2014 CSE

Q.1 Find the curvature vector at any point of the curve $\vec{r}(t) = t\cos t\,\hat{i} + t\sin t\,\hat{j}$; $0 \le t \le 2\pi$. Give its magnitude also.

Soln: The position vector $\vec{r}$ at any point of time (t)

$\Rightarrow \vec{r} = t\cos t\,\hat{i} + t\sin t\,\hat{j}$

$\vec{r} = (t\cos t, t\sin t)$

Diffn wrt 's': $\frac{d\vec{r}}{ds} = (\cos t - t\sin t, \sin t + t\cos t)\frac{dt}{ds}$ — (1)

Now, we know: $\hat{t} = \frac{d\vec{r}}{ds}$ ... ($\hat{t} \rightarrow$ unit tangent vector)

$\Rightarrow |\hat{t}| = \left|\frac{d\vec{r}}{ds}\right| = \sqrt{[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2]}\left(\frac{dt}{ds}\right)^2$

$= \left[\sqrt{(\cos^2 t + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t - 2\cancel{\cos t}\sin t.t + 2t\cancel{\cos t}\sin t)}\right]. \frac{dt}{ds}$

$= \sqrt{1+t^2}\,.\frac{dt}{ds}$

$\Rightarrow \frac{ds}{dt} = \sqrt{1+t^2}$ ..... $[\because |\hat{t}| = 1]$

$\therefore \hat{t} = \frac{d\vec{r}}{ds} = \frac{1}{\sqrt{1+t^2}}(\cos t - t\sin t, \sin t + t\cos t)$

Diffn wrt 's', we get:

$\Rightarrow \frac{d\hat{t}}{ds} = \frac{d\hat{t}/dt}{ds/dt} = \frac{1}{\sqrt{1+t^2}}\cdot\left(\frac{d\hat{t}}{dt}\right)$

$$\Rightarrow \frac{d\hat{t}}{dt} = \left[\left(\frac{(t\sin t - \sin t - t\cos t)\sqrt{1+t^2} - (\cos t - t\sin t)\cdot\frac{t}{\sqrt{1+t^2}}}{(1+t^2)}\right),\right.$$

$$\left.\left(\frac{(\cos t + \cos t - t\sin t)\cdot\sqrt{1+t^2}) - (\sin t + t\cos t)\cdot\frac{t}{\sqrt{1+t^2}}}{(1+t^2)}\right)\right]$$

$$= \left[\frac{(-2\sin t - t\cos t)(1+t^2) + t(\cos t - t\sin t)}{(1+t^2)^{3/2}}, \frac{(2\cos t - t\sin t)(1+t^2) - (\sin t + t\cos t)t}{(1+t^2)^{3/2}}\right]$$

$$= \left[\frac{-(2+t^2)(\sin t + t\cos t)}{(1+t^2)^{3/2}}, \frac{(2+t^2)(\cos t - t\sin t)}{(1+t^2)^{3/2}}\right]$$

$$\Rightarrow \frac{d\hat{t}}{ds} = \frac{1}{\sqrt{1+t^2}}\frac{d\hat{t}}{dt} = \frac{1}{(1+t^2)^2}\left[-(2+t^2)(\sin t + t\cos t), (2+t^2)(\cos t - t\sin t)\right]$$

$$\Rightarrow \frac{d\hat{t}}{ds} = k\hat{N} = -\frac{(2+t^2)}{(1+t^2)^2}(\sin t + t\cos t)\hat{i} + \frac{(2+t^2)}{(1+t^2)^2}(\cos t - t\sin t)\hat{j}$$

$$|k| = |k\hat{N}| \qquad \ldots [\because |\hat{N}| = 1]$$

$$= \sqrt{\left[\frac{(2+t^2)}{(1+t^2)^2}\right]^2\left[(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2\right]}$$

$$= \frac{2+t^2}{(1+t^2)^2}\sqrt{\sin^2 t + t^2\cos^2 t + \cos^2 t + t^2\sin^2 t}$$

$$= \frac{(2+t^2)}{(1+t^2)^2}\sqrt{1+t^2} = \frac{2+t^2}{(1+t^2)^{3/2}}$$

Q.2 Evaluate by Stokes theorem $\int_\Gamma y\,dx + z\,dy + x\,dz$ where Γ is the curve given by $x^2+y^2+z^2-2ax-2ay=0$, $x+y=2a$, starting from $(2a,0,0)$ and then going below the z-plane.

Soln:- The given sphere $x^2+y^2+z^2-2ax-2ay=0$ has the centre $(a,a,0)$. The plane passes through the centre and therefore, their intersection is a circle C of radius $\sqrt{2}a$. ...[C is great circle of sphere]

Now acc. to Stokes theorem:

$$\int F\cdot dR = \iint (\nabla\times F)\cdot\hat{n}\,ds \quad \text{——— (1)}$$

Now, $\int_C y\,dx + z\,dy + x\,dz = \int_C (y\hat{i} + z\hat{j} + x\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= \int_C (y\hat{i} + z\hat{j} + x\hat{k})\cdot dR$

$\therefore F = y\hat{i} + z\hat{j} + x\hat{k}$

Now, $\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} = -(\hat{i} + \hat{j} + \hat{k})$

$\because$ S is surface of plane $x+y=2a$, then a vector normal to S is grad $[x+y-2a] = \hat{i}+\hat{j}$.

$\therefore \hat{n}$ = unit normal vector to $S = \dfrac{\text{grad}(x+y)}{|\text{grad}(x+y)|} = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$.

$\therefore$ from (1):

$$\int_C y\,dx + z\,dy + x\,dz = \iint_S (\nabla \times F)\cdot\hat{n}\,ds$$

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) ds$$

$$= -\sqrt{2}\iint_S ds$$

$$= -\sqrt{2}\,(\text{area of circle with } r = a\sqrt{2})$$

$$= -\sqrt{2}\times\pi\times(a\sqrt{2})^2$$

$$= -2\sqrt{2}\,\pi a^2.$$