IAS MATHEMATICS (OPT.)

PAPER - I : ODE (2007)

IAS-2007

2007

6.

12M 2007 → **5(a).** Solve the ordinary differential equation

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2}\sin 6x + \sin^2 3x, \quad 0 < x < \pi/2$$

Soln: The given differential equation is

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2}\sin 6x + \sin^2 3x.$$

Dividing by $\cos 3x$, we get

$$\frac{dy}{dx} - (3\tan 3x)y = \frac{1}{2}\frac{2\sin 3x\cos 3x}{\cos 3x} + \frac{\sin^2 3x}{\cos 3x}$$

$$\Rightarrow \frac{dy}{dx} - (3\tan 3x)y = \sin 3x + \tan 3x \sec 3x. \quad \text{—— (1)}$$

which is a linear equation

$$I.F. = e^{-\int 3\tan 3x\,dx} = e^{\log \cos 3x} = \cos 3x$$

∴ The solution of (1) is

$$y(I.F) = \int I.F\,(\sin 3x + \tan 3x \sec 3x)\,dx + C$$

$$\Rightarrow y\cos 3x = \int \cos 3x(\sin 3x + \tan 3x \sec 3x)\,dx + c$$

$$= \int (\sin 3x\cos 3x + \tan 3x)\,dx + c$$

$$= \frac{1}{2}\int \sin 6x\,dx + \int \tan 3x\,dx + c$$

$$= \frac{1}{2}\left(\frac{\cos 6x}{-6}\right) + \frac{1}{3}\log \sec 3x + C$$

$$\therefore \boxed{y\cos 3x = -\frac{1}{12}\cos 6x + \frac{1}{3}\log \sec 3x + c.}$$

which is the required solution.

12) 2007

5(b). → Find the solution of the equation

$$\frac{dy}{y} + xy^2 dx = -4x\,dx$$

Sol: The given differential equation is

$$\frac{dy}{y} + xy^2 dx = -4x\,dx$$

$$\Rightarrow dy + (xy^3 + 4xy)dx = 0$$

$$\Rightarrow dy + xy(y^2+4)\,dx = 0$$

$$\Rightarrow \frac{dy}{y(y^2+4)} + x\,dx = 0$$

Integrating, we get

$$\int \frac{dy}{y(y^2+4)} + \int x\,dx = C$$

$$\frac{1}{4}\int\left[\frac{1}{y} - \frac{1}{y^2+4}\right]dy + \frac{x^2}{2} = C$$

$$\Rightarrow \boxed{\frac{1}{4}\left[\log y - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)\right] + \frac{x^2}{2} = C}$$

which is the required solution.

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6(a). 15M, Determine the general and singular solutions of the equation

$$y = x\frac{dy}{dx} + a\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1/2}, \text{ 'a' being constant.}$$

Sol: Given that

$$y = x\frac{dy}{dx} + a\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1/2}$$

i.e, $y = xp + \frac{ap}{\sqrt{1+p^2}}$; where $p = \frac{dy}{dx}$ — (1)

Clearly which is in

∴ Its general solution is

$$y = cx + \frac{ac}{\sqrt{1+c^2}} \quad \text{— (P)}$$

$$\Rightarrow (y-cx)\sqrt{1+c^2} = ac$$

Let $\phi(x,y,c) = (y-cx)\sqrt{1+c^2} - ac = 0$ (2)

Differentiating w.r.t c, we get

$$\frac{\partial\phi}{\partial c} = (y-cx)\frac{1}{2}(1+c^2)^{-1/2}\cdot 2c - x\sqrt{1+c^2} - a = 0$$

$$\Rightarrow \frac{(y-cx)c}{\sqrt{1+c^2}} - x\sqrt{1+c^2} - a = 0 \quad \text{— (3)}$$

$$\Rightarrow \frac{ac}{\sqrt{1+c^2}}\frac{c}{\sqrt{1+c^2}} - x\sqrt{(1+c^2)} - a = 0 \quad (\text{by (1)})$$

$$\Rightarrow \frac{ac^2}{1+c^2} - x\sqrt{1+c^2} - a = 0$$

$$\Rightarrow ac^2 - x(1+c^2)^{3/2} - a(1+c^2) = 0$$

$$\Rightarrow -x(1+c^2)^{3/2} = a$$

$$\Rightarrow (1+c^2)^{3/2} = -\frac{a}{x}$$

$$\Rightarrow \boxed{(1+c^2)^{1/2} = \left(-\frac{a}{x}\right)^{1/3}} \quad \text{— (4)}$$

from (1)

$$y = c\left[x + \frac{a}{\sqrt{1+c^2}}\right]$$

$$= c\left[x + \frac{a}{(-a/x)^{1/3}}\right] \quad (\because \text{by (4)})$$

$$= c\left[x - a^{2/3}x^{1/3}\right]$$

$$\Rightarrow y = cx^{1/3}\left[x^{2/3} - a^{2/3}\right]$$

$$\Rightarrow \boxed{c = \frac{y}{x^{1/3}\left[x^{2/3} - a^{2/3}\right]}} \quad \text{———} (8)$$

Now from (4), we have

$$(1+c^2)^{3/2} = -\frac{a}{x}$$

$$\Rightarrow \left[1 + \frac{y^2}{x^{2/3}\left[x^{2/3} - a^{2/3}\right]^2}\right]^{3/2} = -\frac{a}{x}$$

$$\Rightarrow \left\{1 + \frac{y^2}{x^{2/3}\left[x^{2/3} - a^{2/3}\right]^2}\right\}^3 = \left(\frac{a}{x}\right)^2$$

$$\Rightarrow 1 + \frac{y^2}{x^{2/3}(x^{2/3} - a^{2/3})^2} = \left(\frac{a}{x}\right)^{2/3}$$

$$\Rightarrow \frac{y^2}{\cancel{x^{2/3}}(x^{2/3} - a^{2/3})^2} = \frac{a^{2/3} - x^{2/3}}{\cancel{x^{2/3}}}$$

$$\Rightarrow y^2 = (a^{2/3} - x^{2/3})(a^{2/3} - x^{2/3})^2$$

$$\Rightarrow y^2 = (a^{2/3} - x^{2/3})^3$$

$$\Rightarrow y^{2/3} = a^{2/3} - x^{2/3}$$

$$\Rightarrow \boxed{x^{2/3} + y^{2/3} = a^{2/3}}$$

which is the required singular solution.

15M → Obtain the general solution of

2007 $(D^3-6D^2+12D-8)y = 12\left(e^{2x}+\frac{9}{4}e^{-x}\right)$, where $D=\frac{d}{dx}$.

6(b).

Sol^n: Given that

$$(D^3-6D^2+12D-8)y = 12\left(e^{2x}+\frac{9}{4}e^{-x}\right) \quad \text{—— (1)}$$

Auxiliary equation of (1) is

$$D^3-6D^2+12D-8=0$$

$$\Rightarrow (D-2)^3=0$$

$$\Rightarrow D=2,2,2$$

∴ The complementary function of (1) is

$$y_c=(c_1+c_2x+c_3x^2)e^{2x}.$$

particular integral of (1) is

$$y_p = \frac{1}{(D-2)^3}\,12\left(e^{2x}+\frac{9}{4}e^{-x}\right)$$

$$= \frac{12}{(D-2)^3}e^{2x} + \frac{27e^{-x}}{(D-2)^3}$$

$$= 12e^{2x}\frac{1}{(D+2-2)^3}e^{0x} + \frac{27}{(-3)^3}$$

$$= 12e^{2x}\frac{1}{D^3}e^{0x} - \frac{27}{27}$$

$$= 12e^{2x}\frac{x^3}{3!}-1$$

$$= 12e^{2x}\frac{x^3}{6}-1$$

$$= 2e^{2x}x^3-1$$

∴ The general solution of (1) is $y=y_c+y_p$

i.e, $y=(c_1+c_2x+c_3x^2)e^{2x}+2e^{2x}x^3-1$

15M → Solve the equation $2x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=x^3$.
2007

6(c).

