# 8 Functions

In this chapter, we propose to introduce the so-called Elementary functions.

$$e^x,\ \log x,\ a^x,\ \sin x,\ \cos x$$

The reader is already familiar with these functions but this acquaintance is based on a treatment which was essentially based on intuitive and less rigorous geometrical considerations. Even the question of existence was ignored.

We shall base the study of these functions on the set of real numbers as a complete ordered field, the notion of limit and the convergence of series. Starting from the definitions of these functions, their basic properties will be studied. No attempt, however, will be made to make the discussion exhaustive.

We shall consciously accept an abuse of language in as much as notation for a function will not be distinguished from that for the functional value. Thus instead of denoting the function by cos we shall denote the same as $\cos x$ and so on.

Functions of bounded variation and the vector-valued functions have been considered towards the end.

As the definitions of function will be based on *Power series*, we start our discussion with a brief (very brief!) study of power series. A detailed discussion will be found in Chapter 14.

## 1. POWER SERIES

The series of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n \text{ or simply } \Sigma a_nx^n$$

are called *power series* (in $x$) and the numbers $a_n$ are their *coefficients*.

For such series we are not concerned simply with the alternatives 'convergent' and 'divergent', but the more precise question: For what values of $x$ is the series convergent and for what values is divergent?

**1.1** Some simple examples have already come before us:

1. The geometric series $\Sigma x^n$ is convergent for $|x| < 1$, divergent for $|x| \geq 1$.
   For $|x| < 1$, indeed, we have absolute convergence.

2. $\Sigma \frac{x^n}{n!}$ is (absolutely) convergent for every real $x$; likewise the series $\Sigma(-1)^n \frac{x^{2n}}{2n!}$ and $\Sigma(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

3. $\Sigma \frac{x^n}{n}$.

   For $x = 1$, diverges (reduces to divergent harmonic series)
   $x = -1$, converges
   $|x| < 1$ absolutely convergent ($\leq |x|^n$, comparison test)
   $|x| > 1$ diverges

4. $\Sigma n^n x^n$ converges for $x = 0$, but diverges for $x \neq 0$.

**1.2** For $x = 0$, obviously every power series $\Sigma a_n x^n$ is convergent, whatever be the value of the coefficient $a_n$. The general case is evidently that in which the power series converges for some values of $x$ and diverges for others; while in special instances, the two extreme cases may occur, in which the series converges for every $x$ or for none $x \neq 0$.

In the first of these special cases, when the series converges for every $x$, we say that the power series is *everywhere convergent*, while in the second (leaving out of account the self-evident point of convergence $x = 0$) when the series converges for no value of $x$ we say that it is *nowhere convergent*. The totality of points $x$ for which the series converges is called its *region of convergence*.

For a power series $\Sigma a_n x^n$, which does not merely converge everywhere or nowhere, a definite positive number $R$ exists such that the series converges for every $|x| < R$ (indeed absolutely), but diverges for every $|x| > R$. The number $R$ is called the *Radius of convergence*, and the interval $]-R, R[$, the *Interval of convergence* of the given series. The behaviour of the series is much more varied at $x = \pm R$ and is beyond the scope of the present discussion.

**1.3** For the power series $\Sigma a_n x^n$, put

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \alpha \text{ and } R = \frac{1}{\alpha}$$

where $R = +\infty$, for $\alpha = 0$ and $R = 0$, for $\alpha = \infty$.

By Cauchy's Root test, it follows that if

(*i*) $R = 0$, the series is nowhere convergent;

(*ii*) $R = \infty$, the series is everywhere convergent;

(*iii*) $0 < R < \infty$, the series converges absolutely for $|x| < R$, and diverges for $|x| > R$, *i.e.*, $R$ is the radius of convergence.

**Ex. 1.** $1 + x + x^2 + \ldots$ converges for $|x| < 1$ and equals $(1 - x)^{-1}$.

CHAPTER 8

## 2. EXPONENTIAL FUNCTIONS

The power series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+... \quad ...(1)$$

is everywhere convergent for real $x$. We proceed now to examine in detail the function represented by this series.

### 2.1 Definition

The function represented by the power series (1) is called the *Exponential function*, denoted, provisionally, by $E(x)$. Thus

$$E(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+... \quad ...(2)$$

$$\therefore \quad E(0)=1$$

and

$$E(1)=1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}+... \quad ...(3)$$

The series on the right hand side of (3) converges to a number which lies between 2 and 3. This number is denoted by $e$, the *Exponential base* and is the same number as represented by

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Thus $E(1)=e$

### 2.2 The Additional Formula

The function $E(x)$, defined by (2), is continuous and differentiable any number of times, for every $x$.

By differentiation, we get

$$E'(x)=E(x)$$

$$E''(x)=E(x)$$

$$\dots\dots\dots\dots$$

$$E^{(n)}(x)=E(x)$$

Further we state (justification may be seen expanding by Taylor's Theorem)* that

$$E(x_1+x_2)=E(x_1).E(x_2)$$

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* $E(x)=E(x_1)+\frac{E(x_1)}{1!}(x-x_1)+...$, for all values of $x$ and $x_1$.

Replacing $x$ by $x_1+x_2$, we get

$$E(x_1+x_2)=E(x_1)\left\{1+\frac{x_2}{1!}+\frac{x_2^2}{2!}+...\right\}=E(x_1)\cdot E(x_2)$$

This formula is called the *Addition formula* for the exponential function. It gives further

$$E(x_1+x_2+x_3)=E(x_1+x_2)\cdot E(x_3)$$
$$=E(x_1)\cdot E(x_2)\cdot E(x_3)$$

and repetition of the process gives, for any positive integer $q$,

$$E(x_1+x_2+\dots+x_q)=E(x_1)\cdot E(x_2)\dots E(x_q) \quad \dots(4)$$

If $x_1=x_2=x_3=\dots=x_q=x$, we get

$$E(qx)=\{E(x)\}^q \quad \dots(5)$$

Hence, for $x=1$

$$E(q)=\{E(1)\}^q=e^q, \text{ for any positive integer } q$$

But since $E(0)=1$, therefore the above relation holds for $q=0$ also.

Hence $E(q)=e^q$ holds for all integers $\geq 0$.

Again replacing each $x$ by $p/q$ in (5), we get

$$E\left(q\frac{p}{q}\right)=\left\{E\left(\frac{p}{q}\right)\right\}^q \text{ for positive integers, } p, q$$

or

$$E(p/q)=\{E(p)\}^{1/q}=e^{p/q} \qquad [\because E(p)=e^p]$$

Hence $E(m)=e^m$, for all rational numbers $m\geq 0$.

For any positive *irrational* number $\xi$ there always exists a sequence $\{x_n\}$ of positive rational terms, converging to $\xi$.

Now for each $n$

$$E(x_n)=e^x n.$$

When $n\to+\infty$, the left hand side tends to $E(\xi)$, and the right hand side to $e^\xi$, so that we get

$$E(\xi)=e^\xi$$

$$\therefore \qquad E(x)=e^x, \text{ for real } x\geq 0 \quad \dots(6)$$

Again by Addition formula,

$$E(x)\cdot E(-x)=E(x-x)=E(0)=1 \quad \dots(7)$$

Thus we conclude that $E(x)\neq 0$, for any real $x$, and that for $x\geq 0$,

$$E(-x)=\frac{1}{E(x)}=\frac{1}{e^x}=e^{-x},$$

Consequently, $E(x)=e^x$ holds for all real $x$.

## 2.3 Monotonicity

By definition,

$$E(x)>0,\ \forall\ x>0$$

so that from (7) it follows that

$$E(-x) > 0, \ \forall \, x > 0$$

Hence, $E(x) > 0$, for all real $x$.

Again by definition, for real $x$,

$$E(x) \to +\infty, \text{ as } x \to +\infty$$

Hence, (7) shows that

$$E(x) \to 0 \text{ as } x \to -\infty$$

Also by definition,

$$0 < x_1 < x_2 \Rightarrow E(x_1) < E(x_2)$$

Also it follows from (7) that

$$E(-x_2) < E(-x_1), \text{ when } -x_2 < -x_1 < 0$$

Hence, the function $E$ is strictly increasing from 0 to $+\infty$ on the whole real line.

**Note:** By definition $e^x > \dfrac{x^{n+1}}{(n+1)!}$, for $x > 0$, so that $x^n e^{-x} < \dfrac{(n+1)!}{x}$.

$\therefore \ \lim\limits_{x \to +\infty} x^n e^{-x} = 0$, for all $n$

This fact we express by saying that $e^x$ *tends to* $+\infty$ *"faster" than any power* of $x$, as $x \to +\infty$.

## 3. LOGARITHMIC FUNCTIONS (*base e*)

Since the exponential function $E$ is strictly increasing on the set $\mathbf{R}$ of real numbers (*i.e.*, $E$: $\mathbf{R} \to \mathbf{R}^+$ is one-one onto), it has inverse function $L$ (or $\log_e$) which is also strictly increasing and whose domain of definition is $\mathbf{R}^+$ $(\equiv E(\mathbf{R}))$, the set of positive reals. Thus $\mathbf{L}$ is defined by

$$\left.\begin{array}{l} E\{L(y)\} = y, \ (y > 0) \\ L\{E(x)\} = x, \ (x \text{ real}) \end{array}\right\} \qquad \text{...(i)}$$

or

or equivalently, for any real $x$,

$$\left.\begin{array}{r} E(x) = y \Rightarrow L(y) = x \\ e^x = y \Rightarrow \log_e y = x \end{array}\right\} \qquad \text{...(ii)}$$

or

Thus the logarithmic function $L$ (or $\log_e$) is defined for positive values only of the variable.

