IAS MATHEMATICS (OPT.)-2011

PAPER - II : SOLUTIONS

2011.

1(a) Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of complex numbers defined by $f_1(z) = z$, $f_2(z) = 1 - z$, $f_3(z) = \frac{z}{(z-1)}$, $f_4(z) = \frac{1}{z}$, $f_5(z) = \frac{1}{1-z}$ and $f_6(z) = \frac{z-1}{z}$ is a non-abelian group of order 6 with respect to composition of mappings.

Soln. Here, $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

Let multiplication $\times^n$ be the composite of the composite or product of two functions.

Let $f: A \to A$ and $g: A \to A$ then $(gf): A \to A$ such that $(gf)(x) = g(f(x))\ \forall\ x \in A$

$\therefore$ The function gf is called composite of the functions g and f.

Now, we construct the composition tables

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

$(f_1 f_1)(z) = f_1(f_1(z)) = f_1(z) = z = f_1$

$(f_1 f_2)(z) = f_1(f_2(z)) = f_1\left(\frac{1}{z}\right) = \frac{1}{z} = f_2$

$(f_1 f_3)(z) = f_1(f_3(z)) = f_1(f_3(z))$

$= f_1(1-z) = 1-z = f_3$

Similarly, $f_1 f_4 = f_4$ and $f_1 f_5 = f_5$; $f_2 f_1 = f_2$, $f_3 f_1 = f_3$, $f_4 f_1 = f_4$, $f_5 f_1 = f_5$ and $f_6 f_1 = f_6$

$(f_2 f_2)(z) = f_2(f_2(z)) = f_2\left(\frac{1}{z}\right) = z = f_1$

$(f_2 f_3)(z) = f_2(f_3(z)) = f_2(1-z) = \frac{1}{1-z} = f_5$

$(f_2 f_4)(z) = f_2\left(\frac{z}{z-1}\right) = \frac{z-1}{z} = f_6$

$(f_2 f_5)(z) = f_2\left(\frac{1}{1-z}\right) = 1-z = f_3$

$(f_2 f_6)(z) = f_2\left(\frac{z-1}{z}\right) = \frac{z}{z-1} = f_4$

$(f_3 f_2)(z) = f_3\left(\frac{1}{z}\right) = 1 - \frac{1}{z} = \frac{z-1}{z} = f_6$

Similarly, we can easily find the other products.

(i) Clearly, from the above table, closure property is satisfied

(ii) Also, functional composition is associative in nature.

Since, Let $f: A \to A$, $g: A \to A$, $h: A \to A$ then $h(gf) = (hg)f$

(iii) $\forall f = f_1, f_2, \ldots, f_6 \in G$, $\exists$ a unique element $f_1 \in G$ such that, $f f_1 = f_1 f = f$.

$\therefore$ Identity property is satisfied.

(iv) Inverse property is also satisfied as we can see from the composition table that,

$f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$, $f_5^{-1} = f_6$,

$f_6^{-1} = f_5$.

(v) The composition is not commutative.

Since, $f_2 f_3 = f_5$ and $f_3 f_2 = f_6$

$\therefore f_2 f_3 \neq f_3 f_2$

$\therefore G$ is group but not commutative group w.r.t the composite composition.

$\Rightarrow \boxed{O(G) = 6}$

1(c) 2011 P-II

If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, subject to the condition $f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{2}$.

Sol. We have $u+iv=f(z)$. $\therefore$ $iu-v=if(z)$.

On adding, we have

$$u-v+i(u+v)=(1+i)f(z)=F(z) \text{ say.}$$

i.e.,
$$(u-v)+i(u+v)=F(z).$$

Let $U=u-v$, and $V=u+v$, then $U+iV=F(z)$ is an analytic function.

Here $$U=\frac{e^y-\cos x+\sin x}{\cosh y-\cos x}=\frac{\cosh y+\sinh y-\cos x+\sin x}{\cosh y-\cos x}$$

$$=1+\frac{\sinh y+\sin x}{\cosh y-\cos x}=\left\{1-\frac{\sin x+\sinh y}{\cos x-\cosh y}\right\}.$$

$$\therefore \quad \frac{\partial U}{\partial x}=\frac{-1-\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_1(x,y)$$

and
$$\frac{\partial U}{\partial y}=\frac{1-\sin x\sinh y-\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_2(x,y)$$

By Milne's method we have

$$F'(z)=[\phi_1(z,0)-i\phi_2(z,0)]$$

$$=-\frac{1}{1-\cos z}-i\frac{1}{1-\cos z}$$

$$=-(1+i)\frac{1}{1-\cos z}=-\tfrac{1}{2}(1+i)\operatorname{cosec}^2\frac{z}{2}$$

Integrating it, we get

$$F(z)=-\tfrac{1}{2}(1+i)\int \operatorname{cosec}^2 z/2+c=(1+i)\cot z/2+c$$

i.e.
$$(1+i)f(z)=(1+i)\cot z/2+c$$

or
$$f(z)=\cot z/2+c_1.$$

But when $z=\dfrac{\pi}{2}, f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{2}$ $\therefore$ $c_1=\dfrac{3-i}{2}-1=\dfrac{1-i}{2}$.

Hence $f(z)=\cot z/2+\tfrac{1}{2}(1-i)$.

1(d) → IAS 2011 P-II

Solve by Simplex method, the following LP Problem: Maximize, $Z = 5x_1 + 3x_2$

Constraints, $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$

Sol^n →

Converting standard form:-

Max. $z = 5x_1 + 3x_2 + 0S_1 + 0S_2$

S.C: $3x_1 + 5x_2 + S_1 = 15$

$5x_1 + 2x_2 + S_2 = 10$

$x_1, x_2, S_1, S_2 \ge 0$

where S_1, S_2 are slack variables.

The IBFS is given by:-

$x_1 = x_2 = 0$, $S_1 = 15$, $S_2 = 10$

Max $z = 0$.

The initial simplex table is given as:

	c_j	5	3	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	3	5	1	0	15	5
0	S_2	(5)	2	0	1	10	2 →
$z_j = \Sigma C_B a_{ij}$		0	0	0	0	0	
$c_j = c_j - z_j$		5 ↑	3	0	0		

Here x_1 is the incoming variable and S_2 is the outgoing variable.

The key element here is 5, making it unity and all other elements in that column to 0.

The new simplex table:-

		5	3	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	b	θ
0	s_1	0	$(19/5)$	1	$3/5$	9	$\frac{45}{19}\rightarrow$
5	x_1	1	$2/5$	0	$1/5$	2	5
	z_j	5	2	0	1	10	
	c_j	0	1	0	-1		
			$\uparrow$				

Here x_2 is the incoming variable and s_1 is the outgoing variable.
The key element here is $(19/5)$, making it unity and all elements in that column to zero.

C_B	Basis	x_1	x_2	s_1	s_2	b
3	x_2	0	1	$\frac{5}{19}$	$\frac{-3}{19}$	$\frac{45}{19}$
5	x_1	1	0	$\frac{-2}{19}$	$\frac{5}{19}$	$\frac{20}{19}$
	z_i	5	3	$\frac{5}{19}$	$\frac{16}{19}$	15
	c_j	0	0	$\frac{-5}{19}$	$\frac{-16}{19}$	

Since all c_j's ≤ 0

Hence optimality obtained where $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

$\boxed{\text{Max. } z = 15}$

(e) Prove that a group of prime order is Abelian.

2011 IAS.

Ans. Let us suppose that the order of group G is a prime.

Let $g \in G$ be a nonidentity element. Then the order of the subgroup $\langle g \rangle$ must be a divisor of the order of G, hence it must be p.

Therefore, we have $G = \langle g \rangle$, and G is a cyclic group and in particular an abelian group.

Thus, A group of prime order is Abelian.

1(e)

(ii) How many generators are there of the cyclic group $(G, \cdot)$ of order 8?

Sol^n We have $G = \langle a \rangle$, $a^8 = e$

All the generators of G are a^1, a^3, a^5, a^7.

i.e., $G = \langle a \rangle = \langle a^3 \rangle = \langle a^5 \rangle = \langle a^7 \rangle$

($\because$ 1, 3, 5, 7 are positive integers less than 8 and prime to 8).

Q. 2(c) 2011

Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic.

Ans. Consider the Klein 4 group: $V = \{e, a, b, c = ab\}$ whose multiplication table is:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Here, e is the identity element. Observe that if H is any subgroup containing more than 2 elements, it must be the entire group, and hence not proper.

The square of any element is the identity, therefore we have subgroups containing 2 elements and hence are isomorphic ~~to~~ Z_2 (hence cyclic). A subgroup with 1 element is, of course, the trivial subgroup which is the cyclic group generated by e. Hence every proper subgroup of V is cyclic. But V itself is not cyclic since it does not contain an element of order 4.

