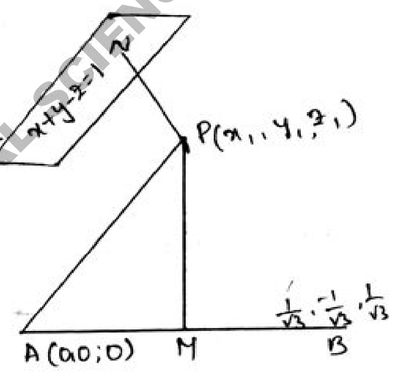
IAS MATHEMATICS (OPT.)

PAPER - I : 3 DIMENSIONAL (2007 to 2000)

IAS-2007

1(e) 2007 12m → Find the locus of the point which moves so that its distance from the plane $x+y-z=1$ is twice its distance from the line $x=-y=z$.

Sol^n: Let (x_1, y_1, z_1) be the given point on the locus, and the given line AB be $\frac{x}{1}=\frac{-y}{1}=\frac{z}{1}$



Let PM and PN be ⊥s from P on the line AB and the given plane

$$x+y-z-1=0 \quad \text{———} \quad (1)$$

Now we are given that

$$PM = 2PN$$

(or) $$PM^2 = 4PN^2 \quad \text{———} \quad (2)$$

Now PN = ⊥ distance of $P(x_1, y_1, z_1)$ from the plane (1)

$$= \frac{x_1+y_1-z_1-1}{\sqrt{1+1+1}} = \frac{x_1+y_1-z_1-1}{\sqrt{3}}$$

Now one point on the given line is $A(0,0,0)$. Join AP. Then

$$AP = \sqrt{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2}$$

$$= \sqrt{x_1^2+y_1^2+z_1^2}$$

The direction cosines of AB, the given line are proportional to $1, -1, 1$.

∴ Its actual d.c.'s are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

∴ AM = Projection of AP on AB

$$= \frac{1}{\sqrt{3}}(x_1-0) - \frac{1}{\sqrt{3}}(y_1-0) + \frac{1}{\sqrt{3}}(z_1-0)$$

| using $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

$$= \frac{x_1-y_1+z_1}{\sqrt{3}}$$

∴ from right angle ΔAMP

$$MP^2 = AP^2 - AM^2$$

$$= x_1^2+y_1^2+z_1^2 - \left(\frac{x_1-y_1+z_1}{\sqrt{3}}\right)^2$$

Hence from ②, we have

$$x_1^2+y_1^2+z_1^2 - \left(\frac{x_1-y_1+z_1}{\sqrt{3}}\right)^2 = 4\left(\frac{x_1+y_1-z_1-1}{\sqrt{3}}\right)^2$$

$$\Rightarrow 3(x_1^2+y_1^2+z_1^2) - (x_1+y_1+z_1)^2 = 4(\underbrace{x_1+y_1-z_1}-1)^2$$

$$\Rightarrow 3x_1^2+3y_1^2+3z_1^2-x_1^2-y_1^2-z_1^2-2x_1y_1-2y_1z_1-2z_1x_1$$

$$= 4\left[(x_1+y_1-z_1)^2 - 2(x_1+y_1-z_1)+1\right]$$

$$\Rightarrow 2x_1^2+2y_1^2+2z_1^2-2x_1y_1-2y_1z_1-2z_1x_1 = 4\left[x_1^2+y_1^2+z_1^2+2x_1y_1-2y_1z_1-2z_1x_1 - 2x_1-2y_1+2z_1+1\right]$$

$$\Rightarrow 2x_1^2+2y_1^2+2z_1^2+10x_1y_1-6y_1z_1-6z_1x_1-8x_1-8y_1+8z_1+4 = 0$$

$$\Rightarrow x_1^2+y_1^2+z_1^2+5x_1y_1-3y_1z_1-3z_1x_1-4x_1-4y_1+4z_1+2=0$$

∴ The locus of (x_1, y_1, z_1) is

$$x^2+y^2+z^2+5xy-3yz-3zx-4x-4y+4z+2=0.$$

2007 12M

1(f). Find the equation of the sphere inscribed in the tetrahedron whose faces are $x=0, y=0, z=0$ and $2x+3y+6z=6$.

