IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

INSTITUTE FOR IAS/IFOS EXAMINATIONS

MATHEMATICS OPTIONAL

By

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Calculus and Real Analysis
Calculus Covers : P-I Section-A
Real Analysis Covers : P-II Section B

WORKSHEET - 1

CALCULUS & REAL ANALYSIS

QLP

(QUICK LEARNING PROGRAMME FOR MAINS-2016)

Scoring Maximum Marks in Main-2016

FEATURES OF MATHEMATICS PROGRAMME

- Class Timings – 9:30 AM – 5:30 PM.
- Atleast one day one module to be discussed (Expected)
- A question based approach to be followed.
- Pin-pointed formulas to be provided.
- 15-20 problems to be provided on each formula set.
- Time-bound problem solving sessions.
- On board discussion sessions to be followed there after.
- Around 50-100 problems expected to be solved per session.
- Only problems to be discussed, no concept explanation.
- Sessions will be from Monday to Friday.

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$f(x) = \begin{cases} \sin^{-1}/x & ; x \neq 0 \\ 0 & , x = 0 \end{cases}$ $\left\langle \frac{1}{n\pi + \pi/2} \right\rangle \to 0$

↳ show discont using sequence . but $f\left(\frac{1}{n\pi + \pi/2}\right) = \sin(n\pi + \pi/2) = \pm 1$ depending on n

but $f(0) = 0$ ∴ Discont

WORKSHEET - 1

Limits & Continuity (Single Variable)

1. Let f be a function on R defined by
$$f = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$
Show that f is discontinuous everywhere.

2. If a function f is continuous in [0, 1], show that
$$\lim_{n \to \infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \frac{\pi}{2} f(0)$$
limit can't be ...

3. Prove that the function f defined by
$$f(x) = \begin{cases} \frac{1}{2}, & if\ x\ is\ rational \\ \frac{1}{3}, & if\ x\ is\ irrational \end{cases}$$
is discontinuous everywhere.

4. Prove that the function f defined by $f(x) = \sin\frac{1}{x}\ \forall x > 0$ is continuous but not uninformly continous on R^+.

$x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$

sin(x) is cont. $1/x$ is cont ⇒ composition ∴ continuous

5. Is the function $f(x) = \frac{x}{(x+1)}$ uniformly continous for $x \in [0, 2]$? Justify your answer.

cont. on bdd set ⇒ UC

6. Define by $g: \mathbb{R} \to \mathbb{R}$ by
$g(x) = 2x$ for x rational
$x + 3$ for x irrational
Find all points at which g is continuous.

$2x = x+3$ ⇒ $\boxed{x = 3}$

7. Determine the points of continuity of the function $f(x) = [x]$.

8. Let $K > 0$ and let $f: \mathbb{R} \to \mathbb{R}^*$ satisfy the condition
$|f(x) - f(y)| \le K|x - y| \forall x, y \in \mathbb{R}$. Show that is continuous at every point $c \in \mathbb{R}$.

Lipschitz

For $\varepsilon > 0$ ∃ $\delta = \frac{\varepsilon}{K}$

9. Let R be the set of real numbers and $f: \mathbb{R} \to \mathbb{R}$ such that for all x and y in $\mathbb{R}$, $|f(x) - f(y)| \le |x - y|^2$ prove that f(x) is a constant function.

10. Let $f: \mathbb{R} \to \mathbb{R}$ be such that
$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0. \\ c & , \text{if } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, & \text{if } x > 0 \end{cases}$$
$\frac{1}{2b} = a + 2 = c$

Determine thev alues of a, b, c for which the function is continuous at x = 0.

11. Prove that the function f defined by
$$f(x) = \begin{cases} 1/2, & \text{if } x \text{ is rational} \\ 1/3, & \text{if } x \text{is irrational} \end{cases}$$
is continuous everywhere.

