Notes

Unit 2: Linear Programming Problems

CONTENTS

Objectives

Introduction

2.1 Basic Terminology

2.2 Application of Linear Programming

2.3 Advantages and Limitations of Linear Programming

2.4 Formulation of LP Models

2.5 Maximization Cases with Mixed Constraints

2.6 Graphical Solutions under Linear Programming

2.7 Minimization Cases of LP

2.8 Cases of Mixed Constraints

2.9 Summary

2.10 Keywords

2.11 Review Questions

2.12 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand what is linear programming
- Locate areas of application with its scope
- Know how to formulate LP models and use graphical procedure to solve them

Introduction

Each and every organization aspires for optimal utilization of its limited scarce resources like men, money, materials, machines, methods and time to reach the targets. The results are generally measured in terms of profits, losses, return on money invested, etc. To achieve these results, the decision-maker has to have thorough knowledge about the tasks or jobs and the relationships among them. Among the popular techniques of Operations Research, Linear Programming deserves mention because it is one of the widely used techniques. And it is a deterministic model. In other words, Linear Programming is one of the important Operations Research tools used to allocate scarce resources in an optimal way so that the allocator can optimize the results either by maximizing the profits or minimizing the costs. The credit of innovating this technique goes to George B. Dantzig. He innovated this technique while he was working for U.S. Air force during World War II, 1947. Initially, this technique was used to solve tough logistic problems like assignment and transportation but instantly the application of this technique has spread to almost every functional area of management, production planning and control, personnel management, advertising and promotion.

Notes

Linear Programming decisions are made obviously under certainty conditions i.e., when the existing situation and the variables are known. The results obtained will be either optimal or nearly optimal. It even helps in cross verification of the results obtained through the process of mere intuition and the one arrived at with the use of Linear Programming technique while an optimum solution is being anticipated.

The general Linear Programming Problem calls for optimizing (maximizing/minimizing) a linear function for variables called the 'objective function' subject to a set of linear equations and/or inequalities called the 'constraints or restrictions.'

2.1 Basic Terminology

The word 'linear' is used to describe the relationship among two or more variables which are directly or precisely proportional.

Programming' means the decisions which are taken systematically by adopting alternative courses of action.

Basic Requirements and their Relationships

1. ***Decision Variables and their Relationships:*** The decision variable refers to any candidate (person, service, projects, jobs, tasks) competing with other decision variables for limited resources. These variables are usually interrelated in terms of utilization of resources and need simultaneous solutions, i.e., the relationship among these variables should be linear.
2. ***Objective Function:*** The Linear Programming Problem must have a well defined objective function to optimize the results. For instance, minimization of cost or maximization of profits. It should be expressed as linear function of decision variables ($Z = X_1 + X_2$, where Z represents the objective, i.e., minimization/maximization, X_1 and X_2 are the decision variables directly affecting the Z value).
3. ***Constraints:*** There would be limitations on resources which are to be allocated among various competing activities. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.
4. ***Alternative Courses of Action:*** There must be presence of alternative solutions for the purpose of choosing the best or optimum one.
5. ***Non-Negativity Restrictions:*** All variables must assume non-negative values. If any of the variable is unrestricted in sign, a tool can be employed which will enforce the negativity without changing the original information of a problem.
6. ***Linearity and Divisibility:*** All relationships (objective function and constraints) must exhibit linearity i.e., relationship among decision variables must be directly proportional. It is assumed that decision variables are continuous, i.e., fractional values of variables must be permissible in obtaining the optimum solution.
7. ***Deterministic:*** In Linear Programming it is assumed that all model coefficients are completely known. For example: profit per unit.

2.2 Application of Linear Programming

LP is a widely used technique of OR in almost every decision of a business and management. However, Linear Programming is exclusively used in the following areas:

1. Production Management

2. Personnel Management
3. Financial Management
4. Marketing Management

Apart from these areas it is used in agricultural operations, farm management and military problems.

1. ***Production Management:*** In the area of production management, Linear Programming is used in the field of:
   - Product planning
   - Research and development
   - Product portfolio management
   - Line expansion and contraction decision
   - Longevity of product life cycle.
2. ***Personnel Management:*** In this area, LP is used in the field of:
   - Recruitment and staffing decisions
   - Wage or salary management
   - Job evaluation and allocation
   - Employee benefits and welfare
   - Overtime and related decisions.
3. ***Financial Management:*** In this area, LP is used in the field of:
   - Portfolio decision
   - Profit planning
   - Alternative capital investment decisions
   - Investment on inventories
   - Allocation of funds to developmental activities.
4. ***Marketing Management:*** In the area of marketing management, it is used in the field of:
   - Media planning and selection
   - Travelling salesman problem
   - Product development
   - Ad and Pro budget
   - Marketing mix decisions.

2.3 Advantages and Limitations of Linear Programming

The LP technique has several advantages and at the same time, it is not free from unmixed blessings. They have been put into T-type classification as under:

Notes

Table 2.1: T-type Classification of Advantages and Limitations

Advantages	Limitations
1. It helps in proper and optimum utilization of the scarce resources	1. The treatment of variables having non-linear relationships is the greatest limitation of this LP
2. It helps in improving the quality of the decisions.	2. It can come out with non-integer solutions too, which would be many a times meaningless.
3. With the use of this technique, the decision-maker becomes more objective and less subjective.	3. It rules out effect of time and uncertainty conditions.
4. It even helps in considering other constraints operating outside the problem.	4. Generally, the objective set will be single and on the contrary, in the real life, there might be several objectives.
5. Many a times it hints the manager about the hurdles faced during the production activities.	5. Large-scale problems tend to be unaccomodative to solve under LP

2.4 Formulation of LP Models

Linear Programming Family

The family of LP consists of:

1. Formulation of Linear Programming Problems. (LPP)
2. LP – Graphical Solutions
3. LP – Simplex Solutions
4. LP – Assignment Problems
5. LP – Transportation Problems.

Steps for Formulating LPP

1. Identify the nature of the problem (maximization/minimization problem).
2. Identify the number of variables to establish the objective function.
3. Formulate the constraints.
4. Develop non-negativity constraints.

Example: A firm manufactures 2 types of products A & B and sells them at a profit or ₹ 2 on type A & ₹ 3 on type B. Each product is processed on 2 machines G & H. Type a requires 1 minute of processing time on G and 2 minutes on H. Type B requires one minute on G & 1 minute on H. The machine G is available for not more than 6 hrs. 40 mins., while machine H is available for 10 hrs. during any working day. Formulate the problem as LPP.

Solution:

Let x_1 be the no. of products of type A

x_2 be the no. of products of type B

Notes

Table 2.2: Showing the Time (minutes) Available and the Profit

Machines	Time on Products (mins.)		Total time available (in minutes)
	Type A	Type B	
G	1	1	400
H	2	1	600
Profit Per Unit	₹ 2	₹ 3	

Since the profit on type A is ₹ 2 per product, $2x_1$ will be the profit on selling x_1 units of type A.

Similarly $3x_2$ will be the profit on selling x_2 units of type B.

Hence the objective function will be,

Maximize 'Z' = $2x_1 + 3x_2$ is subject to constraints.

Since machine 'G' takes one minute on 'A' and one minute on 'B', the total number of minutes required is given by $x_1 + x_2$. Similarly, on machine 'H' $2x_1 + x_2$. But 'G' is not available for more than 400 minutes. Therefore, $x_1 + x_2 \leq 400$ and H is not available for more than 600 minutes, therefore, $2x_1 + x_2 \leq 600$ and $x_1, x_2 \geq 0$, i.e.,

$x_1 + x_2 \leq 400$ (Time availability constraints)

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Example: A company produces 2 types of cowboy hats. Each hat of the first type requires twice as much labour time as the second type. The company can produce a total of 500 hats a day. The market limits the daily sales of first and second types to 150 and 250 hats. Assuming that the profits per hat are ₹ 8 per type A and ₹ 5 per type B, formulate the problem as Linear Programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution:

Let x_1 be the no. of hats of type A.

x_2 be the no. of hats of type B.

$8x_1$ is the total profit for hats of type A.

$5x_2$ is the total profit for hats of type B.

Hence, objective function will be equal to

Maximise 'Z' = $8x_1 + 5x_2$ (Subject to constraints)

$2x_1 + x_2 \leq 500$ (Labour time for total production)

$x_1 \leq 150$ (No. of hats of type A to be sold)

$x_2 \leq 250$ (No. of hats of type B to be sold)

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Notes

Example: A firm can produce 3 types of cloth say A, B and C. Three kinds of wool are required for it say red, green and blue. One unit length of type A cloth needs 2 metres of red wool and 3 metres of blue wool. One unit length of B type cloth needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool; and 1 unit length of type C cloth needs 5 metres of green wool and 4 metres of blue wool. The firm has a stock of 8 metres of red wool, 10 metres of green wool and 15 metres of blue wool, it is assumed that the income obtained from one unit length of type A cloth is ₹ 3, of B ₹ 5 and of C ₹ 4.

Determine how the firm should use the available material so as to maximize the income from the finished cloth. Formulate the above problem as LPP.

Solution:

Let x_1 be the type of cloth A

x_2 be the type of cloth B

x_3 be the type of cloth C

Therefore $3x_1$ is the profit for type A cloth

$5x_2$ is the profit for type B cloth

$4x_2$ is the profit for type C cloth.

Materials	Cloth			Max. Material Available
	A	B	C	
Red	2	3	--	8
Blue	3	2	4	15
Green	--	2	5	10
Profit per unit (₹)	3	5	4	

The Objective function is given by

Maximize $'Z' = 3x_1 + 5x_2 + 4x_3$ (Subject to constraints)

$2x_1 + 3x_2 \le 8$ (Material Constraint)

$3x_1 + 2x_2 + 4x_3 \le 15$ (Material Constraint)

$2x_2 + 5x_3 \le 10$ (Material Constraint)

$x_1, x_2, x_3 \le 0$ (Non-negativity constraints)

Example: A firm manufactures 3 types of products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine from product.

Machine	Product		
	A	B	C
C	4	3	5
D	3	2	4

Machine C and D have 2,000 and 2,500 machine minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's.

Formulate an LPP to maximize the profit.

Solution:

Let x_1 be the type of product A

x_2 be the type of product B

x_3 be the type of product C

Therefore the objective function will be,

Maximize 'Z' = $3x_1 + 2x_2 + 4x_3$ (Subject to constraints)

$4x_1 + 3x_2 + 5x_3 \leq 2{,}000$

$3x_1 + 2x_2 + 4x_3 \leq 2{,}500$ (Machine hour constraints)

$x_1 \leq 150$ or $0 \leq x_1 \leq 150$

$0 \leq x_2 \leq 200$ (Production constraint)

$0 \leq x_3 \leq 50$

$x_1, x_2, x_3 \leq 0$ (Non-negativity constraints)

Self Assessment

Multiple Choice Questions:

1. Linear Programming is a
   (a) constrained optimization technique
   (b) Technique for economic allocation of limited resource
   (c) Mathematical technique
   (d) All of the above
2. A constraint in a LP model restricts
   (a) Value of an objective function (b) Value of a decision variable
   (c) Use of the available resources (d) All of the above
3. Constraints of an LP model represents
   (a) Limitations
   (b) Requirements
   (c) Balancing limitations and requirements
   (d) All of the above

2.5 Maximization Cases with Mixed Constraints

Example: The manager of an oil-refinery must decide on the optimal mix of 2 possible blending processes, of which the input and output per production run are given as follows:

Process	Input (units)		Output (units)	
	Crude 'A'	Crude 'B'	Gasoline X	Gasoline Y
I	5	3	5	8
II	4	5	4	4

Notes

The maximum amount available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process I and process II are ₹ 300 and ₹ 400 respectively. Formulate the above problem as LPP.

Solution:

Let x_1 represent process I

x_2 represent process II

Therefore, 300 x1 represent profit on process I

$400x_2$ represent profit on process II

Hence, the objective function is given by,

Maximize 'Z' = $300x_1 + 400x_2$ (Subject to constraints

$5x_1 + 4x_2 \le 200$

$3x_1 + 5x_2 \le 150$ (Crude Oil constraints)

$5x_1 + 4x_2 \ge 100$

$8x_1 + 4x_1 \ge 80$ (Gasoline constraints)

$x_1, x_2 \le 0$ (Non-negativity constraints)

Example: The management of xyz corporation is currently faced with the problem of determining its product mix for the coming period. Since, the corporation is one of the few suppliers of transformers for laser cover units, only liberal sales ceilings are anticipated. The corporation should not plan on selling transformers more than 200 units of A type, 100 units of B type and 180 units of C type. Contracts call for production of at least 20 units of A type and 70 units of C type. Within these bounds, management is free to establish units production schedules. These are subject to the capacity of the plant to produce without overtime. The production times prevail.

Product	Production Hours per unit				Unit Profit (₹)
	Dept. I	Dept. II	Dept. III	Dept. IV	
A	0.10	0.06	0.18	0.13	10
B	0.12	0.05	---	0.10	12
C	0.15	0.09	0.007	0.08	15
Available Hours	36	30	37	38	

Formulate this as an LPP so as to maximize the total profit.

Solution:

Let x_1 be the units of A type.

x_2 be the units of B type.

x_3 be the units of C type.

Therefore $10x_1$ be the profit of A type.

$12x_2$ be the profit of B type.

$15x_3$ be the profit of C type.

Notes

Hence, the objective function is given by,

Maximize 'Z' = $10x_1 + 12x_2 + 15x_3$ (Subject to constraints)

$0.10x_1 + 0.12x_2 + 0.15x_3 \leq 36$

$0.12x_1 + 0.05x_2 + 0.09x_3 \leq 30$ (Production hours constraints)

$0.18x_1 + 0 + 0.07x_3 \leq 37$

$0.13x_1 + 0.10x_2 + 0.08x_3 \leq 38$

$x_1 \leq 200$

$x_2 \leq 100$ (Sales constraints)

$x_3 \leq 180$

$x_1 \geq 20$ (Production constraints)

$x_2 \geq 70$

OR

$20 \leq x_1 \leq 200$

$70 \leq x_3 \leq 180$ (Sales and Production constraints)

$x_2 \leq 100$

$x_1, x_2, x_3 \leq 0$ (Non-negativity constraints)

Example: An advertising company wishes to plan advertising campaign in 3 different media, television, radio and magazine. The purpose of advertising is to reach as many potential consumers as possible. Results of a marketing study are given below:

Particulars	Television		Radio	Magazines
	Prime Day	Prime Time		
Cost of an Advertising Unit (₹)	40,000	75,000	30,000	15,000
No. of Potential Customers Reached per Unit	4,00,000	9,00,000	5,00,000	2,00,000
No. of Women Customers Reached per Unit	3,00,000	4,00,000	2,00,000	1,00,000

The company doesn't want to spend more than ₹ 8,00,000 on advertising. It further requires that:

a. At least 2 million exposures take place among women.

b. Advertising on television be limited to ₹ 5,00,000.

c. At least 3 advertising units can be bought on prime day and 2 units prime time.

d. The number of advertising units on radio and magazine should each be between 5 and 10.

Formulate the Linear Programming model in order to maximize the total number of potential customers reached.

Notes

Solution:

Let x_1 represent the advertising on prime day on television

x_2 represent the advertising on prime time on television

x_3 represent the campaign on radio

x_4 represent the campaign on magazine.

Therefore $4,00,000x_1$ represent the potential customers on advertising on prime day on television.

$9,00,000x_2$ represent the potential customers on prime time on television.

$5,00,000x_3$ be the potential customers on advertising on Radio.

$2,00,000x_4$ be the potential customers on advertising in Magazines.

Hence the objective function is given by

Maximum 'Z' = $4,00,000x_1 + 9,00,000x_2 + 5,00,000x_3 + 2,00,000x_4$ (Subject to constraints)

$40,000x_1 + 75,000x_2 + 30,000x_3 + 15,000x_4 \leq 8,00,000$ (Advertising constraint)

$40,000x_1 + 75,000x_2 \leq 5,00,000$ (Advertising on television constraint)

$3,00,000x_1 + 4,00,000x_2 + 2,00,000x_3 + 1,00,000x_4 \leq 2$ Million (No. of women customers constraint)

$x_1 \geq 3$

$x_2 \geq 2$ No. of unit constraints)

$5 \leq x3 \leq 10$

$5 \leq x_4 \leq 10$ (Minimum no. of advertisements allowed constraints)

Therefore $x_1, x_2, x_3, x_4 \leq 0$ (Non-negativity constraints)

Example: A city hospital has the following daily requirements of nurses at the minimal level:

Period	Clock Time (24 hours a day)	Minimal no. of nurses required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m. – 6 a.m.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The wants to determine minimal number of nurses to be employed, so that there will be sufficient number of nurses available for each period.

Formulate this as LP model by setting up appropriate constraints and objective function.

Notes

Solution:

Let x_1 be the no. of nurses working during period 1

x_2 be the no. of nurses working during period 2, and

x_3, x_4, x_5 and x_6 be the no. of nurses working during period 3,4,5, and 6 respectively.

Hence, the objective function is given by,

Minimise $\quad$ 'C' = $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ $\quad$ (Subject to constraints)

$$x_1 + x_2 \geq 2$$
$$x_2 + x_3 \geq 7$$
$$x_3 + x_4 \geq 15$$
$$x_4 + x_5 \geq 8$$
$$x_5 + x_6 \geq 20$$
$$x_6 + x_1 \geq 6$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$ (Non-negativity constraints)

Notes Steps of linear programming model formulation are summarized as follows:

Step 1: Identify the decision variables

Step 2: Identify the problem data

Step 3: Formulate the constraints

Step 4: Formulate the Objective Function

Task **Formulate the following as LPP**

One of the interesting problems in Linear Programming is that of balanced diet. Dieticians tell us that a balanced diet must contain certain quantities of nutrients such as proteins, minerals, vitamins, etc. Suppose that you are asked to find out the food that should be recommended from a large number of alternative sources of these nutrients, so, that the total cost of food satisfying minimum requirements of balanced diet is the lowest. The medical experts and the dieticians tell us that it is necessary for an adult to consume at least 75 gms. of proteins, 85 gms. of fats and 300 gms. of carbohydrates daily. The following table gives the different items (which are readily available in the market); Item analysis and their respective costs.

Formulate this problem as LPP

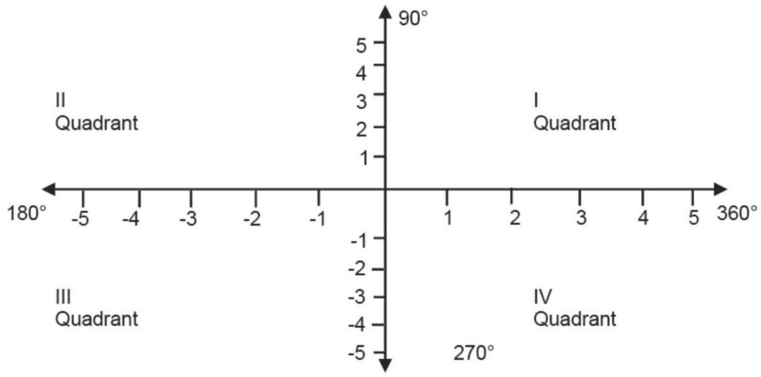
Food	Food volume (per 100 gms.)			Cost/Kg. (₹)
	Proteins	Fats	Carbohydrates	
1	8	1.5	35	1
2	18	15	__	3
3	16	4	7	4
4	4	20	2.5	2
5	5	8	40	1.5
6	2.5	__	25	3
Minimum daily requirements	75	85	300	

Notes

2.6 Graphical Solutions under Linear Programming

Linear programming problems with two variables can be represented and solved graphically with ease. Though in real-life, the two variable problems are practiced very little, the interpretation of this method will help to understand the simplex method.

Following is the portrayal of Cartesian plane.



Cartisian Plane

Steps

1. Consider each inequality constraint as an equation.
2. Plot each equation on the graph as each will geometrically represent a straight line.
3. Plot the feasible region, every point on the line will satisfy the equation on the line.
4. If the inequality constraint corresponding to that line is less than or equal to, then the region below the line lying in the 1st quadrant (as shown in above graph) is shaded (due to non-negativity of variables); for the inequality constraint with greater than or equal to sign, the region above the line in the 1st quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. Hence, it is called feasible region.
5. Identify the co-ordinates of the corner points.
6. Find the 'Z' value by substituting the co-ordinates of corner points to the objective functions.

Example:

Maximize 'Z' = $3x_1 + 5x_2$ (Subject to constraints)

$$x_1 + 2x_2 \leq 2{,}000$$

$$x_1 + x_2 \leq 1{,}500$$

$$x_2 \leq 600$$

$$x_1, x_2 \leq 0$$ (Refer Problem No.4 in the earlier portion of the unit)

Solution:

Step 1: Convert the inequalities into equalities and find the divisibles of the equalities.

Equation	x_1	x_2
$x_1 + 2x_2 = 2{,}000$	2,000	1,000
$x_1 + x_2 = 1{,}500$	1,500	1,500
$x_2 = 600$	---	600

Notes

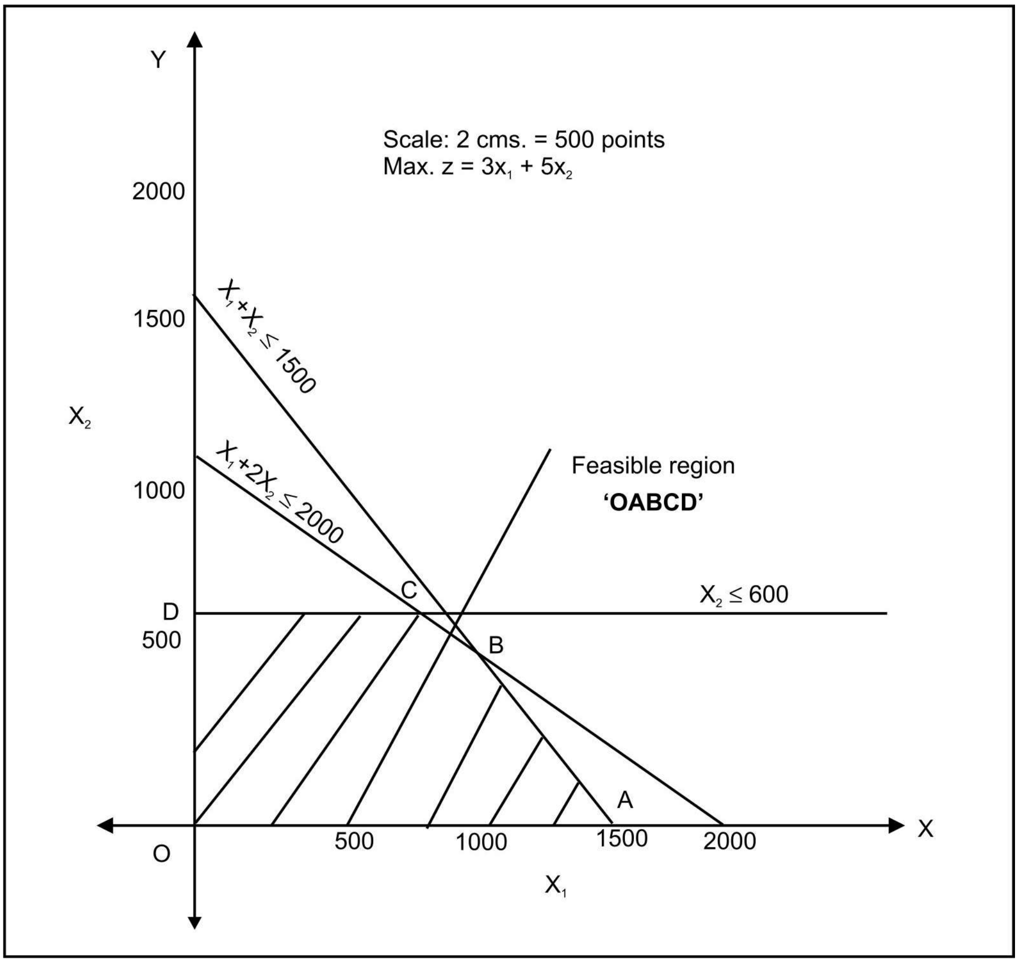
Step 2: Fix up the graphic scale.

Maximum points = 2,000

Minimum points = 600

2 cms = 500 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	X_1	x_2
O	0	0
A	1,500	0
B	1,000	500
C	800	600
D	0	600

At 'B': $x_1 + 2x_2 = 2{,}000$ (1)

$x_1 + x_2 = 1{,}500$ (2)

$x_2 = 500$

Notes

Put $x_2 = 500$ in eq. (1),

$$x_1 + 2(500) = 2{,}000$$

Therefore $x_1 = 2{,}000 - 1{,}000$

Therefore $x_1 = 1{,}000$

At 'C': $x_1 + 2x_2 = 2{,}000$ ....(1)

$x_2 = 600$ ....(2)

Put $x_2 = 600$ in eq. (1),

$$x_1 + 2(600) = 2{,}000$$

$$x_1 = 2{,}000 - 1{,}200$$

Therefore $x_1 = 800$

Step 5: Substitute the co-ordinates of corner points into the objective function.

Maximise 'Z' $= 3x_1 + 5x_2$

At 'O', $Z = 0 + 0 = 0$

At 'A', $Z = 3\ (1{,}500) + 5\ (0) = 4{,}500$

At 'B', $Z = 3\ (1{,}000) + 5\ (500) = 5{,}500$

At 'C', $Z = 3\ (800) + 5\ (600) = 5{,}400$

At 'C', $Z = 3\ (0) + 5\ (600) = 3{,}000$

Inference

A maximum profit of ₹ 5,500 can be earned by producing 1,000 dolls of basic version and 500 dolls of deluxe version.

Example:

Maximise

'Z' $= 2x_1 + 3x_2$ (Subject to constraints)

$$x_1 + x_2 \le 400$$

$$2x_1 + x_2 \le 600$$

$x_1, x_2 \ge 0$ (Non-negativity constraints)

Solution:

Step 1: Find the divisible points on inequalities

Equation	x_1	x_2
$x_1 + x_2 = 400$	400	400
$2x_1 + x_2 = 600$	300	600

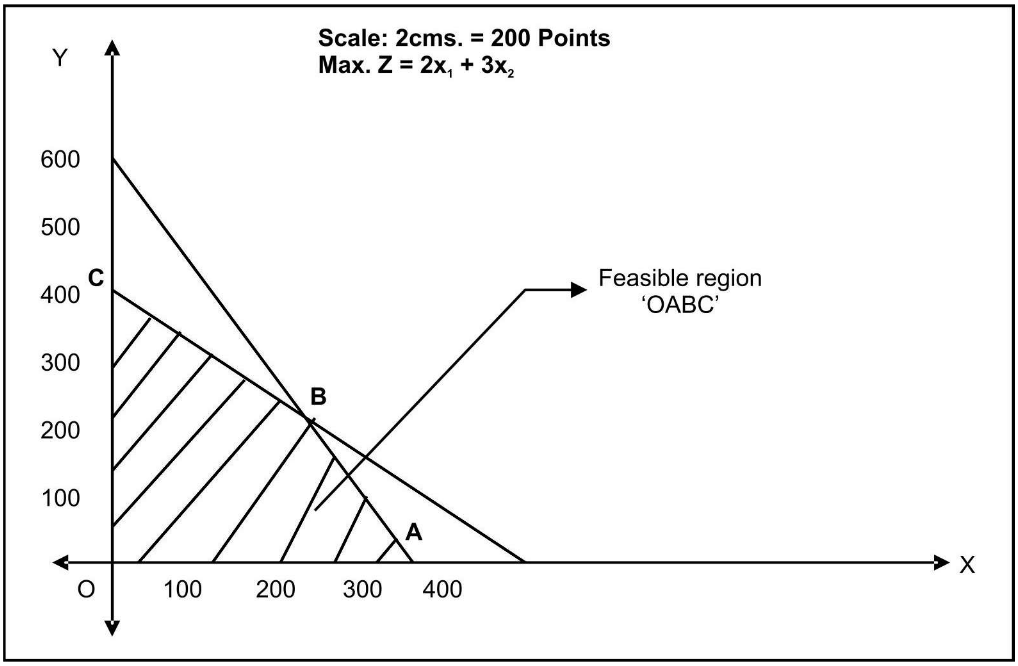
Step 2: Fix up the graphic scale

Maximum points = 600

Minimum points = 300

2 cms. = 200 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	X_1	x_2
O	0	0
A	300	0
B	200	200
C	0	400

Step 5: Subscribe the co-ordinates of the corner points into objective function

Maximise 'Z' = $2x_1 + 3x_2$

At 'O', Z = 2(0) + 3(0) = 0

At 'A', Z = 2(300) + 3(0) = 600

At 'B', Z = 2(200) + 2(200)= 1,000

At 'C', Z = 2(0) + 3(400) = 1,200

Inference

A maximum profit of ₹ 1,200 can be earned by producing 400 units of only type B and none of type A.

Notes

Example:

Maximise $'Z' = 8x_1 + 5x_2$ [Subject to constraints]

$$x_1 \leq 150$$
$$x_2 \leq 250$$
$$2x_1 + x_2 \leq 500$$
$$x_1, x_2 \geq 0$$

Solution:

Step 1: Convert the inequalities into equalities and find the divisibles.

Equation	x_1	X_2
$2x_1 + x_2 = 500$	250	500
$x_1 = 150$	150	-----
$x_2 = 250$	-----	250

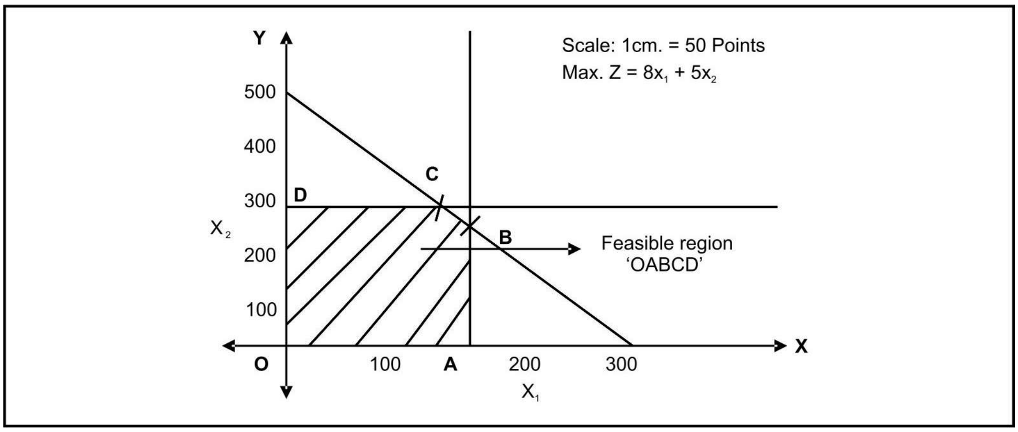
Step 2: Fix up the graphic scale

Maximum points = 500

Minimum points = 150

1 cm. = 50 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points.

Corner Points	x_1	X_2
O	0	0
A	150	0
B	150	200
C	125	150
D	0	250

At 'B': $2x_1 + x_2 = 500$ ....(1)

$x_1 = 150$ ....(2)

Notes

Put $x_1 = 150$ in eq. (1)

$$2(150) + x_2 = 500$$

$$x_2 = 500 - 300$$

$$x_2 = 200$$

$$2x_1 + x_2 = 2500 \quad ....(1)$$

$$x_2 = 2500 \quad ....(2)$$

Put $x_2 = 2500$ in eq. (1),

$$2x_1 + 250 = 500$$

$$2x_1 = 250$$

$$x_1 = 125$$

Step 5: Substitute the co-ordinates of corner points into the objective function.

Maximise $'Z' = 8x_1 + 5x_2$

At 'O', $Z = 8(0) + 5(0) = 0$

At 'A', $Z = 8(150) + 5(0) = 1,200$

At 'B', $Z = 8(150) + 5(200) = 2,200$

At 'C', $Z = 8(0) + 5(250) = 2,250$

At, 'D' $Z = 8(0) + 5(250) = 1,250$

Inference

Hence, to get a maximum profit of ₹ 2,250, the company has to manufacture 125 units of type 1 cowboy hats and 250 units of type 2 hats.

Example:

Maximise $'Z' = 8,000x_1 + 7,000x_2$ (Subject to constraints)

$$3x_1 + x_2 \leq 66$$

$$x_1 \leq 20$$

$$x_2 \leq 40$$

$$x_1 + x_2 \leq 45$$

$$x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}$$

Solution:

Step 1: Convert the inequalities into equalities and find the divisible.

