Set - IV

The Transportation problem

This is a special class of linear programming problems (LPP) in which the objective is to transport a single commodity from various origins to different destinations at a minimum cost.

For example

Suppose Usha sewing machine Co. has two manufacturing centres— one located at meerut and the other at Kanpur. Manufacturing capacity of unit at meerut is 50 machines per day and that at Kanpur is 55 machines per day. At the end of each day, these centres have to supply the machines to their markets located at Delhi, Bhopal and Dehradun. The daily demands of these markets are fixed and are 40 machines at Delhi, 30 at Bhopal and 35 at Dehradun.

We may note that

(i) total supply at the two centres is
$50 + 55 = 105$ machines.

(ii) total demand at three markets is
$40 + 30 + 35 = 105$ machines.

Thus total availability at supply points = total demand at the markets. when this is so, we say that the problem is balanced.

→ suppose availability at Kanpur was 70 machines per day and other data remains unchanged, then availability was greater than total demand. In such a case, we say problem is unbalanced.

Similarly, if demand at Dehradun was 60 and other data is the same, total demand would have more than the total availability. Again, we say the problem is unbalanced.

Thus we see that if total availability is equal to total demand, problem is balanced and if total supply is different from total demand, problem is unbalanced.

Now let us come back to the problem. Our problem under discussion is a balanced one. The machines are to be transported from the centres at Meerut and Kanpur to the markets at Delhi, Bhopal and Dehradun so that demand of each market is exactly met. This is possible only when the centres transport their machines completely.

Now the transport charges Rs. 5/- per piece from Meerut to Delhi, Rs. 15/- per piece from Meerut to Bhopal and Rs. 10/- per piece from Meerut to Dehradun.

Similarly, the transportation charges per piece from Kanpur to Delhi are Rs. 10/-, from Kanpur to Bhopal are Rs. 8/- and from Kanpur to Dehradun are Rs. 20/-. So far, we have given the data of the whole problem.

A transportation problem is concerned with determining the number of machines to transported from Meerut to three markets and also from Kanpur to three markets such that demands of markets are exactly met and the total transportation cost is the minimum.

It may be noted that the problem being balanced, demands of the markets will be exactly met if and only if the manufacturing centres transport all their units to the various markets. what do we expect in case the problem is unbalanced? Obviously, if the availability is more than demand, certain machines will be retained at the centres at Meerut and/or Kanpur and if demand is more than availability, certain markets may not get the total machines required by them (i.e there will be short supply).

In the above introduction, we are introduced to a specimen of a small TP we now gradually take to the world of mathematics, symbols and notations.

(i) The centres like Meerut and Kanpur where machines or goods is stored for transportation are called sources, origins or warehouses. In the present example, there are only two sources but there can be any number of sources say 'm'. These can be ~~any number~~ denoted by $S_1, S_2, \ldots$ etc (or) $O_1, O_2, \ldots$ etc.

In the example, there are three markets located at Delhi, Bhopal and Dehradun where the goods is demanded and is to be transported from the sources. These markets are called Destinations or Markets and usually denoted by $D_1, D_2 \ldots$ etc. Again there can be any number of destinations say 'n'.

Note: 'm' and 'n' can be equal or unequal.

(ii) Availability at Meerut is 50 and at Kanpur is 55. In symbols availability at respective centres is denoted by $a_1 = 50$, $a_2 = 55$. In case, there are 'm' centres, we have 'm' values of availabilities as $a_1, a_2, a_3 \ldots a_m$. Demands at Delhi, Bhopal and Dehradun are respectively 40, 30 and 35. It is denoted as $b_1 = 40$, $b_2 = 30$, $b_3 = 35$. If there are 'n' destinations, we have 'n' values as $b_1, b_2, \ldots b_n$.

(iii) In a LPP, the variables are $x_1, x_2, \ldots x_n$ and costs in objective functions are $c_1, c_2, \ldots c_n$. It is convenient in a TP to have two suffixes with the variables as well as costs. It is because, we have a set of sources and another set of markets.

Let us see how we can do it. As pointed out, a TP is concerned with determining the amount of goods to be sent from each source to various destinations. let x_{ij} be the amount of goods to be transported from i^{th} source to the j^{th} destinations.

Thus x_{12} means amount of goods to be sent from source 1 (Meerut in the example) to destination 2 (Bhopal in example).

x_{23} means amount of goods to be sent from source 2 to destination 3 (for example, it means from Kanpur to Dehradun)

Note that, in the ordered pair of suffixes, first indicates the source number and second the destination number.

The number of variables in the present example are six.

These are x_{11}, x_{12}, x_{13} from meerut to Delhi, Bhopal and Dehradun and x_{21}, x_{22}, x_{23} from Kanpur to three markets.

Thus for a general TP having 'm' sources and 'n' destinations, therefore the number of variables are $m \times n$ (2×3 in the example).

(ii) on the same line, costs are exactly denoted by c_{ij}.

Thus c_{11}, c_{12}, c_{13} are the cost of transportation per unit from Meerut to the markets at Delhi, Bhopal and Dehradun and c_{21}, c_{22}, c_{23} are the costs of transportation per unit from Kanpur to markets at Delhi, Bhopal and Dehradun.

Now we sum up: given the data — availabilities a_i at various sources, demands b_j of the destinations and cost of transportation c_{ij} per unit from various sources to destinations, ~~and~~ ~~cost~~ solving a TP means determining the values of the variables x_{ij} i.e the amount of goods to be transported from various sources to the destinations such that the total transportation cost is the minimum.

It is being assumed here that the TP is balanced.

However, if a TP is unbalanced, then we can formulate a balanced TP, the solution of which gives the solution of the given unbalanced TP.

* **Mathematical formulation of a transportation problem:**

As per notations introduced, let us consider the units of machines supplied from Meerut i.e source S_1 to destinations Delhi (D_1), Bhopal (D_2) and Dehradun (D_3). The units supplied from S_1 to D_1, D_2, D_3 are x_{11}, x_{12}, x_{13} respectively.

Also total units available at Meerut Centre S_1 are $a_1 = 50$.

Since all the units are to the exactly transported, we must have

$$x_{11} + x_{12} + x_{13} = 50 \quad \text{--- (1)}.$$

Similarly, units supplied from Kanpur S_2 to markets at D_1, D_2 and D_3 are denoted by x_{21}, x_{22}, x_{23} respectively.

Also total units available at Kanpur (S_2) are 55 and all the units are to be transported. we must have

$$x_{21}+x_{22}+x_{23}=55 \text{ ——— } (2)$$

Now let us see the problem considering demands at the markets.

Delhi (D_1) gets x_{11} from Meerut S_1 and x_{21} from Kanpur (S_2).

Total demand of Delhi (D_1) is 40 units which is to be exactly met. Thus we have

$$x_{11}+x_{21}=40 \text{ ——— } (3).$$

Total demand of Bhopal (D_2) is 30 units. Bhopal is getting x_{12} from Meerut (S_1) and x_{22} from Kanpur (S_2). Thus we have

$$x_{12}+x_{22}=30 \text{ ——— } (4).$$

Similarly, demand of Dehradun (D_3) is 35 units Dehradun (D_3) is getting x_{13} from Meerut (S_1) and x_{23} from Kanpur (S_2). Thus we have

$$x_{13}+x_{23}=35 \text{ ——— } (5).$$

Thus (1) to (5) are five constraints on the variables and thus the constraints of the T.P.

Also, each source either sends a +ve number of machines to various centres or no machine. It can not send a −ve number of machines.

Thus all the variables are non-negative

i.e $x_{11} \geq 0, x_{12} \geq 0, x_{13} \geq 0,$

$x_{21} \geq 0, x_{22} \geq 0, x_{23} \geq 0.$

i.e $x_{ij} \geq 0$; $i = 1, 2$, $j = 1, 2, 3$. —— (6)

is the restriction on variables.

— The objective is to minimize the total transportation cost. Now c_{11} is cost per unit for sending a machine from Meerut(S_1) to Delhi.

we are transporting x_{11} units

$\therefore$ 1 unit $\longrightarrow c_{11}$

x_{11} —— ?

$\therefore c_{11} x_{11}$ is the cost of transporting x_{11} units from Meerut(S_1) to Delhi.

Sly $c_{12} x_{12}$ is the cost of transporting x_{12} units from Meerut(S_1) to Bhopal.

$c_{13} x_{13}$ is the cost of transporting x_{13} units from Meerut(S_1) to Dehradun.

Similarly we can find the cost of transportation from Kanpur (S_2) to the three destinations.

Thus the total cost of transportation works out to be

$$Z = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}.$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij} \text{ —— } (\cancel{6}\,7).$$

Solving a TP means determining the values of the variables x_{ij}, $i = 1, 2$; $j = 1, 2, 3$ such that satisfy (1) — (7) and minimize Z as given in (7).

We can put the problem in a systematic manner and see that it is an LPP. We may name this problem P-I.

$$\text{Min } Z = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} \quad \text{— (7)}$$

subject to

$$x_{11} + x_{12} + x_{13} = 50 \quad \text{— (1)}$$

$$x_{21} + x_{22} + x_{23} = 55 \quad \text{— (2)}$$

$$x_{11} + x_{21} = 40 \quad \text{— (3)}$$

$$x_{12} + x_{22} = 30 \quad \text{— (4)}$$

$$x_{13} + x_{23} = 35 \quad \text{— (5)}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0 \quad \text{— (6)}$$

i.e. $$\text{Min } Z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij} \quad \text{— (7)}$$

s.to

$$\sum_{j=1}^{3} x_{1j} = 50 \quad \text{— (1)}$$

$$\sum_{j=1}^{3} x_{2j} = 55 \quad \text{— (2)}$$

$$\sum_{i=1}^{2} x_{i1} = 40 \quad \text{— (3)}$$

$$\sum_{i=1}^{2} x_{i2} = 30 \quad \text{— (4)}$$

$$\sum_{i=1}^{2} x_{i3} = 35, \quad \text{— (5)}$$

$$x_{ij} \geq 0; \ i = 1, 2, \ j = 1, 2, 3. \quad \text{— (6)}$$

Now further (1) & (2) can be put in the form $\sum_{j=1}^{3} x_{ij} = a_i, \ i = 1, 2$

(3), (4), (5) can be put as

$$\sum_{i=1}^{2} x_{ij} = b_j, \ j = 1, 2, 3.$$

$$x_{ij} \geq 0; \ i = 1, 2, \ j = 1, 2, 3.$$

In a general TP we may have 'm' sources $S_1, S_2, S_3, \ldots S_m$ with capacities $a_1, a_2, \ldots a_m$. Also 'n' may be number of destinations $D_1, D_2, \ldots D_n$ with requirement of demand $b_1, b_2, \ldots b_n$.

Let x_{ij} be the units (to be determined) to be transported from source S_i to destination D_j's $i = 1, 2, \ldots m$, $j = 1, 2, 3 \ldots n$.

$\boxed{\text{We may name this problem P-II}}$

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, 3, \ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, 3, \ldots n.$$

$$x_{ij} \geq 0; \quad i = 1, 2, 3 \ldots m, \quad j = 1, 2, 3, \ldots n.$$

→ The problem is balanced if the total availability at all the sources is equal to the total demand at all the destinations. For the problem to be balanced, Then for the problem to be balanced, we must have

$$a_1 + a_2 + \ldots + a_m = b_1 + b_2 + \ldots + b_n.$$

$$\text{i.e. } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

For the problem to be unbalanced we must we have

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

i.e $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ or $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$.

→ The sewing machines Co.

TP has 2 sources and three markets. such a problem is usually reffered to as a 2×3 TP; 2 being number of sources and 3 that of destinations.

Now let us try to write Mathematical model of TP.

The required is (1) avalicbilities a_i

(2) demands b_j

(3) costs c_{ij}.

The variables that we introduce are x_{ij}.

— In the following, 'm' denotes number of sources and 'n' number of destinations.

<u>Example:</u>

Write the mathematical model of the following TP

$m = 3, \quad n = 3.$

$a_1 = 25, \quad a_2 = 35, \quad a_3 = 40$

$b_1 = 30 \quad b_2 = 28 \quad b_3 = 42$

$c_{11} = 5 \quad c_{12} = 7 \quad c_{13} = 2$

$c_{21} = 3 \quad c_{22} = 6 \quad c_{23} = 5$

$c_{31} = 1 \quad c_{32} = 12 \quad c_{33} = 4.$

<u>Sol</u>

Min $Z = 5x_{11} + 7x_{12} + 2x_{13} + 3x_{21} + 6x_{22} + 5x_{23} + x_{31} + 12x_{32} + 4x_{33}$.

Subject to

$$x_{11} + x_{12} + x_{13} = 25$$
$$x_{21} + x_{22} + x_{23} = 35$$
$$x_{31} + x_{32} + x_{33} = 40$$
$$x_{11} + x_{21} + x_{31} = 30$$
$$x_{12} + x_{22} + x_{32} = 28$$
$$x_{13} + x_{23} + x_{33} = 42$$
$$x_{ij} \geq 0,\ i = 1,2,3$$
$$j = 1,2,3.$$

* Tabular Representation of a TP :-

In problem P-I, we had two sources S_1 at Meerut and S_2 at Kanpur. Also we had three destinations D_1 at Delhi, D_2 at Bhopal and D_3 at Dehradun.

Thus we have in all 6 links or 6 routes connecting the sources to destinations 3 from S_1 to D_1, D_2, D_3 respectively and 3 from S_2 to D_1, D_2, D_3 respectively.

we represent these 6 links or routes by rectangular (or square) blocks formed by 2 horizontal strips correspond to sources and vertical strips, correspond to destinations or market.