13. A function $f: \mathbb{R} \to \mathbb{R}$ is additive, if
$f(x + y) = f(x) + f(y) \forall x, y \in \mathbb{R}$
prove that if f is continuous at some x_0, then it is continuous at every point of $\mathbb{R}$.

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14. If f is a continuous function of x satisfying the fucntional equaltion

$f(x+y)=f(x)+f(y)$,

Show that $f(x)=ax$ where, a is a constant.

15. Is the function $f(x)=\frac{x}{(x+1)}$ uniformly continuous for $x\in[0,2]$? Justify your answer.

16. Prove that the function f defined on $\mathbb{R}$ by

$f(x)=\frac{1}{x^2+1}, x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

$\frac{|x|}{1+x^2} < 1$ & $\frac{|y|}{1+y^2} < 1$

17. If $f:\mathbb{R}\to\mathbb{R}$ is such that $f(x+y)=f(x)+f(y)$ and f is continuous, then show that $f(x)=xf(1)$ for all $x\in\mathbb{R}$

18. Prove that, the function f defined by

$$f(x)=\begin{cases} \frac{1}{2}, \text{if } x \text{ is rational} \\ \frac{1}{3}, \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous everywhere.

19. Show that $\tan x$ is not continuous at $x=\frac{\pi}{2}$

20. If $f(x)=x^2$ for all $x\in R$, then show that f is uniformly continuous on every closed and finite interval, but is not uniformly continuous on R.

21. (a) Let $f(x)=x$ if x is rational

$= 1-x$ if x is irrational

Show that $f(x)$ is continuous only at $x=\frac{1}{2}$

22. Let f be a real function defined as follows: $f(x)=x,\ -1\le x<1,\ f(x+2)=x,\ \forall x\in\mathbb{R}$. Show that f is discontinuous at every odd integer.

23. Show that the function $f(x)$ defined on R by

$$f(x)=\begin{cases} x \text{ when } x \text{ is irrationa} \\ -x \text{ when } x \text{ is rational}\end{cases}$$

is continuous only at $x=0$.

24. Prove that the function f defined by

$$f(x)=\begin{cases} 1, \text{ when } x \text{ is rational} \\ -1, \text{ when } x \text{ is irrational}\end{cases}$$ is nowhere continuous.

25. Show that if $f:[a,b]\to\mathbb{R}$ is a continuous function then $f([a,b])=[c,d]$ for some real numbers c and $d, c\le d$.

26. Let $S=(0,1)$ and f be defined by $f(x)=\frac{1}{x}$ where $0<x\le 1$ (in $\mathbb{R}$). Is f uniformly continuous on S ? Justify your answer.

27. Show that the function $f(x) = 1/x^2$ is uniformly continuous on $[a,\infty[$, where $a>0$ but not uniformly continuous on $]0,\infty[$.

use $x>a \Rightarrow \frac{1}{x}<\frac{1}{a}$

28. Show that the function

$f(x)=\frac{1}{x}$

is not uniformly continuous on $[0, 1]$.

$x_n=\frac{1}{n}$ $y_n=\frac{1}{n+1}$

29. Show that the function $f(x) = \sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0,\pi)$.

$\sin\frac{1}{x}$ is disc at $x=0$ becau

30. Show that the function defined as

$$f(x)=\begin{cases} \frac{\sin 2x}{x} & \text{when } x\ne 0 \\ 1 & \text{when } x=0\end{cases}$$

$2 = LHL = RHL \ne f(0)=1$

has removable discontinuity at the origin.

31. For a real a, show that if $f:\mathbb{R}\to\mathbb{R}$ is a continuous solution of the equation

$f(x+y)=f(x)+f(y)+axy$, then

$f(x) = \frac{a}{2}x^2 + bx, where b = f(1) - \frac{a}{2}$.

32. Let X = (a, b]. Construct a continuous function f : X → ℝ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on [a + ε, b], a+ε < b? Why?