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$|z-a| < R$, prove that when $0 < r < R$,

2(C) 2011 2015

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$. (Agra 1957, 70)

P-II **Solution.** Since $f(z)$ is analytic in $|z-a| < R$ and $r < R$, it follows that $f(z)$ is also analytic inside the circle C defined by

$$|z-a| = r.$$

Hence by Cauchy's formula for the derivative, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz = f'(a). \qquad \ldots(1)$$

Also $f(z)$ can be expanded as a Taylor's series about $z=a$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m (z-a)^m.$$

Putting $z-a = re^{i\theta}$, we have

$$f(z) = f(a+re^{i\theta}) = \sum_{m=0}^{\infty} a_m\, r^m\, e^{mi\theta}.$$

Then $\overline{f(z)} = \sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}$

$$\therefore \quad \frac{1}{2\pi i}\int \frac{\overline{f(z)}}{(z-a)^2}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}}{r^2\, e^{2i\theta}} \cdot r\, ie^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} \bar{a}_m\, r^{m-1}\int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta$$

$$= 0 \left[\because \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta = 0\right] \qquad \ldots(2)$$

Adding (1) and (2), we have

$$0 + f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\text{real part of } f(a+re^{i\theta})}{r^2\, e^{2i\theta}} \cdot rie^{i\theta}\, d\theta \qquad [\because z = a+re^{i\theta}]$$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

IAS-2011 Q. P-II 3(d)

Find the laurent's series of $f(z) = \frac{1}{1-z^2}$ with centre $|z|=1$.

Sol:

$$f(z) = \frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)}$$

Put $z-1 = u$

$$\therefore f(u) = \frac{-1}{u(u+2)} = \frac{-1}{2u}\left[1+u/2\right]^{-1}$$

$$f(u) = \frac{-1}{2u}\left[1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right]$$

$$\boxed{f(u) = \frac{-1}{2u} + \frac{1}{4} - \frac{u}{8} + \frac{u^2}{16} + \cdots} \quad \text{——— (1)}$$

$$f(u) = \frac{-1}{u(2+u)} = \frac{-1}{u^2(1+2/u)}$$

$$f(u) = \frac{-1}{u^2}\left[1+2/u\right]^{-1} = \frac{-1}{u^2}\left[1 - \frac{2}{u} + \frac{4}{u^2} - \frac{8}{u^3} + \cdots\right]$$

$$f(u) = -\frac{1}{u^2} + \frac{2}{u^3} - \frac{4}{u^4} + \frac{8}{u^5} - \cdots \quad \text{——— (2)}$$

$\left|\frac{2}{u}\right| < 1 \Rightarrow |u| > 2$;

Hence, the above laurent's series (2) is valid for $|u| > 2$.

$\therefore$ from (1)

$$f(u) = -\frac{1}{2u} + \frac{1}{4} - \frac{u}{8} + \frac{u^2}{16} + \cdots \qquad 0 < \left|\frac{u}{2}\right| < 1$$

$$\therefore f(z) = \frac{-1}{2(z-1)} + \frac{1}{4} - \frac{z-1}{8} + \cdots \qquad [0 < u < 2]$$

laurent's series valid for $[0 < |z-1| < 2]$.

from (2)

$$f(z) = \frac{-1}{(z-1)^2} + \frac{2}{(z-1)^3} - \frac{4}{(z-1)^4} + \frac{8}{\cdots}$$

which is valid for $|z-1| > 2$ Ans

3(a) IFoS 2011

Let F be the set of all real valued continuous functions defined on the closed interval $[0,1]$. Prove that $(F,+,\cdot)$ is a commutative ring with unity with respect to addition and multiplication of functions defined pointwise as below:

$(f+g)(x)=f(x)+g(x)$

and $(f\cdot g)(x)=f(x)\cdot g(x)$, $x\in[0,1]$

where, $f,g\in F$.

Ans. Here, $F=\{f:[0,1]\to\mathbb{R} : f$ is real valued and cts $\}$

Let $f,g\in F$, then f and g are the cts functions and $f,g:[0,1]\to\mathbb{R}$.

By the defn of addition and $\times^n$ of functions $f+g$ and $f\cdot g$ are also real valued and cts and, $f+g,\ f\cdot g:[0,1]\to\mathbb{R}$.

$\Rightarrow f+g,\ f\cdot g\in F.$ —(1)

($\because$ Addn and $\times^n$ of cts fns is also cts.)

I. T.S: $(F,+)$ is an abelian group

(i) Closure property: Let $f,g\in F$

then, $f+g\in F$ (From (1))

(ii) Associative property: Let $f,g,h\in F$ $\forall x\in[0,1]$

then $((f+g)+h)(x)=(f+g)(x)+h(x)$

$=f(x)+g(x)+h(x)$

$=f(x)+(g(x)+h(x))$

$= f(x) + (g+h)(x)$

$= (f+(g+h))(x)$

$\Rightarrow (f+g)+h = f+(g+h) \quad \forall f,g,h \in F$

$\Rightarrow (F,+)$ satisfies the associative property.

(III) Identity property →

$\forall f \in F, \exists 0 \in F$ (as zero f^n is real valued and cts on the closed interval $[0,1]$)

such that,

$$f+0 = f = 0+f \quad \forall f \in F$$

∴ Identity property is satisfied.

(IV) Inverse property →

$\forall f \in F, \exists -f \in F$ ($\because$ if f is real valued and cts, then $-f$ is real valued and cts).

Also, $f(x) + (-f)(x) = f(x) - f(x) = 0(x)$

$\Rightarrow \quad (f+(-f))(x) = 0(x), \quad \forall x \in [0,1]$

$\Rightarrow f+(-f) = 0$

$|||^{ly}, \quad 0 = (-f)+f \quad \forall f \in F$

$\Rightarrow$ Inverse property is satisfied.

(V) Abelian :-

As, $\forall f, g \in F$. and $x \in [0,1]$

$\Rightarrow (f+g)(x) = f(x)+g(x) = g(x)+f(x)$

$= (g+f)(x)$

$\Rightarrow f+g = g+f$

$\Rightarrow \quad (F,+)$ is an abelian group.

II. $(F, \times)$ is a semi-group.

(i) Closure prop:-

$\forall f, g \in F$ and $\forall x \in [0,1]$,

$(f \cdot g)(x) = f(x) \cdot g(x) \quad \Rightarrow f \cdot g \in F$

$\forall x \in [0,1] \quad$ (From (1))

(product of cts f^n's is cts).

$\therefore$ Closure property is satisfied.

(ii) Associative property:, $\forall f, g, h \in F$ and $x \in [0,1]$.

$$((f \cdot g) \cdot h)(x) = (f \cdot g)(x) \cdot h(x)$$
$$= f(x) \cdot g(x) \cdot h(x)$$
$$= f(x) \cdot (g(x) \cdot h(x))$$
$$= f(x) \cdot (g \cdot h)(x)$$
$$= (f \cdot (g \cdot h))(x)$$

$\Rightarrow$ Ass. property is satisfied.

(iii) Distributive laws :-

$\forall f, g, h \in F$, and $x \in [0,1]$

(i) $$(f \cdot (g+h))(x) = f(x) \cdot (g+h)(x)$$
$$= f(x) \cdot (g(x) + h(x))$$
$$= f(x) \cdot g(x) + f(x) \cdot h(x)$$
$$= (f \cdot g + f \cdot h)(x)$$

$\Rightarrow f \cdot (g+h) = f \cdot g + f \cdot h$

(ii) Similarly, $(g+h) \cdot f = g \cdot f + h \cdot f$

Also, F is a ring with unity as $\forall f \in F$, $\exists\ 1 \in F$ ($\because 1$ is real valued and cts fn)

then, $(f \cdot 1)(x) = f(x) \cdot 1$
$= f(x) = (1 \cdot f)(x)$

$\Rightarrow f \cdot 1 = f = 1 \cdot f$

$\Rightarrow (F, +, \cdot)$ is a ring with unity.

It is also commutative as: $\forall f, g \in F$

$(f \cdot g)(x) = f(x) \cdot g(x)$
$= g(x) \cdot f(x)$
$= (g f)(x) \quad \forall x \in [0,1]$

$\Rightarrow f \cdot g = g \cdot f$

$\therefore (F, +, \cdot)$ is a commutative ring with unity.

4(a) IAS -2011 Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4 a$. Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$.

Sol^n Let $(G, \cdot)$ be a group.

