Centre of Gravity

§ 1. Centre of Gravity. On account of the attraction of the earth (*known as gravity*) every particle on the surface of the earth is attracted towards the centre of the earth by a force proportional to its mass, called the **weight of the particle.**

A rigid body is considered as a collection of particles, rigidly connected with one another, on which are acting the weights of the particles. Such weights are considered to be parallel forces. The resultant of these forces is called the weight of the body and it always passes through a fixed point. This point is called the *centre of gravity* of the body.

Definition. *The* **centre of gravity** *of a body is the point, fixed relative to the body, through which the line of action of the weight of the body, always passes, whatever be the position of the body, provided that its size and shape remain unaltered.*

Centre of gravity is usually written in brief as C.G.

Note 1. The centre of gravity of a body does not necessarily lie in the body itself.

Note 2. The centre of mass (C.M.) of a body practically coincides with its centre of gravity (C.G.). Sometimes the words 'centroid' or 'centre of inertia' are used in place of the centre of gravity.

§ 2. Determination of the C.G. by integration. If a number of particles of masses $m_1, m_2, m_3, \ldots$ be placed at the points (x_1, y_1), (x_2, y_2), $(x_3, y_3), \ldots$ referred to two rectangular axes, then the coordinates $(\bar{x}, \bar{y})$ of the centre of gravity (C.G.) of the body consisting of those particles are given by

$$\bar{\mathbf{x}} = \frac{\Sigma \mathbf{m}_1 \mathbf{x}_1}{\Sigma \mathbf{m}_1} \text{ and } \bar{\mathbf{y}} = \frac{\Sigma \mathbf{m}_1 \mathbf{y}_1}{\Sigma \mathbf{m}_1}.$$

These results are a simple consequence of a theorem on the moments of a system of parallel forces and their resultant.

In the case of continuous distribution of matter, the summations can be replaced by definite integrals. Then the C.G. of the body, is given by

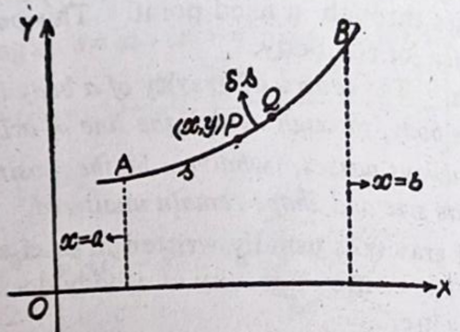
$$\bar{x}=\frac{\int x\,dm}{\int dm} \text{ and } \bar{y}=\frac{\int y\,dm}{\int dm},$$

where (x, y) is the C.G. of an elementary mass dm of the given matter.

Important remark. If a body is symmetrical about a line, the C.G of the body always lies on the line of symmetry.

§ 3. Centre of gravity of an arc. (Meerut 1976)

Let AB be an arc of the curve $y=f(x)$ extending from $x=a$ to $x=b$. Take $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ any two neighbouring points on the arc. Let A be a fixed point on the curve, arc $AP=s$ and arc $PQ=\delta s$. Let ρ be the density of the arc of the



curve at the point P; then ρ may be considered constant from P to Q as arc $PQ=\delta s$ is very small. Hence the mass δm of the elementary arc $PQ=\rho\,\delta s$. Also the centre of gravity of this elementary arc can be approximately taken as the point $P(x, y)$ because the point Q is very close to P and ultimately we have to proceed to the limits as $Q \to P$.

Thus the co-ordinates $(\bar{x}, \bar{y})$ of the centre of gravity the arc AB are given by the formulae

$$\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int x.\rho\,ds}{\int \rho\,ds},$$

and

$$\bar{y}=\frac{\int y\,dm}{\int dm}=\frac{\int y.\rho\,ds}{\int \rho\,ds},$$

where in all the above integrals the limits of integration are to be taken from one end of the arc AB to the other end. If the density ρ is constant, the above formulae take the form

$$\bar{x}=\frac{\int x\,ds}{\int ds},\quad \bar{y}=\frac{\int y\,ds}{\int ds}.$$

The value of the integral in the denominator, *i.e.*, the value of $\int ds$ gives us the length of the arc under consideration.

To perform integration ds will be changed from the following formulae of differential calculus :

$$ds=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx \quad \text{or} \quad ds=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy,$$

if the equation of the curve is in Cartesian co-ordinates;

$$ds=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta \quad \text{or} \quad ds=\sqrt{\left\{r^2\left(\frac{d\theta}{dr}\right)^2+1\right\}}\,dr,$$

if the equation of the curve is in polar coordinates;

or

$$ds=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt$$

if the equations of the curve are in the parametric form $x=f(t),\ y=\phi(t)$.

Note. If the arc under consideration is symmetrical about a straight line, then the centre of gravity will be on the line of symmetry.

Illustrative Examples

Ex. 1. *Find the centre of gravity of the arc of the astroid* $x^{2/3}+y^{2/3}=a^{2/3}$ *lying in the first quadrant.*

(Meerut 1976, 82; Agra 83)

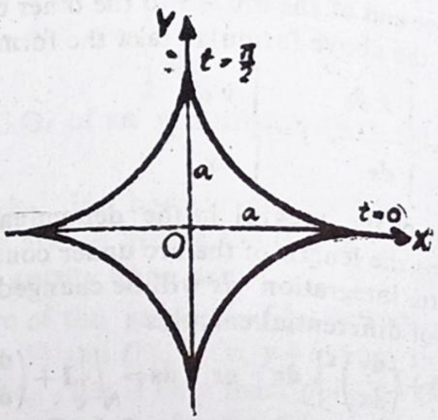
Sol. The equation of the curve is

$$x^{2/3}+y^{2/3}=a^{2/3}. \quad \ldots(1)$$

It meets the x-axis at the points $(\pm a, 0)$ and the y-axis at the points $(0, \pm a)$. The curve (1) is symmetrical about the line $y=x$ and so the arc lying in the first quadrant is symmetrical about the line $y=x$ and its C.G. must lie on this line. Therefore $\bar{x}=\bar{y}$ where $(\bar{x}, \bar{y})$ are the coordinates of the required C.G. of the arc lying in the first quadrant.

Now differentiating (1) w.r.t. x, we get

$$\tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}\frac{dy}{dx}=0, \text{ or } \frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}.$$



$$\therefore \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\frac{y^{2/3}}{x^{2/3}}\right\}}=\sqrt{\left\{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right\}}$$

$$=\sqrt{\left(\frac{a^{2/3}}{x^{2/3}}\right)}, \text{ from (1)}$$

or $\quad ds=(a/x)^{1/3}\,dx.$

$$\text{Hence } \bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_0^a x.(a/x)^{1/3}\,dx}{\int_0^a (a/x)^{1/3}\,dx}=\frac{\int_0^a x^{2/3}\,dx}{\int_0^a x^{1/3}\,dx}$$

[Note that we cannot cancel any term containing x in the numerator and the denominator because x is a variable].

$$=\left[\frac{3}{5}x^{5/3}\right]_0^a\div\left[\frac{3}{2}x^{2/3}\right]_0^a=\frac{2a}{5}=\bar{y}, \text{ (by symmetry).}$$

$\therefore \quad \bar{x}=\bar{y}=2a/5.$

Aliter. The given curve, in parametric form, can be written as

$$x=a\cos^3 t,\ y=a\sin^3 t. \qquad \text{...(2)}$$

Differentiating (2) w.r.t. t, we get

$$dx/dt=-3a\cos^2 t\sin t \text{ and } dy/dt=3a\sin^2 t\cos t.$$

$$\therefore \frac{ds}{dt}=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}=\sqrt{\{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2\}}$$

$$=3a\cos t\sin t\sqrt{(\cos^2 t+\sin^2 t)}=3a\cos t\sin t \qquad \text{...(3)}$$

or $\quad ds=3a\sin t\cos t\,dt.$

Also for the given arc lying in the first quadrant t varies from 0 to $\pi/2$.

$$\text{Hence } \bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_0^{\pi/2} a\cos^3 t\; 3a\cos t\sin t\,dt}{\int_0^{\pi/2} 3a\cos t\sin t\,dt}, \text{ from (2) and (3)}$$

$$=\frac{a\int_0^{\pi/2}\cos^4 t\sin t\,dt}{\int_0^{\pi/2}\cos t\sin t\,dt}=\frac{a\left[-\frac{\cos^5 t}{5}\right]_0^{\pi/2}}{\left[-\frac{\cos^2 t}{2}\right]_0^{\pi/2}}=\frac{2a}{5}.$$

$\therefore$ By symmetry $\bar{y}=\bar{x}=2a/5.$

Ex. 2. *Find the centroid of the arc of the catenary*

$$y=c\cosh(x/c),$$

which is included between the vertex (0, c) *and any point* (x, y).

(Kanpur 1981, 86)

Sol. The given curve is $y=c\cosh(x/c)$. ...(1)

Differentiating (1) w.r.t. x, we get $dy/dx=\sinh(x/c)$.

$$\therefore \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\{1+\sinh^2(x/c)\}}=\cosh(x/c)$$

or $\quad ds=\cosh(x/c)\,dx.$

The vertex of the given catenary is the point $(0, a)$ and (x, y) is any other given point on the curve. We have to find the centroid $(\bar{x}, \bar{y})$ of the arc of the catenary lying between $x=0$ and $x=x$.

$$\text{We have } \bar{x}=\frac{\int_{x=0}^{x} x\,ds}{\int_{x=0}^{x} ds}=\frac{\int_0^x x\cosh\frac{x}{c}\,dx}{\int_0^x \cosh\frac{x}{c}\,dx}, \quad \left[\because ds=\cosh\frac{x}{c}\,dx\right]$$

$$=\frac{\left[x.c\sinh\frac{x}{c}\right]_0^x-c\int_0^x\sinh\frac{x}{c}\,dx}{\left[c\sinh\frac{x}{c}\right]_0^x}=\frac{x\sinh\frac{x}{c}-c\left[\cosh\frac{x}{c}\right]_0^x}{\sinh(x/c)}$$

$$=\frac{x\sinh(x/c)-c\{\cosh(x/c)-1\}}{\sinh(x/c)}=x-c\left\{\frac{c\cosh(x/c)-c}{c\sinh(x/c)}\right\}.$$

If s denotes the length of the arc of the catenary extending from the vertex $(0, c)$ to the point (x, y), then

$$s=\int_{x=0}^{x} ds=\int_0^x\cosh\frac{x}{c}\,dx=\left[c\sinh\frac{x}{c}\right]_0^x=c\sinh\frac{x}{c}.$$

$$\therefore \bar{x}=x-c\left\{\frac{y-c}{s}\right\}, \left[\because y=c\cosh\frac{x}{c} \text{ and } s=c\sinh\frac{x}{c}\right].$$

Also $\bar{y}=\dfrac{\int y\,ds}{\int ds}=\dfrac{\int_0^x c\cosh\frac{x}{c}\cosh\frac{x}{c}\,dx}{\int_0^x \cosh\frac{x}{c}\,dx}=\dfrac{\frac{1}{2}c\int_0^x 2\cosh^2\frac{x}{c}\,dx}{\int_0^x \cosh\frac{x}{c}\,dx}$

$$=\frac{\frac{1}{2}c\int_0^x\left\{1+\cosh\frac{2x}{c}\right\}dx}{\left[c\sinh\frac{x}{c}\right]_0^x}=\frac{1}{2}\,\frac{\left[x+\frac{1}{2}c\sinh\frac{2x}{c}\right]_0^x}{\sinh(x/c)}$$

$$=\frac{1}{2}\left[\frac{x+c\sinh(x/c)\cosh(x/c)}{\sinh(x/c)}\right]\times\frac{c}{c}$$

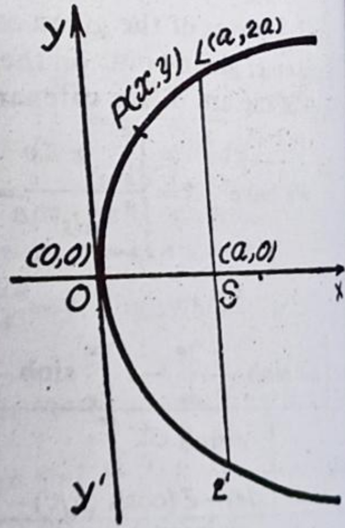
[Note]

$$=\frac{1}{2}\left[\frac{cx+\{c\sinh(x/c)\}\{c\cosh(x/c)\}}{c\sinh(x/c)}\right]=\frac{1}{2}\left[\frac{cx+sy}{s}\right].$$

Hence $\bar{x}=x-\dfrac{c(y-c)}{s}$ and $\bar{y}=\frac{1}{2}\left[\dfrac{cx+sy}{s}\right]$.

Ex. 3. *Find the C.G. of the arc of the parabola $y^2=4ax$ extending from the vertex to an extremity of the latus rectum.*

(Meerut 1979 S)



Sol. The given curve is

$y^2=4ax$. ...(1)

The origin $O(0,0)$ is the vertex and the point $L(a,2a)$ is an extremity of the latus rectum of this parabola. We have to find the C.G. $(\bar{x},\bar{y})$ of the arc OL.

Differentiating (1) w.r.t. 'x', we get

$2y\,(dy/dx)=4a,$

so that $dy/dx=2a/y$

and $dx/dy=y/2a.$

Now

$$\frac{ds}{dy}=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}=\sqrt{\left\{1+\frac{y^2}{4a^2}\right\}}=\frac{1}{2a}\sqrt{(y^2+4a^2)}, \quad ...(2)$$

and
$$\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\frac{4a^2}{y^2}\right\}}$$
$$=\sqrt{\left(\frac{y^2+4a^2}{y^2}\right)}=\sqrt{\left(\frac{4ax+4a^2}{4ax}\right)}$$
$$=\sqrt{(x+a)}/\sqrt{x}. \quad ...(3)$$

We have $\bar{x}=\dfrac{\int x\,ds}{\int ds}$, $\bar{y}=\dfrac{\int y\,ds}{\int ds}$, the limits of integration extending from O to L in each of the integrals.

Now the Nr. of $\bar{x}=\int x\,ds$, between the suitable limits

$=\int_0^a x\dfrac{ds}{dx}\,dx$ [Note]

$=\int_0^a x\dfrac{\sqrt{(x+a)}}{\sqrt{x}}\,dx$, from (3)

$$=\int_0^a \sqrt{x}\sqrt{(x+a)}\,dx=\int_0^a\sqrt{(x^2+ax)}\,dx=\int_0^a\sqrt{\left\{\left(x+\frac{a}{2}\right)^2-\frac{a^2}{4}\right\}}\,dx$$

$$=\left[\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{(x^2+ax)}-\frac{a^2}{8}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{(x^2+ax)}\right\}\right]_0^a$$

$$=[\tfrac{1}{2}.\tfrac{3}{2}a.a\sqrt{2}-\tfrac{1}{8}a^2\log(\tfrac{3}{2}a+a\sqrt{2})-0+\tfrac{1}{8}a^2\log(\tfrac{1}{2}a)]$$

$$=\left[\frac{3}{4}a^2\sqrt{2}-\frac{1}{8}a^2\log\frac{\frac{3}{2}a+a\sqrt{2}}{\frac{1}{2}a}\right]=\frac{3}{4}a^2\sqrt{2}-\frac{1}{8}a^2\log(3+2\sqrt{2})$$

$$=\tfrac{3}{4}a^2\sqrt{2}-\tfrac{1}{8}a^2\log(1+\sqrt{2})^2=\tfrac{3}{4}a^2\sqrt{2}-\tfrac{1}{8}.2a^2\log(1+\sqrt{2})$$

$$=\tfrac{1}{4}a^2.[3\sqrt{2}-\log(1+\sqrt{2})].$$

Also the Dr. of $\bar{x}=\int ds$, between the suitable limits

$=\int_0^{2a}\dfrac{ds}{dy}\,dy$ [Note]

$=\int_0^{2a}\dfrac{1}{2a}\sqrt{(y^2+4a^2)}\,dy$, from (2)

$$=\frac{1}{2a}\left[\tfrac{1}{2}y\sqrt{(y^2+4a^2)}+\tfrac{1}{2}.4a^2\log\{y+\sqrt{(y^2+4a^2)}\}\right]_0^{2a}$$

$$=\frac{1}{2a}\left[a^2.2\sqrt{2}+2a^2\log\frac{2a+2\sqrt{2}a}{2a}\right]=a[\sqrt{2}+\log(1+\sqrt{2})]$$

= the Dr. of $\bar{y}$.

Further the Nr. of $\bar{y}=\int y\,ds$, between the suitable limits

$=\int_0^{2a}y\dfrac{ds}{dy}\,dy$ [Note]

$$=\int_0^{2a}y.\frac{1}{2a}\sqrt{(4a^2+y^2)}\,dy=\frac{1}{2a}.\frac{1}{2}\int_0^{2a}(4a^2+y^2)^{1/2}(2y)\,dy$$

$$=\frac{1}{4a}\left[(4a^2+y^2)^{3/2}.\frac{2}{3}\right]_0^{2a}=\frac{1}{6a}[(8a^2)^{3/2}-(4a^2)^{3/2}]$$

$$=\frac{1}{6a}.a^3.4^{3/2}\,[2^{3/2}-1]=\frac{4}{3}\,a^2\,(2\sqrt{2}-1).$$

$$\therefore\ \bar{x}=\frac{\text{Nr. of }\bar{x}}{\text{Dr. of }\bar{x}}=\frac{\frac{1}{4}a^2\,[3\sqrt{2}-\log(1+\sqrt{2})]}{a\,[\sqrt{2}+\log(1+\sqrt{2})]}=\frac{a}{4}\left[\frac{3\sqrt{2}-\log(1+\sqrt{2})}{\sqrt{2}+\log(1+\sqrt{2})}\right],$$

$$\text{and } \bar{y}=\frac{\text{Nr. of }\bar{y}}{\text{Dr. of }\bar{y}}=\frac{\frac{4}{3}a^2\,(2\sqrt{2}-1)}{a\,[\sqrt{2}+\log(1+\sqrt{2})]}=\frac{4a\,(2\sqrt{2}-1)}{3\,[\sqrt{2}+\log(1+\sqrt{2})]}.$$

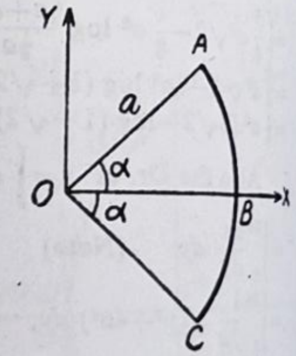
Remark 1. If $(\bar{x}, \bar{y})$ be the C.G. of the arc LOL', then $\bar{y}=0$, by symmetry. Also $\bar{x}$=the x-coordinate of the C.G. of the upper half OL, which we have just found.

Remark 2. If we have to find both $\bar{x}$ and $\bar{y}$ by integration, we should evaluate their numerators and denominators separately because the Dr. of $\bar{x}$ is obviously the same as the Dr. of $\bar{y}$. Thus we avoid the repitition of labour while evaluating the denominator.

Ex. 4. (a) *Find the position of the centre of gravity of an arc of a circle of radius a, which subtends an angle* 2α *at the centre.*

(Kanpur 1976, 78; Jiwaji 81; Delhi 81)

Sol. Take the centre O of the circle as origin. Let the arc ABC subtend an angle 2α at the centre and let the radius OB bisecting the arc ABC be taken as the x-axis so that the arc is symmetrical about the x-axis and $\angle AOB=\alpha$.



The parametric equations of the circle are

$$x=a\cos\theta,\ y=a\sin\theta. \quad \ldots(1)$$

$$\therefore\ dx/d\theta=-a\sin\theta \text{ and } dy/d\theta=a\cos\theta.$$

$$\therefore\ \frac{ds}{d\theta}=\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}=\sqrt{(a^2\sin^2\theta+a^2\cos^2\theta)}=a. \ldots(2)$$

If $(\bar{x}, \bar{y})$ be the centre of gravity of the arc ABC, then by symmetry (about the axis of x) $\bar{y}=0$. Also the x-coordinate of the C.G. of the arc ABC is the same as the x-coordinate of the C.G. of the upper half BA of this arc.

Evidently for the arc BA, θ varies from 0 to α.

$$\therefore\ \bar{x}=\frac{\int_{\theta=0}^{\alpha} x\,ds}{\int_{\theta=0}^{\alpha} ds}=\frac{\int_0^{\alpha} x\,\frac{ds}{d\theta}\,d\theta}{\int_0^{\alpha}\frac{ds}{d\theta}\,d\theta}=\frac{\int_0^{\alpha} a\cos\theta . a\,d\theta}{\int_0^{\alpha} a\,d\theta}$$

[from (1) and (2)]

$$=\frac{a^2\left[\sin\theta\right]_0^{\alpha}}{\left[a\theta\right]_0^{\alpha}}=\frac{a^2\sin\alpha}{a\alpha}=\frac{a\sin\alpha}{\alpha}.$$

Thus the C.G. of a circular arc subtending an angle 2α at the centre lies on the symmetrical radius and its distance from the centre is $(a\sin\alpha)/\alpha$.

Ex. 4. (b) *Find the C.G. of a uniform semi-circular wire.*

(Kanpur 1984; Andhra 76)

Sol. Take the centre O of the circle as origin. Let ABC be the semi-circular arc symmetrical about the central radius OB which is along the x-axis.

Then $\angle AOB=\pi/2$. Now proceed exactly as in Ex. 4. (a). Thus putting $\alpha=\pi/2$ in the result of Ex. 4 (a), we get

$$\bar{x}=\frac{a\sin(\pi/2)}{\pi/2}=\frac{2a}{\pi}, \text{ and } \bar{y}=0.$$

Hence the C.G. *of a semi-circular arc lies on its central radius at a distance* $2a/\pi$ *from the centre.* **(Remember).**

Ex. 4. (c) *Find the C.G. of a uniform circular wire of radius a in the form of a quadrant of a circle.* **(Meerut 73)**

Sol. Proceed as in Ex. 4 (a). Take the arc ABC in the form of a quadrant of a circle and let this arc be symmetrical about the radius OB which is along the x-axis. Here $2\alpha=\pi/2$ or $\alpha=\pi/4$.

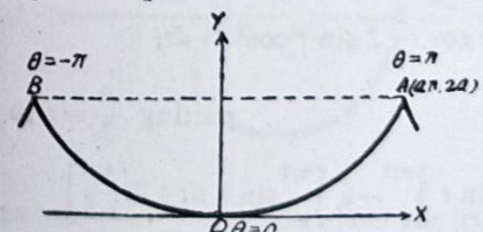
$$\therefore\ \bar{x}=\frac{a\sin(\pi/4)}{\pi/4}=\frac{a\,(1/\sqrt{2})}{\pi/4}=\frac{4a}{\pi\sqrt{2}}=\frac{2\sqrt{2}a}{\pi}$$

and $\bar{y}=0$, (by symmetry).

Ex. 5. *Find the centroid of the cycloid*

$$x=a\,(\theta+\sin\theta),\ y=a\,(1-\cos\theta)$$

which lies in the positive quadrant. **(Jiwaji 1982)**



Sol. The given equations of the cycloid are

$$x=a\,(\theta+\sin\theta),\ y=a\,(1-\cos\theta). \quad \ldots(1)$$

$$\therefore\ dx/d\theta=a\,(1+\cos\theta) \text{ and } dy/d\theta=a\sin\theta.$$

$$\therefore \frac{ds}{d\theta}=\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}=\sqrt{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}}$$
$$=a\sqrt{\{1+2\cos\theta+\cos^2\theta+\sin^2\theta\}}$$
$$=a\sqrt{(2+2\cos\theta)}=2a\cos\tfrac{1}{2}\theta. \qquad ...(2)$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the arc OA lying in the positive quadrant.

For the arc OA, θ varies from 0 to π. We have

$$\bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_0^\pi x\frac{ds}{d\theta}\,d\theta}{\int_0^\pi \frac{ds}{d\theta}\,d\theta}=\frac{\int_0^\pi a(\theta+\sin\theta).2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi 2a\cos\frac{1}{2}\theta\,d\theta}$$

from (1) and (2)

$$=\frac{a\int_0^\pi(\theta+\sin\theta)\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi\cos\frac{1}{2}\theta\,d\theta} \qquad ...(3)$$

and $$\bar{y}=\frac{\int y\,ds}{\int ds}=\frac{\int_0^\pi a(1-\cos\theta).2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi 2a\cos\frac{1}{2}\theta\,d\theta}$$, from (1) and (2)

$$=\frac{2a\int_0^\pi\sin^2\frac{1}{2}\theta\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi\cos\frac{1}{2}\theta\,d\theta}. \qquad ...(4)$$

Now Nr. of $\bar{x}=a\int_0^\pi(\theta+\sin\theta)\cos\frac{1}{2}\theta\,d\theta$

$$=a\int_0^\pi\left\{\theta\cos\tfrac{1}{2}\theta+2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right\}d\theta$$
$$=2a\int_0^{\pi/2}\{2t\cos t+2\sin t\cos^2 t\}\,dt,$$

putting $\frac{\theta}{2}=t$ so that $d\theta=2dt$

$$=2a\left[\left\{2t\sin t\right\}_0^{\pi/2}-2\int_0^{\pi/2}\sin t\,dt+2.\frac{1}{3.1}\right]$$
$$=2a\left[\pi-2.\left\{-\cos t\right\}_0^{\pi/2}+\frac{2}{3}\right]=2a\left[\pi-2+\frac{2}{3}\right]=2a\left[\pi-\frac{4}{3}\right].$$

Nr. of $\bar{y}=2a\int_0^\pi\sin^2\frac{1}{2}\theta\cos\frac{1}{2}\theta\,d\theta$

$$=2a\int_0^{\pi/2}\sin^2\phi\cos\phi.2d\phi, \text{ putting } \tfrac{1}{2}\theta=\phi$$
$$=4a\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi=4a\left[\frac{\sin^3\phi}{3}\right]_0^{\pi/2}=\frac{4a}{3},$$

and Dr. of $\bar{x}$=Dr. of $\bar{y}=\int_0^\pi\cos\frac{1}{2}\theta\,d\theta=\left[2\sin\frac{1}{2}\theta\right]_0^\pi=2(1-0)=2.$

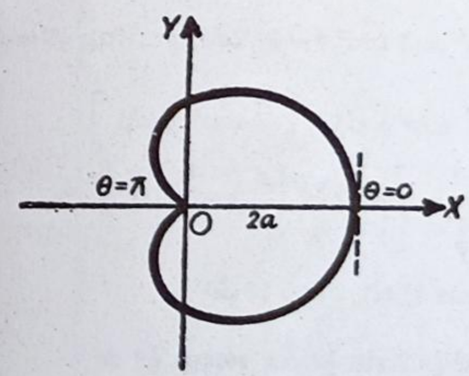
$\therefore$ From (3) and (4), we get

$$\bar{x}=\frac{2a\,[\pi-\frac{4}{3}]}{2}=a\left[\pi-\frac{4}{3}\right] \text{ and } \bar{y}=\frac{4a/3}{2}=\frac{2a}{3}.$$

Remark. If $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of one complete arch BOA of the above cycloid, then by symmetry $\bar{x}=0$. Also y-coordinate of the C.G. of the arc BOA is the same as the y-coordinate of the C.G. of the right half OA of this arc. So to find $\bar{y}$ proceed as in the above example.

Ex. 6. *Find the position of the arc of the cardioid $r=a(1+\cos\theta)$ lying above the initial line.* **(Kanpur 1980; Agra 84, 88)**

Sol. The given curve is
$$r=a(1+\cos\theta). \qquad ...(1)$$
Differentiating (1) w.r.t. θ, we get
$$dr/d\theta=-a\sin\theta.$$
$$\therefore\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{\{a^2(1+\cos\theta)^2+(-a\sin\theta)^2\}}$$
$$=a\sqrt{\{(1+\cos\theta)^2+\sin^2\theta\}}=\sqrt{\{2(1+\cos\theta)\}}=2a\cos\tfrac{1}{2}\theta. \qquad ...(2)$$



The given curve is symmetrical about the initial line and for the arc of the cardioid lying above the initial line θ varies from 0 to π.

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the arc lying above the initial line. Then

$$\bar{x}=\frac{\int_{\theta=0}^{\pi} x\,ds}{\int_{\theta=0}^{\pi} ds}=\frac{\int_0^{\pi} x\,\frac{ds}{d\theta}\,d\theta}{\int_0^{\pi} \frac{ds}{d\theta}\,d\theta}=\frac{\int_0^{\pi} (r\cos\theta).2a\cos\tfrac{1}{2}\theta\,d\theta}{\int_0^{\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta}$$

$[\because\ x=r\cos\theta]$

$$=\frac{\int_0^{\pi} a(1+\cos\theta).\cos\theta\cos\tfrac{1}{2}\theta\,d\theta}{\int_0^{\pi}\cos\tfrac{1}{2}\theta\,d\theta},$$ substituting for r from (1)

and

$$\bar{y}=\frac{\int_{\theta=0}^{\pi} y\,ds}{\int_{\theta=0}^{\pi} ds}=\frac{\int_0^{\pi} y\,\frac{ds}{d\theta}\,d\theta}{\int_0^{\pi} \frac{ds}{d\theta}\,d\theta}=\frac{\int_0^{\pi} r\sin\theta.2a\cos\tfrac{1}{2}\theta\,d\theta}{\int_0^{\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta}$$

$[\because\ y=r\sin\theta]$

$$=\frac{\int_0^{\pi} a(1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta\,d\theta}{\int_0^{\pi}\cos\tfrac{1}{2}\theta\,d\theta},\qquad [\because\ r=a(1+\cos\theta)]$$

Now Nr. of $\bar{x}$

$$=\int_0^{\pi} a(1+\cos\theta)\cos\theta.\cos\tfrac{1}{2}\theta\,d\theta$$

$$=a\int_0^{\pi} 2\cos^2\tfrac{1}{2}\theta.(2\cos^2\tfrac{1}{2}\theta-1).\cos\tfrac{1}{2}\theta\,d\theta$$

$$=2a\int_0^{\pi}\cos^3\tfrac{1}{2}\theta\,(2\cos^2\tfrac{1}{2}\theta-1)\,d\theta$$

$$=2a\int_0^{\pi/2}\cos^3\phi\,(2\cos^2\phi-1).2d\phi,$$ putting $\tfrac{1}{2}\theta=\phi$

$$=4a\left[2\int_0^{\pi/2}\cos^5\phi\,d\phi-\int_0^{\pi/2}\cos^3\phi\,d\phi\right]$$

$$=4a\left[2.\frac{4.2}{5.3.1}-\frac{2}{3.1}\right]=8a\left[\frac{8}{15}-\frac{1}{3}\right]=\frac{8}{5}\,a.$$

Also Nr. of $\bar{y}$

$$=\int_0^{\pi} a\,(1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta\,d\theta$$

$$=a\int_0^{\pi} 2\cos^2\tfrac{1}{2}\theta.2\sin\tfrac{1}{2}\theta.\cos\tfrac{1}{2}\theta.\cos\tfrac{1}{2}\theta\,d\theta$$

$$=4a\int_0^{\pi}\cos^4\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\,d\theta=-4a.2\int_0^{\pi}\cos^4\tfrac{1}{2}\theta\,(-\tfrac{1}{2}\sin\tfrac{1}{2}\theta)\,d\theta$$

$$=-8a\left[\frac{\cos^5\tfrac{1}{2}\theta}{5}\right]_0^{\pi}=-\frac{8a}{5}\,[0-1]=\frac{8a}{5}.$$

Further Dr. of $\bar{x}$=Dr. of $\bar{y}$

$$=\int_0^{\pi}\cos\tfrac{1}{2}\theta\,d\theta=\left[2\sin\tfrac{1}{2}\theta\right]_0^{\pi}=2.$$

$$\therefore\quad \bar{x}=\frac{\frac{8}{5}a}{2}=\frac{4a}{5}\ \text{and}\ \bar{y}=\frac{\frac{8}{5}a}{2}=\frac{4a}{5}.$$

Ex. 7. *Find the C.G. of the whole arc of the cardioid* $r=a\,(1+\cos\theta)$.

Sol. The cardioid is symmetrical about the initial line (*i.e.* x-axis). Therefore the C.G. of the whole arc of the cardioid will be on x-axis. So if $(\bar{x}, \bar{y})$ be the C.G. of the whole arc of the cardioid, we have $\bar{y}=0$ by symmetry. Also by symmetry the x-coordinate of the C.G. of the whole arc of the cardioid is the same as the x-coordinate of the C.G. of the upper half of the arc of this cardioid. Now proceeding as above, we get $\bar{x}=4a/5$.

Ex. 8. *O is the pole of the lemniscate of Bernouli* $r^2=a^2\cos 2\theta$ *and G is the centre of gravity of any arc PQ of the curve; prove that OG bisects the angle POQ.*

(Kanpur 1979, 80, 82; Rohilkhand 79; Agra 86)

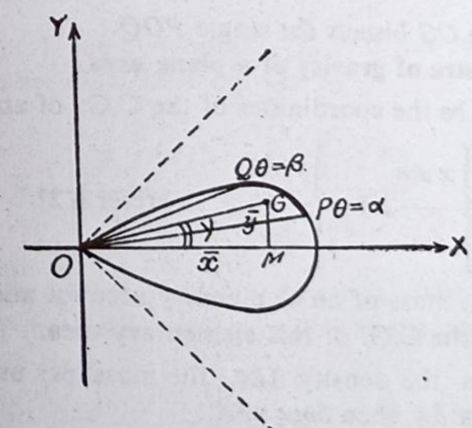
Sol. The given curve is $r^2=a^2\cos 2\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$2r\,(dr/d\theta)=-2a^2\sin 2\theta$$

or

$$dr/d\theta=-(a^2\sin 2\theta)/r.$$



$$\therefore\ \frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{\left\{r^2+\frac{a^4\sin^2 2\theta}{r^2}\right\}}=\sqrt{\left\{\frac{r^4+a^4\sin^2 2\theta}{r^2}\right\}}$$

$$=\frac{1}{r}\sqrt{(a^4\cos^2 2\theta+a^4\sin^2 2\theta)}=\frac{a^2}{r}.\qquad ...(2)$$

Let $(\bar{x}, \bar{y})$ be the cartesian coordinates of the centre of gravity (or the centroid) G of an arc PQ of the curve (1). If the vectorial angles of P and Q be α and β respectively, we have $\angle XOP=\alpha$ and $\angle XOQ=\beta$. Let OG make an angle γ with the initial line OX. Then

$$\tan\gamma=\frac{GM}{OM}=\frac{\bar{y}}{\bar{x}}=\frac{\int y\,ds\,/\int ds}{\int x\,ds\,/\int ds}=\frac{\int y\,ds}{\int x\,ds}$$

$$=\frac{\int_\alpha^\beta y\frac{ds}{d\theta}d\theta}{\int_\alpha^\beta x\frac{ds}{d\theta}d\theta}=\frac{\int_\alpha^\beta r\sin\theta.\frac{a^2}{r}\,d\theta}{\int_\alpha^\beta r\cos\theta.\frac{a^2}{r}\,d\theta}$$

$$\left[\because\ x=r\cos\theta,\ y=r\sin\theta \text{ and } \frac{ds}{d\theta}=\frac{a^2}{r}\right]$$

$$=\frac{\int_\alpha^\beta \sin\theta\,d\theta}{\int_\alpha^\beta \cos\theta\,d\theta}=\frac{\left[-\cos\theta\right]_\alpha^\beta}{\left[\sin\theta\right]_\alpha^\beta}=\frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}=\frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}$$

$$=\tan\tfrac{1}{2}(\alpha+\beta).$$

$$\therefore\quad \gamma=\tfrac{1}{2}(\alpha+\beta)\ i.e.,\ \angle GOX=\tfrac{1}{2}(\alpha+\beta).$$

Now $\angle GOP=\angle GOX-\angle POX=\frac{1}{2}(\alpha+\beta)-\alpha$
$=\frac{1}{2}(\beta-\alpha)=\frac{1}{2}\angle QOP.$

Therefore OG bisects the angle POQ.

