LINEAR ALGEBRA

: IFoS - 2018 :

① Given that $\text{adj} A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\det A = 2$. Find the matrix A.

→ WKT $A^{-1} = \dfrac{\text{adj} A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Then $A = (A^{-1})^{-1}$.

Now: $[A^{-1}|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 5/2 & 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1 \end{array}\right] \underset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1 \end{array}\right]$

$\underset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1 \\ 0 & 3/2 & 1/2 & -1 & 1 & 0 \end{array}\right] \underset{R_1 \to R_1 - 2R_2,\ R_3 \to R_3 - 3R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & -3 \end{array}\right]$

$\underset{R_1 \to R_1 - R_3,\ R_2 \to 2R_2 + R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & -3 \end{array}\right] \underset{R_3 \to -R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{array}\right]$

$\therefore\ A = (A^{-1})^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

② Prove that the eigen values of a Hermitian matrix are all real

→ Let A be a hermitian matrix. Let X be an eigen vector of A corresponding to eigen value λ. Then, we have.

$AX = \lambda X$

Premultiplying by tranjuage of X on both sides,

$X^\theta A X = \lambda X^\theta X$ —— ①

Taking tranjugate both sides, we have.

$(X^\theta A X)^\theta = (\lambda X^\theta X)^\theta \Rightarrow X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$

$\Rightarrow X^\theta A X = \bar{\lambda} X^\theta X$ —— ② $[(X^\theta)^\theta = X$ and $A^\theta = A \to A$ is hermitian$]$

① = ②: $\lambda X^\theta X = \bar{\lambda} X^\theta X \Rightarrow (\lambda - \bar{\lambda}) X^\theta X = 0$.

Since $X \neq 0 \Rightarrow X^\theta X \neq 0$ $\therefore\ \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$.

Hence λ = real. Hence, eigen values of hermitian matrix are real ①

③ Show that the matrices $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ are congruent.

→ By Sylvester's Theorem, two symmetric matrices are congruent iff they have the same rank and signature.

Now: $|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 1(6-1) + 1(-1-3) - 1(1+2)$
$= 5 - 4 - 3 = -2 \neq 0$

$\therefore \rho(A) = 3.$

$|B| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{vmatrix} = 1(-4) + 3(-6) = -22 \neq 0.$

$\therefore \rho(B) = 3.$

∴ The two matrices have the same rank.

Now Char. eqn of $A \Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-1] + 1[-1-(3-\lambda)] - 1(1+2-\lambda) = 0$

$\rightarrow (1-\lambda)[5 - 5\lambda + \lambda^2] - 4 + \lambda - 1 - 2 + \lambda = 0$

$\rightarrow 5 - 5\lambda - 5\lambda + 5\lambda^2 + \lambda^2 - \lambda^3 - 7 + 2\lambda = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 8\lambda + 2 = 0$

It has one negative and two positive roots.

Hence signature of $A = \cancel{3}\ 2-1 = 1$

Char. eqn of $B \Rightarrow |B - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 2-\lambda & 2 \\ 3 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(-\lambda) - 4] + 3[3(\lambda-2)] = 0$

$\rightarrow (1-\lambda)[\lambda^2 - 2\lambda - 4] + 9(\lambda - 2) = 0$

$\rightarrow \lambda^2 - \lambda^3 - 2\lambda + 2\lambda^2 - 4 + 4\lambda + 9\lambda - 2 = 0$

$\rightarrow \lambda^3 - 3\lambda^2 - 11\lambda + 6 = 0.$

It has one negative and two positive roots.

Hence, signature of $B = 2 - 1 = 1$

Hence, signature of A & B are the same.

Therefore A & B are congruent matrices.

④ Show that the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, $\alpha_3 = (0,-3,2)$ form a basis of $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of α_1, α_2 and α_3.

→ Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}$. Then, $|A| = 1[4+3] - 1[-3]$
$= 7+3 = 10 \neq 0$.
$\therefore \rho(A) = 3$.

Hence the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, $\alpha_3 = (0,-3,2)$ are L.I.

Also, wkt any set of three linearly independent vectors belonging to $\mathbb{R}^3$ is a basis of $\mathbb{R}^3$.

