IFOS - 2010

Mechanics

Q5d) The moment of inertia of an ellipse about a tangent with slope $\tan\theta$ is $\frac{5}{4}M(a^2\sin^2\theta + b^2\cos^2\theta) \Rightarrow I_1$

Any tangent $\perp$ to the given tangent will have slope $-\cot\theta$ or $\tan(\frac{\pi}{2}+\theta)$

$\therefore$ Replacing θ by $\frac{\pi}{2}+\theta$ in I_1

$$I_2 = \frac{5}{4}M\left(a^2\sin^2(\theta+\pi/2) + b^2\cos^2(\theta+\tfrac{\pi}{2})\right)$$

$$= \frac{5}{4}M(a^2\cos^2\theta + b^2\sin^2\theta)$$

Adding I_1 and I_2

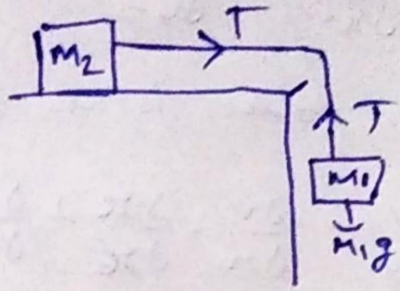
$$I_1+I_2 = \frac{5}{4}M\left(a^2\sin^2\theta + a^2\cos^2\theta + b^2\sin^2\theta + b^2\cos^2\theta\right)$$

$$= \frac{5}{4}M(a^2+b^2) = \text{Constant}$$

$\therefore$ Sum of moments of Inertia about two $\perp$ tangents is always same for ellipse.

Q 8(a)

For the given system if m_1 and m_2 move with acceleration "a" then



$$m_1 g - T = m_1 a \quad -(1)$$

$$T = m_2 a \quad -(2)$$

Adding (1) and (2)

$$m_1 g = (m_1 + m_2) a$$

$$a = \frac{m_1 g}{m_1 + m_2}, \quad \text{Hence } T = \frac{m_1 m_2 g}{m_1 + m_2}$$

Now, If m_2 is doubled.

$$m_1 g - T_1 = m_1 a_1 \quad -(3)$$

$$T_1 = 2m_2 a_1 \quad -(4)$$

Adding (3) and (4)

$$m_1 g = (m_1 + 2m_2) a_1$$

$$a_1 = \frac{m_1 g}{m_1 + 2m_2} \quad \therefore T_1 = \frac{2m_2 m_1 g}{m_1 + 2m_2}$$

It is given that $T_1 = \frac{3}{2} T$

$$\frac{2m_2 m_1 g}{m_1 + 2m_2} = \frac{3}{2} \frac{m_1 m_2 g}{m_1 + m_2} \Rightarrow 4m_1 + 4m_2 = 3m_1 + 6m_2$$

$$\Rightarrow \boxed{\therefore m_1 = 2m_2 \quad \text{or} \quad \frac{m_1}{m_2} = 2:1}$$

Fluid dynamics

Q5(e) Given $\psi = xy$

i) $\frac{\partial \psi}{\partial x} = y$ and $\frac{\partial \psi}{\partial y} = x$

Now, if velocity $q = u\hat{i} + v\hat{j}$

then $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} = -x$

and $v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} = y$

$\therefore q = -x\hat{i} + y\hat{j}$

$$\text{Curl } q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\boxed{\because \text{Curl } q = 0 \quad \therefore \text{ Flow is irrotational}}$

(ii) $\frac{\partial \psi}{\partial x} = y \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial y}{\partial x} = 0$

$\frac{\partial \psi}{\partial y} = x \Rightarrow \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial x}{\partial y} = 0$

$\therefore \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0} \Rightarrow \psi$ satisfies Laplace equation.

Now, $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y}$

$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \Rightarrow \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y \partial x}$

$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$

$\therefore \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$ Hence, ϕ also satisfies the Laplace equation.

