Date : 01.09.2019

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A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

202/250

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-15: IAS(M)/01-SEPT.-2019

Time : 3 Hours **Maximum Marks : 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: UTKARSH KUMAR

Roll No.: 0858343

Test Centre:

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Utkarsh

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			08	
	(c)			08	34
	(d)			02	
	(e)			08	
2	(a)				
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8	(a)			13	
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			Total Marks		

202/250

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SECTION – A

1. (a) Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in IR^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$, is a subspace of IR^4. What is the dimension of this subspace? Find one of its bases. **[10]**

① $(0,0,0,0)$ satisfies given equations and hence belongs to V.

② $\forall\ v_1 = (x_1, x_2, x_3, x_4),\ v_2 = (y_1, y_2, y_3, y_4) \in V,$
$\alpha, \beta \in \mathbb{R},$

$\alpha v_1 + \beta v_2 = (\alpha x_1 + \beta y_1,\ \alpha x_2 + \beta y_2,\ \alpha x_3 + \beta y_3,\ \alpha x_4 + \beta y_4)$

$\alpha x_1 + \beta y_1 + (\alpha x_2 + \beta y_2) + 2(\alpha x_3 + \beta y_3) + \alpha x_4 + \beta y_4 = 0$

and $2(\alpha x_1 + \beta y_1) + 3(\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3) + \alpha x_4 + \beta y_4 = 0$

$\Rightarrow \alpha v_1 + \beta v_2 \in V.$ ($\because v_1, v_2$ satisfy given equations)

By ① and ②, V is a subspace of $\mathbb{R}^4$.

$\forall\ v = (x_1, x_2, x_3, x_4) \in V,$

$x_1 + x_2 + 2x_3 + x_4 = 0 = 2x_1 + 3x_2 - x_3 + x_4$

$\Rightarrow x_1 = 3x_3 - 2x_2,\ x_4 = x_2 - 5x_3.$

$v = (3x_3 - 2x_2,\ x_2,\ x_3,\ x_2 - 5x_3)$

$= x_3(3, 0, 1, -5) + x_2(-2, 1, 0, 1)$

$\therefore \beta = \{(3, 0, 1, -5), (-2, 1, 0, 1)\}$ spans V.

Also β is linearly independent

$\Rightarrow \beta$ forms a basis for V.

$\boxed{\dim V = 2}$

1. (b) Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B = C^{-1}AC$. **[10]**

We find the eigen values of A

Characteristic equation: $\det(A - \lambda I) = 0$

ie $\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[\lambda^2 - 1 - 3] + 2(-2-\lambda) + 2(2+\lambda) = 0$

$\cancel{\Rightarrow -\lambda^3 + 2\lambda^2 + 4\lambda}$

$\Rightarrow (\lambda - 2)(\lambda^2 - 4) = 0 \Rightarrow \lambda = 2, 2, -2.$

$Av = \lambda v$ for eigen vector v

$\Rightarrow A^2 v = \lambda Av = \lambda^2 v$

$\Rightarrow A^3 v = \lambda^3 v.$

$\Rightarrow$ A and A^3 have same eigen vectors and eigen values of A^3 are cubes of those of A.

$\Rightarrow A^3$ has eigenvalues $8, 8, -8.$

$B = C^{-1}AC \Rightarrow B^3 = C^{-1}A^3C$, C being a non singular square matrix

$\Rightarrow B^3 \sim A^3$

$\Rightarrow$ They have same eigenvalues.

$\Rightarrow$ Eigenvalues of B are $8, 8, -8$.

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1. (c) Prove that if $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$, are the real numbers such that

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\ldots+\frac{a_{n-1}}{2}+a_n=0$$

then there exists at least one real number x between 0 and 1 such that
$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0.$ **[10]**

Let $f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + \cdots + a_{n-1}\frac{x^2}{2} + a_n x$,

be a polynomial with real coefficients.

$f(0) = 0$

$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n$

$= 0$ (given). $= f(0)$

$f(x)$ is polynomial and hence continuous in $[0,1]$ and differentiable on $(0,1)$.

$\Rightarrow$ By Rolle's Theorem,

$\exists\ x \in (0,1) : f'(x) = 0$

ie $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$

08

1. (d) Find all the asymptotes of the curve
$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0.$ [10]

① $x^4 - y^4 + 3xy(x+y) + xy = 0$

$\Rightarrow x - y + \dfrac{3xy}{x^2+y^2} + \dfrac{xy}{(x^2+y^2)(x+y)} = 0$

As $y \to \infty$, $\dfrac{3xy}{x^2+y^2} \to 0$, $\dfrac{xy}{(x^2+y^2)(x+y)} \to 0$

$\Rightarrow x - y = 0$ is an asymptote.