Sol'n: The given faces are

$x=0, y=0, z=0$ and $6-2x-3y-6z=0$

Let (α, β, γ) be the centre and r the radius of the inscribed sphere. Then ⊥lar distances of the centre from all the four faces are equal and equal to radius.

$$\therefore \frac{\alpha}{1} = \frac{\beta}{1} = \frac{\gamma}{1} = \frac{6-2\alpha-3\beta-6\gamma}{\sqrt{4+9+36}} = r$$

$\therefore \alpha = \beta = \gamma = r$ and $6-2\alpha-3\beta-6\gamma = 7r$

Eliminating α, β, γ we get

$6-2r-3r-6r=7r$

$\Rightarrow 18r = 6$

$\therefore r = \frac{1}{3}$

$\therefore \alpha = \beta = \gamma = r = \frac{1}{3}$

Hence the centre (α, β, γ) is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and radius r is $\frac{1}{3}$.

Thus the equation of the sphere with centre $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and radius $\frac{1}{3}$ is

$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{1}{3}\right)^2+\left(z-\frac{1}{3}\right)^2=\left(\frac{1}{3}\right)^2$ [using $(x-a)^2+(y-b)^2+(z-c)^2=r^2$]

$\Rightarrow x^2-\frac{2}{3}x+\frac{1}{9}+y^2-\frac{2}{3}y+\frac{1}{9}+z^2-\frac{2}{3}z+\frac{1}{9}=\frac{1}{9}$

$\Rightarrow x^2+y^2+z^2-\frac{2}{3}(x+y+z)+\frac{2}{9}=0$

$\Rightarrow 9(x^2+y^2+z^2)-6(x+y+z)+2=0$

which is the required equation.

2007 P-I 15m

4(b). A line with direction ratios 2, 7, −5 is drawn to intersect the lines.

$$\frac{x}{3}=\frac{y-1}{2}=\frac{z-2}{4} \text{ and } \frac{x-11}{3}=\frac{y-5}{1}=\frac{z}{1}$$

find the Coordinates of the points of intersection and the length intercepted on it.

Sol'n: Given lines are $\frac{x}{3}=\frac{y-1}{2}=\frac{z-2}{4}=r\,(\text{Say})$ — ①

and $\frac{x-11}{3}=\frac{y-5}{1}=\frac{z}{1}=r'\,(\text{Say})$ — ②

Any point on ① is $P(3r, 2r+1, 4r+2)$ and any point on ② is $P'(3r'+11, r'+5, r')$ — ③

∴ the direction ratios of the line PP' are

$[3r-(3r'+11),\ 2r+1-(r'+5),\ 4r+2-r']$

(or) $[3r-3r'-11,\ 2r-r'-4,\ 4r-r'+2]$

But the direction ratios of PP' are given to be 2, 7, −5 so we get.

$$\frac{3r-3r'-11}{2}=\frac{2r-r'-4}{7}=\frac{4r-r'+2}{-5}$$

which gives $7(3r-3r'-11)=2(2r-r'-4)$

and $-5(3r-3r'-11)=2(4r-r'+2)$

(or) $17r-19r'-69=0$ and $-23r+17r'+51=0$

$$\frac{r}{-19(51)+69(17)}=\frac{-r'}{17(51)-69(23)}=\frac{1}{289-23(19)}$$

$$\Rightarrow \frac{r}{204}=\frac{-r'}{720}=\frac{1}{-148}$$

$$\Rightarrow r=\frac{-51}{37},\ r'=\frac{180}{37}$$

Substituting these values in equ'n ③ we get the coordinates of the required points of intersection P and P'.

$$P\left(\frac{-153}{37}, \frac{-65}{37}, \frac{-130}{37}\right) \text{ and } P'\left(\frac{577}{37}, \frac{365}{37}, \frac{180}{37}\right)$$

2007 15M 8.

4(c). Show that the plane $2x-y+2z=0$ cuts the cone $xy+yz+zx=0$ in perpendicular lines.

