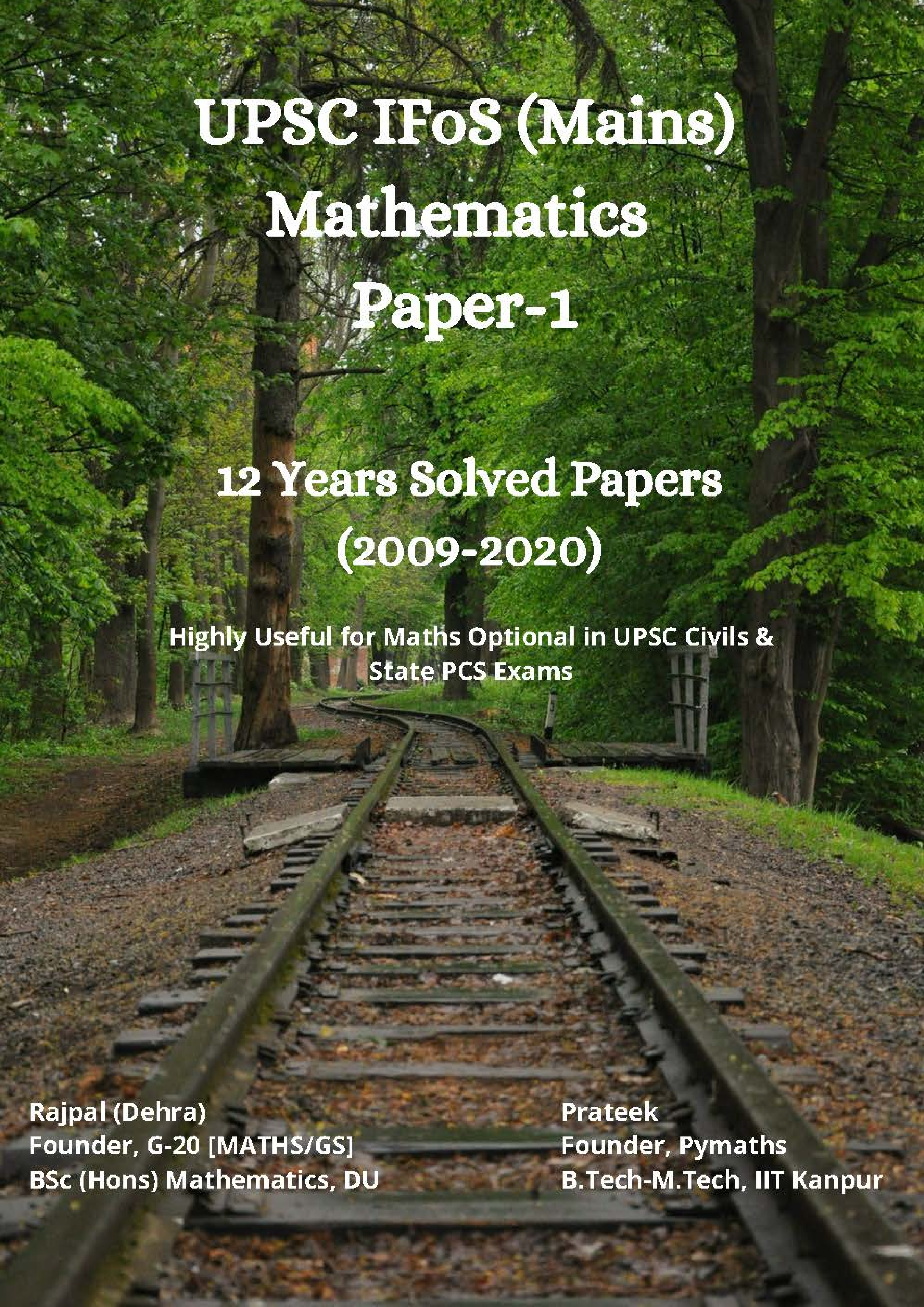

UPSC IFoS (Mains) Mathematics Paper-1

12 Years Solved Papers (2009-2020)

Highly Useful for Maths Optional in UPSC Civils & State PCS Exams

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Preface to the First Edition

This book has been designed for students preparing for the Indian Forest Services (IFoS) examination with Mathematics as the optional subject. The solutions of all the previous year problems from Mathematics Paper-I have been covered in this book for the years 2009-2020. All the solutions are comprehensively covered in a lucid manner.

This book will also be helpful for students preparing for Civil Services Examination (CSE) with Mathematics as the optional subject. We have strived to keep the solutions detailed and free from errors. But as the saying goes, "to err is human", so this book may have a few unwanted errors. We will highly appreciate your feedback regarding our efforts and will progressively incorporate it in our upcoming editions.

-Authors

Acknowledgment

We would like to thank our parents who have always been the guiding light of our journey and our source of inspiration for writing this book. We would also like to express our gratitude towards Shivani Kalyan, Bhavya, Prashant, Mahender and Karan, who helped us a lot in the proofreading, structuring, designing and formatting of the book.

- Authors

Syllabus

Paper-I

Section-A

Linear Algebra: Vector, space, linear dependence and independence, subspaces, bases, dimensions. Finite dimensional vector spaces. Matrices, Cayley-Hamilton theorem, Eigenvalues and Eigenvectors, matrix of linear transformation, row and column reduction, Echelon form, equivalence, congruence and similarity, reduction to canonical form, rank, orthogonal, symmetrical, skew symmetrical, unitary, hermitian, skew-hermitian forms and their eigenvalues. Orthogonal and unitary reduction of quadratic and hermitian forms, positive definite quadratic forms.

Calculus:
Real numbers, limits, continuity, differentiability, mean-value theorems, Taylor's theorem with remainders, indeterminate forms, maxima and minima, asymptotes. Functions of several variables: continuity, differentiability, partial derivatives, maxima and minima, Lagrange's method of multipliers, Jacobian. Riemann's definition of definite integrals, indefinite integrals, infinite and improper integrals, beta and gamma functions. Double and triple integrals (evaluation techniques only). Areas, surface and volumes, centre of gravity.

Analytic Geometry:
Cartesian and polar coordinates in two and three dimensions, second degree equations in two and three dimensions, reduction to canonical forms, straight lines, shortest distance between two skew lines, plane, sphere, cone, cylinder, paraboloid, ellipsoid, hyperboloid of one and two sheets and their properties

Section-B

Ordinary Differential Equations: Formulation of differential equations, order and degree, equations of first order and first degree, integrating factor, equations of first order but not of first degree, Clairaut's equation, singular solution. Higher order linear equations with constant coefficients, complementary function and particular integral, general solution, Euler-Cauchy equation. Second order linear equations with variable coefficients, determination of complete solution when one solution is known, method of variation of parameters.

Dynamics, Statics and Hydrostatics: Degree of freedom and constraints, rectilinear motion, simple harmonic motion, motion in a plane, projectiles, constrained motion, work and energy, conservation of energy, motion under impulsive forces, Kepler's laws, orbits under central forces, motion of varying mass, motion under resistance. Equilibrium of a system of particles, work and potential energy, friction, common catenary, principle of virtual work, stability of equilibrium, equilibrium of forces in three dimensions. Pressure of heavy fluids, equilibrium of fluids under given system of forces Bernoulli's equation, centre of pressure, thrust on curved surfaces, equilibrium of floating bodies, stability of equilibrium, metacentre, pressure of gases.

Vector Analysis: Scalar and vector fields, triple, products, differentiation of vector function of a scalar variable, gradient, divergence and curl in cartesian, cylindrical and spherical coordinates and their physical interpretations. Higher order derivatives, vector identities and vector equations. Application to Geometry: Curves in space, curvature and torsion. Serret-Frenet's formulae, Gauss and Stokes' theorems, Green's identities.

Paper-II

Section-A

Algebra: Groups, subgroups, normal subgroups, homomorphism of groups quotient groups basic isomorphism theorems, Sylow's group, permutation groups, Cayley theorem. Rings and ideals, principal ideal domains, unique factorization domains and Euclidean domains. Field extensions, finite fields

Real Analysis: Real number system, ordered sets, bounds, ordered field, real number system as an ordered field with least upper bound property, Cauchy sequence, completeness, Continuity and uniform continuity of functions, properties of continuous functions on compact sets. Riemann integral, improper integrals, absolute and conditional convergence of series of real and complex terms, rearrangement of series. Uniform convergence, continuity, differentiability and integrability for sequences and series of functions. Differentiation of functions of several variables, change in the order of partial derivatives, implicit function theorem, maxima and minima. Multiple integrals.

Complex Analysis: Analytic function, Cauchy-Riemann equations, Cauchy's theorem, Cauchy's integral formula, power series, Taylor's series, Laurent's Series, Singularities, Cauchy's residue theorem, contour integration. Conformal mapping, bilinear transformations.

Linear Programming: Linear programming problems, basic solution, basic feasible solution and optimal solution, graphical method and Simplex method of solutions. Duality. Transportation and assignment problems. Travelling salesman problems.

Section-B

Partial differential equations: Curves and surfaces in three dimensions, formulation of partial differential equations, solutions of equations of type dx/p=dy/q=dz/r; orthogonal trajectories, Pfaffian differential equations; partial differential equations of the first order, solution by Cauchy's method of characteristics; Charpit's method of solutions, linear partial differential equations of the second order with constant coefficients, equations of vibrating string, heat equation, Laplace equation.

Numerical Analysis and Computer programming: Numerical methods: Solution of algebraic and transcendental equations of one variable by bisection, Regula-Falsi and Newton-Raphson methods, solution of system of linear equations by Gaussian elimination and Gauss-Jordan (direct) methods, Gauss-Seidel(iterative) method. Newton's (Forward and backward) and Lagrange's method. **Numerical integration:** Simpson's one-third rule, trapezoidal rule, Gaussian quadrature formula. **Numerical solution of ordinary differential equations:** Euler and Runge-Kutta methods

Computer Programming: Storage of numbers in Computers, bits, bytes and words, binary system. Arithmetic and logical operations on numbers. Bitwise operations. AND, OR , XOR, NOT, and shift/rotate operators. Octal and Hexadecimal Systems. Conversion to and from decimal Systems. Representation of unsigned integers, signed integers and reals, double precision reals and long integers. Algorithms and flow charts for solving numerical analysis problems. Developing simple programs in Basic for problems involving techniques covered in the numerical analysis.

Mechanics and Fluid Dynamics: Generalised coordinates, constraints, holonomic and non-holonomic, systems. D'Alembert's principle and Lagrange's equations, Hamilton equations, moment of inertia, motion of rigid bodies in two dimensions. Equation of continuity, Euler's equation of motion for inviscid flow, stream-lines, path of a particle, potential flow, two-dimensional and axi-symmetric motion, sources and sinks, vortex motion, flow past a cylinder and a sphere, method of images. Navier-Stokes equation for a viscous fluid.

Contents

Chapter 1

2020

1.1 Section-A

Question-1(a) If A is a skew-symmetric matrix and $I + A$ be a non-singular matrix, then show that $(I - A)(I + A)^{-1}$ is orthogonal.

[8 Marks]

Solution: Given A is a skew-symmetric matrix and $I + A$ is a non-singular matrix.

$$\begin{aligned}
\text{Let} \quad M &= (I - A)(I + A)^{-1} \\
\Rightarrow \quad M^T &= \left[(I - A)(I + A)^{-1}\right]^T \\
&= \left[(I + A)^{-1}\right)^T (I - A)^T \\
&= \left((I + A)^T\right]^{-1} \left(I^T - A^T\right) \\
&= (I - A)^{-1}(I + A) \\
& \qquad [\because \quad A \quad \text{is skew symmetric} \quad \Rightarrow A^T = -A] \\
\therefore \quad M^T M &= (I - A)^{-1}(I + A)(I - A)(I + A)^{-1} \\
&= (I - A)^{-1}(I - A)(I + A)(I + A)^{-1} \\
&= I \cdot I = I \\
& \begin{pmatrix} \because (I + A)(I - A) = I + A - A - A^2 = I - A^2 \\ (I - A)(I + A) = I - A + A - A^2 = I - A^2 \end{pmatrix}
\end{aligned}$$

Thus, $M = (I - A)(I + A)^{-1}$ is orthagonal.

Question-1(b) By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix},$$

reduce it to a row-reduced echelon matrix. Hence find the rank of A.

[8 Marks]

Solution:

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1, \quad R_4 \to R_4 + R_1$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_3, \quad R_1 \to R_1 + R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 0 & 5 \\ 0 & 8 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row-reduced echelon form which has three non-zero rows.
Hence, $\text{rank}(A) = 3$.

Question-1(c) Given that $f(x+y) = f(x)f(y), f(0) \neq 0,$ for all real x, y and $f'(0) = 2$.
Show that for all real $x, f'(x) = 2f(x)$. Hence find $f(x)$.

[8 Marks]

Solution: Given, $f(x+y) = f(x)f(y)$.

Let $x = 0$, $y = 0$.

$$\Rightarrow f(0) = f(0)\cdot f(0)$$
$$f(0)[f(0)-1] = 0$$
$$f(0) = 1 \qquad \ldots(1)[\because f(0)\neq 0 \text{ given}]$$

By definition of derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \\ &= f(x) \quad \lim_{h\to 0}\frac{f(h)-1}{h} \\ &= f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h} \qquad [\text{ using } (1)] \\ &= f(x)\cdot f'(0) = 2f(x) \qquad (\text{ given } f'(0)=2) \\ \therefore f'(x) &= 2f(x) \end{aligned}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int 2dx \Rightarrow \log f(x) = 2x + c$$
$$f(0) = 1 \Rightarrow \log 1 = 2(0) + c \Rightarrow c = 0$$
$$\therefore \log f(x) = 2x \Rightarrow f(x) = e^{2x}$$

Question-1(d) Find the Taylor's series expansion for the function

$$f(x) = \log(1+x), -1 < x < \infty$$

about $x = 2$ with Lagrange's form of remainder after 3-terms.

[8 Marks]

Solution: **Taylor series**: If a function f is defined on $[a, a+h]$ such that
(i) $f, f', f'', \ldots f^{n-1}$ are continuous on $[a, a+h]$
(ii) $f^n(x)$ exists in $(a, a+h)$, then there exists at least one real number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h)$$

Here, $f(x) = \log(1+x) \Rightarrow f(2) = \log 3$.

$$f'(x) = \frac{1}{1+x} \quad \Rightarrow f'(2) = \frac{1}{3}$$
$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(2) = \frac{-1}{9}$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(2) = \frac{2}{27}$$

$$f(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + f'''(2+\theta h)\cdot\frac{(x-2)^3}{3!}$$

$$f(x) = \log 3 + \frac{(x-2)}{3} - \frac{(x-2)^2}{18} + \frac{f'''[2+\theta(x-2))}{81} \times (x-2)^3$$

where $0 < \theta < 1$.
Hence, it is the required form of Taylor expansion of $f(x)$ about $x = 2$ with Lagrange's form of remainder after 3-terms.

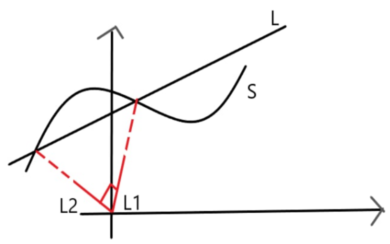
Question-1(e) If the straight lines, joining the origin to the points of intersection of the curve $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ and the straight line $2x + 3y + k = 0$, are at right angles, then show that $6k^2 + 5k + 52 = 0$.

[8 Marks]

Solution:

$$\text{Let } S \equiv 3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0;$$
$$L \equiv 2x + 3y + k = 0$$

By homogenizing equation S with the help of Equation of line L, we get the equation of pair of lines L_1 and L_2 through origin.



$$3x^2 - xy + 3y^2 + (2x - 3y)\frac{(2x+3y)}{-k} + 4\left[\frac{(2x+3y)}{k}\right]^2 = 0$$

If these lines are at right angles then, sum of coefficients of x^2 and $y^2 = 0$

$$3 + 3 + \left(\frac{4-9}{-k}\right) + 4\left(\frac{4+9}{k^2}\right) = 0$$
$$\Rightarrow 6k^2 + 5k + 52 = 0$$

which is the required given condition.

$$[\text{ Pair of lines, } ax^2 + 2hxy + by^2 = 0$$
$$\Rightarrow \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}]$$

Question-2(a) Let $\mathrm{T}:\mathbb{R}^3\rightarrow\mathbb{R}^3$ be defined by $\mathrm{T(x,y,z)=(2x,-3y,x+y)}$, and $B_1=\{(-1,2,0),(0,1,-1),(3,1,2)\}$ be a basis of $\mathbb{R}^3$. Find the matrix representation of T relative to the basis B_1.

[10 Marks]

Solution:

$$\begin{aligned}
\mathrm{T(x,y,z)} &= \mathrm{(2x,-3y,x+y)}\\
B_1 &= \{(-1,2,0),(0,1,-1),(3,1,2)\}\\
\therefore\quad T(-1,2,0) &= (-2,-6,1)\\
&= -1(-1,2,0)-3(0,1,-1)-1(3,1,2)\\
&\qquad\text{[Using Calculator]}\\
T(0,1,-1) &= (0,-3,1)\\
&= \frac{-2}{3}(-1,2,0)-\frac{13}{9}(0,1,-1)-\frac{2}{9}(3,1,2)\\
T(3,1,2) &= (6,-3,4)\\
&= \frac{-5}{3}(-1,2,0)-\frac{10}{9}(0,1,-1)+\frac{13}{9}(3,1,2)
\end{aligned}$$

Hence, matrix representation of T is

$$[T]_{B_1}=\begin{bmatrix}-1 & -3 & -1\\ \frac{-2}{3} & \frac{-13}{9} & \frac{-2}{9}\\ \frac{-5}{3} & \frac{-10}{9} & \frac{13}{9}\end{bmatrix}^{\top}=\begin{bmatrix}-1 & -\frac{2}{3} & -5/3\\ -3 & \frac{-13}{9} & -10/9\\ -1 & -2/9 & 13/9\end{bmatrix}$$

Question-2(b) Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

[15 Marks]

Solution: Let diameter of sphere be D, which is fixed. Let x,y,z be dimensions of rectangular solid inscribed in the sphere.

$$V=xyz$$
$$\text{such that } x^2+y^2+z^2=(D)^2$$

(Diagonal of Rectangular solid = Diameter of the sphere)

Consider the function

$$F(x,y,z)=xyz+\lambda\left(x^2+y^2+z^2-D^2\right)$$

We use Lagrange's multiplier method to maximize $V = xyz$ for critical points,

$$\begin{aligned} \partial F = 0 \quad &\Rightarrow \quad F_x = 0, \quad F_y = 0, F_z = 0 \\ \therefore \quad yz + \lambda(2x) &= 0 \quad \ldots(1) \\ xz + \lambda(2y) &= 0 \quad \ldots(2) \\ xy + \lambda(2z) &= 0 \quad \ldots(3) \end{aligned}$$

Subtracting equation (1) from (2) and (3), we get

$$\begin{aligned} z(x-y) + 2\lambda(y-x) = 0 \quad &\Rightarrow \quad x = y \\ y(x-z) + 2\lambda(z-x) = 0 \quad &\Rightarrow \quad x = z \\ \therefore x = y = z \end{aligned}$$

As all the three dimensions x, y, z should be same for maximum volume of rectargular solid, hence it will be a cube.

$$\begin{aligned} \text{Let} \quad x = y = z = a \\ \therefore x^2 + y^2 + z^2 = 1^2 \Rightarrow 3a^2 = D^2 \\ \Rightarrow \text{ Dimension of cube } = a = \frac{D}{\sqrt{3}} \end{aligned}$$

Question-2(c) Prove that the angle between two straight lines whose direction cosines are given by $l + \mathrm{m} + \mathrm{n} = 0$ and $\mathrm{fmn} + \mathrm{gn}l + \mathrm{hlm} = 0$ is $\frac{\pi}{3}$, if $\frac{1}{\mathrm{f}} + \frac{1}{\mathrm{g}} + \frac{1}{\mathrm{h}} = 0$.

[15 Marks]

Solution: Given $l + m + n = 0 \quad \Rightarrow \quad n = -(1 + m)$ Using it in,

$$\begin{aligned} fmn + gnl + hlm &= 0 \\ -(fm + gl)(l + m) + hlm &= 0 \\ flm + gl^2 + fm^2 + glm - hlm &= 0 \\ gl^2 + \ln(f + g - h) + fm^2 &= 0 \\ g\left(\frac{l}{m}\right)^2 + \frac{l}{m}(f + g - h) + f &= 0 \end{aligned}$$

Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are two roots of the quadratic equation.

$$\therefore \left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right) = \frac{f}{g} \Rightarrow \frac{l_1 l_2}{f} = \frac{m_1 m_2}{g}$$

By symmetry

$$\frac{l_1 l_2}{f} = \frac{m_1 m_2}{g} = \frac{n_1 n_2}{h} = k \quad \text{(Say)}$$

Again,

$$\begin{aligned} \frac{l_1}{m_1} + \frac{l_2}{m_2} &= \frac{-(f + g - h)}{g} \\ \frac{l_1 m_2 + m_1 l_2}{-(f + g + h)} &= \frac{m_1 m_2}{g} = k \end{aligned}$$

$$\begin{aligned}
\therefore (l_1m_2 - l_2m_1)^2 &= (l_1m_2 + l_2m_1)^2 - 4(l_1m_2)(l_2m_1) \\
&= k^2(f+g-h)^2 - 4(kf)(kg) \\
&= k^2\left[f^2+g^2+h^2+2fg-2gh-2fh-4fg\right] \\
&= k^2\left[f^2+g^2+h^2-2(fg+gh+fh)\right] \\
&= k^2\left[f^2+g^2+h^2+2(fg+gh+fh)\right] \\
&\left(\because \frac{1}{f}+\frac{1}{g}+\frac{1}{h}=0 \Rightarrow fg+gh+hf=0\right) \\
&= [k(f+g+h)]^2
\end{aligned}$$

$$\begin{aligned}
\tan^2\theta &= \frac{\sum (l_1m_2 - l_2m_1)^2}{(l_1l_2+m_1m_2+n,n_2)^2} \\
&= \frac{3[k(f+g+h)]^2}{[k(f+g+h)]^2} = 3 \\
\Rightarrow \quad & \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}
\end{aligned}$$

Question-3(a) Find the asymptotes of the curve $x^3+3x^2y-4y^3-x+y+3=0$.

[10 Marks]

Solution: As the coefficients of highest powers of x and y are constant, hence asymptotes parallel to x-axis or y-axis do not exist.

Oblique Asymptotes:

$$\begin{aligned}
&\text{Put } x=1, \quad y=m \\
&\phi_3(m) = 1+3m-4m^3 \text{ (Taking third degree terms)} \\
&\phi_2(m) = 0 \quad \text{(Taking second degree terms)} \\
&\phi_1(m) = -1+m \quad \text{(Taking first degree terms)}
\end{aligned}$$

Slopes of the asymptotes are real roots of eqn, $\phi_3(m)=0$

$$\begin{aligned}
\text{i.e., } 4m^3-3m-1 &= 0 \\
\Rightarrow (m-1)(2m+1)^2 &= 0 \\
\therefore m &= 1, -1/2, -1/2 \\
\text{for } m=1, \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} &= 0
\end{aligned}$$

$\therefore y = mx+c$ i.e. $y=x$ is an asymptote.

For $m=-1/2$ (repeated root), the value of c is given by

$$\begin{aligned}
&\frac{c^2}{2!}\phi_3''(m) + c\cdot\phi_2'(m) + \phi_1(m) = 0 \\
\text{i.e.} \quad &\frac{c^2}{2}(-24m) + c(0) + (m-1) = 0
\end{aligned}$$

$$-12\left(-\frac{1}{2}\right)c^2+\left(-\frac{1}{2}-1\right)=0$$
$$\to 12c^2-3=0$$
$$\Rightarrow c^2=\frac{1}{4}$$
$$\Rightarrow c=+\frac{1}{2},-\frac{1}{2}$$
$$\therefore \text{ Thus, the asymptotes are } y=mx+c$$
$$\text{ie. } y=-\frac{1}{2}x+\frac{1}{2} \text{ and } y=\frac{-1}{2}x-\frac{1}{2}$$

$\therefore$ Three asymptotes are $y-x=0$, $x+2y=1$ and $x+2y=-1$

Question-3(b) When is a matrix A said to be similar to another matrix B? Prove that
(i) if A is similar to B, then B is similar to A.
(ii) two similar matrices have the same eigenvalues.
Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true.

[15 Marks]

Solution: **Similarity of Matrices:** Let A and B be square matrices of order n. Then A is said to be similar to B if there exists a non-singular matrix P such that

$$A=P^{-1}BP$$

i) If A is similar to B, therefore there exists an $n\times n$ non-singular matrix p such that

$$\begin{aligned}&A=P^{-1}BP\\ \Rightarrow\ &PAP^{-1}=P\left(P^{-1}BP\right)\\ \Rightarrow\ &\quad PAP^{-1}=B\end{aligned}$$

i.e. $B=\left(P^{-1}\right)^{-1}A\left(P^{-1}\right)$ P is invertible means P^{-1} is invertible and $\left(P^{-1}\right)^{-1}=P$. It implies that B is similar to A.
(ii) Suppose A and B are similar matrices

$$\text{Then, } B=P^{-1}AP$$

Characteristic polynomial of B is

$$\begin{aligned}|B-\lambda I|&=\left|P^{-1}AP-\lambda P^{-1}P\right|\\&=\left|P^{-1}(A-\lambda I)P\right|\\&=|P^{-1}||A-\lambda I||P|\\&=|A-\lambda I|\quad\left(\because \left|P^{-1}\right||P|\left|P^{-1}P\right|=|I|=1\right)\end{aligned}$$

Which is same as characteristic bolymomial of A. Hence, A and B have same eigenvalues. Finally, consider matrices

$$A=\begin{bmatrix}0&0\\0&0\end{bmatrix} \text{ and } B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

which clearly have the same eigenvalues, but they are not similar because B cannot be obtained by applying any sequence of elemantary transformtions on matrix A.
Hence, converse of (ii) is not true.

Question-3(c) A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel to yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$.

[15 Marks]

Solution: The eqn of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

Let the coordinates of the point P be (α, β, γ). Since the point P lies on plane (1), we have

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \qquad \ldots(2)$$

The d.r.'s of OP are $\alpha-0, \beta-0, \gamma-0$ i.e., α, β, γ.
Hence, the equation of plane passing through point $P(\alpha, \beta, \gamma)$ and perpendicular to OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

or

$$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2 \qquad \ldots(3)$$

The plane (3) meets the axes in the points A, B and c whose cordinates are

$$\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right), \left(0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right), \left(0, 0, \frac{\alpha^2+\beta^2+r^2}{\gamma}\right)$$

Now the eqn of plane through A and parallel to $y =$ plane i.e. plane $x=0$ is

$$x=\frac{\alpha^2+\beta+\gamma^2}{\alpha}$$

Similarly, equations of the other two planes are

$$y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta},$$

$$z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Now, Q is the intersection point of above three planes.
The locus of point Q is obtained by eliminating α, β and γ from eqn (2) with the help of above three equations.

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \text{ gives}$$

$$\frac{1}{a}\left(\frac{\alpha^2+\beta^2+\gamma^2}{x}\right)+\frac{1}{b}\left(\frac{\alpha^2+\beta^2+\gamma^2}{y}\right)+\frac{1}{c}\left(\frac{\alpha^2+\beta^2+\gamma^2}{z}\right)=1$$

$$\Rightarrow \frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Also,

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Hence, required locus is given by:

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$

Question-4(a) Let P be the vertex of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

If the section of this cone made by the plane $z=0$ is a rectangular hyperbola, then find the locus of P.

[10 Marks]

Solution: Let $P(x,y,z)$ be the vertex of enveloping cone.
Eqn of ellipsoid is

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$\therefore \quad S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z^2}{c^2}-1,$$

$$T=\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1$$

Eqn of enveloping cone:

$$SS_1=T^2$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1\right)^2$$

The section of enveloping cone by $z=0$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1\right)^2$$

If it represents rectangular hyperbola, sum of coefficients of x^2 and y^2 will be zero.

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$$

The required locus of $P\left(x_1,y_1,z_1\right)$ is

$$c^2\left(x^2+y^2\right)+\left(a^2+b^2\right)z^2=c^2\left(a^2+b^2\right)$$

Question-4(b) (i) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, hence find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynominl in A.
(ii) Express the vector (1,2,5) as a linear combination of the vectors (1,1,1),(2,1,2) and $(3, 2, 3)$, if possible. Justify your answer.

[15 Marks]

Solution: (i) Given, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
Characteristic polynomial is given by

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} &= 0 \\ \Rightarrow (1-\lambda)(3-\lambda) - 8 &= 0 \\ \Rightarrow \lambda^2 - 4\lambda + 3 - 8 &= 0 \\ \Rightarrow \lambda^2 - 4\lambda - 5 &= 0 \end{aligned}$$

Cayley-Hamilton theorem states that every square matrix satisfies its characteristics eqn.

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-4-5 & 16-16 \\ 8-8 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence Cayley-Hamilton theorem is verified for matrix A. Now,

$$A^2 - 4A - 5I = 0$$

$$\begin{aligned} &\Rightarrow A^5 - 4A^3 - 7A^3 + 11A^2 - A - 10I \\ &= A^5 - 4A^3 - 5A^3 - 2A^3 + 11A^2 - A - 10I \\ &= A^3\left(A^2 - 4A - 5I\right) - 2A(4A + 5I) + 11A^2 - A - 10I \\ &= 0 - 8A^2 - 10 \cdot A + 11A^2 - A - 10I \\ &= 3A^2 - 11A - 10I \\ &= 3(4A + 5I) - 11A - 10I \\ &= 12A + 15I - 11A - 10I \\ &= A + 5I \end{aligned}$$

(ii) Let if possible

$$(1, 2, 5) = a(1, 1, 1) + b(2, 1, 2) + C(3, 3, 3)$$

i.e.

$$\begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 1 & 1 & 2 & : & 2 \\ 1 & 2 & 3 & : & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -1 & : & -1 \\ 0 & 0 & 0 & : & 4 \end{bmatrix}$$

If $AX = b$, then, Rank$(A) = 2$ and Rank $(A : b) = 3$, which are not equal. Hence, above system of equations is inconsistent.
Therefore, vector $(1, 2, 5)$ cannot be expressed as a linear combination of vector $(1, 1, 1)$,$(2, 1, 2)$ and $(3, 2, 3)$

Question-4(c) (i) Evaluate

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$$

(ii) Evaluate the following integral :

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

[15 Marks]

Solution: Let

$$\begin{aligned} l &= \lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} \\ &= \lim_{h\to 0} h\tan\left\{\frac{\pi}{2}(1+h)\right\} \\ &= \lim_{h\to 0}\left[-h\cot\left(\frac{\pi}{2}h\right)\right] \\ &= \lim_{h\to 0}\frac{-h}{\sin\left(\frac{\pi h}{2}\right)}\cdot\left(\cos\frac{\pi h}{2}\right) \\ &= \frac{-2}{\pi}\lim_{h\to 0}\frac{\pi h/2}{\sin\left(\frac{\pi h}{2}\right)}\cdot\lim_{h\to 0}\cos\left(\frac{\pi h}{2}\right) \\ &= \frac{-2}{\pi}\cdot 1\cdot 1 \\ &= \frac{-2}{\pi} \end{aligned}$$

(ii)

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

$$= \lim_{a\to -\infty}\int_a^0 xe^{-x^2}dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

$$= \lim_{a\to -\infty}\left[-\frac{1}{2}e^{-x^2}\right]_a^0 + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b$$

$$= \frac{-1}{2}\left\{\left[e^{-0} - \lim_{a\to -\infty} e^{-a^2}\right] + \left[\lim_{b\to\infty} e^{-b^2} - e^{-0}\right]\right\}$$

$$= \left[-\frac{1}{2}\left(1-e^{-\infty}\right)+\left(e^{-\infty}-1\right)\right]$$

$$= -\frac{1}{2}[1-0+0-1] = 0$$

Hence, the given infinite integral is convergent.

1.2 Section-B

Question-5(a) Solve the initial value problem:

$$\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0, y(1)=2$$

[8 Marks]

Solution:

$$\text{Comparing with } Mdx+Ndy=0,$$

$$M=2x^2y \quad ; \quad N=x^2y-x$$

$$\therefore \frac{\partial M}{\partial y}=1 \quad ; \quad \frac{\partial N}{\partial x}=2xy-1$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so ODE is not exact.
Integrating factor $(I.F.)$:

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1-(2xy-1)}{x^2y-x}$$

$$=\frac{2(1-xy)}{-x(1-xy)}=-\frac{2}{x}=f(x)$$

Which is a function of x only.

$$I.F=e^{\int f(x)dx}=e^{\int -\frac{2}{x}dx}$$

$$=e^{-2\log x}=e^{\log x^{-2}}=\frac{1}{x^2}$$

Multiplying the given Differential Equation with $I.F.=1/x^2$

$$\frac{1}{x^2}\left(2x^2+y\right)dx+\frac{1}{x^2}\left(x^2y-x\right)dy=0$$

$$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$$

Hence, the solution is $\int_{y-\text{ constant}} M dx + \int(\text{ terms in } N \text{ not containing } x) dy = C$

$$\int_{y-\text{constant}} \left(2 + \frac{y}{x^2}\right) dx + \int y dy = C$$

$$2x - \frac{y}{x} + \frac{y^2}{2} = C$$

$$\text{Now, when } x = 1, y = 2$$

$$\therefore \quad 2 - \frac{2}{1} + \frac{(2)^2}{2} = C \Rightarrow C = 2$$

$$\therefore \quad 2x - \frac{y}{x} + \frac{y^2}{2} = 2$$

$$\text{i.e.} \quad 4x^2 + xy^2 - 2y = 4x$$

is the required solution of the initial value problem.

Question-5(b) Solve the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 16x - 12e^{2x}$$

[8 Marks]

Solution: Given $D.E.$ can be written as

$$\left(D^2 - 3D - 4\right) y = 16x - 12e^{2x}$$

Auxiliary Equation:

$$m^2 - 3m - 4 = 0$$

$$m^2 - 4m + m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$\Rightarrow m = 4, -1$$

$$\therefore \quad C \cdot F = C_1 e^{4x} + C_2 e^{-x}$$

$$P \cdot I = \frac{1}{D^2 - 3D - 4}\left(16x - 12e^{2x}\right)$$

$$= 16\frac{1}{D^2 - 3D - 4}x - 12\frac{1}{D^2 - 3D - 4}e^{2x}$$

$$= -\frac{16}{-4}\left[1 - \frac{(D^2 - 3D)}{4}\right]^{-1} x - \frac{12}{(2)^2 - 3(2) - 4}e^{2x}$$

$$= -4\left(1 + \frac{D^2 - 3D}{4} + \ldots\right) x - \frac{12}{(-6)}e^{2x}$$

$$= -4\left[x + \frac{1}{4}(-3)\right] + 2e^{2x}$$

$$= -4x + 2e^{2x} + 3$$

Hence, General Solution is

$$y = C.F. + P.I.$$
$$y = C_1e^{4x} + C_2e^{-x} - 4x + 2e^{2x} + 3$$

Question-5(c) If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector.

[8 Marks]

Solution: Here it is given that radial velocity is proportional to transverse velocity

$$\therefore \quad \frac{dr}{dt} = k\left(r\frac{d\theta}{dt}\right) \qquad \ldots(1)$$

where k is a constant of proportionality

$$\begin{aligned}
\text{i.e.,} \quad \frac{dr}{r} &= kd\theta \\
\text{Integrating,} \quad \log r &= k\theta + \log c \\
\text{or} \quad \log r &= \log e^{k\theta} + \log c \\
\text{or} \quad \log r &= \log c \cdot e^{k\theta} \\
\therefore \quad r &= ce^{k\theta}
\end{aligned}$$

Which is the equation of an equiangular spiral.
Further, radial acceleration is proportional to transverse acceleration. i.e.

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \mu\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \qquad \ldots(2)$$

where μ is a constant. Now, we shall eliminate $\frac{d\theta}{dt}$ between (1) and (2). From (1),

$$\frac{d\theta}{dt} = \frac{1}{kr}.\frac{dr}{dt}$$

Putting the value of $\frac{d\theta}{dt}$ in eq. (2), we get

$$\begin{aligned}
\frac{d^2r}{dt^2} - r\left(\frac{1}{kr}\cdot\frac{dr}{dt}\right)^2 &= \frac{\mu}{r}\cdot\frac{d}{dt}\left[\frac{r^2}{kr}.\frac{dr}{dt}\right] \\
\text{or} \quad \frac{d^2r}{dt^2} - \frac{1}{k^2r}\left(\frac{dr}{dt}\right)^2 &= \frac{\mu}{k}\cdot\frac{1}{r}\left[r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\right] \\
\text{i.e.,} \quad \left(1 - \frac{\mu}{k}\right)\frac{d^2r}{dt^2} &= \left(\frac{1}{k^2} + \frac{\mu}{k}\right)\frac{1}{r}\left(\frac{dr}{dt}\right)^2 \\
\text{or} \quad \frac{d^2r}{dt^2} &= \frac{1+ku}{k(k-\mu)}\frac{1}{r}\left(\frac{dr}{dt}\right)^2
\end{aligned}$$

$$\text{which can be written as } \frac{\frac{d^2r}{dt^2}}{\frac{dr}{dt}} = A\frac{\frac{dr}{dt}}{r} \text{ where } A = \frac{1+k\mu}{k(k-\mu)}$$

$$\text{Integrating,} \qquad \log\frac{dr}{dt} = A\log r + \log c_1 = \log r^A + \log c_1$$

$$\text{Hence} \qquad \frac{dr}{dt} = c_1 r^A$$

$$\therefore \text{ From (1),} \qquad c_1 r^A = kr\frac{d\theta}{dt} \quad \text{i.e., } r\frac{d\theta}{dt} = \frac{c_1}{k}r^A \qquad \ldots(3)$$

$$\therefore \qquad \text{Velocity of the particle } = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2} = \sqrt{(c_1 r^A)^2 + \frac{1}{k^2}(c_1 r^A)^2}$$

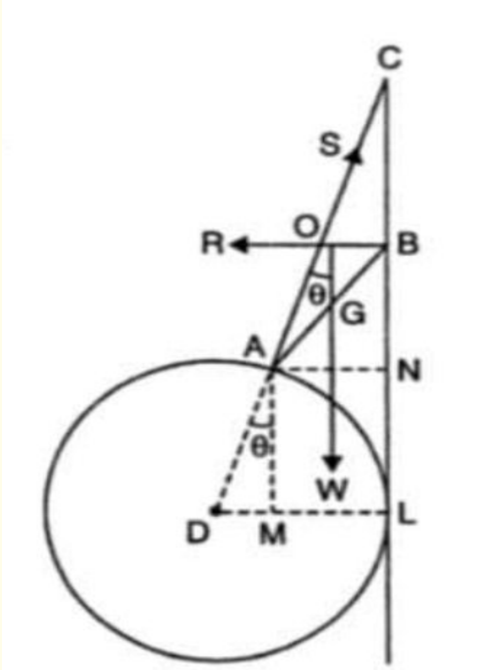
$$= cr^A \text{ where } c = \sqrt{c_1^2 + \frac{c_1^2}{k^2}} \text{ is a constant}$$

Hence the velocity of the particle varies as some power of radius vector.

Question-5(d) A cylinder of radius 'r', whose axis is fixed horizontally, touches a vertical wall along a generating line.
A flat beam of length l and weight $'W'$ rests with its extremities in contact with the wall and the cylinder, making an angle of $45°$ with the vertical.
Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $\frac{W}{2}$.
Also, prove that the ratio of radius of the cylinder to the length of the beam is $5+\sqrt{5} : 4\sqrt{2}$.

[8 Marks]

Solution: Rod AB is in equilibrium on the cylinder and against the wall under the forces:
(i) Reaction R $\perp$ wall at B.
(ii) Reaction S at A passing through the centre D.
(iii) Weight W acting at middle point C of the rod vertically downwards.



Forces are concurrent at O,

$$\angle\text{ABL} = \angle\text{OGB} = 45^\circ \qquad \text{(Given)}$$

Let $\angle\text{AOG} = \theta$,
Applying $m : n$ theorem in $\triangle AOB$

$$(1+1)\cot 45^\circ = 1\cot\theta - 1\cot 90$$
$$\therefore \quad \cot\theta = 2 \qquad \ldots(1)$$

Applying Lami's theorem

$$\frac{\text{R}}{\sin(180-\theta)} = \frac{\text{S}}{\sin 90^\circ} = \frac{\text{W}}{\sin(90+\theta)}$$

$$\text{or} \qquad \frac{\text{R}}{\sin\theta} = \frac{\text{S}}{1} = \frac{\text{W}}{\cos\theta}$$

$$\therefore \qquad R = W\tan\theta = \frac{W}{2} \qquad \text{(using 1)}$$

$$\text{S} = \text{W}\sec\theta = \text{W}\sqrt{1+\tan^2\theta}$$
$$= \text{W}\sqrt{1+\frac{1}{4}}$$
$$= \frac{1}{2}\,\text{W}\sqrt{5}$$

$$\text{Further} \qquad \text{DL} = \text{DM} + \text{ML} = \text{DM} + \text{AN} = r\sin\theta + l\sin 45^\circ$$

$$\therefore \qquad r = r\cdot\frac{1}{\sqrt{5}} + l\cdot\frac{1}{\sqrt{2}} \Rightarrow \frac{r(\sqrt{5}-1)}{\sqrt{5}} = \frac{l}{\sqrt{2}}$$

$$\Rightarrow \qquad \frac{r}{l} = \frac{(5+\sqrt{5})}{4\sqrt{2}}$$

Question-5(e) Prove that for a vector $\overrightarrow{a}$,

$$\nabla(\vec{a}\cdot\vec{r})=\vec{a};$$

where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}, r=|\vec{r}|$ Is there any restriction on $\overrightarrow{a}$?
Further, show that

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

Give an example to verify the above.

[8 Marks]

Solution: Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.
$\therefore \quad \vec{a}\cdot\vec{r}=a_1x+a_2y+a_3z$.

$$\begin{aligned}\text{Now,}\nabla(\vec{a}\cdot\vec{r})&=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\vec{a}\cdot\vec{r})\\&=\hat{i}\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\hat{j}\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\hat{k}\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)\\&=\hat{i}a_1+\hat{j}a_2+\hat{k}a_3=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\end{aligned}$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r})=\vec{a}$$

Condition: Above result is valid only when vector a is a constant vector.
Also,

$$\begin{aligned}\nabla\left(\frac{1}{r}\right)&=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)\\&=-\frac{1}{r^2}\left(\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right)\\&=-\frac{1}{r^2}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)\\&\left[\because r^2=|\vec{r}|^2=x^2+y^2+z^2,\therefore \frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}\right]\\&=-\frac{1}{r^2}\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)=-\frac{\vec{r}}{r^3}\end{aligned}$$

$$\therefore \quad \vec{b}\cdot\nabla\left(\frac{1}{r}\right)=\vec{b}\cdot\left(\frac{-\vec{r}}{r^3}\right)=\left(-\frac{1}{r^3}\right)(\vec{b}\cdot\vec{r})$$

$$\begin{aligned}\text{Also,}\quad \nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)&=\left(-\frac{1}{r^3}\right)\nabla(\vec{b}\cdot\vec{r})+\nabla\left(-\frac{1}{r^3}\right)(\vec{b}\cdot\vec{r})\\&=-\frac{1}{r^3}\vec{b}+\frac{3}{r^4}\left(\hat{i}\frac{\partial\vec{r}}{\partial x}+\hat{j}\frac{\partial\vec{r}}{\partial y}+\hat{k}\frac{\partial\vec{r}}{\partial z}\right)(\vec{b}\cdot\vec{r})\\&[\because \text{ For any vector } \vec{a},\nabla(\vec{a}\cdot\vec{r})=\vec{a}]\end{aligned}$$

$$= -\frac{1}{r^3}\vec{b} + \frac{3}{r^4}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)(\vec{b}\cdot\vec{r})$$

$$= -\frac{\vec{b}}{r^3} + \frac{3\vec{r}(\vec{b}\cdot\vec{r})}{r^5}$$

$$\text{Hence, } \vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\vec{a}\cdot\vec{b}}{r^3} + \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}$$

Verification of above Result: Let us take, $\vec{a} = \hat{i}, \vec{b} = \hat{j}$

$$\vec{r} = xi + yj + zk$$

$$\nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \frac{k\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left(\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right) = \frac{-\vec{r}}{r^3}$$

$$\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \left(\frac{-1}{r^3}\right)j\cdot(\vec{r}) = \frac{-y}{r^3}$$

$$\nabla\left(\frac{-y}{r^3}\right) = i\frac{\partial}{\partial x}\left(\frac{-y}{r^3}\right) + j\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + k\frac{\partial}{\partial z}\left(\frac{-y}{r^3}\right)$$

$$= i\left(\frac{3y}{r^4}\right)\frac{x}{r} + j\left((-1)\frac{1}{r^3} + \frac{3y}{r^4}\cdot\frac{y}{r}\right) + k\left(\frac{3y}{r^4}\right)\frac{z}{r}$$

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = i\cdot\nabla\left(-\frac{y}{r^3}\right) = \frac{3xy}{r^5}$$

$$\frac{3\cdot(\vec{a}\cdot\vec{r})\cdot(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3} = \frac{3(x)(y)}{r^5} - \frac{ij}{r^3} = \frac{3xy}{r^5}$$

Question-6(a) Find one solution of the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

by inspection and using that solution determine the other linearly independent solution of the given equation. Obtain the general solution of the given differential equation.

[10 Marks]

Solution: The $D.E.$ is

$$\frac{d^2y}{dx^2} - \frac{2x}{(x^2+1)}\frac{dy}{dx} + \frac{2}{x^2+1}y = 0 \quad \ldots(1)$$

Which is in the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.
Here, we notice that

$$P + Qx = \frac{-2x}{x^2+1} + \frac{2}{x^2+1}\cdot x = 0$$

$\therefore y = x$ is a part of solution.

$$\text{Consider,} \quad y = vx$$

$$\therefore \frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Putting these values in (1), we get

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - \frac{2x}{(x^2+1)}\left(v + x\frac{dv}{dx}\right) + \frac{2}{x^2+1}(vx) = 0$$

$$x\frac{d^2v}{dx^2} + \left(2 - \frac{2x^2}{1+x^2}\right)\frac{dv}{dx} = 0$$

$$x\frac{d^2v}{dx^2} + \frac{2}{1+x^2}\frac{dv}{dx} = 0 \qquad \ldots(2)$$

$$\text{Let} \quad \frac{dv}{dx} = p \quad \Rightarrow \quad \frac{d^2v}{dx^2} = \frac{dp}{dx}$$

$$\therefore x\frac{dp}{dx} + \frac{2}{1+x^2}p = 0 \text{ (from (2))}$$

$$\Rightarrow \frac{dp}{p} = -\frac{2}{x(1+x^2)}dx$$

$$\frac{dp}{p} = -2\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

$$\text{Integrating,} \qquad \log p = -2\log x + \log\left(1+x^2\right) + \log c_1$$

$$p = \frac{c_1\left(1+x^2\right)}{x^2}$$

$$\frac{dv}{dx} = c_1\frac{\left(1+x^2\right)}{x^2}$$

$$dv = c_1\left(\frac{1}{x^2} + 1\right)dx$$

$$\text{Integrating,} \qquad v = c_1\left(\frac{-1}{x} + x\right) + c_2$$

Hence, the complete solution is

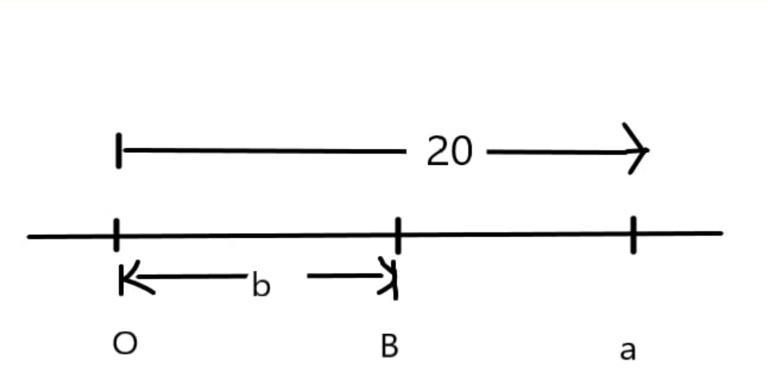
$$y = vx$$

$$y = c_1\left(-1 + x^2\right) + c_2x$$

Question-6(b) A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force 0, it reaches a point B distant b ' from O.
Find the time in reaching from A to B and the velocity at B.
When will the particle reach at the centre?

[15 Marks]

Solution: Let O be the centre of force, and particle starts from rest at point A. Then it reaches point B.



Given $F = \frac{-k}{x^3}$. The differential equation of motion of particle is given by

$$m\frac{d^2x}{dt^2} = \frac{-k}{x^3}$$

Multiplying by $2\frac{dx}{dt}$ on both side and integrating, we get

$$m\left(\frac{dx}{dt}\right)^2 = \frac{k}{x^2} + C$$

$$\text{At } x = 20, v = \frac{dx}{dt} = 0$$

$$0 = \frac{k}{(20)^2} + C \Rightarrow \quad C = \frac{-k}{400}$$

$$\text{and, } m = 5 \text{ units}$$

$$5\left(\frac{dx}{dt}\right)^2 = k\left(\frac{1}{x^2} - \frac{1}{400}\right)$$

$$\frac{dx}{dt} = -\sqrt{\frac{k}{5}}\frac{\sqrt{400 - x^2}}{20x} = \frac{-\mu\sqrt{400 - x^2}}{20x} \qquad \left(\text{for } \mu = \sqrt{\frac{k}{5}}\right)$$

[Negative sign is taken because $v = \frac{dx}{dt}$ is decreasing (central force)]

$$\Rightarrow \int \frac{-20x}{\sqrt{400 - x^2}}dx = \int \mu dt$$

$$\Rightarrow 20\sqrt{400 - x^2} = \mu t + c_1$$

$$\text{When } t = 0, x = 20 \Rightarrow c_1 = 0$$

$$\therefore \quad t = \frac{20}{\mu}\sqrt{400 - x^2}, \mu = \sqrt{k/5}$$

Time taken in reaching from A to B,
i.e. when $x = b$

$$t_B = \frac{20}{\mu}\sqrt{400 - b^2}$$

$$V_B = \frac{-\mu\sqrt{400 - b^2}}{20b}$$

When particle reaches at centre, $x = 0$

$$\therefore \quad t = \frac{20}{\mu}\sqrt{400 - 0} = \frac{400}{\mu}, \mu = \sqrt{\frac{k}{5}}$$

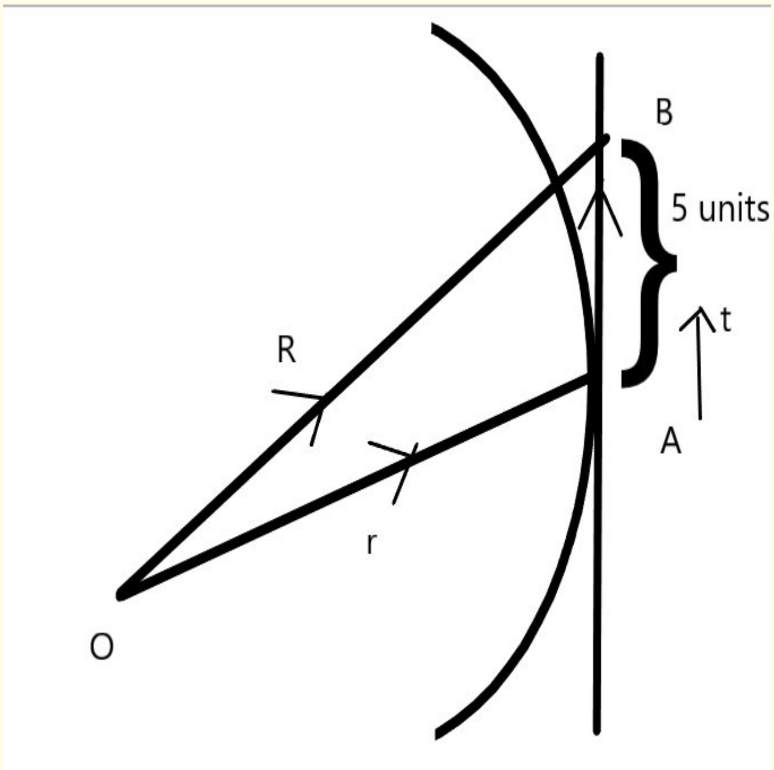
Question-6(c) A tangent is drawn to a given curve at some point of contact. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{\left[25\kappa^2\tau^2\left(1+25\kappa^2\right)+\left\{\kappa+5\frac{d\kappa}{ds}+25\kappa^3\right\}\right]^{1/2}}{\left(1+25\kappa^2\right)^{3/2}}$$

Find the curvature and torsion of the curve $\overrightarrow{\mathrm{r}}=t\hat{\mathrm{i}}+\mathrm{t}^2\hat{\mathrm{j}}+\mathrm{t}^3\hat{\mathrm{k}}$.

[15 Marks]

Solution: Locus of point at distance 5 units from B,



$$\begin{aligned}
R(s) &= r(t)+5t(s)\\
R'(s) &= r'(s)+5t'(s)\\
R''(s) &= r''(s)+5t''(s)\\
|R'(s)| &= |t(s)+5kn(s)| = \sqrt{1+25k^2} \qquad (\because \text{t(s) and n(s) are unit vectors})\\
\Rightarrow R'(s)\times R''(s) &= (r'(s)+5t'(s))\times(r''(s)+5t''(s))\\
&= r'(s)\times r''(s)+r'(s)\times 5t''(s)+5t'(s)\times r''(s)+25t'(s)\times t''(s)\\
&= t(s)\times kn(s)+t(s)\times 5k(-kt(s)+\tau b(s))+25kn(s)\times(-kt(s)+\tau b(s))
\end{aligned}$$

$$\because t'(s) = kn(s)$$

$$\Rightarrow t''(s) = \frac{dk}{ds}n(s) + kn'(s)$$

$$t(s) \times kn(s) + t(s) \times 5k(-kt(s) + \tau b(s)) + 25kn(s) \times (-kt(s) + \tau b(s))$$

$$= kb(s) + (r'(s) + 5t'(s)) \times 5t''(s)$$

$$= kb(s) + (r'(s) + 5t'(s)) \times 5\left(\frac{dk}{ds}n(s) + k(-kt(s) + \tau b(s))\right)$$

$$= kb(s) + 5\left(r'(s) + 5t'(s)\right) \times \left(\frac{dk}{ds}n(s) - k^2t(s) + \tau kb(s)\right)$$

$$= kb(s) + 5\left\{t(s) \times \frac{dk}{ds}n(s) - \tau kn(s) + 5k^3b(s) + 5\tau k^2t(s)\right\}$$

$$= kb(s) + 5\frac{dk}{ds}b(s) + 25k^3b(s) - 5\tau kn(s) + 25\tau k^2t(s)$$

$$= \left(k + 5\frac{dk}{ds} + 25k^3\right)b(s) - 5\tau kn(s) + 25\tau k^2t(s)$$

$$|R'(s) \times R''(s)| = \left\{25\tau^2k^2 + 625\tau^2k^4 + \left(k + 5\frac{dk}{ds} + 25k^3\right)^2\right\}^{1/2}$$

$$(\because k(s) \cdot t(s) = 0)$$

$$= \left\{25\tau^2k^2\left(1 + 25k^2\right) + \left(k + 5\frac{dk}{ds} + 25k^3\right)^2\right\}^{1/2}$$

$$\Rightarrow \frac{|R'(s) \times R''(s)|}{|R'(s)|^3} = \frac{\left\{25\tau^2k^2\left(1 + 25k^2\right) + \left(k + 5\frac{dk}{ds} + 25k^3\right)\right\}^{1/2}}{\{1 + 25k^2\}^{3/2}}$$

Curvature and torsion of the curve $\overrightarrow{\mathrm{r}} = t\hat{\mathrm{i}} + \mathrm{t}^2\hat{\mathrm{j}} + \mathrm{t}^3\hat{\mathrm{k}}$

$$
\begin{aligned}
r(t) &= \left\langle t, t^2, t^3 \right\rangle \\
r'(t) &= \left\langle 1, 2t, 3t^2 \right\rangle \\
r''(t) &= \langle 0, 2, 6t \rangle \\
r'''(t) &= \langle 0, 0, 6 \rangle \\
r'(t) \times r''(t) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle \\
|\ r'(t)\ | &= \sqrt{1 + 4 \cdot t^2 + 9t^4} \\
\textbf{Curvature} &= \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 2t^4)^{3/2}} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \\
\textbf{Torsion} &= \frac{[r' \quad r'' \quad r''']}{|r' \times r''|^2} = r' \cdot [r'' \times r'''] \\
r'' \times r''' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = \langle 12, 0, 0 \rangle \\
\left[\ r' \quad r'' \quad r'''\ \right] &= \left\langle 1, 2t, 3t^2 \right\rangle \cdot \langle 12, 0, 0 \rangle = 12 \\
|r' \times r''|^2 &= \left(36t^4 + 36t^2 + 4\right) = 4\left(1 + 9t^2 + 9t^4\right) \\
\textbf{Torsion} &= \frac{12}{4\left(1 + 9t^2 + 9t^4\right)} = \frac{3}{1 + 9t^2 + 9t^4}
\end{aligned}
$$

Question-7(a) Derive intrinsic equation

$$\mathrm{x} = \mathrm{c}\log(\sec\psi + \tan\psi)$$

of the common catenary, where symbols have usual meanings.
Prove that the length of an endless chain, which will hang over a circular pulley of radius 'a' so as to be in contact with $\frac{2}{3}$ of the circumference of the pulley, is

$$a\left\{\frac{4\pi}{3} + \frac{3}{\log(2+\sqrt{3})}\right\} \qquad (10)$$

[10 Marks]

Solution: Let the uniform flexible string ACB hang freely from two points A and B which are not in a vertical line. Let C be the lowest point of the common catenary. Let P be any point on the portion CA of the string such that are $CP = s$, measured along the curve.

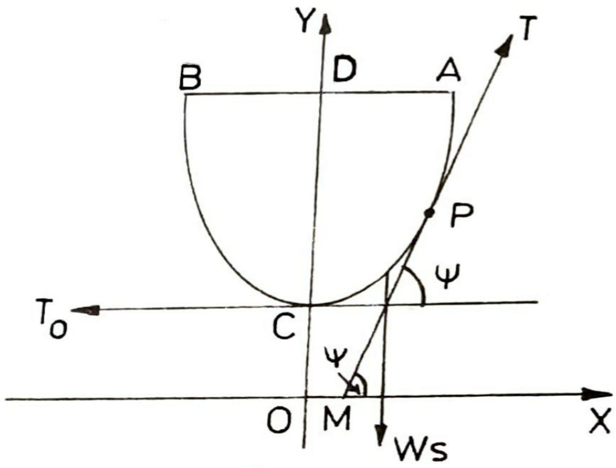


Fig. 4.1

Let w be the weight of the string per unit length of the string, then the weight of the portion $CP = ws$. This weight ws will act vertically downwards through S, the centre of gravity of the arc CP. Let the tangent at P make an angle ψ with the horizontal.
The portion CP of the string is in equilibrium under the action of the following forces:
(i) The tension T_0 at the lowest point C acting horizontally along the tangent to the curve at C.
(ii) The tension T along the tangent at P.
(iii) The weight ws to the portion CP acting vertically downwards through the centre of gravity of the are CP.

Since these three forces are in equilibrium, the line of action of the weight ws must pass through Q, the point of intersection of tangent at C and P.
Resolving these forces horizontally and vertically, we have

$$T\cos\psi = T_0 \qquad \dots(1)$$

$$T\sin\psi = ws \qquad \dots(2)$$

Dividing (2) by (1), we get

$$\tan\psi = \frac{ws}{T_0} \qquad \dots(3)$$

Let the tension T_0 at the lowest point C be taken equal to the weight of a length c of the string, i.e., $T_0 = wc$.
Then from (3),

$$\tan\psi = \frac{ws}{wc} = \frac{s}{c}$$

Hence, $\quad s = c\tan\psi$ is the required intrinsic equation of the catenary, where c is called the parameter of the catenary.

Relation between x and ψ:
We have

$$\frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \frac{dx}{ds}\cdot c\sec^2\psi$$

$$= \cos\psi \cdot c\sec^2\psi \text{ (From differential calculus)}$$
$$= c\sec\psi$$
$$\Rightarrow dx = c\sec\psi \cdot d\psi$$
$$\text{Intergating, } x = c\log(\sec\psi + \tan\psi) + c_2 \qquad \ldots (4)$$

where c_2 is an arbitrary constant. At the lowest point $C, x = 0$ and $\psi = 0$

$$\therefore \qquad c_2 = 0$$
$$\therefore \text{ From (4), } \quad x = c\log(\sec\psi + \tan\psi).$$

Question-7(b) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

[15 Marks]

Solution:

$$\text{Put } x = e^z \Rightarrow z = \log x$$
$$\text{Let } D = x\frac{d}{dx} = \frac{d}{dz}$$

Then given $O.D.E$ reduces to

$$[D(D-1) + 3D + 1]y = \frac{1}{(1-e^z)^2}$$
$$\text{or} \qquad \left(D^2 + 2D + 1\right)y = \frac{1}{(1-e^z)^2}$$

Auxiliary Equation:

$$D^2 + 2D + 1 = 0 \quad \text{ i.e. } (D+1)^2 = 0$$
$$\Rightarrow D = -1, -1$$

$$
\begin{aligned}
\therefore C \cdot F \cdot &= \left(c_1 + c_2 z\right) e^{-z} \\
&= \left(c_1 + c_2 \log x\right) \frac{1}{x} \quad (\because e^z = x) \\
P.I. &= \frac{1}{(D+1)^2} \cdot \frac{1}{\left(1-e^z\right)^2} \\
&= \frac{1}{D+1}\left[\frac{1}{D+1} \cdot \frac{1}{\left(1-e^z\right)^2}\right] \\
&= \frac{1}{D+1} \cdot e^{-z} \int \frac{1}{\left(1-e^z\right)^2} \cdot e^z dz \\
&\left[\because \frac{1}{D-a} X = e^{ax} \int x e^{-ax} dx\right] \\
&= \frac{1}{D+1} \cdot e^{-z} \int (1-t)^{-2} dt, \qquad \text{where } e^z = t \\
&= \frac{1}{D+1} \cdot e^{-z}(1-t) \\
&= \frac{1}{D+1} \cdot \left(\frac{e^{-z}}{1-e^z}\right) \\
&= e^{-z} \int \frac{e^{-z}}{1-e^z} \cdot e^z dz \\
&= e^{-z} \int \frac{dz}{1-e^z} = e^{-z} \int \frac{e^{-z}}{e^{-z}-1} dz \\
&= -e^{-z} \int \frac{-e^{-z}}{e^{-z}-1} dz \\
&= -e^{-z} \log\left(e^{-z}-1\right) \\
&= -\frac{1}{x} \log\left(\frac{1}{x}-1\right) \quad (\because e^z = x) \\
&= -\frac{1}{x} \log\left(\frac{1-x}{x}\right) = \frac{1}{x} \log\left(\frac{x}{1-x}\right)
\end{aligned}
$$

Hence, the complete solution is,

$$
y = \left(c_1 + c_2 \log x\right) \frac{1}{x} + \frac{1}{x} \log\left(\frac{x}{1-x}\right)
$$

Question-7(c) Given a portion of a circular disc of radius 7 units and of height $1 \cdot 5$ units such that $x, y, z \geq 0$.
Verify Gauss Divergence Theorem for the vector field

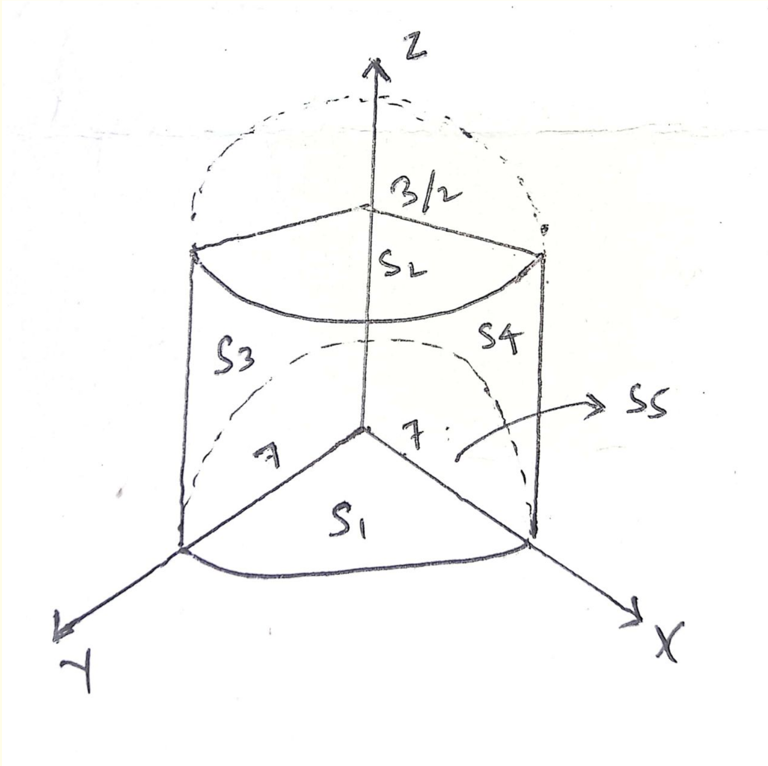
$$
\overrightarrow{\mathrm{f}} = \left(z, x, 3\mathrm{y}^2 z\right)
$$

over the surface of the above mentioned circular disc.

[15 Marks]

Solution: $\vec{F} = z\hat{i} + xj + 3y^2z\hat{k}$.
By Gauss Divergence Theorem,

$$\iint_S F \cdot \hat{n} dS = \iiint_V divF dV$$



Consider RHS,

$$\iiint_V divF dV = \iiint 3y^2 dV$$

$$= \int_0^{\pi/2} \int_0^7 \int_0^{3/2} 3 \text{ r}^3 \sin^2\theta dz dr d\theta \text{ (Changing to polar coordinates)}$$

V is given by:

$$x^2 + y^2 = 7^2$$
$$0 \le z \le 3/2$$
$$x \ge 0, y \ge 0$$

$$\therefore \text{ RHS} = 3\left(\frac{3}{2}\right)\left(\frac{1}{4}\right)(7)^4 \int_0^{\pi/2} \sin^2\theta d\theta$$
$$= \frac{9}{8}(7)^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{21609\pi}{32} \qquad \ldots (I)$$

Consider LHS

For S_1: $z = 0$, $\hat{n} = -\hat{k} \Rightarrow \vec{F} \cdot \hat{n} = -3y^2z$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot \hat{n} dS = \iint_{S_1} 0 dS = 0 \text{ (as } \vec{F} \cdot \hat{n} = 3y^2z = 0 \because z = 0) \qquad \ldots (1)$$

For S_2: $z = 3/2$, $\hat{n} = \hat{k}$, $x^2 + y^2 = 7^2$, $x \geq 0$, $y \geq 0$

$$\Rightarrow \iint\limits_{S_2} \vec{F} \cdot \hat{n} dS = \iint\limits_{S_2} 3y^2 \left(\frac{3}{2}\right) dxdy = \frac{9}{2} \int_0^{\frac{\pi}{2}} \int_0^7 r^3 \sin^2 \theta d\theta$$

$$= \frac{9}{2}\left(\frac{1}{4}\right) 7^4 \times \left(\frac{1}{2}\right) \frac{\pi}{2} = \frac{21609\pi}{32} \qquad \ldots (2)$$

For S_3: $x = 0$, $\hat{n} = -i$, $\vec{F} \cdot \hat{n} = -z$, $0 \leq z \leq 3/2$, $0 \leq y \leq 7$

$$\Rightarrow \iint\limits_{S_3} \vec{F} \cdot \hat{n} dS = \int_0^{3/2} \int_0^7 -zdydz = -\frac{9}{4 \times 2} \times 7 = \frac{-63}{8} \qquad \ldots (3)$$

For S_4: $y = 0$, $\hat{n} = -j$, $\vec{F} \cdot \vec{n} = -x$, $0 \leq z \leq 3/2$, $0 \leq y \leq 7$

$$\Rightarrow \int_{S_4} \vec{F} \cdot \hat{n} dS = \int_0^{1/2} \int_0^x -xdxdz = \left(\frac{3}{2}\right)\left(\frac{7^2}{2}\right) = \frac{-49 \times 3}{4} = \frac{-147}{4} \qquad \ldots (4)$$

For S_5: $\hat{n} = \frac{2xi+2yj+0k}{2\times 7} = \frac{xi+yj}{7}$
Taking projection on yz plane, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3/2$.

$$\iint\limits_{S_5} \vec{F}.\hat{n}dS = \int_0^{3/2} \int_0^7 x(z+y)\frac{dydz}{x}$$

$$= \frac{3}{2} \times \frac{7^2}{2} + 7\left(\frac{9}{8}\right)$$

$$= \frac{147}{4} + \frac{63}{8} \qquad \ldots (5)$$

Adding (1), (2), (3), (4) and (5), we get:

$$0 + \frac{21609\pi}{32} - \frac{63}{8} - \frac{147}{4} + \frac{147}{4} + \frac{63}{8} = \frac{21609\pi}{32} = \text{ RHS from (I)}$$

Therefore, Gauss Divergence Theorem is verified.

Question-8(a) Derive expression of ∇f in terms of spherical coordinates. Prove that

$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f.\nabla g + g\nabla^2 f$$

for any two vector point functions f(r, θ, ϕ) **and** g(r, θ, ϕ)**. Construct one example in three dimensions to verify this identity.**

[10 Marks]

Solution: First we prove that

$$\nabla\phi = \frac{\hat{e}_1}{h_1}\frac{\partial\phi}{\partial u} + \frac{\hat{e}_2}{h_2}\frac{\partial\phi}{\partial v} + \frac{\hat{e}_3}{h_3}\frac{\partial\phi}{\partial w}$$

Consider any scalar point function $\phi(u, v, \omega)$ given in terms of orthogonal curvilinear

coordinates u, v, w.
Regarding u, v, w as functions of x, y, z, we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x} \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y} \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z} \qquad \ldots (3)$$

mutiplying (1) by $\hat{i}$, (2) by $\hat{j}$, (3) by $\hat{k}$ and adding, we get

$$\nabla \phi = \frac{\partial \phi}{\partial u}\nabla u + \frac{\partial \phi}{\partial v}\nabla v + \frac{\partial \phi}{\partial w}\nabla w$$

$$= \frac{\hat{e}_1}{h_1}\frac{\partial \phi}{\partial u} + \frac{\hat{e}_2}{h_2}\frac{\partial \phi}{\partial v} + \frac{\hat{e}_3}{h_3}\frac{\partial \phi}{\partial w} \qquad \ldots (4)$$

$$\left[\because \nabla u = \frac{\hat{e}_1}{h_1} etc\right]$$

The spherical coordinates of (x, y, z) are

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta \\
\therefore dx &= -r\sin\theta\sin\phi d\phi + r\cos\theta\cos\phi d\theta + \sin\theta\cos\phi dr \\
dy &= r\sin\theta\cos\phi d\phi + r\cos\theta\sin\phi d\theta + \sin\theta\sin\phi dr \\
dz &= -r\sin\theta d\theta + \cos\theta dr
\end{aligned}$$

Element of Arc length

$$\begin{aligned}
(ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\
&= (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2
\end{aligned}$$

Thus we have scalar factors $h_1 = h_r = 1$

$$\begin{aligned}
h_2 &= h_\theta = r \\
h_3 &= h\phi = r\sin\theta
\end{aligned}$$

Using these in (4), we get

$$\nabla f = \frac{\hat{e}_r}{1}\frac{\partial f}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{e}_\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Now, we will calculate $\nabla^2(fg)$.

$$\nabla^2(fg) = \nabla\cdot(\nabla fg)$$

$$\Rightarrow \nabla\cdot(\nabla fg) = \nabla\cdot(f\nabla g + g\nabla f)$$

$$\begin{aligned}
\nabla fg &= \left\langle \frac{\partial}{\partial x}fg, \frac{\partial}{\partial y}fg, \frac{\partial}{\partial z}fg \right\rangle \\
&= \left\langle g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z} \right\rangle \\
&= g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle + f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \\
&= g\nabla f + f\nabla g
\end{aligned}$$

$$\Rightarrow \nabla\cdot(f\nabla g+g\nabla f)=\nabla\cdot(f\nabla g)+\nabla\cdot(g\nabla f)$$
$$=(\nabla f)\cdot\nabla g+f(\nabla\cdot\nabla g)+(\nabla g)\cdot\nabla f+g(\nabla\cdot\nabla f)$$
$$=\nabla f\cdot\nabla g+f\nabla^2 g+\nabla g\cdot\nabla f+g\nabla^2 f$$
$$=f\nabla^2 g+2\nabla f\cdot\nabla g+g\nabla^2 f$$

Example to verify above result:
Let $f(r,\theta,\phi)=r;\quad g(r,\theta,\phi)=\theta$
We know that,

$$\nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}$$
$$\therefore\qquad \nabla^2 f=\nabla^2(r)=\frac{2}{r}$$
$$\nabla^2 g=\nabla^2(\theta)=\frac{\cot\theta}{r^2}$$
$$\nabla f=\frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$
$$=\nabla(r)=\hat{r}$$
$$\nabla g=\nabla(\theta)=\frac{1}{r}\hat{\theta}$$
$$\nabla^2(fg)=\nabla^2(r\theta)$$
$$=\frac{2}{r}\theta+\frac{\cot\theta}{r^2}\cdot r=\frac{2\theta}{r}+\frac{\cot\theta}{r}$$
$$f\nabla^2 g+g\nabla^2 f+2\nabla f\cdot\nabla g=r\frac{\cot\theta}{r^2}+\theta\frac{2}{r}+2\hat{r}\cdot\hat{\theta}=\frac{2\theta}{r}+\frac{\cot\theta}{r}$$

Hence, $\nabla^2 fg=f\nabla^2 g+g\nabla^2 f+2\nabla f\cdot\nabla g$ is verified in spherical coordinates.

Question-8(b) Reduce the differential equation

$$xp^2-2yp+x+2y=0,\qquad \left(p=\frac{dy}{dx}\right)$$

to Clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation.

[15 Marks]

Solution: Put $u=x^2$, $v=y-x$ in the given D.E.

$$\text{Let } P=\frac{dv}{du}=\frac{dv}{dx}\cdot\frac{dx}{du}=(p-1)\frac{1}{2x}$$
$$\Rightarrow p=1+2xP$$

using it in given $D \cdot E$, we get

$$\begin{aligned} & x(1+2xP)^2 - 2y(1+2xP) + x + 2y = 0 \\ \Rightarrow & x\left(1+4x^2P^2+4xP\right) - 2y - 4xyP + x + 3y = 0 \\ \Rightarrow & 1 + 4x^2P^2 + 4xP - 4yP + 1 = 0 \\ \Rightarrow & 2 + 4uP^2 - 4Pv = 0 \\ \Rightarrow & v = Pu + \frac{1}{2P} \end{aligned}$$

Which is in Clairaut's form, $y = px + f(p)$. To obtain complete primitive, we replace P with constant c.

$$\therefore \quad v = cu + \frac{1}{2c}$$

$$\text{i.e. } y - x = cx^2 + \frac{1}{2c} \Rightarrow 2c(y-x) = 2c^2x^2 + 1$$

For singular solution, p-discriminant =c-discriminant

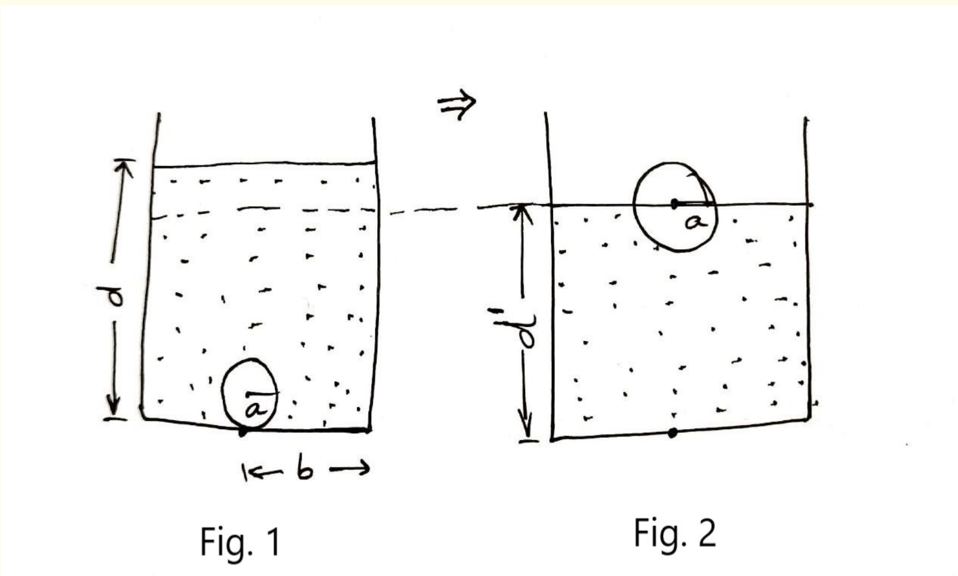
$$\begin{aligned} \text{i.e.,} \quad & (-2y)^2 - 4x(x+2y) = 0 \\ \Rightarrow \quad & y^2 - x^2 - 2xy = 0 \end{aligned}$$

Question-8(c) A sphere of radius $'a'$, and having density half of that of water, is completely immersed at the bottom of a circular cylinder of radius 'b', which is filled with water to depth 'd'. The sphere is set free and takes up its position of equilibrium. Show that the loss of potential energy this way is

$$\mathrm{W}\left(\mathrm{d} - \frac{11}{8}\mathrm{a} - \frac{\mathrm{a}^3}{3\ \mathrm{b}^2}\right)$$

where W is the weight of the sphere.

[15 Marks]



Solution:

Let d' be new height then:

Clearly for floating, height of submerged part of fluid displaced = weight of body

$$\Rightarrow \quad V_1\rho_1 g = V_2\rho_2 g$$

$$V_1\rho g = V_2\tfrac{\rho}{2}g \Rightarrow V_1 = \tfrac{1}{2}V_2$$

$\Rightarrow \frac{1}{2}$of sphere submereged in equilibrium as shown .
Now total vol. of sphere + water $= \pi b^2 d$ and it also equals

$$\pi b^2 d' + \frac{2}{3}\pi a^3$$

$$\Rightarrow \quad \pi b^2 d = \pi b^2 d' + \frac{2}{3}\pi a^3$$

$$\Rightarrow d = d' + \frac{2}{3}\frac{a^3}{b^2}$$

Now,

$$(\triangle PE)_{\text{sphere}} = w\left(d'-a\right)$$

$$= w\left(d - \frac{2a^3}{3b^2} - a\right)$$

, where w is weight of sphere.

Clearly for $(\triangle PE)_{\text{water}}$, water has moved from (α) and (β) in $fig.1$ to (α) and (β) in $fig.2$ leading to decerease in water level d' and enough space for sphere to keep floating in vacant space (β)

$$(\Delta PE) \text{ water due to } (\beta) = w_1'\left[\left(a - \frac{3a}{8}\right) - \left(d' - \frac{3a}{8}\right)\right]$$

$$= w_1'\left(a-d'\right)$$

$$= w\left(a-d'\right)$$

$$= w(a - d' + \frac{2}{3}\frac{a^3}{b^2})$$

[with w_1' = weight of the water in volume shaded by $(\beta) = \frac{1}{2}$ weight of a sphere of water of radius $'a' = \frac{1}{2}\cdot\left(\frac{4}{3}\pi \mathrm{a}^3\,\rho\cdot g\right) = \frac{1}{2}\left(\frac{4}{3}\pi a^3\frac{\rho}{2}g\right)\cdot 2 \ \left(\because w = \frac{4}{3}\pi a^3\cdot\frac{\rho}{2}\cdot g\right) = \frac{1}{2}\left(w\right)\cdot 2 = w$]
Now, $(\triangle PE)_{water}$ due to $(x) = w_2'\left[\left(a+\frac{3a}{8}\right) - \frac{d+d'}{2}\right]$ but $w_1' + w_2'$ = weight of a sphere of water of radius a

$$= \frac{4}{3}\pi a^3\rho g = \left(\frac{4}{3}\pi\times a^3\frac{\rho}{2}\cdot g\right)\cdot 2 = 2w$$

and $w_1' = w$ as shown already.

$$\Rightarrow w_2' = w$$

$\therefore (\triangle PE)_{water}$ due to $(\beta) = w\left[\frac{11a}{8} - \frac{d+d'}{2}\right]$.
$\therefore$ Total $\triangle PE = (\Delta PE)_{\text{sphere}} + (\triangle PE)_{\text{water}}$ due to α and β

$$= w\left(d - \frac{2a^3}{3b^2} - a + a - d + \frac{2a^3}{3b^2} + \frac{11a}{8} - \frac{d+d'}{2}\right)$$

$$= w\left(\frac{11a}{8} - \frac{d+d'}{2}\right)$$

$$\begin{aligned}
\therefore \text{ Loss of PE} &= -\text{ Change in } PE \\
&= \omega\left(\frac{d+d'}{2} - \frac{11a}{8}\right) \\
&= \omega\left(\frac{d}{2} + \frac{d}{2} - \frac{1}{3}\frac{a^3}{b^2} - \frac{11a}{8}\right) \\
&= \omega\left(d - \frac{11a}{8} - \frac{a^3}{3b^2}\right)
\end{aligned}$$

Chapter 2

2019

2.1 Section-A

Question-1(a) **Let** $T : R^3 \to R^3$ **be a linear operator on** R^3 **defined by**

$$T(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix of T **in the basis** $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

[8 Marks]

Solution: Given $T : R^3 \to R^3$ such that Basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$$\begin{aligned}
T(1, 1, 1) &= (3, -3, 3) \\
&= 3(1, 1, 1) - 6, (1, 1, 0) + 6(1, 0, 0) \quad \text{(using calculator)} \\
T(1, 1, 0) &= (2, -3, 3) \\
&= 3(1, 1, 1) - 6(1, 1, 0) + 5(1, 0, 0) \\
T(1, 0, 0) &= (0, 1, 3) \\
&= 3(1, 1, 1) - 2(1, 1, 0) - 1(1, 0, 0)
\end{aligned}$$

$$\begin{aligned}
[T]_B &= \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^\top \\
&= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}
\end{aligned}$$

Question-1(b) **The eigenvalues of a real symmetric matrix** A **are -1, 1 and -2. The corresponding eigenvectors are** $\frac{1}{\sqrt{2}}(-1, 1, 0)^T$**,** $(0, 0, 1)^T$ **and** $\frac{1}{\sqrt{2}}(-1, -1, 0)^T$ **respectively. Find the matrix** A^4.

[8 Marks]

Solution: If a matrix A is diagonalizable, then

$$P^{-1}AP = D$$
$$\therefore \quad A = PDP^{-1}$$

$$\begin{aligned} A^4 &= \left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right) \\ &= PD^4P^{-1} \end{aligned}$$

Where P is diagonalizing matrix consisting of eigenvectors of A.
Also, D is diagonal matrix containing eigenvalues of A at diagonal entries.

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$|P| = -1. \left[\left(\frac{-1}{\sqrt{2}}\right)\cdot\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)\right]$$

[Expanding Along C_2]

$$= -\left(\frac{1}{2} + \frac{1}{2}\right) = -1$$

$$\text{Adj } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{Adj}{|A|} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} (-1)^4 & 0 & 0 \\ 0 & (1)^4 & 0 \\ 0 & 0 & (-2)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Hence, $A^4 = PD^4P^{-1}$

$$\begin{aligned} &= \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -1 & 0 & -16 \\ 1 & 0 & -16 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -17 & -15 & 0 \\ -15 & -17 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{17}{2} & \frac{15}{2} & 0 \\ \frac{15}{2} & \frac{17}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

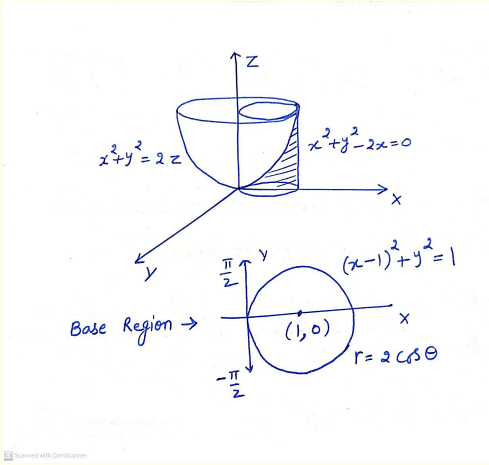
$$\therefore \quad A^4 = PD^4P^{-1} = \frac{1}{2} \begin{bmatrix} 17 & 15 & 0 \\ 15 & 17 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

CHAPTER 2. 2019

Question-1(c) Find the volume lying inside the cylinder $x^2 + y^2 - 2x = 0$ and outside the paraboloid $x^2 + y^2 = 2z$, while bounded by xy-plane.

[8 Marks]

Solution: The required volume is found by integrating $z = \frac{1}{2}(x^2 + y^2)$ over the circle $x^2 + y^2 = 2x$



Changing to polar coordinates in the xy-plane,

$$x = r\cos\theta, y = r\sin\theta$$

$$\therefore z = \frac{1}{2}\left(x^2 + y^2\right) = \frac{r^2}{2}$$

$$r^2\, cos^2\theta + r^2 \sin^2\theta = 2rcos\theta$$

$$r = 2\cos\theta$$

To cover this circle, r varies from 0 to $2\cos\theta$ and θ varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$
$\therefore$ Required volume

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} z \cdot r dr d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{r^3}{2} dr d\theta$$

$$V = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 dr d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left.\frac{r^4}{4}\right|_0^{2cos\theta} d\theta$$

$$= \frac{1}{8}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta d\theta$$

$$= 2 \times 2 \cdot \int_0^{\pi/2} \cos^4\theta d\theta$$

$$\left(\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, if, f(x) = f(-x)\right)$$

$$= 4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \text{(Using Walli's formula)}$$
$$= \frac{3}{4}\pi$$

Question-1(d) Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1, 1]$.

[8 Marks]

Solution:

$$f(x) = x^3 - 3x + k$$

We will prove the result by using method of contradiction.
Let $f(x)$ has two distinct roots a and b in [-1,1] i.e.

$$f(a) = 0 \quad = f(b), -1 \leq a, b \leq 1, a \neq b$$

$f(x)$ is continuous and differentiable over the interval $[a, b]$.
Hence, by Rolle's theorem, there exist some $c \in (a, b)$ s.t.

$$f'(c) = 0$$

i.e.

$$3c^2 - 3 = 0 \quad \Rightarrow \quad c = \pm 1$$

which is contradiction to the fact that a and b lies within [-1,1].
Hence $f(x)$ cannot have two distinct roots in [-1,1] for any value of $'k'$.

Question-1(e) If the coordinates of the points A and B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the line joining A and B is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then show that $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0$

[8 Marks]

Solution: Point $M(x, y)$ divides the line-segment AB in the ratio $b : a$ externally, We take it as $b : -a$ internally.

$$\begin{aligned} x &= \frac{ab\cos\beta - ab\cos\alpha}{b - a} \\ &= \frac{ab}{b-a}(\cos\beta - \cos\alpha) \\ &= \frac{ab}{b-a}\left(-2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \qquad \ldots(1) \end{aligned}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\begin{aligned}
y &= \frac{b\cdot(a\sin\beta)-a(b\sin\alpha)}{b-a}\\
&= \frac{ab}{b-a}(\sin\beta-\sin\alpha)\\
&= \frac{ab}{b-a}\left(2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \quad \ldots(2)\\
&\left[\because \sin C-\sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]
\end{aligned}$$

$$\frac{x}{y} = \frac{-\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)}$$

$$\Rightarrow \quad x\cdot\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=0$$

Question-2(a) Determine the extreme values of the function $f(x,y)=3x^2-6x+2y^2-4y$ in the region $\{(x,y)\in\mathbb{R}^2 : 3x^2+2y^2\le 20\}$

[10 Marks]

Solution: **Method-1:**

First we find the critical points $f(x,y)=3x^2-6x+2y^2-4y$

$$f_x=0\Rightarrow 6x-6=0\Rightarrow x=1$$

$$f_y=0\Rightarrow 4y-4=0\Rightarrow y=1$$

$$f_{xy}=0$$

$\therefore P(1,1)$ is the only critical point. As

$$3(1)^2+2(1)^2=5<20$$

$\Rightarrow P(1,1)$ lies in the given elliptical region.

$$f(1,1)=3-6+2-4=-5 \quad \ldots(1)$$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(4)-0^2=24>0$$

and $f_{xx}=6>0$ at $P(1,1)$

Hence point (1,1) is a point of local minima. Let us check at boundaries of the ellipse i.e. $3x^2+2y^2=20$

$$\begin{aligned}
\therefore f(x,y) &= 3x^2-6x+2y^2-4y\\
&= 20-6x-4y\\
&= 20-6x\pm 2\sqrt{2}\sqrt{20-3x^2}
\end{aligned}$$

Let

$$g(x)=20-6x+2\sqrt{2}\sqrt{20-3x^2}$$

$$g'(x)=-6+2\sqrt{2}\frac{(-6x)}{2\sqrt{20-3x^2}}$$

$g'(x) = 0$ gives $x = \pm 2 \Rightarrow y = \mp 2$

$$\begin{aligned} &\text{At } (2, -2) \\ f(x, y) =& 20 - 6(2) - 4(-2) \\ =& 20 - 12 + 8 = 16 \quad \ldots (2) \end{aligned}$$

$$\begin{aligned} &\text{At } (-2, 2), \\ f(x, y) &= 20 - 6(-2) - 4(2) \\ &= 12 + 12 - 8 = 16 \quad \ldots (3) \end{aligned}$$

Again let

$$h(x) = 20 - 6x - 2\sqrt{2}\sqrt{20 - 3x^2}$$

$$\left(y = \frac{1}{\sqrt{2}}\sqrt{20 - 3x^2}\right)$$

$$h'(x) = -6 + 2\sqrt{2} \cdot \frac{6x}{2\sqrt{20 - 3x^2}}$$

$$h'(x) = 0 \quad \Rightarrow x = \pm 2 \quad \Rightarrow \quad y = \pm 2$$

At

$$(2, 2) \Rightarrow f(x, y) = 20 - 12 - 8 = 0 \quad \ldots (4)$$

At

$$(-2, -2) \quad f(x, y) = 20 + 12 + 8 = 40 \quad \ldots (5)$$

From (1),(2),(3),(4) and (5) ,we get max at $(-2, -2)$,

$$f(x, y) = 40$$

min at $(1, 1)$,

$$f(x, y) = -5$$

Method-2:

Using polar coordinates (elliptical)

$$3x^2 + 2y^2 = 20 \Rightarrow \quad \frac{x^2}{\frac{20}{3}} + \frac{y^2}{10} = 1$$

Let

$$x = 2\sqrt{\frac{5}{3}}r\cos\theta, \quad y = \sqrt{10}\,\mathrm{r}\sin\theta$$

for $0 \leq r \leq 1$, it gives elliptical region $\{3x^2 + 2y^2 \leq 20\}$.

$$\begin{aligned} f(x, y) &= 3x^2 - 6x + 2y^2 - 4y \\ &= 20r^2 - 12\sqrt{\frac{5}{3}}r\cos\theta - 4\sqrt{10} \cdot r\sin\theta \\ &= 20r^2 - 4\sqrt{5}r(\sqrt{3}\cos\theta - \sqrt{2}\sin\theta) \\ &= 20r^2 - 20\,\mathrm{r}\left(\sqrt{\frac{3}{5}}\cos\theta - \sqrt{\frac{2}{5}}\sin\theta\right) \\ &= 20r^2 - 20r(\sin(A - \theta)) \end{aligned}$$

where,

$$\left(\sin A = \sqrt{\frac{3}{5}}, \quad \cos A = \sqrt{\frac{2}{5}}\right)$$

$$f(r,\theta) = 20r[r - \sin(A-\theta)]$$

Max value of $f(r,\theta)$ will occur where

$$\sin(A-\theta) = -1 \text{ and } r = 1$$
$$f(r,\theta) = 20(1)(1-(-1)) = 40$$

for minimum , $sin(A-\theta) = 1$

$$f(r,\theta) = 20r(r-1) = 20(r^2 - r)$$
$$f'(r,\theta) = 20r(r-1) \Rightarrow r = \frac{1}{2}$$
$$\text{min value} \quad 20 \times \frac{1}{2}(\frac{1}{2} - 1) = -5$$

Question-2(b) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1,1,0,0)^T$.
Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$.

[15 Marks]

Solution: Let $\lambda_1 = 4$, $v_2 = (1,1,0,0)^T$

$$Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(2,2,0,0) = (\lambda_2, \lambda_2, 0, 0)$$
$$\Rightarrow \quad \lambda_2 = 2$$

Let the other two eigenvalues be λ_3 and λ_4.

