5(a). Solve the PDE $(D^2-D')(D-2D')Z=e^{2x+y}+xy$

SOLUTION

Given $(D^2-D')(D-2D')Z=e^{2x+y}+xy \ldots(1)$

complementary function corresponding to $(D-2D')$ is given by $\phi(y+2x)$

For the non-linear part $(D^2-D')z = 0$.

Let the trail solution be $Z=Ae^{hx+ky} \ldots(2)$

Put in $(D^2-D')Z = 0$

$$Ae^{hx+ky}(h^2)-Ae^{hx+ky}(k) = 0$$

$$(h^2-k)\,Ae^{hx+ky} = 0$$

$$\therefore \quad h^2 = k$$

$$\therefore \quad \text{C.F.} = \sum Ae^{hx+h^2y}+\phi(y+2x)$$

$$\text{P.I.} = \frac{e^{2x+y}+xy}{(D^2-D')(D-2D')}$$

$$= \frac{e^{2x+y}}{(2^2-1)(D-2D')}+\frac{xy}{(-2D')(1-D/2D')(-D')(1-D^2/D')}$$

$$= \frac{e^{2x+y}}{3(D-2D')}+\frac{(1-D/2D')^{-1}\left(1-\frac{D^2}{D'}\right)^{-1}xy}{2D'^2}$$

$$= \frac{+xe^{2x+y}}{3}+\frac{\left(1-\frac{D}{2D'}\right)^{-1}\left(1+\frac{D^2}{D}+\frac{D^4}{D'^2}\right)xy}{2D'^2}$$

$$= \frac{xe^{2x+y}}{3}+\frac{(1+D/2D')(xy)}{2D'}$$

$$= \frac{x}{3}e^{2x+y}+\frac{1}{2D'^2}\left(xy+\frac{y^2}{4}\right)$$

$$\text{P.I.} = \frac{x}{3}e^{2x+y}+\frac{xy^3}{12}+\frac{y^4}{96}$$

$$\therefore \quad Z=\phi(2x+y)+\sum Ae^{hx+h^2y}+\frac{x}{3}e^{2x+y}+\frac{xy^3}{12}+\frac{y^4}{96}$$

5(b). Find the surface satisfying the PDE $(D^2-2DD'+D'^2)z=0$ and the condition that $bz=y^2$ when $x=0$ and $az=x^2$ when $y=0$.

SOLUTION

Given $\quad (D^2-2DD'+D'^2)Z=0$

$$(D-D')^2 z=0.$$

Since it is homogenous equations.

General solutions

$$\boxed{Z=\phi_1(y+x)+x\phi_2(y+x)}$$

when $x = 0$; $bz = y^2$

$$z = \phi_1(y)+0.\phi_2(y)$$

$$\frac{y^2}{b} = \phi_1(y)$$

$$\therefore \quad \phi_1(x+y) = \frac{(x+y)^2}{b}$$

when $y = 0$; $az = x^2$

$$\frac{x^2}{a} = \frac{(x+0)^2}{b}+x\phi_2(x)$$

$$\phi_2(x) = \frac{x}{a}-\frac{x}{b}$$

$$\phi_2(x+y) = (x+y)\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\therefore \quad Z = \boxed{\frac{(x+y)^2}{b}+x(x+y)\left(\frac{1}{a}-\frac{1}{b}\right)}$$

6(a). Solve the following PDE $pz+qy = x$, $x_0(s)=s$ $y_0(s)=1$ $z_0(s)=2s$ by the method of characterstics.

SOLUTION

$$f(x,y,z,p,q) = pz+qy-x \quad ...(1)$$

given $x_0(s)=s$ $y_0(s)=1$ $z_0(s)=2s$ ...(2)

Solving for p_0, q_0.

$$Z_0'(s) = p_0 X_0'(s) + q_0 y_0'(s)$$

$$2 = p_0 + 0$$

from(1) $\Rightarrow p_0(2s)+q_0(1)-s = 0$

$$q_0 = -3s$$

$$\therefore \quad (x_0,y_0,z_0,p_0,q_0) = (s,1,2s,2,-3s) \quad ...(3)$$

Charastic equations can be written as

$$X'(t) = f_p = z \quad ...(4)$$

$$y'(t) = fq = y \quad ...(5)$$

$$z'(t) = pf_p+qf_q=pz+qy=x \quad ...(6)$$

$$p'(t) = -f_x-pf_z=-(-1)-p(p)=1-p^2 \quad ...(7)$$

$$q'(t) = -f_y-qf_z = -q-pq \quad ...(8)$$

from(5) $\quad y'(t) = y$

$$\frac{dy}{y} = dt \Rightarrow y = c_1e^t$$

$$\Rightarrow \boxed{y=e^t} \quad (\because y_0=1)$$

$$(4)+(5)+(6) \Rightarrow \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt} = x+y+z$$

$$\therefore \quad \frac{d(x+y+z)}{(x+y+z)} = dt$$

$$\therefore \quad (x+y+z) = c_1e^t$$

$$x+z = c_1e^t - e^t \quad ...(9)$$

$$(4)-(6) \Rightarrow \frac{dx}{dt}-\frac{dz}{dt} = z-x$$

$$(x-z) = c_3e^{-t} \quad ...(10)$$

Put initial condition in (9)

$$s+2s = c_1-1$$

$$c_1 = 3s+1$$

$$\therefore \quad (x+z) = 3se^t$$

Putting initial condition is (10)

$$(x-z) = +c_3e^{-t}$$

$$s-2s = c_3$$

$$x-z = -se^{-t}$$

$$\therefore \quad x = \frac{s(3e^t-e^{-t})}{2}; \quad ..(11)$$

$$y = e^t; \quad ..(12)$$

$$z = \frac{s(3e^t + e^{-t})}{2} \quad ..(13)$$

$$\frac{(11)}{(13)} \equiv \frac{x}{z} = \frac{3e^t - e^{-t}}{3e^t + e^{-t}}$$

$$\frac{x}{z} = \frac{3y - 1/y}{3y + 1/y} \quad \boxed{\because e^t = y}$$

$\therefore$ $\boxed{z = x\left(\frac{3y^2 + 1}{3y^2 - 1}\right)}$ required solutions.