Sol'n: Let the plane $2x-y+2z=0$ cut the cone $xy+yz+zx=0$ in a line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Then
$$\left.\begin{array}{l} 2l-m+2n=0 \\ \text{and} \quad lm+mn+nl=0 \end{array}\right\} \text{———} (1).$$

Eliminating m between these relations, we get

$$2l(l+n)+2n(l+n)+nl=0$$

$$\Rightarrow 2l^2+2ln+2ln+2n^2+ln=0.$$

$$\Rightarrow 2l^2+5ln+2n^2=0$$

$$\Rightarrow 2\left(\frac{l}{n}\right)^2+5\left(\frac{l}{n}\right)+2=0 \text{ ———} (2)$$

If the roots of this equation are

$\frac{l_1}{n_1}, \frac{l_2}{n_2}$

$\therefore$ Product of the roots $=\frac{l_1}{n_1}\cdot\frac{l_2}{n_2}=\frac{2}{2}=1$

$$\Rightarrow \frac{l_1 l_2}{1}=\frac{n_1 n_2}{1} \text{ ———} (3)$$

Eliminating l between the relation (1), we get

$\left(\because 2l-m+2n=0 \Rightarrow l=\frac{m-2n}{2}\right)$

$$lm+mn+nl=0$$

$$\Rightarrow m\left(\frac{m-2n}{2}\right)+mn+n\left(\frac{m-2n}{2}\right)=0$$

$$\Rightarrow m^2-\cancel{2mn}+\cancel{2mn}+mn-2n^2=0$$

$$\Rightarrow m^2+mn-2n^2=0$$

$$\Rightarrow \left(\frac{m}{n}\right)^2+\frac{m}{n}-2=0$$

$\therefore$ If the roots of this equation are

$\frac{m_1}{n_1}, \frac{m_2}{n_2}$, then

product of roots $= \frac{m_1 m_2}{n_1 n_2} = \frac{-2}{1}$

$$\Rightarrow \frac{m_1 m_2}{-2} = \frac{n_1 n_2}{1} \quad \text{---} (4)$$

$\therefore$ from (3) and (4)

$$\frac{l_1 l_2}{1} = \frac{n_1 n_2}{1} = \frac{m_1 m_2}{-2}.$$

Now. $l_1 l_2 + m_1 m_2 + n_1 n_2 = 1 - 2 + 1 = 0$

$\therefore$ The lines are perpendicular.

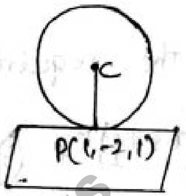
IAS-2006

4(b). 15M → Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.

Sol'n: Given plane $3x+2y-z+2=0$ —— (1)

and the given sphere

$$x^2+y^2+z^2-4x+6y+4=0 \quad \text{—— (2)}.$$



Since the required sphere touches the plane (1) at $P(1,-2,1)$

$\therefore$ Its centre lies on the normal to the plane at 'P'

Now the equations of normal to the plane (1) through $P(1,-2,1)$ are

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}=r \text{ (say)}$$

Any point on this line $C(3r+1, 2r-2, -r+1)$

Let this point be the centre of the required sphere.

Now the radius of the required sphere

$$CP=\sqrt{(3r+1-1)^2+(2r-2+2)^2+(-r+1-1)^2}$$

$$=\sqrt{9r^2+4r^2+r^2}$$

$$=r\sqrt{14}$$

Since the required sphere cuts the sphere (2) orthogonally.

$\therefore$ Square of distance between the centres = Sum of square of their radii —— (4)

Now the centre of the sphere (2) $C'(2,-3,0)$

and radius $=\sqrt{4+9-4}$

$=3$

$$\therefore (4) \equiv (3r+1-2)^2+(2r-2+3)^2+(-r+1-0)^2=9+14r^2$$

$$\Rightarrow r=-3/2$$

$\therefore$ The Centre of the required sphere.

$$C\left(-\tfrac{7}{2}, -5, \tfrac{5}{2}\right)$$

and the radius $CP = -\tfrac{3}{2}\sqrt{14}$

$= \tfrac{3}{2}\sqrt{14}$ (numerically)

$\therefore$ The required sphere is

$$\left(x+\tfrac{7}{2}\right)^2 + (y+5)^2 + \left(z-\tfrac{5}{2}\right)^2 = \left(\frac{3\sqrt{14}}{2}\right)^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12=0$$

15M 4(c). Show that the plane $ax+by+cz=0$ cuts the cone $xy+yz+zx=0$ in perpendicular lines, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Sol'n: The equation of the plane is $ax+by+cz=0$ —— (1)

and the cone is $yz+zx+xy=0$ —— (2).

Comparing (2) with

$ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$

$\therefore a=0,\ b=0,\ c=0$

$\Rightarrow a+b+c=0+0+0=0$

$\therefore$ The cone (2) has three mutually $\perp$lar generators.