② $x^2 + y^2 + \dfrac{3xy}{x-y} + \dfrac{xy}{x^2-y^2} = 0.$

As $y \to \infty$, $\dfrac{3xy}{x-y} \to -3x$, $\dfrac{xy}{x^2-y^2} \to 0$

$\Rightarrow x^2 + y^2 - 3x = 0$ is an asymptote

??

02

1. (e) Find the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has its radius as small as possible. **[10]**

We consider the general equation of sphere

$S: x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + d = 0.$

Passes through $(1,0,0) \Rightarrow 1 + 2f + d = 0$

Passes through $(0,1,0) \Rightarrow 1 + 2g + d = 0$

Passes through $(0,0,1) \Rightarrow 1 + 2h + d = 0.$

$\therefore$ The equation becomes:

$$x^2 + y^2 + z^2 - (d+1)(x+y+z) + d = 0$$

Radius of sphere $= \sqrt{\left(\frac{d+1}{2}\right)^2 + \left(\frac{d+1}{2}\right)^2 + \left(\frac{d+1}{2}\right)^2 - d}$

$$= \sqrt{\frac{3}{4}d^2 + \frac{d}{2} + \frac{3}{4}}$$

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$$= \sqrt{\left(\frac{\sqrt{3}d}{2} + \frac{1}{2\sqrt{3}}\right)^2 + \frac{3}{4} - \frac{1}{12}}$$

$$= \sqrt{\left(\frac{\sqrt{3}d}{2} + \frac{1}{2\sqrt{3}}\right)^2 + \frac{2}{3}}$$

For minimum radius, $\frac{\sqrt{3}d}{2} + \frac{1}{2\sqrt{3}} = 0$

$\Rightarrow d = -1/3.$

$\therefore$ Required sphere is

$$x^2 + y^2 + z^2 - \frac{2x}{3} - \frac{2y}{3} - \frac{2z}{3} - \frac{1}{3} = 0$$

2. (a) (i) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of AdjA.

(ii) If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, determine A^{50}. **[15]**

2. (b) Show that

$$\int_0^{\pi/2} \sin^p\theta \cos^q \theta \, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p,q > -1.$$

Hence evaluate the following integrals :

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$ (ii) $\int_0^1 x^3(1-x^2)^{5/2} dx$ (iii) $\int_0^1 x^4(1-x)^3 dx$ **[15]**

3. (a) (i) Let T be the linear operator in R^3 defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
What is the matrix of T in the standard ordered basis for R^3? what is a basis of range space of T and a basis of null space of T?
(ii) Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible. **[12 + 8 = 20]**

(i) $T(e_1) = T(1,0,0) = (3,-2,-1) = 3e_1 - 2e_2 - e_3$

$T(e_2) = T(0,1,0) = (0,1,2) = e_2 + 2e_3$

$T(e_3) = T(0,0,1) = (1,0,4) = e_1 + 4e_3$

Let $\beta = \{e_1, e_2, e_3\}$ $(e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$

$$[T]_\beta^\beta = [T(e_1) \quad T(e_2) \quad T(e_3)]$$

$$= \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

$\{e_1, e_2, e_3\}$ spans $\mathbb{R}^3 \Rightarrow \{T(e_1), T(e_2), T(e_3)\}$ spans range space.

$\alpha_1 T(e_1) + \alpha_2 T(e_2) + \alpha_3 T(e_3) = 0$

$\Rightarrow 3\alpha_1 + \alpha_3 = 0 = -2\alpha_1 + \alpha_2 = -\alpha_1 + 2\alpha_2 + 4\alpha_3$

$\Rightarrow \alpha_1 = \alpha_2 = 0, \alpha_3 = 0$

$\Rightarrow \{T(e_1), T(e_2), T(e_3)\}$ are linearly independent and form a basis for range space.

ie $\beta' = \{(3,-2,-1), (0,1,2), (1,0,4)\}$ is a basis for range space.

$\dim \mathbb{R}^3 = 3, \quad \dim T(\mathbb{R}^3) = 3.$

$\Rightarrow \dim \text{nullspace}(T) = 0$ (Sylvester's theorem)

$\Rightarrow$ Nullspace (T) contains one element (0,0,0) which is the basis for the nullspace.

(ii) Let T be non invertible.

ie $\exists\ v_1, v_2 \in V,\ v_1 \neq v_2$

$T(v_1) = T(v_2)$

$\Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0$ ($\because$ T is linear operator)

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$T - T^2 = I$

$\Rightarrow (T - T^2)(v_1 - v_2) = I(v_1 - v_2)$.

