Mains Test Series - 2019

Test - 10

Section - A

Ques:- 1(a) Show that S_3 and Z_6 are non-isomorphic groups and for every proper subgroup A of S_3, there exists a proper subgroup B of Z_6 such that $A \cong B$.

Solution:-

We have

$S_3 = [I, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)]$

$Z_6 = [0, 1, 2, 3, 4, 5]$

Since, in Z_6 order of $5 = O(5) = 6$ but in S_3 all elements have either orders 1, 2 or 3.

$[\, O(I) = 1, \; O(1,2) = O(2,3) = O(1,3) = 2$ and
$O(1,2,3) = O(1,3,2) = 3 \,]$

$\therefore$ S_3 & Z_6 are non isomorphic.

Alternatively,

Z_6 is abelian group and S_3 is non-abelian

$\therefore$ They are non-Isomorphic.

Since,

In Z_6, orders of proper subgroups are 2 & 3.

Also, in S_3, orders of proper subgroups are 2 and 3.

So, we can defined a map.

$$\phi : A < S_3 \longrightarrow B < Z_6 \text{ s.t.}$$

$$f(a \cdot b) = f(a) + f(b).$$

which could easily be shown as isomorphic [Please do it yourself].

Ques:- 1(b) If $R = (a, b, c, d)$ is a ring, then complete the multiplication table of R, where;

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

and

·	a	b	c	d
a	a	a	a	a
b	a	b	—	—
c	a	—	—	a
d	a	b	c	—

Is R commutative? Does it have an identity?

Solution:- First note that a is the additive identity and is the zero of the ring. Also $2b = 2c = 2d = a = 0$

We have; $x = -x$ for all $x \in R$;

Now; $(b+c)b = b^2 + cb$ by distributive law

Then; $db = b + cb$; This implies $\boxed{cb = 0 = a}$

as $\boxed{db = b}$

Now; $d(c+d) = dc + d^2 \Rightarrow d^2 = d(d+c) - dc$

$\Rightarrow d^2 = db - c = b - c = b + c = d$

$\Rightarrow c(c+d) = c^2 + cd \Rightarrow c^2 = 0 = a$;

as $c(c+d) = cb = 0 = a = cd$

$\Rightarrow (b+c)c = bc + c^2 \Rightarrow c = dc = (b+c)c = bc + 0 = bc$

$\Rightarrow$ Finally $bd = b(b+c) = b^2 + bc = b + c = d$.

∴ The complete table

·	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	a	a	a
d	a	b	c	d

Clearly, we can see that R is not commutative & it does not have an identity.

Ques: 1(c)) Test for convergence of the series.

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \cdots$$

Solution:-

Adding one more term, say 1, to the beginning of the series because it will not affect the convergence or divergence of the series, we have

$$u_n = \frac{1}{(\log n)^p} \;; \quad \therefore \; u_{n+1} = \frac{1}{[\log(n+1)]^p}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{[\log(n+1)]^p}{(\log n)^p} = \frac{[\log n + \log\{1+(1/n)\}]^p}{(\log n)^p}$$

$$= \frac{\cancel{(\log n)^p}\left[1 + \frac{1}{\log n}\log(1+1/n)\right]^p}{\cancel{(\log n)^p}}$$

$$\boxed{\frac{u_n}{u_{n+1}} = \left[1 + \frac{1}{\log n}\cdot\log(1+1/n)\right]^p}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ and so the ratio test fails to decide the convergence or divergence of the series and we proceed to apply log test.

We have $n\log\frac{u_n}{u_{n+1}} = np\log\left[1 + \frac{1}{\log n}\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$

$$n\log\frac{u_n}{u_{n+1}} = np\log\left[1 + \frac{1}{n\log n} - \frac{1}{2n^2}\cdot\frac{1}{\log n} + \cdots\right]$$

$$= np\left[\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) - \frac{1}{2}\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right)^2 + \cdots\right]$$

$$= \frac{p}{\log n}$$ + all such terms that tend to zero.

$\therefore$ for all p;

$$\boxed{\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = 0 \text{ ; which is } < 1}$$

Hence; by log test, the given series is divergent for all p.

Ques: 1(d) Prove that all the roots of $z^7-5z^3+12=0$ lies between the circles $|z|=1$ and $|z|=2$.

Solve:- given function $\Rightarrow f(z)=z^7-5z^3+12=0$

Let, C_1 be the circle $|z|=1$, and C_2 be the circle $|z|=2$

The given polynomial is

$$z^7-5z^3+12=0$$

Consider, the circle; C_1, i.e $|z|=1$

let, $f(z)=12$ & $g(z)=z^7-5z^3$

we observe that both $f(z)$ and $g(z)$ are analytic within and on C_1.

On C_1, we have $$\frac{|g(z)|}{|f(z)|}=\frac{|z^7-5z^3|}{12}$$

$$\leq \frac{|z|^7+5|z|^3}{12}$$

$$=\frac{1+5}{12}=\frac{6}{12}=\frac{1}{2}$$

Since; $|z|=1$ on C_1

Thus; $\frac{|g(z)|}{|f(z)|}<1$ on C_1.

Hence, by Rouche's theorem $f(z)+g(z)=z^7-5z^2+12$ has the same number of zeros inside C_1 as $f(z)=12$.

But $f(z)=12$ has no zeros.

hence, also; $f(z)+g(z)=z^7-5z^3+12$ has no zeros inside C_1.

Next, consider the circle C_2 i.e $|z|=2$

Let, $f(z) = z^2$ and $g(z) = 12 - 5z^2$.

Plainly $f(z)$ and $g(z)$ are both analytic within and on C_2.

On C_2, we have. $\dfrac{|g(z)|}{|f(z)|} = \dfrac{|12-5z^3|}{|z^7|}$

$$< \frac{|12| + 5|z|^3}{|z|^7} = \frac{12+5\cdot 2^3}{2^7} = \frac{52}{128} = \frac{26}{64}$$

$$= \frac{13}{32} \text{ which is } < 1$$

Thus; $\dfrac{|g(z)|}{|f(z)|} < 1$ on C_2.

Hence, by Rouche's theorem $f(z)+g(z) = z^7 - 5z^3 + 12$ has the same number of zeros as $f(z) = z^7$.

But all the seven zeros of z^7 lie inside $|z|=2$; hence all the seven zeros of $z^7 - 5z^3 + 12$ lie inside $|z|=2$.