The plane (1) will cut the cone (2) in $\perp$lar lines if the normal to the plane (1) through the vertex $(0,0,0)$ [whose d.c's are proportional to a,b,c] lies on the cone (2)

if $bc+ca+ab=0$ ($\because$ direction cosines of the generator satisfy the equation of the cone)

if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

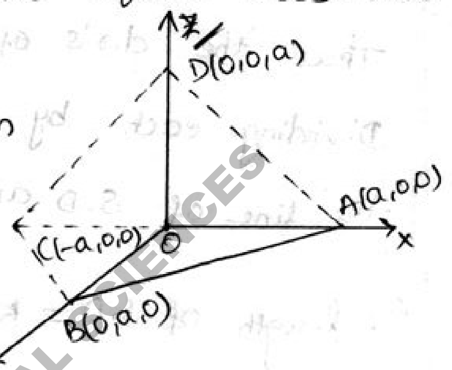
(on dividing throughout by abc)

which is the required condition.

IAS-2005

1(f). A Square ABCD having each diagonal AC & BD of length $2a$, is folded along the diagonal AC so that the planes DAC and BAC are at right angle. Find the shortest distance between AB and DC.

Sol^n: Let O be the centre of the square ABCD. Take OA as the axis of x.



Now planes DAC and BAC are given to be mutualy at right angles.

So, take OB and OD as axes of y and z respectively.

Then the four vertices of the Square are

A(a,0,0), B(0,a,0), C(-a,0,0), and D(0,0,a).

Equations of DC are

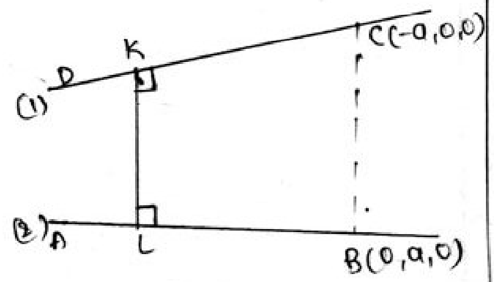
$$\frac{x-0}{-a-0}=\frac{y-0}{0-0}=\frac{z-a}{0-a} \quad \left[\text{using } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right.$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z-a}{1}$$

and the equations of AB are $\frac{x-a}{0-a}=\frac{y-0}{a-0}=\frac{z-0}{0-0}$

$$\Rightarrow \frac{x-a}{1}=\frac{y}{-1}=\frac{z}{0}$$

Let KL be the shortest distance between two lines AB and DC.



let l, m, n be the d.c's of KL

Since KL is ⊥lar to both the lines DC and AB i.e. the lines (1) & (2)

$\therefore\ l + 0.m + n = 0$

$l - m + 0.n = 0$

hence $\frac{l}{0+1} = \frac{m}{1-0} = \frac{n}{-1}$

$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$

Thus the d.r's of the S.D are 1, 1, −1

Dividing each by $\sqrt{1+1+1} = \sqrt{3}$, the actual d.c's of the line of S.D are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

$\therefore$ length of S.D = KL = Projection of BC on KL.

$= \frac{1}{\sqrt{3}}(0+a) + \frac{1}{\sqrt{3}}(a-0) - \frac{1}{\sqrt{3}}(0-0)$

[using $l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$]

$= \frac{a}{\sqrt{3}} + \frac{a}{\sqrt{3}} + 0 = \frac{2a}{\sqrt{3}}$

IAS-2004

15m 2004

1(e). Prove that the locus of a line which meets the lines $y=\pm mx$, $z=\pm c$ and the circle $x^2+y^2=a^2$, $z=0$ is $c^2m^2(cy-mzx)^2+c^2(yz-cmn)^2=a^2m^2(z^2-c^2)^2$.

Sol'n: The given lines are

$$y-mx=0,\quad z-c=0 \quad \text{——— (1)}$$

$$y+mx=0,\quad z+c=0 \quad \text{——— (2)}$$

and the circle is

$$x^2+y^2=a^2;\ z=0 \quad \text{——— (3)}$$

Any line intersecting (1) & (2) is

$$\left.\begin{aligned} y-mx+k_1(z-c)=0 \\ y+mx+k_2(z+c)=0 \end{aligned}\right\} \text{——— (4)}$$

• If it meets the circle (3), we have to eliminate x, y, z from (3) & (4).