Sol^n: Given equation is

$$2x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=x^3$$

$$\Rightarrow \frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}-\frac{3}{2x^2}y=\frac{x}{2} \quad \text{——(1)}$$

Comparing (1) with $y''+Py'+Qy=R$.

$$P=\frac{3}{2x}, \quad Q=\frac{-3}{2x^2}; \quad R=\frac{x}{2}$$

Now the homogeneous equation of (1) is

$$y''+\frac{3}{2x}y'-\frac{3}{2x^2}y=0 \quad \text{——(2)}$$

Now $P+Qx=0$

$\therefore y=x$ is a part of complementary function of (2).

Let $y=uv$ be the general solution of (1)

where $u=x$

then v is given by $\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$ ——(3)

Now since $u=x \Rightarrow \frac{du}{dx}$.

and $P+\frac{2}{u}\frac{du}{dx}=\frac{3}{2x}+\frac{2}{x}(1)$

$$=\frac{7}{2x}$$

$\therefore$ from (3), we have

$$\frac{d^2v}{dx^2}+\frac{7}{2x}\frac{dv}{dx}=\frac{x/2}{x} \quad \text{——(4)}$$

let $\frac{dv}{dx}=q \Rightarrow \frac{d^2v}{dx^2}=\frac{dq}{dx}$.

The equation (4) becomes

$$\frac{dq}{dx} + \frac{7}{2x}q = \frac{1}{2} \quad \text{(5)}$$ which is a linear equation

$$I.F = e^{\int \frac{7}{2x} dx} = e^{\frac{7}{2}\log x} = e^{\log x^{7/2}} = x^{7/2}$$

and its solution is

$$q \cdot x^{7/2} = \int \frac{1}{2} \cdot x^{7/2} dx$$

$$= \frac{1}{2}\frac{x^{9/2}}{9/2} + C_1$$

$$\Rightarrow q x^{7/2} = \frac{x^{9/2}}{9} + C_1$$

$$\Rightarrow q = \frac{x}{9} + C_1 x^{-7/2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{x}{9} + C_1 x^{-7/2}$$

Integrating, we get

$$v = \int \frac{x}{9} dx + \int C_1 x^{-7/2} dx + C_2$$

$$= \frac{x^2}{18} + C_1 \frac{x^{-5/2}}{-5/2} + C_2$$

$$= \frac{x^2}{18} - \frac{2}{5} C_1 x^{-5/2} + C_2$$

$\therefore$ The required general solution is

$$y = uv$$

$$\Rightarrow y = x\left[\frac{x^2}{18} - \frac{2}{5} C_1 x^{-5/2} + C_2\right]$$

$$\boxed{y = \frac{x^3}{18} - \frac{2}{5} C_1 x^{-3/2} + C_2 x.}$$

ISM 2007 **6(d).** Use the method of variation of parameters to find the general solution of the equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=2e^x.$$

Sol^n: Given equation is

$$y''+3y'+2y=2e^x \quad \text{--- (1)}$$

comparing (1) with $y''+Py'+Qy=R$

$$P=3,\ Q=2,\ R=2e^x$$

Consider $y''+3y'+2y=0$

$$(D^2+3D+2)y=0 \quad \text{--- (2)}$$

A.E of (2) is

$$D^2+3D+2=0$$

$$\Rightarrow (D+2)(D+1)=0$$

$$\Rightarrow D=-1,-2$$

$\therefore$ The complementary function of (1) is

$$y_c=c_1e^{-x}+c_2e^{-2x}$$

Now let $u=e^{-x}$, $v=e^{-2x}$ & $R=2e^x$.

Let $y_p=Au+Bv$ be a particular integral of (1)

where A and B are functions of x

and $u=e^{-x}$, $v=e^{-2x}$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix}=uv'-vu'$

$$=e^{-x}(-2e^{-2x})-e^{-2x}(-e^{-x})$$

$$=-2e^{-3x}+e^{-3x}$$

$$=-e^{-3x}\neq 0$$

$$A = \int \frac{-vR}{uv'-u'v}\,dx = -\int \frac{e^{-2x}\,2e^{x}}{-e^{-3x}}\,dx$$

$$= \int 2e^{2x}\,dx$$

$$= e^{2x}$$

and $$B = \int \frac{uR}{uv'-u'v} = \int \frac{e^{-x}\,2e^{x}}{-e^{-3x}}$$

$$= -2\int e^{3x}\,dx$$

$$= -\frac{2}{3}e^{3x}$$

$$\therefore\ y_p = e^{2x}e^{-x} + \left(-\frac{2}{3}e^{3x}\right)e^{-2x}$$

$$= e^{x} - \frac{2}{3}e^{x}$$

$$= \frac{e^{x}}{3}.$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p$$

i.e, $$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^{x}}{3}.$$

2006

IAS-2006

11.

12M 2006

5(a). Find the family of curves whose tangents form an angle $\pi/4$ with the hyperbolas $xy = c, c>0$.

Sol: The given family of curves is $xy = c$ — (1)

Differentiating (1),

we get

$$y + xp = 0 \quad \text{— (2)}$$

Replacing p by $\dfrac{p+\tan\pi/4}{1-p\tan\pi/4}$

i.e, $\dfrac{p+1}{1-p}$ in equation (2)

∴ The differential equation of the desired family of curves is

$$y + x\left(\frac{p+1}{1-p}\right) = 0$$

$$\Rightarrow p = \frac{y+x}{y-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+x}{y-x}$$

$$\Rightarrow (x+y)\,dx + (x-y)\,dy = 0$$

$$\Rightarrow x\,dy + y\,dx + x\,dx - y\,dy = 0$$

$$\Rightarrow d(xy) + d\left(\frac{x^2-y^2}{2}\right) = 0$$

Integrating, we get

$$xy + \frac{x^2-y^2}{2} = c_1$$

$$\Rightarrow \boxed{x^2+2xy-y^2 = c}, \text{ where } c = 2c_1$$

which is the required family of curves

12M 2006 Solve the differential equation

5(b). $\left(xy^2+e^{-1/x^3}\right)dx - x^2y\,dy = 0$

Sol^n: The given equation is

$\left(xy^2+e^{-1/x^3}\right)dx - x^2y\,dy = 0$ ——— (1)

Comparing (1) with $M\,dx + N\,dy = 0$

Here $M = xy^2+e^{-1/x^3}$; $N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad \frac{\partial N}{\partial x} = -2xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \dfrac{2xy+2xy}{-x^2y}$

$= \dfrac{4xy}{-x^2y} = -\dfrac{4}{x} = f(x)$

$\therefore$ I.F $= e^{\int f(x)dx} = e^{-\int \frac{4}{x}dx}$

$= e^{-4\log x} = e^{\log\frac{1}{x^4}} = \frac{1}{x^4}$.