Q7(b) Given $\phi = (x-t)(y-t)$

If fluid velocity $q = u\hat{i} + v\hat{j}$

then $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}[(x-t)(y-t)] = -(y-t)$

$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[(x-t)(y-t)] = -(x-t)$

ϕ will represent two dimensional motion of fluid if equation of continuity is satisfied.

i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\frac{\partial}{\partial x}(t-y) + \frac{\partial}{\partial y}(t-x) = 0$

$\Rightarrow 0 + 0 = 0$

is satisfied. Hence ϕ is possible fluid motion.

The streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow \frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$$

$$\Rightarrow (x-t)\,dx - (y-t)\,dy = 0$$

$$\Rightarrow \frac{(x-t)^2}{2} - \frac{(y-t)^2}{2} = c_1$$

$$\Rightarrow \boxed{(x-t)^2 - (y-t)^2 = c}$$ are the streamlines

where c is constant

Q 8(c) According to Navier's stroke theorem

$$\frac{Dq}{Dt} = \vec{B} + \nu \nabla^2 q - \vec{\nabla} \frac{p}{\rho}$$

As there is no external force $\therefore B = 0$

$$\frac{Dq}{Dt} = \nu \nabla^2 q - \nabla \frac{p}{\rho}$$

$$\Rightarrow \frac{\partial q}{\partial t} + \nabla\left(\frac{q^2}{2}\right) + \Omega \times q = \nu \nabla^2 q - \frac{\nabla p}{\rho}$$

Taking curl of the equation

$$\nabla \times \left(\frac{\partial q}{\partial t} + \nabla\left(\frac{q^2}{2}\right) + (\Omega \times q) \right) = \nabla \times (\nu \nabla^2 q) - \nabla \times \left(\frac{\nabla p}{\rho}\right)$$

Fluid is incompressible $\therefore \rho =$ constant, $\nu =$ constant

$$\nabla \times \frac{\partial q}{\partial t} + \nabla \times \left(\nabla \frac{q^2}{2}\right) + \nabla \times (\Omega \times q) = \nu \nabla \times (\nabla^2 q) - \frac{1}{\rho} \nabla \times (\nabla p)$$

$\because$ Curl (Gradient ϕ) $= 0$

$$\Rightarrow \nabla \times \frac{\partial q}{\partial t} + 0 + \nabla \times (\Omega \times q) = \nu \nabla^2 (\nabla \times q) - 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\nabla \times q) + \nabla \times (\Omega \times q) = \nu \nabla^2 (\nabla \times q)$$

$$\Rightarrow \frac{\partial}{\partial t} (\Omega) + \nabla \times (\Omega \times q) = \nu \nabla^2 (\Omega)$$

$\because \nabla \times (A \times B) = A \operatorname{div} B - B \operatorname{div} A + (B \cdot \nabla) A - (A \cdot \nabla) B$

$$\Rightarrow \frac{\partial}{\partial t} (\Omega) + \Omega (\overset{0}{\nabla \cdot q}) - q (\overset{0}{\nabla \cdot \Omega}) + (q \cdot \nabla)\Omega - (\Omega \cdot \nabla) q$$
$$= \nu \nabla^2 \Omega$$

$$\Rightarrow \cancel{\frac{\partial}{\partial t} (\Omega) + \Omega (\nabla \cdot q) - 0 + (q \cdot \nabla)\Omega}$$

$$\frac{\partial \Omega}{\partial t} + 0 - 0 + (q \cdot \nabla)\Omega - (\Omega \cdot \nabla) q = \nu \nabla^2 \Omega$$

$$\Rightarrow \frac{\partial \Omega}{\partial t} + (q \cdot \nabla) \Omega = \nu \nabla^2 (\Omega) + (\Omega \cdot \nabla) q$$

$$\Rightarrow \boxed{\frac{D\Omega}{Dt} = \nu\nabla^2(\Omega) + (\Omega\cdot\nabla)q}$$

Hence Proved