Hence, all the roots of the equation $z^7 - 5z^3 + 12 = 0$ lie inside $|z|=2$ but outside $|z|=1$; i.e. all the roots of the given equation lie between $|z|=1$ and $|z|=2$.

Ques: 1(c) > Solve the following assignment problems :-

$$\begin{array}{cc} & \text{Man} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Work} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 12 & 30 & 21 & 15 \\ 18 & 33 & 9 & 31 \\ 44 & 25 & 24 & 21 \\ 23 & 30 & 28 & 14 \end{bmatrix} \end{array}$$

Solution :-

Step-I > Subtract minimum element of each row from all the elements of that row.

Thus the reduced matrix is -

$$\begin{bmatrix} 0 & 18 & 9 & 3 \\ 9 & 24 & 0 & 22 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

check whether all the column and rows have at least one zero, if not.

Subtract minimum element of each column from all the elements of that column, we get

$$\begin{bmatrix} 0 & 14 & 9 & 3 \\ 9 & 20 & 0 & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix}$$

Now; cover all the zeros by least number of horizontal and vertical lines. Exactly lines are required to cover all zeros should be $4 = r$.

$$\begin{bmatrix} 0 & 14 & 9 & 3 \\ 9 & 20 & 0 & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix}$$

No. of lines = 4 = r

Hence, we got the optimality, & we can make assignment given by

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline \mathrm{I} & \boxed{0} & 14 & 9 & 23 \\ \mathrm{II} & 9 & 20 & \boxed{0} & 22 \\ \mathrm{III} & 23 & \boxed{0} & 3 & \cancel{0} \\ \mathrm{IV} & 9 & 12 & 14 & \boxed{0} \end{array}$$

By picking zeros from rows containing only one zero and cancel all zeros in its respective column.

$\mathrm{I} \to M_1$, $\mathrm{II} \to M_3$, $\mathrm{III} \to M_2$, $\mathrm{IV} \to M_4$.

$\therefore$ Minimum time $= 12 + 9 + 25 + 14$

$$\boxed{\text{Min. Time} = 60}$$

Which is required solution

Ques:- 2(a)

(i) If $a \in G$. define $N(a) = \{x \in G \mid xa = ax\}$. show that $N(a)$ is a subgroup of G. $N(a)$ is usually called the normalizer or centralizer of a in G?

Solution:- Since ; $ea = ae$

$\therefore \quad e \in N(a)$

$\therefore$ $N(a)$ is non-empty set.

i.e. $N(a) \neq \phi$

Let $x, y \in N(a)$

then $\quad xa = ax \quad \& \quad ya = ay$

Now, we shall show that $y^{-1} \in N(a)$

we have $\quad ya = ay$

$$\Rightarrow (ya)^{-1} = (ay)^{-1}$$

$$\Rightarrow a^{-1}y^{-1} = y^{-1}a^{-1}$$

$$\Rightarrow a(a^{-1}y^{-1}) = a(y^{-1}a^{-1})$$

$$\Rightarrow (aa^{-1})y^{-1} = (ay^{-1})a^{-1} \quad \left[\begin{array}{c}\text{by associative}\\ \text{in } G\end{array}\right]$$

$$\Rightarrow ey^{-1} = (ay^{-1})a^{-1}$$

$$\Rightarrow y^{-1} = (ay^{-1})a^{-1}$$

$$\Rightarrow y^{-1}a = (ay^{-1})a^{-1}a$$

$$\Rightarrow y^{-1}a = (ay^{-1})e$$

$$\boxed{y^{-1}a = ay^{-1}}$$

$$\therefore \quad y^{-1} \in N(a).$$

Now, we shall show that

$$xy^{-1} \in N(a)$$

Since; $\quad y^{-1}a = ay^{-1}$

$$\Rightarrow x(y^{-1}a) = x(ay^{-1})$$

$$= (ax)y^{-1} \quad [\because xa = ax]$$

$$= a(xy^{-1}) \quad [\text{by associative in } G]$$

$$\boxed{\therefore (xy^{-1})a = a(xy^{-1})}$$

$$\therefore xy^{-1} \in N(a)$$

$\therefore N(a)$ is a subgroup of G

Ques: 2(a)(ii) If H is a subgroup of G, then by the centralizer C(H) of H we mean the set $\{x \in G \mid xh = hx \text{ all } h \in H\}$. Prove that C(H) is a subgroup of G.

Solution :-

Since ; $ex = xe \quad \forall \; x \in H$ [$\because$ H is a subgroup of G]

$\therefore \; e \in C(H)$

$\therefore \; C(H) \neq \phi$

Let $a, b \in C(H)$.

then $ax = xa$ and $bx = xb \; \forall \; x \in H$

we shall show that $b^{-1} \in C(H)$

Now we have

$$bx = xb \quad \forall \; x \in H$$

$$b^{-1}(bx) = b^{-1}(xb)$$

$\Rightarrow \; (b^{-1}b)x = (b^{-1}x)b$ (by associative)

$$\Rightarrow \; ex = (b^{-1}x)b$$

$$x = (b^{-1}x)b$$

$$xb^{-1} = (b^{-1}x)bb^{-1}$$

$xb^{-1} = (b^{-1}x)e \;\Rightarrow\; xb^{-1} = b^{-1}x \quad \forall \; x \in H$

$\therefore \; b^{-1} \in C(H)$.

Now, we shall show that $ab^{-1} \in C(H)$

Now, we have; $xb^{-1} = b^{-1}x \quad \forall \; x \in H$

$\Rightarrow \; a(xb^{-1}) = a(b^{-1}x)$

$\Rightarrow \; (ax)b^{-1} = (ab^{-1})x$ (by associative)

$\Rightarrow \; (xa)b^{-1} = (ab^{-1})x$ (by $xa = ax$).

$\Rightarrow \; x(ab^{-1}) = (ab^{-1})x \quad \forall \; x \in C(H)$

$\therefore \; ab^{-1} \in C(H)$.

$\therefore$ C(H) is a subgroup of H.

Ques: 2(iii) Given an example of a group G and a subgroup H, such that $N(H) \neq C(H)$. Is there any containing relation between $N(H)$ and $C(H)$?

Solution: (a) consider,

$$G = S_3 \text{ and}$$
$$H = A_3 = \{ I, (1,2,3), (1,3,2) \}$$

Now,

$$C(H) = \{ x \in G \mid xh = hx \ \forall\, h \in H \}$$

$$\therefore C(H) = A_3 = \{ I, (1,2,3), (1,3,2) \} \quad \text{——} (1)$$

Also,

$$N(H) = \{ x \in G \mid xH \text{ and } Hx \}.$$

$$\therefore N(H) = S_3 = G \quad \text{——} (2)$$

from (1) and (2)

$$N(H) \neq C(H)$$

Hence the result.