Equation	x_1	x_2
$3x_1 + x_2 = 66$	22	66
$x_1 = 20$	20	0
$x_2 = 40$	0	40
$x_1 + x_2 = 45$	45	45

Notes

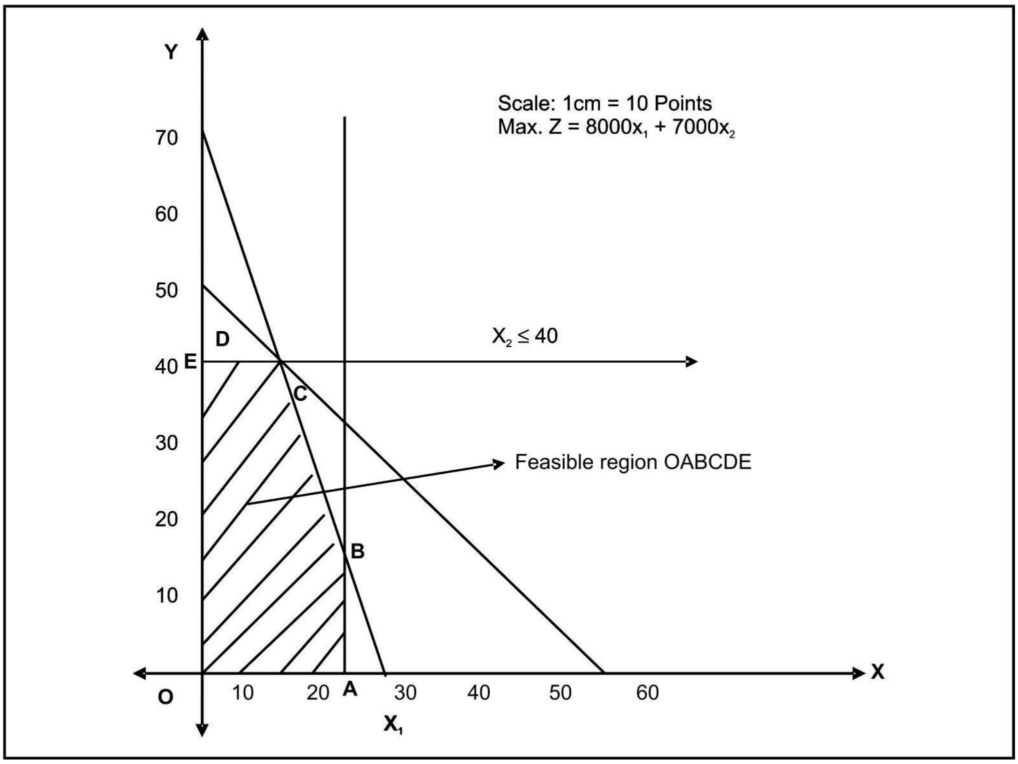
Step 2: Fix up the graphic scale

Maximum points = 66

Minimum points = 20

1 cm. = 10 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner points	x1	x2
O	0	0
A	20	0
B	20	6
C	10.5	34.5
D	5	40
E	0	40

At 'C' = $3x_1 + x_2 = 66$ …..(1)

$(-)\ x_1 + x_2 = 45$ …..(2)

$2x_1 = 21$

Therefore $x_1 = 10.5$

Substituting x_1 10.5 in eq. (2)

Therefore $x_1 + x_2 = 45$

Therefore $10.5 + x_2 = 45$

$x_2 = 34.5$

At 'B': $x_1 = 20$,

Therefore $3x_1 + x_2 = 66$

Therefore $3(20) + x_2 = 66$

Therefore $x_2 = 66 - 60$

Therefore $x_2 = 6$

Step 5: Substituting the co-ordinates of corner points into objective function.

Maximise 'Z' $= 8000x_1 + 7000x_2$

At 'O', Z = 8000(0) + 7000(0) = 0

At 'A', Z = 8000(20) + 7000(0) = 1,60,000

At 'B', Z = 8000(20) + 7000(6) = 2,02,000

At 'C', Z = 8000(10.5) + 7000(34.5) = 3,25,500

At 'D', Z = 8000(5) + 7000(40) = 3,20,000

At 'E', Z = 8000(0) + 7000(40) = 2,80,000

Inference

To maximize the profit, i.e., at ₹ 3,25,500 the company has to manufacture 10,500 bottles of type A medicine and 34,500 bottles of type B medicine.

Task Give the graphical solution for the following LPP

Maximize 'Z' $= 0.50x_2 - 0.10x_1$, (Subject to constraints)

$2x_1 + 5x_2 \leq 80$

$x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0$

Caution It is very much essential to locate the solution point of the LPP with respect to the objective function type (max or min). If the given problem is maximization, zm*ax* then locate the solution point at the far most point of the feasible zone from the origin and if minimization, Zmin then locate the solution at the shortest point of the solution zone from the origin.

Notes

Self Assessment

Fill in the blanks:

4. Graphical method can be used only if there are ………………..decision variables.
5. While solving a LP graphically the area bounded by the constraints is called …………….
6. If the given problem is maximization, Z_{max} then locate the solution point at the ……………point of the feasible zone from the origin

2.7 Minimization Cases of LP

Example: A rubber company is engaged in producing 3 different kinds of tyres A, B and C. These three different tyres are produced at the company's 2 different plants with different production capacities. In a normal 8 hrs working day plant 1 produces 50, 100 and 100 tyres of A, B and C respectively. Plant 2 produce 60, 60 and 200 tyres of type A, B and C respectively. The monthly demand for tyre A, B and C is 2,500, 3,000 and 7,000 units respectively. The daily cost of operation of plant 1 and 2 is ₹ 2,500 and ₹ 3,500 respectively. Find the minimum number of days of operation per month at 2 different plants to minimize the total costs while meeting the demand.

Solution:

Let x_1 be the daily cost of operation in plant 1

x_2 be the daily cost of operation in plant 2

Minimize 'Z' = $2{,}500x_1 + 3{,}500x_2$ (Subject to constraints)

$$50x_1 + 60x_2 \geq 2{,}500$$

$$100x_1 + 60x_2 \geq 3{,}000$$

$$100x_1 + 200x_2 \geq 7{,}000 \qquad \text{(Demand Constraints)}$$

$$x_1, x_2 \geq 0 \qquad \text{(Non-negativity constraints)}$$

Step 1: Find the divisible of the equalities.

Equation	x1	x2
$50x_1 + 60x_2 = 2{,}500$	50	41.67
$100x_1 + 60x_2 = 3{,}000$	30	50
$100x_1 + 200x_2 = 7{,}000$	70	35

Step 2: Fix up the graphic scale

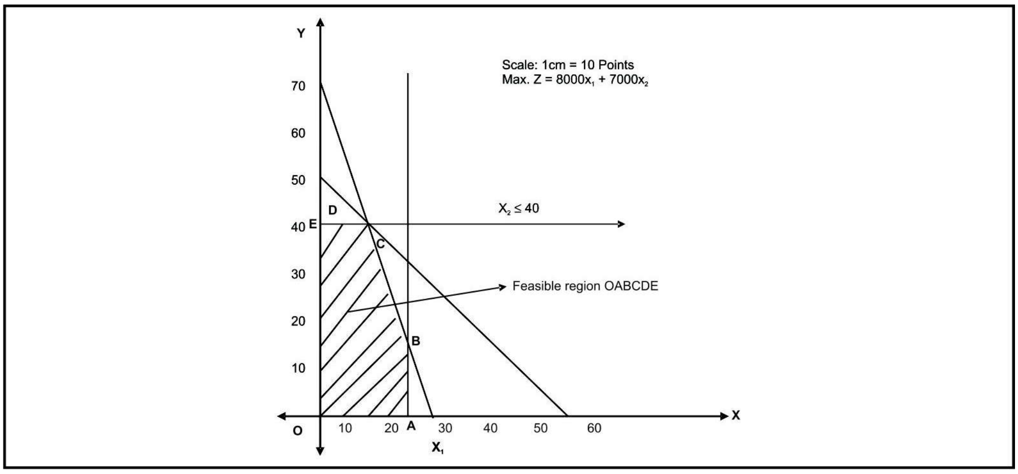
Minimum points = 30

Maximum points = 70

1 cm. = 10 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	70	0
B	20	25
C	10	33.33
D	0	50

At 'B' $100x_1 + 200x_2 = 7{,}000$ …..(1)

$50x_1 + 60x_2 = 2{,}500$ …..(2)

Divide eq. (1) by 2, we get,

$50x_1 + 100x_2 = 3{,}500$

$50x_2 + 60x_2 = 2{,}500$

$40x_2 = 1{,}000$

Therefore $x_2 = 25$

Put $x_2 = 25$ in eq. (2), $50x_1 + 60(25) = 2{,}500$

$50x_1 + 1{,}500 = 2{,}500$

$50x_1 = 1{,}000$

Therefore $x_1 = 20$

At 'C' = $100x_1 + 60x_2 = 3{,}000$ …..(1)

$50x_1 + 60x_2 = 2{,}500$ …..(2)

$50x_1 = 500$

Therefore $x_1 = 10$

Put $x_1 = 10$ in eq. (2)

$50(10) + 60x_2 = 2{,}500$

$60x_2 = 2{,}000$

Therefore $x_2 = 33.33$

Notes

Step 5: Substitute the co-ordinates of the corner points in the objective.

$$\text{Minimise 'Z'} = 2{,}500x_1 + 3{,}000x_2$$

At 'A', Z = 2,500(70) + 3,000(0) = 1,75,000

At 'B', Z = 2,500(20) + 3,000(25) = 1,25,000

At 'C', Z = 2,500(10) + 3,000(33.33) = 1,24,990

At 'D', Z = 2,500(0) + 3,000 (50) = 1,50,000

Inference

Thus, the rubber company can minimize its total cost to ₹ 1,24,990 by producing 10 units of product in plant 1 and 33.33 units in plant 2.

2.8 Cases of Mixed Constraints

Example: A firm that makes products x and y has a total production capacity of 9 tonnes per day, x and y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of x and 3 tonnes of y per day to another company. Each one of x requires 20 machine hrs. Production time and y requires 50 machine hrs production time. The daily maximum possible number of machine hours available is 360. All the firm's output can be sold, and the profit set is ₹ 80 per tonne of x and ₹ 120 per tonne of y. You are required to determine the production schedule to maximize the firm's profit.

Solution:

Let x_1 be the no. of tonnes of product 'X'

x_2 be the no. tonnes of product 'Y'

Hence, the objective function is given by,

Maximize 'Z' = $80x_1 + 120x_2$ (Subject to constraints)

$20x_1 + 50x_2 \leq 360$ (Machine hour constraint)

$x_1 \geq 2$

$x_2 \geq 3$ (Supply constraints)

$x_1 + x_2 \geq 9$ (Production constraint)

$x_1, x_2 \geq 0$ (Non-negativity constraint)

Step 1: Find the divisibles of the equalities.

Equation	x_1	x_2
$20x_1 + 50x_2 = 360$	18	7.2
$x_1 + x_2 = 9$	9	9
$x_1 = 2$	2	0
$x_2 = 3$	0	3

Step 2: Fix up the graphic scale

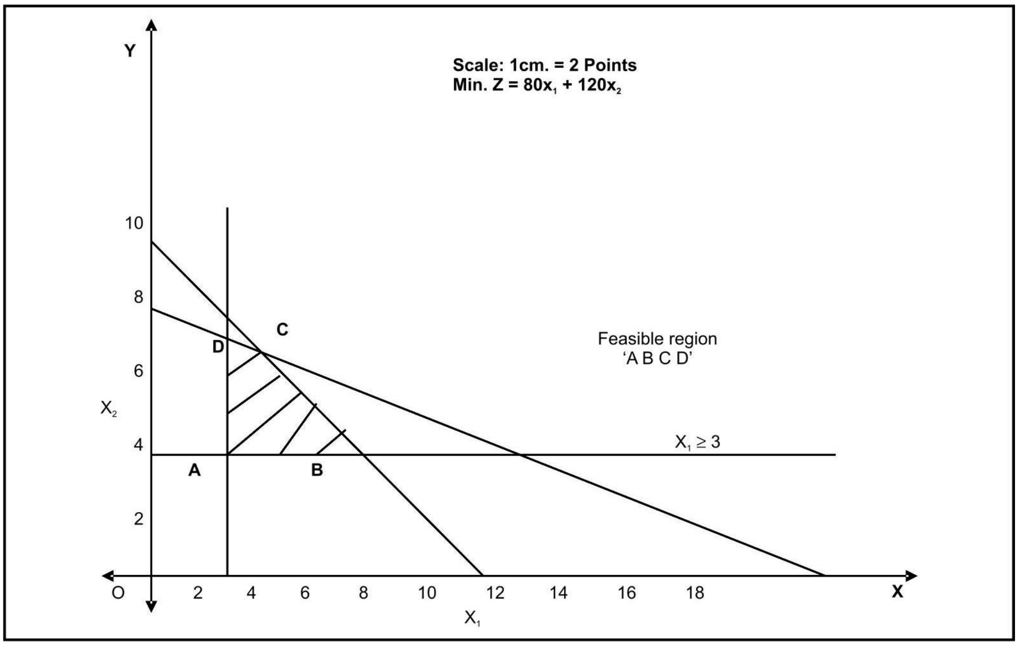
Minimum points = 2

Maximum points = 18

1 cm. = 2 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	2	3
B	6	3
C	3	6
D	2	6.4

At 'C': $20x_1 + 50x_2 = 360$ …..(1)

$x_1 + x_2 = 9$ …..(2)

Multiply eq. (2) by 20 and subtract,

$$20x_1 + 50x_2 = 360$$

$$20x_1 + 20x_2 = 180$$

$$30x_2 = 180$$

Therefore $x_2 = 6$

Put $x_2 = 6$ in eq. (2)

$$x_1 + 6 = 9$$

Therefore $x_1 = 3$

At 'D' $x_1 = 2$ …..(1)

$20x_1 + 50x_2 = 360$ …..(2)

Put $x_1 = 2$ in eq. (2)

$$20(2) + 50x_2 = 360$$

Notes

$$40 + 50x_2 = 360$$

$$50x_2 = 320$$

Therefore $x_2 = 6.4$

Steps 5: Substitute the values of co-ordinates of the corner points to the objective function.

Maximise $'Z' = 80x_1 + 120x_2$

$$\text{At 'A', } Z = 80(2) + 120(3) = 520$$

$$\text{At 'B', } Z = 80(6) + 120(3) = 840$$

$$\text{At 'C', } Z = 80(3) + 120(6) = 960$$

$$\text{At 'D' } Z = 80(2) + 120(6.4) = 928$$

Inference

The company has to produce 3 tonnes of product x and 6 tonnes of product y in order to maximize the profit.

Example:

Maximise $'Z' = 40x_1 + 60x_2$ (Subject to constraints)

$$2x_1 + x_2 \leq 70$$

$$x_1 + x_2 \geq 40$$

$$x_1 + 3x_2 \leq 90$$

$$x_1, x_2 \geq 0$$ (Non-negativity constraints)

Solution:

Step 1: Find the divisibles of the equalities.

Equation	x_1	X_2
$2x_1 + x_2 = 70$	35	70
$x_1 + x_2 = 40$	40	40
$x_1 + 3x_2 = 90$	90	30

Step 2: Fix up the graphic scale

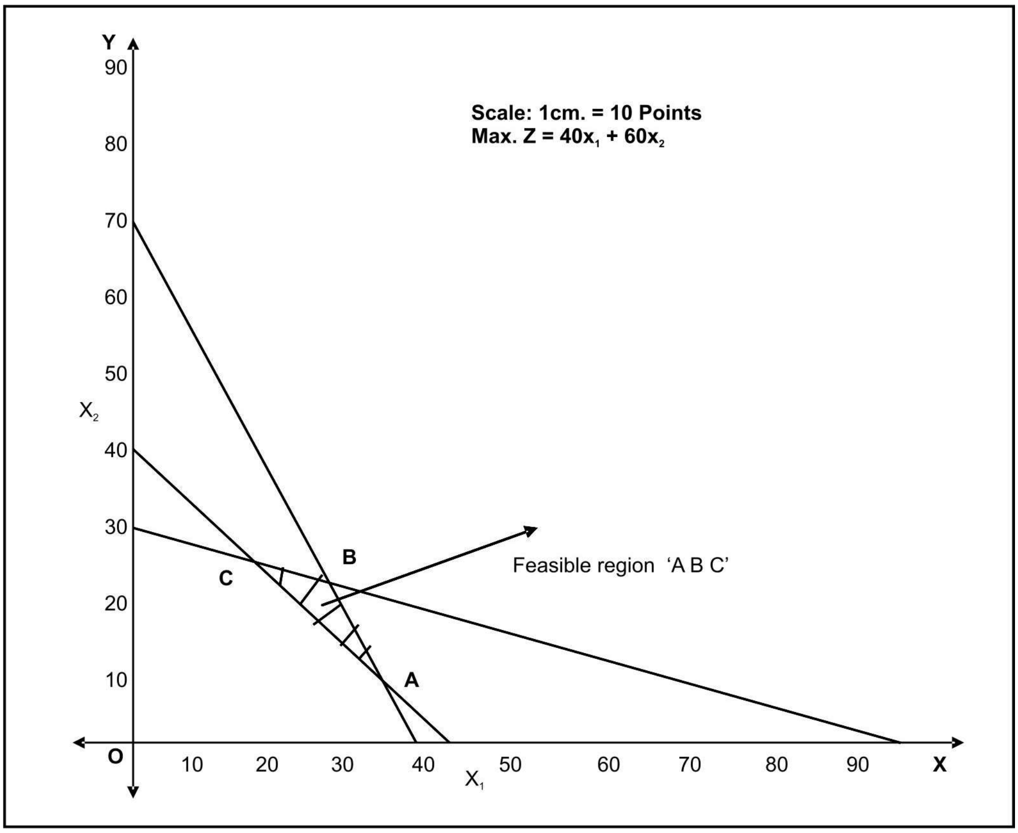
Maximum points = 90

Minimum points = 30

1 cm. = 10 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	30	10
B	24	22
C	15	25

At A: $2x_1 + x_2 = 70$ ….(1)

$x_1 + x_2 = 40$ …..(2)

Put $x_1 = 30$ in eq. (2),

$30 + x_2 = 40$

Therefore $x_2 = 10$

At 'B' = $x_1 + 3x_2 = 90$ …..(1)

$2x_1 + x_2 = 70$ …..(2)

Multiply eq. (1) by 2 and subtract,

$2x_1 + 6x_2 = 180$

$2x_1 + x_2 = 70$

$5x_2 = 110$

Notes

Therefore $x_2 = 22$

Put $x_2 = 22$ in eq. (1),

$x_1 + 3(22) = 90$

$x_1 = 90 - 66$

Therefore $x_1 = 24$

At 'C': $x_1 + x_2 = 40$ …..(1)

$x_1 + 3x_2 = 90$ …..(2)

$-2x_2 = -50$

Therefore $x_2 = 25$

Put $x_2 = 25$ in eq. (1),

$x_1 + 25 = 40$

$x_1 = 15$

Step 5: Substitute the co-ordinates of the corner points to the objective function

Maximise 'Z' = $40x_1 + 60x_2$

At 'A', Z = 40(30) + 60(10) = 1,800

At 'B', Z = 40(24) + 60(22) = 2,280

At 'C', Z = 40(15) + 60(25) = 2,100

Inference

Maximum profit can be obtained by producing 24 units of product A and 22 units of product B.

Example:

Maximise 'Z' = $7x_1 + 3x_2$ (Subject to constraints)

$x_1 + 3x_2 \geq 3$

$x_1 + x_2 \leq 4$

$x_1 \leq 5/2$ or 2.5

$x_2 \leq 3/2$ or 1.5

$x_1, x_2 \geq 0$ (Non-negativing constraints)

Solution:

Step 1: Find the divisibles of the equalities

Equation	x_1	x_2
$x_1 + 3x_2 = 3$	3	1
$x_1 + x_2 = 4$	4	4
$x_1 = 2.5$	2.5	0
$x_2 = 1.5$	0	1.5

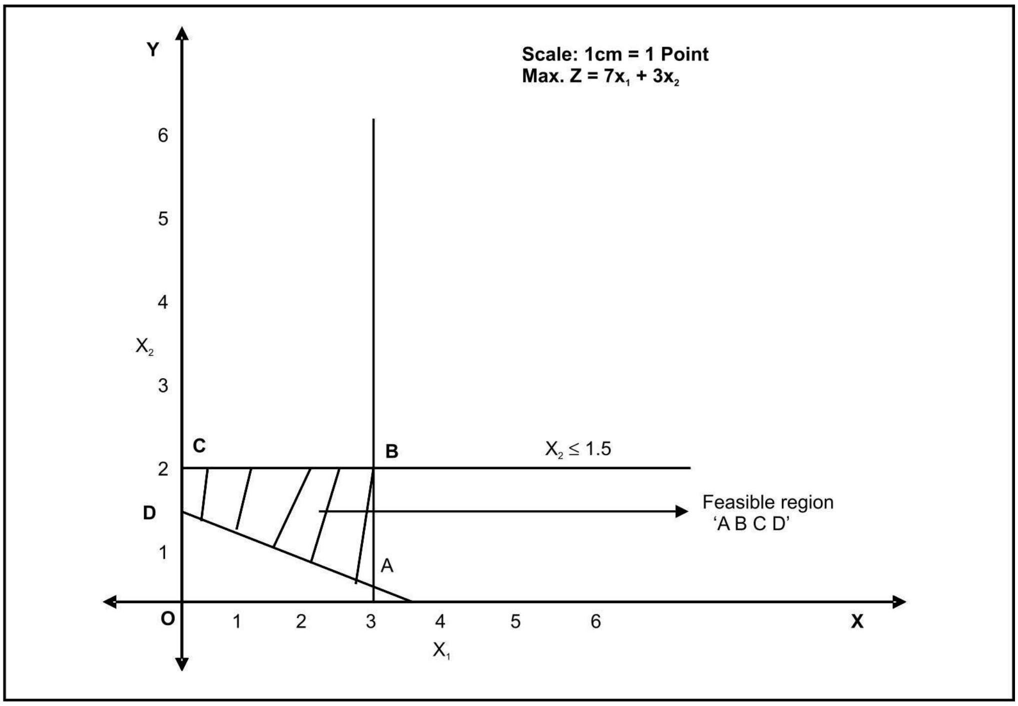
Step 2: Fix up the graphic scale

Maximum points = 4

Minimum points = 1

1 cm. = 1 point

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	2.5	0.17
B	2.5	1.5
C	0	1.5
D	0	1

At A: $x_1 = 2.5$ …..(1)

$x_2 + 3x_2 = 3$ …..(2)

Put $x_1 = 2.5$ in eq. (2) ,

$2.5 + 3x_2 = 3$

$3x_2 = 3 - 2.5$

$3x_2 = 0.5$

$x_2 = 0.166$

Therefore $x_2 = 0.17$

Notes

Step 5: Substitute the co-ordinates of the corner points to the objective function.

Maximise $\quad 'Z' = 7x_1 + 3x_2$

$$\text{At 'A', } Z = 7(2.5) + 3(0.17) = 18.01$$

$$\text{At 'B', } Z = 7(2.5) + 3(1.5) = 22$$

$$\text{At 'C', } Z = 7(0) + 3(1.5) = 4.5$$

$$\text{At 'D', } Z = 7(0) + 3(1) = 3$$

Inference

Hence, the company can get maximum profit by producing 2.5 units of product A and 3 units of product B.

Example:

Maximise $\quad 'Z' = 5x_1 + 3x_2$ (Subject to constraints)

$$x_1 + x_2 \leq 6$$

$$2x_1 + 3x_2 \leq 3$$

$$x_1 \leq 3$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$ (Non-negativity constraints)

Solution:

Step 1: Find the divisible of the equalities.

Equation	x_1	x_2
$x_1 + x_2 = 6$	6	6
$2x_1 + 3x_2 = 3$	1.5	1
$x_1 = 3$	3	0
$x_2 = 3$	0	3

Step 2: Fix up the graphic scale

Maximum points = 6

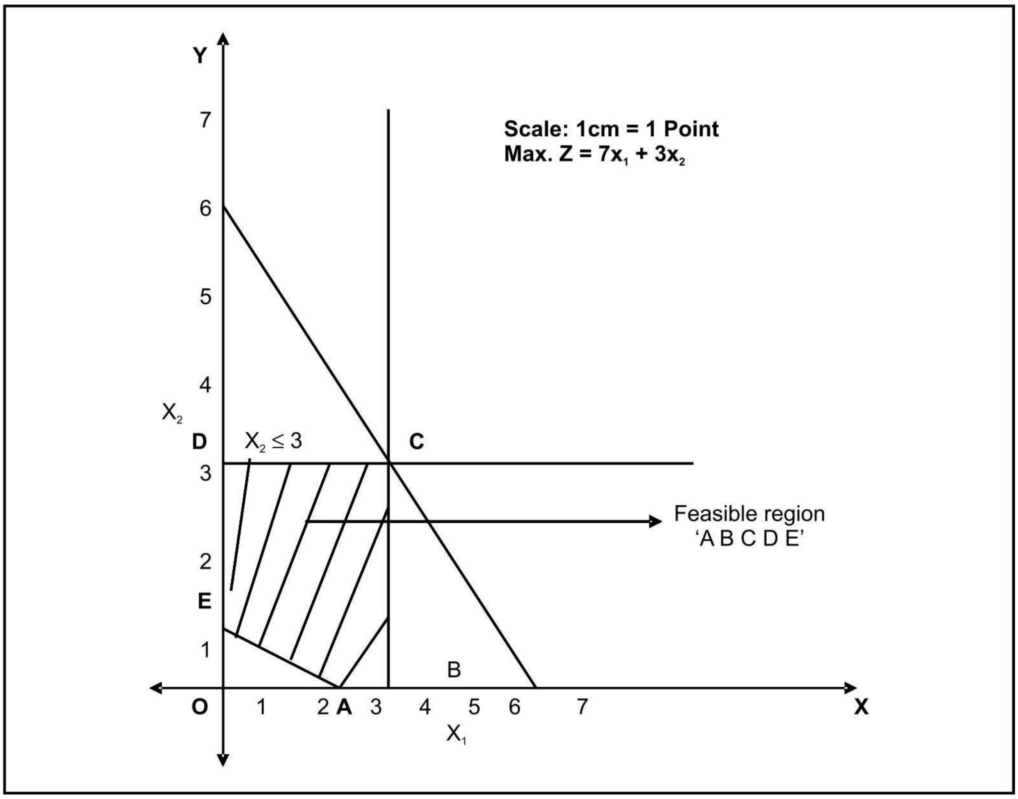
Minimum points = 1

1 cm. = 1 point

Step 3: Graph the data

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x1	x2
A	1.5	0
B	3	0
C	3	3
D	0	3
E	0	1

Step 5: Substitute the co-ordinate of the corner points to the objective function.

$$Z = 5x_1 + 3x_2$$

At 'A', Z = 5(1.5) + 3(0) = 7.5

At 'B', Z = 5(3) + 3(0) = 15

At 'C', Z = 5(3) + 3(3) = 24

At 'D', Z = 5(0) + 3(3) = 9

At 'E', Z = 5(0) + 3(1) = 3

Inference

Maximum profit (₹ 24) can be gained by producing 3 units of product 'M' and 3 units of product 'N'.

Notes

Case Study **Ace Air Lines**

The director of passenger services of Ace Air Lines was trying to decide how many new stewardesses to hire and train over the next six months. He had before him the requirements in number of stewardess flight hours needed.

Month	Hours needed
Jan	8,000
Feb	7,000
March	8,000
April	10,000
May	9,000
June	12,000

It took one month to train a stewardess before she was able to be used on regular flights. Hence, hiring had to be done a month before the need arose. Secondly, training of new stewardess required the time of already trained stewardess. It took approximately 100 hours of regular stewardess time for each trainee during the month of training period. In other words, the number of hours available for flight services by regular stewardesses was cut by 100 hours for each trainee.

The director of passenger services was not worried about January since he had 60 stewardesses available. Company rules required that a stewardess could not work more than 150 hours in any month. This meant that he had a maximum of 9,000 hours available for January, one thousand in excess of his need (stewardesses were not laid off in such cases merely worked fewer hours).

Company record showed that 10% of the stewardesses quit their jobs each month to be married or for other reasons.

The cost of Ace Air lines for a regular stewardesses was ₹ 800 per month for salary and fringe benefits, regardless of how many hours she worked. (She, of course, could not work more than 150 hours.) The cost of a trainee was ₹ 400 per month for salary and fringe benefits.

Question:

Formulate the above as a linear programming design to solve the problem of directory of passenger services at minimum cost. Be sure to identify all the symbols that you use and explain (briefly) all equations.

2.9 Summary

- Linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and some list of requirements represented as linear equations.
- It is a technique to ensure the optimum allocation of scarce resources in order to deliver for the fulfillment of ever increasing demands in the market.
- Linear Programming is used as a helping tool in nearly all functional areas of management.
- The graphical method to solve linear programming problem helps to visualize the procedure explicitly.
- It also helps to understand the different terminologies associated with the solution of LPP.

Notes

2.10 Keywords

Constraints: A condition that a solution to an optimization problem must satisfy.

Feasible Region: The region containing solution.

Feasible Solution: If a solution satisfies all the constraints, it is called feasible solution.

Shadow Price: The amount that the objective function value changes per unit change in the constraint.

2.11 Review Questions

1. Explain the linear programming problem giving two examples.
2. What are the essential characteristics of a linear programming model?
3. What do you understand by 'Graphical Method'? Give its limitations.
4. Explain the graphical method of solving a Linear programming Model involving two variables.
5. Define and explain the following:
   (i) Optimum Solution
   (ii) Feasible Solution
   (iii) Unrestricted Variables
6. A firm manufacturers headache pills in two sizes A and B. Size A contains I grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. It is found by uses that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief.

   Formulate the problem as a standard LPP.
7. Consider a small plant which makes 2 types of automobile parts say A and B. It buys castings that are machined, bored and polished. The capacity of machining is 25 per hour for A and 40 hours for B, capacity of boring is 28 per hours for A and 35 per hour for B, and the capacity of polishing is 35 per hour A and 25 hour of B. Casting for port A costs ₹ 2 each and for part B they cost ₹ 3 each. They sell for ₹ 5 and ₹ 6 respectively. The three machines have running costs of ₹ 20, ₹ 14 and ₹ 17.50 per hour.

   Assuming that any combination of parts A and B can be sold, what product mix maximizes profit?
8. A ship is to carry 3 types of liquid cargo – X, Y and Z. There are 3,000 litres of X available, 2,000 litres of Y available and 1,500 litres of Z available. Each litre of X, Y and Z sold fetches a profit of ₹ 30, ₹ 35 and ₹ 40 respectively. The ship has 3 cargo holds-A, B and C of capacities 2,000, 2,500 and 3,000 litres respectively. From stability considerations, it is required that each hold be filled in the some proportion. Formulate the problem of loading the ship as a linear programming problem. State clearly what are the decision variables and constraints.
9. A company produces two types of pens, say A & B. Pen A is superior in quality while pen B is of lower quality. Net profits on pen A and B are ₹ 5 and ₹ 3 respectively. Raw material required for pen A is twice as that of pen B. The supply of raw material is sufficient only for

1,000 pens of B per day. Pen A requires a special nib and only 400 such nibs are available in a day. For pen B, only 700 nibs are available in a day.

Using graphical method, find the daily product mix so that the company can make maximum profits.

10. G.L Breweries Ltd. has two bottling plants located at Pune and Bangalore. Each plant produces three drinks; whisky, beer and brandy. The number of bottles produced in a day are as follows – whisky 1,500, beer 3,000 and brandy 2,000 at Pune and Whisky 1,500 beer 1,000 and brandy 5,000 at Bangalore. A market survey indicates that during November, there will be demand for 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at Pune and Bangalore are ₹ 600 and ₹ 400 respectively. For how many days each plant be run in November so as to meet the demand at minimum cost?

Answers: Self Assessment

1.	(d)	2.	(d)
3.	(d)	4.	Two
5.	Feasible region	6.	far most

2.12 Further Readings

Books

J.K. Sharma, *Operations Research*, *Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://www.zweigmedia.com/RealWorld/simplex.html

www.math.ucla.edu/

www.math.ncsu.edu

Unit 3: Linear Programming Problem – Simplex Method

Notes

CONTENTS

Objectives

Introduction

3.1 Simplex Method of Linear Programming

3.1.1 Maximisation Cases

3.1.2 Minimization Cases

3.2 Big 'M' Method

3.3 Unconstrained Variables

3.3.1 Change in Objective Function Coefficients and Effect on Optimal Solution

3.3.2 Change in the Right-hand Side Constraints Values and Effect on Optimal Solution

3.4 Special Cases in Linear Programming

3.4.1 Multiple or Alternative Optimal Solutions

3.4.2 Unbounded Solutions

3.4.3 Infeasibility

3.5 Summary

3.6 Keywords

3.7 Review Questions

3.8 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the meaning of word 'simplex' and logic of using simplex method
- Know how to convert a LPP into its standard form by adding slack, surplus and artificial variables
- Learn how to solve the LPP with the help of Big M methodology
- Understand the significance of duality concepts in LPP and ways to solve duality problems

Introduction

In practice, most problems contain more than two variables and are consequently too large to be tackled by conventional means. Therefore, an algebraic technique is used to solve large problems using Simplex Method. This method is carried out through iterative process systematically step by step, and finally the maximum or minimum values of the objective function are attained.

The simplex method solves the linear programming problem in iterations to improve the value of the objective function. The simplex approach not only yields the optimal solution but also other valuable information to perform economic and 'what if' analysis.

Notes

3.1 Simplex Method of Linear Programming

Under 'Graphical solutions' to LP, the objective function obviously should have not more than two decision variables. If the decision variables are more than two, the 'Cartesian Plane' cannot accommodate them. And hence, a most popular and widely used analysis called 'SIMPLEX METHOD', is used. This method of analysis was developed by one American Mathematician by name George B. Dantzig, during 1947.

This method provides an algorithm (a procedure which is iterative) which is based on fundamental theorems of Linear Programming. It helps in moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from one vertex to another vertex is repeated.

Steps:

1. Convert the inequalities into equalities by adding slack variables, surplus variables or artificial variables, as the case may be.

2. Identify the coefficient of equalities and put them into a matrix form AX = B

   Where "A" represents a matrix of coefficient, "X" represents a vector of unknown quantities and B represents a vector of constants, leads to AX = B [This is according to system of equations].

3. Tabulate the data into the first iteration of Simplex Method.

Table 3.1: Specimen

Basic (BV) Variable	C_B	X_B	Y_1	Y_2	S_1	S_2	Minimum Ratio $X_{Bi/Yij}$; $Y_{ij} > 0$
S_1 S_2							
		Z_j C_j					
		$Z_j - C_j$					

   (a) Cj is the coefficient of unknown quantities in the objective function.

   $Zj = \Sigma C_{BiYij}$ (Multiples and additions of coefficients in the table, i.e., $C_{B1} \times Y_{11} + C_{B2} \times Y_{12}$)

   (b) Identify the Key or Pivotal column with the minimum element of Zj - Cj denoted as 'KC' throughout to the problems in the chapter.

   (c) Find the 'Minimum Ratio' i.e., X_{Bi}/Y_{ij}.