By definition,

$$\left.\begin{array}{lcl} E(-x) = \dfrac{1}{y} & \Rightarrow & L\left(\dfrac{1}{y}\right) = -x = -L(y) \\ E(0) = 1 & \Rightarrow & L(1) = 0 = \log_e 1 \\ E(1) = e & \Rightarrow & L(e) = 1 = \log_e e \end{array}\right\} \qquad \text{...(iii)}$$

Again

$$E(x) \to +\infty \text{ as } x \to +\infty$$

$\because$

and

$$E(x) \to 0 \text{ as } x \to -\infty$$

$$L(x) \to +\infty \text{ as } x \to +\infty$$

$\therefore$

$$L(x) \to -\infty \text{ as } x \to 0$$

Writing $u = E(x_1)$, $v = E(x_2)$ or $L(u) = x_1$, $L(v) = x_2$ in (4), we get

$$E(x_1 + x_2) = uv$$

$\Rightarrow$

$$L(uv) = x_1 + x_2 = L(u) + L(v)$$

which is a familiar property of the logarithmic function and which makes logarithms a useful tool for computation.

Since, the function $E$ is differentiable, therefore, its inverse function **L** is also differentiable.

Hence differentiating (*i*), we get

$$L'\{E(x)\} \cdot E(x) = 1$$

Writing $E(x) = y$, we get

$$L'(y) = \frac{1}{y} \qquad \ldots(iv)$$

which implies that

$$L(y) = \int_1^y \frac{dx}{x} \qquad \ldots(8)$$

Quite often (8) is taken as the definition of the logarithmic function and thus the starting point of the theory of the logarithmic and the exponential functions.

**Note:** In theoretical investigations, it is always more convenient to use the so-called *natural logarithms*, that is to say, those with the base $e$. Hence in our further discussion, $\log x$ *shall always stand for* $L(x)$ or $\log_e x$.

## 3.1 Generalised Power Functions

The meaning of $a^x$ is well understood when $a$ is any positive real number and $x$ is any rational number. We shall now give a meaning to $a^x$ when $x$ is any real number whatsoever. We define thus:

*Definition.* $a^x = E(x \log a)$, for all $x$ and $a > 0$.

Evidently the range of $a^x$ is the set $\mathbf{R}^+$ of positive reals, *i.e.*,

$$a^x > 0, \ \forall \ x \qquad \ldots(9)$$

Now $a^x \cdot a^y = E(x \log a) \cdot E(y \log a)$

$$= E\{(x + y) \log a\} = a^{x+y}$$

$\therefore$

$$a^x \cdot a^y = a^{x+y} \qquad \ldots(10)$$

Let us now verify that this definition of $a^x$ is consistent with that already known to us for $x$, an integer or a rational number.

(*i*) Let $x = n$, a positive integer.

$$\therefore \quad a^n = E(n \log a) = E[\log a + \log a + \ldots n \text{ times}]$$
$$= E(\log a) \cdot E(\log a) \ldots n \text{ times}$$
$$= a \cdot a \ldots n \text{ times}$$

(*ii*) Now let $x = -n$, $n$ being a positive integer.

$$\therefore \quad a^{-n} = E(-n \log a)$$
$$= E[(-\log a) + (-\log a) + \ldots n \text{ times}]$$
$$= E\left[\log \frac{1}{a} + \log \frac{1}{a} + \ldots n \text{ times}\right]$$
$$= E\left(\log \frac{1}{a}\right) \cdot E\left(\log \frac{1}{a}\right) \ldots n \text{ times}$$
$$= \frac{1}{a} \cdot \frac{1}{a} \ldots n \text{ times}$$

Thus $E(x \log a)$ has the same meaning as $a^x$ when $x$ is an integer.

(*iii*) Let now $x = p/q$, where $p$, $q$ are integers, and $q$ is positive.

Now

$$E\left(\frac{p}{q} \log a\right) = a^{p/q}$$

$$\therefore \quad \left[E\left(\frac{p}{q} \log a\right)\right]^q = a^p = E(p \log a)$$

so that $E\left(\frac{p}{q} \log a\right)$ is $q$th root of $E(p \log a)$.

Thus $a^{p/q}$ is a $q$th root of $a^p$.

Hence the definition holds good when $x$ is a rational number.

Thus the above definition of $a^x$ agrees with what is already known to us about $a^x$.

## 3.2 Logarithmic Functions (*any base*)

*Definition.* $a^x = y \quad \Leftrightarrow \quad \log_a y = x.$

Since $y$ is always positive, therefore the logarithmic function, $\log_a$, is defined for positive values only of the variable.

Evidently

$$a^{-x} = \frac{1}{y}$$

$$\therefore \quad \log_a \frac{1}{y} = -x = -\log_a y$$

Also, from definition,

$$\log_a 1 = 0, \log_a a = 1$$

It may be easily shown that

$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a (x/y)$$
$$\log_a x^y = y \log_a x$$
$$\log_b x \cdot \log_a b = \log_a x$$
$$\log_b a \cdot \log_a b = 1$$

**Ex.** Show that $\lim\limits_{x \to \infty} \dfrac{\log x}{x^a} = 0,\ a > 0.$

[*Hint*: Use $\lim\limits_{n \to \infty} x^n / e^x = 0.$]

## 4. TRIGONOMETRIC FUNCTIONS

We are now in a position to introduce rigorously the circular functions, employing purely the arithmetical methods. For this purpose, we consider the power series, everywhere convergent (absolutely and uniformly) and the functions represented by them.

*Definition.*

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^2}{2!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \ \forall\, x$$

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \ \forall\, x$$

Each of these series represents a function everywhere continuous and differentiable any number of times in succession. The properties of these functions will be established, taking as starting point their expansions in series form, and it will be seen finally that these coincide with the functions $\cos x$ and $\sin x$ with which we are familiar from elementary studies, *i.e.*, $C(x) \equiv \cos x$ and $S(x) \equiv \sin x$.

### 4.1 Properties of the Functions (C(x), S(x))

(*i*) The functions $C(x)$ and $S(x)$ are continuous and derivable for all $x$; in fact it may easily be seen that

$$C'(x) = -S(x) \text{ and } S'(x) = C(x)$$

Also

$$C^{iv}(x) = C(x) \text{ and } S^{iv}(x) = S(x)$$

(*ii*) From definitions,

$$S(0) = 0, C(0) = 1$$

$$S(-x) = -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \dots + (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} + \dots$$

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\right] = -S(x) \ \forall\ x$$

Similarly, $C(-x) = C(x) \ \forall\ x$

(*iii*) **The Addition Theorems.** These functions, like the exponential function, satisfy simple addition theorems, by means of which they can then be further examined.

*First Method.* By Taylor's expansion for any two variables, $x_1$ and $x_2$ (since the two series converge everywhere absolutely).

$$C(x_1 + x_2) = C(x_1) + \frac{C'(x_1)}{1!}x_2 + \frac{C''(x_1)}{2!}x_2^2 + \ldots$$

$$= C(x_1) - \frac{S(x_1)}{1!}x_2 - \frac{C(x_1)}{2!}x_2^2 + \frac{S(x_1)}{3!}x_2^3 + \ldots$$

As this series is absolutely convergent, we may rearrange it in any way we please.

$$\therefore\ C(x_1 + x_2) = C(x_1)\left\{1 - \frac{x_2^2}{2!} + \frac{x_2^4}{4!} - \ldots\right\} - S(x_1)\left\{x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - \ldots\right\}$$

$$= C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2) \qquad \ldots(11)$$

Similarly,

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2) \qquad \ldots(12)$$

*Second Method.* For any fixed value of $x_2$, consider the functions

$$f(x_1) = S(x_1 + x_2) - S(x_1) \cdot C(x_2) - C(x_1) \cdot S(x_2)$$
$$g(x_1) = C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2)$$

Differentiating with respect to $x_1$, we get

$$f'(x_1) = C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2) = g(x_1)$$
$$g'(x_1) = -S(x_1 + x_2) + S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2) = -f(x_1)$$

$$\therefore \qquad \frac{d}{dx_1}\left[f^2(x_1) + g^2(x_1)\right] = 2f(x_1)f'(x_1) + 2g(x_1)g'(x_1)$$

$$= 2f(x_1)g(x_1) - 2g(x_1)f(x_1) = 0, \ \forall\ x_1$$

$\Rightarrow \quad f^2(x_1) + g^2(x_1)$ is a constant, $\forall\ x_1$

Hence for all $x_1$,

$$f^2(x_1) + g^2(x_1) = f^2(0) + g^2(0) = 0$$

$$\Rightarrow \qquad f(x_1) = 0,\ g(x_1) = 0$$

$$\therefore \qquad C(x_1 + x_2) = C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2)$$

and

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2)$$

The form of these theorems coincides with that of the addition theorems for the functions cosine and sine, with which we are clearly acquainted from an elementary standpoint. With the help of these theorems, we shall now show that the functions $C$ and $S$ satisfy all the other so called purely trigonometrical formulae—in fact $C$ and $S$ are same as the functions cosine and sine. We note, in particular:

(a) Changing $x_2$ to $-x_2$,

$$C(x_1 - x_2) = C(x_1)\cdot C(x_2) + S(x_1)\cdot S(x_2)$$
$$S(x_1 - x_2) = S(x_1)\cdot C(x_2) - C(x_1)\cdot S(x_2)$$

(b) Writing $x_2 = -x_1$, we deduce that

$$C^2(x_1) + S^2(x_1) = 1 \text{ or } C^2(x) + S^2(x) = 1, \ \forall x$$
$$\Rightarrow |S(x)| \le 1, |C(x)| \le 1, \ \forall x$$

(c) Replacing $x_1$ and $x_2$ by $x$,

$$C(2x) = C^2(x) - S^2(x)$$
$$S(2x) = 2S(x)\cdot C(x)$$

## 4.2 The Number π — The Smallest Positive Root of the Equation $C(x) = 0$.

**Theorem 1.** *To prove that there exists a positive number $\pi$ such that*

$$C(\pi/2) = 0 \text{ and } C(x) > 0, \text{ for } 0 \le x < \pi/2 \qquad ...(13)$$

Consider the interval [0, 2].