§ 4. Centre of gravity of a plane area.

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of an area A.

Then $\bar{x}=\dfrac{\int x\,dm}{\int dm}$, $\bar{y}=\dfrac{\int y\,dm}{\int dm}$, [refer § 2]

where δm is the mass of an elementary area δA and (x, y) are the coordinates of the C.G. of this elementary area.

Now if ρ is the density (*i.e.*, the mass per unit area) of the elementary area δA, then $\delta m=\rho\delta A$.

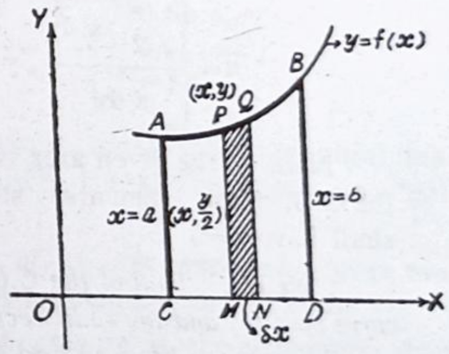
$$\therefore\quad \bar{x}=\frac{\int x\,\rho dA}{\int \rho\,dA} \text{ and } \bar{y}=\frac{\int y\,\rho dA}{\int \rho\,dA}.$$

In case the density ρ **be uniform** (*i.e.*, constant), we have

$$\bar{x}=\frac{\int x\,dA}{\int dA} \text{ and } \bar{y}=\frac{\int y\,dA}{\int dA}.$$

The elementary area δA is chosen according to the nature of the problem. The following cases arise.

Case I. *To find the C.G. of a plane area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$.*



Suppose we have to find the C.G. of the area $ACDB$. Divide this area into elementary strips by drawing lines parallel to the y-axis. Take an elementary strip $PMNQ$ where P and Q are the points (x, y) and $(x+\delta x, y+\delta y)$ respectively. The area δA of this elementary strip is equal to $y\,\delta x$ and the C.G. of this strip can approximately be taken as the middle point $(x, \frac{1}{2}y)$ of PM because the point Q is in the neighbourhood of P and ultimately we have to take limits when $Q\to P$.

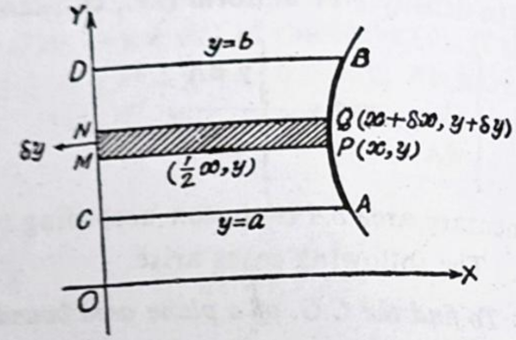
Hence from (1), the coordinates $(\bar{x}, \bar{y})$ of the required centre of gravity of the area $ACDB$ are given by

$$\bar{x}=\frac{\int_a^b x\,y\,dx}{\int_a^b y\,dx},\quad \bar{y}=\frac{\int_a^b \frac{y}{2}.\,y\,dx}{\int_a^b y\,dx}.$$

Note. If the given area is symmetrical about the x-axis, then the C.G. of the elementary strip will be the point $(x\ 0)$ and therefore we shall have $\bar{y}=0$.

Case II. *To find the C.G. of a plane area bounded by the curve $x=f(y)$, the y-axis and the abscissae $y=a$ and $y=b$.*

In this case the elementary area δA is the elementary strip $PMNQ$ parallel to the x-axis with area $x\delta y$ and the centre of gravity at the point $(\frac{1}{2}x, y)$.



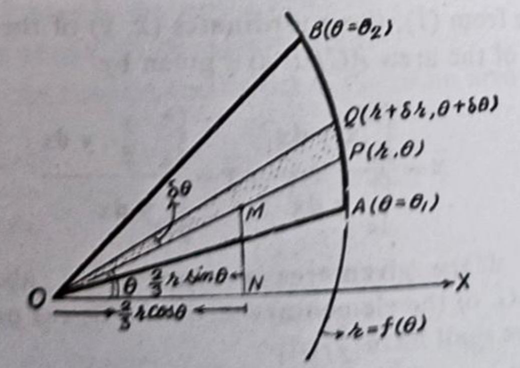
Hence from (1), the coordinates of the required C.G. are

$$\bar{x}=\frac{\int_a^b \frac{x}{2}.x\,dy}{\int_a^b x\,dy},\quad \bar{y}=\frac{\int_a^b y.x\,dy}{\int_a^b x\,dy}.$$

Note. If the given area is symmetrical about the y-axis, then the C.G. of the elementary strip will be $(0, y)$ and therefore we shall have $\bar{x}=0$.

Case III. *To find the C.G. of a sectorial area bounded by the curve $r=f(\theta)$ and the radii vectors $\theta=\theta_1$ and $\theta=\theta_2$.*

Suppose we have to find the C.G. of the sectorial area OAB where OA and OB are the radii vectors $\theta=\theta_1$ and $\theta=\theta_2$ respectively. Divide this area into elementary strips by drawing the radii vectors of the various points of the arc AB of the curve



$r=f(\theta)$. Take an elementary strip OPQ where $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ are any two neighbouring points. The area δA of this strip is equal to $\frac{1}{2}r^2\,\delta\theta$. Also this strip is triangular in shape and the point Q is in the neighbourhood of P. Therefore

the C.G. of this strip OPQ can approximately be taken as the point M on OP such that $OM=\frac{2}{3}OP$. The cartesian coordinates of M are $(\frac{2}{3}r\cos\theta, \frac{2}{3}r\sin\theta)$.

Hence from (1), the cartesian coordinates $(\bar{x}, \bar{y})$ of the required C.G. are given by

$$\bar{x}=\frac{\int_{\theta_1}^{\theta_2} \frac{2}{3}r\cos\theta\,\frac{1}{2}r^2\,d\theta}{\int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta}=\frac{\frac{2}{3}\int_{\theta_1}^{\theta_2} r^3\cos\theta\,d\theta}{\int_{\theta_1}^{\theta_2} r^2\,d\theta}$$

and

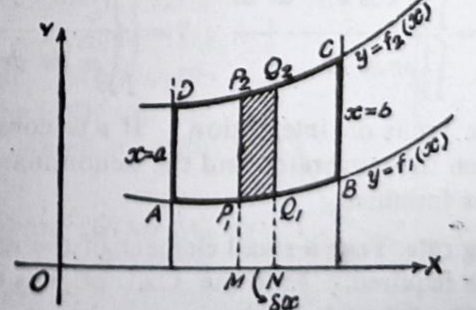
$$\bar{y}=\frac{\int_{\theta_1}^{\theta_2} \frac{2}{3}r\sin\theta.\frac{1}{2}r^2\,d\theta}{\int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta}=\frac{\frac{2}{3}\int_{\theta_1}^{\theta_2} r^3\sin\theta\,d\theta}{\int_{\theta_1}^{\theta_2} r^2\,d\theta}.$$

Note. If the given area is symmetrical about the initial line then the C.G. will lie on the initial line and therefore we shall have $\bar{y}=0$.

Case IV. *To find the C.G. of an area enclosed between two curves $y=f_1(x)$, $y=f_2(x)$ and the ordinates $x=a$ and $x=b$.*

In this case the elementary area δA is the elementary strip $P_1Q_1Q_2P_2$ drawn parallel to the y-axis and enclosed between the two curves $y=f_1(x)$ and $y=f_2(x)$. Let P_1 and P_2 be the points (x, y_1) and (x, y_2) respectively.

Now δA=area of the elementary strip $P_1Q_1Q_2P_2=(y_2-y_1).\delta x$, where $MN=\delta x$.



The C.G. of this strip $P_1Q_1Q_2P_2$ can approximately be taken as the middle point of P_1P_2. Thus the coordinates of the C.G. of the strip are $\left(x, \frac{y_1+y_2}{2}\right)$.

Hence from (1), the coordinates of the required C.G. are given by

$$\bar{x}=\frac{\int x\,dA}{\int dA}=\frac{\int_a^b x.(y_2-y_1)\,dx}{\int_a^b (y_2-y_1)\,dx},$$

and

$$\bar{y}=\frac{\int y\,dA}{\int dA}=\frac{\int_a^b \frac{1}{2}(y_2+y_1).(y_2-y_1)\,dx}{\int_a^b (y_2-y_1)\,dx}.$$

While evaluating the above integrals we shall have to replace y_1 and y_2 in terms of x from the equations $y=f_1(x)$ and $y=f_2(x)$ respectively.

Important Note. In case the area is enclosed between two curves, the limits of integration will be the values of x for the common points of intersection of the given curves.

Case V. Use of double integration. Sometimes it is convenient to consider an elementary area $\delta x\,\delta y$ or $r\,\delta\theta\,\delta r$. Then the C.G. will be given by

$$\bar{x}=\frac{\iint \rho x\,dx\,dy}{\iint \rho\,dx\,dy},\quad \bar{y}=\frac{\iint \rho y\,dx\,dy}{\iint \rho\,dx\,dy}$$

or

$$\bar{x}=\frac{\iint r\cos\theta.\rho r\,d\theta\,dr}{\iint \rho r\,d\theta\,dr},\quad \bar{y}=\frac{\iint r\sin\theta.\rho r\,d\theta\,dr}{\iint \rho r\,d\theta\,dr}$$

under proper limits of integration. If ρ is constant, it can be cancelled from the numerator and the denominator while applying the above formulae.

Working rule. Take a small element of the area whose centre of gravity is required. Find the C.G. of this elementary area. Then the C.G. of the whole area is given by

$$\bar{x}=\frac{\int (\text{abscissa of the C.G. of the elementary area}).\ \text{elementary area}}{\int \text{elementary area}}$$

$$\bar{y}=\frac{\int (\text{ordinate of the C.G. of the elementary area}).(\text{elementary area})}{\int \text{elementary area}}$$

under proper limits of integration; ρ, the density per unit area, being constant.

Remember. If m and n are +ive integers, we have

$$\int_0^{\pi} \sin^n x\,dx=2\int_0^{\pi/2} \sin^n x\,dx;$$

$$\int_0^{\pi} \cos^n x\,dx=0 \text{ or } 2\int_0^{\pi/2} \cos^n x\,dx,$$

according as n is odd or even;

$$\int_0^{\pi} \sin^m x\cos^n x\,dx=0 \text{ or } 2\int_0^{\pi/2} \sin^m x\cos^n x\,dx,$$

according as n is odd or even;

and $$\int_0^{\pi/2} \sin^m x\cos^n x\,dx=\frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots}k,$$

where k is $\pi/2$ if m and n are both even otherwise $k=1$.

Also remember that

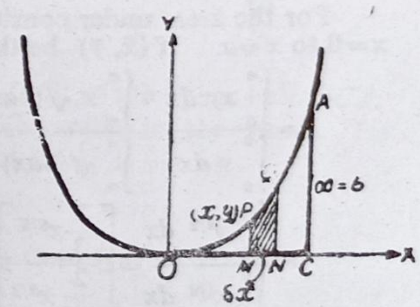
$$\int_0^{\pi/2} \sin^m x\cos^n x\,dx=\frac{\Gamma\{(m+1)/2\}\,\Gamma\{(n+1)/2\}}{2\,\Gamma\{(m+n+2)/2\}},$$

where $m \geqslant 0$ and $n \geqslant 0$.

Illustrative Examples

Ex. 9. *Find the C.G. of the region bounded by the parabola $x^2=4ay$, x-axis and the ordinate $x=b$.* **(Meerut 74)**

Sol. The given parabola is $x^2=4ay$. It is symmetrical about the y-axis. We are required to find the C.G. of the area $OPAC$ for which x varies from 0 to b.



Take $P(x, y)$ and $Q(x+\delta x, y+\delta y)$, the neighbouring points on the arc OA of the parabola $x^2=4ay$. Draw PM and QN perpendiculars to the axis of x. We have $PM=y$, $MN=\delta x$.

Then δA=elementary area $PMNQ=y\,\delta x$ and the C.G. of this elementary area can approximately be taken as the point $(x, \frac{1}{2}y)$ on MP.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^b x.y\,dx}{\int_0^b y\,dx}=\frac{\int_0^b x.\frac{x^2}{4a}\,dx}{\int_0^b \frac{x^2}{4a}\,dx}, \quad \left[\because\ y=\frac{x^2}{4a}\right]$$

$$=\frac{\int_0^b x^3\,dx}{\int_0^b x^2\,dx}=\frac{\left[\frac{x^4}{4}\right]_0^b}{\left[\frac{x^3}{3}\right]_0^b}=\frac{\frac{b^4}{4}}{\frac{b^3}{3}}=\frac{3b}{4}.$$

And $$\bar{y}=\frac{\int_0^b \frac{1}{2}y.y\,dx}{\int_0^b y\,dx}=\frac{\int_0^b \frac{1}{2}\left(\frac{x^2}{4a}\right)^2 dx}{\int_0^b \left(\frac{x^2}{4a}\right) dx}, \quad \left[\because\ y=\frac{x^2}{4a}\right]$$

$$=\frac{\frac{1}{8a}\int_0^b x^4\,dx}{\int_0^b x^2\,dx}=\frac{1}{8a}\frac{\left[\frac{x^5}{5}\right]_0^b}{\left[\frac{x^3}{3}\right]_0^b}=\frac{3}{8\times 5a}\frac{b^5}{b^3}=\frac{3b^2}{40a}.$$

$\therefore$ $\bar{x}=3b/4$, $\bar{y}=3b^2/40a$.

Ex. 10. *Find the C.G. of the area bounded by the parabola $y^2=4ax$, the axis of x and the latus rectum.* (Meerut 198

Sol. The given parabola is $y^2=4ax$. Refer figure of Ex. page 6. The coordinates of one end of the latus rectum are $(a, 2a)$. We are to find the C.G. of the area OSL. Consider an elementary strip of the area OSL parallel to y-axis. Its area is $y\,\delta x$ and its C.G. is the point $(x, \frac{1}{2}y)$.

For the area under consideration the limits of x are from $x=0$ to $x=a$. If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx}=\frac{\int_0^a x.\sqrt{(4ax)}\,dx}{\int_0^a \sqrt{(4ax)}\,dx}$$

$$=\frac{\int_0^a x^{3/2}\,dx}{\int_0^a x^{1/2}\,dx}=\frac{\left[\frac{2}{3}x^{5/2}\right]_0^a}{\left[\frac{2}{3}x^{3/2}\right]_0^a}=\frac{3a^{5/2}}{5a^{3/2}}=\frac{3a}{5},$$

$$\bar{y}=\frac{\int_0^a \frac{y}{2}y\,dx}{\int_0^a y\,dx}=\frac{\frac{1}{2}\int_0^a 4ax.dx}{\int_0^a \sqrt{(4ax)}\,dx}$$

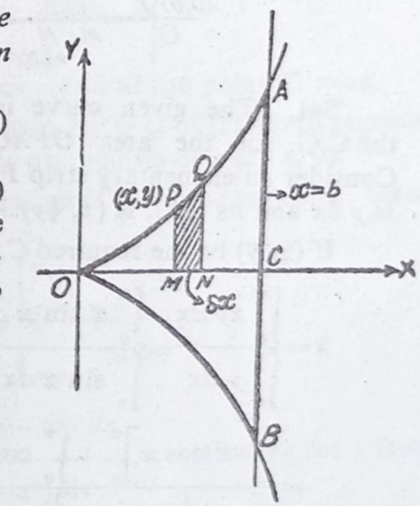
$$=\frac{2a\left[\frac{x^2}{2}\right]_0^a}{2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a}=\frac{3\sqrt{a}}{4}.\frac{a^2}{a^{3/2}}=\frac{3a}{4}.$$

$\therefore$ $\bar{x}=\frac{3}{5}a$, $\bar{y}=\frac{3}{4}a$.

Remark. If $(\bar{x}, \bar{y})$ be the C.G. of the area bounded by the parabola $y^2=4ax$ and the latus rectum, then $\bar{y}=0$ by symmetry. Also the x-coordinate of the C.G. of the whole area is the same as the x-coordinate of the C.G. of the upper half of this area. Now proceeding as above, we find that $\bar{x}=3a/5$.

Ex. 11 (a). *Find the position of the centroid of the area of the curve $ay^2=x^3$ between the origin and $x=b$.*

(Kanpur 1979; Lucknow 70)



Sol. The given curve is

$$ay^2=x^3. \quad \ldots(1)$$

It is symmetrical about the x-axis and passes through the origin. Suppose the line $x=b$ meets the curve (1) at the points A and B and the x-axis at C. Let $(\bar{x}, \bar{y})$ be the C.G. of the area OBA. Then $\bar{y}=0$, by symmetry.

Also the x-coordinate of the C.G. of the whole aera OBA is the same as the x-coordinate of the C.G. of the upper half OCA.

Take an elementary strip $PMNQ$ parallel to the y-axis. Its area is $y\,\delta x$ and its C.G. is the point $(x, \frac{1}{2}y)$.

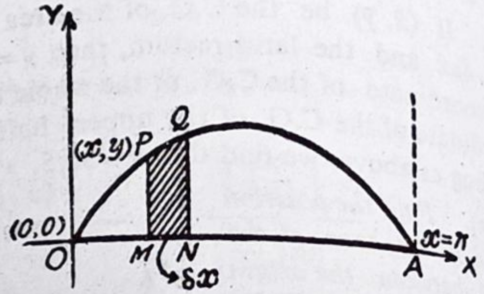
$$\therefore\ \bar{x}=\frac{\int_0^b xy\,dx}{\int_0^b y\,dx}=\frac{\int_0^b x\frac{x^{3/2}}{\sqrt{a}}\,dx}{\int_0^b \frac{x^{3/2}}{\sqrt{a}}\,dx}, \text{ from (1)}$$

$$=\frac{\int_0^b x^{5/2}\,dx}{\int_0^b x^{3/2}\,dx}=\frac{\left[\frac{x^{7/2}}{7/2}\right]_0^b}{\left[\frac{x^{5/2}}{5/2}\right]_0^b}=\frac{5b^{7/2}}{7b^{5/2}}=\frac{5}{7}\ b.$$

$\therefore$ $\bar{x}=\frac{5}{7}b$, $\bar{y}=0$.

Ex. 11 (b). *Find the C.G. of the area enclosed by the parabola $y^2=4ax$, and the double ordinate $x=b$.* (Meerut 1981, 84)

Ex. 12. *Find the position of the centroid of the area bounded by the curve $y=\sin x$, and the line $x=0$ and $x=\pi$.*



Sol. The given curve is $y=\sin x$. We are required to find the C.G. of the area $OPAO$ for which x varies from 0 to π. Consider an elementary strip $PMNQ$ parallel to y-axis. Its area is $y\,\delta x$ and its C.G. is $(x, \frac{1}{2}y)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^\pi xy\,dx}{\int_0^\pi y\,dx}=\frac{\int_0^\pi x\sin x\,dx}{\int_0^\pi \sin x\,dx}, \quad [\because y=\sin x]$$

$$=\frac{\left[-x\cos x\right]_0^\pi+\int_0^\pi \cos x\,dx}{\left[-\cos x\right]_0^\pi}=\frac{\pi+\left[\sin x\right]_0^\pi}{\left[-\cos\pi+\cos 0\right]}$$

$$=\frac{\pi+0}{1+1}=\frac{\pi}{2}.$$

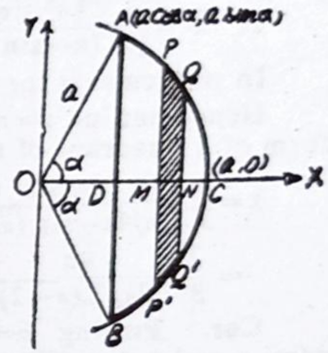
$$\text{Also } \bar{y}=\frac{\int_0^\pi \frac{1}{2}y.y\,dx}{\int_0^\pi y\,dx}=\frac{\frac{1}{2}\int_0^\pi \sin^2 x\,dx}{\int_0^\pi \sin x\,dx}, \quad [\because y=\sin x]$$

$$=\frac{\frac{1}{2}.2\int_0^{\pi/2}\sin^2 x\,dx}{2\int_0^{\pi/2}\sin x\,dx}=\frac{1}{2}\,\frac{\frac{1}{2}.\frac{1}{2}\pi}{1}=\frac{1}{8}\pi$$

$\therefore\ \bar{x}=\frac{1}{2}\pi,\ \bar{y}=\frac{1}{8}\pi.$

Ex. 13. *Find the C.G. of a segment of a circle that subtends an angle 2α at the centre.*

Hence find the centre of gravity of a uniform disc which is of the shape of a quadrant of a circle. **(Lucknow 1981)**



Sol. Referred to the centre as origin let the equation of the circle be

$$x^2+y^2=a^2, \quad \ldots(1)$$

a being the radius of the circle.

Let ACB be the segment of this circle subtending an angle 2α at the centre and let the central radius OC coincide with the x-axis so that $\angle AOC=\alpha$. The segment ACB is symmetrical about the x-axis.

At the point A, $x=a\cos\alpha$ and at the point C, $x=a$.

Take an elementary strip $PP'Q'Q$ of the given segment. Its area is $2y\,\delta x$ and its C.G. is on the axis of x at $(x, 0)$.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G. of the segment, then $\bar{y}=0$ (by symmetry).

$$\text{Also } \bar{x}=\frac{\int_{OD}^{OC} x.2y\,dx}{\int_{OD}^{OC} 2y\,dx}=\frac{\int_{a\cos\alpha}^{a} xy\,dx}{\int_{a\cos\alpha}^{a} y\,dx}$$

$$=\frac{\int_{a\cos\alpha}^{a} x\sqrt{(a^2-x^2)}\,dx}{\int_{a\cos\alpha}^{a}\sqrt{(a^2-x^2)}\,dx}, \text{ substituting for } y \text{ from (1).}$$

Now the Nr. of $\bar{x}$

$$=\int_{a\cos\alpha}^{a} x\sqrt{(a^2-x^2)}\,dx=-\frac{1}{2}\int_{a\cos\alpha}^{a}(a^2-x^2)^{1/2}(-2x)\,dx$$

$$=-\frac{1}{2}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_{a\cos\alpha}^{a}$$

$$=-\frac{1}{3}\left[0-(a^2-a^2\cos^2\alpha)^{3/2}\right]=-\frac{1}{3}\left[0-a^3\sin^3\alpha\right]=\frac{1}{3}a^3\sin^3\alpha.$$

And the Dr. of $\bar{x}=\int_{a\cos\alpha}^{a}\sqrt{(a^2-x^2)}\,dx$

$$=\left[\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a\cos\alpha}^{a}$$

$$=[0+\tfrac12 a^2.\tfrac12\pi-\tfrac12 a^2\sin\alpha\cos\alpha-\tfrac12 a^2\sin^{-1}(\cos\alpha)]$$
$$=\tfrac12 a^2[\tfrac12\pi-\sin^{-1}(\cos\alpha)-\sin\alpha\cos\alpha]$$
$$=\tfrac12 a^2[\cos^{-1}(\cos\alpha)-\sin\alpha\cos\alpha],\quad [\because \tfrac12\pi-\sin^{-1}t=\cos^{-1}t]$$
$$=\tfrac12 a^2[\alpha-\sin\alpha\cos\alpha].$$

$$\therefore\quad \bar{x}=\frac{\frac13 a^3\sin^3\alpha}{\frac12 a^2(\alpha-\sin\alpha\cos\alpha)}=\frac23\cdot\frac{a\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha},\text{ and }\bar{y}=0\ldots(2)$$

In particular if the segment is a quadrant of a circle, $\alpha=\pi/4$. Hence putting $\alpha=\pi/4$ in (2), we get for a uniform disc in the form of a quadrant of a circle,

$$\bar{x}=\frac23\cdot\frac{a\sin^3(\pi/4)}{\{(\pi/4)-\sin(\pi/4)\cos(\pi/4)\}}=\frac23\frac{a(1/\sqrt2)^3}{\{(\pi/4)-(1/\sqrt2)(1/\sqrt2)\}}$$
$$=\frac23\cdot\frac{4a}{2\sqrt2.(\pi-2)}=\frac{2\sqrt2.a}{3(\pi-2)},\text{ and }\bar{y}=0.$$

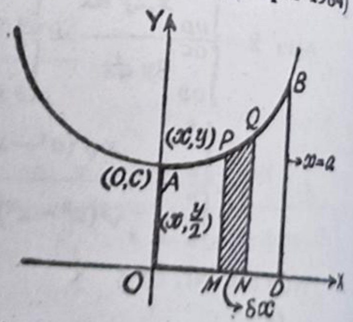
Cor. Putting $\alpha=\pi/2$ in (2), we get the C.G. of a uniform semi-circular disc. Thus for a uniform semi-circular disc,

$$\bar{x}=\frac23\cdot\frac{a\sin^3(\pi/2)}{\{(\pi/2)-\sin(\pi/2)\cos(\pi/2)\}}=\frac23\frac{a}{(\pi/2)}=\frac{4a}{3\pi},\text{ and }\bar{y}=0.$$

Thus the C.G. of a uniform semi-circular disc lies on the symmetrical radius at a distance $4a/3\pi$ from the centre.

Ex. 14. *Find the C.G. of the area between the curve $y=c\cosh(x/c)$, the co-ordinate axes and the ordinate $x=a$.* (Kanpur 1984)

Sol. The given curve is
$y=c\cosh(x/c)$ ...(1)



We are required to find the C.G. of the area $AODBPA$ for which x varies from 0 to a.

Consider an elementary strip $PMNQ$ parallel to the y-axis. Its area is $y\,\delta x$ and its C.G. is $(x, \frac12 y)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^a x.y\,dx}{\int_0^a y\,dx}=\frac{\int_0^a x.c\cosh\frac{x}{c}\,dx}{\int_0^a c\cosh\frac{x}{c}\,dx},\quad\text{from (1)}$$
$$=\frac{\int_0^a x\cosh\frac{x}{c}\,dx}{\int_0^a \cosh\frac{x}{c}\,dx}$$

$$=\frac{\left[cx\sinh\frac{x}{c}\right]_0^a-\int_0^a c\sinh\frac{x}{c}\,dx}{\left[c\sinh\frac{x}{c}\right]_0^a}=\frac{ca\sinh\frac{a}{c}-c^2\left[\cosh\frac{x}{c}\right]_0^a}{c\sinh\frac{a}{c}}$$
$$=\frac{a\sinh(a/c)-c\cosh(a/c)+c}{\sinh(a/c)}=a-c\coth\frac{a}{c}+c\,\text{cosech}\,\frac{a}{c}.$$

Also
$$\bar{y}=\frac{\int_0^a\frac{y}{2}\cdot y\,dx}{\int_0^a y\,dx}=\frac{\frac12\int_0^a c^2\cosh^2\frac{x}{c}\,dx}{\int_0^a c\cosh\frac{x}{c}\,dx}\quad\text{from (1)}$$
$$=\frac{c}{2}\frac{\int_0^a\frac12\left(1+\cosh\frac{2x}{c}\right)dx}{\left[c\sinh\frac{x}{c}\right]_0^a}=\frac{c}{4}\frac{\left[x+\frac{c}{2}\sinh\frac{2x}{c}\right]_0^a}{c\sinh\frac{a}{c}}$$
$$=\frac14\frac{\left[a+\frac{c}{2}\sinh\frac{2a}{c}\right]}{\sinh(a/c)}=\frac14\left[c\cosh\frac{a}{c}+a\,\text{cosech}\,\frac{a}{c}\right]$$
$$[\because \sinh(2a/c)=2\sinh(a/c)\cosh(a/c)]$$

$$\therefore\ \bar{x}=a-c\coth(a/c)+c\,\text{cosech}\,(a/c),$$
$$\bar{y}=\tfrac14[c\cosh(a/c)+a\,\text{cosech}\,(a/c)].$$

****Ex. 15 (a).** *Find the C.G. of the area included between the curve $y^2(2a-x)=x^3$ and its asymptote.*

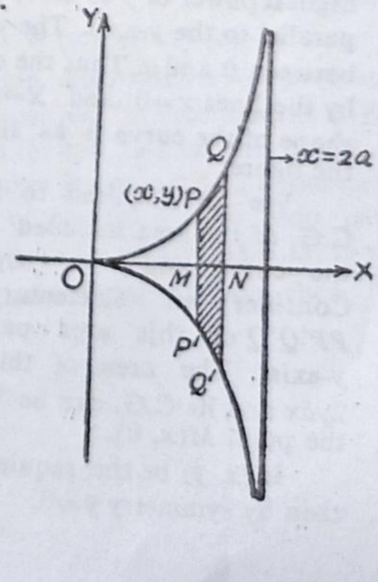
(Meerut 1990; Rohilkhand 88)

Sol. The given curve is $y^2(2a-x)=x^3$. ...(1)

It is symmetrical about the x-axis. It passes through the origin. Equating the coefficient of the highest power of y to zero, we get

$$2a-x=0\quad\text{or}\quad x=2a$$

as an asymptote parallel to the y-axis.



We are required to find the C.G. of the area included between the curve and its asymptote. Consider an elementary strip $PP'Q'Q$ of this area parallel to y-axis. Its area is $2y\,\delta x$ and its C.G. can be taken as the point $M(x, 0)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry $\bar{y}=0$.

Also
$$\bar{x}=\frac{\int_0^{2a}x.y\,dx}{\int_0^{2a}y\,dx}$$

$$=\frac{\int_0^{2a} x.x^{3/2}(2a-x)^{-1/2}\,dx}{\int_0^{2a} x^{3/2}(2a-x)^{-1/2}\,dx}, \text{ from (1).}$$

Now put $x=2a\sin^2\theta$ so that $dx=4a\sin\theta\cos\theta\,d\theta$.

Also when $x=0$, $\theta=0$ and when $x=2a$, $\theta=\pi/2$.

$$\therefore\quad \bar{x}=\frac{\int_0^{\pi/2}(2a\sin^2\theta)^{5/2}(2a\cos^2\theta)^{-1/2}\,4a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2}(2a\sin^2\theta)^{3/2}(2a\cos^2\theta)^{-1/2}.4a\sin\theta\cos\theta\,d\theta}$$

$$=\frac{2a\int_0^{\pi/2}\sin^6\theta\,d\theta}{\int_0^{\pi/2}\sin^4\theta\,d\theta}=\frac{2a.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{1}{2}\pi}{\frac{3}{4}.\frac{1}{2}.\frac{1}{2}\pi}=\frac{5a}{3}.$$

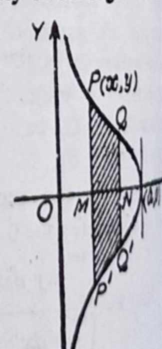
Hence $\bar{x}=\frac{5}{3}a$, $\bar{y}=0$.

Ex. 15 (b). *Find the centre of gravity of the area between the curve and the asymptote, of which the equation is*

$$y^2=b^2(a-x)/x.$$

Sol. The given curve is $y^2x=b^2(a-x)$.

It is symmetrical about the x-axis. It does not pass through the origin. It cuts the x-axis at the point $(a, 0)$ and the tangent there is parrallel to the y-axis. Equating the coefficient of the highest power of y to zero, we get $x=0$ *i.e.*, y-axis as an asymptote parallel to the y-axis. The value of y^2 is positive only when x is between 0 and a. Thus the curve exists only in the region bounded by the lines $x=0$ and $x=a$. The shape of the curve is as shown in the figure.



We are required to find the C.G. of the area included between the curve and its asymptote. Consider an elementary strip $PP'Q'Q$ of this area parallel to y-axis. The area of this strip is $2y\delta x$ and its C.G. can be taken as the point $M(x, 0)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry $\bar{y}=0$.

$$\text{Also}\quad \bar{x}=\frac{\int_0^a x.y\,dx}{\int_0^a y\,dx}$$

$$=\frac{\int_0^a x.b\sqrt{\left(\frac{a-x}{x}\right)}\,dx}{\int_0^a b\sqrt{\left(\frac{a-x}{x}\right)}\,dx}, \qquad \text{putting for } y \text{ from (1).}$$

Now put $x=a\sin^2\theta$ so that $dx=2a\sin\theta\cos\theta\,d\theta$. Also when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore\quad \bar{x}=\frac{\int_0^{\pi/2} a\sin^2\theta.\frac{\cos\theta}{\sin\theta}.2a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2}\frac{\cos\theta}{\sin\theta}.2a\sin\theta\cos\theta\,d\theta}$$

$$=\frac{a\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{\int_0^{\pi/2}\cos^2\theta\,d\theta}=a\,\frac{\frac{1.1}{4.2}.\frac{\pi}{2}}{\frac{1}{2}.\frac{\pi}{2}}$$

$$=\tfrac{1}{4}a.$$

Hence $\bar{x}=\frac{1}{4}a$, $\bar{y}=0$.

Ex. 16. *Find the centre of gravity of the area of the loop of the curve* $y^2(a+x)=x^2(a-x)$ **or** $x(x^2+y^2)=a(x^2-y^2)$.

(Lucknow 1978)

Sol. The given curve is $y^2(a+x)=x^2(a-x)$. ...(1)

It passes through the origin. Also when $x=a$, $y=0$ and when x lies between 0 and a, y^2 is positive. Hence there is a loop between $(0, 0)$ and $(a, 0)$. Thus for the loop of the curve x varies from 0 to a. Also the given curve is symmetrical about the x-axis. Consider an elementary strip of the area of the loop parallel to the y-axis. The area of this strip is $2y\,dx$ and its C.G. is at $(x, 0)$.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G. of the loop, then $\bar{y}=0$ (by symmetry),

$$\text{and}\quad \bar{x}=\frac{\int_0^a x.2y\,dx}{\int_0^a 2y\,dx}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx}.$$

The Nr. of $\bar{x}$

$$=\int_0^a xy\,dx=\int_0^a x^2\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx=\int_0^a \frac{x^2\,(a-x)}{\sqrt{(a^2-x^2)}}\,dx,$$

(rationalizing the numerator)

$$=\int_0^{\pi/2}\frac{a^2\sin^2\theta\,(a-a\sin\theta)}{a\cos\theta}.a\cos\theta\,d\theta, \text{ putting } x=a\sin\theta$$

$$=a^3\int_0^{\pi/2}(\sin^2\theta-\sin^3\theta)\,d\theta=a^3\left[\frac{1}{2}\cdot\frac{\pi}{2}-\frac{2}{3}\right]=\frac{a^3}{12}[3\pi-8],$$

and the Dr. of $\bar{x}$

$$=\int_0^a y\,dx=\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx=\int_0^a \frac{x\,(a-x)}{\sqrt{(a^2-x^2)}}\,dx$$

$$=\int_0^{\pi/2}\frac{(a\sin\theta)\,(a-a\sin\theta)\,a\cos\theta\,d\theta}{a\cos\theta}, \text{ putting } x=a\sin\theta$$

$$=\int_0^{\pi/2}a^2\,(\sin\theta-\sin^2\theta)\,d\theta=a^2.\left[\left(-\cos\theta\right)_0^{\pi/2}-\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

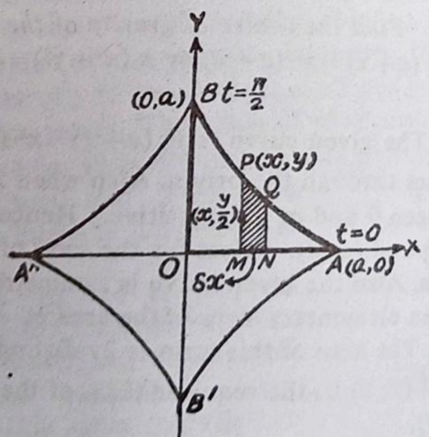
$=a^2\,[1-\frac{1}{4}\pi]=\frac{1}{4}a^2\,[4-\pi]$.