   (d) Identify the key row with the minimum element in a minimum ratio column. Key row is denoted as 'KP'.

   (e) Identify the key element at the intersecting point of key column and key row, which is put into a box $\boxed{}$ throughout to the problems in the chapter.

4. Reinstate the entries to the next iteration of the simplex method.

   (a) The pivotal or key row is to be adjusted by making the key element as '1' and dividing the other elements in the row by the same number.

   (b) The key column must be adjusted such that the other elements other than key elements should be made zero.

Notes

(c) The same multiple should be used to other elements in the row to adjust the rest of the elements. But, the adjusted key row elements should be used for deducting out of the earlier iteration row.

(d) The same iteration is continued until the values of $Z_j - C_j$ become either '0' or positive.

5. Find the 'Z' value given by C_B, X_B.

3.1.1 Maximisation Cases

Example:

Maximise $\quad 'Z' = 5x_1 + 3x_2 \quad$ [Subject to constraints]

$$x_1 + x_2 \le 2$$
$$5x_1 + 2x_2 \le 10$$
$$3x_1 + 8x_2 \le 12$$

Where, $\quad x_1, x_2 \ge 0 \quad$ [Non-negativity constraints]

Solution:

Step 1: Conversion of inequalities into equalities adding slack variables

$$x_1 + x_2 + x_3 = 2$$
$$5x_1 + 2x_2 + x_4 = 10$$
$$3x_1 + 8x_2 + x_5 = 12$$

Where, x_3, x_4 and x_5 are slack variables.

Step 2: Fit the data into the matrix form AX = B

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 2 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: Fit the data into first iteration of Simplex Method

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	2	$\boxed{1}$	1	1	0	0	2/1 = 2(KR) →
S_2	0	10	5	2	0	1	0	10/5 = 2
S_3	0	12	3	8	0	0	1	12/3 = 4
		Z_j	0	0				
		C_j	5	3				
		$Z_j - C_j$	-5	-3				
				(↑ KC)				

Therefore, $\quad Z = C_B X_B$

$$= (0\times2) + (0\times10) + (0\times12)$$
$$= 0$$

Notes

Step 4: Fit the data into second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
Y_1	5	2/1 = 2	1/1 = 1	1/1 = 1	–	–	–	–
S_2	0	10 – 2(5) = 0	5 – 1(5) = 0	2–1(5) = –3	–	–	–	–
S_3	0	12 – 2(3) = 6	3 – 1(3) = 0	8–1(3) = 5	–	–	–	–
		Z_j C_j	5 5	5 3				
		$Z_j - C_j$	0	2				

Therefore, $Z = C_B X_B$

$= (5 \times 2) + (0 \times 0) + (0 \times 6)$

Therefore, $Z = 10$

Therefore, Maximum value of 'Z' = 10

Example:

Maximise $'Z' = 2x_1 + 3x_2$ [Subject to constraints]

$x_1 + x_2 \leq 1$

$3x_1 + x_2 \leq 4$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Conversion of inequalities into equalities by adding slack variables.

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 + x_4 = 4$

Where x_3 and x_4 are slack variables.

Step 2: Identify the coefficients.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	1	1	$\boxed{1}$	1	0	1/1 = 1 (KR)?
S_2	0	4	3	1	0	1	4/1 = 4
		Z_j C_j	0 2	0 3			
		$Z_j - C_j$	-2	-3			

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	**C_B**	**X_B**	**Y_1**	**Y_2**	**S_1**	**S_2**	**Min. Ratio**
y_2	3	1/1 = 1	1/1 = 1	1/1 = 1	--	--	--
S_2	0	4-1(1) = 3	3-1(1) = 2	1-1(1) = 0	--	--	--
		Z_j C_j	3 2	3 3			
		$Z_j - C_j$	1	0			

Therefore, $Z = C_B X_B$

$= (3 \times 1) + (0 \times 3)$

$= 3$

Therefore, Maximum value of 'Z' = 3

Example:

Maximise $'Z' = 4x_1 + 3x_2$ [Subject to constraints]

$2x_1 + x_2 \leq 30$

$x_1 + x_2 \leq 24$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

$2x_1 + x_2 + x_3 = 30$

$x_1 + x_2 + x_4 = 24$

Where x_3 and x_4 are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 30 \\ 24 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	**C_B**	**X_B**	**Y_1**	**Y_2**	**S_1**	**S_2**	**Min. Ratio**
S_1	0	30	$\boxed{2}$		1	0	30/2 = 15 (KR) ?
S_2	0	24	1	1	0	1	24/1 = 24
		Z_j C_j	0 4	0 3			
		$Z_j - C_j$	-4	-3			

(↑ KC)

Notes

Therefore, $Z = (0 \times 30) + (0 \times 24)$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
Y_1	4	30/2 = 15	2/2 = 1	½ = 0.5	-	-	15/0.5 = 30
S_2	0	24 – 15(1) = 9	1 – 1(1) = 2	1 – 0.5(1) = $\boxed{0.5}$	-	-	9/0.5 = 18 (KR)?
		Z_j	4	2			
		C_j	4	3			
		$Z_j – C_j$	0	-1			

(↑ KC)

Therefore, $Z = C_BX_B$

$= (4 \times 15) + (0 \times 9)$

$= 60$

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
Y_1	4	15 – 18(0.5) = 6	1 – 0(0.5) = 1	0.5 – 1(0.5) = 0	-	-	-
S_2	3	9/0.5 = 18	0/0.5 = 0	0.5/0.5 = 1	-	-	-
		Z_j	4	3			
		C_j	4	3			
		$Z_j – C_j$	0	0			

Therefore, $Z = C_BX_B$

$= (4 \times 6) + (3 \times 18)$

$Z = 78$

Example:

Maximise 'Z' $= 5x_1 + 3x_2$ [Subject to constraints]

$x_1 + x_2 \leq 12$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

$x_1 + x_2 + x_3 = 12$

$5x_1 + 2x_2 + x_4 = 10$

$3x_1 + 8x_2 + x_5 = 12$

Where x_3, x_4 and x_5 are slack variables.

Notes

Step 2: Fit the data into a matrix form.

$$A = \begin{array}{c} \begin{array}{ccccc} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad B = \begin{pmatrix} 12 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	12	1	1	1	0	0	12/1 = 12
S_2	0	10	$\boxed{5}$	2	0	1	0	10/5 = 2(KR)?
S_3	0	12	3	8	0	0	1	12/3 = 4
		Z_j	0	0				
		C_j	5	3				
		$Z_j - C_j$	–5	–3				
			(↑ KC)					

Therefore, $Z = C_B X_B$

$= (0 \times 12) + (0 \times 10) + (0 \times 12) = 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	12 –2(1) = 10	1–1(1) = 0	1 – 0.4(1) = 0.6	–	–	–	10/0.6 = 16.67
y_1	5	10/5 = 2	5/5 = 1	2/5 = 0.4	–	–	–	–2/0.4 = 5
S_3	0	12 – 2(3) = 6	3 – 1(3) = 0	8 – 0.4(3) = 6.8	–	–	–	6/6.8 = 0.88(KR) ?
		Z_j	5	2				
		C_j	5	3				
		$Z_j - C_j$	0	–1				
				(↑ KC)				

Therefore, $Z = C_B X_B$

$= (0 \times 10) + (5 \times 2) + (0 \times 6) = 10$

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	10 –0.88 (0.6) = 9.47	0	0.6 –1 (0.6) = 0	-	-	-	-
y_1	5	2 –0.88 (0.4) = 1.698	1	0.4 –1 (0.4) = 0	-	-	-	-
y_2	3	6/6.8 = 0.88	0	6.8/6.8 = 1	-	-	-	-
		Z_j	5	3				
		C_j	5	3				
		$Z_j - C_j$	0	0				

Therefore, $Z = C_B X_B$

$= (0 \times 9.47) + (5 \times 1.698) + (3 \times 0.88)$

Therefore, maximum value of Z = 10.88

Notes

Example:

Maximise 'Z' = $5x_1 + 3x_2$ [Subject to constraints]

$= 3x_1 + 5x_2 + \leq 15$

$= 5x_1 + 2x_2 \leq 10$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities by adding the slack variables.

$3x_1 + 5x_2 + x_3 = 15$

$5x_1 + 2x_2 + x_4 = 10$

Where, x_3 and x_4 are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 3 & 5 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad B = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	15	3	5	1	0	15/3 = 5
S_2	0	10	$\boxed{5}$	2	0	1	10/5 = 2(KR) ?
		Z_j C_j	0 5	0 3			
		$Z_j - C_j$	-5	-3			

(↑ KC)

Therefore, Maximise Z = $C_B X_B$

$= (0 \times 15) + (0 \times 10)$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	15 – 2 (3) = 9	3 –1(3) = 0	5 – 0.4 (3) = $\boxed{3.8}$	--	--	9/3.8 = 2.37 (KR)?
S_2	5	10/5 = 2	5/5 = 1	2/5=0.4	--	--	2/0.4 = 5
		Z_j C_j	5 5	2 3			
		$Z_j - C_j$	0	–1			

(↑ KC)

Therefore, Z = $C_B X_B$

$= (0 \times 9) + (5 \times 2)$

$= 10$

Notes

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	Min. Ratio
y_2	3	9/3.8 = 2.37	0	3.8/3.8 = 1	-	-	-
y_1	5	2 –2.37(0.4) = 1.052	1	0.4 –1(0.4) = 0	-	-	-
		Z_j	5	3			
		C_j	5	3			
		$Z_j – C_j$	0	0			

Therefore, Maximise Z $= C_B X_B$

$= (3 \times 2.37) + (5 \times 1.052)$

Maximum value of 'Z' = 12.37

Notes Real life complex applications usually involve hundreds of constraints and thousands of variables. So, virtually these problems cannot be solved manually. For solving such problems, you will have to rely on employing an electronic computer.

Self Assessment

1. Solve the following LPP problem using simplex method.

   Maximize 'Z' $= 7x_1 + 5x_2$ [Subject to constraints]

   $x_1 + x_2 \leq 6$

   $4x_1 + 3x_2 \leq 12$

   Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

2. Solve the following LPP problem using simplex method.

   Maximise 'Z' $= 5x_1 + 7x_2$ [Subject to constraints]

   $= x_1 + x_2 \leq 4$

   $= 3x_1 - 8x_2 \leq 24$

   $= 10x_1 + 7x_2 \leq 35$

   Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

3.1.2 Minimization Cases

Example:

Minimize 'Z' $= -x_1 - 2x_2$ [Subject to constraints]

$-x_1 + 3x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Notes

Solution:

Step 1: Convert the minimization problem into maximisation case by changing the signs of the decision variables in the objective function.

Therefore, 'Z' = $x_1 + 2x_2$ [Subject to constraints]

Step 2: Convert the inequalities into equalities by adding slack variables.

$$-x_1 + 3x_2 + x_3 = 10$$

$$x_1 + x_2 + x_4 = 6$$

$$x_1 - x_2 + x_5 = 2$$

Where x_3, x_4 and x_5 are slack variables.

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	10	–1	3	1	0	0	10/3 = 3.3 (KR)?
S_2	0	6	1	1	0	1	0	6/1 = 6
S_3	0	2	1	–1	0	0	1	–½ = – 0.5
		Z_j C_j	0 1	0 2				
		$Z_j - C_j$	–1	–2				

(↑ KC)

Therefore, $Z = C_BX_B$

$0 + 0 + 0 = 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
y_2	2	10/3 = 3.33	–1/3 = 0.33	3/3 = 1				3.33/–0.33 = –10.09
S_2	0	6 – 3.33 (1) 2.67	1 – 1(–0.33) (1) = 2.67	1–1(1) = 0 = 1.33				2.67/1.33 = 2.00 (KR) →
S_3	0	2 – 3.33 (–1) = 5.33	1–(–0.33) (–1) = 0.67	–1–1 (–1) = 0				5.33/0.67 = 8.00
		Z_j C_j	–0.66 1	2 2				
		$Z_j - C_j$	–1.66	0				

(↑ KC)

Therefore, $Z = C_BX_B$

$= (2 \times 3.33) + (2 \times 2.67) + (0 \times 5.33)$

$= 6.66 + 0 + 0 = 6.66$

Notes

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
y_2	2	3.33 – 2(–0.33) = 3.99	–0.33 – 1(–0.33) = 0	1	–	–	–	–
y_1	1	2.67/1.35 = 2	1.33/1.33 = 1	0	–	–	–	–
S_3	0	5.33 – 2(0.67) = 3.99	0.67 – 1(0.67) = 0	0	–	–	–	–
		Z_j C_j	1 1	2 2	–			
		$Z_j – C_j$	0	0	–	–		

Max. $Z = C_B X_B$

$= (2 \times 3.99) + (1 \times 2) + (0 \times 3.99)$

$= 7.98 + 2 + 0$

$= .98$

Therefore, min. $Z = –9.98$

Example:

Minimise 'Z' $= – x_1 – 3x_2 + 2x_3$ [Subject to constraints]

$$3x_1 – x_2 + 3x_3 \leq 7$$

$$– 2x_1 + 4x_2 \leq 12$$

$$– 4x_1 + 3x_2 + 8x_3 \leq 10$$

Where, $x_1, x_2, x_3 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Therefore, Maximise $Z = – x_1 + 3x_2 – 2x_2$ [Subject to constraints]

Step 2: Convert of the inequalities into equalities by adding slack variables.

Therefore, $3x_1 – x_2 + 3x_3 + x_4 = 7$

$$– 2x_1 + 4x_2 + x_5 = 12$$

$$– 4x_1 + 3x_2 + 8x_3 + x_6 = 10$$

Where, x_4, x_5 and x_6 are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x \\ 3 & –1 & 3 & 1 & 0 & 0 \\ –2 & 4 & 0 & 0 & 1 & 0 \\ –4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7	3	–1	3	1	0	0	7/–1 = –7
S_2	0	12	–2	4	0	0		0	12/4 = 3 (KR)→
S_3	0	10	–4	3	8	0	0	1	10/3 = 3.33
		Z_j C_j	0 –1	0 3	0 –2				
		$Z_j - C_j$	1	–3	2				

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7–3 (–1) = 10	3 + 0.5 (–1) = 2.5	–1+1 (–1) = 0	3–0 (–1) = 3	–	–	–	10/2.5 = 4 (KR)→
Y_2	3	12/4 = 3	–2/4 = –0.5	4/4 = 1	0	–	–	–	3/–0.5 = -
S_3	0	10 – 3 (3) = 1	–4 + (–0.5) (3) = –5.5	3–1 (3) = 0	8–0 (3) = 8	–	–	–	1/2.5 = -
		Z_j C_j	–1.5 –1	3 –3	0 –2				
		$Z_j - C_j$	–0.5	–0	2				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 0) + (3 \times 3) + (0 \times 1)$

$= 0 + 9 + 0$

$= 9$

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
y_1	–1	10/2.5 = 4	2.5/2.5 = 1	0	3/2.5 = 1.2	–	–	–	–
S_2	3	3–4 (–0.5)= 5	–0.5–1 (–0.5)= 0	1	0	–	–	–	–
S_3	0	1–4 (–2.5)= 11	–2.5–1 (–2.5)= 0	0	8 – 1 (–2.5) = 10.5	–	–	–	–
		Zj Cj	–1 –1	3 3	–1.2 –2				
		Zj–Cj	0	0	0.8				

Therefore, Maximise $'Z' = C_B X_B$

$= -4 + 15 + 0$

$= 11$

Therefore, Minimise $Z = -11$

Example:

$$\text{Minimise} \quad 'Z' = -10x_1 - 12x_2 - 15x_3 \qquad \text{[Subject to constraints]}$$

$$0.10x_1 + 12x_2 + 0.15x_3 \le 36$$

$$0.06x_1 + 0.05x_2 + 0.09x_3 \le 30$$

$$0.18x_1 + x_2 + 0.07x_3 \le 37$$

$$0.13x_1 + 0.10x_2 + 0.08x_3 \le 38$$

$$x_1 \le 200$$

$$x_2 \le 100$$

$$x_3 \le 180$$

$$\text{Where, } x_1, x_2, x_3 \ge 0 \qquad \text{[Non-negativity constraints]}$$

Solution:

Step 1: Conversion of minimization case into maximisation case.

Therefore, Maximise $'Z' = 10x_1 + 12x_2 + 15x_3$ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding slack variables.

$$0.10x_1 + 0.12x_2 + 0.15x_3 + x_4 = 36$$

$$0.06x_1 + 0.05x_2 + 0.09x_3 + x_5 = 30$$

$$0.18x_1 + x_2 + 0.07x_3 + x_6 = 37$$

$$0.13x_1 + 0.10x_2 + 0.08x_3 + x_7 = 38$$

$$x_1 + x_8 = 200$$

$$x_2 + x_9 = 100$$

$$x_3 + x_{10} = 180$$

Where, x_4, x_5, x_6, x_7, x_8, x_9 and x_{10} are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{array}{c} \begin{array}{cccccccccc} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array} \\ \begin{pmatrix} 0.10 & 0.12 & 0.15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.06 & 0.05 & 0.09 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.18 & 1 & 0.07 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.13 & 0.10 & 0.08 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = B = \begin{pmatrix} 36 \\ 30 \\ 37 \\ 200 \\ 200 \\ 100 \\ 180 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	36	0.10	0.12	0.15	1	0	0	0	0	0	0	36/0.15 = 240
S_2	0	30	0.06	0.05	0.09	0	1	0	0	0	0	0	30/0.09 = 33.3
S_3	0	37	0.18	1	0.07	0	0	1	0	0	0	0	37/0.08 = 529
S_4	0	38	0.13	0.10	0.08	0	0	0	1	0	0	0	38/0.08 = 475
S_5	0	200	1	0	0	0	0	0	0	1	0	0	200/0 = –
S_6	0	100	0	1	0	0	0	0	0	0	1	0	100/0 = –
S_7	0	180	0	0	$\boxed{1}$	0	0	0	0	0	0	1	180/1 = 180(KR)?
		Z_j C_j	0 10	0 12	0 15								
		$Z_j - C_j$	–10	–12	–15								

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 36) + (0 \times 30) + (0 \times 37) + (0 \times 38) + (0 \times 200) + (0 \times 100) + (0 \times 180)$

$= 0 + 0 + 0 + 0 + 0 + 0 + 0$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	36 – 180(0.15) = 9	0.10–0(0.15) = 0.10	0.12–0 (0.15) = 0.12	0.15–0 (0.15) = 0	–	–	–	–	–	–	–	9/0.12 = 75
S_2	0	30 – 180(0.09) = 13.8	0.06–0 (0.09) = 0.06	0.05–0 (0.09) = 0.05	0.09 – 1 (0.09) = 0	–	–	–	–	–	–	–	13.80/0.05 = 276
S_3	0	37 – 180 (0.07)= 24.4	0.18–0 (0.07) = 0.18	1–0(0.07) = $\boxed{1}$	0.07–1 (0.07) = 0	–	–	–	–	–	–	–	24.4/1 = 24.4(KR)?
S_4	0	38 – 180 (0.08) = 23.6	0.13–0 (0.08) = 0.13	0.10 – 0 (0.08) = 0.10	0.08 – 1 (0.08) = 0	–	–	–	–	–	–	–	23.6/0.10 = 236
S_5	0	200–180	0.10 (0) = 1	0 – 0 = 0	0	–	–	–	–	–	–	–	(0) = 200
S_6	0	100–180 (0) = 100	0 – 0 = 0	1 – 0 (0)= 1	0	–	–	–	–	–	–	–	100/1= 100
Y_3	15	180/1 = 180	0	0	1/1 = 1	–	–	–	–	–	–	–	
		Z_j C_j	0 10	0 12	15 15								
		$Z_j - C_j$	–10	–12	0								

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 9) + (0 \times 3.8) + (0 \times 24.4) + (0 \times 23.6) + (0 \times 200) + (0 \times 100) + (15 \times 180)$

$= 2700$

Notes

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	9 – 24.4 (0.12) = 6.07	0.10 – 0.18 (0.12= $\boxed{0.078}$	0.12 – 1 (0.12) = 0	0	–	–	–	–	–	–	–	6.072/0.078 = 77.85 (KR) →
S_2	0	13.8–24.4 (0.05)=12.58	0.06 – 18 (0.05) =0.05	0.05 – 1 (0.05) = 0	0	–	–	–	–	–	–	–	12.58/0.051 = 246.67
S_3	0	24.4/1 = 24.4	0.18/1 = 0.18	1/1 = 1	0	–	–	–	–	–	–	–	24.4/0.18 = 135.56
S_4	0	23.6–24.4 (0.10)=21.16	0.13 – 0.18 (0.10) = 0.112	0.10 – 1 (0.10) = 0	0	–	–	–	–	–	–	–	21.16/0.112 = 188.93
S_5	0	200 – 24.4 (0) = 200	1–0.18 (0)=1	0	0	–	–	–	–	–	–	–	200/1 = 200
S_6	0	100 – 24.4 (1) = 75.6	0 – 0.18 (1) = –0.18	1 – 1 (1) =0	0	–	–	–	–	–	–	–	75.6/–0.18 = –420
Y_3	15	180 – 0 =180	0	0	1	–	–	–	–	–	–	–	180/0 = 0
		Z_j C_j	2.16 10	12 12	15 15								
		$Z_j – C_j$	–7.84	0	0								

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 6.072) + (0 \times 12.58) + (12 \times 24.4) + (0 \times 21.16) + (0 \times 200) + (0 \times 75.6) + (15 \times 180)$

$= 2{,}992.8$

Step 7: Fourth iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
y1	10	6.072/0.078 = 77.85	0.078/0.078 = 1	0	0	–	–	–	–	–	–	–	–
S2	0	12.58 – 77.85(0.051)= 8.61	0.051 – 1(0.051) = 0	0	0	–	–	–	–	–	–	–	–
y2	12	24.4 – 77.85(0.18)= 10.387	(0.18) – 1(0.18) = 0	1	0	–	–	–	–	–	–	–	–
S4	0	21.16 – 77.85(0.112)= 12.44	0.112 – 1(0.112) = 0	0	0	–	–	–	–	–	–	–	–
S5	0	200 – 77.85(1)= 122.15	1 – 1(1) = 0	0	0	–	–	–	–	–	–	–	–
S6	0	75.6 – 77.85(–0.18)= 89.613	–0.18 – 1(–0.18) = 0	0	0	–	–	–	–	–	–	–	–
y3	15	180 – 77.85(0) = 180	0	0	1	–	–	–	–	–	–	–	–
		Z_j C_j	10 10	12 12	15 15								
		$Z_j – C_j$	0	0	0								

Notes

Maximise $Z = C_B X_B$

$= (10 \times 77.85) + (12 \times 10.387) + (15 \times 180) + 0 + 0 + 0 + 0$

$= 778.5 + 124.644 + 2{,}700 + 0 + 0 + 0 + 0$

$= 3{,}603.144$

Therefore, Minimise Z = –3,603.144

Steps to Sum-up

1. Prepare the Table No. 1 and find the cost coefficients (Z_j) for different columns of A, i.e., for Y_i, by multiplying C_B column with entries of Y_j and adding the products, i.e., Z_j - $\Sigma C_{Bi} Y_{ij}$ and C_j is the most of coefficients for Z_j in objective function. Then, find the difference of Z_j and C_j (Z_j – C_j) for different columns of A.

   (a) *Incoming Vector:* The j^{th} vector, i.e., y_j enters the basis if $Z_j - C_j$ is minimum where y_j is the j^{th} column of the coefficient matrix 'A'.

   (b) *Outgoing Vector:* Find the ratio of XB_i/y_{ij} (for all $Y_{ij} > 0$) for all the elements of the incoming vector. Then, the vector attached to the row having minimum ratio would be removed from the basis (if y_{ij} is greater than '0', otherwise that item should be neglected).

   (c) *Key Element or Pivotal Element:* The element common to the incoming and outgoing vector is called key element or pivotal element.

2. The incoming vector has the coefficient of objective function in CB. Hence, make the pivotal element as "1" by dividing that row completely by the pivotal element. The other elements of the incoming vector other than pivotal element must be made "0"/"Zero". This can be done by deducting the elements of the respective rows by "K" times the adjusted pivotal row elements completely. The constant 'K' is chosen such that the pivotal columns element(s) is made "0"/"Zero". Then find Z_j and C_j in the usual manner of matrix method and if $Z_j - C_j$ is greater than or equal to zero for all columns, then the basic feasible solution is optimum otherwise the same procedure is to be continued.

Notes **Brief Steps of the simplex method:**

1. Convert the inequalities into equalities.
2. Identify the coefficients of equalities & put them into a matrix form.
3. Tabulate the data into 1st iteration of simplex method.
4. Reinstate entries in the 2nd iteration.
5. Find the 'Z' value.

Self Assessment

Fill in the blanks:

3. The element common to the incoming and outgoing vector is called ……………….. .
4. A ………………. variable represents unused resources and are added to original objective function with zero coefficients.
5. A ………………. Variable represents amount by which solution value exceed a resource.

3.2 Big 'M' Method

Notes

When the Linear Programming problem has greater than or equal to types of equations as constraints, it is obvious that some quantity should be deducted to convert them into equalities. The variables attached to it are known as 'surplus variables'. If the inequality is of the type greater than or equal to then add surplus variables which carry a negative sign and their cost coefficients in the objective function would be zero. As they would be considering slack or artificial variables initially as basic variables, and as the surplus variables carry negative signs, they represent vectors of identify matrix of the form,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ etc.,}$$

which cannot be taken into the basis. Hence, artificial variables along with the surplus variables would be added.

These artificial variables carry large negative values (-m) in the objective function. The artificial variables can also be added to the equation. This helps in choosing the initial variable or variables for the basis. The slack variables then would go to the basis whose cost coefficients are supposed to be zero (0) and the cost coefficients of artificial variables are supposed to be -M for maximization cases and +M for Minimization cases. The procedure for iteration follows when Simplex technique to obtain the optimum solution is used. Since the method involves artificial variables carrying -M as the cost coefficient, where M is a very large number which helps in the optimum solution finding and hence it is known as 'Big M Method'.

Steps:

1. Express the problem in the standard form by using slack, surplus and artificial variables.
2. Select slack variables and artificial variables as the initial basic variables with the cost coefficients as '0' or '-M' respectively.
3. Use simplex procedure for iterations & obtain optimum solution. During the iterations, one can notice that the artificial variables leave the basis first and then the slack variables with improved value of objective function at each iteration to obtain the optimum solution.

Example:

Minimise 'Z' $= 4x_1 + 8x_2 + 3x_3$ [Subject to constraints]

$$x_1 + x_2 \geq 2$$

$$2x_1 + x_3 \geq 5$$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of minimization case into maximisation case.

Therefore, 'Z' $= -4x_1 - 8x_2 - 3x_3$ [Subject to constraints]

Step 2: Conversion of inequalities into equalities adding slack variables and artificial variables.

$$-x_1 + x_2 - x_4 + x_6 = 2$$

$$2x_1 + x_3 - x_5 + x_7 = 5$$

Where, x_4 and x_5 are surplus variables, x_6 and x_7 are artificial variables.

Notes

Step 3: Bring the objective function into a standard form.

Therefore, Maximise 'Z' = $-4x_1 - 8x_2 - 3x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$

Step 4: Find out the matrix form of equalities.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	2	$\boxed{1}$	$\boxed{1}$	0	–1	0	1	0	2/1 = 2(KR)→
a_2	–M	5	2	0	1	0	–1	0	1	5/2 = 2.5
		Z_j C_j	–3M –4	–M 8	–M –3					
		$Z_j - C_j$	–3M+4	–M+8	–M+3					
					(↑ KC)					

Therefore, $Z = C_B X_B$

$= (-M \times 2) + (-M \times 5)$

$= -2M - 5M$

$= -7M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–4	2/1 = 2	1/1 = 1	1	0	–1	0	1	0	2/0 = –
a_2	–M	5 – 2(2) = 1	2 –1 (2) = –2	0–1 (2) = –2	1–0 (2) = 1	–	–	–	–	1/1 = 1 (KR)→
		Z_j C_j	–4 –4	–4 + 2M –8	–M –3					
		$Z_j - C_j$	0	4 + 2M	M+3					
					(↑ KC)					

Therefore, $Z = (-4 \times 2) + (-M \times 1)$

$= -8 - M$

Step 7: Second iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	y_3	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–4	2–1 (0) = 2	1–1 (0) = 1	1 – (–2) (0) = 1	0	–	–	–	–	–
y_3	–3	1/1 = 1	0	–2/1 = –2	1/1 = 1	–	–	–	–	–
		Z_j C_j	–4 –4	2 –8	–3 –3					
		$Z_j - C_j$	0	10	0					

Notes

Therefore, Maximize $Z = C_B X_B$

$= (-4 \times 2) + (-3 \times 1)$

$= -8 - 3$

$= -11$

Therefore, Minimize $Z = 11$

Example:

Minimise $'Z' = 3x_1 + 5x_2$ [Subject to constraints]

$2x_1 + 8x_2 \geq 40$

$3x_1 + 4x_2 \geq 50$

$x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the above minimization case into maximisation case.

Therefore, Maximise $Z = -3x_1 - 5x_2$

Step 2: Convert the inequalities into equalities adding slack variables and artificial variables.

$2x_1 + 8x_2 - x_3 + x_5 = 40$

$3x_1 + 4x_2 - x_4 + x_6 = 50$

Where, x_3 & x_4 are surplus variables and x_5 and x_6 are artificial variables.

Step 3: Bring the objective function into a standard form.

Maximise $'Z' = -3x_1 - 5x_2 - 0x_3 + 0x_4 - Mx_5 - Mx_6$

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} y_1 & y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2 & 8 & -1 & 0 & 1 & 0 \\ 3 & 4 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 40 \\ 50 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	40	2	8	–1	0	1	0	40/8 = 5 (KR)→
a_2	–M	50	3	4	0	–1	0	1	50/4 = 12.5
		Z_j C_j	–5M –3	–12M –5					
		$Z_j - C_j$	–5M + 3	12M + 5					

Therefore, $Z = C_B X_B$

$= (40x - M) + (50x - M)$

Notes

$= -40M - 50M$

$= -90M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_2	–5	40/8 = 5	2/8 = 0.25	8/8 = 1	–	–	–	–	5/0.25 = 20
a_2	–M	50–5(4) = 30	3–0.25(4) = $\boxed{2}$	4–1(4)= 0	–	–	–	–	30/2 = 15(KR)→
		Z_j C_j	–2M – 1.25 –3	–5 –5					
		$Z_j - C_j$	–2M + 1.75	0					

(↑ KC)

Therefore, $Z = (-5 \times 5) + (-M \times 30)$

$= -30M - 25$

Step 7: Third iteration of Simplex Method.

7	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_2	–5	5–15 (0.25) = 1.25	0.25 – 1 (0.25) = 0	1–0 (0.25) = 1	–	–	–	–	–
y_1	–3	30/2 = 15	2/2 = 1	0/2 = 0	–	–	–	–	–
		Z_j C_j	–3 –3	–5 –5					
		$Z_j - C_j$	0	0					

$Z = C_B X_B$

$= -6.25 - 45$

Maximise $Z = -51.25$

Therefore, Minimise $Z = 51.25$

Example:

Minimise 'Z' = $12x_1 + 20x_2$ [Subject to constraints]

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Maximise $Z = -12x_1 - 20x_2$ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding artificial and surplus variables.

$6x_1 + 8x_2 - x_3 + x_5 = 100$

$7x_1 + 12x_2 - x_4 + x_6 = 120$

Where, x_3 and x_4 are surplus variables and x_5 and x_6 are artificial variables.

Notes

Step 3: Bring the objective function into a standard form.

Maximise 'Z' $= -12x_1 - 20x_2 - x_3 + x_4 - Mx_5 - Mx_6$

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 6 & 8 & -1 & 0 & 1 & 0 \\ 7 & 12 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 100 \\ 120 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	100	6	8	–1	0	1	0	100/8 = 12.5
A_2	–M	120	7	12	0	–1	0	1	120/12 = 10(KR)→
		Z_j C_j	–100M 12	–120M –20					
		$Z_j - C_j$	–100M + 12	120M+20					
				(↑ KC)					

Therefore, $Z = C_BX_B$

$= -100M - 120M$

$= -220M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	100–10 (8) = 20	6–0.5(8) = 1.33	8–1(8) = 0	–	–	–	–	20/1.33 = 15.04(KR)→
y_2	–20	120/12 = 10	7/12 = 0.58	12/12 = 1	–	–	–	–	10/0.58 = 17.24
		Z_j C_j	–1.33M–11.6–12	–20 –20					
		$Z_j - C_j$	–1.33M + 0.4	0					
			(↑ KC)						

Therefore, $Z = C_BX_B$

$= (-M \times 20) + (-20 \times 10)$

$= -20M - 200$

Step 7: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–12	20/1.33 = 15.04	1.33/1.33 = 1	0	–	–	–	–	--
y_2	–20	10 – 15.04 (90.58) = 1.28	0.58–1(0.58) = 0	1	–	–	–	–	--
		Zj Cj	–12 –12	–20 –20					
		Zj – Cj	0	0					

$Z = C_BX_B$

Notes

$$= (-12 \times 15.04) + (-20 \times 1.28)$$

$$= -180.48 - 25.6$$

Therefore, Maximise $Z = -206.08$

Therefore, Minimise $Z = 206.08$

Example: Solve the following LPP using Simplex Method.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4$ [Subject to constraints]

$$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$$

$$2x_1 + x_2 + x_3 + 5x_4 \leq 4$$

$$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$$

$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$$

$$\text{Where, } x_1, x_2, x_3, x_4 \geq 0$$

Solution:

Step 1: Convert the inequalities into equalities by adding slack variables and surplus variables.

$$3x_1 + 2x_2 + 4x_3 + x_4 + x_5 = 6$$

$$2x_1 + x_2 + x_3 + 5x_4 + x_6 = 4$$

$$2x_1 + 6x_2 - 8x_3 + 4x_4 + x_7 = 0$$

$$x_1 + 3x_2 - 4x_3 + 3x_4 + x_8 = 0$$

Where, x_5 and x_6 are slack variables and x_7 and x_8 are artificial variables.