This is shown in Table T-I below:

	Delhi D_1	Bhopal D_2	Dehradun D_3	availability $\downarrow$ a_i
Meerut S_1	$c_{11}=5$ (1,1)	$c_{12}=15$ (1,2)	$c_{13}=10$ (1,3)	50
Kanpur S_2	$c_{21}=10$ (2,1)	$c_{22}=8$ (2,2)	$c_{23}=20$ (2,3)	55
Demand b_j	40	30	35	

— Block at the intersection of S_1 and D_1 is called (1,1) cell.
Block at the intersection of S_2 and D_3 is called (2,3) cell and so on.

— Availability of the sources are shown at the end of horizontal rows and demand at the market is shown at the bottom of corresponding column.

— Horizontal strips are hereafter referred to as rows and vertical strips as columns.

— Per unit transportation cost c_{ij} can be put in the north-west top of each cell.

Thus the whole data of TP is put in the form of a table.

— The numbering of the cells is for our understanding only and thus the table (T-I) depicting the data is reproduced below in Table-II:

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5	15	10	50
S_2	10	8	20	55.
$b_j \rightarrow$	40	30	35	Total 105.

Example (1)

Represent the following data of a TP in Tabular form:

$m=3 \quad n=4$

$a_1=15 \quad a_2=20 \quad a_3=30$

$b_1=10 \quad b_2=15 \quad b_3=22 \quad b_4=18$

$c_{11}=5 \quad c_{12}=7 \quad c_{13}=2 \quad c_{14}=1$

$c_{21}=2 \quad c_{22}=4 \quad c_{23}=3 \quad c_{24}=6$

$c_{31}=1 \quad c_{32}=3 \quad c_{33}=7 \quad c_{34}=4$.

Sol

	D_1	D_2	D_3	D_4	$a_i \downarrow$
S_1	5	7	2	1	15
S_2	2	4	3	6	20
S_3	1	3	7	4	30
$b_j \rightarrow$	10	15	22	18	

(2). $m=4 \quad n=3$

$a_1=45 \quad a_2=30 \quad a_3=25 \quad a_4=50$

$b_1=40 \quad b_2=20 \quad b_3=90$

$c_{11}=4 \quad c_{12}=3 \quad c_{13}=2$

$c_{21}=3 \quad c_{22}=7 \quad c_{23}=5$

$c_{31}=7 \quad c_{32}=2 \quad c_{33}=9$

$c_{41}=2 \quad c_{42}=6 \quad c_{43}=7$

	D_1	D_2	D_3	$a_i \downarrow$
S_1	4	3	2	45
S_2	3	7	5	30
S_3	7	2	9	25
S_4	2	6	7	50
$b_j \rightarrow$	40	20	90	

* Special Structure of the transportation problem:

If we examine closely mathematical model of any TP, we will observe some special characteristics of the problem.

Let us consider above example (2).

Its mathematical model is

$$\text{Minimize } z = 4x_{11} + 3x_{12} + 2x_{13} + 3x_{21} + 7x_{22} + 5x_{23} + 7x_{31} + 2x_{32} + 9x_{33} + 2x_{41} + 6x_{42} + 7x_{43}$$

Subject to

$$\begin{array}{llllr}
x_{11} + x_{12} + x_{13} & & & & = 45 \\
 & x_{21} + x_{22} + x_{23} & & & = 30 \\
 & & x_{31} + x_{32} + x_{33} & & = 25 \\
 & & & x_{41} + x_{42} + x_{43} & = 50 \\
x_{11} & + x_{21} & + x_{31} & + x_{41} & = 40 \\
\quad x_{12} & + x_{22} & + x_{32} & + x_{42} & = 20 \\
\qquad x_{13} & + x_{23} & + x_{33} & + x_{43} & = 90
\end{array}$$

$x_{ij} \geq 0, \ i = 1,2,3,4 \ ; \ j = 1,2,3.$

The observations are as follows:

(1) The total number of constraints is 7, namely 4+3 where 4 is number of sources and 3 is number of destinations.

The first 4 constraints – one arising from each availability restriction at the source, are called source constraints or row constraints. and these are exhibited by rows in the table.

The last three constraints – one arising from each demand needed at the destination, are called column constraints or destination constraints. and these are exhibited by columns of table.

It may be noted that the number of variables is $4 \times 3 = 12$.

Thus in a general TP with 'm' sources and 'n' destinations, the total number of constraints is $m+n$ and table consists of m rows and n columns.

The total number of variables is $m \times n (= mn)$

(2) From the above mathematical formulation, we may observe that

(i) each variable appears in exactly two constraints.

(ii) in each constraint, the coefficients of the variables are 1 each (i.e. coefficient of remaining variables are zero).

For example;

Take the variable x_{13}. It occurs in first source constraints and 7^{th} constraint which $(4+3)^{th} = 7^{th}$.

- first four constraints are source constraints as $m=4$ and next three constraints i.e, $(4+1)^{st}$, $(4+2)^{nd}$ and $(4+3)^{rd}$ are destination constraints.

$\therefore x_{13}$ occurs in 1^{st} source constraint and 3^{rd} destination constraint which is the $4+3=7^{th}$ constraint.

- If a TP consists of 'm' sources and 'n' destinations, it has a total of $m+n$ constraints.
- first 'm' constraints are source constraints and next 'n' i.e, $(m+1)^{th}, (m+2)^{th}, \ldots (m+n)^{th}$ are destination constraints.
- A variable x_{ij} occurs in i^{th} source constraints and j^{th} destination constraint which is $(m+j)^{th}$.

(3) Observation (2) above lead us to write activity vector of any variable x_{ij}. for a general TP activity vector of any variable x_{ij} is a '$m+n$' vector.

for example:

x_{13} occurs in 1^{st} constraint with coefficient 1 and in the $(4+3)^{th} = 7^{th}$ constraint with coefficient 1 and with zero coefficient in remaining constraints, its activity vector is given as $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix}0\\0\\1\\0\\0\\0\\1\end{bmatrix} \text{ and } \begin{bmatrix}0\\1\\0\\0\\1\\0\\0\end{bmatrix}$$ are the activity vectors of x_{33} and x_{21} respectively.

We write activity vectors in the order $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{41}, x_{42}, x_{43}$.

the matrix of the coefficient A is 7×12 matrix where 7 is the number of constraints and 12 is the number of variables

$$\therefore A = \begin{bmatrix}1&1&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&0&0&0&1&1&1\\1&0&0&1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1&0&0&1\end{bmatrix}$$

$$A = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\I_{3\times3}&I_{3\times3}&I_{3\times3}&I_{3\times3}\end{bmatrix}$$

where 1 is a row vector of 3 elements with element 1 each. i.e, $1 = [1,1,1]$

$0 = [0\ 0\ 0]$ and $I_{3\times3}$ is a unit matrix of order 3.

For a general TP

1 is a n-vector with element 1 each

0 is a n-vector with element 0 each

and $I_{3\times3}$ is a $m\times n$ unit matrix.

(i) → W.K.T in a general LPP, if the matrix of coefficient A is $m\times n$ and rank of A is m, then the basic feasible solution of the system of equations $Ax=b$ has at the most m +ve variables.

Also, if rank is $k<m$, $m-k$ constraints are reduntant.

This suggests that we must know the rank of matrix A.

If we denote rows of A by $R_1, R_2, R_3, \ldots R_7$ and each is a row vector consisting of 12 elements.

Now multiply $R_1, R_2, R_3 \& R_4$ by 1 and $R_5, R_6 \& R_7$ by -1 and add, we get zero.

i.e., $R_1+R_2+R_3+R_4-(R_5+R_6+R_7)=0$.

This suggests that rows of A are linearly dependent.

i.e., the rank of A is <7.

If we produce a 6×6 non singular submatrix of A, then the rank of A is 6.

To produce this submatrix D, we may omit the last row and take 3^{rd}, 6^{th}, 9^{th}, 12^{th}, 1^{st} & 2^{nd} column of A.

then
$$D = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} I_{4\times 4} & B \\ O & I_{2\times 2} \end{bmatrix}$$

where $I_{4\times 4}$ is a unit matrix of order 4, $I_{2\times 2}$ is a unit matrix of order 2, O is a zero matrix of order 2×4 and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly $|D| = 1 \neq 0$.

Thus the rank of A is $6 = 7-1 = (4+3)-1$.

For a general given $m\times n$ TP, A is $(m+n)\times mn$ matrix and its rank is $(m+n)-1$

This leads us to a very important result that a basic feasible solution of TP consists of $(m+n)-1$ variables.

This means for a basic feasible solution of a TP, at the most $(m+n)-1$ variables are +ve.

Also, we know for an LPP, the optimal solution is a basic feasible solution.

Thus the optimal solution to any TP with 'm' sources and 'n' destinations has at the most $(m+n)-1$ +ve variables. This means, for optimal solution, out of 'mn' routes or links between sources and destinations, we need use transportation on at the most $(m+n)-1$ routes.

For example: If there are 12 sources and 9 destinations, there are $12\times 9 = 108$ routes and variables. Out of these, only $(12+9)-1 = 20$ variables are +ve (basic) and other 88 are zero (non basic). Thus, out of possible 108 routes, we need to transport goods on 20 routes.

An Initial Basic Feasible Solution of the Transportation problem:

Being a special type of LPP, TP could be solved by the simplex method. W.K.T the initial basic feasible solution for this problem can be determined by adding an artificial variable to each constraint and then applying phase I procedure.

Now we will see that such an initial basic feasible solution can be obtained in a better way by exploiting the special structure of the problem. We shall now discuss the methods to obtain an initial basic feasible solution of a TP, viz.

(1) North-West Corner Method (NWC Method)

(2) Least cost or Matrix Minima Method

(3) Vogel's Approximation method (VAM or Penalty Method).

Suppose we solve the TP by simplex method. Observe that in any TP with 'm' sources and 'n' destinations there are $m+n$ constraints in 'mn' variables and each constraint is an equation. Thus, the first step is to add an artificial variable to each of the constraints resulting in $mn+(m+n)$ variables. Next step is to apply Phase-I Method to get an IBFS to the problem. Thus, even for a moderate size problem where 'm' and 'n' are not large, quite a large number of iterations are needed to obtain the IBFS.

Now we shall see that the structure of the problem allows us to write down the IBFS directly and comfortably without adding artificial variable.

North-west Corner Method :-

Let sewing Machines TP, in a tabular form,

Example:

	D_1	D_2	D_3	a_i
S_1	5 (40)	15 (10)	10 x	50
S_2	10 x	8 (20)	20 (35)	55
$b_j \rightarrow$	40	30	35	

In this method,
we start with extreme North-west Corner cell (1,1) connecting source S_1 to Destination D_1. In this route, 50 units are available at S_1 and 40 are needed by D_1.

The minimum of 50 and 40 i.e, $\text{Min}[50, 40] = 40$ can be transported from source S_1 to the market D_1.

$\therefore$ we decide to set $\underline{x_{11} = 40}$, write in the cell (1,1). and encircle it as shown.

Now, demand of D_1 is exactly met and therefore, it doesnot need any supply from S_2.

$\therefore$ $x_{21} = 0$.

we donot write it in the table as it is a non basic variable. So we can put a 'x' in the cell (2,1).

S_1 has still $50 - 40 = 10$ units available with it. Demand of D_2 is 30.

$\text{Min}[10, 30] = 10$.

$\therefore$ 10 units can be supplied by S_1 to D_2.

$\therefore$ set $x_{12}=10$ and encircle it.

Now, availability at S_1 exhausted.

D_2 still needs $30-10=20$ units more. This can be supplied by S_2 only (as availability of S_1 is exhausted).

$\therefore$ set $x_{22}=\min[55,20]=20$ and encircle it.

Now, S_2 has $55-20=35$ units available and only 35 is the need of D_3.

$\therefore$ set $x_{23}=35$ and encircle it.

$\therefore$ we obtain a basic feasible solution of the TP with $x_{11}=40$, $x_{12}=10$, $x_{21}=20$, and $x_{23}=35$. These are basic variables.

All other variables are non-basic and zero where we have put a 'x'.

and the number of basic variables $=(m+n)-1$
$=(2+3)-1$
$=4$.

North-west Corner Method consists of the following steps:

Step 1: Starting with the cell at the upper left (north-west) corner of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied. i.e. $x_{11}=\min(a_1,b_1)$.

Step 2: If $b_1>a_1$, we move down vertically to the 2nd row and make the 2nd allocation of magnitude $x_{21}=\min(a_2, b_1-x_{11})$ in the cell $(2,1)$

- If $b_1 < a_1$, we move right horizontally to the 2nd column and make the 2nd allocation of magnitude $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1,2)
- If $b_1 = a_1$, there is a tie for the 2nd allocation. One can make the 2nd allocation of magnitude

$$x_{12} = \min(a_1 - a_1, b_1) = 0 \text{ in the cell } (1,2)$$

$$\text{or } x_{21} = \min(a_2, b_1 - b_1) = 0 \text{ in the cell } (2,1)$$

Step 3: Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements (a's & b's availability & Demand) are satisfied.

Note: This method doesnot take into account the costs of transportations C_{ij}.

Example-II

	D_1	D_2	D_3	D_4	a_i
S_1	5	7	6	4	70
S_2	2	8	3	1	50
S_3	1	7	4	5	90
$b_j \to$	50	40	50	70	$\Sigma a_i = \Sigma b_j = 210$.

Since $\Sigma a_i = \Sigma b_j = 210$

$\therefore$ It is a Balanced T.P.

Star with N-W. Corner cell (1,1).