Let $a, b \in G$ and 'e' be the identity element of $(G, \cdot)$

$a^2 = e$

$\Rightarrow a = a^{-1} e$ [Multiplying by a^{-1}]

$a = a^{-1}$

$ab = b^4 a$ $(\times ab)$

$(ab)^2 = b^4 a \cdot ab$

$= b^4 a^2 b = b^4 \cdot e \cdot b$ $[O(a^2) = 2]$

$(ab)^2 = b^5$

$\Rightarrow (ab)^{2\times 6} = b^{5\times 6} = b^{30} = e$ $[O(b^6) = e]$

$\Rightarrow (ab)^{12} = e$

$\therefore O(ab) / 12$ $\therefore O(ab) = 1, 2, 3, 4, 6, 12$

Also, $ab = b^4 a$

$\Rightarrow a^{-1}(ab) = a^{-1}(b^4 a)$

$\therefore (a^{-1}a)(b) = a^{-1} b^4 a = b$

ab in terms of $a^m b^n$ and $b^m a^n = D$

$ab = b^4 a$

$\Rightarrow (ab)^{-1} = (b^4 a)^{-1}$

$= a^{-1} (b^4)^{-1}$

$= a (b^4)^{-1}$ $\quad [a^2 = e \Rightarrow a = a^{-1}]$

$= a . b^2$ $\quad [b^6 = e \Rightarrow b^4 = b^{-2}$

$(b^4)^{-1} = b^2]$

(i) $(ab)^{-1} = ab^2$

$m = 1$

$b = 2$

$(ab)^{-1} = b^{-1} a^{-1}$

$= b^{-1} a^{-1}$ $\quad [a = a^{-1}]$

$= b^5 . a$ $\quad \begin{bmatrix} b^6 = e \\ b^5 = b^{-1} \end{bmatrix}$

(ii) $(ab)^{-1} = b^5 . a$

$\Rightarrow m = 5$

$\Rightarrow n = 1$

$(ab)^{-1} = ab^2 = b^5 . a$

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Q. Evaluate by contour integral Method – I:

7(c) $\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$ P–I.

IAS – 2011

Sol^n Given that,

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$$

$$= \int_0^1 \frac{dx}{x\left(\frac{1}{x}-1\right)^{1/3}}$$

Let $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$

$$\therefore \int_0^1 \frac{dx}{x\left(\frac{1}{x}-1\right)^{1/3}} = \int_\infty^1 \frac{t\left(-\frac{1}{t^2}\right)dt}{(t-1)^{1/3}} = \int_1^\infty \frac{dt}{t(t-1)^{1/3}}$$

Let $t=x+1$, $dt=dx$

$$\therefore \int_1^\infty \frac{dt}{t(t-1)^{1/3}} = \int_0^\infty \frac{dx}{x^{1/3}(x+1)} = \int_0^\infty \frac{x^{-1/3}}{(x+1)}dx$$

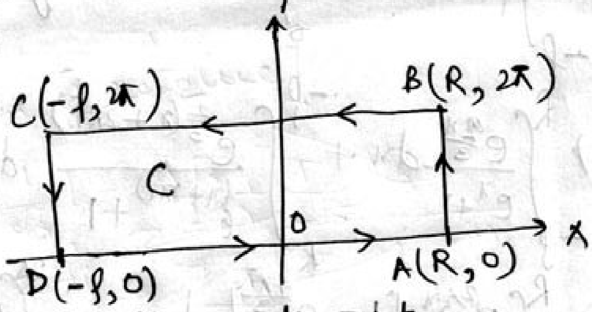
Thus, $\boxed{I = \int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \int_0^\infty \frac{x^{-1/3}}{(x+1)}dx}$ —— (1)

Put $x=e^t$, we have $(dx=e^t dt)$

$$I = \int_{-\infty}^{\infty} \frac{e^{-t/3}e^t dt}{(e^t+1)} = \int_{-\infty}^{\infty} \frac{e^{\frac{2t}{3}}}{e^t+1}dt$$

Consider,

$$\int_C \frac{e^{\frac{2}{3}z}}{e^z+1}dz = \int_C f(z)\,dz$$



where C : Rectangle ABCD with vertices at R, $R+2\pi i$, $-\rho+2\pi i$, $-\rho$.

$f(z)$ has simple poles given by

$$e^z = -1 = e^{\{(2K+1)\pi i\}}$$

$$\Rightarrow z = (2K+1)\pi i \; ; \; K = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now, Residue at $z = \pi i$ (inside contour) is

$$\left[\frac{e^{2/3 z}}{\frac{d}{dz}(e^z+1)}\right]_{z=\pi i}$$

$$= \left.\frac{e^{2/3 z}}{e^z}\right|_{z=\pi i}$$

$$= \frac{e^{\frac{2}{3}(\pi i)}}{e^{\pi i}}$$

$$= -e^{\left(\frac{2\pi i}{3}\right)}$$

By Residue theorem,

$$\int_C f(z) = 2\pi i\,[\text{Sum of residue}]$$

$$\Rightarrow \int_{DA} f(z)dz + \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz = -2\pi i\, e^{\left(\frac{2\pi i}{3}\right)}$$

$$\Rightarrow \int_{-\rho}^{R} f(x)dx + \int_0^{2\pi} f(R+iy)i\,dy + \int_R^{-\rho} f(x+2\pi i)dx + \int_{2\pi}^{0} f(-\rho+iy)i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\rho}^{R} \frac{e^{\frac{2x}{3}}}{e^x+1}dx + \int_R^{-\rho} \frac{e^{\frac{2}{3}(x+2\pi i)}}{e^{(x+2\pi i)}+1}dx + \int_0^{2\pi} f(R+iy)i\,dy + \int_{2\pi}^{0} f(-\rho+iy)i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\rho}^{R} \frac{e^{2/3 x}\left[1 - e^{\frac{4\pi i}{3}}\right]}{(e^x+1)}dx + \int_0^{2\pi} f(R+iy)i\,dy + \int_{2\pi}^{0} f(-\rho+iy)i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}} \quad \text{--- (2)}$$

$$\Rightarrow \left[1-e^{-\frac{4\pi i}{3}}\right] \int_{-R}^{R} \frac{e^{2/3x}}{e^x+1}\,dx = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{2/3x}}{e^x+1}\,dx = \frac{-2\pi i\, e^{\frac{2\pi i}{3}}}{\left[1-e^{-\frac{4\pi i}{3}}\right]}$$

$$\Rightarrow I = \frac{-2\pi i\, \cancel{e^{\frac{2\pi i}{3}}}}{\cancel{e^{+\frac{2\pi i}{3}}}\left[e^{-\frac{2\pi i}{3}} - e^{+\frac{2\pi i}{3}}\right]}$$

$$= \frac{\pi}{\dfrac{e^{-\frac{2\pi}{3}i} - e^{\frac{2\pi}{3}i}}{-2i}}$$

$$= \frac{\pi}{\dfrac{e^{\frac{2\pi}{3}i} - e^{-\frac{2\pi}{3}i}}{2i}} = \frac{\pi}{\left(\sin\frac{2\pi}{3}\right)} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$$

$$\Rightarrow \boxed{I = \frac{2\pi}{\sqrt{3}}} \quad \text{Ans}$$

Now, $\int_0^{2\pi} f(R+iy)\,i\,dy = \int_0^{2\pi} \frac{e^{\frac{2}{3}(R+iy)}}{1+e^{(R+iy)}}\,i\,dy$

$$\Rightarrow \left|\int_0^{2\pi} \frac{e^{\frac{2}{3}(R+iy)}}{1+e^{(R+iy)}}\,i\,dy\right| \leq \int_0^{2\pi}\left|\frac{e^{\frac{2}{3}R}e^{\frac{2}{3}iy}i}{e^R\cdot e^{iy}+1}\right|dy$$

$$= \int_0^{2\pi}\frac{\left|e^{\frac{2}{3}R}e^{\frac{2}{3}iy}i\right|}{\left|e^Re^{iy}-(-1)\right|}dy$$

$$< \int_0^{2\pi}\frac{e^{2/3R}}{|e^Re^{iy}|-|-1|}dy$$

$$= \int_0^{2\pi}\frac{e^{2/3R}}{e^R-1}dy$$

$$= \frac{2\pi e^{\frac{2}{3}R}}{e^R-1} \to 0 \text{ as } R\to\infty$$

Also $\int_{2\pi}^{0} f(-\rho+iy)\,i\,dy = \int_{2\pi}^{0}\frac{e^{\frac{2}{3}(-\rho+iy)}}{1+e^{(-\rho+iy)}}\,i\,dy$

$$\Rightarrow \left|\int_{2\pi}^{0}\frac{e^{-\frac{2}{3}\rho}e^{2/3iy}i}{(1+e^{-\rho}e^{iy})}dy\right| \leq \int_{2\pi}^{0}\frac{\left|e^{-\frac{2}{3}\rho}e^{\frac{2}{3}iy}i\right|}{\left|e^{-\rho}e^{iy}-(-1)\right|}dy$$

$$\leq \int_{2\pi}^{0}\frac{|e^{-2/3\rho}|\,|e^{\frac{2}{3}iy}i|}{|e^{-\rho}e^{iy}|-|-1|}dy$$

$$= \int_{2\pi}^{0}\frac{e^{-\frac{2}{3}\rho}}{e^{-\rho}-1}dy$$

$$= 2\pi\frac{e^{-2/3\rho}}{1-e^{-\rho}} \to 0 \text{ as } \rho\to\infty$$

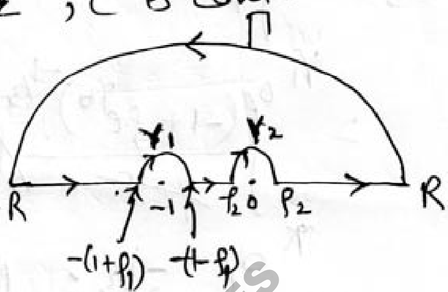
Thus equation (2) becomes.

