\sin\left(\frac{n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2}$$

and

$$I_3 = \int_{2l/3}^{l} (x-l)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \left[\frac{-(x-l)\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_{2l/3}^{l}$$

$$= \frac{-\frac{l}{3}\cos\left(\frac{2n\pi}{3}\right)}{\frac{n\pi}{l}} - \frac{\sin\left(\frac{2n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2}$$

$$E_n = \frac{6\in}{l^2}\left[-\frac{l}{3}\frac{\cos\left(\frac{n\pi}{3}\right)}{\frac{n\pi}{l}} + \frac{\sin\left(\frac{n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2} + \frac{l}{3}\frac{\cos\left(\frac{2n\pi}{3}\right)}{\frac{n\pi}{l}} + \frac{2\sin\left(\frac{2n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2} + \frac{l}{3}\frac{\cos\left(\frac{n\pi}{3}\right)}{\frac{n\pi}{l}} - \frac{2\sin\left(\frac{n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2} - \frac{l}{3}\frac{\cos\left(\frac{2n\pi}{3}\right)}{\frac{n\pi}{l}} - \frac{\sin\left(\frac{2n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{6\in}{l^2}\left[\frac{\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{6\in}{l^2}\cdot\frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(n\pi - \frac{2n\pi}{3}\right)\right]$$

$$= \frac{6\in}{n^2\pi^2}\left[\sin\left(\frac{2n\pi}{3}\right) - \left\{\sin n\pi\cdot\cos\frac{2n\pi}{3} - \cos n\pi\cdot\sin\left(\frac{2n\pi}{3}\right)\right\}\right]$$

$$= \frac{6\in}{n^2\pi^2}\left[1+(-1)^n\right]\sin\left(\frac{2n\pi}{3}\right)$$

$\Rightarrow \quad E_n = 0 \quad$ if $\quad n =$ odd.

and if n = even say $n = 2m$, then

$$E_n = E_{2m} = \frac{6\in}{4m^2\pi^2}\cdot 2\sin\left(\frac{4m\pi}{3}\right)$$

$$= \frac{3\in}{m^2\pi^2}\sin\left(\frac{4m\pi}{3}\right)$$

$$\therefore u(x, t) = \frac{3\in}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\left(\frac{4m\pi}{3}\right)\cdot \sin\left(\frac{6m\pi}{l}x\right)\cos\left(\frac{6m\pi Ct}{l}\right).$$

Clearly for $x = \frac{l}{2}$

$u(x, t) = 0$ *i.e.* mid point will always be at rest.

(c): Let $\quad f(x) = x^3 - 6x + 4$

then $\quad f(0) = 4 \quad$ and $\quad f(1) = -1 < 0.$

i.e., the smallest positive root of $f(x)$ lie between 0 and 1.

Let that root be $x = 0.5$

then by Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 - 6x_n + 4)}{(3x_n^2 - 6)}$$

$$= \frac{3x_n^2 - 6x_n - x_n^3 + 6x_n - 4}{3(x_n^2 - 2)}$$

$$= \frac{2x_n^3 - 4}{3(x_n^2 - 2)}$$

$$x_{n+1} = \frac{2x_n^3 - 4}{3(x_n^2 - 2)}$$

put $x_0 = 0.5$

then, $x_1 = 0.71428$

$x_2 = 0.731898$

$x_3 = 0.732050$

$x_4 = 0.732050$

i.e., smallest positive root of $f(x) = 0$ is 0.732050.

Let $x = \sqrt{3}$

then, $x^2 = 3$

or, $x^2 - 3 = 0$

Let $x = 1.5$ be the initial root of $x^2 - 3 = 0$, then by Newton-Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_{n}^2 - 3)}{2x_n} = \frac{x_n^2 + 3}{2x_n}$$

or, $$x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

put $x_0 = 1.5$

then, $x_1 = 1.75$

$x_2 = 1.7321428$

$x_3 = 1.73205081$

$x_4 = 1.732050808$

Hence, the square root of 3 = 1.7321.

7. (a): The Poiseuille expression gives the rate of flow of liquid through a narrow tube the following assumption.

(*i*) the flow is streamline or steady.

(*ii*) a constant pressure difference is applied.

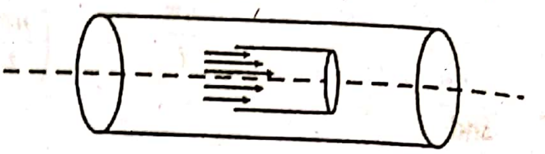
(*iii*) there is no radial motion.

(*iv*) the fluid in contact with the walls are stationary.

Derivation of the formula $\omega(R) = \frac{1}{4}\left(\frac{p}{\mu}\right)(a^2 - R^2)$

let a^2 liquid of coefficient of viscosity μ is flowing through a narrow horizontal tube of radius r and length l.

As the velocity of liquid in contact with the walls of the tube is zero and goes on increasing as the axis is approached, where it is maximum.



Consider a coaxial cylinder of radius x and where all the fluid have velocity v.