Multiplying (1) by $\frac{1}{x^4}$, it becomes

$\left(\frac{y^2}{x^3}+\frac{1}{x^4}e^{-1/x^3}\right)dx - \frac{y}{x^2}dy = 0$ ——— (2)

clearly (2) is exact

Integrating

$\int\left(\frac{y^2}{x^3}+\frac{1}{x^4}e^{-1/x^3}\right)dx = c$

$\Rightarrow y^2\int\frac{1}{x^3}dx - \int\frac{1}{x^4}e^{-1/x^3}dx = c$

$$\Rightarrow y^2\left(-\frac{1}{2x^2}\right) - \int \frac{1}{x^4} e^{-1/x^3} dx = c \quad \text{——— (3)}$$

To integrate $\int \frac{1}{x^4} e^{-1/x^3} dx$

put $\frac{-1}{x^3} = t$

$\Rightarrow \frac{3}{x^4} dx = dt$

$\Rightarrow \frac{dx}{x^4} = \frac{dt}{3}$

$\therefore \int \frac{1}{x^4} e^{-1/x^3} dx = \int e^t \frac{dt}{3}$

$= \frac{e^t}{3} = \frac{e^{-1/x^3}}{3}$

$\therefore$ (3) $\Rightarrow$ we have

$-\frac{y^2}{2x^2} + \frac{e^{-1/x^3}}{3} = c$

$\therefore$ The solution of (1) is

$$\boxed{-\frac{y^2}{2x^2} + \frac{e^{-1/x^3}}{3} = c}$$

$\frac{15M}{2006}$ Solve $(1+y^2)+(x-e^{-\tan^{-1}y})\frac{dy}{dx}=0$.

6(a). Sol^n Given equation is

$$(1+y^2)+(x-e^{-\tan^{-1}y})\frac{dy}{dx}=0$$

$\therefore$ This equation can be written as

$$\frac{dx}{dy}+\frac{x-e^{-\tan^{-1}y}}{1+y^2}=0$$

$$\Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

which is in the form of $\frac{dx}{dy}+P(y)x=Q(y)$

Here $P(y)=\frac{1}{1+y^2}$; $Q(y)=\frac{e^{-\tan^{-1}y}}{1+y^2}$

Now, $\int P\,dy=\int\frac{1}{1+y^2}dy$

$$=\tan^{-1}y.$$

$\therefore$ Integrating factor (I.F) $=e^{\int P dy}=e^{\tan^{-1}y}$

Solution of the given differential equation is

$$x\,(\text{I.F})=\int Q\cdot(\text{I.F})\,dy+C$$

$$x\,e^{\tan^{-1}y}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}\,dy+C$$

$$=\int\frac{1}{1+y^2}dy+C$$

$$=\tan^{-1}y+C$$

$$\therefore \boxed{x\tan^{-1}y=\tan^{-1}y+C}$$

2006
6(b):

6(b). Reduce the equation $x^2p^2+py(2x+y)+y^2=0$ where $p=\frac{dy}{dx}$, to clairaut's form and find its complete primitive and also its singular solution.

Sol^n: Given $x^2p^2+yp(2x+y)+y^2=0$

put $y=u$ and $xy=v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\ ;\quad y+x\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow p=\frac{du}{dx}\ ;\quad y+xp=\frac{dv}{dx}$$

where $p=\frac{dy}{dx}$.

if $P=\frac{dv}{du}$ then

$$P=\frac{dv}{du}=\frac{dv/dx}{du/dx}$$

$$=\frac{y+xp}{p}$$

$$\Rightarrow Pp=y+xp$$

$$\Rightarrow p(P-x)=y$$

$$\Rightarrow p=\frac{y}{P-x} \quad \text{--- (1)}$$

Using (1) the given equation becomes

$$\frac{x^2y^2}{(P-x)^2}+\frac{y^2}{P-x}(2x+y)+y^2=0$$

$$\Rightarrow x^2+(P-x)(2x+y)+(P-x)^2=0$$

$$\Rightarrow Py-xy+P^2=0$$

$$\Rightarrow Pu-v+P^2=0$$

$$\Rightarrow v=Pu+P^2$$

which is of Clairaut's form.

and its solution is

$$v = uc + c^2$$

i.e, $xy = yc + c^2$ —— (2)

which is the required complete primitive.

i.e, $c^2 + cy - xy = 0$ —— (2)

$\therefore$ C-discriminant relation is

$$y^2 + 4xy = 0$$

$$\Rightarrow y(y+4x) = 0 \text{ —— (3)}$$

From the given equation the p-discriminant relation is

$$y^2(2x+y)^2 - 4x^2y^2 = 0$$

$$\Rightarrow y^2[\cancel{4x^2} + y^2 + 4xy - \cancel{4x^2}] = 0$$

$$\Rightarrow y^2[y^2 + 4xy] = 0$$

$$\Rightarrow y \cdot y^2 (y+4x) = 0 \text{ —— (4)}$$

$\therefore$ from (3) & (4) $y=0$ and $y+4x=0$ are both singular solutions since they occur once in both the discriminants and satisfy the given differential equation.

ISM 2006 → Solve the differential equation

6(c). $$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2\frac{y}{x}=10\left(1+\frac{1}{x^2}\right)$$

Sol^n: The given equation can be written as

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$$

$$\Rightarrow (x^3D^3+2x^2D^2+2)y=10\left(x+\frac{1}{x}\right);\ \text{where } D=\frac{d}{dx} \quad \text{—(1)}$$

Let $x=e^z$ so that $z=\log x$ and $D_1=\frac{d}{dz}$ —(2)

Then $xD=D_1$, $x^2D^2=D_1(D_1-1)$, $x^3D^3=D_1(D_1-1)(D_1-2)$ —(3)

Using (2) and (3), 1 reduces to

$$[D_1(D_1-1)(D_1-2)+2D_1(D_1-1)+2]y=10(e^z+e^{-z})$$

$$\Longrightarrow (D_1^3-D_1^2+2)y=10e^z+10e^{-z} \quad \text{—(4)}$$

A.E of (4) is $D_1^3-D_1^2+2=0$

$\Rightarrow (D_1+1)(D_1^2-2D_1+2)=0$ giving $D_1=-1,\ 1\pm i$

$\therefore$ C.F $=c_1e^{-x}+e^z(c_1\cos z+c_2\sin z)$

$=c_1x^{-1}+x(c_1\cos\log x+c_2\sin\log x)$

P.I corresponding to $10e^z$

$$=10\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z=10\frac{1}{2(1-2+2)}e^z=5x,\ \text{by (2)}$$

and P.I corresponding to $10e^{-z}$

$$= 10 \frac{1}{(D_1+1)(D_1^2-2D_1+2)} e^{-z}$$

$$= 10 \frac{1}{(D_1+1)} \cdot \frac{1}{1+2+2} e^{-z}$$

$$= 2 \frac{1}{D_1+1} e^{-z} \cdot 1 = 2e^{-z} \frac{1}{D_1-1+1} \cdot 1 = 2e^{-z} \frac{1}{D_1} \cdot 1$$

$$= 2e^{-z} \cdot z = 2x^{-1} \log x, \text{ by } (2)$$

$$\therefore y = c_1 x^{-1} + x\,(c_1 \cos \log x + c_2 \sin \log x) + 5x + 2x^{-1} \log x.$$

15M, 2006

6(d). Solve the differential equation $(D^2-2D+2)y=e^x\tan x$, where $D=\frac{d}{dx}$; by the method of variation of parameters.

Sol^n: Given equation is

$$(D^2-2D+2)y=e^x\tan x \quad \text{--- (I)}$$

Consider $(D^2-2D+2)y=0$ --- (1)

Auxiliary equation of (1) is

$$D^2-2D+2=0$$

$$D=\frac{2\pm\sqrt{4-8}}{2}$$

$$=\frac{2\pm\sqrt{-4}}{2}$$

$$=\frac{2\pm 2i}{2}=1\pm i$$

$\therefore$ complementary function of (I) is

$$y_C=e^x(c_1\cos x+c_2\sin x)$$

$$=c_1e^x\cos x+c_2e^x\sin x.$$

Now let $u=e^x\cos x$, $v=e^x\sin x$.

and $R=e^x\tan x$.