$\min(a_1, b_1) = \min(50, 70) = 50.$

$\therefore$ Set $x_{11} = 50$.

and set $x_{21} = x_{31} = 0$. and put 'x'.
($\because$ demand of D_1 is exactly met and it doesnot need any supply from S_2 or S_3).

Availability at $S_1 = 70 - 50 = 20$
and Demand for $D_2 = 40$.

Min $(20, 40) = 20$.

$\therefore$ set $x_{12} = 20$.

and set $x_{13} = x_{14} = 0$ and put 'x'
($\because$ Supply at S_1 is exhausted. It cannot supply to markets D_3 & D_4).

D_2 still needs $40 - 20 = 20$ units which are to be supplied by next source S_2.

Min $(20, 50) = 20$.

S_2 can supply all the 20 units need by D_2.

$\therefore$ set $x_{22} = 20$.

and set $x_{32} = 0$ and put 'x'
($\because$ Demand of D_2 is exactly met).

Next consider demand of D_3 which is 50 units.

S_2 is left with $50 - 20 = 30$ units.

Min $(30, 50) = 30$.

S_2 can supply 30 units needed by D_3.

$\therefore$ set $x_{23} = 30$.

and set $x_{24} = 0$ and put a 'x'.

Demand of D_3 is $50 - 30 = 20$ units more.
and S_3 has 90 units

$\min(90, 20) = 20$

$\therefore$ set $x_{33} = 20$

Demand of D_3 is exactly met.

Now consider the demand of D_4.

Demand of $D_4 = 70$,

availability of $S_4 = 90 - 20 = 70$.

$\therefore$ set $x_{34} = 70$.

Hence an initial basic feasible solution to the given T.problem has been obtained and is displayed

	D_1	D_2	D_3	D_4	$a_i \downarrow$
S_1	5 (50)	7 (20)	6 ×	9 ×	70
S_2	2 ×	8 (20)	3 (30)	1	50
S_3	1 ×	7 ×	4 (20)	5 (70)	90
$b_j \rightarrow$	50	40	50	70	

$\therefore x_{11} = 50$, $x_{12} = 20$, $x_{22} = 20$, $x_{23} = 30$.

$x_{33} = 20$, $x_{34} = 70$. These are basic variable.

and the number of basic variables $= (m+n) - 1$

$= (3+4) - 1$

$= 6$.

Non-basic variables are

$x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$

MATRIX MINIMA METHOD (or) LEAST COST METHOD:

Northwest corner method considers only the constraints of availability and supply of material, the cost of transportation is ignored, which is most important factor.

Now, we discuss the method in which we transport starting with cheapest route and using the routes in ascending order of costs of transportation.

For example.

Ex-3 The sewing Machine TP is

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5	15	10	50
S_2	10	8	20	55
$b_j \rightarrow$	40	30	35	

Sol^n: The cheapest route is the one connecting source S_1 to market D_1 where cost per unit is 5. i.e, $C_{11} = 5$.

Availability at S_1 is 50, Demand at D_1 is 40

$\min[50, 40] = 40$.

$\therefore$ S_1 can supply all the 40 units needed by D_1.

Set $x_{11} = 40$ and encircle it.

Set $x_{21} = 0$ and put 'x'.

	D_1	D_2	D_2	
S_1	5 (40)	15	10	50-40=10
S_2	10 x	8	20	55
	~~40~~	30	55	

Next look for next cheapest cost among uncrossed cells.

This route is (2,2) connecting source S_2 to market D_2 with $C_{22} = 8$.

Availability at $S_2 = 55$ and Demand at $D_2 = 30$

$\min(55, 30) = 30$.

Set $x_{22} = 30$ and encircle it.

and set $x_{12} = 0$ and put a 'x'.

	D_1	D_2	D_3	
S_1	5 (40)	15 x	10	$50-40=10$
S_2	10 x	8 (30)	20	$55-30=25$
	40	30	35	

The next cheapest route among uncrossed cells is $c_{13} = 10$ in cell (1,3).

Availability at $S_1 = 50-40 = 10$,

Demand at $D_3 = 35$.

$\min(10, 35) = 10$.

$\therefore$ set $x_{13} = 10$ and encircle it.

	D_1	D_2	D_3	
S_1	5 (40)	15 x	10 (10)	
S_2	10 x	8 (30)	20	25
			$35-10=25$	

The next cheapest and only unused route is (2,3) with $C_{23} = 20$.

Availability at S_2 = Demand at $S_2 = 25$.

$\therefore$ set $x_{23} = 25$ and encircle it.

	D_1	D_2	D_3
S_1	5 (40)	10 x	10 (10)
S_2	10 x	8 (30)	20 (25)

Now, all the rim requirements have been satisfied and hence an initial basic feasible solution is

$\therefore x_{11} = 40,\ x_{13} = 10,\ x_{22} = 30,\ x_{23} = 25$

and other variables are non-basic.

Also the number of basic variables $= (m+n)-1$
$= (2+3)-1 = 4$

Ex-5 → Find the ~~basic~~ feasible solution by Matrix Minima method.

	D_1	D_2	D_3	D_4	a_i ↓
S_1	5	7	6	4	70
S_2	2	8	3	1	50.
S_3	1	7	4	5	90
b_j →	50	40	50	70	

Soln: Minimum cost in the matrix is in the cell (3,1) linking third source S_3 to Destination D_1 and in the cell (2,4) linking source S_2 to Destination D_4.

We can choose any one of the two cell.

Let us choose (3,1)

Availability $a_3 = 90$, Demand $b_1 = 50$.

$\text{Min}(90, 50) = 50$.

$\therefore$ Set $x_{31} = 50$ and encircle it as a basic variable.

As demand of D_1 is exactly met, put a cross 'x' in the cells (1,1) and (2,1) indicating that these are non-basic

i.e., $x_{11} = x_{21} = 0$.

	D_1	D_2	D_3	D_4	$a_i\downarrow$
S_1	5 ×	7	6	4	70
S_2	2 ×	8	3	1	50
S_3	1 (50)	7	4	5	90−50=40.
$b_j\rightarrow$	50	40	50	70	

The next cheapest cost route is (2,4) with $C_{24}=1$

$a_2 = 50,\ b_4 = 70$

$\min(50,70) = 50.$

$\therefore$ set $x_{24} = 50$ and encircle it as a basic variable.

and set $x_{22} = x_{23} = 0$ as these are non-basic variables and put a '×'.

	D_1	D_2	D_3	D_4	
S_1	5 ×	7	6	4	70
S_2	2 ×	8 ×	3 ×	1 (50)	~~50~~
S_3	1 (50)	7	4	5	40
		40	50	70−50 =20	

The next cheapest cost route among the unused routes $C_{14} = C_{33} = 4$ in the cells (1,4) and (3,3).

Choose any one of these cell say (3,3).

~~Capacity~~ availability $a_3 = 40$, Demand $b_3 = 50$

$\min(40,50) = 40$

$\therefore$ set $x_{33} = 40$ and encircle it as a basic variable.

and set $x_{23} = x_{34} = 0$ and put a 'x'.

	D_1	D_2	D_3	D_4	
S_1	5 x	7	6	4	70
S_2	2 x	8 x	3 x	1 (50)	~~0~~
S_3	1 (50)	7 x	4 (40)	5 x	~~40~~
		40	50−40 =10	20	

The next cheapest cost route is (1, 4) with $c_{14} = 4$

Availability $a_1 = 70$, Demand $D_4 = 20$

min $(70, 20) = 20$

$\therefore$ set $x_{14} = 20$ and encircle it as a basic variable

5 x	7	6	4 (20)	70−20=50
2 x	8 x	3 x	1 (50)	
1 (50)	7 x	4 (40)	5 x	
	40	10	~~20~~	

next cheapest cost route is (1, 3) with $c_{13} = 6$.

$a_1 = 50$, $b_3 = 10$

min $(50, 10) = 10$.

$\therefore$ set $x_{13} = 10$ and encircle it as a basic variable.

5 x	7	6 (10)	4 (20)	50−10=40
2 x	8 x	3 x	1 (50)	
1 (50)	7 x	4 (40)	5 x	
	40	~~10~~		

Last unused route is (1, 2) with $c_{12} = 7$

$a_1 = 40$, $b_2 = 40$.

$\therefore$ set $x_{12} = 40$ and encircle it.

	D_1	D_2	D_3	D_4	a_i
S_1	5 ×	7 (40)	6 (10)	4 (20)	70
S_2	2 ×	8 ×	3 ×	1 (50)	50
S_3	1 (50)	7 ×	4 (40)	5 ×	90
b_j	50	40	50	70	

All the rim requirements have been satisfied and hence an initial basic feasible solution is :

$x_{12} = 40,\ x_{13} = 10,\ x_{14} = 20$

$x_{24} = 50,\ x_{31} = 50,\ x_{33} = 40$

and Other variables are non-basic

Also the number of basic variables

$= (m+n) - 1$

$= (3+4) - 1$

$= 6.$

Vogel's Approximation Method:

The Vogel's approximation method takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields a better solution.

Vogel's Approximation Method consists of the following steps:

Step (1): Calculate penalties by taking differences between the minimum and next to minimum unit transportation costs in each row and each column.

Step (2): Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the rim requirements, to the lowest cost cell in that row or column.

In case of a tie allocate to the cell associated with the lower cost.

If the greatest difference corresponds to the i^{th} row/ j^{th} column and c_{ij} is the lowest cost in the i^{th} row allocate as much as possible i.e, $\min(a_i, b_j)$ in the $(i,j)^{th}$ cell and cross off the i^{th} row or j^{th} column.

Step (3): Recalculate the row and column differences for the reduced table and go to step (2).

Step(4): Repeat the procedure till all the rim requirements are satisfied.

Ex-6 → Use Vogel's Approximation method to obtain an initial basic feasible solution of the TP.

	D_1	D_2	D_3	D_4	a_i
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400.
$b_j \rightarrow$	200	225	275	250	

Sol^n: Since $\Sigma a_i = \Sigma b_i = 950$.

$\therefore$ The problem is balanced.

$\therefore$ The differences between the smallest and next to the smallest costs in each row and each column are first computed and displayed inside parenthesis against the respective rows and columns.

11 (200)	13	17	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
200	225	275	250	
(5)	(5)	(1)	(0)	

The largest of these penalties is (5) and is associated with the first column.

Since $c_{11} = 11$, is the minimum cost, we allocate $x_{11} = \min(250, 200) = 200$ in the cell (1,1). This exhausts the requirement of the first column and therefore cross off the first column.

The row and column differences are now computed for the resulting reduced transportation table as follows.

13 (50)	17	14	250-200=50	(1)
18	14	10	300	(4)
24	13	10	400	(3)
225	275	250		
(5)	(1)	(0)		

The largest of these penalties is (5) which is associated with the second column.

Since $c_{12} = 13$, is the minimum cost, we allocate $x_{12} = \min(50, 225) = 50$.

This exhausts the availability of first row and therefore, cross off the first row.

Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are as shown below.

18 (175)	14	10	300	(4)
24	13	10	400	(3)
225-50 =175	275	250		
(6)	(1)	(0)		

14	10 (125)	125	(4)
13	10	400	(3)
275	250		
(1)	(0)		

13 (275)	10 (125)	400
275	125	

finally the initial basic feasible solution is as shown below.

11 (200)	13 (50)	17	14
16	18 (175)	14	10 (125)
21	24	13 (275)	10 (125)

$\therefore x_{11} = 200, \; x_{12} = 50, \; x_{22} = 175,$
$x_{24} = 125, \; x_{33} = 275, \; x_{34} = 125.$

these are basic variables

and other variables are non-basic.

Also the number of basic variables $= (m+n)-1$
$= (3+4)-1$
$= 6.$

we have discussed the methods to find an initial basic feasible solution to the given TP. Now we have to verify the fact that the solution so obtained is a basic feasible solution. So, we now introduce a rule or procedure for checking whether a given feasible solution is basic, i.e. cells are in linearly independent positions. This is called the chain rule.

Closed chain:

For example: Consider the sewing machine TP.

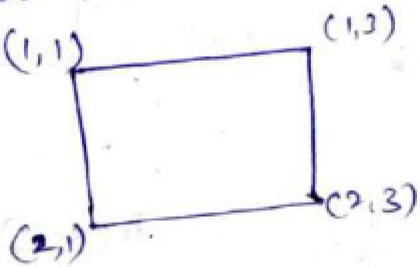
	D_1	D_2	D_3	a_i
S_1	5	15	10	50
S_2	10	8	20	55
b_j	40	30	35	

Consider the set of cells (1,1), (1,3), (2,3), (2,1)

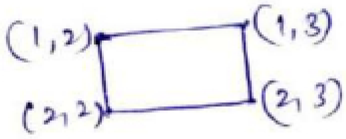
Starting with any one of the cells say (1,1) draw alternately horizontal and vertical lines to reach other cells.

If doing so, we come back to the starting cell, we say that a closed chain is formed.

In this case, the circuit is as follows.

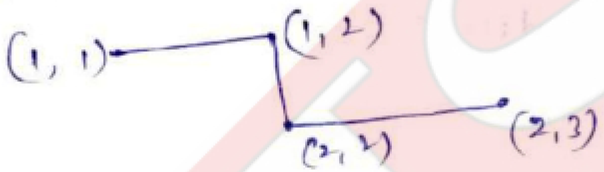


If we consider (1,2), (1,3), (2,3), (2,2) cells, which form a closed chain, as given below



Similarly, (1,1), (1,2), (2,2), (2,1) form a closed chain.

→ Now let us consider the cells (1,1), (1,2) (2,2) (2,3).



