CALCULUS
IAS
$\boxed{2011}$

Q. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3+y^3}$ if it exists.

Sol. (i) $\cancel{f_{xy}(0,0) = \lim\limits_{x\to 0} f_y(0+h, 0) -}$

$$\lim_{x\to 0} \frac{x^2y}{x^3+y^3} = \lim_{y\to} \frac{x^2y}{x^3+y^3} = 0$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} \frac{x^2y}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

$$= \lim_{(x,x)\to(0,0)} \frac{x^3}{x^3+x^3}$$

$$= \lim_{(x,x)\to(0,0)} \frac{\cancel{x^3}}{2\cancel{x^3}} = \frac{1}{2}$$

hence, $\lim\limits_{(x,y)\to(0,0)}$ does not exist.

Q Let f be a function defined on R such that $f(0) = -3$ & $f'(x) \le 5$ for all ~~variables~~ values of x in R. How large can $f(2)$ possibly be?

Sol. Given $f(0) = -3$, $f'(x) \le 5$

$f'(x) \le 5$

Integrating on domain [R]

$f(x) \le 5x + c$

Put $x = 0$

$f(0) \le 0 + c$

hence $c = -3$

$f(x) \leq 5x - 3$

for $x = 2$

$f(2) \leq 5(2) - 3$

$f(2) \leq 10 - 3 = 7$

Hence, largest value of $f(2)$ is 7.

Q. Evaluate

(i) $\lim\limits_{x \to 2} f(x)$, where $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & , x \neq 2 \\ \pi & , x = 2 \end{cases}$

Given, $f(2) = \pi$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= 4$

(ii) $\int_0^1 \ln x \, dx$

$= \int_0^1 1 \cdot \ln x \, dx$ (By Parts)

$= [x]_0^1 \cdot \ln x - \int_0^1 \dfrac{\cancel{x}}{\cancel{x}} \, dx$

$= \ln x - \int_0^1 1 \, dx = \ln x - [x]_0^1$

$= \ln x - 1$

Q. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$.

sol. Given sphere is $x^2+y^2+z^2=4$

let $S = x^2+y^2+z^2-4$

Let (x,y,z) be any point on sphere
& AB is the distance b/w $(3,1,-1)$ & (x,y,z)

$\Rightarrow AB = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

$AB^2 = (x-3)^2+(y-1)^2+(z+1)^2$

let $F = (AB^2) + \lambda S$

$= (x-3)^2+(y-1)^2+(z+1)^2+\lambda(x^2+y^2+z^2-4)$

$F_x = 2(x-3)+2x\lambda = 0$ — ①
$F_y = 2(y-1)+2y\lambda = 0$ — ②
$F_z = 2(z+1)+2z\lambda = 0$ — ③

from ①, ② & ③, we get,

$(x-3)+x\lambda = 0$

$\lambda = \frac{3-x}{x}$

Similarly, $\lambda = \frac{1-y}{y}$, $\lambda = \frac{-(z+1)}{z}$

$\Rightarrow \frac{3-x}{x} = \frac{1-y}{y} = \frac{-(z+1)}{z} = \lambda$

$\Rightarrow \frac{3}{x} = \frac{1}{y}$ $\qquad \frac{1}{y} = \frac{-1}{z}$

$\boxed{3y = x}$ $\qquad \boxed{y = -z}$

$\Rightarrow \boxed{x = 3y = -3z}$

hence, the stationary point is $(x, x/3, -x/3)$

$x^2+y^2+z^2=4$ (Given)

$$(x)^2+\left(\frac{x}{3}\right)^2+\left(\frac{-x}{3}\right)^2=4$$

$$x^2+\frac{x^2}{9}+\frac{x^2}{9}=4$$

$$11x^2=36$$

$$x=\pm\frac{6}{\sqrt{11}}$$

$\Rightarrow$ $y=\pm\frac{2}{\sqrt{11}}$, $z=\mp\frac{2}{\sqrt{11}}$

hence stationary points are $\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)$ & $\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$

Above are the required points.

Q. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$ above xy-plane and inside the cylinder $x^2+y^2=2x$.