Let $y_p=Au+Bv$ be a particular integral of (I)

where A and B are functions of x.

and $u=e^x\cos x$, $v=e^x\sin x$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x\cos x-e^x\sin x \\ e^x\sin x & e^x\sin x+e^x\cos x \end{vmatrix}$

$uv'-vu' = e^{2x}[\cos x\sin x+\cos^2x-\cos x\sin x+\sin^2x]$

$$=e^{2x}(1)=e^{2x}\neq 0.$$

$$\therefore A = \int \frac{-vR}{uv'-u'v}\,dx$$

$$= \int -\frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\,dx$$

$$= \int -\sin x \cdot \tan x\,dx$$

$$= -\int \sin x \cdot \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$= -\left[\log|\sec x + \tan x| - \sin x\right]$$

$$= -\log|\sec x + \tan x| + \sin x$$

$$B = \int \frac{uR}{uv'-u'v}\,dx$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx$$

$$= \int \cos x \cdot \tan x\,dx$$

$$= \int \sin x\,dx$$

$$= -\cos x.$$

$\therefore$ The particular integral of ① is

$$y_p = \left[-\log|\sec x + \tan x| + \sin x\right] e^x \cos x + e^x \sin x(-\cos x)$$

$$= -\log|\sec x + \tan x| + \cancel{e^x \sin x \cos x} - \cancel{e^x \sin x \cos x}$$

$$= -\log|\sec x + \tan x|$$

$\therefore$ The general solution of ① is $y = y_c + y_p$

i.e., $y = e^x (c_1 \cos x + c_2 \sin x) - \log|\sec x + \tan x|$

IAS-2005

2005 12M: Find the orthogonal trajectories of the
5(a). family of co-axial circles $x^2+y^2+2gx+c=0$ where g is the parameter.

Soln: The given family of curves is

$$x^2+y^2+2gx+c=0, \text{ with } g \text{ as parameter.} \quad \text{--- (1)}$$

Differentiating (1) with respect to x, we get

$$2x+2y\frac{dy}{dx}+2g=0$$

$$\Rightarrow g=-\left(x+y\frac{dy}{dx}\right) \quad \text{--- (2)}$$

Eliminating g from (1) & (2), we get

$$x^2+y^2+2x\left(-x-y\frac{dy}{dx}\right)+c=0$$

$$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}+c=0 \quad \text{--- (3)}$$

which is the differential equation of the given family of circles (1).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3),

the differential equation of the required orthogonal trajectories is

$$y^2-x^2+2xy\frac{dx}{dy}+c=0.$$

$$\Rightarrow 2x\frac{dx}{dy}-\frac{1}{y}x^2=\frac{-c}{y}-y \quad \text{--- (4)}$$

which can be reduced to linear differential equation as follows.

putting $x^2 = v$ so that $2x\frac{dx}{dy} = \frac{dv}{dy}$

(4) gives $\frac{dv}{dy} - \frac{v}{y} = \frac{-c}{y} - y$ —— (5)

Integrating factor (I.F) $= e^{-\int \frac{1}{y} dy}$
$= 1/y$.

$\therefore$ solution of (5) is

$$v\frac{1}{y} = \int\left(-\frac{c}{y} - y\right)\frac{1}{y}\,dy + c'$$

$$\Rightarrow \frac{v}{y} = \frac{c}{y} - y + c'$$

$$\Rightarrow x^2 = c - y^2 + c'y \qquad (\because v = x^2)$$

$$\Rightarrow x^2 + y^2 - c'y - c = 0,$$

c' being parameter.

15M 2005 → Solve the differential equation

1(b). $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$ by suitable substitution.

Sol: Given that

$$(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$$

put $x+y=u$, $x^2+y^2=v$.

$$\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx}\ ;\quad 2x+2y\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow 1+p=\frac{du}{dx}\ ;\quad 2x+2yp=\frac{dv}{dx}$$

$$\therefore \frac{dv}{du}=\frac{2(x+yp)}{1+p}$$

$$\Rightarrow P=\frac{2(x+yp)}{1+p};\quad \text{where } P=\frac{dv}{du},\ p=\frac{dy}{dx}$$

$$\Rightarrow P(1+p)=2(x+yp)$$

$$\Rightarrow P+Pp=2x+2yp$$

$$\Rightarrow P-2x=p(2y-P)$$

$$\Rightarrow p=\frac{P-2x}{2y-P}\quad \text{---(1)}$$

Using (1) the given equation becomes

$$(x^2+y^2)\left[1+\frac{P-2x}{2y-P}\right]^2-2(x+y)\left(1+\frac{P-2x}{2y-P}\right)\left(x+y\frac{(P-2x)}{2y-P}\right)$$
$$+\left(x+y\frac{(P-2x)}{2y-P}\right)^2=0$$

$$\Rightarrow (x^2+y^2)\left[\frac{2y-2x}{2y-P}\right]^2-2(x+y)\left[\frac{2y-2x}{2y-P}\right]P\frac{(y-x)}{2y-P}+P^2\left(\frac{y-x}{2y-P}\right)^2=0$$

$$\Rightarrow (x^2+y^2)4(y-x)^2-4(x+y)(y-x)^2P+P^2(y-x)^2=0$$

$\Rightarrow 4(x^2+y^2)-4p(x+y)+p^2=0$

$\Rightarrow 4v-4pu+p^2=0$

$\Rightarrow 4v=4pu-p^2$

$\Rightarrow v=pu-\frac{p^2}{4}$

which is of clairauts form.

and its solution is

$v=uc-\frac{c^2}{4}$

i.e, $\boxed{x^2+y^2=(x+y)c-\frac{c^2}{4}}$

ISM 2008 → Solve the differential equation

6(c). $(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0$; given that $y = \sin x$ is a solution of this equation.

Sol^n: Given that

$$(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0$$

$$\Rightarrow y'' - \frac{x\sin x}{\sin x - x\cos x}y' + \frac{\sin x}{\sin x - x\cos x}y = 0 \quad \text{—— (1)}$$

comparing (1) with $y'' + Py' + Q = R$,

we have

$$P = \frac{-x\sin x}{\sin x - x\cos x}; \quad Q = \frac{\sin x}{\sin x - x\cos x}; \quad R = 0$$

Here given that $y = u = \sin x$ is a part of C.F. of (1). —— (3)

Let the general solution of (1) be $y = uv$ —— (4)

Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{-x\sin x}{\sin x - x\cos x} + \frac{2}{\sin x}(\cos x)\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{-x\sin x}{\sin x - x\cos x} + 2\cot x\right]\frac{dv}{dx} = 0 \quad \text{—— (5)}$$

Let $\frac{dv}{dx} = q \Rightarrow \frac{d^2v}{dx^2} = \frac{dq}{dx}$

Then (5) becomes

$$\frac{dq}{dx}+\left[\frac{-x\sin x}{\sin x-x\cos x}+2\cot x\right]q=0$$

$$\Rightarrow \frac{dq}{q}=\left[\frac{x\sin x}{\sin x-x\cos x}-2\cot x\right]dx$$

Integrating, we get

$$\log q=\log(\sin x-x\cos x)-2\log\sin x+\log c_1$$

$$\Rightarrow q=\frac{\sin x-x\cos x}{\sin^2 x}c_1$$

$$\Rightarrow \frac{dv}{dx}=c_1\left(\frac{\sin x-x\cos x}{\sin^2 x}\right)$$

$$\Rightarrow dv=c_1\left[\frac{\sin x-x\cos x}{\sin^2 x}\right]dx$$

$$\Rightarrow dv=c_1\,d\left(\frac{x}{\sin x}\right)$$

Integrating, we get

$$v=c_1\frac{x}{\sin x}+c_2$$

∴ The required solution is $y=uv$.

i.e, $y=\sin x\left[c_1\frac{x}{\sin x}+c_2\right]$

$\Rightarrow y=c_1x+c_2\sin x$, where c_1 and c_2 are arbitrary constants.