Step 2: The standard form of objective function.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4 \pm 0x_5 \pm 0x_6 - Mx_7 - Mx_8$

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & y_4 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 3 & 2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 6 & -8 & 4 & 0 & 0 & 1 & 0 \\ 1 & 3 & -4 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = B = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6	3	2	4	1	1	0	0	0	6/2 = 3
S_2	0	4	2	1	1	5	0	1	0	0	4/1 = 4
a_1	–M	0	2	6	–8	4	0	0	1	0	0/6 = 0(KR)?
a_2	–M	0	1	3	–4	3	0	0	0	1	0/3 = 0
		Z_j C_j	–3M 1	–9M 1.5	12M 2	–7M 5					
		$Z_j - C_j$	–3M–1	–9M–1.5	+12M–2	–7M–5					

(↑ KC)

Notes

Therefore, $Z = C_B X_B$

$= (0 \times 6) + (0 \times 4) + (-M \times 0) + (-M \times 0)$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6–0 (2)=6 (2)	3– 0.33=2.34	2–1 (2)=0 (2)	4–1 (–1.33)=6.66 (2)	1– (0.67)=–0.34	–	–	–	–	6/–0.34 = –
S_2	0	4–0 (1)=4 (1)	2– 0.33=1.67	1–1 (1)=0 (1)	1– (–1.33)=2.33 (1)	5– (0.67)=0.92	–	–	–	–	4/4.33
y_2	1.5	0/6 = 0	2/6 = 0.33	6/6 = 1	–8/6 = –1.33	4/6 = 0.67	–	–	–	–	0/0.67 = 0
A_2	–M	0–0 (3)=0	1–0.33 (3)=0.01	3–1 (3)=0	–4– (–1.33)(3)= $\boxed{0.01}$	3 – 0.67 (3)= $\boxed{0.99}$	–	–	–	–	0/0.99 = 0 (KR)?→
		Z_j C_j	0.5–0.01 M1	1.5 1.5	–2+0.01 M2	1.01–0.99 M5					
		$Z_j - C_j$	–0.5–0.01M	0	–4+0.01M	–3.99–0.99M					

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0 + 0$

$= 0$

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6–0 (–0.34) = 6	2.34–0.01 (–0.34) = 2.34	0	–0.34–1(–0.34) = 0	6.66– (–1) = $\boxed{6.67}$	–	–	–	–	6/6.67 = 0.9 (KR) →
S_2	0	4 – 0 (4.33) = 4 (4.33)= 1.63	1.67–0.01 (4.33) = 0	0	4.33–1(4.33) = 0	2.33– (–0.01)	–	–	–	–	4/2.37 = 1.69
y_2	1.5	0–0 (0.67)= 0 (0.67)= 0.32	0.33–0.01(0.67) = 0	1	0.67–1 (0.67) = –1.32	–1.33–(–0.01) = –1.32	–	–	–	–	0/–1.32 = –
y_4	5	0/0.99 = 0 9 = 0.01	0.01/0.9 = 1	0	0.99/0.99 = 0–0.01	–0.01/0.09	–	–	–	–	0/–0.01 = –
		Z_j C_j	0.53 1	1.5 1.5	5 5	2.03 2					
		$Z_j - C_j$	0.47	0	0	–4.03					

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 \times 6 + 0 \times 4 + 1.5 \times 0 + 5 \times 0$

$= 0$

Step 7: Fourth iteration of Simplex Method

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
y_3	2	6/6.67 = 0	2.34/6.67 = 0.35	0	6.67/6.67 = 1	0	–	–	–	–	–
S_2	0	4– (0.9) (2.37) = 1.87	1.63 – 0.35 (2.37) = 0.80	0	2.37–1 (2.37)= 0	0	–	–	–	–	–
y_2	1.5	0–0.9 (–1.32) = 1.19	0.32–0.35 (–1.32) = 0.78	1	–1.32 – 1 (–1.32) = 0	0	–	–	–	–	–
y_4	5	0–0.9 (–0.01) = 0.009	0.01–0.35 (–0.01) = 0.014	0	–0.01–1 (–0.01) = 0	1	–	–	–	–	–
		Z_j C_j	3.27 1	1.5 1.5	2 2	5 5					
		$Z_j - C_j$	2.27	0	0	0					

Notes

Therefore,

$$'Z' = C_BX_B$$
$$= (2 \times 0.9) + (0 \times 1.87) + (1.5 \times 1.19) + (5 \times 0.009)$$
$$= 1.8 + 0 + 1.79 + 0.045$$
$$= 3.64$$

Did u know? Big 'M' Method is also known as 'Charnes' 'M' Technique.

Caution If the objective function z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The only visible difference between these two penalty is that the one will be designated by -M for a maximization problem and +M for a minimization problem, where M>0.

Self Assessment

6. Solve the following LPP using the Big M method.

   Maximise $'Z' = 40x_1 + 60x_2$ [Subject to constraints]

   $2x_1 + x_2 \leq 70$

   $x_1 + x_2 \geq 40$

   $x_1 + x_2 \geq 40$

   $x_1 + 3x_2 \leq 90$

   Where, $x_1, x_2 \geq 0$

7. Solve the following LPP using the Big M method.

   Maximise $'Z' = 5x_1 + 3x_2$ [Subject to constraints]

   $x_1 + x_2 \leq 6$

   $2x_1 + 3x_2 \geq 3$

   $x_1 \leq 3$

   $x_2 \leq 3$

   Where, $x_1, x_2 \geq 0$

3.3 Unconstrained Variables

Sensitivity analysis involves 'what if?' questions. In the real world, the situation is constantly changing like change in raw material prices, decrease in machinery availability, increase in profit on one product, and so on. It is important to decision makers for find out how these changes affect the optimal solution. Sensitivity analysis can be used to provide information and to determine solution with these changes.

Sensitivity analysis deals with making individual changes in the coefficient of the objective function and the right hand sides of the constraints. It is the study of how changes in the coefficient of a linear programming problem affect the optimal solution.

Notes

We can answer questions such as,

1. How will a change in an objective function coefficient affect the optimal solution?
2. How will a change in a right-hand side value for a constraint affect the optimal solution?

For example, a company produces two products x_1 and x_2 with the use of three different materials 1, 2 and 3. The availability of materials 1, 2 and 3 are 175, 50 and 150 respectively. The profit for selling per unit of product x_1 is ₹ 40 and that of x_2 is ₹ 30. The raw material requirements for the products are shown by equations, as given below.

$$Z_{max} = 40x_1 + 30x_2$$

Subject to constraints

$$4x_1 + 5x_2 \leq 175 \quad \text{(a)}$$

$$2x_2 \leq 50 \quad \text{(b)}$$

$$6x_1 + 3x_2 \leq 150 \quad \text{(c)}$$

where $x_1, x_2 \geq 0$

The optimal solution is

x_1 = ₹ 12.50

x_2 = ₹ 25.00

Z_{max} = 40 × 12.50 + 30 × 25.00

= ₹ 1250.00

The problem is solved using TORA software and the output screen showing sensitivity analysis is given in Figure 3.1.

Figure 3.1: Sensitivity Analysis Table Output

Variable	Value	Obj Coeff	Obj Val Contrib
x1:	12.50	40.00	500.00
x2:	25.00	30.00	750.00
Constraint	RHS	Slack-/Surplus+	
1 (<)	175.00	0.00	
2 (<)	50.00	0.00	
3 (<)	150.00	0.00	

Sensitivity Analysis

Variable	Current Obj Coeff	Min Obj Coeff	Max Obj Coeff	Reduced Cost
x1:	40.00	24.00	60.00	0.00
x2:	30.00	20.00	50.00	0.00
Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<)	175.00	100.00	175.00	3.33
2 (<)	50.00	50.00	infinity	0.00
3 (<)	150.00	150.00	262.50	4.44

Notes

3.3.1 Change in Objective Function Coefficients and Effect on Optimal Solution

Referring to the current objective coefficient of Figure 2.4, if the values of the objective function coefficient decrease by 16 (Min. obj. coefficient) and increase by 20 (Max. obj. coefficient) there will not be any change in the optimal values of $x_1 = 12.50$ and $x_2 = 25.00$. But there will be a change in the optimal solution, i.e. Z_{max}.

> *Notes* This applies only when there is a change in any one of the coefficients of variables i.e., x_1 or x_2. Simultaneous changes in values of the coefficients need to apply for 100 Percent Rule for objective function coefficients.

For x_1,

$$\begin{aligned}
\text{Allowable decrease} &= \text{Current value} - \text{Min. Obj. coefficient} \\
&= 40 - 24 \\
&= ₹\,16 && \text{(a)} \\
\text{Allowable increase} &= \text{Max. Obj. coefficient} - \text{Current value} \\
&= 60 - 40 \\
&= ₹\,20.00 && \text{(b)}
\end{aligned}$$

Similarly, For x_2,

$$\begin{aligned}
\text{Allowable decrease} &= ₹\,10.00 && \text{(c)} \\
\text{Allowable increase} &= ₹\,20.00 && \text{(d)}
\end{aligned}$$

For example, if coefficient of x_1 is increased to 48, then the increase is 48 – 40 = ₹ 8.00 From (b), the allowable increase is 20, thus the increase in x_1 coefficient is 8/20 = 0.40 or 40%.

Similarly,

If coefficient of x_2 is decreased to 27, then the decrease is 30 – 27= ₹ 3.00.

From (c), the allowable decrease is 10, thus the decrease in x_2 coefficient is 3/10 = 0.30 or 30%.

Therefore, the percentage of increase in x_1 and the percentage of decrease in x_2 is 40 and 30 respectively.

i.e. 40% + 30% = 70%

For all the objective function coefficients that are changed, sum the percentage of the allowable increase and allowable decrease. If the sum of the percentages is less than or equal to 100%, the optimal solution does not change, i.e., x_1 and x_2 values will not change.

But Z_{max} will change, i.e.,

$$= 48(12.50) + 27(25)$$

$$= ₹\,1275.00$$

If the sum of the percentages of increase and decrease is greater than 100%, a different optimal solution exists. A revised problem must be solved in order to determine the new optimal values.

3.3.2 Change in the Right-hand Side Constraints Values and Effect on Optimal Solution

Suppose an additional 40 kgs of material 3 is available, the right-hand side constraint increases from 150 to 190 kgs.

Now, if the problem is solved, we get the optimal values as

$x_1 = 23.61$, $x_2 = 16.11$ and $Z_{max} = 1427.78$

From this, we can infer that an additional resources of 40 kgs increases the profit by = 1427.78 – 1250 = ₹ 177.78

Therefore, for one kg or one unit increase, the profit will increase by

$= 177.78/40$

$= ₹\ 4.44$

Dual price is the improvement in the value of the optimal solution per unit increase in the right-hand side of a constraint. Hence, the dual price of material 3 is ₹ 4.44 per kg.

Increase in material 2 will simply increase the unused material 2 rather than increase in objective function. We cannot increase the RHS constraint values or the resources. If the limit increases, there will be a change in the optimal values.

The limit values are given in Table 2.10, i.e., Min RHS and Max RHS values.

For example, for material 3, the dual price ₹ 4.44 applies only to the limit range 150 kgs to 262.50 kgs.

Where there are simultaneous changes in more than one constraint RHS values, the

100 Per cent Rule must be applied.

Reduced Cost

$$\text{Reduced cost/unit of activity} = \begin{pmatrix}\text{Cost of consumed} \\ \text{resources per unit} \\ \text{of activity}\end{pmatrix} - \begin{pmatrix}\text{Profit per unit} \\ \text{of activity} \\ \end{pmatrix}$$

If the activity's reduced cost per unit is positive, then its unit cost of consumed resources is higher than its unit profit, and the activity should be discarded. This means that the value of its associated variable in the optimum solution should be zero.

Alternatively, an activity that is economically attractive will have a zero reduced cost in the optimum solution signifying equilibrium between the output (unit profit) and the input (unit cost of consumed resources).

In the problem, both x_1 and x_2 assume positive values in the optimum solution and hence have zero reduced cost.

Considering one more variable x_3 with profit ₹ 50

$$Z_{max} = 40x_1 + 30x_2 + 50x_3$$

Subject to constraints,

$$4x_1 + 5x_2 + 6x_3 \le 175 \quad \text{(a)}$$

$$2x_2 + 1x_3 \le 50 \quad \text{(b)}$$

Notes

$$6x_1 + 3x_2 + 3x_3 \leq 150 \qquad (c)$$

where $x_1, x_2, x_3 \geq 0$

The sensitivity analysis of the problem is shown in the computer output below in Figure 3.2.

Figure 3.2: Sensitivity Analysis

Variable	Value	Obj Coeff	Obj Val Contrib
x1:	15.63	40.00	625.00
x2:	0.00	30.00	0.00
x3:	18.75	50.00	937.50

Constraint	RHS	Slack-/Surplus+
1 (<)	175.00	0.00
2 (<)	50.00	31.25-
3 (<)	150.00	0.00

Variable	Current Obj Coeff	Min Obj Coeff	Max Obj Coeff	Reduced Cost
x1:	40.00	33.33	100.00	0.00
x2:	30.00	-infinity	42.50	12.50
x3:	50.00	33.33	60.00	0.00

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<)	175.00	100.00	300.00	7.50
2 (<)	50.00	18.75	infinity	0.00
3 (<)	150.00	87.50	262.50	1.67

The reduced cost indicates how much the objective function coefficient for a particular variable would have to improve before that decision function assumes a positive value in the optimal solution.

The reduced cost of ₹ 12.50 for decision variable x_2 tells us that the profit contribution would have to increase to at least 30 + 12.50 = 42.50 before x_3 could assume a positive value in the optimal solution.

3.4 Special Cases in Linear Programming

Let us discuss special cases in linear programming such as infeasibility and unboundedness. We will examine here how these special problems can be recognised while solving linear programming problems by the simplex method.

3.4.1 Multiple or Alternative Optimal Solutions

In certain conditions, a given LPP may have more than one solution yielding the same optimal function value. Each of such optimal solutions is termed as alternative optimal Solutions.

Example:

Maximise $\quad 'Z' = 3x_1 + 2x_2$

Notes

Sub. to $x_1 \leq 40$

$x_2 \leq 60$

$3x_1 + 2x_2 \leq 180$

$x_1, x_2 \geq 0$

Solution:

Maximise $'Z' = 3x_1 + 2x_2$

Sub. to $x_1 + S_1 = 40$

$x_2 + S_2 = 60$

$3x_1 + 2x_2 + S_3 = 180$

First Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
S_1	0	40	1	0	1	0	0	40/1 = 40 (KR) ?
S_2	0	60	0	1	0	1	0	–
S_3	0	180	3	2	0	0	1	180/3 = 60
		Z_j C_j	0 3	0 2				
		$Z_j - C_j$	–3	–2				

Second Iteration

BV	C_B	X_B	X_1	X_2	Min. Ratio
X_1	3	40	1	0	–
S_2	0	60	0	1	60/1 = 60
S_3	0	180–40 (3) = 60	3–1 (3) = 0	2–0 (3) = 2	60/2 = 30 (KR) ?
		Z_j C_j	3 3	0 2	
		$Z_j - C_j$	0	–2	

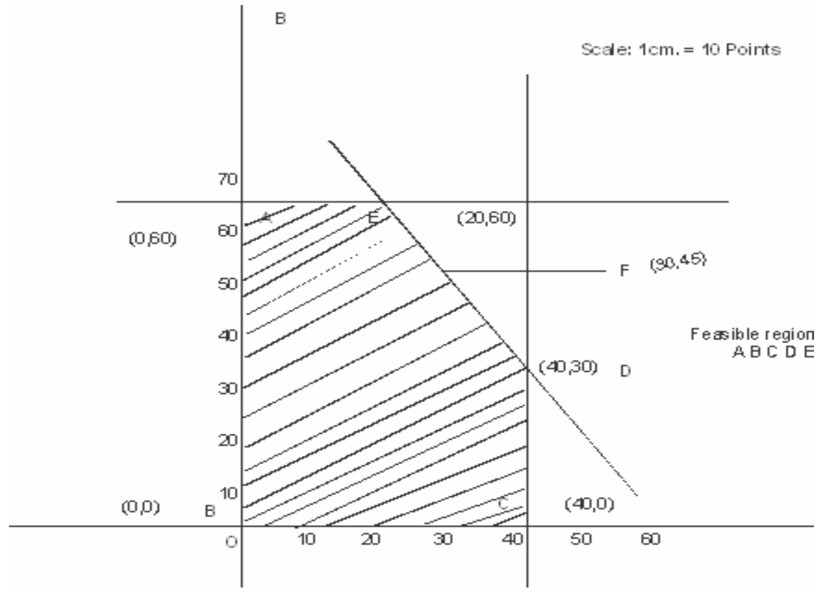
Third Iteration

BV	C_B	X_B	X_1	X_2	Min. Ratio
x_1	3	40	1	0	–
S_2	0	60–30 (1) = 30	0–0 (1) = 0	1–1 (0) = 0	–
S_3	0	60/2 = 30	0	2/2 = 1	–
		Z_j C_j	3 3	2 2	
		$Z_j - C_j$	0	0	

Hence, optimum value for Z = 180 at $x_1 = 40$ and $x_2 = 30$. A graphical representation shows that the points D = (40, 30) and E = (20, 60) are optimal with Z = 180 units. Also, observe that every point on the line DE is optimal. For example, F (30, 45) gives Z = 180 units. Hence, the problem has infinite feasible solutions which are called alternative non-basic feasible solutions.

Notes

Figure 3.3: Graphical Representation of Alternative Non-basic Feasible Solutions



Example:

Maximise $\quad 'Z' = 3x_1 + 2x_2$

Sub. to, $\quad x_1 \le 4$

$x_2 \le 6$

$3x_1 + 2x_2 \ge 18$

$x_1, x_2 \ge 0$

Solution:

Maximise $\quad 'Z' = 3x_1 + 2x_2$

Sub. to $\quad x_1 + S_1 = 4$

$x_2 + S_2 = 6$

$3x_1 + 2x_2 + S_3 = 18$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	$\boxed{1}$	0	1	0	0	4/1 = 4 (KR) ?
S_2	0	6	0	1	0	1	0	–
S_3	0	18	3	2	0	0	1	18/3 = 6
		Z_j	0	0				
		C_j	3	2				
		$Z_j - C_j$	–3	–2				

Notes

Second Iteration

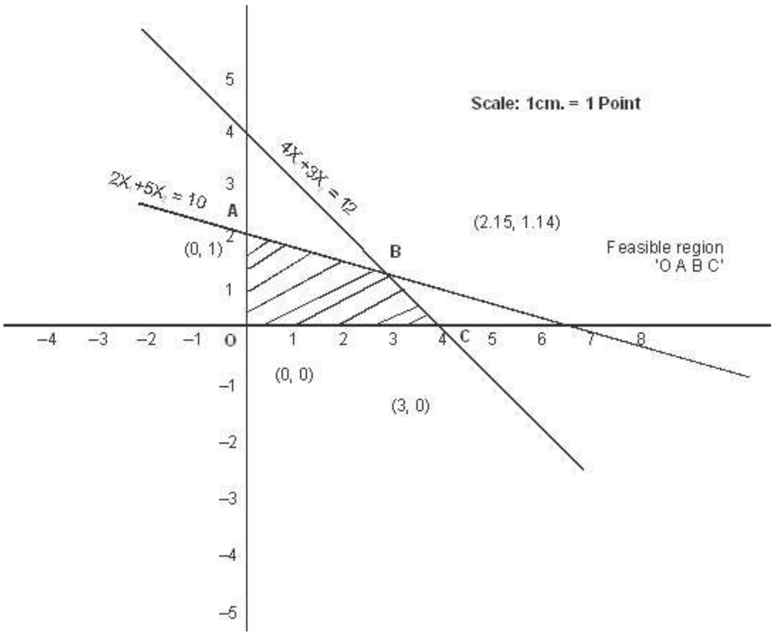
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
x_1	0	4	1	0	–	–	–	–
S_2	0	6	0	1	–	–	–	6/1 = 6
S_3	0	18–4 (3) = 6	3–1 (3) = 0	2–0 (3) = 2	–	–	–	6/2 = 3 (KR) →
		Z_j	3	0				
		C_j	3	2				
		$Z_j - C_j$	0	–2				

Third Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
X_1	3	4	1	0	–	–	–	–
S_2	0	6–3 (1) = 3	0	1–1 (1) = 0	–	–	–	–
x_2	2	6/2 = 3	0	2/2 = 1	–	–	–	–
		Z_j	3	2				
		C_j	3	2				
		$Z_j - C_j$	0	0				

The solution is optimal with Z = 18 at x_1 = 4 and x_2 = 3. A graphical representation of the problem reveals the multiple optimal solutions for the LPP.

Figure 3.4: Graphical Representation of Multiple Optimal Solutions for the LPP



3.4.2 Unbounded Solutions

Sometimes an LP problem will not have a finite solution. This means when no one or more decision variable values and the value of the objective function (maximization case) are permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded.

Notes

Example:

Maximise $\quad 'Z' = 3x + 2y$

Sub. to $\quad x - y \leq 15$

$$2x - y \leq 40$$

$$x, y \geq 0$$

Where, S_1 and S_2 are slack variables.

First Iteration

BV	C_B	X_B	X	y	S_1	S_2	Min. Ratio
S_1	0	15	$\boxed{1}$	–1	1	0	15/1 = 15 (KR) ?
S_2	0	20	2	–1	0	1	20/2 = 10
		Z_j C_j	0 3	0 2			
		$Z_j - C_j$	–3	–2			

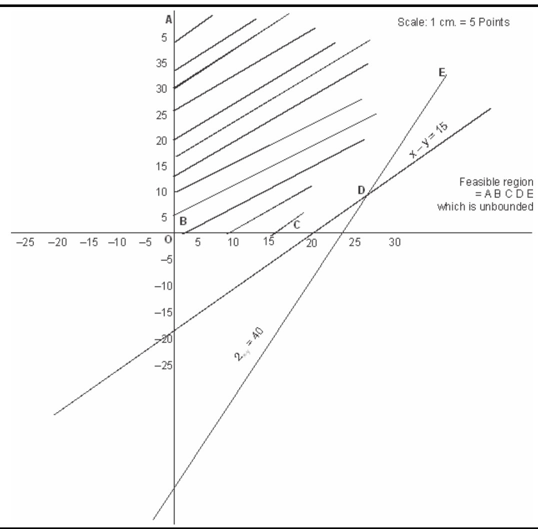
Second Iteration

BV	C_B	X_B	X	Y	S_1	S_2	Min. Ratio
S_1	0	15–10 (1) = 5	1–1 (0) = 0	–1+1/2 (1) –0.5	–	–	y' s are negative
X	3	10	1	–1/2	–	–	
		Z_j C_j	3 3	–1.5 2			
		$Z_j - C_j$	0	–3.5			

At the end of second iteration y should enter the basis to improve the solution of Z. But there is no vector leaving the basis since y's are negative. Hence, the solution is unbounded optimum solution.

Note that the unbounded feasible region by graph is ABCDE. As the point A and E are extended, the value of Z increases. Hence, the solution is unbounded optimal solution.

Figure 3.5: Graphical Representation of Unbounded Optimal Solution



Notes

Example:

Maximise $'Z' = 4x_1 + x_2 + 3x_2 + 5x_4$

Sub. to $-4x_1 + 6x_2 + 5x_3 - 4x_3 \leq 20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Solution:

Maximise $'Z' = 4x_1 + x_2 + 3x_3 + 5x_4$

$-4x_1 + 6x_2 + 5x_3 - 4x_4 + S_1 = 20$

$3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C_B	X_B	X_1	X_2	x_3	x_4	S_1	S_2	S_3	Min. Ratio
S_1	0	20	–4	6	5	–4	–	–	–	
S_2	0	10	3	–2	4	1	–	–	–	10/1 = 10
S_3	0	20	8	–3	3	2	–	–	–	20/2 = 10
		Z_j C_j	0 4	0 4	0 1	0 3	0 5			
		$Z_j - C_j$	–0	4	–1	–3				

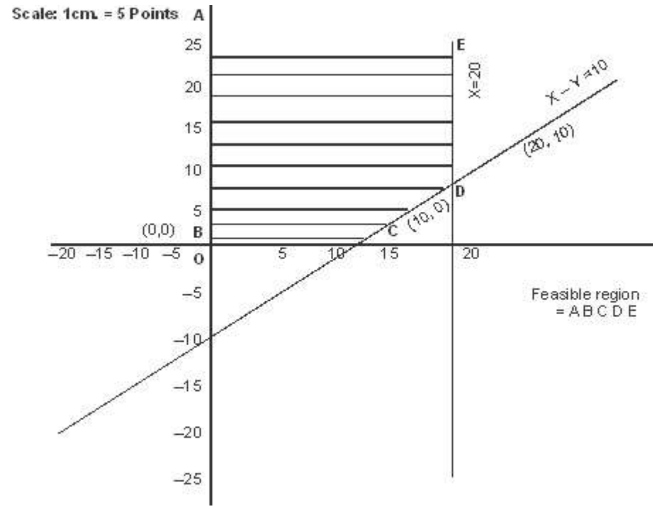
Second Iteration

BV	C_B	X_B	x_1	x_2	x_3	X_4	Min. Ratio
S_1	0	60	12	0	11	0	
S_2	0	10	–1	–0.5	2.5	0	10/1 = 10
S_3	0	10	4	–1.5	1.5	1	20/2 = 10
		Z_j C_j	20 4	–7.5 1	7.5 3	5 5	
		$Z_j - C_j$	16	–8.5	4.5	0	

At the end of second iteration, observe that x_5 should enter the basis. But there is no chance for outgoing vector from the basis since y's are zero or negative. Hence, the solution is unbounded.

Notes

Figure 3.6: Graphical Representation of unbounded solution



3.4.3 Infeasibility

Infeasibility is a condition that arises when constraints are inconsistent (mutually exclusive) i.e. no value of the variables satisfy all the constraint simultaneously. This results in infeasible solution. If two or more constraints of a linear programming problem are mutually conflicting, it does not have a feasible solution. Let us take a problem to illustrate infeasibility.

Example: The Reddin Hardware Ltd. is producing two products, A and B. The profit contribution of product A is ₹ 5 per unit and of product B ₹ 4 per unit. Both the products go through the processing and assembly departments. Product A takes two minutes in the processing department and two minutes in the assembly department. Product B takes one minute in the processing department and one and a half minute in the assembly department. The maximum capacity of the processing department is 10,000 worker-minutes and of the assembly department 12,000 worker-hours. The marketing department has informed that a contract has been made with a hardware chain store for the supply of 6,500 units of product A and that there is no other demand for this product. There is no marketing constraint in the case of product B. What is the optimum product mix for the company?

The linear programming mode for the Reddin Hardware is as follows:

Let

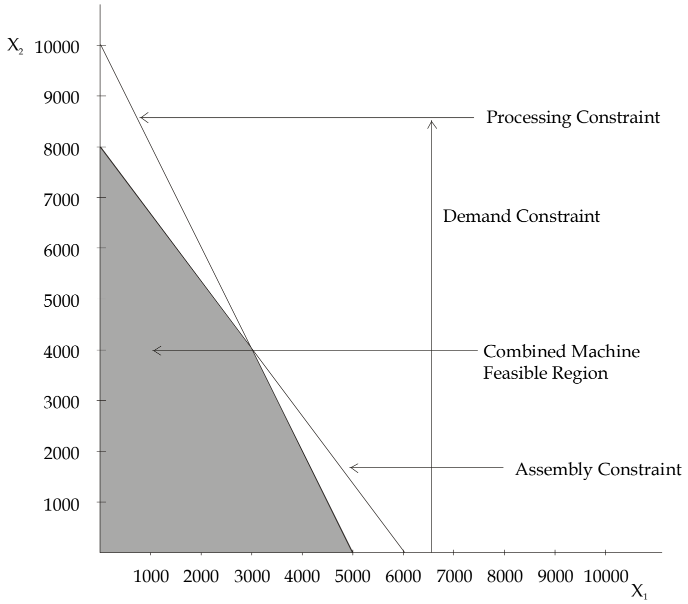
$x_1 =$ product A

$x_2 =$ product B

Maximise	$'Z' = 5x_1 + 4x_2$	Objective Function
	$2x_1 + 1x_2 \leq 10{,}000$	Processing Constraint
	$2x_1 + x_2 \leq 12{,}000$	Assembly Constraint
	$x_1 = 6{,}500$	Demand Constraint
	$x_1 \geq 0$	Non-negativity Constraint

Notes

Figure 3.7: The Reddin Problem—Infeasibility



The graphical presentation of the Reddin Problem in Figure shows that the combined machine feasibility region does not permit the production of 6,500 units of product A. Thus, the demand constraint and the machine capacity constraints are mutually conflicting, resulting in infeasibility.

Let us try to solve this problem by the simplex method. It is formulated below in the standard simplex form.

Row 1: P- $5x_1 - 4x_2$ $-50a_1 = 0$

Row 2: $2x_1 + 1x_2 + s_1 = 10{,}000$

Row 3: $2x_1 + \frac{3}{2}x_2 + s_2 = 12{,}000$

Row 4: $x_1 + a_1 = 6{,}500$

Artificial variable a1 has been assigned a coefficient of 50 in the objective function.

Towards The First Basis

Row 4 × 50: $50x_1 + 50a_1 = 325{,}000$

Old Row 1: $Z - 5x_1 - 4x_2 - 50a_1 = 0$

New Row 1: $Z + 45x_1 - 4x_2 = 325{,}000$

Table 3.2: The First Basis

Row 1: P	$+ 45x_1$	$- 4x_2$				= 325,000
Row 2:	$2x_1$	$+ x_2$	$+ s_1$			= 10,000
Row 3:	$2x_1$	$+ x_2$		$+s_2$		= 12,000
Row 4:	x_1				$+a_1$	= 6,500

Notes

Table 3.3: The Second Basis

Row 1: P +	x_1			$-\ s_2$		$= 357,000$
Row 2:	x_1		$+\ s_1$	$-\ s_2$		$= 2,000$
Row 3:	x_1	$+\ x_2$		$+s_2$		$= 8,000$
Row 4:	x_1				$+a_1$	$= 6,500$

Since there is no variable with a negative coefficient in Row 1, no further improvement in profit is possible. But artificial variable a1 continues to exist in Row 4 and it is also basic variable. This means that the optimum solution has not been reached. The problem has no solution because of infeasibility.

Case Study

Linear Programming & Technical Accounting

A company manufactures 2 products a 10 cu. Ft & 6 cu. ft. refrigerator. The demand for the former has been estimated at 15,000 and for the later at 24,000.

The production process is broken down into 3 stages:

1. Shell production
2. Motor assembly
3. Refrigerator assembly

The production manager estimates that the shell production unit can through put up to 36,000 smaller refrigerators, but only half of this quantity of the 10 cu. ft. product. Similarly, the motor assembly unit can through put 30,000 smaller motors or 80% of this quantity of the other product. The assembly department has no restrictions as labour unskilled is required. The accounting function has produced the standard cost breakdown as follows:

Standard Costs of Production

(in Dollars)

Items	10 cu. ft	6 cu. ft.
Direct materials	30	24
Direct labour	16	10
Variable overhead	14	11
Marginal cost	60	45
Selling price	90	65
Contribution	30	20

Fixed overhead charges are estimated as $ 4,00,000.

Traditional accounting would entail the manufacture of 15,000 of the 10 cu ft. refrigerators before the shell production unit capacity is exceeded only 6,000 smaller refrigerator would be manufactured. On the basis of direct labour, $3,20,000 of fixed overheads would be attributed to 10 cu. ft. & 80,000 to the 6 cu. ft.

Contd...

Notes

The net profit would thus be

Items	10 cu. ft	6 cu. Ft.	Total (in '000 dollars)
Contribution	450	120	570
Overhead allocation	320	80	400
Net profit	130	40	170

But this is an over simplification when a resource is limiting. The scarce resource is shell production and contribution per unit is \$ 20 for the cu. ft & \$ 15 for the 10 cu. ft. This suggests that the 6 cu. ft. is more profitable and should be produced to its demand of 24,000. For this, the net profit would stand at \$2,24,000.

But both are misleading because profitability is not directly related to contribution per unit product or per unit resource. Hence an examination from the linear programming view point was carried out to arrive at a logical solution.

Questions:

1. Formulate the given problem as a linear programming problem.
2. Solving the problem by the revised simplex technique to determine the maximum profit.

Self Assessment

Fill in the blanks:

8. The optimal solution of the primal problem appears under the .... variables in the .... row of the final simplex table associated with the dual problem.
9. The .... analysis involves "what if" questions.
10. The original linear programming problem is known as .... problem.

3.5 Summary

In this unit, you learned the mechanics of obtaining an optimal solution to a linear programming problem by the simplex method. The simplex method is an appropriate method for solving a ≤ type linear programming problem with more than two decision variables. Two phase and M-method are used to solve problems of ≥ or ≤ type constraints. Further, the simplex method can also identify multiple, unbounded and infeasible problems.

3.6 Keywords

Artificial Variables: Temporary slack variables which re used for calculations.

Simplex Method: A method which examines the extreme points in a systematic manner, repeating the same set of steps of the algorithms until an optimal solution is reached.

Slack Variables: Amount of unused resources.

Surplus Variables: A surplus variable represents the amount by which solution exceeds a resource.

Unconstrained Variable: The variable having no non-negativity constraint.