$$\int_{-\rho}^{R}\frac{e^{\frac{2}{3}x}\left[1-e^{\frac{4\pi i}{3}}\right]dx}{e^x+1} = -2\pi i e^{\frac{2\pi i}{3}}.$$

Method - 2

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \int_0^\infty \frac{x^{-1/3}}{x+1}\,dx = I \text{ (say)}$$

Consider $\int_C \frac{z^{-1/3}\,dz}{z+1} = \int_C f(z)\,dz$; C is contour shown



By Cauchy's theorem,

$$\int_C f(z)\,dz = 0$$

But
$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{-(1+\rho_1)} f(r)\,dr + \int_{\gamma_1} f(z)\,dz + \int_{-(1+\rho_1)}^{-\rho_2} f(r)\,dr + \int_{\gamma_2} f(z)\,dz + \int_{\rho_2}^{R} f(x)\,dx = 0$$

$$\Rightarrow I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0$$

where, $I_1 = \int_\Gamma f(z)\,dz \longrightarrow 0$ as $R \to \infty$ [Integral of semi circle about origin]

$$I_2 = \int_{-R}^{-(1+\rho_1)} f(r)\,dr = \int_R^{(1+\rho_1)} f(re^{i\pi})e^{i\pi}\,dr$$

$$I_3 = \int_{\gamma_1} f(z)\,dz$$

$$I_4 = \int_{-(1-\rho_1)}^{-\rho_2} f(r)\,dr = \int_{(1-\rho_1)}^{\rho_2} f(re^{i\pi})\,e^{i\pi}\,dr$$

$I_5 = \int_{\gamma_2} f(z)\,dz \longrightarrow 0$ as $\gamma_2 \to 0$ [Integral semi circ about origin]

$$I_6 = \int_{\rho_2}^{R} f(x)\,dx$$

Thus, We have $I_2 + I_3 + I_4 + I_6 = 0$

Now $I_3 = \int_{\gamma_1} f(z)\,dz$ $\qquad |z+1| = \rho e^{i\theta}$, $dz = \rho i e^{i\theta} d\theta$

$$= \int_{\pi}^{0} \frac{(-1+\rho e^{i\theta})^{-1/3}}{\cancel{\rho e^{i\theta}}} \cancel{\rho} i \cancel{e^{i\theta}} d\theta$$

$$= \int_{\pi}^{0} (-1)^{-1/3} i\, d\theta \quad \text{as } \rho \to 0$$

$$= \left(\int_{\pi}^{0} d\theta\right)(-1)^{-1/3} i \qquad = (-1)^{2/3}\pi i = e^{\frac{i2\pi}{3}}\pi i = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\pi i$$

$$= -\frac{\pi}{2} i - \frac{\sqrt{3}}{2}\pi$$

$$= -\pi(-1)^{-1/3} i \quad \boxed{\cancel{(-1)^{-1/3}\pi i = e^{\frac{i2\pi}{3}}\pi i = \left[\cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right)\right]\pi i}}$$

$$I_2 = \int_{R}^{1+\rho_1} \frac{r^{-1/3} e^{-\frac{i\pi}{3}} e^{i\pi} dr}{re^{i\pi}+1} = \int_{1+\rho_1}^{R} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr$$

$$I_4 = \int_{\rho_2}^{1-\rho_1} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr$$

Thus, $I_2 + I_3 + I_4 + I_6 = 0 \Rightarrow I_2 + I_4 + I_6 = -I_3$

$$\Rightarrow \int_{1+\rho_1}^{R} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr + \int_{\rho_2}^{1-\rho_1} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr + \int_{\rho_2}^{R} \frac{x^{-1/3}}{x+1} dx = \frac{\sqrt{3}\pi}{2} + \frac{i\pi}{2}$$

As $\rho_1, \rho_2 \to 0$, $R \to \infty$, we have

$$\int_{0}^{1} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr + \int_{1}^{\infty} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr + \int_{0}^{\infty} \frac{x^{-1/3}}{x+1} dx = \frac{\sqrt{3}\pi}{2} + \frac{i\pi}{2}$$

$$\int_{0}^{\infty} \frac{r^{-1/3} e^{-\frac{i\pi}{3}}}{re^{i\pi}+1} dr + \int_{0}^{\infty} \frac{x^{-1/3}}{x+1} dx = \frac{\sqrt{3}\pi}{2} + \frac{i\pi}{2}$$

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Replacing t by x, we get

$$\int_0^\infty \frac{x^{-1/3}}{1-x}\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]dx+\int_0^\infty \frac{x^{-1/3}}{x+1}dx=\frac{\sqrt{3}}{2}\pi+i\frac{\pi}{2}$$

$$\Rightarrow \frac{1}{2}\int_0^\infty \frac{x^{-1/3}}{1-x}dx+\int_0^\infty \frac{x^{-1/3}}{x+1}dx+i\int_0^\infty \frac{x^{-1/3}\sin\left(-\frac{\pi}{3}\right)}{(1-x)}dx=\frac{\sqrt{3}\pi}{2}+i\frac{\pi}{2}$$

$$\Rightarrow \left[\int_0^\infty \frac{x^{-1/3}}{x+1}dx+\frac{1}{2}\int_0^\infty \frac{x^{-1/3}}{1-x}dx\right]+i\left(-\frac{\sqrt{3}}{2}\right)\int_0^\infty \frac{x^{-1/3}}{1-x}dx=+\frac{\sqrt{3}}{2}\pi+\frac{\pi}{2}i$$

Equating the real part, we obtain

$$\int_0^\infty \frac{x^{-1/3}}{x+1}dx+\frac{1}{2}\int_0^\infty \frac{x^{-1/3}}{1-x}dx=+\frac{\sqrt{3}}{2}\pi \quad \text{—— (1)}$$

Equating imaginary part, we get

$$-\frac{\sqrt{3}}{2}\int_0^\infty \frac{x^{-1/3}}{1-x}dx=\frac{\pi}{2} \quad \text{—— (2)}$$

Using (2), we get

$$\boxed{\int_0^\infty \frac{x^{-1/3}}{1-x}dx=-\frac{\pi}{\sqrt{3}}} \quad \text{—— (3)}$$

Using (1) & (3), we get

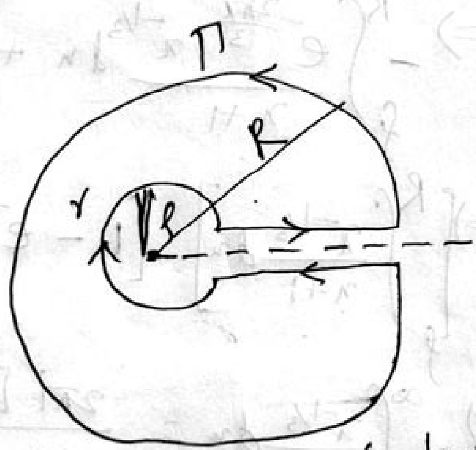
$$\int_0^\infty \frac{x^{-1/3}}{x+1}dx+\frac{1}{2}\left(-\frac{\pi}{\sqrt{3}}\right)=+\frac{\sqrt{3}\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{x^{-1/3}}{x+1}dx=\frac{\pi}{2\sqrt{3}}+\frac{\sqrt{3}\pi}{2}=\frac{\pi}{2\sqrt{3}}+\frac{\sqrt{3}\pi\sqrt{3}}{2\sqrt{3}}=\frac{4\pi}{2\sqrt{3}}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{x^{-1/3}}{x+1}dx=\frac{2\pi}{\sqrt{3}}}. \quad \text{Ans}$$

Method - 3

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \int_0^\infty \frac{x^{-1/3}}{x+1}\,dx$$



Consider.

$$\int_C f(z)\,dz = \int_C \frac{z^{-1/3}}{z+1}\,dz \quad \text{--- (1)}$$

where C is the curve consisting "Dogbone Contour" of

(i) A large circle Γ, $|z|=R$

(ii) Radius vector below lineX ie $\theta = 2\pi$

(iii) A small circle γ, $|z|=\rho$

(iv) Radius vector above line X.