In this case, drawing alternate horizontal and vertical lines, we donot come back to the starting cell (1,1).

Thus, these cells donot form a closed chain. and therefore, form a simply chain.

Similarly (1,1) (1,3), (2,3) form a chain

(1,1), (1,3), (2,3) form a chain.

→ Now we give a method or rule to test, if given any number of cells, these are in linearly independent positions or in linear dependent positions.

Closed Chain Rule:

→ Gives a feasible solution, if we can form a closed circuit involving all the basic cells or any subset or part of basic cells, then the cells are in linearly dependent positions.

- If we can form no circuit involving the basic cells above, the cells are in linearly independent positions.

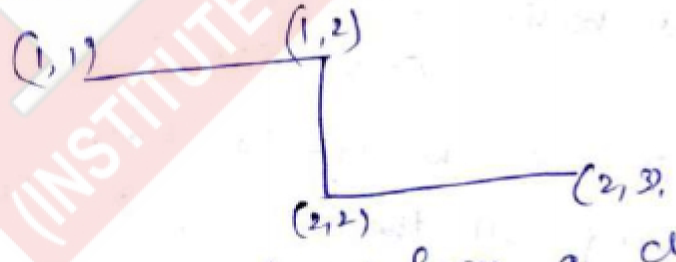
- Applying this rule, we may observe that initial ~~feasible~~ solution obtained by North-west Corner Method, Matrix Minima Method or VAM is indeed a basic feasible solution satisfying the criterion of cells being in independent positions.

For example:

Consider the solution of the sewing machine transportation problem by North west corner method is:

				a_i
	5 (40)	15 (10)	10	50
	10	8 (20)	20 (35)	55
$b_j \rightarrow$	40	30	35	

The basic cells are (1,1) (1,2), (2,2), (2,3)



These cells donot form a closed chain.

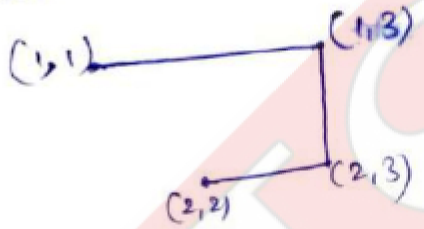
$\therefore$ These are in linearly independent positions.

→ Consider the solution of the sewing machine transportation problem by Matrix minima method is:

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5 (40)	15	10 (10)	50
S_2	10	8 (30)	20 (25)	55
$b_j \rightarrow$	40	30	35	

Basic cells are (1,1), (1,3), (2,3), (2,2)

They donot form a closed chain as shown below.



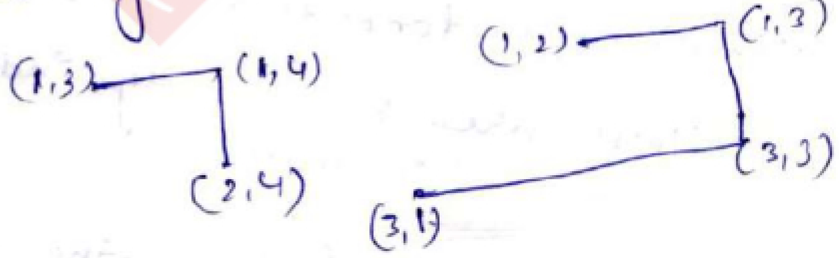
→ consider Example (5)

The Solution of the TP by Matrix Minima method is

5 ×	7 (40)	6 (10)	4 (20)
2 ×	8 ×	3 ×	1 (50)
1 (50)	7 ×	4 (40)	5 ×

The basic cells are (1,2), (1,3), (1,4), (2,4), (3,1), (3,3).

In this even we try to form closed chains involving all the cells, we do not succeed.



Thus the cells are linearly independent positions. Hence the solution is a basic feasible solution.

→ consider example (2)

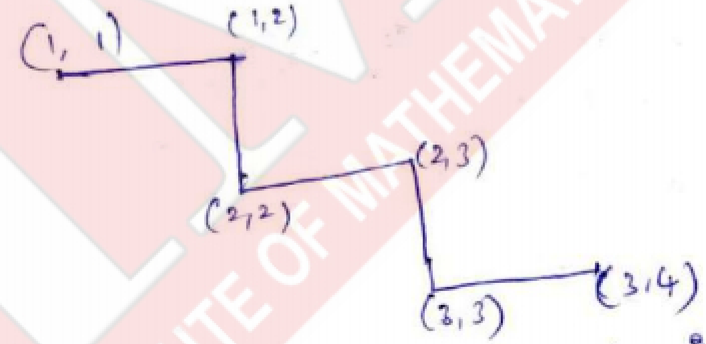
The solution of the TP by NWCM is:

				$a_i \downarrow$
5 (50)	7 (20)	6	4	70
2	8 (20)	3 (30)	1	50
1	7	4 (20)	5 (70)	90
$b_j \rightarrow$ 50	40	50	70	

The basic cells are (1,1), (1,2), (2,2), (2,3), (3,3), (3,4)

and these cells donot form a closed chain

i.e.,



∴ These cells are in linearly independent positions.

→ Now we shall extend the closed chain rule to identify the cells at zero level in case, the solution obtained by any of the methods or even an any stage is a degenerate solution.

Degenerate Basic feasible solution:

In a LPP, a basic feasible solution is said to be degenerate if certain basic variables (one or more) are at zero level. i.e, the number of variables is less than 'm' for the system of equation $Ax=b$, $x \geq 0$, A being $m \times n$ matrix and rank of A is m.

However, in a LPP we were not faced with problem of identifying the basic variables at zero level. we got these variables at zero level in the process of solving the LPP.

But in a Transportation problem while determining basic feasible solution by Northwest corner Method, Matrix Minima method or by Vogel's Approximation Method; we may discover that the number of basic variables are less than required $(m+n)-1$ was number.

For example:

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5	15	10	50
S_2	10	8	20	55.
$b_j \rightarrow$	50	20	35	

By using the North-west Corner Method the initial basic feasible solution is

5 (50)	15 ×	10 ×	50
10 ×	8 (20)	20 (35)	55
50	20	35	

The number of basic variables is 3 which is lessthan $(m+n)-1 = (3+2)-1=4$.

Such a solution in LPP we call a degenerate basic feasible solution.

Now we must identify the additional basic variable at zero level to make it a complete basic feasible solution having $(m+n)-1$ variables.

Rule:

→ Select a variable as a candidate for basic variable at zero level.

Suppose it is x_{12} in the above example. i.e, cell (1,2). It is called a candidate cell. Start joining it by horizontal lines and vertical lines alternately to basic variable cells with circled allocation.

→ In this process, if we can come back to the candidate cell, this is not qualified. because they form linearly dependent set.

Otherwise, we can put a zero in the cell and circle it as 0.

→ In the above example, (1,2) can be joined only to (1,1) by a horizontal line and then (1,1) cannot be joined to any basic cell by a vertical line. So, we cannot come back to (1,2).

∴ (1,2) qualifies and we can put a '0' in the cell (1,2) and circle it. i.e, $x_{12}=0$

Note: (1,3) and (2,1) also qualify to become basic variables at zero level.

But only one cell is to be marked to make $(3+2)-1=4$ variables.

→

	D_1	D_2	D_3	D_4	D_5	a_i ↓
S_1	4	5	3	1	7	20
S_2	10	8	8	6	2	20
S_3	3	6	4	5	4	50
b_j →	15	25	20	10	20	

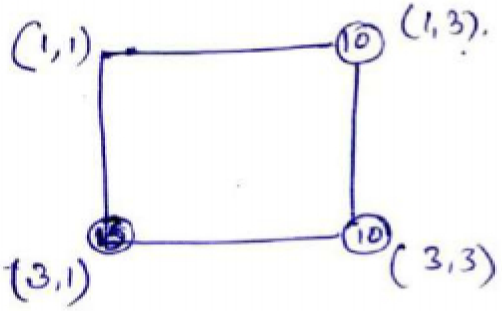
By using the Matrix Minima Method the basic feasible solution is:

4 ×	5 ×	3 (10)	1 (10)	7 ×	20
10 ×	8 ×	8 ×	6 ×	2 (20)	20
3 (15)	6 (25)	4 (10)	5	4 ×	50
15	25	20	10	20	

Number of basic variables needed is $(5+3)-1=7$.

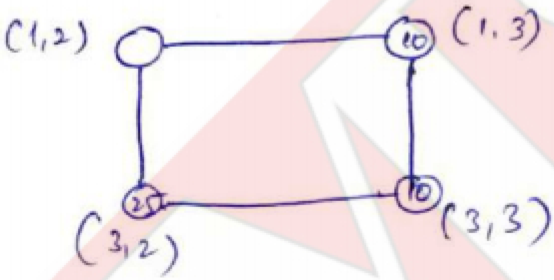
But we find only 6 of these, in the process. Thus an addition basic variable at zero level is to be identified.

Following the rule, we see that (1,1) doesnot qualify as we have the following closed chain.

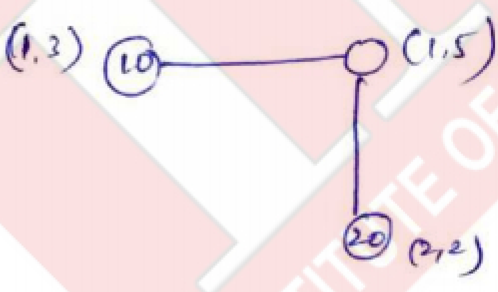


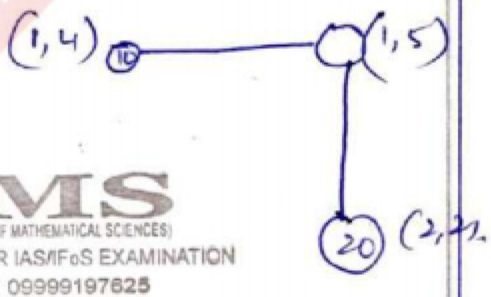
$\therefore$ (1,1) forms a closed chain with basic cells or variables at (1,3), (3,3), and (3,1).

(1,2) also doesnot qualify as it forms a closed chain as shown below



(1,5) qualifies as it doesnot form a closed chain.





Similarly, (2,1), (2,2), (2,3), (2,4) and (3,5) qualify while (3,4) doesnot.

$\therefore$ we can put a zero or (0) in any of these cells (1,5), (2,1), (2,2), (2,3) & (2,4) but not in (1,1), (1,2) and (3,4)

→ But choose only one to make a total of $(5+3)-1 = 7$ basic variables or cells

Computational procedure for the Transportation problem:

Now, we discuss how to proceed from an initial basic feasible solution of the transportation problem to its optimal solution.

Like in linear programming problem it will be ensured that at each step we obtain a basic feasible solution yielding objective function value lesser than or at the most equal to that in the previous step.

We are dealing with a minimization problem, so, we move from one basic feasible solution to a better basic feasible solution, ultimately to reach one yielding the minimum cost of transportation.

To do all this, we have used a computational method both for a balanced ~~on~~ an unbalanced Transportation problem.

The method used for the balanced TP is known as u-v method.

A Balanced transportation problem:

A computational method to solve the balanced transportation problem is known as Modified Distribution Method or MODI Method or u-v method.

The main steps of the u-v method are:

(1) determination of the dual variables.

(2) determination of the net evaluations.

(3) Selecting the cell to enter the basis.

4) Selecting the cell to leave the basis

5) updating the basis and determining a new basic feasible solution, and

(6) identifying the termination stage.

(1) <u>Determination of the dual variables:</u>

Let us consider a transportation problem with 2 sources and 3 destinations, whose tabular form is given as:

	D_1	D_2	D_3	$a_i \downarrow$
S_1	4	3	1	4
S_2	2	6	2	6
$b_j \rightarrow$	2	3	5	

— The rows and columns of this table we associate one variable each and call them dual variables.

— with rows we associate dual variable u_i's and with columns v_j's.

- In the above example dual variables u_1 & u_2 are associated with the first and the second rows respectively whereas dual variables v_1, v_2, v_3 are associated with the three columns respectively.

- Generalising this, we conclude that for a problem with m sources and n destination, we will have $(m+n)$ dual variables.

- An initial basic feasible solution of the given TP by North-west Corner method is

Table-1

4 (3)	3 (2)	1
2	6 (1)	2 (5)

Define $\Delta_{ij} = u_i + v_j - c_{ij}$; for $i = 1, 2, \ldots m$ & $j = 1, 2, \ldots n$.

- The dual variables are determined by the criteria that for each basic cell (i, j)

we must have, $\Delta_{ij} = 0$

i.e, $u_i + v_j - c_{ij} = 0 \Rightarrow u_i + v_j = c_{ij}$ —— (1)

- In case of an $m \times n$ transportation problem, there being $(m+n-1)$ basic cells, we get $(m+n-1)$ equations of the above type.

- In order to determine $(m+n)$ dual variables from these $(m+n-1)$ equations, we give an arbitrary value to one of the dual variables.

- A good convention is to assign value zero to the dual variable corresponding to a row or a column with maximum number of basic cells.

– The remaining $(m+n-1)$ dual variables are then easily determined from the $(m+n-1)$ equations.

– In the above problem there are 4 basic cells, so there will be 4 equations of the type (i). given above.

– As 5 dual variables are to be evaluated, we assign one of these, equal to zero.

– There being 2 basic cells in each of the two rows and also in the 2nd column, we can assign zero to any one of u_1, u_2 or v_2.