Trace (A) = sum of eigenvalues

$$4 + 2 + \lambda_3 + \lambda_4 = -1 + 5 + (-10) + 8$$

$$\lambda_3 + \lambda_4 = -4$$

Also, product of eigenvalues = Det(A)

$$4 \cdot 2 \cdot \lambda_3 \cdot \lambda_4 = 0 \quad \Rightarrow \lambda_3\lambda_4 = 0$$

i.e.

$$\lambda_3(-4-\lambda_3)=0$$
$$\Rightarrow \quad \lambda_3=0 \quad or \quad \lambda_3=-4$$
$$\therefore \quad \lambda_4=-4 \quad or \quad \lambda_4=0$$

Characteristic polynomial

$$\begin{aligned}&\Pi(x-\lambda i)=0\\&(x-4)(x-2)(x+4)(x-0)=0\\&\left(x^2-16\right)(x-2)x=0\\&\left(x^3-16x-2x^2+32\right)x=0\\&x^4-16x^2-2x^3+32x=0\end{aligned}$$

Since, every square matrix satisfies its characteristic equation (Cayley Hamilton Theorem)

$$\therefore \quad A^4-2A^3-16A^2+32A=0$$

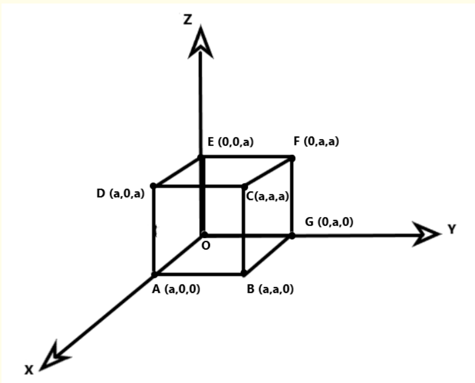
$$\therefore p=-2, q=-16, r=32$$

Question-2(c) A line makes angles $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube. Show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

[15 Marks]

Solution: The D.R. of four diagonals



$$\begin{aligned}AF&=(-a,a,a)\\&=(-1,1,1)\\BE&=(-a,-a,a)\\&=(1,1,-1)\\CO&=(-a,-a,-a)\\&=(1,1,1)\\DG&=(-a,a-a)\\&=(1,-1,1)\end{aligned}$$

Let the D.R.'s of line are $\langle l,m,n\rangle$

$$\cos\alpha=\frac{-l+m+n}{\sqrt{3}\cdot\sqrt{\ell^3+m^2+n^2}};\quad cos\beta=\frac{l+m-n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}}$$

$$\cos\gamma = \frac{\ell + m + n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}} \qquad \cos\delta = \frac{l - m + n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}}$$

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta &= \frac{1}{3\left(\ell^2+m^2+n^2\right)}\left[(-l+m+n)^2 + (l+m-n)^2\right.\\ &\qquad\left. +(l+m+n)^2 + (l-m+n)^2\right]\\ &= \frac{4\left(l^2+m^2+n^2\right)}{3\left(l^2+m^2+n^2\right)} = \frac{4}{3}\end{aligned}$$

Question-3(a) Consider the vectors $x_1 = (1,2,1,-1)$, $x_2 = (2,4,1,1)$, $x_3 = (-1,-2,0,-2)$ and $x_4 = (3,6,2,0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1,x_2,x_3,x_4\}$ is a subspace of $\mathbb{R}^4$, defined as

$$\left\{(\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbf{R}^4 : 2\xi_1 - \xi_2 = 0, \quad 2\xi_1 - 3\xi_3 - \xi_4 = 0\right\}$$

Can this subspace be written as $\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathrm{R}\}$? What is the dimension of this subspace?

[15 Marks]

Solution:

$$x_1 = (1,2,1,-1), \quad x_2 = (2,4,1,1)$$
$$x_3 = (-1,-2,0,-2), \quad x_4 = (3,6,2,0)$$

We find span $\{x_1,x_2,x_3,x_4\}$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -2 \\ 3 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{span}\{(1,2,0,2),(0,0,1,-3)\}$$
$$= \{a(1,2,0,2) + b(0,0,1,-3); a, b \in R\}$$
$$= \{(a, 2a, b, 2a - 3b)\}$$
$$= \{(x,y,z,w) : x = a, y = 2a, z = b, w = 2a - 3b\}$$
$$i.e. \quad y = 2x, w = 2x - 3z\}$$

If we take $a = \alpha, b = \beta$ then above subspace can be written as

$$\{\alpha, 2\alpha, \beta, 2\alpha - 3\beta\}, \quad Dim = 2\ .$$

as α and β are linearly independent.

Question-3(b) The dimensions of a rectangular box are linear functions of time- $l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and the height is decreasing at the rate 3 cm/sec find the rates at which the volume V and with respect to time. If $l(0) = 10$, $w(0) = 8$ and the surface area S are changing $h(0) = 20$, is V increasing or decreasing, when $t = 5$ sec? What about S, when $t = 5$ sec?

[10 Marks]

Solution: Given

$$\frac{dl}{dt} = 2\text{cm/sec} \quad \frac{dw}{dt} = 2 \quad \frac{dh}{dt} = -3$$

$$l = 2t + \ell_0, \quad w = 2w + w_0, \quad h = -3t + h_0$$

Using $l(0) = 10, w(0) = 8, h(0) = 20$

$$l = 2t + 10$$

$$w = 2t + 8$$

$$h = -3t + 20$$

At $t = 5$ sec,

$$l = 20cm, \quad w = 18cm, \quad h = 5cm$$

$$\begin{aligned} V &= lwh \\ &= (2t+10)(2t+8)(-3t+20) \end{aligned}$$

$$\frac{dV}{dt} = 2(2t+8)(-3t+20) + 2(2t+10)(-3t+20)$$

$$(-3)(2t+10)(2t+8)$$

$$\begin{aligned} \left.\frac{dv}{dt}\right|_{t=5} &= 2(18)(5) + 2(20)(5) - 3(20)(18) \\ &= 180 + 200 - 1080 \\ &= -700 < 0 \quad (\text{ Decreasing } V) \end{aligned}$$

$$\text{Surface Area, } S = 2(lw + wh + hl)$$

$$\begin{aligned} \frac{dS}{dt} &= 2\left[(w+h)\frac{dl}{dt} + (l+h)\frac{dw}{dt} + (l+w)\frac{dh}{dt}\right] \\ &= 2[23(2) + 25(2) + 38(-3)] \\ &= 2(46 + 50 - 114) = -36 < 0 \end{aligned}$$

Therefore, S is decreasing.

Question-3(c) Show that the shortest distance between the straight lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

and

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

is $3\sqrt{30}$. Find also the equation of the line of shortest distance.

[15 Marks]

Solution: Let $A(3,8,3)$ and $B(-3,-7,6)$ are points lying on the lines L_1 and L_2.

$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$
$$L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

D.R. of line which is perpendicular to L_1 and L_2 both (i.e. shortest distance line)

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3}$$
$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3}$$
$$< 2, 5, -1 >$$

$$\text{D.R.'s of AB} = \langle 3+3, 8+7, 3-6 \rangle$$
$$= \langle 6, 15, -3 \rangle$$

$$\therefore S.D. = \frac{1}{\sqrt{4+25+1}}(2\cdot 6 + 5\cdot 15 + (-1)(-3))$$
$$= \frac{1}{\sqrt{30}}(12+75+3) = \frac{90}{\sqrt{30}} = 3\sqrt{30}$$

Since, S. D. line is parallel to AB. Hence taking $A(3,8,3)$ as one point its equation is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

[**Alternate Method**]: (When equation of Shortest Distance is asked)
Let

$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$
$$L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Any general point on $L_1$is $P(3a+3, -a+8, a+3)$ and,
any general piint on L_2 is $Q(-3b-3, 2b-7, 4b+6)$.
$\therefore$ D.R.'s of PQ are $\langle P-Q \rangle$ i.e. $\langle 3a+3b+6, -a-2b+15, a-4b-3 \rangle$.

If PQ is the shortest distance line, it will be perpendicular to both the lines L_1 and L_2.

$$\therefore \quad 3(3a+3b+6)-1(-a-2b+15)+1(a-4b-3)=0$$
$$\Rightarrow 11a+7b=0 \qquad \ldots(1)$$
$$Also, -3(3a+3b+6)+2(-a-2b+15)+4(a-4b-3)=0$$
$$\Rightarrow (-9a-9b-18)+(-2a-4b+30)+(4a-16b-12)=0$$
$$\Rightarrow -7a-29b=0 \qquad \ldots(2)$$

From (1) and (2) we get $a=0, b=0$.

$$\therefore \quad P \text{ is } (3,8,3) \text{ and } Q \text{ is } (-3,-7,6)$$

$$\begin{aligned}\text{Shortest Distance,} \quad PQ &= \sqrt{(3+3)^2+(8+7)^2+(3-6)^2} \\ &= \sqrt{36+225+9} \\ &= \sqrt{270} = 3\sqrt{30}\end{aligned}$$

D.R.'s of PQ $\langle 6,15,-3\rangle$ i.e. $\langle 2,5,-1\rangle$

$$\therefore \text{Equation of Shortest Distance is: } \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

.

Question-4(a) Using elementary row operations, reduce the matrix

$$A=\begin{bmatrix}2&1&3&0\\3&0&2&5\\1&1&1&1\\2&1&1&3\end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX=b$, where $X=(x,y,z,u)^T$ and $b=(2,1,0,4)^T$

[15 Marks]

Solution:

$$A = IA$$

$$\begin{bmatrix}2&1&3&0\\3&0&2&5\\1&1&1&1\\2&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}A$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix}1&1&1&1\\3&0&2&5\\2&1&3&0\\2&1&1&3\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}A$$

$$R_2 \to R_2-3R_1, R_3 \to R_3-2R_1, \quad R_4-1R_4-2R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$$R_3 \to R_3 - 3R_2, \quad R_4 \to R_4 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_4 \to R_4 - \frac{R_3}{2}, R_3 \to \frac{R_3}{-4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ 1/2 & -1/2 & -3/2 & 1 \end{bmatrix} A$$

$$R_4 \to -R_4, \quad R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A$$

$$R_3 \to R_3 + 2R_4, \quad R_2 \to R_2 - 2R_4, R_1 \to R_1 - R_4$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1 \\ 0 & -1 & -1 & 2 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix}$$

$$R_2 \to R_2 + R_3, \quad R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -5/4 & 11/4 & 3 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix}$$

$$R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -4 & 3 \\ -1/4 & -1/4 & 5/4 & 0 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A$$

$$AX = b$$

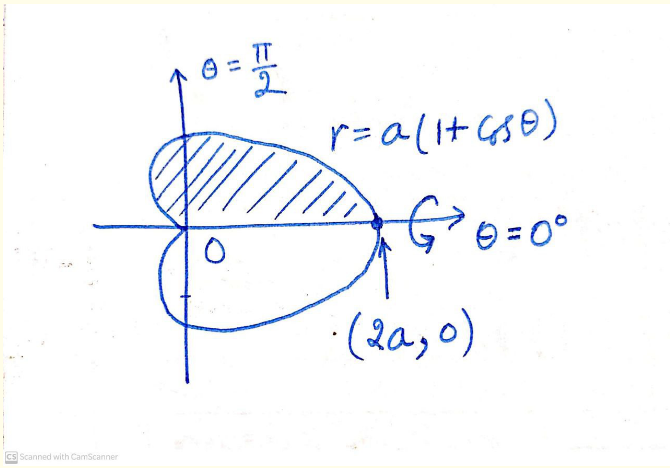
$$X = A^{-1}b$$

$$X = \frac{1}{4} \begin{bmatrix} 4 & -4 & -16 & 12 \\ -1 & -1 & 5 & 0 \\ -1 & 3 & 9 & -8 \\ -2 & 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8-4+48 \\ -2-1+0 \\ -2+3-32 \\ -4+2-16 \end{bmatrix} = \begin{bmatrix} 13 \\ -3/4 \\ -31/4 \\ -9/2 \end{bmatrix}$$

Question-4(b) Find the centroid of the solid generated by revolving the upper half of the cardioid $r = a(1+\cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform.

[10 Marks]



Solution:

As the solid of revolution is symmetric about initial line (x-axis), the centroid will lie on it. ie. y-coordinate will be zero. $\bar{y} = 0$
x-coordinate

$$\bar{x} = \frac{\int x dV}{\int dV}$$

[in polar-coordinates $x = r\cos\theta$

$$dV = 2\pi r^2 \sin\theta d\theta dr,$$

θ varies from 0 to π (upper part)]

$$V = \int_0^{\pi}\int_0^{a(1+cos\theta)} 2\pi r^2 \sin\theta dr d\theta$$

$$= 2\pi\int_0^{\pi} \left.\frac{r^3}{3}\right|_0^{a(1+cos\theta)} \sin\theta d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3\sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}\cdot\left.\frac{(1+cos\theta)^4}{-4}\right|_0^{\pi}$$

$$= \frac{2\pi a^3}{3}\cdot\frac{16}{-4} = \frac{8\pi}{3}a^3$$

$$\begin{aligned}\int xdV &= \int_0^{\pi}\int_0^{a(1+cos\theta)} (rcos\theta)\left(2\pi r^2\sin\theta\right) drd\theta \\ &= 2\pi\int_0^{\pi}\int_0^{a(1+cos\theta)} r^3 cos\theta sin\theta drd\theta \\ &= \frac{2\pi}{4}\int_0^{\pi} a^4(1+cos\theta)^4\cos\theta\sin\theta d\theta \\ &= \frac{2\pi}{4}\int_0^{\pi} a^4(1+cos\theta)^4(\cos\theta+1-1)\sin\theta d\theta \\ &= \frac{\pi}{2}a^4\int_0^{\pi}\left[(1+\cos\theta)^5\sin\theta-(1+cos\theta)^4\sin\theta\right]d\theta \\ &= \frac{\pi}{2}a^4\left[\frac{(1+\cos\theta)^6}{6}-\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} \\ &= \frac{\pi a^4}{2}\left(\frac{64}{6}-\frac{32}{5}\right) = \frac{\pi a^4\times 32}{15} \\ \therefore \bar{x} &= \frac{32\pi a^4}{15}\times\frac{3}{8\pi a^3} = \frac{4a}{5}\end{aligned}$$

$\therefore$ Centroid is $\left(\frac{4a}{5},0\right)$.

Question-4(c) A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A, B and C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

[15 Marks]

Solution: Let the equation of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=p \quad \ldots(1)$$

It meets the axis at points $A(ap,0,0)$, $B(0,bp,0)$, $c(0,0,cp)$ We find equation of sphere passing through origin $O(0,0,0)$ and A, B, C

$$x^2+y^2+z^2-apx-bpy-cpz=0 \quad \ldots(2)$$

Equation (1) and (2) together gives the equation of circle ABC. If we homogenize equation (2) with help of equation (1) , we will get the equation of cone with vertex at origin.

$$x^2+y^2+z^2-(apx+bpy+cpz)\left(\frac{x}{ap}+\frac{y}{bp}+\frac{z}{cp}\right)=0$$

$$\therefore yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

2.2 Section-B

Question-5(a) Solve the differential equation $(D^2+1)\,y = x^2 \sin 2x;\quad D \equiv \dfrac{d}{dx}.$

[8 Marks]

Solution: Auxiliary Equation:

$$D^2+1=0 \Rightarrow D=\pm i$$

$$C\cdot F \quad y = c_1\cos x + c_2 \sin x$$

$$\begin{aligned}
P.I. &= \frac{1}{D^2+1}x^2\sin 2x \\
&= Img \text{ part of } \frac{1}{D^2+1}x^2\cdot e^{i2x} \\
&= \text{Im}\left[e^{i2x}\frac{1}{(D+2i)^2+1}x^2\right]
\end{aligned}$$

$$\left(\because \frac{1}{f(D)}Ve^{ax} = e^{ax}\frac{1}{f(D+a)}V\right)$$

$$\begin{aligned}
&= \text{Im}\left(e^{i2x}\frac{1}{D^2+4iD-4+1}x^2\right) \\
&= \text{Im}\left(\frac{-e^{i2x}}{3}\cdot\left(1-\left(\frac{D^2+4Di}{3}\right)\right)^{-1}x^2\right) \\
&= \text{Im}\left[-\frac{e^{i2x}}{3}\left(1+\frac{D^2+4Di}{3}+\frac{16D^2i^2}{9}+\ldots\right)x^2\right]
\end{aligned}$$

$\{$ using Binomial expansion and neglecting, higher powers of D. $\}$

$$\begin{aligned}
&= \text{Im}\left(-\frac{e^{i2x}}{3}\left(x^2+\frac{(-26)}{9}+\frac{8xi}{3}\right)\right) \\
&= Im\left[-\frac{1}{3}\left(\cos^2 x + i\sin 2x\right)\left(x^2-\frac{26}{9}+i\frac{8x}{3}\right)\right] \\
&= \frac{-1}{3}\left[(\sin 2x)\left(x^2-\frac{26}{9}\right)+(\cos 2x)\frac{8x}{3}\right]
\end{aligned}$$

$\therefore$ complete solution

$$y = (C\cdot F + P\cdot I)$$

$$\Rightarrow y = c_1\cos x + c_2\sin x - \frac{1}{3}\left[(\sin 2x)\left(x^2-\frac{26}{9}\right)+(\cos 2x)\frac{8x}{3}\right]$$

Question-5(b) Solve the differential equation $(px-y)(py+x)=h^2p$, where $p=y'$.

[8 Marks]

Solution:

$$p^2xy+px^2-py^2-xy=h^2p$$

Put,

$$x^2=u, \quad y^2=v$$

$$P=\frac{dv}{du}=\frac{y}{x}p$$

$$(2xdx=du, \quad 2ydy=dv)$$

i.e.

$$p=\frac{x}{y}P=\sqrt{\frac{u}{v}}P$$

$\therefore$ The given $D\cdot E.$ transforms to

$$\frac{u}{v}P^2\sqrt{u}\sqrt{v}+\sqrt{\frac{u}{v}}Pu-\sqrt{\frac{u}{v}}Pv-\sqrt{u}\sqrt{v}$$
$$=h^2\sqrt{\frac{u}{v}}P$$

$$uP^2\sqrt{u}+\sqrt{u}\cdot uP-\sqrt{u}vP-\sqrt{u}v=h^2\sqrt{u}P$$

i.e.

$$uP^2+uP-vP-v=h^2P$$

$$u\left(P^2+P\right)-v(P+1)=h^2P$$

$$uP-v=\frac{h^2P}{P+1}$$

$$\left[v=uP-\frac{h^2P}{P+1}\right]$$

This is in Clairaut's form

$$y=px+f(p)$$

So, replacing P with c, we have general solution

$$v=cu-\frac{ch^2}{c+1}$$

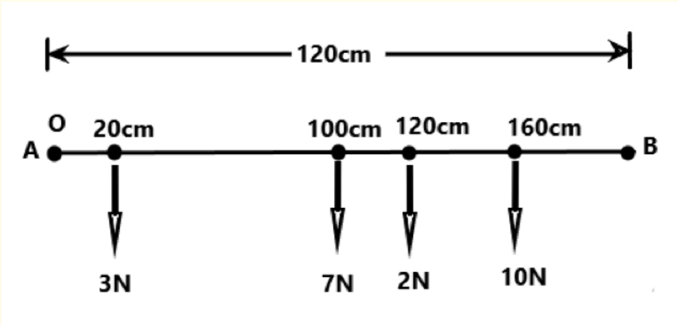
ie.

$$y^2=cx^2-\frac{ch^2}{c+1}$$

Question-5(c) A 2 meters rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hing loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal.

[8 Marks]

Solution: Varignon's Theorem: Moment of a force about a point is equal to the sum of the moments of the forces components about the point.



Let us take moments about point A. Resultant of all forces $= 3 + 7 + 2 + 10 = 22N$

$$\therefore (22 \times r) = 3 \times 20 + 7 \times 100 + 2 \times 120 + 10 \times 160 = 2600$$

$$\therefore \quad r = \frac{2600}{22} = \frac{1300}{11} = 118 \cdot 18cm$$

Hence rod must be supported at a point $118.18cm$ from end A.

Question-5(d) Let $\bar{r} = \bar{r}(s)$ represent a space curve. Find $\dfrac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T}, \bar{N}$ and $\bar{B}$ where $\bar{T}$, $\bar{N}$ and $\bar{B}$ represent tangent, principal normal and binormal respectively. Compute $\dfrac{d\bar{r}}{ds} \cdot \left(\dfrac{d^2\bar{r}}{ds^2} \times \dfrac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion.

[8 Marks]

Solution:

$$\vec{T} = \frac{d\vec{r}}{ds}$$

$$k\vec{N} = \frac{d\vec{T}}{ds} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right) = \frac{d^2\vec{r}}{ds^2}$$

i.e.

$$\frac{d^2\vec{r}}{ds^2} = k\vec{N}$$

$$\Rightarrow \frac{d^3\vec{r}}{ds^3} = k\frac{d\vec{N}}{ds} + \frac{dk}{ds}\vec{N} \ldots (1)$$

$$\frac{d^3\vec{r}}{ds^3} = k(\vec{B}\tau - k\vec{T}) + \frac{d}{ds}\left(\left|\frac{d\vec{T}}{ds}\right|\right) - \vec{N} \ldots (2)$$

(Serret-Frenet)

$$\Rightarrow \frac{d\vec{N}}{ds} = \tau\vec{B} - k\vec{T}$$

$$\begin{aligned}\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} &= k\vec{N} \times \left[k(\vec{B}\tau - k\vec{T}) + \frac{dk}{dS}\vec{N}\right] \qquad [\text{ using } (1)\ ] \\ &= k^2\tau(\vec{N} \times \vec{B}) - k^3(\vec{N} \times \vec{T}) \\ &= k^2\tau\vec{T} - k^3\vec{B}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) &= \vec{T} \cdot \left(k^2\tau\vec{T} - k^3\vec{B}\right) \\ &= k^2\tau \qquad (\because \vec{T} \cdot \vec{B} = 0)\end{aligned}$$

Question-5(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\, dx - 3x^2y^2 dy$ **along the path** $x^4 - 6xy^3 = 4y^2$**.**

[8 Marks]

Solution: The integral is of the form

$$\int_c M dx + N dy$$

where

$$M = 10x^4 - 2xy^3$$
$$N = -3x^2y^2$$
$$\frac{\partial M}{\partial y} = -6xy^2, \quad \frac{\partial N}{\partial x} = -6xy^2$$

Method-1: As

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given integral is path-independent. It means we can use any path.

Let the path consists of straight line L_1 : from $(0,0)$ to $(2,1)$ and then L_2 : from $(2,0)$ to $(2,1)$

Along $L_1 : y = 0 \Rightarrow dy = 0$

Along $L_2 : x = 2 \Rightarrow dx = 0$

Value of integral

$$\int_{x=0}^{2} 10x^4 dx + \int_{y=0}^{1} -3(2)^2 y^2 dy = 2x^5\Big|_0^2 - 4y^3\Big|_0^1$$
$$= 64 - 4$$
$$= 60$$

Method-2: As

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\therefore \left(10x^4 - 2xy^3\right) dx - \left(3x^2y^2\right) dy$$

is an exact differential of $(2x^5 - x^2y^3)$.

$$\therefore \int_{(0,0)}^{(2,1)} \left(10x^4 - 2xy^3\right) dx - 3x^2y^2 dy = \int_{(0,0)}^{(2,1)} d\left(2x^5 - x^2y^3\right)$$
$$= \left(2x^5 - x^2y^3\right)\Big|_{(0,0)}^{(2,1)}$$
$$= 64 - 4$$
$$= 60$$

Question-6(a) Solve by the method of variation of parameters the differential equation

$$x''(t) - \frac{2x(t)}{t^2} = t, \text{ where } 0 < t < \infty$$

[15 Marks]

Solution:

$$\left[D^2 - \frac{2}{t^2}\right] x(t) = t$$

i.e.

$$\left[t^2D^2 - 2\right] x(t) = t^3$$

Put

$$t = e^u \quad \therefore \quad u = \log t$$
$$D' = \frac{d}{du} = tD; \quad D'\left(D' - 1\right) = t^2D^2$$
$$\therefore \left[D'\left(D' - 1\right) - 2\right] x = e^{3u}$$
$$\left(D'^2 - D' - 2\right) x = e^{3u} - (1)$$
$$D' = 2, -1$$
$$C.F = c_1' e^{2u} + c_2' e^{-u}$$

Now, we use the variation of parameters to find complete integral of $D \cdot F$ (1) Replacing c_1', c_2' by unknown functions A and B, the complete solution is

$$y = Ae^{2u} + Be^{-u}$$
$$= Ay_1 + By_2$$

where,

$$y_1 = e^{2u}, \quad y_2 = e^{-u}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2u} & e^{-u} \\ 2e^{2u} & -e^{-u} \end{vmatrix}$$

$$= -e^u - 2e^u = -3e^u \neq 0$$

$\therefore y_1$ & y_2 are independent.

$$\begin{aligned} A &= -\int \frac{y_2 R}{w} du \quad , \quad R = e^{3u} \\ &= -\int \frac{e^{-u} \cdot e^{3u}}{-3e^u} du \\ &= \frac{1}{3} \int e^u du = \frac{e^u}{3} + c_1 \end{aligned}$$

$$\begin{aligned} B &= \int \frac{y_1 R}{w} du \\ &= \int \frac{e^{2u} \cdot e^{3u}}{-3e^u} du \\ &= -\frac{1}{3} \int e^{4u} du = \frac{-e^{4u}}{12} + c_2 \end{aligned}$$

$\therefore$ Complete Solution

$$\begin{aligned} y &= Ay_1 + By_2 \\ &= \left(\frac{e^u}{3} + c_1\right) e^{2u} + \left(\frac{-e^{4u}}{12} + c_2\right) e^{-u} \\ y &= \left(\frac{t}{3} + c_1\right) t^2 + \left(-\frac{t^4}{12} + c_2\right) \frac{1}{t} \end{aligned}$$

Question-6(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force).

[15 Marks]

Solution: $r^2 = a^2 \cos 2\theta$ or $a^2 u^2 \cos 2\theta = 1$, $u = \frac{1}{r} - (1)$ Taking log,

$$2 \log a + 2 \log u + \log \cos 2\theta = 0$$

Differentiating w.r.t. θ

$$0 + \frac{2}{u} \cdot \frac{du}{d\theta} - \frac{2 \sin 2\theta}{\cos 2\theta} = 0$$

$$\frac{du}{d\theta} = u \tan 2\theta$$

$$\begin{aligned}\frac{d^2u}{d\theta^2} &= 2u\sec^2 2\theta + \frac{du}{d\theta}\tan 2\theta\\ &= 2u\sec^2 2\theta + u\tan^2 2\theta\end{aligned}$$

$$\begin{aligned}\therefore \frac{d^2u}{d\theta^2} + u &= 2u\sec^2 2\theta + u\tan^2 2\theta + u\\ &= 3u\sec^2 2\theta = 3u\left(a^2u^2\right)^2 \text{[from (1)]}\\ &= 3a^4u^5\end{aligned}$$

WKT DE of the central orbit in polar form is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$\therefore \frac{F}{h^2u^2} = 3a^4u^5 \Rightarrow F = 3h^2a^4u^7$$

from (1)

$$= k\frac{1}{r^7}$$

i.e.

$$F \propto \frac{1}{r^7}$$

Hence the force varies inversely as the 7^{th} power of the distance from the pole.

Question-6(c) Verify Stokes' theorem for $\bar{V} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

[10 Marks]

Solution: Stokes' Theorem:

$$\oint_C \vec{F}\cdot dr = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}dS$$

Here, the boundary C of S is a circle in xy plane $x^2+y^2=1$
Let

$$x = \cos t, y = \sin t, \quad z = 0$$

$0 \le t \le 2\pi$ be parametric equation of C

$$\oint_c \vec{V}\cdot dr = \oint_c \left[(2x-y)i - yz^2j - y^2zk\right]\cdot[dxi + dyj + dzk]$$

$$= \oint (2x-y)dx - yz^2dy - y^2zdz$$

$$= \phi(2x-y)dx \quad [\because z = 0 \quad \& \quad dz = 0]$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= -\int_0^{2\pi} \sin 2t - \left(\frac{1-cos2t}{2}\right)dt$$

$$= -\left[-\frac{cos2t}{2} - \frac{t}{2} + \frac{\sin^2 t}{4}\right]_0^{2\pi}$$
$$= -\left[\left(-\frac{1}{2} - \frac{2\pi}{2} + 0\right) - \left(\frac{-1}{2} - 0 + 0\right)\right]$$
$$= \pi$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} 1 & J & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$
$$= i(-2yz + 2yz) + j(0 - 0) + k(0 + 1)$$
$$= k - (1)$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dS = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$
$$= \iint_D k \cdot (xi + yj + zk) \frac{dxdy}{z}$$
$$= \iint_D dxdy$$

$[D : x^2 + y^2 \leq 1$ unit circle in xy plane centered at origin$]$ Area of circle D

$$\pi(1)^2 = \pi - (2)$$

From (1) and (2), we see that

$$\oint_C \vec{V} \cdot dr = \iint_S (\nabla \times \vec{V}) \cdot \hat{n} dS$$

Question-7(a) Find the general solution of the differential equation

$$\ddot{x} + 4x = \sin^2 2t$$

Hence find the particular solution satisfying the conditions

$$x\left(\frac{\pi}{8}\right) = 0 \quad \text{and} \quad \dot{x}\left(\frac{\pi}{8}\right) = 0$$

[15 Marks]

Solution: Let

$$D = \frac{d}{dt}, \quad D^2 = \frac{\partial^2}{dt^2}$$
$$\left(D^2 + 4\right) x = \sin^2 2t$$

Auxiliary Eqn: $D^2 + 4 = 0$

$$D = \pm 2i$$
$$C \cdot F = c_1 \cos 2t + c_2 \sin 2t$$

$$\begin{aligned}
P \cdot I &= \frac{1}{D^2+4} \sin^2 2t \\
&= \frac{1}{D^2+4} \left(\frac{1-\cos 4t}{2} \right) \\
&= \frac{1}{2} \frac{1}{D^2+4} \cdot 1 - \frac{1}{2} \frac{1}{D^2+4} \cos 4t \\
&= \frac{1}{2} \frac{1}{D^2+4} e^{0t} - \frac{1}{2} \cdot \frac{\cos ut}{(-16)+4} \\
&= \frac{1}{8} + \frac{1}{24} \cos 4t
\end{aligned}$$

General solution: $\quad x = C \cdot F + P \cdot I$

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{24}(3 + \cos 4t)$$

$$x\left(\frac{\pi}{8}\right) = 0 \Rightarrow c_1 + c_2 = \frac{-\sqrt{2}}{8}$$

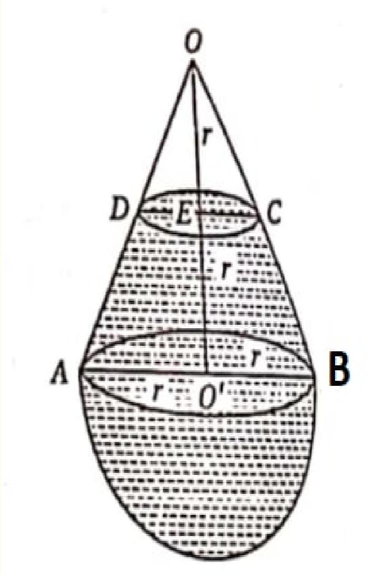
$$\dot{x}\left(\frac{\pi}{8}\right) = 0 \Rightarrow \quad c_2 - c_1 = \frac{\sqrt{2}}{12}$$

$$\therefore \quad c_1 = \frac{-5\sqrt{2}}{48}, \quad c_2 = \frac{-\sqrt{2}}{48}$$

Question-7(b) A vessel is in the shape of a hollow hemisphere surmounted by a cone held with the axis vertical and vertex uppermost. If it is filled with a liquid so as to submerge half the axis of the cone in the liquid and height of the cone be double the radius (r) of its base, find the resultant downward thrust of the liquid on the vessel in terms of the radius of the hemisphere and density (p) of the liquid.

[15 Marks]

Solution: Let r be the radius of the base of the hemisphere or cone so that the height of the surmounting cone is $2r$.



The vessel is filled upto CD so as to submerge half the axis of the cone in the liquid. From similar triangles OEC and $OO'B$, we have

$$\frac{EC}{O'B}=\frac{OE}{OO'}=\frac{r}{2r}=\frac{1}{2}$$

$$\therefore C=\frac{1}{2}OB'=\frac{1}{2}r$$

The resultant downward thrust of the liquid on the vessel = weight of the liquid contained in the vessel = wt. of the liquid in the hemisphere+ wt. of the liquid in the frustum

$$=\frac{2}{3}\pi r^3 w+\left[\frac{1}{3}\pi r^2\cdot 2r-\frac{1}{3}\pi\left(\frac{r}{2}\right)^2\cdot r\right]w$$

$$=\frac{2}{3}\pi r^3 w+\frac{1}{3}\pi r^3 w\left(2-\frac{1}{4}\right)=\frac{1}{3}\pi r^3 w\left(2+\frac{7}{4}\right)=\frac{1}{3}\pi r^3 w\cdot\frac{15}{4}$$

$$=\frac{15}{8}(\frac{2}{3}\pi r^3 w)$$

Question-7(c) Derive the Frenet-Serret formulae. Verify the same for the space curve $x=3\cos t$, $y=3\sin t$, $z=4t$

[10 Marks]

Solution: i) $\dfrac{dT}{ds}=kN$

ii) $\dfrac{dB}{ds}=-\tau N$

iii) $\dfrac{dN}{ds}=\tau B-kT$

where T, N, B are unit vectors along tangent principal normal and binormal directions.

$$|T|=1 \quad\Rightarrow\quad T\cdot T=1$$

$$\Rightarrow \quad 2T \cdot \frac{dT}{ds} = 0$$

$\Rightarrow \dfrac{dT}{ds}$ is $\perp$ to T.
Also, $\dfrac{dT}{ds}$ lies in oscillating plane.

$$\therefore \frac{dT}{ds} \text{ is parallel to } N$$

$$\therefore \frac{dT}{ds} = kN$$

ii) since, $|B| = 1$, unit vector

$$\therefore B \cdot B = 1 \quad \Rightarrow \quad 2B \cdot \frac{dB}{ds} = 0$$

$\Rightarrow dB/ds$ is $\perp$ to B $\quad \ldots \quad (1)$
W.K.T $\dfrac{dB}{dS}$ lies in oscillating plane. Also, since B and T are $\perp$

$$B \cdot T = 0$$

$$\Rightarrow \quad B \cdot \frac{dT}{ds} + T \cdot \frac{dB}{ds} = 0$$

$$B \cdot (kN) + T\frac{dB}{ds} = 0$$

$$(B \cdot N)k + \frac{dB}{ds} \cdot T = 0$$

$$\Rightarrow \frac{dB}{ds} \cdot T = 0$$

$$(\because B \perp N)$$

i.e. $\dfrac{dB}{ds}$ is $\perp$ to T $\quad \ldots \quad (2)$
From (1) and (2), $\dfrac{dB}{ds}$ is parallel to N

$$\Rightarrow \frac{dB}{ds} = -\tau N \quad (\tau = torsion)$$

iii) $B \times T = N$

$$B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$$

$$k(-T) - \tau(-B) = \frac{dN}{ds}$$

$$\therefore \quad \frac{dN}{ds} = \tau B - kT$$

Here, $\quad x = 3\cos t, \quad y = 3\sin t, \quad z = 4t$

$$\vec{R} = (3\cos t)i + (3\sin t)j + (4t)k$$

$$\frac{d\vec{r}}{dt} = (-3\sin t)i + (3\cos t)j + 4k$$

$$\left|\frac{dr}{dt}\right| = \sqrt{9\sin^2 t + 9cos^2 t + 16} = 5$$

Let S be length of arc from $t = 0$ to any point t on the curve, then

$$S = \int_0^t \left| \frac{d\vec{r}}{dt} \right| dt = \int_0^t 5dt = 5t$$

$$\therefore \vec{r} = \left(3\cos\frac{s}{5}\right) i + \left(3\sin\frac{s}{5}\right) j + \left(\frac{4}{5}s\right) k$$

$$T = \frac{d\vec{r}}{ds} = \left(-\frac{3}{5}\sin\frac{s}{5}\right) i + \left(\frac{3}{5}\cdot\sin\frac{s}{5}\right] j + \frac{4}{5}k$$

$$\frac{dT}{ds} = \left(\frac{-3}{25}\cos\frac{s}{5}\right) - \frac{3}{25}\sin\frac{s}{5}j + 0$$

Principal Normal, N is parallel to $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix}$$

$$= i(0 + 12\sin t) + j(-12\cos t + 0) + k\left(+9\sin^2 t + 9\cos^2 t\right)$$

$$= (12\sin t)i - (12\cos t)j + 9k$$

$$\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}}) = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ 12\cos t & -12\sin t & 9 \end{vmatrix}$$

$$= i(27\cos t + 48\cos t) + j(48\sin t + 27\sin t) + k\left(36\sin t\cos t - 12\sin t\cos t\right)$$

$$= (75\cos t)i + (75\sin t)j$$

$$\left[\therefore \quad N = \pm\frac{1}{75}(75\cos ti + 75\sin tj) = -(\cos t)i + (-\sin t)j\right]$$

$$N = -(\cos\frac{s}{5})i + (-sin\frac{s}{5})j$$

(taking -ve sign)

$$\frac{dN}{ds} = -\frac{1}{5}\sin\frac{s}{5} + \frac{1}{5}\cos\frac{s}{5}j$$

Binormal vector B, is parallel to $\dot{\vec{r}} \times \ddot{\vec{r}}$

$$B = \frac{1}{\sqrt{144 + 81}}[12\sin ti - 12\cos tj + 9k]$$

$$B = \frac{12}{15}, \sin ti - \frac{12}{15}\cos tj + \frac{9}{15}k$$

$$B = \frac{4}{5}\sin\frac{s}{5}; -\frac{4}{5}\cdot\cos\frac{s}{5}j + \frac{3}{5}k$$

$$\frac{dB}{ds} = \frac{4}{25}\cos\frac{s}{5}i + \frac{4}{25}\sin\frac{s}{5}j$$

$$k = \left|\frac{dT}{ds}\right| = \frac{3}{25}, \quad \tau = \left|\frac{dB}{ds}\right| = \frac{4}{25}$$

Taking,

$$N = -\cos\frac{s}{5}; -\sin\frac{s}{5}j$$

i)

$$kN = \frac{3}{25}\left(-\cos\frac{s}{5}i - \sin\frac{s}{5}j\right) = \frac{dT}{ds}$$

ii)

$$-\tau N = \frac{-4}{25}\left(-\cos\frac{s}{5}i - \sin\frac{s}{5}j\right) = \frac{dB}{ds}$$

iii)

$$\begin{aligned}\tau B - kT &= \frac{4}{25}\left(\frac{4}{5}\sin\frac{s}{5}i - \frac{4}{5}\cos\frac{s}{5}j + \frac{3}{5}k\right) - \frac{3}{25}\left(\frac{-3}{5}\sin\frac{s}{5}i + \frac{3}{5}\cos\frac{s}{5}j + \frac{4}{5}k\right)\\ &= \frac{1}{5}\sin\frac{s}{5}; -\frac{1}{5}\cos\frac{s}{5}j\\ &= \frac{dN}{ds}\end{aligned}$$

Hence, we see that Frenet-Serret formulae are satisfied by the given curve in space.

Question-8(a) Find the general solution of the differential equation

$$(x-2)y'' - (4x-7)y' + (4x-6)y = 0$$

[10 Marks]

Solution:

$$y'' - \left(\frac{4x-7}{x-2}\right)y' + \left(\frac{4x-6}{x-2}\right)y = 0 - (1)$$

Comparing with:

$$y'' + Py' + Qy = 0$$

.

$$P = \frac{-(4x-7)}{x-2}, \quad Q = \frac{4x-6}{x-2}$$

Let e^{ax} be a solution, then

$$a^2 + aP + Q = 0$$

$$a^2 - \frac{a(4x-7)}{x-2} + \frac{4x-6}{x-2} = 0$$

$$\Rightarrow \quad a^2(x-2) - 4ax + 7a + 4x - 6 = 04$$

i.e.

$$x\left(a^2 - 4a + 4\right) - 2a^2 + 7a - 6 = 0$$

$$\Rightarrow \quad a^2 - 4a + 4 = 0 \quad \Rightarrow (a-2)^2 = 0 \Rightarrow a = 2$$

$$2a^2 - 7a + 6 = 0 \Rightarrow (2a-3)(a-2) = 0$$

i.e.

$$a = \frac{3}{2}, 2$$

$\therefore a = 2$ is common root $\Rightarrow e^{2x}$ is one solution.

$$\text{Consider } y = ve^{2x}$$
$$\frac{dy}{dx} = \frac{dv}{dx}e^{2x} + 2ve^{2x}$$
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^{2x} + 4\frac{dv}{dx}e^{2x} + 4ve^{2x}$$

Putting these values in (1)

$$\left(\frac{d^2v}{dx^2}e^{2x} + 4\frac{dv}{dx}e^{2x} + 4ve^{2x}\right) - \left(\frac{4x-7}{x-2}\right)\left(\frac{dv}{dx}e^{2x} + 2v^2x\right) + \frac{4x-6}{x-2}ve^{2x} = 0$$
$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left[4 - \frac{(4x-7)}{x-2}\right] + 4v - \frac{2(4x+7)}{x-2} + \frac{(4x-6)}{x-2} = 0$$
$$\frac{d^2v}{dx^2} - \frac{1}{x-2}\frac{dv}{dx} = 0$$

Let

$$\frac{dv}{dx} = p \Rightarrow \frac{dp}{dx} - \frac{1}{x-2}\cdot p = 0$$
$$\frac{dp}{p} = \frac{dx}{x-2}$$
$$\log p = \log(x-2) + \log C_1$$
$$\Rightarrow \quad p = c_1(x-2)$$

i.e.,

$$\frac{dv}{dx} = c_1(x-2)$$
$$v = c_1\left(\frac{x^2}{2} + 2x\right) + c_2\Big)$$
$$\therefore y = e^{2x}\cdot v = e^{2x}\left[c_1\left(\frac{x^2}{2} - 2x\right) + c_2\right]$$

Question-8(b) A shot projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h meters above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection.

[15 Marks]

Solution: We know that

$$x = (u\cos\theta)t$$
$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Equation of trajectory,

$$y = x\tan\theta - \frac{g}{2}\cdot\frac{x^2}{u^2\cos^2\theta}$$

When velocity is u, Range, $R = \dfrac{u^2 \sin 2\theta}{g}$ With new velocity (say v), point $p(R, h)$ lies on the equation of trajectory

$$\begin{aligned} h &= R\tan\theta - \frac{g}{2}\cdot\frac{R^2}{v^2\cos^2\theta} \\ &= \frac{u^2\sin 2\theta}{g}\cdot\tan\theta - \frac{g}{2}\left(\frac{u^2\sin 2\theta}{g}\right)^2\frac{1}{v^2\cos^2\theta} \\ &= \frac{2u^2\sin^2\theta}{g} - \frac{2u^4\sin^2\theta}{g\cdot v^2} \\ h &= \frac{2u^2\sin^2\theta}{g}\left(1-\frac{u^2}{v^2}\right) \\ 1-\frac{u^2}{v^2} &= \frac{gh}{2u^2\sin^2\theta} \\ \frac{u}{v} &= \left[1-\frac{gh}{2u^2\sin^2\theta}\right]^{1/2} \end{aligned}$$

i.e.

$$\begin{aligned} v &= u\left(1-\frac{gh}{2u^2\sin^2\theta}\right)^{-1/2} \\ &\simeq u\left(1+\frac{1}{2}\cdot\frac{gh}{2u^2\sin^2\theta}\right) \end{aligned}$$

(Binomial Approximation)

$$\therefore v-u = \frac{gh}{4u\sin^2\theta}$$

Which is the required increase in the velocity of projection with same elevation θ.

Question-8(c) Derive $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right)$ in spherical coordinates.

[15 Marks]

Solution:

$$\nabla^2 F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial F}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial F}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial F}{\partial u_3}\right)\right]$$

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$$

$$h_1 = h_R = 1 \quad , \quad h_2 = h_\theta = r$$

$$h_3 = h_\phi = r\sin\theta$$

$$\begin{aligned}
\nabla^2 F &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\frac{\partial F}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta\cdot 1}{r}\frac{\partial F}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1\cdot r}{r\sin\theta}\cdot\frac{\partial F}{\partial\phi}\right)\right.\\
&= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial F}{\partial\phi}\right)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial F}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 F}{\partial\phi^2}
\end{aligned}$$

$$\begin{aligned}
F &= \frac{x}{(x^2+y^2+z^2)^{3/2}}\\
&= \frac{r\sin\theta cos\phi}{r^3}\\
&= \frac{\sin\theta\cos\phi}{r^2}
\end{aligned}$$

$$\frac{\partial F}{\partial r} = \frac{-2\sin\theta cos\phi}{r^3}$$

$$\frac{\partial F}{\partial\theta} = \frac{\cos\theta cos\phi}{r^2}$$

$$\frac{\partial F}{\partial\phi} = \frac{-\sin\theta\sin\phi}{r^2}$$

$$\frac{\partial^2 F}{\partial\phi^2} = \frac{-\sin\theta cos\phi}{r^2}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) &= \frac{\partial}{\partial r}\left(\frac{-2\cdot r^2\sin\theta\cos\phi}{r^3}\right)\\
&= \frac{-2\sin\theta\cos\phi}{r^2}
\end{aligned}$$

$$\begin{aligned}
\therefore \nabla^2 F &= \frac{1}{r^2}\left(\frac{2\sin\theta cos\phi}{r^2}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta\cos\theta cos\phi}{r^2}-\frac{1}{r^2 sin^2\theta}\frac{sin\theta cos\phi}{r^2}\\
&= \frac{2\sin\theta cos\phi}{24}+\frac{\cos 2\theta\cdot\cos\phi}{r^4\sin\theta}+\frac{cos\phi}{r^4\sin\theta}\\
&= \frac{\cos\phi}{r^4}\left[2sin\theta+\frac{\cos 2\theta-1}{\sin\theta}\right]\\
&= \frac{\cos\phi}{r^4}\left[\frac{2\sin^2\theta+\left(1-2\sin^2\theta\right)-1}{\sin\theta}\right]\\
&= 0
\end{aligned}$$

Chapter 3

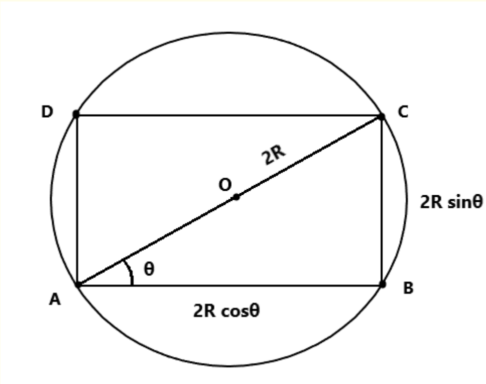
2018

3.1 Section-A

Question-1(a) Show that the maximum rectangle inscribed in a circle is a square.

[8 Marks]

Solution: Let $ABCD$ be the rectangle inscribed in a circle of radius R.



Let

$$\begin{aligned} \angle BAC &= \theta \\ AB &= 2R\cos\theta \\ BC &= 2R\sin\theta \\ \text{Area } A &= (2R\cos\theta)(2R\sin\theta) \\ &= 2R^2\cdot\sin 2\theta \end{aligned}$$

$$\text{For max. area, } \frac{dA}{d\theta} = 0 \Rightarrow \quad 4R^2\cos 2\theta = 0$$

$$\cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{in } [0, 2\pi]$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4} \text{ in } [0, \pi]$$

But no rectangle is possible for $\theta = 3\frac{\pi}{4}$ so, we discard it

$$\frac{d^2A}{d\theta^2} = -8R^2\sin 2\theta < 0 \text{ at } \theta = \frac{\pi}{4}$$

Hence, A is maximum when $\theta = \frac{\pi}{4}$ Then

$$AB = 2R\cos\frac{\pi}{4} = 2R \cdot \frac{1}{\sqrt{2}} = \sqrt{2}R = BC$$

Hence, $ABCD$ becomes square.

Question-1(b) Given that Adj $\mathrm{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **and det** $\mathrm{A} = 2$. **Find the matrix** A.

[8 Marks]

Solution: We know,

$$A^{-1} = \frac{adj A}{|A|}$$

$$\Rightarrow \quad A = |A| \quad (adj A)^{-1}$$

$$|\operatorname{adj} A| = |A|^2 = 4(\because |adj(A)| = |A|^{n-1})$$

we find the adjoint of the given matrix:

$$adj(adj A) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}^{\top} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

$$\therefore A = \mid A \mid (adj A)^{-1}$$

$$= 2 \times \frac{1}{4} \cdot \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Question-1(c) If $\mathrm{f} : [\mathrm{a}, \mathrm{b}] \to \mathrm{R}$ **be continuous in** $[\mathrm{a}, \mathrm{b}]$ **and derivable in** (a, b), **where** $0 < a < b$, **show that for** $c \in (a, b)$

$$f(b) - f(a) = cf'(c)\log(b/a)$$

[8 Marks]

Solution: Cauchy's Mean Value Theorem Two functions f and g are i) continuous on $[a, b]$ ii) derivable in (a, b) iii) $g'(x) \neq 0 \quad \forall x \in (a, b)$, then there exist atleast one point

$C \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Here, take

$$g(x) = \log x \text{ in}[a,b] \quad 0 < a < b$$

Applying Cauchy's MVT $\exists$ some $c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{\log b - \log a} = \frac{f'(c)}{(1/c)}$$

$$\Rightarrow \quad f(b) - f(a) = c \cdot f'(c) \log \frac{b}{a}$$

Hence,proved.

Question-1(d) Find the equations of the tangent planes to the ellipsoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

which pass through the line

$$x - y - z = 0 = x - y + 2z - 9$$

[8 Marks]

Solution: Any plane through the line

$$x - y - z = 0 = x - y + 2z - 9$$

is

$$(x - y - z) + \lambda(x - y + 2z - 9) = 0$$

$$x(1+\lambda) - (1+\lambda)y - (1-2\lambda) = 9\lambda - (1)$$

If it touches the conicoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

i.e.

$$\frac{2}{27}x^2 + \frac{2}{9}y^2 + \frac{1}{9}z^2 = 1$$

then

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$\left(ax^2 + by^2 + cz^2 - 1, \quad (x + my + nz = p)\right.$$

$$\frac{27}{2}(1+\lambda)^2 + \frac{9}{2}(1+\lambda)^2 + 9(1-2\lambda)^2 = (9\lambda)^2$$

$$3\left(\lambda^2 + 2\lambda + 1\right) + \left(\lambda^2 + 2\lambda + 1\right) + 2(4)^2 - 4\lambda + 1\big) = 2 \times 9\lambda^2$$

$$12\lambda^2 + 6 = 18\lambda^2$$

$$6\lambda^2 = 6 \quad \Rightarrow \quad \lambda = 1, -1$$

Hence, from (1) required tangent planes are

$$2x - 2y + z = 9$$

;

$$z = 3$$

Question-1(e) Prove that the eigenvalues of a Hermitian matrix are all real.

[8 Marks]

Solution: Let A be a Hermitian matrix

$$\therefore \quad A^\theta = A - (1). Here, A^\theta = \text{conjugate transpose}$$

Let λ be an eigenvalue of A and x be corresponding eigenvector of λ.

$$\therefore \quad Ax = \lambda x$$

$$\Rightarrow (Ax)^\theta = (\lambda x)^\theta$$

$$\Rightarrow \quad x^\theta \cdot A^\theta = \bar{\lambda} \cdot x^\theta$$

Using eq. (1),

$$\Rightarrow \quad x^\theta A = \bar{\lambda} x^\theta$$

Post mutiplying x both sides,

$$\left(x^\theta A\right) x = \left(\bar{\lambda} x^\theta\right) x$$
$$x^\theta (Ax) = \bar{\lambda}\left(x^\theta x\right)$$
$$x^\theta (\lambda x) = \bar{\lambda}\left(x^\theta x\right)$$
$$\lambda\left(x^\theta x\right) = \bar{\lambda}\left(x^\theta x\right)$$

[λ is a scalar]

$$(\lambda - \bar{\lambda})\left(x^\theta x\right) = 0$$

$$\therefore \lambda - \bar{\lambda} = 0 \quad [\because x \neq 0 \therefore x^\theta x \neq 0]$$

$$\therefore \lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \text{ is real.}$$

Question-2(a) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 4$, $z = 2$.

[10 Marks]

Solution: Let $P(x_1, y_1, z_1)$ be any point on the cylinder then the eqn. of the generator through P are

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3}$$

This generator meets the plane $z = 2$ in the point

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{2 - z_1}{3}$$

i.e.

$$\left[\frac{3x_1 - z_1 + 2}{3}, \frac{3y_1 + 2z_1 - 4}{3}, 2\right]$$

$\therefore$ The generator intersect the given curve if

$$\frac{1}{9}(3x_1 - z_1 + 2)^2 + \frac{1}{9}(3y_1 + 2z_1 - 4)^2 = 4$$

$\therefore$ The locus of P $P(x_1, y_1, z_1)$ or the required eqn of cylinder is

$$(3x - z + 2)^2 + (3y + 2z - 4)^2 = 36$$

$$\left(9x^2 + z^2 + 4 + 6xz - 4z + 12x\right) + \left(9y^2 + 4z^2 + 16 + 12yz - 16z - 24y\right) = 36$$

$$9x^2 + 9y^2 + 5z^2 - 6xz + 12yz - 20z - 24y - 12x$$
$$= 16$$

Question-2(b) **Show that the matrices A** $= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ **are congruent.**

[10 Marks]

Solution: **Sylvestor's Law of Inertia**
Two symmetric $n \times n$ matrices are congruent if and only if their diagonal representations have same rank, index and signature.

Rank = no of non-zero eigen-values
index = no of of positive eigen-values
signature = no of of positive eigen-values- = no of of negative eigen-values

Also, two symmetric matrices (as well as skew-symmetric) are congruent if they have the same rank.

$$|A| = 1(6 - 1) - 1(3 + 1) - (1 + 2)$$
$$= 5 - 4 - 3 = -2 \neq 0$$
$$\therefore P(A) = 3$$

$$(B) = 1(0 - 4) + 3(0 - 6) = -4 - 18 = -22 \neq 0$$

$$\therefore \quad P(B) = 3$$

Hence A and B are congruent.

Question-2(c) If ϕ and ψ be two functions derivable in $[\mathrm{a},\mathrm{b}]$ and $\phi(\mathrm{x})\psi'(\mathrm{x}) - \psi(\mathrm{x})\phi'(\mathrm{x}) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a, b]$, there lies exactly one root of $\psi(x) = 0$.

[10 Marks]

Solution: Let α and β be two consecutive roots of $\phi(x) = 0$ in $[a, b]$ and $\alpha < \beta$.
We are required to prove that only one root of $\psi(x) = 0$ lies between α and β.
If possible, let $\psi(x) = 0$ has no root in (α, β).
Consider the function $F(x) = \dfrac{\phi(x)}{\psi(x)}$

$$F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0 \quad \& \quad F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0$$

$$(\because \phi(\alpha) = 0 = \phi(\beta)$$

and

$$\psi(\alpha) \neq 0, \psi(\beta) \neq 0$$
$$\psi(x) \neq 0, in[\alpha, \beta]$$

$\therefore \quad F(x)$ is continuous in $[\alpha, \beta]$

$$F'(x) = \frac{\phi'(x)\psi(x) - \psi'(x)\phi(x)}{[\psi(x)]^2}$$

exist in$(\alpha, \beta]$.
$\therefore \quad F(x)$ satisfies all condition of Rolle's Theorem in $[\alpha, \beta]$ $\therefore F'(r) = 0$ where $\alpha < r < \beta$
but by given condition

$$\phi'(x)\psi xj - \psi'(x)\phi(x) > 0$$

$\therefore F'(x) \neq 0$ in (α, β) and we get contradiction.
Hence, $\psi(x)$ has atleast one root in (α, β).
By similar argument, it can be shown that between two roots of $\psi(x) = 0$, there is a root of $\phi(x) = 0$.
Now, we prove that there is exactly one root of $\psi(x) = 0$ between α, β.
If possible, let r and δ two roots of $\psi(x) = 0$ in (α, β), i.e.,

$$\alpha < r < \delta < \beta$$

.
Between r and δ, there would exist a root of $\phi(x) = 0$. This contradicts that roots of α and β are consecutive roots of $\phi(x) = 0$.
Hence, there is only one root of $\varphi(x) = 0$ between α and β.

Question-2(d) Show that the vectors $\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$

[10 Marks]

Solution: Let

$$x_1(1,0,-1) + x_2(1,2,1) + x_3(0,-3,2) = (0,0,0)$$

$$(x, +x_2, 2x_2 - 3x_3, -x_1 + x_2 + 2x_3) = (0,0,0)$$

where;

$$x_1, x_2, x_3 \in R$$

solving these, we get $x_1 = x_2 = x_3 = 0$

$$\Rightarrow \alpha_1, \alpha_2, \alpha_3 \text{ are L.I.}$$

Again, let $\beta = (x,y,z) \in R^3$ and

$$\beta = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$$

$$(x,y,z) = (a+b, 2b-3c, -a+b+2c)$$

$$a + b = x$$

$$2b - 3c = y$$

$$-a + b + 2c = z$$

$$\Rightarrow a = \frac{1}{10}(7x - 2y - 3z)$$

$$\Rightarrow b = \frac{1}{10}(3x + 2y + 3z)$$

$$\Rightarrow c = \frac{1}{5}(x - y + z)$$

(elimentary row operations)

$$\therefore (x,y,z) = \frac{1}{10}(7x-2y-3z)(1,0,-1) + \frac{1}{10}(3x+2y+3z)(1,2,1) + \frac{1}{5}(x-y+z)(0,-3,2)$$

$\forall \beta \in R^3$ Using this, we write

$$(1,0,0) = \frac{7}{10}(1,0,-1) + \frac{3}{10}(1,2,1) + \frac{1}{5}(0,-3,2)$$

similarly (0,1,0) and (0,0,1)

Question-3(a) Find the equation of the tangent plane that can be drawn to the sphere

$$x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$$

through the straight line

$$3x - 4y - 8 = 0 = y - 3z + 2$$

[10 Marks]

Solution: The eqn of any plane through given line

$$3x - 4y - 8 + \lambda(y - 3z + 2) = 0$$

$$3x - (y - \lambda)y - 3\lambda z = 8 - 2\lambda - (1)$$

If this plane touches the sphere then. length of prependicular from centre of sphere to plane= Radius of sphere Centre $(1, -3, -1)$ radius $= \sqrt{(1 + 9 + 1 - 8)} = \sqrt{3}$

$$\frac{3(1) - (4 - \lambda)(-3) - 3\lambda(-1) - 8 + 2\lambda}{\sqrt{9 + (4 - \lambda)^2 + 9\lambda^2}} = \pm\sqrt{3}$$

$$-5 + 12 - 3\lambda + 3\lambda + 2\lambda = \pm\sqrt{3} \cdot \sqrt{9 + 16 + \lambda^2 - 8\lambda + 9\lambda^2}$$

$$(2\lambda + 7)^2 = 3\left(10\lambda^2 - 8\lambda + 25\right)$$

$$4\lambda^2 + 28\lambda + 49 = 30\lambda^2 - 24\lambda + 75$$

$$26\lambda^2 - 52\lambda + 26 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1$$

Hence, Required Eqn of plane from (1)

$$3x - 3y - 3z = 6$$

i.e.,

$$x - y - z = 2$$

Question-3(b) If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(u^2 + v^2\right)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

[10 Marks]

Solution: Chain Rule

$$\frac{\partial f(u, v)}{\partial(x)} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f(u,v)}{\partial x^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,

$$\frac{\partial^2 f(u,v)}{\partial y^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$u = e^x \cdot cosy$$
$$\frac{\partial u}{\partial x} = e^x \cdot \cos y = u$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} = u$$
$$\frac{\partial u}{\partial y} = -e^x \cdot \sin y = -v$$
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial y} = -e^x \cos y = -u$$
$$v = e^x \cdot \sin y$$
$$\frac{\partial v}{\partial x} = e^x \cdot \sin y = v$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x} = v$$
$$\frac{\partial v}{\partial y} = e^x \cdot \cos y = u$$
$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial u}{\partial y} = -e^x \sin y = -v$$

Using these values

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 f}{\partial u^2}\cdot\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$
$$+\frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$
$$= \frac{\partial f}{\partial u}(u-u) + \frac{\partial^2 f}{\partial u^2}\left(u^2+v^2\right) + \frac{\partial f}{\partial v}(v-v) + \frac{\partial^2 f}{\partial v^2}\left(u^2+v^2\right)$$
$$= \left(u^2+v^2\right)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Question-3(c) Let $\mathrm{T}:\mathrm{V}_2(\mathrm{R}) \to \mathrm{V}_2(\mathrm{R})$ **be a linear transformation defined by** $\mathrm{T}(\mathrm{a},\mathrm{b}) = (\mathrm{a},\mathrm{a}+\mathrm{b})$. **Find the matrix of** T, **taking** $\{\mathrm{e}_1,\mathrm{e}_2\}$ **as a basis for the domain and {(1,1),(1,-1)} as a basis for the range.**

[10 Marks]

Solution:

$$\text{Let } (x, y) = x_1(1,1) + x_2(1,-1)$$
$$(x, y) = (x_1 + x_2, x_1 - x_2)$$
$$x_1 + x_2 = x$$
$$x_1 - x_2 = y$$
$$\Rightarrow x = \frac{x+y}{2}$$
$$\Rightarrow x_2 = \frac{x-y}{2}$$
$$\therefore (x, y) = \frac{(x+y)}{2}(1,1) + \left(\frac{x-y}{2}\right)(1,-1)$$
$$T(e_1) = T(1,0) = (1, 1+0) = (1,1)$$
$$= \frac{1+1}{2}(1,1) + \frac{1-1}{2}(1,-1),$$
$$= 1 \cdot (1,1) + 0 \cdot (1,-1)$$
$$T(e_2) = T(0,1) = (0, 0+1) = (0,1)$$
$$= \frac{0+1}{2}(1,1) + \frac{(0-1)}{2}(1,-1)$$
$$= \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$$

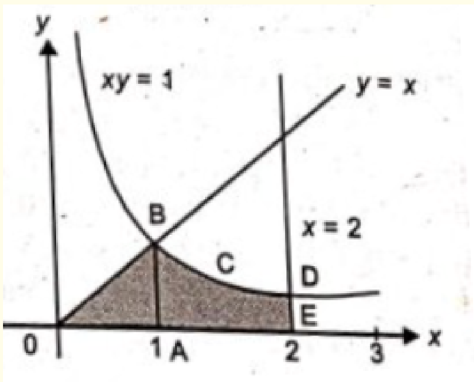
Matrix of $L.T.$ is represented by writing coordinatis of $T(e_1)$ and $T(e_2)$ as columns of matrix.

$$[T] = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

Question-3(d) Evaluate $\iint_{\mathrm{R}} (\mathrm{x}^2 + \mathrm{xy})\, \mathrm{dx}$ **dy over the region** R **bounded by** $\mathrm{xy} = 1$, $\mathrm{y} = 0$ $y = x$ **and** $x = 2$

[10 Marks]

Solution: The shaded region is shown in the region of integration.



Hence,

$$\iint\limits_{R}\left(x^2+xy\right)\cdot dx\cdot dy=\iint\limits_{OAB}\left(x^2+xy\right)\cdot dx\cdot dy+\iint\limits_{ABCDEA}\left(x^2+xy\right)\cdot dxdy$$
$$=\int_0^1 dx\int_0^x\left(x^2+xy\right)\cdot dy+\int_1^2 dx\cdot\int_0^{1/x}\left(x^2+xy\right)\cdot dy$$
$$=\int_0^1\left[x^2\cdot[y]_0^x+x\left[\frac{y^2}{2}\right]_0^x\right]\cdot dx+\int_1^2\left[x^2\cdot[y]_0^{1/x}+x\cdot\left[\frac{y^2}{2}\right]_0^{1/x}\right]\cdot dx$$
$$=\int_0^1\left[x^3+\frac{x^3}{2}\right]dx+\int_1^2\left[x+\frac{1}{2x}\right]dx$$
$$=\frac{3}{8}\left[x^4\right]_0^1+\left[\frac{x^2}{2}+\frac{1}{2}\ln(x)\right]_1^2$$
$$=\frac{3}{8}+\left(2-\frac{1}{2}\right)+\frac{1}{2}\ln 2$$
$$=\frac{15}{8}+0.34657=2.22157$$

Question-4(a) Find the equations of the straight lines in which the plane $2x+y-z=0$ cuts the cone $4\mathrm{x}^2-\mathrm{y}^2+3\mathrm{z}^2=0$. Find the angle between the two straight lines.

[10 Marks]

Solution: Equation of plane : $2x+y-z=0$
Equation of cone : $4x^2-y^2+3z^2=0$
Let l,m,n be the direction cosines of any one line of section

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Since, it lies in the plane and on the cone,
We have, $2l+m-n=0-(1)$ and $4l^2-m^2+3n^2=0-(2)$ From equation (1), $n=2l+m$
Putting in equation (2), we have

$$4l^2-m^2+3(2l+m)^2=0$$
$$4l^2-m^2+12l^2+3m^2+12lm=0$$
$$8l^2+m^2+6lm=0$$
$$(4l+m)(2l+m)=0$$
$$m=-4l\text{ or }-2l$$

On solving, we get

$$l=1,m=-2,n=0$$
$$\text{and, }l=-1,m=4,n=2$$

Hence, equation of lines:

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

Or

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}$$

and angle between two lines:

$$\cos\theta = \frac{l_1 \cdot l_2 + m_1 \cdot m_2 + n_1 \cdot n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\therefore \cos\theta = \frac{1(-1) + 4(-2) + 0}{\sqrt{(1)^2 + (-2)^2 + 0} \cdot \sqrt{(-1)^2 + (4)^2 + (2)^2}}$$

$$\cos\theta = \frac{-1-8}{\sqrt{5} \cdot \sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{105}}\right) = 151.74^\circ$$

Question-4(b) Show that the functions $u = x + y + z, v = xy + yz + zx$ and $\mathrm{w = x^3 + y^3 + z^3 - 3xyz}$ are dependent and find the relation between them.

[10 Marks]

Solution: From question, we have and

$$u = x + y + z$$
$$v = xy + yz + zx$$
$$w = x^3 + y^3 + z^3 - 3xyz$$

Now,

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = (y + z)$$

$$\frac{\partial v}{\partial y} = (x + z)$$

$$\frac{\partial v}{\partial z} = (y + x)$$

$$\frac{\partial w}{\partial x} = 3x^2 - 3yz$$

$$\frac{\partial w}{\partial y} = 3y^2 - 3xz$$

$$\frac{\partial w}{\partial z} = 3z^2 - 3xy$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ (y+z) & (z+x) & (x+y) \\ 3\left(x^2-yz\right) & 3\left(y^2-zx\right) & 3\left(z^2-xy\right) \end{vmatrix}$$

$$\begin{aligned} &= (z+x)\left(3\left(z^2-xy\right)\right) - 3\left(y^2-zx\right)(x+y) \\ &-3\left(z^2-xy\right)(y+z) + 3\left(x^2-yz\right)(x+y) \\ &+3\left(y^2-zx\right)(y+z) - 3\left(x^2-yz\right)(z+x) \\ &= 3\left\{\left(z^2-xy\right)(x-y) + \left(y^2-zx\right)(z-x)\right. \\ &\left.+\left(x^2-yz\right)(y-z)\right\} \\ &= 3\left\{z^2x - x^2y - z^2y + x^2y + y^2z - z^2x - xy^2\right. \\ &\left.+zx^2 + x^2y - y^2z - x^2z + yz^2\right\} \\ &= 0 \end{aligned}$$

Hence, u, v and w are functionally dependent.

We know that:

$$\begin{aligned} x^3+y^3+z^3-3xyz &= (x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right) \\ &= (x+y+z)\left[(x+y+z)^2 - 3(xy+yz+zx)\right] \\ \therefore w &= u(u^2-3v) \\ &= u^3 - 3uv \end{aligned}$$

Hence, relation between u, v and w is given by:

$$w = u^3 - 3u.v$$

Question-4(c) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$.

[10 Marks]

Solution: Let the equation of the hyperbolic paraboloid be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z \quad \ldots \quad (1)$$

The equations of the generator which belong to λ -system are

$$\frac{x}{a} - \frac{y}{b} = \lambda z \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}$$

$$or, \frac{x}{a} - \frac{y}{b} - \lambda z = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} + 0, z - \frac{2}{\lambda} = 0$$

Let (l_1, m_1, n_1) be the direction ratios of the generator, then we have

$$\frac{l_1}{a} - \frac{m_1}{b} - \lambda n_1 = 0 \quad \text{and} \quad \frac{l_1}{a} + \frac{m_1}{b} + 0.n_1 = 0$$

Solving for l_1, m_1, n_1, we have

$$\frac{l_1}{\frac{\lambda}{b}} = \frac{m_1}{\frac{-\lambda}{a}} = \frac{n_1}{\frac{2}{ab}}$$

or

$$\frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2} \quad \ldots \quad (2)$$

Similarly, if l_2, m_2, n_2, be the direction-ratios of any generator

$$\frac{x}{a} - \frac{y}{b} = \frac{2}{\mu} \text{ and } \frac{x}{a} + \frac{y}{b} = \mu z$$

of μ -system, then proceeding as above, we have

$$\frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2} \quad \ldots \quad (3)$$

Since the two generators (2) and (3) are perpendicular

$$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

i.e.,

$$a^2\lambda\mu - b^2\lambda\mu + 4 = 0 \Rightarrow \left(a^2 - b^2\right)\lambda\mu + 4 = 0$$

$$\Rightarrow \quad \left(a^2 - b^2\right)\left(\frac{2}{z}\right) + 4 = 0$$

$\left[\because \text{ Point of intersection of two generators are } x = a\frac{\lambda+\mu}{\lambda\mu}, y = b\frac{\mu-\lambda}{\lambda\mu}, z = \frac{2}{\lambda\mu}\right]$

$$\Rightarrow \quad a^2 - b^2 + 2z = 0$$

Hence, the required locus is the curve of intersection of the hyperbolic paraboloid and the plane $a^2 - b^2 + 2z = 0$.

Question-4(d) If $(\mathrm{n}+1)$ vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set.

[10 Marks]

Solution: Consider

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n + a_{n+1}\alpha = 0$$

Where $a_1, a_2, \ldots, a_{n+1} \in R$.
Claim: $a_{n+1} \neq 0$ Let, if possible, $a_{n+1} = 0$

$$\therefore \quad a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

But, it is given that $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \alpha$ are linearly independent.

$$a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_n = 0 \ .$$

which implies that $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha$ are linearly independent, which is a contradiction. (Given $\alpha_1, \alpha_2, \dots \alpha_n, \alpha$ are linearly dependent.)
Hence, $a_{n+1} \neq 0$.

$$\therefore \quad a_{n+1}\alpha = -\left(a_1\alpha_1 + a_2\alpha_2 + \cdots a_n\alpha_n\right)$$
$$\Rightarrow \alpha = -\frac{a_1}{a_{n+1}}\alpha_1 - \frac{a_2}{a_{n+1}}\alpha_2 + \dots - \frac{a_n}{a_{n+1}}\alpha_n$$

3.2 Section-B

Question-5(a) Find the complementary function and particular integral for the equation

$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

and hence the general solution of the equation.

[8 Marks]

Solution: Given ODE is

$$\left(D^2 - 1\right) y = xe^x + \cos^2 x$$

Auxiliary Equation :

$$m^2 - 1 = 0 \Rightarrow m = 1, -1$$
$$\therefore \quad C \cdot I = c_1 e^x + c_2 e^{-x}$$

$$\begin{aligned}
P.I. &= \frac{1}{D^2-1}\left(xe^x + \cos^2 x\right) \\
&= \frac{1}{D^2-1} xe^x + \frac{1}{D^2-1}\cos^2 x \\
&= e^x \frac{1}{(D+1)^2-1} x + \frac{1}{D^2-1}\left(\frac{1+\cos 2x}{2}\right) \\
&= e^x \frac{1}{D^2+2D} x + \frac{1}{(D^2-1)} \cdot \frac{1}{2} + \frac{1}{D^2-1}\frac{\cos 2x}{2} \\
&= e^x \cdot \frac{1}{2D\left(1+\frac{D}{2}\right)} x + \frac{1}{D^2-1}\frac{1}{2}e^{0x} + \frac{\cos 2x}{2(-4-1)} \\
&= e^x \cdot \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x + \frac{1}{2(0-1)} + \frac{\cos 2x}{-10} \\
&= e^x \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \dots\right) x - \frac{1}{2} - \frac{\cos 2x}{10} \\
&= \frac{e^x}{2}\left(\frac{1}{D} - \frac{1}{2} + \frac{D}{4}\right) x - \frac{1}{2} - \frac{1}{10}\cos 2x \\
&= \frac{e^x}{2}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) - \frac{1}{2} - \frac{1}{10}\cos 2x
\end{aligned}$$

Hence, General Solution is given by

$$y = C.I. + P.I.$$
$$y = C_1e^x + C_2e^{-x} + \frac{e^x}{8}(2x^2 - 2x + 1) - \frac{1}{10}\cos 2x - \frac{1}{2}$$

Question-5(b) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \log x (x > 0)$ by the method of variation of parameters.

[8 Marks]

Solution: Given

$$(D^2 - 2D + 1)y = xe^x \log x$$

Auxiliary Equation:

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$y_c = (C_1 + C_2x)e^x$$

Let $u = e^x, \quad v = xe^x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & 1 \cdot e^x + xe^x \end{vmatrix}$$
$$= e^{2x}[1 + x - x] = e^{2x} \neq 0$$

Hence Solutions are Independent.

$$P.I. = Au + Bv$$
$$A = -\int \frac{vR}{W}dx$$
$$A = -\int \frac{xe^x \cdot xe^x \log x}{e^{2x}}dx$$
$$= -\int x^2 \log x dx = -\int (\log x)x^2 dx$$
$$= -\left[(\log x)\frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}dx\right] \text{ (by parts)}$$
$$= \frac{-x^3}{3}\log x + \frac{x^3}{9} = \frac{-1}{9}x^3(3\log x - 1)$$

$$\begin{aligned}
B &= \int \frac{uR}{W} dx \\
&= \int \frac{e^x \cdot xe^x \log x}{e^{2x}} dx \\
&= \int x \log x dx = \int (\log x) x dx \\
&= (\log x)\frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \qquad \text{(by parts)} \\
&= \frac{x^2}{2} \log x - \frac{x^2}{4} = \frac{1}{4} x^2 (2 \log x - 1) \\
\therefore \quad y_p &= \frac{-e^x . x^3}{9} (3 \log x - 1) + \frac{x^3 . e^x}{4} (2 \log x - 1) \\
&= x^3 . e^x \left[\frac{1}{6} \log x - \frac{5}{36} \right]
\end{aligned}$$

General Solution:

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x) e^{2x} + \frac{x^3 . e^x}{36} (6 \log x - 5)$$

Question-5(c) If the velocities in a simple harmonic motion at distances a, b and c from a fixed point on the straight line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by

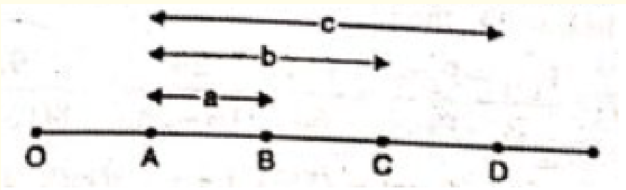
$$\frac{4\pi^2}{\mathrm{T}^2}(\mathrm{b}-\mathrm{c})(\mathrm{c}-\mathrm{a})(\mathrm{a}-\mathrm{b}) = \begin{vmatrix} \mathrm{u}^2 & \mathrm{v}^2 & \mathrm{w}^2 \\ \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 1 & 1 \end{vmatrix}$$

[8 Marks]

Solution: Let O be the centre of the force and A be the fixed point such that

$$AB = a, AC = b, AD = c$$

and let $OA = s$ and amplitude be A.



$$\begin{aligned}
\therefore \quad OB &= s + a \\
OC &= s + b \\
\text{and} \quad OD &= s + c
\end{aligned}$$

Velocities at B, C and D are u, v, w respectively

$$\begin{aligned}
\therefore u^2 &= \mu \left[A^2 - (s+a)^2 \right] - (1) \\
v^2 &= \mu \left[A^2 - (s+b)^2 \right] - (2) \\
w^2 &= \mu \left[A^2 - (s+c)^2 \right] - (3)
\end{aligned}$$

or

$$\frac{u^2}{\mu} = \left(A^2 - s^2\right) - 2as - a^2$$
$$\frac{v^2}{\mu} = \left(A^2 - s^2\right) - 2bs - b^2$$
$$\frac{w^2}{\mu} = \left(A^2 - s^2\right) - 2cs - c^2$$

or

$$\left(\frac{u^2}{\mu} + a^2\right) + 2as + s^2 - A^2 = 0 - (4)$$
$$\left(\frac{v^2}{\mu} + b^2\right) + 2bs + s^2 - A^2 = 0 - (5)$$
$$\left(\frac{w^2}{\mu} + b^2\right) + 2cs + s^2 - A^2 = 0 - (6)$$

From (4),(5) and (6) eliminating s and $s^2 - A^2$ using determinants, we get

$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

Property of determinant

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$
$$\Rightarrow -\mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Solving the determinant, we get

$$\mu(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} - (7)$$

But,

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

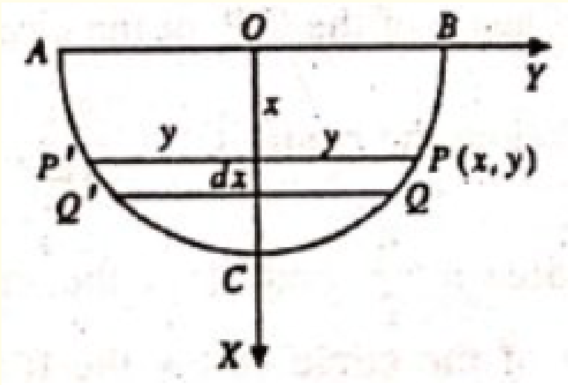
Putting μ in (7), we get

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Question-5(d) From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Find the depth of the centre of pressure of the remainder part.

[8 Marks]

Solution: Let OC be the axis of x



By symmetry, it is evident that the C.P. lies on OX. Consider an elementary strip $PQQ'P'$ of width dx at a depth x below O. Then $dS =$ area of the strip $= 2ydx$, $p =$ intensity of pressure at any point of the strip $= \rho gx$ If $\bar{x}$ be the depth of the C.P. of the semicircular lamina below O, we have

$$\bar{x} = \frac{\int xpdS}{\int pdS} = \frac{\int_0^a x\rho gx \cdot 2ydx}{\int_0^a \rho gx \cdot 2ydx} = \frac{\int_0^a x^2ydx}{\int_0^a xydx}$$

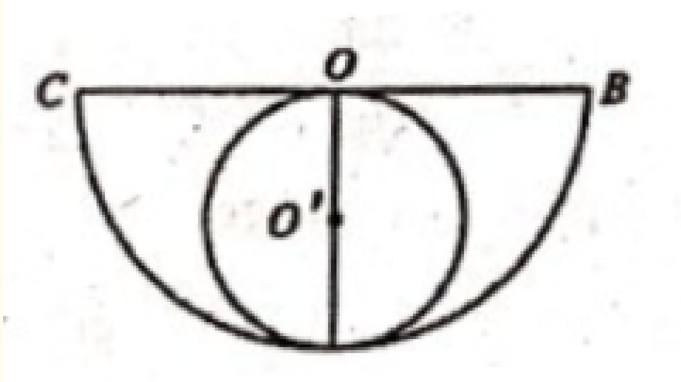
The parametric equations of the circle are

$$x = a\cos t$$
$$y = a\sin t$$
$$\therefore \quad dx = -a\sin tdt$$
$$\therefore \bar{x} = \frac{\int_{\pi/2}^a a^2\cos^2 t \cdot a\sin t(-a\sin tdt)}{\int_{\pi/2}^a a\cos t \cdot a\sin t(-a\sin tdt)}$$

$$= \frac{a\int_0^{\pi/2}\cos^2 t\sin^2 tdt}{\int_0^{\pi/2}\cos t\sin^2 tdt} = \frac{a\left(\frac{1.1}{4.2}\frac{\pi}{2}\right)}{\frac{1}{3.1}} = \frac{3}{16}\pi a$$

Again, $x_1 =$ depth of the C.P. of the semi-circle below

$$O = \frac{3\pi}{16}a$$



$$P_1 = \text{ Pressure on the semi-circle}$$
$$= w \cdot \frac{1}{2}\pi a^2 \cdot \frac{4a}{3\pi} = \frac{2}{3}a^3 w$$

Again depth of the C.P. of the circle of radius $\frac{1}{2}a$ below the centre O' is $\frac{A^2}{4H}$, where A is its radius $= \frac{a}{2}$ and H is the depth of the centre of the circle below the free surface

$$= OO' = \frac{a}{2}$$

$\therefore \quad \frac{A^2}{4H} = \frac{(a/2)^2}{4(a/2)} = \frac{a}{8} \therefore x_2 =$ depth of the C.P. of circle below

$$O = \frac{a}{2} + \frac{a}{8} = \frac{5a}{8}$$

$P_2 =$ pressure on the circle

$$= w \cdot \pi \left(\frac{a}{2}\right)^2 \cdot \frac{a}{2} = \frac{1}{8} w\pi a^3$$

If $\bar{x}$ be the depth of the C.P. of the remainder below O, then

$$\bar{x} = \frac{P_1 x_1 - P_2 x_2}{P_1 - P_2} = \frac{3a\pi}{64} \cdot \frac{24}{(16 - 3\pi)} = \frac{9\pi a}{8(16 - 3\pi)}$$

Question-5(e) f $\overrightarrow{\mathrm{r}} = \mathrm{x}\hat{\mathrm{i}} + \mathrm{y}\hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}$ **and** $\mathrm{f(r)}$ **is differentiable, show that**

$$\mathrm{div}[\mathrm{f(r)}\overrightarrow{\mathrm{r}}] = \mathrm{rf}'(\mathrm{r}) + 3\mathrm{f(r)}$$

Hence or otherwise show that $\mathrm{div}\left(\frac{\vec{r}}{r^3}\right) = 0$

[8 Marks]

Solution: We know that

$$\mathrm{div}(\phi A) = (\mathrm{grad}\,\phi) \cdot A + \phi\,\mathrm{div}(A)$$

$$\begin{aligned}\therefore \nabla\cdot(f(r)\vec{r}) &= [\nabla f(r)]\cdot\vec{r} + f(r)\nabla\cdot\vec{r}\\ &= [f'(r)\nabla r]\cdot\vec{r} + f(r)[1+1+1]\\ &= \left[f'(r)\frac{\vec{r}}{r}\right]\cdot\vec{r} + 3f(r)\\ &= rf'(r) + 3f(r) \quad [\because \vec{r}\cdot\vec{r} = r^2]\end{aligned}$$

Now, taking $f(r) = \dfrac{1}{r^3}$,

$$\begin{aligned}\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) &= r\left(\frac{-3}{r^4}\right) + 3\cdot\frac{1}{r^3}\\ &= \frac{-3}{r^3} + \frac{3}{r^3} = 0.\end{aligned}$$

Question-6(a) Solve the differential equation $(y^2+2x^2y)\,dx + (2x^3 - xy)\,dy = 0.$

[10 Marks]

Solution: We have $(y^2+2x^2y)\,dx + (2x^3-xy)\,dy = 0 - (1)$ Here,

$$M = y^2 + 2x^2y$$

$$\text{and} \qquad N = 2x^3 - xy$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y + 2x^2$$

and

$$\frac{\partial N}{\partial x} = 6x^2 - y$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is not exact. (1)

$$\Rightarrow \left(y^2dx - xydy\right) + \left(2x^2ydx + 2x^3dy\right) = 0$$

$$\Rightarrow y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$$

$$\Rightarrow x^0y^1(1.ydx - 1xdy) + x^2y^0(2ydx + 2xdy) = 0$$

Comparing it with

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$$

we have $a = 0, b = 1, m = 1, n = -1, c = 2\ d = 0, p = 2, q = 2$

$$\text{Also} \quad \frac{m}{n} = \frac{1}{-1} = -1$$

$$\text{and} \quad \frac{p}{q} = \frac{2}{2} = 1$$

$$\therefore \quad \frac{m}{n} \neq \frac{p}{q}$$

$$\text{Let} \quad \text{I.F.} = x^{\alpha} y^{\beta}$$
$$\therefore \quad \frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n}$$
$$\text{and} \quad \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$$

$\Rightarrow$
$$\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$$

and
$$\frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$$

$\Rightarrow$
$$\alpha+1 = -\beta-2$$

and
$$\alpha+3 = \beta+1$$

$\Rightarrow$
$$\alpha+\beta = -3$$

and
$$\alpha-\beta = -2$$

Solving, we get,
$$\alpha = -\frac{5}{2}, \beta = -\frac{1}{2}$$

$$\therefore I.F = x^{\alpha} y^{\beta} = x^{-5/2} y^{-1/2}$$

Multiplying (1) by $x^{-5/2}y^{-1/2}$, we get
$$x^{-5/2}y^{-1/2}\left(y^2+2x^2y\right)dx + x^{-5/2}y^{-1/2}\left(2x^3-xy\right)dy = 0$$

$\Rightarrow$
$$\left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2}\right)dy = 0$$

This equation is exact. $\therefore$ The general solution
$$\int^{x}\left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx = c$$

$\Rightarrow$
$$y^{3/2}\cdot\frac{x^{-3/2}}{-3/2} + 2y^{1/2}\cdot\frac{x^{1/2}}{1/2} = c$$
$$\Rightarrow \quad -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$$

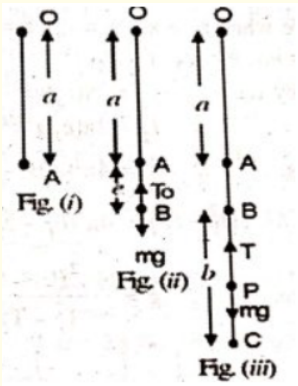
$\Rightarrow$
$$-x^{-3/2}y^{3/2} + 6x^{1/2}y^{1/2} = c'$$

$\left(\text{ Putting } c' = \frac{3}{2}c\right)$

Question-6(b) Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_1 and C_2 are the statical extensions due to these weights and g is the acceleration due to gravity. Show that $g = \dfrac{4\pi^2\left(C_1 - C_2\right)}{T_1^2 - T_2^2}$.

[15 Marks]

Solution: Let one end of an elastic string of natural length a and modulus of elasticity λ be attached to the fixed point O and with the other end A, a mass m be attached .(Refer figure (i))



Due to weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB = e$ then tension T_0 in the string will balance the weight of the particle.(Refer figure (ii)) Thus, at B, we get or

$$\begin{aligned} mg &= T_0 \\ mg &= \lambda(e/a) - (1) \end{aligned}$$

Let the particle be now pulled down a further distance $BC(= b$, say) and released. Let P be the position of the particle at any subsequent time t. Let $BP = x$ and let T be tension in the string. Then equation of motion of the particle is or

$$\begin{aligned} m\left(d^2x/dt^2\right) &= mg - T = mg - \lambda(e+x)/a \\ &= mg - \lambda(e/a) - \lambda(x/a) \\ \text{or} \quad m\left(d^2x/dt^2\right) &= -\lambda(x/a), \text{ using (1)} \\ \text{or} \quad d^2x/dt^2 &= -(\lambda/am)x - (2) \end{aligned}$$

which is of standard form $d^2x/dt^2 = -\mu x$ of S.H.M., where $\mu = \lambda/am$. Here centre of oscillation is B, from which x is measured and amplitude $= BC = b$. The periodic time T of S.H.M. represented by (2) is given by

$$\begin{aligned} T &= 2\pi/\mu^{1/2} = 2\pi/(\lambda/am)^{1/2} \\ &= 2\pi(am/\lambda)^{1/2} \\ &= 2\pi(e/g)^{1/2}, \text{ by (1)} \quad - (3) \end{aligned}$$

Equation (3) by taking $c(= AB)$ as statical extension corresponding to mass m. Then, time period T is given by

$$T = 2\pi(e/g)^{1/2} - (i)$$

Here when $m = m_1, e = c_1, T = t_1$ and when

$$m = m_2, e = c_2, T = t_2$$

So by (i),

$$t_1 = 2\pi\left(c_1/g\right)^2$$

and

$$t_2 = 2\pi\left(c_2/g\right)^{1/2}$$

Thus,

$$t_1^2 - t_2^2 = 4\pi^2\left(c_1/g - c_2/g\right)$$

or

$$g\left(t_1^2 - t_2^2\right) = 4\pi^2\left(c_1 - c_2\right)$$

Hence,

$$g = \frac{4\pi^2\left(C_1 - C_2\right)}{T_1^2 - T_2^2}$$

Question-6(c) Show that $\overrightarrow{\mathrm{F}} = \left(2\mathrm{xy} + \mathrm{z}^3\right)\hat{\mathrm{i}} + \mathrm{x}^2\hat{\mathrm{j}} + 3\mathrm{xz}^2\hat{\mathrm{k}}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1, -2, 1)$ to $(3, 1, 4)$.

[15 Marks]

Solution: Here,

$$\vec{\nabla} \times \overrightarrow{F} = \left|\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{array}\right|$$

$$= \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}x^2\right) + \hat{j}\left[\frac{\partial}{\partial t}\left(2xy + z^3\right) - \frac{\partial}{\partial x}3xz^2\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}\left(2xy + z^3\right)\right]$$

$$= 0 + \hat{j}\left(3z^2 - 3z^2\right) + \hat{k}(2x - 2x) = 0$$

For a conservative force field, $\vec{\nabla} \times \overrightarrow{F} = 0$. Work done

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \left(F_x \cdot dx + F_y \cdot dy + F_z dz\right)$$

$$= \int_A^B \left(2xy + z^3\right) dx + x^2 dy + 3xz^2 dz$$

$$\int_{(1,-2,1)}^{(3,1,4)} \left(2xydx + x^2 dy\right) + \left(z^3 dx + 3xz^2 dz\right)$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\left(x^2 y\right) + d\left(xz^3\right) = \int_{(1,-2,1)}^{(3,1,4)} d\left(x^2 y + xz^3\right)$$

$$= \left[x^2 y + xz^3\right]_{1,-2,1}^{3,1,4}$$

$$= 9 + 3 \cdot (4)^3 - 1(-2) - 1(1)^3$$

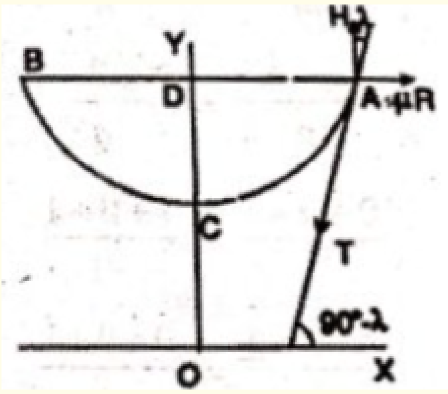
$$= 201 + 2 - 1 = 202$$

Question-7(a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \frac{1+(1+\mu^2)^{\frac{1}{2}}}{\mu}$$

[10 Marks]

Solution: Let AB be the maximum span. Hence the end links A and B are in limiting equilibrium each under three forces namely the normal reaction $R \perp$ to AB (upwards) the force of friction μR along the fixed horizontal *rod* outwards and the tension T along the tangent at $A(or B)$



If S is the resultant of R and μR at A(say) inclined at λ (the angle of friction) to R, then the tension at A must balance S and therefore it is inclined at $(90° - \lambda)$ to the horizon. That is ψ at

$$A = (90° - \lambda)$$

, and

$$\tan\lambda = \frac{\mu R}{R} = \mu$$

$\therefore$ Maximum span $AB = 2x$

$$= 2c\log(\sec\psi + \tan\psi)$$

$$= 2c\log\{\sec(90-\lambda) + \tan(90-\lambda)\}$$

$$= 2c\log\{\operatorname{cosec}\lambda + \cot\lambda\}$$

$$= 2c\log\left\{\sqrt{\left(1+\frac{1}{\mu^2}\right)} + \frac{1}{\mu}\right\}$$

$$= 2c\log\left\{\frac{\sqrt{(\mu^2+1)}+1}{\mu}\right\}$$

And the length of chain $ACB = 2s = 2c$ tan $\psi = 2c$ tan $(90-\lambda)$

$$= 2c\cot\lambda = \frac{2c}{\mu} - (2)$$

$\Rightarrow$

$$\frac{\text{Maximum span } AB}{\text{Length of the chain } ACB}$$
$$= \mu \log \left\{ \frac{1 + \sqrt{(1 + \mu^2)}}{\mu} \right\}$$

[from (1) and (2) by division]

Question-7(b) Solve:

$$\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$$

[10 Marks]

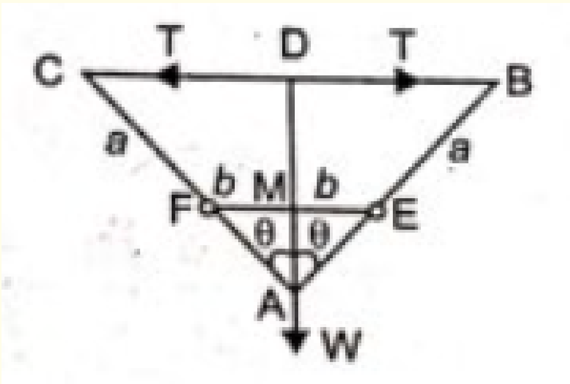
Solution:

$$\frac{dy}{dx} = \frac{2(2x + 3y) + 5}{3y + 2x + 4}$$
$$\text{let} \quad 2x + 3y = t$$
$$2 + 3\frac{dy}{dx} = \frac{dt}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$$
$$\frac{1}{3}\frac{dy}{dx} - \frac{2}{3} = \frac{2t + 5}{t + 4}$$
$$\frac{1}{3}\frac{dt}{dx} = \frac{2t + 5}{t + 4} + \frac{2}{3}$$
$$\Rightarrow = \frac{6t + 15 + 2t + 8}{3(t + 4)}$$
$$\frac{1}{3}\frac{dt}{dx} = \frac{8t + 23}{3(t + 4)}$$
$$\int \left(\frac{t + 4}{8t + 23}\right) dt = \int dx$$
$$\frac{1}{8} \int \left(\frac{8t + 32}{8t + 32}\right) dt = x + c$$
$$= \frac{1}{8} \int \left(\frac{8t + 23 + 9}{8t + 23}\right) dt$$
$$= x + c$$
$$= \frac{1}{8} \int dt + \int \frac{9}{8t + 23} dt$$
$$= x + c$$
$$= \frac{1}{8}|t| + \frac{9}{8} \int \frac{du}{u}$$
$$= \frac{1}{8}\left[2x + 3y + \frac{9}{8}\ln(8t + 23)\right] = x + c$$

Question-7(c) A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance $2\mathrm{b}$ apart. A weight W is suspended from A. Find the thrust in the rod BC.

[10 Marks]

Solution: ABC is framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2b$. Each of the rods AB and AC is of length a.



Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let, $\angle BAD = \theta = \angle CAD$. Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes. The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

$$\text{We have, } BC = 2BD = 2AB\sin\theta$$
$$= 2a\sin\theta$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME\cot\theta = b\cot\theta$$

The equation of virtual work is

$$T\delta(2a\sin\theta) + W\delta(b\cot\theta) = 0$$

or

$$2aT\cos\theta\delta\theta - bW\operatorname{cosec}^2\theta\delta\theta = 0$$

or

$$\left(2aT\cos\theta - bW\operatorname{cosec}^2\theta\right)\delta\theta = 0$$

or

$$2aT\cos\theta - bW\operatorname{cosec}^2\theta = 0$$
$$[\because \delta\theta \neq 0]$$

or

$$2aT\cos\theta = bW\operatorname{cosec}^2\theta$$

or

$$T = \frac{Wb}{2a}\operatorname{cosec}^2\theta\sec\theta$$

But in the position of equilibrium,

$$BC = \frac{3}{2}a \text{ and so } BD = \frac{3}{4}a$$

$$\text{Therefore, } \sin\theta = \frac{BD}{AB} = \frac{\frac{3}{4}a}{4} = \frac{3}{4}$$

$$\text{and } \cos\theta = \sqrt{(1-\sin^2\theta)}$$

$$= \sqrt{\left(1-\frac{9}{16}\right)} = \frac{1}{4}\sqrt{7}$$

$$\therefore \quad = \frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}} = \frac{32}{9\sqrt{7}}\frac{b}{a}W$$

Question-7(d) Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of a circle).