$(\theta = 0)$

According to Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of Residue}] \quad \text{--- (2)}$$

$$\Rightarrow \int_\Gamma f(z)\,dz + \int_R^\rho \frac{(re^{i2\pi})^{-1/3}}{(re^{i2\pi}+1)} e^{i2\pi}\,dr + \int_\gamma f(z)\,dz + \int_\rho^R \frac{x^{-1/3}}{x+1}\,dx$$

$$= 2\pi i\,[\text{Sum of residue}] \quad \text{--- (3)}$$

Clearly, $\int_\Gamma f(z)\,dz = 0$ as $R \to \infty$

and $\int_\gamma f(z)\,dz = 0$ as $\rho \to 0$

Thus, we have $\int_R^\rho \frac{e^{-\frac{2\pi}{3}i}\, r^{-1/3}}{r+1}\,dr + \int_\rho^R \frac{x^{-1/3}}{x+1}\,dx = 2\pi i\,[\text{Sum of Re}$

$$\Rightarrow -\int_{\rho}^{R} \frac{e^{-\frac{2\pi i}{3}} x^{-1/3}}{x+1}dx + \int_{\rho}^{R} \frac{x^{-1/3}}{x+1}dx = 2\pi i\,[\text{Sum of residue}]$$

$$\Rightarrow \left[\int_{\rho}^{R} \frac{x^{-1/3}}{x+1}dx\right]\left[1 - e^{-2\pi i/3}\right] = 2\pi i\,[\text{Sum of residue}]$$

$$\Rightarrow \int_0^\infty \frac{x^{-1/3}}{x+1}dx = \frac{2\pi i\,[\text{Sum of residue}]}{\left[1-e^{-\frac{2\pi i}{3}}\right]} \quad [\text{As } \rho\to 0,\ R\to\infty] \qquad \text{——— (4)}$$

$\because f(z)$ has simple pole at $z = -1 = e^{i\pi}$

Residue at $f(z)$ at $z = e^{i\pi}$ is

$$\lim_{z\to e^{i\pi}} \cancel{(z+1)}\frac{z^{-1/3}}{\cancel{(z+1)}} = e^{-\frac{i\pi}{3}}$$

$$\therefore [\text{Sum of residue}] = e^{-i\frac{\pi}{3}} \qquad \text{——— (5)}$$

Using (4) & (5), we get

$$\int_0^\infty \frac{x^{-1/3}}{x+1}dx = \frac{2\pi i\, e^{-i\pi/3}}{1-e^{-\frac{2\pi i}{3}}}$$

$$= \frac{\pi\, \cancel{e^{-i\pi/3}}}{\cancel{e^{-i\pi/3}}\left[\frac{e^{i\pi/3} - e^{-i\pi/3}}{2i}\right]}$$

$$= \frac{\pi}{\sin(\pi/3)}$$

$$= \frac{2\pi}{\sqrt{3}} \quad \text{Ans}$$

Method-4

$$\int_0^1 \frac{1}{(x^2-x^3)^{1/3}}\,dx$$

$$= \int_0^1 \frac{1}{x\left(\frac{1}{x}-1\right)^{1/3}}\,dx$$

Put $\left(\frac{1}{x}-1\right) = t^3 \Rightarrow x = \frac{1}{t^3+1}$ & $-\frac{1}{x^2}dx = 3t^2 dt$

$$= \int_\infty^0 \frac{-3t^2\,dt}{(t^3+1)\,t}$$

$$= \int_0^\infty \frac{3t}{t^3+1}\,dt \qquad \text{Put } t^3 = x^2 \Rightarrow t = x^{2/3}$$

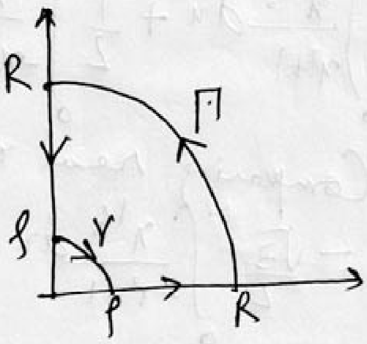
$$dt = \frac{2}{3}x^{-1/3}\,dx$$

$$= \int_0^\infty \frac{3x^{2/3}\cdot\frac{2}{3}x^{-1/3}\,dx}{x^2+1}$$

$$= 2\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$$

$$= 2\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$$

$$= 2I$$



$$I = \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx = \int_C \frac{z^{1/3}}{z^2+1}\,dz$$

By Cauchy's theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of residue}] \quad \text{——(1)}$$

Residue at $z = i = \lim_{z\to i} \cancel{(z-i)}\frac{z^{1/3}}{\cancel{(z-i)}(z+i)}$

$$= \frac{i^{1/3}}{2i} = \frac{e^{i\frac{\pi}{6}}}{2i} = \frac{1}{2}e^{-i\frac{\pi}{3}}$$

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From ①, we have

$$\underbrace{\int_\Gamma f(z)\,dz}_{\to 0} + \int_{Ri}^{\rho i} f(z)\,dz + \underbrace{\int_{\gamma} f(z)\,dz}_{\to 0} + \int_\rho^R \frac{x^{1/3}}{x^2+1}\,dx = 2\pi i\,[\text{Sum of Residue}]$$

(as $R\to\infty$) (as $\rho\to 0$)

$$\Rightarrow \int_R^\rho f(re^{i\pi/2})\,e^{i\pi/2}\,dr + \int_\rho^R \frac{x^{1/3}}{x^2+1}\,dx = 2\pi i\,[\text{Sum of Residue}]$$

$$\Rightarrow \int_R^\rho \frac{r^{1/3} e^{i\pi/6}\cdot e^{i\pi/2}\,dr}{(re^{i\pi/2})^2+1} + \int_\rho^R \frac{x^{1/3}}{x^2+1}\,dx = \cancel{2}\pi i\,\frac{1}{\cancel{2}}\,e^{-i\pi/3}$$

$$\Rightarrow -\int_\rho^R \frac{r^{1/3} e^{\frac{2\pi}{3}}}{-r^2+1}\,dr + \int_\rho^R \frac{x^{1/3}}{x^2+1}\,dx = \pi i\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \frac{\pi}{2}i + \frac{\pi\sqrt{3}}{2}$$

$$\Rightarrow \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx - \int_0^\infty \frac{x^{1/3}}{-x^2+1}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)dx = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{2}i$$

$$\Rightarrow \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx + \frac{1}{2}\int_0^\infty \frac{x^{1/3}}{-x^2+1}\,dx - i\frac{\sqrt{3}}{2}\int_0^\infty \frac{x^{1/3}}{-x^2+1}\,dx = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{2}i$$

Comparing real & imaginary part, we get

$$-\frac{\sqrt{3}}{2}\int_0^\infty \frac{x^{1/3}}{-x^2+1}\,dx = +\frac{\pi}{2} \Rightarrow \boxed{\int_0^\infty \frac{x^{1/3}}{-x^2+1}\,dx = -\frac{\pi}{\sqrt{3}}}$$

And $$\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx + \frac{1}{2}\int_0^\infty \frac{x^{1/3}}{-x^2+1}\,dx = \frac{\pi\sqrt{3}}{2}$$

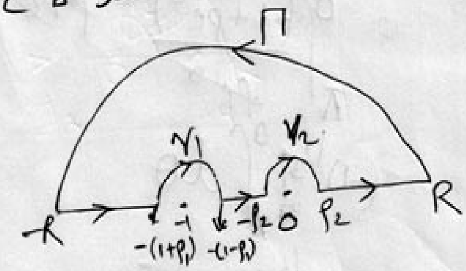
$$\Rightarrow \int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx = \frac{\pi\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\pi}{\sqrt{3}}\right) = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{2\sqrt{3}}$$

$$= \frac{4\pi}{2\sqrt{3}}$$

$$= \frac{2\pi}{\sqrt{3}} \quad \underline{\text{Ans}}$$

Q. $\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \int_0^\infty \frac{x^{-1/3}}{x+1}\,dx$

Soln Consider $\int_C \frac{z^{-1/3}}{z+1}\,dz = \int_C f(z)\,dz$, C is the contour shown