Let us put $u_1 = 0$; using this in the 4 equations

we have

$$u_1 + v_1 = 4 \Rightarrow v_1 = 4$$

$$u_1 + v_2 = 3 \Rightarrow v_2 = 3$$

$$u_2 + v_2 = 6 \Rightarrow u_2 = 6 - 3 = 3$$

$$u_2 + v_3 = 2 \Rightarrow v_3 = 2 - 3 = -1$$

$(\because u_i + v_j = c_{ij}$ as (i,j) a basic cell.)

→ If there are less than $(m+n-1)$ occupied cells in a basic feasible solution, then we treat as many unoccupied cells as zero-entry occupied cells as are necessary to raise the total number of occupied cells to $(m+n-1)$.

<u>For example:</u> if there are 6 occupied cells in a basic feasible solution of a 4×5 transportation problem, then we have to treat 2 unoccupied cells as zero entry occupied cells.

- While selecting such zero-entry cells, utmost care has to be taken so that all the $(m+n-1)$ occupied cells (zero-entry or otherwise) in the basic feasible solution are linearly independent.

- For this one has to ensure non-existence of a closed-chain formed by the occupied cells.

(2) Determination of the net-evaluations:

- Corresponding to each cell (i, j) we associate a quantity $\Delta_{ij} = u_i + v_j - c_{ij}$ and call it the net evaluation for the cell (i,j).

Note: While determing the dual variables we have already taken the net-evaluations for all the basic cells to be zero.

So our interest now is confined to the determination of Δ_{ij}'s for the non-basic cells.

In the above example, from Table-1 we find net evaluations for all the non-basic cells as below.

Table-2

			$u_i \downarrow$
4 (2)	3 (2)	1	0
2	6 (1)	2 (5)	3
4	3	-1	v_j

Using $\Delta_{ij} = u_i + v_j - c_{ij}$, we have

$$\Delta_{13} = u_1 + v_3 - c_{13}$$
$$= 0 + (-1) - 1$$
$$= -2$$

and $\Delta_{21} = u_2 + v_1 - c_{12}$
$$= 3 + 4 - 2 = 5.$$

we denote the Δ_{ij} values in the right-bottom corner of cell (i, j) as shown below.

Table-3

			$u_i \downarrow$
4 (2)	3 (2)	1 (-2)	0
2 (5)	6 (1)	2 (3)	3
$v_j \rightarrow$ 4	3	-1	

Here we do not record Δ_{ij}'s for the basic (occupied) cells as we know that $\Delta_{ij} = 0$ for all (i, j) basic-cells.

(3) Selecting the cell to enter the basis:

– For determining the non-basic cell to enter the basis,

determine $\text{Max}\{\Delta_{ij} / \Delta_{ij} > 0\} = \Delta_{rs}$ (say)

Then the unoccupied cell (r, s) enters the basis. i.e, (r, s) is the non-basic cell with most +ve Δ_{ij} entry.

– If maximum is achieved for more than one unoccupied cell, then we can arbitrarily pick any one of these and enter it in the basis.

In the above example, as there is only one non-basic cell i.e; (2,1) with +ve net-evaluation $\Delta_{21} = 5$.

$\therefore$ the cell (2,1) enter the basis.

(4) Selecting the cell to leave the basis:

In the above step we have selected the cell (r, s) to enter the basis. In order to select the cell to leave the basis, we proceed as follows.

(a) with (r, s) as the starting and ending cell, determine a closed chain with all its other corner on occupied cells.

– In the closed chain, a vertical line should be followed by a horizontal line and a horizontal line should be followed by a vertical line. i.e., two consecutive vertical lines or horizontal lines are not allowed.

(b) Put $x_{rs} = \theta$. Subtract and add θ alternately from x_{ij}'s in the corner cells of the closed chain determined in (a) above, starting by subtracting θ from the corner cell adjacent to cell (r, s) in the closed chain.

→ Even after adding and subtracting θ, as explained above, the row-sums and column-sums of allocations in the table continue to be equal to the availabilities and demands of various sources and destinations respectively.

(c) From all the cells from which θ is subtracted, choose the cell with minimum allocation. Let this be the cell (p, q), then cell (p, q) leaves the basis.

From the example in Table-3, the closed chain as desired in (a) above, is shown as below.

Table-4

4 $2-\theta$	3 $2+\theta$	1 (-2)
2 θ (5)	6 $1-\theta$	2 (5)

Allocation in the entering cell (2,1) is put equal to θ. i.e, $x_{21} = \theta$. Then closed chain is followed, and θ is subtracted and added alternately to the corner cells.

i.e, θ is subtracted from 2 in the cell (1,1), added to 2 in cell (1,2), subtracted from 1 in cell (2,2).

As θ has been subtracted from the cells (1,1) and (2,2) and as out of these entry 1 in the cell (2,2) is least, so we make the cell (2,2) leave the basis.

(5) Updating the basis and determining the new basic feasible solution:

The value of θ is then substituted in all the occupied cells as well as the entering cell (r, s).

If (p,q) be the cell leaving the basis, then by putting $\theta = x_{pq}$, allocation in no cell becomes -ve. The cell (or one of the cells) where the updated allocation becomes zero is treated as non-basic cell for the new basic feasible solution.

From the example in Table-4,

put $\theta = x_{22} = 1$.

Substituting this value of θ in all the occupied cells as well as the entering cell $(2,1)$.

$\therefore$ we get the updated basic feasible solution as

Table-5

4 1	3 3	1
2 1	6	2 5

(6) Identification of the optimal stage:

Continuing the process we move from one basis to another in every step.

At some stage if we find $\Delta_{ij} \le 0$ for all non-basic cells, we declare optimality.

i.e, the basic feasible solution in hand is the optimal solution yielding the minimum cost of transportation.

for the given TP, the following basic feasible solution is optimal, as $\Delta_{ij} \le 0$ for all non-basic cells.

				$u_i \downarrow$
	4 (−5)	3 (3)	1 (1)	0
	2 (2)	6 (0)	2 (4)	3
$v_j \rightarrow$	−1	3	1	

$u_1 + v_2 = 3$
$u_1 + v_3 = 1$
$u_2 + v_1 = 2$
$u_2 + v_3 = 4$
Let $\boxed{u_1 = 0}$
$\Rightarrow \boxed{v_2 = 3}$
$\boxed{v_3 = 1}$
$\boxed{u_2 = 3}$
$\boxed{v_1 = -1}$

Hence $x_{12} = 3$, $x_{13} = 1$, $x_{21} = 2$, $x_{23} = 4$.

is the optimal feasible solution.

$\therefore$ The minimum cost of transportation is given by $(3\times 3) + (1\times 1) + (2\times 2) + (2\times 4) = 9+1+4+8$
$= \underline{\underline{22}}$.

Working procedure for transportation problems

Various steps involved in solving transportation problem may be summarized in the following iterative procedure.

(1) Find the initial basic feasible solution by using any of the three methods NWCM or LCM or VAM.

(2) Check the number of occupied cells. If there are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\epsilon (\approx 0)$ (or assign zero (0)) in suitable independent positions, so that the

number of occupied cells is exactly equal to $m+n-1$.

(3) For each occupied cell in the current solution, solve the system of equations

$u_i + v_j = c_{ij}$,

Starting initially with some $u_i = 0$ or $v_j = 0$ corresponding to a row or a column with maximum number of basic cells (or allocations). and entering the successively the values of u_i and v_j in the transportation table margins.

(4) Compute the net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unoccupied basic cells. and enter them in the right bottom corners of the corresponding cells.

(5) Examine the sign of each Δ_{ij}

If all $\Delta_{ij} \le 0$, then the current basic feasible solution is optimal.

If at least one $\Delta_{ij} > 0$, select the unoccupied cells, having the largest positive net evaluation to enter the basis.

(6) Let the unoccupied cell (r, s) enter the basis. Allocate an unknown quantity, say 'θ', to the cell (r, s).

Identify a loop (or closed chain) that starts and ends at the cell (r, s) and connects some of the basic cells.

Add and subtract alternately to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

(7) Assign a maximum value to θ in such a way that the value of one basic variable becomes zero and the other basic variables remain non-negative.

The basic cell whose allocation has been reduced to zero, leaves the basis.

(8). Return to step (3) and repeat the process until an optimum basic feasible solution has been obtained.

→ Solve the cost minimizing transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i ↓
S_1	10	12	13	8	14	19	18
S_2	15	18	12	16	19	20	22
S_3	17	16	13	14	10	18	39
S_4	19	18	20	21	12	13	14
b_j →	10	11	13	20	24	15	

Sol^n: Using North-west corner Rule, an initial basic feasible solution is given as

Table -1

	D_1	D_2	D_3	D_4	D_5	D_6	
S_1	10 (10)	12 (8)	13	8	14	19	~~18~~ ~~8~~
S_2	15	18 (3)	12 (13)	16 (6)	19	20	~~22~~ ~~19~~ 6
S_3	17	16	13	14 (14)	10 (24)	18 (1)	~~39~~ ~~25~~ 1
S_4	19	18	20	21	12	13 (14)	14
b_j	~~10~~	~~11~~ 3	~~13~~	~~20~~ ~~14~~	~~24~~	~~15~~ 14	

Now finding the values of u_i and v_j:

— As maximum number of basic cells exist in the 2nd and 3rd rows, we can start by putting either u_2 or u_3 equal to zero.

Let us put $\boxed{u_2 = 0}$

— As (2,2), (2,3), (2,4) are the basic cells in this row and for basic cells we know $\Delta_{ij} = u_i + v_j - c_{ij} = 0$

i.e, $u_i + v_j = c_{ij}$

$\therefore\; u_2 + v_2 = 18 \Rightarrow \boxed{v_2 = 18}$

$u_2 + v_3 = 12 \Rightarrow \boxed{v_3 = 12}$

$u_2 + v_4 = 16 \Rightarrow \boxed{v_4 = 16}$

$u_1 + v_2 = 12 \Rightarrow u_1 + 18 = 12 \Rightarrow \boxed{u_1 = -6}$

$u_3 + v_4 = 14 \Rightarrow u_3 + 16 = 14 \Rightarrow \boxed{u_3 = -2}$

$u_3 + v_5 = 10 \Rightarrow -2 + v_5 = 10 \Rightarrow \boxed{v_5 = 12}$

$u_3 + v_6 = 18 \Rightarrow -2 + v_6 = 18 \Rightarrow \boxed{v_6 = 20}$

$u_4 + v_6 = 13 \Rightarrow u_4 + 20 = 13 \Rightarrow \boxed{u_4 = -7}$

$u_1 + v_1 = 10 \Rightarrow -6 + v_1 = 10 \Rightarrow \boxed{v_1 = 16}$

Table-2

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	u_i
S_1	10 (10)	12 (8)$-\theta$	13 (−7)	8 θ (2)	14 (−8)	19 (−5)	18	−6
S_2	15 (1)	18 (3)$+\theta$	12 (13)	16 (6)$-\theta$	19 (−7)	20 (0)	22	0
S_3	17 (−3)	16 (0)	13 (−3)	14 (14)	10 (24)	18 (1)	39	−2
S_4	19 (−10)	18 (−7)	20 (−15)	21 (−12)	12 (−7)	13 (14)	14	−7
$b_j \rightarrow$	10	11	13	20	24	15		
$v_j \rightarrow$	16	18	12	16	12	20		

The net evaluations for the unoccupied cells (non-basic) are now calculated as below:

for the cell (1, 3)

$\Delta_{13} = u_1 + v_3 - C_{13} = -6 + 12 - 13 = -7$

and for the cell (1, 4)

$$\Delta_{14} = u_1 + v_4 - C_{14} = -6 + 16 - 8 = 2$$

In this way Δ_{ij} for all non-basic cells are evaluated and recorded in right bottom of each cell, as shown in Table-2.

1st iteration:

Determining the cell to enter the basis:

Calculate Max $\{\Delta_{ij} / \Delta_{ij} > 0\}$.

Clearly $\Delta_{14} = 2$ is the most +ve.

$\therefore$ cell (1, 4) enters the basis.

- we allocate an unknown quantity θ, to this cell (1,4) and identify a closed-chain (loop) involving basic cells around this entering cell.
- Add and substract θ, alternately to and from the transition cells of the loop subject to the rim requirements.
- Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain ≥ 0.

if we put $\theta = 6$ in Table – 2,

x_{24} becomes zero.

i.e, the cell (2, 4) leaves the basis.

- putting $\theta = 6$ in Table-2 and turning (2, 4) as non-basic cell. we get the new basic feasible solution as shown in Table – 3.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
S_1	10 (10)	12 (2)	13	8 (6)	14	19	18
S_2	15	18 (9)	12 (13)	16	19	20	22
S_3	17	16	13	14 (14)	10 (24)	18 (1)	39
S_4	19	18	26	21	12	13 (14)	14
b_j	10	11	13	20	24	15	

This completes one iteration and we get an improved basic feasible solution.

