MODERN ALGEBRA
IFS PYQs

2019

1. 1a

Q1. (a) Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. *8*

2. 2a

Q2. (a) Let I and J be ideals in a ring R. Then prove that the quotient ring $(I + J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. *10*

3. 3a

Q3. (a) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find the order of b. *10*

4. 4a

Q4. (a) Show that the smallest subgroup V of A_4 containing $(1, 2)(3, 4)$, $(1, 3)(2, 4)$ and $(1, 4)(2, 3)$ is isomorphic to the Klein 4-group. *10*

2018

5. 1a

1. *(a)* Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. 8

6. 2a

2. *(a)* Find all the homomorphisms from the group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$.

7. 2d

(d) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. 10

8. 3b

(b) Show by an example that in a finite commutative ring, every maximal ideal need not be prime. 10

9. 4c

(c) Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. 10

2017

10. 1a

1. (a) Prove that every group of order four is Abelian.

11. 2a

2. (a) Let G be the set of all real numbers except -1 and define $a*b = a + b + ab$ $\forall\ a, b \in G$. Examine if G is an Abelian group under $*$. 10

12. 2b

2. (b) Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh$ $\forall h \in H$ and $k \in K$. 10

13. 2c

2. (c) Let A be an ideal of a commutative ring R and
$B = \{x \in R : x^n \in A$ for some positive integer $n\}$.
Is B an ideal of R ? Justify your answer. 10

14. 2d

2. (d) Prove that the ring
$\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z},\ i = \sqrt{-1}\}$ of Gaussian integers is a Euclidean domain. 10

2016

15. 1a

1.(a) Prove that the set of all bijective functions from a non-empty set X onto itself is a group with respect to usual composition of functions. 8

16. 2a

2.(a) Show that any non-abelian group of order 6 is isomorphic to the symmetric group S_3. 15

17. 2b

2.(b) Let G be a group of order pq, where p and q are prime numbers such that $p > q$ and $q \nmid (p-1)$. Then prove that G is cyclic. 15

18. 2c

2.(c) Show that in the ring $R = \{a + b\sqrt{-5} \mid a, b \text{ are integers}\}$, the elements $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but $\alpha\gamma$ and $\beta\gamma$ have no g.c.d in R, where $\gamma = 7(1 + 2\sqrt{-5})$. 10

2015

19. 1a

Q1. (a) If in a group G there is an element a of order 360, what is the order of a^{220} ? Show that if G is a cyclic group of order n and m divides n, then G has a subgroup of order m. *10*

20. 2a

Q2. (a) If p is a prime number and e a positive integer, what are the elements 'a' in the ring $\mathbb{Z}_{p^e}$ of integers modulo p^e such that $a^2 = a$? Hence (or otherwise) determine the elements in $\mathbb{Z}_{35}$ such that $a^2 = a$. *14*

21. 3a

Q3. (a) What is the maximum possible order of a permutation in S_8, the group of permutations on the eight numbers {1, 2, 3, ..., 8} ? Justify your answer. (Majority of marks will be given for the justification). *13*

2014

22. 1a

(a) If G is a group in which $(a \cdot b)^4 = a^4 \cdot b^4$, $(a \cdot b)^5 = a^5 \cdot b^5$ and $(a \cdot b)^6 = a^6 \cdot b^6$, for all $a, b \in G$, then prove that G is Abelian. 8

23. 2a

2. *(a)* Let J_n be the set of integers mod n. Then prove that J_n is a ring under the operations of addition and multiplication mod n. Under what conditions on n, J_n is a field? Justify your answer. 10

24. 3a

3. *(a)* Let R be an integral domain with unity. Prove that the units of R and $R[x]$ are same. 10

2013

25. 1b

(b) Prove that if every element of a group (G, 0) be its own inverse, then it is an abelian group.

26. 2a

(a) Show that any finite integral domain is a field.

27. 2b

(b) Every field is an integral domain — Prove it.

28. 3b

(b) Prove that :

(i) the intersection of two ideals is an ideal.

(ii) a field has no proper ideals.

2012

29. 1b

(b) Show that every field is without zero divisor. 10

30. 2a

(a) Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. 13

31. 3c

(c) If R is an integral domain, show that the polynomial ring R[x] is also an integral domain. 14

2011

32. 1a

(a) Let G be a group, and x and y be any two elements of G. If $y^5 = e$ and $yxy^{-1} = x^2$, then show that $O(x) = 31$, where e is the identity element of G and $x \neq e$. 10

33. 1b

(b) Let Q be the set of all rational numbers. Show that

$$Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in Q\}$$

is a field under the usual addition and multiplication. 10

34. 2a

(a) Let G be the group of non-zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication. 13

35. 3b

(b) Let G be a group of order $2p$, p prime. Show that either G is cyclic or G is generated by $\{a, b\}$ with relations $a^p = e = b^2$ and $bab = a^{-1}$. 13

2010

36. 1a

(a) Let

$$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \, a \in \mathbb{R}, \, a \neq 0 \right\}$$

Show that G is a group under matrix multiplication. 10

37. 1b

(b) Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. 10

38. 2b

(b) Prove or disprove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. 13

39. 3a

(a) Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. 13

40. 4a

4. (a) Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1) + n \geq 0$. Show that the function $d_n : R - \{0\} \to S$, where S is the set of all negative integers defined by $d_n(a) = d(a) + n$ for all $a \in R - \{0\}$ is a Euclidean valuation. 13

2009

41. 1a

(a) Prove that a non-empty subset H of a group G is normal subgroup of $G \Leftrightarrow$ for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$. 10

42. 1d

(d) If G is a finite Abelian group, then show that $O(a, b)$ is a divisor of l.c.m. of $O(a)$, $O(b)$. 10

43. 2c

(c) Find the multiplicative inverse of the element

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

of the ring, M, of all matrices of order two over the integers. 13

44. 4a

4. (a) Show that $d(a) < d(ab)$, where a, b be two non-zero elements of a Euclidean domain R and b is not a unit in R. 13

45. 4b

(b) Show that a field is an integral domain and a non-zero finite integral domain is a field. 13