By Cauchy's theorem,

$$\int_C f(z)\,dz = 0.$$

But $\int_C f(z)\,dz = \cancel{\int_\Gamma f(z)\,dz} + \int_R^{1+\rho_1} f(re^{i\pi})e^{i\pi}\,dr + \int_{\gamma_1} f(z)\,dz$

$$+ \int_{+(1-\rho_1)}^{+\rho_2} f(re^{i\pi})e^{i\pi}\,dr + \cancel{\int_{\gamma_2} f(z)\,dz} + \int_{\rho_2}^{R} f(z)\,dz = 0$$

Thus, $I_1 + I_2 + I_3 + I_4 = 0$;

$I_1 = \int_R^{1+\rho_1} f(re^{i\pi})e^{i\pi}\,dr = \cancel{e^{i\pi}\int_R^{1+\rho_1} \frac{r^{-1/3}e^{-i\pi/3}}{re^{i\pi}+1}\,dr}$

$I_2 = \int_{\gamma_1} f(z)\,dz$

$I_3 = \int_{1-\rho_1}^{\rho_2} f(re^{i\pi})e^{i\pi}\,dr$

$I_4 = \int_{\rho_2}^{R} f(x)\,dz$

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$$I_2 = \int_{\gamma_1} f(z)\, dz \qquad |z+1| = \rho e^{i\theta}, \quad dz = \rho i e^{i\theta} d\theta$$

$$= \int_{\pi}^{0} \frac{(-1+\rho e^{i\theta})^{1/3}}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta$$

$$= (-1)^{1/3} i \int_{\pi}^{0} d\theta \qquad \text{As } \rho \to 0.$$

$$= (-1)^{1/3}(-\pi) i$$

$$= -\pi e^{i\pi/3} i$$

$$= -\pi e^{i\frac{\pi}{3}} \cdot i = -\pi\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) i = -\frac{\pi}{2} i + \pi\frac{\sqrt{3}}{2}$$

$$I_4 = \int_0^{\infty} \frac{x^{-1/3}}{1+x}\, dx$$

$$I_3 = \int_{1-\rho_1}^{\rho_2} f(re^{i\pi}) e^{i\pi} dr$$

$$= \int_{1-\rho_1}^{\rho_2} \frac{r^{-1/3} \cdot e^{-i\frac{\pi}{3}} e^{i\pi}}{re^{i\pi}+1} dr$$

$$= \int_{\rho_2}^{1-\rho_1} \frac{r^{-1/3} e^{-i\frac{\pi}{3}}}{re^{i\pi}+1} dr$$

$$I_1 = \int_R^{1+\rho_1} \frac{r^{-1/3} e^{-i\frac{\pi}{3}} e^{i\pi}}{re^{i\pi}+1} dr = \int_{1+\rho_1}^{R} \frac{r^{-1/3} e^{-i\frac{\pi}{3}}}{(re^{i\pi}+1)} dr$$

$$\therefore \int_{\rho_2}^{1-\rho_1} \frac{r^{-1/3} e^{-i\frac{\pi}{3}}}{re^{i\pi}+1} dr + \int_{1+\rho_1}^{R} \frac{r^{-1/3} e^{-i\pi/3}}{(re^{i\pi}+1)} dr + \int_0^{\infty} \frac{x^{-1/3}}{1+x} dx = \frac{\pi\sqrt{3}}{2} - \frac{\pi}{2} i$$

As $\rho_2, \rho_1 \to 0$, $R \to \infty$.

$$\int_0^1 \frac{r^{-1/3} e^{-i\pi/3}}{re^{i\pi}+1}\,dr + \int_1^\infty \frac{r^{-1/3}e^{-i\pi/3}}{(re^{i\pi}+1)}\,dr + \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx = -\frac{\pi}{2}\sqrt{3} + i\frac{\pi}{2}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{r^{-1/3}e^{-i\pi/3}}{re^{i\pi}+1}\,dr + \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx = -\frac{\pi}{2}\sqrt{3} + i\frac{\pi}{2}.}$$

$$\int_0^\infty \frac{x^{-1/3}}{(-x+1)}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]dx + \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx = -\frac{\pi}{2}\sqrt{3} + i\frac{\pi}{2}$$

$$\left(\int_0^\infty \frac{x^{-1/3}}{-x+1}\cos\left(\frac{\pi}{3}\right)dx + \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx\right) - i\int_0^\infty \frac{x^{-1/3}}{-x+1}\cdot\sin\left(\frac{\pi}{3}\right)dx = -\frac{\pi}{2}\sqrt{3} + i\frac{\pi}{2}$$

$$\boxed{\int_0^\infty \frac{x^{-1/3}}{1+x}\,dx + \frac{1}{2}\int_0^\infty \frac{x^{-1/3}}{-x+1}\,dx = -\frac{\pi}{2}\sqrt{3}.} \quad \text{——— (1)}$$

$$\boxed{\frac{\sqrt{3}}{\cancel{2}}\int_0^\infty \frac{x^{-1/3}\,dx}{-x+1} = -\frac{\pi}{\cancel{2}}}$$

$$\boxed{\int_0^\infty \frac{x^{-1/3}}{-x+1}\,dx = -\frac{\pi}{\sqrt{3}}.} \quad \text{——— (2)}$$

$$\int_0^\infty \frac{x^{-1/3}}{1+x}\,dx + \frac{1}{2}\left(\frac{-\pi}{\sqrt{3}}\right) = \frac{\pi}{2}\sqrt{3}.$$

$$\Rightarrow \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx = \cancel{\frac{\pi}{2\sqrt{3}} - \frac{\pi\sqrt{3}}{2}} = \cancel{\frac{\pi}{2\sqrt{3}} - \frac{\pi\sqrt{3}\sqrt{3}}{2\sqrt{3}}}$$

$$= \frac{\pi}{2}\sqrt{3} + \frac{\pi}{2\sqrt{3}}. \qquad = \frac{\pi}{2}\sqrt{3} - \frac{\pi}{2\sqrt{3}} \qquad = \cancel{\frac{\pi\sqrt{3}(\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{2\sqrt{3}}}$$

$$= \frac{4\pi}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}} \quad \underline{\text{Ans}}$$

4(c) IAS 2011 P-II → Write down the dual of the following LP problem and hence solve it by graphical method:-

Minimize, $Z = 6x_1 + 4x_2$

Constraints, $2x_1 + x_2 \geq 1$

$3x_1 + 4x_2 \geq 1.5$

$x_1, x_2 \geq 0$

Sol^n → Given LPP is of minimisation type and constraints are '$\geq$' type.

(i) $\because$ Dual of given LPP is given by

Maximise $z' = y_1 + 1.5 y_2$

ST; $2y_1 + 3y_2 \leq 6$

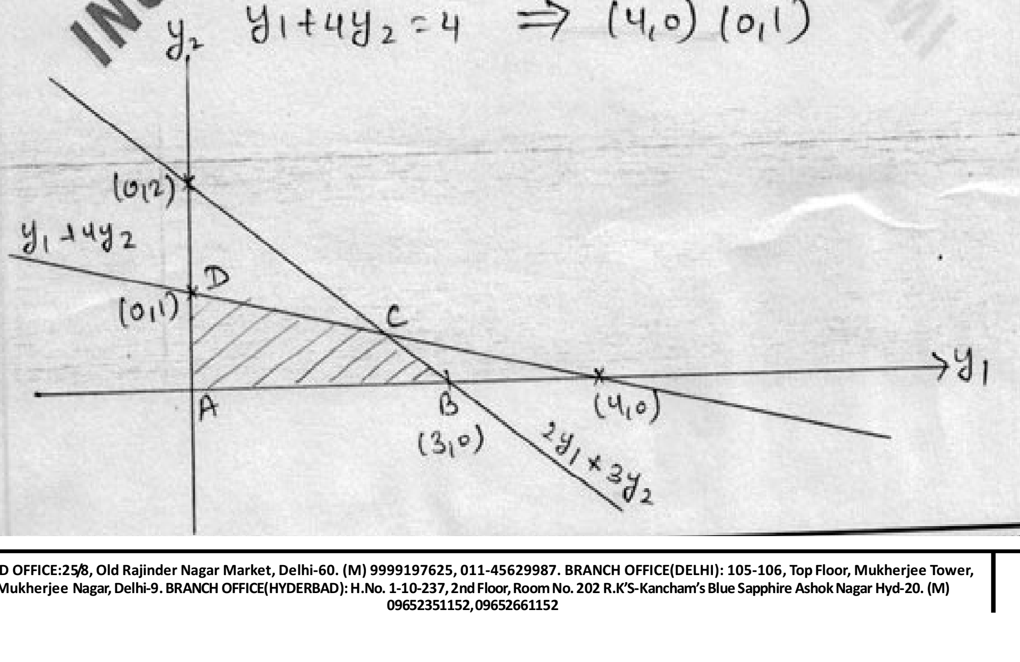
$y_1 + 4y_2 \leq 4$

$y_1, y_2 \geq 0$

(ii) Plot the equations of the constraints in a graph.