We know $C(0) = 1 > 0$, we shall now show that $C(2) < 0$.

Now

$$C(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + ...$$
$$= 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) - \frac{2^6}{6!}\left(1 - \frac{2^2}{7.8}\right) - ...$$

Since the brackets are all positive, we have

$$C(2) < 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) = -\frac{1}{3}$$

so that $C(2)$ is negative.

Thus, the continuous function $C(x)$ is positive at 0 and negative at 2.

∴ $C(x)$ vanishes at least once between 0 and 2 (by the Intermediate-value theorem). Further, since $S(x)$ is positive in [0, 2], where

$$S(x) = x\left(1 - \frac{x^2}{2.3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6.7}\right) + ...$$

therefore, the derivative $(-S(x))$ of $C(x)$ is always negative for all values of $x$ between 0 and 2. Consequently $C(x)$ is a (strictly) monotonic decreasing function in [0, 2], and can therefore vanish at only one point in [0, 2].

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Thus there exists one and only one root of the equation $C(x) = 0$ lying between 0 and 2. Denoting this root by $\pi/2$, we see that $\pi/2$ is the least positive root of the equation $C(x) = 0$.

Clearly $C(x) > 0$, when $0 \le x < \pi/2$.

Using the above results, we deduce that

(*a*) $S(x) > 0$, when $0 < x \le \pi/2$.

Since the derivative of $S(x)$ is non-negative in $[0, \pi/2]$, therefore $S(x)$ is a strictly monotonic increasing function. Also since $S(0) = 0$, therefore $S(x)$ is positive for $0 < x \le \pi/2$.

(*b*) As $C^2(\pi/2) + S^2(\pi/2) = 1$ and $C(\pi/2) = 0$,

$$\therefore \quad S^2(\pi/2) = 1 \Rightarrow S(\pi/2) = \pm 1$$

But, by Lagrange's Mean Value Theorem,

$$S(\pi/2) - S(0) = (\pi/2)C(\alpha) > 0, \text{ where } 0 < \alpha < \pi/2$$

$$\Rightarrow \quad S(\pi/2) = 1$$

(*c*) $C(\pi) = 2C^2(\pi/2) - 1 = -1$

$$S(\pi) = 2S(\pi/2)\,C(\pi/2) = 0$$

(*d*) $C(2\pi) = 1,\ S(2\pi) = 0$

(*e*) $C(\pi/2) = 2C^2(\pi/4) - 1$

$$\therefore \quad C(\pi/4) = 1/\sqrt{2}$$

Rejecting the negative sign, as $C(\pi/4)$ is positive.

Similarly, $S(\pi/4) = 1/\sqrt{2}$

(*f*) It finally follows from the addition theorems that for all $x$,

$$S\left(\frac{1}{2}\pi - x\right) = C(x),\quad C\left(\frac{1}{2}\pi - x\right) = S(x)$$

$$S\left(\frac{1}{2}\pi + x\right) = C(x),\quad C\left(\frac{1}{2}\pi + x\right) = -S(x)$$

$$S(\pi + x) = -S(x),\quad C(\pi + x) = -C(x)$$

$$S(\pi - x) = S(x),\quad C(\pi - x) = -C(x)$$

$$S(2\pi + x) = S(x),\quad C(2\pi + x) = C(x)$$

Thus, we see that the functions $C(x)$ and $S(x)$ exactly coincide with the functions $\cos x$ and $\sin x$ respectively, and so we shall henceforth use $\cos x$ and $\sin x$ in place of $C(x)$ and $S(x)$ respectively.

## 4.3 The Functions tan *x* and cot *x*

The function $\tan x$ and $\cot x$ are defined as usual by the ratios:

$$\tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{\cos x}{\sin x}$$

and as functions they, therefore, represent nothing new. The expansions in power series for these functions are also not so simple. A few of the coefficients of the expansions could be easily obtained by division, but that gives us no insight into any relationships.

Clearly $\tan x$ is defined, continuous and derivable for all values of $x$ except those for which the denominator, $\cos x$, vanishes, which is the case for $x = \frac{1}{2}(2n+1)\pi$, $n$ being any integer, positive, negative or zero.

From § 4.2 ($f$), we have

$$\tan(\pi + x) = \tan x,$$

so that, $\tan x$ is a periodic function with period $\pi$.

Also we may easily show that when $x \neq \frac{1}{2}(2n+1)\pi$,

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x}$$

**Theorem 2.** *Show that*

$$\lim_{x \to \frac{1}{2}\pi - 0} \tan x = \infty, \quad \lim_{x \to \frac{1}{2}\pi + 0} \tan x = -\infty$$

Let $k$ be any positive number.

As $\lim_{x \to \pi/2} \sin x = 1$, $\exists\, \delta_1 > 0$ such that (taking $\varepsilon = \frac{1}{2}$),

$$\frac{1}{2} < \sin x, \ \forall\, x \in \left]\frac{1}{2}\pi - \delta_1, \frac{1}{2}\pi + \delta_1\right[ \qquad \text{...}(i)$$

Again, since $\lim_{x \to \pi/2} \cos x = 0$, therefore $\exists\, \delta_2 > 0$ such that

$$-\frac{1}{2k} < \cos x < \frac{1}{2k}, \ \forall\, x \in \left]\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi + \delta_2\right[$$

As $\cos x$ is positive for $x \in [0, \pi/2[$, and negative for $x \in\, ]\pi/2, \pi]$, we have

$$0 < \cos x < \frac{1}{2k}, \ \forall\, x \in \left]\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi\right[ \qquad \text{...}(ii)$$

$$-\frac{1}{2k} < \cos x < 0, \ \forall\, x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta_2\right[ \qquad \text{...}(iii)$$

Let $\delta = \min(\delta_1, \delta_2)$ therefore from ($i$) and ($ii$),

$$\tan x = \frac{\sin x}{\cos x} > k, \ \forall\, x \in \left]\frac{1}{2}\pi - \delta, \frac{1}{2}\pi\right[$$

and from ($i$) and ($iii$),

$$\tan x = \frac{\sin x}{\cos x} < -k, \ \forall\, x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta\right[$$

## 4.4 Inverse Trigonometric Functions $\cos^{-1} y$, $\sin^{-1} y$, $\tan^{-1} y$

**$\cos^{-1} y$.** Since, as may be easily seen, $\cos x$ strictly decreases from $+1$ to $-1$ as $x$ increases from 0 to $\pi$, the function cos is invertible and its inverse, denoted as $\cos^{-1}$, is a function with domain $[-1, 1]$ and range $[0, \pi]$. We write

$$y = \cos x \quad \Leftrightarrow \quad x = \cos^{-1} y.$$

*Definition.* Given $y$ (where $-1 \le y \le 1$), $\cos^{-1} y$ is that $x$ which lies between 0 and $\pi\,(0 \le x \le \pi)$ and $\cos x = y$.

$\cos^{-1} y$ is derivable in the open interval $]-1, 1[$ with $-1/\sqrt{1-y^2}$ as its derivative. In fact, we have

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = 1, \text{ and } x = \cos^{-1} y,\ y = \cos x$$

$$\frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{\dfrac{d}{dx}\cos x} = -\frac{1}{\sin x} = \frac{-1}{\sqrt{1-y^2}},\ y \ne \pm 1$$

**$\sin^{-1} y$.** Since $\sin x$ is a strictly increasing function in $[-\pi/2, \pi/2]$ with range $[-1, 1]$, therefore the function sin is invertible and its inverse function is denoted by $\sin^{-1}$, with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

Also

$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$

*Definition.* Given $y$ where $-1 \le y \le 1$, $\sin^{-1} y$ is that $x$ which lies between $-\pi/2$ and $\pi/2$, $(-\pi/2 \le x \le \pi/2)$ and $\sin x = y$.