Then, from Newton's law of viscous flow, the backward dragging force on the imaginary liquid shell is given by

$F = \mu A \frac{dv}{dx}$, where A is the surface area of the cylinder which is equal to $2\pi x l$. Now, if the pressure difference across the two ends of the tube is P, the force on the liquid shell accelerating it forwards $= P.\pi x^2$ (πx^2 = cross-sectional area of the shell).

For steady flow,

driving force = backward dragging force

i.e., $$P.\pi x^2 = -(2\pi x l)\ \mu \frac{dv}{dx}$$

or, $$\int_0^v dv = \frac{-P}{2\mu l}\int_a^x x\,dx$$

$$v(x) = \frac{P}{4\mu l}(a^2 - x^2)$$

$$\Rightarrow \quad v(x) = \frac{p}{4\mu}(a^2 - x^2)$$

where, $p = \frac{P}{l}$ = Pressure gradient

$$\Rightarrow \quad v(R) = \frac{p}{4\mu}(a^2 - R^2)$$

Hence the result.

(b): The given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

Here, the auxiliary equation is given by,

$$m^2 - 1 = 0$$

$$\Rightarrow \qquad m = \pm 1.$$

$\therefore$ Complementary function $= \phi_1(x + y) + \phi_2(x - y)$;

Now, the particular integral is given by

$$z = \frac{1}{D^2 - D_1^2}(x - y)$$

$$= \frac{1}{D^2\left(1 - \frac{D_1^2}{D^2}\right)}(x - y)$$

$$= \frac{1}{D^2}\left(1 - \frac{D_1^2}{D^2}\right)^{-1}(x - y)$$

$$= \frac{1}{D^2}\left(1 + \frac{D_1^2}{D^2} + \frac{D_1^4}{D^4} + \ldots\right)(x - y)$$

$$= \frac{1}{D^2}(x - y) = \frac{x^3}{6} - \frac{xy^2}{2}.$$

Hence, the general solution is given by,

$$z = \phi_1(x + y) + \phi_2(x - y) + \frac{x^3}{6} - \frac{xy^2}{2}.$$

(c): The velocity of the particle at time t is given by

t(second) :	0	2	4	6	8	10	12
v(m/sec) :	4	6	16	36	60	94	136

As, $\qquad \frac{ds}{dt} = v$

$$\Rightarrow \qquad \int ds = \int v\, dt$$

or, $\qquad \int_0^s ds = \int_0^{12} V\, dt$

Now, by the Simpson's one third rule, we know that

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4\cdot(y_1 + y_3 + \ldots y_{n-1}) + 2\cdot y_2 + y_4 + \ldots y_{n-2})$$

here, $\qquad h = 2, \qquad n = 6$

$\qquad x_0 = 0, \qquad x_n = 12$

$$\therefore \qquad \int_0^{12} v\,dt = \frac{2}{3}[(4 + 136) + 4(6 + 36 + 94) + 2.(16 + 60)]$$

$$= \frac{2}{3}[140 + 544 + 152]$$

$$= 557.33 \text{ m}$$

2nd part

t	v	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	4	2	8	2	–8	20	–36
2	6	10	10	–6	12	–16	
4	16	20	4	6	–4		
6	36	24	10	2			
8	60	34	8				
10	94	42					
12	136						

Now, $$\left|\frac{dv}{dt}\right|_{t_0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