$2y_1 + 3y_2 = 6 \Rightarrow (3,0)\ (0,2)$

$y_1 + 4y_2 = 4 \Rightarrow (4,0)\ (0,1)$



$\therefore$ The corner points are $(0,0)$, $(0,1)$, $(3,0)$, $(2\cdot4, 0\cdot4)$.

Now the value of z'

z' at $(0,0) = 0$

at $(0,1) = 0 + 1\cdot5 = 1\cdot5$

at $(3,0) = 3 + 0 = 3$

at $(2\cdot4, 0\cdot4) = 2\cdot4 + 0\cdot6 = 3$

$\therefore$ Maximum value is available at 2 corner points Hence optimal solution is available at all points on the line $2y_1 + 3y_2 = 6$ with $2\cdot4 \le y_1 \le 3$ and $0 \le y_2 \le 0\cdot4$

Hence Maximum $z' = 3$.

$\therefore$ Minimum $Z = 3$ $[6x_1 + 4x_2]$

5(a). Solve the PDE $(D^2-D'^2+D+3D'-2)z = e^{(x-y)}-x^2y$

SOLUTION

Given equation can be re written as

$$[(D+D_1'(D-D_1')+2(D+D_1')-(D-D')-2)z=e^{x-y}-x^2y$$

$$(D-D'+2)(D+D'+1)z=e^{x-y}-x^2y$$

C.F. $= e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)$

ϕ_1,ϕ_2 being arbitrary constants.

P.I. corresponding to e^{x-y}

$$\Rightarrow \quad \frac{1}{(D^2-D'^2+D+3D'-2)}e^{x-y} = \frac{1}{1-1+1+(3)-2}e^{x-y}$$

$$= \frac{e^{x-y}}{2}$$

P.I. corresponding to $(-x^2y)$

$$\Rightarrow \quad \frac{1}{D^2-D_3{}^2+D+3D'-2}\left(-x^2y\right)=\frac{1}{2}\left(1+\frac{(D-D')}{2}\right)^{-1}(1-(D+D'))^{-1}x^2y$$

$$= \frac{1}{2}\left(1+\frac{(D-D')}{2}\right)^{-1}\left(1+(D+D')+(D+D')^2\ldots\right)x^2y$$

$$= \frac{1}{2}\left(1+\left(\frac{D-D'}{2}\right)^{-1}\left(x^2y+2xy+x^2+xy+4x+6\right)\right)$$

$$= \frac{1}{2}\left(1-\left(\frac{D-D'}{2}\right)+\left(\frac{D-D'}{2}\right)^2-\left(\frac{D-D'}{2}\right)^3\left(x^2y+2xy+x^2+2y+4x+6\right)\right)$$

$$= \frac{1}{2}\left(1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}+\frac{D'^2}{4}-\frac{DD'}{2}+\frac{3D'D^2}{8}\ldots\right)\left(x^2y+2xy+x^2+2y+4x+6\right)$$

$$= \frac{1}{2}\left(x^2y+2xy+x^2+2y+4x+6-\frac{1}{2}(2xy+2y+2x+4)\right)$$

$$+\frac{1}{2}\left(x^2+2x+2\right)+\frac{1}{4}\left(2y+2\right)+\frac{1}{4}(0)-(2x+2)\frac{3}{8}(2)$$

P.I.$= \frac{1}{2}\left(x^2y+xy+3x^2/2+3y/2+3x+21/4\right)$

$\therefore$ General solution $= e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)\frac{+1}{2}e^{x-y} + \frac{1}{2}\left(x^2y+xy+\frac{3x^2}{2}+\frac{3y}{2}+3x+21/4\right)$

5(b). Solve the PDE $(x+2z)\frac{\partial z}{\partial x}+\left(4zx-y\right)\frac{\partial z}{\partial y}=2x^2+y$.

SOLUTION

Lagranges auxiliary equations are given by $\frac{dx}{x+2z}=\frac{dy}{4xz-y}=\frac{dz}{2x^2+y}$ .....(1)

Choose y,x,–2z as multiplirs

$$(1) = \frac{ydx+xdy-2zdz}{y(x+2z)+x(4xz-y)-2z(2x^2+y4)}=\frac{d(xy-z^2)}{0}$$

$$= \boxed{xy-z^2=c_1} \quad .....(2)$$

Choosing 2x, –1, –1 as multipliers

$$\frac{2xdx-dy-dz}{2x(x+2z)-(4xz-y)-(2x^2+y4)}=\frac{d(x^2-y-z)}{0}$$

$x^2-y-z=c_2$ .....(3)

from (2) and (3). solution is given by

$\phi(xy-z^2, x^2-y-z)=0$

ϕ being arbitrary functions.

Q. Calculate $\int_2^{10} \frac{dx}{1+x}$ (upto 3 decimal places) by dividing the range into 8 equal parts by Simpson's $\frac{1}{3}$ rule.

5(b)

Sol:- $f(x) = \frac{1}{1+x}$

Here ; $a=2$, $b=10$

$h = \frac{b-a}{n} = \frac{10-2}{8} = 1$

x	2	3	4	5	6	7	8	9	10
$y=f(x)=\frac{1}{1+x^2}$	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100	0.091

Now, Applying Simpson 1/3 rule

$$\int_2^{10} f(x)dx = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$=\frac{h}{3}\left[(0.333+0.091)+4(0.642)+2(0.454)\right]$$

put $h=1$.

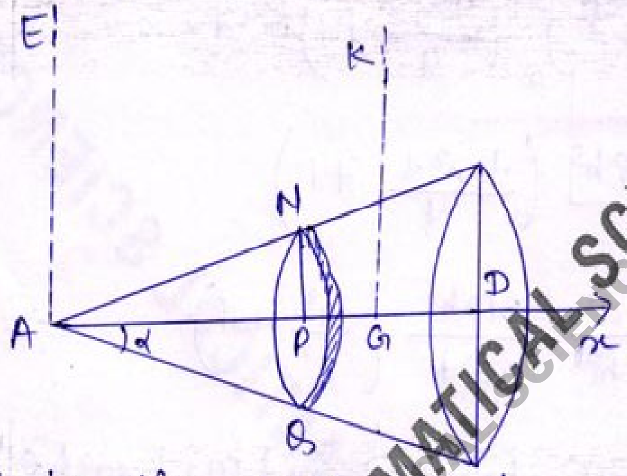
$$\int_2^{10} f(x)\,dx = \frac{1}{3}\left[0.424 + 2.568 + 0.908\right]$$

$$= \frac{1}{3}\left[3.900\right] = \frac{1}{3}\times 3.9$$

$$\int_2^{10} f(x)\,dx = 1.3 \text{ Ans}$$

5(e) IAS 2011 P-II → Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis.

Sol^n →



Let α be the semivertical angle of the cone, Then

$$\tan\alpha = a/h \quad \text{—— (1)}$$

M is the mass of the cone.

$$\therefore M = \frac{1}{3}\pi a^2 h \rho \quad \text{—— (2)}$$

Consider an elementary disc of thickness δx at a distance $x = AP$ from A.

Radius of the disc $= PN = x\tan\alpha$

and mass of the disc $= \rho\pi x^2 \tan^2\alpha\, \delta x$.

M I of the disc about AD

$$= \rho\pi x^2 \tan^2\alpha \cdot \frac{x^2\tan^2\alpha}{2}\,\delta x$$

$$= \frac{1}{2}\rho\pi \tan^4\alpha\, x^4\, \delta x$$

Hence M.I. of the cone about AD

$$= \frac{\rho\pi\tan^4\alpha}{2}\int_0^h x^4 dx = \frac{\pi\tan^4\alpha\,\rho h^5}{10} = \frac{3}{10}Ma^2 \quad \text{—— (3)}$$

(using (2))

Also M.I. of the cone about AF, a line through the vertex A perpendicular to AD.

$$= \int_0^h \pi x^2 \tan^2\alpha \cdot \rho\, dx \left[\frac{x^2 \tan^2\alpha}{4} + x^2\right] \quad \text{(By theorem of parallel axis)}$$

$$= \pi \tan^2\alpha \cdot \rho \int_0^h \left(\frac{\tan^2\alpha}{4} + 1\right) x^4\, dx$$

$$= \frac{\pi \tan^2\alpha\, \rho h^5}{5} \left(\frac{\tan^2\alpha}{4} + 1\right)$$

$$= \frac{\pi}{20} \cdot \frac{a^2}{h^2} \cdot \frac{3mh^5}{\pi a^2 h} \left(\frac{a^2}{h^2} + 4\right)$$

$$\left[\because \tan\alpha = \frac{a}{h} \text{ from (1) and } m = \frac{1}{3}\pi a^2 h \rho \text{ from (2)}\right]$$

$$= \frac{3m}{20}(a^2 + 4h^2) \quad \text{——— (4)}$$

Product of inertia of the cone about AD, AF is clearly zero.