15M 2005 → Solve the differential equation

6(d). $x^2y''-2xy'+2y = x\log x$, $x>0$., by variation of parameters

Sol^n: Given equation is

$$x^2y''-2xy'+2y = x\log x.$$

$$\Rightarrow y''-\frac{2}{x}y'+\frac{2}{x^2}y = \frac{\log x}{x} \quad \text{--- (1)}$$

Comparing (1) with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

$$P=-\frac{2}{x},\ Q=\frac{2}{x^2};\ R=\frac{\log x}{x}.$$

Now the homogeneous equation of (1) is

$$y''-\frac{2}{x}+\frac{2}{x^2}y=0 \quad \text{--- (2)}$$

Now $P+Qx=0$

$\therefore y=x$ is a part of complementary function of (2).

Let $y=uv$ be the general solution of (2) where $u=x$.

then v is given by $\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=0$ --- (3)

Now since $u=x \Rightarrow \frac{du}{dx}=1$

$$\therefore P+\frac{2}{u}\frac{du}{dx} = -\frac{2}{x}+\frac{2}{x}(1)=0.$$

$\therefore$ (3) $\Rightarrow \frac{d^2v}{dx^2}=0$

$$\Rightarrow \frac{dv}{dx}=c_1$$

$$\Rightarrow v = c_1x+c_2.$$

$\therefore$ The solution of (2) is $y=uv$

i.e, $y=c_1x^2+c_2x$.

Let $y_p = Au + Bv$ be a particular integral of (1).

where A and B are functions of x.

and $u = x^2$, $v = x$.

$R = \frac{\log x}{x}$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - u'v$

$= x^2(1) - 2x^2$

$= -x^2 \neq 0$

$\therefore A = \int \frac{-vR}{uv' - u'v} dx$

$= \int \frac{-x \frac{\log x}{x}}{-x^2} dx$

$= \int \frac{\log x}{x^2} dx$

$= \int \underset{g}{x^{-2}} \underset{f}{\log x}$

$= \log x \cdot \left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx$

$= -\frac{1}{x}\log x - \frac{1}{x}$

and $B = \int \frac{uR}{uv' - u'v} = \int \frac{x^2 \frac{\log x}{x}}{-x^2}$

$= -\int \frac{\log x}{x} = I$ (say)

Then $I = -\log x \cdot \log x - \int \frac{1}{x} \log x$

$I = -(\log x)^2 - I$

$\Rightarrow 2I = -(\log x)^2$

$\Rightarrow I = -\frac{1}{2}(\log x)^2 = B$

$\therefore y_p = x^2\left[-\frac{1}{x}\log x - \frac{1}{x}\right] - x\,\frac{1}{2}(\log x)^2$

$= -x\log x - x - \frac{x}{2}(\log x)^2$

$\therefore$ The general solution of (1) is $y = y_c + y_p$

i.e., $y = c_1 x^2 + c_2 - x - x\log x - \frac{1}{2}x(\log x)^2$

IAS-2004

2004

2

12M, 2004

5(a). Find the solution of the following differential equation $\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin 2x$.

Soln: Given that

$$\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin 2x \quad \text{——(1)}$$

which is in the form of

$$\frac{dy}{dx}+P(x)y=Q(x)\,y. \quad \text{(linear equation)}$$

where $P=\cos x$; $Q=\frac{1}{2}\sin 2x$.

$$I.F=e^{\int P\,dx}=e^{\int\cos x\,dx}$$
$$=e^{\sin x}.$$

The solution of (1) is

$$y(I.F)=\int I.F.\tfrac{1}{2}\sin 2x\,dx$$

$$=\int e^{\sin x}\tfrac{1}{2}.2\sin x\cos x\,dx$$

$$ye^{\sin x}=\int e^{\sin x}\sin x\cos x\,dx$$

put $\sin x=t$

$\cos x\,dx=dt$

$$=\int e^{t}t\,dt$$
$$=e^{t}(t-1)+C$$

$$\Rightarrow ye^{\sin x}=e^{\sin x}(\sin x-1)+C$$

$$\Rightarrow \boxed{y=(\sin x-1)+C.e^{-\sin x}}$$

which is the required solution.

12M 004 → Solve $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0$

5(b).

Sol: The given differential equation is

$$y(xy+2x^2y^2)\,dx + x(xy-x^2y^2)\,dy = 0 \quad \text{--- (1)}$$

It is in the form of $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

comparing (1) with $M\,dx + N\,dy = 0$,

we have

$$M = y(xy+2x^2y^2) \quad ; \quad N = x(xy-x^2y^2).$$

now $Mx - Ny = xy(xy+2x^2y^2) - xy(xy-x^2y^2)$

$$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$$

$$= 3x^3y^3.$$

$\therefore$ I.F. of (1) is $\dfrac{1}{Mx-Ny} = \dfrac{1}{3x^3y^3}$

On multiplying (1) by $\dfrac{1}{3x^3y^3}$, we have

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}\,dx + \frac{x(xy-x^2y^2)}{3x^3y^3}\,dy = 0$$

$$\Rightarrow \frac{1+2xy}{3x^2y}\,dx + \frac{1-xy}{3xy^2}\,dy = 0$$

which is must be exact

$\therefore$ The required solution is

$$\int\left(\frac{1}{3x^2y} + \frac{2xy}{3x^2y}\right)dx - \int \frac{1}{3y}\,dy = 0$$

$$\Rightarrow \int \frac{1}{3x^2y}\,dx + \frac{2}{3x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = 0$$

$$\Rightarrow -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$\xrightarrow{\text{ISM}}$ 2004 solve $(D^4-4D^2-5)y = e^x(x+\cos x)$.

6(a). Sol^n: Given that $(D^4-4D^2-5)y = e^x(x+\cos x)$ ——(1)

The auxiliary equation is

$$D^4-4D^2-5=0$$

$$\Rightarrow D^4-5D^2+D^2-5=0$$

$$\Rightarrow D^2(D^2-5)+1(D^2-5)=0$$

$$\Rightarrow (D^2+1)(D^2-5)=0$$

$$\Rightarrow D=\pm i;\ D=\pm\sqrt{5}$$

∴ The complementary function of (1) is

$$y_C = C_1\cos x + C_2\sin x + C_3\cosh\sqrt{5}x + C_4\sinh\sqrt{5}x$$

Particular Integral (P.I)

$$y_p = \frac{1}{D^4-4D^2-5}e^x(x+\cos x) \text{ ——1}$$

$$= \frac{1}{D^4-4D^2-5}xe^x + \frac{1}{D^4-4D^2-5}e^x\cos x \text{ ——(1)}$$

P.I corresponding to xe^x

$$= \frac{1}{D^4-4D^2-5}xe^x$$

$$= e^x\frac{1}{(D+1)^4-4(D+1)^2-5}x$$

$$= e^x\frac{1}{D^4+4D^3+2D^2-4D-8}x$$

$$= e^x\frac{1}{-8\left[1-\frac{(-4D+2D^2+4D^3+D^4)}{8}\right]}x$$

$$= \frac{e^x}{-8}\left[1+\frac{(-4D+2D^2+4D^3+D^4)}{8}+\dots\right]x$$

$$= \frac{e^x}{-8}\left[x-\frac{4}{8}\right] = \frac{e^x}{-8}\left[x-\frac{1}{2}\right]$$

P.I corresponding to $e^{x}\cos x$.

$$= \frac{1}{D^4-4D^2-5}e^{x}\cos x$$

$$= e^{x}\frac{1}{(D+1)^4-4(D+1)^2-5}\cos x$$

$$= e^{x}\frac{1}{D^4+4D^3+2D^2-4D-8}\cos x$$

$$= e^{x}\frac{1}{1-4D-2-4D-8}\cos x$$

$$= e^{x}\frac{1}{-(8D+9)}\cos x$$

$$= (-e^{x})\frac{8D-9}{64D^2-81}\cos x$$

$$= -e^{x}\frac{8D-9}{-145}\cos x$$

$$= \frac{e^{x}}{145}(-8\sin x-9\cos x)$$

$$= -\frac{e^{x}}{145}(8\sin x+9\cos x)$$

∴ The solution of (1) is

$$y = y_c + y_p$$

$$= c_1\cos x + c_2\sin x + c_3\cosh\sqrt{5}x + c_4\sinh\sqrt{5}x - \frac{e^{x}}{145}(8\sin x+9\cos x)$$

IM 2004 → Reduce the equation $(px-y)(py+x)=2p$, where

6(b). $p=\frac{dy}{dx}$ to Clairaut's form and hence solve it.

