MATHEMATICS

PAPER - I

SECTION A

1. Answer any four of the following:

(a) Let V = $P_3(R)$ be the vector space of polynomial functions over real of degree- at most 3. Let D : V → V be the differentiation operator defined by

$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2,\ x \in \mathbb{R}.$

(i) Show that B $\{1, x, x^2, x^3\}$ is a basis for V.

(ii) Find the matrix $[D]_B$ with respect to B of D.

(iii) Show that B′ $\{1, (x + 1), (x + 1)^2, (x + 1)^3\}$ is a basis for V.

(iv) Find the matrix $[D]_{B'}$ with respect to B′ of D.

(v) Find the matrix $[D]_{B',B}$ of D relative to B′ and B.

(10)

(b) Find the eigen values and the corresponding eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

(10)

(c) Let $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$

Show that f is differentiable at each point of reals but f′(x) is not continuous at x = 0.

(10)

(d) Show that $f : R^2 \to R$ defined by f(x, y) $2x^2 - 6xy + 3y^2$ has a critical point at (0, 0) and that it is a saddle point.

(10)

(e) Find the equations of the generators of the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ through any point of the principal elliptic section $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} 1; z = 0$.

(10)

2. (a) Show that the vectors $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (0, 0, 1)$ form a basis for $\mathbf{R}^{(3)}$. Express $v = (3, 1, -4)$ as a linear combination of v_1, v_2 and v_3. Is the set S = $\{v, v_1, v_2, v_3\}$ linearly independent?

(10)

(b) Determine a non-singular matrix P such that P^t A P is a diagonal matrix, where P^t denotes the transpose of P, and $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$.

(10)

(b) Show that the real quadratic form

$\phi = n\left(x_1^2 + x_2^2 + \ldots.. + x_n^2\right) - \left(x_1 + x_2 + \ldots.. + x_n\right)^2$

in n variables is positive semi-definite.

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(10)

3. (a) (i) Using Taylor's theorem with remainder show that

$$x-\frac{x^3}{6}\le \sin x\le x-\frac{x^3}{6}+\frac{x^5}{120} \text{ for all } x\ge 0.$$

(5)

(ii) Let $f:\mathbf{R}^2\to\mathbf{R}$ be defined by

$$f(x,y)=\frac{xy}{x^2-y^2} \text{ if } x\ne \pm y$$
$$=0 \quad \text{if } x=\pm y.$$

Show that $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ does not exist.

(5)

(b) Show that the curve given by

$$x^3-4x^2y+5xy^2-2y^3+3x^2-4xy+2y^2-3x+2y-1=0$$

has only one asymptote given by

$$y=\frac{1}{2}x+3.$$

(10)

(c) Find the extremum values of

$f(x,y)=2x^2-8xy+9y^2$ on $x^2+y^2-1=0$

using Lagrange multiplier method.

(10)

(d) A solid cuboid C in $\mathbf{R}^3$ given in spherical coordinates by $R=[0,a]$, $\theta=[0,2\pi]$, $\varphi=[0,\pi/4]$ has a density function $\rho(\mathbf{R},\theta,\varphi)=4\mathbf{R}\sin\frac{\theta}{2}\cos\varphi$. Find the total mass of C.

(10)

4. (a) A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}.$$

(10)

(b) Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.

(10)

(c) Planes are drawn through a fixed point (α,β,γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0.$$

(10)

(d) Show that the enveloping cylinder of the conicoid

$ax^2+by^2+cz^2=1$

with generators perpendicular to z- axis meets the plane $z=0$ in parabolas.

(10)

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SECTION B

5. Answer any four of the following

(a) Form the differential equation that represents all parabolas each of which has latus rectum 4a and whose axes are parallel to the x-axis.

(10)

(b) (i) The auxiliary polynomial of a certain homogeneous linear differential equation with constant coefficients in factored form is $P(m)=m^4(m-2)^6(m^2-6m+25)^3$. What is the order of the differential equation and write a general solution?

(5)

(ii) Find the equation of the one- parameter family of parabolas given by $y^2=2cx+c^2$, c real and show that this family is self-orthogonal.

(5)

(c) A circular wire of radius a and density ρ attracts a particle according to

$\gamma\frac{m_1m_2}{(\text{distance})^2}$.