(3) ≡ Putting $z=0$ in (4) we get

$$y-mx+k_1(-c)=0$$

$$y+mx+k_2(c)=0$$

Solving

$$\frac{y}{-mk_2c+mk_1c}=\frac{x}{-ck_1-ck_2}=\frac{1}{m+m}$$

$$\Rightarrow x=\frac{-(k_1+k_2)c}{2m}$$

$$y=\frac{c(k_1-k_2)}{2}$$

Putting these values of x, y in (3), we get

$$\frac{c^2(k_1+k_2)^2}{4m^2}+\frac{c^2(k_1-k_2)^2}{4}=a^2$$

$$\Rightarrow c^2(k_1+k_2)^2+c^2m^2(k_1-k_2)^2=4a^2m^2 \quad \text{——— (5)}$$

To find the locus,
eliminate k_1, k_2 from (4) & (5)

$$\therefore (4) \equiv k_1 = \frac{-(y-mx)}{z-c} = \frac{mx-y}{z-c}$$

$$k_2 = \frac{-(y+mx)}{z+c}$$

Substituting these values in (5)

$$\therefore (5) \equiv c^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{-mx-y}{z+c}\right)\right]+c^2m^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right]$$

$$= 4a^2m^2$$

on simplification we get

$$c^2m^2(cy-mxz)^2+c^2(yz-mx)^2=a^2m^2(z^2-c^2)^2$$

which is the required locus.

12M

1(f). Find the equations of tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$, which are parallel to the plane $2x+y-z=4$.

Sol'n: Equation of sphere is $x^2+y^2+z^2-4x+2y-6z+5=0$.

Its Centre $(2,-1,3)$

and radius $=\sqrt{4+1+9-5}=3$

Any plane parallel to the plane $2x+y-z=4$ is

$$2x+y-z=k \quad \text{——— (1)}$$

If it touches the sphere, then length of perpendicular from the centre of sphere must be equal to the radius of the sphere.

$$\therefore \frac{|2(2)+1(-1)-1(3)-k|}{\sqrt{4+1+1}}=3$$

$$\Rightarrow |4-1-3-k|=3\sqrt{6}$$

$$\Rightarrow k=\pm 3\sqrt{6}$$

from (1), we have.

$$2x+y-z=-3\sqrt{6}$$
$$2x+y-z=3\sqrt{6}$$

$\therefore$ The required tangent planes are.

$$2x+y-z+3\sqrt{6}=0 \text{ and } 2x+y-z-3\sqrt{6}=0.$$

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2004

4(c). Prove that the lines of intersection of pairs of tangent planes to $ax^2+by^2+cz^2=0$ which touch along perpendicular generators lie on cone

$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0$.

Sol^n: Let the tangent planes along two perpendicular generators of the cone meet in the line.

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \quad \text{——— (1)}$$

Therefore the equation of the plane containing the two generators is

$$alx+bmy+cnz=0 \quad \text{——— (2)}$$

If $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ is one of the generators.

$$al\lambda+bm\mu+cn\nu=0 \quad \text{——— (3)}$$

$$a\lambda^2+b\mu^2+c\nu^2=0 \quad \text{——— (4)}$$

Eliminating ν from (3) and (4), we get

$$a\lambda^2(cn^2+al^2)+2ablm\lambda\mu+b(cn^2+bm^2)\mu^2=0.$$

$$\therefore \frac{\lambda_1\lambda_2}{\mu_1\mu_2}=\frac{b(cn^2+bm^2)}{a(cn^2+al^2)}$$

$\therefore$ By symmetry.

$$\frac{\lambda_1\lambda_2}{(cn^2+bm^2)/a}=\frac{\mu_1\mu_2}{(cn^2+al^2)/b}=\frac{\nu_1\nu_2}{(al^2+bm^2)/c}$$

Since the generators are at right angles.,

$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0.$$

$$\therefore \frac{cn^2+bm^2}{a}+\frac{cn^2+al^2}{b}+\frac{al^2+bm^2}{c}=0$$

(or) $a^2(b+c)l^2+b^2(c+a)m^2+c^2(a+b)n^2=0$

$\therefore$ Locus of (1) is

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

IAS-2003

2003 → 12M.

1(f). Find the equation of the two straight lines through the point (1,1,1) that intersect the line $x-4 = 2(y-4) = 2(z-1)$ at angle of 60°.

Sol: Any line through the point (1,1,1) is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n} = r \quad \text{--- (1)}$$

The given line is

$$\frac{x-4}{2} = \frac{y-4}{1} = \frac{z-1}{1} \quad \text{--- (2)}$$

Any point on the line (1) is $P(lr+1, mr+1, nr+1)$.