$\therefore\quad \bar{x}=\dfrac{\frac{1}{12}a^3\,[3\pi-8]}{\frac{1}{4}\,a^2\,[4-\pi]}=\dfrac{a\,[3\pi-8]}{3\,[4-\pi]}$ and $\bar{y}=0$.

Ex. 17. *Find the position of the C.G. of the area of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ lying in the positive quadrant.* **(Kanpur 1980;** Agra 86)

Sol. The given curve is

$$x^{2/3}+y^{2/3}=a^{2/3} \qquad \text{...(1)}$$



The parametric equations of (1) are

$$x=a\cos^3 t,\ y=a\sin^3 t. \qquad \text{...(2)}$$

The curve (1) is symmetrical about both the axes. We are required to find the C.G. of the area of the curve lying in the Ist quadrant. This area is symmetrical about the line $y=x$ and therefore the required C.G. $(\bar{x}, \bar{y})$ will lie on the line $y=x$. Thus, we have $\bar{x}=\bar{y}$.

For the given area x varies from 0 to a. Also when $x=0$, $t=\pi/2$ and when $x=a$, $t=0$.

Consider an elementary strip $PMNQ$ parallel to the y-axis. Its area is $y\,\delta x$ and its C.G. is $(x, y/2)$.

$$\text{Now}\quad \bar{x}=\frac{\int_0^a x.y\,dx}{\int_0^a y\,dx}$$

$$=\frac{\int_{\pi/2}^0 xy\frac{dx}{dt}\,dt}{\int_{\pi/2}^0 y\frac{dx}{dt}\,dt}=\frac{\int_{\pi/2}^0 a\cos^3 t.a\sin^3 t.(-3a\cos^2 t\sin t)\,dt}{\int_{\pi/2}^0 a\sin^3 t.(-3a\cos^2 t\sin t)\,dt}, \qquad \text{from (2)}$$

$$=\frac{a\int_0^{\pi/2}\cos^5 t\sin^4 t\,dt}{\int_0^{\pi/2}\cos^2 t\sin^4 t\,dt}=a\,\frac{\frac{4.2.3.1}{9.7.5.3.1}}{\frac{1.3.1}{6.4.2}\,\frac{\pi}{2}}=a\,\frac{\frac{8}{315}}{\frac{\pi}{32}}$$

$=\dfrac{8a}{315}\times\dfrac{32}{\pi}=\dfrac{256a}{315\pi}$.

$\therefore\quad \bar{x}=\bar{y}=(256a)/(315\pi)$.

Ex. 18 (a). *Find the position of the centroid of the area of the curve $(x/a)^{2/3}+(y/b)^{2/3}=1$ lying in the positive quadrant.* **(Agra 75)**

Sol. The given curve is

$$(x/a)^{2/3}+(y/b)^{2/3}=1.$$

The parametric equations of this curve are

$$x=a\sin^3\theta,\ y=b\cos^3\theta. \qquad \textbf{(Note).} \qquad \text{...(1)}$$

The curve is symmetrical about both the axes. We are required to find the C.G. of the area of the curve lying in the first quadrant. Consider an elementary strip of this area parallel to the y-axis. Its area is $y\,\delta x$ and its C.G. is at $(x, y/2)$. For the given area x varies from 0 to a and θ varies from 0 to $\pi/2$. Obviously, when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx},$$

and $\bar{y}=\dfrac{\int_0^a \frac{1}{2}y.y\,dx}{\int_0^a y\,dx}=\dfrac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx}$.

Now using the parametric equations (1), we have

the Nr. of $\bar{x}$

$$=\int_0^a xy\,dx=\int_0^{\pi/2} a\sin^3\theta.\,b\cos^3\theta.3a\sin^2\theta\cos\theta\,d\theta$$

$$=3a^2b\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta=3a^2b\,\frac{4.2.3.1}{9.7.5.3.1}=\frac{8a^2b}{105},$$

the Nr. of $\bar{y}$

$$=\tfrac{1}{2}\int_0^a y^2\,dx=\tfrac{1}{2}\int_0^{\pi/2} b^2\cos^6\theta.3a\sin^2\theta\cos\theta\,d\theta$$

$$=\frac{3ab^2}{2}\int_0^{\pi/2}\sin^2\theta\cos^7\theta\,d\theta=\frac{3ab^2}{3}\,\frac{1.6.4.2}{9.7.5.3.1}=\frac{8ab^2}{105}.$$

and the Dr. of $\bar{x}$=the Dr. of $\bar{y}$

$$=\int_0^a y\,dx=\int_0^{\pi/2} b\cos^3\theta.3a\sin^2\theta\cos\theta\,d\theta$$

$$=3ab\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta=3ab\,\frac{1.3.1}{6.4.2}\,\frac{\pi}{2}=\frac{3\pi ab}{32}.$$

$$\therefore\quad \bar{x}=\frac{8a^2b/105}{3ab\pi/32}=\frac{256a}{315\pi}\ \text{and}\ \bar{y}=\frac{8ab^2/105}{3ab\pi/32}=\frac{256b}{315\pi}.$$

Ex. 18 (b). *Find the centre of gravity of a plane lamina of uniform density in the form of a quadrant of an ellipse.*

(Delhi 1980; Rohilkhand 81, Meerut 82, 84, 85)

Sol. Let the equation of the ellipse be

$$x^2/a^2+y^2/b^2=1. \qquad \ldots(1)$$

The parametric equations of (1) are

$$x=a\cos t,\ y=b\sin t. \qquad \ldots(2)$$

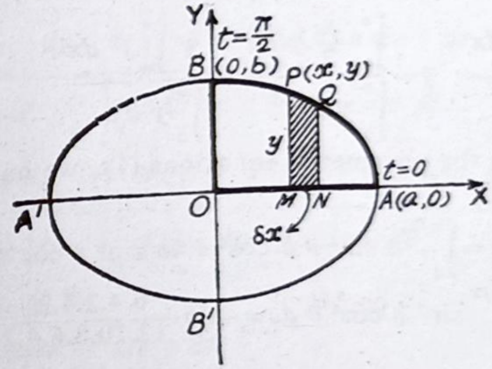
At the point $A\,(a, 0)$, $t=0$ and at the point $B\,(0, b)$, $t=\pi/2$.

Let $(\bar{x}, \bar{y})$ be the C.G. of the area in the form of the quadrant OAB of the ellipse (1). Take an elementary strip $PMNQ$ of this area parallel to the y-axis. The area of this strip is $y\,\delta x$ and its C.G. can be taken as the middle point $(x, \frac{1}{2}y)$ of $P.M.$ We have

$$\bar{x}=\frac{\int_{x=0}^{a} xy\,dx}{\int_{x=0}^{a} y\,dx}\quad\text{and}\quad \bar{y}=\frac{\int_{x=0}^{a}\frac{1}{2}y.y\,dx}{\int_{x=0}^{a} y\,dx}=\frac{\frac{1}{2}\int_{x=0}^{a} y^2\,dx}{\int_{x=0}^{a} y\,dx}.$$

The Nr. of $\bar{x}$

$$=\int_{\pi/2}^{0} xy\,\frac{dx}{dt}\,dt=\int_{\pi/2}^{0} a\cos t.b\sin t.(-a\sin t)\,dt,\ \text{from (2)}$$



$$=a^2b\int_0^{\pi/2}\cos t\sin^2 t\,dt=a^2b\,\frac{1}{3.1}=\tfrac{1}{3}a^2b,$$

the Nr. of $\bar{y}$

$$=\tfrac{1}{2}\int_{\pi/2}^{0} y^2\frac{dx}{dt}\,dt=\tfrac{1}{2}\int_{\pi/2}^{0} b^2\sin^2 t.(-a\sin t)\,dt$$

$$=\tfrac{1}{2}ab^2\int_0^{\pi/2}\sin^3 t\,dt=\tfrac{1}{2}ab^2\,\frac{2}{3.1}=\tfrac{1}{3}ab^2,$$

and the Dr. of $\bar{x}$=the Dr. of $\bar{y}$

$$=\int_{\pi/2}^{0} y\,\frac{dx}{dt}\,dt=\int_{\pi/2}^{0} b\sin t.(-a\sin t)\,dt$$

$$=ab\int_0^{\pi/2}\sin^2 t\,dt=ab.\tfrac{1}{2}.\frac{\pi}{2}=\frac{\pi}{4}\,ab.$$

$$\therefore\quad \bar{x}=\frac{\frac{1}{3}a^2b}{\frac{1}{4}\pi ab}=\frac{4a}{3\pi},\quad \bar{y}=\frac{\frac{1}{3}ab^2}{\frac{1}{4}\pi\,ab}=\frac{4b}{3\pi}.$$

Ex. 23. *Find the C.G. of the area between the curve*

$$(x/a)^{1/2}+(y/b)^{1/2}=1$$

and the coordinate axes.

Sol. The given equation of the curve is $(x/a)^{1/2}+(y/b)^{1/2}=1$. Its parametric equations are

$$x=a\sin^4\theta,\ y=b\cos^4\theta, \qquad \textbf{(Note)} \qquad \ldots(1)$$

where θ is the parameter.

The given curve is a parabola which touches the x-axis at $(a, 0)$ and the y-axis at $(0, b)$. We are required to find the C.G. of the area between the given curve and the co-ordinate axes. Consider an elementary strip parallel to y-axis. Its area is

$y\,\delta x$ and its C.G. is at $(x, y/2)$. For the given area x varies from 0 to a and θ varies from 0 to $\pi/2$.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^a x.y\,dx}{\int_0^a y\,dx};\quad \bar{y}=\frac{\int_0^a \frac{y}{2}.y\,dx}{\int_0^a y\,dx}=\frac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx}.$$

Now using the parametric equations (1), we have the Nr. of $\bar{x}$

$$=\int_0^a xy\,dx=\int_0^{\pi/2} a\sin^4\theta . b\cos^4\theta . 4a\sin^3\theta\cos\theta\,d\theta$$

$$=4a^2b\int_0^{\pi/2}\sin^7\theta\cos^5\theta\,d\theta=4a^2b\,\frac{6.4.2.4.2}{12.10.8.6.4.2}=\frac{a^2b}{30},$$

the Nr. of $\bar{y}$

$$=\tfrac{1}{2}\int_0^a y^2\,dx=\tfrac{1}{2}\int_0^{\pi/2} b^2\cos^8\theta\, 4a\sin^3\theta\cos\theta\,d\theta$$

$$=2ab^2\int_0^{\pi/2}\sin^3\theta\cos^9\theta\,d\theta=2ab^2\,\frac{2.8.6.4.2}{12.10.8.6.4.2}=\frac{ab^2}{30},$$

and the Dr. of $\bar{x}$=the Dr. of $\bar{y}$

$$=\int_0^a y\,dx=\int_0^{\pi/2} b\cos^4\theta . 4\theta\sin^4\theta\cos\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\sin^3\theta\cos^5\theta\,d\theta=4ab\,\frac{2.4.2}{8.6.4.2}=\frac{ab}{6}.$$

$$\therefore\quad \bar{x}=\frac{a^2b/30}{ab/6}=\frac{a}{5},\quad \bar{y}=\frac{a\,b^2/30}{ab/6}=\frac{b}{5}.$$

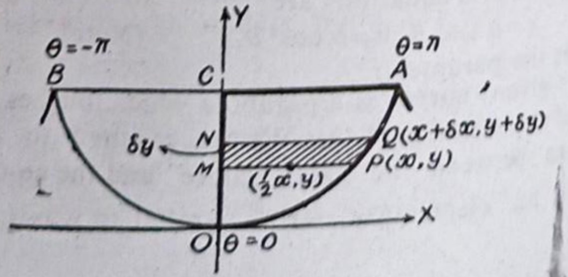
***Ex. 20.** *Find the C.G. of the area bounded by the axis of y, the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ and its base.*

(Kanpur 1981, 85)

Sol. The given parametric equations of the cycloid are

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta). \qquad \text{...(1)}$$

We are required to find the C.G. of the area $OPACO$. Consider an elementary strip $PMNQ$ parallel to the x-axis. Its area



is $x\,\delta y$ and its C.G. can be taken as the middle point $(\frac{1}{2}x, y)$ of MP. For the given area θ varies from 0 to π.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_{\theta=0}^{\pi}\frac{1}{2}x.x\,dy}{\int_{\theta=0}^{\pi}x\,dy}=\frac{\frac{1}{2}\int_0^{\pi}a^2(\theta+\sin\theta).a\sin\theta\,d\theta}{\int_0^{\pi}a(\theta+\sin\theta)^2.\,a\sin\theta\,d\theta},$$

and
$$\bar{y}=\frac{\int_{\theta=0}^{\pi}y.x\,dy}{\int_{\theta=0}^{\pi}y.x\,dy}=\frac{\int_0^{\pi}a(1-\cos\theta).a(\theta+\sin\theta).a\sin\theta\,d\theta}{\int_0^{\pi}a(\theta+\sin\theta)\,a\sin\theta\,d\theta}.$$

Now the Nr. of $\bar{x}$

$$=\tfrac{1}{2}a^3\int_0^{\pi}(\theta+\sin\theta)^2\sin\theta\,d\theta$$

$$=\tfrac{1}{2}a^3\left[\int_0^{\pi}\theta^2\sin\theta\,d\theta+2\int_0^{\pi}\theta\sin^2\theta\,d\theta+\int_0^{\pi}\sin^3\theta\,d\theta\right]. \qquad \text{...(2)}$$

We have

$$\int_0^{\pi}\theta^2\sin\theta\,d\theta=\Big[\theta^2.(-\cos\theta)\Big]_0^{\pi}-\int_0^{\pi}2\theta.(-\cos\theta)\,d\theta$$

$$=\pi^2+2\int_0^{\pi}\theta\cos\theta\,d\theta$$

$$=\pi^2+2\Big[\theta\sin\theta\Big]_0^{\pi}-2\int_0^{\pi}\sin\theta\,d\theta=\pi^2+2(0-0)-4\int_0^{\pi/2}\sin\theta\,d\theta$$

$$=\pi^2-4\Big[-\cos\theta\Big]_0^{\pi/2}=\pi^2+4\Big[\cos\theta\Big]_0^{\pi/2}=\pi^2+4(0-1)=\pi^2-4.$$

Again, let $u=\int_0^{\pi}\theta\sin^2\theta\,d\theta$. Then by a property of definite integrals,

$$u=\int_0^{\pi}(\pi-\theta)\sin^2(\pi-\theta)\,d\theta=\int_0^{\pi}(\pi-\theta)\sin^2\theta\,d\theta.$$

$$\therefore\quad 2u=\int_0^{\pi}\pi\sin^2\theta\,d\theta=\pi\int_0^{\pi}\sin^2\theta\,d\theta=2\pi\int_0^{\pi/2}\sin^2\theta\,d\theta$$

or
$$u=\pi.\tfrac{1}{2}.\tfrac{1}{2}\pi=\tfrac{1}{4}\pi^2.$$

Thus $\int_0^{\pi}\theta\sin^2\theta\,d\theta=\frac{1}{4}\pi^2$.

Also $\int_0^{\pi}\sin^3\theta\,d\theta=2\int_0^{\pi/2}\sin^3\theta\,d\theta=2.\frac{2}{3.1}=\frac{4}{3}.$

Substituting the values of these integrals in (2), we get the Nr. of $\bar{x}$

$=\frac{1}{2}a^3\,[\pi^2-4+2.\frac{1}{4}\pi^2+\frac{4}{3}]=\frac{1}{2}a^3\,[\frac{3}{2}\pi^2-\frac{8}{3}]=\frac{1}{12}a^3\,[9\pi^2-16].$

Again the Nr of $\bar{y}$

$$=a^3\int_0^{\pi}(1-\cos\theta)\,(\theta\sin\theta+\sin^2\theta)\,d\theta$$

$$=a^3\int_0^{\pi}[\theta\sin\theta+\sin^2\theta-\theta\sin\theta\cos\theta-\cos\theta\sin^2\theta]\,d\theta$$

$$=a^3\left[\int_0^{\pi}\theta\sin\theta\,d\theta+\int_0^{\pi}\sin^2\theta\,d\theta-\tfrac{1}{2}\int_0^{\pi}\theta\sin 2\theta\,d\theta\right.$$
$$\left.-\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta\right]$$

We have $\int_0^{\pi}\theta\sin\theta\,d\theta=\Big[\theta.(-\cos\theta)\Big]_0^{\pi}+\int_0^{\pi}\cos\theta\,d\theta=$ [illegible]

$$\int_0^{\pi}\sin^2\theta\,d\theta=2\int_0^{\pi/2}\sin^2\theta\,d\theta=2.\,\frac{1}{2}.\,\frac{\pi}{2}=\frac{\pi}{2},$$

$$\int_0^{\pi}\theta\sin 2\theta\,d\theta=\Big[\theta.(-\tfrac{1}{2}\cos 2\theta)\Big]_0^{\pi}+\tfrac{1}{2}\int_0^{\pi}\cos 2\theta\,d\theta$$
$$=-\frac{\pi}{2}+\tfrac{1}{2}\Big[\tfrac{1}{2}\sin 2\theta\Big]_0^{\pi}=-\frac{\pi}{2},$$

and $\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta=0.$

$\therefore$ the Nr. of $\bar{y}$

$=a^3\,[\pi+\frac{1}{2}\pi-\frac{1}{2}.(-\frac{1}{2}\pi)-0]=a^3\,(\pi+\frac{1}{2}\pi+\frac{1}{4}\pi)$

$=\frac{7}{4}\pi a^3.$

Further the Dr. of $\bar{x}$=the Dr. of $\bar{y}$

$=a^2\int_0^{\pi}(\theta\sin\theta+\sin^2\theta)\,d\theta=a^2\left[\int_0^{\pi}\theta\sin\theta\,d\theta+\int_0^{\pi}\sin^2\theta\,d\theta\right]$

$=a^2\,[\pi+\frac{1}{2}\pi]$, substituting the values of these integrals found above

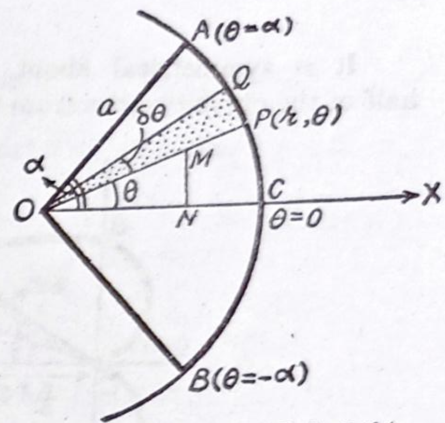
$=\frac{3}{2}\pi a^2.$

$\therefore\quad \bar{x}=\dfrac{\frac{1}{12}a^3\,(9\pi^2-16)}{\frac{3}{2}\pi a^2}=\dfrac{9\,(9\pi^2-16)}{18\pi}$ and $\bar{y}=\dfrac{\frac{7}{4}\pi a^3}{\frac{3}{2}\pi a^2}=\frac{7}{6}a.$

***Ex. 21.** *Find the C.G. of the sector of a circle subtending an angle 2α at the centre of the circle.*

Sol. Referred to the centre as pole, the polar equation of the circle of radius a is $r=a$.

Let the sector AOB subtend an angle 2α at the centre O of the circle and let the x-axis be along the symmetrical radius OC.



We have $\angle AOC=\alpha$. Consider an elementary strip POQ. Its area $=\frac{1}{2}r^2\,\delta\theta$ and its C.G. can be taken as the point M on OP such that $OM=\frac{2}{3}OP=\frac{2}{3}r$. The co-ordinates of M are

$(\frac{2}{3}r\cos\theta,\ \frac{2}{3}r\sin\theta)$.

$\therefore$ if $(\bar{x},\bar{y})$ be the required C.G., then by symmetry $\bar{y}=0$.

Also the x-coordinate of the C.G. of the area AOB is the same as the x-coordinate of the C.G. of the upper half AOC of this area.

$$\therefore\quad \bar{x}=\frac{\int_0^{\alpha}\frac{2}{3}r\cos\theta.\frac{1}{2}r^2\,d\theta}{\int_0^{\alpha}\frac{1}{2}r^2\,d\theta}=\frac{\frac{2}{3}\int_0^{\alpha}r^3\cos\theta\,d\theta}{\int_0^{\alpha}r^2\,d\theta}$$

$$=\frac{\frac{2}{3}\int_0^{\alpha}a^3\cos\theta\,d\theta}{\int_0^{\alpha}a^2\,d\theta},\qquad [\because\ r=a,\text{ from (1)}]$$

$$=\frac{2}{3}a\Big[\sin\theta\Big]_0^{\alpha}\div\Big[\theta\Big]_0^{\alpha}=\frac{2}{3}a.\frac{\sin\alpha}{\alpha}.$$

$\therefore\quad \bar{x}=\frac{2}{3}(a\sin\alpha)/\alpha$ and $\bar{y}=0.$ **(Remember)**

Thus the C.G. of the sector of a circle of radius a lies on the central radius and its distance from the centre of the circle is $\frac{2}{3}\dfrac{a\sin\alpha}{\alpha}$.

***Ex. 22.** *Find the centre of gravity of a uniform semi-circular disc.*
(Meerut 75; Jiwaji 80; Gorakhpur 78; Kanpur 81; Rohilkhand 82)

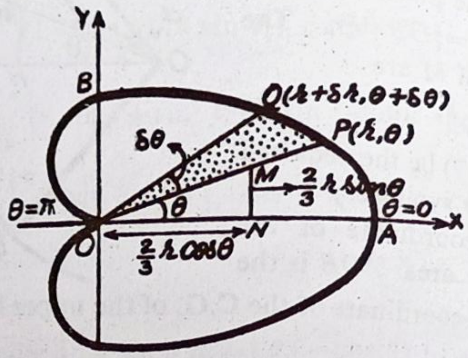
Sol. Proceed exactly as in Ex. 21 taking $2\alpha=\pi$ or $\alpha=\pi/2$. Thus take ACB as a semi-circular area, the symmetrical radius OC being along the x-axis. We have $\angle AOC=\pi/2$.

We shall get $\bar{x}=\dfrac{2}{3}\,\dfrac{a\sin\frac{1}{2}\pi}{\frac{1}{2}\pi}=\dfrac{4a}{3\pi}$, and $\bar{y}=0$.

***Ex. 23.** *Find the position of the centroid of the area of the cardioid $r=a\,(1+\cos\theta)$.* **(Kanpur 1976; Gorakhpur 79; Agra 87; Lucknow 80; Meerut 82)**

Sol. The given curve is $r=a\,(1+\cos\theta)$. ...(1)

It is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π.



If $(\bar{x}, \bar{y})$ be the coordinates of the required C.G. of the whole area of the cardioid then by symmetry $\bar{y}=0$. Also by symmetry the x-coordinate of the C.G. of the whole area of the cardioid is the same as the x-coordinate of the C.G. of the upper half ABO of this area. Consider an elementary strip POQ of the area ABO. The area of this strip is $\frac{1}{2}r^2\,\delta\theta$ and the x-coordinate of its C.G. can be taken as $\frac{2}{3}r\cos\theta$.

$$\therefore\quad \bar{x}=\frac{\int_0^{\pi}\frac{2}{3}r\cos\theta.\frac{1}{2}r^2\,d\theta}{\int_0^{\pi}\frac{1}{2}r^2\,d\theta}=\frac{\int_0^{\pi}\frac{2}{3}r^3\cos\theta\,d\theta}{\int_0^{\pi}r^2\,d\theta}$$

$$=\frac{\frac{2}{3}\int_0^{\pi}a^3(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^{\pi}a^2(1+\cos\theta)^2\,d\theta},\quad\text{[from (1)]}$$

$$=\frac{\frac{2a}{3}\int_0^{\pi}(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^{\pi}(1+\cos\theta)^2\,d\theta}.$$

Now the Nr. of $\bar{x}$

$$=\tfrac{2}{3}a\int_0^{\pi}(1+\cos\theta)^2\cos\theta\,d\theta$$

$$=\tfrac{2}{3}a\int_0^{\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\cos\theta\,d\theta$$

$$=\tfrac{2}{3}a\int_0^{\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta$$

$$=\tfrac{2}{3}a.2\int_0^{\pi/2}(3\cos^2\theta+\cos^4\theta)\,d\theta \qquad \text{[Note]}$$

$\left[\because \int_0^{\pi}\cos^n\theta\,d\theta=0 \text{ or } 2\int_0^{\pi/2}\cos^n\theta\,d\theta \text{ according as } n \text{ is odd or even}\right]$

$$=\frac{4a}{3}\left[3.\frac{1}{2}.\frac{\pi}{2}+\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]=\frac{5\pi a}{4}.$$

Also the Dr. of $\bar{x}$

$$=\int_0^{\pi}(1+\cos\theta)^2\,d\theta=\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=\Big[\theta\Big]_0^{\pi}+0+2\int_0^{\pi/2}\cos^2\theta\,d\theta=\pi+2.\tfrac{1}{2}.\frac{\pi}{2}=\frac{3\pi}{2}.$$

$\therefore$ from (2),

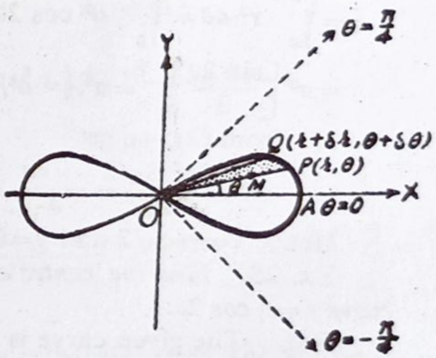
$$\bar{x}=\frac{\frac{5}{4}\pi a}{\frac{3}{2}\pi}=\frac{5a}{6}.\quad\text{Also } \bar{y}=0, \text{ as found above.}$$

Ex. 24. *Find the centre of gravity of the area of one loop of the lemniscate $r^2=a^2\cos 2\theta$.*

(Agra 1984; Kanpur 80, 83; Rohilkhand 83; Raj. 71)

Sol. The given curve is

$r^2=a^2\cos 2\theta$. ...(1)



It is symmetrical about the initial line and about the pole. Putting $r=0$ in (1), we get $\cos 2\theta=0$ *i.e.*, $2\theta=\pm\frac{1}{2}\pi$ *i.e.*, $\theta=\pm\frac{1}{4}\pi$.

For one loop of the curve θ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$ and this loop is symmetrical about the initial line. For the upper half of this loop θ varies from 0 to $\pi/4$.

If $(\bar{x}, \bar{y})$ be the coordinates of the required C. G. of the area of one loop, then $\bar{y}=0$ (by symmetry about the x-axis).

Also the x-coordinate of the C.G. of the whole area of the loop is the same as the x-coordinate of the C.G. of the upper half of the area of this loop. Now take an elementary strip OPQ of the area of the upper half of the loop. The area of this strip is $\frac{1}{2}r^2\,\delta\theta$ and its C.G. can be taken as the point M on OP such that $OM=\frac{2}{3}OP=\frac{2}{3}r$. The x-coordinate of M is $\frac{2}{3}r\cos\theta$.

$$\therefore \quad \bar{x}=\frac{\int_0^{\pi/4} \frac{2}{3} r \cos \theta . \frac{1}{2} r^2 \, d\theta}{\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta}=\frac{\frac{2}{3} \int_0^{\pi/4} r^3 \cos \theta \, d\theta}{\int_0^{\pi/4} r^2 \, d\theta}. \qquad \ldots(2)$$

Now the Nr. of $\bar{x}$

$=\frac{2}{3} \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos \theta \, d\theta$, substituting for r from (1)

$=\frac{2a^3}{3} \int_0^{\pi/4} (1-2 \sin^2 \theta)^{3/2} \cos \theta \, d\theta$ [Note]

Put $\sqrt{2} \sin \theta=\sin \phi$ so that $\sqrt{2} \cos \theta \, d\theta=\cos \phi \, d\phi$.

Also when $\theta=0$, $\phi=0$, and when $\theta=\pi/4$, $\phi=\pi/2$.

$\therefore$ the Nr. of $\bar{x}$

$=\frac{2a^3}{3} \int_0^{\pi/2} (1-\sin^2 \phi)^{3/2} . \frac{\cos \phi}{\sqrt{2}} \, d\phi$

$=\frac{2a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^4 \phi \, d\phi=\frac{2a^3}{3\sqrt{2}} \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}=\frac{\pi a^3}{8\sqrt{2}}.$

Also the Dr. of $\bar{x}$

$=\int_0^{\pi/4} r^2 \, d\theta=\int_0^{\pi/4} a^2 \cos 2\theta \, d\theta,$ [from (1)]

$=a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}=a^2 . \frac{1}{2}=a^2/2.$

$\therefore$ from (2), we get

$$\bar{x}=\frac{\pi a^3/8\sqrt{2}}{a^2/2}=\frac{\pi a}{4\sqrt{2}}=\frac{\pi a\sqrt{2}}{8}.$$

Hence $\bar{x}=\frac{1}{8}\pi a\sqrt{2}$ and $\bar{y}=0$.

Ex. 25. *Find the centre of gravity of the area of a loop of the curve $r=a \cos 2\theta$.* **[Kanpur 1988;** Agra 75]

Sol. The given curve is

$$r=a \cos 2\theta. \qquad \ldots(1)$$

It is symmetrical about the initial line. Putting $r=0$, we get $\cos 2\theta=0$ *i.e.,* $2\theta=\pm\frac{1}{2}\pi$ *i.e.,* $\theta=\pm\frac{1}{4}\pi$.

$\therefore$ for one loop of the curve θ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$ and this loop is symmetrical about the initial line. Draw the figure of this loop.

If $(\bar{x}, \bar{y})$ be the coordinates of the required C.G. of the area of one loop, then $\bar{y}=0$ (by symmetry about x-axis).

Now consider an elementary strip of area $\frac{1}{2}r^2 \delta\theta$ of the given loop of the curve. Clearly the x-coordinate of the C. G. of this strip can be taken as $\frac{2}{3} r \cos \theta$.

$$\therefore \quad \bar{x}=\frac{\int_{-\pi/4}^{\pi/4} \frac{2}{3} r \cos \theta . \frac{1}{2} r^2 \, d\theta}{\int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta}=\frac{\frac{2}{3} \int_{-\pi/4}^{\pi/4} r^3 \cos \theta \, d\theta}{\int_{-\pi/4}^{\pi/4} r^2 \, d\theta} \qquad \ldots(2)$$

Now the Nr. of $\bar{x}$

$=\frac{2}{3} \int_{-\pi/4}^{\pi/4} r^3 \cos \theta \, d\theta=\frac{2}{3} \int_{-\pi/4}^{\pi/4} (a \cos 2\theta)^3 \cos \theta \, d\theta$, from (1)

$=\frac{4a^3}{3} \int_0^{\pi/4} \cos^3 2\theta \cos \theta \, d\theta=\frac{4a^3}{3} \int_0^{\pi/4} (1-2 \sin^2 \theta)^3 \cos \theta \, d\theta.$

Put $\sqrt{2} \sin \theta=\sin \phi$ so that $\sqrt{2} \cos \theta \, d\theta=\cos \phi \, d\phi$.

Also when $\theta=0$, $\phi=0$ and when $\theta=\pi/4$, $\phi=\pi/2$.

$\therefore$ the Nr. of $\bar{x}$

$=\frac{4a^3}{3} \int_0^{\pi/2} (1-\sin^2 \phi)^3 \frac{\cos \phi}{\sqrt{2}} \, d\phi$

$=\frac{4a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^6 \phi . \cos \phi \, d\phi=\frac{4a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^7 \phi \, d\phi$

$=\frac{4a^3}{3\sqrt{2}} . \frac{6}{7} . \frac{4}{5} . \frac{2}{3}=\frac{64a^3}{105\sqrt{2}}.$

Again the Dr. of $\bar{x}$

$=\int_{-\pi/4}^{\pi/4} r^2 \, d\theta=\int_{-\pi/4}^{\pi/4} a^2 \cos^2 2\theta \, d\theta,$ from (1)

$=2a^2 \int_0^{\pi/4} \cos^2 2\theta \, d\theta=2a^2 \int_0^{\pi/2} \cos^2 t . \frac{1}{2} dt$, putting $2\theta=t$

$=a^2 \int_0^{\pi/2} \cos^2 t \, dt=a^2 . \frac{1}{2} . \frac{1}{2}\pi=\frac{a^2\pi}{4}.$

Therefore from (2), we get

$$\bar{x}=\frac{64a^3/(105\sqrt{2})}{a^2\pi/4}=\frac{64 \times 4 \times a}{105\sqrt{2}\pi}=\frac{128\sqrt{2}.a}{105\pi}.$$

Hence $\bar{x}=128\sqrt{2}a/105\pi$ and $\bar{y}=0$.

Remark. To find $\bar{x}$ we can either integrate over the whole loop or we can integrate only over the upper half of this loop. Note that by symmetry about the x-axis, the x-coordinate of the C.G. of the whole loop is the same as the x-coordinate of the C.G. of the upper half of this loop.

Ex. 26. *Find the C.G. of the loop of the curve $r=a \cos 3\theta$ containing the initial line.* **(Agra 1988; Kanpur 85, 87)**

Sol. The given curve is $r=a \cos 3\theta$. $\qquad \ldots(1)$

It is symmetrical about the initial line. Putting $r=0$, we get $\cos 3\theta=0$ *i.e.,* $3\theta=\pm\frac{1}{2}\pi$ *i.e.,* $\theta=\pm\frac{1}{6}\pi$.

$\therefore$ for one loop of the curve θ varies from $-\frac{1}{6}\pi$ to $\frac{1}{6}\pi$ and this loop is symmetrical about the initial line. We are to find the C.G. of this loop. Draw the figure of this loop as in Ex. 24.

If $(\bar{x}, \bar{y})$ be the coordinates of the required C.G. of the area of this loop, then $\bar{y}=0$ (by symmetry about x-axis).