It may be shown as before that $\sin^{-1} y$ is derivable in the open interval $]-1, 1[$ and

$$\frac{d}{dy}\left(\sin^{-1} y\right) = \frac{1}{\sqrt{1-y^2}},\ y \ne \pm 1$$

**$\tan^{-1} y$.** Since $\tan x$ is strictly monotonic with domain $]-\pi/2, \pi/2[$ and range $]-\infty, \infty[$, the function is invertible, we have

$$y = \tan x \quad \Leftrightarrow \quad x = \tan^{-1} y$$

so that $\tan^{-1} y$ is a function with domain $]-\infty, \infty[$ and range $]-\pi/2, \pi/2[$.

*Definition.* For any number $y$, $\tan^{-1} y$ is that $x$ which lies between $-\pi/2$ and $\pi/2\,(-\pi/2 < x < \pi/2)$ and $\tan x = y$.

It may be seen that

$$\frac{d}{dy}\tan^{-1} y = \frac{1}{1+y^2},\quad \forall\, y$$

## 5. FUNCTIONAL EQUATIONS

In this section we shall discuss the solutions of certain functional equations which, together with continuity, suffice to characterize the so-called elementary functions.

First we note that the continuous function $f(x) = cx,\ x \in \mathbf{R}$, $c$ being constant, satisfies the *functional equation*,

$$f(x+y) = f(x) + f(y),\ \forall\, x, y \in \mathbf{R} \qquad ...(1)$$

The interesting fact is that *every continuous function satisfying the functional equation* (1) *is of this form*.

In order to prove this, let $f : \mathbf{R} \to \mathbf{R}$ be continuous and satisfies the functional equation (1). Then we have,

$$f(0) = 0, \text{ and } f(-x) = -f(x), \text{ for all } x \in \mathbf{R}$$

Also, for each positive integer $n$, we have by induction on $n$,

$$f(nx) = nf(x), \text{ for all } x \in \mathbf{R}$$

Replacing $x$ by $x/n$ in above, we obtain

$$f(x/n) = (1/n)\, f(x), \text{ for all } x \in \mathbf{R}$$

Thus, for any pair of integers $p, q$ ($q$ being positive), we have

$$f\left(\frac{p}{q}x\right) = \frac{p}{q} f(x), \text{ for all } x \in \mathbf{R}$$

In other words, $f(rx) = rf(x)$, for any rational number $r$.

If we put $x = 1$ and $f(1) = c$ in above, then we obtain

$f(r) = cr$, for every rational number $r$.

Now, let $\xi$ be any *irrational* number, then there always exists a sequence $\{r_n\}$ of rational numbers, converging to $\xi$.

$$\therefore \qquad f(\xi) = \lim_{x \to \infty} f(r_n) = \lim_{x \to \infty} cr_n = c\,\xi, \text{ by continuity of } f.$$

Hence, $f(x) = cx$, for all $x \in \mathbf{R}$.

**Note:** If $f : \mathbf{R} \to \mathbf{R}$ satisfies the functional equation (1), then the assumption that $f$ is continuous at a *single point* implies that $f$ is continuous *everywhere* on $\mathbf{R}$.

Now, for $a > 0$, the continuous function $f(x) = a^x,\ x \in \mathbf{R}$, satisfies the *functional equation*.

$$f(x+y) - f(x) f(y), \text{ for all } x,\ y \in \mathbf{R} \qquad ...(2)$$

1. *The only non-zero continuous function f on* $\mathbf{R}$ *that satisfies the functional equation* (2) *is the exponential function.*

   By hypothesis $f(x) \neq 0$, for any $x$. Moreover, from (2), we obtain

$$f(x) = [f(x/2)]^2 > 0, \text{ for all } x \in \mathbf{R}.$$

We may, therefore, take the logarithm of $f(x)$, (say, to the base $e$), and so taking $g(x) = \log_e f(x)$, $x \in \mathbf{R}$, it follows that $g(x+y) = g(x) + g(y)$, $\forall\ x, y \in \mathbf{R}$ and $g$, being composition of two continuous functions $f(x)$ and $\log_e x$, is itself continuous on $\mathbf{R}$.

Thus $g(x) = cx$, $\forall\ x \in \mathbf{R}$, $c$ being constant.

Hence $f(x) = e^{cx} = (e^c)^x = a^x$, for all $x \in \mathbf{R}$.

The continuous function $f(x) = \log_a x$ $(a > 0, a \neq 1)$, $x > 0$, satisfies the *functional equation*

$$f(xy) = f(x) + f(y), \text{ for all } x, y > 0 \qquad \text{...(3)}$$

This relation, together with continuity, is enough to characterize the logarithmic function.

2. *The only non-zero function f continuous on* $\mathbf{R}^+$ *that satisfies the functional equation* (3) *is the logarithmic function.*

For $x > 0$, we put $x = e^t$, *i.e.*, $t = \log_e x$, where $t \in \mathbf{R}$.

Moreover, we define another function $g$ on $\mathbf{R}$ by, $g(t) = f(e^t)$, $t \in \mathbf{R}$.

Then $g$ is continuous on $\mathbf{R}$ and satisfies, for all $t$, $s \in \mathbf{R}$,

$$g(t+s) = f(e^{t+s}) = f(e^t e^s) = f(e^t) + f(e^s) = g(t) + g(s),$$

and so $g(t) = ct$, $t \in \mathbf{R}$, $c$ being constant. Thus $f(x) = c \log e^x$, $x \in \mathbf{R}^+$. But $c$ cannot be zero, since $f$ is non-zero. Therefore, by taking $a = e^{1/c}$, we get

$$f(x) = \log_a x, x \in \mathbf{R}^+ \ (a > 0, a \neq 1).$$

The proofs of the following are left as an exercise to the reader.

3. *The only non-zero continuous functions on* $\mathbf{R}$ *that satisfy the functional equation,*

$$f(x+y) + f(x-y) = 2f(x)f(y), \ \forall\ x, y \in \mathbf{R}$$

*are the trigonometric and the hyperbolic cosines.*

4. *The only non-zero continuous functions on* $\mathbf{R}$ *that satisfy the functional equation,*

$$f(x+y)f(x-y) = (f(x))^2 - (f(y))^2, \text{ for all } x, y \in \mathbf{R}$$

*are the scalar* $(f(x) = cx)$, *the trigonometric and the hyperbolic sines.*

## 6. FUNCTIONS OF BOUNDED VARIATION

In this section we shall discuss the concept of the functions of bounded variation. The concept is closely associated to that of monotonic functions and has wide application in Mathematics. Presently we shall use these functions in Riemann-Stieltjes integrals and Fourier series.

A finite set $P$ of points, $x_0, x_1, x_2, \ldots, x_n$, where

$$a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b$$

is called a *partition* of the interval $[a, b]$. Clearly any number of partition of $[a, b]$ are possible.

The intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are the sub-intervals of the partition. The $i$th sub-interval $[x_{i-1}, x_i]$, as also its length $x_i - x_{i-1}$, is denoted by $\Delta x_i$.

Let a function $f$ be defined on an interval $[a, b]$ and $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a, b]$. Consider the sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

Since one such sum corresponds to each partition of $[a, b]$, the set of these sums is infinite. *If this set of sums is bounded above, the function f is said to be of bounded variation and the supremum of the set is called the total variation of f on* $[a, b]$, *and is denoted by the symbol,* $V(f, a, b)$. Thus

$$V(f, a, b) = \text{Sup} \sum \left| f(x_i) - f(x_{i-1}) \right|,$$

the supremum being taken over all partitions of $[a, b]$.

If there is no scope for confusion and the interval in question is clear from the context, we shall abbreviate the symbol to $V(f)$.

Thus the function $f$ is said to be of bounded variation on $[a, b]$ if and only if its total variation is finite, *i.e.*,

$$V(f, a, b) < +\infty$$

**Note:** Since for $x \le c \le y$,

$$|f(y) - f(x)| \le |f(y) - f(c)| + |f(c) - f(x)|$$

the sum $\Sigma \left| f(x_i) - f(x_{i-1}) \right|$ cannot decrease (it can, in fact only increase) by the introduction of additional points to a partition of the interval.

## 6.1 Illustrative Examples

**Example 1.** A bounded monotonic function is a function of bounded variation.

- Let $f$ be monotonic increasing on an interval $[a, b]$, and $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a, b]$. Then

$$\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{i=1}^{n} \left\{ f(x_i) - f(x_{i-1}) \right\} = f(b) - f(a)$$

$$\therefore \quad V(f, a, b) = \text{Sup}\, \Sigma \left| f(x_i) - f(x_{i-1}) \right| = f(b) - f(a)$$

Thus, a monotone increasing bounded function is of bounded variation on $[a, b]$.

Similarly, it may be shown that a monotone decreasing bounded function is of bounded variation, with total variation $= f(a) - f(b)$.

Thus for a bounded monotonic $f$,

$$V(f) = |f(b) - f(a)|$$

**Example 2.** To show that a continuous function may not be a function of bounded variation, consider a function $f$, where

$$f(x) = \begin{cases} x \sin \pi/x, & \text{when } 0 < x \le 1 \\ 0, & \text{when } x = 0 \end{cases}$$

- Clearly $f$ is continuous on $[0, 1]$.