If the line (1) meets the line (2), let it meet at P so that co-ordinates of P satisfy (2).

$$\therefore \frac{lr+1-4}{2} = \frac{mr+1-4}{1} = \frac{nr+1-1}{1}$$

$$\Rightarrow \frac{lr-3}{2} = \frac{mr-3}{1} = \frac{nr}{1}$$

from first two members,

$$lr-3 = 2mr-6 \Rightarrow r(2m-l) = 3 \quad \text{--- (3)}$$

from the last two members,

$$mr-3 = nr \Rightarrow r(m-n) = 3 \quad \text{--- (4)}$$

Dividing (3) and (4), we have

$$\frac{2m-l}{m-n} = \frac{3}{3} = 1$$

$$\Rightarrow 2m-l = m-n$$

$$\Rightarrow l-m-n = 0 \quad \text{--- (5)}$$

Given that the angle between the lines ① & ② is $60°$.

$$\therefore \cos 60 = \pm \frac{2l+m+n}{\sqrt{4+1+1}\sqrt{l^2+m^2+n^2}}$$

$$\frac{1}{2} = \pm \frac{2l+m+n}{\sqrt{6(l^2+m^2+n^2)}}$$

cross multiplying and squaring, we get

$$6(l^2+m^2+n^2) = 4(2l+m+n)^2$$

$$6l^2+6m^2+6n^2 = 4(4l^2+m^2+n^2+4lm+2mn+4ln)$$

$$\Rightarrow 10l^2-2m^2-2n^2+16lm+8mn+16ln=0$$

$$\Rightarrow 5l^2-m^2-n^2+8lm+8ln+4mn=0$$

$$\Rightarrow 5l^2-m^2-n^2+8l(m+n)+4mn=0 \quad \text{—⑥}$$

The equations ⑤ and ⑥ determine d.c's of the required line ①.

from ⑤, $l=m+n$.

putting it in ⑥, we get

$$5(m+n)^2-m^2-n^2+8(m+n)^2+4mn=0$$

$$\Rightarrow 12m^2+30mn+12n^2=0$$

$$\Rightarrow 2m^2+5mn+2n^2=0$$

$$\Rightarrow (2m+n)(m+2n)=0$$

$\Rightarrow 2m+n=0$

i.e, $0l+2m+n=0$

Also from ⑤ $l-m-n=0$

$$\therefore \frac{l}{-2+1} = \frac{m}{1-0} = \frac{n}{0-2}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{2}$$

or $m+2n=0$

i.e, $0l+m+2n=0$

$l-m-n=0$

$$\therefore \frac{l}{-1+2} = \frac{m}{2+0} = \frac{n}{0-1}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{2} = \frac{n}{-1}$$

Putting these values in ① the required lines are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$.

15M 2003

4(a). Find the volume of the tetrahedron formed by the four planes $lx+my+nz=p$, $lx+my=0$, $my+nz=0$ and $nz+lx=0$

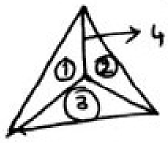
Sol^n

The given plane equations are

$my+nz=0$ — (1) $\quad lx+my=0$ — (3)

$nz+lx=0$ — (2) $\quad$ & $lx+my+nz=p$ (4)

Now solving the above equations, taking three planes at a time, We get the vertices of the tetrahedron.



Now from (1), (2) & (3) $x=y=z=0$

$\therefore$ one vertex of the tetrahedron is $O(0,0,0)$

To solve (1) (2) & 4

Now substitute (1) & (2) in (4) We get,

$lx=p \Rightarrow \boxed{x=\frac{p}{l}}$; $\boxed{y=\frac{p}{m}}$

(1) $\equiv m\left(\frac{p}{m}\right)+nz=0$

$\Rightarrow \boxed{z=\frac{-p}{n}}$

$\therefore \left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)$ is the second vertex of the tetrahedron

Similarly $\left(-\frac{p}{l}, -\frac{p}{m}, \frac{p}{n}\right)$ & $\left(\frac{p}{l}, -\frac{p}{m}, \frac{p}{n}\right)$ are the other vertices of the tetrahedron.