Also the x-coordinate of the C.G. of the whole area of the loop is the same as the x-coordinate of the C.G. of the upper half of the area of the loop. Now take an elementary strip OPQ of the area of the upper half of the loop. The area of this strip is $\frac{1}{2}r^2\,\delta\theta$ and its C.G. can be taken as the point M on OP such that $OM=\frac{2}{3}OP=\frac{2}{3}r$.

The x-coordinate of M is $\frac{2}{3}r\cos\theta$.

$$\therefore \quad \bar{x}=\frac{\int_0^{\pi/6}\frac{2}{3}r\cos\theta.\frac{1}{2}r^2\,d\theta}{\int_0^{\pi/6}\frac{1}{2}r^2\,d\theta}=\frac{\frac{2}{3}\int_0^{\pi/6}r^3\cos\theta\,d\theta}{\int_0^{\pi/6}r^2\,d\theta} \qquad ...(2)$$

Now the Nr. of $\bar{x}$

$=\frac{2}{3}\int_0^{\pi/6}a^3\cos^3 3\theta\cos\theta\,d\theta$, substituting for r from (1)

$=\frac{2a^3}{3}\int_0^{\pi/6}\left(\frac{\cos 9\theta+3\cos 3\theta}{4}\right)\cos\theta\,d\theta$,

using the formula $\cos 3x=4\cos^3 x-3\cos x$

$$=\frac{a^3}{12}\int_0^{\pi/6}(2\cos 9\theta\cos\theta+3.2\cos 3\theta\cos\theta)\,d\theta$$

$$=\frac{a^3}{12}\int_0^{\pi/6}\left[\cos 10\theta+\cos 8\theta+3(\cos 4\theta+\cos 2\theta)\right]d\theta$$

$$=\frac{a^3}{12}\left[\frac{\sin 10\theta}{10}+\frac{\sin 8\theta}{8}+\frac{3\sin 4\theta}{4}+\frac{3\sin 2\theta}{2}\right]_0^{\pi/6}$$

$$=\frac{a^3}{12}\left[-\frac{1}{10}.\frac{\sqrt{3}}{2}-\frac{1}{8}.\frac{\sqrt{3}}{2}+\frac{3}{4}.\frac{\sqrt{3}}{2}+\frac{3}{2}.\frac{\sqrt{3}}{2}\right]$$

$$=\frac{a^3}{12}.\frac{\sqrt{3}}{2}\left(-\frac{1}{10}-\frac{1}{8}+\frac{3}{4}+\frac{3}{2}\right)=\frac{a^3}{12}.\frac{\sqrt{3}}{2}.\frac{81}{40}=\frac{a^3}{12}.\frac{81\sqrt{3}}{80}$$

Also the Dr. of $\bar{x}$

$=\int_0^{\pi/6}r^2\,d\theta=\int_0^{\pi/6}a^2\cos 3\theta\,d\theta$, [from (1)]

$=a^2\int_0^{\pi/2}\cos^2 t\,\frac{dt}{3}$, putting $3\theta=t$, so that $3d\theta=dt$

$$=\frac{a^2}{3}\int_0^{\pi/2}\cos^2 t\,dt=\frac{a^2}{3}.\frac{1}{2}.\frac{\pi}{2}=\frac{a^2\pi}{12}.$$

$\therefore$ from (2), we get

$$\bar{x}=\frac{a^3}{12}\frac{81\sqrt{3}}{80}\times\frac{12}{a^2\pi}=\frac{81\sqrt{3}}{80\pi}a.$$

Hence $\bar{x}=\frac{81\sqrt{3}}{80\pi}a$, and $\bar{y}=0$.

Ex. 27. *Find the co-ordinates of the centre of gravity of the area of the loop of the curve $r=a\sin 2\theta$ which lies in the positive quadrant, the density being supposed uniform.* **(Agra 1985)**

Sol. The given curve is

$$r=a\sin 2\theta. \qquad ...(1)$$

The curve (1) is not symmetrical about the initial line.

Putting $r=0$, we get

$\sin 2\theta=0$ *i.e.* $2\theta=0, \pi$ *i.e.,* $\theta=0, \pi/2$.

$\therefore$ for one loop of the curve θ varies from 0 to $\pi/2$ and this loop is symmetrical about the line $\theta=\pi/4$ *i.e.,* about the line $y=x$. Therefore the C.G. of this loop must lie on the line $y=x$.

Therefore if $(\bar{x}, \bar{y})$ be the required C.G. of the area of this loop, then $\bar{x}=\bar{y}$. Now draw the figure of one loop and to find $\bar{x}$ take an elementary strip of area $\frac{1}{2}r^2\,\delta\theta$.

We have

$$\bar{x}=\frac{\frac{2}{3}\int_0^{\pi/2}r^3\cos\theta\,d\theta}{\int_0^{\pi/2}r^2\,d\theta}=\frac{\frac{2}{3}\int_0^{\pi/2}a^3\sin^3 2\theta\cos\theta\,d\theta}{\int_0^{\pi/2}a^2\sin^2 2\theta\,d\theta}, \text{ from (1).}$$

The Nr. of $\bar{x}$

$$=\frac{2}{3}a^3\int_0^{\pi/2}\sin^3 2\theta.\cos\theta\,d\theta=\frac{2}{3}a^3\int_0^{\pi/2}(2\sin\theta\cos\theta)^3\cos\theta\,d\theta$$

$$=\frac{16}{3}a^3\int_0^{\pi/2}\sin^3\theta.\cos^4\theta\,d\theta=\frac{16}{3}a^3\frac{2.3.1}{7.5.3.1}=\frac{32a^3}{105}.$$

Also the Dr. of $\bar{x}$

$$=a^2\int_0^{\pi/2}\sin^2 2\theta\,d\theta=a^2\int_0^{\pi/2}(2\sin\theta\cos\theta)^2\,d\theta$$

$$=4a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta=4a^2\,\frac{1.1}{4.2}.\frac{\pi}{2}=\frac{\pi a^2}{4}.$$

Therefore $\bar{x}=\frac{32a^3/105}{\pi a^2/4}=\frac{128a}{105\pi}$.

Hence $\bar{x}=\bar{y}=128a/105\pi$.

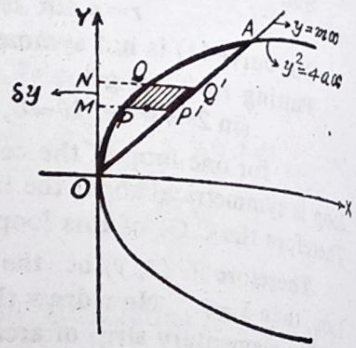
Remark. If we do not make use of the symmetry about the line $y=x$, we can find $\bar{y}$ by integration as we have found $\bar{x}$.

$$\text{Then}\quad \bar{y}=\frac{\int_0^{\pi/2}\frac{2}{3}r\sin\theta.\frac{1}{2}r\,l)}{\int_0^{\pi/2}\frac{1}{2}r^2\,d\theta}.$$

C.G. of an enclosed area.

***Ex. 28.** *Find the centre of gravity of the area cut off from the parabola $y^2=4ax$ by the straight line $y=mx$.*

or *Find the locus of the centroid of the area of the parabola cut off by a variable straight line passing through the vertex.*



Sol. The given parabola is

$$y^2=4ax. \qquad \text{...(1)}$$

Its vertex is (0, 0). Equation of the straight line passing through the vertex is

$$y=mx. \qquad \text{...(2)}$$

Solving (1) and (2) we get the coordinates of the point of intersection A as $(4a/m^2, 4a/m)$.

Take a line PP' parallel to the x-axis where $P(x_1, y)$ is a point on the arc OA of the parabola $y^2=4ax$ and $P'(x_2, y)$ a point on the line $y=mx$. Then

$$y^2=4ax_1 \text{ and } y=mx_2. \qquad \text{...(3)}$$

We are required to find the C.G. of the area $OPAP'O$.

Consider an elementary strip $PP'Q'Q$ parallel to x-axis, of the enclosed area $OPAP'O$. Then area of this strip$=(x_2-x_1)\,\delta y$ and the C.G. of this strip can be taken as the middle point $\left(\frac{x_1+x_2}{2}, y\right)$ of PP'.

The limits of y for the area enclosed by the parabola and the straight line are 0 to $4a/m$.

If $(\bar{x}, \bar{y})$ be the C.G. of the area $OPAP'O$, then

$$\bar{x}=\frac{\int_0^{4a/m}\frac{x_1+x_2}{2}.(x_2-x_1)\,dy}{\int_0^{4a/m}(x_2-x_1)\,dy}=\frac{\frac{1}{2}\int_0^{4a/m}(x_2^2-x_1^2)\,dy}{\int_0^{4a/m}(x_2-x_1)\,dy}$$

$$\text{and}\quad \bar{y}=\frac{\int_0^{4a/m}y\,(x_2-x_1)\,dy}{\int_0^{4a/m}(x_2-x_1)\,dy}.$$

Now the Nr. of $\bar{x}$

$$=\tfrac{1}{2}\int_0^{4a/m}(x_2^2-x_1^2)\,dy=\tfrac{1}{2}\int_0^{4a/m}\left(\frac{y^2}{m^2}-\frac{y^4}{16a^2}\right)dy, \qquad \text{from (3)}$$

$$=\tfrac{1}{2}\left[\frac{y^3}{3m^2}-\frac{y^5}{80a^2}\right]_0^{4a/m}=\tfrac{1}{2}\left[\frac{64a^3}{3m^5}-\frac{64a^3}{5m^5}\right]=\frac{64a^3}{15m^5},$$

the Nr. of $\bar{y}$

$$=\int_0^{4a/m}y\,(x_2-x_1)\,dy=\int_0^{4a/m}y.\left(\frac{y}{m}-\frac{y^2}{4a}\right)dy, \qquad \text{from (3)}$$

$$=\int_0^{4a/m}\left(\frac{y^2}{m}-\frac{y^3}{4a}\right)dy=\left[\frac{y^3}{3m}-\frac{y^4}{16a}\right]_0^{4a/m}$$

$$=\left[\frac{64a^3}{3m^4}-\frac{16a^3}{m^4}\right]=\frac{16a^3}{3m^4},$$

and the Dr. of both $\bar{x}$ and $\bar{y}$

$$=\int_0^{4a/m}(x_2-x_1)\,dy=\int_0^{4a/m}\left(\frac{y}{m}-\frac{y^2}{4a}\right)dy, \qquad \text{from (3)}$$

$$=\left[\frac{y^2}{2m}-\frac{y^3}{12a}\right]_0^{4a/m}=\frac{8a^2}{m^3}-\frac{16a^2}{3m^3}=\frac{8a^2}{3m^3}.$$

$$\text{Thus}\quad \bar{x}=\frac{64a^3/15m^5}{8a^2/3m^3}=\frac{8a}{5m^2} \text{ and } \bar{y}=\frac{16a^3/3m^4}{8a^2/3m^3}=\frac{2a}{m}.$$

Hence the coordinates of the required C.G. are

$$(8a/5m^2, 2a/m).$$

Now as the st. line through the vertex changes, m changes. Therefore to find the locus of the centroid, eliminate the variable m between the vulues of $\bar{x}$ and $\bar{y}$. Thus

$$(2a/\bar{y})^2=(8a/5\bar{x}) \text{ or } \bar{y}^2=\tfrac{5}{2}a\bar{x}.$$

Generalising $(\bar{x}, \bar{y})$, we get the required locus as $y^2=\frac{5}{2}ax$, which is a parabola.

***Ex. 29 (a).** *Find the centre of gravity of the area enclosed by the curves $y^2=ax$ and $x^2=by$.* **(Jiwaji 1982)**

Sol. The given curves are the parabolas $y^2=ax$ ...(1)

and $x^2=by$. ...(2)

Solving (1) and (2), we get the coordinates of the points of intersection as (0, 0) and $(a^{1/3}\,b^{2/3}, b^{1/3}\,a^{2/3})$.

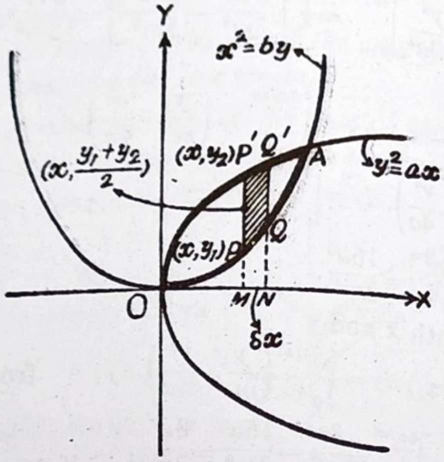
Take a straight line PP' parallel to the y-axis where $P(x, y_1)$ is a point on the arc OA of the parabola $x^2=by$ and $P'(x, y_2)$ is a point on the arc OA of the parabola $y^2=ax$.

Then we have $x^2=by_1$ and $y_2^2=ax$. ...(3)

We are required to find the C.G. of the area $OPAP'O$.

Consider an elementary strip $PP'Q'Q$ of this area parallel to the y-axis.

Then the area of the strip $=(y_2-y_1)\,\delta x$ and the C.G. of the strip can be taken as the point $(x, \frac{1}{2}(y_1+y_2))$.



For the given enclosed area x varies from 0 to $a^{1/3}b^{2/3}$.
If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int x.(y_2-y_1)\,dx}{\int (y_2-y_1)\,dx}$$

and $$\bar{y}=\frac{\int \frac{1}{2}(y_1+y_2).(y_2-y_1)\,dx}{\int (y_2-y_1)\,dx}=\frac{\frac{1}{2}\int (y_2^2-y_1^2)\,dx}{\int (y_2-y_1)\,dx},$$

between the suitable limits of integration.

Now the Nr. of $\bar{x}$

$$=\int_0^{a^{1/3}b^{2/3}} x\,(y_2-y_1)\,dx=\int_0^{a^{1/3}b^{2/3}} x.\left[\sqrt{(ax)}-\frac{x^2}{b}\right]dx,$$ from (3)

$$=\left[\frac{2}{5}a^{1/2}x^{5/2}-\frac{x^4}{4b}\right]_0^{a^{1/3}b^{2/3}}=\left[\frac{2}{5}a^{1/2}.a^{5/6}b^{5/3}-\frac{a^{4/3}b^{8/3}}{4b}\right]$$

$$=\frac{3}{20}a^{4/3}b^{5/3},$$

the Nr. of $\bar{y}$

$$=\frac{1}{2}\int_0^{a^{1/3}b^{2/3}}(y_2^2-y_1^2)\,dx=\frac{1}{2}\int_0^{a^{1/3}b^{2/3}}\left(ax-\frac{x^4}{b^2}\right)dx,$$ from (3)

$$=\frac{1}{2}\left[\frac{ax^2}{2}-\frac{x^5}{5b^2}\right]_0^{a^{1/3}b^{2/3}}=\frac{1}{2}\left[\frac{a.a^{2/3}b^{4/3}}{2}-\frac{a^{5/3}b^{10/3}}{5b^2}\right]$$

$$=\frac{3}{20}a^{5/3}b^{4/3},$$

and the Dr. of both $\bar{x}$ and $\bar{y}$ is

$$=\int_0^{a^{1/3}b^{2/3}}\left[\sqrt{(ax)}-\frac{x^2}{b}\right]dx=\left[\frac{2}{3}a^{1/2}x^{3/2}-\frac{x^3}{3b}\right]_0^{a^{1/3}b^{2/3}}$$

$$=[\tfrac{2}{3}a^{1/2}.a^{1/2}b-(ab^2/3b)]=[\tfrac{2}{3}ab-\tfrac{1}{3}ab]=\tfrac{1}{3}ab.$$

Thus $\bar{x}=\dfrac{\frac{3}{20}a^{4/3}b^{5/3}}{\frac{1}{3}ab}=\dfrac{9}{20}a^{1/3}b^{2/3}$ and $\bar{y}=\dfrac{\frac{3}{20}a^{5/3}b^{4/3}}{\frac{1}{3}ab}=\dfrac{9}{20}a^{2/3}b^{1/3}$.

∴ the required C.G. is
$(\frac{9}{20}a^{1/3}b^{2/3}, \frac{9}{20}a^{2/3}b^{1/3})$.

Ex. 29 (b). *Determine the C.G. of the area bounded by the parabolas* $y^2=8x$ *and* $x^2=8y$. **(Meerut 71)**

Sol. Proceed exactly as in Ex. 29 (a). Here $a=8$ and $b=8$.

The required C.G. is $(\frac{18}{5}, \frac{18}{5})$.

Ex. 29 (c). *Find the position of the centroid of the area enclosed by the parabolas* $y^2=x$ *and* $x^2=2y$.

Sol. Proceed exactly as in Ex. 29 (a). Here $a=1$ and $b=2$.

The required C.G. is $(\frac{9}{20}\,2^{2/3}, \frac{9}{20}\,2^{1/3})$.

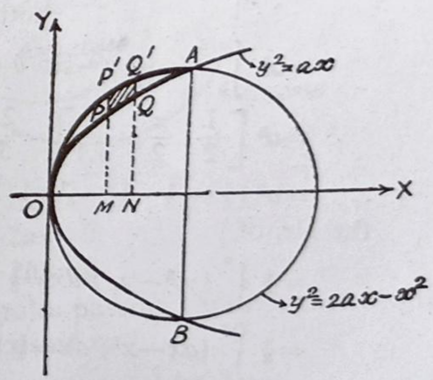
Ex. 30. *Find the position of the centre of gravity of the area enclosed by the curves* $y^2=ax$ *and* $y^2=2ax-x^2$ *on the positive side of the axis of* x. **(Rohilkhand 1980; Kanpur 85, 86)**

Sol. The given curves are

$y^2=ax,$ ...(1)

and $y^2=2ax-x^2$ ...(2)

The curve (1) is a parabola and the curve (2) is a circle with a as radius and $(a, 0)$ as its centre.



Solving (1) and (2) we get the coordinates of the points of intersection as $(0, 0)$, $(a, \pm a)$.

Take a straight line PP' parallel to the y-axis where $P(x, y_1)$ is a point on the arc OA of the parabola $y^2=ax$ and $P'(x, y_2)$ is a point on the arc OA of the circle $y^2=2ax-x^2$.

Then we have $y_1^2=ax$ and $y_2^2=2ax-x^2$. ...(3)

We are required to find the C.G. of the area $OPAP'O$ which is enclosed by the curves (1) and (2) and lies above the axis of x.

Consider an elementary strip $PP'Q'Q$ parallel to the y-axis. The area of this strip is $(y_2-y_1)\,\delta x$ and the C.G. of this strip can be taken as the middle point $\left(x, \dfrac{y_1+y_2}{2}\right)$ of PP'. Also for the given enclosed area x varies from 0 to a.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_0^a x.(y_2-y_1)\,dx}{\int_0^a (y_2-y_1)\,dx}$$

and
$$\bar{y}=\frac{\int_0^a \tfrac{1}{2}(y_1+y_2).(y_2-y_1)dx}{\int_0^a (y_2-y_1)\,dx}=\frac{\tfrac{1}{2}\int_0^a (y_2^2-y_1^2)\,dx}{\int_0^a (y_2-y_1)\,dx}.$$

Now the Nr. of $\bar{x}$

$$=\int_0^a x\,(y_2-y_1)\,dx$$

$$=\int_0^a x\,[\sqrt{(2ax-x^2)}-\sqrt{(ax)}]\,dx, \quad \text{from (3)}$$

$$=\int_0^a [x\sqrt{\{a^2-(a-x)^2\}}]\,dx-\int_0^a a^{1/2}\,x^{3/2}\,dx \quad \text{[Note]}$$

$$=\int_{\pi/2}^0 a\,(1-\sin\theta)\,a\cos\theta.(-a\cos\theta)\,d\theta-a^{1/2}.\left[\frac{2}{5}\,x^{5/2}\right]_0^a$$

putting $a-x=a\sin\theta$ in the first integral

$$=a^3\int_0^{\pi/2}(\cos^2\theta-\sin\theta\cos^2\theta)\,d\theta-\frac{2}{5}\,a^{1/2}\,a^{5/2}$$

$$=a^3\left[\frac{1}{2}.\frac{\pi}{2}-\frac{1}{3.1}\right]-\frac{2}{5}\,a^3=a^3\left[\frac{\pi}{4}-\frac{1}{3}\right]-\frac{2a^3}{5}$$

$$=a^3\left[\tfrac{1}{4}\pi-\tfrac{1}{3}-\tfrac{2}{5}\right]=\tfrac{1}{60}a^3\,[15\pi-44],$$

the Nr. of $\bar{y}$

$$=\tfrac{1}{2}\int_0^a (y_2^2-y_1^2)\,dx=\tfrac{1}{2}\int_0^a [(2ax-x^2)-ax]\,dx$$

$$=\tfrac{1}{2}\int_0^a (ax-x^2)\,dx=\tfrac{1}{2}\left[\frac{ax^2}{2}-\frac{x^3}{3}\right]_0^a=\tfrac{1}{2}\left[\frac{a^3}{2}-\frac{a^3}{3}\right]=\frac{a^3}{12},$$

and the Dr. of both $\bar{x}$ and $\bar{y}$

$$=\int_0^a (y_2-y_1)\,dx$$

$$=\int_0^a [\sqrt{(2ax-x^2)}-\sqrt{(ax)}]\,dx, \quad \text{from (3)}$$

$$=\int_0^a [\sqrt{\{a^2-(a-x)^2\}}-a^{1/2}\,x^{1/2}]\,dx$$

$$=\int_{\pi/2}^0 a\cos\theta.(-a\cos\theta)\,d\theta-a^{1/2}\left[\frac{2}{3}\,x^{3/2}\right]_0^a,$$

putting $a-x=a\sin\theta$ in the first integral

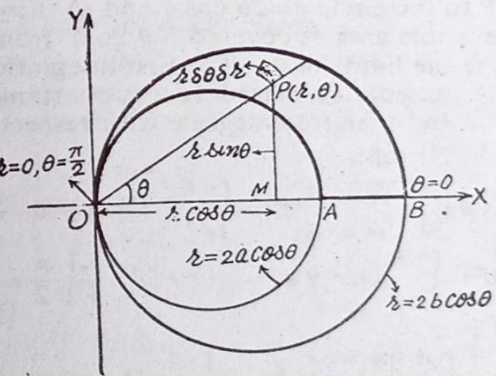
$$=a^2\int_0^{\pi/2}\cos^2\theta\,d\theta-\frac{2}{3}\,a^{1/2}.a^{3/2}=\left[a^2.\frac{1}{2}.\frac{\pi}{2}-\frac{2a^2}{3}\right]=\frac{a^2\,(3\pi-8)}{12}.$$

Thus $$\bar{x}=\frac{\tfrac{1}{60}\,a^3\,[15\pi-44]}{\tfrac{1}{12}\,a^2\,[3\pi-8]}=\frac{a\,(15\pi-44)}{5\,(3\pi-8)}$$

and $$\bar{y}=\frac{\tfrac{1}{12}\,a^3}{\tfrac{1}{12}a^2\,(3\pi-8)}=\frac{a}{(3\pi-8)}.$$

$\therefore$ the required C.G. is the point $\left(\dfrac{a\,(15\pi-44)}{5\,(3\pi-8)}, \dfrac{a}{(3\pi-8)}\right)$.

Ex. 31. *Find the position of the centre of gravity of the area enclosed by the curves $x^2+y^2-2ax=0$ and $x^2+y^2-2bx=0$ on the positive side of the axis of x.* **(Meerut 1990)**



Sol. The given curves are

$$x^2+y^2-2ax=0, \quad \text{...(1)}$$

and
$$x^2+y^2-2bx=0. \quad \text{...(2)}$$

Both these curves are circles passing through the origin and having their centres on the axis of x.

Changing to polars, the equation (1) becomes

$$x^2+y^2=2ax$$

or $$r^2=2ar\cos\theta \quad [\because\ x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2]$$

or $$r=2a\cos\theta. \quad \text{...(3)}$$

Similarly the equation (2) becomes

$$r=2b\cos\theta. \quad \text{...(4)}$$

The diameter of the circle (3) is $OA=2a$ and that of the circle (4) is $OB=2b$. We have taken $b>a$.

Let $(\bar{x}, \bar{y})$ be the C.G. of the area enclosed by the circles (3) and (4) and lying above the x-axis. Take a small element $r\delta\theta\,\delta r$ of this area at the point $P(r, \theta)$ lying within this area. The element $r\delta\theta\,\delta r$ being very small, its C.G. can be taken as the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

$$\therefore\quad \bar{x}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=2a\cos\theta}^{2b\cos\theta} r\cos\theta . r\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=2a\cos\theta}^{2b\cos\theta} r\,d\theta\,dr},$$

$$\bar{y}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=2a\cos\theta}^{2b\cos\theta} r\sin\theta . r\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=2a\cos\theta}^{2b\cos\theta} r\,d\theta\,dr}.$$

[Note that to cover the area under consideration first we regard θ as fixed. For that fixed value of θ, r goes from the circle $r=2a\cos\theta$ to the circle $r=2b\cos\theta$ and so these are the limits of r. Now the whole area is covered if θ goes from 0 to $\pi/2$ which are therefore the limits of θ. The first integration must be performed with respect to r regarding θ as constant whose limits are in terms of θ and then we integrate with respect to θ.]

Now the Nr. of $\bar{x}$

$$=\int_{\theta=0}^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_{2a\cos\theta}^{2b\cos\theta}d\theta=\int_0^{\pi/2}\tfrac{8}{3}(b^3\cos^3\theta-a^3\cos^3\theta)\cos\theta\,d\theta$$

$$=\tfrac{8}{3}(b^3-a^3)\int_{\theta=0}^{\pi/2}\cos^4\theta\,d\theta=\tfrac{8}{3}(b^3-a^3).\frac{3.1}{4.2}\frac{\pi}{2}=\frac{\pi(b^3-a^3)}{2},$$

the Nr. of $\bar{y}$

$$=\int_{\theta=0}^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_{2a\cos\theta}^{2b\cos\theta}d\theta=\tfrac{8}{3}\int_0^{\pi/2}(b^3\cos^3\theta-a^3\cos^3\theta)\sin\theta\,d\theta$$

$$=\tfrac{8}{3}(b^3-a^3)\int_0^{\pi/2}\cos^3\theta\sin\theta\,d\theta=\tfrac{8}{3}(b^3-a^3).\frac{2}{4.2}=\tfrac{2}{3}(b^3-a^3),$$

the Dr. of $\bar{x}$ = the Dr. of $\bar{y}$

$$=\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{2a\cos\theta}^{2b\cos\theta}d\theta$$

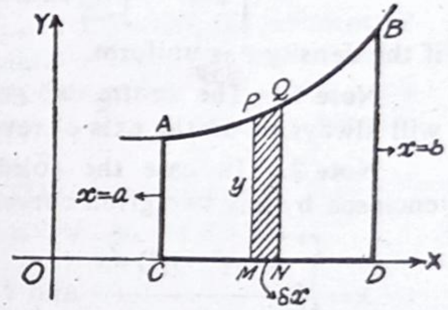
$$=\tfrac{1}{2}\int_0^{\pi/2}4(b^2-a^2)\cos^2\theta\,d\theta=2(b^2-a^2)\,\tfrac{1}{2}.\,\frac{\pi}{2}=\frac{\pi}{2}(b^2-a^2).$$

$$\therefore\quad \bar{x}=\frac{\frac{1}{2}\pi(b^3-a^3)}{\frac{1}{2}\pi(b^2-a^2)}=\frac{(b-a)(a^2+ab+b^2)}{(b-a)(b+a)}=\frac{(a^2+ab+b^2)}{a+b},$$

$$\bar{y}=\frac{\frac{2}{3}(b^3-a^3)}{\frac{1}{2}\pi(b^2-a^2)}=\frac{4(a^2+ab+b^2)}{3\pi(a+b)}.$$

§ 5. Centre of gravity of a solid of revolution.

To find the centre of gravity of a solid formed by revolving the curve $y=f(x)$ about the x-axis and cut off between the plane ends $x=a$ and $x=b$. **(Rohilkhand 1981)**



Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be any two neighbouring points on the arc AB of the curve $y=f(x)$. Draw PM and QN perpendiculars to the x-axis. If we revolve the area between the curve and the x-axis about the x-axis, the elementary strip $PMNQ$ generates a disc of small thickness δx and circular base of area πy^2. Thus volume of the elementary disc$=\pi y^2\delta x$ and its mass$=\pi y^2\delta x\,\rho$, ρ being the density per unit volume.

The C.G. of this disc may be taken as the point $M(x, 0)$ because $MN=\delta x$ is very small.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{y}=0$, (by symmetry). [Note that the solid of revolution formed by revolving the curve about the x-axis is symmetrical about the x-axis and so its C.G. must lie on the x-axis.]

$$\text{Also}\quad \bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int_a^b x.\rho\pi\,y^2\,dx}{\int_a^b \rho\pi\,y^2\,dx}$$

$$=\frac{\int_a^b xy^2\,dx}{\int_a^b y^2\,dx},\ \text{if the density } \rho \text{ is uniform.}$$

$$\textbf{Thus}\quad \bar{x}=\frac{\int_a^b xy^2\,dx}{\int_a^b y^2\,dx}\ \textbf{and}\ \bar{y}=0,\ \textbf{if the density is uniform.}$$

In case the solid is formed by revolving the curve about the axis of y and cut off between the planes $y=a$ and $y=b$, then the volume of the elementary circular disc formed by revolving the elementary strip about the y-axis$=\pi x^2\,\delta y$ and the C.G. of this disc may be supposed to be at $(0, y)$ because its thickness δy is very small.

Then $x=0$, (by symmetry about the y-axis), and $$\bar{y}=\frac{\int y\,dm}{\int dm}=\frac{\int_{y=a}^{b} y.\rho\pi\, x^2\, dy}{\int_{y=a}^{b} \rho\pi\, x^2\, dy}=\frac{\int_a^b yx^2\, dy}{\int_a^b x^2\, dy}$$

if the density ρ is uniform.

Note 1. The centre of gravity of the solid of revolution will always lie on the axis of revolution.

Note 2. In case the solid is formed by revolving the area enclosed by any two given curves **about the x-axis,** then

$$\bar{x}=\frac{\int_a^b x\,(y_2^2-y_1^2)\, dx}{\int_a^b (y_2^2-y_1^2)\, dx} \text{ and } \bar{y}=0,$$

where the points $P_1(x, y_1)$ and $P_2(x, y_2)$ are on the respective arcs of the given curves intervened between their common points; a and b are the abscissae of the common points of intersection.

If the axis of revolution is y-axis, then

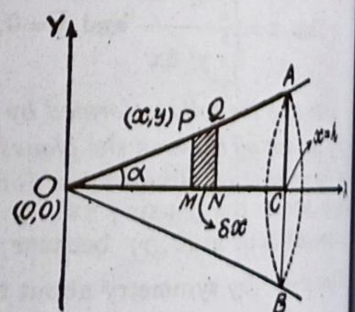
$$\bar{x}=0 \text{ and } \bar{y}=\frac{\int_a^b y\,(x_2^2-x_1^2)\, dy}{\int_a^b (x_2^2-x_1^2)\, dy},$$

where a and b are the ordinates of the common points of intersection; $P_1(x_1, y)$ and $P_2(x_2, y)$ being two points on the respective arcs of the two curves intervened between their points of intersection.

Examples on the C.G. of a solid of revolution.

Ex. 32. *Find the centre of gravity of a solid right circular cone of height h.* **(Rohilkhand 1991; Delhi 81; Meerut 79)**

Sol. Take a right angled triangle OCA in which $\angle OCA=90°$. The side OC is of length h and is along the **x-axis.** If we revolve the triangular area OCA about the **x-axis,** a solid right circular cone of height h is generated, the axis of the cone being along the **x-axis** and its vertex being



at O. Let $\angle AOC=\alpha$. Then the equation of the line OA is
$$y=x\tan\alpha. \qquad \textbf{(Note)} \quad ...(1)$$

Take an elementary strip $PMNQ$ of small width δx parallel to the y-axis. of the area of the triangle OCA. When the area OCA revolves about the line OC (*i.e.*, the x-axis), the elementary strip $PMNQ$ generates a disc of small thickness δx and circular base of area πy^2. The volume of this elementary disc$=\pi y^2\,\delta x$ and the C.G. of this disc may be supposed to be at $M(x, 0)$ because the thickness $MN=\delta x$ is very small.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G. of the solid cone, then $\bar{y}=0$ (by symmetry about OC *i.e.*, the x-axis).

Also $$\bar{x}=\frac{\int_0^h x.\pi y^2\, dx}{\int_0^h \pi y^2\, dx}=\frac{\int_0^h xy^2\, dx}{\int_0^h y^2\, dx}$$

$$=\frac{\int_0^h x.(x\tan\alpha)^2\, dx}{\int_0^h x^2\tan^2\alpha\, dx}, \qquad [\because\ y=x\tan\alpha, \text{ from (1)}]$$

$$=\frac{\int_0^h x^3\, dx}{\int_0^h x^2\, dx}=\frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h}=\frac{\frac{1}{4}h^4}{\frac{1}{3}h^3}=\frac{3h}{4}.$$

Hence for the required C.G., $\bar{x}=\frac{3}{4}h$ and $\bar{y}=0$.

Thus the C.G. of a solid right circular cone lies on its axis at a distance $\frac{3}{4}h$ from the vertex, h being the height of the cone.

Ex. 33. *Find the C.G. of a solid uniform hemi-sphere of radius a.* **(Delhi 1979, Gorakhpur 82)**

Sol. A hemi-sphere is formed by revolving the quadrant of a circle about one of the bounding radii.

Let the equation of the generating circle be
$$x^2+y^2=a^2. \qquad ...(1)$$

For the quadrant of the circle x varies from 0 to a.

If $(\bar{x}, \bar{y})$ be the C.G. of the hemi-sphere formed by revolving the quadrant of the circle about the **x-axis,** then $\bar{y}=0$, by symmetry.

Also $$\bar{x}=\frac{\int_0^a x\,\pi y^2\, dx}{\int_0^a \pi y^2\, dx}$$

$$=\frac{\int_0^a x\,(a^2-x^2)\,dx}{\int_0^a (a^2-x^2)\,dx}, \text{ [from (1)]}$$

$$=\frac{\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a}{\left[a^2x-\frac{x^3}{3}\right]_0^a}=\frac{\left[\frac{a^4}{2}-\frac{a^4}{4}\right]}{\left[a^3-\frac{a^3}{3}\right]}=\frac{\frac{1}{4}a^4}{\frac{2}{3}a^3}=\frac{3a}{8}.$$

$\therefore\ \bar{x}=\frac{3}{8}a$ and $\bar{y}=0$.

Hence the C.G. of a solid hemi-sphere of radius a lies on the symmetrical radius at a distance $\frac{3}{8}a$ from the plane base of the hemisphere.

Ex. 34 (a). *Find the centre of gravity of the segment of a sphere of radius a cut off by a plane at a distance h from the centre.* **(Kanpur 1983, Rohilkhand 78)**

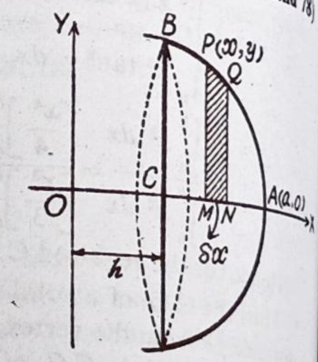
Sol. Suppose the spherical segment is formed by revolving the area CAB of the circle

$$x^2+y^2=a^2$$

about the x-axis.