Let us choose the partition

$$P = \left\{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \ldots, \frac{2}{5}, \frac{2}{3}, 1\right\}$$

$$\therefore \sum_i \left|f(x_i) - f(x_{i-1})\right| = \left|f(1) - f\left(\frac{2}{3}\right)\right| + \left|f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right)\right| + \ldots + \left|f\left(\frac{2}{2n+1}\right) - f(0)\right|$$

$$= \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(\frac{2}{5} + \frac{2}{7}\right) + \ldots + \left(\frac{2}{2n-1} + \frac{2}{2n+1}\right) + \frac{2}{2n+1}$$

$$= 4\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n+1}\right]$$

Since the infinite series $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ is divergent, its partial sums sequence $\{S_n\}$, where

$$S_n = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{2n+1}$$

is not bounded above.

Thus $\sum_i \left|f(x_i) - f(x_{i-1})\right|$ can be made arbitrarily large by taking $n$ sufficiently large.

Consequently, $V(f, 0, 1) \to \infty$ and so $f$ is not of bounded variation on [0, 1].

**Remark:** It may also be seen that a function of bounded variation is not necessarily continuous. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer not greater than $x$, is a function of bounded variation on [0, 2] but is not continuous.

**Example 3.** If the derivative $f'$ exists and is bounded on $[a, b]$, then the function $f$ is of bounded variation on $[a, b]$.

- Since $f'$ is bounded on $[a, b]$, therefore there exists $K$ such that

$$|f'(x)| \le K, \ \forall\, x \in [a, b]$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$.

$$\therefore \quad \sum_i \left|f(x) - f(x_{i-1})\right| = \sum_i (x_i - x_{i-1})\,|f'(\xi)|, \ \xi \in \left]a, b\right[$$

$$\le K\,(b-a)$$

$\Rightarrow$ $V(f, a, b)$ is finite and therefore $f$ is of bounded variation.

**Note:** Boundedness of $f'$ is a sufficient condition. It is not necessary.

**Example 4.** A function of bounded variation is necessarily bounded.

- Let a function $f$ be of bounded variation on $[a, b]$. For any $x \in [a, b]$, consider the partition $\{a, x, b\}$, consisting of just three points. Now

$$|f(x) - f(a)| + |f(b) - f(x)| \leq V(f, a, b)$$

$\Rightarrow$ $$|f(x) - f(a)| \leq V(f, a, b)$$

Again

$$|f(x)| = |f(a) + f(x) - f(a)|$$
$$\leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + V(f, a, b)$$

$\Rightarrow$ $f$ is bounded on $[a, b]$.

## 6.2 Some Properties of Functions of Bounded Variation

We shall see later (§ 6.3, Th. 3) that there is a close relation between functions of bounded variation and monotonic functions but there is one difference which is worthy of note. The functions of bounded variation are closed with respect to the arithmetic operations of addition and multiplication whereas the sum or the product of two monotonic functions need not be monotonic. For example, $x$ and $x^2$ are monotonic in $[0, 1]$, but $x - x^2$ is not. Similarly, $x$ is monotonic in $[-1, 1]$ but $x^2$ is not. In this section, we shall consider some properties of the functions of bounded variation, and in particular the arithmetic operations of addition and multiplication on them.

1. *The sum (difference) of two functions of bounded variation is also of bounded variation.*

Let $f$ and $g$ be two functions of bounded variation on $[a, b]$.

For any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$, we have

$$\sum_i |(f+g)(x_i) - (f+g)(x_{i-1})| = \sum_i |\{f(x_i) + g(x_i)\} - \{f(x_{i-1}) + g(x_{i-1})\}|$$
$$\leq \sum_i |f(x_i) - f(x_{i-1})| + \sum_i |g(x_i) - g(x_{i-1})|$$
$$\leq V(f, a, b) + V(g, a, b)$$

$\Rightarrow$ $$V(f+g, a, b) \leq V(f, a, b) + V(g, a, b)$$

and $(f+g)$ is of bounded variation.

Similarly, it may be shown that $(f-g)$ is of bounded variation and its total variation,

$$V(f-g) \leq V(f) + V(g)$$

**Corollary.** *If $f$ and $g$ are monotonic increasing on $[a, b]$, then $f - g$ is of bounded variation on $[a, b]$.*

The converse of the corollary is also true (see § 6.3).

**Note:** If $C$ is a constant, the sums $\Sigma |f(x_i) - f(x_{i-1})|$ and therefore the total variation function, $V(f)$ are same for $f$ and $f - C$.

2. *The product of two functions of bounded variation is also of bounded variation.*

Let $f$ and $g$ be two functions of bounded variation on $[a, b]$.

Evidently $f$ and $g$ are bounded and therefore a number $k$ exists such that

$$|f(x)| \le k, |g(x)| \le k, \ \forall x \in [a, b]$$

For any partition $\{a = x_0, x_1, \ldots, x_n = b\}$, we have

$$\sum_i |(fg)(x_i) - (fg)(x_{i-1})| = \sum_i |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|$$

$$= \sum_i |f(x_i)\{g(x_i) - g(x_{i-1})\} + g(x_{i-1})\{f(x_i) - f(x_{i-1})\}|$$

$$\le \sum_i \left[|f(x_i)||g(x_i) - g(x_{i-1})| + |g(x_{i-1})||f(x_i) - f(x_{i-1})|\right]$$

$$\le k \sum_i |g(x_i) - g(x_{i-1})| + k \sum_i |f(x_i) - f(x_{i-1})|$$

$$\le kV(g) + kV(f)$$

$\Rightarrow$ the function $(fg)$ is of bounded variation on $[a, b]$.

**Note:** Theorems like the above, could not be applied to quotients of functions because the reciprocal of a function of bounded variation need not be of bounded variation. For example, if $f(x) \to 0$ as $x \to x_0$, then $1/f$ will not be bounded and therefore cannot be of bounded variation on any interval containing $x_0$. Therefore, to consider quotient, we avoid functions whose values become arbitrarily close to zero.

3. *If $f$ is a function of bounded variation on $[a, b]$ and if there exists a positive number $k$ such that $|f(x)| \ge k$, for all $x \in [a, b]$, then $1/f$ is also of bounded variation on $[a, b]$.*

For any partition $\{a = x_0, x_1, \ldots, x_n = b\}$, we have

$$\sum_i \left|\left(\frac{1}{f}\right)(x_i) - \left(\frac{1}{f}\right)(x_{i-1})\right| = \sum_i \left|\frac{1}{f(x_i)} - \frac{1}{f(x_{i-1})}\right|$$

$$= \sum_i \left|\frac{f(x_{i-1}) - f(x_i)}{f(x_i)f(x_{i-1})}\right|$$

$$\le \frac{1}{k^2} \Sigma |f(x_i) - f(x_{i-1})|$$

$$\le V(f, a, b)/k^2$$

$\Rightarrow$ $1/f$ is of bounded variation on $[a, b]$.

4. *If $f$ is of bounded variation on $[a, b]$ then it is also of bounded variation on $[a, c]$ and $[c, b]$, where $c$ is a point of $[a, b]$, and conversely. Also*

$$V(f, a, b) = V(f, a, c) + V(f, c, b)$$

(i) Let, first, $f$ be of bounded variation on $[a, b]$.

Let $P_1 = \{a = x_0, x_1, \ldots, x_m = c\}$

and

$$P_2 = \{c = y_0, y_1, \ldots, y_n = b\}$$

be any two partitions of $[a, c]$ and $[c, b]$ respectively. Evidently,

$$P = P_1 \cup P_2 = \{a = x_0, \ldots, x_m, y_0, \ldots, y_n = b\}$$

is a partition of $[a, b]$.

We have

$$\left\{\sum_{i=1}^{m} \left|f(x_i) - f(x_{i-1})\right| + \sum_{i=1}^{n} \left|f(y_i) - f(y_{i-1})\right|\right\} \le V(f, a, b)$$

$$\Rightarrow \quad \sum_{i=1}^{m} \left|f(x_i) - f(x_{i-1})\right| \le V(f, a, b)$$

and

$$\sum_{i=1}^{n} \left|f(y_i) - f(y_{i-1})\right| \le V(f, a, b)$$

$\Rightarrow$ $f$ is of bounded variation on $[a, c]$ and $[c, b]$ both.

(ii) Let, now, $f$ be of bounded variation on $[a, c]$ and $[c, b]$ both.

Let $P = \{a = z_0, z_1, z_2, \ldots, z_n = b\}$ be any partition of $[a, b]$. If it does not contain the point $c$, let us consider the partition $P^* = P \cup \{c\}$. Let $c \in \Delta z_r$, *i.e.*, $z_{r-1} \le c \le z_r$, $r < n$.

We have

$$\begin{aligned}
\sum_{i=1}^{n} \left|f(z_i) - f(z_{i-1})\right| &= \sum_{i=1}^{r-1} \left|f(z_i) - f(z_{i-1})\right| + \left|f(z_r) - f(z_{r-1})\right| \\
&\quad + \sum_{i=r+1}^{n} \left|f(z_i) - f(z_{i-1})\right| \\
&\le \left\{\sum_{i=1}^{r-1} \left|f(z_i) - f(z_{i-1})\right| + \left|f(c) - f(z_{r-1})\right|\right\} \\
&\quad + \left\{\left|f(z_r) - f(c)\right| + \sum_{i=r+1}^{n} \left|f(z_i) - f(z_{i-1})\right|\right\} \\
&\le V(f, a, c) + V(f, c, b)
\end{aligned}$$

$\Rightarrow$ $f$ is of bounded variation on $[a, b]$ if it is of bounded variation on $[a, c]$ and $[c, b]$ both, and then

$$V(f, a, b) \le V(f, a, c) + V(f, c, b) \qquad \ldots(1)$$

(iii) Let $\varepsilon > 0$ be an arbitrary number.