Sol^n - 32) Let us consider $f : (a, b] \to \mathbb{R}$

s.t. $f(x) = \frac{1}{x-a} \; \forall x \in (a, b]$

Clearly $f(x)$ is cont on $(a, b]$.

But NOT bounded on $(a, b]$.

$\because a < x \le b \Rightarrow 0 < x-a \le b-a$

$\Rightarrow \infty > \frac{1}{x-a} \ge \frac{1}{b-a}$

$\Rightarrow f(x)$ does not have upper bound.

Also, it is NOT UC on $(a, b]$

↳ Let $x_n = a + \frac{b-a}{n} \in (a, b]$

$y_n = a + \frac{b-a}{n+1} \in (a, b]$

s.t. $|x_n - y_n| \to 0$

& $|f(x_n) - f(y_n)| \to \frac{1}{b-a} \neq 0$ as $n \to \infty$

$\therefore$ not UC

(ii) Consider $[a+\varepsilon, b]$

$\because f$ is cont. in $[a+\varepsilon, b]$

$\Rightarrow f$ is UC in $[a+\varepsilon, b]$

WORKSHEET - 2

Differentiability (Single Variable)

1. A function f is defined on $(-1, 1)$ by

$$f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}}, x \neq 0$$
$$= 0, x = 0.$$

Prove that

i) if $0 < \beta < \alpha - 1$, f' is continuous at 0;

ii) if $0 < \alpha - 1 \leq \beta$, f' is discontinuous at 0.

i) find f' ii) check condition on α, β for cont & diff

2. Find the values of a, b and c such that

$$\lim_{x \to 0} \frac{x(a + b - \cos x) - c \sin x}{x^5} = 1.$$

3. Show that between any two roots of $e^x \cos x - 1 = 0$, there exists at least one root of $e^x \sin x - 1 = 0$.

$e^x \sin x - e^x$ ← integrate

$f(x) = e^{-x} - \cos x = 0$

4. Prove that

$$\frac{x}{1+x} < \log(1+x) < x \text{ for all } x > 0.$$

Deduce that

$$\log \frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \log 2, n$$ being a positive integer.

5. Let $f(x), (x \in (-\pi, \pi))$ be defined by $f(x) = \sin|x|$. Is continuous on $(-\pi, \pi)$? If it is continuous, then is it differentiable on $(-\pi, \pi)$?

Prove not differentiable at '0'.

$|x|$ is cont. $\sin x$ is cont $\Rightarrow \sin|x|$ ✓

6. Find the values of a and b such that

$$\lim_{x \to 0} \frac{a \sin^2 x + b \log \cos x}{x^4} = \frac{1}{2}$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \frac{\sin(a+1)x + \sin x}{x}, \text{ if } x < 0$$

$$c, \text{ if } x = 0$$

$$\frac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, \text{ if } x > 0$$

Determine the values of a, b, c for which the function is continuous at x = 0.

$a + 2 = c = \frac{1}{2}$

b independent

8. $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}(b^2 - a^2) & \text{for } 0 < x < a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x} & \text{for } a < x < b \\ \frac{1}{3}\frac{b^3 - a^3}{x} & \text{for } x > b \end{cases}$$

Prove that $f(x)$ and $f'(x)$ are continuous but $f''(x)$ is discontinuous.

9. For all real numbers $x, f(x)$ is given as:

$$f(x) = \begin{cases} e^x + a \sin x, & x < 0 \\ b(x-1)^2 + x - 2 & x \geq 0 \end{cases}$$ find the values of a and b for which f is differentiable at x = 0.

10. Find the value of $\lim_{x \to 1} \ln(1-x) \cot \frac{\pi x}{2}$.

11. Use the mean value theorem and show that $|\tan^{-1} x - \tan^{-1} y| < |x - y|$, for all x & y in R.

$f(t) = \tan^{-1} t$

12. For what choice of a and b, if any, will the function

$$f(x)=\begin{cases} ax-6, & \text{if } x>1 \\ bx^2, & \text{if } x\le 1 \end{cases}$$

become differentiable at x = 1 ?