[10 Marks]

Solution: Consider $\vec{r} + \frac{1}{k}\hat{N}$, where $\vec{r}(s)$ is a unit-speed curve with s as arc length parameter.

$$\frac{d}{ds}\left(\vec{r} + \frac{1}{k}\hat{N}\right) = \frac{d\vec{r}}{ds} + \frac{1}{k}\frac{d\hat{N}}{ds}$$

$$= \hat{T} + \frac{1}{k}(\tau\hat{B} - k\hat{T})$$

$$= \frac{\tau}{k}\hat{B} \qquad \left[\begin{array}{l}\text{Serret-Frenet} \\ \frac{d\hat{N}}{ds} = \tau\hat{B} - k\hat{T}\end{array}\right.$$

$$= 0 \qquad (\because \text{ Torsion } = 0)$$

It implies that vector $\left(\vec{r} + \frac{1}{k}\hat{N}\right)$ is a constant rector, say, $\vec{a}$.

$\therefore \vec{r} + \frac{1}{k}\hat{N} = \vec{a}$
$\Rightarrow |\vec{r} - \vec{a}| = \left|\frac{-1}{k}\hat{N}\right| = \frac{1}{k}$
Since curvature is constant, let $\frac{1}{\kappa} = c \Rightarrow |\vec{r} - \vec{a}| = c$

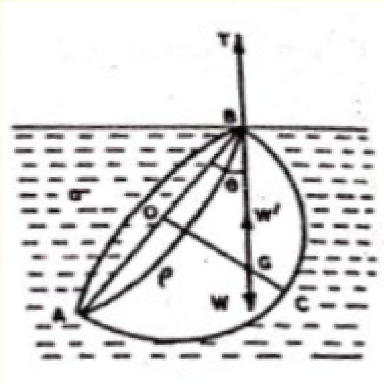
It is the equation of a sphere. Since, torsion is 0, hence curvature α lies in a plane, i.e., α is a part of circle.

Question-8(a) A solid hemisphere floating in a liquid is completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and tension of the string.

[15 Marks]

Solution: ACB is the hemisphere of radius a and density ρ. Density of liquid is σ. Since the hemisphere is completely immersed in the liquid, the weight of the body and force of buoyancy act at the same point G. Here all the forces W, W' and T act along the same vertical line BG.

$$OG = \frac{3a}{8}; OB = a$$



(i) In

$$\Delta BOG, \quad \tan\theta = \frac{OG}{OB} = \frac{\frac{3a}{8}}{a} = \frac{3}{8}$$

$$\therefore \quad \theta = \tan^{-1}\frac{3}{8}$$

(ii) Further $T = W - W'$

$$= \frac{2}{3}\pi a^3 \rho g - \frac{2}{3}\pi a^3 \sigma g = \frac{2}{3}\pi a^3 (\rho - \sigma)_g$$

Question-8(b) A snowball of radius r(t) melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process.

[15 Marks]

Solution: Let $\frac{dr}{dt} = k$ (uniform) , $density = \rho$, fixed.

$$\Rightarrow M = \left(\frac{4}{3}\pi r^3\right)\rho$$

$$\Rightarrow \frac{dM}{dt} = (4\pi\rho)r^2\frac{dr}{dt}$$

$$= 4\pi\rho\left(\frac{3M}{4\pi p}\right)^{2/3} k$$

$$= k_1 M^{2/3},$$

$$\text{where } k_1 = \frac{4\pi\rho \cdot 3^{2/3} \cdot k}{(4\pi\rho)^{1/3}}$$

$$\Rightarrow M^{-2/3}dM = k_1 dt$$

Integrating, we get:

$$3M^{1/3} = k_1 t + k_2$$

Let M_0 be initial mass of snow ball. $\therefore M(0) = M_0$ and $M(1) = \frac{M_0}{2}$

$\Rightarrow k_2 = 3M_0^{1/3}$ and $k_1 = 3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}$

We want to calculate time t when $M = 0$
ie.

$$0 = \left[3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}\right) t + 3M_0^{1/3}$$

$$\Rightarrow t = \frac{-3}{3/2^{1/3} - 3} = \frac{-1}{2^{-1/3} - 1} = 4.85$$

Therefore, the it will take 4.85 hours for the entire mass of the snowball to melt.

Question-8(c) For a curve lying on a sphere of radius a and such that the torsion is never 0**, show that**

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = a^2$$

[10 Marks]

Solution: Let vector point $r(s)$ lies on a sphere with centre r_0 and radius a.

$$\therefore |(r - r_0)| = a$$

$$|r - r_0|^2 = a^2 \quad \Rightarrow \quad (r - r_0)\cdot(r - r_0) = a^2 \quad \ldots(*)$$

Differentiating w.r.t s

$$\frac{dr}{ds}(x - r_0) + (r - r_0)\cdot\frac{dr}{ds} = 0$$

$$2\frac{dr}{ds}\cdot(r - r_0) = 0$$

$$\Rightarrow \quad (r - r_0)\cdot t = 0 \quad \ldots(1)$$

Again differentiating w.r.t s

$$\begin{aligned}\frac{dr}{ds} \cdot t + (r - r_0) \cdot \frac{dt}{ds} &= 0 \\ t \cdot t + (r - r_0) \cdot (kn) &= 0 \\ 1 + (r - r_0) \cdot (kn) &= 0 \qquad \text{(serret-frenet)} \\ (r - r_0) \cdot n &= \frac{-1}{k} \qquad \ldots (2)\end{aligned}$$

Again differentiating w.r.t s

$$\begin{aligned}\frac{dr}{ds} \cdot n + (r - r_0) \cdot \frac{dn}{ds} &= \frac{1}{k^2} \cdot k' \\ t \cdot n + (r - r_0) \cdot (\tau b - kt) &= \frac{k'}{k^2} \qquad \text{(serret-frenet)} \\ 0 + (r - r_0) \cdot (\tau b) - (r - r_0) \cdot (kt) &= \frac{k'}{k^2} \\ (r - r_0) \cdot b &= \frac{k'}{k^2\tau} \quad [\text{using}(1)] \quad \ldots (3)\end{aligned}$$

From $(1), (2), (3)$ we see that The components of $(r - r_0)$ with respect to t, n, b are $0, -\frac{1}{k}$ and $\frac{k'}{k^2\tau}$ Hence,

$$r - r_0 = \frac{-1}{k} n + \left(\frac{k'}{k^2\tau}\right) b$$

From $(*)$ we get,

$$\begin{aligned}a^2 = (r - r_0) \cdot (r - r_0) &= \left(\frac{-1}{k} n + \frac{k'}{k^2\tau} b\right) \cdot \left(\frac{-1}{k} n + \frac{k'}{k^2\tau} b\right) \\ &= \frac{1}{k^2} n \cdot n - \frac{k'}{k^3\tau} b \cdot n - \frac{k'}{k^3\tau} n \cdot b + \left(\frac{k'}{k^2\tau}\right)^2 b \cdot b \\ &= \frac{1}{k^2} + \left(\frac{k'}{k^2 T}\right)^2 \quad [\because n \cdot n = 1 = b \cdot b \Rightarrow n \cdot b = 0]\end{aligned}$$

Chapter 4

2017

4.1 Section-A

Question-1(a) Let A **be a square matrix of order 3 such that each of its diagonal elements is '**a**' and each of its off-diagonal elements is 1. If** B = bA **is orthogonal, determine the values of** a **and** b**.**

[8 Marks]

Solution: Given,

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \quad \therefore B = bA = \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix}$$

B is orthogonal i.e., $BB^T = I$

$$\therefore \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad b^2a^2 + 2b^2 = 1$$

and

$$2b^2a + b^2 = 0$$

$$b^2\left(a^2 + 2\right) = 1$$

and

$$b^2\,(2a + 1) = 0$$

$$\therefore b^2 = 0$$

or

$$\left(2a^2 + 1\right) = 0$$

But $b = 0$ is not possible, as first equation will not be satisfied

$$\therefore \quad a = \frac{-1}{2} \quad \Rightarrow \quad b^2\left(\frac{1}{4} + 2\right) = 1 \Rightarrow b = \pm\frac{2}{3}$$

.

Question-1(b) Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where (i) W contains all 2×2 matrices with zero determinant. (ii) W consists of all 2×2 matrices A such that $\mathrm{A}^2 = \mathrm{A}$.

[8 Marks]

Solution: (i)

$$V = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] : a, b, c, d \in R \right\} = M_2(R)$$

It is a vector space over field R. W= Set of all 2×2 matrices with determinant zero. Let

$$w_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right]$$

and

$$w_2 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right]$$

det

$$(w_1) = 0 = \det(\omega_2) \quad \therefore \omega_1, w_2 \in W$$

But

$$w_1 + w_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

$$\det(\omega_1 + \omega_2) = 1 \neq 0$$

$$\therefore w_1 + w_2 \notin W$$

(ii) W consists of all 2×2 matrices A such that $A^2 = A$. Let,

$$A = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \quad , \quad B = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right]$$

$$A^2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], B^2 = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right]$$

$$\therefore \quad A, B \in W$$

But

$$A + B = \left[\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right]$$

$$(A + B)^2 = \left[\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{cc} 4 & 3 \\ 0 & 1 \end{array} \right] \neq (A + B)$$

$$\therefore (A + B) \notin W$$

Hence, W is not a subspace of V

Question-1(c) Using the Mean Value Theorem, show that (i) $f(x)$ is constant in $[a, b]$, if $f'(x) = 0$ in $[a, b]$
(ii) $f(x)$ is a decreasing function in (a, b), if $f'(x)$ exists and is < 0 everywhere in (a, b)

[8 Marks]

Solution: (i) Let x_1 and x_2 be two distinct points in interval $[a, b]$, and let

$$x_1 < x_2$$

.

$$\therefore [x_1, x_2] \subseteq [a, b]$$

Then f is continuous on $[x_1, x_2]$ and f is differential on $[x_1, x_2]$ Using LMVT, there exist some,

$$c \in [x_1, x_2]$$

such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

ie.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad [\because f'(x) = 0 \quad \forall x \in [a, b]]$$

$\Rightarrow$

$$f(x_2) - f(x_1) = 0$$

ie

$$f(x_1) = f(x_2)$$

Hence, $f(x)$ is constant function. as x_1 and x_2 were arbitrary in $[a, b]$.
(ii) Let x_1 and x_2 be any two distinct points in $[a, b]$ and $x_1 < x_2$

$$\therefore [x_1, x_2] \subseteq [a, b]$$

f is differentiable on (a, b), hence it is differentiable on

$$(x_1, x_2) \subseteq (a, b)$$

and continuous also. Using LMVT, there exist some $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

Now, since

$$x_2 > x_1 \quad \therefore \quad (x_2 - x_1) > 0$$

and $f'(x) < 0$ on (a, b)

$$\therefore f'(c) < 0$$

as

$$c \in (x, x_2) \subseteq (a, b)$$

$$\therefore f(x_2) - f(x_1) < 0$$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

$\therefore f(x)$ is decreasing function on (a, b)

Question-1(d) Jacobian $\mathrm{J} = \dfrac{\partial(\mathrm{u}, \mathrm{v})}{\partial(\mathrm{x}, \mathrm{y})}$, and hence show that u, v. are independent unless

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$$

[8 Marks]

Solution:

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2ax + 2hy & 2hx + 2by \\ 2Ax + 2Hy & 2Hx + 2By \end{vmatrix}$$

$$= 4(ax + 2hy)(Hx + By) - 4(Ax + Hy)(hx + by)$$

$$= 4\left[\left(aHx^2 + xy(hH + aB) + hBy^2\right) - \left(Ahx^2 + xy(Hh + Ab) + bHy^2\right)\right]$$

$$= -4\left[(aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2\right]$$

u and v are independent, if $J = 0$ ie

$$aH - Ah = 0; \quad aB - Ab = 0, \quad Bh - bH = 0$$

$$\therefore \quad \frac{a}{A} = \frac{h}{H}; \quad \frac{a}{A} = \frac{b}{B}, \quad \frac{h}{H} = \frac{b}{B}$$

ie

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$$

Question-1(e) Find the equations of the planes parallel to the plane $3x - 2y + 6z + 8 = 0$ and at a distance 2 from it.

[8 Marks]

Solution: Equation of any plane parallel to given plane is

$$3x - 2y + 6z + k = 0$$

Distance between two planes

$$\frac{|k - 8|}{\sqrt{9 + 4 + 36}} = 2$$

ie

$$|k - 8| = 14$$

$$\therefore \quad k - 8 = 14 \quad or \quad k - 8 = -14$$

$$k = 22 \quad or \quad k = -6$$

Hence, required equations of planes are

$$3x - 2y + 6z + 22 = 0$$

or

$$3x - 2y + 6z - 6 = 0$$

Question-2(a) State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \textbf{ Hence find } A^{-1}$$

[10 Marks]

Solution: Every square matrix satisfies its characteristic equation, given by, $|A - \lambda I| = 0$ from the given matrix

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left(\lambda^2 + \lambda - 1\right) = 0$$

$\Rightarrow$

$$\lambda^3 - 2\lambda + 1 = 0 - (1)$$

Now,

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & +1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 0$$

Hence, A satisfies its characteristic equation, given by (1).

Now, To find A^{-1},

$$A^3 - 2A + I = 0$$

$$A^{-1}A^3 - 2A^{-1}A + A^{-1} = 0$$

$$A^{-1} = -A^2 + 2I$$

$$A^{-1} = -\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Question-2(b) Show that

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p, q > -1$$

Hence, evaluate the following integrals:

(i) $\int_0^{\pi/2} \sin^4 x\cos^5 x dx$
(ii) $\int_0^1 x^3\left(1-x^2\right)^{5/2} dx$
(iii) $\int_0^1 x^4(1-x)^3 dx$

[10 Marks]

Solution: We define,

$$T(m) = \int_0^\infty x^{m-1}\cdot e^{-x}dx, m > 0$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m,n > 0$$

We put,

$$\sin^2\theta = x \quad \therefore \quad 2\sin\theta\cos\theta d\theta = dx$$

$$\int_0^{\pi/2} \sin^2\theta\cos^2\theta d\theta = \int_0^{\pi/2} \left(\sin^2\theta\right)^{\frac{p-1}{2}}\cdot\left(cos^2\theta\right)^{\frac{q-1}{2}}\cdot\sin\theta\cos\theta d\theta$$

$$= \int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}\cdot\frac{dx}{2}$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

(i)

$$\int_0^{\pi/2} \sin^4 x \cos^5 x dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma(3)}{\Gamma(\frac{1}{2})}$$

$$\left(\because \Gamma(m) = (m-1)!, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(m) = (m-1)\Gamma(m-1)\right)$$

$$= \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \cdot 2!}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2})}$$

$$= \frac{8}{315}$$

(ii) Put,

$$x^2 = y \quad \Rightarrow \quad 2xdx = dy$$

or

$$dx = \frac{1}{2x} dy$$

$$\int_0^1 x^3 \left(1 - x^2\right)^{5/2} dx = \int_0^1 y(1-y)^{5/2} \frac{dy}{2}$$

$$= \frac{1}{2}\beta\left(1+1, \frac{5}{2}+1\right) = \frac{1}{2}\beta\left(2, \frac{7}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(2) \cdot \Gamma(\frac{7}{2})}{\Gamma\left(2 + \frac{7}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma(2)\Gamma(\frac{7}{2})}{\Gamma(\frac{11}{2})}$$

$$= \frac{1}{2} \cdot \frac{(1)!\Gamma(\frac{7}{2})}{\frac{9}{2} \cdot \frac{7}{2} \cdot \Gamma(\frac{7}{2})} = \frac{2}{63}$$

(iii)

$$\int_0^1 x^4(1-x)^3 dx = \beta(4+1, 3+1) = \beta(5, 4)$$

$$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} = \frac{\Gamma(5) \cdot 3!}{8 \times 7 \times 6 \times 5 \cdot \Gamma(5)}$$

$$= \frac{1}{280}$$

Question-2(c) Find the maxima and minima for the function

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

Also find the saddle points (if any) for the function.

[10 Marks]

Solution:

$$f_x = 3x^2 - 3, \quad f_y = 3y^2 - 12$$

for critical points, $f_x = 0$ and $f_y = 0$

$$\therefore x = \pm 1, \quad y = \pm 2$$

The function has four stationary points

$$(1,2), (-1,2), (1,-2), (-1,-2)$$

Now,

$$f_{xx} = 6x \quad , \quad f_{yy} = 6y \quad , f_{xy} = 0$$

At

$$(1,2), \quad f_{xx} = +6 > 0, \quad f_{yy} = 12 > 0, f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy} = 6 \cdot 12 - 0 = 72 > 0$$
$$\therefore \quad (1,2) \text{is point of minima}$$

. At $(-1,2)$

$$f_{xx} = -6 < 0, \quad f_{yy} = 12 > 0, \quad f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy}^2 = -6 \times 12 = -72 < 0$$

function is neither maximum, nor minimum at $(-1,2)$ At (1,-2)

$$f_{xx} = 6 > 0, \quad fy_y = -12 < 0, \quad f_{xy} = 0$$
$$F_{xx} \cdot f_{yy} - f_{xy}^2 = -72 < 0$$

function is neither maximum, nor minimum at $(1,-2)$ $At(-1,-2)$

$$f_{xx} = -6 < 0, \quad f_{yy} = -12 < 0, \quad f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 72 > 0$$

(-1,-2) is print of maxima. Point of Maxima is (-1,-2) Point of Minima is (1,2) Stationary points like (-1,2) and (1,-2) which are not extreme points are saddle points. Result Used:

$$f(a,b) \text{ is an extreme value of } f(x,y),$$

if

$$f_x(a,b) = 0 = f_y(a,b),$$

and

$$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0 \text{ at } (a,b)$$

and this extreme value is maximum if

$$f_{xx}(a,b) < 0$$

or minimum if

$$f_{xx}(a,b) > 0$$

Question-2(d) Show that the angles between the planes given by the equation $2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$ is $\tan^{-1}\dfrac{\sqrt{50}}{4}$.

[10 Marks]

Solution: Let the equations of two planes be

$$a_1x + b_1y + z = 0$$

and

$$a_2x + b_2y + 3z = 0$$

($\because$ planes passes through origin).
Now the combined equation is

$$(a_1x + b_1y + z)(a_2x + b_2y + 3z) = 0$$

$$a_1a_2x^2 + b_1b_2y^2 + 3z^2 + xy\,(a_1b_2 + a_2b_1) + xz\,(3a_1 + a_2) + yz\,(b_2 + 3b_1) = 0$$

Comparing the coefficients in the given eqn

$$a_1a_2 = 2, \quad b_1b_2 = -1, a_1b_2 + a_2b_1 = -1$$

$$3a_1 + a_2 = 7, \quad 3b_1 + b_2 = 2$$

$\Rightarrow$

$$a_1 = 2, \quad a_2 = 1 \quad ; \quad b_1 = 1, \quad b_2 = -1$$

Equations of planes are

$$2x + y + z = 0; \quad x - y + 3z = 0$$

Angle between the planes

$$\cos\theta = \frac{2\cdot 1 + 1(-1) + 1(3)}{\sqrt{4+1+1}\sqrt{1+1+9}} = \frac{4}{\sqrt{6}\cdot\sqrt{11}} = \frac{4}{\sqrt{66}}$$

$$\therefore \tan\theta = \frac{\sqrt{50}}{4} \quad \therefore \quad \theta = \tan^{-1}\frac{\sqrt{50}}{4}$$

Alternate solution: Let θ be the angle between pair of planes given by the general homogeneous equation of second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow \tan\theta = \tfrac{2\sqrt{(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}$$

Here, equation of pair of planes is,

$$2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$$

$$a = 2, b = -1, c = 3, f = 1, g = \frac{7}{2}, h = \frac{-1}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{\left(1 + \frac{49}{4} + \frac{1}{4} + 2 + 3 - 6\right)}}{2 - 1 + 3}$$

$$= \frac{2}{4}\sqrt{\left(\frac{50}{4}\right)} = \frac{\sqrt{50}}{4}$$

$$\therefore \quad \theta = \tan^{-1}\frac{\sqrt{50}}{4}$$

Question-3(a) Reduce the following matrix to a row-reduced echelon form and hence find its rank:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

[10 Marks]

Solution:

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1 \quad ; \quad R_4 \to R_4 + R_1$$

$$A \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Above form is row echelon form of A. we have 3 non-zero rows

$$\therefore \quad R\,\text{ank}(A) = 3$$

.

Question-3(b) Given that the set $\{u, v, w\}$ is linearly independent, examine the sets
$(i)\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$
$(ii)\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}-2\mathbf{v}+2\mathbf{w}m\}$ **for linear independence.**

[10 Marks]

Solution: Set of vectors $\{v_1, v_2, \ldots v_n\}$ is L.I. if

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

implies that

$$c_1 = c_2 \ldots. = c_n = 0$$

(i) Let $a, b, c \in R$.

Consider,

$$a(u+v)+b(v+w)+c(\omega+u)=0$$

i.e.

$$(a+c)u+(a+b)v+(b+c)w=0$$

Since $\{u,v,\omega\}$ are L.I.

$$\therefore a+c=0, \quad a+b=0 \quad , \quad b+c=0$$

Adding these three,

$$2(a+b+c)=0$$

. Hence,

$$a=0, b=0, c=0$$

$\therefore$ Given set is L.I.
(ii) Again let $a,b,c \in R$ and consider

$$a(u+v)+b(u-v)+c(u-2v+2w)=0$$

$$(a+b+c)u+(a-b-2c)v+2cw=0$$

$$\Rightarrow a+b+c=0 \qquad [\because \{u,v,\omega\} \text{ are L.I}]$$
$$a-b-2c=0$$
$$\therefore 2c=0 \qquad \therefore \quad c=0 \Rightarrow a=0; b=0$$

Hence, given set is $L.I.$

Question-3(c) Evaluate the integral $\int_0^\infty \int_0^\infty \mathrm{e}^{-\left(x^2+y^2\right)} dxdy$, by changing to polar coordinates. Hence show that $\int_0^\infty \mathrm{e}^{-\mathrm{x}^2}\mathrm{dx} = \dfrac{\sqrt{\pi}}{2}$.

[10 Marks]

Solution: The region of integration is first quadrant of xy-plane.
Hence $'r'$ varies from 0 to ∞ and θ varies from 0 to $\dfrac{\pi}{2}$.

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r dr d\theta = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-r^2}\right]_0^\infty d\theta \\ &= \frac{-1}{2} \int_0^{\pi/2} (0-1) d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4} \end{aligned}$$

Also,

$$\begin{aligned} I &= \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy \\ &= \left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4} \end{aligned}$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Question-3(d) Find the angle between the lines whose direction cosines are given by the relations $l+\ \mathrm{m}+\mathrm{n}=0$ and $2\mathrm{lm}+2\,\mathrm{ln}-\mathrm{mn}=0$.

[10 Marks]

Solution: Eliminating n between the given relations

$$\begin{aligned} n &= -(l+m) \\ 2lm-2l(l+m)+m(l+m) &= 0 \\ 2lm-2l^2-2lm+ml+m^2 &= 0 \\ \text{or} 2\ell^2-lm-m^2 &= 0 \\ 2l^2-2lm+1m-m^2 &= 0 \\ 2l(l-m)+m(l-m) &= 0 \\ (l-m)(2l+m) &= 0 \end{aligned}$$

If $2l+m=0$, then from $l+m+n=0$, $\quad n=l$

$$\therefore \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1} \qquad \ldots(1)$$

If $l-m=0$, then $l+m+n=0$, $n=-2m$

$$\frac{l}{1}=\frac{m}{1}=\frac{n}{-2} \qquad \ldots(2)$$

Hence, angle between the lines with direction ratios given by (1) and (2) is

$$\cos\theta=\frac{1\cdot 1+(-2)\cdot 1+1\cdot(-2)}{\sqrt{1+4+1}\cdot\sqrt{1+1+4}}=\frac{-3}{6}=\frac{-1}{2}$$

$$\cos\theta=\frac{-1}{2} \quad \therefore \quad \theta=120^\circ$$

Question-4(a) Find the eigenvalues and the corresponding eigenvectors for the matrix $A=\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $\mathrm{D}=\mathrm{P}^{-1}\,\mathrm{A}P$.

[10 Marks]

Solution: Characteristic eqn,

$$c\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}=0 \quad \Rightarrow \quad \begin{matrix} \lambda^2-3\lambda+2=0 \\ (\lambda-1)(\lambda-2)=0 \end{matrix}$$

Eigen values, $\lambda=1, \quad \lambda=2$

For $\lambda = 1$, Let eigenvector be, $v \begin{bmatrix} x \\ y \end{bmatrix}$

$$\therefore Av = \lambda v$$

i.e, $(A - \lambda)v = 0$

$$\therefore \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie.

$$x + 2y = 0 \quad \therefore \text{ Eigen-vector is } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For

$$\lambda = 2, \quad \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie

$$x + y = 0 \quad \therefore \text{Eigen vector is } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Here, algebraic multiplicity of each eigenvalue is equal to geometric multiplicity. $\therefore A$ is diagonizable.

$$P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$\therefore D = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Question-4(b) A function $f(x, y)$ is defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & \textbf{if } (x,y) \neq (0,0) \\ 0, & \textbf{if } (x,y) = (0,0) \end{cases}$$

Show that $\mathrm{f_{xy}}(0,0) = \mathrm{f_{yx}}(0,0)$

[10 Marks]

Solution:

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} - (1)$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

$$fy(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{1}{k} \cdot \frac{h^2k^2}{(h^2+k^2)}$$
$$= 0$$

Hence, from (1), $\quad f_{xy}(0,0) = \lim_{h\to 0} \dfrac{0-0}{h} = 0$ Now,

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} - (2)$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^2k^2}{h^2+k^2}$$
$$= 0$$
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Hence, from(2), $f_{yx} = \lim_{k\to 0} \dfrac{0-0}{k} = 0$

$$\therefore \quad f_{xy}(0,0) = f_{yx}(0,0)$$

Question-4(c) Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios $(1,-2,2)$.

[10 Marks]

Solution: The vertex of the cone is O(0,0,0) and since its axis makes equal angles with the coordinate axes, so the equations of its axis can be taken as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} \quad (\because l = m = n)$$

Also, d.r.'s of its generator are (1,-2,2) If θ is the semi-vertical angle of the cone, then

$$\cos\theta = \frac{1\cdot 1 + 1\cdot(-2) + 1\cdot 2}{\sqrt{1+1+1}\cdot\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}} - (1)$$

If $P(x,y,z)$ is any general point on the cone, then OP is a generator and d.r.'s of OP are $(x-0, y-0, z-0)$ ie, (x,y,z) Also OP makes an angle θ with the axis

$$\therefore \cos\theta = \frac{x\cdot 1 + y\cdot 1 + z\cdot 1}{\sqrt{1+1+1}\cdot\sqrt{x^2+y^2+2^2}} - (2)$$

From (1) and (2) ,

$$l\frac{x+y+z}{\sqrt{3}\cdot\sqrt{x^2+y^2+z^2}} = \frac{1}{3\sqrt{3}} \text{ or } 9(x+y+z)^2 = x^2+y^2+z^2$$
$$\text{or } 4\left(x^2+y^2+z^2\right) + 9(xy+yz+zx) = 0$$

Question-4(d) Find the shortest distance and the equation of the line of the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

and

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

[10 Marks]

Solution: Any point on first line $P(3r+3,-r+8,r+3)$ Any point on second line $Q(-3t-3,2t-7,4t+6)$ D.r.'s of PQ are

$$(3r+3t+6,r-2t+15,r-4t-3) \quad -(1)$$

If PQ is the shortest distance (SD) between the given lines, then PQ is perpendicular both lines.

$$\therefore 3(3r+3t+6)-1(-r-2t+15)+1(r-4t-3)=0$$

and

$$-3(3r+3t+6)+2(-r-2t+5)+4(r-4t-3)=0$$

or

$$11r+7t=0 \text{and} 7r+29t=0$$

Solving, we get $r=0,t=0$

$$\therefore Points, P(3,+8,3)\text{and}Q(-3,-7,6)$$

D.r's of PQ are $(6,15,-3)$ or $(2,5,-1)$

$$\begin{aligned} SD=PQ &= \sqrt{(3+3)^2+(8+7)^2+(3-6)^2} \\ &= \sqrt{36+225+9}=3\sqrt{30} \end{aligned}$$

Also, PQ is line through $P(3,8,3)$ and with d.r.'s $(2,5,-1)$, so it equation is

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

4.2 Section-B

Question-5(a) Solve

$$\left(2D^3-7D^2+7D-2\right)y=e^{-8x}.where, D=\frac{d}{dx}$$

[8 Marks]

Solution: Let

$$2D^3 - 7D^2 + 7D - 2 = f(D) \quad -(1)$$

$$\therefore \quad f(D)y = e^{-8x} - (2)$$

Auxiliary equation of (2) is, $f(D) = 0$ or

$$2D^3 - 7D^2 + 7D - 2 = 0$$

roots are , $D = 1, 2, \frac{1}{2}$.

Complementary function (CF) of (2) is

$$y = c_1e^x + c_2e^{2x} + c_3e^{x/2} - (3)$$

For finding particular integral $(PI)-$

$$f(D)y = Q(x)$$

where,

$$Q(x) = e^{-8x}$$

$$\begin{aligned}
PI &= \frac{1}{f(D)}Q(x) \\
&= \frac{1}{(D-1)(D-2)\left(D-\frac{1}{2}\right)2} \cdot \left(e^{-8x}\right) \\
&= \frac{1}{(-8-1)(-8-2)(-16-1)} \cdot e^{-8x} \\
&= \frac{1}{-1530}e^{-8x} - (4)
\end{aligned}$$

General solution, $y = y_c + y_p$

$$y = c_1e^x + c_2e^{2x} + c_3e^{x/2} - \frac{e^{-8x}}{1530}$$

Question-5(b) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

[8 Marks]

Solution:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

Using the substitution,

$$x = e^z, \text{i.e}, z = \log x$$

The equation becomes

$$(D_1(D_1-1)-2D_1-4)y=e^{4z}, \quad D_1=\frac{d}{dz}$$
$$\therefore \left(D_1^2-3D_1-4\right)y=e^{4z}-(2)$$

Auxiliary equation of (2) is

$$D_1^2-3D_1-4=0 \therefore D_1=4,-1$$
$$\therefore C\cdot F=y_c=c_1e^{-z}+c_2e^{4z}-(3)$$

$$\begin{aligned}
PI&=\frac{1}{D_1^2-3D_1-4}\left(e^{4z}\right)\\
&=\frac{1}{(D_1-4)(D_1+1)}\left(e^{4z}\right)\\
&=\frac{1}{(D_1-4)}\cdot\frac{e^{4_2}}{(4+1)}\\
&=\frac{z}{5}\cdot e^{4z}-(4)
\end{aligned}$$

$$\begin{aligned}
\therefore y&=y_c+y_p\\
&=c_1e^{-z}+c_2e^{4z}+\frac{ze^{4z}}{5}\\
&=c_1e^{-\log x}+c_2e^{4\log x}+\log x\cdot\frac{e^{4\log x}}{5}\\
&=\frac{c_1}{x}+c_2x^4+\frac{1}{5}x^4\cdot\log x
\end{aligned}$$

Question-5(c) A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position $P(OP=b)$ with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\frac{\mathrm{T}}{\pi}\tan^{-1}\left(\frac{\mathrm{vT}}{2\pi \mathrm{b}}\right)$.

[8 Marks]

Solution:

$$x=a\sin\omega t, v=a\omega\cos\omega t, \quad \omega=\frac{2\pi}{T}$$

At

$$\begin{aligned}
x&=b\\
b&=a\sin\omega t_b\\
\Rightarrow t_b&=\frac{1}{\omega}\sin^{-1}\frac{b}{a}\\
&=\frac{T}{2\pi}\sin^{-1}\frac{b}{a}\\
t_a&=\frac{T}{4}=\frac{2\pi}{\omega}\cdot\frac{1}{4}
\end{aligned}$$

$$
\begin{aligned}
v &= a\omega \cos \omega t_b \\
\therefore \quad v &= a\omega \left(1 - \sin^2 \omega t_b\right)^{1/2} \\
&= a\omega \left(1 - \frac{b^2}{a^2}\right)^{1/2} \\
&= \omega\sqrt{a^2 - b^2} \\
\Rightarrow a^2 &= \frac{v^2}{\omega^2} + b^2 \\
&= \left(\frac{vT}{2\pi}\right)^2 + b^2
\end{aligned}
$$

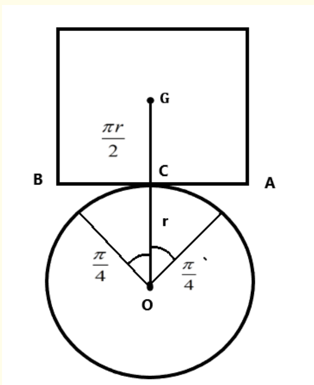
Time required,

$$
\begin{aligned}
t = 2\left(t_a - t_b\right) &= 2\left(\frac{T}{4} - \frac{T}{2\pi}\sin^{-1}\frac{b}{a}\right) \\
&= \frac{T}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{b}{a}\right) = \frac{T}{\pi}\cos^{-1}\frac{b}{a} \\
&= \frac{T}{\pi}\cdot\tan^{-1}\frac{\sqrt{a^2 - b^2}}{b} \\
&= \frac{T}{\pi}\tan^{1}\left(\frac{Tv}{2\pi b}\right) \\
&\left[\because a^2 - b^2 = \left(\frac{vT}{2\pi}\right)^2\right]
\end{aligned}
$$

Question-5(d) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is $90°$.

[8 Marks]

Solution: If G is the centre of gravity of the cube, then for equilibrium the line OCG must be vertical. First we show that the equilibrium of the cube is stable.



Here, P_1 = the radius of curvature of the upper body at the point of contact = ∞ and, P_2 = the radius of curvature of the lower body at the point of contact = r h = height of the centre of gravity, G of the upper body above the point $C =$ half of the edge of cube

$$= \frac{\pi r}{4}$$

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{P_1} + \frac{1}{P_2} \quad \text{i.e,}\ \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

ie.

$$\frac{4}{\pi r} > \frac{1}{r} \quad i.e, \quad 4 > \pi$$

which is true.

Hence, the equilibrium is stable. So, if the cube is slightly displaced, it will tend to come back to its original position of equilibrium.
During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.
If θ is the angle through which the cube twins when the right hand corner A of the lowest edge comes in contact within sphere,
$\therefore \quad r\theta =$ half the edge of the cube $= \dfrac{\pi r}{4}$
$\therefore \quad \theta = \pi/4$

Similarly, the cube can turn through an angle of $\pi/4$ to the left side on the sphere.
Hence the total angle through which the cube can swing (or rock) without falling is $2\dfrac{\pi}{4}$ ie, $\dfrac{\pi}{2}$.

Question-5(e) Prove that

$$\nabla^2 r^n = n(n+1)r^{n-2}$$

and that $\mathrm{r}^{\mathrm{n}}\overrightarrow{\mathrm{r}}$ **-is irrotational, where** $\mathrm{r} = |\overrightarrow{\mathrm{r}}| = \sqrt{\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2}$.

[8 Marks]

Solution:

$$\begin{aligned}
\nabla^2 r^n &= \nabla \cdot (\nabla r^n) = \text{div}\,(gradr^n) \\
&= \text{div}\left(nr^{n-1} gradr\right) \\
&= \text{div}\left(nr^{n-1}\frac{\vec{r}}{r}\right) = \text{div}\left(nr^{n-2}\vec{r}\right) \\
&= \left(nr^{n-2}\right)\text{div}\,\vec{r} + \vec{r}\cdot\left(g\,\text{rad}\,nr^{n-2}\right) \\
&= 3nr^{n-2} + \vec{r}\cdot\left[n(n-2)r^{n-3} gradr\right] \\
&= 3nr^{n-2} + \vec{r}\cdot\left[n(n-2)r^{n-3}\cdot\frac{\vec{r}}{r}\right] \\
&= 3nr^{n-2} + \vec{r}\cdot\left(n(n-2)r^{n-4}\vec{r}\right] \\
&= nr^{n-2}(3+n-2) \\
&= n(n+1)r^{n-2}
\end{aligned}$$

Now,

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^n\vec{r} = r^n\{xi+yj+zk\}$$

$$\text{curl}(\vec{r}) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i(0) + j(0) + k(0) = 0$$

$$[\because \text{curl}(\phi A) = (grad\phi)\times A + \phi\,\text{curl}\,A)]$$

Hence, $r^n\vec{r}$ is irrotational.

Question-6(a) Solve the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2$$

[15 Marks]

Solution:

$$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2 - (1)$$

Put

$$\frac{dy}{dx} = p$$

$$p^2 + 2py\cot x = y^2$$

$\Rightarrow$

$$p^2 + 2py\cot x + y^2\cot^2 x = y^2 + y^2\cot^2 x$$

$\Rightarrow$

$$(p + y\cot x)^2 = y^2\,\text{cosec}^2\,x$$

$$\therefore \quad p + y\cot x = \pm y\,\text{cosec}\,x$$

i.e.,

$$\left.\begin{aligned}\frac{dy}{dx}+y(\cot x-\operatorname{cosec}x)=0\\ \frac{dy}{dx}+y(\cot x+\operatorname{cosec}x)=0\end{aligned}\right\}\text{ components of eqn .}$$

$$\frac{dy}{y}+(\cot x-\operatorname{cosec}x)dx=0$$

Integrating

$$\log y+\log\sin x-\log\left(\tan\frac{x}{2}\right)=\log C$$

$$\therefore\log y=\log C+\log\tan\frac{x}{2}-\log\sin x$$

$$y=\frac{c\times\tan x/2}{\sin x}=\frac{c\cdot\sin\frac{x}{2}}{\cos\frac{x}{2}\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$y=\frac{c}{2\cos^2\frac{x}{2}}=\frac{c}{1+\cos x}$$

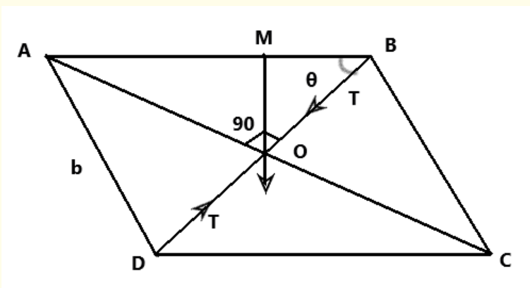
ie $y(1+\cos x)=C$ is one solution. Similarly, solving the second equation, we get, $y(1-cosx)=C$

$$\therefore\text{ General solution of }(1)is,\quad[y(1+\cos x)-c][y(1-cosx)-C]=0$$

Question-6(b) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\dfrac{2\mathrm{W}\left(2\mathrm{b}^2-\mathrm{a}^2\right)}{\mathrm{b}\sqrt{4\mathrm{b}^2-\mathrm{a}^2}}$.

[10 Marks]

Solution: Let T be the tension in the string BD. The total weight out of the rods AB, BC, CD and DA can be taken as acting of the point of intersection O of the diagonals AC and BD. We have, $\angle AOB=90^\circ$



Let $\angle ABO=\theta$, Draw $OM\perp toAB$ Give the system a small symmetrical displacement in which θ change from to $\theta+\delta\theta$. The line AB remains fixed. The points O, C and D change. The angle AOB will remain 90°. $BD=2BO=2ABcos\theta=2bcos\theta$ ($\because$ length $BD=a$ at equilibrinm. It changes during displacement, and depends on angle θ) The depth of O below the fixed line

$$AB=MO=(BO)\sin\theta=(ABcos\theta)\sin\theta$$

ie

$$MO = b\sin\theta\cos\theta$$

By the principle of virtual work,

$$-T\delta(2bcos\theta) + 4W\delta(b\sin\theta\cos\theta) = 0$$

or

$$2bT\sin\theta\delta\theta + 4bW\left(cos^2\theta - \sin^2\theta\right)\delta\theta = 0$$

or

$$2b\left[T\sin\theta - 2W\left(\sin^2\theta - cos^2\theta\right)\delta\theta = 0\right.$$

or

$$T\sin\theta - 2W\left(\sin^2\theta - \cos^2\theta\right) = 0 \quad (\because \delta\theta \neq 0)$$

or

$$T = \frac{2W\left(\sin^2\theta - cos^2\theta\right)}{\sin\theta} = \frac{2W\left(1 - 2\cos^2\theta\right)}{\sqrt{1-\cos^2\theta}}$$

In the position of equilibrium,

$$l \therefore BD = a \text{ or } BO = \frac{a}{2}$$

$$\therefore \quad \cos\theta = \frac{BO}{AB} = \frac{a}{2b}$$

$$\therefore T = \frac{2W\left(1 - 2\cdot\frac{a^2}{4b^2}\right)}{\sqrt{1 - \frac{a^2}{4b^2}}}$$

$$\therefore T = \frac{2W\left(2b^2 - a^2\right)}{b\sqrt{4b^2 - a^2}}$$

Question-6(c) Using Stokes' theorem, evaluate

$$\int_{\mathrm{C}} [(x+y)dx + (2x-z)dy + (y+z)dz]$$

where C is the boundary of the triangle with vertices at $(2,0,0)$**,** $(0,3,0)$ **and** $(0,0,6)$.

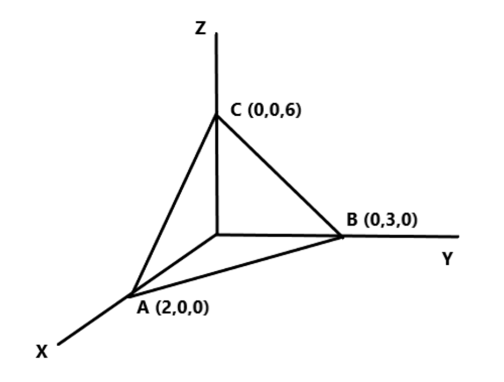
[10 Marks]

Solution: The given integral is of the form $\oint_c F \cdot dr$, where

$$F = (x+y)i + (2x-z)j + (y+z)k$$

$$dx = idx + jdy + kdz$$

C: Boundary of Triangle ABC S: Area of Triangle ABC



$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \dfrac{2}{2x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2i + k - (1)$$

Using Stokes' theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}\cdot\hat{n})dS$$

Here

$$\hat{n}, \text{is unit normal vector to} \frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

$$\hat{n} = \frac{6}{\sqrt{14}}\left(\frac{i}{2}+\frac{j}{3}+\frac{k}{6}\right) = \frac{1}{\sqrt{14}}(3i+2j+k) - (2)$$

$$\text{curl}(\vec{F})\cdot\hat{n} = \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}} \quad [\text{from}(1)\&(2)]$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \frac{7}{\sqrt{14}}\iint_S dS = \frac{7}{\sqrt{14}}(\text{ Area of } \triangle ABC)$$

$$= \frac{7}{\sqrt{14}} \times 3\sqrt{14} = 21$$

$$[Area(\triangle ABC) \Rightarrow \Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$$

$$\Delta^2 = \left(\frac{1}{2}\times 3\cdot 6\right)^2 + \left(\frac{1}{2}\cdot 2\cdot 6\right)^2 + \left(\frac{1}{2}\cdot 2\cdot 3\right)^2 = 126]$$

Question-7(a) Solve the differential equation

$$\mathrm{e}^{3\mathrm{x}}\left(\frac{\mathrm{dy}}{\mathrm{dx}} - 1\right) + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^3 \mathrm{e}^{2\mathrm{y}} = 0$$

[10 Marks]

Solution: Let $e^x = X, \quad e^y = Y$

$$\therefore e^x dx = dx, \quad e^y dy = dy$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{e^y}{e^x}\frac{dy}{dx}$$

$$\Rightarrow P = \frac{Y}{X}p$$

$$\Rightarrow p = \frac{X}{Y}P$$

The given ODE becomes

$$e^{3x}\left(\frac{dy}{dx} - 1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0$$

$$X^3\left(\frac{X}{Y}P - 1\right) + \left(\frac{X}{Y}P\right)^3 Y^2 = 0$$

$$XP - Y + P^3 = 0$$

$$\Rightarrow Y = XP + P^3$$

, which is in Clairaut's form $y = xp + f(p)$
Hence, the general solution is

$$Y = Xc + c^3$$

$$\Rightarrow e^y = ce^x + c^3$$

Question-7(b) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that-the velocity then is $\frac{2\pi \text{ ae}}{\text{T}\sqrt{1-\text{e}^2}}$, where 2a is the major axis, e is the eccentricity and T is the periodic time.

[10 Marks]

Solution: The polar equation of the elliptic orbit B

$$\frac{l}{a} = 1 + e\cos\theta \quad d \quad lu = 1 + e\cos\theta \quad -(1)$$

We know,

$$h^2 = r^2\dot{\theta} \quad d \quad \dot{\theta} = hu^2 \quad \left(u = \frac{1}{r}\right) - (2)$$

Also,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\left(hu^2\right) = -h\frac{du}{d\theta}$$

for maximum value of $\frac{dr}{dt}$, we have

$$\frac{d}{d\theta}\left(\frac{dr}{dt}\right) = 0 \quad d \quad \frac{d}{d\theta}\left(-h\frac{\partial u}{d\theta}\right) = 0 \quad d \quad \frac{d^2u}{d\theta^2} = 0$$

$$(\because \text{h is constant})$$

From (1) ,

$$\frac{du}{d\theta} = \frac{-e}{l}\sin\theta \quad \& \quad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = 0 \Rightarrow \frac{-e}{l}\cos\theta = 0 \Rightarrow \cos\theta = 0, \text{i.e}, \theta = \frac{\pi}{2}$$

This proves the first part. For maximum value of $\frac{dr}{dt}$,

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\frac{\pi}{2} = -\frac{e}{l} \quad -(3)$$

From

$$(2)\&(3), \left(\frac{dr}{dt}\right)_{\max} = \frac{he}{l} = \sqrt{\mu l}\cdot\frac{e}{l} = \int\frac{\mu}{l}e - (4)$$

As,

$$l = \frac{b^2}{a} = a\left(1-e^2\right)$$

and

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$\therefore \sqrt{l} = \sqrt{a\left(1-e^2\right)}$$

and

$$\sqrt{\mu} = \frac{2\pi a^{3/2}}{T}$$

Substituting in (4)

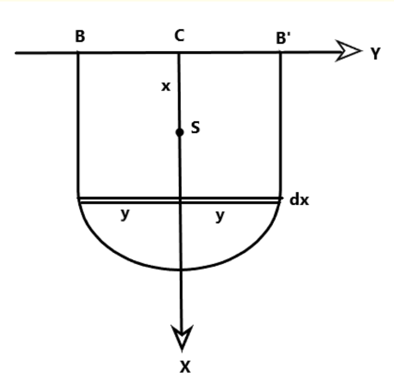
$$\left(\frac{dr}{dt}\right)_{\max} = \frac{2\pi a^{3/2}e}{T\sqrt{a\left(1-e^2\right)}} = \frac{2\pi ae}{T\sqrt{1-e^2}}$$

[l = semi-latus rectum]

Question-7(c) A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth. If the minor axis lies on the surfare, then find the eccentricity in order that the focus may be the centre of pressure.

[10 Marks]

Solution: BAB' is the semi-ellipse immersed in a liquid with minor axis BB' in the surface. Consider the elementary strip of width dx at a distance x from c.



$$\therefore P = \rho g x = kx \cdot gx = kgx^2$$

$$ds = 2y dx$$

But,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ie.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore ds = \frac{2b}{a}\sqrt{a^2 - x^2}dx$$

$$\therefore \bar{x} = \frac{\int_0^a xpds}{\int_0^a pds} = \frac{\int_0^a x \cdot \text{kg}\, x^2 \cdot \frac{2b}{a}\sqrt{a^2 - x^2}dx}{\int_0^a \text{kg}\, x^2 \frac{2b}{a}\sqrt{a^2 - x^2}dx}$$

$$= \frac{\int_0^a x^3\sqrt{a^2 - x^2}dx}{\int_0^a x^2\sqrt{a^2 - x^2}dx}$$

Put,

$$x = a\sin\theta, \quad dx = acos\theta d\theta$$

$$\therefore \quad \bar{x} = \frac{\int_0^{\pi/2} a^3 \sin^3\theta \cdot a^2 \cos^2\theta d\theta}{\int_0^{\pi/2} a^2 \sin^2\theta \cdot a^2 \cos^2\theta d\theta}$$

$$= \left(\frac{2 \cdot 1}{5 \cdot 3}a\right) / \left(\frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{32}{15\pi}a$$

$$CS = \text{ Distance of focus from } C$$

$$= ae = \frac{32a}{15\pi}$$

$$\therefore e = \frac{32}{15\pi}$$

Question-7(d) Evaluate

$$\iint_S (\nabla \times \overrightarrow{\mathrm{f}}) \cdot \hat{\mathrm{n}}\mathrm{dS}$$

$$y = 3 = \frac{-3}{2}(3 \cdot 0)$$

where S is the surface of the cone, $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane and

$$\vec{f} = (x - z)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\hat{k}$$

[10 Marks]

Solution: The xy-plane cuts the surface S of cone in the circle, C whose equation is $x^2 + y^2 = 4; z = 0$. parametric eqn: $x = 2\cos t$, $y = 2\sin t$ By Stokes' theorem,

$$\iint\limits_S (\nabla \times \vec{f}) \cdot \hat{n} dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_c (x-2)dx + \left(x^3 + yz\right) dy + \left(-3xy^2\right) dz$$

$$= \int_c xdx + x^3 dy \quad (\because z = dz = 0)$$

$$= \int_{t=0}^{2\pi} \left[x\frac{dx}{dt} + x^3 \frac{dy}{dt}\right] dt$$

$$= \int_{t=0}^{2\pi} \left(2\cos t(-2\sin t) + 8\cos^3 t \cdot 2\cos t\right] dt$$

$$= \int_{t=0}^{2\pi} \left(-2(\sin 2t) + 16\cos^4 t\right) dt$$

$$= -2\int_0^{2\pi} \sin 2t dt + 16\int_0^{2\pi} \left(\frac{1 + cos2t}{2}\right)^2 dt$$

$$= [\cos 2t]_0^{2\pi} + 16\left\{\frac{1}{32}\left[\sin 4x\right]_0^{2\pi} - \frac{1}{8}[n]_0^{2\pi} + \frac{1}{2}[x]_0^{2\pi} + \frac{1}{4}[\sin 2x]_0^{2\pi}\right\}$$

$$= 0 + 16\left\{0 - \frac{2\pi}{8} + \frac{2\pi}{2} + 0\right\}$$

$$= 12\pi$$

Question-8(a) Solve $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}\mathrm{x}^2} + 4\mathrm{y} = \tan 2\mathrm{x}$ **by using the method of variation of parameter.**

[10 Marks]

Solution: Comparing with,

$$y_2 + Py_1 + Qy = R$$

$$P = 0, \quad Q = 4, R = \tan 2x$$

Auxiliary

$$\text{eqn}\left(D^2 + 4\right) = 0 \quad \therefore \quad D = \pm 2i$$

$$y_c = c_1\cos 2x + c_2\sin 2x \quad -(1)$$

Using method of variation of parameters, let the complete solution be given by

$$y = A\cos 2x + B\sin 2x$$

, where A and B are functions of x.
Then,

$$u(x) = \cos 2x, v(x) = \sin 2x, \quad R(x) = \tan 2x$$

.

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$\begin{aligned} A &= \int \frac{-vR}{w} dx = \int \frac{-\sin 2x \cdot \tan 2x}{2} dx \\ &= \int \frac{-\sin^2 2x}{2\cos 2x} dx = \frac{-1}{2} \int \frac{1-\cos^2 2x}{\cos 2x} dx \\ &= \frac{-1}{2} \int (\sec 2x - \cos 2x) dx \\ &= -\frac{1}{4} (\log|\sec 2x + \tan 2x| - \sin 2x) \end{aligned}$$

$$\begin{aligned} \text{and, } B &= \int \frac{uR}{w} dx = \int \frac{\cos 2x \cdot \tan 2x}{2} dx \\ &= \frac{1}{2} \int \sin 2x dx = -\frac{1}{4} \cos 2x \end{aligned}$$

$$\therefore y = \frac{-1}{4} (\log \mid \sec 2x + \tan 2x \mid - \sin 2x) \cos 2x - \frac{1}{4} \cos 2x \cdot \sin 2x - (2),$$

where y is the general solution of the given DE.

Question-8(b) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu \left(\frac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ -after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$.

[10 Marks]

Solution: Acceleration,

$$\frac{d^2x}{dt^2} = -\mu \cdot \frac{a^{5/3}}{x^{2/3}}$$

$$2\left(\frac{d^2x}{dt^2}\right) \cdot \frac{dx}{dt} = -2\mu \frac{a^{5/3}}{x^{2/3}} \cdot \frac{dx}{dt}$$

Integrating both sides w.r.t $\left(\frac{dx}{dt}\right)$ from rest to final point (0).

$$\left(\frac{dx}{dt}\right)^2 \Bigg|_0^{v_0} = -\, 2\mu a^{5/3} \frac{x^{1/3}}{1/3}\Bigg|_a^0$$

$$v_0^2 = 6\mu a^{5/3} \cdot a^{1/3} = 6\mu a^2$$

$$v_0 = a\sqrt{6\mu}$$

$$\left(\frac{dx}{dt}\right)^2 \Bigg|_0^{dx/dt} = -\, 6\mu a^{5/3} \cdot x^{1/3}\big|_a^x$$

$$\left(\frac{dx}{dt}\right)^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$$

$$\frac{dx}{\sqrt{a^{1/3} - x^{1/3}}} = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}dt$$

Put,

$$x^{1/3} = a^{1/3}\sin^2\theta \Rightarrow x = a\sin^6\theta \Rightarrow dx = 6a\sin^5\theta\cos\theta$$

$$x = 0 \to \theta = \pi/2$$

$$x = 0 \to \theta = 0$$

$$\int_{\pi/2}^0 \frac{6a\sin^5\theta\cos\theta}{a^{1/6}\left(1 - \sin^2\theta\right)^{1/2}}d\theta = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}dt$$

$$\sqrt{6\mu a^{5/3}}t_0 = \int_0^{\pi/2} 6a^{5/6}\sin^5\theta d\theta$$

$$\sqrt{6\mu}a^{5/6}t_0 = 6a^{5/6}\cdot\frac{4\cdot 2}{1\cdot 3\cdot 5} = \frac{16}{5}a^{5/6}$$

$$\therefore t_0 = \frac{16}{5}\cdot\frac{1}{\sqrt{6\mu}}$$

$$t_0 = \frac{8}{15}\cdot\sqrt{\frac{6}{\mu}}$$

Question-8(c) Find the curvature and torsion of the circular helix

$$\overrightarrow{\mathrm{r}} = \mathrm{a}(\cos\theta, \sin\theta, \theta\cot\beta),$$

β is the constant angle at which it cuts its generators.

[10 Marks]

Solution: Curvature,

$$\kappa = \frac{\left|\frac{d\vec{r}}{d\theta}\times\frac{d^2\vec{r}}{d\theta^2}\right|}{\left|\frac{d\vec{r}}{d\theta}\right|^3} - (1)$$

Torsion,

$$\tau = \frac{\left[\frac{d\vec{r}}{d\theta}\,\frac{d^2\vec{r}}{d\theta^2}\,\frac{d^3\vec{r}}{d\theta^3}\right]}{\left|\frac{d\vec{r}}{d\theta}\times\frac{d^2\vec{r}}{d\theta^2}\right|^2} - (2)$$

$$\vec{r} = a(\cos\theta i + \sin\theta j + \theta\cot\beta k)$$

$$\frac{d\vec{r}}{d\theta} = a(-\sin\theta i + \cos\theta j + \cot\beta k)$$

$$\frac{d^2\vec{r}}{d\theta^2} = a(-\cos\theta i - \sin\theta j)$$

$$\frac{d^3\vec{r}}{d\theta^3} = a(\sin\theta i - cos\theta j)$$

$$\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & acot\beta \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= i\left(a^2\sin\theta\cot\beta\right) - j\left(a^2\cos\theta\cot\beta\right)$$

$$+k\left(a^2\sin^2\theta + a^2\cos^2\theta\right)$$

$$= a^2[(\sin\theta\cot\beta)i - (\cos\theta\cot\beta)j + k]$$

$$\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = a^2\sqrt{(\sin\theta\cot\beta)^2 + (\cos\theta cot\beta)^2 + 1^2}$$

$$= a^2\sqrt{1+\cot^2\beta} = a^2\operatorname{cosec}\beta$$

$$\left|\frac{\partial\vec{r}}{d\theta}\right| = a\sqrt{\sin^2\theta + \cos^2\theta + \cot^2\beta} = a\operatorname{cosec}\beta$$

$$\therefore \quad \kappa = \frac{a^2\cdot\operatorname{cosec}\beta}{(a\operatorname{cosec}\beta)^3} = \frac{1}{a}\sin^2\beta$$

For torsion, scalar triple product is

$$\left[\frac{d\vec{r}}{d\theta}\frac{d^2\vec{r}}{d\theta^2}\frac{d^3\vec{r}}{d\theta^3}\right] = \begin{vmatrix} -a\sin\theta & acos\theta & a\cot\beta \\ -a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & -a\cos\theta & 0 \end{vmatrix}$$

$$= a\cot\beta\left(a^2\cos^2\theta + a^2\sin^2\theta\right)$$

$$= a^3\cot\beta$$

$$\tau = \frac{a^3\cot\beta}{\left(a^2\operatorname{cosec}\beta\right)^2}$$

$$= \frac{1}{a}\cdot\frac{\cos\beta}{\sin\beta}\times\sin^2\beta$$

$$= \frac{1}{a}\sin\beta\cos\beta$$

Question-8(d) If the tangent to a curve makes a constant angle α, with a fixed line, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$ Conversely, if $\frac{\mathrm{K}}{\mathrm{t}}$ is constant, then show that the tangent makes a constant angle with a fixed direction.

[10 Marks]

Solution: Let e, be the unit vector parallel to the given fixed line so that as given

$$t\cdot e = \cos\alpha \qquad \ldots(1)$$

Differentiating. we get

$$\frac{dt}{ds}\cdot e = 0 \qquad \kappa n \cdot e = 0 \quad (\text{ frenet's first })$$
$$\therefore \quad n\cdot e = 0 \qquad \ldots(2)$$

Hence, n is $\perp$ to e. Thus, the vectors b, t, e are coplanar.

$$\therefore \quad b\cdot e = \pm\sin\alpha \qquad \ldots(3)$$

Differentiating (2) and applying Frenet-Serret formula,

$$\frac{dn}{ds}\cdot e = 0$$

ie

$$-(\kappa t + \tau b)\cdot e = 0$$

$$\therefore \quad \kappa\cos\alpha \pm \tau\sin\alpha = 0$$

from (1)&(3) Conversely: het

$$\frac{\kappa}{\tau} = \frac{1}{a},$$

a is some scalar constant. or

$$\frac{1}{\kappa} = \frac{a}{\tau}$$

ie,

$$\sigma = ap$$

As

$$\frac{dt}{ds} = \frac{1}{p}n and \frac{db}{ds} = \frac{1}{\sigma}n$$

$$\therefore p\frac{dt}{ds} = n = \sigma\frac{db}{ds}$$

or

$$\frac{dt}{ds} = \frac{\sigma}{p}\cdot\frac{db}{ds} = a\frac{db}{ds}$$

Integrating, we get

$$t = ab + c\ ,$$

where c is a constant vector.
Multiplying scalarly with t, we get

$$t\cdot t = ab\cdot t + c\cdot t$$

$$1 = 0 + ct, i.e \quad t\cdot c = 1$$

Hence, the tangent makes a constant angle with a fixed direction.

Chapter 5

2016

5.1 Section-A

Question-1(a) Let $\mathrm{T} : \mathrm{B}^3 \to \mathrm{B}^4$ be given by $T(x, y, z) = (2x - y, 2x + z, (z) + 2z, x + y + z)$. Find the matrix of T with respect to standard basis of EB^3 and R^4 (i.e., $(1, 0, 0)$, $(0, 1, 0)$, etc. Examine if T is a linear map.

[8 Marks]

Solution: Given $T : R^3 \to R^4$,

$$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$$

$$\begin{aligned} T(1,0,0) &= (2,2,1,1) \\ &= 2(1,0,0,0) + 2(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1) \\ T(0,1,0) &= (-1,0,0,1) \\ &= -1(1,0,0,0) + 0(0,1,0,0) + 0(0,0,1,0) + 1(0,0,0,1) \end{aligned}$$

$$\begin{aligned} T(0,0,1) &= (0,1,2,1) \\ &= 0(1,0,0,0) + 1(0,1,0,0) + 2(0,0,1,0) + 1(0,0,0,1) \end{aligned}$$

]

$$\therefore [T]_\alpha^\beta = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}^{\top} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Let $a = (x_1, y_1, z_1)$, $\quad b = (x_2, y_2, z_2) \quad \& \quad k$ is constant.

$$\begin{aligned} T(a+b) =& T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= \begin{bmatrix} 2(x_1 + x_2) - (y_1 + y_2), 2(x_1 + x_2) + (z_1 + z_2), \\ (x_1 + x_2) + 2(z_1 + z_2), (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (2x_1 - y_1) + (2x_2 - y_2), (2x_1 + z_1) + (2x_2 + z_2) \\ (x_1 + 2z_1) + (x_2 + 2z_2) + (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \end{bmatrix}$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) = T(a) + T(b)$$

Similary,

$$T(kx_1) = k \cdot T(x_1).$$

Hence T is linear.

Question-1(b) Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ **for** $x > 0$.

[8 Marks]

Solution: Consider the function,

$$f(x) = \log(1+x) - \frac{x}{1+x}$$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{1+x^2} > 0$$

$\therefore f(x)$ is increasing function,

$$\therefore \quad If \quad x > 0 \Rightarrow f(x) > f(0)$$

ie

$$\log(1+x) - \frac{x}{1+x} > \log(1+0) - \frac{0}{1+0}$$

ie

$$\log(1+x) > \frac{x}{1+x} - (1)$$

Again, let

$$g(x) = x - \log(1+x)$$

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \quad \forall x > 0$$

$\therefore g(x)$ is increasing function $\therefore$

$$for, x > 0 \quad \Rightarrow \quad f(x) > f(0)$$

ie

$$x - \log(1+x) > 0 - \log(1+0)$$
$$x > \log(1+x) - (2)$$

Combining (1) and (2),

$$\frac{x}{1+x} < \log(1+x) < x$$

Question-1(c) Examine if the function $f(x,y) = \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0)$ **and** $f(0,0) = 0$ **is continuous at** $(0,0)$**. Find** $\dfrac{\partial f}{\partial x}$ **and** $\dfrac{\partial f}{\partial y}$ **at points other than origin.**

[8 Marks]

Solution:

$$f(x) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

We show that limit does not exist at $(0,0)$.
Along the curve $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{x(mx)}{x^2+(mx)^2} = \frac{m}{1+m^2}$$

Which is different for different values of x. Hence, limit does not exist and to f(x) is not continuous at (0,0).
For the points, other than origin

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y\left(x^2+y^2\right)-2x(xy)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^3-x^2y}{\left(x^2+y^2\right)^2} = \frac{y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

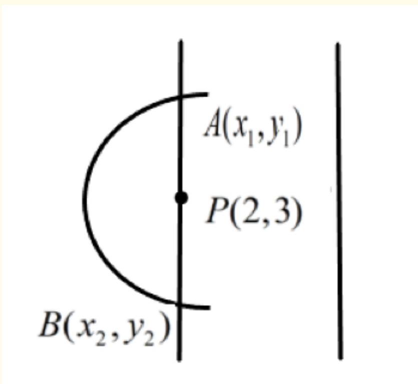
Similarly,

$$\frac{\partial F}{\partial y} = \frac{x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

Question-1(d) If the point (2,3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord.

[8 Marks]

Solution: Let two points on the parabola be $A\left(x_1, y_1\right)$ & $B\left(x_2, y_2\right)$ where chord cut the parabola and $P(2,3)$ be the mid-point.



$$\therefore \quad y_1^2 = 4x_1 - (1) \quad \& \quad y_2^2 = 4x_2 - (2)$$

$$\frac{x_1+x_2}{2} = 2 \quad , \quad \frac{y_1+y_2}{2} = 3$$

As

$$y_2^2 - y_1^2 = 4x_2 - 4x_1$$

$$\left(y_1+y_2\right)\left(y_2-y_1\right) = 4\left(x_2-x_1\right)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4}{y_1 + y_2}$$
$$= \frac{4}{6} = \frac{2}{3}$$

Slope of

$$AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2}{3}$$

$\therefore$ Eqn of Chord:

$$y - 3 = 2/3(x - 2)$$
$$3y - 9 = 2x - 4$$
$$2x - 3y + 5 = 0$$

Question-1(e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$**, obtain the eigenvalue and get the value of** $A^4 + 3A^3 - 9A^2$.

[8 Marks]

Solution: Here, $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 + 3\lambda^2 - 9\lambda - 27 = 0$$
$$\Rightarrow \quad (\lambda + 3)\left(\lambda^2 - 9\right) = 0$$

$\therefore \lambda = -3, 3, 3$ are the eigenvalues. By Cayley-Hamilton Theorem.

$$A^3 + 3A^2 - 9A - 27I = 0$$
$$\Rightarrow \quad A^4 + 3A^3 - 9A^2 - 27A = 0$$

$$\therefore \quad A^4 + 3A^3 - 9A^2 = 27A = \begin{bmatrix} -27 & 54 & 54 \\ 54 & -27 & 54 \\ 54 & 54 & -27 \end{bmatrix}$$

Question-2(a) After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin nx\,dx\,dy$ **show that** $\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$.

[10 Marks]

Solution:

$$I = \int_0^\infty \int_0^\infty \sin nx \cdot e^{-xy} \cdot dydx$$
$$= \int_0^\infty \sin nx \cdot \left(\frac{e^{-xy}}{-x}\right)_{y=0}^{\infty} dx$$
$$= \int_0^\infty \sin nx \left(0 + \frac{1}{x}\right) dx = \int_0^\infty \frac{\sin nx}{x} dx - (1)$$

Now, first integrating w.r.t x,

$$I = \int_0^\infty \left[-\frac{1}{y} e^{-xy} \cdot \sin nx \Big|_{x=0}^{\infty} + \int_0^\infty \frac{1}{y} e^{-xy} \cdot ncosnxdx \right] dy$$
$$= \int_0^\infty \left[\frac{n}{y} \left(-\frac{1}{y} e^{-xy} \cos nx \Big|_{x=0}^{\infty} - \int_0^\infty \frac{e^{-xy}}{y} n \sin nx \right) \right] dy$$
$$= \int_0^\infty \frac{n}{y} \left(0 + \frac{1}{y} - \frac{n}{y} I' \right) dy$$
$$= \int_0^\infty \left(\frac{n}{y^2} - \frac{n^2}{y^2} I \right) dy$$
$$\therefore \qquad \frac{n}{y^2} - \frac{n^2}{y^2} I = I \Rightarrow I \left(1 + \frac{n^2}{y^2}\right) = \frac{n}{y^2}$$
$$I = \frac{n}{n^2 + y^2}$$
$$\therefore \int_0^\infty \frac{n}{n^2 + y^2} dy = \frac{1}{n} \cdot n \, tan^{-1} \frac{y}{n} \Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$
$$\therefore I = \int_0^\infty \frac{\sin nx}{x} = \pi/2$$

Question-2(b) A perpendicular is drawn from the centre of ellipse Q $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given-by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

[10 Marks]

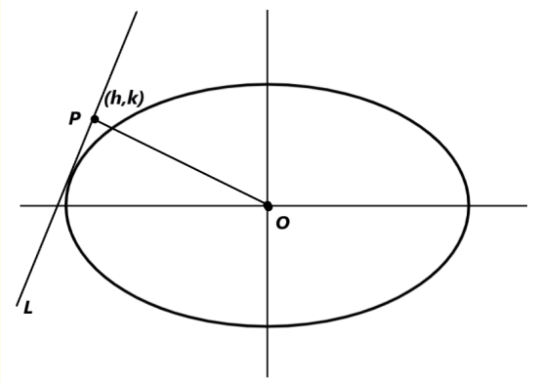
Solution: The tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$y = mx \pm \sqrt{a^2m^2 + b^2} - (1)$$

for any value of m.



Slope of line $OP = \dfrac{k-0}{h-0} = \dfrac{k}{h}$, Slope of tangent line $= -\dfrac{h}{k} \quad (OP \perp L)$

$\therefore$ Eqn of tangent line

$$y - k = -\frac{h}{k}(x-h)$$
$$y = -\frac{h}{k}x + \frac{h^2}{k} + k$$
$$y = -\frac{h}{k}x + \left(\frac{h^2+k^2}{k}\right) - (2)$$

Comparing Eqn (1) with (2)

$$\pm\sqrt{a^2m^2+b^2} = \frac{h^2+k^2}{k}$$

$$\left(a^2\left(\frac{-h}{k}\right)^2 + b^2\right) = \left(\frac{h^2+k^2}{k}\right)^2 \quad \left(\because m = \frac{-h}{k}\right)$$

$$\therefore a^2h^2 + b^2k^2 = \left(h^2+k^2\right)^2$$

Hence required locus:

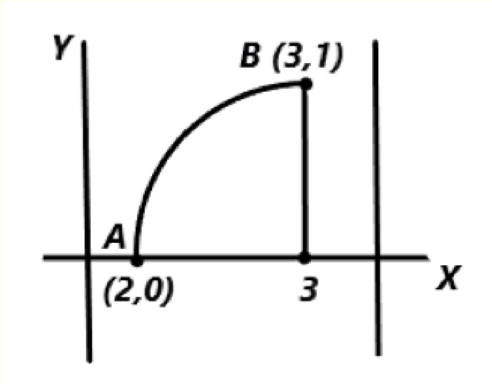
$$\left(x^2+y^2\right)^2 = a^2x^2 + b^2y^2$$

Question-2(c) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on $[2,3]$, where the tangent (is, parallel to the chord joining the end points of the curve.

[10 Marks]

Solution:

$$y = \sqrt{x-2}, \quad x \in [2,3]$$
$$y^2 = x - 2$$



End points are $A(2,0)$ and $B(3,1)$

$$y = \sqrt{x-2}, \text{is continuous on}[2,3]$$

$$y = \sqrt{x-2}, \text{is differentiable on}(2,3)$$

Hence, by Lagrange's mean value theorem (LMVT), there exists some $c \in (2,3)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c-1}} = \frac{f(3)-f(2)}{3-2} = \frac{1-0}{1}$$

$$\Rightarrow 2\sqrt{c-1} = 1$$

$$\text{ie. } c-2 = \frac{1}{4} \Rightarrow c = \frac{9}{4}$$

Hence, at

$$x = 9/4, y = \sqrt{\frac{9}{4}-2} = \frac{1}{2},$$

tangent to the curve is parallel to the chord joining the end points as slopes are equal there.

Question-2(d) Let T be a linear map such that $\mathrm{T} : \mathrm{v}_3 \to \mathrm{v}_2$ defined by

$$\mathrm{T}(\mathrm{e}_1) = 2\mathrm{f}_1 - \mathrm{f}_2,$$

$$\mathrm{T}(\mathrm{e}_2) = \mathrm{f}_1 + 2\mathrm{f}_2,$$

$$\mathrm{T}(\mathrm{e}_3) = 0\mathrm{f}_1 + 0\mathrm{f}_2,$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ and $\mathrm{f}_1, \mathrm{f}_2$ are standard basis in V_3 and V_2.
Find the matrix of T relative to these basis.
Further take two other basis $\mathrm{B}_1[(1,1,0),(1,0,1),(0,1,1)]$ and $\mathrm{B}_2[(1,1),(1,-1)]$. Obtain the matrix T_1 relative to B_1 and B_2.

[10 Marks]

Solution:

$$T(e_1) = 2f_1 - f_2$$

$$T\left(e_2\right) = f_1 + 2f_2$$
$$T\left(e_3\right) = 0f_1 + 0f_2$$
$$T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}^\top = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$
$$T(a,b,c) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+b \\ -a+2b \end{bmatrix}$$
$$T(1,1,0) = (3,1) = x_1(1,1) + y_1(1,-1)$$
$$T(1,0,1) = (2,-1) = x_2(1,1) + y_2(1,-1)$$
$$T(0,1,1) = (1,2) = x_3(1,1) + y_3(1,-1)$$
$$\therefore \quad x_1 = 2, y_1 = 1, \quad x_2 = \frac{1}{2}, y_2 = \frac{3}{2}, \quad x_3 = \frac{3}{2}, y_3 = \frac{-1}{2}$$
$$\therefore \quad [T]_{B_1}^{B_2} = \begin{bmatrix} 2 & 1 \\ 1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix}^\top = \begin{bmatrix} 2 & 1/2 & 3/2 \\ 1 & 3/2 & -1/2 \end{bmatrix}$$

Question-3(a) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$**, find two non-singular matrices** P **and** Q **such that** $\mathrm{PAQ} = \mathrm{I}$**. Hence find** A^{-1}.

[10 Marks]

Solution:

$$IAI = A$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$
$$R_2 \to R_2 - \frac{2}{3}R_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix}$$
$$R_3 \to R_3 - R_2$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix}$$
$$C_2 \to C_2 + C_1, C_3 \to C_3 - 4/3C_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$C_3 \to C_3 + \frac{4}{3}C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$R_1 \to R_1/3, \quad R_2 \to -R_2, R_3 \to -3R_3$$

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAQ = I$$

$$A = P^{-1}Q^{-1}$$

$$A^{-1} = QP$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Question-3(b) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ **which is closest to the point** $(1, 0, 0)$.

[10 Marks]

Solution: Let the required point be (x, y, z). Now we have to maximize

$$f(x, y, z) = (x-1)^2 + y^2 + z^2 - (1)$$

subject to

$$2x + 3y + 4z = 5 \quad -(2)$$

Let

$$g(x, y, z) = 2x + 3y + 4z - 5$$

Let λ be the Lagrange's multiplier,

$$f + \lambda g = F(x, y, z)$$

For critical points, $\partial F = 0$

$$dx = 2(x-1) + 2\lambda = 0 \quad \Rightarrow \quad x = -\lambda + 1$$

$$dy = 2y + 3\lambda = 0 \quad \Rightarrow \quad y = -\frac{3\lambda}{2}$$

$$dz = 2z + 4\lambda = 0 \quad \Rightarrow \quad z = -2\lambda$$

Using Eqn (2)

$$2(-\lambda + 1) + 3\left(-\frac{3\lambda}{2}\right) + 4(-2\lambda) = 5$$

$$\frac{-29}{2}\lambda = 3 \Rightarrow \lambda = -\frac{6}{29}$$

$$\therefore x = \frac{6}{29} + 1 = \frac{35}{29}, \quad y = \frac{9}{29}, z = \frac{12}{29}$$

Hence, the required point is

$$\left(\frac{35}{29}, \frac{9}{29}, \frac{12}{29}\right)$$

(which is the foot of the $\perp$ also).

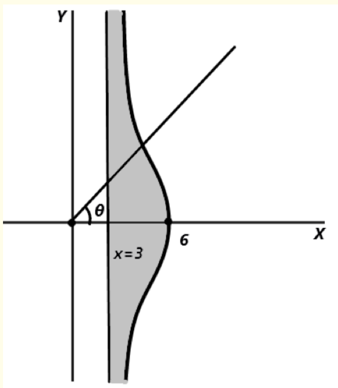
Question-3(c) Obtain the area between the curve $x = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$.

[10 Marks]

Solution: The curve is symmetrical about the initial line and has an asymptote

$$r = 3\sec\theta$$

In the upper half of the curve θ varies from 0 to $\pi/2$.



$\therefore$The required area

$$= 2\int_0^{\pi/2}\int_{3sec\theta}^{3(sec\theta+cos\theta)} rdrd\theta$$

$$= 2\int_0^{\pi/2} \left.\frac{r^2}{2}\right|_{3\sec\theta}^{3(\sec\theta+cos\theta)} d\theta$$

$$= 2\cdot\frac{9}{2}\int_0^{\pi/2}(\sec\theta + cos\theta)^2 - \sec^2\theta d\theta$$

$$= 9\int_0^{\pi/2}\left(2 + \cos^2\theta\right) d\theta$$

$$= 9\left[(2\theta)_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = 9\cdot\frac{\pi}{2}\left(2 + \frac{1}{2}\right)$$

$$= \frac{45}{4}\pi \text{ sq. unit.}$$

Question-3(d) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle.

[10 Marks]

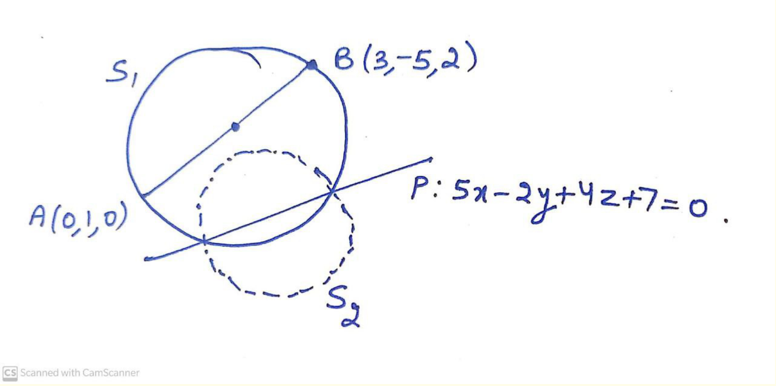
Solution:

$$r = \sqrt{\frac{9}{4} + 9 + 1} = \frac{7}{2}$$

Equation of S_1

$$\left(x - \frac{3}{2}\right)^2 + (y+2)^2 + (z-1)^2 = \frac{49}{4}$$

$$x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$$



Equation of S_2 is :$S_1 + \lambda P = 0$

$$\left(x^2 + y^2 + z^2 - 3x + 4y - 2z - 5\right) + \lambda(5x - 2y + 4z + 7) = 0$$

$$x^2 + y^2 + z^2 + (-3 + 5\lambda)x + (4 - 2\lambda)y$$
$$+(-2 + 4\lambda)z - 5 + 7\lambda = 0$$

Centre

$$\left(\frac{3 - 5\lambda}{2}, -2 + \lambda, 1 - 2\lambda\right)$$

lies on P

$$5\left(\frac{3 - 5\lambda}{2}\right) - 2(-2 + \lambda) + 4(1 - 2\lambda) + 7 = 0$$

$$\lambda = 1$$

$\therefore$ Eqn of S_2

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$$

with centre (-1,-1,-1) and radius 1 .

Question-4(a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature.

[10 Marks]

Solution: The given quadratic form can be written is:

$$\left(4x^2 + xy - 2xz\right) + \left(yx - y^2 - yz\right) + \left(-2zx - zy + 2z^2\right)$$

The matrix of this quadratic form is:

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$ Which is a symmetric square matrix of order 3×3

First we reduce it to its diagonal (canonical) form by writing $A = IAI$

$$\begin{bmatrix} 4 & 1 & -2 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To avoid fraction,

$$R_2 \to 4R_2, R_3 \to 2R_3$$

$$\begin{bmatrix} 4 & 1 & -2 \\ 4 & -4 & -4 \\ -4 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perform corresponding column operations,

$$C_2 \to 4C_2, C_3 \to 2C_3$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 4 & -16 & -8 \\ -4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Apply,

$$R_2 \to R_2 - R_1, R_3 \to R_3 + R_1 \quad 4 \quad C_2 \to C_2 - C_1, C_3 \to C_3 + C_1$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & -4 \\ 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{5}R_2, \quad C_3 \to C_3 - C_2/5$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 24/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 6/5 & -4/5 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 6/5 \\ 0 & 4 & -4/5 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonal form,

$$4x^2 - 20y^2 + \frac{24}{5}z^2$$

Rank (r) of given quadratic form= No. of non zero terms in diagonal form(canonical/normal form)=3.
Signature (S) of given quadratic form = No. of positive terms − No. of negative terms=2-1=1
The index of the given quadratic form= No. of positive terms in normal form=2
Since, $r = S$ here, the given quadratic form is not positive definite.

Question-4(b) Show that the integral $\int_0^\infty \mathrm{e}^{-x}\mathrm{x}^{\alpha-1}\mathrm{dx}, \alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$.

[10 Marks]

Solution:

$$I = \int_0^\infty e^{-x} \cdot x^{\alpha-1} dx = \int_0^1 e^{-x} x^{\alpha-1} dx (\text{Let } I_1) + \int_1^\infty e^{-x} \cdot x^{\alpha-1} dx (\text{Let } I_2)$$

For $\alpha \geq 1$,I_1 is a proper integral
while I_2 is improper

$$I_2 = \int_1^\infty e^{-x} \cdot \alpha^{\alpha-1} dx,$$

let

$$f(x) = x^{\alpha-1} \cdot e^{-x}$$

and take

$$g(x) = \frac{1}{x^2}$$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\alpha-1} \cdot e^{-x}}{1/x^2} = \lim_{x\to\infty} x^{\alpha+1} \cdot e^{-x}$$

$$= \lim_{x\to\infty} \frac{x^{\alpha+1}}{e^x} \left(\frac{\infty}{\infty} form\right)$$

$$= \frac{(\alpha+1)!}{e^x} = 0, \quad \Rightarrow convergent$$

pence I exists for $\alpha \geq 1$. For $0 < \alpha < 1$ I_1 is an improper integral & I_2 is an improper integral & point of non-convergence, $x = 0$

$$I_1 = \int_0^1 e^{-x} \cdot x^{\alpha-1} dx,$$

let

$$f(x) = \frac{e^{-x}}{x^{1-\alpha}}$$

& $\quad g(x) = \dfrac{1}{x^{1/2}}$ where $\int_0^1 \dfrac{1}{x^u} du$ is congt for $0 < u < 1$

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{e^{-x}}{x^{1-\alpha}} x^u = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha-4}}$$

$$= 0$$

$\therefore$ The integral is convergent

$$I_2 = \int_0^\infty e^{-x} \cdot x^{\alpha-1} dx, 0 < \alpha < 1$$

take

$$g(x) = \frac{1}{x^2}$$

$$\lim_{x\to\infty}\frac{e^{-x}}{x^{1-\alpha}}x^2=\frac{e^{-x}}{x^{1-\alpha-2}}=x^{1+\alpha}\cdot e^{-x}$$
$$=\frac{x^{1+\alpha}}{e^x}=\frac{(1+\alpha)x^\alpha}{e^x}=0\quad\left(\frac{0}{0}form\right)$$

Hence we get it convergent by Comparision Test hence integral exist for $0<\alpha<1$

Question-4(c) **Prove that** $\Gamma(2z)=\dfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\dfrac{1}{2}\right)$

[10 Marks]

Solution: We know that

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},m>0,n>0$$

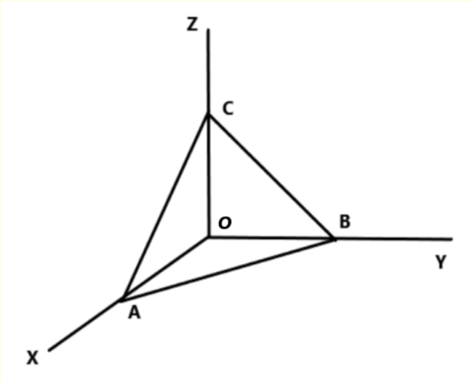
Take $m=n$

$$\beta(n,n)=\frac{(\Gamma(n))^2}{\Gamma(2n)}=2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2n-1}\theta d\theta$$
$$=\int_0^1 x^{n-1}(1-x)^{n-1}dx\quad\left[\begin{array}{l}x=\sin^2\theta\\ dx=\sin 2\theta d\theta\end{array}\right]$$
$$B(n,n)=2\int_0^{\pi/2}(\sin\theta\cdot\cos\theta)^{2n-1}d\theta=\frac{2}{2^{2n-1}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}d\theta$$
$$=\frac{1}{2^{2n-1}}\int_0^{\pi}(\sin\alpha)^{2n-1}d\alpha\quad\left[\begin{array}{l}\text{let } 2\theta=\alpha\\ 2d\theta=d\alpha\end{array}\right]$$
$$=\frac{2}{2^{2n-1}}\cdot\int_0^{\pi/2}\sin^{2n-1}\alpha\cdot d\alpha\left[\begin{array}{c}\int_0^{2a}f(x)dx=2\int_0^a f(x)dx\\ \text{if } f(2a-x)=f(x)\end{array}\right]$$
$$=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}\alpha\cdot\cos^0\alpha d\alpha$$
$$=\frac{1}{2^{2n-2}}\cdot\frac{\Gamma(n)\cdot\Gamma(1/2)}{2\Gamma(n+1/2)}\quad\left[\ 2n-1=0\Rightarrow n=1/2\ \right]$$
$$\therefore\ \frac{\Gamma(n)\cdot\Gamma(n)}{\Gamma(2n)}=\frac{1}{2^{2n-2}\cdot 2}\cdot\frac{\sqrt{\pi}\cdot\Gamma(n)}{\Gamma(n+1/2)}$$
$$\therefore\quad\Gamma(2n)=\Gamma(n)\cdot\Gamma\left(n+\frac{1}{2}\right)\frac{2^{2n-1}}{\sqrt{\pi}}$$

Question-4(d) A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=a_2$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C.

[10 Marks]

Solution: Let $A(a,0,0)$ $B(0,b,0)$, $C(0,0,c)$ Let Eqn of sphere passing through O,A,B,C be



$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$$\therefore d=0; \quad u=-\frac{a}{2}, \quad v=-\frac{b}{2}, \quad w=-\frac{c}{2}$$

$$\therefore \quad x^2+y^2+2^2-ax-by-cz=0 \qquad \ldots(1)$$

plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ $\quad \ldots(2)$.

The equation of the required cone is obtained by making eqn (1) homogeneous with the help of eqn (2).

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$x^2+y^2+z^2-\left(x^2+\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy+y^2+\frac{b}{c}yz\right.$$

$$\left.+\frac{c}{a}zx+\frac{c}{b}zy+z^2\right)=0$$

$$\Rightarrow \quad xy\left(\frac{a}{b}+\frac{b}{a}\right)+yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)=0$$

which is the required eqn of cone.

5.2 Section-B

Question-5(a) Obtain the curve which passes through (1,2) and has a slope $=\frac{-2xy}{x^2+1}$. Obtain one asymptote to the curve.

[8 Marks]

Solution: Given, $\frac{dy}{dx} = -\frac{2xy}{x^2+1}$ and curve passes through (1,2) separate variables

$$\frac{dy}{y} = -\frac{2x}{x^2+1}dx$$

Integrate on both sides

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1}dx + c$$

$$\text{Put,} \quad x^2+1 = t$$
$$2xdx = dt$$
$$\therefore \quad \log y = -\int \frac{dt}{t} + c$$
$$\log y = -\log t + c$$
$$\log y = -\log\left(x^2+1\right) + c$$

Put,

$$x = 1, y = 2$$
$$\log 2 = -\log(2) + c$$
$$\Rightarrow \quad c = 2\log 2 = \log 4$$
$$\therefore \quad \log y = -\log\left(x^2+1\right) + \log 4$$
$$\Rightarrow \quad y = \frac{4}{x^2+1}$$

Question-5(b) Solve the ode to get the particular integral of

$$\frac{\mathrm{d}^4\mathrm{y}}{\mathrm{dx}^4} + 2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + \mathrm{y} = \mathrm{x}^2\cos\mathrm{x}$$

[8 Marks]

Solution: Sol. The auxiliary equation is $m^4+2m^2+1=0$, or

$$\left(m^2+1\right)^2 = 0$$

giving

$$m = \pm i, \pm i$$

$$\therefore \quad \text{C.F.} = (c_1+c_2x)\cos x + (c_3+c_4x)\sin x, \because e^{0x} = 1$$

And

$$P.I. = \frac{1}{D^4+2D^2+1}x^2\cos x$$
$$= \text{ Real part of } \frac{1}{\left(D^2+1\right)^2}x^2e^{ix}, \quad \left[\because e^{ix} = \cos x + i\sin x\right]$$
$$= \text{ R.P. of } e^{ix}\frac{1}{\left\{(D+i)^2+1\right\}^2}x^2$$
$$= \text{ R.P. of } e^{ix}\frac{1}{\left(D^2+2iD\right)^2}x^2 \quad \left[\because i^2 = -1\right]$$

$$= \text{ R.P. of } e^{ix} \frac{1}{4i^2 D^2 [1 + (D/2i)]^2} x^2$$

$$= \text{ R.P. of } -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + \frac{D}{2i}\right]^{-2} x^2 \quad [\because i^2 = -1]$$

$$= \text{ R.P. of } -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 - \frac{1}{2} iD\right]^{-2} x^2, \quad \left[\because \frac{1}{i} = -i\right]$$

$$= \text{ R.P. of } -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + 2 \cdot \frac{1}{2} iD + 3 \cdot \frac{1}{4} i^2 D^2 + \ldots\right] x^2$$

(Expanding by binomial theorem)

$$= \text{R.P. of } -\frac{1}{4} e^{ix} \frac{1}{D^2} \left[1 + iD - \frac{3}{4} D^2 + \ldots\right] x^2$$

$$= \text{ R.P. of } -\frac{1}{4} e^{ix} \left[\frac{1}{D^2} + \frac{i}{D} - \frac{3}{4} + \text{ terms in } D, D^2 \text{ and so on}\right] x^2$$

$$= \text{ RP of } -\frac{1}{4} e^{ix} \left[\frac{1}{3} \frac{x^4}{4} + i \frac{1}{3} x^3 - \frac{3}{4} x^2 + \text{terms in } x^1, x^0\right]$$

($\because 1/D$ stands for integration w.r.t)x

$$= \text{ R.P. of } -\frac{1}{4} (\cos x + i \sin x) \left\{(1/12) x^4 + \frac{1}{3} i x^3 - \frac{3}{4} x^2 + \text{ terms in } x^1, x^0\right\}$$

$$= -\frac{1}{4} \left\{(1/12) x^4 - (3/4) x^2\right\} \cos x + \frac{1}{4} \left(\frac{1}{3} x^3\right) \sin x + \text{ terms already included in the C. F.}$$

$$= (-1/48) \left(x^4 - 9x^2\right) \cos x + (1/12) x^3 \sin x$$

(neglecting the terms already included in the C.F.)
Hence the complete solution is

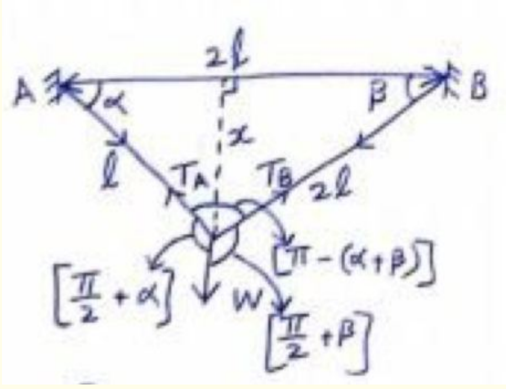
$$y = (\text{C.F.}) + (\text{P.I.})$$

$$y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$$
$$-(1/48) \left(x^4 - 9x^2\right) \cos x + (1/12) x^3 \sin x$$

Question-5(c) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings.

[8 Marks]

Solution:



Lami's theorem,

$$\frac{w}{\sin(\pi-(\alpha+\beta))} = \frac{T_A}{\sin\left(\frac{\pi}{2}+\beta\right)} \cdot \frac{T_B}{\sin\left(\frac{\pi}{2}+\alpha\right)}$$

$$\Rightarrow \frac{W}{\sin(\alpha+\beta)} = \frac{T_A}{\cos\beta} = \frac{T_B}{\cos\alpha} \qquad \ldots(1)$$

Using the Sine rule,

$$\frac{\sin\alpha}{2l} = \frac{\sin\beta}{\ell} = \frac{\sin(\alpha+\beta)}{2l} \qquad \ldots(2)$$

Also,

$$\cos\alpha = \frac{(2l)^2+l^2-(2l)^2}{2(2l)(l)} = \frac{1}{4} \Rightarrow \sin\alpha = \frac{\sqrt{15}}{4}$$

$$\cos\beta = \frac{(2l)^2+(2l)^2-l^2}{2(2l)(2l)} = \frac{3}{8} \Rightarrow \sin\beta = \frac{\sqrt{55}}{8}$$

$\therefore$ From (2),

$$\sin(\alpha+\beta) = \sin\alpha = \frac{\sqrt{15}}{4}$$

Putting above values in (1), we get

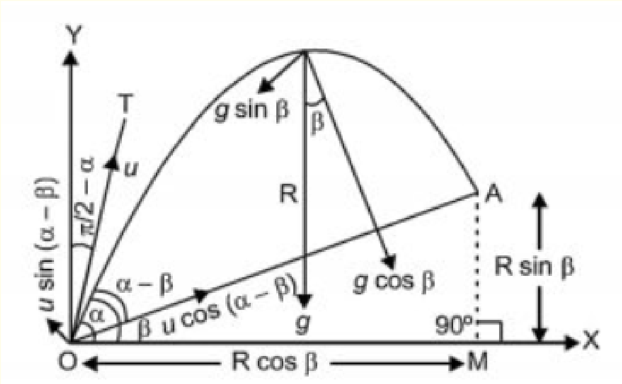
$$\Rightarrow T_A = \frac{\frac{3}{8}W}{\sqrt{15/4}} = \frac{1}{2}\sqrt{\frac{3}{5}}W$$

$$T_B = \frac{\frac{1}{4}W}{\sqrt{15/4}} = \frac{W}{\sqrt{15}}$$

Question-5(d) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot\beta = 2\tan(\alpha-\beta)$.