(b) Clearly, the centralizer $C(H)$ is a subset of the Normalizer $N(H)$.

i.e $C(H) \subseteq N(H)$

But the converse is not true.

Further; $C(H)$ is a subgroup of $N(H)$

i.e $C(H) < N(H)$

Ques: 2(b)) (i) Let $f(x) = \log x$, $x \in (0, \infty)$. Show that 'f' is uniformly continuous on $[a, \infty]$, where $a > 0$.

Solution)

given; $f(x) = \log x$

Let; $x_1, x_2 \in [a, \infty)$, where $a > 0$

If $x_1 < x_2$ then $\frac{x_2}{x_1} > 1$

and therefore;

$$0 < \log \frac{x_2}{x_1} < \frac{x_2}{x_1} - 1 \leq \frac{x_2 - x_1}{a}$$

Since, $\log(1+x) < x$ if $x > 0$

If $x_2 < x_1$ then $\frac{x_1}{x_2} > 1$, and therefore

$$0 < \log \frac{x_1}{x_2} < \frac{x_1}{x_2} - 1 \leq \frac{x_1 - x_2}{a}$$

Since. $\log(1+x) < x$ if $x > 0$

In either case; $|\log x_2 - \log x_1| \leq \frac{1}{a}|x_2 - x_1|$

This shows that 'f' is a Lipschitz function on $[a, \infty)$ with $M = \frac{1}{a}$ and therefore f is uniformly continuous on $[a, \infty)$.

Ques: 2(b) ii) A function 'f' is defined on $[0,1]$ by

$$f(x) = \sqrt{1-x^2} \quad ; x \in [0,1] \cap Q$$
$$= 1 - x \quad ; x \in [0,1] - Q$$

Show that 'f' is not integrable on $[0,1]$.

Solve). Given; function 'f' is defined on $[0,1]$ by.

$$f(x) = \begin{cases} \sqrt{1-x^2} & ; x \in [0,1] \cap Q \\ 1-x & ; x \in [0,1] - Q \end{cases}$$

f is bounded on $[0,1]$. For all $x \in (0,1)$

$\sqrt{1-x^2} > 1-x$.

Let ; $I = [0,1]$.

$f/(I \cap Q)$ is monotone decreasing on $I \cap Q$,

$f/I-Q$ is monotone decreasing on $I - Q$.

Let us take a partition P_n of $[0,1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$; where $x_r = \frac{r}{n}$.

$$\left.\begin{aligned} \text{Let; } M_r &= \sup_{x \in [x_{r-1}, x_r]} f(x) \\ m_r &= \inf_{x \in [x_{r-1}, x_r]} f(x) \end{aligned}\right\} \text{ for } f(x); \text{ for } r = 1, 2, 3, \ldots, n.$$

Since, $f/(I \cap Q)$ is monotone decreasing on $[x_{r-1}, x_r] \cap Q$

$$\sup_{x \in [x_{r-1}, x_r] \cap Q} f(x) = f(x_{r-1}) = \sqrt{1 - x_{r-1}^2} = \sqrt{1 - \left(\frac{r-1}{n}\right)^2}$$

Since; $f/(I-Q)$ is monotone decreasing on $[x_{r-1}, x_r]-Q$ and x_{r-1} is rational

$$\sup_{x\in[x_{r-1}, x_r]-Q} f(x) = \lim_{n\to\infty} f(u_n);$$

where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $\boxed{x_{r-1} = 1 - x_{r-1} = 1 - \frac{r-1}{n}}$

Since; $1 - \frac{r-1}{n} < \sqrt{1-\left(\frac{r-1}{n}\right)^2}$

$$\sup_{x\in[x_{r-1}, x_r]} f(x) = \sqrt{1-\left(\frac{r-1}{n}\right)^2}$$

That is; $\boxed{M_r = \sqrt{1-\left(\frac{r-1}{n}\right)^2}}$

Since, $f/(I\cap Q)$ is monotone decreasing on $[x_{r-1}, x_r]\cap Q$

$$\inf_{x\in[x_{r-1}, x_r]\cap Q} f(x) = f(x_r) = \sqrt{1-\left(\frac{r}{n}\right)^2}$$

Since; $f/(I-Q)$ is monotone decreasing on $[x_{r-1}, x_r]-Q$ and x_r is rational

$$\inf_{x\in[x_{r-1}, x_r]} f(x) = \lim_{n\to\infty} f(v_n)$$

Where $\{v_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $\boxed{x_r = 1 - x_r = 1 - \frac{r}{n}}$

Since; $1 - \frac{r}{n} \leq \sqrt{1-\left(\frac{r}{n}\right)^2}$

$$\inf_{x \in [x_{r-1}, x_r]} f(x) = 1 - \frac{r}{n}$$

$$\therefore \boxed{m_r = 1 - \frac{r}{n}}$$

$$U(P_n, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

$$= \frac{1}{n}\left\{\sqrt{1-\left(\frac{0}{n}\right)^2} + \sqrt{1-\left(\frac{1}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n-1}{n}\right)^2}\right\}$$

$$L(P_n, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$

$$= \frac{1}{n}\left\{\left(1-\frac{1}{n}\right) + \left(1-\frac{2}{n}\right) + \cdots + \left(1-\frac{n}{n}\right)\right\}$$

$$= \frac{1}{n^2}\{1+2+3+\cdots+(n-1)\} = \frac{\cancel{n}(n-1)}{2 \cdot n^{\cancel{2}}} = \frac{n-1}{2n}.$$

Let, us consider the sequence of partition $\{P_n\}$ of $[0,1]$.

Here; $\|P_n\| = \frac{1}{n}$ and $\lim_{n\to\infty} \|P_n\| = 0$

$$\overline{\int_0^1} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \sqrt{1-x^2}\,dx$$

Since, $\sqrt{1-x^2}$ is integ. ble on $[0,1]$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

$$\boxed{\therefore \overline{\int_0^1} f = \lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2} = \frac{\pi}{4}}$$

and $\boxed{\underline{\int_0^1} f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}}$.

since; $\overline{\int_0^1} f \neq \underline{\int_0^1} f$; f is not integrable on $[0,1]$.