Since $V(f, a, c)$ and $V(f, c, b)$ are the total variations of $f$ on $[a, c]$ and $[c, b]$ respectively, there exist partitions

$$P_1 = \{a = x_0, x_1, x_2, \ldots, x_m = c\}$$
$$P_2 = \{c = y_0, y_1, y_2, \ldots, y_n = b\}$$

of $[a, c]$ and $[c, b]$, respectively, such that

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| > V(f, a, c) - \frac{1}{2}\varepsilon$$

$$\sum_{i=1}^{n} |f(y_i) - f(y_{i-1})| > V(f, c, b) - \frac{1}{2}\varepsilon \quad \ldots(2)$$

From (2) and (3), we have ...(3)

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |f(y_i) - f(y_{i-1})| > V(f, a, c) + V(f, c, b) - \varepsilon$$

$$\Rightarrow V(f, a, b) \geq V(f, a, c) + V(f, c, b) - \varepsilon$$

But since $\varepsilon$ is an arbitrary positive number, we get

$$V(f, a, b) \geq V(f, a, c) + V(f, c, b)$$

Thus, from equation (1) and (4), we get ...(4)

$$V(f, a, b) = V(f, a, c) + V(f, c, b)$$

## 6.3 Variation Function

Let $f$ be a function of bounded variation on $[a, b]$ and $x$, a point of $[a, b]$. Then the total variation of $f$, $V(f, a, x)$ on $[a, x]$, which clearly is a function of $x$, is called the *total variation function* or simply the *variation function* of $f$ and is denoted by $v_f(x)$, and when there is no scope for confusion, simply by $v(x)$. Thus,

$$v_f(x) = V(f, a, x), \; a \leq x \leq b$$

If $x_1, x_2$ are two points of $[a, b]$ such that $x_2 > x_1$, then

$$0 \leq |f(x_2) - f(x_1)| \leq V(f, x_1, x_2)$$
$$= V(f, a, x_2) - V(f, a, x_1) = v_f(x_2) - v_f(x_1)$$
$$\Rightarrow \quad v_f(x_2) \geq v_f(x_1)$$

*i.e.*, $v_f(x)$ is a monotone increasing function on $[a, b]$.

**Remark:** If $f$ is of bounded variation on $[a, b]$, then $v_f \pm f$ is a monotone increasing function on $[a, b]$.

The following theorem characterizes the functions of bounded variation.

**Theorem 3.** ***Jordan Theorem.*** *A function of bounded variation is expressible as a difference of two monotone increasing functions.*

*or*

*If $f$ is a function of bounded variation on $[a, b]$, then there exist monotone increasing functions $p$ and $q$ on $[a, b]$ such that for $a \leq x \leq b$,*

$$f(x) = p(x) - q(x) \quad \text{...(1)}$$

$$v_f(x) = p(x) + q(x) \quad \text{...(2)}$$

Let us define $p$ and $q$ by

$$p = \frac{1}{2}(v_f + f)$$

$$q = \frac{1}{2}(v_f - f)$$

so that equation (1) and (2) hold.

Now for $x_2 > x_1$, we have

$$p(x_2) - p(x_1) = \frac{1}{2}[v_f(x_2) + f(x_2)] - \frac{1}{2}[v_f(x_1) + f(x_1)]$$

$$= \frac{1}{2}[v_f(x_2) - v_f(x_1)] + \frac{1}{2}[f(x_2) - f(x_1)]$$

$$= \frac{1}{2}[V(f, x_1, x_2) + \{f(x_2) - f(x_1)\}]$$

But, since $V(f, x_1, x_2) \geq |f(x_2) - f(x_1)|$, we get

$$p(x_2) - p(x_1) \geq 0 \quad \Rightarrow \quad p(x_2) \geq p(x_1)$$

so that $p$ is monotone increasing on $[a, b]$.

Again, we have

$$q(x_2) - q(x_1) = \frac{1}{2}[v_f(x_2) - f(x_2)] - \frac{1}{2}[v_f(x_1) - f(x_1)]$$

$$= \frac{1}{2}[V(f, x_1, x_2) - \{f(x_2) - f(x_1)\}] \geq 0$$

$$\Rightarrow \quad q(x_2) \geq q(x_1)$$

so that $q$ is monotone increasing on $[a, b]$.

Hence the result.

With the help of the result proved above and that of § 6.1, Example 1 *and* § 6.2, we may state that *a function is of bounded variation on an interval if and only if it can be expressed as the difference of two monotone increasing functions.*

**Note:** The functions $p$ and $q$ are respectively called the *positive* and the *negative* variation function of $f$.
We shall now prove that the variation function of a continuous function of bounded variation is itself continuous, and conversely.

**Theorem 4.** ***Variation function of a continuous function.*** *The variation function of a function $f$ of bounded variation is continuous if and only if $f$ is a continuous function.*

*Necessary.* Let the variation function $v(x)$ of $f$ be continuous at a point $c$ of $[a, b]$.

Let $\varepsilon > 0$ be an arbitrary number.

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Because of continuity of $v(x)$ at $c$, $\delta > 0$ exists such that

$$|v(x) - v(c)| < \varepsilon, \text{ for } |x - c| < \delta \quad ...(1)$$

Also

$$|f(x) - f(c)| \le v(x) - v(c) \quad \text{if } x > c \quad ...(2)$$

$$|f(x) - f(c)| \le v(c) - v(x) \quad \text{if } x < c \quad ...(3)$$

Hence, from equation (1), (2) and (3), we get

$$|f(x) - f(c)| < \varepsilon, \text{ for } |x - c| < \delta$$

which implies that $f$ is continuous at $c$.

*Sufficient.* Let now $f$ be continuous at $c$, so that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{1}{2}\varepsilon, \text{ for } |x - c| < \delta$$

Again since $V(f, c, b)$ is total variation of $f$ on $[c, b]$, there exists a partition $P = \{c = x_0, x_1, ..., x_n = b\}$ of $[c, b]$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > V(f, c, b) - \frac{1}{2}\varepsilon \quad ...(4)$$

Let us assume that the length of the first sub-interval $x_1 - c$ is less than $\delta$, for, otherwise we make it so, by introducing additional points to the partition $P$. Also, (§ 6, note) the sum $\sum_i |f(x) - f(x_{i-1})|$ cannot decrease by the introduction of more points to $P$, so that the inequality (4) remains unaffected.

Thus, let $0 < x_1 - c < \delta$ so that

$$|f(x_1) - f(c)| < \frac{1}{2}\varepsilon \quad ...(5)$$

Again (4) gives on using (5),

$$V(f, c, b) - \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| \le \frac{1}{2}\varepsilon + V(f, x_1, b)$$

$\Rightarrow$ $\quad V(f, c, b) - V(f, x_1, b) < \varepsilon$

or

$$V(f, c, x_1) < \varepsilon$$

$\therefore$ $\quad v(x_1) - v(c) = V(f, a, x_1) - V(f, a, c) = V(f, c, x_1) < \varepsilon$

Thus

$$-\varepsilon < 0 < v(x_1) - v(c) < \varepsilon, \text{ when } 0 < x_1 - c < \delta$$

$\Rightarrow$ $\quad \lim_{x \to c+} v(x) = v(c)$

Similarly, it can be shown that

$$\lim_{x \to c-} v(x) = v(c)$$

Hence, $v(x)$ is continuous at $c$.

Again, since $c$ is any point of $[a, b]$, we deduce that $v_f(x)$ is continuous on $[a, b]$ if and only if $f$ is continuous on $[a, b]$.

**Corollary.** Continuity of $f$ implies the continuity of $v_f(x)$ and therefore of the positive and negative variation functions $p$ and $q$ conversely.

Thus, *a continuous function is of bounded variation if and only if it can be expressed as a difference of two continuous monotonic increasing functions.*

**Example 5.** Determine whether or not the following function $f$ is of bounded variation on $[0, 1]$.

$$f(x) = x^2 \sin(1/x), \text{ if } x \neq 0,\ f(0) = 0$$

- Clearly, $f$ is continuous on $[0, 1]$, and

$$f'(x) = 2x \sin(1/x) - \cos(1/x), \text{ if } x \neq 0$$

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x} = 0$$

Also

$$|f'(x)| \leq 3, \text{ for all } x \in [0, 1]$$

Thus we find that $f'$ exists and is bounded on $[0, 1]$. So that (by the hypothesis of example 3, § 6.1) the function $f$ is of bounded variation on $[0, 1]$.

**Example 6.** $f$ is a function of bounded variation on $[a, b]$, and $p$ and $q$ are its positive and negative variation functions. If $p_1$, $q_1$ are two monotone increasing functions on $[a, b]$ such that $f = p_1 - q_1$, then show that $V(p) \leq V(p_1)$ and $V(q) \leq V(q_1)$, where $V$ denotes total variation on $[a, b]$.