The next two iterations are shown in Table-4, Table-5 and Table-6

Table-4:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	u_i
O_1	10 (10)	12 (2) $-\theta$	13 (−7)	8 (6) $+\theta$	14 (−10)	19 (−7)	18	$0 = u_1$
O_2	15 (1)	18 (9)	12 (13)	16 (−2)	19 (−9)	20 (−2)	22	6
O_3	17 (−1)	16 θ (+2)	13 (−1)	14 (14) $-\theta$	10 (24)	18 (1)	39	6
O_4	19 (−8)	18 (−5)	20 (−13)	21 (−12)	12 (−7)	13 (14)	14	1
$b_j \to$	10	11	13	20	24	15		
$v_j \to$	10	12	6	8	4	12		

$u_2 + v_2 = 18$, $u_2 = 18 - 12 = 6$

$u_3 + v_4 = 14$, $u_3 = 14 - 8 = 6$

$u_3 + v_5 = 10$, $v_5 = 4$

$u_3 + v_6 = 18$, $v_6 = 12$

Table-5:

$\theta = 2$

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	u_i
S_1	10 (10) $-\theta$	12 (−2)	13 (−9)	8 (8) $+\theta$	14 (−10)	19 (−7)	18	−6
S_2	15 θ (3)	18 (9) $-\theta$	12 (13)	16 (0)	19 (−7)	20 (0)	12	2
S_3	17 (−1)	16 (2) $+\theta$	13 (−3)	14 (12) $-\theta$	10 (24)	18 (1)	39	0
S_4	19 (−8)	18 (−7)	20 (−15)	21 (−12)	12 (−7)	13 (14)	14	−5
$b_j \to$	10	11	13	20	24	15		
$v_j \to$	16	16	10	14	10	18		

Let $u_3 = 0$

$v_2 = 16$, $v_4 = 14$, $v_5 = 10$, $v_6 = 18$

$u_2 + v_2 = 18$, $u_2 = 2$

$u_2 + v_3 = 12$, $v_3 = 10$

$u_1 + v_4 = 8$, $u_1 = -6$

$u_4 + v_6 = 13$, $u_4 = -5$

$u_1 + v_1 = 10$, $v_1 = 16$

Table-6

Let θ = 9

						$a_i \downarrow$	$u_i \downarrow$
10 (1)	12 (−)	13 (−)	8 (17)	14 (−)	19 (−)	18	−6
15 (9)	18 (−)	12 (13)	16 (−)	19 (−)	20 (−)	22	−1
17 (−)	16 (11)	13 (−)	14 (3)	10 (24)	18 (1)	39	0
19 (−)	18 (−)	20 (−)	21 (−)	12 (−)	13 (14)	14	−5
$b_j \rightarrow$ 10	11	13	20	24	15		
$v_j \rightarrow$ 16	16	13	14	10	18		

In Table-6.

we have $\Delta_{ij} \le 0$ for all non-basic cells.

$\therefore$ The current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$x_{11} = 1$, $x_{14} = 17$, $x_{21} = 9$, $x_{23} = 13$,

$x_{32} = 11$, $x_{34} = 3$, $x_{35} = 24$, $x_{36} = 1$

and $x_{46} = 14$.

and the minimum cost of transportation is given by

$(10\times1) + (8\times17) + (15\times9) + (12\times13) + (16\times11) + (14\times3)$
$+ (10\times24) + (8\times1) + (14\times13) = 1095.$

→ Solve the following transportation problem:

Destination

Source	A	B	C	D	Availability (a_i)
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Requirement (b_j)	6	10	12	15	43

by finding the initial solution by Vogel's Approximation Method.

Sol^n: Since $\sum a_i = \sum b_j = 43$.

$\therefore$ The problem is balanced.

Find the initial basic feasible solution:

Using Vogel's Approximation method, the initial basic feasible solution is:

The differences between the smallest and next to the smallest costs in each row and each column are first computed and displayed inside paranthesis against the respective rows and columns.

Table-1

21	16	25	13 (11)	~~11~~	(3)
17	18	14	23	13	(3)
32	27	18	41	19	(9)
6	10	12	~~15~~ 4		
(4)	(2)	(4)	(10)		

The largest of these differences is (10) which is associated with the fourth column.

Since $c_{14} = 13$, is the minimum cost, we allocate $x_{14} = \min(11, 15) = 11$. in the cell $(1,4)$.

This exhausts the availability of the first row. and therefore we cross it.

Table -2

17	18	14	23 (4)	13 9	(3)
32	27	18	41	19	(9)
32	10	12	4		
(15)	(9)	(6)	(18)		

The row and column differences are now computed for reduced table -2 and displayed with in brackets.

The largest of these is (18) which is against the fourth column.

Since $c_{14} = 23$ is the minimum cost, we allocate $x_{14} = \min(13, 4) = 4$. in the cell $(1,4)$.

This exhausts the availability of fourth column. and therefore cross it.

Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are as shown below.

17 (6)	18	14	9 3	(3)
32	27	18	19	(9)
	10	12		
(15)	(9)	(4)		

18 (3)	14	3	(4)
27	18	19	(9)
7 10	12		
(9)	(4)		

27 (7)	18 (12)	19
7	12	

∴ Finally the initial basic feasible solution is as shown below.

				$a_i \downarrow$
21	16	25	13 (11)	11
17 (6)	18 (3)	14	23 (4)	13
32	27 (7)	18 (12)	41	19
$b_j \rightarrow$ 6	10	12	15	

The number of allocations $= m+n-1$
$= 3+4-1$
$= 6$ (basic variables)

<u>Now, finding the values of u_i and v_j:</u>

As the maximum number of basic cells exist in the 2nd row.

∴ Let $u_2 = 0$.

we have $u_2 + v_1 = 17 \Rightarrow v_1 = 17$

$u_2 + v_2 = 18 \Rightarrow v_2 = 18$

$u_2 + v_4 = 23 \Rightarrow v_4 = 23$

$u_1 + v_4 = 13 \Rightarrow u_1 = -10$

$u_3 + v_2 = 27 \Rightarrow u_3 = 9$

$u_3 + v_3 = 18 \Rightarrow v_3 = 9$

Table-3

				u_i
21 (−)	16 (−)	25 (−)	13 (11)	−10
17 (6)	18 (3)	14 (−)	23 (4)	0
32 (−)	27 (7)	18 (12)	41 (−)	9
$v_j \rightarrow$ 17	18	9	23	

The net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$, for all unoccupied cells are.

$$\Delta_{11} = u_1 + v_1 - c_{11} = 17 - 10 - 21 = -14.$$

$$\Delta_{12} = -8, \Delta_{13} = -26, \Delta_{23} = -5, \Delta_{34} = -9.$$

The values of Δ_{ij}'s are recorded in the right bottom of the each cell. as shown in Table - 3

Since all the net evaluations are negative. ∴ the current basic feasible solution is optimal.

Hence the optimal allocation is given by

$x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$ and $x_{33} = 12$

∴ The optimal (minimum) transportation cost $= 11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 23 + 7 \times 27 + 12 \times 18 =$ Rs. 796.

→ A company has three cement factories located in cities 1, 2, 3 which supply cement to four projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truck loads of cement daily respectively and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truck loads. The transportation costs per truck load of cement (in hundres of rupees) from each plant to each project site are as follows:

Project sites

Factories	1	2	3	4
1	2	3	11	7
2	1	0	6	1
3	5	8	15	9

Determine the optimal distribution for the Company so as to minimize the total transportation cost.

Sol:

Express the supply from the factories, demands at sites and the unit shipping cost in the form of the following transportation table.

Table-1 project sites

Factories	1	2	3	4	Supply (a_i) ↓
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand (b_j) →	7	5	3	2	17

Here the supply being equal to the demand

i.e $\Sigma a_i = \Sigma b_j = 17$.

∴ The problem is balanced.

Find the initial basic feasible solution:

Using the VAM, the initial basic feasible solution is shown in Table-2.

Table-2

2 (1)	3 (5)	11	7	6
1	0	6	1 (1)	1
5 (6)	8	15 (3)	9 (1)	10
7	5	3	2	

The no. of allocations $= m+n-1 = 3+4-1 = 6$ (basic variables)

i.e, $x_{11}=1$, $x_{12}=5$, $x_{24}=1$, $x_{31}=6$, $x_{33}=3$ and $x_{34}=1$

$\therefore$ The transportation cost according to this route is given by, Rs $(1\times 2+5\times 3+1\times 1+6\times 5+3\times 15+1\times 9)\times 100 =$ Rs 10,200.

Now finding the values of u_i & v_j:

As the maximum no. of allocations (basic cells) exist in the 3rd row.

$\therefore$ Let $u_3 = 0$

we have $u_3+v_1 = 5 \Rightarrow v_1 = 5$

$u_3+v_3 = 15 \Rightarrow v_3 = 15$

$u_3+v_4 = 9 \Rightarrow v_4 = 9$

$u_1+v_1 = 2 \Rightarrow u_1 = -3$

$u_1+v_2 = 3 \Rightarrow v_2 = 6$

$u_2+v_4 = 1 \Rightarrow u_2 = -8$

Also the net evaluations $\Delta_{ij} = u_i+v_j-c_{ij}$ for all unoccupied cells are exhibited in Table-3.

				$u_i\downarrow$
2 (1)	3 (5)	11 (1)	7 (-1)	-3
1 (-4)	0 (-2)	6 (1)	1 (1)	-8
5 (6)	8 (-2)	15 (3)	9 (1)	0
$v_j\rightarrow$ 5	6	15	9	

- Since the net evaluations in two cells are +ve, a better solution can be found. (i.e., the current basic feasible solution is not optimal).

- Choose the unoccupied cell with the maximum Δ_{ij}. In case of a tie, select the one with lower original cost.

In Table-3, cells (1,3) & (2,3) each have $\Delta_{ij} = 1$ and out of these, cell (2,3) has lower original cost 6.

$\therefore$ cell (2,3) enters the basis.

- We allocate an unknown quantity θ, to this cell (2,3) and identify a loop involving basic cells around this entering cell.

- Add and subtract θ, alternately to and from the transition cells of the loop subject to the rim requirements. as shown in the table-4

Table-4

2 (1)	3 (5)	11	7
1	0	6 θ	1 (1)$-\theta$
5 (6)	8	15 (3)$-\theta$	9 (1)$+\theta$

Now, assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain ≥ 0.

If we put $\theta = 1$ in Table-4

x_{24} becomes zero.

i.e, $x_{24} = 0$ (non-basic).

i.e, the cell (2,4) leaves the basis

∴ The new basic feasible solution is as shown in Table – 5.

2 (1)	3 (5)	11	7
1	0	6 (1)	1
5 (6)	8	15 (2)	9 (2)

∴ $x_{11} = 1$, $x_{12} = 5$, $x_{23} = 1$, $x_{31} = 6$, $x_{33} = 2$ & $x_{34} = 2$

∴ The total transportation cost of this revised solution

$= Rs.(1\times2 + 5\times3 + 1\times6 + 6\times5 + 2\times15 + 2\times9)\times100$

$= Rs\ \underline{10,100}$.

As the no. of allocations in table – 5

$= m+n-1 = 3+4-1 = 6$.

we compute the net evaluations which are shown in Table – 6.

Table – 6

				$u_i \downarrow$
2 (1)	3 (5)	11 (1)	7 (−1)	−3
1 (−5)	0 (−3)	6 (1)	1 (−1)	−9
5 (6)	8 (−2)	15 (2)	9 (2)	0
$v_j \rightarrow$ 5	6	15	9	

Since the cell (1,3) has a +ve value,

∴ the current basic feasible solution is not optimal.

∴ The cell (1, 3) enters the basis.

∴ we allocate 'θ' in the cell (1, 3) and draw a closed path beginning and ending at θ-cell (i.e., (1, 3) cell).

Add and subtract θ, alternately to and from the transition cells of the loop subject to the rim requirements, as shown in table – 7

2 (1)−θ	3 (5)	11 θ	7
1	0	6 (1)	1
5 (6)+θ	8	15 (2)−θ	9 (2)

Taking θ = 1; x_{11} becomes zero (i.e., $x_{11}=0$)

∴ the cell (1,1) leaves the basis.

∴ The new basic feasible solution is shown in Table – 8.

Table – 8

2	3 (5)	11 (1)	7
1	0	6 (1)	1
5 (7)	8	15 (1)	9 (2)

The no. of allocations = m+n−1 = 6 (as shown in table – 8)

Now compute the net evaluations which are shown in Table – 9.

Table – 9

				u_i
2 (−1)	3 (5)	11 (1)	7 (−2)	−4
1 (−5)	0 (−2)	6 (1)	1 (−1)	−9
5 (7)	8 (−1)	15 (1)	9 (2)	0
v_j → 5	7	15	9	

Since all the net evaluations are ≤ 0.

∴ The current basic feasible solution is optimal.

∴ The optimal (minimum) transportation cost

= Rs. (5×3 + 1×11 + 1×6 + 7×5 + 1×15 + 2×9) ×100

= Rs 10,000

→ Solve the following transportation problem:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i ↓
S_1	9	12	9	6	9	10	5
S_2	7	3	7	7	5	5	6
S_3	6	5	9	11	3	11	2
S_4	6	8	11	2	2	10	9
b_j →	4	4	6	2	4	2	

Sol^n:

The total supply and total demand being equal

i.e. $\sum a_i = \sum b_j = 22$.

∴ The transportation problem is balanced.

Using the vogel's approximation method, the initial basic feasible solution is as shown in Table-1.

Table-1

9	12	9 (5)	6	9	10	5	(3)	(3)	(0)	(0)	(0)	(0)
7	3 (4)	7	7	5	5 (2)	~~6~~ 4	(2)	(2)	(2)	(4)	6	
6 (1)	5	9 (1)	11	3	11	~~2~~ 1	(2)	(2)	(2)	(1)	(3)	(3)
6 (3)	8	11	2 (2)	2 (4)	10	~~9~~ ~~7~~ 3	(0)	(0)	(4)	(2)	(5)	
~~4~~ 1	4	~~6~~ 1	2	4	2							
(0)	(2)	(2)	(4)	(1)	(5)							
(0)	(2)	(2)	(4)	(1)								
(0)	(2)	(2)		(1)								
(0)	(2)	(2)										
(0)		(2)										
(3)		(0)										

number of allocations (basic cells) which is less than $m+n-1 = 4+6-1=9$.