Ques:- 2(c)) By method of contour integration

Prove that $\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2} = \pi \cdot \frac{1-p+p^2}{1-p}$; $0<p<1$

Solution:

Let $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta+p^2} = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta}{1-2p\cos 2\theta+p^2} d\theta$

$I = \frac{1}{2}$ real part of $\int_0^{2\pi} \frac{1+e^{6i\theta} \, d\theta}{1-p(e^{2i\theta}+e^{-2i\theta})+p^2}$

$= \frac{1}{2}$ real part of $\int_C \frac{1+z^6}{1-p\left(z^2+\frac{1}{z^2}\right)+p^2} \frac{dz}{iz}$

$\left[\because e^{i\theta}=z;\ d\theta = \frac{dz}{iz}\right]$

Where 'c' denotes the unit circle $|z|=1$.

$= \frac{1}{2}$ real part of $\frac{1}{i}\int_C \frac{z(1+z^6)}{(1-pz^2)(z^2-p)} dz$

$= \frac{1}{2}$ real part of $\frac{1}{i}\int_C f(z)\,dz$.

where; $f(z) = \frac{z(1+z^6)}{(1-pz^2)(z^2-p)}$

Poles of $f(z)$ are given by $(1-pz^2)(z^2-p)=0$

Thus; $z=\pm\sqrt{p}$ and $z=\pm\frac{1}{\sqrt{p}}$ are the simple poles.

The only poles which lie with C are

$z=\pm\sqrt{p}$ as $p<1$

Residue at $z=\sqrt{p}$

$= \lim_{z\to\sqrt{p}} \cancel{(z-\sqrt{p})} \frac{z(1+z^6)}{(1-pz^2)\cancel{(z-\sqrt{p})}(z+\sqrt{p})}$

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$$= \lim_{z \to \sqrt{p}} \frac{z(1+z^6)}{(1-pz^2)(z+\sqrt{p})}$$

$$= \frac{\cancel{\sqrt{p}}(1+p^3)}{2\cancel{\sqrt{p}}(1-p^2)} = \frac{1}{2} \times \frac{1+p^3}{1-p^2}.$$

Now; residue at $z \Rightarrow -\sqrt{p}$

$$= \lim_{z \to -\sqrt{p}} \cancel{(z+\sqrt{p})} \frac{z(1+z^6)}{(1-pz^2)(z-\sqrt{p})\cancel{(z+\sqrt{p})}}$$

$$= \lim_{z \to -\sqrt{p}} \frac{z(1+z^6)}{(1-pz^2)(z-\sqrt{p})} = \frac{\cancel{-\sqrt{p}}(1+p^3)}{\cancel{-2\sqrt{p}}(1-p^2)}$$

$$= \frac{1}{2} \cdot \frac{(1+p^3)}{1-p^2}.$$

$\therefore$ Total sum of residue $= \left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1+p^3}{1-p^2}\right) = \frac{1+p^3}{1-p^2}$

Hence, By Cauchy's residue theorem, we have.

$\int_c f(z)\,dz = 2\pi i \times$ Sum of residues within the contour

$$= 2\pi i \cdot \frac{1+p^3}{1-p^2}.$$

$\therefore$ $I = \frac{1}{2}$ real part of $\frac{1}{i}\int_c f(z)\,dz = \frac{1}{\cancel{i}} \times 2\pi \cancel{i} \times \frac{1+p^3}{1-p^2}$

$$= \frac{1}{\cancel{2}} \text{ real part of } \cancel{2}\pi \cdot \frac{1+p^3}{1-p^2}$$

$$= \pi \frac{1+p^3}{1-p^2} = \pi \frac{\cancel{(1+p)}(1-p+p^2)}{(1-p)\cancel{(1+p)}}$$

$$= \pi \frac{(1-p+p^2)}{1-p}$$ Hence proved.

$$\boxed{\therefore \int_0^{2\pi} \frac{\cos^3 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p}}$$

Ques: 3(a)(i) Find the elements in Z_{12} which are zero divisors.

Solution :- Verify that

in Z_{12}, $[0]=[2][6]$
$[0]=[3][4]$
$[0]=[8][3]$
$[0]=[9][4]$
$[0]=[10][6]$

Thus, $[2], [3], [4], [6], [8], [9]$ and $[10]$ are zero divisors of Z_{12}.

For ; $k=1, 5, 7$ or 11

$[k]$ is a unit in Z_{12} as in these cases

$\gcd(k, 12)=1$.

So these are not zero divisor.

(ii) Is there any integral domain which has six elements?

Solution: Let, R be an Integral domain

If $|R|=6$; then $(R,+)$ is an abelian group of order 6.

Then; $(R,+)$ is cyclic.

Suppose, it is generated by $a \in R$.

Then $2a, 3a \neq 0$ but $2a \cdot 3a = 6a^2 = 0$

as the order of $(R,+)$ is 6.

Thus; there is no integral domain with six elements.

Ques: 3(b)(i) Show that $\int_0^\infty \frac{b\sin ax - a\sin bx}{x^2}dx = ab\log\left(\frac{b}{a}\right)$
$0 < b < a$.

Solution)

given function; $I = \int_0^\infty \frac{b\sin ax - a\sin bx}{x^2}dx$

Let, $\phi(x) = \frac{\sin x}{x}$; $x > 0$

Then ϕ is continuous on $(0,\infty)$

Let $\lim_{x\to 0^+} \phi(x) = 1$; $\lim_{x\to\infty} \phi(x) = 0$.

Therefore;

$$\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}dx = [1-0]\log\left[\frac{b}{a}\right]$$

or. $$\int_0^\infty \frac{\frac{\sin ax}{ax} - \frac{\sin bx}{bx}}{x}dx = \log\left[\frac{b}{a}\right]$$

$$\int_0^\infty \frac{b\sin ax - a\sin bx}{abx \cdot x}dx = \log\left[\frac{b}{a}\right]$$

$$= \frac{1}{ab}\int_0^\infty \frac{b\sin ax - a\sin bx}{x^2}dx = \log\left[\frac{b}{a}\right]$$

$$\Rightarrow \boxed{\int_0^\infty \frac{b\sin ax - a\sin bx}{x^2}dx = ab\log\left[\frac{b}{a}\right]}$$

Hence, proved.

Ques: 3(b) ii) Prove that the series $\sum (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in any closed and bounded interval $[a,b]$, but does not converge absolutely for any real x.