Let BC be the line $x=h$. Then for the area BCA, x varies from h to a.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G. then $\bar{y}=0$, (by symmetry).



Also

$$\bar{x}=\frac{\int_h^a x\pi y^2\,dx}{\int_h^a \pi y^2\,dx}=\frac{\int_h^a x\,(a^2-x^2)\,dx}{\int_h^a (a^2-x^2)\,dx}, \ [\because\ y^2=a^2-x^2 \text{ from the circle}]$$

$$=\frac{\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_h^a}{\left[a^2x-\frac{x^3}{3}\right]_h^a}=\frac{\left[\left(\frac{a^4}{2}-\frac{a^4}{4}\right)-\left(\frac{a^2h^2}{2}-\frac{h^4}{4}\right)\right]}{\left[\left(a^3-\frac{a^3}{3}\right)-\left(a^2h-\frac{h^3}{3}\right)\right]}=\frac{\frac{a^4+h^4-2a^2h^2}{4}}{\frac{2a^3-3a^2h+h^3}{3}}$$

$$=\frac{3\,(a^2-h^2)^2}{4\,(2a+h)\,(a^2-2ah+h^2)}=\frac{3\,(a-h)^2\,(a+h)^2}{4\,(2a+h)\,(a-h)^2}=\frac{3\,(a+h)^2}{4\,(2a+h)}.$$

$\therefore\ \bar{x}=\frac{3}{4}\{(a+h)^2/(2a+h)\}$, $\bar{y}=0$.

Note. In case the segment is a hemisphere, then $h=0$.
$\therefore$ the C.G. of a hemi-sphere is the point $(\frac{3}{4}a^2/2a,\ 0)$ *i.e.*, $(\frac{3}{8}a, 0)$.

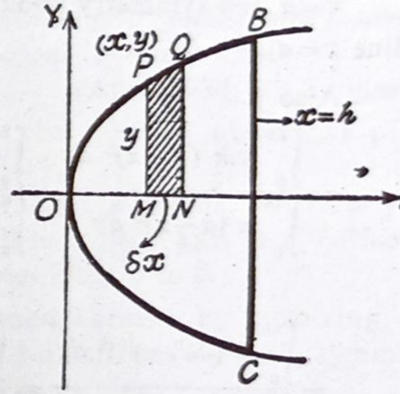
Ex. 34 (b). *Find the centre of gravity of a segment of height h, of a sphere of radius a, and deduce the position of the C.G. of a hemisphere.* **(Agra 1983, 88)**

Sol. Proceed as in Ex. 34 (a). Here $CA=h$ so that $OC=a-h$. The equation of the line BC is $x=a-h$ and for the area BCA x varies from $a-h$ to a.

Ex. 35. *Find the centre of gravity of the volume formed by the revolution of the portion of the parabola $y^2=4ax$ cut off by the ordinate $x=h$ about the axis of x.* **(Lucknow 1979)**

Sol. The portion BOC of the parabola $y^2=4ax$ cut off by the ordinate $x=h$ revolves about the x-axis. For this portion x varies from 0 to h.

If $(\bar{x}, \bar{y})$ be the C.G. of the solid formed; then $\bar{y}=0$, (by symmetry).



Also

$$\bar{x}=\frac{\int_0^h x\pi\,y^2\,dx}{\int_0^h \pi\,y^2\,dx}=\frac{\int_0^h x.4ax\,dx}{\int_0^h 4ax\,dx}, \ [\because\ y^2=4ax]$$

$$=\frac{\int_0^h x^2\,dx}{\int_0^h x\,dx}=\frac{\left[\frac{x^3}{3}\right]_0^h}{\left[\frac{x^2}{2}\right]_0^h}=\frac{\frac{h^3}{3}}{\frac{h^2}{2}}=\frac{2h}{3}.$$

$\therefore\ \bar{x}=\frac{2}{3}h$, $\bar{y}=0$.

****Ex. 36.** *Find the centre of gravity of the solid formed by the revolution of the area bounded by the parabola $y^2=4ax$, the axis of x and the latus rectum, about the latus rectum.* **(Meerut 1983)**

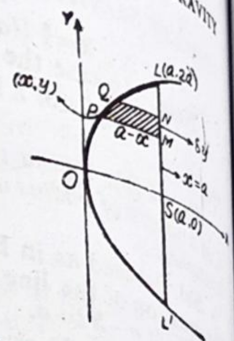
Sol. The given parabola is

$$y^2=4ax. \qquad \ldots(1)$$

Let O be the vertex and LSL' be the latus rectum. We have to revolve the area LOS about the line LS.

Take an elementary strip $PMNQ$ of small width δy perpendicular to the latus rectum and terminated by the latus rectum. We have $PM=a-x$.

When the area bounded by the parabola $y^2=4ax$, the axis of x and the latus rectum is revolved about the latus rectum LL', the elementary strip $PMNQ$ generates a disc of small thickness δy and circular base of area $\pi(a-x)^2$.



Thus volume of this elementary disc$=\pi(a-x)^2\,\delta y$ and the C.G. of this disc may be supposed to be at $M(a, y)$ because the thickness $MN=\delta y$ is very small. Also for the given area y varies from 0 to $2a$.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{x}=a$, (by symmetry about the axis of revolution LL' i.e., the line $x=a$).

Also $\bar{y}$

$$=\frac{\int_0^{2a} y.\pi\,(a-x)^2\,dy}{\int_0^{2a} \pi\,(a-x)^2\,dy}=\frac{\int_0^{2a} y.\left(a-\frac{y^2}{4a}\right)^2 dy}{\int_0^{2a}\left(a-\frac{y^2}{4a}\right)^2 dy}, \quad \text{from (1)}$$

$$=\frac{\int_0^{2a} y\,(4a^2-y^2)^2\,dy}{\int_0^{2a}(4a^2-y^2)^2\,dy}=\frac{\int_0^{2a} y\,(16a^4-8a^2y^2+y^4)\,dy}{\int_0^{2a}(16a^4-8a^2y^2+y^4)\,dy}$$

$$=\frac{\left[16a^4\,\frac{y^2}{2}-8a^2\,\frac{y^4}{4}+\frac{y^6}{6}\right]_0^{2a}}{\left[16a^4\,y-8\,\frac{a^2y^3}{3}+\frac{y^5}{5}\right]_0^{2a}}=\frac{\frac{32a^6}{3}}{\frac{256a^5}{15}}=\frac{5a}{8}.$$

Hence the required C.G. is given by $\bar{x}=a$, $\bar{y}=\frac{5}{8}a$.

Ex. 37. *A quadrant of the ellipse $(x^2/a^2)+(y^2/b^2)=1$ revolves about the major axis. Find the centre of gravity of the solid thus generated.*

Sol. The given ellipse is

$$x^2/a^2+y^2/b^2=1. \qquad \ldots(1)$$

Its quadrant (say the positive quadrant) is revolved about the major axis *i.e.*, the x-axis. For this quadrant of the ellipse x varies from 0 to a.

If $(\bar{x}, \bar{y})$ be the C.G. of the solid formed by revolving the quadrant of the ellipse (1) about the x-axis, then $\bar{y}=0$, (by symmetry).

Also
$$\bar{x}=\frac{\int_0^a x\,\pi y^2\,dx}{\int_0^a \pi y^2\,dx}=\frac{\int_0^a x.\,\frac{b^2}{a^2}\,(a^2-x^2)\,dx}{\int_0^a \frac{b^2}{a^2}\,(a^2-x^2)}, \qquad \text{[from (1)]}$$

$$=\frac{\left[a^2.\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a}{\left[a^2x-\frac{x^3}{3}\right]_0^a}=\frac{\left[\frac{a^4}{2}-\frac{a^4}{4}\right]}{\left[a^3-\frac{a^3}{3}\right]}=\frac{3a}{8}.$$

Hence the required C.G. is the point $(\frac{3}{8}a, 0)$.

***Ex. 38.** *Find the centre of gravity of a solid figure formed by revolving a quadrant of an ellipse about its minor axis.*

(Agra 1984, 87; Kanpur 78)

Sol. Let the quadrant (say the positive quadrant) of the ellipse
$$x^2/a^2+y^2/b^2=1, \qquad \ldots(1)$$
be revolved about its minor axis *i.e.*, the y-axis. Obviously the C.G. of the solid thus formed will lie on the axis of y. Also for this quadrant of the ellipse y varies from 0 to b.

If $(\bar{x}, \bar{y})$ be the C.G. of the solid formed by revolving the quadrant of the ellipse about the **y-axis**, then $\bar{x}=0$, (by symmetry about the axis of revolution *i.e.*, the y-axis).

And
$$\bar{y}=\frac{\int_0^b y\pi x^2\,dy}{\int_0^b \pi x^2\,dy}=\frac{\int_0^b y\,\frac{a^2}{b^2}\,(b^2-y^2)\,dy}{\int_0^b \frac{a^2}{b^2}\,(b^2-y^2)\,dy}, \qquad \text{from (1)}$$

$$=\frac{\int_0^b (b^2\,y-y^3)\,dy}{\int_0^b (b^2-y^2)\,dy}=\frac{\left[b^2\,\frac{y^2}{2}-\frac{y^4}{4}\right]_0^b}{\left[b^2y-\frac{y^3}{3}\right]_0^b}=\frac{\frac{1}{4}b^4}{\frac{2}{3}b^3}=\frac{3b}{8}.$$

Hence the required C.G. is the point $(0, \frac{3}{8}b)$.

Ex. 39. *Find the centroid of the volume formed by the revolution of the cycloid $x=a\,(\theta+\sin\theta)$, $y=a\,(1-\cos\theta)$ about the axis of y.*

Sol. The given cycloid is
$$x=a\,(\theta+\sin\theta),\ y=a\,(1-\cos\theta). \qquad \ldots(1)$$

[For figure see Ex. 20 on page 32]

It is revolved about the axis of y. For the requisite portion of the curve to be revolved y varies from 0 to $2a$ and θ varies from 0 to π.

If $(\bar{x}, \bar{y})$ be the C.G. of the solid formed by the revolution of the given cycloid about the y-axis, then $\bar{x}=0$, (by symmetry about the axis of revolution *i.e.*, the y-axis).

Also

$$\bar{y}=\frac{\int_{y=0}^{2a} y\,\pi x^2\,dy}{\int_{y=0}^{2a} \pi x^2\,dy}=\frac{\int_0^{\pi} y x^2 \frac{dy}{d\theta}\,d\theta}{\int_0^{\pi} x^2 \frac{dy}{d\theta}\,d\theta}. \quad \ldots(2)$$

Now the Nr. of $\bar{y}$

$$=\int_0^{\pi} a\,(1-\cos\theta).a^2\,(\theta+\sin\theta)^2.a\sin\theta\,d\theta, \quad \text{[from (1)]}$$

$$=a^4\int_0^{\pi} (\theta^2+2\theta\sin\theta+\sin^2\theta)\,(1-\cos\theta)\sin\theta\,d\theta$$

$$=a^4\int_0^{\pi} (\theta^2-\theta^2\cos\theta+2\theta\sin\theta-2\theta\sin\theta\cos\theta+\sin^2\theta-\sin^2\theta\cos\theta)\sin\theta\,d\theta$$

$$=a^4\left[\int_0^{\pi}\theta^2\sin\theta\,d\theta-\int_0^{\pi}\theta^2\cos\theta\sin\theta\,d\theta+2\int_0^{\pi}\theta\sin^2\theta\,d\theta-2\int_0^{\pi}\theta\sin^2\theta\cos\theta\,d\theta+\int_0^{\pi}\sin^3\theta\,d\theta-\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta\right] \ldots(3)$$

We have

$$\int_0^{\pi}\theta^2\sin\theta\,d\theta=\Big[\theta^2.(-\cos\theta)\Big]_0^{\pi}-\int_0^{\pi}2\theta.(-\cos\theta)\,d\theta$$

$$=\pi^2+2\int_0^{\pi}\theta\cos\theta\,d\theta$$

$$=\pi^2+2\Big[\theta\sin\theta\Big]_0^{\pi}-2\int_0^{\pi}\sin\theta\,d\theta=\pi^2-4\int_0^{\pi/2}\sin\theta\,d\theta=\pi^2-4.$$

$$\left[\text{Note that }\int_0^{\pi/2}\sin\theta\,d\theta=1.\right]$$

Again, let $u=\int_0^{\pi}\theta\sin^2\theta\,d\theta.$

Then $u=\int_0^{\pi}(\pi-\theta)\sin^2(\pi-\theta)\,d\theta$

$$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$=\int_0^{\pi}(\pi-\theta)\sin^2\theta\,d\theta.$$

$$\therefore\ 2u=\int_0^{\pi}\pi\sin^2\theta\,d\theta=2\pi\int_0^{\pi/2}\sin^2\theta\,d\theta$$

or $u=\pi.\tfrac{1}{2}.\tfrac{1}{2}\pi=\tfrac{1}{4}\pi^2.$

Thus $\int_0^{\pi}\theta\sin^2\theta\,d\theta=\tfrac{1}{4}\pi^2. \quad \ldots(4)$

Further $\int_0^{\pi}\theta^2\sin\theta\cos\theta\,d\theta=\left[\theta^2.\frac{\sin^2\theta}{2}\right]_0^{\pi}-\int_0^{\pi}2\theta.\frac{\sin^2\theta}{2}\,d\theta$

$$=0-\int_0^{\pi}\theta\sin^2\theta\,d\theta=-\tfrac{1}{4}\pi^2, \quad \text{by (4).}$$

Also $\int_0^{\pi}\theta\sin^2\theta\cos\theta\,d\theta=\left[\theta.\frac{\sin^3\theta}{3}\right]_0^{\pi}-\int_0^{\pi}\frac{\sin^3\theta}{3}\,d\theta$

$$=0-\tfrac{1}{3}\int_0^{\pi}\sin^3\theta\,d\theta$$

$$=-\frac{2}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta=-\frac{2}{3}\cdot\frac{2}{3.1}=-\frac{4}{9},$$

$$\int_0^{\pi}\sin^3\theta\,d\theta=2\int_0^{\pi/2}\sin^3\theta\,d\theta=2.\frac{2}{3.1}=\frac{4}{3},$$

and $\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta=0.$

Substituting the values of all these integrals in (3), we get the Nr. of $\bar{y}$

$$=a^4\,[\pi^2-4-(-\tfrac{1}{4}\pi^2)+2.\tfrac{1}{4}\pi^2-2.(-\tfrac{4}{9})+\tfrac{4}{3}-0]$$

$$=a^4\,[\pi^2-4+\tfrac{1}{4}\pi^2+\tfrac{1}{2}\pi^2+\tfrac{8}{9}+\tfrac{4}{3}]$$

$$=\tfrac{1}{36}a^4\,(63\pi^2-64). \quad \ldots(5)$$

Again the Dr. of $\bar{y}$

$$=\int_0^{\pi}a^2\,(\theta+\sin\theta)^2.a\sin\theta\,d\theta=a^3\int_0^{\pi}(\theta^2\sin\theta-2\theta\sin^2\theta+\sin^3\theta)\,d\theta$$

$$=a^3\left[\int_0^{\pi}\theta^2\sin\theta\,d\theta+2\int_0^{\pi}\theta\sin^2\theta\,d\theta+\int_0^{\pi}\sin^3\theta\,d\theta\right]$$

$$=a^3\,[\pi^2-4+2.\tfrac{1}{4}\pi^2+\tfrac{4}{3}],$$

substituting the values of the integrals as found above

$$=a^3\,(\pi^2+\tfrac{1}{2}\pi^2-4+\tfrac{4}{3})=\tfrac{1}{6}a^3\,(9\pi^2-16). \quad \ldots(6)$$

Substituting the values from (5) and (6) in (2), we get

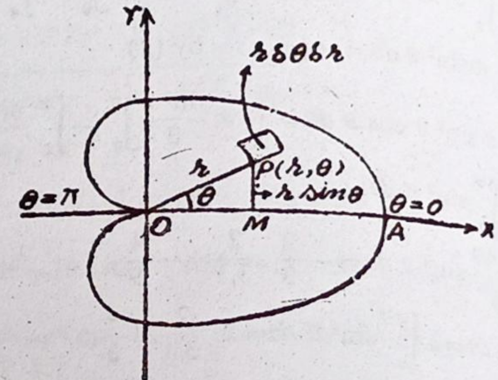
$$\bar{y}=\frac{\tfrac{1}{36}a^4\,(63\pi^2-64)}{\tfrac{1}{6}a^3\,(9\pi^2-16)}=\frac{a\,(63\pi^2-64)}{6\,(9\pi^2-16)}.$$

Hence for the required C.G. $(\bar{x}, \bar{y})$,

$$\bar{x}=0 \text{ and } \bar{y}=\frac{a\,(63\pi^2-64)}{6\,(9\pi^2-16)}.$$

***Ex. 40.** *Find the centroid of the volume formed by the revolution of the cardioid $r=a\,(1+\cos\theta)$ about the x-axis.*
(Rohilkhand 1988; Kanpur 88; Jiwaji 89)

Sol. Obviously the upper half of the cardioid generates the same volume while revolving about the x-axis as is the volume generated by the revolution of the whole of the cardioid about the x-axis.



Consider an elementary area $r\delta\theta\,\delta r$ at the point $P(r, \theta)$ lying within the upper half area of the cardioid. When the cardioid is revolved about the x-axis, this elementary area will generate a ring of radius $PM=r\sin\theta$ and of thickness $r\delta\theta\delta r$. Volume of this elementary ring$=(2\pi r\sin\theta)\,r\delta\theta\,\delta r$ and its C.G. can be taken as the point M on the x-axis whose cartesian coordinates are $(r\cos\theta, 0)$. Note that the thickness $r\delta\theta\delta r$ of the ring is very small.

To cover the whole area of the upper half of the cardioid the limits of r are 0 to $a\,(1+\cos\theta)$ and the limits of θ are 0 to π.

If $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry about the axis of revolution, $\bar{y}=0$.

Also

$$\bar{x}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a\,(1+\cos\theta)} r\cos\theta.(2\pi r\sin\theta)\,r\,d\theta\,dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a\,(1+\cos\theta)}(2\pi r\sin\theta)\,r\,d\theta\,dr}$$

$$=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a\,(1+\cos\theta)} r^3\sin\theta\cos\theta\,d\theta\,dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a\,(1+\cos\theta)} r^2\sin\theta\,d\theta\,dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a\,(1+\cos\theta)}.\sin\theta\cos\theta\,d\theta}{\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}\sin\theta\,d\theta}$$

$$=\frac{3a}{4}\frac{\int_0^{\pi}(1+\cos\theta)^4\sin\theta\cos\theta\,d\theta}{\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta}.$$

Now put $1+\cos\theta=t$ so that $-\sin\theta\,d\theta=dt$.
Also when $\theta=0$, $t=2$ and when $\theta=\pi$, $t=0$.

$$\therefore\quad \bar{x}=\frac{3a}{4}\frac{\int_2^0 t^4\,(t-1)\,(-dt)}{\int_2^0 t^3\,(-dt)}=\frac{3a}{4}\frac{\int_0^2 (t^5-t^4)\,dt}{\int_0^2 t^3\,dt}$$

$$=\frac{3a}{4}\frac{\left[\frac{t^6}{6}-\frac{t^5}{5}\right]_0^2}{\left[\frac{t^4}{4}\right]_0^2}=\frac{3a}{4}\frac{\left[\frac{2^6}{6}-\frac{2^5}{5}\right]}{\frac{2^4}{4}}=3a.\frac{2^5}{2^4}\left[\frac{1}{3}-\frac{1}{5}\right]$$

$$=3a\times 2\left(\frac{1}{3}-\frac{1}{5}\right)=6a.\frac{2}{15}=\frac{4a}{5}.$$

$\therefore$ the required C.G. is the point $(\frac{4}{5}a, 0)$.

Ex. 41. *Find the centre of gravity of the volume formed by revolving the area bounded by the parabolas $y^2=4ax$ and $x^2=4by$ about the axis of x.* **(Kanpur 1984)**

Sol. Draw figure similar to that on page 44 of this chapter.

The given parabolas are

$$y^2=4ax, \quad \text{...(1)}$$

and

$$x^2=4by. \quad \text{...(2)}$$

Solving (1) and (2), we get

$(x^2/4b)^2=4ax$, which gives $x=0$ or $4a^{1/3}\,b^{2/3}$.

From (2), $x=0$ gives $y=0$ and $x=4a^{1/3}\,b^{2/3}$ gives $y=4a^{2/3}\,b^{1/3}$.

$\therefore$ the points of intersection of the given parabolas are $(0, 0)$ and $(4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})$.

Take a straight line PP' parallel to the y-axis where $P\,(x, y_1)$ is a point on the arc OA of the parabola $x^2=4by$ and $P'\,(x, y_2)$ is a point on the arc OA of the parabola $y^2=4ax$. Then we have $x^2=4by_1$ and $y_2{}^2=4ax$.

Now consider an elementary strip $PP'Q'Q$ parallel to the y-axis and of width δx. The volume of the ring formed by revolving this strip about the x-axis$=\pi y_2^2\delta x-\pi y_1^2\,\delta x=\pi\,(y_2^2-y_1^2)\,\delta x$. Also the C.G. of this ring can be taken as the point M $(x, 0)$ on the axis of x.

If $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the solid formed by revolving the area bounded by the parabolas $y^2=4ax$ and $x^2=4by$ about the axis of x, then $\bar{y}=0$ (by symmetry).

Also $\bar{x}=\dfrac{\int_0^{4a^{1/3}b^{2/3}} x\pi\,(y_2^2-y_1^2)\,dx}{\int_0^{4a^{1/3}b^{2/3}} \pi\,(y_2^2-y_1^2)\,dx}$, here $y_2^2=4ax$ and $x^2=4by_1$

$$=\frac{\int_0^{4a^{1/3}b^{2/3}} x\left(4ax-\frac{x^4}{16b^2}\right)dx}{\int_0^{4a^{1/3}b^{2/3}}\left(4ax-\frac{x^4}{16b^2}\right)dx}=\frac{\int_0^{4a^{1/3}b^{2/3}}\left[4ax^2-\frac{x^5}{16b^2}\right]dx}{\int_0^{4a^{1/3}b^{2/3}}\left[4ax-\frac{x^4}{16b^2}\right]dx}$$

$$=\frac{\left[4a.\frac{x^3}{3}-\frac{1}{16b^2}.\frac{x^6}{6}\right]_0^{4a^{1/3}b^{2/3}}}{\left[\frac{x^2}{2}-\frac{1}{16b^2}.\frac{x^5}{5}\right]_0^{4a^{1/3}b^{2/3}}}=\frac{\left[x^3\left(\frac{4a}{3}-\frac{x^3}{96b^2}\right)\right]_0^{4a^{1/3}b^{2/3}}}{\left[x^2\left(2a-\frac{x^3}{80b^2}\right)\right]_0^{4a^{1/3}b^{2/3}}}$$

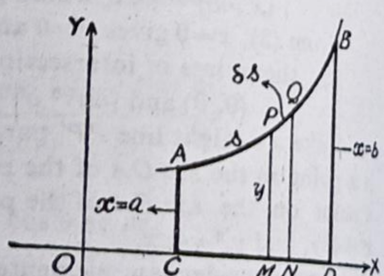
$$=\frac{4a^{1/3}b^{2/3}\left[\frac{4a}{3}-\frac{64ab^2}{96b^2}\right]}{\left[2a-\frac{64ab^2}{80b^2}\right]}=\frac{4a^{1/3}b^{2/3}\left[\frac{4}{3}-\frac{2}{3}\right]}{\left[2-\frac{4}{5}\right]}=\frac{\frac{2}{3}.4a^{1/3}b^{2/3}}{\frac{6}{5}}$$

$=\frac{20}{9}a^{1/3}b^{2/3}$.

Hence the required C.G. is the point $(\frac{20}{9}a^{1/3}b^{2/3}, 0)$.

§ 6. Centre of gravity of surface of revolution.

To find the centre of gravity of the surface formed by revolving the curve $y=f(x)$ about the x-axis and cut off between the planes $x=a$ and $x=b$.



Let the arc AB of the curve $y=f(x)$ lying between the lines $x=a$ and $x=b$ be revolved about the x-axis. Take an element PQ $(=\delta s)$ of the arc AB. When the arc AB revolves about the x-axis, the elementary arc PQ $(=\delta s)$ generates an elementary surface area $2\pi y\ \delta s$ of mass $2\pi y\ \delta s\ \rho$, where ρ is the density per unit area of the surface. The C.G. of this elementary surface area may be supposed to be at the point M $(x, 0)$ because δs is very small.

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{y}=0$, (by symmetry about the axis of revolution *i.e.*, the x-axis) and

$$\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int x.\rho\ 2\pi y\ ds}{\int \rho\ 2\pi y\ ds}, \text{ between the suitable limits}$$

$$=\frac{\int xy\ ds}{\int y\ ds}, \text{ if the density } \rho \text{ is uniform.}$$

To perform integration, we put $ds=\sqrt{\{1+(dy/dx)^2\}}\ dx$, for cartesian equation of the curve and we adjust the limits of x suitably.

If the equation of the curve is in the polar form $r=f(\theta)$, then we put

$ds=\sqrt{\{r^2+(dr/d\theta)^2\}}\ d\theta$ and we adjust the limits of θ.

If the equation of the curve is in the parametric form $x=f(t)$, $y=\phi\ (t)$ then we put

$$ds=\sqrt{\{(dx/dt)^2+(dy/dt)^2\}}\ dt$$

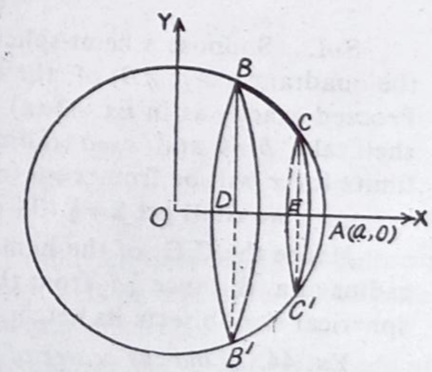
and we adjust the corresponding limits of t.

If the surface is formed by revolving the curve about the y-axis, the corresponding formulae will be

$$\bar{x}=0,\ \bar{y}=\frac{\int y.\rho\ 2\pi x\ ds}{\int \rho\ 2\pi x\ ds}=\frac{\int xy\ ds}{\int x\ ds}, \text{ if the density } \rho \text{ is uniform.}$$

Ex. 42 (a). *Find the C.G. of a zone of a sphere.*

(Meerut 1991)



Sol. Suppose a zone of a sphere is generated by revolving an arc (say, BC) of the circle

$$x^2+y^2=a^2, \qquad \ldots(1)$$

about the x-axis.

Then the axis of the zone (*i.e.*, the height of the zone) will be along the x-axis.

Differentiating (1), we get

$$dy/dx=-(x/y).$$

$$\therefore \quad \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left(1+\frac{x^2}{y^2}\right)}=\sqrt{\left(\frac{x^2+y^2}{y^2}\right)}=\frac{a}{y} \quad ...(2)$$

Also let BD be the line $x=b$ and CE be the line $x=c$.

If $(\bar{x}, \bar{y})$ be the required C.G. of the zone, then $\bar{y}=0$, (by symmetry about the axis of revolution *i.e.*, the x-axis).

$$\text{Also } \bar{x}=\frac{\int x\, 2\pi\, y\, ds}{\int 2\pi\, y\, ds}=\frac{\int_{x=b}^{c} xy.\frac{ds}{dx}dx}{\int_{x=b}^{c} y\,\frac{ds}{dx}\,dx}=\frac{\int_b^c x.y.\frac{a}{y}\,dx}{\int_b^c y.\frac{a}{y}\,dx}, \text{ from (2)}$$

$$=\frac{\int_b^c x\,dx}{\int_b^c dx}=\frac{\left[\frac{x^2}{2}\right]_b^c}{\left[x\right]_b^c}=\frac{\frac{1}{2}(c^2-b^2)}{(c-b)}=\frac{1}{2}(c+b).$$

Thus the C.G. of the zone is the middle point of its height DE. Hence the required C.G. bisects the height of the zone of the sphere.

Ex. 42 (b). *Show that the C.G. of the surface of a spherical segment bisects its height.* **(Gorakhpur 1977)**

Sol. Proceed exactly as in Ex. 42 (a). Here $c=a=$the radius of the generating circle. Start like this :

Suppose a spherical segment is generated by revolving an arc (say, AB) of the circle $x^2+y^2=a^2$ about the x-axis.

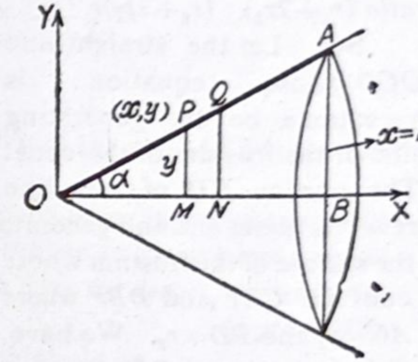
Ex. 43. *Find the C.G. of a thin uniform hemispherical shell.* **(Meerut 1983; Jiwaji 81; Agra 75)**

Sol. Suppose a hemi-spherical shell is formed by revolving the quadrant (say, AB) of the circle $x^2+y^2=a^2$ about the x-axis. Proceed exactly as in Ex. 42 (a). In the case of a hemispherical shell take $b=0$ and $c=a$ (radius). Therefore in this case the limits for x will be from $x=0$ to $x=a$.

$\therefore$ we shall get $\bar{x}=\frac{1}{2}(0+a)=\frac{1}{2}a$ and $\bar{y}=0$.

Hence the C.G. of the hemi-spherical shell is on the central radius at a distance $\frac{1}{2}a$ from the centre *i.e.*, the C.G. of a hemi-spherical shell bisects its height.

Ex. 44. *Find the centre of gravity of a thin right conical shell of uniform thickness and density.*



Sol. Suppose a thin conical shell is generated by the revolution of the line OA about the line OB taken along the x-axis. Let O be the origin and let the line OA be inclined at an angle α to the x-axis. Then the equation of the line OA is

$$y=mx, \quad ...(1)$$

where $\quad m=\tan\alpha.$

From (1), $(dy/dx)=m$.

$$\therefore \quad \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{(1+m^2)}. \quad ...(2)$$

Take an elementary arc PQ $(=\delta s)$ of the line OA. When the line OA revolves about the x-axis, the arc PQ generates an elementary surface of area $2\pi y\,\delta s$ whose C.G. is at the point $M(x, 0)$. If h be the height of the cone (*i.e.*, $OB=h$), then for the conical shell x varies from $x=0$ to $x=h$.

If $(\bar{x}, \bar{y})$ be the required C.G , then $\bar{y}=0$, (by symmetry about the axis of revolution *i.e.*, the x-axis).

$$\text{Also } \bar{x}=\frac{\int x.2\pi y\, ds}{\int 2\pi y\, ds}=\frac{\int_{x=0}^{h} xy\,\frac{ds}{dx}\,dx}{\int_{x=0}^{h} y\,\frac{ds}{dx}\,dx}$$

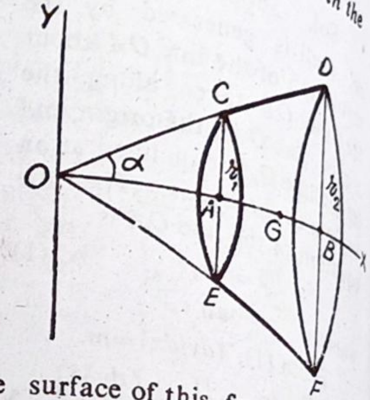
$$=\frac{\int_0^h x.mx.\sqrt{(1+m^2)}\,dx}{\int_0^h mx.\sqrt{(1+m^2)}\,dx}, \text{ from (1) and (2)}$$

$$=\frac{\int_0^h x^2\,dx}{\int_0^h x\,dx}=\frac{\left[\frac{x^3}{3}\right]_0^h}{\left[\frac{x^2}{2}\right]_0^h}=\frac{\frac{1}{3}h^3}{\frac{1}{2}h^2}=\frac{2h}{3}.$$

Hence the C.G. of the thin right conical shell of uniform thickness and density divides the axis of the cone in the ratio $2:1$, the major portion lying towards the vertex.

Ex. 44 (b). *If A and B be the centres and r_1 and r_2 the radii of plane ends of the frustum of hollow right circular cone of uniform* *density, show that its centre of gravity divides the axis AB in the ratio* $(r_1+2r_2):(r_2+2r_1)$.

Sol. Let the straight line OCD whose equation is $y=x\tan\alpha$ be the generating line of the frustum of the cone. The portion CD of this line revolves about OX and generates the surface of the frustum whose ends are CAE and DBF where $AC=r_1$ and $BD=r_2$. We have, $OA=r_1\cot\alpha$ and $OB=r_2\cot\alpha$.



If $(\bar{x},\bar{y})$ be the coordinates of the centre of gravity G of the surface of this frustum, then $\bar{y}=0$, by symmetry about OX.

$$\text{Also } \bar{x}=\frac{\int_{r_1\cot\alpha}^{r_2\cot\alpha} x.\left(2\pi y\dfrac{ds}{dx}\right)dx}{\int_{r_1\cot\alpha}^{r_2\cot\alpha}\left(2\pi y\dfrac{ds}{dx}\right)dx}.$$

Now from $y=x\tan\alpha$, we have $dy/dx=\tan\alpha$.

$$\therefore\quad ds/dx=\sqrt{\{1+(dy/dx)^2\}}=\sqrt{(1+\tan^2\alpha)}=\sec\alpha.$$

$$\therefore\quad \bar{x}=OG=\frac{\int_{r_1\cot\alpha}^{r_2\cot\alpha} x.x\tan\alpha\sec\alpha\,dx}{\int_{r_1\cot\alpha}^{r_2\cot\alpha} x\tan\alpha.\sec\alpha\,dx}$$

$$=\frac{\left[\dfrac{x^3}{3}\right]_{r_1\cot\alpha}^{r_2\cot\alpha}}{\left[\dfrac{x^2}{2}\right]_{r_1\cot\alpha}^{r_2\cot\alpha}}=\frac{2}{3}\cot\alpha\frac{r_2^3-r_1^3}{r_2^2-r_1^2}$$

$$=\frac{2}{3}\cot\alpha\frac{r_1^2+r_1r_2+r_2^2}{r_1+r_2}.$$

$$\therefore\quad \frac{AG}{GB}=\frac{OG-OA}{OB-OG}$$

$$=\frac{\dfrac{2}{3}\cot\alpha\dfrac{r_1^2+r_1r_2+r_2^2}{r_1+r_2}-r_1\cot\alpha}{r_2\cot\alpha-\dfrac{2}{3}\cot\alpha\dfrac{r_1^2+r_1r_2+r_2^2}{r_1+r_2}}$$

$$=\frac{2(r_1^2+r_1r_2+r_2^2)-3r_1(r_1+r_2)}{3r_2(r_1+r_2)-2(r_1^2+r_1r_2+r_2^2)}$$

$$=\frac{2r_2^2-r_1r_2-r_1^2}{r_2^2+r_1r_2-2r_1^2}=\frac{(r_2-r_1)(r_1+2r_2)}{(r_2-r_1)(r_2+2r_1)}$$

$$=\frac{r_1+2r_2}{r_2+2r_1}.$$

Hence the centre of gravity G divides the axis AB in the ratio $(r_1+2r_2):(r_2+2r_1)$.