$\therefore$ The solution is feasible, but not basic feasible.

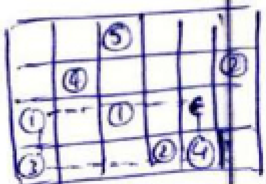
i.e, the solution is degenerate.

- In order to complete the basic and thereby remove degeneracy, we require only one more non-negative basic variable.

- To break degeneracy, we allocate a very small +ve quantity $\epsilon(\approx 0)$ (or allocate ⓪) to occupied cell with minimum cost.

Minimum entry in unoccupied position is in cell (3,5)

If we allocate small quantity 'ϵ' to cell (3,5) then $m+n-1$ allocations (basic cells) will not be independent. because allocation at cell (3,5) forms a closed loop.



So small allocation 'ϵ' cannot be made at cell (3,5).

Next, higher cost in unoccupied cells is 5 in cell (3,2) and (2,5).

Let us make small positive quantity 'ϵ' allocation in cell (2,5), say.

$\therefore$ we have $m+n-1 (=9)$ allocations which are independent because no closed loop is formed.

Table-2

9	12	9 (5)	6	9	10	5
7	3 (4)	7	7	5 (ε)	5 (2)	6+ε =6
6 (1)	5	9 (1)	11	3	11	2
6 (3)	8	11	2 (2)	2 (4)	10	9
4	4	6	2	4+ε =4	2	

Now find the values of u_i & v_j.

As the maximum no. of allocations (basic cells) exist in the 2^{nd} and 4^{th} rows.

putting either $u_2=0$ or $u_4=0$.

Let $u_2=0$

we have
$$u_2+v_2=3 \Rightarrow v_2=3$$
$$u_2+v_5=5 \Rightarrow v_5=5$$
$$u_2+v_6=5 \Rightarrow v_6=5$$
$$u_4+v_5=2 \Rightarrow u_4=-3$$
$$u_4+v_1=6 \Rightarrow v_1=9$$
$$u_4+v_4=2 \Rightarrow v_4=5$$
$$u_3+v_1=6 \Rightarrow u_3=-3$$
$$u_3+v_3=9 \Rightarrow v_3=12$$
$$u_1+v_3=9 \Rightarrow u_1=-3$$

and also the net evaluations $\Delta_{ij}=u_i+v_j-c_{ij}$ for all unoccupied cells are exhibited in Table-3.

Table-3

						$u_i \downarrow$
9 (-)	12 (-)	9 (5)	6 (-)	9 (-)	10 (-)	-3
7 (2)	3 (4)	7 (5)	7 (-)	5 (ε)	5 (2)	0
6 (1)	5 (-)	9 (1)	11 (-)	3 (-)	11	-2
6 (3)	8 (-)	11 (-)	2 (2)	2 (4)	10 (-)	-3
$v_j \rightarrow$ 9	3	12	5	5	5	

Since the net evaluations in two cells are +ve,

$\therefore$ the current basic feasible solution is not optimal.

Choose unoccupied cell with the maximum Δ_{ij}.

Clearly $\Delta_{23} = 5$ is the most +ve.

$\therefore$ the cell (2,3) enters the basis.

- we allocate an unknown quantity θ, to this cell (2,3) and identify a loop involving basic cells around this entering cell.
- Add and subtract θ, alternately to and from the transition cells of the loop subject to the rim requirements as shown in the table-4

Table - 4

9	12	9 (5)	6	9	10
7	3 (4)	7 θ	7	5 (ε)$-\theta$	5 (2)
6 (1)$-\theta$	5	9 (1)$+\theta$	11	3	11
6 (3)$+\theta$	8	11	2 (2)$-\theta$	2 (4)$+\theta$	10

Now, assign maximum value θ so that one basic variable becomes zero and the other basic variables ≥ 0.

Taking $\theta = \epsilon (\to 0)$ in table - 4

x_{23} becomes zero.,

ie, $x_{23} = 0$.

∴ The cell (2,3) leaves the basis.

∴ The new basic feasible solution is shown in Table – 5.

Table – 5

9	12	9 (5)	6	9	10
7	3 (4)	7 (ε)	7	5	5 (2)
6 (1)	5	9 (1)	11	3	11
6 (3)	8	11	2 (2)	2 (4)	10

The no. of allocations = m+n−1 = 9.

Now compute the net evaluations which are shown in Table – 6.

						u_i ↓
9 (−)	12 (−)	9 (5)	6 (−)	9 (−)	10 (−)	2
7 (−)	3 (4)	7 (ε)	7 (−)	5 (−)	5 (2)	0
6 (1)	5 (0)	9 (1)	11 (−)	3 (−)	11 (−)	2
6 (3)	8 (−)	11 (−)	2 (2)	2 (4)	10 (−)	2
v_j → 4	3	7	0	0	5	

Since all the net evaluations ≤ 0.

∴ The current basic feasible solution is optimal.

Hence, the optimum solution is

$x_{13}=5$, $x_{22}=4$, $x_{26}=2$, $x_{31}=1$, $x_{33}=1$, $x_{41}=3$,

$x_{44}=2$, $x_{45}=4$. $x_{23}=\epsilon=0$.

∴ The minimum transportation cost is

$= (5\times 9)+(4\times 3)+[7\times(\epsilon=0)]+(2\times 5)+(1\times 6)+(1\times 9)+$
$(3\times 6)+(2\times 2)+(4\times 2) = 112.$

Unbalanced Transportation problems:

Till now we have dealt with the transportation problem, assuming that the total demand of all the destinations is equal to the total availability at all the sources. i.e, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

But a situation may arise when the total available supply is not equal to the total requirement. Such type of T.P's are called unbalanced transportation problem.

Case I when $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

Here the total availability at all sources is less than the total demand of all the destinations. In such cases we do the following.

(i) Create an artificial sources S_{m+1} with availability $a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$

This gives us a new problem with $(m+1)$ sources and n destinations, and of course is balanced.

(ii) Set c_{ij}'s in cells corresponding to this artificial source S_{m+1}, equal to zero.

i.e, $c_{m+1,j} = 0$ for $j = 1, 2, \ldots n$.

(iii) Solve the balanced TP thus obtained by u-v method. optimal solution of this balanced TP with variables $x_{m+1\,j}$; $j = 1, 2, \ldots n$ corresponding to the artificial sources S_{m+1} deleted give an optimal solution for the given unbalanced TP.

case(2): when $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

Here the total availability at all sources is more than the total demand of all the destinations.

In such cases we do the following.

(i) Create an artificial destination D_{n+1} with demand $b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$.

This gives us a new problem with m sources and n+1 destinations, and of course it is balanced.

(ii) Set c_{ij}'s for all cells corresponding to the artificial destination D_{n+1}, equal to zero.

i.e, $c_{i,n+1} = 0$ for $i = 1, 2, \dots, m$.

(iii) Solve the above balanced TP by u-v method. Optimal solution of this problem, with variables $x_{i,n+1}$; $i = 1, 2, \dots m$; corresponding to the artificial destination D_{n+1} deleted give an optimal solution for the given unbalanced TP.

→ Solve the TP.

	D_1	D_2	D_3	D_4	a_i
S_1	25	17	25	14	30
S_2	15	10	18	24	50
S_3	16	20	8	13	60
	30	30	50	50	

Sol^n: In the given TP,

Total demand = 160

Total availability = 140.

Since the total demand is more than total availability.

∴ The given problem is unbalanced.

∴ we introduce an artificial source S_4 with availability $a_4 = 160 - 140 = 20$.

costs C_{ij}'s for all cells corresponding to artificial source S_4 are taken as zeros.

By this we get the balanced TP.

	D_1	D_2	D_3	D_4	$a_i \downarrow$
S_1	25	17	25	14	30
S_2	15	10	18	24	50
S_3	16	20	8	13	60
S_4	0	0	0	0	20
$b_j \rightarrow$	30	30	50	50	

Solving this balanced transportation problem by the u-v method,

The optimal solution is given as

	D_1	D_2	D_3	D_4	$a_i \downarrow$	$u_i \downarrow$
S_1	25 (-11)	17 (-8)	25 (-16)	14 (30)	30	14
S_2	15 (20)	10 (30)	18 (-8)	24 (-9)	50	15
S_3	16 (-3)	20 (-12)	8 (50)	13 (10)	60	13
S_4	0 (10)	0 (-5)	0 (-5)	0 (10)	20	0
$b_j \rightarrow$	30	30	50	50		
$v_j \rightarrow$	0	-5	-5	0		

from this, the optimal solution of the original unbalanced TP is given by:

$x_{14} = 30$, $x_{21} = 20$, $x_{22} = 30$, $x_{33} = 50$, $x_{34} = 10$.

and the minimum cost of transportation is

$$= (14\times30) + (15\times20) + (10\times30) + (8\times50) + (13\times10)$$

$$= \underline{1550}.$$

→ Solve the TP:

	D_1	D_2	D_3	a_i ↓
S_1	4	3	2	10
S_2	1	5	0	13
S_3	3	8	6	12
b_j →	8	5	4	

The Assignment problem

An assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. The objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

for example:

A factory manager may wish to assign three different jobs to three machines in such a way that the total cost is minimized.

There are $3! = 6$ ways of assigning 3 jobs to 3 machines. However, a problem of assigning 10 jobs to 10 machines requires $10!$ assignments to be examined, which clearly is not a simple task. Hence, the need to evolve efficient method to solve an assignment problem.

However, a much more efficient method of solving such problems is available. This is the Hungarian method. ~~which we now~~

Formulation of an Assignment problem:

Let us consider the case of a factory which has 3 jobs to be done on the 3 available machines. Each machine is capable of doing any of the three jobs. For each job the machining-cost depends on the machine to which it is assigned.

Costs incurred by doing various jobs on different machines are given below:

	M-I	M-II	M-III
J-I	3	4	2
J-II	1	3	7
J-III	2	5	4

The problem of assigning jobs to machines, one to each, so as to minimize the total cost of doing all the jobs, is an assignment problem.

→ Each job-machine combination which associates all jobs to machines on one-to-one basis is called assignment.

→ Every assignment corresponds to a total-cost.

→ The assignment problem, proposes to determine that assignment which corresponds to minimum cost. This will be called an optimal assignment.

→ In the example above,

let us write all the possible assignments.

Job-I can be assigned to any of the three available machines.

So there are 3 ways in which it can be done.

Now, Job-II cannot be assigned to the machine to which Job-I has already been assigned.

So, Job-II can associated to any of the two remaining machines, which can be done in two ways.

Ultimately Job-III has only one machine left for it and so has only one way.

Combining the above, there are $3 \times 2 \times 1 = 6$ ways in which all the jobs can be assigned to various machines, one to each.

i.e, there are $3!$ number of possible assignments.

The following table enumerates all these possible assignments and also mention the 'total-cost' corresponding to each one of these.

No.	Assignment	Total cost
1.	J-I — M-I J-II — M-II J-III — M-III	$3+3+4=10$
2.	J-I — M-I J-II — M-III J-III — M-II	$3+7+5=15$
3.	J-I — M-II J-II — M-III J-III — M-I	$4+7+2=13$
4.	J-I — M-II J-II — M-I J-III — M-III	$4+1+4=9$
5.	J-I — M-III J-II — M-I J-III — M-II	$2+1+5=8$
6.	J-I — M-III J-II — M-II J-III — M-I	$2+3+2=7$.

Out of all these, the minimum total-cost i.e 7 corresponds to the assignment

Job I —— Machine-III

Job II —— Machine-II

Job III —— Machine-I

which is called the optimal assignment.

Let us now generalise it and formulate it for n jobs

There be 'n' jobs which are to be processed on 'n' machines on one job-one machine basis:

Let $J_1, J_2, \ldots J_n$ be the 'n' jobs and let $M_1, M_2, \ldots M_n$ be the 'n' machines.

Let c_{ij} be the cost of processing i^{th} job J_i on the machine M_j.

Let us formulate the problem of assigning jobs to machines so as to minimize the overall cost.

Let us define variable x_{ij} as follows,

$$x_{ij} = \begin{cases} 0 & \text{if } i^{th} \text{ job is not assigned to } j^{th} \text{ machine} \\ 1 & \text{if } i^{th} \text{ job is assigned to } j^{th} \text{ machine.} \end{cases}$$

No job remains unprocessed and no machine remains idle.

Note that the number of jobs is equal to the number of machines.

The hypothesis of one job one machine implies

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots n)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots n)$$

In each of these summations, only one term on the LHS has variable x_{ij} equal to one and the rest are zeros.

The assignment problem is mathematically stated as:

$$\left.\begin{aligned} &\text{Minimize } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij} \\ &\text{subject to } \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots n) \\ &\qquad\qquad \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots n). \end{aligned}\right\} \text{—— (1)}$$

with $x_{ij} = 0$ or 1 ; $i = 1, 2, \ldots\ldots n$
$j = 1, 2, \ldots\ldots n$.

An assignment problem is known from its cost-matrix $[c_{ij}]$, which is given as

$$\begin{bmatrix} c_{11} & c_{12} & \cdots\cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots\cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & & & & & \\ c_{i1} & c_{i2} & \cdots\cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & & \\ c_{n1} & c_{n2} & \cdots\cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix}$$

If each row refers to a job and each column refers to a machine, then c_{ij} is the cost of processing i^{th} job on j^{th} machine.

Note: An Assignment problem (1) could be solved by simplex method. It also happens to be an $n \times n$ transportation problem with each $a_i = b_j = 1$. However, as an assignment problem is highly degenerate, it will be frustrating to attempt to solve it by simplex method or transportation method.

