Date : ....

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2022

(JUNE *to* SEP.-2022)

IAS/IFoS

MATHEMATICS

Under the guidance *of* K. Venkanna

LINEAR ALGEBRA, CALCULUS AND THREE DIMENSIONAL GEOMETRY

TEST CODE: TEST-1: IAS(M)/12-JUNE-2022

Time: 3 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 56 pages and has 39 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name	
Roll No.	
Test Centre	
Medium	

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

_ _

Signature of the Candidate

I have verified the information filled by the candidate above

_ _

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

DO NOT WRITE ON THIS SPACE

SECTION – A

1. (a) Let $V = \{x \mid x \in \mathbb{R}, x > 0\}$. Define addition and scalar multiplication as follows :

For $x \in V$, $y \in V$, define $x \oplus y = xy$

For $r \in \mathbb{R}$, $x \in V$, define $r \odot x = x^r$.

Is V with these operations a vectors space ? Justify your answer. **[10]**

1. (b) Determine the values of k so that the following system in unknowns x, y, z has : (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions :

$x + y - z = 1$

$2x + 3y + kz = 3$

$x + ky + 3z = 2$ **[10]**

1. (c) Evaluate $\lim_{y \to 0^+} \left[\cos(2y)\right]^{1/y^2}$ **[10]**

1. (d) Find the asymptotes of the curve $x^2y^2 (x^2 - y^2)^2 = (x^2 + y^2)^3$. **[10]**

1. (e) Find the magnitude and the equations of the line of shortest distance between the lines :

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7} \quad \ldots\ldots(i)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \quad \ldots\ldots(ii)$$

[10]

2. (a) (i) Is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$? Justify.

(ii) Let V = C_{nn} and let W = {n × n hermitian matrices}. Is W a subspace of V ?

[16]

2. (b) (i) If a function f is such that its derivative f' is continuous on [a, b] and derivable on]a, b[, then show that there exists a number c between a and b such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{1}{2}(b-a)^2 f''(c)$$

(ii) Show that $\frac{\tan x}{x}>\frac{x}{\sin x}$, for $0<x<\frac{\pi}{2}$ **[16]**

2. (c) (i) Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{1}{2}x = \frac{1}{3}y = -\frac{1}{6}z$.

(ii) Prove that the locus of the line of intersection of tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ which touch along perpendicular generators is the cone $a^2(b + c)\, x^2 + b^2\,(c + a)\, y^2 + c^2\,(a + b)\, z^2 = 0$ **[18]**

3. (a) (i) Let W be the vector space of 3×3 antisymmetric matrices over K. Show that $\dim W = 3$ by exhibiting a basis of W.

(ii) Find a basis and dimension of the subspace W of V spanned by the polynomials $v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$, $v_3 = t^3 + 6t - 5$, $v_4 = 2t^3 - 5t^2 + 7t + 5$.

[18]

3. (b) (i) Find the maxima and minima of $f(x, y) = x^4 + y^4 - 2(x - y)^2$.

(ii) Examine the convergence of the integral

$$\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}.$$

[6+10=16]

3. (c) (i) Find the locus of the centre of the sphere of constant radius which passes through a given point and touches the given line.

(ii) Show that the spheres $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally and find their point of contact. **[16]**

4. (a) Let $A=\begin{pmatrix}0 & 1 & 1\\ 1 & -2 & 2\\ 1 & 2 & -1\end{pmatrix}$, a symmetric matrix. Find a nonsingular matrix P such that $P^T AP$ is diagonal and find the diagonal matrix $P^T AP$. **[16]**

4. (b) Find the maximum and minimum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, when $lx + my + nz = 0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Interpret the result geometrically. **[16]**

4. (c) Find the locus of the point of intersection of perpendiculars of a hyperboloid of one sheet. **[18]**

SECTION – B

5. (a) Let $R_3[x] = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$.

Define $T : R_3[x] \to R_3[x]$ by $T(f(x)) = \frac{d}{dx} f(x)$,

For all $f(x) \in R_3[x]$. Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1 + x, 1 + x + x^2\}$.

[10]

5. (b) (i) If A be an $n \times n$ matrix, prove that
$|\text{adj } A| = |A|^{n-1}$.

(ii) If α is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of Adj A. **[10]**

5. (c) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$. **[10]**

5. (d) A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equations

$x = y + a = z$; $x + a = 2y = 2z$;

Find the co-ordinates of each of the points of intersection. **[10]**

5. (e) Prove that the enveloping cylinder of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ whose generators are parallel to the line $\frac{x}{0} = \frac{y}{\pm\sqrt{(a^2 - b^2)}} = \frac{z}{c}$ meet the plane $z = 0$ in circles. **[10]**

6. (a) Let $U = \text{span}\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$
$W = \text{Span}\{(1, 2, 2, -2), (2, 3\ 2, -3), (1, 3, 4, -3)\}$ be the subspaces of R^4.
Find a basis and the dimension of $U + W$, U, W and $U \cap W$. **[15]**

6. (b) (i) Let $V = P_\infty$ be the vector space of polynomials. Is the set $\{1 + x + x^2, 1 - x, 1 - x^3\}$ linearly independent ? Prove your claim.

(ii) Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with $L([1, -1, 0]) = [2, 1]$, $L([0, 1, -1]) = [-1, 3]$ and $L([0, 1, 0]) = [0, 1]$. Find $L([-1, 1, 2])$. Also, give a formula for $L([x, y, z])$, for any $[x, y, z] \in \mathbb{R}^3$.

[18]

6. (c) (i) If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence determine A^{50}.

(ii) Is the matrix $A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ similar over the field **R** to a diagonal matrix ? Is A similar over the field **C** to a diagonal matrix ? **[17]**

7. (a) Show that the function f, where

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & if\ x^2+y^2\neq 0\\ 0, & if\ x=y=0\end{cases}$$

is continuous, possesses partial derivative but is not differentiable at the origin. **[12]**

7. (b) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, $x \neq y$ show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(1-4\sin^2 u\right)\sin 2u.$ **[14]**

7. (c) Prove that $\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2} = \frac{1}{2}\pi \log\left\{\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \beta^\beta}\right\}$. **[12]**

7. (d) Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

[12]

8. (a) (i) Obtain the equations of the spheres which pass through the circle $y^2 + z^2 = 4$, $x = 0$ and are cut by the plane $2x + 2y + z = 0$ in a circle of radius 3.

(ii) If $x/1 = y/1 = z/2$ be one of a set of three mutually perpendicular generators of the cone $3yz - 2zx - 2xy = 0$. Find the equations of other two generators.

[16]

8. (b) Show that the surface represented by the equation

$$x^2 + y^2 + z^2 - yz - zx - xy - 3x - 6y - 9z + 21 = 0$$

is a paraboloid of revolution the coordinates of the focus being (1, 2, 3) and the equations to axis are $x = y - 1 = z - 2$. **[16]**

8. (c) Through a fixed point (k, 0, 0) pairs of perpendicular lines are drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$. Show that the plane through any pair touches the cone

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$ **[18]**

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