Sol^n: Given differential equation is

$$(px-y)(py+x)=2p.$$

put $x^2=u$, $y^2=v$

$2x\,dx=du$; $2y\,dy=dv$

$$\Rightarrow \frac{dv}{du}=\frac{2y}{2x}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{du}=\frac{y}{x}\frac{dy}{dx}$$

$$\Rightarrow P=\frac{y}{x}p$$

$$\Rightarrow p=\frac{xP}{y} \quad \text{where } p=\frac{dy}{dx} \text{ and } P=\frac{dv}{du}.$$

Using $p=\frac{xP}{y}$, the given equation becomes

$$\left(\frac{x}{y}Px-y\right)\left(\frac{x}{y}Py+x\right)=2\left(\frac{xP}{y}\right)$$

$$\Rightarrow (Px^2-y^2)(P+1)\frac{x}{y}=2\frac{xP}{y}$$

$$\Rightarrow (Px^2-y^2)(P+1)=2P$$

$$\Rightarrow (Pu-v)(P+1)=2P \qquad (\because x^2=u,\ y^2=v)$$

$$\Rightarrow Pu-v=\frac{2P}{P+1}$$

$$\Rightarrow v=Pu-\frac{2P}{P+1}$$

which is a Clairaut's equation.

$\therefore$ The general solution is

$$v=cu-\frac{2c}{c+1}$$

i.e., $y^2=cx^2-\frac{2c}{c+1}$

$$(c+1)cx^2-y^2(c+1)-2c=0$$

$$\Rightarrow \boxed{c^2x^2+(x^2-y^2-2)c-y^2=0}$$

6(c). (15M) Solve $(x+2)\dfrac{d^2y}{dx^2}-(2x+5)\dfrac{dy}{dx}+2y=(x+1)e^x$.

Sol: Dividing by $x+2$, the given equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\frac{dy}{dx}+\frac{2}{x+2}y=\frac{x+1}{x+2}e^x \quad \text{—— (1)}$$

Comparing (1) with $y''+Py'+Qy=R$,

we have $P=-\dfrac{2x+5}{x+2}$; $Q=\dfrac{2}{x+2}$; $R=\dfrac{(x+1)}{(x+2)}e^x$. —— (2)

Here

$$a^2+aP+Q=4+2\left(\frac{-(2x+5)}{x+2}\right)+\frac{2}{x+2}$$

$$=\frac{4(x+2)-2(2x+5)+2}{x+2}=0$$

Showing that a part of C.F of (1) is $y=u=e^{2x}$. —— (3)

Let the general solution of (1) be $y=uv$. —— (4)

Then v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{(2x+5)}{x+2}+\frac{2}{e^{2x}}\left(2e^{2x}\right)\right]\frac{dv}{dx}=\frac{(x+1)e^x}{(x+2)e^{2x}}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[4-\frac{2x+5}{x+2}\right]\frac{dv}{dx}=\frac{x+1}{(x+2)e^x} \quad \text{—— (5)}$$

Let $\dfrac{dv}{dx}=q$ so that $\dfrac{d^2v}{dx^2}=\dfrac{dq}{dx}$ —— (6)

Then (5) becomes $\dfrac{dq}{dx}+\dfrac{2x+3}{x+2}q=\dfrac{x+1}{(x+2)e^x}$ —— (7)

which is linear equation

Now

$$I.F = e^{\int \frac{2x+3}{x+2}dx} = e^{\int\left(2 - \frac{1}{x+2}\right)dx}$$

$$= e^{2x - \log(x+2)}$$

$$= e^{2x}(x+2)^{-1} = \frac{e^{2x}}{x+2}$$

$\therefore$ The solution of (7) is

$$q \cdot \frac{e^{2x}}{x+2} = \int \frac{e^{2x}}{x+2} \cdot \frac{x+1}{(x+2)e^{x}}\, dx + C_1$$

$$q\frac{e^{2x}}{x+2} = \int \frac{(x+1)e^{x}}{(x+2)^2}dx + C_1$$

$$= \int \frac{[(x+2)-1]e^{x}}{(x+2)^2}dx + C_1$$

$$= \int e^{x}\left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right]dx + C_1$$

$$q\frac{e^{2x}}{x+2} = e^{x}\frac{1}{x+2} + C_1 \quad \left[\because \int e^{x}(f(x)+f'(x)) = e^{x}f(x)\right]$$

$$q = e^{-x} + C_1(x+2)e^{-2x}$$

$$\Rightarrow \frac{dv}{dx} = e^{-x} + C_1(x+2)e^{-2x} \quad \left(\because q = \frac{dv}{dx}\right)$$

integrating

$$\Rightarrow v = -e^{-x} + C_1\int (x+2)e^{-2x}dx + C_2$$

$$\Rightarrow v = -e^{-x} + C_1\left[(x+2)\left(-\frac{1}{2}e^{-2x}\right) - \int 1 \cdot \left(-\frac{1}{2}\right)e^{-2x}dx\right] + C_2$$

$$= -e^{-x} + C_1\left[(x+2)\left(-\frac{1}{2}e^{-2x}\right) - \frac{1}{4}e^{-2x}\right] + C_2$$

$$= -e^{-x} + C_1\left[\frac{-2x-5}{4}\right]e^{-2x} + C_2$$

$$v = -e^{-x} - \frac{C_1}{4}(2x+5)e^{-2x} + C_2$$

$\therefore$ The required solution is $y = uv$

i.e., $y = e^{2x}\left[-e^{-x} - \frac{C_1}{4}(2x+5)e^{-2x} + C_2\right]$

$$= -e^{x} - \frac{C_1}{4}(2x+5) + C_2e^{2x}$$

$$y = -e^{x} + C_3(2x+5) + C_2e^{2x} \quad \left(\text{where } C_3 = \frac{-C_1}{4}\right)$$

IAS-2003

5(a).

→ Show that the orthogonal trajectory of a system of confocal ellipses is self orthogonal.

2003 24.

Sol. The given family of curves is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \text{ with } \lambda \text{ as parameter.} \qquad ...(1)$$

Differentiating (1) with respect to x, we get

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0 \quad \text{or} \quad \frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0$$

or $$x(b^2+\lambda)+y(a^2+\lambda)\frac{dy}{dx}=0 \quad \text{or} \quad \lambda\left(x+y\frac{dy}{dx}\right)=-\left(b^2x+a^2y\frac{dy}{dx}\right)$$

$$\therefore \qquad \lambda=-\frac{b^2x+a^2y\,(dy/dx)}{x+y\,(dy/dx)}$$

$$\therefore \qquad a^2+\lambda=a^2-\frac{b^2x+a^2y\,(dy/dx)}{x+y\,(dy/dx)}=\frac{(a^2-b^2)\,x}{x+y\,(dy/dx)}$$

and $$b^2+\lambda=b^2-\frac{b^2x+a^2y\,(dy/dx)}{x+y\,(dy/dx)}=\frac{-(a^2-b^2)\,y\,(dy/dx)}{x+y\,(dy/dx)}.$$

Putting the above values of $(a^2+\lambda)$ and $(b^2+\lambda)$ in (1), we have

$$\frac{x^2\{x+y\,(dy/dx)\}}{(a^2-b^2)\,x}-\frac{y^2\{x+y\,(dy/dx)\}}{(a^2-b^2)\,y\,(dy/dx)}=1$$

or $$\{x+y\,(dy/dx)\}\,\{x-y\,(dx/dy)\}=a^2-b^2, \qquad ...(2)$$

which is the differential equation of the given family of curves (1). Replacing dy/dx by $(-dx/dy)$ in (2), the differential equation of the required orthogonal trajectories is

$$\{x+y\,(-dx/dy)\}\,\{x-y\,(-dy/dx)\}=a^2-b^2$$

or $$\{x+y\,(dy/dx)\}\,\{x-y\,(dx/dy)\}=a^2-b^2, \qquad ...(3)$$

which is the same as the differential equation (2) of the given family of curves (1). Hence the system of given curves (1) is self orthogonal, *i.e.*, each member of the given family of curves intersects its own members orthogonally.