Ex. 45. *A parabola revolves round its axis; find the centroid of the portion of the surface between the vertex and a plane perpendicular to the axis at a distance h from the vertex, 4a being the latus rectum.* **(Agra 1986)**

Sol. Let the equation of the parabola be

$$y^2=4ax. \qquad \text{...(1)}$$

Differentiating (1), we get $dy/dx=2a/y$.

$$\therefore\ \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\frac{4a^2}{y^2}\right\}}=\sqrt{\left\{1+\frac{4a^2}{4ax}\right\}}=\sqrt{\left(\frac{x+a}{x}\right)}. \qquad \text{...(2)}$$

Take an elementary arc δs at any point $P(x,y)$ on the arc of the portion of the parabola to be revolved. The area of the elementary surface generated by the revolution of this arc δs about the x-axis$=2\pi y\,\delta s$ and the C.G. of this elementary surface can be taken as the point $(x,0)$ on the axis of x. Note that this elementary surface is symmetrical about the axis of revolution *i.e.*, x-axis.

$\therefore$ if $(\bar{x},\bar{y})$ be the required C.G., then $\bar{y}=0$, (by symmetry about the axis of revolution *i.e.*, x-axis).

$$\text{Also } \bar{x}=\frac{\int x\,2\pi y\,ds}{\int 2\pi y\,dx}=\frac{\int_0^h xy\dfrac{ds}{dx}dx}{\int_0^h y\dfrac{ds}{dx}dx}$$

$$=\frac{\int_0^h x\sqrt{(4ax)}.\sqrt{\left(\dfrac{x+a}{x}\right)}dx}{\int_0^h\sqrt{(4ax)}.\sqrt{\left(\dfrac{x+a}{x}\right)}dx},\quad\text{from (1) and (2)}$$

$$=\frac{\int_0^h x(a+x)^{1/2}dx}{\int_0^h (a+x)^{1/2}dx}=\frac{\left[\dfrac{x(a+x)^{3/2}}{3/2}\right]_0^h-\int_0^h\dfrac{2}{3}(a+x)^{3/2}dx}{\left[\dfrac{2}{3}(a+x)^{3/2}\right]_0^h}$$

$$=\frac{\dfrac{2}{3}h(a+h)^{3/2}-\left[\dfrac{2}{3}.\dfrac{2}{5}(a+x)^{5/2}\right]_0^h}{\dfrac{2}{3}[(a+h)^{3/2}-h^{3/2}]}=\frac{h(a+h)^{3/2}-\dfrac{2}{5}\{(a+h)^{5/2}-a^{5/2}\}}{[(a+h)^{3/2}-a^{3/2}]}.$$

Ex. 46. *Find the centre of gravity of the surface formed by the revolution of the cycloid* $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ *about the axis of y.* **(Meerut 1990)**

Sol. The given parametric equations of the cycloid are $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$. ...(1)

[For figure refer Ex. 20 on page 32]

Differentiating (1) w.r.t. θ, we have

$$(dx/d\theta)=a(1+\cos\theta) \text{ and } (dy/d\theta)=a\sin\theta.$$

$$\therefore \frac{ds}{d\theta}=\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}=\sqrt{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}}$$

$$=a\sqrt{\{1+2\cos\theta+\cos^2\theta+\sin^2\theta\}}=a\sqrt{[2+2\cos\theta]}$$

$$=a\sqrt{[2.2\cos^2\tfrac{1}{2}\theta]}=2a\cos\tfrac{1}{2}\theta. \quad ...(2)$$

The cycloid is symmetrical about the y-axis and for the portion of the cycloid lying in the positive quadrant θ varies from 0 to π. Obviously the surface formed by revolving the whole cycloid about the y-axis is the same as that formed by revolving the portion of the cycloid lying in the positive quadrant.

Take an elementary arc δs at any point (x, y) of the portion of the cycloid lying in the positive quadrant. The area of the elementary surface generated by the revolution of the arc δs about the y-axis$=2\pi x\,\delta s$ and the C.G. of this elementary surface is the point $(0, y)$ on the axis of rotation.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{x}=0$, by symmetry about the axis of rotation *i.e.* the y-axis.

Also

$$\bar{y}=\frac{\int y.2\pi x\,ds}{\int 2\pi x\,ds}=\frac{\int_0^\pi yx\frac{ds}{d\theta}d\theta}{\int_0^\pi x\frac{ds}{d\theta}d\theta}$$

$$=\frac{\int_0^\pi a(1-\cos\theta).a(\theta+\sin\theta).2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi a(\theta+\sin\theta).2a\cos\frac{1}{2}\theta\,d\theta}, \text{ from (1) and (2)}$$

$$=\frac{a\int_0^\pi(1-\cos\theta)(\theta+\sin\theta)\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi(\theta+\sin\theta)\cos\frac{1}{2}\theta\,d\theta}. \quad ...(3)$$

Now the Nr. of $\bar{y}=a\int_0^\pi 2\sin^2\frac{1}{2}\theta.(\theta+2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta).\cos\frac{1}{2}\theta\,d\theta$ (No

$$=2a\int_0^{\pi/2}\sin^2\phi(2\phi+2\sin\phi\cos\phi)\cos\phi.2d\phi, \text{ putting } \frac{\theta}{2}=\phi$$

$$=8a\left[\int_0^{\pi/2}\phi\sin^2\phi\cos\phi\,d\phi+\int_0^{\pi/2}\sin^3\phi\cos^2\phi\,d\phi\right]$$

$$=8a\left[\left(\phi.\frac{\sin^3\phi}{3}\right)_0^{\pi/2}-\int_0^{\pi/2}1.\frac{\sin^3\phi}{3}d\phi+\frac{2.1}{5.3.1}\right]$$

$$=8a\left\{\frac{\pi}{2}.\frac{1}{3}-\frac{1}{3}.\frac{2}{3.1}+\frac{2}{15}\right\}=8a\left\{\frac{\pi}{6}-\frac{2}{9}+\frac{2}{15}\right\}=\frac{8a}{90}(15\pi-20+12)$$

$$=\frac{4a}{45}(15\pi-8),$$

and the Dr. of $\bar{y}=\int_0^\pi(\theta+\sin\theta)\cos\frac{1}{2}\theta\,d\theta$

$$=\int_0^\pi(\theta\cos\tfrac{1}{2}\theta+2\sin\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta)\,d\theta$$

$$=\int_0^{\pi/2}(2\phi\cos\phi+2\sin\phi\cos^2\phi).2d\phi, \text{ putting } \frac{\theta}{2}=\phi$$

$$=4\left[\int_0^{\pi/2}\phi\cos\phi\,d\phi+\int_0^{\pi/2}\sin\phi\cos^2\phi\,d\phi\right]$$

$$=4\left\{\left[\phi.\sin\phi\right]_0^{\pi/2}-\int_0^{\pi/2}1.\sin\phi\,d\phi+\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}\right\}$$

$$=4\left\{\frac{\pi}{2}+\left[\cos\phi\right]_0^{\pi/2}+\frac{1}{3}\right\}=4\left\{\frac{\pi}{2}-1+\frac{1}{3}\right\}=\frac{2}{3}(3\pi-4).$$

Therefore form (3), we get

$$\bar{y}=\frac{\frac{4}{45}a(15\pi-8)}{\frac{2}{3}(3\pi-4)}=\frac{2a(15\pi-8)}{15(3\pi-4)}.$$

Hence the required C.G. is given by

$$\bar{x}=0, \bar{y}=\{2a(15\pi-8)/15(3\pi-4)\}.$$

***Ex. 47.** *Find the centre of gravity of the surface formed by the revolution of the cardioid* $r=a(1+\cos\theta)$ *about its axis.*

(Rohilkhand 1990; Kanpur 81)

Sol. For figure of the cardioid refer page 36 of this chapter.

The given cardioid is $r=a(1+\cos\theta)$. ...(1)

Differentiating (1) w.r.t. θ. we get $dr/d\theta=-a\sin\theta$.

$$\therefore \frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}}$$

$$=a\sqrt{(2+2\cos\theta)}=a\sqrt{\{2.2\cos^2\tfrac{1}{2}\theta\}}=2a\cos\tfrac{1}{2}\theta \quad ...(2)$$

The given cardioid is symmetrical about its axis (*i.e.* the x-axis) and for the portion of cardioid above the x-axis θ varies from 0 to π. Obviously to form the surface under consideration it is sufficient to revolve the portion of the cardioid lying above the x-axis. Take an elementary arc δs of the cardioid at any point (x, y) on the arc of the cardioid lying above the x-axis. The area of the elementary surface generated by the revolution of this arc about the x-axis$=2\pi y\delta s$ and its G.G. is the point $(x, 0)$ on the axis of rotation *i.e.*, the axis of x.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{y}=0$, by symmetry about the axis of rotation *i.e.*, the x-axis.

Also $$\bar{x}=\frac{\int x.2\pi y\, ds}{\int 2\pi y\, ds}=\frac{\int_0^\pi r\cos\theta . r\sin\theta . 2a\cos\frac{1}{2}\theta\, d\theta}{\int_0^\pi r\sin\theta . 2a\cos\frac{1}{2}\theta\, d\theta}$$

[$\because$ $x=r\cos\theta$, $y=r\sin\theta$ and $ds=2a\cos\frac{1}{2}\theta\, d\theta$]

$$=\frac{\int_0^\pi r^2\cos\theta\sin\theta\cos\frac{1}{2}\theta\, d\theta}{\int_0^\pi r\sin\theta\cos\frac{1}{2}\theta\, d\theta}$$

$$=\frac{\int_0^\pi a^2(1+\cos\theta)^2\cos\theta\sin\theta\cos\frac{1}{2}\theta\, d\theta}{\int_0^\pi a(1+\cos\theta)\sin\theta\cos\frac{1}{2}\theta\, d\theta}, \text{ from (1)}$$

$$=\frac{a\int_0^\pi 4\cos^4\frac{1}{2}\theta.(2\cos^2\frac{1}{2}\theta-1).2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta.\cos\frac{1}{2}\theta\, d\theta}{\int_0^\pi 2\cos^2\frac{1}{2}\theta.2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta.\cos\frac{1}{2}\theta\, d\theta}$$

$$=\frac{2a\int_0^\pi \sin\frac{1}{2}\theta\cos^6\frac{1}{2}\theta\,(2\cos^2\frac{1}{2}\theta-1)\, d\theta}{\int_0^\pi \sin\frac{1}{2}\theta\cos^4\frac{1}{2}\theta\, d\theta}$$

$$=\frac{2a\int_0^{\pi/2}\sin\phi\cos^2\phi\,(2\cos^2\phi-1).2d\phi}{\int_0^{\pi/2}\sin\phi\cos^4\phi.2d\phi}, \text{ putting } \frac{\theta}{2}=\phi$$

$$=\frac{2a\left[2\int_0^{\pi/2}\sin\phi\cos^8\phi\, d\phi-\int_0^{\pi/2}\sin\phi\cos^6\phi\, d\phi\right]}{\int_0^{\pi/2}\sin\phi\cos^4\phi\, d\phi}$$

$$=2a\left\{2\left[-\frac{\cos^9\phi}{9}\right]_0^{\pi/2}+\left[\frac{\cos^7\phi}{7}\right]_0^{\pi/2}\right\}\div\left[-\frac{\cos^5\phi}{5}\right]_0^{\pi/2}$$

$=2a\,\{\frac{2}{9}-\frac{1}{7}\}/(\frac{1}{5})=2a.(5/63).5=\frac{50}{63}a.$

$\therefore$ the required C.G. is given by $\bar{x}=\frac{50}{63}a$, $\bar{y}=0$.

Ex. 48. *Find the centre of gravity of the surface formed by the revolution of one loop of the lemniscate of Bernoulli $r^2=a^2\cos 2\theta$ about the initial line.* **(Agra 1987)**

Sol. The given equation of the curve is

$$r^2=a^2\cos 2\theta. \qquad \ldots(1)$$

Differentiating (1) w.r.t. θ, we get

$$2r\,(dr/d\theta)=-2a^2\sin 2\theta \text{ or } (dr/d\theta)=-(a^2\sin 2\theta)/r.$$

$$\therefore \frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{\left\{r^2+\frac{a^4\sin^2 2\theta}{r^2}\right\}}$$

$$=\frac{\sqrt{\{r^4+a^4\sin^2 2\theta\}}}{r}=\frac{\sqrt{(a^4\cos^2 2\theta+a^4\sin^2 2\theta)}}{r}, \text{ form (1)}$$

$$=a^2\sqrt{(\cos^2 2\theta+\sin^2 2\theta)}/r=a^2/r. \qquad \ldots(2)$$

The given curve is symmetrical about the initial line (*i.e.* the x-axis).

Putting $r=0$, we get

$$\cos 2\theta=0 \text{ i.e., } 2\theta=\pm\tfrac{1}{2}\pi \text{ i.e., } \theta=\pm\tfrac{1}{4}\pi.$$

$\therefore$ for one loop of the curve θ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$.

For the figure of the lemniscate refer page 37 of this chapter.

Obviously to form the surface under consideration it is sufficient to revolve the portion of the loop lying above the x-axis and for this portion θ varies from 0 to $\pi/4$.

Take an elementary arc δs at any point (x, y) of the arc of the portion of the loop lying above the x-axis.

The area of the elementary surface generated by the revolution of this arc δs about the initial line (*i.e.*, the x-axis)$=2\pi y\,\delta s$ and its C.G. is at the point $(x, 0)$ lying on the axis of rotation.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{y}=0$, by symmetry about the axis of rotation *i.e.*, the x-axis.

Also

$$\bar{x}=\frac{\int x.2\pi y\, ds}{\int 2\pi y\, ds}=\frac{\int_{\theta=0}^{\pi/4} x.y\,\frac{ds}{d\theta}\, d\theta}{\int_{\theta=0}^{\pi/4} y\,\frac{ds}{d\theta}\, d\theta}$$

$$=\frac{\int_0^{\pi/4} r\cos\theta . r\sin\theta . \frac{a^2}{r} . d\theta}{\int_0^{\pi/4} r\sin\theta . \frac{a^2}{r} d\theta}=\frac{\int_0^{\pi/4} r\sin\theta\cos\theta\, d\theta}{\int_0^{\pi/4}\sin\theta\, d\theta} \quad ...(3)$$

[$\because$ $x=r\cos\theta$, $y=r\sin\theta$ and $ds/d\theta=a^2/r$]

Now the Nr. of $\bar{x}$

$=\int_0^{\pi/4} a(\cos 2\theta)^{1/2}.\sin\theta\cos\theta\, d\theta$, from (1)

$=\frac{a}{2}\int_0^{\pi/4}(\cos 2\theta)^{1/2}\sin 2\theta\, d\theta=-\frac{a}{4}\int_1^0 t^{1/2}\, dt$, putting $\cos 2\theta=t$

$=-\frac{a}{4}\left[\frac{t^{3/2}}{3/2}\right]_1^0=-\frac{a}{4}\left[-\frac{2}{3}\right]=\frac{a}{6}$,

and the Dr. of $\bar{x}$

$=\int_0^{\pi/4}\sin\theta\, d\theta=\left[-\cos\theta\right]_0^{\pi/4}=-\left[\cos\frac{\pi}{4}-\cos 0\right]$

$=-[(1/\sqrt{2})-1]=-(1-\sqrt{2})/\sqrt{2}=(\sqrt{2}-1)/\sqrt{2}.$

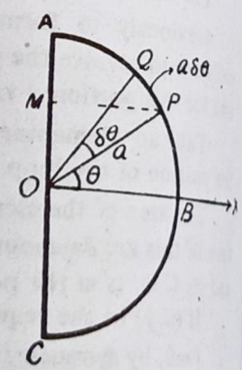
Therefore from (3), we get

$$x=\frac{a/6}{(\sqrt{2}-1)/\sqrt{2}}=\frac{a\sqrt{2}}{6(\sqrt{2}-1)}=\frac{a(\sqrt{2}+1)}{3\sqrt{2}}.$$

$\therefore$ the required C.G. is given by $\bar{x}=a(2+1)/3\sqrt{2}$, $\bar{y}=0$.

Ex. 49. *Show that the centre of gravity of a lune of a sphere of angle 2α is at a distance $\frac{1}{4}\pi(a\sin\alpha)/\alpha$ from its axis.*

Sol. The lune of a sphere of angle 2α is the surface generated by the revolution of the semi-circular arc ABC about the diameter AC through an angle 2α. The diameter AC is the axis if this lune.



Consider an elementary arc PQ $(=a\,\delta\theta)$ of the arc ABC. If the arc PQ is revolved through an angle 2α about the diameter AC, it would generate a curved surface in the form of an arc of a circle of radius PM $(=a\cos\theta)$ and having its centre at M. The mass of this surface is $2\alpha.(a\cos\theta).a\,\delta\theta.\rho$ and the distance of its C.G. from AC

$=$ the radius $PM\times\frac{\sin\alpha}{\alpha}=a\cos\theta.\frac{\sin\alpha}{\alpha}$. [Note]

$\therefore$ if $\bar{x}$ be the distance of the C.G. of the lune from its axis AC, then

$$\bar{x}=\frac{\int_{-\pi/2}^{\pi/2}\frac{a\cos\theta.\sin\alpha}{\alpha}.\rho\, 2\alpha.(a\cos\theta).a\, d\theta}{\int_{-\pi/2}^{\pi/2}\rho\, 2\alpha.(a\cos\theta).a\, d\theta}$$

$$=\frac{a\sin\alpha}{\alpha}.\frac{\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta}{\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta}=\frac{a\sin\alpha}{\alpha}\cdot\frac{2\int_0^{\pi/2}\cos^2\theta\, d\theta}{2\int_0^{\pi/2}\cos\theta\, d\theta}$$

$$=\frac{a\sin\alpha}{\alpha}\cdot\frac{\frac{1}{2}.\frac{1}{2}\pi}{[\sin\frac{1}{2}\pi-\sin 0]}=\frac{a\sin\alpha}{\alpha}.\frac{\pi}{4}.$$

Hence the C.G. of a lune of a sphere of angle 2α is at a distance $\frac{1}{4}\pi(a\sin\alpha)/\alpha$ from its axis.

Remark. *A lune* of a sphere is a part of its surface bounded by any two planes passing through a fixed diameter. This fixed diameter is called the axis of the lune and the angle between the bounding planes is called **'the angle of the lune'**.

§ 7. **Centre of gravity when the density varies.** If we are required to find the C.G. of a body when its density ρ varies from point to point, we cannot cancel ρ from the integrals occuring in the numerators and the denominators of $\bar{x}$ and $\bar{y}$ because now ρ is not a constant. Further if the body is in the form of a plane lamina and the density is not uniform we cannot take an element of the body in the form of a strip. Obviously we shall not be able to write the mass of the strip because its density varies from point to point. Here we shall divide the lamina into infinitesimal elements of the second order and thus we shall make use of the double integrals. The procedure to be adopted is as given below :

(i) **In the case of cartesian curves.** Take a small element of area $\delta x\,\delta y$ at any point $P(x, y)$ lying in the area whose C.G. is to be found. If ρ be the density at the point P, then the mass of the elementary area $\delta x\,\delta y$ is $\rho\,\delta x\,\delta y$, because the density at every point of this elementary area can be taken the same as that at P. Also the C.G. of this small element $\delta x\,\delta y$ can be taken as the point $P(x, y)$. If $(\bar{x}, \bar{y})$ be the required C.G. of the whole area under consideration, we have

$$\bar{x}=\frac{\iint x.\rho\, dx\, dy}{\iint \rho\, dx\, dy} \text{ and } \bar{y}=\frac{\iint y.\rho\, dx\, dy}{\iint \rho\, dx\, dy},$$

where the limits of integration are to be so chosen that the whole area under consideration is covered.

(ii) In the case of polar curves. Take a small element of area $r\,\delta\theta\,\delta r$ at any point $P\,(r, \theta)$ lying within the area whose C.G. is to be found. If ρ be the density at the point P, then the mass of the elementary area $r\,\delta\theta\,\delta r$ is $\rho.r\,\delta\theta\,\theta r$. Also the C.G. of this small element $r\,\delta\theta\,\delta r$ can be taken as the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$. If $(\bar{x}, \bar{y})$ be the required C.G. of the whole area under consideration, we have

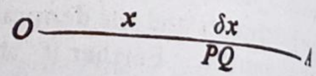
$$\bar{x}=\frac{\iint r\cos\theta.\rho r\,d\theta\,dr}{\iint \rho r\,d\theta\,dr} \text{ and } \bar{y}=\frac{\iint r\sin\theta.\rho r\,d\theta\,dr}{\iint \rho r\,d\theta\,dr},$$

where the limits of integration are to be so taken as to cover the whole area under consideration.

A similar procedure is adopted in the case of solids and surfaces of revolution when the density varies. The whole procedure will be clear from the worked out examples which follow.

Ex. 50. *Find the centre of mass of a rod whose density varies as the square of the distance from one end.*

Sol. Let OA be a rod of length l. Take O as origin. Let $PQ=\delta x$ be an elementary portion of this rod, where $OP=x$. According to the question the density of the rod at the point $P=\lambda.OP^2=\lambda x^2$, where λ is some constant.



$\therefore$ the mass of the element $PQ=\lambda x^2\,\delta x$ and the centre of mass of this element can be taken as the point P whose distance from O is x. If $\bar{x}$ be the distance of the centre of mass of the whole rod OA from O, then

$$\bar{x}=\frac{\int_0^l x.\lambda x^2\,dx}{\int_0^l \lambda x^2\,dx}=\frac{\left[\frac{x^4}{4}\right]_0^l}{\left[\frac{x^3}{3}\right]_0^l}=\tfrac{3}{4}l.$$

Hence the centre of mass of the rod is at a distance $\frac{3}{4}l$ from the given end, where l is the length of the rod.

Ex. 51. *The mass per unit length at any point of a straight beam of length l is $\rho\,(1+x/l)$, where x is the distance of the point from one end of the beam. Find (i) the mass of the beam, (ii) the position of its centre of mass.*

Sol. Refer figure of Ex. 50.

Let OA be a beam of length l. Take O as origin. Let $PQ=\delta x$ be an elementary portion of this beam, where $OP=x$.

According to the question the density of the beam at the point $P=\rho\,(1+x/l)$.

$\therefore$ the mass of the element $PQ=\rho\,(1+x/l)\,\delta x$ and the centre of mass of this element can be taken as the point P whose distance from O is x.

Let $\bar{x}$ be the distance of the centre of mass of the whole beam from O. We have,

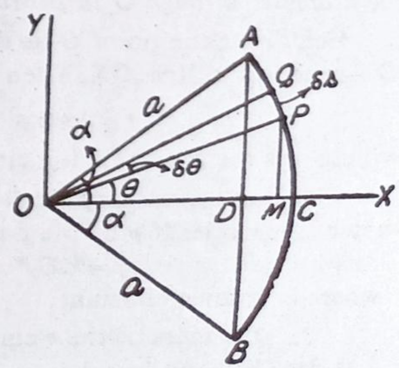
$$\text{the mass of the beam } OA=\int_0^l \rho\left(1+\frac{x}{l}\right)dx =\rho\left[x+\frac{x^2}{2l}\right]_0^l=\rho\left[l+\frac{l}{2}\right] =\tfrac{3}{2}\rho l.$$

$$\text{Also } \bar{x}=\frac{\int_0^l x.\rho\left(1+\frac{x}{l}\right)dx}{\int_0^l \rho\left(1+\frac{x}{l}\right)dx}=\frac{\left[\frac{x^2}{2}+\frac{x^3}{3l}\right]_0^l}{\left[x+\frac{x^2}{2l}\right]_0^l}=\frac{\frac{l^2}{2}+\frac{l^2}{3}}{l+\frac{l}{2}} =\frac{\frac{5}{6}l^2}{\frac{3}{2}l}=\frac{5}{6}\cdot\frac{2}{3}\,l=\frac{5}{9}\,l.$$

Hence the centre of mass of the beam is at a distance $\frac{5}{9}l$ from the given end.

Ex. 52. *If the density at any point of an arc of a uniform circular wire varies as its distance from the central radius, prove that the centre of mass is at a point on the central radius, mid-way between the chord and the arc.*

Sol. Let ACB be an arc of a circle whose centre is O and radius a, OC being the central radius of the arc and D being the middle point of the chord AB. Let $\angle AOC=\angle BOC=\alpha$.



We have

$OD=a\cos\alpha$, $OC=a$.

Take O as origin and OC as the axis of x. Let $PQ=\delta s$ be an elementary portion of the arc, where $\angle COP=\theta$ and $\angle POQ=\theta+\delta\theta$.

We have $\delta s = a\,\delta\theta$. If ρ be the density of the arc at P, then according to the question $\rho = k.PM = ka \sin\theta$. Therefore the mass of the elementary arc $PQ = a\,\delta\theta . ka \sin\theta = ka^2 \sin\theta\,\delta\theta$. The centre of mass of this elementary arc PQ can be taken as the point P whose x-coordinate$=OM=a\cos\theta$.

The arc ACB is symmetrical about OX. The density of the arc is also symmetrical about OX. Therefore the centre of mass of the arc ACB must lie on OX. If $(\bar{x}, \bar{y})$ be the coordinates of the centre of mass of the arc ACB, we have $\bar{y}=0$. Also the x-coordinate of the centre of mass of the whole arc ACB is the same as the x-coordinate of the centre of mass of the upper half AC of this arc.

Therefore

$$\bar{x} = \frac{\int_0^{\alpha} a\cos\theta . ka^2 \sin\theta\, d\theta}{\int_0^{\alpha} ka^2 \sin\theta\, d\theta} = \frac{a\int_0^{\alpha} \sin\theta\cos\theta\, d\theta}{\int_0^{\alpha} \sin\theta\, d\theta}$$

$$= a\frac{\left[\dfrac{\sin^2\theta}{2}\right]_0^{\alpha}}{\left[-\cos\theta\right]_0^{\alpha}} = \frac{a}{2}\,\frac{\sin^2\alpha}{1-\cos\alpha} = \frac{a}{2}\,\frac{1-\cos^2\alpha}{1-\cos\alpha}$$

$$= \frac{a}{2}(1+\cos\alpha) = \tfrac{1}{2}(a + a\cos\alpha) = \tfrac{1}{2}(OC+OD).$$

Hence the centre of mass is at a point on the central radius midway between the points D and C *i.e.*, midway between the chord and the arc.

Ex. 53. *If the density of a circular arc varies as the square of the distance from a point O on the arc, show that the centroid divides the diameter through O in the ratio 3 : 1.*

Sol. Take the point O as the pole and the diameter through O as the initial line OX. Then the equation of the circle is

$$r = 2a\cos\theta,$$

where a is the radius of the circle.

Take an elementary arc $PQ=\delta s$ at any point $P(r, \theta)$ on the arc of the circle. If ρ be the density at the point P, then, as given,

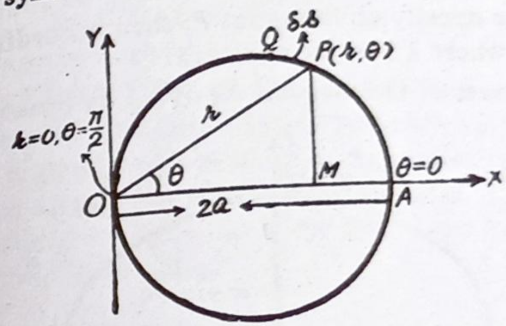
$$\rho = k.OP^2 = kr^2,$$

where k is some constant.

$\therefore$ the mass of the elementary arc $PQ = \rho\,\delta s = kr^2\,\delta s$.

Its C.G. can be taken as the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

The whole circular arc is symmetrical about OX and its density is also symmetrical about OX as can be easily seen by



taking two points on the circular arc on opposite sides of OX and equidistant from OX.

Therefore if $(\bar{x}, \bar{y})$ be the C.G. of the whole circular arc, then $\bar{y}=0$, by symmetry about OX.

Also $$\bar{x} = \frac{\int r\cos\theta . kr^2\, ds}{\int kr^2\, ds} = \frac{\int_{-\pi/2}^{\pi/2} r^3\cos\theta . \dfrac{ds}{d\theta}\, d\theta}{\int_{-\pi/2}^{\pi/2} r^2 . \dfrac{ds}{d\theta}\, d\theta}.$$

Now $r = 2a\cos\theta$, so that $dr/d\theta = -2a\sin\theta$.

$$\therefore\quad \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\{4a^2\cos^2\theta + 4a^2\sin^2\theta\}} = 2a.$$

$$\therefore\quad \bar{x} = \frac{\int_{-\pi/2}^{\pi/2} 8a^3\cos^3\theta . \cos\theta . 2a . d\theta}{\int_{-\pi/2}^{\pi/2} 4a^2\cos^2\theta . 2a\, d\theta} = \frac{2a.2\int_0^{\pi/2}\cos^4\theta\, d\theta}{2.\int_0^{\pi/2}\cos^2\theta\, d\theta}$$

$$= 2a.\frac{\frac{3}{4}.\frac{1}{2}.\frac{1}{2}\pi}{\frac{1}{2}.\frac{1}{2}\pi} = 2a.\tfrac{3}{4} = \tfrac{3}{4}.2a = \tfrac{3}{4}(OA).$$

Thus if G be the C.G. of the whole circular arc, we have $OG = \bar{x} = \frac{3}{4}OA$ and $GA = OA - OG = OA - \frac{3}{4}OA = \frac{1}{4}.(OA)$.

Therefore $OG : GA = \frac{3}{4}OA : \frac{1}{4}OA = 3 : 1$.

Ex. 54. *Find the centre of gravity of a plate in the form of a quadrant of an ellipse, the density at any point of the plate varying as the product of the distance of the point from the major and minor axes.*

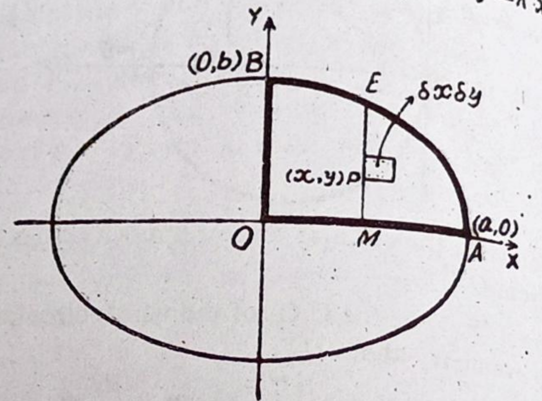
Sol. Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$.

Let AOB be a quadrant of this ellipse with OA and OB as the semi-axes of the ellipse.

Take a small element of area $\delta x\,\delta y$ at any point $P(x, y)$ lying within the area OAB whose C.G. is to be found.

If ρ be the density at the point P, then according to the question $\rho=\lambda\,xy$, where λ is some constant.

$\therefore$ the mass of the element $\delta x\,\delta y=\rho\,\delta x\,\delta y=\lambda\,xy\,\delta x\,\delta y$.



The element $\delta x\,\delta y$ being very small, its C.G. can be taken as the point P whose coordinates are (x, y).

To cover the area of the quadrant OAB, we draw the straight lines x=constant. On any such line, say on the line ME, y goes from OX to the arc of the ellipse. Thus to cover the area OAB the limits of y are 0 to $b\sqrt{\{1-(x^2/a^2)\}}$ and the limits of x are 0 to a.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} x.\lambda xy\,dx\,dy}{\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} \lambda\,xy\,dx\,dy}$$

$$=\frac{\int_0^a x^2\left[\frac{y^2}{2}\right]_0^{b\sqrt{(1-x^2/a^2)}}dx}{\int_0^a x\left[\frac{y^2}{2}\right]_0^{b\sqrt{(1-x^2/a^2)}}dx}=\frac{\int_0^a \frac{x^2}{2}\,b^2\left(1-\frac{x^2}{a^2}\right)dx}{\int_0^a \frac{x}{2}\,b^2\left(1-\frac{x^2}{a^2}\right)dx}$$

$$=\left[\frac{x^3}{3}-\frac{1}{a^2}\cdot\frac{x^5}{5}\right]_0^a\div\left[\frac{x^2}{2}-\frac{1}{a^2}\cdot\frac{x^4}{4}\right]_0^a$$

$$=\left[\frac{1}{3}a^3-\frac{1}{5}a^3\right]\div\left[\frac{1}{2}a^2-\frac{1}{4}a^2\right]=\left(\frac{2}{15}a^3\right)\div\left(\frac{1}{4}a^2\right)=\frac{8}{15}a.$$

$\therefore\ \bar{x}/a=\frac{8}{15}$.

But x/a and y/b are symmetrically placed in the equation of the curve. Therefore, by symmetry, $\bar{y}/b=\frac{8}{15}$.

$\therefore$ the required C.G. $(\bar{x}, \bar{y})$ is given by

$$\bar{x}=\tfrac{8}{15}a,\ \bar{y}=\tfrac{8}{15}b.$$

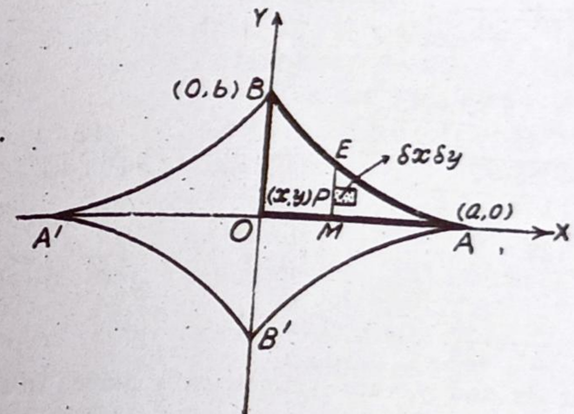
Ex. 55. *Find the co-ordinates of the centre of gravity of a lamina in the shape of a quadrant of the curve*

$$(x/a)^{2/3}+(y/b)^{2/3}=1$$

the density being given by $\rho=kxy$.

(Kanpur 1985, 87, Rohilkhand 91, Meerut 86, 91 P)

Sol. The quadrant OAB of the given curve is shown in the figure. Take a small element of area $\delta x\,\delta y$ at any point $P(x, y)$ lying within the area OAB. If ρ be the density at P, then according to the question, $\rho=kxy$. Therefore the mass of the element $\delta x\,\delta y=\rho\,\delta x\,\delta y=kxy\,\delta x\,\delta y$.



The element $\delta x\,\delta y$ being very small, its C.G. can be taken as the point P whose coordinates are (x, y).