Notes

3.7 Review Questions

1. Explain the simplex procedure to solve the Linear Programming Problem.
2. How do you recognize optimality in the simplex method?
3. Can a vector that is inserted at one iteration in simplex method be removed immediately at the next iteration? When can this occur and when is it impossible?
4. Explain briefly the term 'Artificial' variables.
5. Explain the use of artificial variables in L.P..
6. Maximize: $Z = x_1 + x_2$

   Subject to $x_1 + 5x_2 \le 5$

   $2x_1 + x_2 \le 4$ with x_1, x_2 non-negative.
7. Minimize: $Z = 3x_1 + 4x_2$

   Subject to $2x_1 + x_2 \le 6$

   $2x_1 + 3x_2 \ge 9$

   With, x_1, x_2 non-negative.
8. Minimize: $Z = x_1 + 2x_2$

   Subject to $x_1 + 3x_2 \le 11$

   $2x_1 + x_2 \ge 9$

   With, x_1, x_2 non-negative.
9. Maximize $Z = -x_1 - x_2$

   Subject to $x_1 + 2x_2 \ge 5000$

   $5x_1 + 3x_2 \ge 12000$

   With, x_1, x_2 non-negative
10. Maximize $Z = 2x_1 + 3x_2 + 4x_3$

    $x_1 + x_2 + x_3 \ge 1$

    $x_1 + x_2 + 2x_3 = 2$

    $2x_1 + 2x_2 + x_3 \ge 4$

    With all variables non-negative
11. Minimize: $Z = 14x_1 + 13x_2 + 11x_3 + 13x_4 + 13x_5 + 12x_6$

    Subject to $x_1 + x_2 + x_3 = 1200$

    $x_4 + x_5 + x_6 = 1000$

    $x_1 + x_4 = 1000$

    $x_2 + x_5 = 700$

    $x_3 + x_6 = 500$

    With all variables non-negative.

Notes

12. Discuss the role of sensitivity analysis in linear programming.
13. In sensitivity analysis, explain
    (a) The effect of change of objective function coefficients.
    (b) The effect of change in the right hand side of constraints.

Answers: Self Assessment

1.	z = 21	2.	z = 28
3.	key element or pivotal element	4.	slack
5.	surplus	6.	z = 2,100
7.	z = 7.45 – 3.51m	8.	slack, last
9.	sensitivity	10.	Primal

3.8 Further Readings

Books

J.K. Sharma, Operations Research, Theory and Applications, MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, Operations Research, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, Operations Research: A Practical Introduction, CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, Operations Research Models and Methods, John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, Schaum's Outline of Theory and Problems of Operations Research, McGraw-Hill Professional; 1997

Online links

home.fit.ba

www.maa.org

www.lionhrtpub.com

Notes

Unit 4: Linear Programming – Duality

CONTENTS

Objectives

Introduction

4.1 Concept of Duality

4.2 Dual Formulation

4.3 Advantages of Duality

4.4 Summary

4.5 Keywords

4.6 Review Questions

4.7 Further Readings

Objectives

After studying this unit, you will be able to:

- Analyze the importance of duality in linear programming
- Understand dual formation and solution
- Learn the significance and advantages of duality in LPP

Introduction

For every linear programming problem, there is an associated linear programming problem. The former problem is called primal and the latter is called its dual and vice versa. The two problems may appear to have superficial relationship between each other but they possess very intimately related properties and useful one, so that the optimal solution of one problem gives complete information about the optimal solution to the other. In other words, the optimal solutions for both the problems are same.

The concept of duality is very much useful to obtain additional information about the variations in the optimal solutions when certain changes are effected in the constraint coefficients, resource availabilities and objective function coefficients. This is termed as post-optimality or sensitivity analysis.

4.1 Concept of Duality

One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*. In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Notes

Caution The necessary and sufficient condition for any LPP and its dual to have optimal solutions is that both have feasible solutions.

4.2 Dual Formulation

The following procedure is adopted to convert primal problem into its dual. Simplex method is applied to obtain the optimal solution for both primal and dual form.

Step 1: For each constraint, in primal problem there is an associated variable in the dual problem.

Step 2: The elements of right-hand side of the constraints will be taken as the coefficients of the objective function in the dual problem.

Step 3: If the primal problem is maximization, then its dual problem will be minimization and vice versa.

Step 4: The inequalities of the constraints should be interchanged from ^{3} to ≤ and vice versa and the variables in both the problems are non-negative.

Step 5: The rows of primal problem are changed to columns in the dual problem. In other words, the matrix A of the primal problem will be changed to its transpose (A) for the dual problem.

Step 6: The coefficients of the objective function will be taken to the right hand side of the constraints of the dual problem.

Example 1: Write the dual of the following problem.

Maximise $\quad$ 'Z' $= -6x_1 + 7x_2$

Subject to,

$$-x_1 + 2x_2 \le -5$$
$$3x_1 + 4x_2 \le 7$$
$$x_1, x_2 \ge 0$$

Solution: The given problem is considered as primal linear programming problem. To convert it into dual, the following procedure is adopted.

Step 1: There are 2 constraints and hence the dual problem will have 2 variables. Let us denote them as y_1 and y_2.

Step 2: The right hand side of the constraints are –5 and 7 which are taken as the coefficients of the variables y_1 and y_2 in the objective function.

Step 3: The primal objective problem is maximization and hence the dual seeks minimization for the objective function. Hence, the objective functions for the dual problem is given by

$$\text{Minimise } Z = -5y_1 + 7y_2$$

Step 4: The inequalities of the constraints for the primal problem are of the type (≤). Hence, the inequalities for the dual constraints will be of the type (≥).

Step 5: The coefficient matrix for the primal problem is

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$

Notes

The transpose of this matrix which serves as the coefficient matrix for the dual problem is given by.

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

Step 6: The coefficients of the objective function for the given primal are –6 and 7. They are taken on the right hand side of the constraints for the dual problem. Hence, the constraints for the dual problem are represented as

$$-y_1 + 3y_2 \geq -6$$
$$2y_1 + 4y_2 \geq 7$$
$$y_1, y_2 \geq 0$$

Example 2: Find the dual of the following problem:

Minimise $\quad Z = 3x_1 + 4x_2 + 5x_3$

Subject to $\quad x_1 + x_3 \geq 3$

$x_2 + x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Solution: Let the given problem be denoted as primal. The objective of the primal is minimization and hence the objective of the dual is maximization. There are two constraints for the primal problem. Hence, the problem has 2 variables namely, y_1 and y_2. The right hand side of the constraints are 3 and 4 which are the coefficients for y_1 and y_2 in the objective function for the dual problem given as

Maximise $\quad Z = 3y_1 + 4y_2$

The coefficient of the objective function in the primal problem are 3, 4 and 5 which serve as the right hand side of the constraints for the dual problem. The inequalities of the constraints of the type (≥) are converted as (≤) type for the dual problem. The coefficient matrix for primal problem is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

And hence the coefficient matrix for the dual problem is

$$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence, the dual problem is represented as

Maximise 'Z' = $3y_1 + 4y_2$

Subject to

$y_1 + y_2 \leq 3 \quad$ is $\quad y_1 \leq 3$

$y_1 + y_2 \leq 4 \quad\quad y_2 \leq 4$

$y_1 + y_2 \leq 5 \quad\quad y_1 + y_2 \leq 5$

$y_1, y_2 \geq 0 \quad\quad y_1, y_2 \geq 0$

Example 3: Write the dual of the following problem. Show that the optimal solution for primal and dual are same.

Minimise 'Z' = $4x_1 + 3x_2 + 6x_3$

Subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Solution:

The dual for the given problem can be written as

Maximise Z = $2y_1 + 5y_2$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Subject to $y_1 \leq 4$

$y_2 \leq 3$

$y_1 + y_2 \leq 6$

$$A^{|} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$y_1, y_2 \geq 0$

Let us use simple technique to obtain the optimum solution.

Maximise 'Z' = $2y_1 + 5y_2$

Subject to $y_1 + y_3 = 4$

$y_2 + y_4 = 3$

$y_1 + y_2 + y_5 = 6$

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Where, y_3, y_4 and y_5 are slack variables.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	1	0	1	0	0	–
S_2	0	3	0	$\boxed{1}$	0	1	0	3/1 = 3 (KR) ?
S_3	0	6	1	1	0	0	1	6/1 = 6
		Z_J C_J	0 2	0 5	0 0	0 0	0 0	
		$Z_J - C_J$	–2	–5	0	0	0	

(↑ KC)

Notes

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	1	0	1	0	0	4/1 = 4
y_2	5	3	0	1	0	1	0	--
S_3	0	6–3 (1)= 3	1–0 (1)= $\boxed{1}$	1–1 (1)= 0	0–0 (1)= 0	0–1 (1)= –1	1–0 (1)= 1	3/1 = 3 KR→
		Z_J	0	5	0	5	0	
		C_J	2	5	0	0	0	
		$Z_J - C_J$	–2	0	0	5	0	

(↑ KC)

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4–3 (1) = 1	1–1 (1) = 0	0–0 (1) = 0	1–0 (1) = 1	0+1 (1) = 1	0–1 (1) = –1	-
y_2	5	3	0	1	0	1	0	-
y_1	2	3	1	0	0	–1	1	-
		Z_J	2	5	0	3	2	
		C_J	2	5	0	0	0	
		$Z_J - C_J$	0	0	0	3	2	

Z = 15 + 6

Z = 21

$Z_J - C_J = 0$ for all V_j, hence the solution is optimal with Z = 21 at $y_1 = 3$ and y_2 for the dual form. Now, let us take the primal form and obtain the optimal solution.

Minimise $Z = 4x_1 + 3x_2 + 6x_2$

Subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Maximise $Z' = -4x - 3x_2 - 6x_3$

Subject to $x_1 + x_3 - x_4 + a_1 = 2$

$x_2 + x_3 - x_5 + a_2 = 5$

Where x_4 and x_5 are surplus variables; a_1 and a_2 are artificial variables.

BV	C_B	X_B	y_1	y_2	y_3	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	2	1	0	$\boxed{1}$	–1	0	1	0	2/1 = 2 KR→
a_2	–M	5	0	1	1	0	–1	0	1	5/1 = 5
		Z_j	–M	–M	–2M					
		C_j	–4	–3	–6					
		$Z_j - C_j$	–M+4	–M+3	–2M+6					

(↑ KC)

Z = –7M

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min Ratio
y_3	–6	2	1	0	1	-	-	-	-	-
a_2	M	5–2 (1) = 3	0–1 (1)= –1	1–0 (1)= $\boxed{1}$	1–1 (1)= 0	-	-	-	-	4/1 = 4→
		Z_j	M–6	–M	–6					
		C_j	–4	–3	–6					
		$Z_j – C_j$	M–2	–M+3	0					

(↑ KC)

Z = –12 – 3m

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min Ratio
y_3	–6	2	1	0	1	–	–	–	–	–
y_2	–3	3	–1	1	0	–	–	–	–	–
		Z_j	–3	–3	–6					
		C_j	–4	–3	–6					
		$Z_j – C_j$	1	0	0					

$Z_j – C_j \geq 0$ hence, solution is optimal.

Maximise 'Z_1' = –21

Therefore Minimise Z = 21

Hence, the optimal solution for primal problem is Z = 21, where $x_1 = 0$, $x_2 = 3$ and $x_3 = 2$. Observe that the optimal solution for primal and dual is same i.e., Z = 21

Primal Form	Dual Form
Min. Z = $4x_1 + 3x_2 + 6x_3$	Max. Z = $2y_1 + 5y_2$
Min. Z = 21	Max. Z = 21
$x_1 = 0$	$y_1 = 3$
$x_2 = 3$	$y_2 = 3$
$x_3 = 2$	

The primal solution can be obtained from the last table of the dual problem without simplex iteration. Notice that $Z_j – C_j$ entries for S_1, S_2 and S_3 are respectively 0, 3 and 2 in the final iteration of the dual problem there are optimum value of the variables in the primal problem. Hence, $x_1 = 0$, $x_2 = 3$ and $x_3 = 2$ and Z = 21 which tallies with the solution obtained by simplex iteration of the primal problem.

Notes The study of duality is very important in LP. Knowledge of duality allows one to develop increased insight into LP solution interpretation. Also, when solving the dual of any problem, one simultaneously solves the primal. Thus, duality is an alternative way of solving LP problems. However, given today's computer capabilities, this is an infrequently used aspect of duality. Therefore, we concentrate on the study of duality as a means of gaining insight into the LP solution.

Notes

Task Find the dual to following problem:

Maximise $Z = 4x_1 + 5x_2 + 3x_3 + 6x_4$

Subject to $x_1 + 3x_2 + x_3 + 2x_4 \leq 2$

$$3x_1 + 3x_2 + 2x_3 + 2x_4 \leq 4$$

$$3x_1 + 2x_2 + 4x_3\ 5x_4 \leq 6$$

$$x_1, x_2, x_3, x_4 \geq 0$$

4.3 Advantages of Duality

1. It is advantageous to solve the dual of a primal having less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.
2. It avoids the necessity for adding surplus or artificial variables and solves the problem quickly (the technique is known as the primal-dual method).
3. The dual variables provide an important economic interpretation of the final solution of an LP problem.
4. It is quite useful when investigating changes in the parameters of an LPP (the technique known as sensitivity analysis).
5. Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible.

Did u know? The dual of the dual problem is again the primal problem.

Task Find out the managerial significance of Duality with the help of some real life cases.

Self Assessment

State true or false:

1. One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*.
2. In the primal problem, the objective function is a exponential combination of *n* variables.
3. In the dual problem, the dual vector multiplies the constants that determine the positions of the constraints in the primal.
4. Duality is quite useful when investigating changes in the parameters of an LPP(the technique known as sensitivity analysis).
5. It is advantageous to solve the dual of a primal having less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.

Notes

Case Study

Minimum Dietary Requirement

A dietician wishes to design a minimum-cost diet to meet minimum daily requirements for calories, protein, carbohydrate, fat, vitamin A and vitamin B dietary needs. Several different foods can be used in the diet, with data as specified in the following table.

	Content and costs per pound consumed						
	Food 1	Food 2	....	Food j	....	Food n	Daily requirements
Calories	a_{11}	a_{12}		a_{1j}		a_{1n}	b_1
Protein (grams)	a_{21}	a_{22}		a_{2j}		a_{2n}	b_2
Carbohydrate (grams)	a_{31}	a_{32}		a_{3j}		a_{3n}	b_3
Fat (grams)	a_{41}	a_{42}		a_{4j}		a_{4n}	b_4
Vitamin A (milligrams)	a_{51}	a_{52}		a_{5j}		a_{5n}	b_5
Vitamin B (milligrams)	a_{61}	a_{62}		a_{6j}		a_{6n}	b_6
Costs (dollars)	c_1	c_2		c_j		c_n	

Questions:

1. Formulate a linear program to determine which foods to include in the minimum cost diet. (More than the minimum daily requirements of any dietary need can be consumed.)

2. State the dual to the diet problem, specifying the units of measurement for each of the dual variables. Interpret the dual problem in terms of a druggist who sets prices on the dietary needs in a manner to sell a dietary pill with b_1, b_2, b_3, b_4, b_5 and b_6 units of the given dietary needs at maximum profit.

4.4 Summary

One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*. In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Notes

4.5 Keywords

Dual Problem: In the dual problem, the dual vector multiplies the constants that determine the positions of the constraints in the primal.

Duality principle: In optimization theory, the duality principle states that optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem.

Primal Problems: In the primal problem, the objective function is a linear combination of *n* variables.

4.6 Review Questions

Determine the duals of the given problems.

1. Minimise $Z = 12x_1 + 26x_2 + 80x_3$

$$2x_1 + 6x_2 + 5x_3 \geq 4$$
$$4x_1 + 2x_2 + x_3 \geq 10$$
$$x_1 + x_2 + 2x_3 \geq 6$$

With all variables non-negative.

2. Minimise $Z = 3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5$

Subject to:
$$2x_1 + 5x_2 + x_4 + x_5 \geq 6$$
$$4x_2 - 2x_3 + 2x_4 + 3x_5 \geq 5$$
$$x_1 - 6x_2 + 3x_2 + 7x_4 + 5x_5 \geq 7$$

With all variables non-negative.

3. Maximise $Z = 6x_1 - x_2 + 3x_3$

Subject to
$$7x_1 + 11x_2 + 3x_3 \leq 25$$
$$2x_1 + 8x_2 + 6x_3 \leq 30$$
$$6x_1 + x_2 + 7x_3 \leq 35$$

With all variables non-negative.

4. Minimise $Z = 10x_1 + 15x_2 + 20x_3 + 25x_4$

Subject to:
$$8x_1 + 6x_2 - x_3 + x_4 \geq 16$$
$$3x_1 + 2x_3 - x_4 \geq 20$$

With all variables non-negative.

5. Minimise $Z = x_1 + 2x_2 + x_3$

Subject to
$$x_2 + x_3 = 1$$
$$3x_1 + x_2 + 3x_3 = 4$$

With all variables non-negative.

Notes

Answers: Self Assessment

1. True
2. False
3. True
4. True
5. True

4.7 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://web.mit.edu/15.053/

http://agecon2.tamu.edu/

http://lucatrevisan.wordpress.com/

Notes

Unit 5: Transportation Problem

CONTENTS

Objectives

Introduction

5.1 Modeling of Transportation Problem

5.1.1 Mathematical Representation

5.1.2 Network Representation of Transportation Model

5.1.3 General Representation of Transportation Model

5.2 Use of Linear Programming to Solve Transportation Problem

5.3 Minimizing Case

5.4 Maximization Transportation Problem

5.5 Balanced Transportation Problem

5.6 Unbalanced Transportation Problem

5.6.1 Demand Less than Supply

5.6.2 Demand Greater than Supply

5.7 Initial Feasible Solution

5.7.1 Algorithm for North-West Corner Method (NWC)

5.7.2 Algorithm for Row and Column Minima Method or Least Cost Method (LCM)

5.7.3 Algorithm for Vogel's Approximation Method (VAM)

5.8 Degeneracy in Transportation Problems

5.9 Summary

5.10 Keywords

5.11 Review Questions

5.12 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the meaning of operations research
- Know about the history of operations research
- Discuss the scope and application of operations research
- Explain the various types of models used in operations research

Introduction

Notes

Transportation problem is a particular class of linear programming, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of sources (e.g., factory) to a set of destinations (e.g., warehouse) to meet the specific requirements. In other words, transportation problems deal with the transportation of a product manufactured at different plants (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. In addition, we must also know the location, to find the cost of transporting one unit of commodity from the place of origin to the destination. The model is useful for making strategic decisions involved in selecting optimum transportation routes so as to allocate the production of various plants to several warehouses or distribution centers.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model.

5.1 Modeling of Transportation Problem

A transportation problem can be expressed in two ways.

1. Mathematical representation
2. Network representation

Obviously the method used for solving the problems is the formulation of transportation problem through mathematical methods. But for understanding of the readers, network representation is equally important.

Let us understand each of them one by one.

5.1.1 Mathematical Representation

The transportation problem applies to situations where a single commodity is to be transported from various sources of supply **(origins)** to various demands **(destinations).**

Let there be *m* sources of supply $S_1, S_2, \ldots\ldots\ldots\ldots S_m$ having a_i (i = 1, 2,......m) units of supplies respectively to be transported among *n* destinations $D_1, D_2 \ldots\ldots\ldots D_n$ with b_j (j = 1,2…..n) units of requirements respectively. Let C_{ij} be the cost for shipping one unit of the commodity from source i, to destination j for each route. If x_{ij} represents the units shipped per route from source i, to destination j, then the problem is to determine the transportation schedule which minimizes the total transportation cost of satisfying supply and demand conditions.

The transportation problem can be stated mathematically as a linear programming problem as below:

Minimize $$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Notes

Subject to constraints,

$$\sum_{j=1}^{n} x_{ij} = a_{i,} \quad i = 1,2,\ldots..m \text{ (supply constraints)}$$

$$\sum_{i=1}^{n} x_{ij} = b_{j,} \quad j = 1,2,\ldots..m \text{ (demand constraints)}$$

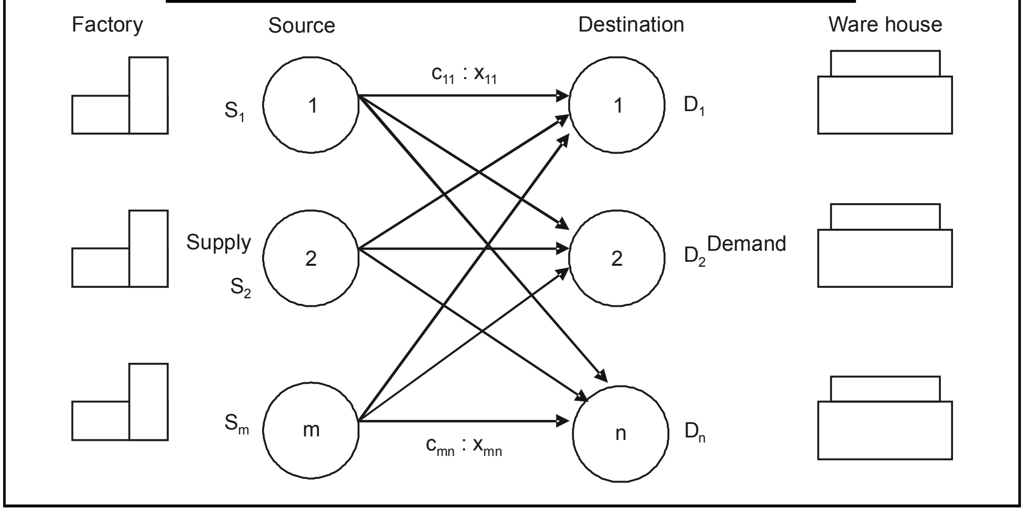
and x_{ij} ³ 0 for all *i = 1,2,…..m and,*

j = 1,2,…..m

5.1.2 Network Representation of Transportation Model

The transportation model is represented by a network diagram in Figure 3.1

Figure 5.1: Network Representation of Transportation Model



where,

m be the number of sources,

n be the number of destinations,

S_m be the supply at source m,

D_n be the demand at destination n,

c_{mn} be the cost of transportation from source m to destination n, and

x_{mn} be the number of units to be shipped from source m to destination n.

The objective is to minimize the total transportation cost by determining the unknowns x_{mn}, i.e., the number of units to be shipped from the sources and the destinations while satisfying all the supply and demand requirements.

5.1.3 General Representation of Transportation Model

Notes

The Transportation problem can also be represented in a tabular form as shown in Table 5.1

Let c_{ij} be the cost of transporting a unit of the product from i^{th} origin to j^{th} destination.

a_i be the quantity of the commodity available at source i,

b_j be the quantity of the commodity needed at destination j, and

x_{ij} be the quantity transported from i^{th} source to j^{th} destination

Table 5.1: Tabular Representation of Transportation Model

		To			Supply
	D_1	D_2	$\cdots$	D_3	
From					a_1
S_1	C_{11} X_{11}	C_{12} X_{12}	$\cdots$	C_{1n}	a_1
S_2	C_{21} X_{21}	C_{22} X_{22}	$\cdots$	C_{2n}	a_2
. . .	. . .	. . .	$\cdots$ $\cdots$ $\cdots$	. . .	. . .
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	$\cdots$	C_{mn}	a_m
b_j Demand	b_1	b_2	$\cdots$	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

If the total supply is equal to total demand, then the given transportation problem is a balanced one.

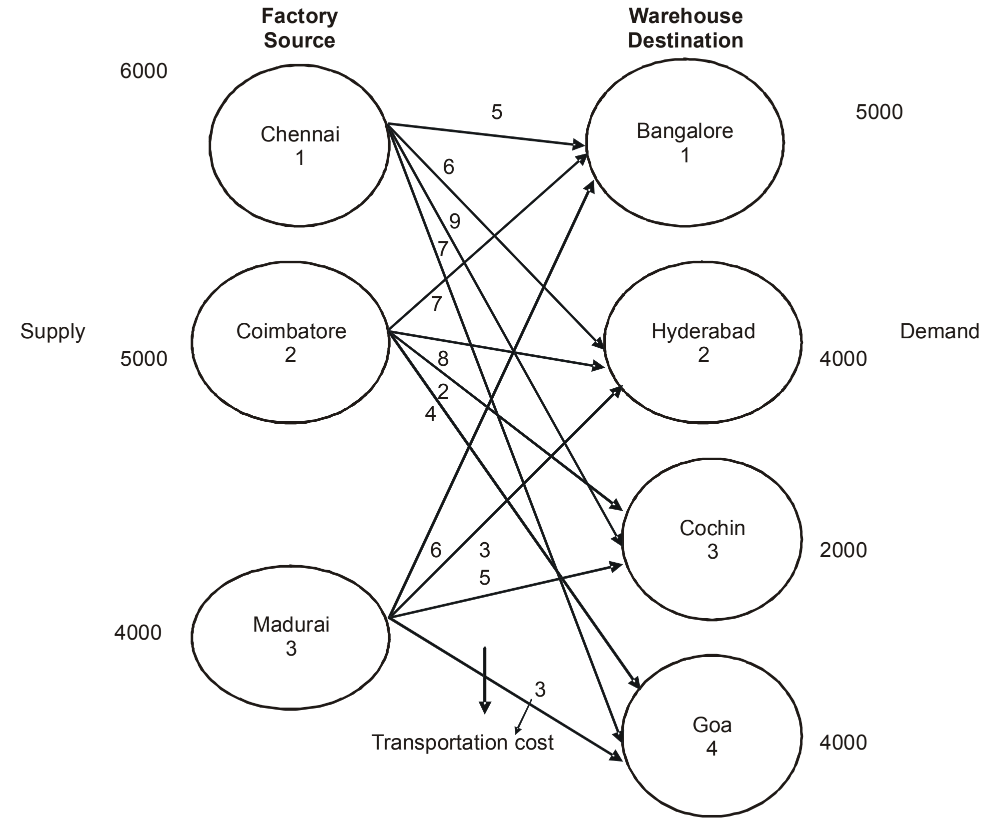
5.2 Use of Linear Programming to Solve Transportation Problem

The network diagram shown in Figure 5.2 represents the transportation model of GM Textiles units located at Chennai, Coimbatore and Madurai. GM Textiles produces ready-made garments at these locations with capacities 6000, 5000 and 4000 units per week at Chennai, Coimbatore and Madurai respectively. The textile unit distributes its ready-made garments through four of its wholesale distributors situated at four locations Bangalore, Hyderabad, Cochin and Goa. The weekly demand of the distributors are 5000, 4000, 2000 and 4000 units for Bangalore, Hyderabad, Cochin and Goa respectively.

The cost of transportation per unit varies between different supply points and destination points. The transportation costs are given in the network diagram.

Notes

Figure 5.2: Network Model of Transportation Problem



The management of GM Textiles would like to determine the number of units to be shipped from each textile unit to satisfy the demand of each wholesale distributor. The supply, demand and transportation cost are as follows:

Table 5.2 (a): Production Capacities

Supply	Textile Unit	Weekly Production (Units)
1	Chennai	6000
2	Coimbatore	5000
3	Madurai	4000

Table 5.2 (b): Demand Requirements

Destination	Wholesale Distributor	Weekly Demand (Units)
1	Bangalore	5000
2	Hyderabad	4000
3	Cochin	2000
4	Goa	4000

Notes

Table 5.2 (c): Transportation Cost per Unit

Supply	Destination			
	Bangalore	Hyderabad	Cochin	Goa
Chennai	5	6	9	7
Coimbatore	7	8	2	4
Madurai	6	3	5	3

A linear programming model can be used to solve the transportation problem.

Let,

X_{11} be number of units shipped from source1 (Chennai) to destination 1 (Bangalore).

X_{12} be number of units shipped from source1 (Chennai) to destination 2 (Hyderabad).

X_{13} number of units shipped from source 1 (Chennai) to destination 3 (Cochin).

X_{14} number of units shipped from source 1 (Chennai) to destination 4 (Goa).

:

and so on.

X_{ij} = number of units shipped from source i to destination j, where i = 1,2,……..m and,

j = 1,2,………n.

5.3 Minimizing Case

Objective Function

The objective is to minimize the total transportation cost. Using the cost data table, the following equation can be arrived at:

- Transportation cost for units shipped from Chennai = $5x_{11}+6x_{12}+9x_{13}+7x_{14}$
- Transportation cost for units shipped from Coimbatore = $7x_{21}+8x_{22}+2x_{23}+4x_{24}$
- Transportation cost for units shipped from Madurai = $6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Combining the transportation cost for all the units shipped from each supply point with the objective to minimize the transportation cost, the objective function will be,

Minimize $Z = 5x_{11}+6x_{12}+9x_{13}+7x_{14}+7x_{21}+8x_{22}+2x_{23}+4x_{24}+6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Constraints: In transportation problems, there are supply constraints for each source, and demand constraints for each destination.

Supply constraints:

For Chennai, $x_{11}+ x_{12}+ x_{13}+ x_{14} \leq 6000$

For Coimbatore, $x_{21}+ x_{22}+ x_{23}+ x_{24} \leq 5000$

For Madurai, $x_{31}+ x_{32}+ x_{33}+ x_{34} \leq 4000$

Demand constraints:

For B'lore, $x_{11}+ x_{21}+ x_{31} = 5000$

For Hyderabad, $x_{12} + x_{22} + x_{32} = 4000$

Notes

For Cochin, $x_{13} + x_{23} + x_{33}$ = 2000

For Goa, $x_{14} + x_{24} + x_{34}$ = 4000

The linear programming model for GM Textiles will be write in the next line. Minimize $Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Subject to constraints,

$X_{11}+x_{12}+x_{13}+x_{14} \leq 6000$ (i)

$X_{21}+x_{22}+x_{23}+x_{24} \leq 5000$ (ii)

$X_{31}+x_{32}+x_{33}+x_{34} \leq 4000$ (iii)

$X_{11}+x_{21}+x_{31} = 5000$ (iv)

$X_{12}+x_{22}+x_{32} = 4000$ (v)

$X_{13}+x_{23}+x_{33} = 2000$ (vi)

$X_{14}+x_{24}+x_{34} = 4000$ (vii)

Where, $x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3, 4.

Example: Consider the following transportation problem (Table 3.3) and develop a linear programming (LP) model.

Table 5.3: Transportation Problem

Source	Destination			
	1	**2**	**3**	**Supply**
1	15	20	30	350
2	10	9	15	200
3	14	12	18	400
Demand	250	400	300	

Solution:

Let x_{ij} be the number of units to be transported from the source i to the destination j, where i = 1, 2, 3,…m and j = 1, 2, 3,…n.

The linear programming model is

Minimize $Z = 15x_{11}+20x_{12}+30x_{13}+10x_{21}+9x_{22}+15x_{23}+14x_{31}+12x_{32}+18x_{33}$

Subject to constraints,

$x_{11}+x_{12}+x_{13} \leq 350$ (i)

$x_{21}+x_{22}+x_{23} \leq 200$ (ii)

$x_{31}+x_{32}+x_{33} \leq 400$ (iii)

$x_{11}+x_{12}+x_{31} = 250$ (iv)

$x_{12}+x_{22}+x_{32} = 400$ (v)

$x_{13}+x_{23}+x_{33} = 300$ (vi)

$x_{ij} \geq 0$ for all i and j.

Notes

In the above LP problem, there are m x n = 3 x 3 = 9 decision variables and m + n = 3 + 3 = 6 constraints.

5.4 Maximization Transportation Problem

In this type of problem, the objective is to maximize the total profit or return. In this case, convert the maximization problem into minimization by subtracting all the unit cost from the highest unit cost given in the table and solve.

Example: A manufacturing company has four plants situated at different locations, all producing the same product. The manufacturing cost varies at each plant due to internal and external factors. The size of each plant varies, and hence the production capacities also vary. The cost and capacities at different locations are given in the following table:

Table 5.4: Cost and Capacity of Different Plants

Particulars	Plant			
	A	B	C	D
Production cost per unit (Rs.)	18	17	15	12
Capacity	150	250	100	70

The company has five warehouses. The demands at these warehouses and the transportation costs per unit are given in the Table 5.5 below. The selling price per unit is ₹ 30.

Table 5.5: Transportation Problem

Warehouse	Transportation cost (Rs) — Unit-wise				Demand
	A	B	C	D	
1	6	9	5	3	100
2	8	10	7	7	200
3	2	6	3	8	120
4	11	6	2	9	80
5	3	4	8	10	70

1. Formulate the problem to maximize profits.
2. Find the total profit.

Solution:

1. The objective is to maximize the profits. Formulation of transportation problem as profit matrix table is shown in Table 5.6. The profit values are arrived as follows.

   Profit = Selling Price –Production cost –Transportation cost

Notes

Table 5.6: Profit Matrix

	Destination				
	A	**B**	**C**	**D**	**Demand**
1	6	4	10	15	100
2	4	3	8	11	200
3	10	7	12	10	120
4	1	7	13	9	80
5	9	9	7	8	70
Supply	150	250	100	70	570

Converting the profit matrix to an equivalent loss matrix by subtracting all the profit values in Table 5.6 from the highest value 13. Subtracting all the values from 13, the loss matrix obtained is shown in the Table 5.7.

