Civils - 2015

① $v(x,y) = \ln(x^2+y^2) + x + y$.

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1.$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)2 - 2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1, \quad \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

We can see that $\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$. Hence, v is harmonic function.

Since $f(z) = u(x,y) + iv(x,y)$ is analytic. ~~by~~ by CR equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

So we have $\frac{\partial u}{\partial x} = \frac{2y}{x^2+y^2} + 1$.

$$\Rightarrow u = \int\left(2\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}\right)dx + x + f(y)$$

$$u = 2\tan^{-1}\left(\frac{x}{y}\right) + x + f(y)$$

$$\frac{\partial u}{\partial y} = 2\frac{\left(\frac{-x}{y^2}\right)}{1+\left(\frac{x}{y}\right)^2} + f'(y).$$

$$\frac{\partial u}{\partial y} = \frac{-2x}{x^2+y^2} + f'(y)$$

Since, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

then $\frac{-2x}{x^2+y^2} + f'(y) = -\left(\frac{2x}{x^2+y^2} + 1\right)$

$\Rightarrow f'(y) = -1$.

$f(y) = -y$.

$\Rightarrow$ $\boxed{u = 2\tan^{-1}\left(\frac{x}{y}\right) + x - y}$.

As per Milne Thomas Equation.

$f(z) = \int \left(\phi_1(z,0) + i\,\phi_2(z,0)\right) dz + C$.

$\phi_1(x,y) = \frac{\partial v}{\partial y}$ $\quad \phi_2(x,y) = \frac{\partial v}{\partial x}$

$f(z) = \int \left(1 + i\left(\frac{2z}{z^2} + 1\right)\right) dz + C$.

$f(z) = \int \left(1 + i\left(\frac{2}{z} + 1\right)\right) dz + C$.

$f(z) = z + i(2\ln z + z) + C$.

$\boxed{f(z) = z + i(\ln z^2 + z) + C.}$ is the analytic function.

② $f(z) = \dfrac{2z-3}{z^2-3z+2}$. at $z=0$

<u>taylor series expansion</u> :

Singularities of $f(z)$ are . $z^2-3z+2 = (z-1)(z-2)$

$z = 1, 2$.

As per defination . For a function analytic in $|z-z_0| < R$.
$f(z)$ can be expressed as taylor series .

for $|z| < 1$.

$$f(z) = \frac{2z-3}{z^2-3z+2} = \left[\frac{1}{z-1} + \frac{1}{z-2}\right] .$$

Expanding $\dfrac{1}{z-1} = \dfrac{-1}{1-z} = -1(1-z)^{-1} = -\sum_{n=0}^{\infty} (z)^n$. $|z| < 1$.

Expanding $\dfrac{1}{z-2} = \dfrac{-1}{2}\left[\dfrac{1}{1-\frac{z}{2}}\right] = \dfrac{-1}{2}\left(1-\dfrac{z}{2}\right)^{-1} = \dfrac{-1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\dfrac{(z)^n}{2^{n+1}}$ $|z| < 2$

<u>Combining both</u> : $\forall\ |z| < 1$.

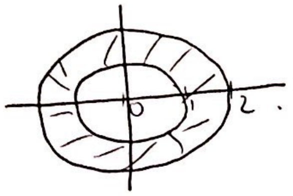
$$f(z) = -\sum_{n=0}^{\infty}(z)^n - \sum_{n=0}^{\infty}\frac{(z)^n}{2^{n+1}}$$

$$\boxed{f(z) = -\sum_{n=0}^{\infty} z^n\left(1+\frac{1}{2^{n+1}}\right) \quad |z| < 1}$$

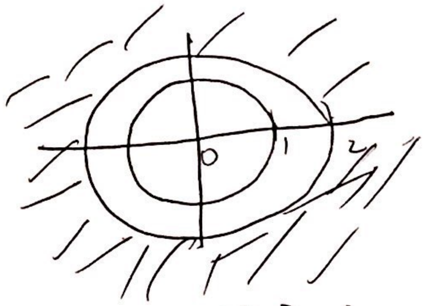
Scanned with CamScanner

Now, Laurent Series:

two possibilities.



$1 < |z| < 2$



$|z| > 2$.

Case I.

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2}$$

Expanding $\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n.$ $\left|\frac{1}{z}\right| < 1 \Rightarrow 1 < |z|$.

Expanding $\frac{1}{z-2}$ as done in last part $\Rightarrow -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ $|z| < 2$.

Combining both and writing in Laurent form we have.

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \quad 1 < |z| < 2$$

$$\boxed{f(z) = \underbrace{\sum_{n=1}^{\infty}(z)^{-n}}_{\text{principal part}} - \underbrace{\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}}_{\text{analytical part}} \quad 1 < |z| < 2}$$

Case II.

$|z| > 2$.

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2}.$$

Expanding $\frac{1}{z-1}$ we have $\Rightarrow$ similar as in last case $\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(z)^{n+1}}$. $|z| > 1$

Expanding $\frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$, $|z| > 2$.

Adding both we get $\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(z)^{n+1}} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$ $|z| > 2$.

Writing in laurent's standard form.

$$f(z) = \sum_{n=1}^{\infty} (z)^{-n} + (z)^{-n} 2^n \Rightarrow \boxed{f(z) = \sum_{n=1}^{\infty} (z)^{-n} (1+2^n)} \cdot |z| > 2.$$

No analytical part.

③ $$\oint_{C,\ |z|=2} \frac{e^z+1}{z(z+1)(z-i)^2}\,dz.$$

Cauchy's residue theorem states that -

$\oint_C \frac{f(z)}{(z-\alpha)(z-\beta)(z-\gamma)}$ along a closed curve $= 2\pi i \sum$ Residue at each singular point.

Singularities of given $f(z)$ are.

$z=0,\ z=-1,\ z=i$ ↳ order two.

Residue at $z=0$.

$$\Rightarrow \lim_{z\to 0} \frac{e^z+1}{(z+1)(z-i)^2} = \frac{2}{1(-1)} = -2.$$

Residue at $z=-1$.

$$\lim_{z\to -1} \frac{e^z+1}{z(z-i)^2} = \frac{e^{-1}+1}{(-1)(2i)} = \frac{i(1+e^{-1})}{2}.$$

Residue at $z=i$

$$\lim_{z\to i} \frac{d}{dz} \frac{e^z+1}{z(z+1)} \Rightarrow \lim_{z\to i}\left[\frac{(z)(z+1)(e^z)-(e^z+1)(2z+1)}{((z)(z+1))^2}\right]$$

$$\lim_{z\to i} = \left[\frac{e^z(z^2+z-2z-1)-(2z+1)}{((z)(z+1))^2}\right] = \left[\frac{e^i(-1-i-1)-(2i+1)}{(-1)(2i)}\right]$$

$$\Rightarrow \frac{i[e^i(-2-i)-(2i+1)]}{(-1)(2i)\,i} = \frac{ie^i(-2-i)-(i-2)}{2}$$

$$\oint_C f(z)\,dz = 2\pi i\left[-2+\frac{i(1+e^{-1})}{2}+\frac{ie^i(-2-i)+(2-i)}{2}\right]$$

$$\oint_C f(z)\,dz = \left[\pi i\left[-4+\cancel{\pi}+ie^{-1}+ie^{i}(-2-i)+2-\cancel{\pi}\right]\right]$$

$$\boxed{\oint_C \frac{e^z+1}{z(z+1)(z-i)^2}\,dz = \pi i\left[-2+ie^{-1}+ie^{i}(-2-1)\right]}$$