To cover the area OAB the limits of y are

$$0 \text{ to } b\,\{1-(x/a)^{2/3}\}^{3/2}$$

and the limits of x are 0 to a.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G. of the area OAB, then

$$\bar{x}=\frac{\int_{x=0}^{a}\int_{y=0}^{b\,\{1-(x/a)^{2/3}\}^{3/2}} x.k\,xy\,dx\,dy}{\int_{x=0}^{a}\int_{y=0}^{b\,\{1-(x/a)^{2/3}\}^{3/2}} k\,xy\,dx\,dy}$$

$$=\frac{\int_0^a x^2\left[\frac{y^2}{2}\right]_0^{b\,\{1-(x/a)^{2/3}\}^{3/2}}dx}{\int_0^a x.\left[\frac{y^2}{2}\right]_0^{b\,\{1-(x/a)^{2/3}\}^{3/2}}dx}$$

$$=\frac{\int_0^a \frac{1}{2}x^2 b^2\left\{1-\left(\frac{x}{a}\right)^{2/3}\right\}^3 dx}{\int_0^a b^2 \frac{1}{2}x\left\{1-\left(\frac{x}{a}\right)^{2/3}\right\}^3 dx}.$$

Now put $x=a\sin^3\theta$ so that $dx=3a\sin^2\theta\cos\theta\,d\theta$.
Also when $x=0$, $\theta=0$ and when $\bar{x}=a$, $\theta=\pi/2$.

$$\therefore\ \bar{x}=\frac{\int_0^{\pi/2} a^2\sin^6\theta\,(1-\sin^2\theta)^3.3a\sin^2\theta\cos\theta\,d\theta}{\int_0^{\pi/2} a\sin^3\theta\,(1-\sin^2\theta)^3.3a\sin^2\theta\cos\theta\,d\theta}$$

$$=\frac{a\int_0^{\pi/2}\sin^8\theta\cos^7\theta\,d\theta}{\int_0^{\pi/2}\sin^5\theta\cos^7\theta\,d\theta}$$

$$=\frac{a\,\dfrac{7.5.3.1.6.4.2.}{15.13.11.9.7.5.3.1}}{\dfrac{4.2.6.4.2}{12.10.8.6.4.2}}=a\,\frac{6.4.2}{15.13.11.9}\times\frac{12.10.8}{4.2}$$

$$=\frac{128}{429}\,a.$$

$$\therefore\ \frac{\bar{x}}{a}=\frac{128}{429}.$$

But x/a and y/b are symmetrically placed in the equation of the curve. Therefore by symmetry $\bar{y}/b=128/429$.

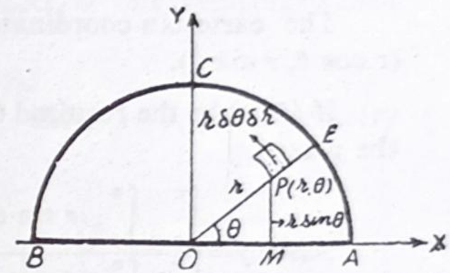
$\therefore$ The required C.G. $(\bar{x}, \bar{y})$ is given by

$$\bar{x}=\frac{128}{429}\,a,\ y=\frac{128}{429}\,b.$$

Ex. 56. *Find the centre of gravity of a semi-circular lamina of radius a when the density at any point*

(*i*) *varies as the square of the distance from the centre,*
(*ii*) *varies as the cube of the distance from the centre,*
(*iii*) *varies as the distance from the centre,*
(*iv*) *varies as the square of the distance from the diameter,*
*(*v*) *varies as* $\sqrt{(a^2-r^2)}$ *where r is the distance of the point from the centre.*

Sol. (*i*) Take the semi-circular area ACB whose central radius OC is along the y-axis and the centre O is at the pole. The diameter BA is along the x-axis.



The equation of the bounding circle is $r=a$, a being the radius.

Take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r, \theta)$ lying within the area ACB whose C.G. is to be found. If ρ be the density at the point P then according to the question $\rho=\lambda r^2$, where λ is some constant.

$\therefore$ the mass of the element $r\,\delta\theta\,\delta r=\rho.\,r\,\delta\theta\,\delta r=\lambda r^2.r\delta\theta\,\delta r$
$=\lambda r^3\,\delta\theta\,\delta r$.

The element $r\,\delta\theta\,\delta r$ being very small, its C.G. can be taken as the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

The semi-circular lamina ACB is symmetrical about OY and its density is also symmetrical about OY as can be easily seen by taking two points of this lamina on opposite sides of OY and at equal distances from it.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G. of the semi-circular lamina ACB, then $\bar{x}=0$, (by symmetry about the y-axis).

$$\text{Also}\ \bar{y}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r\sin\theta\,\lambda r^3\,d\theta\,dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a}\lambda r^3\,d\theta\,dr}$$

$$=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^4\sin\theta\,d\theta\,dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\,d\theta\,dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^5}{5}\right]_0^a\sin\theta\,d\theta}{\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^a d\theta}=\frac{4a^5}{5a^4}\,\frac{\int_0^{\pi}\sin\theta\,d\theta}{\int_0^{\pi}d\theta}=\frac{\left[-\cos\theta\right]_0^{\pi}}{5\left[\theta\right]_0^{\pi}}$$

$$=\frac{4a\left[-(\cos\pi-\cos 0)\right]}{5\pi}=\frac{8a}{5\pi}.$$

$\therefore$ the required C.G. is given by $\bar{x}=0$, $\bar{y}=8a/5\pi$.

Hence the C.G. of the lamina lies on the central radius at a distance $8a/5\pi$ from the centre, a being the radius of the lamina.

(*ii*) Proceed as in part (*i*). In this case the density varies as the cube of the distance from the centre *i.e.*, $\rho=\lambda r^3$.

$\therefore$ the mass of the element $=\rho r\ \delta\theta\ \delta r=\lambda r^3 . r\ \delta\theta\ \delta r=\lambda r^4\ \delta\theta\ \delta r$.

The cartesian coordinates of the C.G. of this element are $(r\cos\theta, r\sin\theta)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{x}=0$, (by symmetry about the y-axis).

$$\text{Also}\quad \bar{y}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r\sin\theta . \lambda r^4\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} \lambda r^4\, d\theta\, dr}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^5 \sin\theta\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^4\, d\theta\, dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^6}{6}\right]_0^a \sin\theta\, d\theta}{\int_0^{\pi}\left[\frac{r^5}{5}\right]_0^a d\theta}=\frac{\frac{1}{6}a^6\int_0^{\pi}\sin\theta\, d\theta}{\frac{1}{6}a^5\int_0^{\pi} d\theta}=\frac{5a}{6}.\frac{\left[-\cos\theta\right]_0^{\pi}}{\left[\theta\right]_0^{\pi}}$$

$=\frac{5}{6}a.(2/\pi)=5a/3\pi$.

$\therefore$ the required C.G. is given by $\bar{x}=0$, $\bar{y}=5a/3\pi$.

(*iii*) Proceed as in part (i). In this case the density varies as the distance from the centre *i.e.*, $\rho=\lambda r$.

$\therefore$ the mass of the element $=\rho . r\ \delta\theta\ \delta r=\lambda r . r\ \delta\theta\ \delta r=\lambda r^2\ \delta\theta\ \delta r$ and the cartesian coordinates of the C.G. of this elementary area are $(r\cos\theta, r\sin\theta)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{x}=0$, (by symmetry).

$$\text{Also}\quad \bar{y}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r\sin\theta . (\lambda r^2\, d\theta\, dr)}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} \lambda\, r^2\, d\theta\, dr}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3 \sin\theta\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\, d\theta\, dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^a \sin\theta\, d\theta}{\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta}=\frac{\frac{1}{4}a^4\int_0^{\pi}\sin\theta\, d\theta}{\frac{1}{3}a^3\int_0^{\pi} d\theta}=\frac{3a}{4}.\frac{\left[-\cos\theta\right]_0^{\pi}}{\left[\theta\right]_0^{\pi}}$$

$$=\frac{3a}{4}.\frac{[-(\cos\pi-\cos 0)]}{[\pi-0]}=\frac{3a}{4}.\frac{2}{\pi}=\frac{3a}{2\pi}.$$

$\therefore$ the required C.G. is given by $\bar{x}=0$, $\bar{y}=3a/2\pi$.

(*iv*) Proceed as in part (i). In this case the density varies as the square of the distance from the diameter BOA *i.e.*,

$$\rho=\lambda . PM^2=\lambda\,(r\sin\theta)^2. \qquad \text{[Note]}$$

$\therefore$ the mass of the element $r\ \delta\theta\ \delta r$

$$=\rho . r\ \delta\theta\ \delta r=\lambda r^2\sin^2\theta . r\ \delta\theta\ \delta r=\lambda r^3\sin^2\theta\ \delta\theta\ \delta r$$

and the cartesian coordinates of the C.G. of this elementary area are

$$(r\cos\theta, r\sin\theta).$$

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{x}=0$, (by symmetry about the y-axis).

Also

$$\bar{y}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r\sin\theta . \lambda r^3\sin^2\theta\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} \lambda r^3\sin^2\theta\, d\theta\, dr}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^4\sin^3\theta\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin^2\theta\, d\theta\, dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^5}{5}\right]_0^a \sin^3\theta\, d\theta}{\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^a \sin^2\theta\, d\theta}=\frac{\frac{1}{5}a^5\int_0^{\pi}\sin^3\theta\, d\theta}{\frac{1}{4}a^4\int_0^{\pi}\sin^2\theta\, d\theta}$$

$$=\frac{4a}{5}.\frac{2\int_0^{\pi/2}\sin^3\theta\, d\theta}{2\int_0^{\pi/2}\sin^2\theta\, d\theta}=\frac{4a}{5}.\frac{\frac{2}{3.1}}{\frac{1}{2}.\frac{1}{2}\pi}=\frac{4a}{5}.\frac{8}{3\pi}=\frac{32a}{15\pi}$$

$\therefore$ the required C.G. is given by $\bar{x}=0$, $\bar{y}=32a/15\pi$.

(*v*) Proceed exactly in the same way as in part (i). In this case the density ρ at the point $P\,(r, \theta)$ is given by $\rho=\lambda\sqrt{(a^2-r^2)}$, where λ is some constant.

$\therefore$ the mass of the element $r\ \delta\theta\ \delta r$ at P

$$=\rho r\ \delta\theta\ \delta r=\lambda\sqrt{(a^2-r^2)} . r\ \delta\theta\ \delta r.$$

The cartesian coordinates of the C.G. of this element are

$$(r\cos\theta, r\sin\theta).$$

The semi-circular lamina ACB is symmetrical about OY and its density is also symmetrical about OY. Therefore if $(\bar{x}, \bar{y})$ be the required C.G. of the area ACB, then

$\bar{x}=0$, by symmetry about the y-axis.

Also

$$\bar{y}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a} r\sin\theta . \lambda r\sqrt{(a^2-r^2)}\, d\theta\, dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a} \lambda r\sqrt{(a^2-r^2)}\, d\theta\, dr}$$

$$=\frac{\int_{r=0}^{a}\left[-\cos\theta\right]_0^{\pi} r^2\sqrt{(a^2-r^2)}\, dr}{\int_{r=0}^{a}\left[\theta\right]_0^{\pi} r\sqrt{(a^2-r^2)}\, dr},$$

first integrating w.r.t. θ because the limits of both r and θ are constants

$$=\frac{\int_0^a 2r^2\sqrt{(a^2-r^2)}\,dr}{\int_0^a \pi r\sqrt{(a^2-r^2)}\,dr}$$

$$=\frac{2\int_0^{\pi/2} a^2\sin^2 t.a\cos t.a\cos t\,dt}{\pi\int_0^{\pi/2} a\sin t\, a\cos t.a\cos t\,dt}, \text{ putting } r=a\sin t$$

$$=\frac{2a\int_0^{\pi/2}\sin^2 t\cos^2 t\,dt}{\pi\int_0^{\pi/2}\sin t\cos^2 t\,dt}=\frac{2a}{\pi}\cdot\frac{\frac{1.1}{4.2}\cdot\frac{\pi}{2}}{\frac{1}{3.1}}=\frac{2a}{\pi}\cdot\frac{\pi}{16}.3=\frac{3}{8}a.$$

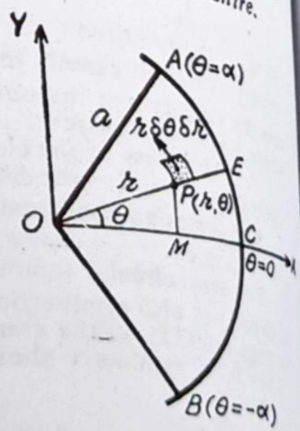
$\therefore$ $\bar{x}=0$, $\bar{y}=3a$. Hence the C.G. lies on the symmetrical radius at a distance $3a/8$ from the centre.

Ex. 57. *Find the centre of gravity of a sector of a circle in which the surface density varies as the distance from the centre.*

Sol. Referred to the centre O as pole, the polar equation of a circle of radius a is

$$r=a. \quad ...(1)$$

Let the sector AOB subtend an angle 2α at the centre O and let the x-axis be along the symmetrical radius OC. We have $\angle AOC=\alpha$.



The sector AOB is symmetrical about the initial line OX. Since the density of the area AOB at any point varies as the distance from the centre O, therefore the density of the area AOB is also symmetrical about OX. Hence the C.G. of the area AOB lies on OX. If $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the area AOB, we have $\bar{y}=0$, by symmetry about OX. Also the x-coordinate of the C.G. of the whole area AOB is the same as the x-coordinate of the C.G. of the upper half AOC of this area.

Now take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r,\theta)$ lying within the area AOC. If ρ be the density at the point P, then according to the question $\rho=\lambda r$.

$\therefore$ the mass of the element $r\delta\theta\,\delta r=r\delta\theta\,\delta r.\rho=\lambda r^2\,\delta\theta\,\delta r$.

The element $r\delta\theta\,\delta r$ being very small, its C.G. can be taken at the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

We have

$$\bar{x}=\frac{\int_{\theta=0}^{\alpha}\int_{r=0}^{a} r\cos\theta.\lambda r^2\,d\theta\,dr}{\int_{\theta=0}^{\alpha}\int_{r=0}^{a}\lambda r^2\,d\theta\,dr}$$

[Note that on the line OPE, r varies from 0 to a and to cover the area AOC, θ varies from 0 to α].

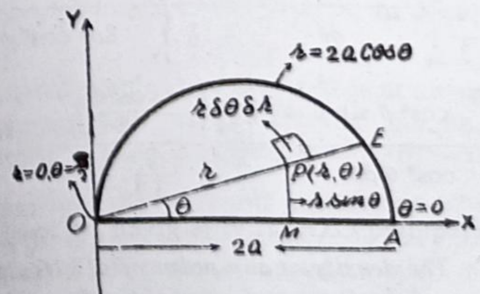
$$=\frac{\int_0^\alpha\left[\frac{r^4}{4}\right]_0^a\cos\theta\,d\theta}{\int_0^\alpha\left[\frac{r^3}{3}\right]_0^a d\theta}=\frac{3}{4}a\,\frac{\int_0^\alpha\cos\theta\,d\theta}{\int_0^\alpha d\theta}$$

$$=\frac{3}{4}a\,\frac{\left[\sin\theta\right]_0^\alpha}{\left[\theta\right]_0^\alpha}=\frac{3}{4}a\,\frac{\sin\alpha}{\alpha}.$$

Hence the C.G. of the sector lies on the symmetrical radius at a distance $(\frac{3}{4}a\sin\alpha)/\alpha$ from the centre, where a is the radius and 2α the angle of the sector.

Ex. 58. *Find the C.G. of a semi-circular area when the density varies as the distance from one end of the bounding diameter.*

Sol. Let OEA be a semi-circular area whose density at any point varies as the distance from the end O of the bounding diameter OA. Take the point O as the pole and the diameter



OA as the initial line. Then the equation of the circle OEA is $r=2a\cos\theta$, a being the radius of the circle.

Take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r, \theta)$ lying within the area OEA whose C.G. is to be found. If ρ be the density at P, then according to the question $\rho=\lambda r$.

$\therefore$ the mass of the element $r\,\delta\theta\,\delta r$ at P

$$=\rho r\,\delta\theta\,\delta r=\lambda r.r\,\delta\theta\,\delta r=\lambda r^2\,\delta\theta\,\delta r.$$

To cover the area OEA we draw the straight lines $\theta=$ const. On any such line, say on the line OPE, r goes from O to the arc of the circle $r=2a\cos\theta$. Thus to cover the area OEA the limits of r are from 0 to $2a\cos\theta$ and the limits of θ are from 0 to $\pi/2$.

$\therefore$ if $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cos\theta.\lambda r^2\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\lambda r^2\,d\theta\,dr}=\frac{\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta}\cos\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta}$$

$$=\frac{\frac{1}{4}\int_0^{\pi/2}16a^4\cos^4\theta.\cos\theta\,d\theta}{\frac{1}{3}\int_0^{\pi/2}8a^3\cos^3\theta\,d\theta}=\frac{4a\int_0^{\pi/2}\cos^5\theta\,d\theta}{\frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta}=\frac{3a}{2}\cdot\frac{\frac{4}{5}\cdot\frac{2}{3}}{\frac{2}{3}}=\frac{6a}{5},$$

and

$$\bar{y}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\sin\theta.\lambda r^2\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\lambda r^2\,d\theta\,dr}$$

$$=\frac{\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta}\sin\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta}=\frac{\frac{1}{4}\int_0^{\pi/2}16a^4\cos^4\theta\sin\theta\,d\theta}{\frac{1}{3}\int_0^{\pi/2}8a^3\cos^3\theta\,d\theta}$$

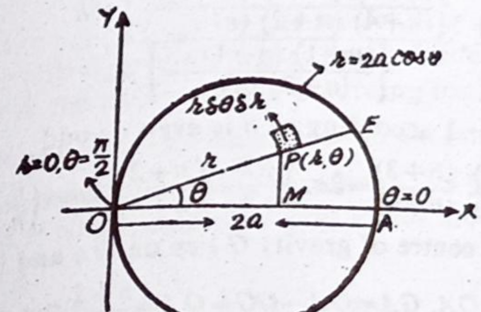
$$=\frac{3a}{2}\cdot\frac{\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta}{\int_0^{\pi/2}\cos^3\theta\,d\theta}=\frac{3a}{2}\cdot\frac{\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2}}{\frac{2}{3.1}}=\frac{3a}{2}\cdot\frac{3}{2}\cdot\frac{1}{5}=\frac{9a}{20}.$$

$\therefore$ the required C.G. $(\bar{x}, \bar{y})$ is given by $\bar{x}=\frac{6}{5}a$, $\bar{y}=\frac{9}{20}a$.

***Ex. 59.** *The density at any point of a circular lamina varies as the nth power of the distance from a point O on the circumference. Show that the centre of gravity of the lamina divides the diameter through O in the ratio n+2 : 2.*

(Rohilkhand 1990; Kanpur 82; Agra 85)

Sol. Let the given point O be taken as the pole and the diameter OA through O as the initial line. Then the equation of the circle is $r=2a\cos\theta$, where the diameter $OA=2a$.



The circular lamina is symmetrical about the diameter OA. Its density is also symmetrical about OA as can be seen by taking two points of the lamina on opposite sides of OA and at equal distances from it.

Therefore the C.G. of the lamina lies on the diameter OA.

If $(\bar{x}, \bar{y})$ be the coordinates of the C.G. of the whole lamina, then $\bar{y}=0$, by symmetry about the x-axis.

Also the x-coordinate of the C.G. of the whole lamina is the same as the x-coordinate of the C.G. of the upper half of this lamina.

Now take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r, \theta)$ lying inside the upper half of the circular lamina. If ρ be the density at the point P, then, as given, $\rho=\lambda\,(OP)^n=\lambda r^n$.

$\therefore$ the mass of the element $r\,\delta\theta\,\delta r$ at $P=(r\,\delta\theta\,\delta r).\lambda r^n$ $=\lambda r^{n+1}\,\delta\theta\,\delta r$. The C.G. of this element is the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

We have

$$\bar{x}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cos\theta.\lambda r^{n+1}\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}\lambda r^{n+1}\,d\theta\,dr}$$

$$=\frac{\int_0^{\pi/2}\left[\frac{r^{n+3}}{n+3}\right]_0^{2a\cos\theta}\cos\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^{n+2}}{n+2}\right]_0^{2a\cos\theta}d\theta}=\frac{(n+2).2a\int_0^{\pi/2}\cos^{n+4}\theta\,d\theta}{(n+3)\int_0^{\pi/2}\cos^{n+2}\theta\,d\theta}$$

$$=\frac{2a\,(n+2)}{(n+3)}\cdot\frac{\left\{\frac{(n+3)\,(n+1)\,(n-1)\ldots}{(n+4)\,(n+2)\,(n)\ldots}\right\}\times k}{\left\{\frac{(n+1)\,(n-1)\ldots}{(n+2)\,(n)\ldots}\right\}\times k},$$

where k is $\pi/2$ or 1 according as n is even or odd

$$=2a.\frac{(n+2)}{(n+3)}\cdot\frac{(n+3)}{(n+4)}=2a.\frac{n+2}{n+4}=\left(\frac{n+2}{n+4}\right).2a=\left(\frac{n+2}{n+4}\right).OA.$$

Hence the centre of gravity G lies on OA and is such that

$$OG=\frac{n+2}{n+4}\,OA,\ GA=OA-OG=OA-\frac{n+2}{n+4}\,OA=\frac{2}{(n+4)}\,OA.$$

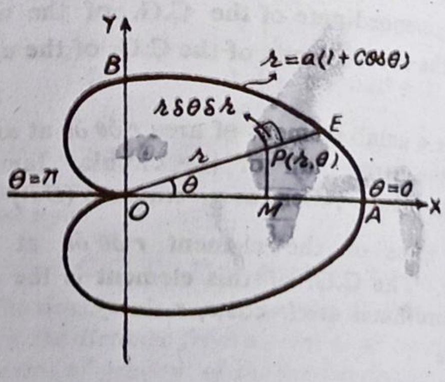
$\therefore\quad OG:GA=(n+2):2$

i.e., G divides OA in the ratio $(n+2):2$.

Ex. 60. *Find the distance of the centre of gravity of the area of the cardioid $r=a\,(1+\cos\theta)$ from the cusp, when the density varies as the nth power of the distance from O.*

Sol. The given cardioid is $r=a\,(1+\cos\theta)$.

It has a cusp at the pole O.



The given curve is symmetrical about the initial line OX. The density of the area of the cardioid is also symmetrical about OX. Therefore the C.G. of the whole area of the cardioid lies on

OX. If $(\bar{x},\bar{y})$ be the coordinates of the C.G. of the whole area of the cardioid, we have $\bar{y}=0$, by symmetry about OX.

Also the x-coordinate of the C.G. of the whole area of the cardioid is the same as the x-coordinate of the C.G. of the upper half of the area of the cardioid.

Now take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r,\theta)$ lying within the area of the upper half of the cardioid. If ρ be the density at the point P, then according to the question $\rho=\lambda r^n$.

$\therefore$ the mass of the element $r\,\delta\theta\,\delta r=r\,\delta\theta\,\delta r.\rho=\lambda r^{n+1}\,\delta\theta\,\delta r$.

The element $r\,\delta\theta\,\delta r$ being very small, its C.G. can be taken at the point P whose cartesian coordinates are $(r\cos\theta,\ r\sin\theta)$.

We have

$$\bar{x}=\frac{\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)}r\cos\theta.\lambda r^{n+1}\,d\theta\,dr}{\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)}\lambda r^{n+1}\,d\theta\,dr}$$

$$=\frac{\int_0^{\pi}\left[\frac{r^{n+3}}{n+3}\right]_0^{a(1+\cos\theta)}\cos\theta\,d\theta}{\int_0^{\pi}\left[\frac{r^{n+2}}{n+2}\right]_0^{a(1+\cos\theta)}d\theta}$$

$$=\frac{a^{n+3}.(n+2)}{a^{n+2}.(n+3)}\cdot\frac{\int_0^{\pi}(1+\cos\theta)^{n+3}\cos\theta\,d\theta}{\int_0^{\pi}(1+\cos\theta)^{n+2}\,d\theta}$$

$$=a.\frac{n+2}{n+3}\,\frac{\int_0^{\pi}2^{n+3}\cos^{2n+6}\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta}{\int_0^{\pi}2^{n+2}\cos^{2n+4}\frac{\theta}{2}\,d\theta}$$

$$=2a.\frac{n+2}{n+3}\,\frac{\int_0^{\pi/2}\{2\cos^{2n+8}\phi-\cos^{n+6}\phi\}\,d\phi}{\int_0^{\pi/2}\cos^{2n+4}\phi\,d\phi},\ \text{putting}\ \frac{\theta}{2}=\phi$$

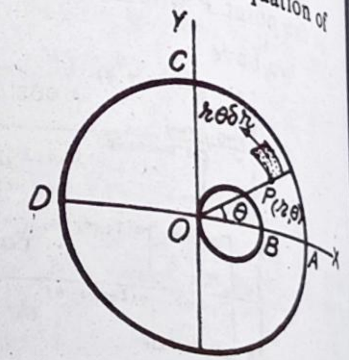
$$=2a.\frac{n+2}{n+3}\,\frac{\left[2.\frac{(2n+7)\,(2n+5)\,(2n+3)\ldots}{(2n+8)\,(2n+6)\,(2n+4)\ldots}\frac{\pi}{2}-\frac{(2n+5)(2n+3)\ldots}{(2n+6)(2n+4)\ldots}\frac{\pi}{2}\right]}{\frac{(2n+3)\,(2n+1)\ldots}{(2n+4)\,(2n+2)\ldots}\cdot\frac{\pi}{2}}$$

$$=2a.\left(\frac{n+2}{n+3}\right).\left(\frac{2n+5}{2n+6}\right)\left[2.\frac{2n+7}{2n+8}-1\right]$$

$$=2a.\left(\frac{n+2}{n+3}\right)\left(\frac{2n+5}{2n+6}\right).\left(\frac{2n+6}{2n+8}\right)=a\left(\frac{n+2}{n+3}\right).\left(\frac{2n+5}{n+4}\right).$$

Ex. 61. *A circular disc, of radius a, whose density is proportional to the distance from the centre, has a hole cut in it bounded by a circle of diameter b, which passes through the centre. Show that the distance from the centre of the disc of the centre of gravity of the remaining portion is* $\dfrac{6b^4}{15\pi a^3-10b^3}$.

Sol. Referred to the centre O as pole, the polar equation of the bounding circle of the disc is $r=a$. Take the radius OA as the x-axis or the initial line.



Let $OB=b$. The circle drawn on OB as diameter has been removed from the disc, the equation of this circle is $r=b\cos\theta$. We are to find the distance of the C.G. of the remaining portion from O.

Both the circles are symmetrical about the x-axis and their density is also symmetrical about the x-axis. Therefore their centres of gravity must lie on the x-axis. By symmetry, the centre of gravity of the bigger circle is at its centre O.

The density ρ at any point $P\ (r, \theta)$ lying within the area of the disc is λr.

$\therefore$ the mass of the small element of area $r\ \delta\theta\ \delta r$ at P

$$=\rho . r\ \delta\theta\ \delta r=\lambda r . r\ \delta\theta\ \delta r=\lambda r^2\ \delta\theta\ \delta r.$$

We have,

M_1=mass of the bigger circle

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\lambda r^2\ d\theta\ dr=\int_0^{2\pi}\lambda\left[\frac{r^3}{3}\right]_0^a d\theta$$

$$=\frac{\lambda a^3}{3}\int_0^{2\pi}d\theta=\frac{2\pi\lambda a^3}{3},$$

M_2=mass of the smaller circle

$$=2\int_{\theta=0}^{\pi/2}\int_{r=0}^{b\cos\theta}\lambda r^2\ d\theta\ dr=2\int_0^{\pi/2}\lambda\left[\frac{r^3}{3}\right]_0^{b\cos\theta}d\theta$$

$$=\frac{2\lambda b^3}{3}\int_0^{\pi/2}\cos^3\theta\ d\theta=\frac{2\lambda\ b^3}{3}.\frac{2}{3.1}=\frac{4}{9}\lambda b^3,$$

x_1=the distance of the C.G. of the bigger circle from O=0,

x_2=the distance of the C.G. of the smaller circle from O

=x-coordinate of the C.G. of the smaller circle

$$=\frac{\int_{-\pi/2}^{\pi/2}\int_0^{b\cos\theta}r\cos\theta.\lambda r^2\ d\theta\ dr}{\int_{-\pi/2}^{\pi/2}\int_0^{b\cos\theta}\lambda r^2\ d\theta\ dr}$$

$$=\frac{\int_{-\pi/2}^{\pi/2}\lambda\left[\frac{r^4}{4}\right]_0^{b\cos\theta}\cos\theta\ d\theta}{\int_{-\pi/2}^{\pi/2}\lambda\left[\frac{r^3}{3}\right]_0^{b\cos\theta}d\theta}=\frac{\frac{\lambda b^4}{4}\int_{-\pi/2}^{\pi/2}\cos^5\theta\ d\theta}{\frac{\lambda b^3}{3}\int_{-\pi/2}^{\pi/2}\cos^3\theta\ d\theta}$$

$$=\frac{3}{4}b\frac{2\int_0^{\pi/2}\cos^5\theta\ d\theta}{2\int_0^{\pi/2}\cos^3\theta\ d\theta}=\frac{3}{4}b\frac{\frac{4.2}{5.3.1}}{\frac{2}{3.1}}=\frac{3}{4}b.\frac{8}{15}.\frac{3}{2}$$

$$=\frac{3}{5}b.$$

If $\bar{x}$ be the distance from O of the C.G. of the remainder portion left after removing the smaller circle from the bigger one, then

$$\bar{x}=\frac{M_1x_1-M_2x_2}{M_1-M_2}=\frac{\frac{2}{3}\pi\ \lambda a^3.0-\frac{4}{9}\lambda b^3.\frac{3}{5}b}{\frac{2}{3}\pi\ \lambda a^3-\frac{4}{9}\lambda b^3}$$

$$=\frac{-\frac{12}{5}b^4}{6\pi a^3-4b^3}=-\frac{6b^4}{15\pi a^3-10b^3}.$$

The negative sign shows that the C.G. lies on the negative side of the x-axis. Hence the distance of the C.G. of the remaining portion from the centre of the disc is $6b^4/(15\pi a^3-10b^3)$.

Ex. 62. *If the density at any point of a circular disc, whose radius is a, vary directly as the distance from the centre and a circle described on a radius as diameter be cut out, prove that the centre of inertia of the remainder will be at a distance* $6a/(15\pi-10)$ *from the centre.*

Sol. Proceed as in Ex. 61. Here the circle drawn on OA as diameter has been removed from the disc. If we put $b=a$ in the result of Ex. 61, we get the result of Ex. 62.

***Ex. 63.** *Find the C.G. of a solid hemi-sphere when the density at any point :*

(*i*) *is proportional to the* n^{th} *power of the distance from the centre. Hence show that the C.G. divides the radius perpendicular* *to its plane surface in the ratio* $(n+3):(n+5)$, (Rohilkhand 1977)

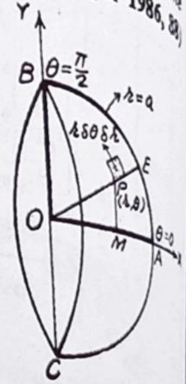
(ii) *varies directly as the distance from the centre,*

(iii) *varies inversely as the distance from the centre,*

(iv) *varies directly as the square of the distance from the centre,* (Rohilkhand 1980)

(v) *varies as the square of the distance from the centre of the whole sphere.* (Kanpur 1986, 88)

Sol. (i) Let the hemisphere of radius a be generated by revolving the quadrant OAB of the circle $r=a$ about the x-axis. The line OB will generate the plane base of the hemisphere and the line OA will be the axis of the hemisphere. The centre of the hemisphere is at the pole O.



Take a small element of area $r\delta\theta\delta r$ at any point $P(r,\theta)$ lying within the area of the quadrant OAB. When the elementary area $r\,\delta\theta\,\delta r$ is revolved about OA, a circular ring of radius $PM\ (=r\sin\theta)$ is generated.

The volume of this elementary ring $=(2\pi\,PM)\,r\,\delta\theta\,\delta r$
$$=(2\pi.r\sin\theta)\,r\,\delta\theta\,\delta r=2\pi\,r^2\sin\theta\,\delta\theta\,\delta r.$$

The distance of each point of this ring from the centre O is r. So according to the question, the density ρ at each point of it is λr^n.

$\therefore$ the mass of this ring
$$=2\pi\,r^2\sin\theta\,\delta\theta\,\delta r.\rho=2\pi\,r^2\sin\theta\,\delta\theta\,\delta r\,(\lambda r^n)$$
$$=2\pi\,\lambda r^{n+2}\sin\theta\,\delta\theta\,\delta r.$$

Its C.G. can be taken at the point M whose cartesian coordinates are $(r\cos\theta,0)$.

If $(\bar{x},\bar{y})$ be the required C.G. of the hemisphere, then $\bar{y}=0$, (by symmetry about the axis of rotation *i.e.*, the line OX).

Also
$$\bar{x}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\cos\theta.2\pi\lambda r^{n+2}\sin\theta\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}2\pi\,\lambda r^{n+2}\sin\theta\,d\theta\,dr}$$
$$=\frac{\int_0^{\pi/2}\left[\frac{r^{n+4}}{n+4}\right]_0^a\cos\theta\sin\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^{n+3}}{n+3}\right]_0^a\sin\theta\,d\theta}=\frac{(n+3)\,a}{(n+4)}\frac{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2}\sin\theta\,d\theta}$$

$$=\frac{(n+3)\,a}{(n+4)}\frac{\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}}{\left[-\cos\theta\right]_0^{\pi/2}}=\frac{(n+3)\,a}{(n+4)}\cdot\tfrac{1}{2}=\frac{(n+3)\,a}{2\,(n+4)}.$$

$\therefore$ the required C.G. is given by
$$\bar{x}=\{(n+3)\,a/2\,(n+4)\},\ \bar{y}=0.$$

Also if G be the C.G., then
$$OG=\bar{x}=\{(n+3)\,a/2\,(n+4)\}$$
and
$$GA=OA-OG=a-\frac{a}{2}\cdot\frac{n+3}{n+4}=\frac{a}{2}\cdot\frac{n+5}{n+4}.$$
$$\therefore\quad OG:GA=(n+3):(n+5).$$

(ii) **Proceed exactly as in part** (i). Here $\rho=\lambda r$, *i.e.*, put $n=1$ in part (i) and get the required C.G. as $\bar{x}=\frac{2}{5}a$, $\bar{y}=0$.

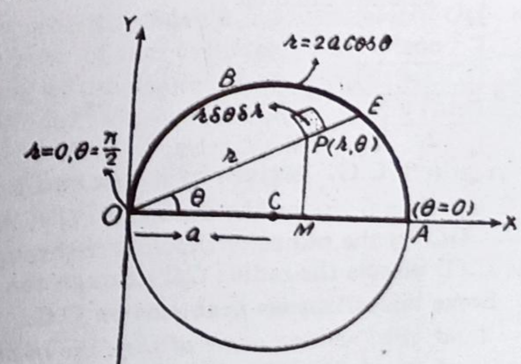
(iii) **Proceed exactly as in part** (i). In this case $\rho\propto 1/r$, *i.e.*, $\rho=\lambda r^{-1}$ *i.e.*, put $n=-1$ in part (i). Thus the required C.G. is given by $\bar{x}=\frac{1}{3}a$, $\bar{y}=0$.

(iv) **Proceed exactly as in part** (i). Here $\rho=\lambda r^2$ *i.e.*, put $n=2$ in part (i) and obtain the required C.G. as $\bar{x}=\frac{5}{12}a$, $\bar{y}=0$.

(v) It is the same as part (iv).