6(b). Reduce the following 2nd order PDE into canonical form and find its general solutions $xu_{xx} + 2x^2u_{xy} - u_x = 0.$

SOLUTION

Given $$xr + 2x^2s - p = 0 \qquad \text{...(1)}$$

comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$R = x,\ S = 2x^2,\ T = 0$

$S^2 - 4RT = 4x^4 - 0 = 4x^4 > 0$. Hyperbolic λ-quadratic is given by $R\lambda^2 + S\lambda + T = 0$.

$$x\lambda^2 + 2x^2\lambda = 0$$

$$\therefore \quad \lambda = 0 \quad \lambda = -2x$$

Hence characteristic equation

$$\frac{dy}{dx} + \lambda_1 = 0\ ; \qquad \frac{dy}{dx} + \lambda_2 = 0$$

$$\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} - 2x = 0$$

$$y = c_1, \qquad y - x^2 = c_2$$

$\therefore$ Let
$$u = y \qquad \text{...(2)}$$
$$v = y - x^2 \qquad \text{...(3)}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(0) + \frac{\partial z}{\partial v}(-2x) \qquad \text{...(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \text{...(5)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-2x\frac{\partial z}{\partial v}\right)$$

$$= -2\frac{\partial z}{\partial v} - 2x\left[\frac{\partial^2 z}{\partial u \partial v}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right]$$

$$= -2\frac{\partial z}{\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} \qquad \text{...(6)}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}$$

$$= -2x\left(\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots(7)$$

Using (4)(5)(6)(7) given equation (1) transform to

$$x\left[-2\frac{\partial z}{\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2}\right] + 2x^2\left[-2x\left(\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)\right] - \left[-2x\frac{\partial z}{\partial v}\right] = 0$$

$\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$ Required canonical form.

$$\therefore \quad \frac{\partial z}{\partial u} = \phi_1(u)$$

$$z = \int \phi_1(u) + \phi_2(v)$$

$\boxed{z = \int \phi_1(y)dy + \phi_2(y - x^2)}$ general solution

6(c). Solve the following heat equations

$u_t - u_{xx} = 0$, $0<x<2$, $t>0$, $u(0, t) = u(2,t) = 0$, $t>0$. $u(x,0) = x(2-x)$, $0 \le x \le 2$.

SOLUTION

Heat flow equation $u_t - u_{xx} = 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Boundary conditions $u(0,t)=u(2,t)=0$..(2) $t>0$.

Initial conditions $\quad u(x,0) = x(2-x) \qquad ...(3)$

Let the trail solution be

$$u(x,t) = X(x)T(t)$$

From(2) $\quad X(0)T(t) = X(2)T(t) = 0$

For some $t > 0$, there exist t such that $T(t) \neq 0$.

$\therefore \quad X(0) = X(2) = 0. \qquad ...(4)$

From(1) $\quad XT' = X''T$

$$\frac{T'}{T} = \frac{X''}{X} = \mu \qquad \text{(say)}$$

Solving $\quad X'-\mu x = 0;$

$$X(0)=X(2) = 0$$

Case(i)

$\mu = 0$ $X'' = 0 \Rightarrow X = Ax+B.$

Putting(4) $A = 0$, $B = 0$

$\therefore$ we reject $\mu=0$.

Case(ii) $\mu = \lambda^2$ $X''-\lambda^2 X=0 \Rightarrow X=Ae^{\lambda x}+Be^{-\lambda x}$

Putting (4) $\Rightarrow A=0$; $B=0$

$\therefore$ we reject $\mu = \lambda^2$

Case(iii) $\mu = -\lambda^2$, $\lambda \neq 0$

$X'' + \lambda^2 X = 0$

$$\Rightarrow \quad X = A\cos\lambda x + B\sin\lambda x$$

$$X(0) = A + B(0) = 0$$

$$X(2) = B\sin(2\lambda) = 0$$

$$\therefore \quad 2\lambda = n\pi$$

$$\lambda = n\pi/2$$

$$\therefore \quad X_n = B_n \sin\left(\frac{n\pi x}{2}\right)$$

Corresponding

$$\frac{T'}{T} = \mu$$

$$\frac{T'}{T} = -\lambda^2$$

$$\therefore \quad T = ce^{-\lambda^2 t} \quad T_n = c_n e^{-n^2\pi^2 t/4}$$

$$\therefore \quad u_n(x,t) = X_n(x)T_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t/4}$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{2}\right) = x(2-x)$$

$$D_n = \frac{2}{2}\int u(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$D_n = \frac{2}{2}\int_0^2 x(2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$D_n = \frac{16}{(n\pi)^3}\left[1-(-1)^n\right]$$

$$\boxed{u(x,t) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\right)\sin\left(\frac{n\pi x}{2}\right).e^{-n^2\pi^2 t/4}}$$