If the particle is placed on the axis of the wire at a distance b from the centre find its velocity when it is at a distance x. If it is placed at a small distance from the centre on the axis show that the time of a complete oscillation is

$a\sqrt{\frac{2\pi}{\gamma\rho}}$.

(10)

(d) A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that the tension in the string is $W\sqrt{3}$.

(10)

(e) For the curve

$\vec{r}=a\left(3t-t^3\right)\vec{i}+3at^2\vec{j}+a\left(3t+t^3\right)\vec{k}$, a

being a constant. Show that the radius of curvature is equal to its radius of torsion.

(10)

6. (a) Solve and examine for singular solution the following equation.

$$p^2(x^2-a^2)-2pxy+y^2-b^2=0.$$

(10)

(b) Solve the differential equation

$$\frac{d^2y}{dx^2}+9y=\sec 3x.$$

(10)

(c) Given $y=x+\frac{1}{x}$ is one solution, solve the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

by reduction of order method.

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(10)

(d) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - 3y = 2e^x - 10\sin x$$

by the method of undetermined coefficients.

(10)

7. (a) If two particles are projected in the same vertical plane with velocities u and u' at angles α and α'' with the horizontal, show that the interval between their transits through the other point common to their paths is $\frac{2}{g}\frac{uu'\sin(\alpha-\alpha')}{u\cos\alpha+u'\cos\alpha'}$ assuming $\alpha > \alpha$.

(10)

(b) Two particles of masses m and M move under the force of their mutual attraction, if the orbit of m relative to M is a circle of radius a described with velocity v, show that

$$v = \left[\frac{G(M+m)}{a}\right]^{1/2}$$

(10)

(c) Show that the length of an endless chain which will hang over a circular pulley of radius r so as to be in contact with two- thirds of the circumference of the pulley is

$$r\left\{\frac{3}{\log\left(2+\sqrt{3}\right)}+\frac{4\pi}{3}\right\}.$$

(10)

(d) Prove that a circular cylinder of radius a and length $\frac{a}{n}$ cannot float upright in water in stable equilibrium if its specific gravity lies between

$$\frac{1}{2}\left\{1-\sqrt{1-2n^2}\right\} \text{ and } \frac{1}{2}\left\{1+\sqrt{1-2n^2}\right\}.$$

What will happen if $2n^2 > 1$?

(10)

8. (a) Find f(r) if f(r) $\bar{r}$ is both solenoidal and irrotational.

(10)

(b) Evaluate $\iint_S \bar{F}\,d\bar{s}$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

(10)

(c) Verify the divergence theorem for $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$

(10)

(d) By using vector methods, find an equation for the tangent plane to the surface $z=x^2+y^2$ at the point (1, -1, 2).

(10)

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MATHEMATICS

PAPER -II

SECTION A

1. Answer any four parts

(10 x 4 = 40)

(a) Show that the set of cube roots of unity is a finite Abelian group with respect to multiplication.

(b) Evaluate the double integral $\iint_R x^2 dxdy$ where R is the region bounded by the line $y = x$ and the curve $y = x^2$.

(c) Show that the function f defined by

$$f(x) = \frac{1}{x}, x \in [1, \infty)$$

is uniformly continuous on $[1, \infty)$.

(d) If f analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

(e) Find the basic feasible solutions of the following system of equations in a linear programming problem

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$
$$x_j \geq 0, j = 1, 2, 3$$

2. (a) Show that the set $S = \{1, 2, 3, 4\}$ forms an Abelian group for the operation of multiplication modulo 5.

(14)

(b) Show that

(i) $h(x) = \sqrt{x + \sqrt{x}}, x \geq 0$ is continuous on $[0, \infty)$;

(ii) $h(x) = e^{\sin x}$ is continuous on R.

(13)

(c) If $f(x, y) = xy \dfrac{x^2 - y^2}{x^2 + y^2}$, when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, show that at $(0, 0)$

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

(13)

3. (a) Prove that the set of all real numbers of the form $a + b\sqrt{2}$, where a and b are real numbers, is a field under the usual addition and multiplication.

(13)

(b) Show that the transformation

$$w = \frac{5 - 4z}{4z - 2}$$

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maps unit circle $|z| = 1$ onto a circle of radius unity and centre at $-\frac{1}{2}$.

(13)

(c) Use contour integration technique to find the value of

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

(14)

4. (a) If R is commutative ring with unit element and M is an ideal in R, then show that M is maximal ideal if R/M is a field.