Solve:- given; series $\sum (-1)^n \frac{x^2+n}{n^2}$

Let $u_n = (-1)^n$, $v_n = \frac{x^2+n}{n^2}$; $x \in [a,b]$

Let; $S_n = u_1 + u_2 + u_3 + \cdots + u_n$.

Then, the sequence $\{S_n\}$ is bounded

$$v_{n+1} - v_n = \frac{x^2+n+1}{(n+1)^2} - \frac{x^2+n}{n^2} =$$

$$= x^2\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] + \left[\frac{1}{n+1} - \frac{1}{n}\right] < 0$$

for any $x \in [a,b]$

This shows that $\{v_n\}$ is a monotone decreasing sequence for each x in $[a,b]$.

$$\lim_{n\to\infty} v_n(x) = 0 \quad \text{for all } x \in [a,b]$$

Thus, the sequence of functions $\{v_n\}$ is such that each v_n is continuous on $[a,b]$, the sequence converges to a continuous function on $[a,b]$ and $\{v_n\}$ is a monotone decreasing sequence on $[a,b]$.

By Dini's theorem, the sequence $\{v_n\}$ is uniformly convergent on $[a,b]$.

Since;
(i) The sequence $\{S_n\}$ is uniformly bounded on $[a,b]$, &
(ii) the sequence $\{V_n\}$ is monotone decreasing sequence on $[a,b]$ and converges uniformly to 0.

$\therefore$ The series $\sum (-1)^n \frac{x^2+n}{n^2}$ is uniformly convergent on $[a,b]$ by Dirichlet's test.

Second part) let, the series be $(-1)^n V_n(x)$
For each real x, the series $\sum V_n(x)$ is a series of positive terms.

Let, $u_n = \frac{1}{n}$

Then; $\lim_{n\to\infty} \frac{V_n}{u_n} = 1.$

By comparision test, the series $\sum V_n(x)$ is divergent for each real value of x.

Since, $\{V_n\}$ is a monotone decreasing sequence for each $x \in R$, and
$\lim_{n\to\infty} V_n(x) = 0$ for each $x \in R$.

The series $\sum (-1)^{n-1} V_n(x)$ is convergent for each $x \in R$.

$\therefore$ The series $\sum (-1)^{n-1} \frac{x^2+n}{n^2}$ does not converge absolutely for $x \in R$.

Ques:-3(c)) Using simplex Method solve the LP problem:

$$\text{Max}\cdot Z = 3x_1 + 5x_2 + 4x_3.$$

$$\text{s.c.} \quad 2x_1 + 3x_2 \le 8$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$2x_2 + 5x_3 \le 10$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

Solution:

Converting in standard form:-

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3.$$

$$\text{s.c.} \quad 2x_1 + 3x_2 + s_1 = 8$$
$$3x_1 + 2x_2 + 4x_3 + s_2 = 15$$
$$2x_2 + 5x_3 + s_3 = 10$$
$$\text{and } x_1, x_2, x_3 \ge 0, \; s_1, s_2, s_3 \ge 0.$$

where ; s_1, s_2, s_3 are slack variables.

Now, the IBFS is -

$$x_1 = x_2 = x_3 = 0 \;;\; s_1 = 8, \; s_2 = 15, \; s_3 = 10$$

The initial simplex table is :-

C_B	c_j / Basis	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	b	θ
0	S_1	2	(3)	0	1	0	0	8	$8/3 \rightarrow$
0	S_2	3	2	4	0	1	0	15	$15/2$
0	S_3	0	2	5	0	0	1	10	5
	$Z_j = \Sigma C_B a_{ij}$	0	0	0	0	0	0		
	$C_j = C_i - Z_j$	3	5 $\uparrow$	4	0	0	0		

Here; s_1 is outgoing variable & x_2 is incoming variable.

The key element is 3, making it unity and all other elements in that column to zero. The New table is

C_B	c_j / basis	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	b	θ
5	x_2	2/3	1	0	1/3	0	0	8/3	–
0	s_2	5/3	0	4	-2/3	1	0	29/3	9/12
0	s_3	-4/3	0	(5)	-2/3	0	1	19/3	14/15 →
	z_j	10/3	5	0	5/3	0	0	40/3	
	c_j	-1/3	0	4 ↑	-5/3	0	0		

Here ; coming variable is x_3 & outgoing is s_3. The key element = 5, making it unity & all other elements in that column as '0'.

C_B	c_j / basis	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	b	θ
5	x_2	2/3	1	0	1/3	0	0	8/3	4
0	s_2	41/15	0	0	-2/15	1	-4/5	89/15	89/41 →
4	x_3	-4/15	0	0	-2/15	0	1/5	14/15	–
	z_j	34/15	5	4	17/15	0	4/5.		
		11/15 ↑	0	0	-17/15	0	-4/5		

$x_1 \to$ incoming ; $s_2 \to$ outgoing ; $\frac{41}{15}$ Key element. The new simplex Table.

C_B		3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	b
5	x_1	0	1	0	15/41	-10/41	-2/41	50/41
3	x_2	1	0	0	-23/41	15/41	-12/41	89/41
4	x_3	0	0	1	-2/41	4/41	5/41	62/41
	z_j	3	5	4	15/41	11/41	24/41	$\frac{765}{41}$
		0	0	0	-15/41	-11/41	-24/41	

Since; all $c_j = 0$; hence optimality obtained.

$x_1 = \frac{89}{41}$;

$x_2 = \frac{50}{41}$

$x_3 = 62/41$

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$= 3 \times \frac{89}{41} + 5 \times \frac{50}{41} + 4 \times \frac{62}{41}$$

$$= \frac{247 + 250 + 248}{41} = \frac{765}{41}.$$

$\boxed{\therefore \text{ Max } Z = \frac{765}{41}}$ which is required solution.

Ques: 4(a) Let R be a commutative ring with identity; $1 \neq 0$. Then an ideal M of R is maximal if and only if R/M is a field.

Solution:- Suppose;

M is a maximal ideal of R.

Since; R is a commutative ring with identity, it follows that the quotient ring R/M is a commutative ring with identity $1+M$.

If $1+M = 0+M$; then $1 \in M$

which implies that $M=R$,

So; M cannot be a maximal ideal.

Hence, $1+M \neq 0+M$.

Now, prove that R/M is a field.

We show that every non-zero element of R/M is a unit.

Let; $a+M$ be a non-zero element of R/M.

Then; $a+M \neq 0+M$ implies that $a \notin M$.

Let $J = \{ra+m \mid r \in R \text{ and } m \in M\}$

It is easy to see that J is an ideal of R and $m = 0a+m \in J$ for all $m \in M$.