5(b).

12M 2003 → Solve $x\frac{dy}{dx} + y\log y = xye^{x}$.

Sol^n: Given that

$$x\frac{dy}{dx} + y\log y = xye^{x}$$

Dividing with xy, we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^{x} \quad \text{--- (1)}$$

Now let $\log y = t$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$$

∴ from (1), we have

$$\frac{dt}{dx} + \frac{1}{x}t = e^{x} \quad \text{--- (2)}$$

which is a linear equation.

$$I.F = e^{\int\frac{1}{x}dx}$$
$$= e^{\log x} = x$$

∴ The solution of (2) is

$$tx = \int xe^{x} + C$$
$$= e^{x}(x-1) + C$$

$$\Rightarrow (\log y)x = e^{x}(x-1) + C \qquad (\because t = \log y)$$

$$\Rightarrow x\log y = e^{x}(x-1) + C$$

which is the required solution.

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15M 2003

6(a). → solve $(D^5-D)y = 4(e^x+\cos x+x^3)$, where $D=\frac{d}{dx}$.

Sol: Given equation is

$$(D^5-D)y = 4(e^x+\cos x+x^3) \quad \text{—①}$$

A.E of ① is

$$D^5-D=0$$

$$\Rightarrow D(D^4-1)=0$$

$$\Rightarrow D=0,\ D^4-1=0$$

$$\Rightarrow (D^2-1)(D^2+1)=0$$

$$\Rightarrow D=\pm1,\ \pm i$$

$$\therefore D=0,\ \pm1,\ \pm i.$$

∴ The complementary function of ① is

$$y_c = c_1+c_2e^x+c_3e^{-x}+(c_4\cos x+c_2\sin x)$$

particular integral (P.I) of ① is

$$y_p = \frac{1}{D^5-D}\,4(e^x+\cos x+x^3)$$

P.I. corresponding to $4e^{x}$

$$\Rightarrow \quad = \frac{1}{D^5-D}4e^x$$

$$= \frac{1}{D(D^2-1)(D^2+1)}4e^x$$

$$= \frac{4}{1(2)}\frac{1}{D^2-1}e^x$$

$$= 2\cdot\frac{1}{D^2-1}e^x$$

$$= 2\frac{1}{(D+1)}\frac{1}{(D-1)}e^x$$

$$= 2\left(\frac{1}{2}\right)\frac{1}{D-1}e^x$$

$$= 1(x)e^x$$

$$= xe^x.$$

P.I corresponding to $4\cos x$

$$= \frac{4}{D^5-D}\cos x.$$

$$= 4\,\frac{1}{D(D^2-1)(D^2+1)}\cos x$$

$$= 4\,\frac{1}{D(D^2+1)}\left(\frac{1}{-2}\right)\cos x$$

$$= -2\,\frac{1}{D(D^2+1)}\cos x$$

$$= -2\,\frac{1}{D^2+1}\left[\frac{1}{D}\cos x\right]$$

$$= -2\,\frac{1}{D^2+1}\sin x \qquad \left(\because \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax\right)$$

$$= -2\left(-\frac{x}{2}\cos x\right)$$

$$= x\cos x.$$

P.I corresponding to $4x^3$

$$= \frac{1}{D^5-D}\,4x^3$$

$$= \frac{4}{-D[1-D^4]}(x^3)$$

$$= \frac{4}{-D}\left[1-D^4\right]^{-1}(x^3)$$

$$= \frac{4}{-D}\left[1+D^4+(D^4)^2+\cdots\right](x^3)$$

$$= -\frac{4}{D}\left[x^3\right]$$

$$= -4\cdot\left[\frac{x^4}{4}\right]$$

$$= -x^4$$

(or) $\frac{4}{D(D^2-1)(D^2+1)}(x^3)$

$= \frac{-4}{D(1-D^2)(1+D^2)}(x^3)$

$= -(4)\frac{1}{D}[1-D^2]^{-1}[1+D^2]^{-1}x^3$

$= -4\left(\frac{1}{D}\right)[1+D^2+D^4+\dots][1-D^2+D^4+\dots]x^3$

$= -4\left(\frac{1}{D}\right)[1+\cancel{D^2}+D^4-\cancel{D^2}+D^4+\dots]x^3$

$= -\frac{4}{D}[x^3]$

$= -4\left(\frac{x^4}{4}\right) = -x^4$

$\therefore$ Particular integral of (1) is

$$y_p = xe^x + x\cos x - x^4.$$

$\therefore$ The solution of (1) is

$$y = y_c + y_p$$

i.e, $y = C_1 + C_2e^x + C_3e^{-x} + C_4\cos x + C_5\sin x + xe^x + x\cos x - x^4.$

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6(b). 15M 2003

Solve the differential equation $(px^2+y^2)(px+y)=(p+1)^2$ where $p=\frac{dy}{dx}$ by reducing it to clairaut's form using suitable substitution.

Sol^n: Given differential equation is

$$(px^2+y^2)(px+y)=(p+1)^2$$

put $u=xy$; $v=x+y$

$$\Rightarrow \frac{du}{dx}=y+x\frac{dy}{dx} \; ; \; \frac{dv}{dx}=1+\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx}=y+xp \; ; \; \frac{dv}{dx}=1+p \quad \text{where } p=\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{du}=\frac{1+p}{y+xp}$$

$$\Rightarrow P=\frac{1+p}{y+xp} \quad \text{where } P=\frac{dv}{du} : p=\frac{dy}{dx}$$

$$\Rightarrow P(y+xp)=1+p$$

$$\Rightarrow p(xP-1)=1-Py$$

$$\Rightarrow p=\frac{1-Py}{xP-1} \quad \text{---(1)}$$

Using (1) the given equation becomes

$$\left[\frac{(1-Py)}{xP-1}x^2+y^2\right]\left[\frac{(1-Py)}{xP-1}x+y\right]=\left[\frac{1-Py}{xP-1}+1\right]^2$$

$$\Rightarrow \left[x^2(1-Py)+y^2(xP-1)\right]\left[(1-Py)x+y(xP-1)\right]=\left[1-Py+xP-1\right]^2$$

$$\Rightarrow \left[x^2-Px^2y+y^2xP-y^2\right]\left[x-P\cancel{xy}+P\cancel{xy}-y\right]=P^2(x-y)^2$$

$$\Rightarrow \left\{(x^2-y^2)-Pxy[x-y]\right\}[x-y]=P^2(x-y)^2$$

$$\cancel{(x-y)}\left[(x+y)-pxy\right]\cancel{(x-y)} = p^2\cancel{(x-y)^2}$$

$$\Rightarrow x+y-pxy = p^2$$

$$\Rightarrow v-pu = p^2$$

$$\Rightarrow v = up+p^2$$

which is a Clairaut's equation

$\therefore$ The general solution is

$$v = cu+c^2$$

i.e, $x+y = cxy+c^2$

$$\Rightarrow c^2+xyc-(x+y)=0 \quad \text{—— (2)}$$

from (2), the c-discriminant relation is

$$(xy)^2+4(x+y)=0$$

$$\Rightarrow \boxed{x^2y^2+4(x+y)=0.}$$

which is the required singular solution.