13. Determine the constants a, b, c and d so that f is differentiable on IR.

$$f(x)=\begin{cases} ax+b & \text{if} \quad x\le 1 \\ ax^2+c & \text{if} \quad 1<x\le 2 \\ \dfrac{dx^2+1}{x} & \text{if} \quad x>2 \end{cases}$$

14. Determine where the following function from $\mathbb{R}\to\mathbb{R}$ is differentiable and find derivative $f(x)=|x|+|x+1|$

15. Apply Lagrange's mean value theorem to the function $\log(1+x)$ to show that $0<[\log(1+x)]^{-1}-x^{-1}<1\ \forall x>0$.

$\frac{x}{1+x} < \log(1+x) < x$

16. Show that $f(x)=x\tan^{-1}(1/x)$ for $x\neq 0$ and $f(0)=0$ is continuous but not differentiable at x=0.

17. Prove that $x<\sin^{-1}x<\frac{x}{\sqrt{1-x^2}}$ if $0<x<1$.

18. Use the mean value theorem to prove

$\frac{x}{1+x^2}<\tan^{-1}x<x$, if $x>0$.

19. (i) Using Taylor's theorem, show that

$$1+x+\frac{x^2}{2}<e^{x}<1+x+\frac{x^2e^x}{2}, x>0.$$

(ii) Evaluate $\underset{x\to 0+}{Lt}\left(\frac{1}{x}-\cot x\right)$.

20. (i) Let f be a function defined on IR such that $f(x+y)=f(x)+f(y), x,y\in\mathbb{R}$

If f is differentiable at one point of $\mathbb{R}$, then prove thst f is differentiable on $\mathbb{R}$.

(ii) Determine the values of p and q for which $\lim_{x\to 0}\frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

$p = q - 1$
$q - 3p = 6$

21. Shwo that $\frac{\tan x}{x}>\frac{x}{\sin x}$, for $0<x<\frac{\pi}{2}$.

22. Show that there is no real number k for which the equation $x^3-3x+k=0$ has two distinct roots in [0, 1].

No two roots are possible

23. Does the function $f(x)=|x-2|$ satisfy the conditions of Rolle's theorem in the interval [1, 3] ? Justify your answer with correct reasoning.

f' doesn't exist in $(1,3)$

24. Evaluate $\underset{x\to 0}{Lt}\left(\cot^2 x-\frac{1}{x^2}\right)$

25. Prove that between any two real roots of the equation $e^{\boxed{\sin x}}+1=0$ there is at least one real root of the equation $\tan x+1=0$.

26. Find a and b such that

$$\underset{x\to 0}{Lt}\frac{x(1+a\cos 2x)+b\sin 2x}{x^3}=1$$

27. Prove that

$$x-\frac{x^2}{2}+\frac{x^3}{3(1+x)}<\log(1+x)<x-\frac{x^2}{2}+\frac{x^3}{3} \text{ for } x>0.$$

28. A function $f(x)$ is defined as follows:

$f(x)=e^{-\frac{1}{x^2}}\sin\frac{1}{x}; x\neq 0$
$=0 ; x=0$

Examine whether or not $f(x)$ is differentiable at $x = 0$

29. Use Rolle's theorem to establish that under suitable conditions (to be stated)

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}, a<\xi<b$$

Hence or otherwise deduce the inequality $nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$ where a > b and n > 1.

30. If $f:\mathbb{R}\to\mathbb{R}$ is such that $f(x+y)=f(x)f(y)$ for all x, y in $\mathbb{R}$ and $f(x)\neq 0$ for any x in $\mathbb{R}$, show that $f'(x)=f(x)$ for all x in $\mathbb{R}$ given that $f'(0)=f(0)$ and the function is differentiable for all x in $\mathbb{R}$

31. Prove that $f(x)=x^2\sin\frac{1}{x}, x\neq 0$ and $f(x)=0$ for $x=0$ is continuous and differentiable at $x=0$ but its derivative is not continuous there.