Hence; J contains M.

Now, let $a = 1a+0 \in J$ and $a \notin M$

implies, $M \subset J$

Since, M is a maximal ideal, it follows that $\boxed{J=R}$

Now; $1 \in R = J$

Hence, $1 = ra + m$ for some $r \in R$ and $m \in M$.

Then; $1+M = (ra+m)+M = (ra+M)+(m+M) =$

$$(r+M)(a+M)+(0+M) = (r+M)(a+m)$$

This shows that $a+M$ is a unit in R/M

and hence R/M is a field.

Conversely; assume R/m is a field.

Since; R/M contains at least two elements $M \subset R$

Let, J be an ideal of R; such that $M \subset J \subseteq R$.

Then, there exists $u \in J$ such that $u \notin M$.

Hence; $u+M \neq 0+M$ in the field R/m. Then there exists $v+M$ in R/M such that

$$\boxed{(v+M)(u+M) = 1+M}$$

This shows that $vu + M = 1+M$

i.e $1 - vu \in M \subset J$.

Now, $u \in J$; Hence $uv \in J$

From this, it follows that

$$\boxed{1 = (1-vu)+vu} \in J.$$

and hence; $\boxed{J=R}$

So; M is a maximal ideal.

Alternative Proof :-

Let, R be a commutative ring with identity M be an ideal of R.

Then, the quotient ring R/M is also commutative and contains identity.

There is a canonical epimorphism of R onto R/M.

Thus,

M is Maximal ideal.

$\Leftrightarrow$ There is no proper ideal (other than M) of R containing M.

$\Leftrightarrow$ R/M is simple

$\Leftrightarrow$ R/M is a field.

Ques: 4(b) If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}\left(u_n + \frac{9}{u_n}\right)$ for $n \geq 1$, prove that the sequence $\{u_n\}$ converges to 3.

Solution:-

given; If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}\left(u_n + \frac{9}{u_n}\right)$; $n \geq 1$.

$$\Rightarrow \left(\frac{u_n^2 + 9}{u_n}\right) = 2u_{n+1} \Rightarrow u_n^2 - 2u_n u_{n+1} + 9 = 0$$

This is a quadratic equation in u_n having real roots. Therefore $4u_{n+1}^2 - 36 \geq 0$

This implies ; $u_{n+1} \geq 3$ for all $n \geq 1$.

Since; $u_{n+1} > 0$ for all $n \geq 1$.

$$u_n - u_{n+1} = u_n - \frac{1}{2}\left(u_n + \frac{9}{u_n}\right) = \frac{1}{2}\left(u_n - \frac{9}{u_n}\right)$$

$$= \frac{1}{2}\left(\frac{u_n^2 - 9}{u_n}\right) \geq 0 \text{ for all } n \geq 2.$$

Therefore; $u_{n+1} \leq u_n$ for all $n \geq 2$.

This shows that the sequence $\{u_n\}_{n=2}^{\infty}$ is monotone decreasing sequence bounded below and hence the sequence is convergent.

Let $\lim u_n = l$

$$u_{n+1} = \frac{1}{2}\left[u_n + \frac{9}{u_n}\right] \text{ for } n \geq 1;$$

$$\Rightarrow l = \lim_{u_n \to \infty} \frac{1}{2}\left[u_n + \frac{9}{u_n}\right] \Rightarrow \frac{1}{2}\left(l + \frac{9}{l}\right)$$

$$\Rightarrow 2l \cdot l = l^2 + 9 \Rightarrow l^2 = 9 \Rightarrow l = \pm 3.$$

$\therefore \boxed{l = 3}$; since $l \geq 0$ i.e $u_n \geq 0$

Ques:- 4 (c)(i) for the function $f(z)$, defined by

$$f(z) = \begin{cases} \left(\frac{\bar{z}}{z}\right)^2 & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

show that the C-R equations are satisfied at $(0,0)$ but the function is not differentiable at $0+i0$.

Solution:- given; $f(z) = \begin{cases} \left(\frac{\bar{z}}{z}\right)^2 & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$f'(0) = \lim_{z\to 0} \left(\frac{\bar{z}}{z}\right)^2 = \lim_{(x,y)\to(0,0)} \left(\frac{x-iy}{x+iy}\right)^2$$

$$f'(0) = \begin{cases} 1 & , z \to 0 \text{ along the x-axis} \\ 1 & ; z \to 0 \text{ along the y-axis} \\ -1 & ; z \to 0 \text{ along the line } x = y \end{cases}$$

As, $f'(0)$ turns up with different values for different paths, $f'(0)$ does not exist; i.e. f is not differentiable at $0+i0$.

To check validity of the C-R equation at $(0,0)$ write $f(z) = u(x,y) + iv(x,y)$ and.

$$f(0) = 0 + i0$$

$$u(0,0) = 0 \quad ; \quad v(0,0) = 0.$$

Then; $f(z) = \frac{(\bar{z})^3}{z \cdot \bar{z}} = \frac{(x - iy)^3}{x^2+y^2} = \frac{x^3 - i3x^2y - 3xy^2 + iy^3}{x^2+y^2}$

and $u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$; $v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2}$

Then; $u_x = \lim_{\Delta x \to 0} \dfrac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = 1$

$$u_y = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = 0$$

$$v_x = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = 0$$

$$v_y = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y} = 1$$

Thus; we observe that

$$u_x = v_y$$
$$v_x = -u_y \quad \text{at } z = (0,0)$$

To make the C-R equation as sufficient, an additional condition on partial derivative is induced and this is the condition of continuity.

Ques: 4(c)-(ii) If a function $f(z)$ is analytic for all finite values of z and as $|z| \to \infty$

$$\boxed{|f(z)| = A(|z|^k)}$$

then $f(z)$ is a polynomial of degree $\leq k$.

Solution:-

Since; $f(z)$ is an analytic in the finite part of plane, therefore by Taylor's series.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad ; \text{ where } |z| < R.$$

Now; if max. $|f(z)| - M(r)$ on the circle $|z| = 1, (r-R)$,

Then by Cauchy's inequality, we have

$$|a_n| \leq \frac{M(r)}{r^n} \text{ for all values of } n.$$

$$|a_n| = \frac{A(|z|^k)}{r^n} ; \text{ Since; } M(r) = |f(z)| = A|z|^k$$
when $z \to \infty$

$$|a_n| = \frac{A r^k}{r^n} \quad (\text{since, on the circle } |z| = r)$$

$$|a_n| = A r^{k-n} ; \text{ where } r \text{ is large.}$$

So, when $r \to \infty$, the right hand side tends to zero if $n > k$

i.e $\boxed{a_n = 0 \text{ for values of } n > k}$

i.e. all the co-efficients a_n for which $n > k$ become zero.