In fact a very convenient iterative procedure is available for solving an Assignment problem. It is called the Hungarian Method. Before we discuss this method, let us take up the following results.

→ The optimal solution of an assignment problem remains the same, if constant is added or subtracted from any row or column of the cost matrix.

for example:

Let us take the assignment problem.

$$\begin{array}{c|cc} & M_1 & M_2 \\ \hline J_1 & 5 & 3 \\ J_2 & 2 & 6 \end{array}$$

In order to ensure that no element of the cost-matrix becomes negative, subtract the minimum element of each row from all the elements of that row.

we get,

$$\begin{array}{c|cc} & M_1 & M_2 \\ \hline J_1 & 2 & 0 \\ J_2 & 0 & 4 \end{array}$$

In the above reduced cost-matrix, the optimal assignment yielding total cost zero is J_1M_2, J_2M_1.
So for the original problem the optimal assignment is J_1M_2, J_2M_1, yielding optimal value $3+2 = \underline{5}$.

→ Let us consider

$$\begin{array}{c|cc} & M_1 & M_2 \\ \hline J_1 & 5 & 3 \\ J_2 & 6 & 2 \end{array}$$

Subtracting the minimum element of each row from all elements of that row, the reduced matrix obtained is,

$$\begin{array}{c|cc} & M_1 & M_2 \\ \hline J_1 & 2 & 0 \\ J_2 & 4 & 0 \end{array}$$

Now, subtracting the minimum element of each column from all elements of that column, the cost-matrix is further reduced to

	M_1	M_2
J_1	0	0
J_2	2	0

Keeping in mind the one job-one machine basis, the optimal assignment yielding total cost zero is J_1M_1, J_2M_2. from the original cost-matrix of this problem the optimal assignment J_1M_1, J_2M_2 corresponds to total cost $5+2=7$.

→ Sometimes, even after this, optimal assignments in the reduced cost-matrix yielding zero total-cost cannot be located.

→ for example:

Consider the problem

	M_1	M_2	M_3
J_1	2	4	2
J_2	5	2	3
J_3	4	2	5

After subtracting the minimum of each row (column) from all elements of that row (column) the reduced matrix so obtained is

	M_1	M_2	M_3
J_1	0	2	0
J_2	3	0	1
J_3	2	0	3

It can be seen that on one job-one machine basis, an optimal assignment yielding total-cost zero cannot be obtained from this reduced matrix.

such a situation can be systematically identified by observing that all the zeros in the above reduced matrix can be covered by a minimum of two lines only (shown as dotted lines below).

$$\begin{matrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 3 \end{matrix}$$

This can be resolved by creating new zeros from amongst the elements uncovered by these two dotted lines. For this minimum of the uncovered elements i.e, 1 is subtracted from all uncovered elements; added to the elements at intersection of the dotted lines, leaving other covered elements unchanged. The cost-matrix further reduces to,

	M_1	M_2	M_3
J_1	0	3	0
J_2	2	0	0
J_3	1	0	2

Now, optimal assignment yielding zero total-cost can be made as J_1M_1, J_2M_3, J_3M_2 which corresponds to the total cost $2+3+2=7$.

→ Now, it can be observed that we can make optimal assignments yielding zero total cost in the reduced cost matrix, only when the minimum number of dotted horizontal and vertical lines needed to cover all the zeros is equal to the order of the given assignment problem. (3×3 matrix)

In the first two problems, each of order 2, we could make optimal assignments only when the minimum number of lines required to cover all the zeros was two.

In the third problem of order 3, the optimal assignment could be made only when the minimum number of lines required to cover all zeros in the reduced matrix is equal to three.

All the above observations contribute to the following steps of the Hungarian-Method for solving an $n\times n$ assignment problem.

*Hungarian Method:

step 1:

(i) Subtract the minimum element of each row from all elements of that row

(ii) subtract the minimum element of each column from all elements of that column.

The reduced matrix thus obtained, contains atleast one zero in each row and each column.

step 2:

Cover all the zeros in the reduced cost-matrix by minimum number of horizontal and vertical lines. Let the least number of such lines needed to cover all the zeros be r.

If $r=n$, an optimal assignment can be made at this stage. In this case go to step 4.

If $r<n$, an optimal assignment cannot be made at this stage. In this case go to step 3.

Step(3):

Here, the least number of lines needed to cover all the zeros is less than the order of the assignment problem.

pick the minimum element not covered by these r covering-lines and,

(i) Subtract it from all uncovered elements

(ii) add to all elements at intersection of two covering lines, and

(iii) leave all other covered elements unchanged.

Thus we get a new reduced matrix. Go to step(2)

Step(4):

Here the minimum number of lines needed to cover all the zeros is exactly equal to the order of the assignment.

An optimal assignment shall be made now.

(i) Examine the rows successively until a row with exactly one zero is found. Encircle this zero and cross all other zeros in its column

(ii) Similarly, examine the columns successively until a column with exactly one zero is found. Encircle this zero and cross all other zeros in its row.

Repeating the above steps either of the following situations encountered.

a) Each row and each column has an encircled zero. In this case an optimal assignment has been made and the process terminates.

b) There lie more than one zero in some rows and columns which are not encircled.

In such a case encircle any one of the zeros which is not encircled arbitrarily and cross all other zeros in its row and column, both.

Continuing in this way, we shall have exactly one encircled zero in each row and each column.

Assignments are made corresponding to each encircled zero.

Step(5):
For obtaining the minimum cost, refer to the original cost-matrix of the given problem. optimum cost is obtained by adding costs C_{ij}'s at all the encircled-zero positions.

Note: If the cost matrix is not square i.e, $m \neq n$, make it square by adding suitable number of dummy rows (or columns) with 0 cost elements problem.

→ Solve the cost-minimizing assignment problem.

	I	II	III	IV
A	10	12	9	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Step 1
Subtracting the minimum element of each row from all elements of that row.

we get

	I	II	III	IV
A	1	3	0	2
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

(ii) Subtracting the minimum elements of each column from elements of that column.

we get.

	I	II	III	IV
A	1	0	0	2
B	0	2	2	3
C	1	0	2	0
D	0	4	3	1

Step(2)

Cover all the zeros by minimum number of horizontal and vertical lines.

A systematic approach for this is to look for a row or column containing the maximum number of zeros.

See that we can cover all the zeros by 3 lines only.

So, $r = 3 < 4 = n$.

So go to step (3)

1	0	0	2
0	2	2	3
1	0	2	0
0	4	3	1

Step(3):

1 is the least uncovered element.

(i) Subtract 1 from all the uncovered elements.

(ii) add 1 to elements at intersection of the covering lines namely 1 at position (1,1) and 1 at position (3,1).

(iii) leave other covered elements. unchanged.

The reduced cost-matrix so obtained is,

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 2 & 0 & 0 & 2 \\ B & 0 & 1 & 1 & 2 \\ C & 2 & 0 & 2 & 0 \\ D & 0 & 3 & 2 & 0 \\ \hline \end{array}$$

Again, cover the zeros by minimum number of horizontal and vertical lines.

we require exactly 4 lines to cover all the zeros.

As $r = 4 = n$; optimal assignment can be made at this stage;

So go to to step(4).

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 2 & 0 & 0 & 2 \\ B & 0 & 1 & 1 & 2 \\ C & 2 & 0 & 2 & 0 \\ D & 0 & 3 & 2 & 0 \\ \hline \end{array}$$

Step(4):

For making assignments, proceed as follows.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 2 & \cancel{0} & 0 & 2 \\ B & 0 & 1 & 1 & 2 \\ C & 2 & 0 & 2 & \cancel{0} \\ D & \cancel{0} & 3 & 2 & 0 \\ \hline \end{array}$$

(i) 2^{nd} row has only ~~one~~ zero in position (2,1). So encircle this zero and cross all other zeros in its column (i.e., 1^{st} column).

(ii) Now, the 4^{th} row has only one zero in position (4,4), so encircle it and cross all ~~other~~ other zeros in its column (i.e., 4^{th} column).

(iii) 3^{rd} column contains only one zero in position (1,3), so, encircle it and cross all other zero in its row. (i.e, 1^{st} row)

(iv) There is only one zero in 3^{rd} row, so encircle it.

Now, since each row and each column has a single encircled zero.

i.e, each row and each column has one and only one assignment, so an optimal assignment is reached.

$\therefore$ The optimal assignment is

$A \to III$, $B \to I$, $C \to II$, $D \to IV$.

Step5: The minimum assignment cost is

$$C_{13} + C_{21} + C_{32} + C_{44} = 9+5+14+9$$
$$= 37.$$

→ Solve the cost-minimizing assignment problem with the cost-matrix.

machine

Job	I	II	III	IV	V
A	11	10	18	5	9
B	14	13	12	19	6
C	5	3	4	2	4
D	15	18	17	9	12
E	10	11	19	6	14.

EXAMPLE 2

Solve the cost-minimizing assignment problem whose cost matrix is given below,

	M_1	M_2	M_3	M_4
J_1	2	5	7	9
J_2	4	9	10	1
J_3	7	3	5	8
J_4	8	2	4	9

SOLUTION : Step 1

i) Subtracting the minimum element of each row from all elements of that row, the reduced cost-matrix is,

	M_1	M_2	M_3	M_4
J_1	0	3	5	7
J_2	3	8	9	0
J_3	4	0	2	5
J_4	6	0	2	7

ii) subtracting the minimum element of each column from all elements of that column, we get

	M_1	M_2	M_3	M_4
J_1	0	3	3	7
J_2	3	8	7	0
J_3	4	0	0	5
J_4	6	0	0	7

Step 2

Cover all the zeros by least number of horizontal and vertical lines. Exactly 4 lines are required to cover all the zeros. So, $r = 4$.

	M_1	M_2	M_3	M_4
J_1	0	3	3	7
J_2	3	8	7	0
J_3	4	0	0	5
J_4	6	0	0	7

As r = 4 = n, we can straightway go to step 4, and make the optimal assignment.

Step 4

i) There is only one zero in 1st row in position (1, 1), so encircle this zero, and cross other zeros (if any) in its column i.e. 1st column.

ii) There is only one zero in 2nd row in position (2, 4), so encircle this zero and cross other zeros (if any) in is column i.e. 4th column.

	M_1	M_2	M_3	M_4
J_1	0	3	3	7
J_2	3	8	7	0
J_3	4	0	$\cancel{0}$	5
J_4	6	$\cancel{0}$	0	7

iii) Now, observe that 3rd and 4th rows as well as 2nd and 3rd columns contain two zeros each. To break this, and make an assignment, we pick any zero arbitrarily. Say, we pick zero in position (3, 2) and encircle it. Now, cross all zeros in its row i.e. 3rd row as well as its column i.e. 2nd column.

iv) There is only one zero left in position (4, 3), Encircle it to get the optimal assignment as $J_1 M_1$, $J_2 M_4$, $J_3 M_2$, $J_4 M_3$.

v) $J_1 M_1$, $J_2 M_2$, $J_3 M_3$, $J_4 M_2$ is an alternative optional assignment.

Step 5

For determining minimum total cost, refer to the original cost-matrix of this problem and add the costs corresponding to $J_1 M_1$, $J_2 M_4$, $J_3 M_3$, $J_4 M_3$. This gives the minimum assignment cost as.

$$2 + 1 + 3 + 4 = 10.$$

Note that any arbitrary choice in step 4 (iii), of the zero to be encircled, would yield the same minimum total-cost.

EXERCISE 2: Solve the cost-minimizing assignment problem.

	I	II	III	IV	V	VI
A	7	8	3	7	6	2
B	3	7	9	3	1	6
C	5	3	7	5	6	3
D	8	4	8	7	2	2
E	6	7	8	6	9	4
F	5	7	7	5	5	7

EXAMPLE 3

The owner of a small machine shop has 4 mechanists available to do 4 jobs. Jobs are offered with expected profits for each mechanist as follows:

Machanists

		I	II	III	IV
	A	6	7	5	2
	B	4	3	2	8
Jobs	C	2	4	9	4
	D	5	3	1	7

Find by using the assignment method, the assignment of mechanists to jobs that will result in a maximum profit.

SOLUTION

From linear programming we know that a maximization problem can be converted into a minimization problem by replacing the costs with their negatives. It is also known that an assignment problem is a linear programming problem. So, we can convert the above maximizing assignment problem into the usual minimizing assignment problem, by replacing costs with their negatives and proceed with the Hungarian Method.
The corresponding minimizing assignment problem has the cost-matrix given below:

	I	II	III	IV
A	–6	–7	–5	–2
B	–4	–3	–2	–8
C	–2	–4	–9	–4
D	–5	–3	–1	–7

Step 1

i) Substract the minimum element –7 from all elements of 1st row. Similarly, subtract –8, –9 and –7 respectively from all elements of 2nd, 3rd and 4th rows. The reduced matrix is,

	I	II	III	IV
A	1	0	2	5
B	4	5	6	0
C	7	5	0	5
D	2	4	6	0

(Note that this step amounts to subtracting each element of the original matrix (of the profit maximizing assignment problem) from the corresponding maximum element of their rows respectively. In other words, substract all elements of 1st row from the maximum element i.e. 7 of the 1st row. Similarly, for the other rows).

ii) Subtract the minimum element of each column from all elements of that column. The reduced matrix so obtained is,

	I	II	III	IV
A	0	0	2	5
B	3	5	6	0
C	6	5	0	5
D	1	4	6	0

Step 2

Cover all the zeros by minimum number of horizontal and vertical lines. Observe that only 3 lines can cover all the zeros. So, $r = 3$. As $3 = r < n = 4$, so we go to step 3.