- With usual notation, we have

$$p = \frac{1}{2}(v_f + f), \text{ and } q = \frac{1}{2}(v_f - f)$$

Given that $f = p_1 - q_1$, therefore for $x \in [a, b]$, we have

$$2p(x) = V(f, a, x) + p_1(x) - q_1(x)$$

$$\therefore \quad 2p(b) - 2p(a) = V(f, a, b) + p_1(b) - p_1(a) - \{q_1(b) - q_1(a)\}$$

*i.e.,*

$$2V(p) = V(f) + V(p_1) - V(q_1) \quad \text{...(1)}$$

But

$$V(f) = V(p_1 - q_1) \leq V(p_1) + V(q_1) \quad \text{...(2)}$$

From (1) and (2), it follows that

$$2V(p) \leq 2V(p_1), \text{ and so } V(p) \leq V(p_1)$$

Similarly, $V(q) \leq V(q_1)$.

**Example 7.** Compute the positive, negative and the total variation functions of

$$f(x) = 3x^2 - 2x^3, \text{ for } -2 \leq x \leq 2$$

- Here

$$f'(x) = 6x(1 - x)$$

which vanishes for $x = 0, 1$ and is negative for $-2 \leq x < 0$ or $1 < x \leq 2$, and positive for $0 < x < 1$.

Hence, for $-2 \leq x \leq 0$, (when $f$ is monotone decreasing)

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$$v_f(x) = V(f, -2, x) = f(-2) - f(x)$$
$$= 28 - 3x^2 + 2x^3$$

$$\therefore \quad V(f, -2, 0) = 28$$

For $0 \le x \le 1$ (when $f$ is monotone increasing)

$$V(f, 0, x) = f(x) - f(0) = 3x^2 - 2x^3$$
$$\therefore \quad v_f(x) = V(f, -2, x) = V(f, -2, 0) + V(f, 0, x)$$
$$= 28 + 3x^2 - 2x^3$$

and

$$V(f, -2, 1) = 29$$

For $1 \le x \le 2$ (when $f$ is monotone decreasing)

$$V(f, 1, x) = f(1) - f(x) = 1 - 3x^2 + 2x^3$$
$$\therefore \quad v_f(x) = V(f, -2, x) = V(f, -2, 1) + V(f, 1, x)$$
$$= 30 - 3x^2 + 2x^3$$

Thus the total variation function on $-2 \le x \le 2$ is defined as:

$$v_f(x) = \begin{cases} 28 - 3x^2 + 2x^3, & \text{for } -2 \le x \le 0 \\ 28 + 3x^2 - 2x^3, & \text{for } 0 \le x \le 1 \\ 30 - 3x^2 + 2x^3, & \text{for } 1 \le x \le 2 \end{cases}$$

Since the positive variation function $p$ is defined as:

$$p(x) = \frac{1}{2}\{v_f(x) + f(x)\}$$

$$\therefore \quad p(x) = \begin{cases} 14, & \text{for } -2 \le x \le 0 \\ 14 + 3x^2 - 2x^3, & \text{for } 0 \le x \le 1 \\ 15, & \text{for } 1 \le x \le 2 \end{cases}$$

Similarly, the negative variation function,

$$q(x) = \begin{cases} 14 - 3x^2 + 2x^3, & \text{for } -2 \le x \le 0 \\ 14, & \text{for } 0 \le x \le 1 \\ 15 - 3x^2 + 2x^3, & \text{for } 1 \le x \le 2 \end{cases}$$

**Example 8.** Compute the positive, negative and the total variation functions of $f$, where

$$f(x) = [x] - x \quad (0 \le x \le 2)$$

- The function $f$ is monotone decreasing from 0 to 1 and from 1 to 2 and has discontinuities at 1 and 2. It may be restated as

$$f(x) = \begin{cases} -x, & \text{for } 0 \le x < 1 \\ 1 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x = 1, 2 \end{cases}$$

For $0 \le x < 1$,

$$v_f(x) = V(f, 0, x) = f(0) - f(x) = x \equiv [x] + x$$

$$V(f, 0, 1) = \underset{0<x<1}{\text{l.u.b.}} \left[ V(f, 0, x) + V(f, x, 1) \right]$$

$$= \underset{0<x<1}{\text{l.u.b.}} \left[ x + |f(1) - f(x)| \right]$$

$$= \underset{0<x<1}{\text{l.u.b.}} \left[ x + |0 - x| \right] = 2$$

$$\therefore \quad \left[ v_f(x) \right]_{x=1} = V(f, 0, 1) = 2$$

For $1 \le x < 2$,

$$V(f, 1, x) = f(1) - f(x) = -1 + x$$

$$\therefore \quad v_f(x) = V(f, 0, x) = V(f, 0, 1) + V(f, 1, x)$$

$$= 1 + x \equiv [x] + x$$

$$V(f, 1, 2) = \underset{1<x<2}{\text{l.u.b.}} \left[ V(f, 1, x) + V(f, x, 2) \right]$$

$$= \underset{1<x<2}{\text{l.u.b.}} \left[ -1 + x + |f(2) - f(x)| \right]$$

$$= \underset{1<x<2}{\text{l.u.b.}} \left[ -1 + x + |0 - (1 - x)| \right] = 2$$

$$\therefore \quad \left[ v_f(x) \right]_{x=2} = V(f, 0, 2) = V(f, 0, 1) + V(f, 1, 2) = 4$$

Thus the total variation function is defined on $[0, 2]$ as

$$v_f(x) = [x] + x, \text{ for } 0 \le x \le 2$$

Accordingly the positive and negative variation functions, $p$ and $q$, are defined as:

$$p(x) = [x], \text{ for } 0 \le x \le 2$$

$$q(x) = x, \text{ for } 0 \le x \le 2$$

**Ex. 1.** Show that $\sin x$ and $\cos x$ are of bounded variation over a finite interval.

**Ex. 2.** Show that a polynomial function is of bounded variation over any finite interval. Describe a method for finding the total variation of $f$ on $[a, b]$, if the zeros of the derivative $f'$ are known.

**Ex. 3.** Determine whether or not $f$ is of bounded variation on $[0, 1]$, where,

(*i*) $f(x) = \sqrt{x} \sin(1/x)$, if $x \ne 0$, $f(0) = 0$, and

(*ii*) $f(x) = x^2 \sin(1/x^2)$, if $x \ne 0$, $f(0) = 0$.

## 7. VECTOR-VALUED FUNCTIONS

### 7.1 The Euclidean Space $\mathbf{R}^n$

The set of all ordered $n$-tuples of real numbers is denoted by $\mathbf{R}^n$. Thus

$$\mathbf{R}^n = \left\{ (t_1, t_2, \ldots, t_n) : t_j \in \mathbf{R} \text{ for } j = 1, 2, \ldots, n \right\}$$

The $n$-tuples $(t_1, t_2, ..., t_n)$, where $t_1, t_2, ..., t_n$ are real numbers, are *members* or *points* of $\mathbf{R}^n$, and $t_1$, $t_2, ..., t_n$ the *components* or the *coordinates* of the points of $\mathbf{R}^n$. Members of $\mathbf{R}^n$ will be denoted by bold faced letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ..., so that each of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. stands for an ordered $n$-tuple of real numbers.

$\mathbf{0}$ denotes the point $(0, 0, ..., 0)$.

If $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be two points of $\mathbf{R}^n$, then we define *addition* and *scalar multiplication* by

(*i*) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$,

(*ii*) $c\mathbf{x} = (cx_1, cx_2, ..., cx_n)$, $c \in \mathbf{R}$.

(*iii*) Distance between two points $\mathbf{x}$ and $\mathbf{y}$, denoted symbolically by $d(\mathbf{x}, \mathbf{y})$, is defined as

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

$$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

If we define a non-negative number $|\mathbf{x}|$, where

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

then $|\mathbf{x}|$ denotes the distance between $\mathbf{x}$ and $0$, and is called the Norm of $\mathbf{x}$. Also

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$$

The set $\mathbf{R}^n$ equipped with the properties (*i*), (*ii*) and (*iii*), mentioned above, is called the **Euclidean space $\mathbf{R}^n$** of $n$ dimension.

## Cauchy-Schwarz Inequality

If $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be two points of $\mathbf{R}^n$, then

$$\left| \sum_{i=1}^{n} x_i y_i \right| \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2}$$

We note that

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left( x_i y_j - x_j y_i \right)^2$$

$$= \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2 - 2 \sum_{i=1}^{n} x_i y_i \sum_{j=1}^{n} y_j x_j + \sum_{j=1}^{n} x_j^2 \sum_{i=1}^{n} y_i^2$$

$$= 2 \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - 2 \left( \sum_{i=1}^{n} x_i y_i \right)^2$$

It follows that

$$\left( \sum_{i=1}^{n} x_i y_i \right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

Taking positive square root of both sides, the Cauchy-Schwarz inequality is established.

Ex. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$, show that

(i) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \geq 0$,

(ii) $d(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$,

(iii) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ (Triangle inequality).