$\boxed{\text{Hence; } f(z) \text{ is a polynomial of degree } n \text{ become } n \leq k.}$

Ques:-4)d) A firm manufactures two products A&B on which the profit earned per unit are ₹ 3 and ₹4 respectively. Each product is processed on two machines M_1 & M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 while processing of product M_2 requires 1 minute on M_1 and 1 minute on M_2. M_1 is not available for more than 7 hours 30 minutes while M_2 is available for 10 hours during any working day. Find the number of units of products A and B need to be manufactured to get maximum profit. Formulate this as an LP problem model and solve by graphical method.

Solution:- A firm manufactures two products 'A' and 'B' on which the profit earned per unit are ₹ 3 and ₹ 4 respectively.

i.e $\boxed{\text{Max. } Z = 3A + 4B}$

Working time of Machine one (M_1) is not more than 7 hours 30 minutes i.e 450 minutes

Product A takes 1 minute and so 1 minute for B.

$$\text{i.e. } M_1 \Rightarrow A + B \leq 450 \quad \text{—(1)}$$

Working time of Machine two (M_2) is 10 hours i.e 600 minutes.

A takes 2 minutes and B takes 1 minute on M_2.

$$\therefore M_2 \Rightarrow 2A + B = 600 \quad \text{—(2)}$$

and No. of manufactured product should ~~not be~~

be more than zero;

$$i.e \quad A, B \geq 0 \quad \text{—— (3)}$$

Hence, The LPP is

$$\text{Max. } z = 3A + 4B$$

$$\text{s.c.} \Rightarrow \quad A + B \leq 450$$
$$2A + B = 600$$
$$A, B \geq 0$$

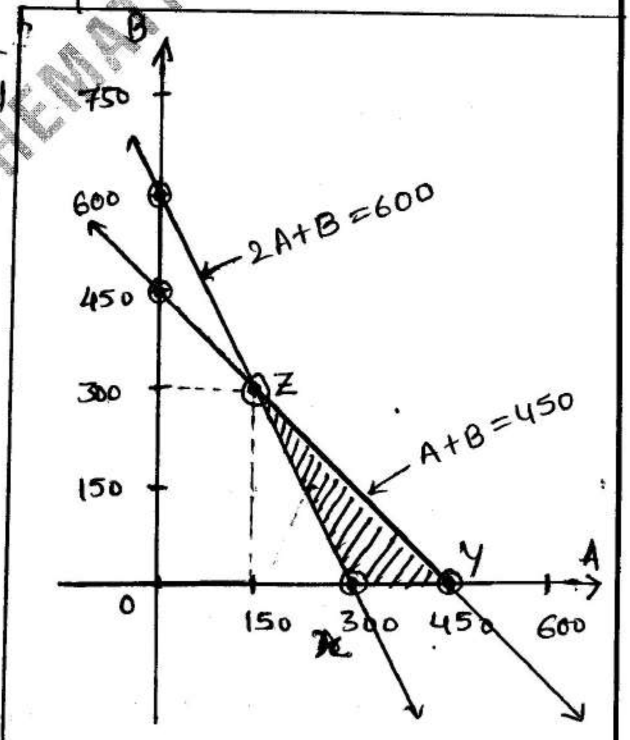
for graphical Method

$A+B = 450$ is the line and below it the region it will cover i.e optimal region.

$2A+B = 600$ - is a line.

Hence from these two equation.

The graphical representation, shows the shaded optimal region.



This consist of three optimal points

$x \rightarrow (300, 0)$

$y \rightarrow (450, 0)$

$z \rightarrow (150, 300)$

$z_{max \text{ at } x} = 3 \times 300 = 900$

$z_{max \text{ at } y} = 3 \times 450 = 1350$

$z_{max \text{ at } z} = 3 \times 150 + 4 \times 300$

$= 450 + 1200 = 1650$

$\therefore$ z_{max} at $(150, 300)$ gives us maximum value.

$$\boxed{\therefore \quad z_{max} (\text{Profit}) = 1650 \; ; \text{ with } A = 150, B = 300}$$

Section-B

Ques: 5(a) find the partial differentiable equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

Solution:-

Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; be the equation of the plane intercept from, so that $a+b+c=1$

Thus ; we have ;

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b}=1 \quad \text{——} (1)$$

Differentiating eqⁿ (1) w.r.t x & y, we have

$$\frac{1}{a}+\frac{p}{1-a-b}=0 \quad \text{or} \quad \frac{p}{1-a-b}=-\frac{1}{a} \quad \text{——} (2)$$

and $$\frac{1}{b}+\frac{q}{1-a-b}=0 \quad \text{or} \quad \frac{q}{1-a-b}=\frac{-1}{b} \quad \text{——} (3)$$

from eqⁿ (2) and (3), we get

$$\boxed{\frac{p}{q}=\frac{b}{a}} \quad \text{——} \quad (4)$$

from eqⁿ (2) and (4), we get

$$pa=a+b-1=a+\frac{p}{q}a-1.$$

$$a\left(1+\frac{p}{q}-p\right)=1$$

Therefore;

$$\boxed{a = q/(p+q-pq)} \quad \text{——— (5)}$$

similarly from Eqn (3) & (4), we have.

$$\boxed{b = p/(p+q-pq)} \quad \text{——— (6)}$$

Substituting the values of a and b from Eqs (5) and (6) respectively to Eq.(1), we have;

$$\frac{p+q-pq}{q}x + \frac{p+q-pq}{p}y + \frac{p+q-pq}{-pq}z = 1$$

Or.

$$\left(\frac{x}{q} + \frac{y}{p} - \frac{z}{pq}\right)(p+q-pq) = 1$$

$$\Rightarrow \quad \frac{x}{q} + \frac{y}{p} - \frac{z}{pq} = \frac{1}{p+q-pq}$$

$$\boxed{px + qy - z = \frac{pq}{p+q-pq}}$$

Which is the required PDE.