[8 Marks]

Solution: Suppose the particle strike the inclined plane at A. Let $OA = R$. Let T be the time of flight from O to A. As shown in the figure, the components of initial velocity of the particle along and perpendicular to the inclined plane are $u\cos(\alpha-\beta)$ and $u\sin(\alpha-\beta)$ respectively. Again, the component of g along the inclined is $g\sin\beta$ (down the plane)



and the component of g perpendicular to the inclined plane is $g\sin\beta$ (along the downward normal to the plane OA). Let time taken from O to A be T. While moving from O to A, the displacement of the particle perpendicular to OA is zero. So, considering motion of the particle from O to A perpendicular to OA and using the formula

$$s = ut + (1/2)ft^2$$

We have

$$s = u.t + \frac{1}{2}a.t^2$$

$$0 = u\sin(\alpha-\beta)\cdot T - (1/2)g\cos\beta\cdot T^2 \text{ or } T\{g$$
$$\cos\beta\cdot T - 2u\sin(\alpha-\beta)\} = 0$$

Since $T = 0$ gives time from O to O, hence time from O to A is given by $\therefore T =$ time of flight up the inclined plane

$$= \frac{2u\sin(\alpha-\beta)}{g\cos\theta} - (1)$$

Since the particle strikes the plane OA at right angles at A, hence the direction of velocity of the particle at A is perpendicular to OA and so the component of velocity of the particle at A along OA is zero. So, considering the motion of the particle from O to A along OA and using the formula.

$$V = u + a.t$$
$$O = u\cos(\alpha-\beta) - g\sin\beta.T$$
$$T = \frac{u}{g}\cdot\frac{\cos(\alpha-\beta)}{\sin\beta} \quad -(ii)$$

From (i) and (ii), we have

$$\frac{2u}{g}\cdot\frac{\sin(\alpha-\beta)}{\cos\beta} = \frac{u}{g}\cdot\frac{\cos(\alpha-\beta)}{\sin\beta}$$
$$2\tan(\alpha-\beta) = \cot\beta$$

Question-5(e) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \overline{\mathrm{F}} \cdot d\ \mathrm{S}$, where S is the surface bounding the volume E and $\bar{F}$=$(zx \sin yz + x^3)\,\hat{i} + \cos yz\hat{j} + \left(3zy^2 - e^{\lambda^2+y^2}\right)\hat{k}$.

[8 Marks]

Solution: Given that

$$\overrightarrow{F} = \left(zx \sin yz + x^3\right)\hat{i}$$
$$+\cos yz\hat{j} + \left(3zy^2 - e^{x^2+y^2}\right)\hat{k}$$

$$divF = \frac{\partial}{\partial x}\left(xz\sin(yz) + x^3\right) + \frac{\partial}{\partial y}(\cos(yz))$$
$$+\frac{\partial}{\partial z}\left(3zy^2 - e^{x^2+y^2}\right)$$
$$= \left(z\sin(yz) + 3x^2\right) + (-z\sin(yz))$$
$$+ \left(3y^2\right) = 3x^2 + 3y^2$$

Thus, we have from the divergence theorem

$$\iint_S F.dS = \iiint_E \operatorname{div} F dV$$
$$= \iint_D \int_0^{4-x^2-y^2} \left(3x^2 + 3y^2\right) dzdA$$

where D is the disk $x^2 + y^2 \leq 4$ in the xy -plane. Thus, we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\iint_S F.dS = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} \left(3r^2\right) rdzdrd\theta$$
$$= \int_0^{2\pi}\int_0^2 \left(12r^3 - 3r^5\right) drd\theta$$
$$= \int_0^{2\pi}\left[3r^4 - \frac{1}{2}r^6\right]_0^2 d\theta$$
$$= \int_0^{2\pi}(48 - 32)d\theta = 32\pi$$

Question-6(a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet?

[10 Marks]

Solution: Let u be the initial velocity of projection. since the greatest height is $40m$, we have

$$0 = u^2 - 2g \cdot 40$$
$$\therefore u = \sqrt{2g \times 40} = 28m$$

Let T be the time after the first stone starts before the two stones meet. Then, the distance traversed by the first stone in time T = distance traversed by the second stone in time $(T - 2)$

$$\therefore 28T - \frac{1}{2}gT^2 = 28(T-2) - \frac{1}{2}g(T-2)^2$$
$$= 28T - 56 - \frac{1}{2}g\left(T^2 - 4T + 4\right)$$
$$\therefore \quad 56 = \frac{1}{2}g(4T-4) = 4.9(4T-4)$$
$$\therefore \quad T = 3\frac{6}{7} \text{ seconds.}$$

Also, the height at which they meet

$$= 28 \times \frac{27}{7} - \frac{1}{2} \times 9.8 \times \left(\frac{27}{7}\right)^2$$
$$= 108 - 72.9 = 35.1m$$

The first stone will be coming down and the second stone going upwards.

Question-6(b) Using the method of variation of parameters, solve

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$$

[10 Marks]

Solution: Let, $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}$$
$$\text{and } \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Now,

$$x^2 \cdot \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$
$$x^2 \cdot m(m-1) \cdot x^{m-2} + x \cdot mx^{m-1} - x^m = 0$$
$$x^m\{m(m-1) + m - 1\} = 0$$
$$x^m\left\{m^2 - 1\right\} = 0$$
$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

The general solution is then

$$y = c_1e^{-x} + c_2 \cdot e^x$$

Question-6(c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line.

[10 Marks]

Solution: Given Data: Diameter of pipe:

$$d = 80mm = 0.08m$$

Gauge pressure of water:

$$p = 60kPa = 60 \times 10^3 pa \text{ or } N/m^2$$

Mean velocity of water:

$$V = 2m/s$$

Datum head:

$$z = 7m$$

According to Bernoulli's equation: Total head of water:

$$\begin{aligned} H &= \frac{p}{\rho g} + \frac{V^2}{2g} + z \\ &= \frac{60 \times 10^3}{1000 \times 9.81} + \frac{(2)^2}{2 \times 9.81} + 7 \\ &= 6.11 + 0.20 + 7 \\ &= 13.31m \text{ of water} \end{aligned}$$

Question-6(d) Evaluate $\iint_S (\nabla \times \bar{f}), \hat{n}dS$ for $\bar{f} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane.

[10 Marks]

Solution:

$$\begin{aligned} \int_C F \cdot dr &= \oint_C (F_x dx + F_y dy + E_z dz) \\ &= \oint_C \{(2x - y)dx - yz^2 dy - y^2 z dz\} \end{aligned}$$

But the boundary C of S is a circle in the xy -plane of radius unity and centre at (0,0,0) Hence the parametric equations of C are $x = \cos\theta, y = \sin\theta, z = 0$ where θ varies from 0 to 2π. Thus,

$$\begin{aligned}\int_C F \cdot dr &= \int_{\theta=0}^{2\pi} \{(2\cos\theta - \sin\theta)(-\sin\theta d\theta) - 0 - 0\} \\ &= \int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta \\ &= \int_0^{2\pi} \left(\sin 2\theta - \sin^2\theta\right) d\theta \\ &= \int_0^{2\pi} \left\{\sin 2\theta - \frac{1-\cos 2\theta}{2}\right\} d\theta \\ &= -\left[\frac{\cos 2\theta}{2} - \frac{\theta}{2} + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi\end{aligned}$$

Further

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix} = k$$

Hence,

$$\iint_S (\nabla \times A) \cdot ds = \iint_S k.ds = \iint_R dxdy$$

where R is the projection of S on $xy - plane$ and $k \cdot ds = dxdy =$ projection of ds on xy -plane. Thus, R is $x^2 + y^2 = 1$

$$\begin{aligned}\therefore \iint_R dxdy &= 4\int_0^1 \int_0^1 \sqrt{(1-x^2)\,dxdy} \\ &= 4\int_0^1 \sqrt{(1-x^2)}dx \\ &= 4\left[\frac{x}{2}\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1} x\right]_0^1 \\ &= 4\left[\frac{\pi}{4}\right] = \pi\end{aligned}$$

Thus, from above, we have $\int_C A.dr = \iint_S (\nabla \times A) \cdot ds$ and hence Stokes' Theorem is verified.

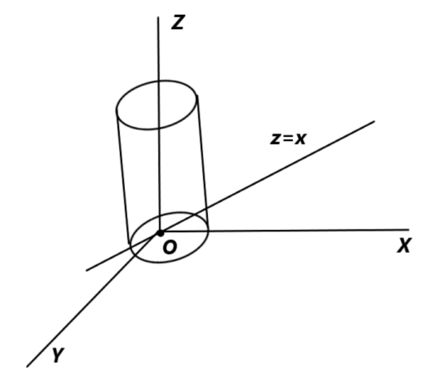
Question-7(a) State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{\mathrm{f}} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected cone.

[15 Marks]

Solution: Stokes' Theorem: Let S be a closed surface, bounded by curve C, then

$$\oint_c \vec{F} \cdot dr = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds$$

$\hat{n}$ is outward unit normal the surface.



Here,

$$\vec{F} = xi + zj + 2yk$$
$$\vec{r} = xi + yj + zk$$
$$d\vec{r} = dxi + dyj + dzk$$
$$\vec{F} \cdot d\vec{r} = xdx + zdy + 2ydz$$

Surface S is intersection of cylinder $x^2 + y^2 = 1$ and plane $x = 2$ (passing through y -axis)
Boundary curve

$$C : x^2 + y^2 = 1 \quad \& z = x$$

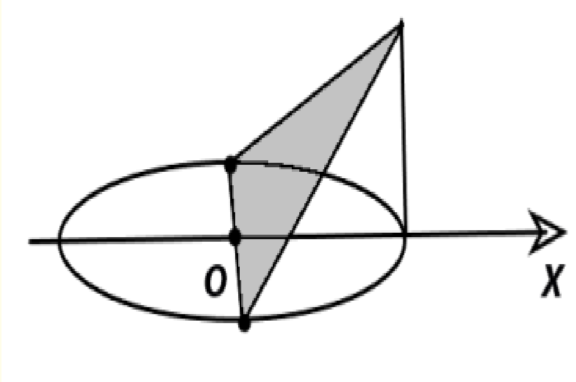
parameterizing

$$C : x = \cos\theta, y = \sin\theta$$
$$0 \le \theta < 2\pi$$

$$\oint_C F \cdot dx = \oint xdx + zdy + 2ydz$$
$$= \int_1^{2\pi} (\cos\theta)(-\sin\theta)d\theta + \cos\theta \cdot \cos\theta d\theta + 2\sin\theta(-\sin\theta)d\theta$$
$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left[\frac{-1}{2}\sin 2\theta + \left(\frac{1+\cos 2\theta}{2}\right) - 2\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$
$$= \int_0^{2\pi} \left(\frac{-1}{2}\sin 2\theta + \frac{3}{2}\cos 2\theta - \frac{1}{2}\right) d\theta$$
$$= \left[\frac{1}{4}\cos 2\theta + \frac{3}{4}\sin 2\theta - \frac{\theta}{2}\right]_0^{2\pi}$$
$$= -\pi$$

Now,

$$\nabla \times \vec{F} = \begin{vmatrix} 1 & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & 2y \end{vmatrix}$$
$$= i(2-1) + j(0-0) + k(0-0)$$
$$= i$$



S: $x - z = 0$

$$\hat{n} = \frac{\frac{\nabla S}{|\nabla S|}}{= \frac{1}{\sqrt{2}}(i - k)}$$

$$\iint_S (\nabla \times F) \cdot \hat{n} dS$$

$$= \iint_D i \cdot \left(\frac{i-k}{\sqrt{2}}\right) \frac{dxdy}{(\hat{n} \cdot k)}$$

(Taking Projection on xy -plane)

$$D : x^2 + y^2 \leq 1$$

$$= \int\int_D \frac{1}{\sqrt{2}} \cdot \frac{dxdy}{-1/\sqrt{2}} = -\iint_D dxdy$$

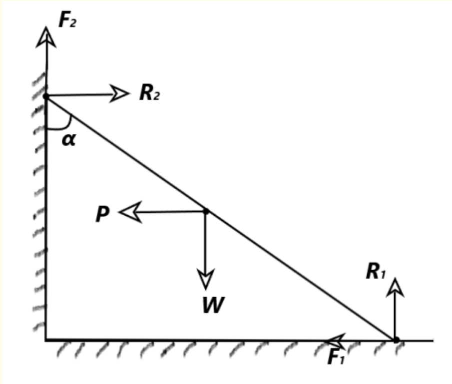
=-Area of unit circle D

$$= -\pi(1)^2 = -\pi$$

Question-7(b) A uniform rod of weight W is resting against an equally rough horizon and a wall, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W\tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α?

[15 Marks]

Solution: $\mu = \tan\lambda$ Let length (say) $F_1 = \mu R_1 - (1)$ $F_2 = \mu R_2 - (2)$



Force:

$$R_1 + F_2 = W - (3)$$
$$F_1 + P = R_2 - (4)$$

Moments about O:

$$\rightarrow R_1(a\sin\alpha) = R_2(a\cos\alpha) + F_1(a\cos\alpha) + F_2(\sin\alpha)$$

$$\Rightarrow R_1(\sin\alpha - \mu\cos\alpha) = R_2(\cos\alpha + \mu\sin\alpha)$$

From Eqn (3) and (4)

$$\Rightarrow R_2 = R_1 \times \frac{(\tan\alpha - \mu)}{(1+\mu\tan\alpha)}$$
$$\Rightarrow R_2 = R_1\tan(\alpha-\lambda) - (5)(\because \mu = \tan\lambda)$$
$$(3) \equiv R_1 + \mu R_2 = W$$

and

$$(4) \equiv \mu R_1 + P = R_2$$

Using (5),

$$\Rightarrow R_1 + \mu\tan(\alpha-\lambda)R_1 = W - (6)$$
$$\& \quad \mu R_1 + P = R_1\tan(\alpha-\lambda)$$
$$\Rightarrow P = R_1(\tan(\alpha-\lambda)-\mu) - (7)$$
$$\frac{(7)}{(6)} \Rightarrow \frac{P}{W} = \frac{(\tan(\alpha-\lambda)-\mu)}{1+\mu\tan(\alpha-\lambda)}$$
$$\Rightarrow \quad P = W\tan(\alpha-2\lambda) \quad (\because \mu = \tan\lambda)$$

condition is that P should be the +ve

$$\Rightarrow \alpha > 2\lambda$$

Question-7(c) Obtain the singular solution of the differential equation

$$y^2 - 2pxy + p^2\left(x^2-1\right) = m^2, p = \frac{dy}{dx}$$

[10 Marks]

Solution:

$$y^2 - 2pxy + p^2x^2 = m^2 + p^2$$
$$(y - px)^2 = p^2 + m^2$$
$$y = px \pm \sqrt{p^2 + m^2}$$

It is in Clairaut's form: $y = px + f(p)$ To get the solution, we replace p by arbitrary constant c.

$$y = cx \pm \sqrt{c^2 + m^2}$$

or

$$y^2 - 2cxy + c^2x^2 = c^2 + m^2$$
$$c^2\left(x^2 - 1\right) - 2cxy + y^2 - m^2 = 0$$

C-Discriminant:

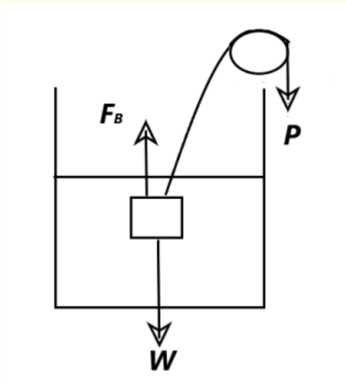
$$B^2 - 4AC$$
$$A = x^2 - 1, \quad B = -2xy, C = y^2 - m^2$$
$$\begin{aligned} \therefore B^2 - 4AC &= (-2xy)^2 - 4\left(x^2 - 1\right)\left(y^2 - m^2\right) \\ &= 4x^2y^2 - 4x^2y^2 + 4y^2 + 4x^2m^2 - 4m^2 \\ &= 4\left(y^2 + m^2\left(x^2 - 1\right)\right) \end{aligned}$$

$B^2 - 4AC = 0$ i.e. $y^2 + m^2\left(x^2 - 1\right)$ is the required singular solution of the given $D \cdot E$.

Question-8(a) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio $3:2$?

[10 Marks]

Solution: Let ρ_s = density of body, V = volume of body. ρ_l = density of liquid. W = Weigh of body = $\rho_s V g$ & F_B = Buyount force Body immersed in liquid

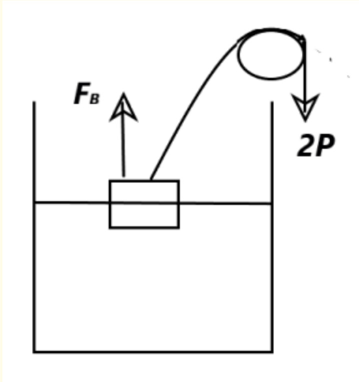


$$W = \rho_s V g$$
$$F_B = \rho_l V g$$

Balancing Forces

$$\rho_l V g + P = \rho_s V g - (1)$$

Body half immersed in liquid



$$W = \rho_s V g$$
$$F_B = \rho_l \frac{V}{2} g$$

Balancing Forces

$$\rho_l \frac{V}{2} g + 2P = \rho_s V g - (2)$$

Subtract(1) by (2)

$$P = \rho_l \frac{V}{2} g - (3)$$

Putting (3)in (1)

$$3\rho_l \frac{V}{2} g = \rho_s V g - (2)$$
$$\therefore \frac{\rho_s}{\rho_l} = \frac{3}{2}$$

Hence, Proved.

Question-8(b) Solve the differential equation

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x)$$

[10 Marks]

Solution:

$$\Rightarrow \quad \frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \sin x + \cos x$$

It is Bernoulli's equation. Let

$$\frac{1}{y} = z \quad , \quad \frac{-1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$
$$\therefore \quad \frac{dz}{dx} + z = \sin x + \cos x$$

I.F.$= e^{\int 1 dx} = e^x$ solution:

$$z \cdot e^x = \int e^x(\sin x + \cos x)dx$$

$$ze^x = \int e^x \sin x dx + \int e^x \cos x dx$$

$$= (\sin x)e^x - \int (\cos x)e^x dx + \int e^x \cos x dx$$

(integrating by parts)

$$= e^x \sin x + c$$

$$z = \sin x + ce^{-x}$$

i.e.

$$y\left(\sin x + ce^{-x}\right) - 1 = 0$$

is the required general solution of ODE.

Question-8(c) Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \vec{b}) \times \vec{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$.

[10 Marks]

Solution:

$$(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$$

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

First, If $b = 0$, then $a \times (b \times c) = 0$ and $(a \times b) \times c = 0$, hence true. If c is collinear with a i.e. $c = \lambda a$

$$a \times (b \times c) = a \times [b \times (\lambda a)]$$

$$= [a \cdot (\lambda a)]b - [a \cdot b](\lambda a)$$

$$= \lambda \left[|a|^2 b - (a \cdot b)a\right]$$

$$(a \times b) \times c = (a \times b) \times (\lambda a)$$

$$= (a \cdot (\lambda a))b - (b \cdot (\lambda a))a$$

$$= \lambda \left(|a|^2 b - (a \cdot b)a\right]$$

$\therefore \quad a \times (b \times c) = (a \times b) \times c$. If b is $\perp$ to a and c both

$$b \cdot a = 0, \quad b \cdot c = 0$$

$$\begin{aligned} a \times (b \times c) &= (a \cdot c)b - (a \cdot b)c \\ &= (a \cdot c)b \\ (a \times b) \times c &= (a \cdot c)b - (b \cdot c)a \\ &= (a \cdot c)b \\ \therefore (a \times b) \times c &= a \times (b \times c) \end{aligned}$$

Conversely, Let

$$(a \times b) \times c = a \times (b \times c)$$

ie.

$$(a \cdot c)b - (a \cdot b)c = (a \cdot c)b - (b \cdot c)a$$

$$(b.c)a - (a \cdot b)c = 0$$

$$b \times (a \times c) = 0$$

This is possible, when either of the condition is met.
i) $b = 0$
ii) c is collinear with a, then $a \times c = 0$
iii) $b \cdot a = 0$ & $b \cdot c = 0$ i.e. b is perpendicular to both a and c.

Question-8(d) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary.

[10 Marks]

Solution: Given,

$$\frac{d^2y}{dt^2} = \lambda y$$

$$\Rightarrow \quad 2\frac{dy}{dt} \cdot \frac{d^2y}{dt^2} = 2\lambda \cdot y\frac{dy}{dt}$$

[multiplying by $2\frac{dy}{dt}$ and integrating]

$$\left(\frac{dy}{dt}\right)^2 = \lambda y^2 + C_1$$

When $t = 0$, $\frac{dy}{dt} = 0$ and $y = a$ (initial velocity is 0 in y-direction)

$$\therefore \quad C_1 = -\lambda a^2$$

$$\left(\frac{dy}{dt}\right)^2 = \lambda\left(y^2 - a^2\right)$$

$$\frac{dy}{dt} = \sqrt{\lambda}\sqrt{y^2 - a^2} - (1)$$

Also, In x -direction, $\frac{d^2x}{dt^2} = 0$ [No acceleration in x -direction]

$$\frac{dx}{dt} = C_2; t = 0, \frac{dx}{dt} = a\sqrt{\lambda} \Rightarrow C_2 = a\sqrt{\lambda}$$

$$\therefore \frac{dx}{dt} = a\sqrt{\lambda} - (2)$$

Dinding (1) by (2),

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - a^2}}{a}$$

ie

$$\frac{dy}{\sqrt{y^2 - a^2}} = \frac{dx}{a} \Rightarrow \cosh^{-1}\frac{y}{a} = \frac{x}{a} + C_3$$

Initially, $x = 0$ and

$$y = a \Rightarrow c_3 = \cosh^{-1}(1) = 0$$

$$\therefore \quad y = a\cosh(x/a)$$

Eqn of catenary.

Chapter 6

2015

6.1 Section-A

Question-1(a) Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$

[8 Marks]

Solution:

Let upper triangular matrix,

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$
$$\Longrightarrow \mathrm{A}^2 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$
$$= \begin{bmatrix} x^2 & xy + yz \\ 0 & z^2 \end{bmatrix}$$
$$\Longrightarrow A^3 = A^2 \cdot A$$
$$= \begin{bmatrix} x^2 & xy + yz \\ 0 & z^2 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$
$$= \begin{bmatrix} x^3 & x^2y + xyz + yz^2 \\ 0 & z^3 \end{bmatrix}$$

It is given that

$$A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} x^3 & x^2y + xyz + yz^2 \\ 0 & z^3 \end{bmatrix} = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$
$$\therefore x^3 = 8 \Rightarrow x = 2,$$
$$z^3 = 27 \Rightarrow z = 3,$$
$$x^2y + xyz + yz^2 = -57 \Rightarrow 4y + 6y + 9y = -57 \Rightarrow y = 3$$
$$\therefore A = \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}$$

Question-1(b) Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

[8 Marks]

Solution:

$$\text{Given, } G(x, y, z) = (2y + z, x - 4y, 3x)$$
$$\text{Basis, } S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

$$\begin{aligned}\text{Let } (x, y, z) =& a(1, 1, 1) + b(1, 1, 0) + c(1, 0, 0)\\ =&(a + b + c, a + b, a)\end{aligned}$$

$$\begin{aligned}
&\therefore \quad x = a + b + c, \quad y = a + b, \quad z = a\\
&\text{i.e. } a = z, \quad b = y - z, \quad c = x - y\\
\therefore \quad &(x, y, z) = z(1, 1, 1) + (y - z)(1, 1, 0) + (x - y)(1, 0, 0)
\end{aligned}$$

$$\begin{aligned}
G(1, 1, 1) &= (3, -3, 3)\\
&= 3(1, 1, 1) + (-6)(1, 1, 0) + 6(1, 0, 0)\\
G(1, 1, 0) &= (2, -3, 3)\\
&= 3(-1, 1, 1) + (-6)(1, 1, 0) + (-1)(1, 0, 0)\\
G(1, 0, 0) &= (0, 1, 3)\\
&= 3(1, 1, 1) + (-2)(1, 1, 0) + (-3)(1, 0, 0)
\end{aligned}$$

$$\therefore [M]_S = \begin{bmatrix} 3 & -6 & 6\\ 3 & -6 & -1\\ 3 & -2 & -3 \end{bmatrix}^{\top} = \begin{bmatrix} 3 & 3 & 3\\ -6 & -6 & -2\\ 6 & -1 & -3 \end{bmatrix}$$

Question-1(c) Let $f(x)$ be a real-valued function defined on the interval (-5,5) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1}dt$ for all $x \in (-5, 5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

[8 Marks]

Solution:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(t)}, \text{ where } f(t) = x$$

.

Here,

$$e^{-x}f(x) = 2 + \int_0^x \sqrt{x^4 + 1}dt \qquad \ldots(1)$$

Differentiating both sides w.r.t x,

$$-e^{-x}f(x) + e^{-x}f'(x) = 0 + \sqrt{x^4+1}$$

Put, $x = 0$

$$-f(0) + f'(0) = 1$$

Also, ptting x=0 in (1),($f(0) = 2 + 0 \Rightarrow f(0) = 2$

$$\therefore f'(0) = 3$$

$$\therefore \frac{d}{dx}f^{-1}(x)\bigg|_{x=2} = \frac{1}{f'(0)} = \frac{1}{3}$$

Question-1(d) For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. Evaluate $f(e) + f\left(\frac{1}{e}\right)$

[8 Marks]

Solution:

$$I_1 = f(e) = \int_1^e \frac{\log t}{1+t}dt$$

$$I_2 = f\left(\frac{1}{e}\right) = \int_1^{1/e} \frac{\log t}{1+t}dt$$

$$= \int_1^e \frac{\log(1/y)}{1+\frac{1}{y}} \cdot \left(-\frac{dy}{y^2}\right) \quad \left[\begin{array}{c}\text{Puting} \\ t = \frac{1}{y}\end{array}\right]$$

$$\int_1^e \frac{\log y}{1+y} \cdot \frac{dy}{y} = \int_1^e \frac{\log t}{(1+t)} \cdot \frac{dt}{t}$$

$$\therefore I_1 + I_2 = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log t}{1+t} + \frac{\log t}{1+t}\frac{dt}{t}$$

$$= \int_1 \frac{\log t}{1+t}\left(1 + \frac{1}{t}\right)dt$$

$$= \int \frac{\log t}{t}dt = \frac{(\log t)^2}{2}\bigg|_1^e = \frac{1}{2}$$

Question-1(e) The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

[8 Marks]

Solution: Equation of the tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

is

$$\Rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1...(i)$$

The joint equation of the lines. joining the points of intersection of (i) and the auxiliary circle $x^2 + y^2 = a^2$ to the origin, which is the center of the circle, is

$$x^2 + y^2 = a^2\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2$$

Since, these lines are at right angles co-efficient of x^2 + co-efficient of $y^2 = 0$

$$\Rightarrow 1 - a^2\left(\frac{\cos^2\theta}{a^2}\right) + 1 - a^2\left(\frac{\sin^2\theta}{b^2}\right) = 0$$

$$\Rightarrow \sin^2\theta\left(1 - \frac{a^2}{b^2}\right) + 1 = 0$$

$$\Rightarrow \sin^2\theta\left(b^2 - a^2\right) + b^2 = 0$$

$$\Rightarrow \sin^2\theta\left[a^2\left(1 - e^2\right) - a^2\right] + a^2\left(1 - e^2\right) = 0$$

$$\Rightarrow \left(1 + \sin^2\theta\right)a^2e^2 = a^2$$

$$\Rightarrow e = \frac{1}{\sqrt{\left(1 + \sin^2\theta\right)}}$$

Question-2(a) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$.

[10 Marks]

Solution:

$$Dim(U + W) = \mathrm{Dim}(U) + \mathrm{Dim}(W) - \mathrm{Dim}(U \cap W)$$
$$= 4 + 4 - \mathrm{Dim}(U \cap W)$$
$$\therefore \mathrm{Dim}(U \cap W) = 8 - \mathrm{Dim}(U + W)$$

$U + W$ is a subspace of V $\therefore \dim(U + W) \leq \dim(V)$

$$\therefore \quad \dim(U + W) \leq 6$$

$\Rightarrow$ Dim $(\cup \cap w) \geq 8 - 6$ ie, $\dim(U \cap W) \geq 2$ Also, $U \cap W$ is a subspace of U

$$\therefore \dim(U \cup W)' \leq \dim(U)$$

i.e

$$\dim(U \cap W) \leq 4$$

Hence, Possible values of $\dim(U \cup W)$ are 2,3 or 4 . Result: Intersection of two subspaces is a subspace.

Question-2(b) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution:

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$$

[10 Marks]

Solution:

$$Ax = B$$

$$[A : B] \quad \sim \quad \begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 + R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & -5a+2b+c \end{bmatrix}$$

Now this system has solution of

$$\text{Rank}(A; B) = \text{Rank}(A) = 2$$

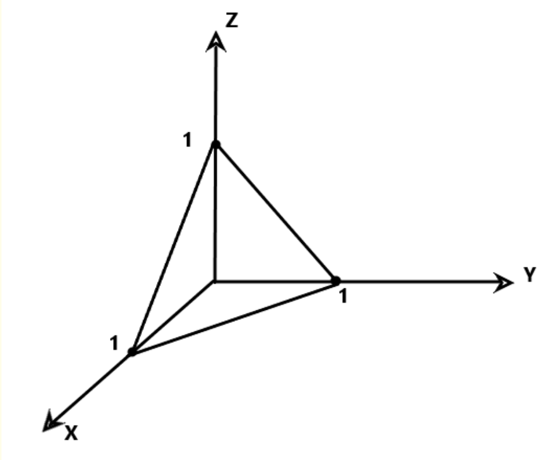
which is possible only if

$$-5a + 2b + c = 0$$

Question-2(c) Consider the three-dimensional region R bounded by $x+y+z = 1, y = 0, z = 0$. Evaluate $\iiint_R (x^2 + y^2 + z^2)\, dxdydz$.

[10 Marks]

Solution: Let R be the region bounded by the given tetrahedron.



$$
\begin{aligned}
I &= \int_0^1 \int_0^1 \int_0^{1-x} \left(x^2+y^2+z^2\right) dzdydx \\
&= \int_0^1 \int_0^{1-x} 2\left(x^2+y^2\right) z + \left.\frac{z^3}{3}\right|_0^{1-x-y} dydx \\
&= \int_0^1 x^2(1-x)y - x^2 \cdot \frac{y^2}{2} + (1-x)\frac{y^3}{3} - \frac{y^4}{4} - \left.\frac{(1-x-y)^4}{12}\right|_0^{1-x} dx \\
&= \int_0^1 x^2(1-x)^2 - \frac{1}{2}x^2(1-x)^2 + \frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4 + \frac{1}{12}(1-x)^4 dx \\
&= \int \frac{1}{2}\left(x^2+x^4-2x^3\right) + \frac{2}{12}(1-x)^4 dx \\
&= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^y}{5} - \frac{2x^4}{4}\right) - \left.\frac{1}{30}(1-x)^5\right|_0^1 \\
&= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right) - \frac{1}{30}(0-1) \\
&= \frac{1}{20}
\end{aligned}
$$

Question-2(d) Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid

$$\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$$

[10 Marks]

Solution: The intersection of plane $z = 2$ with the ellipsoid is given by

$$\frac{x^2}{25} + y^2 + \frac{(2)^2}{5} = 1 \quad \Rightarrow \frac{x^2}{25} + y^2 = \frac{1}{5}$$

ie

$$\frac{x^2}{5} + \frac{y^2}{1/5} = 1$$

(say S_1) in space. $Z = 2$. The area enclosed by this true is an ellipse. We take projection on $xy - plane$.

$$\begin{aligned}
A &= \iint_D \sqrt{z_x^2 + z_y^2 + 1dA} \\
&= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{1}{5}\sqrt{5-x^2}}^{\frac{1}{5}\sqrt{5-x^2}} 1.dydx \\
&= \int_{-\sqrt{5}}^{\sqrt{5}} \frac{2}{5}\sqrt{5-x^2}dx \quad \Big| \quad \begin{matrix} \text{Put } x = \sqrt{5}\sin\theta \\ dx = \sqrt{5}\,\text{cssos}\,\theta \end{matrix} \\
&= \frac{2}{5} \times 2 \int_0^{\pi/2} \sqrt{5 - 5\sin^2\theta}\sqrt{5}\cos\theta d\theta \\
&= 4\int_0^{\pi/2} \cos^2\theta d\theta \\
&= 4 \times \frac{1}{2} \times \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

Question-3(a) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$

[10 Marks]

Solution: Minimal polynomial of a matrix is a monic polynomial of least degree such that

$$p(A) = 0$$

First, let us find the characteristic polynomial

$$\begin{vmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(3+\lambda)(\lambda-3)+8) + 2(18 - 6\lambda - 12) + 2(-12 + 9 + 3\lambda) = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda + 10 = 0$$

$$\Rightarrow (\lambda+1)\left(\lambda^2 - 5\lambda + 10\right) = 0$$

Hence, we have 3 possibilities for minimal polynomial,

$$(\lambda+1), -\left(\lambda^2 - 5\lambda + 10\right)$$

or

$$\lambda^3 - 4\lambda^2 + 5\lambda + 10$$

. Let us check one by one. clearly

$$A + I \neq 0$$

$$A^2 - 5A + 10I = \begin{bmatrix} 0 & 4 & -4 \\ -12 & 14 & -8 \\ -6 & 4 & 2 \end{bmatrix} \neq 0$$

By Cayley-Hamilton theorem,

$$A^3 - 4A^2 + 5A + 10I = 0$$

Here, minimal polynomial is

$$x^3 - 4x^2 + 5x + 10$$

Question-3(b) If $\sqrt{x+y} + \sqrt{y-x} = c$, find $\dfrac{d^2y}{dx^2}$.

[10 Marks]

Solution:

$$\begin{aligned}(\sqrt{y+x} + \sqrt{y-x})^2 &= c^2 \\ (y+x) + (y-x) + 2\sqrt{y^2-x^2} &= c^2 \\ 2y - c^2 &= -2\sqrt{y^2-x^2} \\ 4y^2 - 4c^2y + c^4 &= 4\left(y^2 - x^2\right) \\ -4c^2y &= -4x^2 - c^4 \\ y &= \frac{1}{c^2}x^2 + \frac{c^2}{4}\end{aligned}$$

Differentiating wrt x,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{c^2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{2}{c^2}\end{aligned}$$

Question-3(c) A rectangular box, open at the top, is said to have a volume of 32 cubic meters. Find the dimensions of the box so that the total surface is a minimum.

[10 Marks]

Solution:

$$v = xyz = 32 \quad \text{(given)}$$

$$S = xy + 2yz + 2zx, \quad \text{where } x, y \text{ z are dimension}$$

$$S = xy + 2y \cdot \frac{32}{xy} + 2x - \frac{32}{xy}$$
$$= xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$
$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}; \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$
$$r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad S = \frac{\partial^2 S}{\partial x \partial y} = 1 \quad , \quad t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

for stationary points,

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = 0$$
$$y - \frac{64}{x^2} = 0$$

&

$$x - \frac{64}{y^2} = 0$$
$$y = \frac{64}{x^2} \Rightarrow x \cdot \left(\frac{64}{x^2}\right)^2 = 64 \quad \Rightarrow \quad x = 4$$
$$\therefore \quad (4, 4)$$

is stationary point

$$\Rightarrow \quad y = 4$$

Also,

$$rt - s^2 = 4 - 1 = 3 > 0$$

&,

$$r > 0$$

$\therefore (4, 4)$ is a point of minima.

$$\therefore \quad x = 4, \quad y = 4, \quad z = \frac{32}{4 \times 4} = 2$$

Question-3(d) Find the equation of the plane containing the straight line $y + z = 1, x = 0$ and parallel to the straight line $x - z = 1, \; y = 0$.

[10 Marks]

Solution: Eqn of plane through the line

$$y + z - 1 = 0, x = 0$$

is

$$\lambda x + y + z - 1 = 0$$

other line ie.

$$x - z = 1, \quad y = 0$$

ie.

$$\frac{x}{1} = \frac{z+1}{1} = \frac{y}{0}$$

Plane is parallel to this line

$$\therefore \quad \lambda \cdot 1 + 1.1 + 1 \cdot 0 = 0$$
$$\lambda = -1$$

Hence eqn of plane:

$$-x + y + z - 1 = 0$$

.

Question-4(a) Find a 3×3 orthogonal matrix whose first two rows are

$$\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$$

and

$$\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

.

[10 Marks]

Solution: Let given two rows vectors are

$$\vec{u} = \left[\begin{array}{ccc} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{array}\right]$$
$$\vec{v} = \left[\begin{array}{ccc} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}\right]$$

Let $\vec{w}$ be the third row vector which makes the given matrix orthogonal.
Then $\vec{w}$ is obtained by the cross product of $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} i & j & k \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} \\
&= i\left(\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right) + j\left(0 + \frac{1}{3\sqrt{2}}\right) + k\left(\frac{1}{3\sqrt{2}} - 0\right) \\
&= \frac{-4i}{3\sqrt{2}} + \frac{j}{3\sqrt{2}} + \frac{k}{3\sqrt{2}} \\
&= \frac{1}{3\sqrt{2}}(-4i + j + k)
\end{aligned}$$

Hence, unit vector, $w = \left[\begin{array}{ccc} \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{array}\right]$
Hence, the orthogonal matrix will be

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

Question-4(b) Find the locus of the variable straight line that always intersects $x = 1$, $y = 0$; $y = 1$, $z = 0$; $z = 1$, $x = 0$.

[10 Marks]

Solution: The plane passes through given lines are respectively, given by

$$x - 1 + py = 0$$
$$y - 1 + qz = 0$$
$$z - 1 + rx = 0$$

These planes will intersect in a lines if the determinant formed the coefficients of x, y and z is 0, i.e.,

$$\begin{vmatrix} 1 & p & 0 \\ 0 & 1 & q \\ r & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & p & 0 \\ 0 & 1 & q \\ r & 0 & 1 \end{vmatrix} = 0$$

$$1 + pqr = 0$$

$$1 + \left(\frac{1-x}{y}\right)\left(\frac{1-y}{z}\right)\left(\frac{1-z}{x}\right) = 0$$

$$xyz + (1-x)(1-y)(1-z) = 0$$

Question-4(c) Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$.

[10 Marks]

Solution: Any normal to the parabola $y^2 = 4ax$ is $\ldots(i)$

$$y = mx - 2am - am^3 \quad \ldots(ii)$$

Let, $(x_1, \dot{y}_1)$. be the pole of (2) with respect to (i), then (ii) is the polar of (x_1, y_1) w.r.t. (i) i.e.

$$yy_1 = 2a\,(x + x_1)\,...(iii)$$

comparing (ii) and (iii), we get

$$\frac{2a}{m} = \frac{y_1}{1}$$

$$= \frac{(2ax_1)}{(-2am - am^3)}$$

Hence, we get

$$x_1 = -2a - am^2 \ldots(iv)$$

And,

$$y_1 = 2a/m...(v)$$

Eliminating m between $(iv)\&(v)$ we get

$$y_1^2(x_1 + 2a) + 4a^3 = 0$$

$\therefore$ The required locus of (x_1, y_1) is

$$(x + 2a)y^2 - 4a^3 = 0$$

Question-4(d) Evaluate

$$\lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x} - \frac{3}{x^4}\right)$$

[10 Marks]

Solution:

$$\begin{aligned}
L &= \lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x} - \frac{3}{x^4}\right)\\
&= \lim_{x\to 0}\frac{2x + x\cos x - 3\sin x}{x^4\sin x}\\
&= \lim_{x\to 0}\frac{x(2+\cos x) - 3\sin x}{x^5}\cdot\frac{x}{\sin x}\\
&= \lim_{x\to 0}\frac{2x + x\cos x - 3\sin x}{x^5}\cdot 1\\
&= \lim_{x\to 0}\frac{2 + \cos x - x\sin x - 3\operatorname{css} x}{5x^4}\\
&= \lim_{x\to 0}\frac{2\sin x - \sin x - x\cos x}{20x^3}\\
&= \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{60x^2}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{60} = \left(\frac{1}{60}\right).
\end{aligned}$$

6.2 Section-B

Question-5(a) Reduce the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0, \quad p = \dfrac{dy}{dx}$ **to Clairaut's form. Hence, find the singular solution of the equation.**

[8 Marks]

Solution:

$$x^2p^2 + yp(2x + y) + y^2 = 0$$

Put $y = u$ and $xy = v$
$\therefore dy = du$ and $xdy + ydx = dv$

$$P = \frac{dv}{du} = \frac{xdy+ydx}{dy} = x + y\frac{dx}{dy}$$
$$\Rightarrow P = x + y\left(\frac{1}{p}\right)$$
$$\Rightarrow \quad p = \frac{y}{P-x}$$

The given DE transforms to

$$x^2\frac{y^2}{(P-x)^2} + y\frac{y}{(P-x)}(2x + y) + y^2 = 0$$
$$x^2 + (2x + y)(P - x) + (P - x)^2 = 0$$
$$x^2 + (2xP + yP - 2x^2 - xy) + (P^2 + x^2 - 2xP) = 0$$
$$P^2 + Py - xy = 0$$
$$P^2 + Pu - V = 0$$
$$V = Pu + P^2$$

Which is in Claiuraut's form

$$y = px + f(p)$$

General solution is given by

$$v = cu + c^2$$
$$\Rightarrow xy = cy + c^2$$
$$\Rightarrow c^2 + yc - xy = 0$$

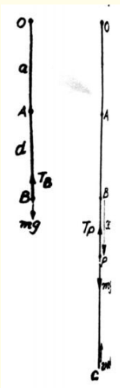
To obtain the singular solution, we equate the discriminant of above equation to 0

$$\Rightarrow y^2 - 4(-xy) = 0$$
$$\Rightarrow y^2 + 4xy = 0$$
$$\Rightarrow y + 4x = 0$$

Question-5(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point.

[8 Marks]

Solution: Let, OA $= a$ be the natural length of the string whose one end O is fixed. Let B be the position of equilibrium of a particle of mass m attached to the other end A of the string and let AB $= e$. Then at B, the weight of the particle = the tension T_o in the string.



$\therefore$

$$mg = \lambda(e/a) = (mg)(e/a) \text{ as}$$
$$\lambda = mg \qquad \ldots(i) \text{ (given)}$$

Thus, here

$$e = a \qquad \ldots(ii)$$

Now, the particle is pulled down to a point C such that OC $= 4a$ (given) and then let go. The particle will start to move towards B from rest from C
Let P be its position after time t, where BP $= x$ At P, the forces acting on the particle are its weight mg acting vertically downwards and tension T $= \lambda(x+e)/a$ acting vertically upwards.
Then, the equation of motion of the particle at P is Or,

$$\frac{md_x^2}{dt^2} = mg - \mathrm{T} = mg - \lambda(e+x)/a$$
$$= mg - \lambda(e/a) - \lambda(x/a)$$
$$\frac{md_x^2}{dt^2} = -\lambda(x/a)$$

i.e.

$$\frac{d_x^2}{dt^2} = -(\lambda/am)x,$$

using Or,

$$\frac{d_x^2}{dt^2} = -(mg/am)x$$
$$= -(g/a)x$$

as

$$\lambda = mg \qquad \ldots(iii)$$

Or,

$$v\left(\frac{dv}{dx}\right) = -(g/a)x$$

Or,

$$2vdv = -(2g/a)xdx$$

Integrating,

$$v^2 = -(g/a)x^2 + K \qquad \ldots(iv)$$

Where, K being a constant. At the point C, when

$$x = \mathrm{BC} = \mathrm{OC} - \mathrm{OB}$$
$$= 4a - (a+e)$$
$$= 4a - (a+a)$$

i.e.,

$$x = 2a, \quad v = 0$$

Hence, (iv) reduces to

$$0 = -(g/a)(2a)^2 + K$$

So that,

$$K = 4ag$$

. From

$$(iv), \quad v^2 = 4ag - (g/a)x^2$$

Or,

$$(dx/dt) = (g/a)\left(4a^2 - x^2\right)$$
$$\cdots (v)$$

When the particle reaches A, let the velocity of the particle be V. Then, putting $x = -a$ and $v = V$ in (v), we get Or,

$$\mathrm{V}^2 = (g/a)\left(4a^2 - a^2\right)$$
$$\mathrm{V} = (3ag)^{1/2} ... (vi)$$

From

$$(v), \quad \frac{dx}{dt} = \left(\frac{g}{a}\right)^{1/2}\left(4a^2 - x^2\right)$$

Or,

$$dt = \left(\frac{g}{a}\right)^{1/2} \frac{dx}{\left(4a^2 - x^2\right)^{1/2}} \ldots (vii)$$

Where we have taken negative sign on R.H.S. due to the fact that in moving from C towards B, x decreases as t increases. Let, t_1 be the time taken from C to A. Then integrating (vii) between $t = 0$ to $t = t_1$ and corresponding limits $x = -2a$ to $x = -a$ we get,

$$\int_0^1 dt = \left(\frac{a}{g}\right)^{1/2} \int_{2a}^{a} \frac{dx}{\left(4a^2 - x^2\right)^{1/2}}$$
$$= \left(\frac{a}{g}\right)^{1/2} \left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$
$$\text{Or,} \quad t_1 = (a/g)^{1/2}\left\{\cos^{-1}(-1/2) - \cos^{-1}\right\}$$
$$= (a/g)^{1/2}\left\{\pi - \cos^{-1}(1/2)\right\}$$
$$\text{Thus,} \quad t_1 = (a/g)^{1/2}(\pi - \pi/3)$$
$$= (a/g)^{1/2}(2\pi/3)$$

Thus, particle has velocity V in upward direction and it moves above A. But thee string becomes slack in upward motion from A so the S.H.M. ceases at A and the particle moves vertically upwards freely under gravity till its velocity V is destroyed. Let t_2 be the time taken by the particle from A till its velocity V becomes zero. Then, using formula

$$v = u - gt, \text{ we get}$$
$$0 = \mathrm{V} - gt_2$$

Or,

$$t_2 = \mathrm{V/g}$$
$$= (3ag)^{1/2}/g$$
$$= (3a/g)^{1/2}$$

Conditions being the same, the particle will take time t_2 in falling freely back to A. Again from A to C the time taken by the particle will be t_1 (which was taken by it to move from C to A). So, the required time

$$= 2\,(t_1 + t_2) = 2\{a/g)^{1/2}(2\pi/3) + (3a/g)^{1/2}\}$$

$$= (a/g)^{1/2}(4\pi/3 + 2\sqrt{3})$$

Question-5(c) Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, z = bt.$

[8 Marks]

Solution: The position vector $(\vec{r})$ of the curve at any point of the time (t) can be given as:

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath} + bt\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -a\cos t\hat{i} - a\sin t\hat{j}$$

$$\Rightarrow \frac{d^3\vec{r}}{dt^3} = a\sin t\hat{i} - a\cos t\hat{j}$$

(1) **Curvature**:

$$\kappa = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{\partial^2\vec{r}}{\partial t^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} \qquad \ldots(1)$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & -a\cos t & 0 \\ -acost & -asint & 0 \end{vmatrix} = ab\sin t\hat{i} + ab\cos t\hat{j} + a^2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$$

$$= \sqrt{a^2b^2\left(\sin^2 t + \cos^2 t\right) + a^4}$$

$$= \sqrt{a^2\left(a^2 + b^2\right)}$$

$$= a\left(a^2 + b^2\right)^{1/2}$$

$$\Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \left(a^2 + b^2\right)^{1/2}$$

$$\therefore \quad \text{From (1)}, \quad \kappa = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} = \frac{a\left(a^2 + b^2\right)^{1/2}}{\left(a^2 + b^2\right)^{3/2}}$$

$$\Rightarrow \kappa = \frac{a}{a^2 + b^2}$$

(2) **Torsion**:

$$\tau = \frac{\left[\frac{dr}{dt}\,\frac{d^2r}{\partial t^2}\,\frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

$$\left[\frac{dr}{dt}\,\frac{d^2r}{dt^2}\,\frac{d^3r}{dt^3}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \text{-asint} & \text{acost} & b \\ \text{-acost} & \text{-asint} & 0 \\ \text{asint} & \text{-acost} & 0 \end{vmatrix}$$

$$= b\left(a^2\cos^2 t + a^2\sin^2 t\right)$$

$$= a^2 b$$

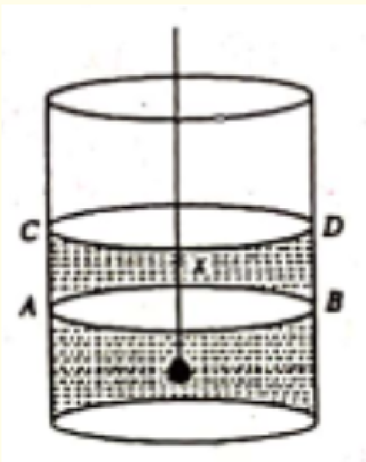
$$\Rightarrow \tau = \frac{a^2 b}{(a^2\left(a^2+b^2\right))}$$

$$\Rightarrow \tau = \frac{b}{a^2+b^2}$$

Question-5(d) A cylindrical vessel on a horizontal circular base of radius a is filled with a liquid of density w with a height h. If a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface.

[8 Marks]

Solution: Let the level of the liquid in the vessel be AB before the immersion of the sphere. After the sphere is immersed, let the level of the liquid be CD. If x be the increased height when the level is raised the $AC = BD = x$.



Since the volume of the liquid displaced by the sphere must be equal to the volume of the sphere, so we have $\Rightarrow$

$$\pi a^2 x = \frac{4}{3}\pi c^3$$

$$x = \frac{4}{3}\left(\frac{c^3}{a^2}\right)$$

Now, the whole pressure on the curved surface before immersion

$$\begin{aligned} &= \mathrm{P}_1 = 2\pi ah \cdot \tfrac{1}{2}h \cdot wg \\ &= \pi ah^2 wg \end{aligned}$$

Whole pressure on the curved surface after

$$\begin{aligned} \text{immersion } &= \mathrm{P}_2 \\ &= 2\pi a(h+x) \cdot \frac{1}{2}(h+x)wg \\ &= \pi a(h+x)^2 wg \end{aligned}$$

$\therefore$ Increase of whole pressure on the curved surface $= \mathrm{P}_2 - \mathrm{P}_1$

$$\begin{aligned} &= \pi awg\left[(h+x)^2 - h^2\right] \\ &= \pi awg\left(x^2 + 2hx\right) \\ &= \pi awgx(x+2h) \end{aligned}$$

$$= \pi awg\frac{4}{3} \cdot \left(\frac{c^3}{a^2}\right)\left(\frac{4}{3} \cdot \frac{c^3}{a^2} + 2h\right)$$

$$= \frac{8\pi}{3a} wgc^3\left(h + \frac{2c^3}{3a^2}\right) \cdots$$

Question-5(e) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

[8 Marks]

Solution: The given equation is

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

Put $x = e^z$ so that $z = \log x$ and let $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation reduces to

$$[\mathrm{D}(\mathrm{D}-1) + 3\mathrm{D} + 1]y = \frac{1}{\left(1-e^x\right)^2}$$

$$\left(\mathrm{D}^2 + 2\mathrm{D} + 1\right)y = \frac{1}{\left(1-e^x\right)^2}$$

Auxiliary equation is $D^2 + 2D + 1 = 0$.
$\Rightarrow (\mathrm{D}+1)^2 = 0 \quad \Rightarrow \quad \mathrm{D} = -1, -1 \;\therefore \quad$ C.F. $= (c_1 + c_2 z)\, e^{-z} = (c_1 + c_2 \log x)\, \frac{1}{x}$
Now,

$$P.I. = \frac{1}{\mathrm{D}^2 + 2\mathrm{D} + 1} \cdot \frac{1}{\left(1-e^z\right)^2} = \frac{1}{(\mathrm{D}+1)^2} \cdot \frac{1}{\left(1-e^z\right)^2}$$

$$= \frac{1}{(D+1)(D+1)} \cdot \frac{1}{\left(1-e^z\right)^2} == \frac{1}{(D+1)}\left[\frac{1}{(D+1)}\left(1-e^z\right)^2\right]$$

$$= \frac{1}{\mathrm{D}+1} e^{-z} \int \frac{1}{(1-e^z)^2} \cdot e^z dz \quad \left[\because \frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx\right]$$

$$= \frac{1}{\mathrm{D}+1} \cdot e^{-z} \int (1-t)^{-2} dt, \text{ where } e^x = t$$

$$= \frac{1}{\mathrm{D}+1} . e^{-z} \left[- \int (1-t)^{-2} (-dt)\right]$$

$$= \frac{1}{D+1} . e^{-z} \frac{-(1-t)^{-1}}{-1} \left[\because \int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{D+1} \cdot e^{-z} \cdot \frac{1}{1-t} = \frac{1}{D+1} \cdot e^{-z} \frac{1}{1-e^z} \left[\because t = e^t\right]$$

$$= \frac{1}{\mathrm{D}+1} \cdot \frac{e^{-z}}{1-e^z}$$

$$= e^{-z} \int \frac{e^{-z}}{1-e^z} \cdot e^z dz$$

$$= e^{-z} \int \frac{dz}{1-e^z}$$

$$= e^{-z} \int \frac{e^{-z}}{e^{-z}-1} dz \text{ (Mult. num and den by } e^{-z})$$

$$= -e^{-z} \int \frac{-e^{-z}}{e^{-z}-1} dz$$

$$= -e^{-z} \log \left(e^{-z}-1\right)$$

$$= -\frac{1}{x} \log \left(\frac{1}{x}-1\right)$$

$$= -\frac{1}{x} \log \left(\frac{1-x}{x}\right) = \frac{1}{x} \log \left(\frac{1-x}{x}\right)^{-1}$$

$$= \frac{1}{x} \log \frac{x}{1-x}$$

Hence the complete solution is

$$y = (c_1 + c_2 \log x) \frac{1}{x} + \frac{1}{x} \log \frac{x}{1-x}$$

Question-6(a) Solve

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$$

by changing the independent variable.

[10 Marks]

Solution:

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - \left(4x^2\right)y = 8x^2\sin x^2$$

Comparing it with

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + \varphi y = R$$

$$p = -\frac{1}{x}, \quad Q = -4x^2, \quad R = -8x^2\sin x^2$$

Let

$$\left(\frac{dz}{dx}\right)^2 = \pm a^2 Q = -4x^2$$

(for $a = 1$)

$$\frac{dz}{dx} = 2x \quad \therefore \quad z = x^2$$

(Note that $\frac{dz}{dx} = e^{-\int Pdx}$ is not working here)

$$\text{Now,} \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(\frac{-1}{x}\right)2x}{4x^2}$$

$$Q_1 = \frac{Q}{(dz)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{R}{(d2/dx)^2} = \frac{8x^2\sin x^2}{4x^2} = 2\sin x^2$$

Transformed Eqn is:

$$\frac{d^2y}{dz^2} + p_1\frac{dy}{dz} + \varphi_1 y = R_1$$

$$\frac{d^2y}{dz^2} - 1(y) = 2\sin z$$

$$\text{ie.} \quad \left(D'^2 - 1\right)y = 2\sin 2$$

Auxiliary Eqn: $\quad D'^2 - 1 = 0$

$$D' = 1, -1$$

$$c \cdot F\cdot = c_1e^2 + c_2e^{-2}$$

$$= c_1e^{x^2} + c_2e^{-x^2}$$

$$p \cdot I_1 = \frac{1}{D'^2 - 1}2\sin z$$

$$= \frac{2}{(-1^2) - 1}\sin z = -\sin z$$

$$= -\sin x^2$$

Hence, complete solution is:

$$y = CF + PI$$

$$y = c_1e^{x^2} + c_2e^{-x^2} - \sin x^2$$

Question-6(b) The forces P, Q and R act along three straight lines $y = b$, $z = -c$, $z = c$, $x = -a$ and $x = a$, $y = -b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action.

[10 Marks]

Solution: The forces X, Y, Z act along the lines

$$y = b, z = -c; \quad z = c, x = -a; \quad x = a, \quad y = -b$$

The equations of these lines are

$$\frac{x-0}{1} = \frac{y-b}{0} = \frac{z+c}{0}, \frac{x+a}{0} = \frac{y-0}{1} = \frac{z-c}{0}, \frac{x-a}{0} = \frac{y+b}{0} = \frac{z-0}{1}$$

The forces acting on the body are as follows : (i) A force X acting at the point $(0, b, -c)$ along the line whose d.c's a $< 1, 0, 0 >$ (ii) A force Y acting at the point $(-a, 0, c)$ along the line whose de 'c $< 0, 1, 0 > < 0.0, 1 >$ $\therefore$ The components of the forces parallel to the axes are

$$X_1 = X.1 = X, \quad X_2 = Y.0 = 0, \quad X_3 = Z.0 = 0$$

$$Y_1 = X.0 = 0, \quad Y_2 = Y.1 = Y, \quad Y_3 = Z.0 = 0$$
$$Z_1 = X.0 = 0, \quad Z_2 = Y.0 = 0, \quad Z_3 = Z.1 = Z$$

If the system reduces to a single force $R = (X.Y.Z\ )$ arting at Ω. couple $G = (L, M, N)$, then

$$X = \Sigma X_1 = X_1 + X_2 + X_3 = X + 0 + 0 = X$$
$$Y = \Sigma Y_1 = 0 + Y + 0 = Y$$

and

$$Z = \Sigma Z_1 = 0 + 0 + Z = Z$$

To find L, M, N (i) Consider

$$\hat{\imath}L_1 + j\hat{M}_1 + \hat{k}N_1 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & b & -c \\ X & 0 & 0 \end{vmatrix}$$

$$= \hat{\imath}(0) - \hat{\jmath}(cX) + \hat{k}(-bX)$$

$$\therefore \quad L_1 = 0, M_1 = -cX, N_1 = -bX$$

$$\therefore \hat{\imath}L_2 + jM_2 + \hat{k}N_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{h} \\ -a & 0 & c \\ 0 & Y & 0 \end{vmatrix}$$

$$= \hat{\imath}(-cY) - \hat{\jmath}(0) + \hat{k}(-aY)$$

$$\therefore \quad L_2 = -cY, \quad M_2 = 0; \quad N_2 = -aY$$

$$\text{(iii) } \hat{\imath}L_3 + \hat{\jmath}M_3 + \hat{k}N_3 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_3 & y_3 & z_3 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & -b & 0 \\ 0 & 0 & Z \end{vmatrix}$$

$$= \hat{\imath}(-bZ) - \hat{\jmath}(aZ) + \hat{k}(0)$$

$$\therefore \quad L_3 = -bZ, M_3 = -aZ, N_3 = 0$$

Here

$$L = \Sigma L_1 = -(bZ + cY), M = \sum M_1 = -(cX + aZ)$$

and

$$N = \Sigma N_1 = -(bX + aV$$

The system is equivalent to a single form if

$$LX + MY + NZ = 0$$

Substituting the values of L, M in

$$- \mid (bZ + cY)X + (cX + aZ)Y + (bX + aY)Z1 - n$$

or

$$2|aYZ + bZX + cXY| = 0$$

or

$$\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0$$

which is the required condition. The equations of the line of action of the single force is of the central axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = 0$$

$\therefore$ The equations of the line of action of the single resultant force are any two of the following three:

$$L - yZ + zY = 0, M - zX + xZ = 0, N - xY + yX = 0$$

or

$$\begin{gathered} -(bZ + cY) - yZ + zY = 0 \\ -(cX + aZ) - zX + xZ = 0 \\ -(bX + aY) - xY + yX = 0 \end{gathered}$$

Dividing these equations by YZ, ZX and XY respectively, we get

$$\begin{gathered} -\left(\tfrac{b}{Y} + \tfrac{c}{Z}\right) - \tfrac{y}{Y} + \tfrac{z}{Z} = 0, -\left(\tfrac{c}{Z} + \tfrac{a}{X}\right) - \tfrac{z}{Z} + \tfrac{x}{X} = 0 \\ -\left(\tfrac{b}{Y} + \tfrac{a}{X}\right) - \tfrac{x}{X} + \tfrac{y}{Y} = 0 \end{gathered}$$

Using (2), we have

$$\frac{a}{X} - \frac{y}{Y} + \frac{z}{Z} = 0; \frac{b}{Y} - \frac{z}{Z} + \frac{x}{X} = 0, \frac{c}{Z} - \frac{x}{X} + \frac{y}{Y} = 0$$

or

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

Hence, the equations to its line of action are any two of the three

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \quad \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \quad \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

Question-6(c) Solve

$$\left(D^4+D^2+1\right) y=e^{-x/2} \cos \left(\frac{x\sqrt{3}}{2}\right),$$

where $D \equiv \dfrac{d}{dx}$.

[10 Marks]

Solution: Solve the differential equation

$$\left(D^4+D^2+1\right) y=e^{-x/2} \cos \left(x \frac{\sqrt{3}}{2}\right)$$

Sol. Here the auxiliary equation is $m^4+m^2+1=0$ or

$$\left(m^2+1\right)^2-m^2=0$$

$$\left(m^2+m+1\right)\left(m^2-m+1\right)=0$$

$$\begin{aligned}
m &= \frac{-1 \pm \sqrt{(1-4)}}{2}, \frac{1 \pm \sqrt{(1-4)}}{2} \\
&= -\frac{1}{2} \pm \frac{1}{2} \sqrt{3} i, \frac{1}{2} \pm \frac{1}{2} \sqrt{3} i \\
\therefore \quad C.F. &= c_1 e^{-x/2} \cos \left(\frac{1}{2} \sqrt{3} x+c_2\right)+c_3 e^{x/2} \cos \left(\frac{1}{2} \sqrt{3} x+c_4\right)
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \text{Also P.I. } &= \frac{1}{D^4+D^2+1} e^{-x/2} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= e^{-x/2} \frac{1}{\left(D-\frac{1}{2}\right)^4+\left(D-\frac{1}{2}\right)^2+1} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= e^{-x/2} \frac{1}{D^4-2 D^3+\frac{5}{2} D^2-\frac{3}{2} D+\frac{21}{16}} \cos \left(\frac{1}{2} \sqrt{3} x\right)
\end{aligned}$$

We observe that $D^4-2 D^3+\frac{5}{2} D^2-\frac{3}{2} D+\frac{21}{16}$ becomes zero on putting $D^2=-3/4$.
So $D^2+\frac{3}{4}$ must be one of its factors.
By actual division, we get the other factor.

$$\begin{aligned}
\text{So the P.I. } &= e^{-x/2} \frac{1}{\left(D^2+\frac{3}{4}\right)\left(D^2-2 D+\frac{7}{4}\right)} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= e^{-x/2} \frac{1}{\left(-\frac{3}{4}-2 D+\frac{7}{4}\right)\left(D^2+\frac{3}{4}\right)} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= e^{-x/2} \frac{1+2 D}{\left(1-4 D^2\right)\left(D^2+\frac{3}{4}\right)} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= e^{-x/2} \frac{1+2 D}{[1-4(-3/4)]\left(D^2+\frac{3}{4}\right)} \cos \left(\frac{1}{2} \sqrt{3} x\right) \\
&= \frac{1}{4} e^{-x/2} \frac{1}{D^2+\frac{3}{4}}(1+2 D) \cos \left(\frac{1}{2} \sqrt{3} x\right)
\end{aligned}$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+\frac{3}{4}}\left[\cos\left(\frac{1}{2}\sqrt{3}x\right)-\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+\frac{3}{4}}\cos\left(\frac{1}{2}\sqrt{3}x\right)-\frac{1}{4}\sqrt{3}e^{-x/2}\frac{1}{D^2+4}\sin\left(\frac{1}{2}\sqrt{3}x\right)$$

$$= \frac{1}{4}e^{-x/2}\cdot\frac{x}{2\cdot\left(\frac{1}{2}\sqrt{3}\right)}\sin\left(\frac{1}{2}\sqrt{3}x\right)-\frac{1}{4}\sqrt{3}e^{-x/2}\left[-\frac{x}{2.\left(\frac{1}{2}\sqrt{3}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$= \frac{x}{4\sqrt{3}}e^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right)+\frac{x}{4}e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$= \frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right)+\frac{x}{4}e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

Hence the general solution is $y = (\text{ C.F. }) + (\text{ P.I. })$

$$y = c_1e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_2\right)+c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_4\right)$$

$$+\frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right)+\frac{1}{4}xe^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$y = e^{-x/2}\left[c_1\cos\left(\frac{1}{2}\sqrt{3}x\right)+c_2\sin\left(\frac{1}{2}\sqrt{3}x\right)\right]+c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_4\right)$$

$$+\frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right)+\frac{1}{4}xe^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$y = e^{-x/2}\left[\left(\frac{1}{4}x+c_1\right)\cos\left(\frac{1}{2}\sqrt{3}x\right)+\left(\frac{1}{12}\sqrt{3}x+c_2\right)\sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$+c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_4\right)$$

Question-6(d) Examine if the vector field defined by $\vec{F} = 2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F} = \text{grad}\,\phi$.

[10 Marks]

Solution: $\vec{F}$ is irrotational if Curl $\mid F = \nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= i\left(3x^2z^2-3x^2z^2\right)+j\left(6xyz^2-6xyz^2\right)$$
$$+k\left(2xz^3-2xz^3\right) = 0$$

$\therefore \vec{F}$ is irrotational.

$$\vec{F} = grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$$

$$\left.\begin{array}{l}\therefore \frac{\partial\phi}{\partial x} = 2xyz^3 \\ \frac{\partial\phi}{\partial y} = x^2z^3 \\ \frac{\partial\phi}{\partial z} = 3x^2yz^2\end{array}\right\} \Rightarrow \phi = x^2yz^3 + c$$

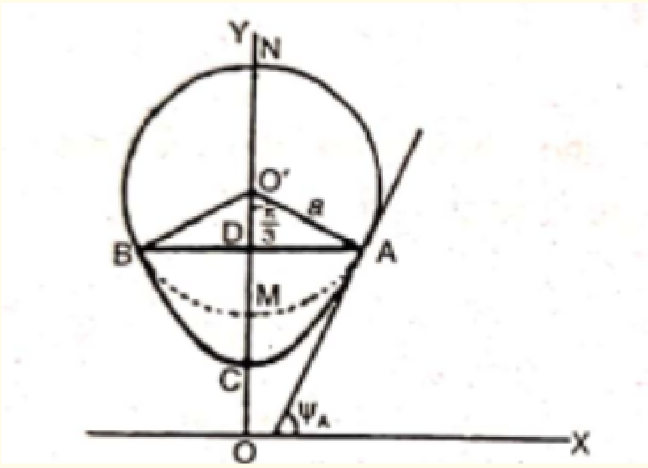
$\therefore$ Scalar potential,

$$\phi = x^2yz^3 + C$$

Question-7(a) Determine the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley.

[15 Marks]

Solution: Let, $ANBMA$ be the circular pulley of radius a and $ANBCA$ the endless chain hanging over it.



Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain.

$$= \frac{2}{3}(\text{circumference of the pulley})$$

$$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a$$

Let, the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e., the vertex, CO'N the axis and OX the directrix of this catenary.
Let, OC $= c =$ the parameter of the catenary. The tangent at A will be perpendicular to the radius O$'$A $\therefore$ If the tangent at A is inclined at an angle ψ_{A} to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B)$$

$$= \frac{1}{2}\left(\frac{1}{3}\cdot 2\pi\right) = \frac{\pi}{3}$$

From the triangle $AO'D$, we have

$$\text{DA} = \text{O}'\text{A}\sin\frac{1}{3}\pi = a\sqrt{3/2}$$

$\therefore$ From $x = c\log(\tan\psi + \sec\psi)$; for the point A, we have

$$x = \mathrm{DA} = c\log\cdot(\tan\psi_{\mathrm{A}} + \sec\psi_{\mathrm{A}})$$

$$\text{Or, } \frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$

$$= c\log(\sqrt{3}+2)$$

$$\therefore \quad c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $s = c$ tan ψ applied for the point A, we have

$$\mathrm{arc\,CA} = c\tan\psi_{\mathrm{A}} = c\tan\frac{1}{3}\pi = c\sqrt{3}$$

$$= \frac{3a}{2\log(2+\sqrt{3})}$$

Hence, the total length of the chain = arc ABC+ length of the chain in contact with the pulley

$$= 2\cdot(\mathrm{arc\,CA}) + \frac{4}{3}\pi a$$

$$= 2\frac{3a}{2\log(2+\sqrt{3})} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$$

Question-7(b) Using divergence theorem, evaluate

$$\iint\limits_S \left(x^3dydz + x^2ydzdx + x^2zdydx\right)$$

where S is the surface of the sphere $x^2+y^2+z^2=1$.

[15 Marks]

Solution: Divergence Theorem states that

$$\iint\limits_S (\vec{F}\cdot\hat{n})dS = \iiint\limits_V (\nabla\cdot\vec{F})dV$$

Here,

$$I = \iint\limits_S \left(x^3dydz + x^2ydzdx + x^2zdydx\right)$$

$$= \iint\limits_S \left(x^3i + x^2yj + x^2zk\right)\cdot\hat{n}dS$$

$$\therefore \vec{F} = x^3i + x^2yj + x^2z\hat{k}$$

$$\Rightarrow \nabla \cdot \vec{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$
$$= 3x^2 + x^2 + x^2 = 5x^2$$

Hence,

$$I = \iiint\limits_V 5x^2 dV$$

where V is volume of sphere $x^2 + y^2 + z^2 = 1$.
Converting to spherical corrdinates

$$x = r \sin\theta \cos\phi$$
$$dV = r^2 \sin\theta dr d\theta d\phi$$

Limits:
r varies from 0 to 1
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\therefore \iiint\limits_V 5x^2 dV = 5 \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (r \sin\theta \cos\phi)^2 r^2 \sin\theta dr d\theta d\phi$$
$$= 5 \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta \cos^2\phi dr d\theta d\phi$$
$$= 5 \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta \left[\frac{1 + \cos 2\phi}{2}\right] dr d\theta d\phi,$$
$$= \frac{5}{2} \int_0^1 \int_0^\pi r^4 \sin^3\theta \left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi} dr d\theta$$
$$= \frac{5}{2} \times \left[\frac{r^5}{5}\right]_0^1 [2\pi + 0] \int_0^\pi \sin^3\theta d\theta$$
$$= \frac{1}{2} \times 2\pi \times 2 \times \int_0^{\pi/2} \sin^3\theta d\theta$$
$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a - x) = f(x)\right]$$
$$= 2\pi \times \frac{2}{3} = \frac{4\pi}{3} (\text{ Using Walli's formula for definite integral})$$

Walli's Formula:

$$I = \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$$

If n is even,

$$I = \frac{(n-1)(n-3)(n-5)\ldots 3 \cdot 1}{n(n-2)(n-4) \cdots 4 \cdot 2} \times \frac{\pi}{2}$$

If n is odd,

$$I = \frac{(n-1)(n-3)(n-5) \cdots \cdot 4 \cdot 2}{n(n-2)(n-4) \cdot \cdots \cdot 1}$$

Question-7(c) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t.

[10 Marks]

Solution: Let, the particle start from rest from a fixed point O. Let, P be the position of the particle at any time t such that OP $= x$.
Let, v be its velocity at P. Here, the force of resistance is μv (given) which acts in vertical upward direction.
The weight mg of the particle acts in vertically downward direction.

Then, the equation of motion of the particle at any time t is

$$m\ddot{x} = mg - \mu v$$
$$\frac{dv}{dt} = g - \left(\frac{\mu}{m}\right) v$$

Or,

$$dt = \left\{\frac{\frac{1}{(g-\mu v)}}{m}\right\} dv$$

On Integration, we have,

$$t = -\left(\frac{m}{\mu}\right) \log\left(g - \frac{\mu v}{m}\right) + \text{A}$$

where A is a constant.
Initially, at Point O, when $t = 0, v = 0$. Hence,

$$A = \left(\frac{m}{\mu}\right) \log g$$
$$\therefore \quad t = -\left(\frac{m}{\mu}\right) \log\left(g - \frac{\mu v}{m}\right) + \left(\frac{m}{\mu}\right) \log g$$
$$= -\left(\frac{m}{\mu}\right) \log\left(1 - \frac{\mu v}{gm}\right)$$

Or,

$$\log\left(1 - \frac{\mu v}{gm}\right) = -\frac{\mu t}{m}$$
$$\text{Or,} \quad 1^2 - \left(\frac{\mu}{gm}\right) v = e^{-\mu t/m}$$

Or,

$$v = \frac{dx}{dt} = \left(\frac{gm}{\mu}\right) \left\{1 - e^{-\frac{\mu t}{m}}\right\}$$

Or,

$$dx = \left(\frac{gm}{\mu}\right) \left(1 - e^{-\frac{\mu t}{m}}\right) dt$$

On integration, we have

$$x = \left(\frac{gm}{\mu}\right) \left\{t + \left(\frac{m}{\mu}\right) e^{-\mu t/m}\right\} + \text{B}$$

where B is a constant.
Initially, at Point O, when $t=0, x=0$

$$\Rightarrow \mathrm{B}=-\frac{gm^2}{\mu^2}$$

Then,

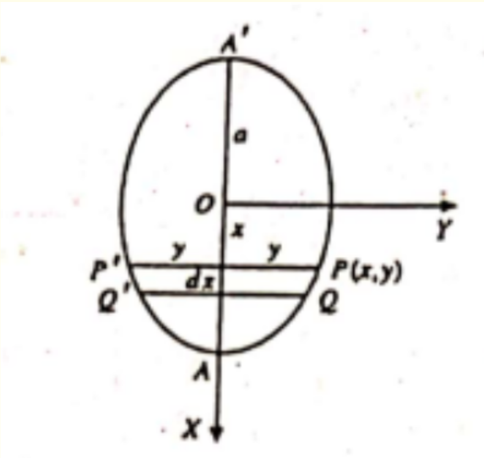
$$x=\left(\frac{gm}{\mu}\right)\left\{t+\left(\frac{m}{\mu}\right)e^{-\mu t/m}\right\}-\frac{gm^2}{\mu^2}$$
$$\text{or}\quad x=\left(\frac{gm^2}{\mu^2}\right)\left\{e^{-(\mu/m)t}-1+\frac{\mu t}{m}\right\}$$

Question-8(a) An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse.

[15 Marks]

Solution: Take the major axis and minor axis respectively as the axes of x and y Then the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



By symmetry it is clear that the C.P. $(\bar{x}, \bar{y})$ will lie on the line AOA$'$ i.e., x -axis. $\therefore$

$$\bar{y}=0$$

Take an elementary strip PQQ$'$P$'$ at a depth x below O, the centre of the ellipse, and of width dx. Then d S $=$ area of the elementary strip $=2ydx$ $p=$ intensity of pressure at any point of the strip $=\rho g(a+x)$, where ρ is the density of the liquid. $\therefore \bar{x}=$ depth of the C.P. of the ellipse below Point,

$$\mathrm{O}=\frac{\int xpd\,\mathrm{S}}{\int pd\,\mathrm{S}},$$

between suitable limits

$$\frac{\int_{-a}^{a}x\rho g(a+x)2ydx}{\int_{-a}^{a}\rho g(a+x)2ydx}=\frac{\int_{-a}^{a}xy(a+x)dx}{\int_{-a}^{a}y(a+x)dx}$$

The parametric equations of the ellipse (1) are

$$x = a\cos t, y = b\sin t$$
$$\therefore \quad dx = -a\sin t dt$$

Also when, $x = a, \cos t = 1 \Rightarrow t = 0$ and when $x = -a, \cos t = -1 \Rightarrow t = \pi$

$$\begin{aligned}
\therefore \bar{x} &= \frac{\int_{\pi}^{0} a\cos t \cdot b\sin t(a + a\cos t)(-a\sin t dt)}{\int_{\pi}^{0} b\sin t(a + a\cos t)(-a\sin t dt)} \\
&= \frac{a\int_0^{\pi} \cdot \left(\cos t\sin^2 t + \cos^2 t\sin^2 t\right) dt}{\int_0^{\pi}\left(\sin^2 t + \cos t\sin^2 t\right) dt} \\
&= \frac{a\left[\int_0^{\pi}\cos t\sin^2 t dt + \int_0^{\pi}\cos^2 t\sin^2 t dt\right]}{\int_0^{\pi}\sin^2 t dt + \int_0^{\pi}\cos t\sin^2 t dt} \\
&= \frac{a\left[0 + 2\int_0^{\pi/2}\cos^2 t\sin^2 t dt\right]}{2\int_0^{\pi/2}\sin^2 t dt + 0}
\end{aligned}$$

$$= \frac{a\left(\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right)}{\frac{1}{2}\cdot\frac{\pi}{2}} = \frac{a}{4}$$

Now, the C.P. of the ellipse will coincide with the focus, if

$$\bar{x} = ae$$

i.e., if

$$\frac{a}{4} = ae$$
$$e = \frac{1}{4}$$

Question-8(b) If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

[10 Marks]

Solution: The boundary C of the surface S is the circle $x^2 + y^2 = a^2$, $z = 0$. Suppose, $x = a\cos t$, $y = a\sin t$, $z = 0$ are parametric eqns of C, where $0 \le t \le 2\pi$ By Stokes Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C [y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C ydx + (x - 2xz)dy + xydz$$

$$= \int_C ydx + xdy \quad (\because \text{ On } C_1, z = 0, dz = 0)$$

Converting to parametric form:

$$\iint_s (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_C y dx + x dy$$

$$= \int_0^{2\pi} [a \sin t(-a \sin t) + \text{a} \cos t(a \cos t)] dt$$

$$= \int_0^{2\pi} a^2 \left(\cos^2 t - \sin^2 t\right) dt$$

$$= a^2 \int_0^{2\pi} \cos dt = a^2 \left[\frac{\sin 2t}{2}\right]_0^{2\pi} = 0$$

Question-8(c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path.

[15 Marks]

Solution: Here, the central acceleration varies inversely as the cube of the distance *i.e.* $\text{P} = \frac{\mu}{r^3} = \mu u^3$, where μ is a constant. If V is the velocity of a particle along a circle of radius a, then Or,

$$\frac{\text{V}^2}{a} = [\text{P}]_{r=a} = \frac{\mu}{a^3}$$

$$\text{V} = \sqrt{\left(\frac{\mu}{a^2}\right)}$$

$\therefore$ The velocity of projection $v_1 = \sqrt{2}\ \text{V}$

$$= \sqrt{\left(\frac{2\mu}{a^2}\right)}$$

The differential equation of the path is

$$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\text{P}}{u^2} = \frac{\mu u^3}{u^2} = \mu u$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$= \mu u^2 + \text{A}...(i)$$

Where, A is a constant. But initially when $r = a$ i.e.,

$$u = \frac{1}{a}, \frac{du}{d\theta} = 0 \text{ (at an apse}$$

and

$$v = v_1 = \sqrt{\left(\frac{2\mu}{a^2}\right)}$$

$\therefore$ From equation, we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + \text{A}$$

$$\therefore h^2 = 2\mu \text{ and } \text{A} = \frac{\mu}{a^2}$$

Substituting the values of h^2 and A in (1) we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2$$

Or,

$$= \frac{1 - a^2u^2}{a^2}$$

Or,

$$\sqrt{2}a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)}$$

Or,

$$\frac{d\theta}{\sqrt{2}} = \frac{adu}{\sqrt{(1 - a^2u^2)}}$$

On integration, we have,

$$\left(\frac{\theta}{\sqrt{2}}\right) + \text{B} = \sin^{-1}(au),$$

where B is a constant.
But initially, when

$$u = \frac{1}{a}, \theta = 0$$

$$B = \sin^{-1} 1 = \frac{1}{2}\pi$$

$$\therefore \quad \left(\frac{\theta}{\sqrt{2}}\right) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$$

For,

$$au = \frac{a}{r}$$

$$\frac{a}{r} = \sin\left\{\frac{1}{2}\pi + \left(\frac{\theta}{\sqrt{2}}\right)\right\}$$

or,

$$a = r\cos\left(\frac{\theta}{\sqrt{2}}\right),$$

which is the required equation of the path.

Chapter 7

2014

7.1 Section-A

Question-1(a) Show that $\mathrm{u}_1 = (1, -1, 0), \mathrm{u}_2 = (1, 1, 0)$ and $\mathrm{u}_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Express (5,3,4) in terms of $\mathrm{u}_1, \mathrm{u}_2$ and u_3.

[8 Marks]

Solution: Consider

$$xu_1 + yu_2 + zu_3 = 0, \text{ where } x, y, z \in R$$

$$x(1, -1, 0) + y(1, 1, 0) + z(0, 1, 1) = (0, 0, 0)$$
$$(x + y, -x + y + z, z) = (0, 0, 0)$$
$$\Rightarrow x + y = 0, -x + y + z = 0 \quad z = 0$$
$$x = 0, \quad y = 0, \quad z = 0$$

Hence, u_1, u_2 and u_3 are linearly independent. Again, let

$$(a, b, c) \in \mathbb{R}^3$$

and

$$xu_1 + yu_2 + zu_3 = (a, b, c)$$
$$\Rightarrow \quad x + y = a, \quad -x + y + z = b, \quad z = c$$
$$x + y = a, x - y = c - b$$
$$\therefore x = \frac{a - b + c}{2}, \quad y = \frac{a + b - c}{2}, \quad z = c$$
$$\therefore (a, b, c) = \left(\frac{a - b + c}{2}\right) u_1 + \left(\frac{a + b - c}{2}\right) u_2 + cu_3$$

Taking $(a, b, c) = (5, 3, 4)$

$$(5, 3, 4) = \left(\frac{5 - 3 + 4}{2}\right) u_1 + \left(\frac{5 + 3 - 4}{2}\right) u_2 + 4u_3$$
$$= 3u_1 + 2u_2 + 4u_3$$
$$= 3(1, -1, 0) + 2(1, 1, 0) + 4(0, 1, 1)$$

Question-1(b) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. **Prove that**

$$A^n = A^{n-2} + A^2 - I, n \geq 3$$

[8 Marks]

Solution: Consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2 - 1\right) = 0$$
$$\Rightarrow \lambda^2 - \lambda^3 - 1 + \lambda = 0$$
$$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

$$\therefore A^3 - A^2 - A + I = 0$$
$$\Rightarrow \quad A^3 = A^2 + A - I$$

Hence given relation is true for $n = 3$.
Now, assume that this statement holds true for n.
We have to prove that it also holds true for $n + 1$.
Let $A^n = A^{n-2} + A^2 - I$. Multiply both sides by A,

$$\begin{aligned} A^{n+1} &= A^{n-1} + A^3 - A \\ &= A^{n-1} + \left(A^2 + A - I\right) - A \quad \text{using (1)} \\ &= A^{(n+1)-2} + A^2 - I \end{aligned}$$

Hence given statement is true for $n + 1$ also.
Using principle of Mathematical Induction (PMI), we infer that the statement holds true for all natural numbers greater than or equal to 3.

Question-1(c) Show that the function given by

$$f(x) = \begin{cases} \dfrac{x\left(e^{1/x} - 1\right)}{\left(e^{1/x} + 1\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$.

[8 Marks]

Solution:

$$f(x)=\begin{cases}\frac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0\\ 0 & ,x=0\end{cases}$$

$$RHL=\mathrm{Lim}_{x\to 0^+}$$

$$f(x)=\lim_{x\to 0^+}\frac{x\left(e^{1/x}-1\right)}{e^{1/x}+1}$$

$$=\lim_{x\to 0^+}\frac{x\left(1-e^{-1/x}\right)}{1+e^{-1/x}}$$

$$=\frac{0(1-0)}{1+0}=0$$

$$\left(\because \text{ as } x\to 0^+, -\frac{1}{x}\to-\infty\Rightarrow e^{-1/x}+0\right)$$

$$LHL=\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{x\left(e^{1/x}-1\right)}{e^{1/x}+1}$$

$$=\lim_{h\to 0^+}\frac{(0-h)\left(e^{\frac{1}{(0-h)}-1}\right)}{\left(e^{\frac{1}{0-h}+1}\right)}$$

$$=\lim_{h\to 0^+}\frac{(-h)\left(e^{-1/h}-1\right)}{e^{-1/h}+1}$$

$$=\frac{0(0-1)}{0+1}=0$$

Hence, $LHL=RHL=f(0)\therefore f$ is continuous at $x=0$. For differentiability,

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h\left(e^{1/h}-1\right)}{e^{1/h}+1}-0\right]$$

$$=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$$

This limit does not exist as

$$RHL=\lim_{h\to 0^+}\frac{e^{1/h}-1}{e^{1/h}+1}=\lim_{h\to 0^+}\frac{1-e^{-1/h}}{1+e^{-1/h}}$$

$$=\frac{1-0}{1+0}=1$$

$$LHL=\lim_{h\to 0^-}\frac{e^{1/h}-1}{e^{1/h}+1}=\frac{0-1}{0+1}=-1$$

Hence, $f(x)$ is not differentiable at $x=0$.

Question-1(d) Evaluate $\iint_R y\dfrac{\sin x}{x}dxdy$ over R where $R = \{(x, y) : y \le x \le \pi/2, \quad 0 \le y \le \pi/2\}$.

[8 Marks]

Solution:

$$
\begin{aligned}
I &= \iint\limits_R y\frac{\sin x}{x}dxdy \\
&= \int_{x=0}^{\pi/2}\int_{y=0}^{x}\frac{\sin x}{x}ydydx \\
&= \int_0^{\pi/2}\frac{\sin x}{x}\left[\frac{y^2}{2}\right]_0^x dx \\
&= \frac{1}{2}\int_0^{\pi/2}\frac{\sin x}{x}\cdot\left(x^2-0\right)dx \\
&= \frac{1}{2}\int_0^{\pi/2} x\cdot\sin x dx \\
&= \frac{1}{2}\left[\int_0^{\pi/2}[x(-\cos x)]_0^{\pi/2}+\int_0^{\pi/2}1\cdot\cos x dx\right] \\
&= \frac{1}{2}\left[(0-0)+[\sin x]_0^{\pi/2}\right] \\
&= \frac{1}{2}(1-0)=\frac{1}{2}
\end{aligned}
$$

Question-1(e) Prove that the locus of a variable line which intersects the three lines:

$$y = mx, z = c; \quad y = -mx, \quad z = -c; \quad y = z, mx = -c$$

is the surface $y^2 - m^2x^2 = z^2 - c^2$.

[8 Marks]

Solution: The given lines are

$$
\begin{aligned}
&y - mx = 0, z - c = 0 ...(i) \\
&y + mx = 0, z + c = 0 ...(ii) \\
&y - z = 0, mx + c = 0 ...(iii)
\end{aligned}
$$

Any line intersecting (i) and (ii) is

$$y - mx - k_1(z - c) = 0, y + mx - k_2(z + c) = 0 ...(iv)$$

If it intersects (iii) also, we have to eliminate x, y, z from (iii) and (iv).

Now putting $y = z$ and $mx = -c$ from (iii) in (iv), we get

$$z + c - k_1(z - c) = 0$$

$$z - c - k_2(z + c) = 0$$

$$z\,(1 - k_1) + c\,(1 + k_1) = 0$$

$$z\,(1 - k_2) - c\,(1 + k_2) = 0$$

Equating the two values of z, we get

$$\frac{c\,(1 + k_1)}{k_1 - 1} = \frac{c\,(1 + k_2)}{1 - k_2}(= z)$$

$$(1 + k_1)\,(1 - k_2) = (1 + k_2)\,(k_1 - 1)$$

$$1 + k_1 - k_2 - k_1k_2 = k_1 + k_1k_2 - 1 - k_2$$

$$2k_1k_2 - 2 = 0$$

$$k_1k_2 = 1$$

To find the locus, we have to eliminate k_1, k_2 from (iv) and (v) From (iv)

$$k_1 = \frac{y - mx}{z - c}$$

$$k_2 = \frac{y + mx}{z + c}$$

Putting these values in (v), we get

$$\left(\frac{y - mx}{z - c}\right)\left(\frac{y + mx}{z + c}\right) = 1$$

$$\frac{y^2 - m^2x^2}{z^2 - c^2} = 1$$

$$y^2 - m^2x^2 = z^2 - c^2$$

which is the required locus.

Question-2(a) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors of B viewed as a matrix over:

(i) the real field R
(ii) the complex field C.

[10 Marks]

Solution:

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 1 - \lambda & -1 \\ 2 & -1 - \lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda + 1) + 2 = 0$$
$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = i, -i$$

v is the eigenvector

$$Bv = \lambda v$$
$$\Rightarrow \quad (B - \lambda I)v = 0$$
$$\lambda = i$$
$$\Rightarrow \begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{matrix} (1-i)x - y = 0 \\ 2x - (1+i)y = 0 \end{matrix} \Rightarrow y = (1-i)x$$
$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1-i)x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$
$$\lambda = -i$$
$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$(1+i)x - y = 0$$
$$2x - y(1-i) = 0$$
$$y = (1+i)x$$
$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (1+i)x \end{pmatrix} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} x$$

When B is viewed as matrix over the complex field, then eigenvectors are

$$\begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ and } [1+i]$$

When B is viewed as matrix over the real field the eigenvectros

$$v = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad for \quad \lambda = i \quad \text{i.e.} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$a + ib \to \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$
$$v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad for \quad \lambda = -i \quad \text{i.e.} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Question-2(b) If $xyz = a^3$ then show that the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

[10 Marks]

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$g(x, y, z) = xyz - a^3$$

Let

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$g(x, y, z) = xyz - a^3$$

To get maximum value of f, we use Lagrange's multiplier method.
Consider,

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\left(xyz - a^3\right)$$

For critical points,

$$dF = 0, \text{ i.e. } F_x = 0, \quad F_y = 0, \quad F_z = 0$$

$$\begin{aligned} 2x + \lambda(yz) = 0 \quad &\Rightarrow 2x^2 + \lambda xyz = 0 \\ 2y + \lambda(xz) = 0 \quad &\Rightarrow 2y^2 + \lambda xyz = 0 \\ 2z + \lambda(xy) = 0 \quad &\Rightarrow 2z^2 + \lambda xyz = 0 \end{aligned}$$

$$\therefore \quad x^2 = y^2 = z^2$$

Also, given $xyz = a^3$ which implies $(x \cdot x + x)a = a^3$ ie. $x^3 = a^3$

$$= \quad x^2 = y^2 = z^2 = a^2$$

Hence, minimum value of $x^2 + y^2 + z^2$ will be

$$a^2 + a^2 + a^2 = 3a^2$$

Question-2(c) Prove that every sphere passing through the circle

$$x^2 + y^2 + 2ax + r^2 = 0, \quad z = 0$$

cut orthogonally every sphere through the circle

$$x^2 + z^2 = r^2 \quad y = 0$$

[10 Marks]

Solution: Equations of two spheres can be taken as

$$\begin{aligned} &S_1 : x^2 + y^2 + z^2 - 2ax + r^2 + \lambda z = 0 \\ &S_2 : x^2 + y^2 + z^2 - r^2 + \mu y = 0 \end{aligned}$$

condition of orthogonality

$$2\left(u_1u_2+v_1v_2+w_1w_2\right)=d_1+d_2$$

$$2\left[a\cdot 0+0\cdot\left(-\frac{\mu}{2}\right)+\left(-\frac{\lambda}{2}\right)\cdot 0\right]=r^2-(r)^2$$

$$2(0+0+0)=0$$

$$0=0$$

which is true for all values of parameters λ and μ. Hence proved.

Question-2(d) Show that the mapping $\mathrm{T}:\mathrm{V}_2(\overline{\mathrm{R}})\to\mathrm{V}_3(\overline{\mathrm{R}})$ defined as $\mathrm{T(a,b)}=\mathrm{(a+b,a-b,b)}$ is a linear transformation. Find the range, rank and nullity of $\overline{T}$.

[10 Marks]

Solution:

$$\mathrm{T}(a)=(a+b,a-b,b),\forall a,b\in\mathrm{R}$$

Let $\alpha_1=(a_1,b_1)$ and $\alpha_2=(a_2,b_2)$ be any two elements of $\mathrm{V}_2(\mathrm{R})$ then

$$\left.\begin{array}{r}\mathrm{T}\left(\alpha_1\right)=\mathrm{T}\left(a_1,b_1\right)=\left(a_1+b_1,a_1-b_1,b_1\right)\\ \text{and }\mathrm{T}\left(\alpha_2\right)=\mathrm{T}\left(a_2,b_2\right)=\left(a_2+b_2,a_2-b_2,b_2\right)\end{array}\right\}\text{from (i)}$$

Now,

$$a,b\in\mathrm{R}\Rightarrow a\alpha_1+b\alpha_2\in\mathrm{V}_2(\mathrm{R})$$

$$\begin{aligned}\therefore\mathrm{T}\left(a\alpha_1+b\alpha_2\right)&=\mathrm{T}\left[a\left(a_1,b_1\right)+b\left(a_2,b_2\right)\right]\\&=\mathrm{T}\left[aa_1+ba_2,ab_1+bb_2\right]\\&=\left(aa_1+ba_2+ab_1+bb_2,aa_1+ba_2-ab_1-bb_2,ab_1+bb_2\right)\quad(by def. of T)\\&=\left[a\left(a_1+b_1\right)+b\left(a_2+b_2\right),a\left(a_1-b_1\right)+b\left(a_2-b_2\right)\ ab_1+bb_2\right]\\&=a\left(a_1+b_1,a_1-b_1,b_1\right)+b\left(a_2+b_2,a_2-b_2,b_2\right)\\&=a\ \mathrm{T}\left(\alpha_1\right)+b\ \mathrm{T}\left(\alpha_2\right)\end{aligned}$$

which proves that $T:V_2(R)\to V_3(R)$ as defined in (i) is a linear transformation.
Now, we will calculate the null space of T.
If $\alpha=(a,b)$, then

$$N(T)=\{\alpha\in V_2(R);T(\alpha)=0\in V_3(R)\}$$

Now,

$$T(\alpha)=T(a,b)=(a+b,a-b,b)=(0,0,0)$$

$$\Rightarrow a+b=0,a-b=0,b=0$$

$$\Rightarrow a=0,b=0$$

$$\therefore\alpha=(a,b)=(0,0)\in\mathrm{N(T)}$$

showing that null space consists of only zero vector of $\mathrm{V}_2(\mathrm{R})$ i.e. domain or in other words null space of T is the zero subspace of $\mathrm{V}_2(\mathrm{R})$ i.e. nullity of T $=\dim[\mathrm{N(T)}]=0$.

Range space of T. We have

$$R(T) = \{\beta \in V_3(R) : \beta = T(\alpha), \alpha \in V, (R)\}$$

Now $\{(1,0),(0,1)\}$ is the basis of $V_2(R)$.

Also $T(1,0) = (1+0, 1-0, 0) = (1,1,0)$ and $T(0,1) = (0+1, 0-1, 1) = (1,-1,1)$

Hence the range space of T is a sub-snace ***of*** $V_3(R)$ generated by $(1,1,0)$ and $(1,-1,0)$.