$\Rightarrow T(v_1 - v_2) - T(T(v_1 - v_2)) = v_1 - v_2$

$\Rightarrow 0 - T(0) = v_1 - v_2$

$\Rightarrow 0 = v_1 - v_2 \Rightarrow v_1 = v_2$

Contradiction.

$\therefore$ T is invertible.

3. (b) (i) For x > 0, let $f(x)=\int_1^x \frac{\ln t}{1+t}dt.$

Evaluate $f(e)+f\left(\frac{1}{e}\right)$

(ii) A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum. **[16]**

(i) $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$

$t = 1/y \Rightarrow dt = -\frac{1}{y^2}dy$

$f(x) = \int_1^{1/x} \frac{-\ln y}{1+\frac{1}{y}} \cdot \frac{-dy}{y^2} = \int_1^{1/x} \frac{\ln y}{y(y+1)}dy$

Putting $x = \frac{1}{x}$

$f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln y}{y(y+1)}dy = \int_1^x \left(\frac{\ln y}{y} - \frac{\ln y}{1+y}\right)dy$

$= \left.\frac{(\ln y)^2}{2}\right]_1^x - f(x)$

$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = \left.\frac{(\ln y)^2}{2}\right]_1^x = \frac{1}{2}(\log x)^2$

$x = e \Rightarrow f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}.$

(ii) Let the length, breadth, height of box be l, b, h respectively.

We need to minimize

$S = lb + 2(l+b)h$

subject to $V = lbh = 32$

$\Rightarrow \quad h = \frac{32}{lb}$.

minimise $\quad S = lb + 2 \times \frac{32}{lb}(l+b) \quad , l, b \geq 0$

$= lb + \frac{64}{l} + \frac{64}{b}$.

For 3 positive reals,
arithmetic mean $\geq$ geometric mean

$\Rightarrow \quad \dfrac{lb + \frac{64}{l} + \frac{64}{b}}{3} \geq \left(lb \cdot \frac{64}{l} \cdot \frac{64}{b}\right)^{1/2}$

$\Rightarrow \quad \frac{S}{3} \geq 16$.

$\Rightarrow \quad S \geq 48$.

For $S = 48$, $\quad lb = \frac{64}{l} = \frac{64}{b}$

$\Rightarrow \quad l = b = 4, \quad h = \frac{32}{16} = 2$.

Thus the dimensions of the box should be :
length : 4m,
breadth : 4m,
height : 2m.

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3. (c) Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2 + by^2 = 2z$ is given by
$ab(x^2 + y^2) - 2(a + b)z - 1 = 0$ **[14]**

Let (α, β, γ) be the point of intersection of the three tangents.

Let tangent line be represented as:

$$L: \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = \lambda$$

$$\Rightarrow x = \alpha + l\lambda, \; y = \beta + m\lambda, \; z = \gamma + n\lambda.$$

Substituting these in the paraboloid

$$a(\alpha + l\lambda)^2 + b(\beta + m\lambda)^2 = 2(\gamma + n\lambda)$$

L is tangent to paraboloid

$\Rightarrow$ this equation has coincident roots

Discriminant $= \left[2\left(al\alpha + bm\beta - n\right)\right]^2 - 4\left(al^2 + bm^2\right)\left(a\alpha^2 + b\beta^2 - 2\gamma\right) = 0$

$\Rightarrow \left(al\alpha + bm\beta - n\right)^2 - \left(al^2 + bm^2\right)\left(a\alpha^2 + b\beta^2 - 2\gamma\right) = 0$

Replacing $\dfrac{l}{m} = \dfrac{x-\alpha}{y-\beta}$, $\dfrac{n}{m} = \dfrac{z-\gamma}{y-\beta}$,

$\left[a\alpha(x-\alpha) + b\beta(y-\beta) - (z-\gamma)\right]^2 - \left[a(x-\alpha)^2 + b(y-\beta)^2\right]\left(a\alpha^2 + b\beta^2 - 2\gamma\right) = 0$

This is a cone with vertex at (α, β, γ) and the paraboloid tangency as guiding condition.

Three mutually perpendicular tangents

$\Rightarrow$ This cone has three mutually perpendicular generators

$\Rightarrow$ Coefficient of x^2 + coefficient of y^2 + coefficient of $z^2 = 0$

$\Rightarrow a^2\alpha^2 + b^2\beta^2 + 1 - (a+b)\left(a\alpha^2 + b\beta^2 - 2\gamma\right) = 0$

$\Rightarrow ab\left(\alpha^2 + \beta^2\right) - 2(a+b)\gamma - 1 = 0$

Replacing $\alpha \to x$, $\beta \to y$, $\gamma \to z$,

we get required locus as

$$\boxed{ab\left(x^2 + y^2\right) - 2(a+b)z - 1 = 0}$$

4. (a) Consider the vector space

$X = \{p(X)/p(X)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$,

over the real field IR. Define the map $D: X \to X$ by $(Dp)(X) := p_1 + 2p_2X + 3p_3X^2$

where $p(X) = p_0 + p_1X + p_2X^2 + p_3X^3$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the basis $\{1, X, X^2, X^3\}$ for X. **[18]**

Consider $p(x) = p_0 + p_1X + p_2X^2 + p_3x^3 \in X$

$q(x) = q_0 + q_1X + q_2X^2 + q_3X^3 \in X$

~~$(D(p+q))(X) =$~~ and $\alpha, \beta \in \mathbb{R}$.