32. Determine the set of all points where the function $f(x)=\frac{x}{1+|x|}$ is differentiable

33. Show that $\frac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1}b-\sin^{-1}a \leq \frac{b-a}{\sqrt{1-b^2}}$ for 0 < a < b < 1.

34. Let $f(x)=\begin{cases} x^P\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

obtain condition on P such that

(i) f is continuous at $x=0$ and

(ii) f is differentiable at $x=0$.

35. Show that $x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$.

36. Find a and b so that $f'(2)$ exists, where

$$f(x)=\begin{cases} \frac{1}{|x|}, & \text{if } |x|>2 \\ a+bx^2, & \text{if } |x|\leq 2 \end{cases}$$

37. Suppose that f'' is continuous on [1, 2] and that f has three zeroes in the interval (1, 2). show that f'' has at least one zero in the interval (1, 2).

38. Evaluate the following limit:

$$L = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}. \Rightarrow \log L = \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$\infty \cdot 0 = \frac{\log(2-x/a)}{\cot(\pi x/2a)}$$

39. Let f be a function defined on [0, 1] by

$$f(x)=\begin{cases} 0, \text{ if } x \text{ is irrational} \\ \frac{1}{q}, \text{ if } x=\frac{p}{q}, q\neq 0 \text{ and p, q are relatively} \end{cases}$$

Prime +ve integers. Show that f is continuous at each irrational point and discontinuous at each rational point $\frac{p}{q}$.

40. If a continuous function of x satisfies the functional equation $f(x+y)=f(x)+f(y)$, then show that $f(x)=\alpha x$ where α is constant.

41. A twice differentiable function f is such that $f(a)=f(b)=0$ and $f(c)>0$ for $a<c<b$. Prove that there is at least one value $\xi, a<\xi<b$ for which $f''(\xi)<0$.

42. Using Lagrange's mean value theorem, show, that $|\cos b-\cos a| \leq |b-a|$

43. For $x>0$, show $\frac{x}{1+x} < \log(1+x) < x$.

44. Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}, x\in\left(0,\frac{\pi}{2}\right)$

45. State Rolle's theorem. Use it to prove that between two roots of $e^x\cos x=1$ there will be a root of $e^x\sin x=1$.

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46. Let $f(x)=\begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{|x|}{2}+1 & \text{if } 2\le x \end{cases}$

What are the points of dissontinuity of f, if any ? What are the points where f is not differentiable, if any ? justify yours answers.

47. Define the function

$f(x)=x^2\sin\frac{1}{x}$, if $x\neq 0$

$=0$, if $x=0$

Find $f'(x)$. Is $f'(x)$ continuous at $x=0$? Justify your answer.

48. Using Lagranges mean value theorem, show that $1-x<e^{-x}<1-x+\frac{x^2}{2}, x>0$

49. Let $f(x) = x^2\sin\frac{1}{x}$ for $x\neq 0$

$=0$ for $x=0$

Show that f is differentiable at each point of reals but $f(x)$ is not continuous at $x=0$.

50. (i) Using Taylor's theorem with remainder show that

$x-\frac{x^3}{6}x\le\sin x\le x-\frac{x^3}{6}+\frac{x^5}{120}$ for all $x\ge 0$

50. Find the values of a and b, so that $\underset{x\to 0}{Lt}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$.

51. Prove that between any two real roots of $e^x\sin x=1$, there is at least one real root of $e^x\cos x+1=0$.

52. Let f be a function defined on IR such that $f(x+y)=f(x)+f(y), x,y\in IR$

If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$.

53. Show that the function given by

$f(x)=\begin{cases}\frac{x(e^{1/x}-1)}{(e^{1/x}+1)}, & x\neq 0\\ 0, & x=0\end{cases}$

is continuous but not differentiable at $x=0$.

OOO