Table 5.7: Loss Matrix

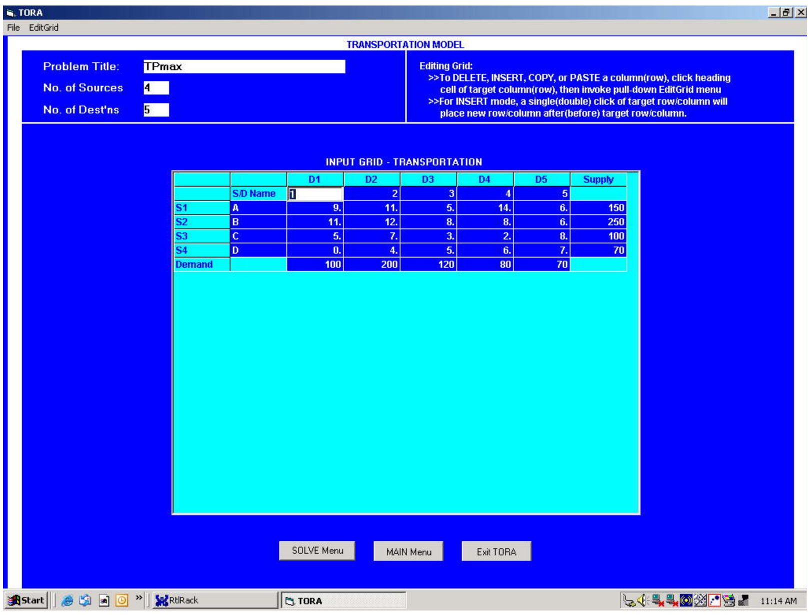
Destination

	A	**B**	**C**	**D**	**Demand**
1	9	11	5	0	100
2	11	12	7	4	200
3	5	8	3	5	120
4	14	8	2	6	80
5	6	6	8	7	70
Supply	150	250	100	70	570

2. To determine the initial solution using TORA

Input Screen:

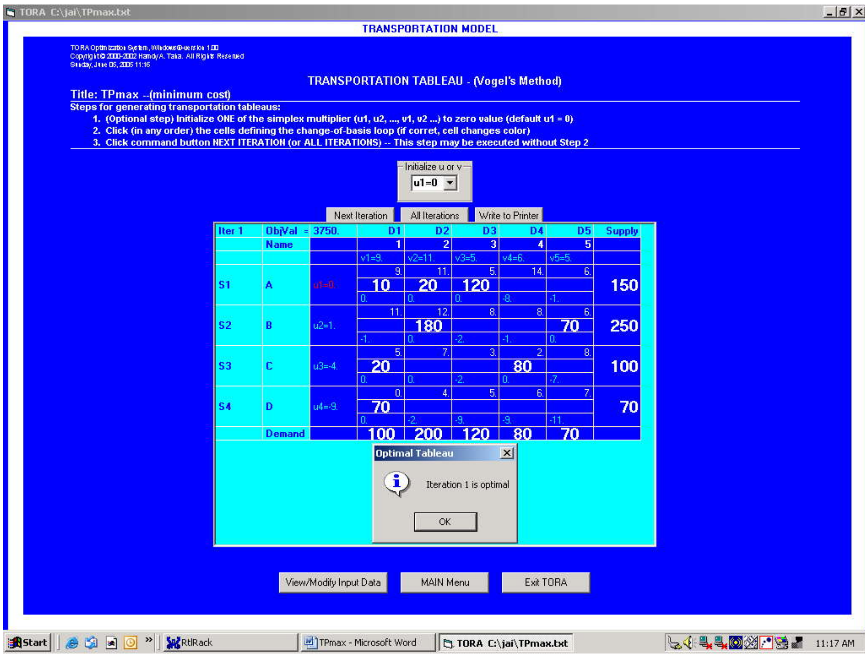
Table 5.8 (a): TORA, Input Screen for TP Max Problem



Notes

Output Screen:

Table 5.8 (b): TORA Output Screen (Vogel's Method)



The first iteration itself is optimal, hence optimality is reached.

3. To find the total cost:

The total maximization profit associated with the solution is

Total Profit = (6 × 10) + (4 × 20) + (10 × 120) + (3 × 180) + (9 × 70) + (10 × 20) + (13 × 80) + (15 × 70)

= 60 + 80 + 1200 + 540 + 630 + 200 + 1040 + 1050

= ₹ 4800.00

5.5 Balanced Transportation Problem

When the total supplies of all the sources are equal to the total demand of all destinations, the problem is a ***balanced transportation problem.***

Total supply = Total demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The problem given in Example 3.1 represents a balanced transportation problem.

5.6 Unbalanced Transportation Problem

When the total supply of all the sources is not equal to the total demand of all destinations, the problem is an ***unbalanced transportation problem.***

Notes

Total supply ≠ Total demand

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

5.6.1 Demand Less than Supply

In real-life, supply and demand requirements will rarely be equal. This is because of variation in production from the supplier end, and variations in forecast from the customer end. Supply variations may be because of shortage of raw materials, labour problems, improper planning and scheduling. Demand variations may be because of change in customer preference, change in prices and introduction of new products by competitors. These unbalanced problems can be easily solved by introducing dummy sources and dummy destinations. If the total supply is greater than the total demand, a dummy destination (dummy column) with demand equal to the supply surplus is added. If the total demand is greater than the total supply, a dummy source (dummy row) with supply equal to the demand surplus is added. The unit transportation cost for the dummy column and dummy row are assigned zero values, because no shipment is actually made in case of a dummy source and dummy destination.

Example:

Check whether the given transportation problem shown in Table 5.9 is a balanced one. If not, convert the unbalanced problem into a balanced transportation problem.

Table 5.9: Transportation Model with Supply exceeding Demand

Source	**Destination**			**Supply**
	1	**2**	**3**	
1	25	45	10	200
2	30	65	15	100
3	15	40	55	400
Demand	200	100	300	

Solution:

For the given problem, the total supply is not equal to the total demand.

$$\sum_{i=1}^{3} a_i \neq \sum_{j=1}^{3} b_j$$

since,

$$\sum_{i=1}^{3} a_i = 700 \text{ and } \sum_{j=1}^{3} b_j = 600$$

The given problem is an unbalanced transportation problem. To convert the unbalanced transportation problem into a balanced problem, add a dummy destination (dummy column).

i.e., the demand of the dummy destination is equal to,

$$\sum_{i=1}^{3} a_i - \sum_{j=1}^{3} b_j$$

Notes

Thus, a dummy destination is added to the table, with a demand of 100 units. The modified table is shown in Table 5.10 which has been converted into a balanced transportation table. The unit costs of transportation of dummy destinations are assigned as zero.

Table 5.10: Dummy Destination Added

Source	Destination				Supply
	1	2	3	4	
1	25	45	10	0	200
2	30	65	15	0	100
3	15	40	55	0	400
Demand	200	100	300	100	700/700

Similarly,

If $\sum_{j=1}^{n} b_j > \sum_{i=1}^{m} a_i$ then include a dummy source to supply the excess demand.

5.6.2 Demand Greater than Supply

Example: Convert the transportation problem shown in Table 3.6 into a balanced problem.

Table 5.11: Demand Exceeding Supply

Source	Destination				Supply
	1	2	3	4	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
Demand	100	200	450	250	1000/800

Solution:

The given problem is, $\sum_{j=1}^{4} b_j > \sum_{i=1}^{3} a_j$

$\sum_{i=1}^{3} a_i = 800$ and $\sum_{j=1}^{4} b_j = 1000$

The given problem is an unbalanced one. To convert it into a balanced transportation problem, include a dummy source (dummy row) as shown in Table 5.12

Notes

Table 5.12: Balanced TP Model

Source	Destination				
	1	**2**	**3**	**4**	**Supply**
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
4	0	0	0	0	200
Demand	100	200	450	250	**1000/1000**

5.7 Initial Feasible Solution

Step 1: Formulate the Problem

Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

Step 2: Obtain the Initial Feasible Solution

The initial feasible solution can be obtained by any of the following three methods.

1. Northwest Corner Method (NWC)
2. Row and Column Minima Method (RCMM)
3. Vogel's Approximation Method (VAM)

The transportation cost of the initial basic feasible solution through Vogel's approximation method, VAM will be the least when compared to the other two methods which gives the value nearer to the optimal solution or optimal solution itself. Algorithms for all the three methods to find the initial basic feasible solution are given.

5.7.1 Algorithm for North-West Corner Method (NWC)

1. Select the North-west (i.e., upper left) corner cell of the table and allocate the maximum possible units between the supply and demand requirements. During allocation, the transportation cost is completely discarded (not taken into consideration).
2. Delete that row or column which has no values (fully exhausted) for supply or demand.
3. Now, with the new reduced table, again select the North-west corner cell and allocate the available values.
4. Repeat steps (2) and (3) until all the supply and demand values are zero.
5. Obtain the initial basic feasible solution.

Notes

Example: Find out the initial feasible solution for transportation cost involved in the problem shown through Table 5.13.

Table 5.13

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Solution:

Table 5.14

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3^{20}	5^{20}	7^{10}	6	~~50~~
2	2	5	8^{40}	2^{35}	~~75~~
3	3	6	9	2^{25}	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

As under the process of NWC method, we allocate x_{11} = 20. Now demand for the first column is satisfied, therefore, eliminate that column.

Proceeding in this way, we observe that

$$x_{12} = 20, x_{13} = 10, x_{23} = 40, x_{24} = 35, x_{34} = 25.$$

Delete the row if supply is exhausted.

Delete the column if demand is satisfied.

Here, number of retail shops (n) = 4, and

Number of factories (m) = 3

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6.

Initial basic feasible solution:

$$20 \times 3 + 20 \times 5 + 10 \times 7 + 40 \times 8 + 35 \times 2 + 25 \times 2 = 670$$

5.7.2 Algorithm for Row and Column Minima Method or Least Cost Method (LCM)

1. Select the smallest transportation cost cell available in the entire table and allocate the supply and demand.
2. Delete that row/column which has exhausted. The deleted row/column must not be considered for further allocation.
3. Again select the smallest cost cell in the existing table and allocate. (*Note:* In case, if there are more than one smallest costs, select the cells where maximum allocation can be made)
4. Obtain the initial basic feasible solution.

Notes

Example: Solve example 5 by least cost method.

Solution:

To make our understanding better, let us go through the Table 5.15.

Table 5.15

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Applying the least cost method-

Table 5.16

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5 [20]	7 [30]	6	~~50~~
2	2 [20]	5	8	2 [55]	~~75~~
3	3	6	9 [20]	2 [5]	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

We observe that $c_{21} = 2$, which is the minimum transportation cost. So, $x_{21} = 20$.

Proceeding in this way, we observe that $x_{24} = 55$, $x_{34} = 5$, $x_{12} = 20$, $x_{13} = 30$, $x_{33} = 20$.

Number of basic variables = m + n –1 = 3 + 4 – 1 = 6.

The initial basic feasible solution:

$$= 20 \times 2 + 55 \times 2 + 5 \times 2 + 20 \times 5 + 30 \times 7 + 20 \times 9$$

$$= 650.$$

5.7.3 Algorithm for Vogel's Approximation Method (VAM)

1. Calculate penalties for each row and column by taking the difference between the smallest cost and next highest cost available in that row/column. If there are two smallest costs, then the penalty is zero.
2. Select the row/column, which has the largest penalty and make allocation in the cell having the least cost in the selected row/column. If two or more equal penalties exist, select one where a row/column contains minimum unit cost. If there is again a tie, select one where maximum allocation can be made.
3. Delete the row/column, which has satisfied the supply and demand.
4. Repeat steps (1) and (2) until the entire supply and demands are satisfied.
5. Obtain the initial basic feasible solution.

Notes

Caution The initial solution obtained by any of the three methods must satisfy the following conditions.

1. The solution must be feasible, i.e., the supply and demand constraints must be satisfied (also known as rim conditions).
2. The number of positive allocations, N must be equal to m + n – 1, where m is the number of rows and n is the number of columns.

Example: Find the initial feasible solution of the transportation problem illustrated through Table 5.17

Table 5.17

	Destination				
Origin	**1**	**2**	**3**	**4**	**Supply**
1	20	22	17	4	120
2	24	37	9	7	75
3	34	37	20	15	25
Demand	60	40	30	110	240

Solution:

Solving the problem through Vogel's Approximation Method, we get the Table 5.18

Table 5.18

	Destination					
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**
1	20	22^{40}	17	4	~~120~~ 80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
Demand	60	~~40~~	30	110	240	
Penalty	4	15	8	3		

The highest penalty occurs in the second column. The minimum c_{ij} in this column is c_{12} (i.e., 22). Hence, x_{12} = 40 and the second column is eliminated.

Now again calculate the penalty.

Table 5.19

Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**
1	20	22^{40}	17	4^{80}	~~120~~	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
Demand	60	~~40~~	30	110	240	
Penalty	4		8	3		

The highest penalty occurs in the first row. The minimum cij in this row is c_{14} (i.e., 4). So x_{14} = 80 and the first row is eliminated.

Notes

Final table:

Now assuming that you can calculate the values yourself, we reach the final table as in Table 5.20

Table 5.20

Destination											
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**					
1	20	**22** ^{40}	17	**4** ^{80}	~~120~~	3	13	-	-	-	-
2	**24** 10	37	**9** ^{30}	**7** ^{30}	~~70~~	2	2	2	17	24	24
3	**32** 50	37	20	15	~~50~~	5	5	5	17	32	-
Demand	~~60~~	~~40~~	~~30~~	~~110~~	240						
Penalty	4	15	8	3							
	4	-	8	3							
	8	-	11	8							
	8	-	-	8							
	8	-	-	-							
	24	-	-	-							

The initial basic feasible solution:

$= 22 \times 40 + 4 \times 80 + 24 \times 10 + 9 \times 30 + 7 \times 30 + 32 \times 50$

$= 3520$

Self Assessment

State true or false:

1. Transportation problem applies to situations where a set of commodities is to be transported from source to another.
2. The cost of transportation per unit varies between different supply points,
3. In transportation problems, there are supply constraints for each destination.
4. The most important objective of a transportation problem is to maximize the cost of shipping.
5. The transportation problem is an extension of transshipment problem.
6. There may be routes that are unavailable to transport units from one source to one or more destinations.
7. In transshipment problem, each node makes supplies to the other.
8. Degeneracy is the condition when the number of filled cells is less than the number of rows plus the number of columns minus one.

5.8 Degeneracy in Transportation Problems

Degeneracy involves two steps:

1. ***Check for degeneracy:*** The solution that satisfies the above said conditions N = m + n – 1 is a non-degenerate basic feasible solution otherwise, it is a degenerate solution. Degeneracy may occur either at the initial stage or at subsequent iterations.

   If number of allocations, N = m + n – 1, then degeneracy does not exist, one has to go to the next step.

   If number of allocations, N = m + n – 1, then degeneracy does exist, and has to be resolved before going to the next step.

2. ***Resolving Degeneracy:*** To resolve degeneracy at the initial solution, allocate a small positive quantity e to one or more unoccupied cell that have lowest transportation costs, so as to make m + n – 1 allocations (i.e., to satisfy the condition N = m + n – 1). The cell chosen for allocating e must be of an independent position. In other words, the allocation of e should avoid a closed loop and should not have a path.

The following Table 5.21 shows independent allocations.

Table 5.21: Independent Allocations

*	*	*
	*	

*	*		
		*	
	*		*

The following Tables 5.22 (a), (b) and (c) show non-independent allocations.

Table 5.22: (a) Independent Allocations, (b) and (c)

	*	*
	*	*

(a)

	*		*
	*		*

(b)

*	*		
	*		
*	*		

(c)

Notes

Self Assessment

Fill in the blanks:

9. Degeneracy involves ………………steps.
10. To resolve degeneracy at the initial solution, allocate a small positive quantity e to one or more unoccupied cell that have ………………..transportation costs.
11. …………………..may occur either at the initial stage or at subsequent iterations.

Case Study

South India Soaps Ltd.

The South India Soaps Ltd. (SISOL) operated 3 factories from which it shipped soaps to regional warehouses. In 2007, the demand for soaps was 24,000 tonnes distributed as follows:

Region	Demand in 000′ tones
Cochin	5
Nellore	4
Salem	4
Madurai	11
	24

One-shift production capacity in each of the 3 factories was as follows:

Region	Demand in 000′ tones
Chennai	12
Coimbatore	7
Bangalore	7
	26

Estimated transport costs (in hundred rupees per 000′ tones) are given below:

Factory	Regional warehouses			
	Cochin	Nellore	Salem	Madurai
Chennai	95	105	80	15
Coimbatore	115	180	40	30
Bangalore	155	180	95	70

SISOL followed a policy of decentralization under which each of the four regional warehouses was under the direct supervision of a regional sales manager and he was responsible for the profitability of operation under his control.

Over a period of time, this procedure lead to increasing friction in the organization. There were questions whether this procedure achieved the objectives of minimizing transport costs; also there was no coordination. For instance in 2005, the sales manager of Madurai & Nellore placed their orders with the Chennai factory which did not have the capacity to meet all demands. This led to inefficient and duplicate orders, friction etc. The final pattern that emerged in 2005 was as follows:

Contd...

	Cochin	Nellore	Salem	Madurai
Chennai	0	1	2	9
Coimbatore	3	0	0	2
Bangalore	2	3	2	0

The general manager of SISOL called meeting of the executives at the central office. Some executives proposed that all orders should be routed through the central office which would determine the optimal programme. Others protested that this would seriously conflict with the firm's philosophy of decentralization.

Question:

You have been hired as a consultant by the general manager. Prepare a minimum cost distribution schedule for SISOL. Compare this schedule with the present schedule of 2005; which is better?

5.9 Summary

- A transportation problem basically deals with that problem which aims to find the best way to fulfill the demand of various demand sources using the capacities of various supply points.
- Although the formation can be used to represent more general assignment and scheduling problems as well as transportation and distribution problems, it gets its name from its application to problems involving transporting products from several sources to several destinations.
- The two common objectives of such problems are either to minimize the cost of shipping commodity to various destinations or to maximize the profit of shipping it various destinations.
- Transportation problems are often used in, surprise, transportation planning. But it should always be remembered that the transportation problem is only a special topic of the linear programming problems.

5.10 Keywords

Balanced Transportation Problem: when the total supplies of all the sources are equal to the total demand of all destinations.

Basic Feasible Solution: A feasible solution to an 'm' origin and 'n; destination is said to be basic, if the number of positive allocations are (m+n+1).

Degeneracy: When the number of filled cells is less than the number of rows plus the number of columns minus one.

Notes

5.11 Review Questions

1. Find the initial basic feasible solution for the transportation problem given in following table.

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

2. Obtain an optimal solution for the transportation problem by MODI method given in this table:

		Destination				
		D_1	D_2	D_3	D_4	Supply
Source	S_1	19	30	50	10	7
	S_2	70	30	40	60	9
	S_3	40	8	70	20	18
	Demand	5	8	7	14	

3. Solve the transportation problem

		Destination			
		1	2	3	Supply
Source	1	3	5	7	10
	2	11	8	9	8
	3	13	3	9	5
Demand		5	9	11	23 / 25

4. Develop a network representation of the transportation problem for a company that manufactures products at three plants and ships them to three warehouses. The plant capacities and warehouse demands are shown in the following table:

   The transportations cost per unit (in ₹) is given in matrix.

Plant	Warehouse			Plant Capacity (no. of units)
	W1	W2	W3	
P1	22	18	26	350
P2	12	12	10	450
P3	14	20	10	200
Warehouse demand (no. of units)	250	450	300	

5. Determine whether a dummy source or a dummy destination is required to balance the model given.

   (a) Supply $a_1 = 15, a_2 = 5, a_3 = 4, a_4 = 6$

   Demand $b_1 = 4, b_2 = 15, b_3 = 6, b_4 = 10$

Notes

(b) Supply $a_1 = 27, a_2 = 13, a_3 = 10$

Demand $b_1 = 30, b_2 = 10, b_3 = 6, b_4 = 10$

(c) Supply $a_1 = 2, a_2 = 3, a_3 = 5$

Demand $b_1 = 3, b_2 = 2, b_3 = 2, b_4 = 2, b_5 = 1$.

6. A state has three power plants with generating capacities of 30, 40 and 25 million KWH that supply electricity to three cities located in the same state. The demand requirements (maximum) of the three cities are 35, 40 and 20 million KWH. The distribution cost (₹ in thousand) per million unit for the three cities are given in the table below:

City

		1	2	3
Plant	1	60	75	45
	2	35	35	40
	3	55	50	45

(a) Formulate the problem as a transportation model.

(b) Determine an economical distribution plan.

(c) If the demand is estimated to increase by 15%, what is your revised plan?

(d) If the transmission loss of 5% is considered, determine the optimal plan.

7. Find the initial transportation cost for the transportation matrix given using North-West corner method, Least cost method and Vogel's Approximation method.

Destination

	1	2	3	4	**Supply**
A	5	6	7	8	25
Source B	7	5	4	2	75
C	6	1	3	2	15
Demand	50	30	20	15	

8. The Sharp Manufacturing Company produces three types of monoblock pumps for domestic use. Five machines are used for manufacturing the pumps. The production rate varies for each machine and also the unit product cost. Daily demand and machine availability are given below.

Demand Information

Product			
	A	**B**	**C**
Demand (units)	2000	15000	700

Machine Availability Details

Machine capacity (units)					
	1	**2**	**3**	**4**	**5**
Available	700	1000	1500	1200	800

Notes

Unit Product Cost

	Product		
Machine	**A**	**B**	**C**
1	150	80	75
2	120	95	60
3	112	100	60
4	121	95	50
5	125	75	50

Determine the minimum production schedule for the products and machines.

9. A company has plants at locations A, B and C with the daily capacity to produce chemicals to a maximum of 3000 kg, 1000 kg and 2000 kg respectively. The cost of production (per kg) are ₹ 800, ₹ 900 and ₹ 7.50 respectively. Customer's requirement of chemicals per day is as follows:

Customer	Chemical Required	Price offered
1	2000	200
2	1000	215
3	2500	225
4	1000	200

Transportation cost (in rupees) per kg from plant locations to customer's place is given in table.

Customer

		1	2	3	4
	A	5	7	10	12
Plant	B	7	3	4	2
	C	4	6	3	9

Find the transportation schedule that minimizes the total transportation cost.

10. A transportation model has four supplies and five destinations. The following table shows the cost of shipping one unit from a particular supply to a particular destination.

Source	**Destination**					**Supply**
	1	**2**	**3**	**4**	**5**	
1	13	6	9	6	10	13
2	8	2	7	7	9	15
3	2	12	5	8	7	13
Demand	10	15	7	10	2	

The following feasible transportation pattern is proposed:

$x_{11} = 10, x_{12} = 3, x_{22} = 9, x_{23} = 6, x_{33} = 9, x_{34} = 4, x_{44} = 9, x_{45} = 5.$

Test whether these allocations involve least transportation cost. If not, determine the optimal solution.

Notes

11. A company has four factories situated in four different locations in the state and four company showrooms in four other locations outside the state. The per unit sale price, transportation cost and cost of production is given in table below, along with weekly requirement.

Factory	Showrooms				Cost of production (₹)
	1	2	3	4	
A	9	4	5	3	12
B	4	4	4	4	17
C	4	6	5	6	19
D	8	7	7	4	17

Factory	Weekly Capacity (units)	Weekly demand (units)
A	15	10
B	20	14
C	25	20
D	20	22

Determine the weekly distribution schedule to maximize the sales profits.

12. A computer manufacturer has decided to launch an advertising campaign on television, magazines and radio. It is estimated that maximum exposure for these media will be 70, 50, and 40 million respectively. According to a market survey, it was found that the minimum desired exposures within age groups 15-20, 21-25, 26-30, 31-35 and above 35 are 10, 20, 25, 35 and 55 million respectively. The table below gives the estimated cost in paise per exposure for each of the media. Determine an advertising plan to minimize the cost.

Media	Age Groups				
	15-20	21-25	26-30	31-35	above 35
TV	14	9	11	11	12
Radio	11	7	6	7	8
Magazine	9	10	7	10	8

Solve the problem and find the optimal solution, i.e., maximum coverage at minimum cost.

13. A garment manufacturer has 4 units I, II, III, and IV, the production from which are received by 4 direct customers. The weekly production of each manufacturing unit is 1200 units and all the units are of the same capacity. The company supplies the entire production from one unit to one supplier. Since the customers are situated at different locations, the transportation cost per unit varies. The unit cost of transportation is given in the table. As per the company's policy, the supply from unit B is restricted to customer 2 and 4, and from unit D to customer 1 and 3. Solve the problem to cope with the supply and demand constraints.

Manufacturing unit	1	2	3	4
A	4	6	8	3
B	4	–	5	–
C	6	5	5	9
D	–	7	–	6

Notes

14. A company dealing in home appliances has a sales force of 20 men who operate from three distribution centers. The sales manager feels that 5 salesmen are needed to distribute product line I, 6 to distribute product line II, 5 for product line III and 4 to distribute product line IV. The cost (in ₹) per day of assigning salesmen from each of the offices are as follows:

		Product Line			
		I	II	III	IV
Source	A	10	12	13	9
	B	9	11	12	13
	C	7	8	9	10

Currently, 8 salesmen are available at center A, 5 at center B and 7 at center C. How many salesmen should be assigned from each center to sell each product line, in order to minimize the cost? Is the solution unique?

15. Solve the following degenerate transportation problem:

		Destination			
		I	II	III	Supply
Source	A	7	3	4	2
	B	2	1	3	3
	C	3	4	6	5
	Demand	4	1	5	

Answers: Self Assessment

1. False
2. False
3. False
4. False
5. False
6. True
7. False
8. True
9. Two
10. Lowest
11. Degeneracy

5.12 Further Readings

Books

G Dantzig, *Linear Programming and Extensions,* Princeton University Press.

G Dantzig, M N Thapa, *Linear Programming 2: Theory and Extensions,* Princeton University Press.

Robert J. Vanderbei, *Linear Programming: Foundations and Extensions,* Princeton University Press.

S Jaisanker, *Quantitative Techniques for Management*, Excel Books.

Online links

www.adbook.net

www.mathresources.com

Unit 6: Transportation Problem – Optimality Tests

Notes

CONTENTS

Objectives

Introduction

6.1 Techniques of Finding Optimal Solution

6.2 Stepping Stone Method

6.3 Modified Distribution Method (MODI)

6.4 Procedure for Shifting of Allocations

6.5 Prohibited Routes Problem

6.6 Transshipment Problem

6.7 Summary

6.8 Keywords

6.9 Review Questions

6.10 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the significance of optimality tests in transportation problem
- Learn to drive optimal solution using Modified Distribution method and Stepping stone Method
- Construct the transshipment transportation table for transshipment problem
- Examine multiple optimal solutions, and prohibited routes in the transportation problem

Introduction

Once the initial feasible solution is reached, the next step is to check the optimality. An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost. In this process, if there is no negative opportunity cost, the solution is an optimal solution.

6.1 Techniques of Finding Optimal Solution

Optimality can be tested by two ways namely:

1. Stepping Stone Method
2. Modified Distribution Method

Let us understand each of them one by one.

Notes

6.2 Stepping Stone Method

It is a method for computing optimum solution of a transportation problem.

Steps Involved:

Step 1: Determine an initial basic feasible solution using any one of the following:

(a) North West Corner Rule

(b) Matrix Minimum Method

(c) Vogel Approximation Method

Step 2: Make sure that the number of occupied cells is exactly equal to m+n–1, where m is the number of rows and n is the number of columns.

Step 3: Select an unoccupied cell.

Step 4: Beginning at this cell, trace a closed path using the most direct route through at least three occupied cells used in a solution and then back to the original occupied cell and moving with only horizontal and vertical moves. The cells at the turning points are called "Stepping Stones" on the path.

Step 5: Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, starting with the plus sign at unoccupied cell to be evaluated.

Step 6: Compute the net change in the cost along the closed path by adding together the unit cost figures found in each cell containing a plus sign and then subtracting the unit costs in each square containing the minus sign.

Step 7: Check the sign of each of the net changes. If all the net changes computed are greater than or equal to zero, an optimum solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost.

Step 8: Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to a cell marked with a minus sign on the closed path corresponding to this cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

Step 9: Repeat the procedure until you get an optimum solution

Example: Consider the following transportation problem (cost in rupees). Find the optimum solution

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

Solution: First, we find out an initial basic feasible solution by Matrix Minimum Method

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6^{450}	8	6^{250}	700
B	3^{50}	5	2^{350}	5	400
C	3^{350}	9	6	5^{250}	600
Requirement	400	450	350	500	1700

Here, m + n – 1 = 6. So the solution is not degenerate.

The cell AD (4) is empty so allocate one unit to it. Now draw a closed path from AD.

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4^{+1}	6^{450}	8	6^{249}	700
B	3^{50}	5	2^{350}	5	400
C	3^{349}	9	6	5^{251}	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is + 4 – 6 + 5 – 3 = 0.

This indicates that every unit allocated to route AD will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

Choose another unoccupied cell. The cell BE is empty so allocate one unit to it.

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6^{449}	8	6^{251}	700
B	3^{49}	5^{+1}	2^{350}	5	400
C	3^{351}	9	6	5^{249}	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is + 5 – 6 + 6 – 5 + 3 – 3 = 0

This indicates that every unit allocated to route BE will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

The allocations for other unoccupied cells are:

Unoccupied cells	**Increase in cost per unit of reallocation**	**Remarks**
CE	+ 9 – 6 + 6 – 5 = 4	Cost Increases
CF	+ 6 – 3 + 3 – 2 = 4	Cost Increases
AF	+ 8 – 6 + 5 – 3 + 3 – 2 = 5	Cost Increases
BG	+ 5 – 5 + 3 – 3 = 0	Neither increase nor decrease

Since all the values of unoccupied cells are greater than or equal to zero, the solution obtained is optimum.

Minimum transportation cost is:

6 × 450 + 6 × 250 + 3 × 250 + 2 × 250 + 3 × 350 + 5 × 250 = ₹ 7350

Notes

Self Assessment

Multiple Choice Questions:

1. When total supply is equal to the total demand in a transportation problem, the problem is said to be

   (a) Balanced (b) Unbalanced

   (c) Degenerate (d) All of the above

2. Which of the following methods is used to verify the optimality of the current solution of the transportation problem?

   (a) Least Cost Method (b) Vogel's Approximation Method

   (c) MODI method (d) All of the above

3. MODI method and stepping stone method is used for this purpose

   (a) To find out initial basic feasible solution

   (b) Optimal solution

   (c) Multiple solution

   (d) All of the above

6.3 Modified Distribution Method (MODI)

Another method to check optimality is that of MODI.

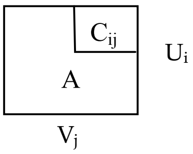
Steps Involved:

Step 1: Row 1, row 2,…, row i of the cost matrix are assigned with variables $U_1, U_2, \ldots, U_i$ and the column 1, column 2,…, column j are assigned with variables $V_1, V_2, \ldots, V_j$ respectively.

Step 2: Initially, assume any one of U_i values as zero and compute the values for $U_1, U_2, \ldots, U_i$ and $V_1, V_2, \ldots, V_j$ by applying the formula for occupied cell.

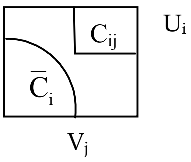
For occupied cells,

$C_{ij} + U_i + V_j = 0$



Step 3: Obtain all the values of C_{ij} for unoccupied cells by applying the formula for unoccupied cell. For unoccupied cells,

Opportunity Cost, $= C_{ij} + U_i + V_j$



Notes

If $\bar{C}_{ij}$ values are > 0 then, the basic initial feasible solution is optimal.

If $\bar{C}_{ij}$ values are = 0 then, the multiple basic initial feasible solution exists.

If $\bar{C}_{ij}$ values are < 0 then, the basic initial feasible solution is not optimal.

Notes The MODI method is based on the concept of Duality.

Did u know? MODI method is also known as u-v method or method of multipliers.

6.4 Procedure for Shifting of Allocations

Select the cell which has the most negative $\bar{C}_{ij}$ value and introduce a positive quantity called 'θ' in that cell. To balance that row, allocate a '– θ' to that row in occupied cell. Again, to balance that column put a positive 'θ' in an occupied cell and similarly a '–θ' to that row. Connecting all the 'θ's and '–θ's, a closed loop is formed.

Two cases are represented in Table 6.1 (a) and 6.1 (b). In Table 6.1 (a) if all the θ allocations are joined by horizontal and vertical lines, a closed loop is obtained.

The set of cells forming a closed loop is

CL = {(A, 1), (A, 3), (C, 3), (C, 4), (E, 4), (E, 1), (A, 1)}

The loop in Table 4.22(b) is not allowed because the cell (D3) appears twice.

Table 6.1(a)

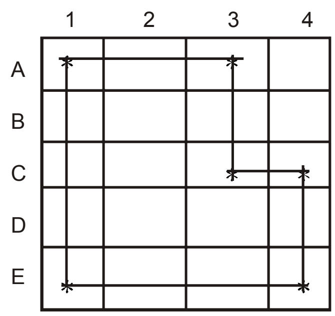
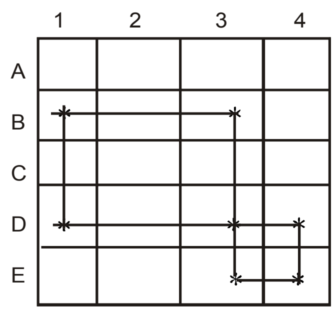


Table 6.1(b)



Notes

Conditions for Forming a Loop

1. The start and end points of a loop must be the same.
2. The lines connecting the cells must be horizontal and vertical.
3. The turns must be taken at occupied cells only.
4. Take a shortest path possible (for easy calculations).

Notes

1. Every loop has an even number of cells and at least four cells.
2. Each row or column should have only one '+' and '-' sign.
3. Closed loop may or may not be square in shape. It can also be a rectangle or a stepped shape.
4. It doesn't matter whether the loop is traced in a clockwise or anti-clockwise direction.

Take the most negative '– q' value, and shift the allocated cells accordingly by adding the value in positive cells and subtracting it in the negative cells. This gives a new improved table. Then go to step 5 to test for optimality.

Calculate the Total Transportation Cost

Since all the $\bar{C}_{ij}$ values are positive, optimality is reached and hence the present allocations are the optimum allocations. Calculate the total transportation cost by summing the product of allocated units and unit costs.

Task Find the initial basic solution for the transportation problem and hence solve it.

Source	**Destination**				**Supply**
	1	**2**	**3**	**4**	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	**1200**

6.5 Prohibited Routes Problem

In practice, there may be routes that are unavailable to transport units from one source to one or more destinations. The problem is said to have an unacceptable or prohibited route. To overcome such kind of transportation problems, assign a very high cost to prohibited routes, thus preventing them from being used in the optimal solution regarding allocation of units.