(13)

(b) Solve the linear programming problem

Find mm $(8x_1 + 6x_2)$

subject to the constraints

$4x_1 + 3x_2 \geq 18$

$2x_1 + 5x_2 \geq 16$

$x_1, x_2 \geq 0$

using graphical method. Show that more than one feasible solution will yield the minimum of the objective function. Interpret this fact geometrically.

(13)

(c) Use simplex method to solve the following linear programming problem .

(14)

Maximize $Z = 2x_1 - x_2 + 3x_3$

subject to the constraints

$3x_1 + x_2 - 2x_3 \leq 6$

$2x_1 + 5x_2 + x_3 \leq 14$

$x_1 + 4x_2 + 2x_3 \leq 8$

$x_1, x_2, x_3 \geq 0$

(b) Show that the transformation

SECTION B

5. Answer any four parts :

(10×4 = 40)

(a) Apply Charpit's method to solve the equation

$2z + p^2 + qy + 2y^2 = 0$

(b) Perform four iterations of the bisection method to obtain a positive root of the equation

$f(x) = x^3 - 5x + 1 = 0$

(c) Evaluate $\int_0^1 \sqrt{1+2x}\,dx$ by applying Gaussian quadrature formula, namely

$$\int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{n} A_i f(t_i)$$

where the coefficients A_i and the roots t_i are given below for n = 4 as

$t_1 = -0.8611 \quad A_1 = A_4 = 0.3478$

$t_2 = -0.3399 \quad A_2 = A_3 = 0.6521$

$t_3 = 0.3399$

$t_4 = 0.8611$

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(d) Evaluate the following expressions:

(i) 78 OR 87

(ii) 78 XOR 87

(iii) 78 AND 87

(iv) Shift 87 left by 2

(v) Rotate 78 right by 2

(e) Find the moment of inertia of a uniform triangular lamina about one side.

6. (a) Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ given that

(i) u = 0, when x 0 for all t

(ii) u = 0, when x = l for all t

(iii) $\left.\begin{aligned} u &= \frac{bx}{a}, \quad 0 < x < a \\ &= \frac{b(l-x)}{l-a}, \quad a < x < l \end{aligned}\right\} \text{at } t = 0$

(iv) $\frac{\partial u}{\partial t} = 0$ at $t = 0$, x in $(0, l)$

(13)

(b) Solve:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$$

(13)

(c) A solid circular cylinder of radius a rotating about its axis is placed gently with its axis horizontal on a rough plane, whose inclination to the horizon is a. Initially, the friction acts up the plane and the coefficient of friction is .t. Show that the cylinder will move upwards, if $\mu > \tan \alpha$. Also, show that the time that lapses before rolling commences, is

$$\frac{a\Omega}{g(3\mu \cos \alpha - \sin \alpha)}$$

where Ω is the initial angular velocity of the cylinder.

(14)

7. (a) Apply Gauss-Seidel iterative method for five iterations to solve the equations :

(14)

$$\begin{bmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 27 \\ -9 \end{bmatrix}$$

(b) A two-dimensional flow field is given by ψ = xy. Then

(i) show that the flow is irrotational;

(ii) find the velocity potential;

(iii) verify that P and its complex conjugate p satisfy the Laplace equation;

(iv) find the streamlines and potential lines.

(13)

(c) Write a BASIC program to evaluate a definite integral

$$\int_0^1 \left(x^3 + \sin x\right) dx$$

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by Simpson's one-third rule. Indicate the lines which are to be modified for a different problem.

(13)

8. (a) Write a program in BASIC to solve the equation

$$x^3 - 4x^2 + x + 6 = 0$$

by Newton-Raphson method by taking the initial approximation as $x_0 = 5$. Indicate which lines are to be changed for a different equation.

(13)

(b) Apply Runge-Kutta method of fourth order to find an approximate value of y when $x = 0.2$, given that

$$\frac{dy}{dx} = x + y^2,\ y = 1 \text{ when } x = 0$$

(14)

(c) Determine the restrictions on f_1, f_2 and f_3, if

$$\frac{x^2}{a^2} f_1(t) + \frac{y^2}{b^2} f_2(t) + \frac{z^2}{c^2} f_3(t) = 1$$

is a possible boundary surface of a liquid.

(13)

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