$\therefore$ the required volume of the tetrahedron:

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -\frac{p}{l} & \frac{l}{m} & \frac{l}{n} & 1 \\ \frac{p}{l} & -\frac{l}{m} & \frac{l}{n} & 1 \\ \frac{p}{l} & \frac{l}{m} & -\frac{l}{n} & 1 \end{vmatrix} = \frac{1}{6}\frac{p}{l}\frac{p}{m}\frac{p}{n}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix}$$

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2003 15M 16

4(b). → A sphere of constant radius r passes through the origin O and cuts the co-ordinate axes at A, B & C. Find the locus of the foot of the perpendicular from O to the plane ABC.

Sol'n: Let the co-ordinates of the points A, B, C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

Then the equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0$$

Its radius $= \sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2}$

$= r$ (given)

$$\Rightarrow a^2+b^2+c^2=4r^2 \quad \text{——— (1)}$$

Now the equation of the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{——— (2)}$$

Direction ratio's of the ⊥lar to this plane (2) are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

∴ Equations of the line through O(0,0,0) and ⊥lar to the plane (2) are

$$\frac{x-0}{1/a}=\frac{y-0}{1/b}=\frac{z-0}{1/c}$$

$$\Rightarrow ax=by=cz \quad \text{——— (3)}$$

To find the locus of foot of ⊥lar from 'O' on the plane (2), i.e, the locus of the point of intersection of the plane (2) and line (3), we have to eliminate the unknown constants a, b, c from (1), (2) & (3).

Now from (3),

Let $ax = by = cz = \lambda$ (say)

$\Rightarrow a = \frac{\lambda}{x},\ b = \frac{\lambda}{y},\ c = \frac{\lambda}{z}$

putting these values in (1) we get.

$$\lambda^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4r^2$$

$$\Rightarrow \lambda^2(x^{-2} + y^{-2} + z^{-2}) = 4r^2 \text{ ——— } (4)$$

and putting the values of a, b, c in (2), we get

$$\frac{1}{\lambda}(x^2 + y^2 + z^2) = 1$$

$$\Rightarrow \frac{1}{\lambda^2}(x^2 + y^2 + z^2)^2 = 1$$

multiplying (4) & (5), we get

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2}) = 4r^2$$

which is the required locus.

IAS-2001

1(f). 12M → Find the Shortest Distance b/w the axis of z and the line $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$.

Sol'n: Now the equations of the line on z-axis are

$$x=0=y \quad \text{—— (1)}$$

and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ —— (2)

Now the equations of the planes through the lines (1) and (2) are

$$x+\lambda_1 y=0 \quad \text{—— (3)}$$

and $(ax+by+cz+d)+\lambda_2(a'x+b'y+c'z+d')=0$

$\Rightarrow (a+a'\lambda_2)x+(b+b'\lambda_2)y+(c+c'\lambda_2)z+(d+d'\lambda_2)=0$ —— (4)

If these two planes (3) and (4) are parallel then

$$\frac{a+a'\lambda_2}{1}=\frac{b+b'\lambda_2}{\lambda_1}=\frac{c+c'\lambda_2}{0}$$

$\Rightarrow a+a'\lambda_2=1$

$\Rightarrow \boxed{\lambda_2=\frac{1-a}{a'}}$ —— (i)

$\Rightarrow b+b'\lambda_2=\lambda_1$

$\Rightarrow \lambda_1=b+b'\lambda_2$

$\Rightarrow \boxed{\lambda_1=b-\frac{c}{c'}b'}$ —— (ii)

$\Rightarrow c+c'\lambda_2=0$

$\Rightarrow \boxed{\lambda_2=\frac{-c}{c_1}}$ —— (iii)

∴ from (3) we have

$$x+\left(b-\frac{c}{c'}b'\right)y=0$$

$\Rightarrow c'x+(bc'-cb')y=0$ —— (5)

from (4), we have

$$\left(a-\frac{ca'}{c'}\right)x+\left(b-\frac{cb'}{c'}\right)y+\left(c-\frac{cc'}{c'}\right)z+\left(d-\frac{cd'}{c'}\right)=0$$