6.6 Transshipment Problem

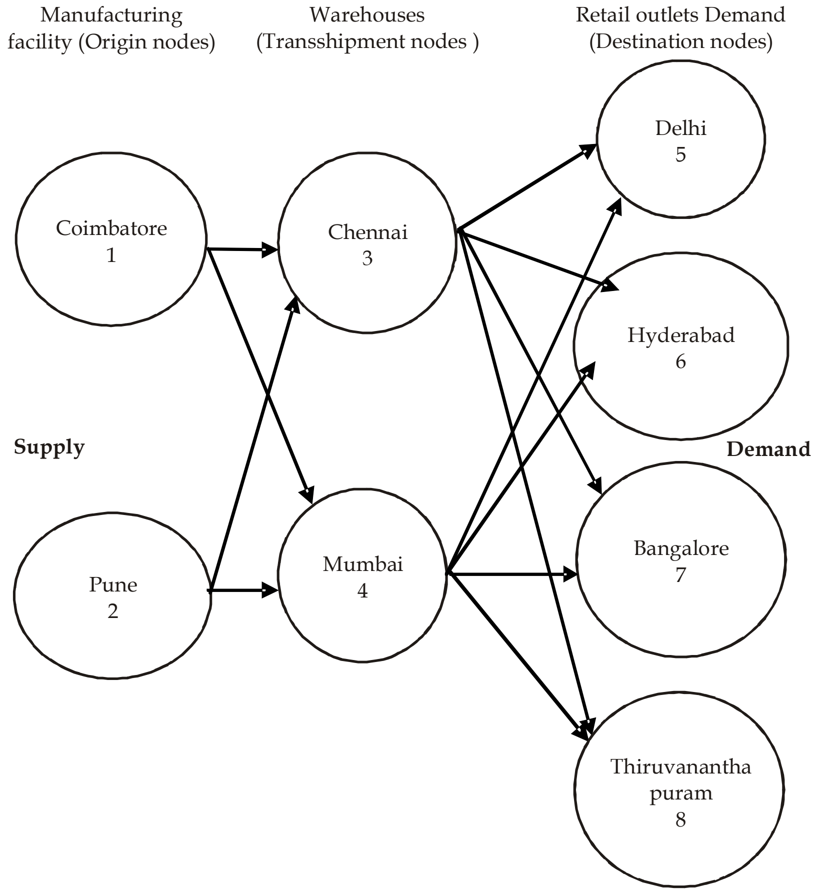
The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

Notes

Each of these nodes in turn supply to other destinations. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum transportation cost.

Considering a company with its manufacturing facilities situated at two places, Coimbatore and Pune. The units produced at each facility are shipped to either of the company's warehouse hubs located at Chennai and Mumbai. The company has its own retail outlets in Delhi, Hyderabad, Bangalore and Thiruvananthapuram. The network diagram representing the nodes and transportation per unit cost is shown in Figure 6.1. The supply and demand requirements are also given.

Figure 6.1: Network Representation of Transshipment Problem



Solving Transshipment Problem using Linear Programming

Let

X_{ij} be the number of units shipped from node i to node j,

X_{13} be the number of units shipped from Coimbatore to Chennai,

X_{24} be the number of units shipped from Pune to Mumbai, and so on

Table 6.2 shows the unit transportation cost from sources to destination.

Notes

Table 6.2: TP of the Shipment

	Warehouse	
Facility	Chennai	Mumbai
Coimbatore	4	7
Pune	6	3

Warehouses	Retail outlets			
	Delhi	Hyderabad	Bangalore	Thiruvananthapuram
Chennai	7	4	3	5
Mumbai	5	6	7	8

Objective: The objective is to minimize the total cost

Minimize $Z = 4X_{13} + 7X_{14} + 6X_{23} + 3X_{24} + 7X_{35} + 4X_{36} + 3X_{37} + 5X_{38} + 5X_{45}6X_{46} + 7X_{47} + 8X_{48}$

Constraints: The number of units shipped from Coimbatore must be less than or equal to 800. Because the supply from Coimbatore facility is 800 units. Therefore, the constraints equation is as follows:

$$X_{13} + X_{14} \leq 800 \quad \text{(i)}$$

Similarly, for Pune facility

$$X_{23} + X_{24} \leq 600 \quad \text{(ii)}$$

Now, considering the node 3,

Number of units shipped out from node 1 and 2 are,

$$X_{13} + X_{23}$$

Number of units shipped out from node 3 is,

$$X_{35} + X_{36} + X_{37} + X_{38}$$

The number of units shipped in must be equal to number of units shipped out, therefore

$$X_{13} + X_{23} = X_{35} + X_{36} + X_{37} + X_{38}$$

Bringing all the variables to one side, we get

$$-X_{13} - X_{23} + X_{35} + X_{36} + X_{37} + X_{38} = 0 \quad \text{(iii)}$$

Similarly for node 4

$$-X_{14} - X_{24} + X_{45}X_{46} + X_{47} + X_{48} = 0 \quad \text{(iv)}$$

Now considering the retail outlet nodes, the demand requirements of each outlet must be satisfied. Therefore for retail node 5, the constraint equation is

$$X_{35} + X_{45} = 350 \quad \text{(v)}$$

Similarly for nodes 6, 7, and 8, we get,

$$X_{36} + X_{46} = 200 \quad \text{(vi)}$$

$$X_{37} + X_{47} = 400 \quad \text{(vii)}$$

$$X_{38} + X_{48} = 450 \quad \text{(viii)}$$

Notes

Linear Programming formulation,

Minimize $Z = 4X_{13}+7X_{14}+6X_{22}+3X_{24}+7X_{35}+4X_{36}+3X_{37}+5X_{38}+5X_{45}+6X_{46}+7X_{47}+8X_{48}$

Subject to constraints,

$$\left.\begin{array}{l} X_{13}+X_{14} \le 800 \\ X_{23}+X_{23} \le 600 \end{array}\right\} \text{origin constraints}$$

$$-X_{13} - X_{23} + X_{35} + X_{36} + X_{37} + X_{38} = 0$$

$$-X_{14} - X_{24} + X_{45} + X_{46} + X_{47} + X_{48} = 0$$

$$\left.\begin{array}{l} X_{35} + X_{45} = 350 \\ X_{36} + X_{46} = 200 \\ X_{37} + X_{47} = 400 \\ X_{38} + X_{48} = 450 \end{array}\right\} \text{Destination constraints}$$

Caution **Don't confuse between transportation and transshipment problem**

A transportation problem can be converted into a transshipment problem by relaxing the restrictions on the receiving and sending the units on the origins and destinations respectively. A m-origin, n-destination, transportation problem, when expressed as transshipment problem, shall become an enlarged problem: with m + n origins and an equal number of destinations. With minor modifications, this problem can be solved using the transportation method.

Case Study

Transportation for Manufacturing Unit

A toy manufacturer wants to open a third warehouse that will supply three retail outlets. The new warehouse will supply **500 units** of backyard play sets per week. Two locations are being studied, N1 and N2. Refer to the table below for transportation costs per play set from each warehouse to each store locations.

	Store Location		
Warehouse	**A**	**B**	**C**
N1	$6	$8	$7
N2	$10	$6	$4

The existing system is shown on the following table.

	Store Location			
Warehouse	**A**	**B**	**C**	**Capacity**
1	$8	$3	$7	500
2	$5	$10	$9	400
Demand	400	600	400	

Question:

Which location would result in the lower transportation cost for the system?

Notes

Self Assessment

Fill in the blanks:

4. ………………………. is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.
5. A very ……………cost is applied to prohibited routes.
6. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum …………………..

6.7 Summary

- An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost.
- Optimality can be tested by two ways namely:
  - Stepping Stone Method
  - Modified Distribution Method
- Stepping Stones is method for computing optimum solution of a transportation problem.
- The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

6.8 Keywords

MODI Method: The modified distribution method, also known as MODI method or (u – v) method provides a minimum cost solution to the transportation problem.

Optimal Solution: A feasible solution is said to be optimal if it minimizes the total transportation cost.

Stepping Stone Method: In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Transshipment: When it is possible to ship both into and out of the same node.

6.9 Review Questions

1. What are the conditions for forming a closed loop?
2. How are the maximization problems solved using transportation model?
3. How is optimality tested in solving transportation problems?
4. In what ways is a transshipment problem different from a transportation problem?

Notes

5. Solve the following transportation problems.

(a)

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

(b)

From	To			Available
	A	B	C	
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Demand	6	10	15	

6. A potato chip manufacturer has three plants and four warehouses. Transportation cost for shipping from plants to warehouses, the plant availability and warehouses requirements are as follows:

Plants	Warehouses				Plant Availability (quintals)
	W_1	W_2	W_3	W_4	
F_1	7	4	3	5	235
F_2	6	8	7	4	280
F_3	5	6	9	10	110
Requirements (quintals)	125	160	110	230	625

Find optimum shipping schedule.

7. Solve

	Destination				Available
	1	2	3	4	
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Requirement	6	10	12	15	43

8. Find the optimal solution.

	D_1	D_2	D_3	D_4	Available
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Required	22	35	25	41	123

The cell entries are unit transportation cost.

Notes

9. Six plants with capacities of 100, 70, 80, 20, 55 and 95 respectively, supply goods to five stores requiring 50, 200, 30, 60 and 80 units respectively. The transportation costs matrix are given below. Find the optimal solution:

Plants	Stores				
	1	2	3	4	5
1	10	8	6	3	5
2	2	9	9	8	0
3	0	10	15	8	7
4	6	3	12	15	0
5	8	9	6	2	1
6	20	0	8	5	11

10. Find the minimum transportation cost.

Source	Destination					Available
	D_1	D_2	D_3	D_4	D_5	
S_1	4	7	3	8	2	4
S_2	1	4	7	3	8	7
S_3	7	2	4	7	7	9
S_4	4	7	2	4	7	2
Requirement	8	3	7	2	2	-

11. Solve the following transportation problem for maximum profit:

Warehouse	Per unit Profit (Rs.) Market				Availability of Warehouses
	A	B	C	D	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand in the Market	180	320	100	400	

Answers: Self Assessment

1. (a)
2. (c)
3. (b)
4. Transshipment
5. High
6. Transportation Cost

6.10 Further Readings

Notes

Books

J.K. Sharma, *Operations Research*, *Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://myopm.net/courses/common/dt_module/tp_tutorial.pdf

www.nos.org

www.hkbu.edu.hk/

Notes

Unit 7: Assignment Problem – Balanced

CONTENTS

Objectives

Introduction

7.1 Application of Assignment Problem

7.2 Types of Assignment Problem

7.3 Mathematical Model of Assignment Problem

7.4 Hungarian Approach

7.5 Summary

7.6 Keywords

7.7 Review Questions

7.8 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the nature of assignment problem
- Analyze the mathematical formulation
- Learn the methods of solutions

Introduction

It is not uncommon to see Business Organisations confronting the conflicting need for optimal utilization of their limited resources among competing activities. In recent years, Linear Programming has received wider acclaim among the decision makers as a tool for achieving the business objectives. Out of various Quantitative Techniques developed over the past three decades, Linear Programming (LP) has found application in a wider screen. LP relates to the problems concerning distribution of scarce resources (satisfying some constraints which can be algebraically represented as Linear equations) so as to maximize profit or minimize cost. Under LP, decisions are arrived at under certainty conditions i.e., the information available on resources and relationship between variables are known. Hence, the course of action chosen will invariably lead to optimal or nearly optimal results.

The prominent problems which gained much importance under the house of LP are:

(1) Assignment problems.

(2) Transportation problems.

L.P is used in solving problems faced in assigning the 'equal number of jobs to equal number of workers so as to maximise profit or minimize cost'. Hence it is called one-to-one assignment. Say for instance, there are 'n' jobs to be performed and 'n' number of persons are available for doing these jobs and each person can do one job at a time though with varying degree of efficiency. Say let C_{ij} be the total cost : here C=cost, I = individual and j = job. So, a problem arises

Notes

as to which worker is to be assigned which job as to minimize the total job cost. The simple matrix would go like this:

Cost Matrix Job

Individual		1	2	J	n
	1	C_{11}	C_{12}	C_{1j}	C_{1n}
	2	C_{21}	C_{22}	C_{2j}	C_{2n}
	i	C_{i1}	C_{i2}	C_{ij}	C_{in}
	n	C_{n1}	C_{n2}	C_{nj}	C_{nn}

Although these types of problems could be solved by using transportation algorithm but a more efficient method called the assignment algorithm is used to solve such typical problems.

$C_{ij} \rightarrow$ indicates the cost of assigning i^{th} job to j^{th} individual

$X_{ij} \rightarrow$ (reference index) which indicates whether i^{th} job is assigned to j^{th} person or not.

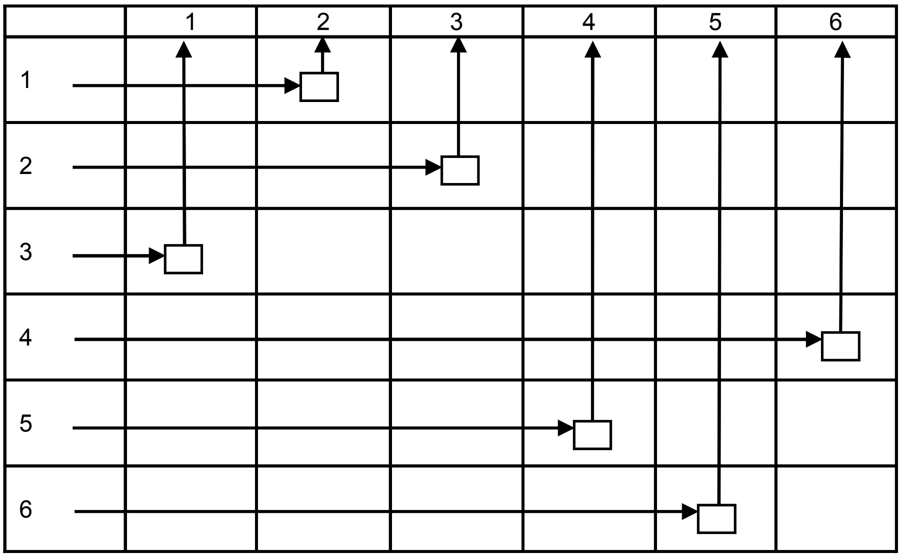
X_{ij} = 1 if i^{th} job are assigned to j^{th} person.

0 if i^{th} job are assigned to j^{th} person.

Consider the example:

There are 6 persons and 6 jobs to be allotted.

Let the assignment of jobs be as shown below:



The assignments are $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1, 4 \rightarrow 6, 5 \rightarrow 4, 6 \rightarrow 5$.

Here, x_{11} = 0 since the first job is not allotted to first person.

X_{12} = 1 since the first job is allotted to second person.

Similarly, $x_{21} = x_{22} = x_{32} = x_{34} = 0$, and

$x_{31} = x_{46} = x_{54} = x_{65} = 1$

Sum of all jobs assigned to first person = $\sum_{i=1}^{6} X_{i1} = 1$

Notes

Since only one job can be allotted to a person.

$\sum_{j=1}^{6} X_{ij}$ = Sum of all persons with first job = 1, since a job can be assigned to only one person.

In general, sum of all jobs assigned to j^{th} person = 1

i.e., $$\sum_{i=j}^{n} X_{ij} = 1$$

And sum of all persons with ith job = 1

i.e., $$\sum_{i=j}^{n} X_{ij} = 1$$

And initial basic feasible solution can be found out by following:

1. Reduction Theorem
2. Hungarian Approach

Similarly, many real life problems can be solved such as assigning number of classes, for number of rooms, number of drivers to number of trucks or vice versa, number of teachers to number of classes, etc.

Reduction Theorem can be used for solving assignment problems with an objective of minimization of costs. For such maximization assignment problems, commonly used rules are:

1. Blind fold assignment/assignment by intuition.
2. Converting the maximization problem into minimization by considering the largest element in the whole matrix.
3. Converting the maximization problem into minimization by using negative signs for all the elements in the profit matrix.

> *Notes* Assignment Problem is a variation of the transportation problem with two characteristics:
>
> 1. Cost matrix is a square matrix
> 2. The optimal solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix

7.1 Application of Assignment Problem

Few applications of assignment problem are as follows:

1. Assignment of employees to machines.
2. Assignment of operators to jobs.
3. Effectiveness of teachers and subjects.
4. Allocation of machines for optimum utilization of space.

5. Allocation of salesmen to different sales areas.
6. Allocation of clerks to various counters.

In all the cases, the objective is to minimize the total time and cost or otherwise maximize the sales and returns.

Self Assessment

Multiple Choice Questions:

1. An assignment problem will have the following solution
   (a) optimal (b) unique
   (c) multiple (d) all of the above
2. Maximization assignment problem is transformed into minimization problem by
   (a) Adding each entry in a column from the maximum value in that column
   (b) Subtracting each entry in a column from maximum value in that column
   (c) Subtracting each entry in the table from the maximum value in that table
   (d) Any one of the above
3. The main objective of an assignment problem is to
   (a) Minimize the total cost (b) Maximize the sales and returns.
   (c) Both (a) and (b) (d) None

7.2 Types of Assignment Problem

The assignment problems are of two types. It can be either

(i) Balanced or

(ii) Unbalanced.

If the number of rows is equal to the number of columns or if the given problem is a square matrix, the problem is termed as a *balanced assignment problem*. If the given problem is not a square matrix, the problem is termed as an *unbalanced assignment problem*.

If the problem is an unbalanced one, add dummy rows / dummy columns as required so that the matrix becomes a square matrix or a balanced one. The cost or time values for the dummy cells are assumed as zero.

Self Assessment

Fill in the blanks:

4. An assignment problem is said to be unbalanced when ....
5. When the number of rows is equal to the number of columns then the problem is said to be .... assignment problem.
6. For solving an assignment problem the matrix should be a .... matrix.

Notes

7.3 Mathematical Model of Assignment Problem

Minimize the total cost which is given by,

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij}$$

Where, $i = 1, 2, 3,\ldots\ldots\ldots n$

$j = 1, 2, 3,\ldots\ldots\ldots n$

Subject to restriction

$X_{ij} = 1$ (One job is done by one worker)

$= 0$ (No job is assigned)

$\Sigma X_{ij} = 1$ (only one job be assigned to one person)
Where, $j = 1,2, 3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots n$

$\Sigma X_{ij} = 1$ (only one person can do one job at a time)

Where, $i=1, 2, 3,\ldots\ldots\ldots\ldots n$

$C_{ij} \rightarrow$ indicates the cost of assigning i^{th} job to j^{th} individual or vice versa, $Vi_{ij} = 1$ to n.

$X_{ij} \rightarrow$ indicates whether i^{th} job is assigned to j^{th} person or not.

$X_{ij} = 1$ if i^{th} job is assigned to j^{th} person '0' otherwise.

Theorem Statement

It states that in an assignment problem if we add or subtract a constant to every element of any row or column of the cost matrix (C_{ij}), then an assignment that minimizes the total cost on one matrix will also minimize the total cost on the other matrix.

Proof:

Let $X_{ij} = X_{ij}$ X_{ij} = elements of first cost matrix.

X_{ij} = elements of second cost matrix.

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij} : X_{ij} \geq 0$$

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n}\left(C_{ij} \pm X_{ij} \pm V_{j}\right)X_{ij}$$

$X_{ij} = i = 1.2.3\ldots\ldots\ldots\ldots\ldots n$

$j = 1.2.3\ldots\ldots\ldots\ldots\ldots n$

$U_{i} \rightarrow i^{th}$ row constant taken for reduction.

$V_{j} \rightarrow j^{th}$ column constant taken for reduction.

Where, U_i and V_j are considered to be constant.

$$Z_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij} - \sum U_{i}\sum_{i-1}^{n} X_{ij} - \sum_{j=1}^{n} V_{j}\sum_{j=1}^{n} X_{ij}$$

Notes

$$Z_1 = Z - \sum_{i-1}^{n} U_i - \sum_{j=1}^{n} V_j$$

$$\sum_{j=1}^{n} X_{ij} = \sum_{i=1}^{n} x_{ij} = 1$$

Since, terms that are subtracted from 'Z' to give 'Z_1' are independent of x_{ij}, it follows that 'Z' is minimized whenever 'Z_1' is minimized and it can be proved conversely.

Notes In an assignment problem if a constant is added to or subtracted from every element of any row or column of the cost matrix, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix.

Example: **Reduction Theorem**

Find the minimum cost for the following problem:

		Persons			
		I	**II**	**III**	**IV**
Tasks	A	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

(i) Row-wise reduction

	I	II	III	IV
A	0	2	9	1
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

(ii) Column-wise reduction

	I	II	III	IV
A	0	$\boxed{0}$	7	1
B	0	3	$\boxed{0}$	3
C	1	1	0	0
D	$\boxed{0}$	5	1	1

Hence, the assignment is

A	→	II	12
B	→	III	07
C	→	IV	11
D	→	I	08
			₹ 38

Here minimized cost $= \Sigma\Sigma\ C_{ij}X_{ij}$ = ₹ 38

Example: Solve the following assignment problem and find the minimum cost.

Notes

Solve the following assignment problem and find the minimum cost.

Jobs		I	II	III	IV
Workers	A	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Solution:

Using Reduction Rules

Step 1: Row-wise Reduction of the matrix.

Jobs		I	II	III	IV
Workers	A	0	2	9	1
	B	0	5	2	3
	C	1	3	2	0
	D	0	7	3	1

Step 2: Column-wise Reduction of the matrix.

Jobs		I	II	III	IV
Workers	A	0	0	7	1
	B	0	3	0	3
	C	1	1	0	0
	D	0	5	1	1

Step 3: Assignment of jobs to workers.

Jobs		I	II	III	IV
Workers	A	0	$\boxed{0}$	7	1
	B	0	3	$\boxed{0}$	3
	C	1	1	0	$\boxed{0}$
	D	$\boxed{0}$	5	1	1

Step 4: Calculation of the Minimum total job cost associated with the assignment.

Workers	Jobs	Total Cost (Rs.)
A	II	12
B	III	7
C	IV	11
D	I	8
Therefore Total job cost		38

Inference

The minimum total cost required to complete the assignment is ₹ 38.

Notes

Example: A departmental head has 4 subordinates and 4 tasks are to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effective matrix. How the tasks should be allocated to each person so as to minimize the total man hours?

Subordinates		**I**	**II**	**III**	**IV**
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution:

Using Reduction Theorem Rules

Step 1: Row-wise reduction of the matrix.

Subordinates		**I**	**II**	**III**	**IV**
Tasks	A	0	18	9	3
	B	9	24	0	22
	C	23	4	3	0
	D	9	16	14	0

Step 2: Column-wise reduction of the matrix.

Subordinates		**I**	**II**	**III**	**IV**
Tasks	A	0	14	9	3
	B	9	20	0	22
	C	23	0	3	0
	D	9	12	14	0

Step 3: Assignment of tasks to subordinates.

Subordinates		**I**	**II**	**III**	**IV**
Tasks	A	$\boxed{0}$	14	9	3
	B	9	20	$\boxed{0}$	22
	C	23	$\boxed{0}$	3	0
	D	9	12	14	$\boxed{0}$

Step 4: Calculation of the minimum total man hours associated with the assignment.

Subordinates	**Tasks**	**Total Man Hours**
I	A	8
II	C	19
III	B	4
IV	D	10
		41

Notes

Inference

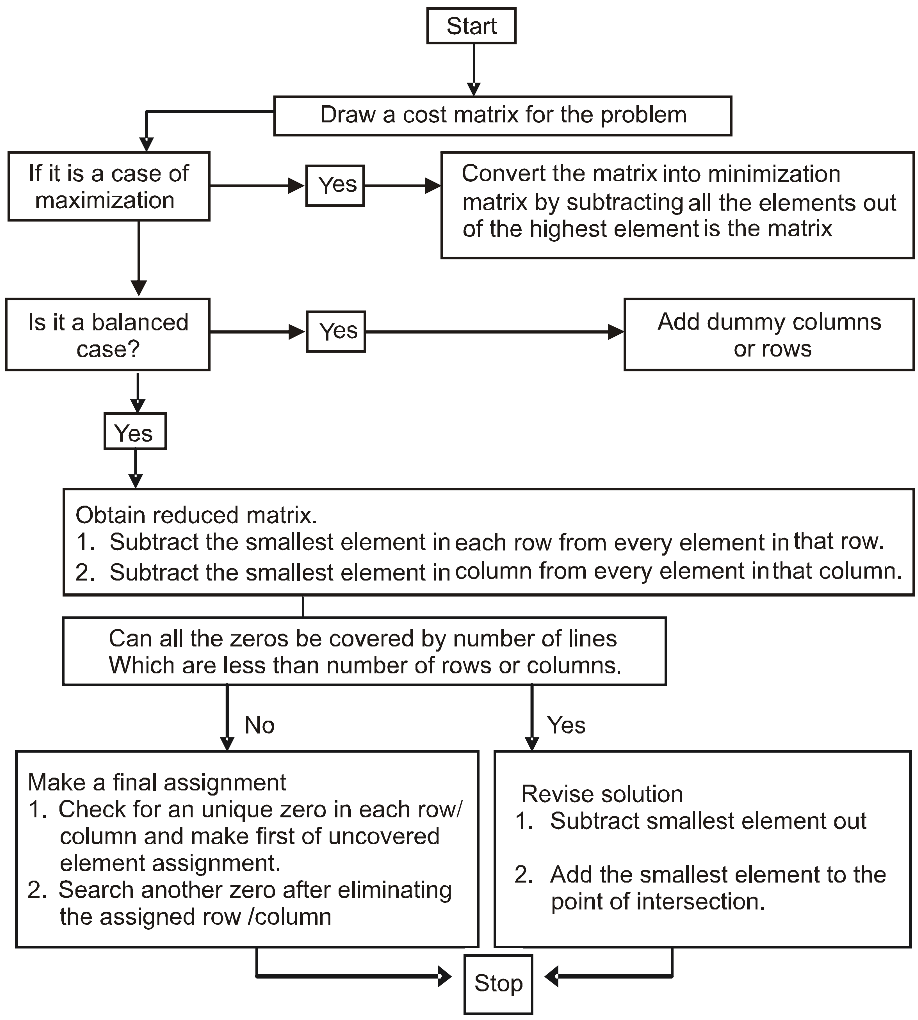
Therefore, the minimum man hours required to complete the assignment are 41.

Task Show that the optimal solution of an assignment problem is unchanged if we add or subtract the same constant to the entries of any row or column of the cost matrix.

7.4 Hungarian Approach

The systematic procedure to be followed while solving assignment problems under Hungarian Approach is picturized in the chart shown below:

Figure 5.1: Diagrammatic Representation of Hungarian Approach



The credit of developing an algorithm goes to Hungarian mathematician D. Konig and later on, it became popular as Hungarian method. This technique is used to solve more typical problems, viz., assignment of chines to jobs, assignment of salesmen to sales territories, assignment of workers to various tasks, vehicles to routes, contracts to bidders, etc.

Hungarian method of solving assignment problems is based on two important properties:

1. In given cost matrix, if a constant quantity is added or subtracted from every element of any row or column, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other.
2. For an assignment problem having all non-negative cost, a solution having zero total cost is an optimum solution.

Example: **Hungarian Approach**

Let U_i denote the i^{th} row constant which is added to i^{th} row elements and V_j denote the j^{th} column constant which is added to elements of j^{th} column.

Tasks

	I	II	III	IV	U_i
A	10	12	19	11	$U_1 = 10$
B	5	10	7	8	$U_2 = 5$
C	12	14	13	11	$U_3 = 8$
D	8	15	11	9	$U_4 = 2$
V_j	$V_1 = 5$	$V_2 = 8$	$V_3 = 7$	$V_4 = 9$	

$$\text{Total } U_i = \Sigma U_i = 25$$

$$\text{Total } V_j = \Sigma V_j = 29$$

First of all, we will add U_{is} to all the rows.

	I	II	III	IV
A	20	22	29	21
B	10	15	12	13
C	20	22	21	19
D	10	17	13	11
Vj	V1 = 5	V2 = 8	V3 = 7	V4 = 9

Now, we shall add V_{js} to all the columns.

	I	II	III	IV
A	25	20	36	30
B	15	23	19	22
C	25	30	28	28
D	15	25	20	20

Now, we will find the minimum cost of assignment for this problem.

	I	II	III	IV
A	0	5	11	5
B	0	8	4	7
C	0	5	3	3
D	0	10	5	5

Notes

Column reduction.

	I	II	III	IV
A	0	0	7	2
B	0	3	1	4
C	0	0	0	0
D	0	5	2	2

L_2

L_1

Here, we use Hungarian method to find the optimum solution.

	I	II	III	IV
A	1	$\boxed{0}$	8	2
B	0	2	$\boxed{0}$	3
C	1	0	0	$\boxed{0}$
D	$\boxed{0}$	4	1	1

Here, assignment is

A → II 30

B → III 19

C → IV 28

D → I 15

The minimum cost is ₹ 92

Hence, $Z_1 = 92$

We got the cost matrix for Z_1 by adding U_1 to different rows and V_j to different columns of cost matrix for Z.

Hence, $Z_1 = Z + \Sigma U_i + \Sigma V_j$

i.e., $92 = 38 + 25 + 29$

Observe that the assignment for both the matrix is same. Hence, we can say that Z is minimized whenever Z_1 is minimized. However, the solution (minimised cost for Z = and Z_1) will be different due to cost elements (C_{ij}).

Example: In a textile sales emporium, 4 sales girls Arpitha (A1), Archana (A2), Aradhana (A3) and Aakansha (A4) are available to 4 sales counters M, N, O and P. Each sales girl can handle any counter. The service of each sales counter [in hours] when carried out by each sales girl is given below:

Sales girls		**A_1**	**A_2**	**A_3**	**A_4**
Sales counters	M	41	72	39	52
	N	22	29	49	65
	O	27	39	60	51
	P	45	50	48	52

How to allocate the appropriate sales counters to sales girls so as to minimize the service time?

Solution:

Notes

Using Reduction Rules

Step 1: Row-wise reduction of the matrix.

Sales girls		A1	A2	A3	A4
Sales counters	M	2	33	0	13
	N	0	7	27	43
	O	0	12	33	24
	P	0	5	3	7

Step 2: Column-wise reduction of the matrix.

Sales girls		A1	A2	A3	A4
Sales counters	M	$\boxed{2}$	28	0	6
	N	0	2	27	36
	O	0	7	33	17
	P	$\boxed{0}$	0	3	0
		L_1			L_2

Step 3: Hungarian Approach used.

Sales girls		A1	A2	A3	A4
Sales counters	M	4	28	$\boxed{0}$	6
	N	0	$\boxed{0}$	25	34
	O	$\boxed{0}$	5	31	15
	P	2	0	3	0

Step 4: Trial assignment.

Sales girls		A1	A2	A3	A4
Sales counters	M	4	28	$\boxed{0}$	6
	N	0	$\boxed{0}$	25	34
	O	$\boxed{0}$	5	31	15
	P	2	0	3	$\boxed{0}$

Step 5: Calculation of total time on optimal assignment.

Sales Counters	Sales girls	Total sales Time
M	A3	39
N	A2	29
O	A1	27
P	A4	52
		147

Notes

Step 6: Requirement 2: By intuition.

Sales girls		A1	A2	A3	A4
Sales counters	M	41	72	$\boxed{39}$	52
	N	$\boxed{22}$	29	49	65
	O	27	$\boxed{39}$	60	51
	P	45	50	48	$\boxed{52}$

Step 7: Calculation of total time on optimal assignment.

Sales Counters	Sales girls	Total sales Time
M	A3	39
N	A1	22
O	A2	39
P	A4	52
		147

Inference

The test of optimality can be confirmed when an assignment is carried out by intuition. As it is apparent from the above revealed results, that the assignment by intuition would come out with 152 hours. As against hours when Hungarian approach is used.

Example: A project consists of 4 major jobs for which 4 contractors have submitted their tenders. The tender amount quoted in lakhs of rupees are given in the matrix below:

Jobs		A	B	C	D
Contractors	1	10	24	30	15
	2	16	22	28	12
	3	12	20	32	10
	4	9	26	34	16

Find the assignment which minimizes the total project cost [each contractor has to be assigned at least one job].

Solution:

Using reduction rules

Step 1: Row-wise reduction of the matrix.

Jobs		A	B	C	D
Contractors	1	0	14	20	5
	2	4	10	16	0
	3	2	10	22	0
	4	0	17	25	7

Step 2: Calculation-wise reduction of the matrix.

Jobs		A	B	C	D
Contractors	1	0	4	4	5
	2	4	0	0	0
	3	2	0	6	0
	4	0	7	9	7

Step 3: Applying Hungarian approach.

Jobs		A	B	C	D
Contractors	1	0	4	4	5
	2	4	0	0	0
	3	2	0	6	0
	4	0	7	9	7

Step 4: Trial assignment.

Jobs		A	B	C	D
Contractors	1	0	0	$\boxed{0}$	1
	2	8	0	0	$\boxed{0}$
	3	6	$\boxed{0}$	6	0
	4	$\boxed{0}$	3	5	2

Step 5: Calculation of total minimum cost.

Job	Contractors	Total Cost (Rs.)
A	4	09
B	3	20
C	1	30
D	2	12
		71

Inference

Therefore, the total minimum cost required to complete the assignment, i.e., 4 jobs by 4 contractors is ₹ 71 (i.e., ₹ 71,00,000).

Did u know? An assignment problem can be solved by the following four methods:

1. Enumeration Method
2. Simplex Method
3. Transportation Method
4. Hungarian Method

Notes

Caution Assignment Problem may not necessarily have an optimal solution. It may also have multiple solutions to a given problem.

7.5 Summary

- An assignment problem seeks to minimize the total cost assignment of n workers to n jobs.
- It assumes all workers are assigned and each job is performed.
- An assignment problem is a special case of a transportation problem in which all supplies and all demands are equal to 1; hence assignment problems may be solved as linear programs.
- Applications of assignment problems are varied in the real world.
- It can be useful for the classic task of assigning employees to tasks or machines to production jobs, but its uses are more widespread.

7.6 Keywords

Allocate: To distribute according to a plan; allot

Assignment Problem: Assignment problem is used to assign *n* number of resources to *n* number of activities so as to minimize the total cost or to maximize the total profit of allocation in such a way that the measure of effectiveness is optimized.