$(D(\alpha p + \beta q))(X) = D(\alpha p_0 + \beta q_0 + \alpha p_1 X + \beta q_1 X + \alpha p_2 X^2 + \beta q_2 X^2 + \alpha p_3 X^3 + \beta q_3 X^3)$

~~$= (p_1 + q_1) + 2(p_2 + q_2)X + 3(p_3 + q_3)X^2$~~

$= (\alpha p_1 + \beta q_1) + 2(\alpha p_2 + \beta q_2)X + 3(\alpha p_3 + \beta q_3)X^2$

$= \alpha(p_1 + 2p_2X + 3p_3X^2) + \beta(q_1 + 2q_2X + 3q_3X^2)$

$= \alpha(Dp)(x) + \beta(Dq)(x)$

$\Rightarrow D$ is a linear transformation on X.

We now find the matrix representation for D with respect to basis $\beta = \{1, X, X^2, X^3\}$

$D(1) = 0, \quad D(X) = 1, \quad D(X^2) = 2X,$

$D(X^3) = 3X^2.$

$$[D]_{\beta}^{\beta} = \begin{bmatrix} D(1) & D(x) & D(x^2) & D(x^3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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4. (b) By means of the substitution

$x+y+z=u,\ y+z=uv, z=uvw;$

evaluate $\iiint (x+y+z)^n xyz\,dxdydz$ taken over the volume bounded by

$x=0, y=0, z=0, x+y+z=1.$ **[15]**

$$I = \iiint (x+y+z)\, xyz\; dx dy dz$$

$$x+y+z = u,\quad y+z = uv,\quad z = uvw$$

$$\Rightarrow \quad x = u(1-v),\quad y = uv(1-w),\quad z = uvw$$

$$\text{Jacobian} = \frac{\partial[x,y,z]}{\partial[u,v,w]} = \begin{bmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{bmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$$= u^2 v$$

$(x+y+z)^n$

$$I = \int_0^1\int_0^1\int_0^1 (u)^n \left[u^3(1-v)(1-w)v^2 w\right] u^2 v \; du\, dv\, dw$$

$$= \left(\int_0^1 u^6\, du\right)\left(\int_0^1 v^3(1-v)\, dv\right)\left(\int_0^1 w(1-w)\,dw\right)$$

$$= \left[\frac{u^7}{7}\right]_0^1 \cdot \left[\frac{v^4}{4} - \frac{v^5}{5}\right]_0^1 \cdot \left[\frac{w^2}{2} - \frac{w^3}{3}\right]_0^1$$

$$= \frac{1}{7} \times \left(\frac{1}{4} - \frac{1}{5}\right) \times \left(\frac{1}{2} - \frac{1}{3}\right)$$

08

$$= \frac{1}{7} \times \frac{1}{20} \times \frac{1}{6}$$

$$= \frac{1}{840} \Rightarrow \frac{1}{120(n+6)}$$

4. (c) (i) Find the equation of the plane through the point (2, 1, 1), (1, – 2, 3) and parallel to the x-axis.

(ii) If $x/1 = y/2 = z/3$ represent one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. **[17]**

(i) General equation of plane is

$$lx + my + nz = d.$$

① It passes through (2, 1, 1) :

$\Rightarrow 2l + m + n = d$

② It passes through (1, -2, 3)

$\Rightarrow l - 2m + 3n = d$.

③ Plane is parallel to x-axis with direction ratios (1, 0, 0).

$\Rightarrow (l, m, n) \cdot (1, 0, 0) = 0$

$\Rightarrow l = 0.$

$\Rightarrow \quad m+n = d, \quad -2m+3n = d$

$\Rightarrow \quad n = \frac{3d}{5}, \quad m = \frac{2d}{5}$

$\therefore$ Equation of plane is $\frac{2d}{5}y + \frac{3d}{5}z = d$

or $\boxed{2y + 3z = 5}$

(ii) Let (l, m, n) represent direction ratios of other lines.