Now

$$\begin{aligned} a(1,1,0) + b(1,-1,1) &= (0,0,0) \forall a, b \in R \\ \Rightarrow (a+b, a-b, b) &= (0,0,0) \\ \Rightarrow a+b = 0, a-b &= 0, b = 0 \\ \Rightarrow a = 0, b &= 0 \end{aligned}$$

Therefore $(1,1,0),(1,-1,1)$; elements of $R(T)$ are L.I. and generates $R(T)$ Hence, $\{(1,1,0),(1,-1,1)\}$ is the basis of R(T)

$$\therefore \dim(\mathrm{R(T)}) = \mathrm{rank(T)} = 2$$

Question-3(a) Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$ **is diagonalizable. Find all eigenvalues. Then obtain a matrix** P **such that** $P^{-1}AP$ **is a diagonal matrix.**

[10 Marks]

Solution: Characterestic Equation of a square matrix A is given by :

$$| A - \lambda I | = 0$$

i.e.

$$\begin{aligned} \lambda^3 - (\text{ trace of A})\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| &= 0 \\ \mathrm{trace}(A) &= -2 + 1 + 0 = -1 \\ C_{11} + C_{22} + C_{33} &= (0-12) + (0-3) + (-2-4) \\ &= -12 - 3 - 6 = -21 \\ |A| &= 45 \end{aligned}$$

$\therefore$ Characterestic Equation: $\quad \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\Rightarrow \lambda = 5, -3, 3 \qquad \text{(Use Calci)}$$

Now, let us find the corresponding eigen-vectors for each eigen-values.

$$\text{(i) } \lambda = 5$$

$$\therefore (A - \lambda I)X = 0 \Rightarrow (A - 5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1; \quad R_3 \to R_3 - 7R_1$$

$$R_2 \to R_2 - 2R_1; \quad R_1 \to \frac{R_1}{-8}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 + 2R_2$$

$$\begin{aligned} -x - z &= 0 \\ y + 2z &= 0 \\ \text{i.e.} \quad x = -z; &\quad y = -2z \\ X = (x, y, z) &= (-z, -2z, z) = z(-1, -2, 0) \end{aligned}$$

$\therefore X_1 = (-1, -2, 0)$ is the given vector corresponding to eigen value $\lambda = 5$

(ii) $\lambda = -3, \quad (A + 3I)X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} [\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $x + 2y - 3z = 0$ i.e. $x = -2y + 3z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y + 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

X_2 and X_3 are eigen-vectors, corresponding to eigen value $\lambda = -3$.
We notice that for each eigen-value, algebraic multiplicity (number of same roots) is equal to geometric multiplicity ie. number of independent eigen-vectors.
Hence, A is diagonalizable. Now, for $P^{-1}AP = D$,
Transformation matrix P is obtained by placing eigen-vectors as columns

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and Diagonal matrix, D consists of eigen-values placed at diagonal positions

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

We can verify that

$$P^{-1} = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P^{-1}AP = D$$

Question-3(b) A moving plane passes through a fixed point $(2, 2, 2)$ and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C.

[10 Marks]

Solution: Let the eqn of plane be given by:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Let the points on coordinate axes through which the plane passes be given by $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$$\Rightarrow \frac{2}{a} + \frac{2}{b} + \frac{2}{c} = 1 \qquad \ldots(1)$$

Let general eqn of sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Origin O(0,0,0) lies on it $\Rightarrow d = 0$
$A(a, 0, 0)$ lies on it $\Rightarrow a^2 + 2ua = 0 \Rightarrow 2u = -a$.
Similarly, $B(0, b, 0)$ gives $2v = b$
$C(0, 0, c)$ gives $2\omega = c$

$$\therefore \quad x^2 + y^2 + z^2 + ax - by - cz = 0$$

Centre

$$x_1 = \frac{a}{2}, \quad y_1 = \frac{b}{2}, \quad z_1 = \frac{c}{2}$$

Using (1),

$$\frac{2}{2x_1} + \frac{2}{2y_1} + \frac{2}{2z_1} = 1$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

is the required locus.

Question-3(c) Evaluate the integral

$$I = \int_0^\infty 2^{-ax^2} dx$$

using Gamma function.

[10 Marks]

Solution:

$$\begin{aligned} I &= \int_0^\infty 2^{-ax^2} dx \\ &= \int_0^\infty e^{-ax^2(\log 2)} dx \quad \left[t = e^{\log t}\right] \\ \text{Let } a(\log 2)x^2 &= y \\ \Rightarrow a(\log 2)2x dx &= dy \\ I &= \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)x} \\ &= \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)\sqrt{y}} \times \sqrt{a \log 2} \\ &= \frac{1}{2\sqrt{a \log_2 2}} \int_0^\infty e^{-y} \cdot y^{-1/2} dy \\ &= \frac{1}{2\sqrt{a \log 2}} \cdot T\left(\frac{-1}{2} + 1\right) \\ &= \frac{\sqrt{\pi}}{2\sqrt{a \log 2}} \end{aligned}$$

Question-3(d) Prove that the equation:

$$4x^2 - y^2 + z^2 + 2xy - 3yz + 2xy + 12x - 11y + 6z + 4 = 0$$

represents a cone with vertex at $(-1, -2, -3)$

[10 Marks]

Solution: Making the given equation homogeneous, we get

$$F(x, y, z, t) \equiv 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0$$

$\frac{\partial F}{\partial x} = 0$ gives $8x + 2y + 12t = 0$ or $4x + y + 6t = 0$
$\frac{\partial F}{\partial y} = 0$ gives $-2y + 2x - 3z - 11t = 0$ or $2x - 2y - 3z - 11t = 0$
$\frac{\partial F}{\partial z} = 0$ gives $4z - 3y + 6t = 0$ or $3y - 4z - 6t = 0$
$\frac{\partial F}{\partial t} = 0$ gives $12x - 11y + 6z + 8t = 0$

Putting $t = 1$, these equations become

$$4x + y + 6 = 0 \qquad \ldots(i); \qquad 2x - 2y - 3z - 11 = 0 \ldots(ii)$$

$$3y - 4z - 6 = 0 \qquad \ldots(iii); \qquad 12x - 11y + 6z + 8 = 0 \ldots(iv)$$

From (ii), we get $4x - 4y - 6z - 22 = 0$

Subtracting (i) from it, we get $5y + 6z + 28 = 0$, or

$$10y + 12z + 56 = 0$$

Multiplying (iii) by, 3 we get $9y - 12z - 18 = 0 \qquad \ldots(vi)$.

Adding (v) and (vi), we get $19y + 38 = 0$ or $y = -2$.

$\therefore$ From (iii), we get $3(-2) - 4z - 6 = 0$ or $z = -3$

From (i), we get $4x + (-2) + 6 = 0$ or $x = -1$.

These values, i.e., $x = -1, y = -2, z = -3$ satisfy (iv) and so the given equation represents a cone and its vertex is (-1,-2,-3).

Question-4(a) Let f be a real valued function defined on [0,1] as follows:

$$f(x) = \begin{cases} \dfrac{1}{a^{r-1}}, & \dfrac{1}{a^r} < x \leq \dfrac{1}{a^{r-1}}, r = 1, 2, 3 \ldots\ldots \\ 0 & x = 0 \end{cases}$$

where a is an integer greater than 2. Show that $\int_0^1 f(x)dx$ exists and is equal to $\dfrac{a}{a+1}$.

[10 Marks]

Solution:

$$f(x) = \begin{cases} \frac{1}{a^{1-1}} = 1, \frac{1}{a} < x \leq 1 \\ \frac{1}{a}, \frac{1}{a^2} < x \leq \frac{1}{a} \\ \frac{1}{a^2}, \frac{1}{a^3} < x \leq \frac{1}{a^2} \\ \frac{1}{a^{r-1}}, \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \\ 0, x = 0 \end{cases}$$

Clearly $f(x) \in [0, 1]$ for all $x \in [0, 1] \Rightarrow f$ is bounded on [0,1] as $a > 2$

Also, it is continuous on [0,1] except at points $0, \frac{1}{a}, \frac{1}{a^2}, a^3$

The set of points of discontinuities has only one limit point O and hence, f is integrable on $[0,1]$.

$$\begin{aligned}
\int_{\frac{1}{a^r}}^{1} f(x)dx &= \int_{\frac{1}{a}}^{1} f dx + \int_{\frac{1}{a^2}}^{\frac{1}{a}} f dx + \ldots.. + \int_{\frac{1}{a^r}}^{\frac{1}{a^{r-1}}} f dx \\
&= \int_{\frac{1}{a}}^{1} 1.dx + \int_{\frac{1}{a^2}}^{\frac{1}{a}} \frac{1}{a}dx + \int_{\frac{1}{a^3}}^{\frac{1}{a^2}} \frac{1}{a^2}dx + \ldots.. + \int_{\frac{1}{a^r}}^{\frac{1}{a^{r-1}}} \frac{1}{a^{r-1}}dx \\
&= \left(1 - \frac{1}{a}\right) + \left(\frac{1}{a} - \frac{1}{a^2}\right)\frac{1}{a} + \left(\frac{1}{a^2} - \frac{1}{a^3}\right)\frac{1}{a^2} + \ldots\ldots + \left(\frac{1}{a^8 - 1} - \frac{1}{a^8}\right) \cdot \frac{1}{a^8 - 1} \\
&= \left(1 - \frac{1}{a}\right)\left[1 + \frac{1}{a^2} + \frac{1}{a^4} + \ldots + \frac{1}{a^2(h-1)}\right] \\
&= \frac{a-1}{a} \times \frac{1\left[1 - \left(\frac{1}{a^2}\right)^r\right]}{1 - \frac{1}{a^2}} \\
S_n &= \frac{a\left(1 - \lambda^n\right)}{1 - r} \\
&= \frac{a}{a+1} \cdot \left(1 - \frac{1}{a^2}\right)
\end{aligned}$$

Taking limit $r \to \infty$

$$\int_0^1 f(x)dx = \frac{a}{a+1}$$

Question-4(b) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

[10 Marks]

Solution: Let

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

be the line of section.
$\therefore l, m, n$ satisfies equation of cone and plane.

$$\therefore \quad lm + mn + \ln = 0$$

and

$$al + bm + cn = 0 \ldots (i)$$
$$n = \frac{al+bm}{-c}$$

Substituting we get

$$lm - \frac{m}{c}(al+bm) - \frac{l}{c}(al+bm) = 0$$
$$\therefore \quad al^2 + lm(a+b-c) + bm^2 = 0$$
$$\therefore \quad a\left(\frac{l}{m}\right)^2 + (a+b-c)\left(\frac{l}{m}\right) + b = 0$$

which is a quadratic in $\frac{l}{m}$.
Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ be the two roots of this equation.
$\therefore$ Product of the roots $= \frac{b}{a}$ i.e.

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

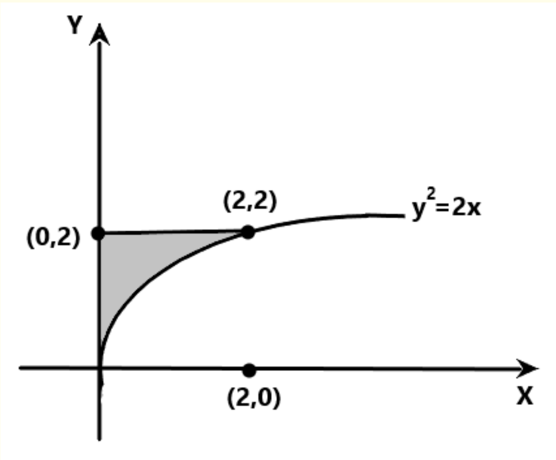
$$\therefore \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

by symmetry.
$\therefore$ *perpendicular* if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Question-4(c) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}} dxdy$ **over the region** R **bounded between** $0 \le x \le \frac{y^2}{2}$ **and** $0 \le y \le 2$.

[10 Marks]



Solution:

$$
\begin{aligned}
I &= \int_R \frac{y}{\sqrt{x^2+y^2+1}} dxdy \\
&= \int_{x=0}^{2} \int_{y=\sqrt{2x}}^{2} \frac{2y}{2\sqrt{1+x^2+y^2}} dydx \\
&= \int_{x=0}^{2} \left(1+x^2+y^2\right)^{1/2} \Big|_{y=\sqrt{2x}}^{y=2} dx \\
&= \int_0^2 \left(\sqrt{1+4+x^2} - \sqrt{1+x^2+2x}\right) dx \\
x &= 0^2\sqrt{5+x^2} - (1+x)dx \\
&= \int_0^2 \sqrt{5+x^2} - (1+x)dx \\
&= -\frac{\left[\frac{1}{2}x\sqrt{5+x^2} + \frac{5^2}{2}\log\left|x+\sqrt{5+x^2}\right|_0^2\right.}{-\left[x+\frac{x^2}{2}\right]_0^2} \\
&= \left(\frac{2x\sqrt{9}}{2} + \frac{5}{2}\log(2+\sqrt{9}) - \frac{5}{2}\log\sqrt{5}\right) - \left(2+\frac{4}{2}\right) \\
I &= \frac{5}{4}\log 5 - 1
\end{aligned}
$$

Question-4(d) **Consider the linear mapping** $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}^2$ **given as** $\mathrm{F(x,y)} = (3\mathrm{x}+4\mathrm{y}, 2\mathrm{x}-5\mathrm{y})$ **with usual basis. Find the matrix associated with the linear transformation relative to the basis** $\mathrm{S} = \{\mathrm{u_1}, \mathrm{u_2}\}$ **where** $u_1 = (1,2), u_2 = (2,3)$**.**

[10 Marks]

Solution: $F : R^2 \to R^2$

$$
F(x,y) = (3x+4y, 2x-5y)
$$
$$
s = \{(1,2),(2,3)\}
$$

$$
\begin{aligned}
F(1,2) &= (3+8, 2-10) = (11,-8) \\
&= -49(1,2) + 30(2,3) \text{ [Using calculator]} \\
F(2,3) &= (6+12, 4-15) = (18,-11) \\
&= -76(1,2) + 47(2,3)
\end{aligned}
$$

$\therefore$ Matrix of LT wrt. Basis S

$$
[M] = \begin{bmatrix} -49 & 30 \\ -76 & 47 \end{bmatrix}' = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}
$$

7.2 Section-B

Question-5(a) Solve the differential equation :

$$y = 2px + p^2 y, p = \frac{dy}{dx}$$

and obtain the non-singular solution.

[8 Marks]

Solution: We have

$$y = 2xp + yp^2 \qquad \ldots(i)$$

$$\Rightarrow \quad 2xp = y - yp^2$$

$$\Rightarrow \quad x = \frac{y}{2p} - \frac{py}{2} \qquad \ldots(ii)$$

Differentiating (ii) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{p} \cdot 1 + y \cdot -\frac{1}{p^2}\frac{dp}{dy}\right) - \frac{1}{2}\left(p \cdot 1 + y\frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2p} - \frac{y.}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} - \frac{p}{2} = \left(\frac{y}{2} - \frac{y}{2p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -\left(\frac{p}{2} - \frac{1}{2p}\right) = \frac{y}{p}\left(\frac{p}{2} - \frac{1}{2p}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -1 = \frac{y}{p}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating,

$$\log p + \log y = \log c$$

$$\Rightarrow py = c$$

$\Rightarrow$

$$p = c/y$$

Putting the value of p in (i), we get

$$y = 2x\left(\frac{c}{y}\right) + y\left(\frac{c}{y}\right)^2$$

$$\Rightarrow \quad y^2 = 2cx + c^2$$

$$\Rightarrow y^2 - 2cx - c^2 = 0$$

Question-5(b) Solve :

$$\frac{d^4y}{dx^4} - 16y = x^4 + \sin x$$

[8 Marks]

Solution: Auxiliary Egn: $D^4 - 16 = 0$ ie. $(D^2 - 4)(D^2 + 4)$

$$D = \pm 2, \pm 2i$$
$$C \cdot F \cdot = c_1 e^{2x} + c_2 e^{-2x} + c_3' \cos 2x + c_4' \sin 2x$$
$$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(2x + (4).$$

$$P.I_1 = \frac{1}{D^4 - 16}\left(x^4 + \sin x\right)$$
$$= \frac{1}{D^4 - 16} x^4 + \frac{1}{D^4 - 16} \sin x$$
$$= \frac{-1}{16}\left(1 - \frac{D^4}{16}\right)^{-1} x^4 + \frac{1}{(-1^2)(-1^2) - 16} \sin x$$
$$\left(D^4 = D^2 \cdot D^2\right)$$
$$= \frac{-1}{16}\left(1 + \frac{D^4}{16}\right) x^4 - \frac{11}{15} \sin x$$
$$= \frac{-1}{16}\left(x^4 + \frac{\cancel{4} \cdot 3 \cdot x \cdot 1}{+6x_2}\right) - \frac{1}{15} \sin x$$

Hence, complete solution,

$$y = C.F. + P.I.$$
$$y = ce^{2x} + c_2 e^{-2x} + C_3 \operatorname{cs}(2x + c_4) \frac{-x^4}{16} - \frac{3}{32} - \frac{1}{15} \sin x$$

Question-5(c) A particle whose mass is m, is acted upon by a force $m\mu\left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

[8 Marks]

Solution: Given

$$\frac{d^2x}{dt^2} = -\mu\left[x + \frac{a^4}{x^3}\right], \quad \ldots (i)$$

the -ve sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + C$$

When $x = a, dx/dt = 0$, so that $C = 0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

$$\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4 - x^4)}}{x}$$

the -ve sign is taken because the particle is moving in the direction of x decreasing. If t_1 be the time taken to reach the origin, then integrating (ii), we get

$$\begin{aligned} t_1 &= -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4 - x^4)}}dx \\ &= \frac{1}{\sqrt{\mu}}\int_0^t \frac{xdx}{\sqrt{(a^4 - x^4)}} \end{aligned}$$

Put $x^2 = a^2 \sin\theta$ so that

$$2xdx = a^2 \cos\theta d\theta$$

When $x = 0, \theta = 0$ and when

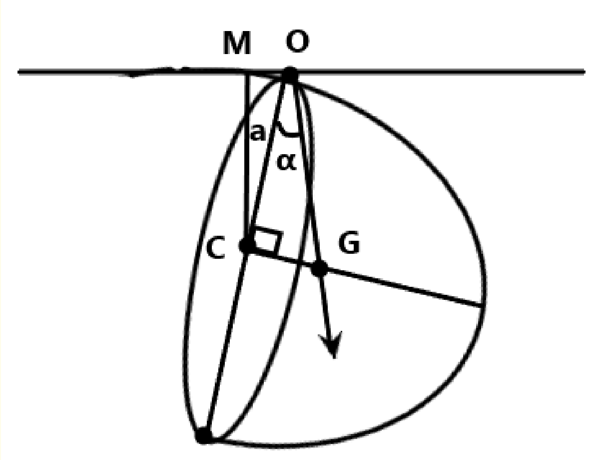
$$x = a, \theta = \frac{\pi}{2}$$

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{\mu}}\int_0^{\pi/2} \frac{\frac{1}{2}a^2\cos\theta d\theta}{a^2\cos\theta} \\ &= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2} d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2} \\ &= \frac{1}{2\sqrt{\mu}} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{4\sqrt{\mu}} \end{aligned}$$

Question-5(d) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12 : \sqrt{73}$.

[8 Marks]

Solution: Let a' be the radius of the hemisphere and O the point of rim from which it is suspended.



Let G be the CG (centre of gravity) of the hemisphere, then $CG = \frac{3}{8}a$ and OG must be vertical.
If α be the inclination of the base to the vertical, then

$$\tan\alpha = \frac{3}{8} \qquad \ldots(1)$$

$$\text{The whole pressure (thrust) on the base } = w \cdot \pi a^2 \cdot (a \operatorname{css} \alpha)$$

Here, w = weight per unit volume of liquid.

Depth of the center of gravity of the boy below surface of liquid = CM=$a\cos\alpha$.

$$\text{Weight of the liquid contained } = w \cdot \left(\frac{2}{3}\pi a^3\right)$$

$$\therefore \text{ Required ratio is } = \frac{w \cdot \pi a^2(a\cos\alpha)}{w \cdot \frac{2}{3}\pi a^3}$$

$$= \frac{3}{2} \cdot \frac{8^4}{\sqrt{73}}$$

$$[\text{From (1)}, \ \tan\alpha = \frac{3}{8} \Rightarrow \cos\alpha = \frac{8}{\sqrt{73}}]$$

$$= \frac{12}{\sqrt{73}}$$

Question-5(e) For three vectors show that:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

[8 Marks]

Solution: We know that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Hence,

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$
$$= [(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}] + [(\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}] + [(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}]$$
$$= 0$$

Question-6(a) Solve the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x \tan \frac{y}{x^2}$$

[10 Marks]

Solution: Put $\frac{y}{x^2} = t$ ie, $y = tx^2$

$$\frac{dy}{dx} = 2tx + x^2 \cdot \frac{dt}{dx}$$

Now DE becomes

$$2tx + x^2\frac{dt}{dx} = \frac{2tx^2}{x} + \frac{x^3}{tx^2} + x + \tan t$$
$$x\frac{dt}{dx} = \frac{1}{t} + \frac{\cos t + t\sin t}{t\cos t}$$
$$\int \frac{t\cos t dt}{\cos t + t\sin t} = \int \frac{dx}{x}$$
$$\log(\cos t + t\sin t) = \log x + \log c$$
$$\Rightarrow \quad \cos t + t\sin t = cx$$

i.e.

$$\cos\left(\frac{y}{x^2}\right) + \frac{y}{x^2}\sin\left(\frac{y}{x^2}\right) = cx$$

which is the required solution.

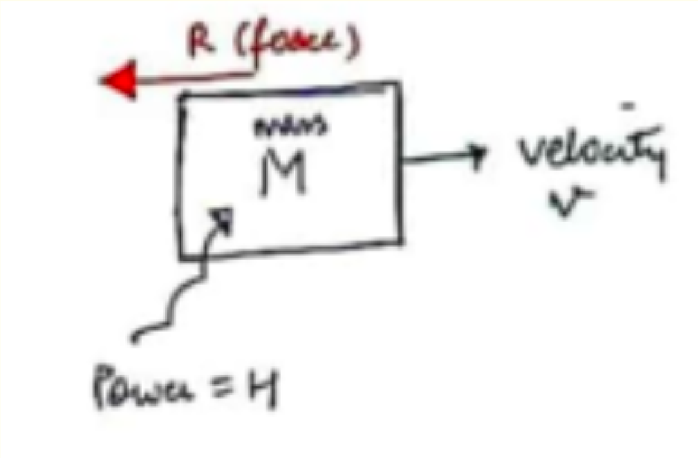
Question-6(b) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{\mathrm{MH}}{\mathrm{R}^2}\left(\log 2 - \frac{1}{2}\right)$

[10 Marks]

Solution: Energy equation for time dt,

$$\text{Energy supplied} = \mathrm{Hdt}$$

$$\text{Energy lost due to the resistance} = \text{Force} \times \text{distance} == Rvdt$$



Assuming no change of PE;

$$\triangle PE = 0$$

$$\sum \text{energy supplied} - \sum \text{enegy lost} = \Delta(k.E.)$$

$$Hdt - RVdt = d \cdot \left(\frac{1}{2}\mathrm{mv}^2\right)$$

$$Hdt - Rvdt = mvdv$$

$$H - Rv = mv\frac{dv}{dt}$$

For max. velocity,

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \text{accelaration} = 0$$

$$H - Rv = 0$$

$$V_{\max} = H/R$$

Now, integrating

$$H - Rv = mvd\frac{d}{dt}$$

$$
\begin{aligned}
dt &= m\left(\frac{vdv}{\beta tRv}\right)\\
&= \frac{m}{R}\left(\frac{kv}{\mu - Rv}\right)dv\\
dt &= \frac{m}{R}\left(\frac{Rv - H + H}{H - Rv}\right)dv\\
&= \frac{m}{R}\left(\frac{H}{H + Rv} - 1\right)dv\\
\int_0^t dt &= \int_0^{\frac{V_{\max}}{2}} \frac{M}{R}\left(\frac{H}{H - Rv} - 1\right)dv\\
t &= \int_0^{H/2R} \frac{M}{R}\left(\frac{M}{H - Rv} - 1\right)dv\\
&= \frac{M}{R}\left[\frac{-H}{R}\log(H - Rv) - v\right]_0^{H/2R}\\
&= \frac{M}{R}\left[\frac{-H}{R}\log\left(H - R\cdot\frac{H}{2R}\right) - \frac{H}{2R} + \frac{H}{R}\log(H - R\cdot 0) - 0\right]\\
&= \frac{M}{R}\left[-\frac{H}{R}\log\frac{H}{2} - \frac{H}{2R} + \frac{H}{R}\log H\right]\\
&= \frac{M}{R}\left[\left(\frac{-H}{R}\log H + \frac{H}{R}\log 2\right) - \frac{H}{2R} + \frac{H}{R}\log H\right]\\
&= \frac{MH}{R^2}\left[(\log 2) - \frac{1}{2}\right]
\end{aligned}
$$

Question-6(c) Solve by the method of variation of parameters:

$$y'' + 3y' + 2y = x + \cos x$$

[10 Marks]

Solution: D.E.

$$\Rightarrow \quad \left(D^2 + 3D + 2\right)y = x + \cos x$$

Auxiliary Eqn:

$$
\begin{gathered}
D^2 + 3D + 2 = 0\\
(D+1)(D+2) = 0\\
D = -1, -2\\
C\cdot F = C_1' e^{-x} + C_2' e^{-2x}\\
y_1 = e^{-x} \quad , \quad y_2 = e^{-2x}\\
W = \begin{vmatrix} y_1 & y_2\\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x}\\ -e^{-x} & -2e^{-2x} \end{vmatrix}\\
= -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0
\end{gathered}
$$

$\therefore y_1$ & y_2 are linearly independent.

To get complete solution by variation of parameters, we replace C_1' and C_2' in C.F. by functions A and B.

$$\begin{aligned}
y &= Ae^{-x} + Be^{-2x} \\
&= Ay_1 + By_2 \\
A &= -\int \frac{Ry_2}{w} dx \\
&= -\int \frac{(x+\cos x)e^{-2x}}{-e^{-3x}} dx \\
&= \int e^{x}(x+\cos x)dx \\
&= \int x \cdot e^{x} dx + \int e^{x} \cos x dx \\
&= xe^{x} - \int e^{x} dx + \frac{1}{2} e^{x}(\cos x + \sin x) + c_1 \\
&= xe^{x} - e^{x} + \frac{e^{x}}{2}(\cos x + \sin x) + c_1 \\
\because \int e^{ax} \sin axdx &= \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx), \text{ and} \\
\int e^{ax} \cos bxdx &= \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) \\
\text{Also, } B &= \int \frac{y_1 R}{w} dx \\
&= \int \frac{e^{-x}(x+\cos x)}{-e^{-3x}} dx \\
&= \left[\int e^{-2x} xdx + \int e^{-2x} \cos xdx \right] \\
&= -\left[x \cdot \frac{e^{-2x}}{-2} - \int 1 \frac{e^{-2x}}{-2} dx + \frac{e^{-2x}}{4+1}(-2\cos x + \sin x) \right] \\
&= \frac{x}{2} e^{-2x} + \frac{1}{4} e^{-2x} + \frac{e^{-2x}}{5}(-2\cos x + \sin x) + C_2
\end{aligned}$$

Hence, complete solution is given by

$$y = Ay_1 + By_2$$

$$y = \left[xe^{x} - e^{x} + \tfrac{e^{x}}{2}(\cos x + \sin x) + c_1 \right] e^{-x} + \left[\tfrac{x}{2} e^{-2x} + \tfrac{e^{-2x}}{4} + \tfrac{e^{-2x}}{5}(-2\sin x + \sin x) + C_2 \right] e^{-2x}$$

Question-6(d) For the vector $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\vec{A} = \nabla\phi$

[10 Marks]

Solution: $\vec{A}$ is irrotational if

$$\nabla \times \vec{A} = 0$$

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix} \quad (r^2 = x^2 + y^2 + z^2)$$

$$= i\left(\frac{-2z}{r^3} \cdot 2y + \frac{2y}{r^3}2z\right) + j\left(\frac{-2x}{r^3}2z + \frac{2}{x^3}2x\right) + k\left(\frac{-2y}{r^3}2x + \frac{2x}{r^3}2y\right)$$

$$= 0$$

$\therefore \vec{A}$ is irrotational vector.

$$\vec{A} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial phi}{\partial x} = \frac{x}{x^2 + y^2 + 2^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$$

$$\frac{\partial\phi}{\partial z} = \frac{2}{x^2 + y^2 + 2^2}$$

$$\Rightarrow \phi = \frac{1}{2}\log\left(x^2 + y^2 + z^2\right) + C$$

$\therefore$ Scalar potential ϕ is such that $\vec{A} = \nabla\phi$

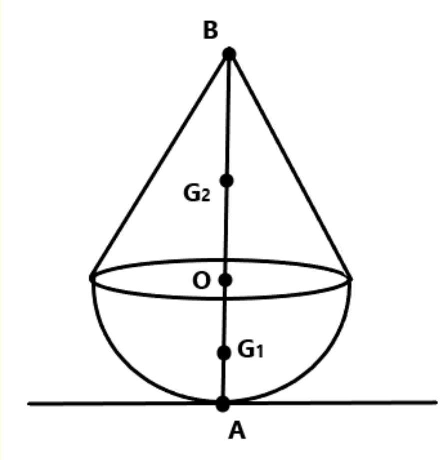
Question-7(a) A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$ radius of hemisphere.

[15 Marks]

Solution: Let us first try to find out the $C \cdot G \cdot$ of the whole body.
As we know, CG of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its base, where $'a'$ is radius of sphere

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

.



$$\begin{aligned}\omega_1 &= \text{ weight of hemisphere}\\ &= \frac{2}{3}\pi a^3 \rho q\\ x_2 &= \text{ distance of centre of gravity of cone from table}\\ &= AO + OG_2\\ &= a + \frac{H}{4}\\ \omega_2 &= \text{ weight of cone}\\ &= \frac{1}{3}\pi a^2 \text{Hpg}\end{aligned}$$

$$\begin{aligned}h &= \text{ distance of C.G of combined body from horizontal plane}\\ &= \frac{w_1 x_1 + \omega_2 x_2}{w_1 + w_2}\\ &= \frac{\frac{2}{3}\pi a^3 \text{ pg } \frac{5a}{8} + \frac{1}{3}\pi a^2 \text{ Heg } \left(a + \frac{1}{4}\right)}{\frac{2}{3}\pi a^3 pg + \frac{1}{3} i + a^2 + 1g}\\ &= \frac{\frac{5}{4}a^2 + 1 + \left(a + \frac{11}{4}\right)}{2a + H}\\ &= \frac{5a^2 + 11(4a + 11)}{4(2a + 11)}\end{aligned}$$

Let $R =$ radius of lower surface $= \infty$ & $r =$ radius of upper furface $= a$

For stable equilibrium,

$$\begin{aligned}\frac{1}{h} &> \frac{1}{r} + \frac{1}{R}\\ \frac{4(2a + H)}{5a^2 + H(4a + H)} &> \frac{1}{a} + \frac{1}{\infty}\\ a(8a + 4H) &> 5a^2 + 4aH + H^2\\ 3a^2 &> H^2\\ H &< \sqrt{3}a\end{aligned}$$

Question-7(b) Evaluate $\iint_S \nabla \times \vec{A} \cdot \hat{n} dS$ for $\vec{A} = (x^2 + y - 4)\,\hat{i} + 3xy\hat{j} + (2xz + z^2)\,\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane.

[15 Marks]

Solution: The boundary C of the surface S is the circle $x^2 + y^2 = a^2, z = 0$. Suppose $x = a\cos t,\ y = a\sin t, z = 0, 0 \le t \le 2\pi$ are the parametric equations of C. By Stokes' theorem, we have

$$\int\int_S (\nabla \times \overline{\mathrm{A}}) \cdot \bar{n} ds = \int_C \overline{\mathrm{A}} \cdot \bar{d}r$$

$$= \int_C [\left(x^2 + y - 4\right)\hat{i} + 3xy\hat{j} + \left(2xz + z^2\right)\hat{k}] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C \left(x^2 + y - 4\right) dx + 3xy \cdot dy + \left(2xz + z^2\right) \cdot dz$$

$$= \int_C \left(x^2 + y - 4\right) dx + 3xy \cdot dy \quad [\because z = 0 \therefore dz = 0]$$

$$= \int_0^{2\pi} \left(a^2\cos^2 t + a\sin t - 4\right)(-a\sin t) \cdot dt + 3a\cos t \cdot a\sin t(a\cos t) \cdot dt$$

$$= \int_0^{2\pi} \left[2a^3 \cdot \cos^2 t \cdot \sin t - \frac{a^2}{2}(1 - \cos 2t) + 4a\sin t\right] \cdot dt$$

$$= \frac{-2a^3}{3}\left(\cos^3 t\right)_0^{2x} - \frac{a^2}{2}[t]_0^{2\pi} + \frac{a^2}{4}[\sin 2t]_0^{2\pi} - 4a[\cos t]_0^{2\pi}$$

$$= 0 - a^2 \cdot \pi + 0 - 0$$

$$= -16\pi \quad \left(\because a = 4 \Rightarrow a^2 = 16\right)$$

Question-7(c) Solve the D.E.:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

[10 Marks]

Solution:

$$\left(D^3 - 3D^2 + 4D - 2\right) y = e^x + \cos x$$

Auxiliary eqn:

$$D^3 - 3D^2 + 4D - 2 = 0$$

$$(D - 1)\left(D^2 - 2D + 2\right) = 0$$

$$D = 1, \frac{2 \pm \sqrt{4 - 8}}{2}$$

i.e.

$$D = 1, 1 \pm i$$

$$\begin{aligned} CF &= c_1e^x + e^x\left(c_2^1\cos x + c_3'\sin x\right) \\ &= c_1e^x + c_2e^x\operatorname{css}\left(x + c_3\right) \end{aligned}$$

$$\begin{aligned} P.I. &= \frac{1}{D^3-3D^2+4D-2}\left(e^x+\cos x\right) \\ &= x\cdot\frac{1}{3D^2-6D+4}e^x + \frac{1}{D\left(-1^2\right)-3\left(-1^2\right)+4D-2}\cos x \\ &= \frac{x\cdot e^x}{3-6+4} + \frac{1}{(3D+1)}\cdot\frac{3D-1}{3D-1}\cos x \\ &= x'e^x + \frac{1}{9D^2-1}\left((3D-1)-(\cos x)\right. \\ &= xe^x + \frac{1}{9\left(-1^2\right)-1}(-3\sin x - \cos x) \\ &= xe^x + \frac{1}{10}(3\sin x + \cos x) \end{aligned}$$

$\therefore$ Complete solution, $y = CF + PI$

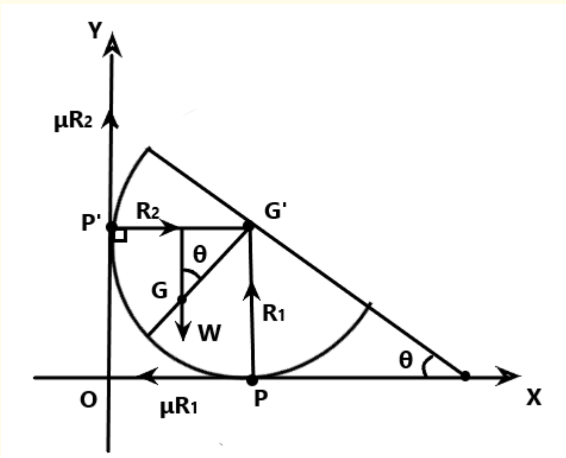
$$y = ge^x + c_2e^x\cos\left(x+c_3\right) + xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Question-8(a) A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coefficient of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$$\sin^{-1}\left(\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}\right)$$

[10 Marks]

Solution: Let the disc's diameter makes angle θ with the x-axis (horizontal)
At equilibrium ie. before motion



$$\sum F_x = 0$$

$$\Rightarrow R_2 - \mu R_1 = 0 \quad \ldots \quad (1)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \mu R_2 + R_1 - W = 0 \quad \ldots \quad (2))$$

Taking moments about G'

$$(\mu R_2)\, r + (\mu R_1)\, r - W\,(GG' \sin\theta) = 0$$

Where r is radius of disc.

$$GG' = \frac{4}{3\pi} r \text{ [A result from chapter on center of gravity]}$$

$$\mu r\,(R_1 + R_2) = w\frac{4r}{3\pi}\sin\theta \quad \ldots \quad (3)$$

Using (1) and (2) in (3), we get

$$R_2 = \mu R_1$$

$$\mu R_2 + R_1 - W = 0$$

$$\Rightarrow \mu^2 R_1 + R_1 - W = 0$$

$$R_1 = \frac{W}{1+\mu^2}$$

$$\therefore \mu\left(\frac{W}{1+\mu^2} + \frac{\mu W}{1+\mu^2}\right) = \frac{4W\sin\theta}{3\pi}$$

$$\frac{\mu}{1+\mu^2}(1+\mu) = \frac{4}{3\pi}\sin\theta$$

$$\Rightarrow \theta = \sin^{-1}\left[\frac{3\pi}{4}\cdot\left(\frac{\mu+\mu^2}{1+\mu^2}\right)\right]$$

Hence, proved.

Question-8(b) A body floating in water has volumes V_1, V_2 and V_3 above the surface when the densities of the surrounding air are ρ_1, ρ_2, ρ_3 respectively. Prove that:

$$\frac{\rho_2 - \rho_3}{\mathrm{V}_1} + \frac{\rho_3 - \rho_1}{\mathrm{V}_2} + \frac{\rho_1 - \rho_2}{\mathrm{V}_3} = 0$$

[10 Marks]

Solution: Let V be the volume and W the weight of the body. Then the volumes immersed in water in the three faces are

$$(V - V_1)\,, (V - V_2) \text{ and } (V - V_3)$$

Let ρ be the density of water.
For equilibrium,
weight of the body = weight of water displaced + weight of air displaced

$$\therefore \quad W = (V - V_1)\,\rho g + V_1\rho_1 g \text{ or } W - V\rho g = V_1 g\,(\rho_1 - \rho)$$

$$\frac{W - V\rho g}{V_1} = g\left(\rho_1 - \rho\right) \qquad \ldots(1)$$

Similarly,

$$\frac{W - V\rho g}{V_2} = g\left(\rho_2 - \rho\right) \qquad \ldots(2)$$

and

$$\frac{W - V\rho g}{V_3} = g\left(\rho_3 - \rho\right) \qquad \ldots(3)$$

Multiplying (1) by $(\rho_2 = \rho_3)$, (2) by $(\rho_3 - \rho_1)$ and (3) by $(\rho_1 - \rho_2)$ and adding, we get

$$(W - V\rho g)\left[\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3}\right] = 0$$
$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0$$

Hence, proved.
Note that the above result can also be put in the form

$$V_2V_3\left(\rho_2 - \rho_3\right) + V_3V_1\left(\rho_3 - \rho_1\right) + V_1V_2\left(\rho_1 - \rho_2\right) = 0$$
$$\rho_1V_1\left(V_2 - V_3\right) + \rho_2V_2\left(V_3 - V_1\right) + \rho_3V_3\left(V_1 - V_2\right) = 0$$

Question-8(c) Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

[10 Marks]

Solution: The divergence theorem is

$$\iiint\limits_V \nabla \cdot \bar{A}dv = \iint\limits_S \mathrm{A} \cdot \hat{n}ids$$

Now, volume integral

$$= \iiint \left\{\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial x}\hat{j} + \frac{\partial}{\partial x}\hat{k}\right\} \cdot \left\{4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right\} dxdydz$$
$$= \iiint\limits_V \left(\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)\right) dxdydz$$
$$= \int_{x=-2}^{2}\int_{-y_1}^{y_1}\int_{z=0}^{3}(4 - 4y + 2z)dxdydz$$
$$= \int_{r=-2}^{2}\int_{z=0}^{3}\int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}}(4 - 4y + 2z)dxdzdy$$
$$= \int_{x=-2}^{2}\int_{z=0}^{3}\left[4y - 2y^2 + 2zy\right]_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dxdy$$
$$= 2\int_{-2}^{2}\int_{0}^{3}\left[(4 + 2z)y\right]_0^{\sqrt{(4-x^2)}} dxdz$$

$$= 2\int_{-2}^{2}\int_{0}^{3}\left\{(4+2z)\sqrt{(4-x^2)}\right\}dxdz$$
$$= 4\int_{0}^{2}\int_{0}^{2}(4+2z)\sqrt{(4-x^2)}dxdz$$
$$= 4\left[\left\{4z+z^2\right\}_0^3\left\{\frac{x}{2}\sqrt{(4-x^2)}+2\sin^{-1}\left(\frac{x}{2}\right)\right\}_0^2\right]$$
$$= 4[(12+9)(\pi)]$$
$$= 84\pi$$

Now, we proceed to find the surface integral.
The surface S of the cylinder consists of a base $S_1(z=0)$, the top $S_2(z=3)$ and the convex portion

$$S_3\left(x^2+y^2=4\right)$$

For S_1 : Normal is towards $-\hat{k}$ direction and $z=0$

$$\therefore \iint_{S_1}(\vec{A}\cdot\vec{n})dS = \iint_{S_1}\left(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\right)\cdot(-\hat{k})dS$$
$$= \iint_{S_1} -z^2 dS = 0$$

For S_2 : Normal is towards $\hat{k}$ direction and $z=3$

$$\therefore \iint_{S_2}(4x\hat{i}-2y^2\hat{j}+9\hat{k})\cdot(\hat{k})ds = \iint_{S_2} 9dS$$
$$= 9(2\pi r^2) \qquad [\because \text{area of } S_2 = 2\pi r^2 = 4\pi]$$
$$= 36\pi$$

For S_3 : Vector normal to S_3 i.e., $x^2+y^2=4$

$$\therefore \hat{n} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \frac{2x\hat{i}+2y\hat{j}}{4} = \frac{x\hat{i}+y\hat{j}}{2} \qquad [\because x^2+y^2=4 \quad \text{on } S_3]$$

$$\therefore \text{ On } S_3, (\vec{A}\cdot\vec{n}) = (4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$
$$= 2x^2-y^3$$

Also, dS = elementary area on the surface S_3

$$= 2d\theta dz \qquad \ldots[\text{Polar Coordinates } dS = rd\theta dz \text{ and } r=z]$$

$$\therefore \iint\limits_{S_3} (\vec{A}.\vec{n})dS = \iint\limits_{S_3} (2x^2 - y^3)2d\theta dz \qquad \ldots [x = 2\cos\theta, y = 2\sin\theta]$$

$$= \int_{F=0}^{F=3} \int_{\theta=0}^{2\pi} 2(B\cos^2\theta - 8\sin^3\theta)d\theta dF$$

$$= 16 \int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)[F]_0^3 d\theta$$

$$= 16.3 \int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)d\theta$$

$$= 48 \left[\int_{\theta=0}^{2\pi} \cos^2\theta d\theta - \int_{\theta=0}^{2\pi} \sin^2\theta d\theta \right]$$

$$= 48 \left[\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right)_0^{2\pi} - 0 \right] \qquad \ldots [\sin\theta \text{ is a odd function}]$$

$$= 48.\frac{2\pi}{2} = 48\pi$$

$$\therefore \quad \iint\limits_{S} \vec{A}\cdot\vec{n}dS = \left(\iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} \right) (\vec{A}\cdot\vec{n})dS$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$$\therefore \iint\limits_{S} (\vec{A}\cdot\vec{n})dS = \iiint\limits_{v} (\nabla\cdot\vec{A})dv$$

Hence, the divergence theorem is proved.

Chapter 8

2013

8.1 Section-A

Question-1(a) Find the dimension and a basis of the solution space W of the system $x + 2y + 2z - s + 3t = 0$, $x + 2y + 3z + s + t = 0$, $3x + 6y + 8z + s + 5t = 0$.

[8 Marks]

Solution: The matrix form of the given homogeneous system of linear equations is

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & +1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2, \quad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -5 & 7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the required row reduced echelon form.

$$x + 2y - 5s + 7t = 0$$
$$z + 2s - 2t = 0$$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2y + 5s - 7t \\ y \\ -2s + 2t \\ s \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Dimension of Solution Space $(W) = 3$.
Basis of Solution Space $= \{(-2,1,0,0,0), (5,0,-2,1,0), (-7,0,2,0,1)\}$.

Question-1(b) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ **and hence find the matrix represented by:**

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

[8 Marks]

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Characterstic Equation of a square matrix is given by :$\mid A - \lambda I \mid = 0$ i.e.

$$\lambda^3 - (\text{ trace of A})\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| = 0$$
$$\text{trace}(A) = 2 + 1 + 2 = 5$$
$$C_{11} + C_{22} + C_{33} = (2-0) + (4-1) + (2-0)$$
$$= 7$$
$$|A| = 2(2-0) + 0 + 1(0-1) = 3$$

$\therefore$ Characteristic Equation: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation.

$$\therefore A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(*)$$

We have to find,

$$\begin{aligned} &A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ &= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + \left(A^4 - 5A^3 + 7A^2 - 3A\right) \\ &+ A^2 + A + I \\ &= A^5 \cdot 0 + A \cdot 0 + A^2 + A + I \quad (\text{using}(*)) \\ &= A^2 + A + I \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} \end{aligned}$$

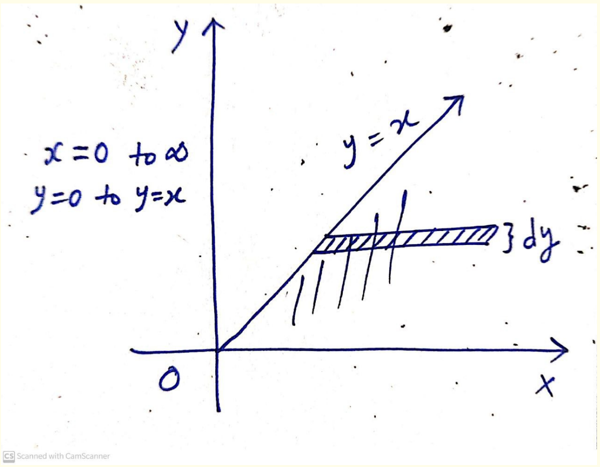
Question-1(c) Evaluate the integral $\int_0^\infty \int_0^x xe^{-x^2/y} dydx$ by changing the order of integration.

[8 Marks]

Solution: Let

$$1 = \int_0^\infty \int_0^x xe^{-x^2/y} dydx$$

Here the limits of integration show that the integration is done first with respect to y from $y = 0$ to $y = x$ and then with respect to x from $x = 0$ and $x = \infty$, i. *e.*, the strip is taken parallel to y -axis in the region bounded by these curves.



On changing the order of integration, we find that the strip parallel to x -axis varies form $x = y$ to $x = \infty$ and then y varies from $y = 0$ to $y = \infty$ to cover the whole region (fig.) Hence on changing the order of integration, we have figure

$$\begin{aligned}
I &= \int_0^\infty \int_{x=y}^\infty xe^{-x^2/y} dxdy \\
&= \int_0^\infty \left[-\frac{y}{2} e^{-x^2/y}\right]_{x=y}^\infty dy \\
&= \frac{1}{2} \int_0^\infty ye^{-y} dy \\
&= \frac{1}{2} \left([y(-e^{-y})]_0^\infty - \int_0^\infty 1 \cdot (-e^{-y}) dy \right) \\
&= \frac{1}{2} \left[\lim_{y\to\infty} \frac{-y}{e^y} - 0 \right] - \frac{1}{2} \left[e^{-y}\right]_0^\infty \qquad \left(\frac{0}{0}\text{form}\right) \\
&= \frac{1}{2} \left[\lim_{y\to\infty} \frac{-1}{e^y} \right] - \frac{1}{2} [0 - 1] \qquad \text{(Using L-Hospital)} \\
&= \frac{1}{2}
\end{aligned}$$

Question-1(d) Find the surface generated by the straight line which intersects the lines $y = z = a$ and $x + 3z = a = y + z$ and is parallel to the plane $x + y = 0$.

[8 Marks]

Solution: The equation of the given lines are

$$y - a = 0, z - a = 0 \qquad \ldots (i)$$

$$x + 3z - a = 0, y + z - a = 0 \qquad \ldots (ii)$$

The equation of any plane through the lines (i) and (ii) are

$$(y - a) - \lambda_1(z - a) = 0$$

$$\Rightarrow y - \lambda_1 z - a + a\lambda_1 = 0 \qquad \ldots (iii)$$

and

$$(x + 3z - a) - \lambda_2(y + z - a) = 0$$

$$(x - \lambda_2 y) + (3 - \lambda_2)\, z - a + a\lambda_2 = 0 \ldots (iv)$$

Any line intersecting the line (i) and (ii) is given by the intersection of the plane (iii) and (iv).
Let λ, μ, v are its dr's, then,

$$0.\lambda + 1.\mu - \lambda_1 \cdot v = 0$$

and

$$1.\lambda - \lambda_2 \cdot \mu + (3 - \lambda_2) \cdot v = 0$$

$$\therefore \frac{\lambda}{3 - \lambda_2 - \lambda_1\lambda_2} = \frac{\mu}{-\lambda_1} = \frac{v}{-1}$$

Now, the line with dr's λ, μ, v is parallel to the plane $x + y = 0$, i.e., this line is perpendicular to the normal to the plane $x + y = 0$, whose dr's are $1, 1, 0$ So, we have

$$1.\, (3 - \lambda_2 - \lambda_1\lambda_2) + 1\, (-\lambda_1) + 0.(-1) = 0$$

$$3 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2 = 0$$

The required locus of the line is obtained by eliminating λ_1 and λ_2 between $(iii), (iv)$ and (v) hence is given by

$$3 - \frac{y - a}{z - a} - \frac{x + 3z - a}{y + z - a} - \frac{y - a}{z - a} \cdot \frac{x + 3z - a}{y + z - a} = 0$$

$$3(y + z - a)(z - a) - (y - a)(y + z - a) - (z - a)(x + 3z - a) - (y - a)(x + 3z - a) = 0$$

$$-yz - y^2 + 2az - xz + 2ax - xy = 0$$

$$yz - y^2 + 2az - xz + 2ax - xy = 0$$

$$yz + y^2 + xz + xy = 2az + 2ax$$

$$(y + z)(x + y) = 2a(x + z)$$

Question-1(e) Find C of the Mean value theorem, if $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$ and C has usual meaning.

[8 Marks]

Solution:

$$f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

$$\therefore \quad f(a) = f(0) = 0$$

and

$$\begin{aligned} f(b) &= f\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right) \\ &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \\ &= \frac{3}{8} \end{aligned}$$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{4}$$

Also

$$f'(x) = 3x^2 - 6x + 2$$

so that

$$f'(c) = 3c^2 - 6c + 2$$

Substituting these values for Lagrange's mean value theorem,

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= f'(c), (a < c < b) \\ \frac{3}{4} &= 3c^2 - 6c + 2 \end{aligned}$$

$$12c^2 - 24c + 5 = 0$$

$$\begin{aligned} c &= \frac{24 \pm \sqrt{(24)^2 - 4 \cdot 12 \cdot 5}}{2 \times 12} \\ &= \frac{24 \pm \sqrt{576 - 240}}{24} \\ &= \frac{24 \pm 4\sqrt{21}}{24} \\ &= 1 \pm \frac{\sqrt{21}}{6} \\ c &= 1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right) \quad \text{Using Calculator} \end{aligned}$$

Question-2(a) Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $\mathrm{M} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ **Let** $\mathrm{F} : \mathrm{V} \to \mathrm{V}$ **be the linear map defined by** $\mathrm{F(A)} = \mathrm{MA}$**. Find a basis and the dimension of (i) the kernel of W of** F **(ii) the image U of F.**

[10 Marks]

Solution:

$$T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= \begin{bmatrix} x-z & y-w \\ -2x+2z & -2y+2w \end{bmatrix}$$

$$= (x-z)\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} + (y-\omega)\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= k_1\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad k_1, k_2 \in \mathbb{R}$$

$\therefore$ Range (T)

$$w = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\right\}$$

Dimension $(w) = 2$

($\because$ two vectors $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ are not multiples of each other), hence independent.

For kernel $T(A) = 0$, i.e.

$$T\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x-z & y-w \\ -2x+2z & -2y+2w \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x - z = 0 - 2x + 2z = 0$$

$$y - \omega = 0$$

$$-2y + 2w = 0$$

$$\text{i.e. } x = z \quad \textit{and} \quad y = \omega$$

$$\therefore \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ x & y \end{bmatrix}$$

$$= x\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Since vectors $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are not multiples of each other, hence they are independent therefore they form the basis of kernel (T). Dim (ker T) $= 2$.

Question-2(b) Locate the stationary points of the function $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature.

[10 Marks]

Solution: We have

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y \qquad \ldots(2)$$

For extreme values, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$$x^3 - x + y = 0$$

$$\therefore y^3 + x - y = 0$$

Adding (1) and (2), we have

$$x^3 + y^3 = 0$$
$$(x+y)\left(x^2 - xy + y^2\right) = 0$$

$\therefore$ For real $x, x + y = 0$ is the only possibility. Putting $y = -x$ in (1), we get

$$x^3 - x - x = 0$$
$$x^3 - 2x = 0$$
$$x\left(x^2 - 2\right) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Hence, the extreme points are $(0,0), (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

$$\text{A} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$\text{B} = \frac{\partial^2 f}{\partial y \partial x} = 4$$

and

$$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

At $(0,0)$: $\quad A = -4, \quad B = 4, \quad C = -4$

$$\therefore \quad AC - B^2 = 16 - 16 = 0$$

$\therefore$ At$(0,0)$, further investigation is required. For small h, k and $h \neq k$, we have

$$f(h,k) - f(0,0) = h^4 + k^4 - 2h^2 + 4hk - 2k^2$$
$$= -2(h-k)^2 < 0 \quad \left[\text{ Neglecting } h^4, k^4 ash, k \text{ are small} \right.$$

For $h = k$, we have

$$f(h,k) - f(0,0) = h^4 + h^4 - 2h^2 + 4h^2 - 2h^2$$
$$= 2h^4 > 0$$

As $f(h,k) - f(0,0)$ does not keep the same sign for all small values of h and k, so the point (0,0) is a saddle point.

$$At(\sqrt{2}, -\sqrt{2}) : A = 20, \quad B = 4, \quad C = 20$$

$$\therefore \mathrm{AC} - \mathrm{B}^2 > 0 \text{ and } \mathrm{A} > 0$$

$\Rightarrow f$ has a minimum at $(\sqrt{2}, -\sqrt{2})$

$$\begin{aligned}\text{Minimum value } &= f(\sqrt{2}, -\sqrt{2}) \\ &= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 \\ &= 4 + 4 - 4 - 8 - 4 = -8\end{aligned}$$

At $(-\sqrt{2}, \sqrt{2})$: $\quad A = 20, B = 4, C = 20$
$\therefore C - B^2 = 400 - 16 = 384 > 0$ and $A = 20 > 0$

$\therefore \quad f(x,y)$ has a minimum at $(-\sqrt{2}, \sqrt{2})$ Minimum value $= f(-\sqrt{2}, \sqrt{2}) = -8$

Question-2(c) Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form

$$q(x,y) = 2x^2 - 4xy + 5y^2$$

[10 Marks]

Solution:

$$q(x,y) = 2x^2 - 2xy - 2yx + 5y^2$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

First we diagonalize this matrix by finding eigenvectors.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(\lambda - 2)(x - 5) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda = 1, 6$$

For $\lambda = 1$:

$$\Rightarrow \quad \begin{bmatrix} 2-1 & -2 \\ -2 & 5-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - 2y = 0.$$

$$x = 2y$$

$\therefore$ Eigenvector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix} = y\begin{bmatrix} 2/\sqrt{5} \\ 1/5 \end{bmatrix}$$

For $\lambda = 6$:

$$\begin{bmatrix} 2-6 & -2 \\ -2 & 5-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-2x - y = 0 \quad \Rightarrow \quad y = -2x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Hence diagonalizing matrix is

$$M = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

such that

$$M^{-1}AM = D$$

Orthogonal transformation is

$$x = \frac{2}{\sqrt{5}}u + \frac{1}{\sqrt{5}}v$$
$$y = \frac{1}{\sqrt{5}}u - \frac{2}{\sqrt{5}}v$$

Question-2(d) Discuss the consistency and the solutions of the equations

$$x + ay + az = 1, ax + y + 2az = -4, ax - ay + 4z = 2$$

for different values of a.

[10 Marks]

Solution: Matrix eqn. $Ax = B$, therefore,

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & 2a \\ a & -a & 4 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{aligned}\det(A) &= 1\left(4 + 2a^2\right) - a\left(4a - 2a^2\right) + a\left(-a^2 - a\right) \\ &= 4 + 2a^2 - 4a^2 + 2a^3 - a^3 - a^2 \\ &= a^3 - 3a^2 + 4 \\ &= (a+1)(a-2)^2\end{aligned}$$

Case 1: When $a \neq -1$ and $a \neq 2$

$$\mid A \mid \neq 0 \quad \Rightarrow A^{-1}\text{exist.}$$

Hence, system has unique solution.

Case 2: When $a = -1$

$$[A:B] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -2 & -4 \\ -1 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x - y = 2, \quad z = 1$$

$$\begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} y+2 \\ y \\ 1 \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence system has infinitly many solutions.

Case 3: When $a = 2$.

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & -4 \\ 2 & -2 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & -3 & 0 & 6 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_4 \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

The $\text{Rank}(A) = 2$ & $\text{Rank}(A \cdot B) = 3$

Both are not equal, hence system is inconsistent for $a = 2$.

Question-3(a) Prove that if $a_0, a_1, a_2, \ldots\ldots, a_n$ are the real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0$$

then there exists at least one real number x between 0 and 1 such that

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + f_{n-1} x + a_n = 0$$

[10 Marks]

Solution: Consider the function

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + a_2 \frac{x^{n-1}}{n-1} + \ldots a_{n-1} \frac{x^2}{2} + a_n x$$

over the interval $[0, 1]$.

$$\begin{aligned} f(0) &= 0; \\ f(1) &= \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n \\ &= 0 \quad \text{(given)} \end{aligned}$$

Being a polynomial function, $f(x)$ is continuous and differentiate over interval [0,1]. Hence, Using Rolle's theorem, there exists $C \in (0, 1)$ such that

$$f'(c) = 0$$

$$\text{or} \quad a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + \ldots + a_{n-1} c + a_n = 0$$

Hence, Proved

Question-3(b) Reduce the following equation to its canonical form and determine the nature of the conic $4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$

[10 Marks]

Solution:

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

General equation of second degree:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

here

$$a = 4, b = 1, c = 5, g = -6, f = -3, h = 2$$

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2\\ &= 20 + 72 - 36 - 36 - 20\\ &= 0\end{aligned}$$

and

$$ab - h^2$$

$$4 \times 1 - (2)^2 = 0$$

Hence, given equation will represent pair of parallel straight lines.

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

$$(2x + y)^2 - 6(2x + y) + 5 = 0$$

$$(2x + y - 5)(2x + y - 1) = 0$$

$$2x + y - 5 = 0$$

and

$$2x + y - 1 = 0$$

Question-3(c) Let F be a subfield of complex numbers and T a function from $\mathrm{F}^3 \rightarrow \mathrm{F}^3$ defined by $\mathrm{T}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = (\mathrm{x}_1 + \mathrm{x}_2 + 3\mathrm{x}_3, 2\mathrm{x}_1 - \mathrm{x}_2, -3\mathrm{x}_1 + \mathrm{x}_2 - \mathrm{x}_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T? Find the nullity of T.

[10 Marks]

Solution: $N_A(T) = \{(x_1, x_2, x_3) \in F \mid T(x_1, x_2, x_1) = (0, 0, 0)\}$ Let $(a, b, c) \in N_A(T)$. Then, $T(a, b, c) = (0, 0, 0)$. ie. $(a + b + 3c, 2a - b, -3a + b - c) = (0, 0, 0)$

$$\begin{array}{lll} \Rightarrow a + b + 3c = 0, & 2a - b = 0, & -3a + b - c = 0 \\ \quad\downarrow & 2a = b \rightarrow & -3a + 2a - c = 0 \\ & & \Rightarrow c = -a. \\ a + b + 3c = 0 & \Rightarrow a + 2a - 3a = 0 & \text{hence it satisfies the values formed} \end{array}$$

$\therefore$ The required conditions are $b = 2a, c = -a$.
ie. $N_A(T) = \{(a, 2a, -a)/a \in \mathbb{F}\}$.
Clearly, the basis of $N_A(T) = \{(1, 2, -1)\}$.
$\therefore$ Nullity $(T) = 1$.

Question-3(d) Find the equations to the tangent planes to the surface $7x^2 - 3y^2 - z^2 + 21 = 0$, which pass through the line $7x - 6y + 9 = 0$, $z = 3$.

[10 Marks]

Solution: Eqn of a plane passing through given line

$$7x - 6y + 9 + \lambda(z-3) = 0$$

$$7x - 6y + \lambda z + (9 - 3\lambda) = 0$$

Equation of tangent plane to given surface at a point (α, β, γ), lying on surface is

$$7\alpha x - 3\beta y - \gamma z + 21 = 0 \quad - \quad (2)$$

then

$$\frac{7\alpha}{7} = \frac{-3\beta}{-6} = \frac{-\gamma}{\lambda} = \frac{+21}{9-3\lambda}$$

(α, β, γ) lies on given surface

$$\therefore 7\left(\frac{1}{3-\lambda}\right)^2 - 3\left(\frac{14}{3-\lambda}\right)^2 - \left(\frac{-7\lambda}{3-\lambda}\right)^2 + 21 = 0$$

$$2\lambda^2 + 9\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda = -4, \frac{-1}{2}$$

Hence, equation of tangent planes are

$$7x - 6y - 4z + 21 = 0$$
$$14x - 12y - z + 21 = 0$$

Question-4(a) Evaluate

$$\int_0^{\pi/2} \frac{x \sin x \cos x dx}{\sin^4 x + \cos^4 x}$$

[10 Marks]

Solution: Using the formula

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^{\pi/2} \frac{\pi/2 \cdot \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} - \int_0^{\pi/2} \frac{x \cdot \sin x \cos x}{\sin^4 x + \cos^4 x} dx (= I)$$

$$\therefore 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx$$

$$I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x}{1 + \tan^4 x} dx \text{ (dividing by } \cos^4 x \text{ in numerator and denominator.)}$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

$$\begin{aligned} I &= \frac{\pi}{4} \times \frac{1}{2} \int \frac{dt}{1+t^2} \\ &= \frac{\pi}{8} \tan^{-1} t \Big|_0^{\infty} \\ &= \frac{\pi}{8} \left(\frac{\pi}{2} - 0 \right) \\ &= \boxed{\frac{\pi^2}{16}} \end{aligned}$$

Question-4(b) Let $H = \begin{bmatrix} 1 & \mathbf{i} & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ **be a Hermitian matrix. Find a non-singular matrix** P **such that** $\mathrm{P^t H \overline{P}}$ **is diagonal and also find its signature.**

[10 Marks]

Solution: Let $H = IHI$

$$\begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} H \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & \\ 0 & 0 & 1 \end{bmatrix}$$

Row-operations applied on pre-factor and column operations on post-factor on R H S.

$$R_2 \to R_2 + iR_1, \quad R_3 \to R_3 + (-2+i)R_y$$

$$C_2 \to C_2 - iC_1, \quad C_3 \to C_3 - (2+i)C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ -2+1 & 0 & 1 \end{bmatrix} H \begin{bmatrix} 1 & -i & -2-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + iR_2, \quad C_3 \to C_3 - iC_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ -3+i & i & 1 \end{bmatrix} \cdot H \begin{bmatrix} 1 & -i & -3-i \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T H \bar{P} = D$$

$$\Rightarrow P = \begin{bmatrix} 1 & i & -3+i \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

Rank$(H) = 3$

Index $(H) = 2$ (Positive diagonal entries)

Signature $(H) =$ No. of positive diagonal entries - No. of the negative diagonal entries $= 2 - 1 = 1$.

Question-4(c) Find the magnitude and the equations of the line of shortest distance between the lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$$

and

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

[10 Marks]

Solution: Two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Here

$$\begin{vmatrix} 15-8 & 29-(-9) & 5-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}=\begin{vmatrix} 7 & 38 & -5 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$
$$=1176\neq 0$$

Hence given two lines are not coplanar and therefore, not intersecting.

Let $A(3a+8,-16a-9,7a+10)$ and $B(3b+15,8b+29,-5b+5)$ be two general points on the given lines.
Also, let $P(8,-9,10),Q(15,29,5)$ are two given points on the given lines.

$$\therefore \text{D.r of } AB=\langle 3a-3b-7,-16a-8b-38,7a+5b+5\rangle$$

If AB is line of shortest distance, it will be perpendicular to both the lines.

$$\therefore \quad 3(3a-3b-7)-16(-16a-8b-38)+7(7a+5b+5)=0$$
$$157a+77b+311=0$$

&

$$3(3a-3b-7)+8(-16a-8b-38)-5(7a+5b+5)=0$$
$$154a+98b+350=0.$$

Solving, we get $a=-1,b=-2$

$$\therefore A(-3+8,16-9,-7+10) \text{ i.e. } (5,7,3)$$
$$B(-6+15,-16+29,10+5) \text{ i.e. } (9,13,15)$$
$$\begin{aligned}(AB)&=\sqrt{(9-5)^2+(13-7)^2+(15-3)^2}\\&=\sqrt{16+36+144}\\&=\sqrt{196}\\&=14\end{aligned}$$

eqn of AB,

$$\frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12}$$

i.e.

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

Question-4(d) Find all the asymptotes of the curve

$$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$$

[10 Marks]

Solution: As coefficients of highest power of x and y are constants, hence the given curve has no asymptotes parallel to x-axis or y-axis.

So, we will find only the oblique asymptotes .
Let eqn of asymptote: $y = mx + c$.

$$\phi_4 = x^4 - y^4 \qquad \phi_3 = 3x^2y + 3xy^2$$

Putting $x = 1, \quad y = m$

$$\phi_4(m) = 1 - m^4$$

$$\phi_4(m) = 0 \quad \Rightarrow \quad m = 1, -1$$

Also,

$$\begin{aligned} c &= \frac{-\phi_3(m)}{\phi_4'(m)} \\ &= \frac{-3(m)(1+m)}{-4m^3} \\ &= \frac{3(1+m)}{4m^2} \end{aligned}$$

For, $m = 1 \Rightarrow c = \dfrac{3}{2}$

For $m = -1 \Rightarrow c = 0$

Hence, equations of asymptotes are $y = x + 3/2 \quad \& \quad y = -x$

8.2 Section-B

Question-5(a) Solve:

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

[8 Marks]

Solution:

$$\frac{dy}{dx} + x\cdot\sin 2y = x^3\cdot\cos^2 y$$

Dividing both sides by $\cos^2 y$, we have

$$\sec^2 y\cdot\frac{dy}{dx} + \tan y\cdot(2x) = x^3$$

Let $\tan y = t$ then

$$\sec^2 y\cdot\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \quad \frac{dt}{dx} + 2x\cdot t = x^3$$

$$P = 2x, \quad Q = x^3$$

$$\text{I.F. } \equiv e^{\int p\cdot dx}$$

$$= e^{\int 2x\cdot dx}$$

$$= \frac{e^{2x}}{2}$$

$\therefore$ Solution of the differential equation is given as

$$t\ .(\text{I.F.}) \ = \int \text{Q}\ .(\text{I.F.})\ dx + c$$

where c is integration constant

$$t\cdot\frac{e^{2x}}{2} = \int x^3\cdot\frac{e^{2x}}{2}\cdot dx + c$$

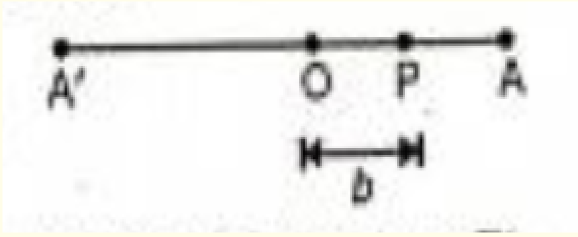
$$- = \frac{e^{2x}}{4}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c$$

$$2\tan y e^{2x} = e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c.$$

Question-5(b) A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where $OP = (b$ with velocity v in the direction of OP. Find the time which elapses before it returns to P.

[8 Marks]

Solution: We have to find time taken from P to A and d then A to P.



$$t = 2(\text{ time from } A \text{ to } P)$$
$$= 2\int_0^{\prime} dt$$
$$= 2\int_a^p \frac{dx}{\sqrt{u}\sqrt{a^2-x^2}}$$
$$(\text{ Ignoring -ve sign }) \left(\frac{dx}{dt} = \sqrt{u}\sqrt{a^2-x^2}\right)$$
$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$
$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}\frac{a}{b}\right]$$
$$= \frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$
$$\Rightarrow t = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)$$
$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$
$$= \frac{2}{\frac{2\pi}{\mathrm{T}}}\tan^{-1}\left[\frac{v}{b\left(\frac{2\pi}{\mathrm{T}}\right)}\right]$$
$$= \frac{\mathrm{T}}{\pi}\tan^{-1}\left[\frac{vT}{2\pi b}\right]$$

$$v^2 = \mu\left(a^2-b^2\right)$$
$$\Rightarrow \quad v = \sqrt{a}\sqrt{(a^2-b^2)}$$
$$\Rightarrow \quad \frac{v}{\sqrt{\mu}} = \sqrt{a^2-b^2}$$
$$\mathrm{T} = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \frac{2\pi}{\mathrm{T}}$$

Proved.

Question-5(c) $\overrightarrow{F}$ **being a vector, prove that curl curl** $\overrightarrow{F} = grad\, \text{div}\, \overrightarrow{F} - \nabla^2 \overrightarrow{F}$ **where** $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

[8 Marks]

Solution: Proof

$$\text{Let} \quad \mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}.$$

$$\text{Then} \quad \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore \nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \partial A_2 - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^3 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1^2}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

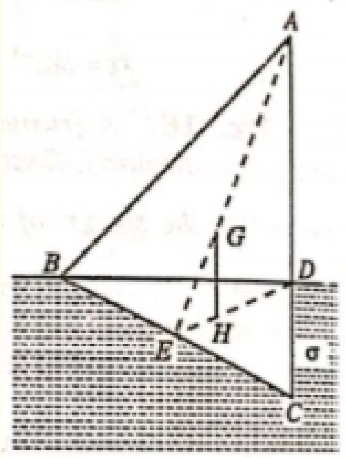
$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot \mathbf{A}) - (\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot \mathbf{A})\right\}\mathbf{i}\right] - \nabla^2 \Sigma A_1 \mathbf{i} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

Question-5(d) A triangular lamina ABC **of density** ρ **floats in a liquid of density** σ **with its plane vertical, the angle** B **being in the surface of the liquid, and the angle** A **not immersed. Find** p/σ **in terms of the lengths of the sides of the triangle.**

[8 Marks]

Solution: The portion BCD of the ΔABC is immersed in the liquid with BD in contact with the surface Let G and H be the centres of gravity and buoyancy respectively. E is the mid-point of BC The conditions of equilibrium are :
(i) The line GH must be vertical.
(ii) The weight of the lamina must be equal to the weight of the liquid displaced.



Since $EG = \frac{1}{3}EA, EH = \frac{1}{3}ED, GH$ is parallel to AD. But GH is vertical from the first condition so AC must be vertical.

From the second condition of equilibrium, we have

$$\Delta ABC \rho g = \Delta BDC \sigma g$$

$$\begin{aligned}
\therefore \quad \frac{\rho}{\sigma} &= \frac{\Delta BDC}{\Delta ABC} \\
&= \frac{\frac{1}{2}BD \cdot DC}{\frac{1}{2}BD \cdot AC} \\
&= \frac{DC}{AC} \\
&= \frac{BC \cos C}{AC}
\end{aligned}$$

But

$$\begin{aligned}
\frac{AC}{\sin B} &= \frac{BC}{\sin A} \\
BC &= \frac{AC \sin A}{\sin B}
\end{aligned}$$

Hence

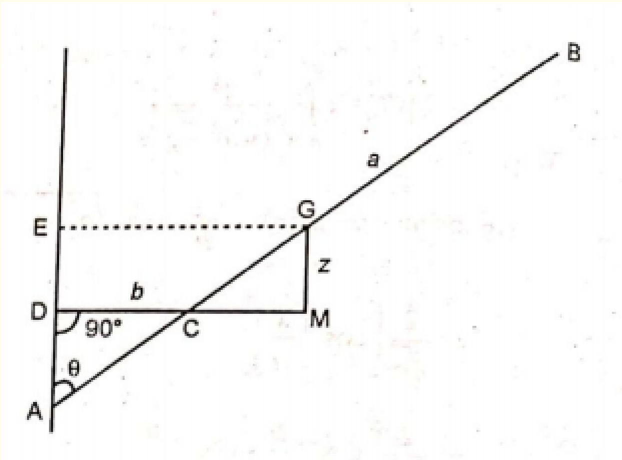
$$\begin{aligned}
\frac{\rho}{\sigma} &= \frac{AC \sin A \cos C}{AC \sin B} \\
&= \frac{\sin A \cos C}{\sin B} \\
&= \frac{a}{b} \cdot \frac{a^2 + b^2 - c^2}{2ab} \\
&= \frac{a^2 + b^2 - c^2}{2b^2}
\end{aligned}$$

Question-5(e) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium.

[8 Marks]

Solution: Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e.,

$$\text{CD} = b.$$



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of above the fixed peg C, i.e., GM $= z$. We shall express z in terms of θ. We have,

$$\begin{aligned} z &= \text{GM} = \text{ED} = \text{AE} - \text{AD} \\ &= \text{AG}\cos\theta - \text{CD}\cot\theta \\ &= a\cos\theta - b\cot\theta \\ \therefore \quad dz/d\theta &= -a\sin\theta + b\,\text{cosec}^2\,\theta \end{aligned}$$

and

$$\frac{d^2z}{d\theta^2} = -a\cos\theta - 2b\,\text{cosec}^2\,A$$

For equilibrium of the rod, we have

$$\frac{dz}{d\theta} = 0$$

i.e.,

$$-a\sin\theta + b\,\text{cosec}^2\,\theta = 0$$

$$a\sin\theta = b\,\text{cosec}^2\,\theta$$

$$\sin^3\theta = b/a$$

$$\sin\theta = (b/a)^{1/3}$$

$$\theta = \sin^{-1}\cdot(b/a)^{1/3}$$

This gives the position of equilibrium of the rod. Again

$$\begin{aligned} \frac{d^2z}{d\theta^2} &= -\left(a\cos\theta + 2b\,\text{cosec}^2\,\theta\cot\theta\right) \\ &= \text{ negative for all acute values of } \theta \end{aligned}$$

Thus $\frac{d^2z}{d\theta^2}$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Question-6(a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2 - 1\right) y = -3e^{x^2} \sin^4 2x$$

by changing the dependent variable.

[13 Marks]

Solution: We have,

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2 - 1\right) y = -3e^{x^2} \sin 2x$$

Here

$$P = -4x, Q = 4x^2 - 1$$
$$R = -3e^{x^2} \sin 2x$$

In order to remove the first derivative

$$\begin{aligned} v &= e^{-\frac{1}{2}\int pdx} \\ &= e^{-\frac{1}{2}\int -4xdx} \\ &= e^{2\int xdx} \\ &= e^{x^2} \end{aligned}$$

On putting $y = av$, the normal equation is $\frac{d^2u}{dx^2} + \mathrm{Q}_1 u = \mathrm{R}_1$ where

$$\begin{aligned} Q_1 &= Q - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4} \\ &= \left(4x^2 - 1\right) - \frac{1}{2}(-4) - \frac{16x^2}{4} \\ &= 4x^2 - 1 + 2 - 4x^2 \\ &= 1 \\ \mathrm{R}_1 &= \frac{\mathrm{R}}{v} \\ &= \frac{-3e^{x^2} \sin 2x}{e^{x^2}} \\ &= -3\sin 2x \end{aligned}$$

Equation (ii) becomes

$$\begin{aligned} &\frac{d^2u}{dx^2} + u = -3\sin 2x \\ \Rightarrow &\left(D^2 + 1\right) u = -3\sin 2x \end{aligned}$$

A.E. is

$$m^2 + 1 = 0$$

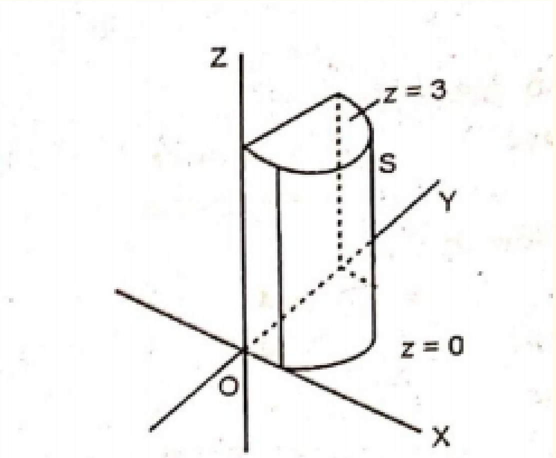
$$\Rightarrow m = \pm i$$

$$\Rightarrow \text{ C.F. } = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D_2+1}(-3\sin 2x) \\
&= \frac{-3\sin 2x}{-4+1} \\
&= \sin 2x \\
u &= c_1 \cos x + c_2 \sin x + \sin 2x \\
y &= u.v \\
&= (c_1 \cos x + c_2 \sin x + \sin 2x)\, e^{x^2}
\end{aligned}$$

Question-6(b) Evaluate $\int_S \overrightarrow{\mathrm{F}} \cdot \mathrm{d}\overrightarrow{\mathrm{s}}$, where $\overrightarrow{\mathrm{F}} = 4\mathrm{xi} - 2\mathrm{y}^2\mathrm{j} + z^2\overrightarrow{\mathrm{k}}$ and s is the surface bounding the region $x^2+y^2=4, z=0$ and $z=3$.

[13 Marks]



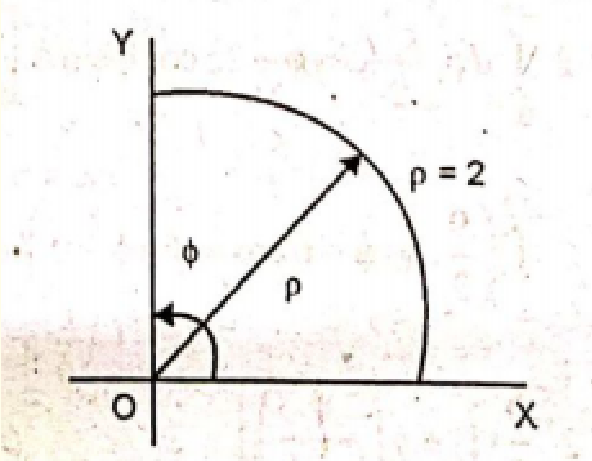
Solution:

Surface S is closed and let us assume that the volume enclosed by it is V.
Then, by Gauss divergence theorem

$$\int_S \vec{F}\cdot d\vec{S} = \iiint_V div(\vec{F})dV, \text{ where } V = \text{Volume enclosed by the surface}$$

$$\begin{aligned}
\text{div}(\vec{F}) &= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) \\
&= 4 - 4y + 2z = 2(2-2y+2)
\end{aligned}$$

$$\therefore \iiint_V \text{div}\,\vec{F}dV = \iiint_V 2(2-2y+z)dV$$



Converting integral to cylindrical co-ordinates.

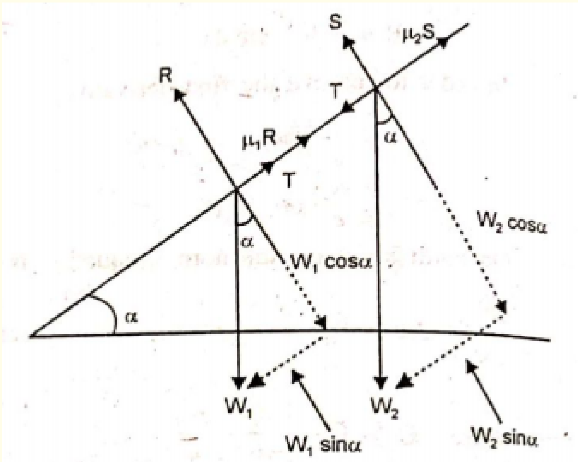
$$z = z, x^2 + y^2 = r^2, \quad x = r\sin\theta, y = r\cos\theta$$
$$r^2 = 4 \Rightarrow 0 \le r \le 2$$
$$\text{and} \quad 0 \le \theta \le 2\pi, \quad \text{also} \quad 0 \le z \le 3$$
$$\text{and} \quad V = rdrd\theta dz$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2 - 2r\sin\theta + z)rdrd\theta dz$$
$$= 2\int_0^2\int_0^{2\pi}\left|2z - 2r\sin\theta z + \frac{z^2}{2}\right|_0^3 rdrd\theta$$
$$= 2\int_0^2\int_0^{2\pi}\left(6 - 6r\sin\theta + \frac{9}{2}\right)rdrd\theta$$
$$= 2\int_0^2\left|6\theta + 6r\cos\theta + \frac{9}{2}\theta\right|_0^{2\pi} rdr$$
$$= 2\int_0^2 [6(2\pi) + 6r(1-1) + \frac{9}{2}(2\pi)]rdr$$
$$= 2\int_0^2 \frac{21}{2}(2\pi)rdr = 42\pi\int_0^2 rdr = 42\pi\left|\frac{r^2}{2}\right|_0^2$$
$$= 42\pi\left(\frac{4}{2} - 0\right) = 84\pi$$
$$\therefore \int_S \vec{F}d\vec{s} = \iiint_V \int \text{div}\,\vec{F}dV = 84\pi$$

Question-6(c) Two bodies of weights w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other.

[14 Marks]

Solution: R and S are normal reactions and μ_1R and μ_2 S are forces of friction.



Let T be the tension in the string.

Let α be the inclination of plane to the horizontal. For W_1 : For limiting equilibrium, Horizontally

$$\mu_1 R + T = W_1 \sin\alpha$$

$$\Rightarrow \quad \mathrm{T} = \dot{\mathrm{W}}_1 \sin\alpha - \mu_1 \mathrm{R} ...(i)$$

Vertically

$$\mathrm{R} = \mathrm{W}_1 \cos\alpha ...(ii)$$

From (i) and (ii), we get

$$\mathrm{T} = \mathrm{W}_1 \sin\alpha - \mu_1 \ \mathrm{W}_1 \cos\alpha ...(iii)$$

For W_2: For limiting equilibrium, Horizontally

$$\mathrm{T} + \mathrm{W}_2 \sin\alpha = \mu_2 \ \mathrm{S}$$

$$\Rightarrow \quad \mathrm{T} = \mu_2 \ \mathrm{S} - \mathrm{W}_2 \sin\alpha ...(iv)$$

Vertically,

$$S = W_2 \cos\alpha ...(v)$$

From (iv) and (v), we get

$$\mathrm{T}^{\circ} = \mu_2 \ \mathrm{W}_2 \cos\alpha - \mathrm{W}_2 \sin\alpha ...(vi)$$

From (iii) and (vi), we get,

$$\begin{aligned}
\mathrm{W}_1 \sin\alpha - \mu_1\ \mathrm{W}_1 \cos\alpha &= \mu_2\ \mathrm{W}_2 \cos\alpha - \mathrm{W}_2 \sin\alpha \\
\Rightarrow \mathrm{W}_1 \sin\alpha + \mathrm{W}_2 \sin\alpha &= \mu_1\ \mathrm{W}_1 \cos\alpha + \mu_2\ \mathrm{W}_2 \cos\alpha \\
\Rightarrow (\mathrm{W}_1 + \mathrm{W}_2) \sin\alpha &= (\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2) \cos\alpha \\
\Rightarrow \quad \tan\alpha &= \frac{\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2}{\mathrm{W}_1 + \mathrm{W}_2} \\
\Rightarrow \quad \alpha &= \tan^{-1}\left(\frac{\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2}{\mathrm{W}_1 + \mathrm{W}_2}\right)
\end{aligned}$$

Question-7(a) Solve

$$\left(D^3 + 1\right) y = e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

where $D = \dfrac{d}{dx}$

[13 Marks]

Solution: Auxiliary Eqn:

$$\begin{aligned}
D^3 + 1 &= 0 \\
D &= -1, \frac{1 \pm \sqrt{3} i}{2} \\
C.F. &= C_1 e^{-x} + e^{x/2}\left(C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right) \\
P.I. &= \frac{1}{D^3 + 1} e^{x/2} \sin\frac{\sqrt{3}x}{2} \\
&= e^{x/2} \frac{1}{\left(D + \frac{1}{2}\right)^3 + 1} \sin\frac{\sqrt{3}x}{2} \\
&\left(\because \frac{1}{f(D)} e^{ax} V = e^x \cdot \frac{1}{f(D+a)} V\right) \\
&= e^{x/2} \frac{1}{D^3 + \frac{1}{8} + \frac{3}{2} D^2 + \frac{3D}{4} + 1} \sin\frac{\sqrt{3}x}{2} \\
f(D) &= D^3 + \frac{1}{8} + \frac{3}{2} D^2 + \frac{3D}{4} + 1 \\
f\left(-\frac{3}{4}\right) &= f\left(-a^2\right) \\
&= D\left(\frac{-3}{4}\right) + \frac{3}{2}\left(\frac{-3}{4}\right) + \frac{3D}{4} + \frac{9}{8} \\
&= 0
\end{aligned}$$

Hence, we take derivative of denominator and multiply by x

$$= xe^{x/2}\frac{1}{3D^2+3D+3/4}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3}\frac{1}{\left(\frac{-3}{4}\right)+D+\frac{1}{4}}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3}\cdot\frac{1}{D-\frac{1}{2}}\cdot\frac{D+1/2}{D+1/2}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3\left(\frac{-3}{24}-1/4\right)}\left(D+\frac{1}{2}\right)\sin\frac{\sqrt{3}x}{2}$$
$$= -1/3xe^{x/2}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}x}{2}+\frac{1}{2}\sin\frac{\sqrt{3}x}{2}\right)$$
$$P.I. = \frac{-x}{3}e^{x/2}\cdot\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}x}{2}\right)$$

Question-7(b) A body floating in water has volumes v_1, v_2 and v_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Find the value of:

$$\frac{\rho_2-\rho_3}{v_1}+\frac{\rho_3-\rho_1}{v_2}+\frac{\rho_1-\rho_2}{v_3}$$

[13 Marks]

Solution: Suppose the volume and the density of the body be V and ρ respectively.

Now, weight of the body = weight of air displaced + weight of water displaced Hence,

$$\mathrm{V}'\rho g = \mathrm{V}_1\rho_1 g + (\mathrm{V}-\mathrm{V}_1)\times 1\times g ...(i)$$
$$\mathrm{V}\rho g = \mathrm{V}_2\rho_2 g + (\mathrm{V}-\mathrm{V}_2)\times 1\times g ...(ii)$$
$$\mathrm{V}\rho g = \mathrm{V}_3\rho_3 g + (\mathrm{V}-\mathrm{V}_3)\times 1\times g \quad ......(iii)$$

These relations give,

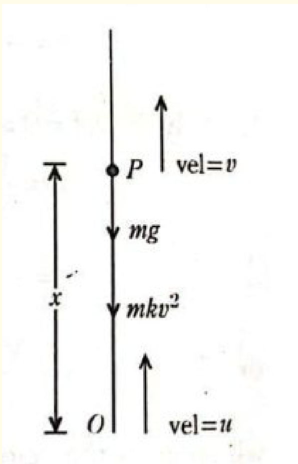
$$V_1 = \frac{\rho-1}{\rho_1-1}\mathrm{V}$$
$$\frac{1}{\mathrm{V}_1} = \frac{\rho_1-1}{(p-1)\mathrm{V}}$$
$$V_2 = \frac{\rho-1}{\rho_2-1}V$$
$$\frac{1}{V_2} = \frac{\rho_2-1}{(p-1)V}$$
$$\mathrm{V}_3 = \frac{\rho-1}{\rho_3-1}\mathrm{V}$$
$$\frac{1}{\mathrm{V}_3} = \frac{\rho_3-1}{(\rho-1)\mathrm{V}}$$

$$\begin{aligned}
\therefore \frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{\mathrm{V}_2}+\frac{\rho_1-\rho_2}{\mathrm{V}_3} &= \frac{(\rho_1-1)}{(p-1)\mathrm{V}}(\rho_2-\rho_3)+\frac{(\rho_2-1)}{(\rho-1)\mathrm{V}}(\rho_3-\rho_1) \\
&= \frac{1}{(\rho-1)\mathrm{V}}\left[(\rho_1-1)(\rho_2-\rho_3)+(\rho_2-1)(\rho_3-\rho_1)\right. \\
&\qquad \left. +(\rho_3-1)(\rho_1-\rho_2)\right] \\
&= 0
\end{aligned}$$

Question-7(c) A particle is projected vertically upwards with a velocity u, **in a resisting medium which produces a retardation** $\mathrm{k}v^2$ **when the velocity is** v. **Find the height when the particle comes to rest above the point of projection.**

[14 Marks]

Solution: Let a particle of mass m be projected vertically upwards from the point O with velocity u. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are:
(i) The force mkv^2 due to resistance acting against the direction of motion i.e., acting vertically downwards.
(ii) The weight mg of the particle also acting vertically downwards.



Both these forces act in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$\text{Or} \quad \frac{d^2x}{dt^2} = -g\left(1+\frac{k}{g}v^2\right)$$

Let V be the terminal velocity of the particle during its downwards motion i.e., the velocity when the resultant acceleration of the particle during its downwards motion is zero. Then

$$0 = mg - mkV^2 \text{ or } k = g/V^2$$

Putting this value of k in the above equation of motion of the particle, we get

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right)$$

$$\text{or} \qquad \frac{d^2x}{dt^2} = \frac{-g}{V^2}\left(V^2 + v^2\right). \qquad \ldots(1)$$

Relation between v and x : Equation (1) can be written as

$$v\frac{dv}{dx} = \frac{-g}{V^2}\left(V^2 + v^2\right) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

$$\text{or} \quad \frac{-2g}{V^2}dx = \frac{2vdv}{V^2 + v^2}, \qquad \text{separating the variables.}$$

Integrating, $\frac{-2gx}{V^2} = \log\left(V^2 + v^2\right) + A$, where A is a constant. Initially at $O, x = 0$ and $v = u$

$$\therefore \qquad 0 = \log\left(V^2 + u^2\right) + A$$

$$\text{or} \qquad A = -\log\left(V^2 + u^2\right)$$

$$\therefore \qquad \frac{-2gx}{V^2} = \log\left(V^2 + v^2\right) - \log\left(V^2 + u^2\right)$$

$$\text{or} \qquad x = \frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2 + \nu^2} \qquad \ldots(2)$$

which gives the velocity of the particle in any position. If H is the greatest height attained by the particle, then putting $x = H$ and $v = 0$ in (2), we get

$$H = \frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2}.$$

Question-8(a) Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} - y = 2\left(1 + e^x\right)^{-1}$$

[13 Marks]

Solution: Given DE Eqn:

$$\left(D^2 - 1\right)y = 2\left(1 + e^x\right)^{-1}$$

Auxiliary Eqn:

$$D^2 = 1 = 0 \quad \Rightarrow D = \pm 1$$

$$C \cdot F \cdot \quad = C_1e^x + c_2e^{-x}$$

To find complete solution, we replace constants c_1 and c_2 with functions A and B.

$$\begin{aligned} y &= Ae^x + Be^{-x} \\ &= Ay_1 + By_1 \end{aligned}$$

where $y_1 = e^x, \quad y_2 = e^{-x}$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 \\ &= -2 \neq 0 \end{aligned}$$

$\Rightarrow y_1 \& y_2$ are independent.

$$\begin{aligned} A &= -\int \frac{y_2 R}{w} dx \\ &= -\int \frac{e^{-x} \cdot 2\left(1+e^x\right)^{-1}}{-2} dx \\ &= \int \frac{dx}{e^x\left(1+e^x\right)} \\ &\begin{pmatrix} \text{put} \quad e^x = t \\ e^x dx = dt \end{pmatrix} \\ &= \int \frac{dt}{t^2(1+t)} \end{aligned}$$

$$\begin{aligned} \frac{1}{t^2(1+t)} &= \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t} \\ 1 &= At(t+1) + B(1+t) + ct^2 \\ 1 &= t^2(A+C) + t(A+B) + B \end{aligned}$$

$A + C = 0, \quad A + B = 0, \quad B = 1 \quad \Rightarrow \quad A = -1, B = 1, \quad C = 1$

$$\begin{aligned} A &= \int \left(\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right) dt \\ &= -\log t - \frac{1}{t} + \log(t+1) + c_1' \\ &= \log\left(\frac{t+1}{t}\right) - \frac{1}{t} + c_1' \\ &= \log\left(1+e^{-x}\right) - e^{-x} + c_1' \end{aligned}$$

$$\begin{aligned} B &= \int \frac{y_1 R}{w} dx \\ &= \int \frac{e^x \cdot 2\left(1+e^x\right)^{-1}}{-2} dx \\ &= -\int \frac{e^x}{1+e^x} dx \\ &= -\log\left(1+e^x\right) + c_2' \end{aligned}$$

Hence, complete general solution is

$$\begin{aligned} y &= Ay_1 + By_2 \\ &= e^x\left[\log\left(1+e^{-x}\right) - e^{-x} + c_1^1\right] + e^{-x}\left[-\log\left(1+e^x\right) + c_2^1\right] \\ y &= e^x \log\left(1+e^{-x}\right) - 1 + e^x \cdot c_1' - e^{-x}\log\left(1+e^x\right) + e^{-x} c_2' \end{aligned}$$

Question-8(b) Verify the divergence theorem for the vector function

$$\overrightarrow{\mathrm{F}} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - xz\right)\vec{j} + \left(z^2 - xy\right)\bar{k}$$

taken over the rectangular parallelopiped

$$0 \le x \le a, 0 \le y \le b, 0 \le z \le c$$

[14 Marks]

Solution: To verify Gauss divergence theorem, we have to show that

$$\iiint\limits_V div\vec{F}dv = \iint\limits_s \vec{F}\cdot\hat{n}\cdot ds$$

Firstly,

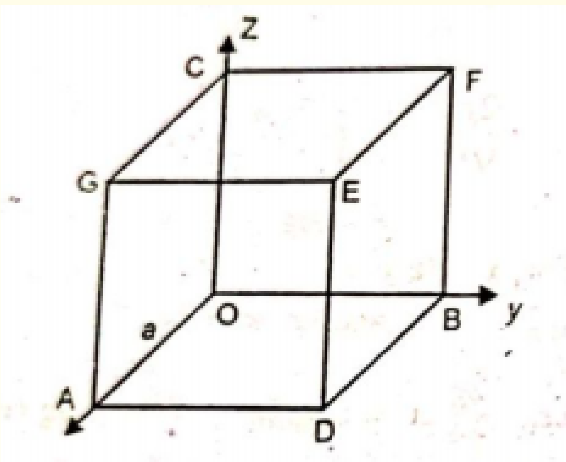
$$\iiint\limits_v \operatorname{div}\overrightarrow{\mathrm{F}}dv = \int_0^c\int_{00}^{ba}\int_0^a\left[\frac{\partial}{\partial x}\left(x^2-yz\right)+\frac{\partial}{\partial y}\left(y^2-zx\right)+\frac{\partial}{\partial z}\left(z^2-xy\right)\right]dxdydz$$

$$= \int_{000}^{1ba}\int_0^b 2(x+y+z)dxdydz$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

Now to calculate $\iint_s \overrightarrow{\mathrm{F}}.nds$, we divide the surface s of the parallelopiped $0 \le x \le a,\ 0 \le y \le b, 0 \le z \le c$ into six parts.