We have $h = 2$

$t_0 = 2 \qquad \Delta v_0 = 10$

$\Delta^2 v_0 = 10, \qquad \Delta^3 v_0 = -6,$

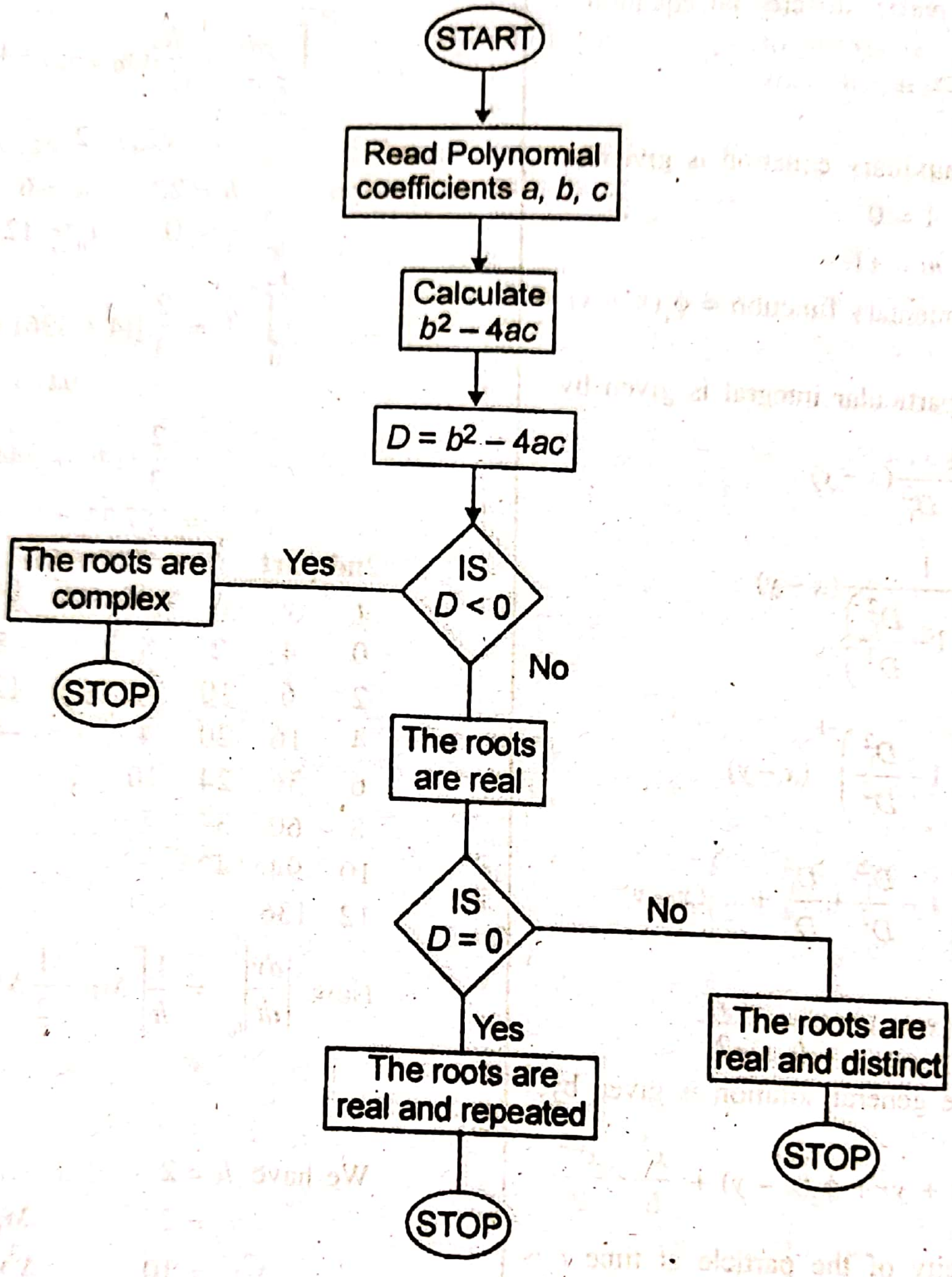
$\Delta^4 v_0 = 12, \qquad \Delta^5 v_0 = -16$

$$\therefore \left.\frac{dv}{dt}\right|_{t=2} = \frac{1}{2}\left[10 - \frac{1}{2}\times 10 + \frac{1}{3}\times(-6) - \frac{1}{4}\cdot 12 - \frac{1}{5}\cdot 16\right]$$

$$= \frac{1}{2}\left[10 - 5 - 2 - 3 - \frac{16}{5}\right] = -1.6 \text{ m/sec}^2$$

i.e., acceleration at $t = 2$ second $= -1.6$ m/sec^2.

8. (b):



(c): We know that $\nabla \cdot \bar{q} = 0$

<by continuity equation for an incompressible fluid motion>

or, $$k^2\left\{\frac{-\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\}=0$$

or, $$k^2\left\{\frac{2xy}{(x^2+y^2)^2}-\frac{2xy}{(x^2+y^2)^2}\right\}=0$$

i.e., Equation of continuity for an incompressible fluid is satisfied and hence is a possible motion.

The equation of the stream lines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$=\frac{dx}{-\frac{k^2y}{(x^2+y^2)}}=\frac{dy}{\frac{k^2x}{(x^2+y^2)}}=\frac{dz}{o}$$

or, $xdx + ydy = 0$, $dz = 0$

By integrating, we have

$x^2 + y^2 =$ constant, $z =$ constant

i.e., Thus, the streamlines are circles, whose centres are on z-axis, their planes being perpendicular to the axis

$$\text{Also, } \bar{\nabla}\times\bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-k^2 y}{x^2 - y^2} & \frac{k^2 x}{x^2 + y^2} & 0 \end{vmatrix}$$

or,

$$\bar{\nabla}\times\bar{q} = \hat{k}\left[\frac{\partial}{\partial x}\left\{\frac{k^2 x}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{k^2 y}{x^2+y^2}\right\}\right]$$

$$\text{or, } \bar{\nabla}\times\bar{q} = \hat{k}K^2\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

$$= 0$$

Thus, the flow is of potential kind, so we can determine $\phi(x, y, z)$ such that

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \qquad ...(1)$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2} \qquad ...(2)$$

$$\frac{\partial\phi}{\partial z} = -w = 0$$

which shows that ϕ is independent of z, hence, $\phi = \phi(x, y)$.

Integrating the relation (1), we have

$$\phi(x, y) = f(y) + k^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2}$$

using relation (2), we get

$$f'(y) = 0$$

$$\Rightarrow \quad f(y) = \text{constant}$$

Therefore,

$$\phi(x, y) = k^2 \tan^{-1}\left(\frac{x}{y}\right).$$

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