① Line is contained in cone

$\Rightarrow \quad 5mn - 8nl + 3\cancel{lm} = 0$

② Line is orthogonal to given line

$\Rightarrow \quad (l, m, n) \cdot (1, 2, 3) = 0$

$\Rightarrow \quad l + 2m + 3n = 0$

Eliminating l from ① and ②,

$5mn + (8n + 3m)(2m + 3n) = 0$

$\Rightarrow \quad 6m^2 + 30mn + 24n^2 = 0$

$\Rightarrow \quad m^2 + 5mn + 4n^2 = 0 \Rightarrow \frac{m}{n} = -1, -4$

$\frac{m}{n} = -1 \Rightarrow \frac{l}{n} = -1$

$\frac{m}{n} = -4 \Rightarrow \frac{l}{n} = 5$

$\therefore$ The other two lines are

$\frac{x}{-1} = \frac{y}{+1} = \frac{z}{-1}$ and

$\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$

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SECTION – B

5. (a) Find the orthogonal trajectories of the family of curve $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, λ being the parameter. **[10]**

Differentiating implicitly,

$$\frac{2x}{a^2}+\frac{2yp}{b^2+\lambda}=0, \quad p=\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{b^2+\lambda}=\frac{-x}{py\,a^2}$$

So, given equation becomes,

$$\frac{x^2}{a^2}-\frac{xy}{pa^2}=1$$

To get orthogonal trajectories, we replace p with $-1/p$.

i.e. $$\frac{x^2}{a^2}+\frac{xy}{a^2}\cdot p=1$$

$$\Rightarrow y\frac{dy}{dx}=\frac{1-x^2/a^2}{x/a^2}=\frac{a^2}{x}-x.$$

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$$\Rightarrow \frac{y^2}{2}=a^2\log x-\frac{x^2}{2}+\frac{\mu}{2}$$

$$\Rightarrow x^2+y^2=2a^2\log x+\mu.$$

This is the family of orthogonal trajectories with μ being the parameter.

5. (b) Solve $y = 2xp - yp^2$ and examine for singular solutions. [10]

$$y = 2xp - yp^2$$

$$\Rightarrow x = \frac{y(1+p^2)}{2p} = \frac{y}{2p} + \frac{yp}{2}$$

Differentiating,

$$1 = p \cdot \frac{1}{2p} - \frac{y}{2p^2} \cdot p' + \frac{p^2}{2} + \frac{yp'}{2}$$

$$\Rightarrow p^2 + yp' = 1 + \frac{yp'}{p^2}$$

$$\Rightarrow (p^2 + yp')\left(1 - \frac{1}{p^2}\right) = 0$$

Ignoring $1 - \frac{1}{p^2} = 0$,

we get $p^2 + yp' = 0$

$$\Rightarrow \frac{p'}{p} = -\frac{p}{y}$$

$$\cancel{\Rightarrow -\frac{1}{p} = \log y + \log c \Rightarrow p =}$$

$$\Rightarrow \log p = \log c - \log y \Rightarrow p = \frac{c}{y}$$

$\therefore$ General solution is

$$y = 2x \cdot \frac{c}{y} - y \cdot \frac{c^2}{y^2} = \frac{2cx}{y} - \frac{c^2}{y}$$

For singular solution,

we differentiate with respect to c,

$$0 = \frac{2x}{y} - \frac{2c}{y} \Rightarrow c = x$$

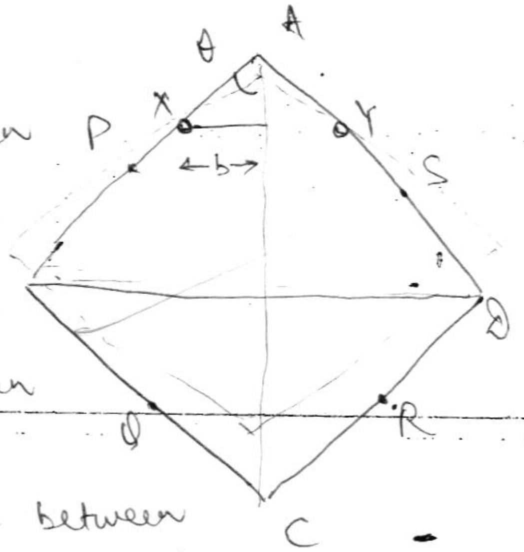
$$y = \frac{2x^2 - x^2}{y} \Rightarrow y^2 = x^2$$

$y = x \Rightarrow p = 1 \Rightarrow 2xp - p^2 y = x = y$

$y = -x \Rightarrow p = -1 \Rightarrow 2xp - p^2 y = -x = y$

$\therefore$ Two singular solutions exist: $y = x,\ y = -x$.