IAS-2002

6(c). By using the method of variation of parameters find the solution of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$ with $y(0)=0$ and $\left(\frac{dy}{dx}\right)_{x=0}=0$.

Sol: Given $(D^2-2D+1)y=xe^x\sin x$, $D\equiv d/dx$ ——— (1)

Consider $(D^2-2D+1)y=0$ or $(D-1)^2y=0$ ... (2)

Auxiliary equation of (2) is $(D-1)^2=0$ so that $D=1,1$.

$\therefore$ C.F. of (1) $=(C_1+C_2x)e^x=C_1e^x+C_2xe^x$ ... (3)

Let $u=e^x$, $v=xe^x$ and $R=xe^x\sin x$ ... (4)

Here $W=\begin{vmatrix}u & v\\ u_1 & v_1\end{vmatrix}=\begin{vmatrix}e^x & xe^x\\ e^x & e^x+xe^x\end{vmatrix}=e^{2x}\neq 0$ ... (5)

Then, P.I. of (1) $=u\,f(x)+v\,g(x)$ ... (6)

where $f(x)=-\int\frac{vR}{W}dx=-\int\frac{xe^x(xe^x\sin x)}{e^{2x}}dx=-\int x^2\sin x\,dx$

$$=-\{x^2(-\cos x)-(2x)(-\sin x)+(2)(\cos x)\}$$

and $g(x)=\int\frac{uR}{W}dx=\int\frac{e^x(xe^x\sin x)}{e^{2x}}dx=\int x\sin x\,dx$

$$=(x)(-\cos x)-(1)(-\sin x)=\sin x-x\cos x$$

P.I. of (1) $=e^x(x^2\cos x-2x\sin x-2\cos x)+xe^x(\sin x-x\cos x)$, by (6)

$$=-xe^x\sin x-2e^x\cos x$$

Hence the general solution of (1) is $y=$ C.F. + P.I.,

$=C_1e^x+C_2xe^x-xe^x\sin x-2e^x\cos x=e^x(C_1+C_2x-x\sin x-2\cos x)$ ... (7)

Given that $y=0$ when $x=0$. Hence (7) gives $0=C_1-2$ or $C_1=2$.

Putting $C_1=2$ in (7) we get

$$y=e^x(2+C_2x-x\sin x-2\cos x) \quad ...(8)$$

$\Rightarrow dy/dx=e^x(2+C_2x-x\sin x-2\cos x)+e^x\{C_2-(\sin x+x\cos x)+2\sin x\}$

Given that $dy/dx=0$ when $x=0$. So the above equation gives $0=C_2$.

Putting $C_1=2$ and $C_2=0$ in (8) the required solution is

$$y=e^x(2-x\sin x-2\cos x)$$

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Using the method of variation of parameters,

6(d).

Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Sol: Given $$y_2 + 4y = 4\tan 2x. \qquad \ldots (1)$$

Consider $y_2 + 4y = 0$ or $(D^2+4)y = 0$ ... (2)

Auxiliary equation of (2) is $D^2 + 4 = 0$ so that $D = \pm 2i$.

$\therefore$ C.F. of (1) $= C_1 \cos 2x + C_2 \sin 2x$, C_1, C_2 being arbitrary constants. (3)

Let $u = \cos 2x$, $v = \sin 2x$ and $R = 4\tan 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0 \qquad \ldots (5)$$

Then, P.I. of (1) $= uf(x) + vg(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W} dx = -4\int \frac{\sin 2x \tan 2x}{2} dx = -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= 2\int (\cos 2x - \sec 2x) dx = \sin 2x - \log(\sec 2x + \tan 2x)$$

and $$g(x) = \int \frac{uR}{W} dx = 4\int \frac{\cos 2x \tan 2x}{2} dx = -\cos 2x$$

$\therefore$ P.I. of (1)

$$= (\cos 2x)\{\sin 2x - \log(\sec 2x + \tan 2x)\} + (\sin 2x)(-\cos 2x), \text{ by (6)}$$

or $$\text{P.I. of (1)} = -\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the general solution of (1) is y = C.F. + P.I. *i.e.,*

$$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$$

IAS-2000

15M 2000

6(a). Reduce $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q, R are functions of x, to the normal form. Hence solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$.

Sol Given equation is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— (1)

Let $y = uv$ —— (2) be the g.s of (1)

where u, v are functions of x.

Now $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ —— (3)

and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$ —— (4)

(1) $\equiv u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$.

$\Rightarrow u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$ —— (5)

To remove the first derivative $\frac{dv}{dx}$ in (5)

choose u such that $Pu + 2\frac{du}{dx} = 0$ —— (6)

$\Rightarrow \frac{du}{dx} = -\frac{P}{2}u$

$\Rightarrow \frac{du}{u} = -\frac{1}{2}P(x)\,dx$

$\Rightarrow \log u = -\frac{1}{2}\int P(x)\,dx$.

$\Rightarrow u = e^{-\frac{1}{2}\int P\,dx}$

(6) $\equiv \boxed{\frac{du}{dx} = -\frac{1}{2}Pu}$ —— (7)

$\frac{d^2u}{dx^2} = -\frac{1}{2}P\frac{du}{dx} - \frac{1}{2}u\frac{dP}{dx}$

$\boxed{\frac{d^2u}{dx^2} = -\frac{1}{2}P\left(-\frac{1}{2}Pu\right) - \frac{1}{2}u\frac{dP}{dx}}$ —— (8)

(5) $\equiv u\dfrac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\dfrac{dP}{dx}\right)uv = R$

$\Rightarrow \dfrac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\dfrac{dP}{dx}\right)v = \dfrac{R}{u}$

$\Rightarrow \boxed{\dfrac{d^2v}{dx^2} + Iv = S}$ ——— (9)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\dfrac{dP}{dx}$; $S = \dfrac{R}{u}$

which is the required normal form of (1).

$\therefore$ The g.s of (1) is $y = uv$
where $u = e^{-\frac{1}{2}\int P\,dx}$ and
v is given by (9).

To solve $y'' - 4xy' + (4x^2-1)y = -3e^{x^2}\sin 2x$. ——— (A)

comparing (A) with
$y'' + P(x)y' + Q(x)y = R(x)$

$P = -4x$; $Q = 4x^2-1$; $R = -3e^{x^2}\sin 2x$.

To remove the first derivative we choose $u = e^{-\frac{1}{2}\int P\,dx}$

$= e^{-\frac{1}{2}\int(-4x)\,dx}$

$= e^{x^2}$ ——— (B)

Let $y = uv$ ——— (C) be the g.s of (A)

then v is given by the

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normal form $\frac{d^2v}{dx^2} + Iv = S$ —— (D)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$S = \frac{R}{u}$

Now $I = 4x^2 + 1 - \frac{1}{4}(16x^2) - \frac{1}{2}(-4)$

$= 1.$

and $S = -3\sin 2x.$

$\therefore$ (D) $\equiv \frac{d^2v}{dx^2} + v = -3\sin 2x$

$\Rightarrow (D^2+1)v = -3\sin 2x$ —— (E)

AE of is $D^2+1=0$

$\Rightarrow D = \pm i$

$\therefore$ C.F $= c_1\cos x + c_2\sin x.$

$P.I = \frac{1}{D^2+1}(-3\sin 2x)$

$= \frac{-3}{-3}\sin 2x.$

$= \sin 2x.$

$\therefore$ G.S of (E) is $v = c_1\cos x + c_2\sin x + \sin 2x$

$\therefore$ (C) $\equiv y = e^{x^2}(c_1\cos x + c_2\sin x) + \sin 2x.$

which is the required g.s of (A)