(i) For the face OADB, we have

$$\hat{n} = -\hat{k}, z = 0$$

Therefore,

$$\int_{OADB}\overrightarrow{\mathrm{F}}\cdot\hat{n}\cdot ds = \int_{OADB}\left(x^2\hat{i}+y^2\hat{j}-xy\hat{k}\right)\cdot(-\hat{k})ds$$

$$= \int_{00}^{ba} xydxdy$$

$$= \frac{a^2b^2}{4}$$

(ii) For the face $CGEF$, we have $z = c$ -

$$\hat{n} = \hat{k}$$

$$z = \int_{(\text{GEF}} \left[\left(x^2 - cy \right) \hat{i} + \left(y^2 - cx \right) \hat{j} + \left(c^2 - xy \right) \hat{k} \right] \cdot \hat{k} ds$$
$$= \int_0^{ba} \int_0^a \left(c^2 - xy \right) dxdy$$
$$= abc^2 - \frac{a^2 b^2}{4}$$

(iii) For the face $ADEG$, we have $\hat{n} = \hat{i}, x = a$ and $dx = 0$. Therefore,

$$\int_{\text{ADEG}} \int_0 \overrightarrow{\text{F}}.n \cdot ds = \int_0^{c_0 b} \int_0^2 \left(a^2 - yz \right) dydz$$
$$= a^2 bc - \frac{b^2 c^2}{4}$$

(iv) For the face OBFC, we have $\hat{n} = -\hat{i}, x = 0 \; dx = 0$, Therefore,

$$\iint_{\text{OBFC}} \overrightarrow{\text{F}} \cdot \hat{n} \cdot ds = \int_0^{ab} \int yzdydz$$
$$= \frac{b^2 c^2}{4}$$

(v) For the face $OAGC$, we have $\hat{n} = -\hat{j}, y = 0 \; dy = 0$, Therefore,

$$\iint_{OAGC} \vec{F} \cdot \hat{n}.ds = \int_0^{ab} \int_0^b zxdzdx$$
$$= \frac{a^2 c^2}{4}$$

(vi) For the face DBFE, we have $\hat{n} = \hat{j}, y = b \; dy = 0$ Therefore,

$$\iint_{\text{DBFE}} \overrightarrow{\text{F}} \cdot \hat{n} \cdot ds = \int_0^{ab} \int_0^b \left(b^2 - zx \right) dzdx$$
$$= ab^2 c - \frac{a^2 c^2}{4}$$

Hence adding the values of the above integrals, we get

$$\iint_s \overrightarrow{\text{F}} \cdot \hat{n} \cdot ds = abc(a + b + c)$$

Hence,

$$\iiint_V \int \text{div} \, \vec{F} dv = \iint_s \overrightarrow{\text{F}} \cdot \hat{n}.ds$$

which verifies the Gauss's divergence theorem.

Question-8(c) A particle is projected with a velocity v along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time t.

[13 Marks]

Solution: Here since particle is moving in a horizontal plane, the weight mg of the particle will not act. Hence the only force acting on the particle is that due to resistance and is equal to $-m\mu v^3$.
The equation of motion of the particle is

$$m\,(dv/dt) = -m\mu v^3 \quad \text{or} \quad -\left(dv/v^3\right) = \mu dt$$

Integrating, $\frac{1}{2v^2} = \mu t + C$, where C is a constant of integration.
Initially when $t = 0, v = V$,

$$\therefore \quad \frac{1}{2v^2} = \mu t + \frac{1}{2V^2} \text{ or } \frac{1}{v^2} = \frac{2\mu t V^2 + 1}{V^2}$$
$$\text{or} \quad v = V/\sqrt{(1 + 2\mu t V^2)} \quad \ldots (1)$$

If x be the distance descrihed by ine particle in time t, then equation (1) may be written as

$$\frac{dx}{dt} = \frac{V}{\sqrt{1 + 2\mu t V^2}} \quad \text{or} \quad dx = \frac{V}{\sqrt{1 + 2\mu t V^2}} dt$$

Integrating,

$$x = \frac{1}{\mu V}\sqrt{(1 + 2\mu t V^2) + C'} \quad \ldots (2)$$

Initially when $t = 0, x = 0, \quad \Rightarrow \quad C' = -1/\mu V$. Hence equation (2) becomes

$$x = \frac{1}{\mu V}\sqrt{(1 + 2\mu t V^2)} - \frac{1}{\mu}$$
$$\text{or} \quad x = \frac{1}{\mu V}\left[\sqrt{(1 + 2\mu t V^2)} - 1\right] \quad \ldots (3)$$

Equations (1) and (3) give required results.

Chapter 9

2012

9.1 Section-A

Question-1(a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2), \alpha_2 = (0,1,3)$ $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)\cdot$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

[8 Marks]

Solution: Let $W_1 = (\alpha_1, \alpha_2) = a(1,1,2) + b(0,1,3) = (a, a+b, 2a+3b)$
Let $W_2 = \sin(\alpha_3, \alpha_4) = c(2,4,5) + d(-1,0,-1) = (2c-d, 4c, 5c-d)$
Let (x,y,z) be an element of intersection of W_1 and W_2 i.e. $(x,y,z) \in W_1 \cap W_2$.

Then,

$$(x,y,z) = (a, a+b, 2a+3b) = (2c-d, 4c, 5c-d)$$
$$\Rightarrow (a, a+b, 2a+3b) - (2c-d, 4c, 5c-d) = (0,0,0)$$
$$\Rightarrow (a-2c+d, a+b-4c, 2a+3b-5c+d) = (0,0,0)$$

Let,

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & -4 & 0 \\ 2 & 3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & -4 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

$$R_1 \to 5R_1 + 2R_3, R_2 \to 5R_2 + 2R_3 \quad R_1 \to R_1/5, R_2 \to R_2/5, R_3 \to R_3/5$$

$$\sim \begin{bmatrix} 5 & 0 & 0 & 9 \\ 0 & 5 & 0 & -1 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 9/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

$$\therefore \quad a + \frac{9}{5}d = 0, b - \frac{1}{5}d = 0, \quad c + \frac{2}{5}d = 0$$

$$a = \frac{-9}{5}d, \quad b = \frac{1}{5}d, \quad c = -\frac{2}{5}d.$$

$$\begin{aligned}(x,y,z) = (a, a+b, 2a+3b) &= \left(\frac{-9}{5}d, -\frac{9}{5}d + \frac{1}{5}d, 2\left(\frac{-9}{5}d\right) + 3\left(\frac{1}{5}d\right)\right) \\ &= d\left(-\frac{9}{5}, -\frac{8}{5}, -3\right) \\ &= k(-9,-8,-15) \\ &= k_1(9,8,15)\end{aligned}$$

$\therefore$ Basis of $w_1 \cap \omega_2$ is $\{(9,8,15)\}$.

Question-1(b) Show that the set of all functions which satisfy the differential equation, $\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space.

[8 Marks]

Solution: Let W be the set of all functions which satisfy the differential equation,

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$

$$\therefore \mathrm{W} = \left\{ f : \frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0 \right\}$$

Let $y = f(x)$ Obviously $f(x) = 0$ or $y = 0$ satisfy the given differential equation and as such it belongs to W and thus $W \neq \phi$ Now let $y_1, y_2 \in \mathrm{W}$, then

$$\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx} = 0$$

and

$$\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx} = 0$$

Let $a, b \in \mathrm{R}$. If W is to be a subspace then we should show that $ay_1 + by_2$ also belongs to W i.e., it is a solution of the given differential equation. We have

$$\begin{aligned} \frac{d^2}{dx^2}(ay_1 + by_2) + 3\frac{d}{dx}(ay_1 + by_2) &= a\frac{d^2 y_1}{dx^2} + b\frac{d^2 y_2}{dx^2} + 3a\frac{dy_1}{dx} + 3b\frac{dy_2}{dx} \\ &= a\left(\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx}\right) + b\left(\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx}\right) \\ &= a(0) + b(0) \\ &= 0 \end{aligned}$$

using (1) and (2)

Thus $ay_1 + by_2$ is a solution of the given differential equation and so it belongs to W.

Hence, W is the subspace. Thus, W is a vector space.

Question-1(c) If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$. Show that,

$$\left(\frac{\partial P}{\partial T}\right)_V \cdot \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T \cong -1$$

[8 Marks]

Solution: Given $f(P, V, T) = 0$ When V is constant;
Taking P as function of T, we have

$$\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial P}}$$

Similarly,

$$\left(\frac{\partial T}{\partial V}\right)_p = -\frac{\frac{\partial f}{\partial V}}{\frac{\partial f}{\partial T}}; \left(\frac{\partial V}{\partial P}\right)_T = -\frac{\frac{\partial f}{\partial P}}{\frac{\partial f}{\partial V}}$$

Multiplying the three, we get

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$$

Question-1(d) If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$.

[8 Marks]

Solution: $u = \mathrm{A}e^{-gx}\sin(nt - gx)$, where A, g, n positive constants.

This expression satisfies the heat conduction equation.

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$$

First, finding $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from give expression of u, we get

$$\frac{\partial u}{\partial t} = n\ \mathrm{A}e^{-gx}\cos(nt - gx)$$

and

$$\frac{\partial u}{\partial x} = \mathrm{A}\left(\ -ge^{-gx}\cos(nt - gx) - ge^{-tx}\sin(nt - gx)\ \right)$$

$$= -\text{Age}^{-gx}[\cos(nt-gx)+\sin(nt-gx)]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\text{Ag}\begin{pmatrix} e^{-gx}[(g\sin(nt-gx)) \\ -g\cos(nt-gx)] \\ -ge^{-gx}[\cos(nt-gx) \\ +\sin(nt-gx)] \end{pmatrix}$$

$$\frac{\partial^2 u}{\partial x^2} = -\text{Ag}^2 e^{-gx}[\sin(nt-g\dot{x})-\cos(nt-gx)$$
$$-\sin(nt-gx)-\cos(nt-gx)]$$

$$\frac{\partial^2 u}{\partial x^2} = 2\text{Ag}^2 e^{-gx}\cos(nt-gx)$$

Substituting values of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from (2) and (3) in (1), we get

$$n\ \text{Ae}^{-gx}\cos(nt-gx) = 2\text{Ag}^2 e^{-gx}\mu[\cos(nt-gx)$$

$$n = 2\mu g^2$$

$\therefore g = \sqrt{\left(\frac{n}{2\mu}\right)}$

Question-1(e) Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$

[8 Marks]

Solution: Let one of the lines of intersection of the plane

$$2x+y-z=0 \qquad \ldots(1)$$

and the cone

$$4x^2-y^2+3z^2=0 \qquad \ldots(2)$$

be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(3)$$

The line (3) lies in the plane (1) and on the cone (2).

$$\therefore \qquad 2l+m-n=0 \qquad \ldots(4)$$

and

$$4l^2-m^2+3n^2=0 \qquad \ldots(5)$$

Eliminating n between (4) and (5) we get

$$4l^2-m^2+3(2l+m)^2=0$$

$$\Rightarrow 16l^2+12lm+2m^2=0$$

$$\Rightarrow 8l^2+6lm+m^2=0$$

$$\Rightarrow (4l+m)(2l+m)=0$$

$$4l+m=0, \quad 2l+m=0$$

$$m=-4l, \quad m=-2l$$

when $m = -4l$, then from $(4), n = -2l$ and when $m = -2l$, then from $(4), n = 0$

Hence, In first case we rearrange as

$$\frac{l}{1} = \frac{m}{-4} = \frac{n}{-2}$$

and in second case, we rearrange as

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{0}$$

Thus, the equation of the lines in which the given plane cuts the given cone are:

$$\frac{x}{1} = \frac{y}{-4} = \frac{z}{-2}$$

and

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

Question-2(a) Let $f : \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a+b, 0)$. Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0), e_2' = (0, 1, 1)$ $e_3' = (1, 1, 1)$.
Also, show that there exists an invertible matrix P such that

$$B = P^{-1}AP$$

[10 Marks]

Solution: $S_1 = \{e_1, e_2, e_3\}$ where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3(0, 0, 1)$ is the standard basis of R^3.

$$T(e_1) = (1, 1, 0) = e_1 + e_2 + 0e_3$$

$$T(e_2) = (0, 1, 0) = 0e_1 + e_2 + 0e_3$$

$$T(e_3) = (0, 0, 0) = 0e_1 + 0e_2 + 0e_3$$

$\therefore$ Matrix of T wrt standand basis is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Now: $S_2 = \{e_1', e_2', e_3'\}$ where $e_1' = (1, 1, 0), e_2'(0, 1, 1)$ and $e_3' = (1, 1, 1)$.

Let $(x, y, z) = ae_1' + be_2' + ce_3' = (a + c, a + b + c, b + c)$
$a + c = x, b + c \leq z, a + b + c = y$.
On comparing,

$$a + x = c \qquad \ldots (1)$$

$$b + c = z \qquad \ldots (2)$$

$$a + b + c = y \qquad \ldots (3)$$

From (1), (2) and (3), we get:
$a = y - z$,
$b = y - x$,
$c = x - y + z$

$$\begin{aligned}
\therefore (x, y, z) &= (y - z)(1, 1, 0) + (-x + y)(0, 1, 1) + (x - y + z)(1, 1, 1) \\
&= (y - z)e_{\prime} + (-x + y)e_2' + (x - y + z)e_3' \\
T(e_1') &= T(1, 1, 0) = (1, 2, 0) = 2e_1' + 1 \cdot e_2' + (-1)e_3' \\
T(e_2') &= T(0, 1, 1) = (0, 1, 0) = 1 \cdot e_1' + 1 \cdot e_2' + (-1)e_3' \\
T(e_3') &= T(1, 1, 1) = (01, 2, 0) = 2 \cdot e_1' + 1 \cdot e_2' + (-1)e_3'
\end{aligned}$$

$\therefore$ Matrix of T wrt basis S_2 is $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

To prove that $B = P^{-1}AP$ for some non-singular matrix P, we need to show that A and B are similar, i.e., the characteristic equation and the roots of A and B are the same.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \begin{matrix} (1-\lambda)(1-\lambda)(-\lambda) = 0 \\ \Rightarrow \lambda = 1, 1, 0 \end{matrix}$$

Also,

$$B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \quad |B - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ -1 & -1 & -(1+\lambda) \end{vmatrix} = \begin{matrix} (2-\lambda)(\lambda^2 - 1) + 1 = 0 \\ \Rightarrow \lambda = 1, 1, 0 \end{matrix}$$

$\therefore$ A and B are similar.
Hence, $\exists$ a non-singular matrix P such that $B = P^{-1}AP$.

Question-2(b) Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

[10 Marks]

Solution: Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation. Now, for matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$
$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

By Cayley-Hamilton theorem the matrix A must satisfy (1).
$\therefore$ We have to verify that

$$A^2 - 4A - 5I = 0$$

Now,

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now

$$\begin{aligned} \mathrm{A}^2 - 4\ \mathrm{A} - 5\mathrm{I} &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

Hence, $A^2 - 4A - 5I = 0$ Thus, Cayley-Hamilton theorem verified. Now we have to compute A^{-1}. Multiply (2) by A^{-1} we get $A - 4I - 5A^{-1} = 0$

$$\begin{aligned} \Rightarrow \quad \mathrm{A}^{-1} &= \frac{1}{5}(\ \mathrm{A} - 4\mathrm{I}) \\ &= \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix} \\ \therefore \mathrm{A}^{-1} &= \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} \end{aligned}$$

Now from (2), we get

$$\mathrm{A}^2 = 4\ \mathrm{A} + 5\mathrm{I} \ldots (3)$$

Multiplying both sides of (3) by A, we get

$$A^3 = 4A^2 + 5A \ldots (4)$$
$$A^4 = 4A^3 + 5A^2 \ldots (5)$$

and

$$\mathrm{A}^5 = 4\ \mathrm{A}^4 + 5\ \mathrm{A}^3 \ldots (6)$$

Now,

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

is calculated by substituting for A^5 from (6)

$$
\begin{aligned}
A^5-4A^4-7A^3+11A^2-A-10I &= \left(4\ \mathrm{A}^4+5\ \mathrm{A}^3\right)-4\ \mathrm{A}^4-7\ \mathrm{A}^3+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}\\
&= -2\ \mathrm{A}^3+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}\\
&= -2\left(4\ \mathrm{A}^2+5\ \mathrm{A}\right)+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}[u\,\mathrm{sing}(4)]\\
&= 3\ \mathrm{A}^2-11\ \mathrm{A}-10\mathrm{I}\\
&= 3(4\ \mathrm{A}+5\mathrm{I})-11\ \mathrm{A}-10\mathrm{I} \quad [using(3)]\\
&= \mathrm{A}+5\mathrm{I}
\end{aligned}
$$

,
which is a linear polynomial in A

Question-2(c) Find the equations of the tangent plane to the ellipsoid

$$2x^2+6y_1^2+3z^2=27$$

which passes through the line

$$x-y-z=0=x-y+2z-9$$

[10 Marks]

Solution: **Method 1:**

Ellipsoid, $2x^2+6y^2+3z^2=27 \qquad \ldots(1).$
Equation of plane passing through the line

$$x-y-z=0=x-y+2z-9$$

is given by:

$$x-y+2z-9+k(x-y-z)=0$$

ie. $(k+1)x-(k+1)y+(-k+2)z=9 \qquad \ldots(2).$
The equation of tangent plane at point (a,b,c) to the ellipsoid (1) is

$$2ax+6by+3cz=27 \qquad \ldots(3)$$

If equations (2) and (3) are idential, then

$$\frac{2a}{k+1}=\frac{6b}{-(k+1)}=\frac{3c}{-k+2}=\frac{27}{9}$$

ie. $a=\frac{3}{2}(k+1), \quad b=-\frac{1}{2}(k+1), \quad c=-k+2.$

Point (a,b,c) lies on ellipsoid (1),
$\therefore \quad 2\cdot\frac{9}{4}(k+1)^2+6\cdot\frac{1}{4}(k+1)^2+3(-k+2)^2=27$

$$x \Rightarrow k=\pm 1$$

When $k=1$, tangent plane: $2x-2y+2=9$
When $k=-1$, tangent plane: $z=3$.

Method 2:
The equation of the plane passing through the line

$$x - y - z = 0 = x - y + 2z - 9$$

is

$$x - y - z + k(x - y + 2z - 9) = 0$$
$$\Rightarrow (1+k)x - (1+k)y + (2k-1)z - 9k = 0$$

Compare it with the general equation of the plane $Lx + my + nz = p$,
we get

$$l = 1 + k, m = -(1+k)$$
$$n = 2k - 1, p = 9k$$

Now, using the condition of tangency to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by the plane $Lx + my + nz = p$, is

$$a^2l^2 + b^2m^2 + c^2n^2 = p^2$$

Here, we are given the cquation of the ellipsoid as

$$2x^2 + 6y^2 + 3z^2 = 27$$
$$\Rightarrow \frac{x^2}{\left(\frac{27}{2}\right)} + \frac{y^2}{\left(\frac{27}{6}\right)} + \frac{z^2}{\left(\frac{27}{3}\right)} = 1$$
$$\therefore \quad a^2 = \frac{27}{2}, b^2 = \frac{27}{6}, c^2 = \frac{27}{3}$$

On substituting the values in (2), we get

$$\frac{27}{2}(1+k)^2 + \frac{27}{6}[-(1+k)]^2 + \frac{27}{3}(2k-1)^2 = (9k)^2$$
$$\Rightarrow 18(1+k)^2 + 9(2k-1)^2 = 81k^2$$
$$\Rightarrow 2(1+k)^2 + (2k-1)^2 = 9k^2$$
$$\Rightarrow 2 + 2k^2 + 4k + 4k^2 + 1 - 4k = 9k^2$$
$$\Rightarrow 3k^2 = 3 \Rightarrow k = \pm 1$$

Putting the values of k in (1), we get two equations of the tangent planes to the given ellipsoid as when $k = 1$

$$\Rightarrow 2x - 2y + z - 9 = 0$$

when

$$k = -1 \Rightarrow -3z + 9 = 0$$
$$\Rightarrow z = 3$$

Question-2(d) Show that there are three real values of λ for which the equations:

$$(a-\lambda)x + by + cz = 0,$$
$$bx + (c-\lambda)y + az = 0,$$
$$cx + ay + (b-\lambda)z = 0$$

are simultaneously true and that the product of these values of λ is $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

[10 Marks]

Solution: The given equations are:

$$(a-\lambda)x + by + cz = 0$$
$$bx + (c-\lambda)y + az = 0$$
$$cx + ay + (b-\lambda)z = 0$$

The above system of equations are simultaneously true when the determinant of the coefficient matrix is zero i.e.,

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (a+b+c)\lambda^2 - \left(a^2+b^2+c^2-ab-bc-ca\right)\lambda + (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) = 0$$

This is a cubic equation in λ.
Hence, product of its roots $= \lambda_1\lambda_2\lambda_3$

$$= \frac{(-1)^3(\text{ Constant term })}{(\text{ Coefficient of } \lambda^3)}$$
$$= \frac{-(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)}{(1)}$$

(Using the fact that in $Ax^3 + Bx^2 + Cx + D = 0$, product of roots $= (-1)^3\frac{D}{A}$)

$$\therefore \lambda_1\lambda_2\lambda_3 = -\left(a^3+b^3+c^3-3abc\right)$$
$$= 3abc - a^3 - b^3 - c^3$$

$$Also, \quad D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc-a^2) + b(ac-b^2) + c(ab-c^2)$$
$$= -(a^3+b^3+c^3-3abc)$$
$$= \lambda_1\lambda_2\lambda_3$$

Hence, verified.

Question-3(a) Find the matrix representation of linear transformation T on $V_3(IR)$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ corresponding to the basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.

[10 Marks]

Solution: Given basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$and

$$T(a, b, c) = (2b + c, a - 4b, 3a)$$

Let $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1, 0, 0)$

By definition of T, we have

$$\mathrm{T}\left(\alpha_1\right) = \mathrm{T}(1, 1, 1) = (2(1) + 1, 1 - 4, 3)$$
$$\Rightarrow \mathrm{T}\left(\alpha_1\right) = (3, -3, 3)$$

Similarly,

$$\mathrm{T}\left(\alpha_2\right) = \mathrm{T}(1, 1, 0) = (2, -3, 3)$$

and

$$\mathrm{T}\left(\alpha_3\right) = \mathrm{T}(1, 0, 0) = (0, 1, 3)$$

Now our aim is to express $\mathrm{T}\left(\alpha_1\right), \mathrm{T}\left(\alpha_2\right)$ and $T\left(\alpha_3\right)$ as linear combination of the vectors in the basis $\mathrm{B}\left[\alpha_1, \alpha_2, \alpha_3\right]$

Let

$$(x, y, z) = p\alpha_1 + q\alpha_2 + r\alpha_3$$
$$(x, y, z) = p(1, 1, 1) + q(1, 1, 0) + r(1, 0, 0)$$
$$(x, y, z) = (p + q + r, p + q, p)$$

$\therefore \quad x = p + q + r,\ y = p + q$ and $z = p$

Solving these equations, we get

$$p = z, q = y - z, r = x - y$$

Putting $x = 3, y = -3, z = 3$, we get

$$p = 3, q = -6, r = 6$$
$$\therefore T\left(\alpha_1\right) = 3\alpha_1 - 6\alpha_2 + 6\alpha_3 \ldots (1)$$

Similarly, on putting $x = 2, y = -3, z = 3$, we get

$$p = 3, q = -6, r = 5$$
$$\therefore \mathrm{T}\left(\alpha_2\right) = 3\alpha_1 - 6\alpha_2 + 5\alpha_3$$

Similarly, on putting $x = 0, y = 1, z = 3$, we get

$$p = 3, q = -2, r = -1$$
$$\therefore \mathrm{T}\left(\alpha_3\right) = 3\alpha_1 - 2\alpha_2 - \alpha_3$$

From $(1), (2)$ and (3), we see that the matrix of T relative to the basis

$$\{\alpha_1, \alpha_2, \alpha_3\} = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Question-3(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm.

[10 Marks]

Solution: Let the dimensions of the rectangular box be x, y and z where these represent length, breadth and height respectively.
Then volume, V $= xyz$ and the surface area of the rectangular box (open at the top) $= xy + 2z(x + y) = 432$ (given)

Define a Lagrangian function

$$\mathrm{F} = xyz + \lambda(xy + 2z(x + y) - 432)$$

Then for extremum value $d\,\mathrm{F} = 0$

$$\Rightarrow d\,\mathrm{F} = [yz + \lambda(y + 2z)]dx + [xz + \lambda(x + 2z)]dy + [xy + \lambda(2(x + y))]d_{z_z}$$

Now equating the coefficients, we

$$yz + \lambda(y + 2z) = 0$$
$$xz + \lambda(x + 2z) = 0$$
$$xy + 2\lambda(x + y) = 0$$

Subtracting (2) from (1) we get 0,

$$\Rightarrow (y - x)z + \lambda(y - x) = 0$$

$$\Rightarrow (y - x)(z + \lambda) = 0$$

$$\Rightarrow y - x = 0,$$

other factors cannot be zero.

$$\therefore y = x$$

Now multiplying equation (2) by 2 and then subtracting the resulting equation from equation (3), we get

$$x(y - 2z) + 2\lambda(x + y - x - 2z) = 0$$

$$\Rightarrow (x + 2\lambda)(y - 2z) = 0$$

$$\Rightarrow y = 2z$$

$\therefore$ The dimensions of the box are of the form

$$x = y = 2z$$
$$xy + 2z(x + y) = 432$$

$$\Rightarrow \quad 12z^2 = 432$$
$$\Rightarrow z^2 = 36$$
$$z = 6$$

Hence, the dimensions of the box are (12,12,6) cm respectively.

Question-3(c) If $2C$ is the shortest distance between the lines

$$\frac{x}{l} - \frac{z}{n} = 1, \quad y = 0$$

and

$$\frac{y}{m} + \frac{z}{n} = 1, \quad x = 0$$

then show that

$$\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}$$

[10 Marks]

Solution: The equations of the given lines are:

$$\frac{x}{l} - \frac{z}{n} = 1, y = 0 \qquad \ldots(1)$$

and

$$\frac{y}{m} + \frac{z}{n} = 1, x = 0 \qquad \ldots(2)$$

The equation of the line (1) being put in symmetrical form as

$$\frac{x-l}{l} = \frac{y}{0} = \frac{z}{n} \qquad \ldots(I)$$

The equation of any plane through the line (2) is

$$\left(\frac{y}{m} + \frac{z}{n} - 1\right) + \lambda x = 0$$

$$\Rightarrow \lambda x + \left(\frac{1}{m}\right) y + \left(\frac{1}{n}\right) z - 1 = 0 \qquad \ldots(3)$$

If the plane (3) is parallel to the line (I), then the normal to the plane (3) whose d.c.'s are $\lambda, \frac{1}{m}, \frac{1}{n}$ will be perpendicular to the line (I), and so we have

$$l\lambda + 0\left(\frac{1}{m}\right) + n\left(\frac{1}{n}\right) = 0$$

$$\lambda = \frac{-1}{l}$$

Putting this value of λ in (3), the equation of the plane containing the line (2) and parallel to the line (I) is

$$-\frac{x}{l} + \frac{y}{m} + \frac{z}{n} - 1 = 0$$

$$\frac{x}{l} - \frac{y}{m} - \frac{z}{n} + 1 = 0 \qquad \ldots(4)$$

Clearly, $(l, 0, 0)$ is a point on the line (I) [i.e., (1)]. Hence, the length $2c$ or shortest distance = perpendicular distance of $(l, 0, 0)$ from the plane (4).

$$\therefore \quad 2c = \frac{\left|l\left(\frac{1}{l}\right) - 0 - 0, +1\right|}{\sqrt{\left(\frac{1}{l}\right)^2 + \left(\frac{1}{m}\right)^2 + \left(\frac{1}{n}\right)^2}}$$

$$= \frac{2}{\sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}} = \frac{1}{c}$$

$$\text{Hence, } \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}$$

Question-3(d) Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \textbf{when } x \neq 0 \\ 1 & \textbf{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

[10 Marks]

Solution:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x}$$

$$= \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 \cdot$$

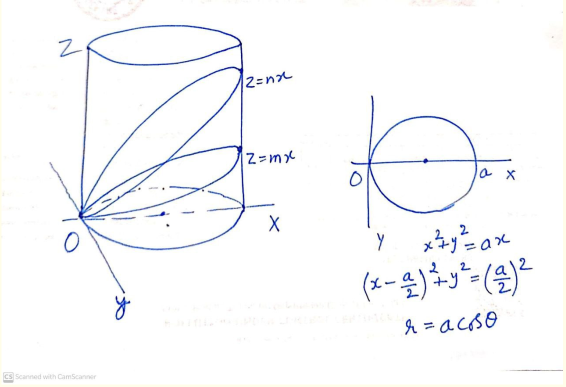
$$= 2$$

$$\text{So that } \lim_{x \to 0} f(x) \neq f(0)$$

Hence, the limit exists but is not equal to the value of the function at the origin. Thus, the function has a removable discontinuity at the origin.

Question-4(a) Find by triple integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$.

[10 Marks]



Solution:
Required Volume

$$V = \iint_R (nx - mx)dR$$

Changing to polar co-ordinates

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{a\cos\theta} (n-m)r\cos\theta(rdrd\theta) \\
&= (n-m)\int_0^{2\pi}\cos\theta\left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta \\
&= \frac{(n-m)a^3}{3}\int_0^{2\pi}\cos^4\theta d\theta \\
&= \frac{2\times 2(n-m)a^3}{3}\int_0^{\pi/2}\cos^4\theta d\theta \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx,\ \text{if } f(2a-x)=f(x)\right] \\
&= \frac{4}{3}(n-m)a^3 \times \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} \\
&= \frac{1}{4}(n-m)\pi a^3
\end{aligned}$$

Question-4(b) Show that all the spheres that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2+y^2+2fx+2gy+2hz=0$$

if $af+bg+ch=0$

[10 Marks]

Solution: The equation of spheres passing through the origin is

$$x^2+y^2+z^2+2ux+2vy+2wz=0$$

Now, the planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ is given as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$ (where k is any constant) The x -intercept of the above plane is given as

$$\frac{x_{\text{intercept}}}{a}+0+0=k$$

$$x_{\text{intercept}} = ak$$

$\therefore$ Coordinates of the point is $(ak, 0, 0)$ Similarly, y intercept is bk and z intercept is ck Thus, the four points through which the set of spheres passes are

$$(0,0,0),(ak,0,0),(0,bk,0)(0,0,ck)$$

Putting these values one by one in equation (1) we get

$$u = \frac{-ak}{2}, v = \frac{-bk}{2}, w = \frac{-ck}{2}$$

Hence, the equation of a system-spheres passing through the origin and each set of points where planes parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ cut the coordinate axes is

$$x^2 + y^2 + z^2 - k(ax + by + cz) = 0$$

The equation of other sphere cut orthogonally by the above system of spheres is given as

$$x^2 + y^2 + 2fx + 2gy + 2hz = 0$$

Thus, by the condition of orthogonally, i.e.,

$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

Putting the values, we get

$$2\left(\frac{-ak}{2}\right)(f) + 2\left(\frac{-bk}{2}\right)(g) + 2\left(\frac{-ck}{2}\right)(h) = 0 + 0$$

$$\Rightarrow -afk - bgk - chk = 0$$

$$\Rightarrow k(af + bg + ch) = 0$$

either $k = 0$ or $af + bg + ch = 0$. But $k \neq 0$. (as it will represent the given plane itself, not the plane parallel to the given plane.) Hence,

$$af + bg + ch = 0$$

Question-4(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$. Find its equation.

[10 Marks]

Solution: Let the equation of the plane, making equal intercepts on the positive parts of the axes, be

$$x + y + z = k$$

(where $k > 0$ and indicate the value of the intercept).

Now, it is given that the above plane touch the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 36$$

Therefore, by using the condition of tangency,

$$\left(\begin{array}{c}\text{i.e., when the plane b } x+my+nz= \\ \text{touches the ellipsoid } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1\end{array}\right)$$

given by

$$a^2l^2+b^2m^2+c^2n^2=p^2$$

we have [from (1)] Here, $l=m=n=1$ and $p=k$ Also, rearranging the given equation of ellipsoid as

$$\frac{x^2}{36}+\frac{4y^2}{36}+\frac{9z^2}{36}=1$$
$$\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}+\frac{z^2}{(2)^2}=1$$

$\therefore$ We get the values as

$$a=6, b=3, c=2$$

. Now, putting values in equation (2) we get

$$36(1)+9(1)+4(1)=k^2$$
$$\Rightarrow k^2=49$$
$$\Rightarrow \quad k=\pm 7$$

But

$$k \neq -7(\text{ as } k>0)$$
$$\therefore \quad k=7$$

Hence, the equation of the required plane is

$$x+y+z=7$$

Question-4(d) Evaluate the following in terms of Gamma function:

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}\,dx$$

[10 Marks]

Solution: Let

$$\mathrm{I}=\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}dx$$

Let $x^3=a^3\sin^2\theta$ when $x\to 0, \theta\to 0$

$$\Rightarrow \quad x=a\sin^{2/3}\theta \quad \text{when } x\to a, \theta\to\frac{\pi}{2}$$

$$\therefore \quad dx=\frac{2}{3}a\sin^{-1/3}\theta\cos\theta d\theta$$

$$\therefore \quad \mathrm{I} = \int_0^{\pi/2} \sqrt{\frac{a^3 \sin^2\theta}{a^3 - a^3\sin^2\theta}} d\frac{2a}{3}\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\frac{2}{3}a\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \frac{2}{3}a\int_0^{\pi/2}\sin^{2/3}\theta d\theta$$

Now, using formula

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{\sqrt{\left(\frac{p+1}{2}\right)}\sqrt{\left(\frac{q+1}{2}\right)}}{2\sqrt{\left(\frac{p+q+2}{2}\right)}}$$

$$\therefore \mathrm{I} = \frac{2}{3}a\frac{\sqrt{\frac{\left(\frac{2}{3}+1\right)}{2}}\sqrt{\left(\frac{0+1}{2}\right)}}{2\sqrt{\left(\frac{\frac{2}{3}+0+2}{2}\right)}}$$

$$\left[i.e., \text{ putting } p = \frac{2}{3} \text{ and } q = 0\right]$$

$$I = \frac{2}{3}a\frac{\sqrt{\frac{5}{6}}\sqrt{\frac{1}{2}}}{2\sqrt{\frac{4}{3}}}$$

$$= \frac{\frac{\sqrt{\pi}a}{3}\sqrt{\frac{5}{6}}}{\sqrt{\left(\frac{1}{3}+1\right)}}$$

$$= \frac{a\sqrt{\pi}}{3}\frac{\sqrt{\frac{5}{6}}}{\frac{1}{3}\sqrt{\frac{1}{3}}}\left(\text{using } \sqrt{n+1} = n\sqrt{n}\right)$$

$$\therefore I = a\sqrt{\pi}\frac{\sqrt{\frac{5}{6}}}{\sqrt{\frac{1}{3}}}$$

9.2 Section-B

Question-5(a) **Solve** $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x\sec y$

[8 Marks]

Solution:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$$

This is the general form of first degree linear differential equation. It can be rearranged in the form of

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where P and Q are function of x or constants.

Dividing by $(\sec y)$ to both sides, we get

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y}\left(\frac{1}{1+x}\right) = (1+x)e^x$$

$$\Rightarrow \cos y\frac{dy}{dx} - \sin y\left(\frac{1}{1+x}\right) = e^x(1+x)\ldots(1)$$

Let $\sin y = t$ On differentiation, we get

$$\cos y\frac{dy}{dx} = \frac{dt}{dx}$$

Putting in equation (1) we get

$$\frac{dt}{dx} - \frac{1t}{(1+x)} = e^x(1+x)$$

which is the general form of first order and first degree linear differential equation. Now, solving this linear differential equation

$$\text{Integrating factor (I.F.)} = \int_e \frac{-1}{(1+x)}dx$$
$$= e^{-\ln|1+x|}$$
$$\text{IF.} = \frac{1}{(1+x)}$$

$\therefore$ Solution of the differential equation (2) is given as

$$t(\text{L.F.}) = \int \text{Q}(\text{I}\cdot\text{F}\cdot)dx + \text{C}$$

where C is a constant of integration and Q is the right side of equation (2) Putting values of Q and I.F. we get

$$\frac{t}{1+x} = \int e^x(1+x)\cdot\frac{1}{(1+x)}dx + \text{C}$$
$$= \int e^x dx + \text{C} = e^x + \text{C}$$

since, the original differential equation is a function of x and y $\therefore$ Replace t by a function of y (which we let) Hence,

$$\frac{\sin y}{1+x} = e^x + C$$

Thus, the required solution is

$$\frac{\sin y}{1+x} - e^x = \text{C}$$

Question-5(b) Solve and find the singular solution of $x^3p^2 + x^2py + a^3 = 0$.

[8 Marks]

Solution: The given equation is $x^3p^2 + x^2py + a^3 = 0$ solving for y,

$$y = -xp - \frac{a^3}{x^2p}$$

Differentiating (2) with respect to (x) writing p for $\frac{dy}{dx}$, we have

$$\begin{aligned}
&p = -p - x\frac{dp}{dx} - a^3\left(\frac{-2}{x^3p} - \frac{1}{x^2p^2}\frac{dp}{dx}\right)\\
&\Rightarrow 2p + x\frac{dp}{dx} - \frac{2a^3}{x^3p} - \frac{a^3}{x^2p^2}\frac{dp}{dx} = 0\\
&\Rightarrow 2p\left(1 - \frac{a^3}{x^3p^2}\right) + x\frac{dp}{dx}\left(1 - \frac{a^3}{x^3p^2}\right) = 0\\
&\left(1 - \frac{a^3}{x^3p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0
\end{aligned}$$

Omitting the first factor since it does not involve $\dfrac{dp}{dx}$, we get

$$2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{p}dp + \frac{2}{x}dx = 0$$

Integrating, we get

$$\log p + 2\log x = \log \mathrm{C}$$

(where $\log \mathrm{C}$ is an integration constant)

$$\Rightarrow \quad \log\left(px^2\right) = \log C$$

$$\Rightarrow \quad px^2 = C$$

$$p = \frac{C}{x^2}\ldots(3)$$

Eliminating p between (1) and (3), the required general solution is

$$x^3\frac{C^2}{x^4} + x^2y\left(\frac{C}{x^2}\right) + a^3 = 0$$

$$\Rightarrow \quad \frac{C^2}{x} + Cy + a^3 = 0$$

$$\Rightarrow \quad C^2 + xyC + a^3x = 0$$

By (4), C-discriminant relation is

$$-(4)(xy)^2 - 4(1)\left(a^3x\right) = 0$$

$$\Rightarrow \quad x\left(xy^2 - 4a^3\right) = 0$$

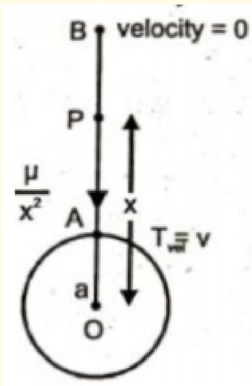
Now, $x = 0$ and $xy^2 - 4a^3 = 0$ both satisfy equation (1) and hence required singular solutions are $x = 0$ and $xy^2 - 4a^3 = 0$

Question-5(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

[8 Marks]

Solution:



Let O be the centre of the earth and A be the point of projection on the earth's surface. If P be the position of the particle at any time t, such that OP $= x$, then the acceleration at

$$\text{P} = \frac{\mu}{x^2}$$

directed towards 0.

$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^2}$$

(Negative sign indicates that acceleration acts in the direction of x decreasing.) But at the point A, on the surface of the earth, $x = a$. and $\frac{d^2x}{dt^2} = -g$

$$\therefore \quad -g = \frac{-\mu}{2} \text{ or } \mu = a^2 g$$

$$-g = \frac{-\mu}{a^2}$$

$$\frac{d^2x}{dt^2} = \frac{-a^2 g}{x^2}$$

Multiplying by $2\left(\frac{dx}{dt}\right)$ and integrating with respect to (t) we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + \text{C}$$

where C is a constant. But when

$$x \to \infty, \frac{dx}{dt} \text{ (velocity) } \to 0$$

$$\therefore C = 0$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x}$$

$$\therefore \mathrm{C} = 0$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x}$$

(Here +ve sign is taken because the particle is moving in the direction of x increasing)

$$\Rightarrow \quad \frac{dx}{dt} = a\sqrt{\frac{2g}{x}}$$

Separating the variables, we have

$$dt = \frac{1}{a\sqrt{2g}}\sqrt{x}dx$$

Integrating between the limits $x = a$ to $x = a + h$, the required time t to reach height h is given by

$$t = \frac{1}{a\sqrt{2g}}\int_a^{a+h}\sqrt{x}dx = \frac{1}{a\sqrt{2g}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$= \frac{1}{3a}\sqrt{\frac{2}{g}}\left[(a+h)^{3/2} - a^{3/2}\right]$$

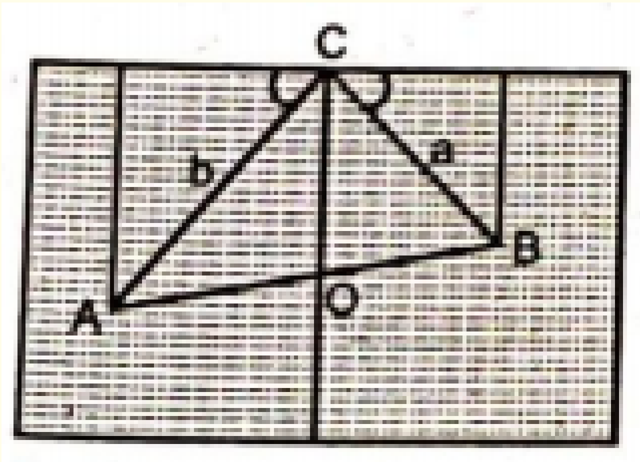
$$= \frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right]$$

Question-5(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3 + 3ab^2 : a^3 + 3a^2b$ where a and b are the sides BC & AC respectively.

[8 Marks]

Solution: Let the vertical through C meets AB at O. then

$$\angle \mathrm{ACO} = \angle \mathrm{BCO} = \frac{1}{2}\angle \mathrm{C}$$



Area of ΔAOC $= \frac{1}{2}\mathrm{AC}\cdot OC\sin\angle\mathrm{ACO}$ & Area of ΔBOC $= \frac{1}{2}\mathrm{BC}\cdot\mathrm{OC}\sin\angle\mathrm{BCO}$

The depth of the centre of gravity (C.G.) of ΔAOC below the surface of the liquid

$$= \frac{1}{3}(\mathrm{AC}\cos\angle\mathrm{ACO} + \mathrm{OC})$$

and the depth of the C.G of ΔBOC below the surface of the liquid

$$= \frac{1}{3}(\mathrm{BC}\cos\angle\mathrm{BCO} + \mathrm{OC})$$

$$\therefore \frac{\text{Pressure on } \Delta\mathrm{AOC}}{\text{Pressure on } \Delta\mathrm{BOC}} \frac{1}{2}AC\cdot OC\sin\angle ACO\cdot\frac{1}{3}(AC\cos\angle ACO$$
$$= \frac{1+\mathrm{OC})\cdot w}{\frac{1}{2}\mathrm{BC}\cdot\mathrm{OC}\sin\angle\mathrm{BCO}\cdot\frac{1}{3}(\mathrm{BC}\cos\angle\mathrm{BCO}} + \mathrm{OC})\cdot w$$
$$= \frac{\left(\frac{1}{2}bOC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(b\cos\frac{C}{2}+OC\right)\right)}{\left(\frac{1}{2}aOC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(a\cos\frac{C}{2}+OC\right)\right)}$$
$$= \frac{b\left(b\cos\frac{C}{2}+OC\right)}{a\left(a\cos\frac{C}{2}+OC\right)}$$

From Δ 's BCO and ACO, we have

$$\frac{\mathrm{CO}}{\sin\mathrm{B}} = \frac{\mathrm{OB}}{\sin\frac{\mathrm{C}}{2}} \text{ and } \frac{\mathrm{CO}}{\sin\mathrm{A}} = \frac{\mathrm{AO}}{\sin\frac{\mathrm{C}}{2}} \ldots(1)$$

Also

$$\frac{\mathrm{AO}}{b} = \frac{\mathrm{OB}}{a}$$
$$= \frac{\mathrm{AO}+\mathrm{OB}}{b+a}$$
$$= \frac{c}{b+a}\ldots(2)$$

$$\begin{aligned}
\therefore \text{ The required ratio } &= \frac{b\left(b\cos\frac{C}{2} + \frac{OB\sin B}{\sin\frac{C}{2}}\right)}{a\left(a\cos\frac{C}{2} + \frac{AO\sin A}{\sin\frac{C}{2}}\right)} \\
&= \frac{b(b\sin C + 2OB\sin B)}{a(a\sin C + 2OA\sin A)} \\
&= \frac{b\left(b\sin C + 2OB\frac{b\sin C}{c}\right)}{a\left(a\sin C + 2OAa\frac{\sin C}{c}\right)} \\
&= \frac{b^2}{a^2}\cdot\left(\frac{c+2OB}{c+2OA}\right) \\
&= \frac{b^2}{a^2}\cdot\frac{\left(c+\frac{2ac}{b+a}\right)}{\left(c+\frac{2bc}{b+a}\right)}\left[\text{using } = \frac{b^2}{a^2}\cdot\left[\frac{c(a+b)+2ac}{c(a+b)+2bc}\right]\right] \\
&= \frac{b^2(3a+b)}{a^2(a+3b)} = \frac{b^3+3ab^2}{a^3+3a^2b}
\end{aligned}$$

Question-5(e) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, prove that grad u, grad v and grad w are coplanar.

[8 Marks]

Solution: Given $u = x+y+z, v = x^2+y^2+z^2$, and $w = yz+zx+xy$

$$\begin{aligned}
\operatorname{grad} u &= \nabla u \\
&= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z) \\
&= \hat{i}\frac{\partial}{\partial x}(x+y+z) + \hat{j}\frac{\partial}{\partial y}(x+y+z) + \hat{k}\frac{\partial}{\partial z}(x+y+z) \\
\nabla u &= \hat{i}+\hat{j}+\hat{k}
\end{aligned}$$

Now,

$$\begin{aligned}
\operatorname{grad} v &= \hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) \\
\nabla v &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}
\end{aligned}$$

Now,

$$\begin{aligned}
\operatorname{grad} w &= \hat{i}\frac{\partial}{\partial x}(yz+zx+xy) + \hat{j}\frac{\partial}{\partial y}(yz+zx+xy) + \hat{k}\frac{\partial}{\partial z}(yz+zx+xy) \\
\nabla w &= (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}
\end{aligned}$$

To prove that $\nabla u, \nabla v$ and ∇w coplanar, we must have the following condition to be true. i.e.,

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

On carrying out operations on LHS, we get $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, we get

$$\text{LHS} = \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ y-x & z-y & x+y \end{vmatrix}$$

Solving the determinant we get

$$\begin{aligned} \text{LHS} &= 1[2(x-y)(z-y) - 2(y-z)(y-x)] \\ &= 2[(x-y)(z-y) - (x-y)(z-y)] \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Hence, we can say that grad u, grad v and grad w are coplanar.

Question-6(a) Solve:

$$x^2 y \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

[10 Marks]

Solution:

$$x^2 y \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

The given equation can be rewritten as

$$x^2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - \left[2xy\frac{dy}{dx} - y^2\right] = 0$$

$$\Rightarrow \left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - \frac{\left[2xy\left(\frac{dy}{dx}\right) - y^2\right]}{x^2} = 0$$

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) - \frac{d}{dx}\left(\frac{y^2}{x}\right) = 0$$

Integrating, we get

$$y\frac{dy}{dx} - \frac{y^2}{x} = C_1$$

This is Bernoulli form $\therefore$ Putting $y^2 = v$, so that

$$2y\frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ (1) becomes

$$\frac{1}{2}\frac{dv}{dx} - \frac{v}{x} = C_1 \Rightarrow \frac{dv}{dx} - \frac{2v}{x} = 2C_1$$

This is the first order linear differential equation. Its I.F.

$$\begin{aligned} \text{I.F.} &= e^{-\int \frac{2}{x} dx} \\ &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

Hence, solution is

$$\begin{aligned} v\left(\frac{1}{x^2}\right) &= 2C_1 \int \frac{1}{x^2} dx + C_2 \\ \frac{y^2}{x^2} &= \frac{-2C_1}{x} + C_2 \\ \Rightarrow \quad y^2 &= x\left(C_2 x - 2C_1\right) \end{aligned}$$

Question-6(b) Find the value of $\iint_s (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{ds}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when
$\vec{F} = (y^2 + z^2 - x)\,\vec{i} + (z^2 + x^2 - y^2)\,\vec{j} + (x^2 + y^2 - z^2)\,\vec{k}$

[10 Marks]

Solution: By Stokes' Theorem

$$I = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

Here,

$$\vec{F} = \left(y^2 + z^2 - x^2\right) i + \left(z^2 + x^2 - y^2\right) j + \left(x^2 + y^2 - z^2\right) k$$
$$d\vec{r} = i dx + j dy + k dz$$

Surface $S : x^2 + y^2 - 2ax + az = 0$ with bounding curve lying on $z = 0$.

$$\begin{aligned} \therefore \quad \text{Boundary} \quad C : \quad & x^2 + y^2 - 2ax = 0; z = 0 \\ \text{i.e.} \quad (r\cos\theta)^2 + (r\sin\theta)^2 &- 2ar\cos\theta = 0 \\ r = 2a\cos\theta, &\quad r = 0 \end{aligned}$$

r varies from 0 to $2a\cos\theta$ and θ varies from 0 to 2π.
Hence,

$$\begin{aligned} I &= \int_C \left(y^2 + z^2 - x^2\right) dx + \left(z^2 + x^2 - y^2\right) dy + \left(x^2 + y^2 - z^2\right) \\ &= \int_C \left(y^2 - x^2\right) dx + \left(x^2 - y^2\right) dy \qquad (\because z = 0 \text{ on } C) \\ &= \int_C \left(x^2 - y^2\right)(dy - dx) \end{aligned}$$

$$\begin{aligned} \text{Now, C:} \quad & (x-a)^2 + y^2 = a^2 \\ \therefore \quad x - a = a\cos\theta \quad &; \quad y = a\sin\theta \\ x = a + a\cos\theta \quad &; \quad y = a\sin\theta \end{aligned}$$

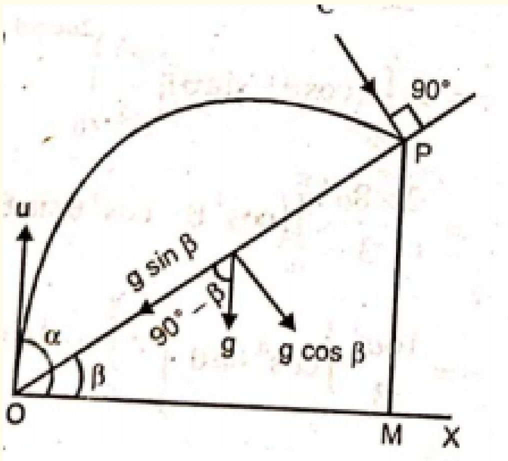
$$\begin{aligned}
\Rightarrow I &= \int_0^{2\pi} \left[a^2(1+\cos\theta)^2 - a^2\sin^2\theta\right]\left[a\cos\theta + a\sin\theta\right]d\theta \\
&= \int_0^{2\pi} a^3\left(1+\cos^2\theta + 2\cos\theta - \sin^2\theta\right)(\cos\theta+\sin\theta)d\theta \\
&= a^3\int_0^{2\pi}\left(2\cos^2\theta + 2\cos\theta\right)(\cos\theta+\sin\theta)d\theta \\
&= 2a^3\int_0^{2\pi}\left[\cos^3\theta + \cos^2\theta + \left(\cos^2\theta+\cos\theta\right)\sin\theta\right]d\theta \\
&= 2a^3\left[2\int_0^{\pi}\left(\cos^3\theta+\cos^2\theta\right)d\theta + \int_0^{2\pi}\left(\cos^3\theta+\sin\theta\right)\sin\theta d\theta\right] \\
&= 2a^3\left[2\times 2\int_0^{2}\cos^2\theta d\theta + 0\right] \\
&= 8a^3\times\frac{1}{2}\times\frac{\pi}{2} = 2\pi a^3
\end{aligned}$$

Question-6(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is $2u\ g\sqrt{\left(1+3\sin^2\beta\right)}$ range on the plane is $\frac{2u^2}{g}\cdot\frac{\sin\beta}{1+3\sin^2\beta}$ and the vertical height of the point struck is $\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$ above the point of projection.

[10 Marks]

Solution: Let O be the point of projection, u be the velocity of projection, α be the angle of projection and P be the point where the particle strikes the plane at right angles. Let T be the time of flight from O to P. Then by the formula for the time of flight in an inclined plane, we have

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$



Since the particle strikes the inclined plane at right angle at P, therefore the velocity of the particle at P along inclined plane is zero.

Also, the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the incline plane is $g\sin\beta$ downwards. So, considering the motion of the particle from O to P along the inclined plane and using the formula $v = u + at$, we have

$$0 = u\cos(\alpha-\beta) - g\sin\beta \mathrm{T}$$

$$\mathrm{T} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

Equating the values of T from (1) and (2) we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\tan(\alpha-\beta) = \frac{1}{2}\cot\beta$$

The condition for striking the plane at right angles.

(i) To prove

$$\mathrm{T} = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

Proof: From (2) we have

$$\begin{aligned}
\mathrm{T} &= \frac{u}{g\sin\beta}\cos(\alpha-\beta)\\
&= \frac{u}{g\sin\beta\sec(\alpha-\beta)}\\
&= \frac{u}{g\sin\beta\sqrt{1+\tan^2(\alpha-\beta)}}\\
&= \frac{u}{g\sin\beta\sqrt{1+\frac{1}{4}\cot^2\beta}} [\text{ substituting value from (3)}]\\
&= \frac{2u\sin\beta}{g\sin\beta\sqrt{4\sin^2\beta+\cos^2\beta}}\\
&= \frac{2u}{\sqrt[3]{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}
\end{aligned}$$

$$\therefore \mathrm{T} = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

(ii) Range, on the plane

$$R = \frac{2u^2}{8}\frac{\sin\beta}{1+3\sin^2\beta}$$

Proof: Let R be the range on the inclined plane then $R = OP$ considering the motion from O to P along the inclined plane and using the formula $v^2 = u^2 + 2as$, we have

$$\begin{aligned}
0 =& u^2\cos^2(\alpha-\beta) - 2g\sin\beta \mathrm{R} \\
\mathrm{R} =& \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} \\
=& \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)} \\
=& \frac{u^2}{2g\sin\beta\left[1+\tan^2(\alpha-\beta)\right]} \\
=& \frac{u^2}{2g\sin\beta\left[1+\frac{1}{4}\cot^2\beta\right]} \quad [\text{ From } (3)] \\
=& \frac{4u^2\sin^2\beta}{2g\sin\beta\left(4\sin^2\beta+\cos^2\beta\right)}
\end{aligned}$$

Hence, Range, $\mathrm{R} = \dfrac{2u^2\sin\beta}{g\left(1+3\sin^2\beta\right)}$

(iii) The vertical height of the point struck is

$$\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$$

Proof:

$$\begin{aligned}
\text{The vertical height of P above O} &= \mathrm{PM} \\
&= \mathrm{OP}\sin\beta \\
&= \mathrm{R}\sin\beta \\
&= \frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}
\end{aligned}$$

Question-6(d) Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2\mathrm{c}.$

[10 Marks]

Solution: Let $D \equiv \frac{d}{dx}$, then the given differential equation becomes

$$\left(D^4 + 2D^2 + 1\right) y = x^2\cos x$$

This equation is the differential equation of first order with constant coefficients. It is solved by the following method. The auxiliary equation is

$$\begin{aligned}
m^4 + 2m^2 + 1 &= 0 \\
\Rightarrow \quad \left(m^2+1\right)^2 &= 0 \\
\Rightarrow \quad m &= \pm i
\end{aligned}$$

Thus, the complementary function is given by

$$y = (\mathrm{C}_1 + \mathrm{C}_2x)\cos x + (\mathrm{C}_3 + \mathrm{C}_4x)\sin x$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Now, the particular integral is given by

$$\begin{aligned}
y &= \frac{1}{(1+2D^2+D^4)} x^2 \cos x \\
&= \frac{1}{(D^2+1)^2} x^2 \cos x \\
y &= \text{ Real part of } \left(\frac{1}{(D^2+1)^2} x^2 e^{ix} \right) \left[\because e^{ix} = \cos x + i \sin x \right]
\end{aligned}$$

Now, solving

$$\frac{1}{(\mathrm{D}^2+1)^2} x^2 e^{ix} = e^{ix} \frac{1}{[(\mathrm{D}+i)^2+1]^2} x^2 \left(\begin{array}{c} \text{Using formula } \frac{1}{f(\mathrm{D})} e^{ax} \text{ V} \\ = e^{ax} \cdot \frac{1}{f(\mathrm{D}+a)} \mathrm{V} \end{array} \right)$$

where, V is any function of x

Here $\mathrm{V} = x^2$ $f(\mathrm{D}) = (\mathrm{D}^2+1)^2$ & $a = i$

$$\begin{aligned}
&= e^{ix} \frac{1}{[\mathrm{D}^2+i^2+2i\mathrm{D}+1]^2} x^2 \\
&= e^{ix} \frac{1}{(\mathrm{D}^2+2i\mathrm{D})^2} x^2 \quad (\because i^2 = -1) \\
&= e^{ix} \frac{1}{(2iD)^2 \left[1+\frac{D^2}{2iD}\right]^2} x^2 \\
&= e^{ix} \frac{1}{-4D^2} \left[1+\frac{D}{2i}\right]^{-2} x^2 \\
&= \frac{-1}{4} e^{ix} \frac{1}{\mathrm{D}^2} \left(1+(-2)\left(\frac{\mathrm{D}}{2i}\right) + \frac{(-2)(-2-1)}{2!} \left(\frac{\mathrm{D}}{2i}\right)^2 + \dots \right) x^2 \\
&\left[\text{using expansion of } (1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\dots \right] \\
&= \frac{-1}{4} e^{ix} \frac{1}{D^2} \left(1-\frac{D}{i}-\frac{3}{4}D^2+\dots\right) x^2 \\
&= \frac{-e^{ix}}{4} \frac{1}{D^2} \left[x^2-\frac{1}{i}(2x)-\frac{3}{4}(2)+0+0+\dots\right] \\
&= \frac{-e^{ix}}{4} \frac{1}{D^2} \left[\left(x^2-\frac{3}{2}\right)+i(2x)\right] \\
&= \frac{-e^{ix}}{4} \left[\frac{1}{D} \int \left(x^2-\frac{3}{2}\right) dx + 2i\frac{1}{D} \int x dx\right] \left[\because \frac{1}{D} = \int dx\right] \\
&= \frac{-e^{ix}}{4} \left[\int \left(\frac{x^3}{3}-\frac{3x}{2}\right) dx + 2i \int \frac{x^2}{2} dx\right] \\
&= \frac{-e^{ix}}{4} \left[\frac{x^4}{12}-\frac{3x^2}{4}+2i\left(\frac{x^3}{6}\right)\right] \\
&= \frac{-e^{ix}}{4} \left[\frac{x^4}{12}-\frac{3x^2}{4}+\frac{ix^3}{3}\right]
\end{aligned}$$

Note: While we want the real part of (1), we must open e^{ix} as $(\cos x + i \sin x)$

$\therefore$ (1) equation can be arranged as

$$= \frac{-1}{4}(\cos x + i\sin x)\left[\frac{x^4 - 9x^2}{12} + \frac{i}{3}x^3\right]$$
$$= \left(\frac{9x^2 - x^4}{48} - \frac{i}{12}x^3\right)(\cos x + i\sin x)$$
$$= \left[\left(\frac{9x^2 - x^4}{48}\right)\cos x + \frac{1}{12}x^3\sin x\right] + i\left[\frac{-1}{12}x^3\cos x + \frac{\sin x}{48}\left(9x^2 - x^4\right)\right]$$

The real part of this is the particular integral
$\therefore$ Particular Integral,
$$y = \frac{x^2}{48}\cos x\left(9 - x^2\right) + \frac{1}{12}x^3\sin x$$
Thus, the general solution is given by
$$y = \text{C.F} + \text{P.I}$$
$$\therefore y = (\mathrm{C}_1 + \mathrm{C}_2 x)\cos x + (\mathrm{C}_3 + \mathrm{C}_4 x)\sin_2 + \frac{x^2}{48}\left(9 - x^2\right)\cos x + \frac{x^3}{12}\sin x$$
is the required solution.

Question-7(a) A particle is moving with central acceleration $\mu\left[r^5 - c^4 r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4 + y^4 = c^4$.

[14 Marks]

Solution: Here, the central acceleration,
$$p = \mu\left[r^5 - c^4 r\right] = \mu\left[\frac{1}{u^5} - \frac{c^4}{u}\right]\left(\because r = \frac{1}{u}\right)$$
$\therefore$ The differential equation of the path is
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \frac{\mu}{u^2}\left[\frac{1}{u^5} - \frac{c^4}{u}\right]$$
$$\Rightarrow u^2 = h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \mu\left[\frac{1}{u^7} - \frac{c^4}{u^3}\right]$$
Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$, we get
$$\dot{h}^2\left[2\left(\frac{du}{d\theta}\right)u + 2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$
$$\frac{h^2 d}{d\theta}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

Now, integrating above equation with respect to $'\theta'$, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\int\frac{p}{u^2}du+\mathrm{A}$$

where A is a constant

$$v^2=h\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\int\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right)+A$$

$$\begin{aligned} v^2&=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]\\ &=\mu\left(\frac{-1}{3u^6}+\frac{c^4}{u^2}\right)+\mathrm{A}\end{aligned}$$

But initially when $r=c$ i.e. $u=\frac{1}{c}$, $\frac{du}{d\theta}=0$ (at apse) and $v=c^3\sqrt{\frac{2\mu}{3}}$. $\therefore$ From (1) we have

$$\frac{2\mu c^6}{3}=h^2\cdot\frac{1}{c^2}=\mu\left[\frac{-c^6}{3}+c^6\right]+\mathrm{A}$$

$$\therefore h^2=\frac{2}{3}\mu c^8,\ \mathrm{A}=0$$

Substituting the values of h^2 and A, in (1) we have

$$\frac{2}{3}\mu c^8\cdot\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{-1}{3u^6}+\frac{c^4}{u^2}\right]$$

$$\begin{aligned} c^8\left(\frac{du}{d\theta}\right)^2&=\frac{-1}{2u^6}+\frac{3c^4}{2u^2}-c^8u^2\\ &=\frac{1}{u^6}\left[\frac{-1}{2}+\frac{3}{2}c^4u^4-c^8u^8\right]\\ \Rightarrow c^8\left(\frac{du}{d\theta}\right)^2&=\frac{1}{u^6}\left[\frac{-1}{2}-\left(c^8u^8-\frac{3}{2}c^4u^4\right)\right]\\ &=\frac{1}{u^6}\left[\frac{-1}{2}-\left(c^4u^4-\frac{3}{4}\right)^2+\frac{9}{16}\right]\\ c^8\left(\frac{du}{d\theta}\right)^2&=\frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2\right]\\ \therefore c^4u^3\frac{du}{d\theta}&=\sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}\\ d\theta&=\frac{c^4u^3du}{\sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}}\end{aligned}$$

Putting $c^4u^4-\frac{3}{4}=z$, so that $4c^4u^3du=dz$ we have

$$4d\theta=\frac{dz}{\sqrt{\left(\frac{1}{4}\right)^2-z^2}}$$

Integrating,

$$4\theta + \mathrm{B} = \sin^{-1}\left(\frac{z}{1/4}\right)$$

$$\Rightarrow \quad 4\theta + \mathrm{B} = \sin^{-1}(4z)$$

where B is a constant

$$\Rightarrow \quad 4\theta + \mathrm{B} = \sin^{-1}\left(4c^4u^4 - 3\right)$$

But initially when $u = \frac{1}{c}, \theta = 0$

$$\therefore \quad B = \sin^{-1}(1)$$

$$\Rightarrow \quad B = \frac{\pi}{2}$$

$$\therefore \quad 4\theta + \frac{\pi}{2} = \sin^{-1}\left(4c^4u^4 - 3\right)$$

$$\Rightarrow \sin\left(\frac{\pi}{2} + \dot{4}\theta\right) = 4c^4u^4 - 3$$

$$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$$

$$\Rightarrow \quad 4c^4u^4 = 3 + \cos 4\theta$$

$$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$$

$$\Rightarrow 4c^4u^4 = 3 + \cos 4\theta$$

$$\Rightarrow \quad \frac{4c^4}{r^4} = 3 + \cos 4\theta$$

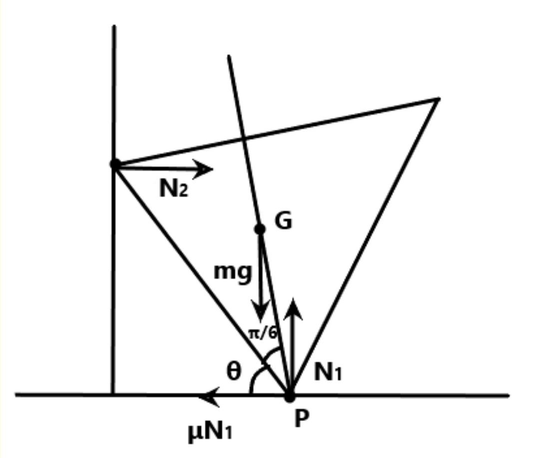
$$\begin{aligned}
\Rightarrow \quad 4c^4 &= r^4\left[3 + 2\cos^2 2\theta - 1\right] \\
&= 2r^4\left[1 + \cos^2 2\theta\right] \\
&= 2r^4\left[\left(\cos^2\theta + \sin^2\theta\right)^2 + \left(\cos^2\theta - \sin^2\theta\right)^2\right] \\
&= 4r^4\left(\cos^4\theta + \sin^4\theta\right) \\
\therefore \quad c^4 &= r^4\left(\cos^4\theta + \sin^4\theta\right) \\
\Rightarrow \quad c^4 &= (r\cos\theta)^4 + (r\sin\theta)^4 \\
\Rightarrow \quad c^4 &= x^4 + y^4 (\because x = r\cos\theta \text{ and } y = r\sin\theta)
\end{aligned}$$

Hence, $x^4 + y^4 = c^4$ is the equation of path.

Question-7(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}\ \mu$, being the coefficient of friction.

[14 Marks]

Solution: Let the side of equilateral triangular plate be $'a'$ and G be its center of gravity. N_1 = Normal reaction by rough horizontal plane.



$$N_1 = \text{mg}, \text{ where m is mass of plate.}$$

$N_2 =$ Normal reaction by small vertical wall

$$N_2 = \mu N_1 = \mu(mg)$$

Taking moments about point P

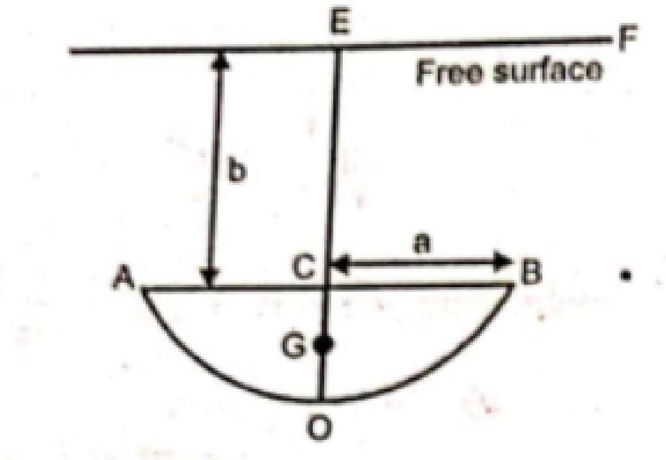
$$mg - CAP\cos\left(\theta + \frac{\pi}{6}\right) = N_2 \times a\sin\theta$$
$$\text{mg} \cdot \frac{\alpha}{\sqrt{3}}\left(\cos\theta \cdot \frac{\sqrt{3}}{2} - \sin\theta \frac{1}{2}\right) = \alpha\sin\theta.\mu mg$$

$$\sqrt{3}\cos\theta - \sin\theta = 2\sqrt{3}\mu\sin\theta$$
$$\sqrt{3}\cos\theta = (1 + 2\sqrt{3}\mu)\sin\theta$$
$$\Rightarrow \cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

Question-7(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2 + 4b^2\right) + 32ab}{4a + 3\pi b}$$

[13 Marks]



Solution:

Depth of the centre of pressure of the semicircular area $= \dfrac{k^2}{h}$, where k is the radius of gyration about the line EF on the free surface and $h =$ depth of CG of the lamina below EF = EG

$$k^2 = "k^2$$

about parallel axis through

$$G + (\text{EG})^2$$

Now,

$$\text{CG} = \frac{4a}{3\pi}$$

and hence

$$\text{EG} = b + \frac{4a}{3\pi}$$

$$\Rightarrow \text{EG} = h = \frac{4a + 3b\pi}{3\pi} \ldots (1)$$

$$\therefore k^2 = "k^2"$$

about

$$\begin{aligned}\text{AB} - (\text{CG})^2 + (\text{EG})^2 &= \frac{a^2}{4} - \left(\frac{4a}{3\pi}\right)^2 + \left(\frac{4a+3b\pi}{3\pi}\right)^2 \\ &= \frac{9\pi^2a^2 + 36b^2\pi^2 + 96ab\pi}{36\pi^2} \\ \therefore k^2 &= \frac{3\pi\left(a^2+4b^2\right)+32ab}{12\pi} \ldots (2)\end{aligned}$$

From (1) and (2) we get Depth of the centre of pressure

$$\begin{aligned} &= \frac{k^2}{h} \\ &= \left(\frac{3\pi\left(a^2+4b^2\right)+32ab}{12\pi}\right) / \left(\frac{4a+3b\pi}{3\pi}\right) \\ &= \frac{1}{4}\left(\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}\right)\end{aligned}$$

Question-8(a) Solve $x = y\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$.

[10 Marks]

Solution: Solving the given differential equation fot x, we get

$$x = py + ap^2 \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = p + y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

$$\text{or} \quad \frac{1-p^2}{p} = y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

$$\text{or} \quad \frac{1-p^2}{p}\frac{dy}{dp} - y = 2ap, \qquad \text{multiplying both sides by dy/dp}$$

$$\text{or} \quad \frac{dy}{dp} - \frac{1}{p^2-1}y = -\frac{2ap^2}{p^2-1} \qquad \ldots(2)$$

which is a linear differential equation.
Here the I.F. $= e^{\int (p/(p^2-1))dp} = e^{\frac{1}{2}\log(p^2-1)} = (p^2-1)^{1/2}$ $\therefore$ the solution of (2) is

$$\begin{aligned}
y\left(p^2-1\right)^{1/2} &= \int \frac{-2ap^2}{p^2-1}\left(p^2-1\right)^{1/2} dp + c \\
&= -2a\int \frac{(p^2-1)+1}{\sqrt{(p^2-1)}}dp + c \\
&= -2a\int \left[\sqrt{(p^2-1)} + \frac{1}{\sqrt{(p^2-1)}}\right] dp + c \\
&= -2a\left[\frac{1}{2}p\sqrt{(p^2-1)} - \frac{1}{2}\cosh^{-1}p + \cosh^{-1}p\right] + c \\
&= -ap\sqrt{(p^2-1)} - a\cosh^{-1}p + c \\
\text{or} \quad y &= \frac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap\,. \qquad \ldots(3)
\end{aligned}$$

Substituting this value of y in (1), we get,

$$x = \left(\frac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap\right) + ap^2$$

$$\Rightarrow x = \frac{p(c - a\cosh^{-1}p)}{\sqrt{(p^2-1)}} \qquad \ldots(4)$$

The equations (3) and (4) constitute the parametric equations of the required solution.

Question-8(b) Find the value of the line integral over a circular path given by
$x^2 + y^2 = a^2, z = 0$ **where the vector field,** $\vec{F} = (\sin y)\vec{i} + x(1 + \cos y)\vec{j}$.

[10 Marks]

Solution: The line integral over a circular path given by C over vector field

$$\overrightarrow{\mathrm{F}} = \int_{\mathrm{C}} \overrightarrow{\mathrm{F}} \cdot dr$$

Here, C is given as $x^2 + y^2 = a^2, z = 0$ and

$$\overrightarrow{\mathrm{F}} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$$

As we know that $\vec{r}$ is a position vector and is given as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\therefore \quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\begin{aligned}
\text{Thus, the required integral value } &= \oint_{\mathrm{C}} [\sin y\hat{i} + x(1 + \cos y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
&= \oint_{\mathrm{C}} \sin y dx + x(1 + \cos y)dy \\
&= \oint_{\mathrm{C}} \mathrm{M}dx + \mathrm{N}dy
\end{aligned}$$

Now, by Green's theorem in plane we have

$$\iint\limits_{\mathbb{R}} \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y} \right) dxdy = \oint_{\mathrm{C}} \mathrm{M}dx + \mathrm{N}dy$$

Here $M = \sin y, N = x(1 + \cos y)$

$$\therefore \frac{\partial \mathrm{M}}{\partial y} = \cos y,$$
$$\frac{\partial \mathrm{N}}{\partial x} = 1 + \cos y$$

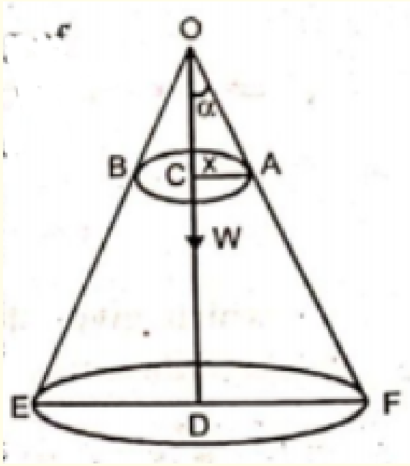
Hence, the given line integral is equal to $= \iint_R (1 + \cos y - \cos y)dxdy = \iint_R dxdy =$ Area of the circle $C = \pi a^2$

Question-8(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$$

[10 Marks]

Solution: OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical.



A heavy elastic string of natural length $2\pi a$ placed round the cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x.

The weight W of the string acts at its centre of gravity C.

Let T be the tension in this string. Give the string a small displacement in which x changes to $x+\delta x$. The point O remains fixed, the point C is slightly displaced.

$\angle\alpha$ is fixed and the length of the string slight changed. We have the length of the string AB in the form of a circle of radius x is $2\pi x$ and so the work done by the tension T of this string is $-\mathrm{T}\delta(2\pi x)$.
Also, the depth of the point of application C of the weight W below the fixed point O

$$\mathrm{OC}=\mathrm{AC}\cot\alpha=x\cot\alpha$$

the work done by the weight W during this small displacement $=\mathrm{W}\delta(x\cot\alpha)$

Since the reactions at the various points of contact do work, thus by the principle of virtual work,

$$\begin{aligned}-\mathrm{T}\delta(2\pi x)+\mathrm{W}\delta(x\cot\alpha)&=0\\ \Rightarrow -2\pi\mathrm{T}\delta x+\mathrm{W}\cot\alpha\delta x&=0\\ (-2\pi\mathrm{T}+\mathrm{W}\cot\alpha)\delta x&=0\\ \Rightarrow\quad -2\pi\mathrm{T}+\mathrm{W}\cot\alpha&=0(\because\delta x\neq 0)\end{aligned}$$

$$\mathrm{T}=\frac{\mathrm{W}\cot\alpha}{2\pi}$$

Now, by Hooke's law the tension T in the elastic string AB is given by

$$\mathrm{T} = \lambda\frac{(2\pi x - 2\pi a)}{2\pi a}$$
$$\mathrm{T} = \lambda\frac{x - a}{a}$$

Equating the two values of T we get

$$\frac{\mathrm{W}\cot\alpha}{2\pi} = \lambda\frac{(x-a)}{a}$$
$$\Rightarrow \quad x - a = \frac{a}{2\pi\lambda}\mathrm{W}\cot\alpha$$
$$\Rightarrow \quad x = a\left(1 + \frac{w}{2\pi\lambda}\cot\alpha\right)$$

which gives the radius of the string in equilibrium.

Question-8(d) **Solve** $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1-x)^{-2}$.

[10 Marks]

Solution: Putting $x = e^z$ and denoting d/dz by D', the given differenitial equation becomes

$$\left[D'\left(D'-1\right)+3D'+1\right]y=\frac{1}{\left(1-e^z\right)^2}$$

$$\text{or}\quad \left(D'+1\right)^2 y=\frac{1}{\left(1-e^z\right)^2}$$

$$\text{A.E. is}\quad (m+1)^2=0.\quad\Rightarrow\quad m=-1,-1$$

$$\therefore\quad C.F.=\left(c_1+c_2z\right)e^{-z}=\left(c_1+c_2\log x\right)\cdot x^{-1}$$

$$\text{P.L.}\ =\frac{1}{\left(D'+1\right)^2}\frac{1}{\left(1-e^z\right)^2}=\frac{1}{\left(D'+1\right)}\cdot\frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^z\right)^2}\right]$$

$$\text{Let}\ \frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^z\right)^2}\right]=v\ \text{or}\ \left(D'+1\right)v=\frac{1}{\left(1-e^z\right)^2}$$

$$\text{or}\quad \frac{dv}{dz}+v=\frac{1}{\left(1-e^z\right)^2},\qquad \text{which is a linear equation.}$$

$$\text{I.F.}=e^{\int dz}=e^z$$

$$\therefore\quad ve^z=\int e^z\left(1-e^z\right)^{-2}dz=\left(1-e^z\right)^{-1}$$

$$\text{or}\qquad v=\frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^z\right)^2}\right]=e^{-z}\left(1-e^z\right)^{-1}$$

$$\begin{aligned}\therefore\quad \text{P.I.}\ &=\frac{1}{\left(D'+1\right)}e^{-z}\left(1-e^z\right)^{-1}\\ &=e^{-z}\int e^ze^{-z}\left(1-e^z\right)^{-1}dz.\\ &=e^{-z}\int\frac{dz}{1-e^z}\\ &=e^{-z}\int\frac{1}{x(1-x)}dx,\ \text{putting}\ x=e^z, dz=(1/x)dx\\ &=e^{-z}\int\left[\frac{1}{x}+\frac{1}{1-x}\right]dx=e^{-z}[\log x-\log(1-x)]\\ &=\frac{1}{x}\log\frac{x}{1-x}.\end{aligned}$$

Hence the complete solution of the given equation is

$$y=\left(c_1+c_2\log x\right)\frac{1}{x}+\frac{1}{x}\log\frac{x}{1-x}$$

Chapter 10

2011

10.1 Section-A

Question-1(a) Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$. Let $\mathrm{W} = \{\mathrm{A} \in \mathrm{V} |\ Trace(\mathrm{A}) = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W.

[10 Marks]

Solution: Given, v is a vector space of 2×2 matrices over R i.e.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \cdot \in R \right\}$$

and W is a subset of V such that Trace $(A) = 0$ when $A \in W$.
Clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ i.e., W is not empty. Now, let $A_1, A_2 \in W$ then,

$$\text{Trace}\,(A_1) = 0 \text{ and } \text{Trace}\,(A_2) = 0$$

then,

$$\begin{aligned} \text{tr}\,(xA_1 + yA_2) &= x\,\text{Tr}\,(A_1) + y\,\text{Tr}\,(A_2) \\ &= x \cdot 0 + y \cdot 0 \\ &= 0 \end{aligned}$$

$$\text{i.e.,} \quad Trace(xA_1 + yA_2) = 0$$

$$\Rightarrow \quad xA_1 + yA_2 \in W$$

If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in W$ then $x + w = 0$ i.e., it can have at maximum three free variables.
Hence, dimension of $W = 4 - 1 = 3$ and the basis of W are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

i.e.,

$$W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Question 1(b) Find the linear transformation from R^3 into R^3 which has its range the subspace spanned by $(1, 0, -1)$, $(1, 2, 2)$.

[10 Marks]

Solution: Let T be the required linear transformation such that the range of it is spanned by $(1, 0, -1), (1, 2, 2)$ As $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ are the standard basis of R_3. Hence, we can assume

$$\begin{aligned} T(1, 0, 0) &= (1, 0, -1) \\ T(0, 1, 0) &= (1, 2, 2) \\ \text{and } T(0, 0, 1) &= (0, 0, 0) \end{aligned}$$

$$\begin{aligned} \text{Also, } (x, y, z) &= x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) \\ \Rightarrow \quad T(x, y, z) &= xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) \\ &= x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0) \\ &= (x + y, 2y, -x + 2y) \\ \text{i.e.,} \quad T(x, y, z) &= (x + y, 2y, -x, +2y) \end{aligned}$$

is the required transformation.

Question-1(c) Show that the function defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y there at.

[10 Marks]

Solution: The given function is

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

The above function is continuous at origin if it is equal to $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0$, irrespective of the path taken by the function to approach origin.

$$\text{Now} \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x - y}$$

Let $y = x - mx^3$ be the path through which this (x, y) approaches to origin, then. Clearly

$y \to 0$ when $x \to 0$. Then,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{x - x + mx^3} \\
&= \lim_{x\to 0} \frac{x^3 + x^3 - 3x^2 \cdot n^2x^3 + 3x \cdot m^2x^6 - m^3x^9}{mx^3} \\
&= \lim_{x\to 0} \frac{2x^3 - 3mx^5 + 3m^2x^7 - m^3x^9}{mx^3} \\
&= \frac{2}{m}
\end{aligned}$$

$\implies$ It approaches to different values depending on the value of m.
$\implies$ The function is discontinuous at the origin.

Again,

$$\begin{aligned}
f(x,y) &= \frac{x^3 + y^3}{x - y} \\
f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h^3/h - 0}{h} = 0 \\
\text{and } f_y(0,0) &= \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} \\
&= \lim_{k\to 0} \frac{\frac{k^3}{-k} - 0}{k} = 0
\end{aligned}$$

$\implies$ $f_x(0,0)$ and $f_y(0,0)$ exist at origin.

Question-1(d) Let the function f **be defined by**

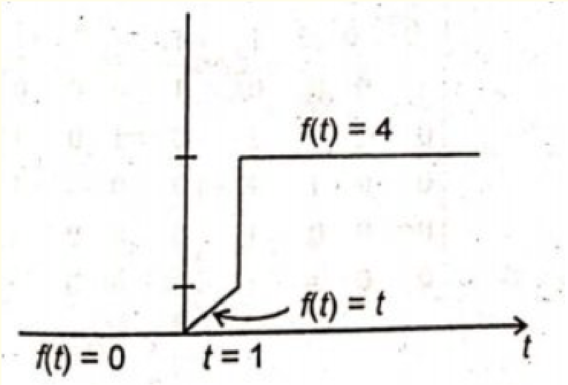
$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } \quad 0 \le t \le 1 \\ 4, & \text{for } t > 1 \end{cases}$$

(i) Determine the function $\mathrm{F(x)} = \int_0^{\mathrm{x}} \mathrm{f(t)dt}$
(ii) Where is F **non-differentiable? Justify your answer.**

[10 Marks]

Solution: The given function f is defined as

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } \quad 0 \le t \le 1 \\ 4, & \text{for } t > 1 \end{cases}$$



Now we have to calculate

$$F(x) = \int_0^x f(t)dt$$

Case I: When $0 < x \leq 1$
then

$$\begin{aligned} F(x) &= \int_0^x f(t)dt \\ &= \int_0^x tdt \\ &= \frac{x^2}{2} \end{aligned}$$

Case II: $x > 1$
then

$$\begin{aligned} F(x) &= \int_0^{x'} f(t)dt \\ &= \int_0^1 f(t)dt + \int_1^x f(t)dt \\ &= \int tdt + \int 4dt \\ &= \frac{1}{2} + 4(x-1) \\ &= 4x - \frac{7}{2} \\ &= 4x - \frac{7}{2} \end{aligned}$$

i.e., $\quad F(x) = 4x - \frac{7}{2} \quad x > 1$

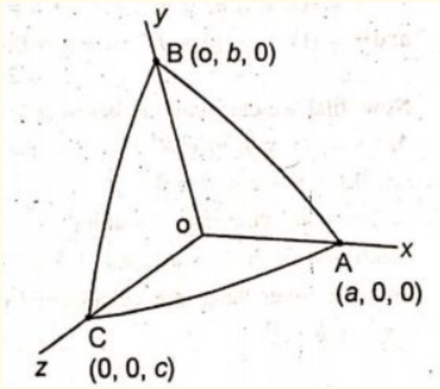
$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } 0 < x \leq 1 \\ 4x - \frac{7}{2}, & \text{for } x > 1 \end{cases}$$

Clearly the function $F(x)$ is not differentiable at $x = 1$.

Question-1(e) A variable plane is at a constant distance p from the origin and meets the axes at $\mathrm{A, B, C}$. Prove that the locus of the centroid of the tetrahedron $OABC$ is $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{16}{p^2}$.

[10 Marks]

Solution: Let the required plane cut the axes at A, B, C such that $A=(a,0,0), B=(0,b,0)$ and $C=(0,0,c)$



Then the equation of this plane is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Now from question, the length of perpendicular to this plane from origin is p. Then,

$$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$$

or,

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Again, let (α,β,γ) be the centroid of the tetrahedron. then, $\alpha=\frac{0+a+0+0}{4}$

$$\beta=\frac{0+0+b+0}{4}$$
$$\gamma=\frac{0+0+0+c}{4}$$

or, $a=4\alpha,\quad b=4\beta,\quad c=4\gamma$

putting a,b,c is equation (2), we get

$$\frac{1}{p^2}=\frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}$$

or,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{16}{p^2}$$

Hence, locus of (α,β,γ) is given by $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$
Hence, proved.

Question-2(a) Let

$$V = \{(x, y, z, u) \in R^4 : y + z + u = 0\}$$

$$W = \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\}$$

be two subspaces of R^4. Find bases for V, W, V + W and $V \cap W$.

[10 Marks]

Solution:

$$\begin{aligned} V &= \{(x, y, z, u) \in R^4 : \quad y + z + u = 0\} \\ &= \{(x, -z-u, z, u) \in R^4\} \\ &= \{x(1,0,0,0) + z(0,-1,1,0) + u(0,-1,0,1)\} \\ &= \text{Span}\{(1,0,0,0), (0,-1,1,0), (0,-1,0,1)\} \\ W &= \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\} \\ &= \{(-y, y, 2u, u) \in R^4\} \\ &= \{y(-1,1,0,0) + u(0,0,2,1)\} \\ &= \text{Span}\{(-1,1,0,0), (0,0,2,1)\} \end{aligned}$$

Now the bases of $V + W$ are given by the number of independent rows in the matrix formed by the bases of V and W in the form of row vector

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., No. of independent rows in above matrix = 4

i.e., Dimension of $V + W = 4$. i.e. The bases of $V + W$ are given by

$$(1,0,0,0), (0,-1,0,1), (0,0,1,-1)(0,0,0,1)$$

Now we calculate the bases of $V \cap W$. Clearly it should satisfy

$$y + z + u = 0$$

$$x + y = 0, z = 2u$$

i.e., there is only one free variable in this subspace. Choose $u = 1$ is the free variable then,

$$z = 2, y = -3, x = 3$$

i.e., $\{(3, -3, 2, 1)\}$ is the basis of $V \cap W$ and the dimension of this subspace is 1.

Question-2(b) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}.

[10 Marks]

Solution: The given matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

Then, the characteristic equation of this polynomial is given by

$$|A - \lambda l| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\{(4-\lambda)(1-\lambda) + 2\} - 1\}2(1-\lambda) + 2\} + 1\{-2 + 4 - \lambda\} = 0$$

$$16 - 20\lambda + 8\lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

Hence, by Cayley-Hamilton Theorem it should be satisfy the by matrix A. i.e.,

$$A^3 - 8A^2 + 20A - 16I = 0$$

or, the characteristic polynomial is given by

$$A^3 - 8A^2 + 20A - 16I = 0$$

from the given expression, it is difficult to calculate A^{10}.

Question-2(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$. **Find an invertible matrix** P **such that** $P^{-1}AP$ **is a diagonal matrix.**

[10 Marks]

Solution: The given matrix A is $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$ The characteristic equation of this matrix is given by

$$|A - \lambda| = 0$$

$$\begin{vmatrix} |-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \mid (\lambda+5)(\lambda-4)+18\} = 0$$

$$(1-\lambda) \mid \lambda^2+\lambda-20+18\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2+\lambda-2) = 0$$

$$(1-\lambda)(\lambda^2+2\lambda-\lambda-2) = 0 \Rightarrow \quad \lambda = 1, 1, -2$$

i.e., $\lambda = 1, 1, -2$ are the eigen values of the matrix A.

Now, for $\lambda = 1$, the eigen vector is given by $[A - I][X] = 0$ where $[X] = [x, yz]^T$

$$\begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3y + 3z = 0$$

$$-y + z = 0$$

. Clearly, this will possess two eigenvectors as there are two free variables satisfying the above condition.
Hence, the eigen vectors corresponding to $\lambda = 1$ is given by, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$

For $\lambda = -2$, the eigenvector is given by

$$[A + 2I][X] = 0$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - y + z = 0$$

$$-y + 2z = 0$$

i.e. It'll possesses only one free variable.

Choose $z = 1$ as the free variable then. $y = 2$ and $x = 1$ i.e., $[121]^T$ is the required eigen vector.

Hence, the invertible matrix (P) is given by

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}^T$$

It will reduce the matrix A to a diagonal matrix by operation $P^{-1}AP = D$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verification As

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow | P |= -1$ Now

$$P^{-1} = \frac{\text{Adj } P}{|P|} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Question-2(d) Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form.

[10 Marks]

Solution: The given quadratic form is

$$5x^2 + 2y^2 + 4xy$$

its associated matrix A can be written as

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_2 \to R_2 - \frac{2}{5}R_1$ and $C_2 \to C_2 - \frac{2}{5}C_1$ we get:

$$\begin{bmatrix} 5 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

Now, apply $R_1 \to R_1 \cdot \frac{1}{\sqrt{5}}$ and $C_i = \frac{1}{\sqrt{5}} \cdot C_1$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix}, A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & 1 \end{bmatrix}$$

Apply $R_2 \to \sqrt{\frac{5}{6}}R_2$ and $C_2 \to \sqrt{\frac{5}{6}}C_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

Hence the orthogonal transformation is

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

Question-3(a) Show that the equation $3^x + 4^x = 5^x$ has exactly one root.

[8 Marks]

Solution: The given equation is

$$3^x + 4^x = 5^x$$

Dividing both the sides by 5^x, we get

$$\left(\frac{3}{5}\right)^x + \left(-\frac{4}{5}\right)^x = 1$$

let $\sin\theta = \frac{3}{5}$ then $\cos\theta = \frac{4}{5}$ hence the equation (2) is reduced to

$$(\sin\theta)^x + (\cos\theta)^x = 1$$

which is true for $x = 2$ i.e. $x = 2$ is the only root of equation (1). This result is also known as known as Fermat theorem. It states that $a^n + b^n \neq c^n$ for $n > 2$ where $a, b, c \in z$ and $n \in N$.

Question-3(b) Test for convergence the integral $\int_0^\infty \sqrt{\mathrm{xe}^{-\mathrm{x}}}\mathrm{dx}$.

[8 Marks]

Solution: The given integral is

$$\int_0^\infty \sqrt{xe^{-x}}dx = \int_0^\infty \sqrt{x}e^{-x/2}dx$$

Let $y = \frac{x}{2}$ then

$$\begin{aligned} x &= 2y \\ dx &= 2dy \\ &= \int_0^\infty \sqrt{2}ye^{-y}2dy \\ &= 2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy \end{aligned}$$

Let

$$f(y) = \sqrt{y}e^{-y} = \frac{e^{-y}}{y^{-1/2}}$$

Clearly the function has an infinite discontinuity at $y = 0$ and $y = \infty$.
Hence, we have to examine the convergence at both $y = 0$ and $y = \infty$.
Consider, $y = \infty$,

$$\int_0^\infty \sqrt{y}e^{-y}dy = \int_0^1 \sqrt{y}e^{-y}dy + \int_1^\infty \sqrt{y}e^{-y}dy$$

We test the two integrals on the right for convergence at 0 and ∞ respectively.
Convergence at $y = 0$:
Let $g(y) = \sqrt{y}$ such that

$$\lim_{y\to 0}\frac{f(y)}{g(y)} = e^{-y} \to 1 \quad \text{as } x \to 0$$

However

$$\int_0^1 g(y)dy = \left.\frac{y^{3/2}}{32}\right|_0^1 \qquad \ldots(1)$$

converges

$$\Rightarrow \int_0^1 \sqrt{y}e^{-y}dy$$

.
Convergence at ∞:
Let $g(y) = \frac{1}{y^2}$ then

$$\begin{aligned} \lim_{y\to\infty}\frac{f(y)}{g(y)} &= \lim_{y\to\infty}\frac{\sqrt{y}e^{-y}}{17y^2} \\ &= \lim_{y\to\infty}\frac{y^{5/2}}{e^y} \to 0 \text{ as } y \to \infty \end{aligned}$$

As $\int_1^\infty g(y)dy$ converges if $g(y) = \frac{1}{y^2}$ i.e., $\int_1^\infty \sqrt{y}e^{-y}dy$ converges $\ldots(2)$

From (1) and (2),
$\int_0^\infty \sqrt{y}e^{-y}dy$ converges.

The given integral

$$
\begin{aligned}
2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy &= 2\sqrt{2}\int_0^\infty y^{\frac{3}{2}-1}e^{-y}dy \\
&= 2\sqrt{2}\left(\frac{3}{2}\right) \\
&= 2\sqrt{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \\
&= \sqrt{2\pi}
\end{aligned}
$$

Question-3(c) Show that the area of the surface of the sphere $x^2+y^2+z^2=a^2$ cut off by $x^2+y^2=ax$ is $2(\pi-2)\mathrm{a}^2$.

[12 Marks]

Solution: The given sphere is $x^2+y^2+z^2=a^2$

$$
\begin{aligned}
\therefore \quad \frac{\partial z}{\partial x} &= -\frac{x}{z} \\
\frac{\partial z}{\partial y} &= -\frac{y}{z} \\
\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} &= \frac{1}{z}\sqrt{x^2+y^2+z^2} \\
&= \frac{a}{\sqrt{a^2-x^2-y^2}}
\end{aligned}
$$

Now the surface area is

$$
\begin{aligned}
\iint ds &= \iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy \\
&= 4\iint \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy
\end{aligned}
$$

over half the circle $x^2+y^2=ax$

let $\quad x=r\cos\theta,\quad y=r\sin\theta$ then

$$x^2+y^2=ax$$

becomes $\quad r=a\cos\theta$ and $\quad dxdy=rd\theta dr$

$$
\begin{aligned}
\therefore \quad S &= 4a\int_0^{\pi/2}\int_0^{a\cos\theta}\frac{rd\theta dr}{a^2-r^2}\\
&= 4a\int_0^{\pi/2}\left[-\sqrt{a^2-r^2}\right]^{n\cos\theta} d\theta\\
&= 4a\int_0^{\pi/2}(1-\sin\theta)d\theta\\
&= 4a^2[\theta+\cos\theta]_0^{\pi/2}\\
&= 4a^2\left[\left(\frac{\pi}{2}+0\right)-(0+1)\right]\\
&= 2a^2(\pi-2) \text{ units}
\end{aligned}
$$

i.e.,

$$S = 2(\pi-2)a^2 \text{ units}$$

Proved.