Now moment of inertia of the cone about slant height which makes an angle α with AX.

$$= \frac{3ma^2}{10} \cos^2\alpha + \frac{3m}{20}(a^2 + 4h^2) \sin^2\alpha$$

$$= \frac{3ma^2}{10} \frac{h^2}{a^2 h^2} + \frac{3m}{20}(a^2 + 4h^2) \frac{a^2}{a^2 + h^2}$$

$$= \frac{3ma^2}{20} \frac{6h^2 + a^2}{a^2 + h^2}$$

$$\left[\because \tan\alpha = a/h,\ \sin\alpha = \frac{a}{\sqrt{a^2+h^2}} \text{ and } \cos\alpha = \frac{h}{\sqrt{a^2+h^2}}\right]$$

Now to find the moment of inertia about a line GK through the centre of Gravity 'G' of the cone and perpendicular to the axis of the cone.

By parallel axis theorem, we have.

M.I. about AE = M.I. about GK + MI about AE of mass M placed at G.

M.I. About GK = M.I. about AE − M.I. about AE of mass M placed at G.

$$= \frac{3M}{20}\left(a^2+4h^2\right) - M.\frac{9h^2}{16}$$

[using (7) also $AG = \frac{3h}{4}$]

$$= \frac{3M}{80}\left[4a^2+16h^2-15h^2\right]$$

$$= \frac{3M}{80}\left[h^2+4a^2\right]$$

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6(a). Find surface satisfying $\frac{\partial^2 z}{\partial x^2} = 6x + 2$ and touching z = x^3+y^3 along its section by the plane x+y+1=0.

SOLUTION

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2$$

$$\frac{\partial p}{\partial x} = 6x + 2$$

$$p = 3x^2 + 2x + \phi(y)$$

$$z = x^3 + x^2 + x\phi(y) + \psi(y) \qquad \text{....(1)}$$

Given surface satisfies $z = x^3 + y^3$ and (x+y+1) = 0. (1) & (2) touch along their common section by x+y+1=0.

The values of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ from (1) & (2) must be the same.

$\therefore$ from (1) $\frac{\partial z}{\partial x} = 3x^2 + 2x + \phi(y)$ .....(4)

from the given surface $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3) = 3x^2$ .....5)

Equating, (4) = (5) $\Rightarrow$

$$3x^2 = 3x^2 + 2x + \phi(y)$$
$$\phi(y) = -2x \text{ as } x+y+1=0.$$
$$\phi(y) = 2y+2 \qquad \text{.....(6)}$$

Similarly $\frac{\partial z}{\partial y} = x\phi'(y) + \psi'(y)$

$\frac{\partial z}{\partial y} = x(2) + \psi'(y)$ from (6) $\phi'(y)=2$

from (2) $\frac{\partial z}{\partial y} = 3y^2$

$$3y^2 = 2x + \psi'(y)$$
$$\psi'(y) = 3y^2 - 2x$$
$$\psi'(y) = 3y^2 + 2(y+1)$$
$$\psi(y) = y^3 + y^2 + 2y + c$$

$\therefore$ $Z = x^3 + x^2 + x(2y+2) + y^3 + y^2 + 2y + c$

$Z = x^3 + y^3 + x^2 + 2xy + y^2 + 2x + 2y + c$

Putting z = x^3+y^3, x + y = –1

Given c = 1

$\therefore$ $\boxed{Z = x^3 + y^3 + (x+y+1)^2}$ Required solution

6(c). Obtain the temperature distribution y(x, t) in a uniform bar of unit length whose one end is kept at 10^0C and the other end is insulated. also it is given that y(x,0) = 1–x, 0<x<1.

SOLUTION

Let the bar be placed along x axis with its one end at origin and other end at x = 1.

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2} \quad \text{....(1) heat equations}$$

Boundary conditions $y_x(1, t) = 0 \quad y(0,t) = 10.$

Initial conditions $y(x,0) = 1 - x \qquad 0 < x < 1$

Taking y(x, t) = u (x, t) + 10

Boundary condition changes to

$$u_x(1,t) = 0 \quad \text{....(2)}$$

$$u(0,t) = 0 \quad \text{....(3)}$$

$$u(x,0) = -(x+9) \quad \text{....(4)}$$

Suppose $u(x,t) = X(x)\,T(t)$

$\therefore \quad XT' = kX''T$

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \mu$$

$\Rightarrow \quad X'' - \mu X = 0 \quad$ .....(5)

$T' - \mu kT = 0 \quad$ .....(6)

From (2) and (3) X'(1)T(t)=0; X(0)T(t)=0

T(t) depends on t and for some t , T(t)≠0.

$\Rightarrow$ X'(1) = 0, X(0) = 0

From (5)

Case(i): $\mu = 0$

$X(x) = Ax + B$

$X'(1) = 0 \Rightarrow A = 0$

$X(0) = 0 \Rightarrow B = 0$

$\therefore$ X(x) = 0 $\therefore$ we reject $\mu = 0$.

Case(ii) $\mu = \lambda^2, \lambda \neq 0$.

$X'' - \mu x = 0$

$X'' - \lambda^2 X = 0$

$\Rightarrow \quad X = A e^{\lambda x} + B e^{-\lambda x}$

$X'(1) = A\lambda e^{\lambda} + Be^{-\lambda}$

$X(0) = A+B$

$X'(1) = 0; X(0) = 0$

$\Rightarrow$ A = 0, B = 0

$\therefore$ we reject $\mu = \lambda^2$

Case(iii) $\mu = -\lambda^2, \lambda \neq 0$

$X'' + \lambda^2 X = 0$

$\therefore \quad X(x) = A\cos\lambda x + B\sin\lambda x$

$X'(1) = -A\sin\lambda + B\lambda\cos\lambda = 0$

$X(0) = A = 0$

$$\cos\lambda = 0 \Rightarrow \lambda = (2n-1)\frac{\pi}{2}$$

$$X_n(n) = B_n \sin\left(\frac{2n-1}{2}\pi x\right)$$

From(6) $\quad T' - \mu k T = 0$

$$T' + \left[\left(\frac{2n-1}{2}\right)\pi\right]^2 \kappa T = 0$$

$\Rightarrow T_n(t) = D_n e^{-c_n^2 t}$

$$\boxed{c_n^2 = \frac{(2n-1)^2}{4}\pi^2 k}$$

$\therefore \quad u_n(x,t) = X(x)T(t)$

$$\Rightarrow \quad u_n(x,t) = E_n\left[\sin.\frac{(2n-1)\pi x}{2}\right]e^{-c_n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} E_n\left[\sin\frac{(2n-1)\pi x}{2}\right]e^{-c_n^2 t}$$

Putting t = 0

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$$

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$$

$$\therefore \quad E_n = \frac{2}{2}\int_0^1 (-x-9)\sin\frac{(2n-1)\pi x}{2}.dx$$

$$\therefore \quad E_n = -2\left[\frac{(x+9)\left(-\cos\left(\frac{2n-1}{2}\right)\pi x\right)}{(2n-1)\pi/2} - \frac{(-1)\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right]_0^1$$

$$\boxed{E_n = \frac{8(-1)^n}{(2n-1)\pi^2} - \frac{36}{(2n-1)\pi}}$$

$$\boxed{y(x,t) = 10 + E_n\left[\sin(2n-1)\frac{\pi x}{2}\right]e^{-c_n^2 t}}$$

Required solutions.

Q. 7(a) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x=0$ and $x=1$ and curve through the points with the following co-ordinates:

x	0.00	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Sol ⇒ From the above chart

$h = 0.25$

$y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$

$y_3 = 0.9089$, $y_4 = 0.8415$

If V is the volume of the solid formed then we know that

$$\boxed{V = \pi \int_0^1 y^2\,dx.}$$

The value of y^2 are tabulated below, correct to 4 decimal places.

x	0	0.25	0.50	0.75	1
y^2	1	0.9793	0.9195	0.8261	0.7081

By Simpson's 1/3 rule

$$V = \pi \frac{h}{3}\left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2\right]$$

$$V = \frac{\pi h}{3}\left[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)\right]$$

$$V = \frac{\pi}{3} \times 0.25\left[1.7081 + \cancel{4}(7.2216 + 1.839\right]$$

$$\boxed{V = 2.8192} \text{ Ans}$$

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