3d) Show that the improper integral

$$\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx \text{ is convergent.}$$

(10)

Let $f(x) = \dfrac{\sin 1/x}{\sqrt{x}}$

We note that $f(x)$ does not keep the same sign near lower limit 0.

Now, $|f(x)| = \left|\dfrac{\sin 1/x}{\sqrt{x}}\right|$

$$= \frac{|\sin 1/x|}{|\sqrt{x}|} \leq \frac{1}{\sqrt{x}} \quad \forall x \in (0,1]$$

$$(\because |\sin 1/x| \leq 1)$$

Since $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx$ is convergent at '0'.

$$(\because n = 1/2 < 1)$$

Hence, by comparision test

$\displaystyle\int_0^1 |f(x)|\,dx$ is convergent at '0'.

Also, abosute convergence implies convergence.

$\therefore \displaystyle\int_0^1 f(x)\,dx$ is convergent.