Question-3(d) Show that the function defined by

$$f(x,y,z) = 3\log\left(x^2+y^2+z^2\right) - 2x^2 - 2y^3 - 2z^3, (x,y,z) \neq (0,0,0)$$

has only one extreme value, $\log\left(\frac{3}{\mathrm{e}^2}\right)$

[12 Marks]

Solution: The given function is

$$f(x,y,z) = 3\log\left(x^2+y^2+z^2\right) - 2x^3 - 2y^3 - 2z^3 \quad (x,y,z) \neq (0,0,0)$$

for extremum value

$$f_x = f_y = f_z = 0$$

Now,

$$
\begin{aligned}
f_x &= 3\cdot\frac{2x}{x^2+y^2+z^2} - 6x^2\\
&= \frac{6x}{x^2+y^2+z^2} - 6x^2\\
&= 0
\end{aligned}
$$

$$\Rightarrow \frac{6x\left[1-x\left(x^2+y^2+z^2\right)\right]}{\left(x^2+y^2+z^2\right)} = 0 \text{ as, } (x,y,z) \neq (0,0,0)$$

$$
\begin{aligned}
\Rightarrow \quad 1 - x\left(x^2+y^2+z^2\right) &= 0\\
x\left(x^2+y^2+z^2\right) &= 1
\end{aligned}
$$

Similarly,

$$f_y = 0$$

$$\Rightarrow \quad y\left(x^2+y^2+z^2\right) = 1$$

and

$$f_z = 0$$

$$\Rightarrow \quad z\left(x^2+y^2+z^2\right)=1$$

From equations (1),(2) and (3), we get $x=y=z$, i.e.,

$$x\left(x^2+x^2+x^2\right)=1$$

$$3x^3=1$$

$$\Rightarrow x=\left(\frac{1}{3}\right)^{\frac{1}{3}}=\frac{1}{3^{1/3}}$$

$$x=y=z=\frac{1}{3^{1/3}}$$

Hence, the value of $f(x,y,z)$ at the point $\left(\frac{1}{3^{1/3}}\cdot\frac{1}{3^{1/3}},\frac{1}{3^{1/3}}\right)$ is given by

$$f\left(\frac{1}{3^{1/3}},\frac{1}{3^{1/3}},\frac{1}{3^{1/3}}\right)=3\log\left(\frac{1}{3^{2/3}}+\frac{1}{3^{2/3}}+\frac{1}{3^{2/3}}\right)-2\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$$

$$=3\log\left(\frac{3}{3^{2/3}}\right)-2$$

$$=3\log 3^{1/3}-2=\frac{3}{3}\log 3-2$$

$$=\log 3-2$$

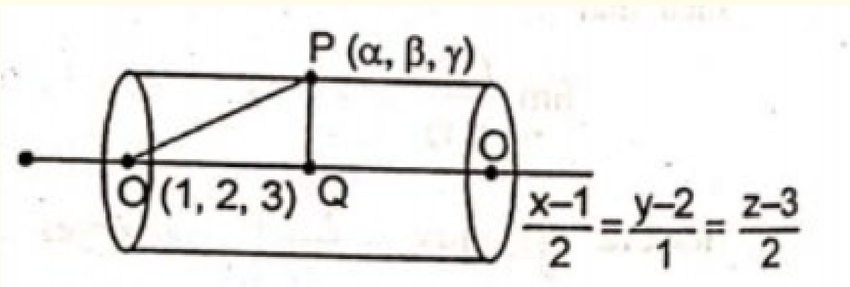
$$=\log\left(\frac{3}{e^2}\right)$$

i.e., the only extreme value of $f(x,y,z)$ is $\log\left(\frac{3}{e^2}\right)$

Question-4(a) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.

[10 Marks]

Solution: Let OO' be the axis of the right circular cylinder which has radius 2 .



From question, The equation of the line OO'. is

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$$

clearly it passes through (1,2,3) let

$$O\equiv(1,2,3)$$

Now, let $P(\alpha, \beta, \gamma)$ be a point which lies on the cylinder, then from the figure, the projection of OP on the line OO' is given by

$$(\alpha - 1)\frac{2}{3} + (\beta - 2) \cdot \frac{1}{3} + (\gamma - 3) \cdot \frac{2}{3}$$

Now in right angled triangle OPQ we have,

$$OP^2 = PQ^2 + OQ^2$$

$$\Rightarrow (\alpha-1)^2+(\beta-2)^2+(\gamma-3)^2 = 2^2+\tfrac{1}{9}\,[2(\alpha-1)+(\beta-2)+(\gamma-3)]^2$$

$$\Rightarrow 9\,[(\alpha-1)^2+(\beta-2)^2+(\gamma-3)^2] = 36 + [2\alpha+\beta+2\gamma-10]^2$$

$$\Rightarrow 9\,[\alpha^2+\beta^2+\gamma^2+14-2\alpha-4\beta-6\gamma]\,36+$$

$$[4\alpha^2+\beta^2+4\gamma^2+100+4\alpha\beta+4\beta\gamma+8\alpha\gamma-40\alpha-20\beta-40\gamma]$$

$$\Rightarrow 5\alpha^2+8\beta^2+5\gamma^2-4\alpha\beta-8\alpha\gamma+2\alpha-16\beta+4\gamma-10=0$$

Hence, equation of right circular cylinder is given by the locus (α, β, γ) i.e.,

$$5x^2+8y^2+5z^2-4xy-4yz-8xz+22x-16y+4z-10=0$$

Question-4(b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$.

[10 Marks]

Solution: The equation of ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

Let equation of tangent plane which is parallel to given plane is

$$lx+my+nz=p \qquad \ldots(2)$$

Let it touches ellipsoid at point (x, y, z). We know that equation of tangent plane at point (x, y, z) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \qquad \ldots(3)$$

If (2) and (3) are identical then

$$\frac{x_1}{a^2l}=\frac{y_1}{b^2m}=\frac{z_1}{c^2n}=\frac{1}{p}$$

$$\Rightarrow \quad x_1=\frac{a^2l}{p}, y_1=\frac{b^2m}{p}, z=\frac{c^2n}{p}$$

Point (x, y, z) lies on ellipsoid (1),

$$\therefore \qquad \frac{1}{a^2}\left(\frac{a^2 l}{p}\right)^2 + \frac{1}{b^2}\left(\frac{b^2 m}{p}\right)^2 + \frac{1}{c^2}\left(\frac{c^2 n}{p}\right)^2 = 1$$
$$\Rightarrow a^2 l^2 + b^2 m^2 + c^2 n^2 = p^2$$

Using (2), equation of plane is

$$lx + my + nz = \pm\sqrt{a^2 l^2 + b^2 m^2 + c^2 n^2}$$

Question-4(c) Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.

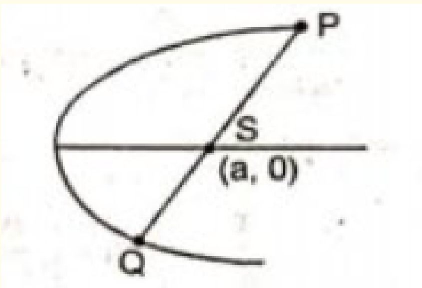
[8 Marks]

Solution: Consider a parabola conic whose equation is

$$y^2 = 4ax$$

then the length of semi-latus rectum $= 2a$

Let $S = (a, 0)$ be the focus of this parabola and PSQ be any focal chord of this parabola,



let $\quad P \equiv (at^2, 2at)$ then

$$Q \equiv \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$
$$SP^2 = a^2\left(1 - t^2\right)^2 + 4a^2t^2$$
$$= a^2\left(1 + t^2\right)^2$$
$$\Rightarrow SP = a\left(1 + t^2\right)$$
$$\text{Similarly, } SQ = a\left(1 + \frac{1}{t^2}\right)$$

Now, the harmonic mean of SP and SQ is given by

$$\frac{2\cdot SP\cdot SQ}{SP+SQ}=\frac{2a^2\left(1+t^2\right)\left(1+\frac{1}{t^2}\right)}{a\left[1+t^2+1+\frac{1}{t^2}\right]}$$
$$=\frac{2a\left[1+t^2+\frac{1}{t^2}+1\right]}{\left[2+t^2+\frac{1}{t^2}\right]}$$
$$=\frac{2a\left[2+t^2+\frac{1}{t^2}\right]}{\left[2+t^2+\frac{1}{t^2}\right]}=2a$$

which is equal to semi-latus rectum.

Question-4(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid.

[12 Marks]

Solution: Let standard equation of paraboloid be

$$2cz=ax^2+by^2$$

and the given points be $P\left(x_1,y_1,z_1\right)$ & $Q\left(x_2,y_2,z_2\right)$.
Tangent planes at P and Q are given by:

$$c\left(z+z_1\right)=ax_1x+by_1y \qquad \ldots(1)$$
$$c\left(z+z_2\right)=ax_2x+by_2y \qquad \ldots(2)$$

Hence, equation of plane passing through line of intersection of (1) and (2) is given by:

$$\left(ax_1\right)x+(by)y-cz-cz+\lambda\left[\left(ax_2\right)x+\left(by_2\right)y-\left(z-cz_2\right)=0 \qquad \ldots(3)$$

Middle point of $p\varphi, m\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{2_1+2_2}{2}\right)$ lies on the above plane.

Hence, we obtain the value of λ as

$$\lambda=\frac{-\left[-ax_1\left(\frac{x_1+x_2}{2}\right)+by_1\left(\frac{y_1+y_2}{2}\right)-c\left(\frac{z+z_2}{2}\right)-\left(z_1\right)\right]}{\left[ax_2\left(\frac{x_1+x_2}{2}\right)+by_2\left(\frac{y_1+y_2}{2}\right)-c\left(\frac{2+z_2}{2}\right)-\left(z_2\right)\right.}$$
$$=\frac{-\left[ax_1x_2+by_1y_2-c\left(z_1+z_2\right)\right]}{\left[ax_2x_1+by_2y_1-c\left(2_1+z_2\right)\right]}$$
$$=-1$$

P and Q lies on paraboloid, therefore

$$ax_1^2+by_1^2-2cz=0$$
$$ax_2^2+by_2^2-2cz_2=0$$

Hence equation of plane (from(3))

$$a\left(x,+x_2\right)x+b(y,-y,)y=c\left(z+z_2\right)$$

D.R of Normal of this plane are

$$\left\langle a\left(x_1+x_2\right),b\left(y_1+y_2\right),0\right\rangle$$

Axis of the paraboloid is $z - axis$ D.R. of z-axis are $(0, 0, 1)$

$$\therefore a\left(x_1 - x_2\right) \times 0 + b\left(y_1 - y_2\right) \times 0 + 0x_1 = 0$$

Therefore, the above plane is parallel to axis of paraboloid.

10.2 Section-B

Question-5(a) Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy = c$.

[10 Marks]

Solution: The given curve is

$$xy = C$$
$$\Rightarrow y = \frac{C}{x}$$
$$\frac{dy}{dx} = \frac{-C}{x^2} = m_2(\text{say})$$

From the question,

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$
$$\tan\frac{\pi}{4} = \frac{\frac{dy}{dx} + \frac{C}{x^2}}{1 - \frac{dy}{dx}\cdot\frac{C}{x^2}}$$
$$\Rightarrow 1 - \frac{dy}{dx}\cdot\frac{C}{x^2} = \frac{dy}{dx} + \frac{C}{x^2}$$
$$\Rightarrow \frac{dy}{dx}\left[1 + \frac{C}{x^2}\right] = 1 - \frac{C}{x^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - C}{x^2 + C}$$
$$= \frac{x^2 + C - 2C}{x^2 + C}$$
$$= 1 - \frac{2C}{x^2 + C}$$
$$\therefore dy = \left(1 - \frac{2C}{x^2 + C}\right) dx$$

Integrating both the sides, we get

$$y = x - \frac{2C}{\sqrt{C}}\tan^{-1}\frac{x}{\sqrt{C}} + C^1$$

($C^1 =$ integration constants)

$$y = x - 2\sqrt{C}\tan^{-1}\frac{x}{\sqrt{C}} + C^1$$

is the required family of curves.

Question-5(b) Solve: $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} + 5y = \sec x \cdot e^x$

[10 Marks]

Solution: Here $P = -2\tan x, Q = -\left(a^2+1\right)$ and $R = e^x \sec x$.
We choose $u = e^{-\frac{1}{2}\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$.
Putting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2\nu}{dx^2} + Xv = Y \qquad \ldots(1)$$

$$\begin{aligned} \text{where } X &= Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = -\left(a^2+1\right) - \frac{1}{2}\cdot\left(-2\sec^2 x\right) - \frac{1}{4}\cdot 4\tan^2 x \\ &= -a^2 - 1 + \sec^2 x - \tan^2 x = -a^2 - 1 + 1 = -a^2 \\ \text{and } Y &= Re^{\frac{1}{2}\int P dx} = e^x \sec x e^{-\int \tan x dx} = \left(e^x \sec x\right)\left(1/\sec x\right) \\ &= e^x \end{aligned}$$

Hence, the normal form (1) of the given differential equation is

$$\frac{d^2v}{dx^2} - a^2 v = e^x, \text{ or } \left(D^2 - a^2\right) v = e^x \qquad \ldots(2)$$

Now (2) is a linear differential equation with constant coefficients.
A.E. is $m^2 - a^2 = 0$, or $m^2 = a^2$ giving $m = \pm a$. $\therefore$ C.F. of the solution of (2) $= c_1 e^{ax} + c_2 e^{-ax}$.

$$P.I. = \frac{1}{D^2 - a^2} e^x = \frac{1}{1^2 - a^2} e^x$$

.
$\therefore$ the solution of (2) is $v = c_1 e^{ax} + c_2 e^{-ax} + \frac{e^x}{1-a^2}$.
Hence the general solution of the given differential equation is

$$y = uv = \left(c_1 e^{ax} + c_2 e^{-ax}\right) \sec x + \frac{e^x \sec x}{1 - a^2}$$

Question-5(c) The apses of a satellite of the Earth are at distances r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2.

[10 Marks]

Solution: Let the satellite of the Earth moves under the inverse square law $= \frac{\mu}{r^2}$
Clearly the satellite will move in elliptical orbit and the velocity at a distance r is given by

$$v^2 = \mu \left[\frac{2}{r} - \frac{1}{a}\right]$$

where $2a$ = major axis of elliptical orbit Now at apse

$$r_1 = a + ae$$

and

$$r_2 = a - ae \quad \Rightarrow 2a = r_1 + r_2$$

Now from question at $r = r_1, v = v_1$

$$\begin{aligned} v_1^2 &= \mu\left[\frac{2}{r_1} - \frac{1}{a}\right]\cdots \\ &= \mu\left[\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right] \\ v_1^2 &= 2\mu\left[\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right] \\ &= 2\mu\left[\frac{r_1 + r_2 - r_1}{r_1(r_1 + r_2)}\right] \\ v_1^2 &= \frac{2\mu r_2}{r_1 + r_2} \\ \Rightarrow \quad v_1 &= \sqrt{\frac{2\mu r_2}{r_1(r_1 + r_2)}} \end{aligned}$$

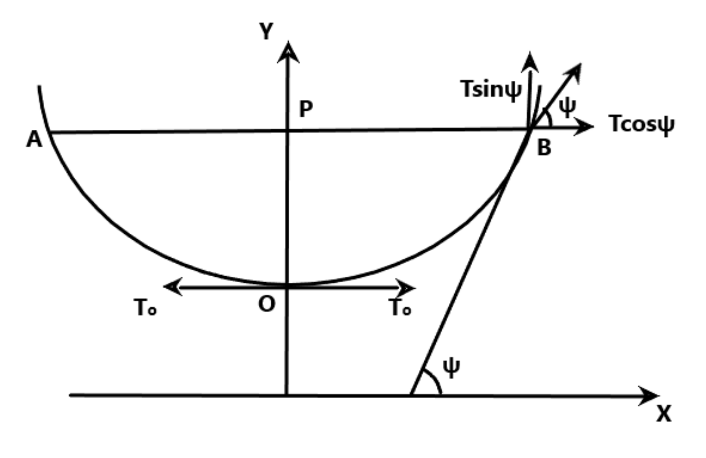
Similarly,

$$v_2 = \sqrt{\frac{2\mu r_1}{r_2(r_1 + r_2)}}$$

Question-5(d) A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120\cosh^{-1}\left(\frac{5}{3}\right)$ and also find the sag.

[10 Marks]

Solution: Weight of the cable $(= 160 \times 2 = 320)$ will act at middle point of AB, i.e., at point O.



In equilibrium,

$$W = 2T\sin\psi$$
$$\implies 3202 \times 200\sin\psi$$
$$\implies \sin\psi = \frac{4}{5}$$

We know that equation of common catenary is given by:

$$y = c\cosh\left(\frac{x}{c}\right) ...(i)$$

and $T = wy$, where $w =$ weight per unit length $= 2kg/m$
At point A or B,

$$T = 200kg$$
$$\implies y = T/w = 100m$$

Also,

$$T = T_0\cos\psi = wc$$
$$\therefore 22 \times \frac{3}{5} = 2c \;\; (\because \sin\psi = \frac{4}{5})$$
$$\implies c = 60m$$

Now, for span, we put y=100 m in equation of catenary (i),

$$\implies 100 = 60\cosh\left(\frac{x}{60}\right) \Rightarrow x = 60\cos h^{-1}\left(\frac{5}{3}\right)$$

$$Span = 2x = 120\cos h^{-1}(5/3)$$
$$Sag = y - c = 100 - 60 = 40 \text{ m}$$

.

Question-5(e) Evaluate the line integral $\oint_C(\sin xdx + y^2dy - dz)$, where C is the circle $x^2 + x^2 = 16, z = 3$, by using Stokes' theorem.

[10 Marks]

Solution: The given line integral is

$$\int_C \left(\sin x dx + y^2 dy - dz\right)$$

where C is the circle $x^2 + y^2 = 16, z = 3$

$$\int_C \left(\sin x\hat{i} + y^2\hat{j} - \hat{k}\right) \cdot (dxi + dy\hat{j} + dz\hat{k}) = \int_C \bar{F} \cdot dr$$

where $\bar{F} = \sin x\hat{i} + y^2\hat{j} - \hat{k}$ Now from the Stokes' theorem

$$\int_C \bar{F} \cdot dr = \oiint_S (\bar{\nabla} \times \bar{F}) \cdot d\bar{s}$$

where S is the surface enclosed by the curves. Now,

$$(\bar{\nabla} \times \bar{F}) \cdot \bar{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & y^2 & -1 \end{vmatrix} \cdot K = 0$$

$$\Rightarrow \quad \oint \bar{F} \cdot d\bar{r} = 0$$

Question-6(a) Solve: $p^2 + 2py\cot x = y^2$ **where** $p = \dfrac{dy}{dx}$

[10 Marks]

Solution:

$$p^2 + 2py\cot x = y^2$$
$$p^2 + 2py\cot x - y^2 = 0$$

Solving the above equation, for the quadratic in p, we get

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$
$$p = -y\cot x \pm y\,\mathrm{cosec}\, x$$

Case I: When $p = -y\cot x + y\,\mathrm{cosec}\, x$
then

$$\frac{dy}{dx} = y(-\cot x - \mathrm{cosec}\, x)$$
$$\frac{dy}{y} = (-\cot x + \mathrm{cosec}\, x)dx$$

$$\frac{dy}{y} = \left(-\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right) dx$$
$$= \left(\frac{2\sin^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) dx$$
$$r\frac{dy}{y} = \tan\frac{x}{2} dx$$

integrating both the sides, we get

$$\log y = 2\log\sec\frac{x}{2} + \log C_1$$
$$< \log C_1 \text{ (integration constant)}$$
$$y = C_1 \sec^2\frac{x}{2}$$
$$y - C_1\sec^2\frac{x}{2} = 0$$

is one solution.
Case II: when $p = -y\cot x - y\,\text{cosec}\, x$
then,

$$\frac{dy}{dx} = -y(\cot x - \text{cosec}\, x)dx$$
$$\frac{dy}{y} = (-\cot x - \text{cosec}\, x)dx$$
$$\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right) dx = \frac{-\cos x/2}{\sin x/2} dx$$

Integrating both the sides, we get

$$\log y = 2\log\text{cosec}\,\frac{x}{2} + \log C_2$$
$$< \log C_2 = \text{ integration constant}$$
$$y - C_2\,\text{cosec}^2\frac{x}{2} = 0$$

is another solution.
Hence, the required solution is given by,

$$\left(y - C_1\sec^2\frac{x}{2}\right)\left(y - C_2\,\text{cosec}^2\frac{x}{2}\right) = 0$$

where C_1 and C_2 are arbitrary constant.

Question-6(b) Solve:

$$(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$$

where $D \equiv \dfrac{d}{dx}$

[15 Marks]

Solution: The given differential equation is

$$\left[x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1 \mid y = \langle 1 + \log x\rangle^2\right.$$

It is solved by putting $x = e^z$ then reducing the above equation in the form of y and

$$x = e^z$$

since,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \\
x\frac{dy}{dx} &= \frac{dy}{dz} \\
x\frac{dy}{dx} &= D_1 y \quad (\text{where } D_1 = \frac{d}{dz}\ ) \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dz}\left(\frac{dy}{dx}\right) \cdot \frac{dz}{dx} \\
&= \frac{d}{dz}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) \cdot \frac{1}{x} \\
&= \frac{1}{x}\left[\frac{1}{x} \cdot \frac{d^2y}{dz^2} - \frac{1}{x}\frac{dy}{dz}\right] \\
x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz} \\
x^2\frac{d^2y}{dx^2} &= D_1\left(D_1 - 1\right) y \\
\text{Similarly, } x^3\frac{d^3y}{dx^3} &= D_1\left(D_1 - 1\right)\left(D_1 - 2\right) y \\
\text{and } x^4\frac{d^4y}{dx^4} &= D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\left(D_1 - 3\right) y
\end{aligned}$$

Putting these values in equation (1), we get

$$\begin{gathered}
\left[D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\left(D_1 - 3\right) + 6D_1\left(D_1 - 1\right)\left(D_1 - 2\right) + \right. \\
\left. 9D_1\left(D_1 - 1\right) + 3D_1 + 1\right] y = \left(1 + e^2\right)^2 \\
\Rightarrow \left[D_1\left(D_1^3 - 6D_1^2 + 11D_1 - 6\right) + 6D_1\left(D_1^2 - 3D_1 + 2\right) + \right. \\
\left. 9D_1\left(D_1 - 1\right) + 3D_1 + 1\right] y = 1 + 2e^2 + e^2
\end{gathered}$$

$$\left(D_1^4 + 2D_1^2 + 1\right) y = 1 + 2e^z + e^{2z}$$

The auxiliary equation is given by

$$\begin{gathered}
m^4 + 2m^2 + 1 = 0 \\
\Rightarrow \left(m^2 + 1\right)^2 = 0 \\
\Rightarrow \quad m = \pm i, \pm i
\end{gathered}$$

Hence, the complementary function is given by

$$y = \left(C_1 + C_2 z\right)\cos z + \left(C_3 + C_4 z\right)\sin z$$

where C_1, C_2, C_3 and C_4 are arbitrary constant Now, the Particular Integral is given by

$$\begin{aligned}
y &= \frac{1}{(1+2D_1^2+D_1^4)}\left(1+2e^z+e^{2z}\right) \\
&= \frac{1}{\left(1+D_1^2\right)^2} + 2\cdot\frac{1}{\left(1+D_1^2\right)^2}e^z + \frac{e^{2z}}{\left(1+D_1^2\right)^2} \\
&= 1 + \frac{2\cdot e^z}{(1+1)^2} + \frac{e^{2z}}{(1+4)^2} = 1 + \frac{e^z}{2} + \frac{e^{2z}}{25}
\end{aligned}$$

i.e., the general solution is given by

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.} \\
&= (C_1 + C_2 z)\cos z + (C_3 + C_4 z)\sin z + 1 + \frac{e^z}{2} + \frac{e^{2z}}{25}
\end{aligned}$$

Putting the value of z, we get

$$y = (C_1 + C_2\log x)\cos(\log x) + (C_3 + C_4\log x)\sin(\log x) + 1 + \frac{x}{2} + \frac{x^2}{25}$$

is the required solution.

Question-6(c) Solve:

$$\left(\mathrm{D}^4 + \mathrm{D}^2 + 1\right)\mathrm{y} = \mathrm{ax}^2 + \mathrm{be}^{-\mathrm{x}}\sin 2\mathrm{x}$$

, where $D = \dfrac{d}{dx}$

[15 Marks]

Solution:

$$\left(D^4 + D^2 + 1\right)y = ax^2 + be^{-x}\sin 2x$$

The auxiliary equation is given by

$$m^4 + m^2 + 1 = 0$$

$$m^4 + 2m^2 + 1 - m^2 = 0$$

$$\left(m^2+1\right)^2 - m^2 = 0$$

$$\Rightarrow \left(m^2+m+1\right)\left(m^2-m+1\right) = 0$$

$$m = \frac{-1\pm\sqrt{1-4}}{2}, \frac{1+\sqrt{1-4}}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$$

and

$$m = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

are the roots of auxiliary equation.// Hence, the complementary function is given by

$$y = e^{-x/2}\left[C_1\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] +$$
$$e^{x/2}\left[C_3\cos\left(\frac{\sqrt{3}x}{2}\right) + C_4\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

Now, the Particular Integral is given by

$$y = \frac{1}{(1+D^2+D^4)}\left\{ax^2 + be^{-x}\sin 2x\right\}$$

Consider

$$\begin{aligned}\frac{a}{(1+D^2+D^4)}x^2 &= a\left\langle 1+D^2+D^4\right\rangle^{-1}x^2 \\ &= a\left\langle 1-\left(D^2+D^4\right)+\left(D^2+D^4\right)^2+\ldots\right\rangle x^2 \\ &= a\left[\left(x^2\right) - D^2(x)^2\right] = a\left[x^2-2\right]\end{aligned}$$

Now Consider,

$$\begin{aligned}\frac{1}{(1+D^2+D^4)}e^{-x}\sin 2x &= e^{-x}\frac{1}{[1+(D-1)^2+(D-1)^4)}\sin 2x \\ &= e^{-x}\frac{1}{[1+D^2-2D+1+D^4-4D^3+6D^2} = 4D+1\Big]^{\sin 2x} \\ &= e^{-x}\frac{1}{(D^4-4D^3+7D^2-6D+3)}\sin 2x \\ &= e^{-x}\frac{1}{\left(D^2\right)^2-4D\left(D^2\right)+7\cdot D^2-6D+3}\sin 2x \\ &= e^{-x}\frac{1}{\left(-2^2\right)^2-4D\left(-2^2\right)+7\left(-2^2\right)-6D+3}\sin 2x \\ &= e^{-x}\frac{1}{16+16D-28-6D+3}\sin 2x \\ &= e^{-x}\frac{1}{10D-9}\sin 2x \\ &= e^{-x}\frac{(10D+9)}{(100D^2-81)}\sin 2x \\ &= e^{-x}\frac{(10D+9)\sin 2x}{-400-81} \\ &= \frac{-e^{-x}}{481}(20\cos 2x + 9\sin 2x) \\ \therefore y &= \frac{1}{(1+D^2+D^4)}\left(ax^2+be^{-x}\sin 2x\right) \\ &= a\left(x^2-2\right) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x)\end{aligned}$$

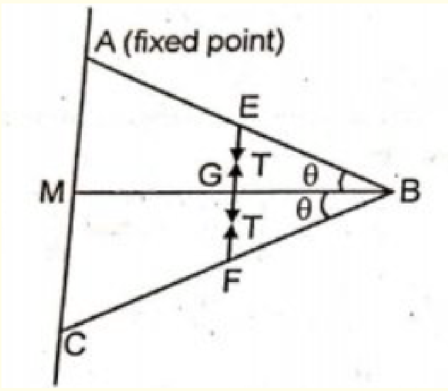
Hence, the general solution is given by,

$$y = e^{\frac{-x}{2}}\left[C_1\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{\frac{x}{2}}\left[C_3\cos\left(\frac{\sqrt{3}}{2}x\right) + C_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$
$$+a\left(x^2-2\right) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x)$$

Question-7(a) One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$.

[13 Marks]

Solution: AB and BC are two rods each of length $2a$ and weight W smoothly jointed together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of rods AB and BC are connected by an clastic string of natural length a.



Let T be the tension in the string EF. The total weight 2 W of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M. Let $\angle \text{ABM} = \theta = \angle \text{CBM}$

Give the system a small symmetrical displacement about BM in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change. We have $\text{EF} = 2\text{EG} = 2\ \text{EB}\sin\theta = 2a\sin\theta$ Also the depth of G below the fixed point

$$\text{A} = \text{AM} = \text{AB}\sin\theta = 2a\sin\theta$$

The equation of virtual work is

$$-T\delta(2a\sin\theta) + 2W\delta(2a\sin\theta) = 0$$

$$(-2aT\cos\theta + 4aW\cos\theta)\delta\theta = 0$$

$$2a\cos\theta(-T + 2W)\delta\theta = 0$$

$$-T + 2W = 0$$

$$[\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$T = 2W$$

Also, by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda \cdot \frac{2a\sin\theta - a}{a} \left(\begin{array}{c} \text{where } \lambda \text{ is the modulus} \\ \text{elasticity of the sthing} \end{array} \right)$$
$$T = 4W(2\sin\theta - 1)$$

Equating the two values of T, we

$$2W = 4W(2\sin\theta - 1)$$
$$1 = 2(2\sin\theta - 1)$$
$$1 = 4\sin\theta - 2$$
$$3 = 4\sin\theta$$
$$\sin\theta = \frac{3}{4}$$
$$\theta = \sin^{-1}\left(\frac{3}{4}\right)$$

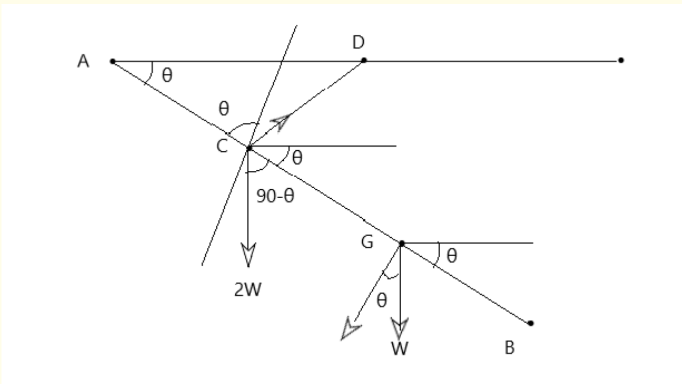
$\therefore$ In equilibrium, the whole angle between AB and BC

$$\implies 2\theta = 2\sin^{-1}\left(\frac{3}{4}\right)$$

Question-7(b) AB is a uniform rod, of length 8 a, which can turn freely about the end A, which is fixed. C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium. Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$W, where W is the weight of the rod.

[14 Marks]

Solution: Given $AD = CD = a$



Force at ring C along the rod AB

$$2W\cos(90-\theta) = T\cos\theta$$
$$2W\sin\theta = T\cos\theta$$

Moments about A

$$\begin{aligned}
2W \times ACCA\theta + W \times AGGS\theta &= T(\sin\theta AC) \\
2W\cos\theta \cdot AC - T\sin\theta(AC) &= -W\cos\theta(AG) \\
(T\sin\theta - 2W\cos\theta)AC &= W\cos\theta \cdot 4a \\
2\phi\left(\frac{\sin^2\theta}{\cos\theta} - \cos\theta\right)AC &= \mu\cos\theta \cdot 4a \\
\frac{\sin^2\theta - \cos^2\theta}{\cos\theta} \cdot 2\pi\cos\theta &= 2a\cos\theta \\
-\cos 2\theta &= \cos\theta \\
\Rightarrow \quad 2\theta &= 1\pi - \theta \\
\Rightarrow \quad \theta &= \frac{\pi}{3}
\end{aligned}$$

$$T = 2W\tan\frac{\pi}{3} \Rightarrow T = 2\sqrt{3}W$$

Horizontal component at A,

$$\begin{aligned}
&= T\cos\theta = T\cos\frac{\pi}{3} \\
&2\sqrt{3} = \frac{1}{3}
\end{aligned}$$

Vertical Component

$$\begin{aligned}
&= 3W - T\sin\theta \\
&= 3W - T\sin\frac{\pi}{3} \\
&= 3W - 2\sqrt{3}W\frac{2\sqrt{3}}{2} \\
&= 0
\end{aligned}$$

So, only action at A is the horizontal force.

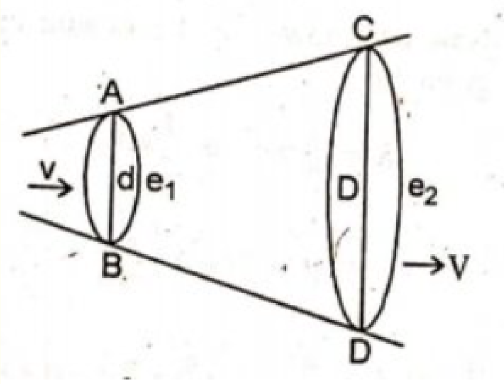
Question-7(c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; If V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{\mathrm{V}} = \frac{\mathrm{D}^2}{\mathrm{d}^2}\mathrm{e}^{\left(v^2 - \mathrm{V}^2\right)/2K}$$

where K is the pressure divided by the density and supposed constant.

[13 Marks]

Solution: Let e_1 and e_2 be the densities of steam at the ends of the conical pipe AB and CD. By the principle of conservation of mass, the mass of the steam that enters and leaves at the ends AB and CD are the same. Thus we have



$$\pi\left(\frac{1}{2}d\right)^2 ve_1 = \pi\left(\frac{1}{2}D\right)^2 Ve_2^-$$

$$\frac{v}{V} = \frac{D^2 e_2}{d^2 e_1}$$

let p be the pressure, e the density and u the velocity at distance r from AB, then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{e}\frac{\partial p}{\partial r}$$

$$p = Ke$$

$$u\frac{\partial u}{\partial r} = -\frac{K}{e}\frac{\partial e}{\partial r}$$

By integrating, we have

$$\frac{1}{2}u^2 = -K\log e + K\log E$$

where E is an arbitrary constant

$$\log\frac{e}{E} = -\frac{u^2}{2K}$$

$$e = E\exp\left(-\frac{u^2}{2K}\right)$$

Again

$$e = e_1 \text{ when } u = v$$

$$c_1 = E\exp\left(-\frac{v^2}{2K}\right)$$

and

$$e = e_2 \text{ when } u = v$$

$$e_2 = E\exp\left(\frac{-V^2}{2K}\right)$$

$$\frac{e_1}{e_2} = \frac{\exp\left(-v^2/2K\right)}{\exp\left(-V^2/2K\right)}$$

from (1) and (2), we have

$$\frac{v}{V} = \frac{D^2}{d^2}\exp\cdot\left(\frac{v^2-V^2}{2K}\right)$$

Proved.

Question-8(a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve: $\vec{\mathbf{r}} = \vec{\mathbf{r}}(\mathrm{u}) = 2\log_{\mathrm{e}} \mathbf{u}\hat{\mathbf{i}} + 4\mathbf{u}\hat{\mathbf{j}} + (2\mathbf{u}^2+1)\,\hat{\mathbf{k}}$

[10 Marks]

Solution: The parametric equation of the given curve is

$$\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + \left(2u^2+1\right)\hat{k}$$

where u is the parameter. Now,

$$\vec{\dot{r}} = \frac{d\bar{r}}{du} = \frac{2.1}{u}\hat{i} + 4\dot{j} + (4ut)\hat{k}$$

and

$$\vec{\ddot{r}} = \frac{d^2r}{du^2} = \frac{-2}{u^2}\hat{i} + 0 + 4\hat{k}$$

and

$$\vec{\dddot{r}} = \frac{d^3r}{du^2} = \frac{4}{u^3}\hat{i}$$

Now, we know that the curvature (k) of the curve is given by

$$K = \frac{|\vec{\dot{r}} \times \vec{\ddot{r}}|}{|\vec{\dot{r}}|^3}$$

$$Now, \vec{\dot{r}} \times \vec{\ddot{r}} = \begin{bmatrix} i & j & k \\ \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \end{bmatrix}$$

$$= 16\hat{i} - \frac{16}{u}\hat{j} + \frac{8}{u^2}\hat{k}$$

$$\Rightarrow |\vec{\dot{r}} \times \vec{\ddot{r}}| = \sqrt{16^2 + \left(\frac{16}{u}\right)^2 + \left(\frac{8}{u^2}\right)^2}$$

$$= \frac{8\left(1+2u^2\right)}{u^2}$$

$$\text{and } |\vec{\dot{r}}| = \left|\frac{2\hat{i} + 4\hat{j} + 4u\hat{k}}{u}\right|$$

$$= 2\sqrt{\left(\frac{1}{u}\right)^2 + 2^2 + (2u)^2}$$

$$= \frac{2\left(1+2u^2\right)}{u}$$

$$\therefore K = \frac{|\vec{\dot{r}} \times \vec{\ddot{r}}|}{|\vec{\dot{r}}|^3}$$

$$= \frac{8\left(1+2u^2\right)}{u^2} \cdot \frac{u^3}{8\left(1+2u^2\right)^3}$$

$$\therefore \quad = \frac{u}{\left(1+2u^2\right)^2}$$

$$\therefore \quad = \frac{u}{\left(1+2u^2\right)^2}$$

and the torsion (T) is given by

$$T = \frac{[\vec{\dot{r}} \quad \vec{\ddot{r}} \quad \vec{\dddot{r}}]}{|\vec{\dot{r}} \times \vec{\ddot{r}}|^2}$$

$$\text{Now, } [\vec{\dot{r}} \quad \vec{\ddot{r}} \quad \vec{\dddot{r}}] = \begin{bmatrix} \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \\ \frac{4}{u^3} & 0 & 0 \end{bmatrix} = \frac{64}{u^3}$$

$$\therefore \quad = \frac{64}{u^3} \cdot \frac{u^4}{64\left(1+2u^2\right)^2}$$

$$= \frac{u}{\left(1+2u^2\right)^2}$$

$$\text{Torsion } (T) = \frac{u}{\left(1+2u^2\right)^2}$$

Now, the arc length S is given by the

$$\int ds = \int \frac{d\vec{r}}{du} \mid \cdot du$$

$$\because \quad \frac{d\vec{r}}{du} = \frac{2}{u}\hat{i} + 4\hat{j} + 4u\hat{k}$$

$$\Rightarrow \left|\frac{d\vec{r}}{du}\right| = \sqrt{\frac{4}{u^2} + 16 + 16u^2}$$

$$= \frac{2\left(1+2u^2\right)}{u}$$

$$\therefore \quad \int ds = \int 2\left(\frac{1+2u^2}{u}\right) du$$

$$= 2\int \frac{1}{u} du + 4\int u du$$

$$\therefore \quad S = 2\log u + 2u^2$$

$$\Rightarrow \quad S = 2\left(u^2 + \log u\right)$$

is the required relation between S and parameter u.

Question-8(b) Prove the vector identity: $curl(\vec{f} \times \vec{g}) = \vec{\mathrm{f}}\ \mathrm{div}\ \vec{\mathrm{g}} - \vec{\mathrm{g}}\ \mathrm{div}\ \vec{\mathrm{f}} + (\vec{\mathrm{g}} \cdot \nabla)\vec{\mathrm{f}} - (\vec{\mathrm{f}} \cdot \nabla)\vec{\mathrm{g}}$ **and verify it for the vectors** $\vec{\mathrm{f}} = \mathrm{x}\hat{\mathrm{i}} + \mathrm{z}\hat{\mathrm{j}} + \mathrm{y}\hat{\mathrm{k}}$ **and** $\vec{\mathrm{g}} = \mathrm{y}\hat{\mathrm{i}} + \mathrm{z}\hat{\mathrm{k}}$.

[10 Marks]

Solution: The given vector identity is

$$\nabla \times (\bar{f} \times \vec{g}) = \bar{f}(\bar{\nabla} \cdot \bar{g}) - \bar{g}(\bar{\nabla} \cdot \bar{f}) + \quad (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

$$\begin{aligned} \text{L.H.S. } &= \bar{\nabla} \times (\bar{f} \times \bar{g}) \\ &= \Sigma i \times \frac{\partial}{\partial x}(\bar{f} \times \bar{g}) \\ &= \sum \bar{i} \times \left[\left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)\right] \\ &= \Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \Sigma \bar{i} \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right) \end{aligned}$$

Now, consider

$$\bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) = (\bar{g} \cdot \bar{i})\frac{\partial \bar{f}}{\partial x} - \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} \left(\begin{array}{c} \text{using relation } \bar{A} \times (\bar{B} \times \bar{C}) \\ = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C} \end{array}\right)$$

$$\begin{aligned} \therefore \quad \Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) &= \Sigma(\bar{g} \cdot \bar{i})\frac{\partial \bar{f}}{\partial x} - \Sigma\left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} \\ &= \bar{g} \cdot \left(\Sigma i \frac{\partial}{\partial x}\right)\bar{f} - \Sigma\left(i \frac{\partial}{\partial x} \cdot \bar{f}\right)\bar{g} \\ &= (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{\nabla} \cdot \bar{f})\bar{g} \\ &= (\bar{g} \cdot \bar{\nabla})\bar{f} - \bar{g}(\bar{\nabla} \cdot \bar{f}) \end{aligned}$$

Similarly for

$$\begin{aligned} \Sigma i \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right) &= \Sigma\left(\bar{i} \cdot \frac{\partial \bar{g}}{\partial x}\right) \cdot \bar{f} - \Sigma(\bar{i} \cdot \bar{f})\frac{\partial \bar{g}}{\partial x} \\ &= \bar{f}(\bar{\nabla} \cdot \bar{g}) - (\bar{f} \cdot \bar{\nabla})\bar{g} \end{aligned}$$

hence

$$\bar{\nabla}(\bar{f} \times \bar{g}) = f(\bar{\nabla} \cdot \bar{g}) - g(\bar{\nabla} \cdot \bar{f}) + (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

which proves the identity.

Now, if $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$ and $\quad \vec{g} = y\hat{i} + z\hat{k}$ then

$$\begin{aligned} \bar{f} \times \bar{g} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & z & y \\ y & 0 & z \end{vmatrix} \\ &= z^2\hat{i} + \left(y^2 - xz\right)\hat{j} - yz\hat{k} \end{aligned}$$

Now

$$\begin{aligned} \bar{\nabla} \times (\bar{f} \times \bar{g}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 - xz & -yz \end{vmatrix} \\ &= \hat{i}\left\{\frac{-\partial}{\partial y}(yz) - \frac{\partial}{\partial z}\left(y^2 - xz\right)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial z}\left(z^2\right)\right\} \\ &\quad + \hat{k}\left\{\frac{\partial}{\partial x}\left(y^2 - xz\right) - \frac{\partial z^2}{\partial y}\right\} \\ &= \hat{i}(-z + x) + \hat{j}(2z) + \hat{k}(-z) \\ &= (x - z)\hat{i} + 2z\hat{j} - z\hat{k} \\ \bar{g} &= (y\hat{i} + z\hat{k}) \end{aligned}$$

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{g} = 1 \text{ and } \quad \bar{f} = (x\hat{i} + z\hat{j} + y\hat{k}) \Rightarrow \quad \bar{\nabla}\cdot\bar{f} = 1$$

Also

$$\begin{aligned}(\bar{g}\cdot\bar{\nabla})\bar{f} &= \left(\frac{y\partial}{\partial x} + \frac{z\partial}{\partial z}\right)(x\hat{i} + z\hat{j} + y\hat{k})\\ &= (y\hat{i} + \vec{j})\\ (\bar{f}\cdot\bar{\nabla})\bar{g} &= \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)(y\hat{i} + z\hat{k})\\ &= (z\hat{i} + y\hat{k})\end{aligned}$$

Hence,

$$\begin{aligned}\bar{f}(\bar{\nabla}\cdot\bar{g}) - \bar{g}(\bar{\nabla}\cdot\bar{f}) + (\bar{g}\cdot\bar{\nabla})\bar{f} - (\bar{f}\cdot\bar{\nabla})\bar{g} &= \vec{f} - \bar{g} + (y\hat{i} + z\hat{j}) - (z\hat{i} + y\hat{k})\\ &= (x\hat{i} + z\hat{j} + y\hat{k}) - (y\hat{i} + z\hat{k}) + (y\hat{i} + z\hat{j}) - (z\hat{i} + y\hat{k})\\ &= (x - z)\hat{i} + 2z\hat{j} - z\hat{k}\end{aligned}$$

The given identity is verified for the vector $\bar{f} = x\hat{i} + z\hat{j} + y\hat{k}$ and $\vec{g} = y\hat{i} + z\hat{k}$

Question-8(c) Verify Green's theorem in the plane for

$$\oint_C \left[\left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy\right]$$

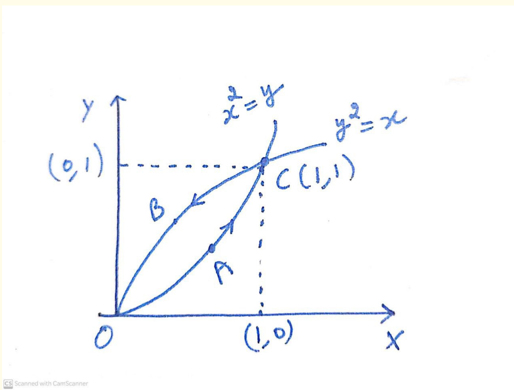
where C **is the boundary of the region enclosed by the curves** $y = \sqrt{x}$ **and** $y = x^2$.

[10 Marks]

Solution: The Green's theorem in a plane is given by

$$\oint_C Mdx + Ndy = \iint_S \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dxdy$$

where S is the area enclosed by the boundary of the curve C shown as shaded portion of the figure:



Now from question, the given curve is $y\ \sqrt{x}$ and $y = x^2$ and we have to verify Green's theorem for

$$\oint_C \left[\left(3x^2-8y^2\right)dx+(4y-6xy)dy\right]=\int_{OAC}\left[3x^2-8y^2\right)dx+(4y-6xy)dy+$$
$$\oint_{CBO}\left(3x^2-8y^2\right)dx+(4y-6xy)dy$$

Consider $\int_{OAC}(3x^2-8y^2)\,dx+(4y-6xy)dy$ along OAC path $y=\sqrt{x}$ or $x=y^2$

$$\therefore \quad dx=2ydy$$

$\therefore$ the integral can be changed to

$$\begin{aligned}\int_{y=0}^{1}\left(3y^4-8y^2\right)2ydy+\left(4y-6y^3\right)dy&=\int_{y=0}^{1}\left(6y^5+16y^2+4y-6y^3\right)dy\\&=\int_{y=0}^{1}\left(6y^5-22y^2+4y\right)dy\\&=\left[y^6-\frac{22}{4}y^4+\frac{4}{2}y^2\right]_0^1\\&=\left(1-\frac{22}{4}+2\right)\\&=3-\frac{11}{2}=\frac{-5}{2}\end{aligned}$$

Consider $\int_{CBO}(3x^2-8y^2)\,dx+(4y-6xy)dy$ along this path $y=x^2$

$$\begin{aligned}y=\int_{x=1}^{0}\left(3x^2-8x^4\right)dx+\left(4x^2-6x^3\right)2xdx&=\int_{x=1}^{0}\left(3x^2-8x^4+8x^3-12x^4\right)dx\\&=\int_{1}^{0}\left(3x^2+8x^3-20x^4\right)dx\\&=3\frac{x^3}{3}+8\frac{x^4}{4}-20\frac{x^5}{5}\bigg|_1^0=1\end{aligned}$$

$$\therefore\int_C\left(3x^2-8y^2\right)dx+(4y-6xy)dy=\frac{-5}{2}+1=\frac{-3}{2}$$

Now consider the integral

$$\begin{aligned}\oiint_S\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dxdy&=\iint(-16y-(-6y))dxdy\\&=-10\iint ydxdy\\&=-10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}ydxdy\\&=-10\int_0^1\frac{y^2}{2}\int_{x^2}^{\sqrt{x}}dx\\&=-5\int_0^1\left(x-4^4\right)dx\\&=-\frac{3}{2}\end{aligned}$$

Hence the Green's theorem is verified.

Question-8(d) The position vector $\overrightarrow{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$$

At time $t = 1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin.

[10 Marks]

Solution: The position vector of the particle of mass 2 unit at time t is given by

$$\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$$

Now we know that the kinetic energy is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}\cdot\vec{v}) \because \quad \vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{t^2}{2}\vec{k}$$

Hence

$$\vec{v} = \frac{dr}{dt} = (2t-2)\hat{i} + t\hat{j} + t\hat{k}$$

$$\therefore \quad \vec{v}\cdot\bar{v} = 4(t-1)^2 + t^2 + t^2$$

$$= 2\left[(t-1)^2 + t^2\right]$$

$$\therefore \quad K = \frac{1}{2}\cdot 2\cdot 2\left[2(t-1)^2 + t^2\right]$$

$\therefore$ At $t = 1$, the K.E. is given by

$$K = 2\left[2(1-1)^2 + 1^2\right] = 2 \text{ Units.}$$

$$\vec{v}\big|_{t=1} = \hat{j} + \hat{k}$$

$$\vec{r}\big|_{t=1} = -\hat{i} + \frac{3}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$\therefore \quad \vec{L}\Big|_{t=1} = \vec{r}\times m\vec{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad \vec{L} = 2(\hat{i} + \hat{j} + \hat{k})$$

Since,

$$\vec{L} = \vec{r}\times m\vec{v}$$

differentiating both sides with respect to t we get

$$\frac{d\bar{L}}{dt} = \frac{dr}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\bar{v}}{dt} = \vec{r} \times m\frac{d\bar{v}}{dt}$$

$$\because \quad \vec{v} = (2t-2)\hat{i} + \hat{t}\hat{j} + t\hat{k}$$

$$\Rightarrow \quad \frac{d\bar{v}}{dt} = 2\hat{i} + \hat{j} + \hat{k}$$

$$\therefore \left.\frac{d\bar{L}}{dt}\right|_{t=1} = 2\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 1 & 1 \end{bmatrix} = 2\hat{i} + 4\hat{j} - 8\hat{k}$$

Finally, the moment of the resultant force is given by,

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\frac{d\bar{v}}{dt}$$

$$\Rightarrow \quad \vec{\tau}|_{t=1} = 2\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 1 & 1 \end{bmatrix}$$

$$= 2\hat{i} + 4\hat{j} - 8\hat{k}$$

Thus, at $t = 1$, Kinetic energy = 2 units.

Angular momentum $(\vec{L}) = 2(\hat{i} + \hat{j} - \hat{k})$ units

Time Rate of change of angular momentum $= (2\hat{i} + 4\hat{j} - 8\hat{k})$ units and moment of the resultant force $= (2\hat{i} + 4\hat{j} - 8\hat{k})$ units.

Chapter 11

2010

11.1 Section-A

Question-1(a) Show that the set

$$\mathrm{P[t]} = \left\{\mathrm{at^2 + bt + c/a, b, c} \in \mathbb{R}\right\}$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space?

[8 Marks]

Solution: From question $\mathrm{P}(t) = \{at^2 + bt + c\}$
Let, $f(t)$ and $g(t) \in p(t)$ then, $f(t) = a_1t^2 + b_1t + c_1$ and $g(t) = a_2t^2 + b_2t + c_2$
then,

$$f(t) + g(t) = (a_1 + a_2)\, t^2 + (b_1 + b_2)\, t + (c_1 + c_2)$$

$$\Rightarrow f(t) + g(t) \in p(t)$$

$\because a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$
Also, $f(t) = g(t)$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$
and, $kf(t) = (ka_1)\, t^2 + (kb_1)\, t + kc_1 = i \in p(t)$

$$\begin{aligned} f(t) + g(t) &= (a_1 + a_2)\, t^2 + (b_1 + b_2)\, t + (c_1 + c_2) \\ &= (a_2 + a_1)\, t^2 + (b_2 + b_1)\, t + (c_2 + c_1) \\ &= g(t) + f(t) \end{aligned}$$

$\Rightarrow$ Set is commutative.
Also, if $b(t) = a_3t^2 + b_3t + c_3$ then $f(t) + (g(t) + h(t)\} = \{f(t) + g(t)\} + h(t)$ Existence of identity $0 = 0.t^2 + 0.t + 0$ i.e., $0 \in p(t) \Rightarrow 0 + f(t) = f(t)$

Existence of additive inverse of each member as $f(t) \in p(t)$ then $-f(t) \in p(t)$ and $-f(t) + f(t) = 0$
$\therefore -f(t)$ is the additive inverse of $f(t)$ i.e. $P(t)$ is an abelian group w.r.t. addition of polynomial of less than or equal to degree. Hence: $p(t)$ is vector space.

Question-1(b) Determine whether the quadratic form is positive definite.

$$q = x^2 + y^2 + 2xz + 4yz + 3z^2$$

[8 Marks]

Solution: The associated symmetric matrix of the given quadratic form can be written as:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{i.e.} q = \begin{bmatrix} x & y & z \end{bmatrix} \mathrm{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to ascertain the positive definite, we have to apply the congruent operation in the above matrix.i.e.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congrucnt operation $R_3 \to R_3 - R_1$ & $C_3 \to C_3 - C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_3 \to R_3 - 2R_2$& $C_3 \to C_3 - 2C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

As all the roots of scalar matrix in the left hand side are not positive. Hence, the given quadratic form is not positive.

Question-1(c) Prove that between any two real roots of $\mathrm{e}^{\mathrm{x}} \sin \mathrm{x} = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

[8 Marks]

Solution: The given function is $f(x) = e^x \sin x - 1 = 0$ or, $\sin x - e^{-x} = 0$
Now, $f(x) = \sin x - e^{-x} = 0$
If x_1 and x_2 are two roots of $f(x) = 0$ then by Rolle's theorem $\exists$ at least one real root of $f'(x) = 0$ lies between x_1 and x_2.

$$\because f'(x) = \cos x + e^{-x} = 0$$

i.e. $e^x \cos x + 1 = 0$ has a root lies between two real roots of $e^x \sin x = 1$

Question-1(d) Let f be a function defined on $\mathbb{R}$ such that

$$f(x+y) = f(x) + f(y), \quad x, y \in \mathbb{R}$$

If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$

[8 Marks]

Solution: Given,

$$f(x+y) = f(x) + f(y) \quad \ldots \quad (1)$$

Let f be differentiable at a and c be any general point.

Then,

$$\begin{aligned} \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} &= \underset{h\to 0}{Lt} \frac{f(a)+f(h)-f(a)}{h} \text{ (from (1))} \\ &= \underset{h\to 0}{Lt} \frac{f(h)}{h} \text{ (exists, } \because f \text{ is diff at } a) \end{aligned}$$

Hence,

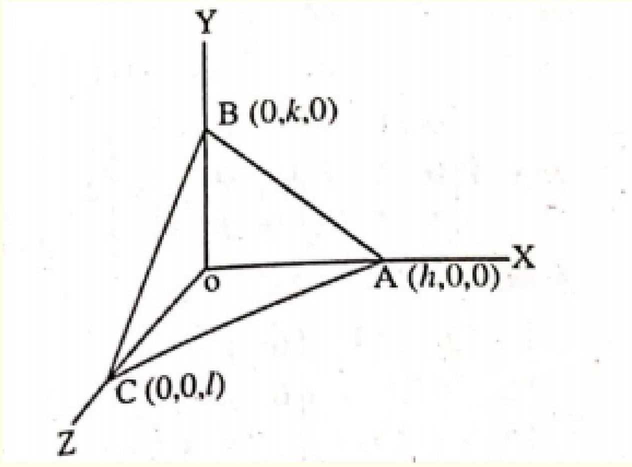
$$\begin{aligned} \underset{h\to 0}{Lt} \frac{f(c+h)-f(c)}{h} &= \underset{h\to 0}{Lt} \frac{f(c)+f(h)-f(c)}{h} \\ &= \underset{h\to 0}{Lt} \frac{f(h)}{h} \text{ exists} \quad \ldots \quad (2) \end{aligned}$$

As c was arbitrary point on $\mathbb{R}$, hence f is differentiable on $\mathbb{R}$.

Question-1(e) If a plane cuts the axes in $\mathrm{A, B, C}$ and $(\mathrm{a, b, c})$ are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

[8 Marks]

Solution: Let the co-ordinate of $\mathrm{A} \equiv (h, 0, 0)$, $\mathrm{B} = (0, k, 0)$ and $\mathrm{C} \equiv (0, 0, l)$ then, equation of plane ABC is $\frac{x}{h} + \frac{y}{k} + \frac{z}{l} = 1$.



Now, (a, b, c) is the centroid of ΔABC then

$$a = \frac{h+0+0}{3}, b = \frac{0+k+0}{3}, c = \frac{0+0+l}{3}$$

$\alpha, \quad h = 3a, k = 3b, l = 3c$
i.e. equation of the plane ABC can be rewritten as

$$\frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$$

Question-1(f) Find the equations of the spheres passing through the circle

$$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, y = 0$$

and touching the plane $3y + 4z + 5 = 0$.

[8 Marks]

Solution: The equation of the given circle is

$$\left.\begin{array}{r} x^2 + y^2 + z^2 - 6x - 2z + 5 = 0 \\ y = 0 \end{array}\right\} \cdots (1)$$

Equation of any sphere passing through the circle (I) is given by

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + \lambda y = 0 \quad \cdots (2)$$

Centre of sphere (2) is $\left(3, -\frac{\lambda}{2}, 1\right)$ and radius of this sphere is $\sqrt{\frac{\lambda^2}{4} + 5}$.
Now, if the plane $3y + 4z + 5 = 0 \ldots (3)$ is a tangent plane to (2), then,

$$\frac{\left|3\left(\frac{-\lambda}{2}\right)+4+5\right|}{5} = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \left|\frac{9-\frac{3\lambda}{2}}{5}\right| = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \frac{3(6-\lambda)}{10} = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \frac{9(6-\lambda)^2}{100} = \frac{\lambda^2+20}{4}$$

$$\Rightarrow 9\left(\lambda^2 - 12\lambda + 36\right) = 25\left(\lambda^2 + 20\right)$$

$$\Rightarrow 25\lambda^2 + 500 = 9\lambda^2 - 108\lambda + 324$$

$$\Rightarrow 16\lambda^2 + 108\lambda + 176 = 0$$

$$\Rightarrow 4\lambda^2 + 27\lambda + 44 = 0$$

$$\Rightarrow 4\lambda^2 + 11\lambda + 16\lambda + 44 = 0$$

$$\Rightarrow \quad \lambda(4\lambda + 11) + 4(4\lambda + 11) = 0$$

$$\Rightarrow (\lambda + 4)(4\lambda + 11) = 0$$

$$\Rightarrow \lambda = -4 \quad \text{or,} \quad \lambda = -\frac{11}{4}$$

Hence, the equation of sphere is given by $x^2 + y^2 + z^2 - 6x - 2z + 5 - 4y = 0$ and $4\left(x^2 + y^2 + z^2 - 6x - 2z + 5\right) - 11y = 0$.

Question-2(a) Show that the following vectors form a basis for $\mathbf{R}^3$

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

Find the components of $(1, 0, 0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

[10 Marks]

Solution: To show that $\alpha_1, \alpha_2, \alpha_3$, form a basis of $\mathbb{R}^3$. It is sufficient to show that they are linearly independent. i.e. $\exists\, ax, y, z \in \mathbb{R}$ such that

$$x\alpha_1 + y\alpha_2 + z\alpha_3 = (0, 0, 0)$$

then $x = y = z = 0$

$$x(1, 0, -1) + y(1, 2, 1) + z(0, -3, 2) = (0, 0, 0)$$

$$(x + y, 2y - 3z - x + y + 2z) = (0, 0, 0)$$

Comparing the co-efficients, we get,

$$\begin{aligned} x + y = 0z \cdots (1) \\ 2y - 3z = 0 \cdots (2) \\ -x + y + 2z = 0 \cdots (3) \end{aligned}$$

(1) and (3) $\Rightarrow 2y + 2z = 0 \cdot (4)$
(2) and (4) $\Rightarrow 5z = 0$ or $z = 0$
$\Rightarrow y = 0$ i.e. $\quad x = y = z = 0$

Hence, $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly independent. Also dimension $= 3$, hence, they form a basis of $\mathbb{R}^3$.

Now, let $(1,0,0) = a\alpha_1 + b\alpha_2 + c\alpha_3$ then,

$$(1,0,0) = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$$

$$\Rightarrow (1,0,0) = (a+b, 2b-3c, -a+b+2c)$$

$$\Rightarrow a+b=1, \quad 2b-3c=0 \quad -a+b+2c=0$$

$$\therefore \quad a+b=1 \Rightarrow a=(1-b)$$

$$\Rightarrow 2b = 3c \Rightarrow c = \tfrac{2}{3}b$$

$$-a+b+2c=0$$

$$\Rightarrow b-1+b+\tfrac{4}{3}b=0$$

$$2b+\tfrac{4}{3}b=1$$

$$\Rightarrow \tfrac{10b}{3}=1$$

$$b=\tfrac{3}{10}$$

$$\therefore a = 1-\tfrac{3}{10}=\tfrac{7}{10}$$

$$\therefore \mathrm{C} = \tfrac{2}{3}\cdot\tfrac{3}{10}=\tfrac{1}{5}$$

$$\therefore \quad (1,0,0) = \tfrac{7}{10}\alpha_1 + \tfrac{3}{10}\alpha_2 + \tfrac{1}{5}\alpha_3$$

Question-2(b) Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley-Hamilton theorem for this matrix and hence find its inverse.

[10 Marks]

Solution: Let the given matrix be A$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

then, the characteristic equation of A is given by,

$$|A-\lambda.I| = 0$$

$$\Rightarrow \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 0 & 1 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow -\lambda \cdot \{\lambda(\lambda-3)-2\}+1(1)=0$$

$$\Rightarrow -\lambda\,(\lambda^2-3\lambda-2\}+1=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+2\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-2\lambda-1=0$$

Now, by Cayley-Hamilton theorem, it should also satisfy the matrix A i.e.

$$A^3-3A^2-2A-I=0\cdots(1)$$

To prove the identity (1), we will calculate A^3 and A^2.

$$A^2=\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}$$

$$A^2=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}$$

&

$$\mathrm{A}^3=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}$$

$$\mathrm{A}^3=\begin{bmatrix}1&3&11\\2&7&25\\3&11&39\end{bmatrix}$$

Now,

$$\mathrm{A}^3-3\ \mathrm{A}^2-2\ \mathrm{A}-\mathrm{I}=\begin{bmatrix}1&3&11\\2&7&25\\3&11&39\end{bmatrix}-3\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}-2\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Hence, Cayley-Hamilton theorem is verified.
Now,

$$A^3-3A^2-2A-I=0$$

$$\Rightarrow I=A^3-3A^2-2A$$

multiply both the sides by $\cdot\ \mathrm{A}^{-1}$, we get

$$A^{-1}=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}-3\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$$

$$\Rightarrow A^{-1}=\begin{bmatrix}-2&1&0\\1&0&0\end{bmatrix}$$

Question-2(c) **Let** $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & .4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. **Find an invertible matrix** P **such that** P^{-1} AP **is a diagonal matrix.**

[10 Marks]

Solution: The such invertible matrix can be formed with the help of eigenvectors of matrix A. The characteristic equation of matrix is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)((\lambda-4)(\lambda+4)+12\} +6\{\lambda+4-6\} - 6\{6-3(4-\lambda)\} = 0$$

$$\Rightarrow 4 - 8\lambda + 5\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)^2 = 0$$

Hence, eigenvalues of matrix A is given by

$$\lambda = 1, 2, 2$$

Now, corresponding to $\lambda = 2$, the eigenvector is obtained through

$$[\mathrm{A} - 2\mathrm{I}]\mathrm{X} = \begin{bmatrix} 5-2 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 3 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - 2x_2 - 2x_3 = 0$$

This imples that there are two free variables. Putting $x_2 = 0, x_3 = 1$, we get the eigenvector $[2, 0, 1]$ and by putting $x_2 = 1, x_3 = 0$, we get the eigenvector $[2, 1, 0]$.

Hence, the two eigenvectors corresponding to $i = 2$ are $[2, 0, 1]$ and $[2, 1, 0]$.

Now, the eigenvector corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 1 & -3/2 & -3/2 \\ 1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_1 \to \frac{1}{4}R_1$, $R_2 \to R_2 - R_1 \& R_3 \to R_3 - 3R_1$, we get:

$$\begin{bmatrix} 1 & -3/2 & -3/2 \\ 0 & 3/2 & 1/2 \\ 0 & -3/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 3x_2 - 3x_3 = 0$$

$$\Rightarrow 3x_2 + x_3 = 0$$

There is only one free variable say $x_2 = 1$ then $\quad x_3 = -3$ & $\quad 2x_1 - 3 + 9 = 0 \quad x_1 = -3$

$\therefore \quad (-3, 1, -3)$

Hence, the invertible matrix P can be written as

$$P = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{bmatrix} \text{ and } |\mathrm{P}| = -6 + 2 + 3 = -1$$

$$\therefore \quad P^{-1} = -\begin{bmatrix} -3 & 1 & -1 \\ 6 & -3 & 2 \\ 5 & -2 & 2 \end{bmatrix}^{\top} = \begin{bmatrix} 3 & -6 & -5 \\ -1 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

Hence,

$$\begin{aligned} \mathrm{P}^{-1}\mathrm{AP} &= \begin{bmatrix} 3 & -6 & -5 \\ -1 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} 2 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

which is a diagonal matrix.

Question-2(d) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

[10 Marks]

Solution: The rank of any matrix is equal to number of non-zero rows in the echelon form of the given matrix. Now, Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $\mathrm{R}_4 \to \mathrm{R}_4 - 3\mathrm{R}_1$ and $\mathrm{R}_5 \to \mathrm{R}_5 - 4\mathrm{R}_1$

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$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Apply $R_3 \to -\frac{1}{4}R_3, R_5 \to R_5 - 5R_3$, $R_3 \to R_3 - 3R_2, R_4 \to R_4 + R_2$ and $R_5 \to R_5 - 2R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows in echelon form $= 3$ i.e. Rank of the given matrix $= 3$.

Question-3(a) Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1 + x^4 \sin^2 x}$$

[10 Marks]

Solution: Consider the integral $I = \int_0^{\pi x} \frac{dx}{1+x^4 \sin^2 x} \propto, \quad I = \sum_{n=1}^{n} \int_{(r-1)x}^{n} \frac{dx}{1+x^4 \sin^2 x}$ Now for $\int_{(r-1)x}^{r} \frac{dx}{1+x^4 \sin^2 x}$.

Let $x = (r-1)\pi + y$ then $\quad dx = dy$

$\therefore$ Above integral reduces to

$$\int_0^\pi \frac{dy}{1 + [(r-1)\pi + y]^4 \sin^2[(r-1)\pi + y]} = \int_n^\pi \frac{dy}{1 + \{(r-1)\pi + y\}^4 \sin^2 y}$$

$$< \int_0^\pi \frac{dy}{1 + \{(r-1)\pi\}^4 \sin^2 y}$$

$$2\int_0^{\pi/2} \frac{\operatorname{cosec}^2 y dy}{\operatorname{cosec}^2 + (r-1)^4\pi^4} = 2\int_0^{\frac{\pi}{2}} \frac{\cos ec^2 y dy}{1 + (r-1)^4\pi^4 + \cos^2 y} \,|$$

$$= 2.\frac{1}{\sqrt{1+(r-1)^4\pi^4}} \cot^{-1} \frac{\cot y}{\sqrt{1+(r-1)^4\pi^4}}$$

$$= \frac{2}{\sqrt{1+(r-1)^4\pi^4} \cdot \frac{\pi}{2}}$$

ie.

$$\int_{(r-1)x}^{n} \frac{dx}{1+x^4\sin^2\alpha} < \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$
$$= \frac{\pi}{(r-1)^2\pi^2} - \frac{1}{r^2\pi^2}$$

i.e.

$$\sum_{r=1}^{n} \int \frac{dx}{1+x^4\sin^2\alpha} < \sum_{r=1}^{n} \frac{1}{\pi^2 r^2}$$
$$\therefore \underset{n\to\infty}{Lt} \int_0^{n\pi} \frac{dx}{1+x^4\sin^2\alpha} < \sum \frac{1}{r^2}$$

which is convergent. Hence,

$$\int_0^{\infty} \frac{dx}{1+x^4\sin^2 x}$$

is convergent.

Question-3(b) Find the extreme value of xyz if $x+y+z=a$.

[10 Marks]

Solution: Define a Lagrangian function $\mathrm{F}(x,y,z,\lambda) = xyz + \lambda(x+y+z-a)$
Then for extremum value

$$d\,\mathrm{F} = 0$$
$$\Rightarrow yzdx + xzdy + xydz + \lambda(dx+dy+dz) = 0$$
$$\Rightarrow (yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz = 0$$

Equating the co-efficients, we get

$$yz+\lambda = 0; \quad xz+\lambda = 0; \quad xy+\lambda = 0$$
$$yz+\lambda - xz - \lambda = 0$$
$$\Rightarrow z(x-y) = 0$$
$$\Rightarrow z = 0 \text{ or } x = y$$

However, $z = 0 \Rightarrow \lambda = 0$ which further led to

$$x = y = 0$$

Hence, $x - y$ is the acceptable solution.
Similarly from $xz + \lambda = 0$ and $xy + \lambda = 0$ we get

$$y = z \text{ i.e. } x = y = z$$

is the condition for extremum of Lagrangian function.

Also,

$$x + y + z = 0 \Rightarrow 3x = a$$

or

$$x = y = z = \frac{a}{3}$$

Hence, the extremum value of

$$xyz = \frac{a^3}{27}$$

Question-3(c) Let

$$f(x,y) = \begin{cases} \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that: $(i) f_{xy}(0,0) \neq f_{yx}(0,0)$
(ii)f **is differentiable at** $(0,0)$.

[10 Marks]

Solution:

$$f_{xy}(0,0) = \underset{k\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{f_x(h,0) - f_x(0,0)}{h}$$

Now,

$$\begin{aligned} f_x(0,k) &= \underset{h\to 0}{Lt} \frac{f(h,k) - f(h,0)}{h} \\ &= \underset{h\to 0}{Lt} \frac{\frac{hk\left(h^2-k^2\right)}{h^2+k^2} - 0}{h} \\ &= \underset{h\to 0}{Lt} \frac{k\left(h^2 - k^2\right)}{h^2 + k^2} \\ &= -k \end{aligned}$$

$$\begin{aligned} f_x(0,0) &= \underset{h\to 0}{Lt} \frac{f(h,0) - f(0,0)}{h} \\ &= \underset{h\to 0}{Lt} \frac{0}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad f_{xy}(0,0) &= \underset{h\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k} \\ &= \underset{k\to 0}{Lt} - \frac{k - 0}{k} \\ &= -1 \end{aligned}$$

Also,

$$\begin{aligned} f_y(h,0) &= \underset{k\to 0}{Lt}\frac{f(h,k)-f(h,0)}{k} \\ &= \underset{k\to 0}{Lt}\frac{h\frac{k\left(h^2-k^2\right)}{h^2+k^2}-0}{k} \\ &= h \end{aligned}$$

$$\begin{aligned} f_y(0,0) &= \underset{k\to 0}{Lt}\frac{f_y(h,0)-f_y(0,0)}{k} \\ &= \underset{k\to 0}{Lt}\frac{0-0}{k} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad f_{yx}(0,0) &= \underset{h\to 0}{Lt}\frac{f_y(h,0)-f_y(0,0)}{h} \\ &= \underset{h\to 0}{Lt}\frac{h-0}{h} \\ &= 1 \end{aligned}$$

i.e.$f_{yx}(0,0) = 1$ also $f_{xy}(0,0) = -1$

Hence,$f_{yx}(0,0) \neq f_{xy}(0,0)$.
Further, $f_x(0,0) = 0 = f_y(0,0)$

Also, when $x^2+y^2 \neq 0$, then

$$\begin{aligned} |f_x| &= \frac{|x^4y+4x^2y^3-y^5|}{(x^2+y^2)^2} \\ &\leq \frac{6\left(x^2+y^2\right)^{5/2}}{(x^2+y^2)^2} \\ &= 6\left(x^2+y^2\right)^{1/2} \end{aligned}$$

Evidently,

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

Thus, f_x is continuous at (0,0) and $f_y(0,0)$ exists $\Rightarrow f$ is differentiable at $(0,0)$.

Question-3(d) Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

[10 Marks]

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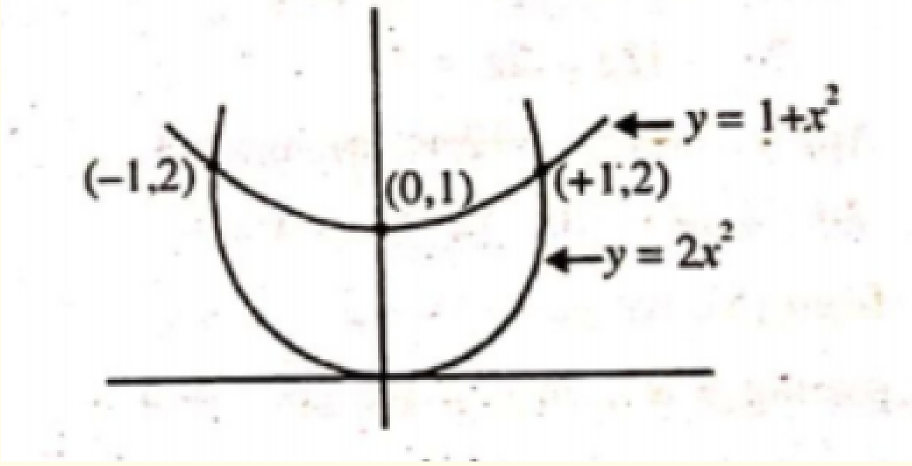


Solution:
We have to calculate

$$\iint (x+2y)dA = \cdot \int_{y=0-1}^{1}\int_{y=0}^{1}(x+2y)dxdy$$
$$= \int_{y=0}^{1}\frac{x^2}{2}+2xy|'\,dy$$
$$= 4\int_{y=0}^{1} ydy = 4\times\frac{1}{2}$$
$$= 2 \text{ units}$$

Question-4(a) Prove that the second degree equation represents a cone

$$x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$$

whose vertex is $(1,-2,3)$.

[10 Marks]

Solution: The given equation is

$$f(x,y,z)=x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$$

Making homogeneous with the help of new variable t, to calculate the vertex of cone. i.e.

$$\mathrm{F}(x,y,z,t)=x^2-2y^2+3z^2+5yz-6z-4xy+8xt-19xt-2z-20t^2=0$$

Now, differentiating partially with respect to x,y,z,t and then putting $t=1$, we get,

$$F_x = 2x-6z-4y+8=0$$

$$\Rightarrow x-2y-3z+4=0$$

$$F_y=-4y+5z-4x-19=0$$

$$\Rightarrow 4x+4y-5z+19=0$$

$$F_z = 6z - 6x + 5y - 2 = 0$$

$$\Rightarrow 6x - 5y - 6z + 2 = 0$$

$$F_t = 8x - 19y - 2z - 40 = 0$$

$$8x - 19y - 2z - 40 = 0$$

Now, if $f(x, y, z) = 0$ represent a cone the value of $x, y; z$ obtained from solving (1),(2) and (3) should satisfy (4) and that value represent the vertex of the cone.

Apply $(2) - 4 \times (1)$, we get

$$12y + 7z + 3 = 0$$

Apply $(3) - 6 \times (1)$ we get

$$7y + 12z - 2z = 0$$

Apply $7 \times (5) - 12 \times (6)$, we get,

$$-95z + 285 = 0 \Rightarrow z = 3$$

from (5), we get,

$$y = -2$$

putting y & z in (1), we get $x = 1$ i.e.

$$(x, y, z) = (1, -2, 3)$$

Now putting this in (4) we get,

$$8 + 38 - 6 - 40 = 0$$

Hence, the given second degree equation represent a cone with vertex $(1, -2, 3)$

Question-4(b) If the feet of three normals drawn from a point P to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ lie in the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, prove that the feet of the other three normals lie in the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} + 1 = 0$.

[10 Marks]

Solution: Let the co-ordinates of the given point be (x_1, y_1, z_1). Now the co-ordinates (α, β, γ) of the feet of six normals from (x_1, y_1, z_1) to given ellipsoid are given by:

$$\alpha = \frac{a^2 x_1}{a^2 + \lambda}, \beta = \frac{b^2 y_1}{b^2 + \lambda}, \gamma = \frac{c^2 z_1}{c^2 + \lambda}$$

where λ is a parameter.
Now, (α, β, γ) lies on ellipsoid.

$$\Rightarrow \frac{a^2 x_1^2}{(a^2 + \lambda)^2} + \frac{b^2 y_1^2}{(b^2 + \lambda)^2} + \frac{c^2 z_1^2}{(c^2 + \lambda)^2} = 1 \ ldots (1)$$

which gives six values of $\lambda^{\cdot}$

Now, if three of six lie on plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ then

$$\frac{ax_1}{a^2+\lambda}+\frac{by_1}{b^2+\lambda}+\frac{cz_1}{c^2+\lambda}-1=0\ldots(2)$$

(satisfied by three value of λ).

Let the other three feet lie on

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}-p'=0$$

then

$$\frac{a^2x_1}{a'\left(a'+\lambda\right)}+\frac{b^2y_1}{b'\left(b^2+\lambda\right)}+\frac{c^2z_1}{c'\left(c^2+\lambda\right)}-p'=0\ldots(3)$$

(2) and (3) in combined form represent a conic passing through the feet of six normals, which is represented by equation (1) also.
Comparing coefficients, we get

$$\frac{a^3}{a'\left(a^2+\lambda\right)^2}=\frac{a^2}{\left(a^2+\lambda\right)^2}$$

$$\Rightarrow\frac{1}{a'}=\frac{1}{a}$$

Similarly $b'=b$, $c'=c$ and $\quad p'=-1$
$\Rightarrow$ The equation of other plane is given by:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$$

Question-4(c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5\text{yz}-8\text{zx}-3\text{xy}=0$, find the equations of the other two.

[10 Marks]

Solution: Let

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{h}$$

represent one of other two generator as this is perpendicular to given generator

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

. Hence,

$$l+2m+3n=0$$

Also

$$5mn-8\ln-3lm=0$$

$$\Rightarrow 5mn-l(3m+8n)=0$$

$$\Rightarrow 5mn + (2m+3n)(3m+8n) = 0 < \text{using } (1) >$$

$$\Rightarrow 6m^2 + 30mn + 24n^2 = 0$$

$$\Rightarrow m^2 + 5mn + 4n^2 = 0$$

$$\Rightarrow m^2 + mn + 4mn + 4n^2 = 0$$

$$\Rightarrow m(m+n) + 4n(m+n) = 0$$

$$\Rightarrow (m+n)(m+4n) = 0$$

$$m + n = 0 \Rightarrow \frac{m}{1} = \frac{n}{-1}$$

$$1 + 2 - 3 = 0 \Rightarrow l = 1$$

then, i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ represent one generator if $m + 4n = 0$, then, $\frac{m}{-4} = \frac{n}{1}$ then, $l - 8 + 3 = 0 \Rightarrow l = 5 \Rightarrow \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$ represent other generator.

Hence, the equation of two other generators are

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

&

$$\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

Question-4(d) Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{\mathrm{x}^2}{\mathrm{a}^2} + \frac{\mathrm{y}^2}{\mathrm{b}^2} + \frac{\mathrm{z}^2}{\mathrm{c}^2} = 1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ is $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}$.

[10 Marks]

Solution: Let (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3) be the end point of conjugate diametrical planes of ellipsoid $\frac{x^2}{\alpha^1} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ then equation of plane parallel to these conjugate diameterical planes are given by,

$$\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} + \frac{zz_1}{\gamma^2} = d_1; \quad \frac{xx_2}{\alpha^2} + \frac{yy_2}{\beta^2} + \frac{zz_2}{\gamma^2} = d_2; \quad \text{and} \frac{xx_3}{\alpha^2} + \frac{yy_3}{\beta^2} + \frac{zz_3}{\gamma^2} = d_3$$

Now, three planes are tangent planes to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

then, by the properties of tangent planes.

$$\frac{a^2x_1^2}{\alpha^4}+\frac{b^2y_1^2}{\beta^4}+\frac{c^2z_1^2}{\gamma^4}=d_1^2\frac{a^2x_2^2}{\alpha^4}+\frac{b^2y_2^2}{\beta^4}+\frac{c^2z_2^2}{\gamma^4}$$
$$=d_2^2\frac{a^2x_3^2}{\alpha^4}+\frac{b^2y_3^2}{\beta^4}+\frac{c^2z_3^2}{\gamma^4}$$
$$=d_3^2$$

adding above three equation we get,

$$\frac{a^2}{\alpha^4}\left(x_1^2+x_2^2+x_3^2\right)+\frac{b^2}{\beta^4}\left(y_1^2+y_2^2+y_3^2\right)+\frac{c^2}{\gamma^4}\left(z_1^2+z_2^2+z_3^2\right)=d_1^2+d_2^2+d_3^2$$

$$\Rightarrow \tfrac{a^2}{\alpha^4}\alpha^2+\tfrac{b^2}{\beta^4}\beta^2+\tfrac{c^2}{\gamma^4}\gamma^2=d_1^2+d_2^2+d_3^2$$

(By properties of conjugate diametrical planes)

$$\Rightarrow \tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}=d_1^2+d_2^2+d_3^2$$

Also,

$$\left(\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}\right)^2+\left(\frac{xx_2}{\alpha^2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}\right)^2+\left(\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_3}{\gamma^2}\right)^2=d_1^2+d_2^2+d_3^2$$

$$\Rightarrow \tfrac{x^2}{\alpha^4}\Sigma x_1^2+\tfrac{y^2}{\beta^4}\Sigma y_1^2+\tfrac{z^2}{\gamma^4}\Sigma z_1^2=d_1^2+d_2^2+d_3^2$$

(other term of equation vanishes)

$$\Rightarrow \tfrac{x^2}{\alpha^4}\alpha^2+\tfrac{y^2}{\beta^4}\beta^2+\tfrac{z^2}{\gamma^4}\gamma^2=\tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}$$

$$\Rightarrow \tfrac{x^2}{\alpha^2}+\tfrac{y^2}{\beta^2}+\tfrac{z^2}{\gamma^2}=\tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}$$

which is the locus of the point of intersection of tangent planes of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1$$

11.2 Section-B

Question-5(a) Show that $\cos(\mathrm{x}+\mathrm{y})$ **is an integrating factor of**

$$ydx+[y+\tan(x+y)]dy=0$$

Hence solve it.

[8 Marks]

Solution: The given differential equation is

$$ydx+[y+\tan(x+y)]dy=0 \quad \cdots(1)$$

Now, if $\cos(x+y)$ is an I.F. of the above equation, then it should reduce it into exact form.

$$y\cos(x+y)dx + \left[\begin{array}{l} y\cos(x+y) \\ +\sin(x+y)]dy = 0 \end{array}\right.$$

Now, if it is exact then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where, $M = y\cos(x+y)$

$$\begin{aligned} \text{N} &= y\cos(x+y) + \sin(x+y) \\ \frac{\partial M}{\partial y} &= \cos(x+y) - y\sin(x+y) \\ \frac{\partial N}{\partial x} &= -y\sin(x+y) + \cos(x+y) \end{aligned}$$

i.e. (1) becomes exact after multiplication by $\cos(x+y)$

Hence, solution of the equation is given by

$$\int y\cos(x+y)dx + \int \{y\cos(x+y) + \sin(x+y)\}dy$$

$$y\sin(x+y) + 0 = c$$

as there is no term independent of x is contained in second integral.

Question-5(b) Solve:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

[8 Marks]

Solution: For complementary function, the auxiliary equation is given by

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad m = 1, 1$$

Hence, complementary function

$$y = (c_1 + c_2 x)\, e^x$$

where, c_1, c_2 are arbitrary constants.
Now, the particular integral is given by,

$$\begin{aligned} y &= \frac{1}{(D-1)^2} xe^x \sin x \\ &= e^x \cdot \frac{1}{(D+1-1)^2} x\sin x \\ &= e^x \frac{1}{D^2} x\sin x = e^x \frac{1}{D}\int x\sin x dx \\ &= e^x \frac{1}{D}[-x\cos x + \sin x] \\ &= e^x \left[\int (\sin x - x\cos x)dx\right] \\ &= e^x[-\cos x - \{x\sin x + \cos x\}] \\ &= -xe^x \sin x - 2\cos x \end{aligned}$$

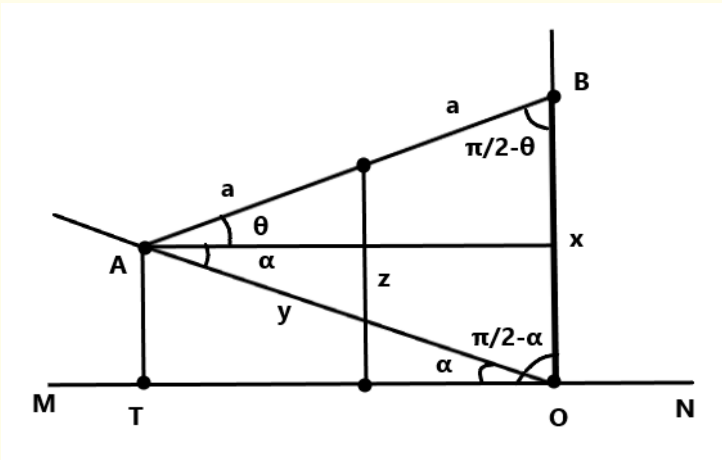
Hence, General solution is given

$$y = (c_1 + c_2 x)\, e^x - x e^x \sin x - 2\cos$$

Question-5(c) A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability.

[8 Marks]

Solution: Let rod AB is resting with one end on inclined plane AO and other end on smooth wall BO.
Let $AO = y, \quad BO = x, \quad AB = 2a.$



In triangle ABO,

$$\frac{2a}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{x}{\sin(\theta + \alpha)} = \frac{y}{\sin\left(\frac{\pi}{2} - \theta\right)}$$

$$\frac{2a}{\cos\alpha} = \frac{x}{\sin(\theta + \alpha)} = \frac{y}{\cos\theta}$$

$$\therefore \quad x = \frac{2a\sin(\theta + \alpha)}{\cos\alpha}; y = \frac{2a\,\mathrm{cs}\,\theta}{\cos\alpha}$$

$z =$ height of centre of gravity
of rod AB° from fined plane mN

$$z = \frac{1}{2}[AT + BO] = \frac{1}{2}[y\sin\alpha + x]$$

$$= \frac{1}{2}\left[\frac{2a\,\mathrm{cs}\,\theta \cdot \sin\alpha}{\mathrm{cs}\,\alpha} + \frac{2a\sin(\theta + \alpha)}{\mathrm{cs}\,\alpha}\right)$$

$$z = \frac{a}{\cos\alpha}[-\cos\theta - \sin\alpha + \sin(-\theta + \alpha)]$$

$$= \frac{a}{\cos\alpha}[\sin\theta - \cos\alpha + 2 - \cos\theta - \sin\alpha]$$

For stability,

$$\frac{dz}{d\theta} = -0$$

$$\frac{a}{\cos\alpha[\cos\theta - \cos\alpha - 2\sin\theta - \sin\alpha]} = 0$$

i.e.

$$\cos\theta - \cos\alpha = 2\sin\theta sin\alpha$$

$$=| \tan\theta = \frac{1}{2}\cot\alpha \quad \ldots \quad (1)$$

$$\frac{dz}{d\theta} = \frac{a}{\cos\alpha}[\cos\theta\cdot\cos\alpha - 2\sin\theta\sin\alpha]$$

$$\begin{aligned}\frac{d^2z}{d\theta^2} &= \frac{a}{\cos\alpha}[-\sin\theta\cos\alpha - 2\cos\theta\sin\alpha] \\ &= -\frac{a}{\cos\alpha}(\sin\theta\cos\alpha + 2\cos\theta\sin\alpha) \\ &= a \text{ negative quantity because } \theta \text{ and } \alpha \text{ are acute angles.}\end{aligned}$$

Thus, in the position of equilibrium, given by condition (1),

$$\frac{d^2z}{d\theta^2}$$

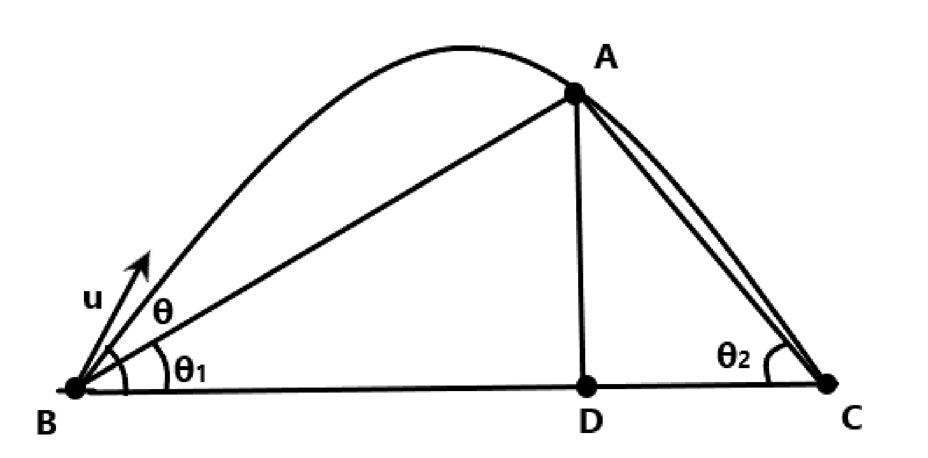
is negative which means z is maximum.
Hence, the equilibrium is unstable.

Question-5(d) A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that,

$$\tan\theta = \tan\theta_1 + \tan\theta_2$$

[8 Marks]

Solution: Given:
1)$\angle ABC = \theta_1$
2) $\angle ACB = \theta_2$
3) Angle of projection $= \theta_3$



Let the initial velocity be $'u'$ and $AD = h$

$$\Rightarrow \tan\theta_1 = \frac{AD}{BD} \Rightarrow BD = h\cot\theta_1 \cdots (1)$$

Again,

$$\tan\theta_2 = \frac{AD}{CD}$$
$$CD = h\cot\theta_2 \cdots (2)$$
$$BC = BD + CD \cdots (3)$$

Putting (1) and (2) in (3) , we get,

$$BC = h \cdot [cot\theta_1 + [\cot\theta_2] \cdots (4)$$

Thus the range of the projectile is given in equation (4) , that is BC

Now, Range, $R = \frac{u^2 \sin 2\theta}{g} - (5)$ where g= gravitational acceleration Using (4) and (5),

$$h\left[(\cot\theta_1 + \cot\theta_2)\right] = \frac{u^2}{g}\sin 2\theta$$
$$\Rightarrow \frac{u^2}{g} = h \cdot \frac{[(\cot\theta + cot\theta_2)]}{\sin 2\theta} \cdots (6)$$

At any instant $'t'$, equation of projectile is given as:

$$y = -u\sin\theta t - \frac{1}{2}yt^2 \quad \text{and } x = \text{ucos}\,\theta$$

$$\Rightarrow \quad y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}...(7)$$

Using (6) in (y) we get:

$$y = x\tan\theta - \frac{sin2\theta.x^2}{2h\left[\cot\theta_1 + \cot\theta_2\right] \cdot \cos^2\theta}....(8)$$

At the point A, $\quad x = h \cdot \cot\theta_1 \quad$ and $\quad y = h$
Hence, putting these values in (8) we get,

$$h = h\cot\theta_1\tan\theta - \frac{2\sin\theta\cos\theta}{2h\cos^2\theta} \cdot \frac{h^2\cot^2\theta_1}{[\cot\theta_1 + \cot\theta_2]}$$

$$1 = \cot\theta_1 \tan\theta - \frac{\tan\theta \cot^2\theta_1}{[60 + \theta_1 + (0 + \theta)}$$

$$1 = \tan\theta \mid \frac{\cot\theta_1 \cot\theta_2}{\cot\theta + \cot\theta}\Big]$$

$$\Rightarrow \quad \tan\theta = \frac{[\cot\theta_1 + \cot\theta_2]}{\cot\theta \cot\theta_2}$$

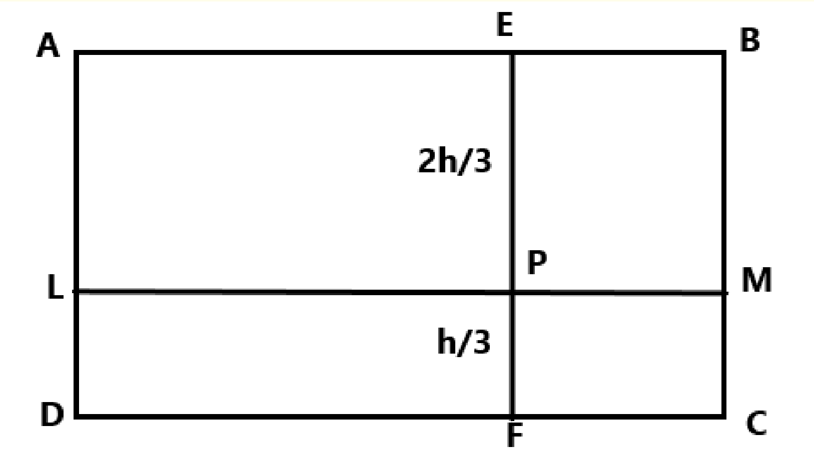
$$\therefore \tan\theta = \tan\theta_1 + \tan\theta_2$$

Hence proved.

Question-5(e) Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which, are in the ratio, $4:5$.

[8 Marks]

Solution: Let LM be the horizontal line through P, the centre of pressure of rectangle $ABCD$ is immersed in liquid with the side AB in the surface.