Given Cylinder , $x^2+y^2=2x$

$\Rightarrow$ $(x-1)^2+y^2=1$ is the shadow of cylinder in xy - plane

Changing to polar coordinates, the shadow of the cylinder is $r^2=2r\cos\theta$ or $r=2\cos\theta$

$$R=\left\{(r,\theta)\,\middle|\,-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\ \&\ 0\le r\le 2\cos\theta\right\}$$

$$V = \iint_R (x^2+y^2)\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4\times 2\int_0^{-\pi/2} \cos^4 d\theta = \cancel{8} \times \frac{\pi}{\cancel{2}} \times \frac{3}{\cancel{4}} \times \frac{1}{2} = \frac{3\pi}{2}$$

$\boxed{2011}$ IFoS

CALCULUS

Q. Show that the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x=y \end{cases}$$

is discont. at origin but possesses partial derivatives f_x & f_y there at.

Sol. ~~put $x = r\cos\theta$, $y = r\sin\theta$~~

$\therefore$ $\cancel{\dfrac{x^3-y^3}{x}}$ $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & x \neq y \\ 0 & x=0 \end{cases}$

Above function is continous at origin if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

whatever the path taken by the function to approach origin.

NOW, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$

let $y = x - mx^3$, $y \to 0$ when $x \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + (x-mx^3)^3}{x - x + mx^3}$$

$$= \lim_{x\to 0} \frac{x^3 + x^3 - 3x^2 \cdot mx^3 + 3x \cdot m^2x^6 - m^3x^9}{mx^3}$$

$$= \lim_{x\to 0} \frac{2x^3 - 3mx^5 + 3m^2x^7 - m^3x^9}{mx^3} = \frac{2}{m}$$

i.e it approaches to diff. values depending on the value of m.

$\Rightarrow$ f is discontinuous at $(0,0)$.

$$f(x,y) = \frac{x^3+y^3}{x-y}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h-0}$$

$$= \lim_{h\to 0} \frac{h^3/h}{h} = 0$$

& $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{k^3/-k}{k} = 0$$

i.e $f_x(0,0)$ & $f_y(0,0)$ exist at origin.

Q. Let f be defined by

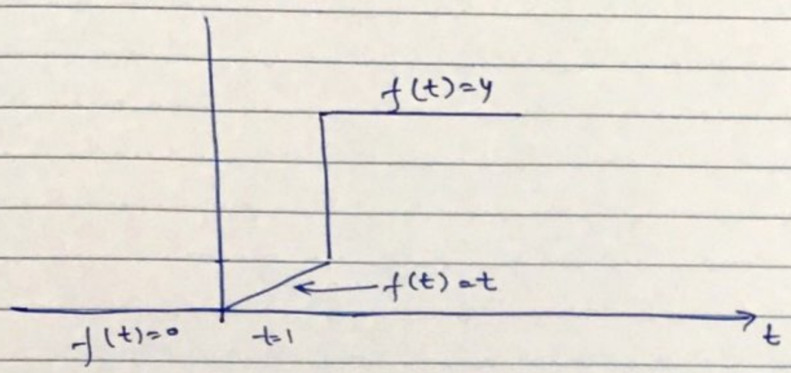
$$f(t) = \begin{cases} 0 & , \text{ for } t<0 \\ t & , \text{ for } 0 \le t \le 1 \\ 4 & , \text{ for } t>1 \end{cases}$$

(i) Determine the function $F(x) = \int_0^x f(t)\,dt$

(ii) Where is F non differentiable? Justify your answer.

(iii) Show that the eqn $3^x + 4^x = 5^x$ has exactly one root

Sol. $f(t) = \begin{cases} 0 & t<0 \\ t & 0 \le t \le 1 \\ 4 & t > 1 \end{cases}$



Now we have to calculate $F(x) = \int_0^x f(t)\,dt$

Case I. when $0 < x \le 1$

then $F(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \frac{x^2}{2}$.

i.e $F(x) = \frac{x^2}{2} \quad 0 < x \le 1$

Case II $x > 1$

then $F(x) = \int_0^x f(t)\,dt$

$= \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$

$= \int_0^1 t\,dt + \int_1^x 4\,dt = \frac{1}{2} + 4(x-1)$

$= 4x - \frac{7}{2}$

i.e, $F(x) = 4x - \frac{7}{2} \quad x > 1$

$F(x) = \frac{x^2}{2} \quad 0 < x \le 1$

$= 4x - \frac{7}{2} \quad x > 1$

hence F is not differentiable at $x=1$