Hence $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

Now, let $S_1 = \{e_1, e_2, e_3\}$ where S is the standard basis of $\mathbb{R}^3$. Then $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

~~Now: $\alpha_1 = 1(1,0,-1) = 1e_1 + 0.e_2 + (-1)$~~

~~Now: $e_1 = (1,0,0)$.~~

Let $(x,y,z) \in \mathbb{R}^3$. Then $(x,y,z) = a\alpha_1 + b\alpha_2 + c\alpha_3$ where $a, b, c \in \mathbb{R}$.

Then $(x,y,z) = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$
$= (a+b,\ 2b-3c,\ -a+b+2c)$

$\therefore x = a+b,\ y = 2b-3c,\ z = -a+b+2c$.

$x+z = 2b+2c \Rightarrow x+z-y = 5c \Rightarrow c = \dfrac{x+z-y}{5}$

$y = 2b-3c \Rightarrow 2b = y+3c = y + \dfrac{3x+3z-3y}{5}$

$\Rightarrow b = \dfrac{1}{2}\left[\dfrac{3x+3z+2y}{5}\right] = \dfrac{3x+2y+3z}{10}$

$x = a+b \Rightarrow a = x-b = x - \dfrac{3x+2y+3z}{10}$

$\Rightarrow a = \dfrac{7x-2y-3z}{10}$

$\therefore (x,y,z) = \left(\dfrac{7x-2y-3z}{10}\right)\alpha_1 + \left(\dfrac{3x+2y+3z}{10}\right)\alpha_2 + \left(\dfrac{x-y+z}{5}\right)\alpha_3$.

→ $e_1 = (1,0,0) = \dfrac{7}{10}\alpha_1 + \dfrac{3}{10}\alpha_2 + \dfrac{1}{5}\alpha_3$.

$e_2 = (0,1,0) = -\dfrac{\alpha_1}{5} + \dfrac{1}{5}\alpha_2 - \dfrac{1}{5}\alpha_3$

$e_3 = (0,0,1) = -\dfrac{3}{10}\alpha_1 + \dfrac{3}{10}\alpha_2 + \dfrac{1}{5}\alpha_3$

⑤ Let $T: V_2(\mathbb{R}) \to V_2(\mathbb{R})$ be a L.T. defined by $T(a,b) = (a, a+b)$. Find the matrix of T, taking (e_1, e_2) as basis for the domain and $\{(1,1), (1,-1)\}$ as a basis for the range.

→ Here $e_1 = (1,0)$ & $e_2 (0,1)$.

Let $(x,y) = a(1,1) + b(1,-1)$

$\Rightarrow (x,y) = (a+b, a-b)$ $\Rightarrow a+b = x, \; a-b = y$

$2a = x+y$ | $2b = x-y$

$a = \frac{x+y}{2}$ | $b = \frac{x-y}{2}$

Now:

$T(e_1) = T(1,0) = (1,1) = 1.(1,1) + 0(1,-1)$

$T(e_1) = 1(1,1) + 0(1,-1)$

$T(e_2) = T(0,1) = (0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

$T(e_2) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

$\therefore$ Matrix required is $\begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$

⑥ If $(n+1)$ vectors $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_n$ provided that $\alpha_1, \alpha_2, \ldots \alpha_n$ form a linearly independent set.

→ Given that: $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha$ is a L.D. set. Then, there exists $a_1, a_2, \ldots a_n, a \in \mathbb{R}$ such that

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + a\alpha = 0$ — ① where $a_1, a_2, \ldots a_n, a$ are not all zeroes.

Since $\alpha_1, \alpha_2, \ldots \alpha_n$ are L.I. set, if $b_1, b_2, \ldots b_n \in \mathbb{R}$ such that

$b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n = 0 \Rightarrow b_1 = b_2 = \ldots = b_n = 0$

Hence in ①, $a \neq 0$ since if $a = 0$, then atleast one among $a_1, a_2, \ldots a_n$ is non-zero which is a contradiction.

Hence $a \neq 0$. Then ① $\equiv a\alpha = -a_1\alpha_1 - a_2\alpha_2 - a_3\alpha_3 \ldots - a_n\alpha_n$

$\Rightarrow \alpha = \left(-\frac{a_1}{a}\right)\alpha_1 + \left(-\frac{a_2}{a}\right)\alpha_2 + \ldots + \left(-\frac{a_n}{a}\right)\alpha_n.$

Hence, α is a L.C. of $\alpha_1, \alpha_2, \ldots \alpha_n$.