$\Rightarrow (ac'-a'c)x+(bc'-cb')y+(dc'-cd')=0$ —— (6)

from (i) & (iii), we have

$$\frac{1-a}{a'}=\frac{-c}{c'}$$

$\Rightarrow c'-ac'=-ca'$

$$\Rightarrow \boxed{c' = ac' - ca'}$$

$\therefore$ from (5), we have

$$(ac' - ca')x + (bc' - cb')y = 0 \quad \text{——} \quad (7)$$

$\therefore$ Shortest Distance = distance between two parallel planes is

$$= \frac{|0 - (dc' - cd')|}{\sqrt{(ac' - a'c)^2 + (bc' - cb')^2}}$$

$$= \frac{|(dc' - cd')|}{\sqrt{(ac' - a'c)^2 + (bc' - cb')^2}}$$

2001 15M

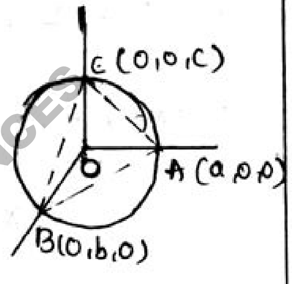
4(a). Find the equation of the circle circumscribing triangle formed by the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$. Obtain also the coordinates of the centre of the circle.

Sol'n: Let the given points be $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ then the circumcircle of ΔABC is the intersection of the plane ABC and the sphere OABC.

Now the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{---} (1)$$

and the equation of the sphere OABC is



$$x^2+y^2+z^2-ax-by-cz=0 \quad \text{---} (2)$$

$\therefore$ The equations of the circle ΔABC are

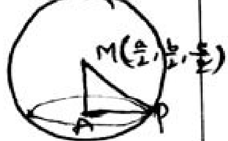
$$x^2+y^2+z^2-ax-by-cz=0$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{---} (2)$$

Equations (1) & (2) taken together represent a circle.

Now the centre of the circle is the foot of $\perp$lar from the centre of the sphere (1) on the plane (2). Here the centre of the sphere is $M\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

Now the d.r.'s of the normal to the plane (2) are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.



$\therefore$ Equations of the line MA, through $M\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ and $\perp$lar to plane (1) are

$$\frac{x-\frac{a}{2}}{1/a}=\frac{y-b/2}{1/b}=\frac{z-c/2}{1/c}.$$

Any point on this line is $\left(\frac{a}{2}+\frac{r}{a},\ \frac{b}{2}+\frac{r}{b},\ \frac{c}{2}+\frac{r}{c}\right)$. Let it be A. ∴ ———— (3)

Since A lies on the plane (1),

$$\therefore \frac{1}{a}\left(\frac{a}{2}+\frac{r}{a}\right)+\frac{1}{b}\left(\frac{b}{2}+\frac{r}{b}\right)+\frac{1}{c}\left(\frac{c}{2}+\frac{r}{c}\right)=1$$

$$\Rightarrow \frac{1}{2}+\frac{r}{a^2}+\frac{1}{2}+\frac{r}{b^2}+\frac{1}{2}+\frac{r}{c^2}=1$$

$$\Rightarrow r(a^{-2}+b^{-2}+c^{-2})=1-\frac{3}{2}=-\frac{1}{2}$$

$$\therefore r=-\frac{1}{2(a^{-2}+b^{-2}+c^{-2})}.$$

Putting this value of r in (7), the point A is

$$\left[\frac{a}{2}-\frac{a^{-1}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b}{2}-\frac{b^{-1}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c}{2}-\frac{c^{-1}}{2(a^{-2}+b^{-2}+c^{-2})}\right].$$

$$\Rightarrow \left[\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}\right]$$

which is the Centre of the Circle.

→

IAS-2000

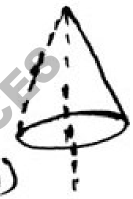
4(d). → Show that the locus of mid-points of chords of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ drawn parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is the plane

$$(al+hm+gn)x+(hl+bm+fn)y+(gl+fm+cn)z=0.$$

Sol'n: Let $P(x_1, y_1, z_1)$ be the midpoint of one of the chords drawn ||lel to the $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

then equation of this chord is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{——} \quad (1)$$



Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the cone.

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2+2f(mr+y_1)(nr+z_1)$$
$$+2g(nr+z_1)(lr+x_1)+2h(lr+x_1)(mr+y_1)=0.$$

$$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)+$$
$$2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)]+$$
$$(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)=0$$

which is a quadratic in r.

Since $P(x_1, y_1, z_1)$ is the midpoint of the chord.

∴ The two values of r should be equal in magnitude but opposite in sign.

∴ Sum of roots $=0$ (or) the coefficients of $r=0$

i.e. $l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$

$\Rightarrow x_1(al+hm+gn)+y_1(hl+bm+fn)+z_1(gl+fm+cn)=0.$

∴ the locus of $P(x_1, y_1, z_1)$ is

$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0$ —— (2).

which is the required plane.