## 7.2 Vector-Valued Functions

If $f_1, f_2, \ldots, f_m$ be $m$ real-valued functions on an interval $[a, b]$, then the corresponding mapping $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ of $[a, b]$ into $\mathbf{R}^m$ is called a real vector valued function* of $\mathbf{R}$ into $\mathbf{R}^m$.

The domain of such a function is a subset of $\mathbf{R}$ and range, a subset of the $m$-dimensional space $\mathbf{R}^m$. The functions $f_1, f_2, \ldots, f_m$ are called its components.

Vector valued functions with domain $\mathbf{R}^k$ and range, $\mathbf{R}^n$ are also defined in a similar way.

The vector valued functions satisfy the following relations.

If $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, $\mathbf{g} = (g_1, g_2, \ldots, g_m)$ be two vector valued functions, then

1. $\mathbf{f} = \mathbf{g}$ if and only if $f_j = g_j$, for $j = 1, 2, \ldots, m$
2. $\mathbf{f} + \mathbf{g} = (f_1 + g_1, f_2 + g_2, \ldots, f_m + g_m)$
3. $a\mathbf{f} = (af_1, af_2, \ldots, af_m)$, $a$ any real number
   $0\mathbf{f} = (0, 0, \ldots, 0) = \mathbf{0}$
4. $\mathbf{f}$ is said to be continuous if and only if each $f_j$ is continuous
5. $\mathbf{f}$ is differentiable when each $f_j$ is differentiable, and then

$$\mathbf{f}' = (f_1', f_2', \ldots, f_m')$$

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## 7.3 Vector-Valued Functions of Bounded Variation

Let $\mathbf{f}$ be a vector-valued function with domain $[a, b]$ and range $\mathbf{R}^m$. If $P = \{a = x_0, x_1, \ldots, x_n = b\}$ is any partition of $[a, b]$ and $\Delta\mathbf{f}_i = \mathbf{f}(x_i) - \mathbf{f}(x_{i-1})$ denotes the oscillation of $\mathbf{f}$ on $\Delta x_i$, then the *total variation* of $\mathbf{f}$ on $[a, b]$ is defined as

$$V(\mathbf{f}, a, b) = \text{Sup} \sum_{i=1}^{n} |\Delta\mathbf{f}_i| = \text{Sup} \sum_{i=1}^{n} \left|\mathbf{f}(x_i) - \mathbf{f}(x_{i-1})\right|,$$

the supremum being taken over all partitions of $[a, b]$.

$\mathbf{f}$ is said to be of *bounded variation* on $[a, b]$ if and only if $V(\mathbf{f}, a, b) < +\infty$, *i.e.*, $V(\mathbf{f}, a, b)$ is finite.

The function

$$v_{\mathbf{f}}(x) = V(\mathbf{f}, a, x),\ a \leq x \leq b$$

is called the *total variation function* of $\mathbf{f}$.

* The reader is already familiar with the vector functions in space (3-dimensional), where a vector function $P$ can be expressed as $P = iP_1 + jP_2 + kP_3$ or equivalently as an ordered triplet $(P_1, P_2, P_3)$. The definition given here is a generalisation of the same.

**Note:** $V(\mathbf{f}, a, b)$ or $v_{\mathbf{f}}(x)$ are non-negative and can be zero only when $\mathbf{f}(x_i) = \mathbf{f}(x_{i-1})$ for all $i$, that is only when $\mathbf{f}$ is a constant function.

We shall now prove some theorems for the vector valued functions of bounded variation. As many properties of such functions can be reduced to that of the real-valued functions, we shall give the outline of the proofs, details may be provided by the reader himself.

1. **f** *is of bounded variation on* $[a, b]$ *if and only if each component function* $f_j$ *is of bounded variation on* $[a, b]$.

For any partition $\{a = x_0, x_1, ..., x_n = b\}$ of $[a, b]$.

$$\left| f_j(x_i) - f_j(x_{i-1}) \right| \le \left| \mathbf{f}(x_i) - \mathbf{f}(x_{i-1}) \right| \le \sum_{j=1}^{m} \left| f_j(x_i) - f_j(x_{i-1}) \right|$$

If we add these inequalities for $i = 1, 2, ..., n$ and take the least upper bound over all partitions of $[a, b]$, we get

$$V(f_j, a, b) \le V(f, a, b) \le \sum_{j=1}^{m} V(f_j, a, b)$$

which proves the theorem.

**Corollary.** Since each real-valued function of bounded variation is necessarily bounded, it follows that each function $f_j$ and consequently **f** is bounded. Hence *a vector valued function of bounded variation is necessarily bounded.*

2. *If* $\mathbf{f}'$ *exists and is bounded on* $[a, b]$ *then* **f** *is of bounded variation on* $[a, b]$.

Since **f** is derivable, therefore each $f_j$ is derivable.

Also, if $\left|\mathbf{f}'(x)\right| \le M$, for all $x \in [a, b]$, then

$$\left| f_j'(x) \right| \le \left| \mathbf{f}'(x) \right| \le M$$

Hence for any partition $\{a = x_0, x_1, ..., x_n = b\}$, we have by Lagrange's Mean Value Theorem,

$$\left| f_j(x_i) - f_j(x_{i-1}) \right| = \left| x_i - x_{i-1} \right| \left| f_j'(\xi) \right|, \; x_{i-1} < \xi < x_i \le M\left(x_i - x_{i-1}\right)$$

$$\therefore \quad \sum_{i=1}^{n} \left| f_j(x_i) - f_j(x_{i-1}) \right| \le M \sum_{i=1}^{n} \left(x_i - x_{i-1}\right) = M(b-a)$$

Taking the supremum over all partition of $[a, b]$, we get

$$V(f_j, a, b) \le M(b-a)$$

so that each function $f_j$ is of bounded variation on $[a, b]$. Consequently (by 1) **f** is of bounded variation on $[a, b]$.

3. *If* **f** *and* **g** *are of bounded variation then* $\mathbf{f} \pm \mathbf{g}$ *is also of bounded variation, and*

$$V(\mathbf{f} \pm \mathbf{g}, a, b) \le V(\mathbf{f}, a, b) + V(\mathbf{g}, a, b)$$

Let $\mathbf{f} = (f_1, f_2, ..., f_m)$, $\mathbf{g} = (g_1, g_2, ..., g_m)$

Since **f** and **g** are both of bounded variation, therefore each $f_j$ and $g_j$ is of bounded variation over $[a, b]$.

$\Rightarrow$ each $f_j \pm g_j$ is of bounded variation on $[a, b]$

$\Rightarrow$ $\mathbf{f} \pm \mathbf{g}$ is of bounded variation on $[a, b]$

For any partition $\{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$, we have

$$\sum_{i=1}^{n}\left|(\mathbf{f} \pm \mathbf{g})(x_i)-(\mathbf{f} \pm \mathbf{g})(x_{i-1})\right|$$

$$=\sum_{i=1}^{n}\left|\{\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\} \pm \{\mathbf{g}(x_i)-\mathbf{g}(x_{i-1})\}\right|$$

$$\leq \sum_{i=1}^{n}\left|\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\right| + \sum_{i=1}^{n}\left|\mathbf{g}(x_i)-\mathbf{g}(x_{i-1})\right|$$

$$\leq V(\mathbf{f}, a, b) + V(\mathbf{g}, a, b)$$

$$\Rightarrow \quad V(\mathbf{f} \pm \mathbf{g}, a, b) \leq V(\mathbf{f}, a, b) + V(\mathbf{g}, a, b)$$

4. *For $a \leq x \leq y \leq b$ prove that*

$$V(\mathbf{f}, a, y) \leq V(\mathbf{f}, a, x) + V(\mathbf{f}, x, y) \qquad ...(1)$$

If $x = a$ or $y = x$, the result is trivial, for, $V(\mathbf{f}, x, x) = 0$.

Let $a < x < y \leq b$.

Let $\varepsilon > 0$ be given.

There exists a partition $P = \{a = x_0, x_1, ..., x_n = y\}$ of $[a, y]$ such that

$$V(\mathbf{f}, a, y) - \varepsilon \leq \sum_{P}\left|\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\right| \leq V(\mathbf{f}, a, y) \qquad ...(2)$$

where $\sum_{P}$ denotes the summation over all points of $P$.

If $x$ is not a member of $P$, we consider the partition $P^* = P \cup \{x\}$, for which (2) still holds. Evidently $P^*$ gives rise to the partitions $P_1$ and $P_2$ of $[a, x]$ and $[x, y]$ respectively such that $P^* = P_1 \cup P_2$.

Now from (2),

$$\sum_{P^*}\left|\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\right| \leq V(\mathbf{f}, a, y)$$

or

$$\sum_{P_1}\left|\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\right| + \sum_{P_2}\left|\mathbf{f}(x_i)-\mathbf{f}(x_{i-1})\right| \leq V(\mathbf{f}, a, y)$$

$$\Rightarrow \quad V(\mathbf{f}, a, x) + V(\mathbf{f}, x, y) \leq V(\mathbf{f}, a, y) \qquad ...(3)$$

From (2) and (3), for any $\varepsilon > 0$, we get

$$V(\mathbf{f}, a, y) - \varepsilon \leq V(\mathbf{f}, a, x) + V(\mathbf{f}, x, y) \leq V(\mathbf{f}, a, y)$$

and since $\varepsilon$ is arbitrary, result (1) follows.