5. (c) Four equal rods, each of length 2a and weight W, are freely jointed to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance 2b apart on the same level. Find the stress in the rod BD. **[10]**



Vertical distance between P, midpoint of AB and X, the peg,

$\alpha =$ Vertical distance between A and P − Vertical distance between X and A

$= a\cos\theta - \dfrac{b}{\tan\theta} = a\cos\theta - b\cot\theta$

Vertical distance between Q, midpoint of BC, and X, the peg,

$\beta =$ Vertical distance between P and Q + Vertical distance between X and P

$= 2a\cos\theta + a\cos\theta - \dfrac{b}{\tan\theta}$

$= 3a\cos\theta - \dfrac{b}{\tan\theta} = 3a\cos\theta - b\cot\theta$

We now replace BD by an elastic rod and deform system such that θ changes to $\theta + \Delta\theta$.

Net work done $= W\cdot\Delta\alpha + W\Delta\alpha + W\Delta\beta + W\Delta\beta$
$+ N\cdot\Delta(BD)$

$= 2W(-a\sin\theta + b\csc^2\theta)\Delta\theta + 2W(-3\sin\theta a + b\csc^2\theta)\Delta\theta$

$+ N \times \Delta(4a\sin\theta)$

$= 2W(-4a\sin\theta + 2b\csc^2\theta)\Delta\theta$

$+ 4Na\cos\theta\,\Delta\theta$.

System is in equilibrium $\Rightarrow$ net work done $= 0$

$\Rightarrow 2W(-4a\sin\theta + 2b\csc^2\theta)\Delta\theta = -4Na\cos\theta\Delta\theta$

$\Rightarrow 2W(-2\sqrt{2}a + 4b) = -4Na\cdot\frac{1}{\sqrt{2}}$

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$\Rightarrow N = (2a - 2\sqrt{2}b)W/a$

$= 2\left(1 - \frac{\sqrt{2}b}{a}\right)W = 2\left(1 - \frac{\sqrt{2}b}{a}\right)W.$

5. (d) Find the curvature and torsion of the curve x = a cost, y = a sin t, z = bt. **[10]**

$x = a\cos t,\ y = a\sin t,\ z = bt.$

$\vec{T} = \frac{d\vec{r}}{dt} = (-a\sin t,\ a\cos t,\ b)$

$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2+b^2} \Rightarrow \hat{T} = \left(\frac{-a\sin t,\ a\cos t,\ b}{\sqrt{a^2+b^2}}\right)$

$\frac{d\hat{T}}{dt} = \frac{(-a\cos t,\ -a\sin t,\ 0)}{\sqrt{a^2+b^2}}$

$\Rightarrow \frac{d\hat{T}}{ds} = \frac{d\hat{T}/dt}{ds/dt} = \frac{1}{a^2+b^2}(-a\cos t,\ -a\sin t,\ 0)$

$= \kappa\hat{N}$

$\Rightarrow$ Curvature, $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{a}{a^2+b^2}$.

$$\hat{N} = \frac{1}{K}\frac{d\hat{T}}{ds} = (-\cos t, -\sin t, 0)$$

$$\hat{B} = \hat{T} \times \hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix} \cdot \frac{1}{\sqrt{a^2+b^2}}$$

$$= \frac{(b\sin t\,\hat{i} - b\cos t\,\hat{j} + a\hat{k})}{\sqrt{a^2+b^2}}$$

$$\frac{d\hat{B}}{ds} = \frac{d\hat{B}/dt}{ds/dt} = \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{(b\cos t\,\hat{i} + b\sin t\,\hat{j})}{\sqrt{a^2+b^2}} = -\tau\hat{N}$$

$$\Rightarrow \tau = \frac{b}{a^2+b^2}$$

$$\text{Curvature} = \frac{a}{a^2+b^2}, \quad \text{Torsion} = \frac{b}{a^2+b^2}$$

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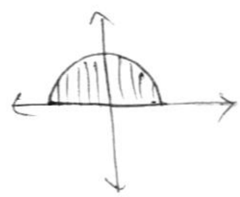
5. (e) Apply Green's theorem in the plane to evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$ where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1 - x^2)^{1/2}$. **[10]**

Comparing to $\int (P\,dx + Q\,dy)$,

$P = 2x^2 - y^2$, $Q = x^2 + y^2$.



By Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

S being area enclosed by C.

$$\therefore I = \int_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$$

$$= \iint_S (2x - 2y)\,dx\,dy$$

$= 2 \iint (x-y)\, dy\, dx$

$x = r\cos\theta, \quad y = r\sin\theta$

$I = 2 \iint r(\cos\theta - \sin\theta) \cdot r\, dr\, d\theta$

$= 2 \int_0^1 r^2\, dr \cdot \int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta$

$= 2 \cdot \frac{1}{3} \cdot \left[\sin\theta + \cos\theta\right]_0^{2\pi}$

$= \frac{2}{3} \left[0 + (-2)\right] = -\frac{4}{3}$

07

4/3

6. (a) Solve $(D^4 + D^2 + 1)y = e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right)$. **[15]**

8. (a) By using Laplace transform, solve $(D^3 - D^2 - D + 1)y = 8te^{-t}$ if $y = D^2 y = 0$, $Dy = 0$ when $t = 0$. **[15]**

$$L(Dy) = sL(y) - y(0) = sL(y)$$

$$L(D^2y) = sL(Dy) - Dy(0) = s^2 L(y)$$

$$L(D^3y) = sL(D^2y) - D^2y(0) = s^3 L(y)$$

$$L(8te^{-t}) = \frac{8}{(s+1)^2}$$

$\therefore$ On using Laplace transform, the equation transforms to:

$$L(D^3y) - L(D^2y) - L(Dy) + L(y) = \frac{8}{(s+1)^2}$$

$$\Rightarrow (s^3 - s^2 - s + 1)L(y) = \frac{8}{(s+1)^2}$$

$$\Rightarrow L(y) = \frac{8}{(s+1)^2(s^3 - s^2 - s + 1)} = \frac{8}{(s+1)^3(s-1)^2}$$

$$L^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \frac{1}{2}t^2e^{-t}$$

$$L^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t$$

$$\therefore L^{-1}\left(\frac{1}{(s+1)^3(s-1)^2}\right) = \int_0^t \frac{1}{2}u^2e^{-u}\cdot(t-u)e^{t-u}\,du$$

$$= \frac{e^t}{2}\int_0^t (tu^2e^{-2u} - u^3e^{-2u})\,du$$

$$= \frac{e^t}{2}\left[\left(-\frac{u^2e^{-2u}}{2} - \frac{ue^{-2u}}{2} - \frac{e^{-2u}}{4}\right)t + e^{-2u}\left(\frac{u^3}{2} + \frac{3}{4}u^2 + \frac{3}{4}u + \frac{3}{8}\right)\right]_0^t$$

$$= \frac{e^t}{2} \cdot e^{-2u}\left(\frac{u^3}{2} + \frac{1}{4}u^2 + \frac{1}{4}u + \frac{1}{8}\right)\Big]_0^t$$

$$= \frac{e^t}{2}\left[e^{-2t}\left(\frac{t^3}{2} + \frac{t^2}{4} + \frac{t}{4} + \frac{1}{8}\right) - \frac{1}{8}\right]$$

$$\Rightarrow y = 8 \cdot L^{-1}\left(\frac{1}{(s+1)^3(s-1)^2}\right)$$

$$= e^{-t}\left(2t^3 + t^2 + t + \frac{1}{2}\right) - \frac{e^t}{2}$$

$$= \frac{e^t}{2}\left[\left(-\frac{t^2}{2} - \frac{t}{2} - \frac{1}{4}\right)e^{-2t} + \frac{1}{4}\right]t$$

$$+ \frac{e^t}{2}\left[\left(\frac{t^3}{2} + \frac{3t^2}{4} + \frac{3t}{4} + \frac{3}{8}\right)e^{-2t} - \frac{3}{8}\right]$$

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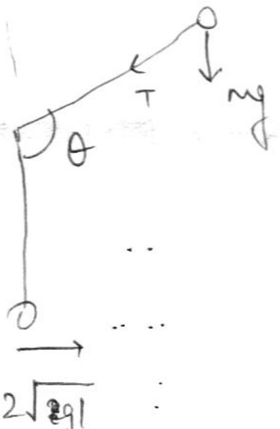
$$= \frac{e^{-t}}{2}\left(\frac{t^2}{2} + \frac{t}{2} + \frac{3}{8}\right) + \frac{e^t}{2}\left(\frac{t}{4} - \frac{3}{8}\right)$$

$$\Rightarrow y = e^{-t}\left(t^2 + 2t + 3/2\right) + e^t\left(t - 3/8\right)2$$

8. (b) A particle of mass m, hanging vertically from a fixed point by a light inextensible cord of length l, is struck by a horizontal blow which imparts to it a velocity $2\sqrt{gl}$. Find the velocity and height of the particle from the level of its initial position when the cord becomes slack. **[15]**



After the particle has covered an angular distance θ as shown in figure,

① Work-energy theorem:

$$\frac{1}{2}m\left(2\sqrt{gl}\right)^2 - \frac{1}{2}mv^2 = mgl(1 - \cos\theta)$$

$$\Rightarrow v^2 = 4gl - 2gl(1-\cos\theta)$$

$$= 2(1+\cos\theta)gl.$$

(2) Force equations:

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$\Rightarrow T = mg\cos\theta + \frac{mv^2}{l}$$

$$= mg\cos\theta + 2mg(1+\cos\theta)$$

$$= mg(2 + 3\cos\theta)$$

When cord becomes slack,

$$T = 0 \Rightarrow \cos\theta = -2/3$$

$$\text{Velocity} = \sqrt{2gl(1+\cos\theta)} = \boxed{\sqrt{\tfrac{2}{3}gl}}$$

Height from initial position

$$= l(1-\cos\theta) = \boxed{\frac{5l}{3}}$$

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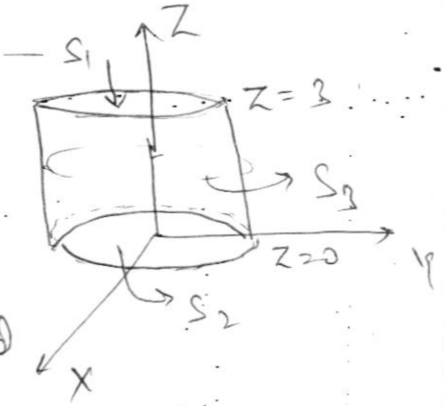
8. (c) Verify the divergence theorem for $\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **[20]**

Divergence theorem:

$$\int_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV$$



where V is volume enclosed by closed surface S.

① $\int \vec{F}\cdot\hat{n}\,dS = I_1 + I_2 + I_3$

S_1: $I_1 = \int_{(z=3)} \vec{F}\cdot\hat{k}\,dS = \iint z^2\,dx\,dy$

$= 9\iint dx\,dy = 9 \times 4\pi$ (area of S_1 = area of circle of radius 2)

$= 36\pi$

S_2: bottom circular surface.

$I_2 = \int \vec{F}\cdot(-\hat{k})\,dS = \iint -z^2\,dx\,dy$

$= \underline{0}$ $(z=0)$

S_3: curved part of the cylinder.

$$\hat{n} = \frac{\text{grad}(x^2+y^2-4)}{|\text{grad}(x^2+y^2-4)|} = \frac{(2x\hat{i} + 2y\hat{j})}{2\sqrt{x^2+y^2}}$$

$x = 2\cos\theta$, $y = 2\sin\theta$

$dS = dz \cdot \sqrt{(dx)^2 + \left(\frac{dy}{dx}\right)^2}$

$= dz \cdot \sqrt{(dy)^2 + \left(\frac{dx}{dy}\right)^2} = 2\,dz\,d\theta$

$$I_3 = \int \vec{F}\cdot\hat{n}\, ds = \iint \frac{(8x^2 - 4y^3)}{2\sqrt{x^2+y^2}}\cdot\frac{2}{y}\, dz\, d\theta$$

$$= \iint \frac{(4x^2 - 2y^3)}{y}\, dz\, dx = 3\int \frac{4x^2-2y^3}{y}\, d\theta$$

$$\cancel{= 3\left[\int \frac{4x^2}{y}\, dx - \int 2y^2\, dx\right]}$$

$$\cancel{\int y^2\, dx = \int (4-x^2)\, dx}$$

$$= 3\int_0^{2\pi} (16\cos^2\theta - 16\sin^3\theta)\, d\theta$$

$$= 48\int_0^{2\pi} \cos^2\theta\, d\theta \qquad \left(\int_0^{2\pi} \sin^3\theta\, d\theta = 0\right)$$

$$= 48\pi.$$

$$\therefore \int \vec{F}\cdot\hat{n}\, dS = 36\pi + 0 + 48\pi = 84\pi.$$

(2) $\operatorname{div}\vec{F} = \sum \frac{\partial}{\partial x}\vec{F} = 4 - 4y + 2z.$

$$I_4 = \iiint \operatorname{div}\vec{F}\, dV = \iiint (4 - 4y + 2z)\, dz\, dx\, dy$$

$$= \iint \Big[(4-4y)z + z^2\Big]_0^3\, dx\, dy = \iint (12(1-y) + 9)\, dx\, dy$$

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$$= \iint (21 - 12y)\, dx\, dy.$$

$x = r\cos\theta,\quad y = r\sin\theta$

$$I_4 = \iint (21 - 12r\sin\theta)\, r\, dr\, d\theta$$

$$= 21\cdot\frac{r^2}{2}\Big|_0^2 2\pi + 12\cdot\frac{r^3}{3}\Big|_0^2\cdot\cos\theta\Big|_0^{2\pi}$$

$$= 84\pi$$

$$\therefore \int \vec{F}\cdot\hat{n}\, ds = \iiint \operatorname{div}\vec{F}\, dV. \quad \underline{\text{Verified}}$$

ROUGH SPACE