Let

$$AB = a$$

and

$$AD = h \Rightarrow EP = 2/3h$$

$$\begin{aligned}
P &= \text{Pressure on area } ABCD\\
&= w \cdot (\text{Area } ABCD) \cdot (\text{depth of its } C.G. \text{ below the free surface}))\\
&= w \cdot (ah)\left(\frac{h}{2}\right)\\
&= \frac{1}{2}wah^2\\
P_1 &= \text{presure on area } ALMB\\
&= w - (\text{Area } ALMB) \cdot (\text{depth of its} C.G. \text{below the free surface})\\
&= -w \cdot \left(a \cdot \frac{2}{3}h\right) \mid \left(\frac{1}{2} \cdot \frac{2}{3}h\right)\\
&= \frac{2}{9}wah^2\\
P_2 &= \text{ Pressure on area } LDCM\\
&= P - P_1\\
&= w\,\text{ah}^2\left(\frac{1}{2} - \frac{2}{9}\right)\\
&= \frac{5}{18}wah^2
\end{aligned}$$

$$\therefore \frac{P_1}{P_2} = \frac{2}{9} \times \frac{18}{5} \cdot \frac{wah^2}{wah^2} = \frac{4}{5}$$

Question-5(f) Find the directional derivate of $\overrightarrow{\mathrm{V}}^2$, where, $\overrightarrow{\mathrm{V}} = \mathrm{xy}^2\overrightarrow{\mathrm{i}} + \mathrm{zy}^2\overrightarrow{\mathrm{j}} + \mathrm{xz}^2\overrightarrow{\mathrm{k}}$ at the point $(2,0,3)$ in the direction of the outward normal to the surface $x^2+y^2+z^2=14$ at the point $(3,2,1)$.

[8 Marks]

Solution: The unit normal vector at point (3,2,1) of the surface

$$x^2+y^2+z^2=14$$

is given by

$$\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}} = \hat{n}(\text{say})$$

Now,

$$\overline{\mathrm{V}} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$$

then,

$$\overline{\mathrm{v}}^2 = \left(x^2y^4 + z^2y^4 + x^2z^4\right)$$

then,

$$\nabla\overline{\mathrm{V}}^2 = \left(2xy^4+2xz^4\right)\hat{i} + \left(4x^2y^3+4y^3z^2\right)\hat{j} + \left(2y^4z+4x^2z^3\right)\hat{k}$$

Hence, required directional derivative at point (2,0,3) is given by:

$$\left[\left(2xy^4+2xz^4\right)\hat{i} + \left(4x^2y^3+4y^3z^2\right)\hat{j} + \left(2y^4z+4x^2z^3\right)\hat{k}\right] \cdot \left[\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right]$$

$$= \frac{81 \times 4 \times 3 + 16 \times 27 \times 4}{\sqrt{14}}$$
$$= \frac{2700}{\sqrt{14}}$$

Question-6(a) Solve the following differential equation

$$\frac{dy}{dx} = \sin^2(x - y + 6)$$

[8 Marks]

Solution: Let $z = x - y + 6$ then,

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

or,

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$
$$1 - \frac{dz}{dx} = \sin^2 z$$

or,

$$\frac{dz}{dx} = \cos^2 z$$

or,

$$\sec^2 z dz = dx$$

After integrating, we get:

$$\tan z = x + c$$

or,

$$\tan(x - y + 6) = x + c$$

where $c =$ arbitrary constant.

Question-6(b) Find the general solution of

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left(x^2 + 1\right) y = 0$$

[12 Marks]

Solution: The above equation is solved by reducing it to normal form. i.e. (removal of 1st derivative). Let, $y = uv$ be the solution of above equation then. The above equation can be reduced to

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+2x\frac{du}{dx}+u\right)v=0\ldots(1)$$

Now, to remove 1st derivative, we should equate

$$P+\frac{2}{u}\frac{du}{dx}=0$$

or

$$\frac{du}{u}+xdx=0$$

then, (1) is reduced to

$$\frac{d^2v}{dx^2}+\mathrm{I}v=0$$

where,

$$I=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dp}{dx}$$

$$Q=\left(x^2+1\right),\quad P=2x$$

$$1=\left(x^2+1\right)-x^2-1=0$$

$$\therefore\frac{d^2v}{dx^2}=0\Rightarrow v=\left(c_1+c_2x\right)$$

where, c_1 and c_2 are arbitrary constant Hence,

$$y=\left(c_1+c_2x\right)e^{-x^{1/2}}$$

$$y=c_1e^{-x^2/2}+c_2xe^{-x^2/2}$$

is the general solution of the given equation.

Question-6(c) Solve

$$\left(\frac{d}{dx}-1\right)^2\left(\frac{d^2}{dx^2}+1\right)^2y=x+e^x$$

[10 Marks]

Solution: The complementary function is given by

$$y=\left(c_1+c_zx\right)e^x+\left(c_3+c_4x\right)\sin x+\left(c_5+c_6x\right)\cos x$$

The particular integral is given by:

$$\begin{aligned} y &= \frac{1}{(D-1)^2\left(D^2+1\right)^2}\left(x+e^x\right) \\ &= \frac{1}{(1-D)^2\left(1+D^2\right)^2}x + \frac{1}{(D-1)^2\left(D^2+1\right)^2}e^x \\ &= \left[1+2D+3D^2+\cdots\right]\left(1-2D^2+3D^4-\cdots\cdots\right)x + \frac{x^2e^x}{2.4} \\ &= (x+2) + \frac{x^2e^x}{8} \end{aligned}$$

Hence, the general solution is given by

$$y = \left(c_1+c_2x\right)e^x + \left(c_3+c_4x\right)\sin x + \left(c_5+c_6x\right)\cos x + (x+2) + \frac{x^2e^x}{8}$$

Question-6(d) Solve by the method of variation of parameters the following equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \left(x^2-1\right)^2$$

[10 Marks]

Solution: The above equation can be written as

$$y_2 - \frac{2x}{x^2-1}y_1 + \frac{2}{x^2-1}y = \left(x^2-1\right) \quad \ldots(1)$$

Clearly, x and x^2+1 is solution of reduced differential equation (i.e. making right hand side to zero).
Let, $y = \mathrm{A}x + \mathrm{B}\left(x^2+1\right)$ be the solution of (1) where A and B are function of x. put a condition $A_1x + B_1\left(x^2+1\right) = 0$.
Now,

$$\begin{gathered} y = \mathrm{Ax} + \mathrm{B}\left(x^2+1\right), \\ y_1 = \mathrm{A} + 2\ \mathrm{B}x, \\ y_2 = \mathrm{A}_1 + 2\ \mathrm{B}_1x + 2\ \mathrm{B} \end{gathered}$$

Putting y, y_1 & y_2 in equation (1), we get

$$A_1 + 2B_1x + 2B - \frac{2x}{x^2-1}(A+2Bx) + \frac{2}{x^2-1}\left[\mathrm{Ax} + \mathrm{B}\left(x^2+1\right)\right] = \left(x^2-1\right)$$

or,

$$\mathrm{A}_1 + 2\ \mathrm{B}_1x = x^2-1$$

also,

$$\mathrm{A}_1x + \mathrm{B}_1\left(x^2+1\right) = 0$$

or,

$$\mathrm{B}_1\left(2x^2-x^2-1\right) = x\left(x^2-1\right)$$

or,

$$B_1 = x \quad \Rightarrow \quad B = \frac{x^2}{2} + c_1$$

also,

$$A_1 + 2x^2 = x^2 - 1$$
$$\Rightarrow A_1 = -\left(x^2 + 1\right)$$
$$\therefore A = -\frac{x^3}{3} - x + c_2$$

$$\begin{aligned}
\therefore y &= \mathrm{Ax} + \mathrm{B}\left(x^2 + 1\right) \\
&= \left(c_2 - x - \frac{x^3}{3}\right)\dot{x} + \left(\frac{x^2}{2} + c_1\right)\left(x^2 + 1\right) \\
&= c_1\left(x^2 + 1\right) + c_2 x - x^2 - \frac{x^4}{3} + \frac{x^4}{2} + \frac{x^2}{2} \\
&= c_1\left(x^2 + 1\right) + c_2 x - \frac{x^2}{2} + \frac{x^4}{6}
\end{aligned}$$

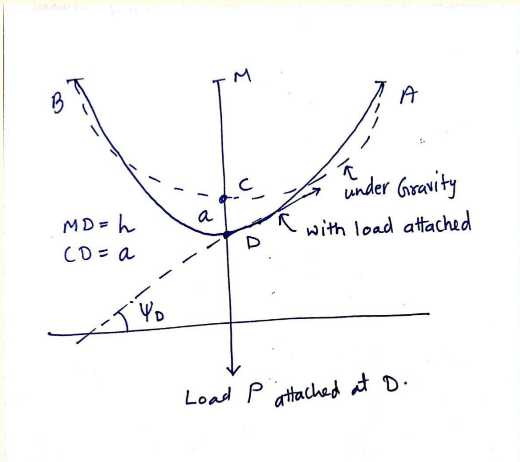
i.e. the general solution is

$$y = c_1\left(x^2 + 1\right) + c_2 x - \frac{x^2}{2} + \frac{x^4}{6}$$

Question-7(a) A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P \cdot \frac{l}{h} + W \cdot \frac{h^2 + i^2}{2hl}\right]$$

[14 Marks]



Solution:

Initially AB hangs under gravity. But when load P is attached to middle point D such that $AD = BD = l$, then let T_D be the tension at D along tangent at D to AD and BD.

Let C be the lowest point of catenary such that $CD = a$.

Sag of catenary $= h$.

Let ψ_D be the angle that T_D at D makes with horizontal.

$$\Rightarrow 2T_D \sin\psi_D = P$$

Also,

$$T_D \sin\psi_D = wS \quad (\because T_x = wC; \quad T_y = ws)$$

Now, since $w = \frac{W}{2l}$ and $s = CD = a$, therefore,

$$T_D \sin\psi_D = \frac{W}{2l}a$$
$$\frac{P}{2} = \frac{W}{2l}a$$
$$\implies a = \frac{P}{W}l$$

Let y_A be the height and s_A be the arc length at A and similarly let y_D be the height and s_D be the arc length at D. Then,

$s_A = l + a$ and $s_D = a$;
$y_D = h = y_A \Rightarrow y_D = y_A - h$

Also, $c^2 + s^2 = y^2$ (given)

$$\Rightarrow c^2 + s_A^2 = y_A^2; c^2 + s_D^2 = y_D^2$$
$$\Rightarrow y_A^2 - y_D^2 = s_A^2 - s_D^2 = (l + a)^2 - a^2$$
$$\Rightarrow y_A^2 - (y_A - h)^2 = (l + a)^2 - a^2$$
$$\Rightarrow y_A = \frac{l^2 + h^2 + 2al}{2h}$$

Also, terminal tension at A or B is given by:

$$\begin{aligned} T &= wy_A \\ &= \frac{W}{2l} \times \frac{l^2 + h^2 + 2al}{2h} \\ &= \frac{W}{4lh}\left[l^2 + h^2 + 2 \times \frac{P}{W}l^2\right] \\ &= \frac{1}{2}\left[P\frac{l}{h} + W\frac{l^2 + h^2}{2lh}\right] \end{aligned}$$

Question-7(b) A particle moves with a central acceleration $\frac{\mu}{(\text{ distance })^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{\mathrm{VR}\left(\mathrm{V}^2-\frac{2\mu}{\mathrm{R}}\right)^{1/2}}\right]$$

[13 Marks]

Solution: If the particle describes a hyperbola under the central acceleration

$$\frac{\mu}{(\text{ distance })^2},$$

then the velocities V of the particle at distance r from centre of force is given by,

$$\mathrm{V}^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$$

where $2a=$ transverse axis.
As particle is projected with velocity V at distance R, then from (1), we have,

$$\mathrm{V}^2=\mu\left(\frac{2}{\mathrm{R}}+\frac{1}{a}\right) \quad \text{or}$$

$$\frac{\mu}{a}=\mathrm{V}^2-\frac{2\mu}{\mathrm{R}}$$

If α is required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h=vp$ we have

$$h=\mathrm{Vp}=\mathrm{VR}\sin\alpha \quad [\because p=r\sin\phi \ \&\ \text{initially } r=R, \phi=\alpha]$$

Also,

$$\begin{aligned} h &= \sqrt{\mu l} \\ &= \sqrt{\mu\cdot b^2/a} \\ &= \sqrt{\mu a} \quad [b=a \text{ for rectangular hyperbola}] \end{aligned}$$

from (3) and (4) we have,

$$\begin{aligned} \mathrm{VR}\sin\alpha &= \sqrt{\mu a} \\ \Rightarrow \sin\alpha &= \frac{\sqrt{\mu a}}{\mathrm{VR}} \\ &= \frac{\mu\sqrt{a}}{\mathrm{VR}\sqrt{\mu}} \\ &= \frac{\mu}{\mathrm{VR}\sqrt{\mu a}} \end{aligned}$$

from (2)

$$\Rightarrow \quad \sin\alpha=\frac{\mu}{\mathrm{VR}\sqrt{\mathrm{V}^2-\frac{2\mu}{\mathrm{R}}}}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\left\{\frac{\mu}{\mathrm{VR}\sqrt{\mathrm{V}^2 - \frac{2\mu}{\mathrm{R}}}}\right\}$$

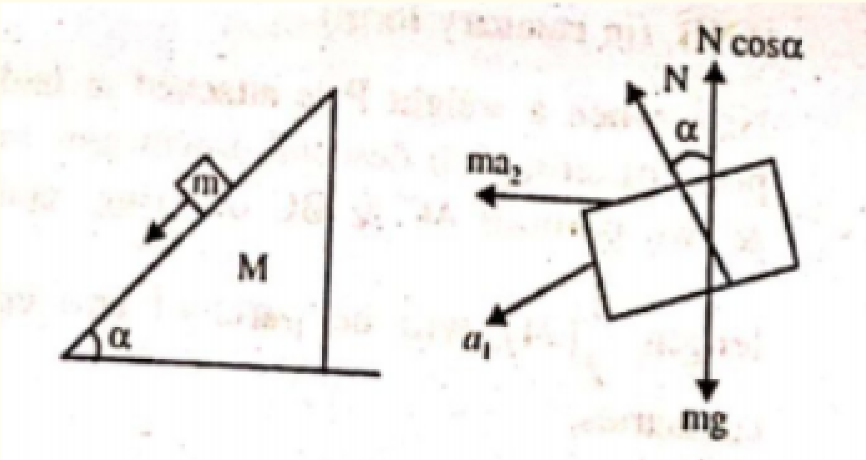
which is required angle of projection.

Question-7(c) A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle α to the horizontal plane, Prove that the acceleration of the wedge is,

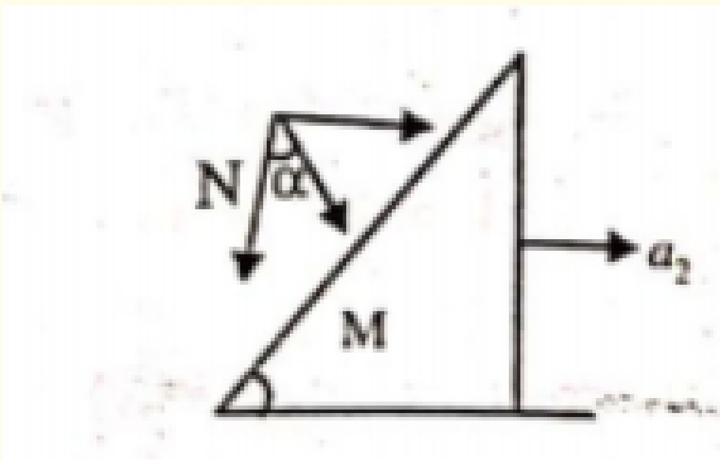
$$\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

[13 Marks]

Solution: Let a_1 and a_2 be the acceleration of m and M respectively.



Then from free body diagram.



$$mg - N\cos\alpha = ma_1\sin\alpha ma_2 + \mathrm{N}\sin\alpha$$
$$= ma_1\cos\alpha$$

Also, $\mathrm{N}\sin\alpha = \mathrm{M}a_2$ (1) $\times\cos\alpha - (2)\times\sin\alpha$ we get,

$$mg\cos\alpha - N - ma_2\sin\alpha = 0$$

putting N from (3), we get,

$$\Rightarrow mg\cos\alpha - \frac{Ma_2}{\sin\alpha} - ma_2\sin\alpha = 0$$
$$\Rightarrow a_2\left(\mathrm{M} + m\sin^2\alpha\right) = mg\sin\alpha\cos\alpha$$
$$\therefore a_2 = \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

Question-8(a) (i) Show that $\overrightarrow{\mathrm{F}} = (2\mathrm{x}y + \mathrm{z}^3)\overrightarrow{\mathrm{i}} + \mathrm{x}^2\overrightarrow{\mathrm{j}} + 3\mathrm{z}^2\mathrm{x}\overrightarrow{\mathrm{k}}$ is a conservative field. Find its scalar potential and also the work done in moving a particle from $(1, -2, 1)$ to $(3, 1, 4)$.

[5 Marks]

Solution: Field F will be conservative then $\bar{\nabla}\times\overline{\mathrm{F}} = 0$ i.e.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

Now

$$\begin{aligned}\bar{\nabla}\times\overline{\mathrm{F}} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix} \\ &= \hat{i}.0 - \hat{j}\cdot\left(3z^2 - 3z^2\right) + \hat{k}(2x - 2x) \\ &= 0 \\ \text{i.e.}\bar{\nabla}\times\overline{\mathrm{F}} &= 0\end{aligned}$$

$\Rightarrow \overrightarrow{\mathrm{F}}$ is conservative field.

Hence, $\overrightarrow{\mathrm{F}}$ can be written as $\overrightarrow{\mathrm{F}} = \nabla\mathrm{U}$ where U is scalar function.

Now,

$$\begin{aligned}\frac{\partial U}{\partial x} &= 2xy + z^3 \\ \Rightarrow U &= x^2y + xz^3 + f_1(y, z)\frac{\partial \mathrm{U}}{\partial y} \\ &= x^2 \\ \Rightarrow \mathrm{U} &= x^2y + f_2(x, z)\frac{\partial U}{\partial z} \\ &= 3z^2x, \\ \Rightarrow \mathrm{U} &= xz^3 + f_3(x, y)\end{aligned}$$

above three expression which represent same potential function, we get,

$$U = x^2y + xz^3$$

. Now. work done in moving a particle from $(1. -2, 1)$ to $(3, 1, 4)$

$$\begin{aligned}\Rightarrow U(3,1,4) - \mathrm{U}(1,-2,1) &= 3.^21 + 3.4^3 - (1(-2)+1)\\ &= 202 \text{ units.}\end{aligned}$$

Question-8(a) (ii) Show that, $\nabla^2 f(r) = \left(\frac{2}{r}\right) f'(r) + f''(r)$, **where**

$$r = \sqrt{x^2 + y^2 + z^2}$$

[5 Marks]

Solution:

$$\begin{aligned}\nabla^2 f(r) &= \bar{\nabla} \cdot (\nabla f(r))\\ &= \bar{\nabla} \cdot \left(f'(r)\frac{\vec{r}}{r}\right)\\ &= \bar{\nabla} \cdot \left(\frac{f'(r)}{r}\vec{r}\right)\\ &= \left(\bar{\nabla}\frac{f'(r)}{r}\right) \cdot \bar{r} + \frac{f'(r)}{r}(\bar{\nabla} \cdot \bar{r}) \left[\frac{f''(r)\bar{r}}{r} + f'(r)\left(-\frac{1}{r^2}\right)\frac{\vec{r}}{r}\right] \cdot \vec{r} + 3\frac{f'(r)}{r}\\ &= \frac{f''(r)}{r^2}(\vec{r} \cdot \vec{r}) - \frac{f'(r)}{r} + \frac{3f'(r)}{r}\\ &= f''(r) + \frac{2f'(r)}{r}\\ \text{i.e.} \nabla^2 f(r) &= f''(r) + \frac{2f'(r)}{r}\end{aligned}$$

Question-8(b) Use divergence theorem to evaluate,

$$\iint_{\mathrm{s}} \left(\mathrm{x}^3\mathrm{dydz} + \mathrm{x}^2\mathrm{ydzdx} + \mathrm{x}^2\mathrm{zdydx}\right)$$

where S is the sphere, $x^2 + y^2 + z^2 = 1$.

[10 Marks]

Solution: By divergence theorem, we have

$$\iint\limits_{s} \mathrm{F}_1 dydz + \mathrm{F}_2 dzdx + \mathrm{F}_3 dxdy = \iiint \left(\frac{\partial \mathrm{F}_1}{\partial x} + \frac{\partial \mathrm{F}_2}{\partial y} + \frac{\partial \mathrm{F}_3}{\partial z}\right) dxdydz$$

$$\Rightarrow \int \left(x^3 dydz + x^2 ydxdz + x^2 zdxdy\right) = \iiint \left\{\frac{\partial x^3}{\partial x} + \frac{\partial \left(x^2 y\right)}{\partial y} + \frac{\partial \left(x^2 z\right)}{\partial z}\right\} dxdydz$$

$$= \iiint\limits_{x^+ + y^2 + z^2 = 1} 5x^2 dxdydz$$

Converting the above integral into polar form, we get,

$$\int_{r=0} \int_{-0}^{\pi} \int_{-0}^{2\pi} \left(5r^2 \cos^2\theta \cos^2\phi\right)\left(r^2 \sin\theta drd\theta d\phi\right) = \int_{r=0}^{1} 5r^4 dr \int \cos^2\theta \sin\theta d\theta \int_0^{2\pi} \cos^2\phi d\phi$$

$$= 5 \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{3}$$

Question-8(c) If $\vec{A} = 2y\vec{i} - z\vec{j} - x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, $z = 6$, evaluate the surface integral,

$$\iint\limits_{\mathrm{S}} \overrightarrow{\mathrm{A}} \cdot \hat{\mathrm{n}} \mathrm{d}\overrightarrow{\mathrm{S}}$$

[10 Marks]

Solution: A vector normal to the parabolic cylinder is given by.

$$\nabla\left(8x - y^2\right) = 8\bar{i} - 2y\hat{y}$$

$$\Rightarrow \hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}}$$

$$= \frac{4\bar{i} - \bar{y}\bar{j}}{\sqrt{16 + y^2}}$$

$$\Rightarrow \iint\limits_{S} \bar{A} \cdot \hat{n} d\bar{S} = \iint \left(2y\bar{i} - z\bar{j} + x^2\hat{k}\right) \cdot \frac{(4\bar{i} - \bar{y})}{\sqrt{16 + y^2}} \cdot \frac{dydz}{|\hat{i}\hat{n}|}$$

$$= \iint \left(2y\bar{i} - z\hat{j} + x^2\hat{k}\right) \frac{(4\bar{i} - \hat{y})}{\sqrt{16 + y^2}} \cdot \frac{dydz}{\frac{4}{\sqrt{16 + y^2}}}$$

$$= \frac{1}{4} \int (8 + z) ydydz = \frac{1}{4} \int_{z=0}^{6} (8 + z) \frac{16}{2} dz$$

$$= \frac{1}{4} \int_0^6 (64 + 8z) dz = \frac{1}{4} \left[64z + 4z^2\right]_{2=0}^{6}$$

$$= \frac{4}{4} \left[16z + z^2\right]_0^6 = 96 + 36$$

$$= 132 \text{ Units.}$$

Question-8(d) Use Green's theorem in a plane to evaluate the integral, $\int_{\mathrm{C}} \left[\left(2\mathrm{x}^2-\mathrm{y}^2\right) \mathrm{dx}+\left(\mathrm{x}^2+\mathrm{y}^2\right) \mathrm{dy}\right]$, where C is the boundary of the surface in the xy-plane enclosed by $y=0$ and the semi-circle, $y=\sqrt{1-x^2}$.

[10 Marks]

Solution: The Green's theorem in a plane is defined as

$$\int \mathrm{M}dx+\mathrm{N}dy=\iint\left(\frac{\partial \mathrm{N}}{\partial x}-\frac{\partial \mathrm{M}}{\partial \mathrm{y}}\right) dxdy$$

$$\int\left(2x^2-y^2\right) dx+\left(x^2+y^2\right) dy=\iint(2x+2y) dydx$$

$$\begin{aligned}
2\int_{x=1}^{1}\int_{0}^{\sqrt{1-x^2}}(x+y)dydx &= 2\int_{-1}^{1}\left[\left(x\sqrt{1-x^2}+\frac{1-x^2}{2}\right)\right]dx\\
&=\frac{2\times 2}{2}\int_0^1\frac{1-x^2}{0}dx \quad [\text{ other integral vanishes }]\\
&=2\left(x-\frac{x^3}{3}\right)_0^1\\
&=\frac{4}{3}
\end{aligned}$$

Chapter 12

2009

12.1 Section-A

Question-1(a); Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by $\{t^3+4t^2-t+3, t^3+5t^2+5, 3t^3+10t^2-5t+5\}$ and $\{t^3+4t^2+6, t^3+2t^2-t+5, 2t^3+2t^2-3t+9\}$ respectively. Find
(i) $dim(U+W)$
(ii) $dim(U\cap W)$

[10 Marks]

Solution: Let $S=\{t^3+4t^2-t+3, t^3+5t^2+5, 3t^3+10t^2-5t+5\}=\{\alpha_1,\alpha_2,\alpha_3\}$, and $T=\{t^3+4t^2+6, t^3+2t^2-t+5, 2t^3+2t^2-3t+9\}$=$\{\beta_1,\beta_2,\beta_3\}$
Since U and W are spanned by sets S and T of all polynomials of degree 3.
$\therefore U$ and W are subspaces of vector space $V(R)$ of all real polynomials of degree ≤ 3.

We know that the set $S_1=\{1,t,t^2,t^3\}$ is a standard basis for $V(R)$.
Now, the coordinte vectors of $\alpha_1,\alpha_2,\alpha_3$ wrt above basis S_1 are given by: $(3,-1,4,1)$, $(5,0,5,1)$ and $(5,-5,10,3)$, and the coordinate vectors of β_1,β_2,β_3 wrt above basis S_1 are given by: $(6,0,4,1)$, $(5,-1,2,1)$ and $(9,-3,2,2)$.

(i) Since U and W are 2 subspaces of $V(R)$.
$\therefore U+W$ is also a subspace of $V(R)$.
$\implies$ $U+W$ is the space generated by all the 6 coordinate vectors.
Now, for the matrix A whose rows are given by 6 coordinate vectors abd reduce it to echelon form.

$$A=\begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \\ 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix}$$

On performing row operations, $R_2\to 3R_2-5R_1$, $R_3\longrightarrow 3R_3-5R_1$, $R_4\to R_4-2R_1$, $R_5\to 3R_5-5R_1$ and $R_6\to R_6-3R_1$ we get:

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & 2 \\ 0 & -10 & 10 & 4 \\ 0 & 2 & -4 & -1 \\ 0 & 2 & -14 & -2 \\ 0 & 0 & -10 & -1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 2R_2$, $R_4 \longrightarrow 5R_4 - 2R_2$, $R_5 \longrightarrow 5R_5 - 2R_2$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -16 \\ 0 & 0 & -10 & -1 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_6$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$, $R_5 \to R_5 - bR_3$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form.
$\therefore$ The echelon form of A has 3 non-zero rows, $\implies dim(U + W) = 3...$ **(a)**

(ii) Now, form the matrix A whose rows are coordinate vectors of $'S'$ and reduce it to echelon form.

$$\implies A = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & -10 & 10 & 4 \end{bmatrix} \quad (R_2 \to 3R_2 - 5R_1)$$

$$\implies A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$\therefore$ The echelon matrix of A has 2 non-zero rows. $\implies dim(U) = 2$ ... **(b)**
Again, form matrix $'A'$ whose rows are coordinate vectors of T and reduce it to an echelon matrix.

$$A = \begin{bmatrix} 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -18 & -24 & 3 \end{bmatrix} \begin{pmatrix} R_2 \to 6R_2 - 5R_1 \\ R_3 \to 6R_3 - 9R_1 \end{pmatrix}$$

$$\implies A \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -6 & -8 & 1 \end{bmatrix} \quad R_3 \to \frac{1}{3}R_3$$

$$\Longrightarrow A \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

It is in echelon form with 2 non-zero rows. $\therefore dim(W) = 2$ ... **(c)**
Now, since $dim(U \cap W) = dim(U) + dim(W) - dim(U + W) = 2 + 2 - 3 = 1$ (Using (a), (b) and (c)) $\therefore dim(U \cap W) = 1$

Question-1(b) Find a linear map $T : R^3 \to R^4$ whose image is generated by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$.

[10 Marks]

Solution: Given that $R(T)$ is spanned by $\{(1, 2, 0, -4), (2, 0, -1, -3)\}$.
Let us include a vector $(0, 0, 0, 0)$ in this set which will not affect the spanning property so that:
$S = \{(1, 2, 0, -4), (2, 0, -1, -3), (0, 0, 0, 0)\}$ Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the standard basis of R^3.
We know that there exists a transformation $'T'$ such that

$$T(\alpha_1) = (1, 2, 0, -4)$$
$$T(\alpha_2) = (2, 0, -1, -3)$$
$$T(\alpha_3) = (0, 0, 0, 0)$$

Now,

$$\begin{aligned} \alpha \in R^3 \Rightarrow \alpha &= (a, b, c) \\ &= a\alpha_1 + b\alpha_2 + c\alpha_3 \end{aligned}$$

Therefore,

$$\begin{aligned} T(\alpha) &= T(a\alpha_1 + b\alpha_2 + c + \alpha_3) \\ &= aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3) \\ &= a(1, 2, 0, -4) + b(2, 0, -1, -3) + c(0, 0, 0, 0) \end{aligned}$$

$\therefore T(a, b, c) = (a + 2b, 2a, -b, -4a - 3b)$ which is the required transformation.

Question-1(c) **(i)Find the difference between the maximum and the minimum of the function** $\left(a - \frac{1}{a} - x\right)(4 - 3x^2)$ **where** a **is a constant and greater than zero.**

[5 Marks]

(ii) If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(\theta h),\ 0 < \theta < 1.$ **Find** θ, **when** $h = 1$ **and** $f(x) = (1 - x)^{5/2}$.

[5 Marks]

Solution: (i) Let $f(x) = \left(a - \frac{1}{a} - x\right)(4 - 3x^2)$ ... (a)
where a is a constant and greater than 0.
$\implies f'(x) = \left(a - \frac{1}{a} - x\right)(-6x) + (-1)(4 - 3x^2)$
$\implies f'(x) = -6ax + \frac{6x}{a} + 6x^2 - 4 + 3x^2$
$\implies f'(x) = 9x^2 - 6\left(a - \frac{1}{a}\right)x - 4 \ldots$ (b)

For maxima or minima, $f' = 0$

$$\implies 9x^2 - 6\left(a - \tfrac{1}{a}\right)x - 4 = 0$$
$$\implies x = \frac{6\left(a-\frac{1}{a}\right) \pm \sqrt{36\left(a-\frac{1}{a}\right)^2 + 36\times 4}}{2\times 9}$$
$$\implies x = \frac{\left(a-\frac{1}{a}\right) \pm \sqrt{\left(a-\frac{1}{a}\right)^2 + 4}}{3}$$
$$\implies x = \frac{\left(a-\frac{1}{a}\right) \pm \left(a+\frac{1}{a}\right)}{3}$$
$$\implies x = \frac{2a}{3}, \frac{-2}{3a}$$

From (b), $f''(x) = 18x - 6\left(a - \frac{1}{a}\right)$ ... (c)
For $x = \frac{2a}{3}$,

$$f''\left(\tfrac{2a}{3}\right) = 18 \times \tfrac{2a}{3} - 6\left(a - \tfrac{1}{a}\right)$$
$$\implies f''\left(\tfrac{2a}{3}\right) = 6a + \tfrac{6}{a} > 0 \because a > 0$$
$$\implies f \text{ is minimum at } x = \tfrac{2a}{3}.$$

$\therefore f_{min} = f\left(\frac{2a}{3}\right) = \left(a - \frac{1}{a} - \frac{2a}{3}\right)\left(4 - 3 \times \frac{4a^2}{9}\right)$

$$\implies f_{min} = \left(\tfrac{a}{3} - \tfrac{1}{a}\right)\left(4 - \tfrac{4a^2}{3}\right)$$
$$\implies f_{min} = \tfrac{4a}{3} - \tfrac{4}{a} - \tfrac{4a^3}{9} + \tfrac{4a}{3}$$
$$\implies f_{min} = \tfrac{8a}{3} - \tfrac{4}{a} - \tfrac{4a^3}{9} \text{ ... (d)}$$

For $x = \frac{-2}{3a}$,

$$f''\left(-\tfrac{2}{3a}\right) = 18\left(\tfrac{-2}{3a}\right) - 6\left(a - \tfrac{1}{a}\right)$$

$$f''\left(-\tfrac{2}{3a}\right) = \tfrac{-12}{a} - 6a + \tfrac{6}{a} = \tfrac{-6}{a} - 6a < 0$$

$$\implies f \text{ is maximum at } x = \tfrac{-2}{3a}$$

$\therefore f_{max} = f\left(-\tfrac{2}{3a}\right) = 4a - \tfrac{8}{3a} + \tfrac{4}{9a^3} \ \ldots$ (e)

Subtracting (e) trom (d) we get,

$$\begin{aligned}\left(4a - \frac{8}{3a} + \frac{4}{9a^3}\right) - \left(\frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9}\right) &= \frac{4a}{3} + \frac{4}{3a} + \frac{4}{a}\left(\frac{1}{a^3} + a^3\right) \\ &= \frac{4}{3}\left(a + \frac{1}{a}\right) + \frac{4}{9}\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2} + 1\right) \\ &= \left(a + \frac{1}{a}\right)\left[\frac{4}{3} + \frac{4}{9}\left(a^2 + \frac{1}{a^2} + 1\right)\right]\end{aligned}$$

which is the required answer.

(ii) Given: $f(x) = (1-x)^{5/2} \implies f(h) = (1-h)^{5/2}$
Now,

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \implies f'(0) = -\frac{5}{2}$$

Also,

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta h) = \frac{15}{4}(1-\theta h)^{1/2}$$

$$\therefore f(h) = f(0) + hf'(0) + \tfrac{h^2}{2!}f''(\theta h),\ 0 < \theta < 1$$

$$\implies (1-h)^{5/2} = 1 + h\left(\tfrac{-5}{2}\right) + \tfrac{h^2}{2!}\tfrac{15}{4}(1-\theta h)^{1/2}$$

When $h = 1$,

$$0 = 1 + \tfrac{-5}{2} + \tfrac{1}{2} \times \tfrac{15}{4}(1-\theta)^{1/2}$$

$$\implies 0 = 1 - \tfrac{5}{2} + \tfrac{15}{8} - (1-\theta)^{1/2}$$

$$\implies 0 = \tfrac{3}{8} - (1-\theta)^{1/2}$$

$$\implies (1-\theta)^{1/2} = 3/8$$

$$\implies (1-\theta) = \tfrac{9}{64}$$

$$\implies \theta = \tfrac{55}{64}$$

Question-1(d) Evaluate:

(i) $\int_0^{\pi/2} \dfrac{\sin^2 x dx}{\sin x + \cos x}$

[6 Marks]

(ii) $\int_1^{\infty} \dfrac{x^2 dx}{(1+x^2)^2}$

[4 Marks]

Solution: (i) $I = \int_0^{\pi/2} \frac{\sin^2 x dx}{\sin x + \cos x}$

$\implies I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)du}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}$ [Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$] ... (a)

$\implies I = \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x}$ ... (b)

Adding (a) and (b), we get:

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx + \int_0^{\pi} \frac{\cos^2 x}{\sin x + \cos x}dx$$

$$\implies I = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$\implies I = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin x/\sqrt{2} + \cos x/\sqrt{2}}$$

$$\implies I = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$\implies I = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sec\left(x-\frac{\pi}{4}\right)dx.$$

$$\implies \frac{1}{2\sqrt{2}}[\log|\sqrt{x}+1| - \log|\sqrt{2}-1|]$$

$$\implies I = \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

(ii) $I = \int_1^{\infty} \dfrac{x^2 dx}{(1+x^2)^2} = \lim_{t\to\infty} \int_1^t \dfrac{x^2 dx}{(1+x^2)^2}$

Let $x = \tan\theta \implies dx = \sec^2\theta d\theta$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \frac{\tan^2\theta \sec^2\theta d\theta}{(1+\tan^2\theta)^2}$$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \frac{\tan^2\theta}{\sec^2\theta} d\theta = \int_{\pi/4}^{\tan^{-1} t} \sin^2\theta d\theta$$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{1}{2} \times \frac{2\tan\theta}{1+\tan^2\theta}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{\tan\theta}{1+\tan^2\theta}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \frac{1}{2}\lim_{t\to\infty}\left[\left(\tan^{-1} t - \frac{t}{1+1^2}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$\implies I = \frac{1}{2}\left[\frac{\pi}{2} - 0 - \frac{\pi}{4} + \frac{1}{2}\right]$$

$$\implies I = \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right] = \frac{\pi}{8} + \frac{1}{4}$$

$$\implies \int_1^\infty \frac{x^2}{(1+x^2)^2}dx \text{ is convergent and its value is } \frac{\pi}{8} + \frac{1}{4}$$

Question-1(e) Show that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z = 2$ in a circle of radius unity and find the equation of the sphere which has this circle as one of its great circles.

[10 Marks]

Solution: The given sphere is. $x^2 + y^2 + z^2 - x - z - 2 = 0 \ldots$ (i)
and the given plane is $x + 2y - z - 4 = 0 \ldots$ (ii)

Centre of sphere (i) is $C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ and
its radius is $CP = \sqrt{\frac{1}{4} + 0 + \frac{1}{4} + 2} = \sqrt{\frac{5}{2}}$
CA is the perpendicular distance from $C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ to plane (ii) and is given by:

$$CP = \frac{\left|\frac{1}{2} + 2(0) - \frac{1}{2} - 4\right|}{\sqrt{1+4+1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

$\therefore$ Radius of circle, $AP = \sqrt{CP^2 - CA^2} = \sqrt{\frac{5}{2} - \frac{3}{2}} = 1$

The plane (ii) meets the sphere (i) in a circle of radius 1. Now, any sphere through the intersection of (i) and (ii) is given by:

$$x^2 + y^2 + z^2 - x + z - 2 + k(x + 2y - z - 4) = 0...(iii)$$

If the circle of intersection of (i) and (ii) is a great circle of sphere (iii), then the centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ lies on plane (ii).

$\therefore \frac{1-k}{k} + 2(-k) - \left(\frac{k-1}{2}\right) - 4 = 0 \implies k = 1$
$\therefore$ required equation of sphere is given by:

$$x^2 + y^2 + z^2 - 2x - 2z + 2x + 2 = 0$$

Question-2(a) Let T be the linear operator on R^3 defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$
(i) Show that T is invertible.
(ii) Find a formula for T^{-1}.

[10 Marks]

Solution: (i) Let $(x, y, z) \in ker(T)$ be arbitrary.

$$\implies T(x, y, z) = (0, 0, 0)$$

$$\implies (2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$$

$$\implies 2x = 0,\ 4x - y = 0 \text{ and } 2x + 3y - z = 0$$

$$\implies x = 0,\ y = 0,\ z = 0$$

$$\implies ker(T) = \{()0, 0, 0)\}$$

Hence, T is invertible.

(ii) Now we shall find T^{-1}. Since T is invertible, therefore T is onto.
For any $(a, b, c) \in R^3$, there exists some $(x, y, z) \in R^3$ such that

$$T(x, y, z) = (a, b, c).$$

$$\implies (2x, 4x - y, 2x + 3y - z) = (a, b, c)$$

$$\implies 2x = a, 4x - y = b, 2x + 3y - z = c$$

$$\implies a = \frac{a}{2}, y = 2a - b$$

From $2x + 3y - z = c$, we get:

$$a\left(\frac{a}{2}\right) + 3(2a - b) - z = c$$

$$\implies a + 6a - 3b - z = c$$

$$\implies z = 7a - 3b - c$$

Hence $T(x, y, z) = (a, b, c)$

$$\implies T^{-1}(a, b, c) = (x, y, z)$$

$$\implies T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a - b, 7a - 3b - c\right) \forall (a, b, c) \in R^3$$

Question-2(b) Find the rank of the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

[10 Marks]

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

Applying the operations $R_2 \to R_2 - R_1$, $R_1 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$ we get:

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 3R_2$ and $R_1 \to R_4 + R_2$ we get;

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form. Now, the number of non-zero rows of this echelon form is 2.
$\therefore$ Rank of A is equal to 2.

Question-2(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. **Is A similar to a diagonal matrix? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.**

[10 Marks]

Solution: Characteristic Equation of A is given by:
$\lambda^3 - Trace(A)\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| = 0$ ... (i)
Now, $Trace(A) = 1 - 5 + 4 = 0$,
$C_{11} + C_{22} + C_{33} = (-20 + 18) + (4 - 18) + (-5 + 9) = -2 - 14 + 4 = -12$, and
$|A| = 16$.

$\therefore$, (i) becomes $\lambda^3 + 0\lambda^2 - 12\lambda - 16 = 0$
$\implies \lambda = 4, -2, -2$ (Use calculator for this step)

Now, we find the eigenvectors corresponding to the above eigenvalues:

i) $\lambda = 4$, $(A - 4I)X = 0$

$$\implies \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 + R_3 + 2R_1$ and $R_1 \to \frac{R_1}{-3}$ we get:

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2/12$ and $R_3 \to R_3/12$ we get:

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$,

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get:

$$\sim \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$\implies x - \frac{z}{2} = 0$, $y - \frac{z}{2} = 0$
$\implies x = \frac{z}{2}$, $y = \frac{z}{2}$

$$\begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = z/2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$X_1 = (1, 1, 2)$ is eigenvector for $\lambda = 4$

ii) $\lambda = -2$, $(A + 2I)x = 0$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\implies x - y + z = 0$
$\implies x = y - z$

$$\therefore \begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\implies x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors for $\lambda = -2$.

Since algebraic multiplicity of each eigenvalue is equal to its geometic multiplicity, hence given matrix A is diagonalizable i.e. similar to some diagonal matrix.

Transformation matrix:

$$p = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Diagonal matrix:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We can verify that $P^{-1}AP = D$, where $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix}$

Question-2(d) Find an orthogonal transformation of coordinates to reduce the quadratic form $g(x,y) = 2x^2 + 2xy + 2y^2$ to a canonical form.

[10 Marks]

Solution: The matrix form of the given quadratic form is given by:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Now, characteristic equation of A is given by $|A - \lambda I| = 0$

$$\implies \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\implies (\lambda - 2)^2 - 1 = 0$$

$$\implies \lambda^2 - 4\lambda + 3 = 0$$

$$\implies (\lambda - 3)(\lambda - 1) = 0$$

$$\implies \lambda = 1, 3$$

Eigenvector for $\lambda = 1$:

$$(A - 1.I)X = 0$$

$$\implies \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies x + y = 0 \text{ and } x = -y$$
$$\implies v_1(x,y) = (1,-1)$$

Eigenvector for $\lambda = 3$:

$$(A - 3I)X = 0$$

$$\Longrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow x = y$$

$$\Longrightarrow v_2 = (x, y) = (1, 1)$$

Since the vectors v_1 and v_2 are orthogonal, $\Longrightarrow v_1 v_2^T = 0$

Modal matrix comprising of eigenvectors is given by $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\Longrightarrow$ Normalized modal matrix is given by $N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\Longrightarrow N^\top = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

To obtain the canonical form, we calculate $N^T(AN)$.

Now,

$$AN = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$$

$$\Longrightarrow N^T(AN) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore$ Canonical form is given by:

$$y_1^2 + 3y_2^2 = \begin{bmatrix} y_1 & y_2 \end{bmatrix} D \begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top$$

Question-3(a) The adiabatic law for the expansion of air is $PV^{1/4} = K$, where K is a constant. If at a given time the volume is observed to be 50c.c. and the pressure is 30kg per square centimetre, at what rate is the pressure changing if the volume is decreasing at the rate of 2 c.c. per second?

[10 Marks]

Solution: $PV^{1.4} = K \Rightarrow K = 30(50)^{1.4}$ $P = KV^{-1.4}$

$$\begin{aligned} \frac{dP}{dt} &= K(-1.4)\mathrm{V}^{-2\cdot 4} \cdot \frac{dV}{dt} \\ &= -30(50)^{1.4} \frac{1.4}{(50)^{2.4}}(-2) \\ &= \frac{30 \times 1.4 \times 2}{50} \\ &= 1.68 \end{aligned}$$

$\therefore$ The pressure in increasing at the rate of 1.68 $kg/cm^2/sec$.

Question-3(b) Determine the asymptotes of the curve $x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$.

[10 Marks]

Solution: Asymptotes parallel to coordinate axes: As the coefficients of highest degree terms of x and y i. x^3 and y^3 are constants, so the curvve has no asymptotes parallel to $x-axis$ or $y-axis$.
Oblique Asymptotes: Put $x=1, y=m$ in the third, second and first degree terms,

$$\phi_3(m) = 1+m-m^2-m^3$$
$$\phi_2(m) = 2m+2m^2$$
$$\phi_1(m) = -3+m$$

Slopes of the asymptotes are real roots of the equation: $\phi_3(n)=0$.

$$\implies 1+m-m^2(1+m)=0$$

$$\implies (1+m)(1-m^2)=0$$

$$\implies m=1,-1,-1$$

For $m=1$:

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{2m+2m^2}{1-2m-3m^2}\right) = \frac{-4}{-4} = 1$$

$$\implies y=x+1$$

For, $m=-1$, c is given by:

$$\frac{c^2}{2!}\phi_3''(m)+c\phi_2'(m)+\phi_1(m)=0$$

$$\implies \frac{c^2}{2}(-2-6m)+c(2+4m)+(-3+m)=0$$

$$\implies \frac{c^2}{2}(4)+c(-2)+(-4)=0 \quad (\because m=-1)$$

$$\implies 2c^2-2c-4=0$$

$$\implies c^2-c-2=0$$

$$\implies (c-2)(c+1)=0$$

$$\implies c=2,-1$$

$$\implies y=-x+2,\ y=-x-1$$

Question-3(c) **Evaluate:** $\iint_D x\sin(x+y)dxdy$, **where** D **is the region bounded by** $0 \leqslant x \leqslant \pi$ **and** $0 \leqslant y \leqslant \dfrac{\pi}{2}$.

[10 Marks]

Solution: Let $I = \int_{x=0}^{\pi}\int_{y=0}^{\pi/2} x\sin(x+y)dxdy$

$\implies I = -\int_0^{\pi} x[\cos(x+y)]_{y=0}^{\pi/2}dx$

$\implies I = -\int_0^{\pi} x\left[\cos\left(\frac{\pi}{2}+x\right) - \cos(0+x)\right]dx$

$\implies I = \int_0^{\pi} x(\sin x + \cos x)dx$

$\implies I = \pi - 2$

Question-3(d) **Evaluate** $\iiint(x+y+z+1)^4 dxdydz$ **over the region defined by** $x \geq 0,\ y \geq 0,\ z \geq 0$ and $x+y+z \leq 1$.

[10 Marks]

Solution: Let $I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(x+y+z+1)^4 dzdydx$

$\implies I = \int_{x=0}^{1}\int_{y=0}^{1-x}\frac{1}{5}\left((x+y+z+1)^5\right]_{z=0}^{1-x-y}dydx$

$\implies I = \frac{1}{5}\int_{x=0}^{1}\int_{y=0}^{1-x}[2^5 - (x+y+1)^5]dydx$

$\implies I = \frac{1}{5}\int_0^1\left[32y - \frac{1}{6}(x+y+1)^6\right]_{y=0}^{y=1-x}dx$

$\implies I = \frac{1}{5}\int_0^1\left[32(1-x) - \frac{1}{6}(2)^6 + \frac{1}{6}(x+1)^6\right]dx$

$\implies I = \frac{1}{5}\int_0^1\left[\frac{64}{3} - 32x + \frac{1}{6}(x+1)^6\right]dx$

$\implies I = \left[\frac{64}{3}x - 16x^2 + \frac{1}{42}(x+1)^7\right]_0^1$

$\implies I = \frac{117}{70}$

Question-4(a) **Obtain the equations of the planes which pass through the point** $(3,0,3)$, **touch the sphere** $x^2+y^2+z^2 = 9$ **and are parallel to the line** $x = 2y = -z$

[10 Marks]

Solution: The given line is $x = 2y = -z \implies \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ ... (i)
Any line parallel to (i) and passing through $(3, 0, 3)$ is given by:

$$\frac{x-3}{2} = \frac{y-0}{1} = \frac{z-3}{-2} \text{ ... (ii)}$$

Now, the general form of the line (ii) is given by:

$$x - 3 = 2y,\ -2y = z - 3$$

$$\implies x - 2y - 3 = 2y + z - 3 \text{ ... (iii)}$$

Now, any plane passing through line (iii) is given by:

$$(x - 2y - 3) + \lambda(2y + z - 3) = 0 \text{ ... (iv)}$$

$$\implies x + (-2 + 2\lambda)y + \lambda z + (-3 - 3\lambda) = 0.$$

Clearly, it will be the tangent plane to the given sphere $x^2 + y^2 + z^2 - 9 = 0$ if the perpendicular distance of the plane form the centre $(0, 0, 0)$ of the sphere is equal to the radius of the sphere, i.e.,

$$\frac{|0 + 0 + 0 - 3 - 3\lambda|}{\sqrt{1 + (-2 + 2\lambda)^2 + \lambda^2}} = 3 \ (\because radius = 3)$$

$$\Rightarrow 1 + \lambda = \sqrt{5 + 5\lambda^2 - 8\lambda}$$

$$\Rightarrow (1 + \lambda)^2 = 5 + 5\lambda^2 - 8\lambda$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 5\lambda^2 - 8\lambda + 5$$

$$\Rightarrow 4\lambda^2 - 10\lambda + 4 = 0$$

$$\Rightarrow 2\lambda^2 - 5\lambda + 2 = 0$$

$$\Rightarrow 2\lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$\Rightarrow (2\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 2, \tfrac{1}{2}$$

If $\lambda = 2$, then from (iv), we get

$$(x - 2y - 3) + 2(2y + z - 3) = 0$$

$$\implies x + 2y + 2z - 9 = 0 \text{ ... (v)}$$

If $\lambda = \frac{1}{2}$, then from (iv), we get

$$(x - 2y - 3) + \tfrac{1}{2}(2y + z - 3) = 0$$

$$\implies 2x - 2y + z - 9 = 0 \text{ ... (vi)}$$

Therefore, the required planes are given by equation (v) and equation (vi) above.

Question-4(b) The section of a cone whose vertex is P and guiding curve is the eilipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $\dfrac{x^2}{a^2}+\dfrac{y^2+z^2}{b^2}=1.$

[10 Marks]

Solution: Let the vertex P be (α,β,γ) and given guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ ... (i)

Now, the equation of any line through $P(\alpha,\beta,\gamma)$ is given by:

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}...(ii)$$

It meets the plane $z=0$

$$\therefore \tfrac{x-\alpha}{l}=\tfrac{y-\beta}{m}=\tfrac{0-\gamma}{n}$$

$$\implies x-\alpha=\tfrac{-l\gamma}{n},\ y-\beta=\tfrac{-m\gamma}{n},\ z=0$$

$$\implies x=\alpha-\tfrac{l\gamma}{n},\ y=\beta-\tfrac{m\gamma}{n},\ z=0$$

This point lies on the ellipse given by (i),

$$\therefore \tfrac{1}{a^2}\left(\alpha-\tfrac{l\gamma}{n}\right)^2+\tfrac{1}{b^2}\left(\beta-\tfrac{m\gamma}{n}\right)^2=1\ ...\ \text{(iii)}$$

Eliminating l, m and n from (ii) and (iii), we get:

$$\tfrac{1}{a^2}\left(\alpha-\tfrac{(x-\alpha)\gamma}{z-\gamma}\right)^2+\tfrac{1}{b^2}\left(\beta-\tfrac{(y-\beta)\gamma}{z-\gamma}\right)^2=1$$

$\frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}\left(\beta z-\gamma y\right)^2=(z-\gamma)^2$, which is the required equation of the cone.

This meets the plane $x=0$.

$$\implies \tfrac{1}{a^2}(\alpha z-0)^2+\tfrac{1}{b^2}\left(\beta z-\gamma y\right)^2=(z-\gamma)^2$$

$$\implies \tfrac{\alpha^2z^2}{a^2}+\tfrac{\beta^2z^2+\gamma^2y^2-2\beta\gamma zy}{b^2}=z^2+\gamma^2-2z\gamma$$

This will represent a rectangular hyperbola in $yz-plane$ if coefficient of y^2+ coefficient of $z^2=0$.

$$\implies \tfrac{\gamma^2}{b^2}+\tfrac{\alpha^2}{a^2}+\tfrac{\beta^2}{b^2}-1=0$$

$$\implies \tfrac{\alpha^2}{a^2}+\tfrac{\beta^2+\gamma^2}{b^2}-1=0$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ is given by:

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1$$

Question-4(c) Prove that the locus of the poles of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is the conicoid $\dfrac{\alpha^2 x^2}{a} + \dfrac{\beta^2 y^2}{b} + \dfrac{\gamma^2 z^2}{c} = 1$.

[10 Marks]

Solution: Let the tangent plane of the conicoid $ax^2 + by^2 + cz = 1$ be given by:

$$lx + my + nz = p \text{ ... (i)}$$

Then, $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$ ... (ii)

Let (a', b', c') be the pole of the plane (ii) w.r.t

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1.$$

$$\implies a'\alpha x + b'\beta y + c'\gamma z = 1 \text{ ... (iii)}$$

Comparing (i) and (iii), we get:

$$\frac{a'\alpha}{l} = \frac{b'\beta}{m} = \frac{c'\gamma}{n} = \frac{1}{p} \text{ ... (iv)}$$

Eliminating l, m and n from (ii) and (iv), we get:

$$\frac{(a'\alpha p)^2}{a} + \frac{(b'\beta p)^2}{b} + \frac{(c'\gamma p)^2}{c} = p^2$$

$\therefore$ The required locus of (a', b', c') is given by:

$$\frac{(\alpha x)^2}{a} + \frac{(\beta y)^2}{b} + \frac{(\gamma z)}{c}\Big)^2 = 1$$

Question-4(d) Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2 + by^2 + cz^2 = 1$ at its points of intersection with the planes $lx + my + nz = p$ generate the cone $p^2\left(\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c}\right) = \left(\dfrac{lx}{a} + \dfrac{my}{b} + \dfrac{nz}{c}\right)^2$.

[10 Marks]

Solution: Let (α, β, γ) be the point of intersection of the conicoid and the given plane, then we have:

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \ldots \text{ (i),and}$$

$$l\alpha + m\beta + n\gamma = p \ldots \text{ (ii)}$$

Also, the equations of the normals to the given conicoid at (α, β, γ) are:

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$

The equation of the line passing through the origin and parallel to this line is given by:

$$\tfrac{x}{a\alpha} = \tfrac{y}{b\beta} = \tfrac{z}{c\gamma} \ldots \text{ (iii)}$$

From (i) and (ii), we have:

$$a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\tfrac{l\alpha+m\beta+n\gamma^2}{p}\right)^2$$

$$\implies p^2\left(a\alpha^2 + b\beta^2 + c\gamma^2\right) = \left(l\alpha + m\beta + n\gamma\right)^2$$

$$\implies p^2\left(\tfrac{a^2\alpha^2}{a} + \tfrac{b^2\beta^2}{b} + \tfrac{c^2\gamma^2}{c}\right) = \left(\tfrac{l(a\alpha)}{a} + \tfrac{m(b\beta)}{b} + \tfrac{n(c\gamma)}{c}\right]^2$$

$$\implies p^2\left[\tfrac{(a\alpha)^2}{a} + \tfrac{(b\beta)^2}{b} + \tfrac{(c\gamma)^2}{c}\right] = \left(\tfrac{l(a\alpha)}{a} + \tfrac{m(b\beta)}{b} + \tfrac{n(c\gamma)}{c}\right]^2$$

Now, eliminating α, β and γ from this equation using (iii), we get:

$$p^2\left(\tfrac{x^2}{a} + \tfrac{y^2}{b} + \tfrac{z^2}{c}\right) = \left(\tfrac{lx}{a} + \tfrac{my}{b} + \tfrac{nz}{c}\right)^2$$

Hence, the line given by (iii) generates the above conicoid.

12.2 Section-B

Question-5(a) Solve: $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$

[10 Marks]

Solution: $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \ldots$ (i)

Let $\tan y = z$

$$\implies \sec^2 y \tfrac{dy}{dx} = \tfrac{dz}{dx}$$

So, equation (i) becomes

$$\tfrac{dz}{dx} + 2xz = x^3$$

$$\implies IF = \int e^{2x} dx = \tfrac{e^x}{2}$$

The general solution of equation (i) is given by:

$$\begin{aligned}
z\cdot\frac{e^{2x}}{2} &= \int x^3\frac{e^{2x}}{2}\dot{d}x \\
&= \frac{1}{2}\int x^3e^{2x}dx \\
&= \frac{1}{2}\left[\frac{e^{2x}}{2}\cdot x^3-\int 3x^2\frac{e^{2x}}{2}dx\right] \\
&= \frac{e^{2x}}{4}x^3-\frac{3}{4}\int x^2e^{2x}dx \\
&= \frac{e^{x}}{4}x^3-\frac{3}{4}\left[\frac{2x}{2}\cdot x^2-\int\frac{e^{2x}}{2}\cdot 2xdx\right] \\
&= \frac{e^{2x}}{4}x^3-\frac{3}{8}x^2e^{2x}+\int xe^{2x}dx \\
&= \frac{x^3e^{2x}}{4}-\frac{3}{8}x^2e^{2x}+\frac{e^{2x}}{2}x-\int\frac{e^{2x}}{2}\alpha \\
&= \frac{x^3e^{2x}}{4}-\frac{3}{8}x^2e^{2x}+\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+c \\
\tan y\frac{e^{2x}}{2} &= \frac{x^3e^{2x}}{4}-\frac{3}{8}x^2e^{2x}+\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}+c
\end{aligned}$$

Question-5(b) Find the 2nd order ODE for which e^x and x^2e^x are solutions.

[10 Marks]

Solution: Let $y_1 = x$ & $y_2 = x^2e^x$
Wronskian is

$$\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} x & x^2e^x \\ 1 & x^2e^x+2xe^x \end{vmatrix} \\
&= x\left(x^2e^x+2xe^x\right)-x^2e^x \\
&= x^3e^x+2x^2e^x-x^2e^x \\
&= x^3e^x+x^2e^x,
\end{aligned}$$

which is not identically equal to 0 on R in $(-\infty,\infty)$. The general solution of the required differential equation can be written as:

$$y = c_1y_1+c_2y_2$$

,
where C_1, C_2 are arbitrary contants.

$$\Rightarrow y = c_1 e^x + c_2 x^2 e^x \ ... \ \text{(i)}$$

Difrerentiating eq (i) wrt x, we get:

$$\frac{dy}{dx} = c_1 e^x + c_2 x^2 e^x + 2c_2 x e^x$$
$$\Rightarrow y' = y + 2c_2 x e^x$$
$$\Rightarrow y' - y = 2c_2 x e^x \ldots (ii)$$

Again differentiating eq (ii) w.rt x, we get

$$y'' - y' = 2c_2 x e^x + 2c_2 e^x = y' - y + 2c_2 e^x$$

$$y'' - 2y' + y = 2c_2 e^x \ \ldots \ \text{(iii)}$$

Now, substituting (iii) in (ii), we get,

$$y' - y = x\,(y'' - 2y' + y)$$
$$= xy'' - 2xy' + xy$$

$\Rightarrow x - y'' - (2x+1)y' + (x+1)y = 0$,
which is the required differential equation.

Question-5(c) A uniform rectangular board, whose sides are $2a$ and $2b$, rests in limiting equilibrium in contact with two rough pegs in the same horizontal line at a distance d apart. Show that the inclination θ of the side $2a$ to the horizontal is given by the equation $d\cos\lambda[\cos(\lambda + 2\theta)] = a\cos\theta - b\sin\theta$ where λ is the angle of friction.

[10 Marks]

Solution: Let $ABCD$ be the rectangle resting on two pegs P and Q.
Suppose that the resultant of the reactions and the frictional forces at P and Q meet at O. Then, the centre of gravity G of the rectangle must be vertically below O.
Let AN be the perpendicular from A on OG. Suppose that the normals at P and Q meet at O'.
The angles OPO' and OQO' are equal. Hence O, P, Q, O' are concyclic. Again, O', P, A, Q are concyclic. Hence, O, O', A, P are concyclic.
It follows that:

$$\angle OAO' = \lambda$$
$$\angle O'AQ = \angle AQP = \angle QAN = \theta$$
$$\angle O'OA = \angle O'PA = \pi/2$$

Also trom the rectangle $O'PAP$, $O'A = PQ$.
We can now find AN in 2 ways:

$$AN = OA\cos(\lambda + 2\theta)$$
$$= O'A\cos\lambda\cos(\lambda + 2\theta)$$
$$= d\cos\lambda\cos(\lambda + 2\theta)$$

Again,

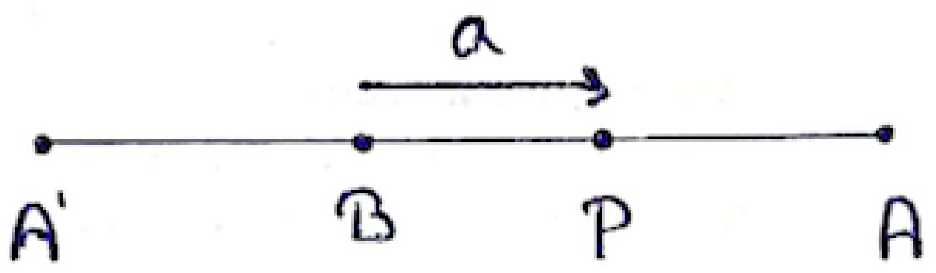
$$\begin{aligned}AN &= AG\cos(\angle GAQ + \angle QAN)\\ &= AG\cos\angle GAQ\cos\theta - AG\sin\angle GAQ\sin\theta\|\\ &= acos\theta - b\sin\theta\end{aligned}$$

Hence, $d\cos\lambda\cos(\lambda+2\theta) = a\cos\theta - b\sin\theta$.

Question-5(d) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'. The particle is slightly displaced towards one of them, show that the time of small oscillation is $\frac{2\pi}{\sqrt{(\mu+\mu')}}$.

[10 Marks]

Solution: Suppose A and A' are two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at B under the attraction of these two centres.



The forces of attraction at B due to the centres. A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions.
As these two forces balance each other, therefore

$$m\mu a = m\mu' a' \text{ ... (i)}$$

Now, suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP = x$.
The attraction at P due to centre A is $m\mu AP = m\mu(a-x)$ in the direction PA, i.e., in the direction of x increasing.
Also, the attraction at P due to centre A' is $m\mu' A'P = m\mu'(a'+x)$ in the direction PA', i.e. in the direction of x decreasing.
Hence, by Newton's 2^{nd} law of motion, the equation of motion of particle at P is given by:

$$m\left(\frac{d^2x}{dt^2}\right) = m\mu(a-x) - m\mu'\left(a'+x\right) \text{ ... (ii)}$$

where the forces in the direction of x increasing has been taken with $+ve$ sign and thee force in the direction of x decreasing has been taken with $-ve$ sign.
Simplifying equation (ii), we get

$$m\left(\frac{d^2x}{dt^2}\right) = m\left(\mu a - \mu x - \mu' a' - \mu' x\right)$$

$$\implies \frac{d^2x}{dt^2} = -\left(\mu + \mu'\right)x \; [\because m\mu a = m\mu' a']$$

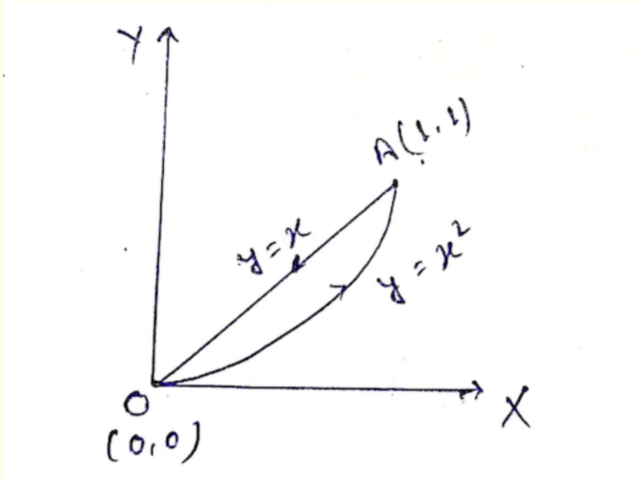
This is the equation of motion with centre at the origin. Hence the motion of particle is SHM with centre at B and its time period is $\frac{2\pi}{\sqrt{\mu+\mu'}}$.

Question-5(e) Verify Green's theorem in the plane for $\oint_c \left[\left(xy + y^2\right)dx + x^2 dy\right]$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

[10 Marks]

Solution: By Green's theorem in plane, we have:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$$



Here, $M = xy + y^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. We have,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R \left(\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(xy + y^2\right)\right) dxdy$$
$$= \iint_R (2x - x - 2y) dxdy$$
$$= \iint_R (x - 2y) dxdy$$
$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y) dydx$$
$$= \int_0^1 [xy - y^2]_{y=x^2}^{x} dx$$
$$= \int_0^1 \left[x^2 - x^2 - x^3 + x^4\right] dx$$
$$= \int_0^1 x^4 - x^3 dx$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{1}{5} - \frac{1}{4} - 0$$
$$= \frac{4-5}{20} = \frac{-1}{20}$$

Now, let us evaluate the line integral along C. The line integral along C = line integral along $y = x^2$ + line integral along $y = X = I_1 + I_2$
Along $y = x^2$, $dy = 2xdx$

$$I_1 = \int_0^1 \left(x.x^2 + x^4\right) dx + x^2 2xdx$$
$$I_1 = \int_0^1 \left(x \cdot x^2 + x^4\right) dx + \int_0^1 x^2 2xdx$$
$$= \int_0^1 \left(x^3 + x^4 + 2x^3\right) dx$$
$$= \int_0^1 (3x^3 + x^4) dx$$
$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$
$$= 3\frac{1}{4} + \frac{1}{5}$$
$$= \frac{19}{20}$$

Along $y = X$, $dy = dx$

$$I_2 = -\int_0^1 \left(x^2 + x^2\right) dx + x^2 dx$$
$$= \int_0^1 3x^2 dx = -\left[x^3\right]_0^1$$
$$= 1$$

$\therefore I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$
Hence, the theorem is verified.

Question-6(a) Solve: $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

[10 Marks]

Solution: $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0 \ldots$ (i)

Equation (i) is a homogeneous equation. Comparing equation (i) with $Mdx + Ndy$, we get: $M = y^3 - 2yx^2$, $N = 2xy^2 - x^3$

$$\begin{aligned} Mx + Ny &= xy^3 - 2yx^3 + 2xy^3 - yx^3 \\ &= 3xy^3 - 3yx^3 \\ &= 3xy\left(y^2 - x^2\right) \\ \frac{1}{Mx + Ny} &= \frac{1}{3xy\,(y^2 - x^2)} \end{aligned}$$

Multiplying eq (i) by $\frac{1}{3xy(y^2-x^2)}$, we get

$$\frac{y^3-2yx^2}{3xy(y^2-x^2)}dx + \frac{2xy^2-x^3}{3xy(y^2-x^2)}dy = 0$$

$$\implies \frac{y^2-2x^2}{3x(y^2-x^2)}dx + \frac{2y^2-x^2}{3y(y^2-x^2)}dy = 0 \ldots \text{(ii)}$$

Comparing eq (ii) with $Pdx + Qdy = 0$, we get

$$P = \frac{y^2-2x^2}{3xy^2-3x^3}, \quad Q = \frac{2y^2-x^2}{3y^3-3x^2y}$$

$$\implies \frac{\partial P}{\partial y} = \frac{6xy}{(3y^2-3x^2)^2}, \quad \frac{\partial Q}{\partial x} = \frac{6xy}{(3y^2-3x^2)^2}$$

$$\implies \frac{\partial p}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ Eq (ii) is exact.
Solution is given by:

$$\int_{y=constant} Pdx + \int (\text{ terms in Q containing x})\, dy = c_1$$

$$\implies \int \frac{y^2-2x^2}{3xy^2-3x^3}dx + \int \frac{2}{3y}dy = c_1$$
$$\implies \int \frac{y^2-2x^2}{3x(y^2-x^2)}dx + \int \frac{2dy}{3y} = c_1$$

$$\implies \int \frac{dx}{3x} - \frac{1}{3}\int \frac{xdx}{y^2-x^2} + \frac{2}{3}\int \frac{dy}{y} = c_1$$

Let $y^2 - x^2 = t$

$$\implies -2xdx = dx$$

$$\implies \tfrac{1}{3}\log x + \tfrac{1}{3}\int \tfrac{dt}{2\cdot t} + \tfrac{2}{3}\log y = c_1$$

$$\implies \tfrac{1}{3}\log x + \tfrac{1}{6}\log t + \tfrac{2}{3}\log y = C_1$$

$$\implies 2\log x + \log\left(y^2 - x^2\right) + 4\log y = 6c_1$$

$$\implies \log x^2\left(y^2 - x^2\right)y^4 = \log(c_2). \text{ where } 6c_1 = \log c_2$$

$$\implies x^2y^4\left(y^2 - x^2\right) = c_2,$$

which is the required solution.

Question-6(b) Solve: $\left(\dfrac{dy}{dx}\right)^2 - 2\dfrac{dy}{dx}\cos hx + 1 = 0$

[8 Marks]

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Solution:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}\cos hx + 1 = 0$$

Let $\dfrac{dy}{dx} = p$

$$p^2 - 2p\cos hx + 1 = 0$$

Solving for p

$$p = \frac{2\cosh x \pm \sqrt{4\cos h^2x - 4}}{2}$$

$$p = \cosh x \pm \sqrt{\cos h^2x - 1}$$

$$p = \cosh x \pm \sin hx$$

$$p = \cos m + \sin hx \quad \& \quad p = \cos hx - \sin hx$$

$$\frac{dy}{dx} = \cosh x + \sin hx \quad \& \quad \frac{dy}{dx} = \cos hx - \sin hx$$

integrating above,

$$y = \sin hx + \cos hx + C_1; y = \sin m_x - \cosh x + C_2$$

Hence, general solution is

$$(y - \sin hx - \cos hx - C_1)\,(y - \sin m_x + \cosh x - C_2) = 0$$

$$\Rightarrow \left(y - \left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right) - C_1\right)\left(y - \left(\frac{e^x - e^{-x}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right) - C_2\right) = 0$$

$$\Rightarrow \left(y - \frac{e^x}{2} - \frac{e^x}{2} - C_1\right)\left(y + \frac{e^{-x}}{2} + \frac{e^{-x}}{2} - C_2\right) = 0$$

$$\Rightarrow (y - e^x - C_1) \quad \left(y + e^{-x} - C_2\right) = 0$$

Question-6(c) **Solve:** $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x^2e^{-x}$

[10 Marks]

Solution: Given: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x^2e^{-x}$

$$\implies (D^3 + 3D^2 + 3D + 1)^2\, y = x^2e^{-x}$$

The auxiliary equation is:

$$\begin{aligned} & D^3 + 3D^2 + 3D + 1 = 0 \\ \implies & (D+1)^3 = 0 \\ \implies & D = -1, -1, -1 \end{aligned}$$

CF is given by:

$$y_c = \left(c_1 + c_2x + c_3x^2\right) e^{-x}$$

Now, we calculate PI (particular integral),

$$\begin{aligned} y_p &= \frac{1}{D^3 + 3D^2 + 3D + 1} x^2e^{-x} \\ &= \frac{1}{(D+1)^3} x^2e^{-x} \\ &= e^{-x}\frac{1}{(D-1+1)^3} x^2 \\ &= e^{x}\frac{1}{D^3} x^2 \\ &= e^{x}\frac{1}{D^2}\frac{x^3}{3} \\ &= \frac{e^x}{3}\frac{1}{D}\frac{x^4}{4} \\ &= \frac{e^x}{12}\frac{1}{D} x^4 \\ &= \frac{e^x}{12}\frac{x^5}{5} \\ &= \frac{x^5e^x}{60} \end{aligned}$$

The complete solution is given by:

$$\begin{aligned} & y = y_c + y_p \\ \implies & y = \left(c_1 + c_2x + c_3x^2\right) e^{-x} + \tfrac{x^5e^x}{60}. \end{aligned}$$

Question-6(d) Show that e^{x^2} is a solution of $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-2)\,y = 0$. Find a second independent solution.

[12 Marks]

Solution: Given: $y = e^{x^2} \Rightarrow y' = e^{x^2} \cdot 2x$

$$y'' = 2\left[e^{x^2} + 2x^2 e^{x^2}\right] = 2e^{x^2}\left(1+2x^2\right)$$

Substituting these in the given differential equation, we get:

$$\begin{aligned}\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2-2\right)y &= 2e^{x^2}(1+2x^2) - 4x \cdot e^{x^2} \cdot 2x + \left(4x^2-2\right)e^{x^2} \\ &= \left(4x^2 - 8x^2 + 4x^2\right)e^{x^2} + \left(2e^{x^2} - 2e^{x^2}\right) \\ &= 0\end{aligned}$$

Hence, $y = e^{x^2}$ is a solution of given DE.
$\therefore y = u = e^{x^2}$ is a part of complimentary function of the given DE.

Comparing given DE with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$P = -4x$, $Q = 4x^2 - 2$, $R = 0$
Let $y = uv$ be the general solution, then v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\cdot\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\Longrightarrow P + \frac{2}{u}\cdot\frac{du}{dx} = -4x + \frac{2}{e^{x^2}}\left(2xe^{x^2}\right) = 0$$

$$\therefore \frac{d^2v}{dx^2} = 0$$

$$\Longrightarrow \frac{dv}{dx} = c_1$$

$$v = c_1x + c_2$$

Hence, the complete solution is:

$$\begin{aligned} & y = uv \\ \Longrightarrow\ & y = e^{x^2}\left(c_1x + c_2\right)\end{aligned}$$

Question-7(a) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the sphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$.

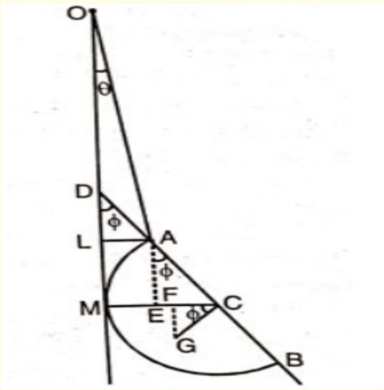
[10 Marks]

Solution: Let O be the point of suspension in the wall, AB the base of the hemisphere, C its centre, G its centre of gravity, M the point of contact of the hemisphere and the wall and OA the string.
Let l be the length of the string OA and let a be the radius of the hemisphere.

$$\therefore \quad CA = a \quad \text{and} \quad CG = \frac{3a}{8}$$

since O is a fixed point, so all the distances will be measured from this point O.

Let d be the depth of G below O.



$$\begin{aligned}
\therefore \quad d &= OM + FG = OL + LM + CG\sin\phi \\
&= l\cos\theta + AC\cos\phi + \frac{3a}{8}\sin\phi \\
\Rightarrow \quad d &= l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi \quad \ldots(1)
\end{aligned}$$

The normal reaction at M is perpendicular to the wall.

$$\therefore MC \text{ is horizontal.}$$

Let the system be given a small virtual displacement such that θ becomes $\theta + \delta\theta$ and ϕ becomes $\phi + \delta\phi$
W, the weight of the hemisphere will be the only force doing work. The reaction at M does not appear in the equation of virtual work.

$$\therefore \text{Equation of virtual work is } W.\delta(d) = 0$$

or

$$\delta(d) = 0 \quad [\because W \neq 0]$$

or

$$\delta\left[l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right] = 0$$

or

$$-l\sin\theta\cdot\delta\theta - a\sin\phi\delta\phi + \frac{3a}{8}\cos\phi\delta\phi = 0$$

$$\therefore l\sin\theta.\delta\theta = \left(\frac{3}{8}\cos\phi - \sin\phi\right)\cdot a\delta\phi\ldots(2)$$

Again,

$$\begin{aligned}a = CM &= CE + EM = CE + AL \quad [\because EM = AL]\\ &= CA\sin\phi + OA\sin\theta\\ &= a\sin\phi + l\sin\theta\end{aligned}$$

or

$$l\sin\theta = a - a\sin\phi$$

Differentiating,

$$l\cos\theta\cdot\delta\theta = -a\cos\phi\delta\phi\ldots(3)$$

Dividing (2) by (3), we get

$$\tan\theta = -\frac{3}{8} + \tan\phi$$

Hence $\tan\phi = \frac{3}{8} + \tan\theta$

Question-7(b) A particle moves with a central acceleration $\mu\left(\gamma + \frac{a^4}{\gamma^3}\right)$ being projected from an apse at a distance a with a velocity $2\sqrt{\mu}a$.
Prove that its path is $\gamma^2(2+\cos\sqrt{3}\theta) = 3a^2$.

[10 Marks]

Solution: Here, $F = \mu\left(r + \frac{a^4}{r^3}\right) = \mu\left(u^{-1} + a^4u^3\right)\ldots(1)$ where $u = \frac{1}{r}$
Differential equation of central orbit is

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{F}{u^2} = \mu\frac{(u^{-1}+a^4u^3)}{u^2} = \mu\left(u^{-3} + u^4u\right)\ldots(2)$$

Multiplying by $2\frac{du}{d\theta}$, we get

$$h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = 2\mu\left[u^{-3} + a^4u\right]\frac{du}{d\theta}$$

Integrating, we get

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left[-\frac{1}{2}u^{-2}+\frac{a^4u^2}{2}\right]+c$$

or

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left(-u^{-2}+a^4u^2\right)+c\ldots(3)$$

Initially, at an apse,

$$u=\frac{1}{a},\frac{du}{d\theta}=0 \text{ and } v=2a\sqrt{\mu} \text{ [Given]}$$

From (3),

$$\therefore \quad 4a^2\mu=\frac{h^2}{a^2}=\mu\left(-a^2+a^2\right)+c$$

$\therefore h^2=4\mu a^4$ and $c=4\mu a^2$

Putting the values of h^2 and c in (3), we get

$$4\mu a^4\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left(-u^{-2}+a^4u^2\right)+4\mu a^2$$

or

$$\begin{aligned}4a^4\left(\frac{du}{d\theta}\right)^2&=-\frac{1}{u^2}+a^4u^2-4a^4u^2+4a^2\\&=\frac{-1+a^4u^4-4a^4u^4+4a^2u^2}{u^2}\\&=\frac{-1-3a^4u^4+4a^2u^2}{u^2}\\&=\frac{-1-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2}{u^2}\\&=\frac{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}{u^2}\end{aligned}$$

or

$$2a^2\frac{du}{d\theta}=\pm\frac{\left[\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2\right]^{1/2}}{u}$$

or

$$-\frac{2\sqrt{3}a^2udu}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}}=\sqrt{3}d\theta$$

[Taking -ve sign] Put $\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}} = t$ so that $2\sqrt{3}a^2udu = dt$

$$\therefore \quad -\frac{dt}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2}} = \sqrt{3}d\theta$$

Integrating, $\cos^{-1}(t\sqrt{3}) = \sqrt{3}\theta + A \ldots (4)$
Initially, when $u = \frac{1}{a}$, i.e., $t = \frac{1}{\sqrt{3}}, \theta = 0 \therefore \quad A = 0$
From (4),

$$\therefore \quad \cos^{-1}(t\sqrt{3}) = \sqrt{3}\theta$$

or

$$t\sqrt{3} = \cos\sqrt{3}0$$

or

$$\sqrt{3}\left(\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}}\right) = \cos\sqrt{3}\theta$$

or

$$3a^2u^2 - 2 = \cos\sqrt{3}\theta$$

or

$$3a^2u^2 = 2 + \cos\sqrt{3}\theta$$

Hence, $3a^2 = r^2[2 + \cos\sqrt{3}\theta]$ which is the required path. $\left[\because u = \frac{1}{r}\right]$

Question-7(c) A shell, lying in a straight smooth horizontal tube, suddenly explodes and breaks into portions of masses m and m'. If d is the distance apart of the masses after a time t, show that the work done by the explosion is $\frac{1}{2}\frac{mm'}{m+m'} \cdot \frac{d^2}{t^2}$.

[10 Marks]

Solution: Since, the shell is lying in the tube, its velocity before explosion is zero. Let u_1 and u_2 be the velocities, of the masses m and m' respectively after explosion. Then, the relative velocity of the masses after explosion is $u_1 + u_2$. Since, the tube is smooth and horizontal, $u_1 + u_2$ will remain constant.

$$\therefore (u_1 + u_2)\,t = d - (i)$$

Also, by the principle of conservation of linear momentum, we have,

$$mu_1 - m'u_2 = 0$$
$$\Rightarrow \quad mu_1 = m'u_2 - (ii)$$

Substituting, for u_2 from (ii) in (i) we get,

$$\left(u_1 + \frac{mu_1}{m'}\right) t = d \; u_1 \left(\frac{m' + m}{m'}\right) t = d$$
$$\Longrightarrow u_1 = \frac{dm'}{(m' + m)\, t}$$

from (ii)

$$\therefore u_2 = \frac{m}{m'} u_1$$
$$= \frac{m}{m'} \frac{m'd}{(m + m')\, t}$$
$$u_2 = \frac{md}{(m + m')\, t}$$

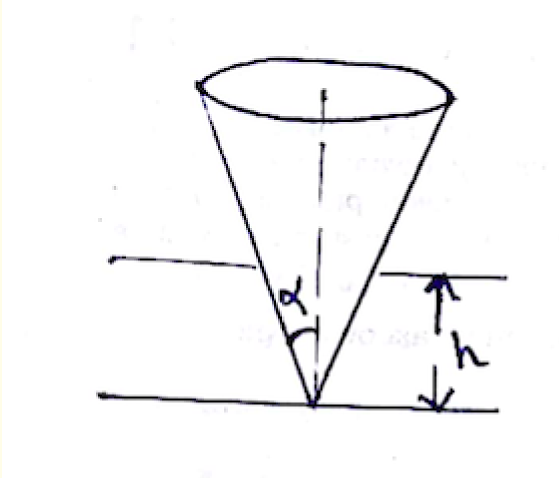
Now, the work done by the explosion = the kinetic energy released due to the explosion

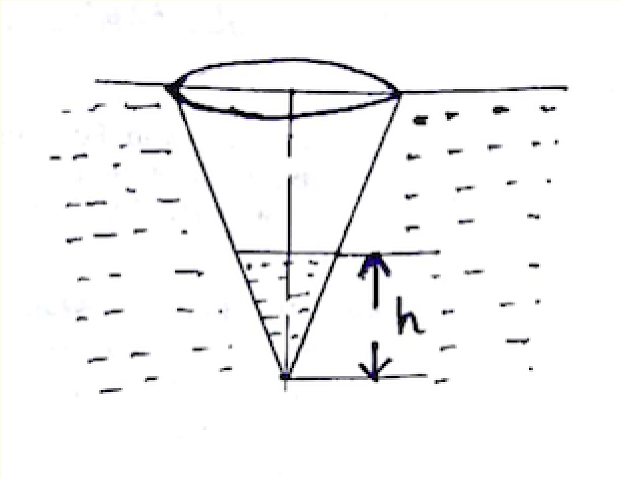
$$= \frac{1}{2} mu_1^2 + \frac{1}{2} m'u_2^2$$
$$= \frac{1}{2} m \left[\frac{m'^2 d^2}{(m + m')^2 t^2}\right] + \frac{1}{2} m' \left[\frac{m^2 d^2}{(m + m')^2 t^2}\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{1}{(m + m')^2} \left[mm'^2 + m'm^2\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{mm'}{(m + m')^2} \left[m' + m\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{mm'}{(m + m')}$$

Question-7(d) A hollow conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed. When water is poured into it up to the level originally immersed, it sinks till its mouth is on a level with the surface of the water. What portion of axis was originally immersed?

[10 Marks]

Solution: According to Law of buoyancy,
Upward force on a body = weight of fluid displaced by immersed part of body.





Let W be weight of cone with height H and semi-vertical angle α, then
upward force $= \frac{1}{3}\pi(htan\alpha)^2 \times h\rho g$
[weight of fluid = vol. of body submerged×density ×g]
$\therefore \quad W = \frac{1}{3}\pi h^3 \tan^2 \alpha\rho g - (1)$

Now,

$$\text{Total weight} = \text{Total upward force}$$

$$\Rightarrow W + \text{ Weight of water} = \text{Vol. of body submerged}\times\text{density}\times\text{g}$$

$$\Rightarrow \quad W + \frac{1}{3}\pi(h\tan\alpha)^2 \times h\rho g = \frac{1}{3}\pi(H\tan\alpha)^2 H\rho g$$

where W =Weight of cone,

$$\frac{1}{3}\pi(htan\alpha)^2 \times h\rho g = \text{Weight of water in cone,}$$

$$\frac{1}{3}\pi(H\tan\alpha)^2 H\rho g = \text{Total upward force}$$

$$\Rightarrow 2 \times \frac{1}{3}\pi h^3 \tan^2\alpha\rho g = \frac{1}{3}\pi H^3 \tan^2\alpha\rho g \text{ [from (1)]}$$

$$\Rightarrow \quad 2h^3 = H^3$$

$$\Rightarrow \quad \frac{h}{H} = \left(\frac{1}{2}\right)^{1/3}$$

Question-8(a) Show that $\bar{A} = (6xy + z^3)\,\hat{\imath} + (3x^2 - z)\,\hat{\jmath} + (3xz^2 - y)\,\hat{k}$ is irrotational. Find a scalar function ϕ such that $\bar{A} = \text{grad}\,\phi$.

[10 Marks]

Solution: Given that $\vec{A} = (6xy + z^3)\,\hat{\imath} + (3x^2 - z)\,\hat{\jmath} + (3xz^2 - y)\,k$

$$\begin{aligned} curl\vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix} \\ &= \hat{\imath}(-1+1) + \hat{\jmath}\left(3z^2 - 3z^2\right) + \hat{k}(6x - 6x) \\ &= \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(0) \\ &= 0 \end{aligned}$$

$\therefore$ The vector $\vec{A}$ is irrotational.

Let $\vec{A} = \text{grad}\,\phi$ i.e $\vec{A} = \nabla\phi$

$$\begin{aligned} \Rightarrow \left(6xy + z^3\right)\hat{\imath} + \left(3x^2 - z\right)\hat{\jmath} + \left(3xz^2 - y\right)\hat{k} &= \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} \\ \frac{\partial\phi}{\partial x} &= 6xy + z^3 - (i) \\ \frac{\partial\phi}{\partial y} &= 3x^2 - z - (ii) \\ \frac{\partial\phi}{\partial z} &= 3xz^2 - y - (iii) \end{aligned}$$

(i) partially w.r.t x treating y, z as constants.

$$\text{(i)} \equiv \phi = 3x^2y + z^3x + f_1(y,z) - (iv)$$

(ii) partially w.r.t y treating x, z as constant.

$$\phi = 3x^2y - zy + f_2(x,z) - (v)$$

iii) partially w.r.t z treating x, y as constant.

$$(iii) \equiv \phi = xz^3 - yz + f_3(x,y) - (vi)$$

(iv), (v), (vi) each represents ϕ.These agree if we choose:

$$f_1(y,z) = -yz, f_2(x,z) = xz^2, f_3(x,y) = 3x^2y$$

$\therefore \phi = 3x^2y + xz^3 - yz + C$ where C is an arbitrary constant.

Question-8(b) Let $\psi(x, y, z)$ be a scalar function. Find grad ψ and $\nabla^2\psi$ in spherical coordinates.

[8 Marks]

Solution: We know that, $\nabla\psi = grad\ \psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}e_3 - (i)$

for spherical coordinates (r, θ, ϕ)
$u_1 = r, \quad u_2 = \theta \quad , \quad u_3 = \phi$
$e_1 = e_r, \quad e_2 = e_\theta \quad , \quad e_3 = e_\phi$
$h_1 = h_r, \quad h_2 = h_\theta \quad , \quad h_3 = h_\phi = r\sin\theta$

$\therefore$ From (i)

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi = \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi$$

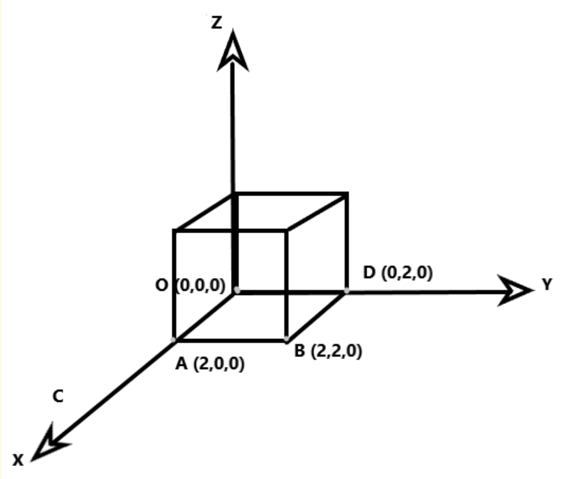
We know that,

$$\begin{aligned}
\nabla^2\Psi &= \frac{1}{h, h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\cdot\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h, h_2}{h_3}\cdot\frac{\partial\psi}{\partial u_3}\right)\right] \\
&= \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta(t)}{r}\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\cdot\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned}$$

Question-8(c) Verify Stokes' theorem for $\bar{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ where $\dot{S}$ is the surface of the cube $x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 2,\ z = 2$ above the xy-plane.

[12 Marks]

Solution: The xy - plane cuts the surface of the cube in a square. Thus, the curve C bounding the surface S is the square.



Say $OABD$, in the xy - plane whose vertices in the xy-plane are the points. $O(0,0), A(2,0), B(2,2), D(0,2)$
Then,

$$\begin{aligned}
\oint \vec{F} \cdot dr &= \int_c [(y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}] \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}) \\
&= \int_c (y - z + 2)dx + (yz + 4)dy - xzdz \\
&= \int_c (y + z)dx + 4dy \quad (\because \text{on } c, z = 0 \& dz = 0) \\
&= \int_{OA} + \int_{AB} + \int_{BD} + \int_{DO} \\
&= I_1 + I_2 + I_3 + I_4 - (i)
\end{aligned}$$

Along OA:

$$y = 0, dy = 0 \quad \& \quad x \text{ varies from } 0 \text{ to } 2$$
$$\therefore I_1 = \int_{OA} (y + 2)dx + 4dy$$
$$\Rightarrow \int_0^2 2 \cdot dx = [2x]_0^2 = 4$$

Along AB :

$$x = 2, dx = 0 \quad \& \quad y \text{ varies from } 0 \text{ to } 2$$
$$\therefore I_2 = \int_{AB} (y + 2)dx + 4dy$$
$$\Rightarrow \int_0^2 4 \cdot dy = [4y]_0^2 = 8$$

Along BD:

$$y = 2, dy = 0 \quad \& \quad x \text{ varies from } 2 \text{ to } 0$$
$$\therefore I_3 = \int_{BD} (y + 2)dx + 4 \cdot dy$$
$$\int_2^0 4 \cdot dx = [4x]_2^0 = -8$$

Along DO:

$$x = 0, dx = 0 \quad \text{\& y varics . From 2 to 0}$$
$$\therefore I_4 = \int_{DO} (y+2)dx + 4 \cdot dy$$
$$\Rightarrow \int_2^0 4 \cdot dy = [4y]_2^0 = -8$$
$$\therefore (i) \equiv \int \vec{F} \cdot dr = I_1 + I_2 + I_3 + I_4$$
$$= 4 + 8 - 8 - 8$$
$$= -4 - (ii)$$

Now,

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$
$$= \hat{\imath}(0 - y) + \hat{\jmath}(-1 + z) + \hat{k}(0 - 1)$$
$$= -y\hat{\imath} + (-1 + z)\hat{\jmath} - \hat{k}$$

$$\hat{n} = \text{ unit normal vector to } S = \hat{k}$$
$$\therefore \quad dS = \frac{dxdy}{|n \cdot \hat{k}|} = dxdy$$
$$(\nabla \times F) \cdot \hat{n} = [(-y\hat{i} + (-1 + z)\hat{\jmath} - \hat{k}] \cdot \hat{k}$$
$$= -1$$
$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = \int_{x=0}^2 \int_{y=0}^2 (-1) dxdy$$
$$= -\int_{x=0}^2 [y]_0^2 dx$$
$$\Rightarrow -2 \int_0^2 dx = -2[x]_0^2 = -4 - (iii)$$

$\therefore$ From (2) and (3)

$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = \int \vec{F} \cdot dr = -4$

Hence the stokes theorem is verified.

Question-8(d) Show that, if $\bar{r} = x(s)\hat{\imath} + y(s)\hat{\jmath} + z(s)\hat{k}$ is a space curve, $\dfrac{d\bar{r}}{ds} \cdot \dfrac{d^2\bar{r}}{ds^2} \times \dfrac{d^3\bar{r}}{ds^3} = \dfrac{\tau}{\rho^2}$, where τ is the torsion and ρ the radius of curvature.

[10 Marks]

Solution: We know that $\tau = \frac{d\vec{r}}{ds}$ and $\kappa N = \frac{d^2\vec{r}}{ds^2}$,
here, κ is the curvature
Now,

$$\begin{aligned}\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} &= \tau \times \kappa N \\ &= \kappa(\tau \times N) \\ &= \kappa B \quad (\because \tau \times N = B)\end{aligned}$$

$$\therefore \quad \kappa = \left| \frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} \right|$$

$$\begin{aligned}\frac{d^3\vec{r}}{ds^2} &= \frac{d}{ds}\left(\frac{d\vec{r}}{ds^2}\right) \\ &= \frac{d}{ds}(\kappa N) \\ &= \kappa \cdot \frac{dN}{ds} + \frac{d\kappa}{ds}N \\ &= \kappa(\tau B - kT) + \frac{d\kappa}{ds}N \left(\because \frac{dN}{ds} = \tau B - \kappa T\right) \\ &= \kappa\tau B - \kappa^2 T + \frac{d\kappa}{ds}N\end{aligned}$$

$$\begin{aligned}\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) &= T \cdot \left[\kappa N \times \left(\kappa\tau B - \kappa^2 T + \frac{d\kappa}{ds}N\right)\right] \\ &= T \cdot \left[\kappa^2(N \times \tau B) - \kappa^3 \cdot (N \times T) + \kappa \cdot \frac{d\kappa}{ds}(N \times N)\right] \\ &= T \cdot \left[\kappa^2\tau(N \times B) - \kappa^3(-B) + \kappa\frac{d\kappa}{ds}(0)\right] (\because N \times T = -B, N \times N = 0) \\ &= T \cdot \left(\kappa^2\tau(T) + \kappa^3 B\right] \quad (\because N \times B = T) \\ &= \kappa^2\tau(T \cdot T) - \kappa^3(T \cdot B)(\because T \cdot T = 1 and T \cdot B = 0) \\ &= \kappa^2\tau(1) - \kappa^3(0) \\ &= \kappa^2\tau.\end{aligned}$$

We know that, radius of the curvature $'\rho'$ is the reciprocal of curvature κ. i.e

$$\rho = \frac{1}{\kappa} \Rightarrow \kappa = \frac{1}{\rho}$$

$$\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) = \frac{1}{\rho^2}\tau$$

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