Ex. 64. *Show that the C.G. of a sphere, the density at any point of which varies inversely as the square of the di tance from a fixed point on the surface of the sphere, bisects the radius through the fixed point.* (Meerut 1983)

Sol. Suppose the sphere is formed by revolving the semi-circle OBA about its bounding diameter OA. Referred to O as



pole and the diameter OA as the initial line, the equation of the generating circle is $r=2a\cos\theta$, $2a$ being the diameter of the circle.

Take a small element of area $r\,\delta\theta\,\delta r$ at any point $P(r, \theta)$ lying within the area OBA. When revolved about OA (*i.e.*, the x-axis), it will generate a circular ring whose volume is $(2\pi r \sin\theta).\,r\,\delta\theta\,\delta r$. The distance of each point of this ring from the fixed point on the surface of the sphere is r. Therefore the density ρ at each point of this elementary ring$=\lambda/r^2$, as given in the question.

$\therefore$ the mas of the ring

$$=(\lambda/r^2).\ (2\pi r \sin\theta)\, r\,\delta\theta\,\delta r = 2\pi\lambda \sin\theta\,\delta\theta\,\delta r.$$

The C.G. of this elementary ring can be taken as the point M whose cartesian coordinates are $(r\cos\theta, 0)$.

The limits of r are 0 to $2a\cos\theta$ and the limits of θ are 0 to $\frac{1}{2}\pi$.

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{y}=0$, (by symmetry about the axis of rotation *i.e.*, the x-axis).

Also

$$\bar{x}=\frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cos\theta.2\pi\lambda\sin\theta\, d\theta\, dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} 2\pi\lambda\sin\theta\, d\theta\, dr}$$

$$=\frac{\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2a\cos\theta}\cos\theta\sin\theta\, d\theta}{\int_0^{\pi/2}\left[r\right]_0^{2a\cos\theta}\sin\theta\, d\theta}$$

$$=\frac{\int_0^{\pi/2} 2a^2\cos^3\theta\sin\theta\, d\theta}{\int_0^{\pi/2} 2a\cos\theta\sin\theta\, d\theta}$$

$$=a.\frac{\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2}}{\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}}=\frac{a}{2}.$$

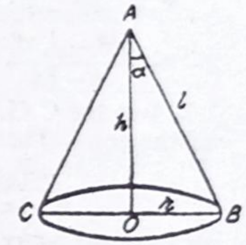
$\therefore$ the required C.G. is given by $\bar{x}=\frac{1}{2}a$ and $\bar{y}=0$. If G is the required C.G., then G lies on the diameter OA. Also $OG=\frac{1}{2}a$ $=\frac{1}{2}OC$, where OC is the radius of the sphere through the point O. Hence the C.G. bisects the radius OC through the point O.

Some miscellaneous problems on C.G.

Ex. 65. *Find the vertical angle of the cone in order that the centre of gravity of its whole surface, including its base, may coincide with the centre of gravity of its volume.*

(Agra 1980, Kanpur 83, Rohilkhand 77)

Sol. ABC is a right circular cone. Let $OA=h$, $OB=r$, $AB=l$ and $\angle OAB=\alpha$. Let ρ be the density of the cone.



We have,

W_1=weight of the curved surface of the cone$=(\pi rl)\,\rho g$,

and W_2=weight of the base of the cone $=\pi r^2\,\rho g$.

The C.G. of the curved surface of the cone lies on OA and its distance from O $=h/3=x_1$, say.

The C.G. of the base of the cone is at its centre O and its distance from $O=0=x_2$, say.

Therefore the C.G. of the whole surface of the cone will lie on OA. If its distance from O be $\bar{x}$, we have

$$\bar{x}=\frac{W_1x_1+W_2x_2}{W_1+W_2}=\frac{\pi rl\rho g.(h/3)+\pi r^2\rho g.0}{\pi rl\,\rho g+\pi r^2\,\rho g}=\frac{hl/3}{l+r}.$$

Again the C.G. of the volume of the cone lies on OA and its distance from $O=h/4$. If the C.G. of the whole surface of the cone coincides with the C.G. of its volume, then

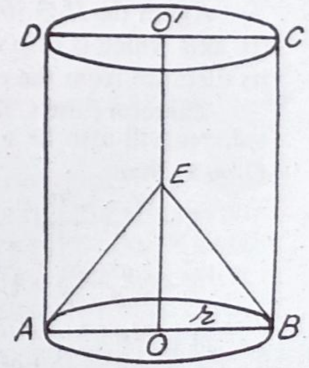
$$\frac{hl/3}{l+r}=\frac{h}{4},\ \text{or}\ \frac{l}{3\,(l+r)}=\frac{1}{4},\ \text{or}\ 4l=3l+r,$$

or $\quad l=3r$, or $r/l=\frac{1}{3}$, or $\sin\alpha=\frac{1}{3}$, or $\alpha=\sin^{-1}\frac{1}{3}$.

Hence the required vertical angle of the cone$=2\sin^{-1}\frac{1}{3}$.

Ex. 66. *Show how to cut out of a uniform cylinder a cone, whose base coincides with that of the cylinder, so that the centre of gravity of the remaining solid may coincide with the vertex of the cone.*

Sol. Let ρ be the density, h be the height OO' and r be the radius of the base of the cylinder. Let x be the height of the cone EAB cut out of the cylinder. We have,



W_1=weight of the cylinder$=\pi r^2 h\rho g$,

and W_2=weight of the cone$=\frac{1}{3}\pi r^2 x\rho g$.

The C.G. of the cylinder lies on OO' and its distance from $O=h/2=x_1$, say.

The C.G. of the cone lies on OE and its distance from $O=x/4=x_2$, say.

Therefore the C.G. of the remainder portion will lie on OO'. If $\bar{x}$ be its distance from O, then

$$\bar{x}=\frac{W_1x_1-W_2x_2}{W_1-W_2}=\frac{\pi r^2h\,\rho g.\tfrac{1}{2}h-\tfrac{1}{3}\pi r^2x\rho g.\tfrac{1}{4}x}{\pi r^2h\rho g-\tfrac{1}{3}\pi r^2x\rho g}$$

$$=\frac{\tfrac{1}{2}h^2-\tfrac{1}{12}x^2}{h-\tfrac{1}{3}x}.$$

But the C.G. of the remainder coincides with E, the vertex of the cone. Therefore $\bar{x}=x$,

or $$\frac{h^2-\tfrac{1}{12}x^2}{h-\tfrac{1}{3}x}=x, \text{ or } hx-\tfrac{1}{3}x^2=\tfrac{1}{2}h^2-\tfrac{1}{12}x^2,$$

or $$x^2-4hx+2h^2=0.$$

$$\therefore\quad x=\frac{4h\pm\sqrt{(16h^2-8h^2)}}{2}=h(2\pm\sqrt{2}).$$

But x cannot be greater than h. Therefore $x=h(2-\sqrt{2})$. Hence the height of the cone is $(2-\sqrt{2})$ times the height of the cylinder.

Ex. 67. *A uniform bowl has for its inner and outer surfaces concentric hemispheres of radii a and b respectively, prove that the distance of its centre of gravity from the geometric centre is*

$$\frac{3(a+b)\,(a^2+b^2)}{8(a^2+ab+b^2)}.$$

Sol. The bowl can be considered as the difference of two concentric solid hemispheres of radii a and b. Let w be the weight of a unit volume of the bowl. We have, W_1=weight of the bigger solid hemisphere$=\tfrac{2}{3}\pi a^3w$, and W_2=weight of the smaller solid hemisphere supposed to be scooped out$=\tfrac{2}{3}\pi b^3w$.

The C.G. of the bigger solid hemisphere lies on its axis (i.e., symmetrical radius) and its distance from the centre O of its base$=\tfrac{3}{8}a=x_1$, say.

Again the C.G. of the smaller solid hemisphere also lies on its axis which is also the axis of the bigger solid hemisphere and its distance from the centre O of its base$=\tfrac{3}{8}b=x_2$, say.

Therefore the C.G. of the difference of the two solid hemispheres will also lie on their common axis. If its distance from O be $\bar{x}$, then

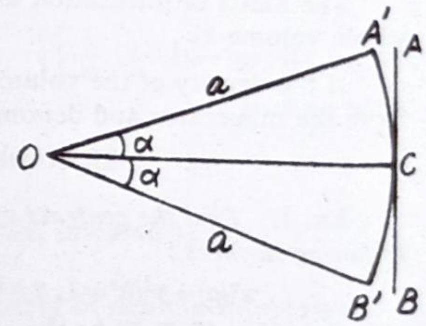
$$\bar{x}=\frac{W_1x_1-W_2x_2}{W_1-W_2}=\frac{\tfrac{2}{3}\pi a^3w.\tfrac{3}{8}a-\tfrac{2}{3}\pi b^3w.\tfrac{3}{8}b}{\tfrac{2}{3}\pi a^3w-\tfrac{2}{3}\pi b^3w}$$

$$=\frac{3}{8}\,\frac{a^4-b^4}{a^3-b^3}=\frac{3}{8}\,\frac{(a-b)\,(a+b)\,(a^2+b^2)}{(a-b)\,(a^2+ab+b^2)}$$

$$=\frac{3(a+b)\,(a^2+b^2)}{8(a^2+ab+b^2)}.$$

Ex. 68. *A thin uniform rod of length l is bent into the form of a circular arc whose radius a is large compared to l, prove that the displacement of the centre of gravity is* $(l^2/24a)$ *approximately.*

Sol. Suppose ACB is a thin uniform rod of length l, C being the middle point of the rod. The rod is bent into the form of a circular arc $A'CB'$ of centre O and radius a, which is large compared to l. We have arc $A'C$=arc $B'C=\tfrac{1}{2}l$. Let $\angle A'OB'=2\alpha$, so that $\angle A'OC=\alpha$.



The C.G. of the arc $A'CB'$ is on the symmetrical radius OC. If its distance from O is $\bar{x}$, then

$$\bar{x}=\frac{a\sin\alpha}{\alpha}\quad\left[\text{Refer Ex. 4 (a), page 8}\right]$$

$$=\frac{a}{\alpha}\left[\alpha-\frac{\alpha^3}{3\,!}+\dots\right],\ \text{expanding } \sin\alpha$$

$$=a-\frac{a\,\alpha^2}{6}+\dots$$

But $\alpha=\dfrac{\text{arc } A'C}{\text{radius } OC}=\dfrac{\tfrac{1}{2}l}{a}=\dfrac{l}{2a}$, which is small for a is large compared to l.

$$\therefore\ \bar{x}=a-\frac{a}{6}\left(\frac{l}{2a}\right)^2,\ \text{neglecting powers of } \alpha \text{ higher than 2}$$

$$=a-\frac{l^2}{24a}\ \text{(nearly).}$$

When the rod was straight, its C.G. was at its middle point C. Hence the displacement of the C.G.

$$=OC-\bar{x}=a-\left(a-\frac{l^2}{24a}\right)=\frac{l^2}{24a}\ \text{(nearly).}$$

§ 8. Use of multiple integrals to find the centre of gravity of any volume.

Suppose we are to find the centre of gravity of any volume V. Divide the volume V into a large number of small elements. Let $dx\,dy\,dz$ be the volume of an elementary portion of V situated at the point $P\,(x, y, z)$. If ρ be the density of V at the point P, then the mass of this small element is $\rho\,dx\,dy\,dz$. The C.G. of this small element can be taken as the point $P\,(x, y, z)$. If $(\bar{x}, \bar{y}, \bar{z})$ be the C.G. of the whole volume V, then

$$\bar{x}=\frac{\iiint \rho x\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\quad \bar{y}=\frac{\iiint \rho y\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},\quad \bar{z}=\frac{\iiint \rho z\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}$$

The limits of integration are to be so taken as to cover the whole volume V.

If the density of the volume V is uniform, ρ can be cancelled from the numerator and denominator in the above formulae.

Illustrative Examples

Ex. 1. *Find the centroid of the volume included between the following surfaces :*

$$x^2/a^2+y^2/b^2=1,\ z=0 \text{ and } lx+my+nz=1.$$

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the volume included between the given surfaces. Then

$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{y}=\frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}.$$

The limits of integration for z are from 0 to $(1-lx-my)/n$, for y are from $(-b/a)\sqrt{(a^2-x^2)}$ to $(b/a)\sqrt{(a^2-x^2)}$ and for x are from $-a$ to a.

$\therefore$ the numerator of $\bar{x}=\iint x\left[z\right]_0^{(1-lx-my)/n} dx\,dy$, integrating w.r.t. z

$$=\int_{x=-a}^{a}\int_{y=-(b/a)\sqrt{(a^2-x^2)}}^{(b/a)\sqrt{(a^2-x^2)}}\frac{x}{n}(1-lx-my)\,dxdy$$

$$=2\int_{x=-a}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}\frac{x}{n}(1-lx)\,dx\,dy,$$ the integral of $(-mxy)/n$ w.r.t. y vanishing because it is an odd function of y

$$=2\int_{-a}^{a}\frac{x}{n}(1-lx).\frac{b}{a}\sqrt{(a^2-x^2)}\,dx,$$ integrating w.r.t. y

$$=-\frac{4bl}{na}\int_0^a x^2\sqrt{(a^2-x^2)}\,dx,$$ by a property of definite integrals

$$=-\frac{4bl}{na}\int_0^{\pi/2} a^2\sin^2\theta.a\cos\theta.a\cos\theta\,d\theta,$$ putting $x=a\sin\theta$, so that $dx=a\cos\theta\,d\theta$

$$=-\frac{4bla^3}{n}\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta=-\frac{4bla^3}{n}.\frac{1.1}{4.2}.\frac{\pi}{2}$$

$$=-\frac{\pi bla^3}{4n}.$$

Again the denominator of $\bar{x}=\iint\frac{1}{n}(1-lx-my)\,dx\,dy$, integrating w.r.t. z

$$=2\int_{x=-a}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}\frac{1}{n}(1-lx)\,dx\,dy$$

$$=2\int_{-a}^{a}\frac{1}{n}.(1-lx)\frac{b}{a}\sqrt{(a^2-x^2)}\,dx,$$ integrating w.r.t. y

$$=\frac{4b}{na}\int_0^a\sqrt{(a^2-x^2)}\,dx,$$ by a property of definite integrals

$$=\frac{4b}{na}\int_0^{\pi/2}a\cos\theta.a\cos\theta\,d\theta,$$ putting $x=a\sin\theta$

$$=\frac{4ab}{n}\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{4ab}{n}.\tfrac{1}{2}.\frac{\pi}{2}=\frac{\pi ab}{n}.$$

$$\therefore\quad \bar{x}=-\frac{\pi bla^3}{4n}.\frac{n}{\pi ab}=-\frac{la^2}{4}.$$

Similarly $\bar{y}=-\dfrac{mb^2}{4}$.

Now the numerator of $\bar{z}=\iint\left[\frac{z^2}{2}\right]_0^{(1-lx-my)/n}dx\,dy$

$$=\tfrac{1}{2}\int_{x=-a}^{a}\int_{y=-(b/a)\sqrt{(a^2-x^2)}}^{(b/a)\sqrt{(a^2-x^2)}}\frac{(1-lx-my)^2}{n^2}\,dx\,dy$$

$$=\frac{1}{2n^2}\int\int_{y=-(b/a)\sqrt{(a^2-x^2)}}^{(b/a)\sqrt{(a^2-x^2)}}(1+l^2x^2+m^2y^2-2lx-2my+2lmxy)\,dxdy$$

$$=\frac{2}{2n^2}\int_{x=-a}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}(1+l^2x^2-2lx+m^2y^2)\,dx\,dy$$

$$=\frac{1}{n^2}\int_{x=-a}^{a}\left[(1+l\,x^2-2lx).\frac{b}{a}\sqrt{(a^2-x^2)}+\frac{m^2}{3}.\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$=\frac{2}{n^2}\int_0^a\left[\frac{b}{a}\sqrt{(a^2-x^2)}+l^2x^2.\frac{b}{a}\sqrt{(a^2-x^2)}+\frac{m^2b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

$$=\frac{2}{n^2}\int_0^{\pi/2}\left[\frac{b}{a}.a\cos\theta+\frac{l^2b}{a}.a^2\sin^2\theta.a\cos\theta+\frac{m^2b^3}{3a^3}.a^3\cos^3\theta\right]a\cos\theta\,d\theta,$$ putting $x=a\sin\theta$

$$=\frac{2}{n^2}\int_0^{\pi/2}\left[ab\cos^2\theta+l\,ba^3\sin^2\theta\cos^2\theta+\frac{m^2b^3a}{3}\cos^4\theta\right]d\theta$$

$$=\frac{2ab}{n^3}\left[\tfrac{1}{3}\cdot\frac{\pi}{2}+l^2a^2\frac{1.1}{4.2}\cdot\frac{\pi}{2}+\frac{m^2b^2}{3}\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}\right]$$

$$=\frac{2ab}{n^2}\left[\frac{\pi}{4}+\frac{\pi l^2a^2}{16}+\frac{\pi m^2b^2}{16}\right]$$

$$=\frac{\pi ab}{8n^2}\,[4+l^2a^2+m^2b^2],$$

and the denominator of $\bar{z}$=the denominator of $\bar{x}=\dfrac{\pi ab}{n}$.

$$\therefore\quad \bar{z}=\frac{\pi ab}{8n^2}\,(4+l^2a^2+m^2b^2).\frac{n}{\pi ab}=\frac{4+l^2a^2+m^2b^2}{8n}.$$

Hence $\bar{x}=-\dfrac{la^2}{4}$, $\bar{y}=-\dfrac{mb^2}{4}$, $\bar{z}=\dfrac{4+l^2a^2+m^2b^2}{8n}$.

Ex. 2. *Find the centre of gravity of the volume cut off from the cylinder $x^2+y^2-2ax=0$ by the planes $z=mx$ and $z=nx$.* **(Agra 1987)**

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume. Obviously the given volume is symmetrical about the plane XOZ *i.e.*, the plane $y=0$. [Note that in the equation $x^2+y^2-2ax=0$, the powers of y are all even]. Therefore $\bar{y}=0$.

We have,
$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},\quad \bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}.$$

The limits of integration for z are from mx to nx, for y are from $-\sqrt{(2ax-x^2)}$ to $\sqrt{(2ax-x^2)}$, and for x are from 0 to $2a$.

[Note that from the equation $x^2+y^2-2ax=0$, we get
$$y=\pm\sqrt{(2ax-x^2)},$$
giving the limits for y, and if we put $y=0$ in this equation, we get $x^2-2ax=0$, giving $x=0$ and $2a$ as the limits for x.]

Now
$$\bar{x}=\frac{\iint x\left[z\right]_{mx}^{nx}dx\,dy}{\iint\left[z\right]_{mx}^{nx}dx\,dy},\ \text{integrating w.r.t. } z$$

$$=\frac{\int_{x=0}^{2a}\int_{y=-\sqrt{(2ax-x^2)}}^{\sqrt{(2ax-x^2)}}(n-m)\,x^2\,dx\,dy}{\int_{x=0}^{2a}\int_{y=-\sqrt{(2ax-x^2)}}^{\sqrt{(2ax-x^2)}}(n-m)\,x\,dx\,dy}$$

$$=\frac{\int_0^{2a}x^2.2\sqrt{(2ax-x^2)}\,dx}{\int_0^{2a}x.2\sqrt{(2ax-x^2)}\,dx},\ \text{integrating w.r.t. } y$$

$$=\frac{\int_0^{2a}x^{5/2}.\sqrt{(2a-x)}\,dx}{\int_0^{2a}x^{3/2}.\sqrt{(2a-x)}\,dx}$$

$$=\frac{\int_0^{\pi/2}(2a\sin^2\theta)^{5/2}\,(2a\cos^2\theta)^{1/2}.4a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2}(2a\sin^2\theta)^{3/2}\,(2a\cos^2\theta)^{1/2}.4a\sin\theta\cos\theta\,d\theta},$$

putting $x=2a\sin^2\theta$, so that $dx=4a\sin\theta\cos\theta\,d\theta$

$$=2a\frac{\int_0^{\pi/2}\sin^6\theta\cos^2\theta\,d\theta}{\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta}=2a\frac{\frac{5.3.1.1}{8.6.4.2}\cdot\frac{\pi}{2}}{\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}}=2a.\frac{5}{8}=\frac{5a}{4}.$$

Again
$$\bar{z}=\frac{\iint x\left[\frac{z^2}{2}\right]_{mx}^{nx}dx\,dy}{\iint\left[z\right]_{mx}^{nx}dx\,dy},\ \text{integrating w.r.t. } z$$

$$=\frac{\iint x.\tfrac{1}{2}(n^2x^2-m^2x^2)\,dx\,dy}{\iint(nx-mx)\,dx\,dy}=\tfrac{1}{2}\,(n+m)\frac{\iint(n-m)\,x^2\,dx\,dy}{\iint(n-m)\,x\,dx\,dy}$$

$$=\tfrac{1}{2}\,(n+m).\bar{x}=\tfrac{1}{2}\,(n+m).\tfrac{5}{4}a=\tfrac{5}{8}\,(n+m)\,a.$$

Hence $\quad\bar{x}=\tfrac{5}{4}a,\ \bar{y}=0,\ \bar{z}=\tfrac{5}{8}a\,(n+m).$

Ex. 3. *Find the centre of gravity of the volume cut off from the cylinder $2x^2+y^2=2ax$ by the planes $z=mx$, $z=nx$.* **(Agra 1985)**

Sol. Proceed as in Ex. 2.

Here $\bar{x}=\tfrac{5}{8}a$, $\bar{y}=0$, $\bar{z}=\tfrac{5}{16}a\,(m+n)$.

Ex. 4. *Find the centre of gravity of the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, which is of constant density.*

(Agra 1984; Garhwal 76)

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume.

We have
$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

where x, y, z have any positive values subject to the condition
$$(x/a)^2+(y/b)^2+(z/c)^2\leqslant 1.$$

Put $(x/a)^2=u$ *i.e.*, $x=au^{1/2}$ so that $dx=a.\tfrac{1}{2}u^{(1/2)-1}\,du$,

$(y/b)^2=v$ *i.e.*, $y=bv^{1/2}$ so that $dy=b.\frac{1}{2}v^{(1/2)-1}\,dv$,

and $(z/c)^2=w$ *i.e.*, $z=cw^{1/2}$ so that $dz=c.\frac{1}{2}w^{(1/2)-1}\,dw$.

Then

$$\bar{x}=\frac{\iiint au^{1/2}.abc.\frac{1}{8}u^{(1/2)-1}\,v^{(1/2)-1}\,w^{(1/2)-1}\,du\,dv\,dw}{\iiint abc.\frac{1}{8}\,u^{(1/2)-1}\,v^{(1/2)-1}\,w^{(1/2)-1}\,du\,dv\,dw},$$

where u, v, w have any positive values subject to the condition $u+v+w \leqslant 1$

$$=a\frac{\iiint u^{1-1}\,v^{(1/2)-1}\,w^{(1/2)-1}\,du\,dv\,dw}{\iiint u^{(1/2)-1}\,v^{(1/2)-1}\,w^{(1/2)-1}\,du\,dv\,dw}$$

$$=a\frac{\Gamma(1)\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+1)}\div\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1)},$$

by Dirichlet's integrals

$$=a.\frac{\Gamma(1)}{\Gamma(3)}\times\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}=a.\ \frac{1}{2.1}.\ \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{\sqrt{\pi}}=\tfrac{3}{8}a.$$

Similarly $\bar{y}=\frac{3}{8}b$, $\bar{z}=\frac{3}{8}c$.

Ex. 5 (a). *Find the centroid of the mass which is in the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, where the density ρ is given by* $\rho=\mu x^p y^q z^r$.

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given mass. We have

$$\bar{x}=\frac{\iiint x\,\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}=\frac{\iiint x\,\mu\,x^p y^q z^r\,dx\,dy\,dz}{\iiint \mu\,x^p y^q z^r\,dx\,dy\,dz}$$

$$=\frac{\iiint x^{p+1}y^q z^r\,dx\,dy\,dz}{\iiint x^p y^q z^r\,dx\,dy\,dz},$$

where x, y, z have any positive values subject to the condition

$$x^2/a^2+y^2/b^2+z^2/c^2 \leqslant 1.$$

Put $x^2/a^2=u$ *i.e.*, $x=au^{1/2}$ so that $dx=a.\frac{1}{2}u^{(1/2)-1}\,du$,

$y^2/b^2=v$ *i.e.*, $y=bv^{1/2}$ so that $dy=b.\frac{1}{2}v^{(1/2)-1}\,dv$,

and $z^2/c^2=w$ *i.e.*, $z=cw^{1/2}$ so that $dz=c.\frac{1}{2}w^{(1/2)-1}\,dw$.

$$\therefore\ \bar{x}=\frac{\iiint a^{p+1}u^{(p+1)/2}b^q v^{q/2}c^r w^{r/2}\ abc.\frac{1}{8}u^{(1/2)-1}v^{(1/2)-1}w^{(1/2)-1}\,du\,dv\,dw}{\iiint a^p u^{p/2}\,b^q v^{q/2}\,c^r w^{r/2}\ abc.\frac{1}{8}u^{(1/2)-1}\,v^{(1/2)-1}\,w^{(1/2)-1}\,du\,dv\,dw}$$

where u, v, w have any positive values subject to the condition

$$u+v+w \leqslant 1$$

$$=a\frac{\iiint u^{[(p+1)/2]-1}\,v^{[(q+1)/2]-1}\,w^{[(r+1)/2]-1}\,du\,dv\,dw}{\iiint u^{[(p+2)/2]-1}\,v^{[(q+1)/2]-1}\,w^{[(r+1)/2]-1}\,du\,dv\,dw}$$

$$=a\frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+4}{2}+1\right)}\div\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+3}{2}+1\right)},$$

by Dirichlet's integrals

$$=a\frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)}.$$

Similarly $$\bar{y}=b\frac{\Gamma\left(\frac{q+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)},$$

and $$\bar{z}=c\frac{\Gamma\left(\frac{r+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)}.$$

Ex. 5 (b). *Find the C.G. of the solid bounded by the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, in the positive octant, if density at any point varies as xy^2z^3.* (Agra 1986)

Sol. Proceed exactly as in part (a).

$$\bar{x}=\frac{63\pi a}{512},\ \bar{y}=\frac{63b}{128},\ \bar{z}=\frac{189\pi c}{1024}.$$

Ex. 6. *Find the centroid of the volume contained in the positive octant by* $(x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3}=1$.

(Agra 1988; Garhwal 77)

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume. We have,

$$\bar{x}=\frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

where x, y, z have any positive values subject to the condition

$$(x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3} \leqslant 1.$$

Put $(x/a)^{2/3}=u$ *i.e.*, $x=au^{3/2}$ so that $dx=a.\frac{3}{2}\, u^{(3/2)-1}\, du$,

$(y/b)^{2/3}=v$ *i.e.*, $y=bv^{3/2}$ so that $dy=b.\frac{3}{2}\, v^{(3/2)-1}\, dv$,

and $(z/c)^{2/3}=w$ *i.e.*, $z=cw^{3/2}$ so that $dz=c.\frac{3}{2}\, w^{(3/2)-1}\, dw$.

$$\therefore\ \bar{x}=\frac{\iiint au^{3/2}.abc\ \frac{27}{8}.u^{(3/2)-1}\, v^{(3/2)-1}\, w^{(3/2)-1}\, du\, dv\, dw}{\iiint abc.\frac{27}{8}\, u^{(3/2)-1}\, v^{(3/2)-1}\, w^{(3/2)-1}\, du\, dv\, dw},$$

where u, v, w have any positive values subject to the condition

$$u+v+w \leqslant 1$$

$$=a\frac{\iiint u^{3-1}\, v^{(3/2)-1}\, w^{(3/2)-1}\, du\, dv\, dw}{\iiint u^{(3/2)-1}\, v^{(3/2)-1}\, w^{(3/2)-1}\, du\, dv\, dw}$$

$$=a\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(3+\frac{3}{2}+\frac{3}{2}+1)} \div \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1)}$$

$$=a\,\frac{\Gamma(3)}{\Gamma(7)}\times\frac{\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})}$$

$$=a\,\frac{2.1}{6.5.4.3.2.1.}\times\frac{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{\frac{1}{2}.\sqrt{\pi}}=\frac{21a}{128}.$$

Similarly

$$\bar{y}=\frac{21b}{128},\ \bar{z}=\frac{21c}{128}.$$

Ex. 7. *Find the centre of gravity of the volume cut off from the first octant by the plane* $x+2y+4z=8$.

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume. We have

$$\bar{x}=\frac{\iiint x\, dx\, dy\, dz}{\iiint dx\, dy\, dz},\quad \bar{y}=\frac{\iiint y\, dx\, dy\, dz}{\iiint dx\, dy\, dz},$$

$$\bar{z}=\frac{\iiint z\, dx\, dy\, dz}{\iiint dx\, dy\, dz},$$

where x, y, z have any positive values subject to the condition

$$x/8+y/4+z/2 \leqslant 1.$$

Put $x/8=u$ *i.e.*, $x=8u$ so that $dx=8du$,

$y/4=v$ *i.e.*, $y=4v$ so that $dy=4dv$,

$z/2=w$ *i.e.*, $z=2w$ so that $dz=2dw$.

$$\therefore\ \bar{x}=\frac{\iiint 8u.64\, du\, dv\, dw}{\iiint 64\, du\, dv\, dw},$$ where u, v, w have any positive values subject to the condition $u+v+w \leqslant 1$

$$=8.\frac{\iiint u^{2-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw}{\iiint u^{1-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw}$$

$$=8.\,\frac{\Gamma(2)\,\Gamma(1)\,\Gamma(1)}{\Gamma(2+1+1+1)}\div\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)}$$

$$=8.\,\frac{\Gamma(2)}{\Gamma(5)}\times\frac{\Gamma(4)}{\Gamma(1)}=8.\,\frac{1}{4.\ 3.\ 2.\ 1}\times\frac{3.\ 2.\ 1}{1}=2.$$

Also $$\bar{y}=\frac{\iiint 4v.64\, du\, dv\, dw}{\iiint 64\, du\, dv\, dw}=\frac{4\iiint v\, du\, dv\, dw}{\iiint du\, dv\, dw}$$

$$=\tfrac{1}{2}\bar{x}=\tfrac{1}{2}.2=1,$$

and $$\bar{z}=\frac{\iiint 2w.64\, du\, dv\, dw}{\iiint 64\, du\, dv\, dw}=\frac{2\iiint w\, du\, dv\, dw}{\iiint du\, dv\, dw}$$

$$=\tfrac{1}{4}\bar{x}=\tfrac{1}{4}.2=\tfrac{1}{2}.$$

$$\therefore\ \bar{x}=2,\ \bar{y}=1,\ \bar{z}=\tfrac{1}{2}.$$

Ex. 8. *Find the centroid of the volume included between the following surfaces* $x^2/a^2+y^2/b^2-z^2/c^2=1$, $x=0$ *and* $z=\pm c$.

(Agra 1983)

Sol. Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume. Obviously the given volume is symmetrical about the planes $y=0$ and $z=0$. Therefore $\bar{y}=0$, $\bar{z}=0$. We have

$$\bar{x}=\frac{\iiint x\, dx\, dy\, dz}{\iiint dx\, dy\, dz}.$$

The limits of integration for y are from

$$-b\sqrt{(1+z^2/c^2-x^2/a^2)} \text{ to } b\sqrt{(1+z^2/c^2-x^2/a^2)},$$

or x from 0 to $a\sqrt{(1+z^2/c^2)}$, and for z from $-c$ to c.

First we are to integrate w.r.t. y, next w.r.t. x and then w.r.t. z.

$$\therefore \quad \bar{x}=\frac{\iint 2x.b\sqrt{\left(1+\frac{z^2}{c^2}-\frac{x^2}{a^2}\right)}\,dx\,dz}{\iint 2b\sqrt{\left(1+\frac{z^2}{c^2}-\frac{x^2}{a^2}\right)}\,dx\,dz}, \text{ integrating w.r.t. } y$$

$$=\frac{\iint\left(-\frac{a^2}{2}\right)\left(-\frac{2x}{a^2}\right)\left(1+\frac{z^2}{c^2}-\frac{x^2}{a^2}\right)^{1/2}dx\,dz}{\iint\frac{1}{a}\sqrt{\left\{a^2\left(1+\frac{z^2}{c^2}\right)-x^2\right\}}\,dx\,dz}$$

$$=\frac{-\frac{a^3}{2}\int_{z=-c}^{c}\left[\frac{2}{3}\left(1+\frac{z^2}{c^2}-\frac{x^2}{a^2}\right)^{3/2}\right]_{x=0}^{a\sqrt{(1+z^2/c^2)}}dz}{\int_{z=-c}^{c}\left[\frac{x}{2}\sqrt{\left\{a^2\left(1+\frac{z^2}{c^2}\right)-x^2\right\}}+\frac{a^2}{2}\left(1+\frac{z^2}{c^2}\right)\sin^{-1}\left\{\frac{x}{a\sqrt{(1+z^2/c^2)}}\right\}\right]_{x=0}^{a\sqrt{(1+z^2/c^2)}}dz}$$

$$=\frac{\frac{a^3}{3}\int_{-c}^{c}\left(1+\frac{z^2}{c^2}\right)^{3/2}dz}{\int_{-c}^{c}\left[\frac{a^2}{2}\left(1+\frac{z^2}{c^2}\right)\sin^{-1}1\right]dz}=\frac{2a}{3}\,\frac{\int_0^c\left(1+\frac{z^2}{c^2}\right)^{3/2}dz}{\int_0^c\left(1+\frac{z^2}{c^2}\right)\frac{\pi}{2}\,dz}$$

$$=\frac{4a}{3\pi}\,\frac{\int_0^{\pi/4}\sec^3\theta.c\sec^2\theta\,d\theta}{\left[z+\frac{z^3}{3c^2}\right]_0^c},$$

putting $z=c\tan\theta$ in the numerator

$$=\frac{4a}{3\pi}\,\frac{c\int_0^{\pi/4}\sec^5\theta\,d\theta}{\frac{4}{3}c}=\frac{a}{\pi}\int_0^{\pi/4}\sec^5\theta\,d\theta.$$

But $\quad \int\sec^n\theta\,d\theta=\frac{\sec^{n-2}\theta\tan\theta}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}\theta\,d\theta.$

$$\therefore \quad \int_0^{\pi/4}\sec^5\theta\,d\theta=\left[\frac{\sec^3\theta\tan\theta}{4}\right]_0^{\pi/4}+\frac{3}{4}\int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$=\frac{2\sqrt{2}}{4}+\frac{3}{4}\left[\left\{\frac{\sec\theta\tan\theta}{2}\right\}_0^{\pi/4}+\frac{1}{2}\int_0^{\pi/4}\sec\theta\,d\theta\right]$$

$$=\frac{2\sqrt{2}}{4}+\frac{3}{4}\,\frac{\sqrt{2}}{2}+\frac{3}{8}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$=\frac{7\sqrt{2}}{8}+\frac{3}{8}\Big[\log(\sec\theta+\tan\theta)\Big]_0^{\pi/4}$$

$$=\frac{1}{8}\{7\sqrt{2}+3\log(1+\sqrt{2})\}.$$

$$\therefore \quad \bar{x}=\frac{a}{8\pi}\{7\sqrt{2}+3\log(1+\sqrt{2})\}.$$