7.7 Review Questions

1. Describe the assignment problem giving a suitable example. Give two areas of its application.
2. Explain the difference between a transportation problem and an assignment problem.
3. Give the mathematical formulation of the assignment problem.
4. Show that the optimal solution of an assignment problem is unchanged if we add or subtract the same constant to the entries of any row or column of the cost matrix.
5. Describe the method of drawing minimum number of lines in the context of assignment problem. Name the method.

Problems

6. A firm plans to begin production of three new products. They own three plants and wish to assign one new plant. The unit cost of producing i at plant j is Cij as given by the following matrix. Find the assignment that minimizes the total unit cost.

$$\begin{array}{cc} & \text{Plants} \\ \text{Product} & \begin{pmatrix} 10 & 8 & 12 \\ 18 & 6 & 14 \\ 6 & 4 & 2 \end{pmatrix} \end{array}$$

Notes

7. A company has 4 machines on which do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

Job	Machine			
	A	B	C	D
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22

Determine the optimum assignment.

8. Solve.

(a)

Jobs	Men			
	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

(b)

Jobs	Men			
	A	B	C	D
I	10	25	15	20
II	15	30	5	15
III	35	20	12	24
IV	17	25	24	20

9. A machine tool company decides to make 4 sub-assemblies through four contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown in table below (in '000 ₹). Assign different assemblies to contractors so as minimize the total cost:

Sub-assembly	Contractor			
	A	B	C	D
I	15	13	14	17
II	11	12	15	13
III	18	12	10	11
IV	15	17	14	16

Notes

10. Solve and find the maximum total expected sale.

Salesman	Area			
	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

11. The following is the cost matrix of assigning 4 clerks to keypunching jobs. Find the optimal assignment if clerk 1 cannot be assigned to job 1.

Clerk	Job			
	1	2	3	4
1	-	5	2	0
2	4	7	5	6
3	5	8	4	3
4	3	6	6	2

12. A student has to select one and only one elective in each semester and the some elective should not be selected in different semesters. Due to various reasons, the expected grades in each subject, if selected in different semester, vary and they are given below.

Semester	Advanced OR	Advanced Statistics	Graph Theory	Discrete Mathematics
I	F	E	D	C
II	E	E	C	C
III	C	D	C	A
IV	B	A	H	H

The grade points are; H = 10, A = 9, B = 8, C = 7, D = 6, E = 5 and F = 4. How will the student select the electives in order to maximize the total expected point's and what will be his maximum expected total points?

13. Kapil Corporation has four plants each of which can manufacture any of the products. Production costs differ from one plant to another plant as does revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

Plant			Sales Revenue (₹ '000) Product			Production Cost (₹ '000) Product		
	1	2	3	4	1	2	3	4
A	50	68	49	62	49	60	45	61
B	60	70	51	74	55	63	45	69
C	55	67	53	60	52	62	49	58
D	68	65	64	69	55	64	48	66

Notes

14. Consider the problem of assigning five jobs to 4 persons.

Person	Job				
	1	**2**	**3**	**4**	**5**
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Answers: Self Assessment

1. (d)
2. (c)
3. (c)
4. number of rows is not equal to the number of columns
5. Balanced
6. Square

7.8 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

www.utdallas.edu/

http://www.usna.edu/

http://businessmanagementcourses.org/

Notes

Unit 8: Assignment Problem – Unbalanced

CONTENTS

Objectives

Introduction

8.1 Variations of the Assignment Problem

8.2 Multiple Optimal Solutions

8.3 Maximization Assignment Problems

8.4 Unbalanced Assignment Problem

8.5 Routing Problems/Travelling Salesman Problems

8.5.1 Steps to Resolve Looping in Travelling Salesman Problem

8.6 Summary

8.7 Keywords

8.8 Review Questions

8.9 Further Readings

Objectives

After studying this unit, you will be able to:

- Know about variations of the assignment problem
- Learn how to solve an unbalanced assignment problem, profit maximization problem etc.
- Understand how to solve a travelling salesman problem

Introduction

The basic objective of an assignment problem is to assign *n* number of resources to *n* number of activities so as to minimize the total cost or to maximize the total profit of allocation in such a way that the measure of effectiveness is optimized.

The problem of assignment arises because available resources such as men, machines, etc., have varying degree of efficiency for performing different activities such as job. Therefore cost, profit or time for performing the different activities is different. Hence the problem is, how should the assignments be made so as to optimize (maximize or minimize) the given objective.

The assignment model can be applied in many decision-making processes like determining optimum processing time in machine operators and jobs, effectiveness of teachers and subjects, designing of good plant layout, etc. This technique is found suitable for routing traveling salesmen to minimize the total traveling cost, or to maximize the sales.

In this unit you will learn how to solve special types of assignment problems such as unbalanced problems, travelling salesman problem etc.

Notes

8.1 Variations of the Assignment Problem

There can be many types of variations in a assignment problem. In this unit we will discuss in detail about the various types of assignment problem you may encounter in the real life scenario.

In this unit, we will discuss on following four variations of the assignment problem:

1. Multiple Optimal Solutions
2. Maximization case in assignment problems
3. Unbalanced Assignment problem
4. Travelling Salesman Problem

8.2 Multiple Optimal Solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number or zeroes. Such a situation indicates multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable solution may be considered by the decision maker.

Notes If the assignment problem has only one solution then the solution is said to be ***Unique solution.***

8.3 Maximization Assignment Problems

In maximization problem, the objective is to maximize profit, revenue, etc. Such problems can be solved by converting the given maximization problem into a minimization problem.

1. Change the signs of all values given in the table or another method is,
2. Select the highest element in the entire assignment table and subtract all the elements of the table from the highest element.

Alpha Corporation has 4 plants, each of which can manufacture any one of the 4 products. Production cost differs from one plant to another plant, so also the sales revenue. Given the revenue and the cost data below, obtain which product each plant should produce to maximize the profit.

Sales revenue (₹ in 000s)

Product		1	2	3	4
Plant	A	50	68	49	62
	B	60	70	51	74
	C	55	67	53	70
	D	58	65	54	69

Notes

Production Cost (₹ in' 000s)

Product		1	2	3	4
Plant	A	49	60	45	61
	B	55	63	45	69
	C	52	62	49	68
	D	55	64	48	66

Solution:

Step 1: Determination of profit matrix.

Product		1	2	3	4
Plant	A	1	8	4	1
	B	5	7	6	5
	C	3	5	4	2
	D	3	1	6	3

Step 2: Conversion of profit matrix into cost matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	3	1	2	3
	C	5	3	4	6
	D	5	7	2	5

Using Reduction Rules

Step 3: Row-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	2	0	1	2
	C	2	0	1	3
	D	3	5	0	3

Step 4: Column-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	5	0	4	5
	B	0	0	1	0
	C	0	0	1	1
	D	1	5	0	1

Step 5: Trial Assignment.

Product		1	2	3	4
Plant	A	5	$\boxed{0}$	4	5
	B	0	0	1	$\boxed{0}$
	C	$\boxed{0}$	0	1	1
	D	1	5	$\boxed{0}$	1

Notes

Step 6: Determination of profit associated with the assignment.

Plant	Product	Total Profit (Rs.)
A	2	8,000
B	4	5,000
C	1	3,000
D	3	6,000
	Total Profit	22,000

When negative signs are used to make the optimal assignment.

Step 1: Profit matrix with (–)ve signs.

Product		1	2	3	4
Plant	A	–1	–8	–4	–1
	B	–5	–7	–6	–5
	C	–3	–5	–4	–2
	D	–3	–1	–6	–3

Step 2: Row-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	2	0	1	2
	C	2	0	1	3
	D	3	5	0	3

Step 3: Column-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	5	0	4	5
	B	0	0	1	0
	C	0	0	1	1
	D	1	5	0	1

Step 4: Trial assignment.

Product		1	2	3	4
Plant	A	5	$\boxed{0}$	4	5
	B	0	0	1	$\boxed{0}$
	C	$\boxed{0}$	0	1	1
	D	1	5	$\boxed{0}$	1

Step 5: Determine of profit associated with the assignment.

Plant	Product	Total Profit (₹)
A	2	8,000
B	4	5,000
C	1	3,000
D	3	6,000
	Total Profit	22,000

Notes

Inference

Hence to get a maximum profit of ₹ 22,000, Alpha Corporation should produce 2 products in plant A, B should manufacture product 4, C plant should manufacture product 1 and D plant should manufacture product 3.

Example: A company has 4 territories and 4 salesmen available for assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	I	II	III	IV
Annual Sales (₹)	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability. It is estimated that working under the same conditions, their yearly sales would be proportionately as follows:

Salesmen	A	B	C	D
Proportion	7	5	5	4

If the criterion is the maximum expected total sales cum the intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest and so on. Verify this answer by assignment model.

Solution:

Step 1: Matrix showing the proportion of sales revenue.

Territory (in'000)		I (6)	II (5)	III (4)	IV (3)
Sales	A (7)	$\boxed{42}$	35	28	21
	B (5)	30	$\boxed{25}$	20	15
	C (5)	30	25	$\boxed{20}$	15
	D (4)	24	20	16	$\boxed{12}$

Step 2: Calculation of sales revenue associated with the assignment based on intuition.

Salesman	Territory	Total Profit (₹)
A	I	42,000
B	II/III	25,000/20,000
C	III/II	20,000/25,000
D	IV	12,000
		99,000

Step 3: Verification of the results intuitively found with the use of assignment model.

Conversion of sales revenue matrix into cost matrix.

Territory		I	II	III	IV
Salesman	A	0	7	14	21
	B	12	17	22	27
	C	12	17	22	27
	D	18	22	26	30

Notes

Step 4: Using reduction rules.

Row-wise reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	7	14	21
	B	0	5	10	15
	C	0	5	10	15
	D	0	4	8	12

Step 5: Column-wise reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	3	6	9
	B	0	1	2	3
	C	0	1	2	3
	D	0	0	0	0

Note: The above matrix which would have been used for making a trial assignment is not an effective matrix because all the zeros either fall on a row or a column and further the matrix cannot be reduced because each row has a zero element and each column has a zero element respectively. Hence, Hungarian approach is to be used to overcome this limitation.

Hungarian Approach

Step 1:

Territory		I	II	III	IV
Salesman	A	0	3	6	9
	B	0	1	2	3
	C	0	1	2	3
	D	$\boxed{0}$	0	0	0

L_1 L_2

Step 2:

Territory		I	II	III	IV
Salesman	A	0	2	5	8
	B	0	0	1	2
	C	0	0	1	2
	D	1	0	0	0

Step 3: Further reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	2	5	8
	B	0	0	1	2
	C	0	0	1	2
	D	$\boxed{0}$	$\boxed{0}$	0	0

L_1 L_2 L_3

Notes

Step 4: Trial assignment.

Territory		I	II	III	IV
Salesman	A	0	2	4	6
	B	0	0	0	1
	C	0	0	0	1
	D	2	0	0	0

Step 5: Calculation of total sales revenue.

Salesman	Territory	Total Sales Revenue (Rs.)
A	I	42,000
B	II/III	25,000/20,000
C	III/II	20,000/25,000
D	IV	12,000
		99,000

Inference

Thus, by following the above assignment schedule for allocating the territories to the 4 salesmen, the sales revenue can be maximized to ₹ 99,000.

Task A marketing manager has 5 salesman and 5 sales districts. Considering the capabilities of the salesman and the nature of the districts, the marketing manager estimates that sales per month (in, 00 ₹) for each salesman in each district would be as follows:

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment that will result in maximum sale.

Self Assessment

Multiple Choice Questions:

1. An assignment problem will have the following solution:

   (a) optimal (b) unique

   (c) multiple (d) all of the above

2. Maximisation assignment problem is transformed into minimization problem by

   (a) adding each entry in a column from the maximum value in that column

   (b) subtracting each entry in a column from maximum value in that column

   (c) subtracting each entry in the table from the maximum value in that table

   (d) any one of the above

Notes

3. When an assignment problem has more than one solution, then it is
   (a) Multiple Optimal solution (b) The problem is unbalanced
   (c) Maximization problem (d) Balanced problem

8.4 Unbalanced Assignment Problem

If the given matrix is not a square matrix, the assignment problem is called an **unbalanced problem.** In such type of problems, add dummy row(s) or column(s) with the cost elements as zero to convert the matrix as a square matrix. Then the assignment problem is solved by the Hungarian method.

Example: A company has 4 machines to do 3 jobs. Each job can be assigned to 1 and only 1 machine. The cost of each job on a machine is given to the following table:

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22

What are the job assignments which will minimize the cost?

Solution:

Step 1: Conversion of the above unbalanced matrix into a balanced matrix.

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22
	D	0	0	0	0

Using reduction rules

Step 2: Row-wise reduction of the matrix.

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	0	6	10	14
	B	0	5	9	10
	C	0	5	9	12
	D	0	0	0	0

Step 3: Column-wise reduction of the matrix.

Note: It is apparent from the above matrix that it is not possible to reduce the matrix column-wise. Hence, Hungarian approach is used.

Notes

Step 4: Hungarian approach is used.

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	0	6	10	14
	B	0	5	9	10
	C	0	5	9	12
	D	$\boxed{0}$	0	0	0

L_1 L_2

Step 5:

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	0	1	5	9
	B	0	0	4	5
	C	0	0	4	7
	D	5	0	0	0

Step 6: Trial assignment.

Machines		**W**	**X**	**Y**	**Z**
Jobs	A	0	1	1	5
	B	0	0	0	1
	C	0	0	0	3
	D	9	4	0	0

Step 7: Calculation of total minimum cost.

Jobs	**Machines**	**Total cost (₹)**
A	W	18
B	X/Y	13/17
C	Y/X	19/15
D	Z	0
	Minimum Time required	50

Inference

Therefore, the minimum cost associated with the assignment of 3 jobs to 4 machines is ₹ 50.

Example: Find the assignment of machines to jobs that will result in a maximum profit and which is the job that should be declined from the following. The owner of a small machine shop has 4 machines available to assign 2 jobs for the day, 5 jobs are offered with the expected profit (₹) per machinist on each job being as follows.

Job		**A**	**B**	**C**	**D**	**E**
Machinist	1	6.20	7.80	5	10.10	8.20
	2	7.10	8.40	6.10	7.30	5.90
	3	8.70	9.20	11.10	7.10	8.10
	4	4.80	6.40	8.70	7.70	8.00

Notes

Solution:

Step 1: Conversion of the unbalanced matrix into a balanced matrix.

Job		A	B	C	D	E
Machinist	1	6.20	7.80	5	10.10	8.20
	2	7.10	8.40	6.10	7.30	5.90
	3	8.70	9.20	11.10	7.10	8.10
	4	4.80	6.40	8.70	7.70	8.00
	5	0	0	0	0	0

Step 2: Conversion of maximisation matrix into a minimization case.

Job		A	B	C	D	E
Machinist	1	–6.20	–7.80	–5	–10.10	–8.20
	2	–7.10	–8.40	–6.10	–7.30	–5.90
	3	–8.70	–9.20	–11.10	–7.10	–8.10
	4	–4.80	–6.40	–8.70	–7.70	–8.00
	5	0	0	0	0	0

Step 3: Conversion of above matrix into an effective matrix.

Job		A	B	C	D	E
Machinist	1	3.90	2.30	5.10	0	1.90
	2	1.30	0	2.30	1.10	2.50
	3	2.40	1.90	0	4.0	3.00
	4	3.90	2.30	0	1.00	0.70
	5	0	0	0	0	0

It is apparent that in the above matrix the reduction rules cannot be applied hence direct Hungarian approach is followed.

Step 4: Hungarian approach is applied.

Job		A	B	C	D	E
Machinist	1	3.90	2.30	5.10	0	1.90
	2	1.30	0	2.30	1.10	2.50
	3	2.40	1.90	0	4.0	3.00
	4	3.90	2.30	0	1.00	0.70
	5	0	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	0
			L_1	L_2	L_3	L_4

Notes

Step 5: Trial assignment.

Job		A	B	C	D	E
Machinist	1	3.20	2.30	5.10	$\boxed{0}$	1.20
	2	0.60	$\boxed{0}$	2.30	1.10	1.80
	3	1.70	1.90	$\boxed{0}$	4.00	2.30
	4	3.20	2.30	0	1.00	$\boxed{0}$
	5	$\boxed{0}$	0.70	0.70	0.70	0.70

Step 6: Calculation of maximum profit for the assignment.

From	To	Total Sales (₹)
A	5	0
B	2	8.40
C	3	11.10
D	1	10.10
E	4	8.00
	Total profit	37.60

Inference

The maximum total profit for the assignment is ₹ 37.60. And the job to be declined is A as the machinist available 5th is supposed to be fictitious.

Self Assessment

Fill in the blanks:

4. If the given matrix is not a …………matrix, the assignment problem is called an unbalanced problem
5. A dummy row(s) or column(s) with the cost elements as ………….. is added to convert the matrix of an unbalanced assignment problem as a square matrix.
6. A given assignment problem can be classified into balanced or unbalanced on the basis of ……………………

8.5 Routing Problems/Travelling Salesman Problems

A salesman, who wishes to travel through his territory visiting various cities, wishes to visit one city only once and wants to come back to the city from where be started and then go to other cities one after other. As he is a single person who has to visit various cities, mere Reduction Theorem Rules or Hungarian approach would not help in arriving at optimum solution always. Hence, a modified solution would be found out to arrive at an optimal solution.

Notes

Example: Solve the following travelling salesman problem with the following matrix.

Travelling Expenses

From \ To

	1	**2**	**3**	**4**	**5**
I	-	6	12	6	4
II	6	-	10	5	4
III	8	7	-	11	3
IV	5	4	11	-	5
V	5	2	7	8	-

Solution:

Apply Reduction Theorem Rules

1. Row-wise Reduced Matrix.

From \ To

	1	**2**	**3**	**4**	**5**
I	-	2	8	2	0
II	2	-	6	1	0
III	5	4	-	8	0
IV	1	0	7	-	1
V	3	0	5	6	-

2. Column-wise Reduced Matrix.

From \ To

	1	**2**	**3**	**4**	**5**
I	-	2	3	1	0
II	1	-	1	0	0
III	4	4	-	7	0
IV	0	0	2	-	1
V	2	0	0	5	
				L_1	L_2

Note: Used Hungarian approach and get the improvised matrices.

3. Modified Matrix – 1.

From \ To

	1	**2**	**3**	**4**	**5**
I	-	1	2	1	0
II	0	-	0	0	0
III	3	3	-	7	0
IV	0	0	2	-	0
V	2	0	0	6	0
				L_1	L_2

Notes

4. Modified Matrix – 2.

From \ To	1	2	3	4	5
I	-	0	1	0	0
II	0	-	0	0	1
III	2	2	-	6	0
IV	0	0	2	-	3
V	2	0	0	6	-

The routes chosen for journey are:

Alternative - I		Alternative - II	
From	To	From	To
I	2	I	4
II	4	II	1
III	5	III	5
IV	1	IV	2
V	3	V	3

Note: In both the solutions, there is looping, Hence, it is not an optimum solution to travelling salesman problem. Hence, it is to be improved further.

8.5.1 Steps to Resolve Looping in Travelling Salesman Problem

1. Choose the next minimum uncovered element in the reduced matrix. If there are more than one such numbers, all the possible alternatives are to be tried until an unique optimum solution is found out.
2. Check whether looping is present in modified matrix after trial assignment. Modified matrix having no looping is an optimum solution.

Did u know? The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips. Slightly modified, it appears as a sub-problem in many areas, such as DNA sequencing. In these applications, the concept city represents, for example, customers, soldering points, or DNA fragments, and the concept distance represents travelling times or cost, or a similarity measure between DNA fragments. In many applications, additional constraints such as limited resources or time windows make the problem considerably harder.

Example: A travelling sales man has to visit 5 cities. He wishes to start form a particular city visit each city once and then return to his starting point. The travelling time (in hours) for each city from a particular city is given below:

Notes

From	To	A	B	C	D	E
	a	∞	4	7	3	4
	b	4	∞	6	3	4
	c	7	6	∞	7	5
	d	3	3	7	∞	7
	e	4	4	5	7	∞

What should be the sequence of visit of the salesman, so that the total travelled time is minimum.

Solution:

Application of Reduction Theorem Rules

(1) ***Row-wise Reduced Matrix***

To / From	A	B	C	D	E
a	∞	1	4	0	1
b	1	∞	2	0	1
c	2	1	∞	2	0
d	0	0	4	∞	4
e	0	0	1	3	∞

(2) ***Column-wise Reduced Matrix***

To / From	A	B	C	D	E
a	∞	1	3	0	1
b	1	∞	1	0	1
c	2	1	∞	2	0
d	0	0	3	∞	4
e	0	0	0	3	∞

(i) *Modified Matrix – 1*

To / From	A	B	C	D	E
a	-	0	$\boxed{1}$	0	0
b	0	-	0	$\boxed{0}$	1
c	2	2	-	6	$\boxed{0}$
d	$\boxed{0}$	0	2	-	3
e	2	$\boxed{0}$	0	6	-

Notes

(ii) *Modified Matrix – 2*

From \ To	A	B	C	D	E
a	-	0	1	0	0
b	0	-	0	0	$\boxed{1}$
c	2	2	-	6	0
d	0	0	2	-	3
e	2	0	0	6	-

Note: This alternative cannot be worked out as no unique zero remains in the third row after choosing next minimum element '1' in the II row.

(3) ***Calculation of Travelling Expenses.***

From	To	Travelling Expenses (₹)
a	3	12
b	4	5
c	5	3
d	1	5
e	2	2
	Total Expenses	27

Inference

The minimum travelling expenses with an unique and optimum solution to the above problem works to be ₹ 27.

(4) ***Modified Matrix-1.***

From \ To	A	B	C	D	E
a	∞	$\boxed{0}$	2	0	1
b	0	∞	0	$\boxed{0}$	1
c	1	0	∞	2	$\boxed{0}$
d	$\boxed{0}$	0	3	∞	5
e	0	0	$\boxed{0}$	4	∞

Note: There are three alternative solutions to the above matrix.

(5) (a) Alternative 1 (b) Alternative - 2 (c) Alternative - 3

From	To	From	To	From	To
a	B	a	D	a	D
b	D	b	A	b	C
c	E	c	E	c	E
d	A	d	B	d	A
e	C	e	C	e	B

Note: Observe there is presence of looping in all the three solutions. Hence, they are not the unique solutions to travelling salesman problem. Further, the matrix is to be modified.

(6) ***Modified Matrix 2.***

To / From	A	B	C	D	E
a	∞	0	2	$\boxed{0}$	$\boxed{1}$
b	0	∞	0	0	1
c	1	$\boxed{0}$	∞	2	0
d	$\boxed{0}$	0	3	∞	5
e	0	0	$\boxed{0}$	4	∞

(7) ***Calculation of Minimum Total Time Required for Travelling.***

From	To	Travelling Time (Hours)
a	E	4
b	D	3
c	B	6
d	A	3
e	C	5
	Total Time	21

Inference

The minimum time required to complete the travel programme with the above unique solution works out to be 21 hours.

Example: A salesman has to visit 4 cities A, B, C, and D. The distance (100 kms) between 4 cities is:

From \ To	A	B	C	D
a	-	4	7	3
b	4	-	6	3
c	7	6	-	7
d	3	3	7	-

If the salesman starts from city 'A' and comes back to city 'A', which route should he select so that total distance travelled by him is minimum?

Solution:

Application of Reduction Theorem Rules

(1) Row-wise Reduced Matrix.

To / From	A	B	C	D
a	-	1	4	0
b	1	-	3	0
c	1	0	-	1
d	0	0	7	-

(2) Column-wise Reduced Matrix.

To / From	A	B	C	D
a	-	1	1	$\boxed{0}$
b	1	-	$\boxed{0}$	0
c	1	$\boxed{0}$	-	1
d	$\boxed{0}$	0	4	-

(3) The Route Chosen for Journey are:

From	To	Journey Time
a	D	3
b	C	6
c	B	6
d	A	3
Total time		18

Note: There is looping in the above solution. Hence, it is not a unique solution to the travelling salesman problem. Hence, requires further modification.

(4) Modified Matrix - 1

	A	B	C	D
a	-	1	1	$\boxed{0}$
b	1	-	$\boxed{0}$	0
c	$\boxed{1}$	0	-	1
d	0	$\boxed{0}$	4	-

(5) Calculation of Minimum Total Time Required to complete the journey.

From	To	Journey Time
a	D	3
b	C	6
c	A	7
d	B	3
	Total time	19

Inference

The unique and optimum solution shows that the minimum time required is 19 hours.

Task For a salesman who has to visit n cities, which of the following are the ways of his tour plan

a. n!

b. (n+1)!

c. (n-1)!

d. n

Self Assessment

State true or false:

7. For a salesman to visit n cities, there are (n+1)! ways to plan his tour.
8. An essential condition in a travelling salesman problem is that a salesman, who wishes to travel through his territory visiting various cities, wishes to visit one city only once.
9. Mere Reduction is not the solution to travelling salesman problem; hence the solution is to find an optimal route that could achieve the objective of the salesman.

8.6 Summary

- Assignment problem is one of the special cases of transportation problems. The goal of the assignment problem is to minimize the cost or time of completing a number of jobs by a number of persons. An important characteristic of the assignment problem is the number of sources is equal to the number of destinations. It is explained in the following way.
  1. Only one job is assigned to person.
  2. Each person is assigned with exactly one job.
- Assignment problem can have various variants such as maximization assignment problem, unbalanced assignment problem, multiple optimal solutions assignment problems and travelling salesman problem.
- The assignment problem where the number of persons is not equal to the number of jobs is called an unbalanced assignment problem. A dummy variable, either for a person or job (as it required) is introduced with zero cost or time to make it a balanced one.
- While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number or zeroes. Such a situation indicates multiple optimal solutions with the same optimal value of objective function.
- In maximization assignment problems, the objective is to maximize profit, revenue, etc. Such problems can be solved by converting the given maximization problem into a minimization problem.
- In a travelling salesman problem is a special type of Assignment problem. In this given a number of cities and the costs of travelling from one to the other, it is required to determine the cheapest route that visits each city once and then returns to the initial city.

8.7 Keywords

An Infeasible Assignment: Infeasible assignment occurs when a person is incapable of doing certain job or a specific job cannot be performed on a particular machine. These restrictions should be taken in to account when finding the solutions for the assignment problem to avoid infeasible assignment.

Balanced Assignment Problem: This is an assignment where the number of persons is equal to the number of jobs.

Dummy Job/ Person: Dummy job or person is an imaginary job or person with zero cost or time introduced in the unbalanced assignment problem to make it balanced one.

Unbalanced Assignment Problem: This is the case of assignment problem where the number of persons is not equal to the number of jobs. A dummy variable, either for a person or job (as it required) is introduced with zero cost or time to make it a balanced one.

Notes

Travelling Salesman Problem: The problem in combinatorial optimization in which, given a number of cities and the costs of travelling from one to the other, it is required to determine the cheapest route that visits each city once and then returns to the initial city.

8.8 Review Questions

1. Discuss the variations in assignment problem.
2. Can there be multiple optimal solutions to an assignment problem? How would you identify the existence of multiple solutions if any?
3. How would you deal with the assignment problems, where the objective function is to be maximized?
4. What is an unbalanced assignment problem? Explain with the help of an example.
5. How is the Hungarian method applied for obtaining a solution if the matrix is rectangular?

Problems

6. A manufacturer of garments plans to add 4 regional warehouses to meet increased demand. The following bids in lakhs of rupees have been for construction of the warehouses.

Contractor	Warehouse			
	A	**B**	**C**	**D**
1	30	27	31	39
2	28	18	28	37
3	33	17	29	41
4	27	18	30	43
5	40	20	27	36

Explain why each warehouse contract cannot simply be given to the contractor who bids cheapest. How will you then determine the optimal contract?

7. 5 operators have to be assigned to 5 machines. The assignment costs are given in the table.

Operator	Machine				
	I	**II**	**III**	**IV**	**V**
A	5	5	-	2	6
B	7	4	2	3	4
C	9	3	5	-	3
D	7	2	6	7	2
E	6	3	7	9	1

Operator A cannot operate machine II and operator C cannot operate machine IV. Find the optimal assignment schedule.

Notes

8. Determine an optimum assignment schedule for the following assignment problem. The cost matrix is –

Job	Machine					
	1	**2**	**3**	**4**	**5**	**6**
A	11	17	8	16	20	15
B	9	7	12	6	15	13
C	13	16	15	12	16	8
D	21	24	17	28	26	15
E	14	10	12	11	15	6

If job C cannot be assigned to machine 6, will the optimum solution change?

9. A company has 6 jobs to be processed by 6 machines. The following table gives the return in rupees when the i^{th} job is assigned to the j^{th} machine (I, J = 1…….6). How should the jobs be assigned to the mechanics so as to maximize the overall return?

Machine	Jobs					
	1	**2**	**3**	**4**	**5**	**6**
1	9	22	58	11	19	27
2	43	78	72	50	63	48
3	41	28	91	37	45	33
4	74	42	27	49	39	32
5	36	11	57	22	25	18
6	13	56	53	31	17	28

10. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are:

Job	Machine			
	A	**B**	**C**	**D**
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

11. A process can be carried out an anyone of 6 machines. The average time taken by any operator on any specific machine is tabulated in the given matrix. It is proposed to buy a new machine to replace one of the existing ones for operations. To carry out the process on this machine average times have been estimated and entered in the matrix. Is it advantageous at this stage to use the new machine? If so, which of the original machines should be replaced and how should the operators be allocated?

Notes

Machine Operators	Avg. time on machines						
	1	2	3	4	5	6	New
A	10	12	8	10	8	12	11
B	9	10	8	7	8	9	10
C	8	7	8	8	8	6	8
D	12	13	14	14	15	14	11
E	9	9	9	8	8	10	9
F	7	8	9	9	9	8	8

12. A section head has five stenotypists and five jobs to complete. The stenos differ in their efficiency and the jobs differ in their intrinsic complexity. The estimate of the time (in hours) each steno would take to perform the task is given in the effectiveness matrix below. How should tasks be allocated, one to a person, so as to maximize the total time taken to complete all the jobs?

Tasks	Steno-typists				
	A	B	C	D	E
I	10	17	15	11	14
II	18	22	14	16	12
III	29	20	12	18	22
IV	14	19	7	17	16
V	9	17	14	12	15

13. A tourist car rental agency has a surplus car in each of the cities A, B, C, D, E and F, and a deficit of one car in each of the cities AA, BB, CC, DD, EE, and FF. The distances between cities with a surplus car and cities with a deficit car are given in the following matrix. How should the cars be dispatched so as to maximize the total distance covered?

From	To					
	AA	BB	CC	DD	EE	FF
A	28	39	47	51	36	42
B	44	49	38	25	29	32
C	59	50	31	31	21	28
D	48	64	80	49	26	38
E	40	71	38	51	50	29
F	39	39	80	59	40	71

14. A domestic airline which operates seven days a week has a time table between two airports as shown below. Crew must have a minimum layover of 2 hours between any two flights. Obtain an optimal pairing of flights to minimize layover time away from HQ. For any given pairing, the crew will be based at the city that results in the smaller layover.

Chennai-Bangalore			Bangalore-Chennai		
Ft. no.	Departure	Arrival	Ft. no.	Departure	Arrival
185	6.30 a.m	7.00 a.m	186	8.00 a.m	8.30 a.m
255	8.00 a.m	8.30 a.m	256	10.00 a.m	10.00 a.m
275	2.00 a.m	2.30 a.m	286	7.00 a.m	1.30 a.m
285	6.00 p.m	6.30 a.m	286	7.00 a.m	7.30 a.m

Notes

15. Carew's Machine shop has four jobs of which three jobs have to be done. Each job can be assigned to one and only one machine. The cost (in ₹) of each on each machine is given below. What are the job assignments which will minimize the total cost?

Job	Machine			
	m_1	m_2	m_3	m_4
J_1	145	160	170	180
J_2	220	132	242	147
J_3	225	237	147	255

16. In response to an advertisement for sale of old vehicles, the bank has received tender bids from four parties for four vehicles. Although anyone can make bids on all four vehicles, the bank has decided to accept only one bid per party. The bids (in ₹) made by the four parties are given below.

Party	Vehicles			
	Premier	Ambassador	Maruti	Jeep
Patel	75,000	87,500	72,500	68,000
Shah	77,500	85,000	73,000	66,000
Vohra	74,000	86,000	72,000	80,000
Trivedi	75,000	88,500	72,000	80,000

Whom should the bank award the vehicles in order to maximize the total revenue?

17. A large consumer marketing company is planning to cover 5 more states for sales promotion. However, the company could recruit four new experienced salesmen. On analyzing the new salesmen's past experience in conjunction with a performance test which was given to them, the company assigned performance ratings to them for each of the state. These ratings are given below. As the company's policy is to attach one salesman to one state, with four salesmen it would not be possible to go for all the five states. Therefore, company would like to know which four states have to be selected in order to maximize the total performance.

Salesman	States				
	Punjab	Haryana	H.P.	Orissa	Bihar
A	46	42	44	39	45
B	45	45	44	42	47
C	47	46	41	43	48
D	40	47	45	41	47

18. A company has 6 machines and five skilled operators to operate them. The performance (in terms of number of units produced in an hour) of these five operators on these six machines is known to the company and the same is given in following table:

Operators	Machines					
	1	**2**	**3**	**4**	**5**	**6**
A	25	20	30	35	40	60
B	50	50	65	70	70	90
C	10	15	20	20	30	50
D	60	70	80	90	90	100
E	20	30	30	45	60	60

How would you assign the operators, one to a machine, so as to maximize the output?

Answers: Self Assessment

1. (d)
2. (c)
3. (a)
4. Square
5. Zero
6. Number of jobs and resources
7. False
8. True
9. True

8.9 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

www.utdallas.edu/

http://www.usna.edu/

http://businessmanagementcourses.org/