Ques:-5(b)) find the complete integral of

$$\boxed{(p^2+q^2)x = pz}$$

Solve:-

given: $f(x,y,z,p,q) = (p^2+q^2)x - pz = 0$ ——(1)

The usual charpit's auxillary equations are—

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \cdots$$

or $$\frac{dp}{\cancel{p^2}+q^2-\cancel{p^2}} = \frac{dq}{-pq} = \cdots$$

So that $\frac{dp}{q\cancel{q}} = \frac{dq}{-p\cancel{q}}$ or $-p\,dp = q\,dq$

$\Rightarrow p\,dp + q\,dq = 0$

Integrating $\Rightarrow$ $p^2+q^2 = a^2$ ——— (2)

solving (1) and (2): $p = a^2x/z$ & $q = \left(\frac{a}{z}\right)\sqrt{z^2-a^2x^2}$ ——(3)

$\therefore dz = p\,dx + q\,dy = \frac{a^2x}{z}dx + \frac{a}{z}\sqrt{z^2-a^2x^2}\,dy.$

$\Rightarrow \frac{z\,dz - a^2x\,dx}{\sqrt{z^2-a^2x^2}} = a\,dy.$

putting $z^2-a^2x^2 = t$

so that; $2(z\,dz - a^2x\,dx) = dt$

$$z\,dz - a^2x\,dx = \frac{dt}{2}$$

we get

$$\frac{\frac{1}{2}dt}{\sqrt{t}} = a\,dy$$

$$\Rightarrow \frac{1}{2}.(t)^{-1/2}dt = a\,dy$$

Integrating both side.

$$t^{1/2} = ay + b$$

put $t = z^2 - a^2x^2$

$$\sqrt{z^2 - a^2x^2} = ay + b$$

Squaring both side.

$$z^2 - a^2x^2 = (ay+b)^2$$

$$\boxed{z^2 = a^2x^2 + (ay+b)^2}$$

Which is required PDE

Ques:-5(c)> For an integral $\int_{-1}^{1} f(x)dx$, show that the two-point Gauss quadrature rule is given by

$$\int_{-1}^{1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}}\right)$$

using this rule, estimate $\int_{2}^{4} 2xe^{x}dx$.

Solution:-

Since any finite interval $[a,b]$ can always be transformed to $[-1,1]$, using the transformation

$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

we consider the integral in the form

$$\boxed{\int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} \lambda_k f(x_k)} \quad \text{--- (1)}$$

where the weight function $w(x) = 1$.

Now considering, two point formula ;(i.e $n=1$)

The formula is given by -

$$\boxed{\int_{-1}^{1} f(x)\, dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)} \quad \text{--- (2)}$$

The method has four unknowns, $x_0, x_1, \lambda_0, \lambda_1$.

Making the method exact for $f(x) = 1, x, x^2, x^3$, we get,

$$f(x) = 1 \quad : \quad 2 = \lambda_0 + \lambda_1 \quad \text{--- 3(a)}$$

$$f(x) = x \quad : \quad 0 = \lambda_0 x_0 + \lambda_1 x_1 \quad \text{--- 3(b)}$$

$$f(x) = x^2 \quad : \quad \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 \quad \text{--- 3(c)}$$

$$f(x) = x^3 \quad : \quad 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 \quad \text{--- 3(d)}$$

Eliminating λ_0 from 3(b), 3(d), we get

$\lambda_1 x_1^3 - \lambda_1 x_1 x_0^2 = 0$

or $\lambda_1 x_1 (x_1 - x_0)(x_1 + x_0) = 0$

Since $\lambda_1 \neq 0$, $x_0 \neq x_1$,

we get $x_1 + x_0 = 0$ or $x_1 = -x_0$

Note that if $x_1 = 0$, then from 3(b)

we get ; $x_0 = 0$ since $\lambda_0 \neq 0$.

Therefore ; $x_1 \neq 0$

Substituting in 3(b), we get

$$\lambda_0 - \lambda_1 = 0 \Rightarrow \lambda_0 = \lambda_1$$

Substituting in 3(a), we get

$$\lambda_0 = \lambda_1 = 1.$$

Using 3(c), we get $x_0^2 = 1/3$ or $x_0 = \pm \frac{1}{\sqrt{3}}$

and $x_1 = \pm \frac{1}{\sqrt{3}}$

Therefore, the two point Gaussian Legendre method is given by -

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Hence proved.

Hence; $I = \int_2^4 2x e^x \, dx.$

convert the limit (2,4) to (-1,1) for gauss-legendre formula.

$$x = \frac{1}{2}(4-2)u + \frac{1}{2}(4+2)$$

$$\boxed{x = u + 3}$$

$$dx = du.$$

Now, two point formula.

$$I = \int_2^4 2x e^x dx = \int_{-1}^{1} 2(u+3) e^{(u+3)} du$$

$$I = \int_{-1}^{1} 2(u+3) e^{u+3} du = 1 \cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right)$$

$$I = 1 \cdot \left[2\left(\frac{-1}{\sqrt{3}} + 3\right) e^{\left(\frac{-1}{\sqrt{3}}+3\right)} \right] + \left[2\left(\frac{1}{\sqrt{3}}+3\right) e^{3+\frac{1}{\sqrt{3}}} \right]$$

$$I = 2\left[\frac{3\sqrt{3}-1}{\sqrt{3}} e^{3-1/\sqrt{3}} + \frac{3\sqrt{3}+1}{\sqrt{3}} e^{3+1/\sqrt{3}} \right]$$

$$I = \frac{2}{\sqrt{3}} e^3 \left[(3\sqrt{3}-1) e^{-1/\sqrt{3}} + (3\sqrt{3}+1) e^{1/\sqrt{3}} \right]$$

$$\therefore \; I = \frac{2}{\sqrt{3}} e^3 \left[(3\sqrt{3}-1) e^{-1/\sqrt{3}} + (3\sqrt{3}+1) e^{1/\sqrt{3}} \right]$$

$$\boxed{I = \int_2^4 2x e^x dx = \frac{2}{\sqrt{3}} e^3 \left[(3\sqrt{3}-1) e^{-1/3} + (3\sqrt{3}+1) e^{1/\sqrt{3}} \right]}$$

Ques:- 5 (d) Convert:

(i) 46655 given to be in the decimal system into one in base 6.

Solve:-

$$\begin{array}{r|l|l} 6 & 46655 & \\ \hline & 7775 & 5 \\ \hline & 1295 & 5 \\ \hline & 215 & 5 \\ \hline & 35 & 5 \\ \hline & 5 & 5 \end{array}$$

$$\boxed{\therefore \ (46655)_{10} \longleftrightarrow (555555)_6}$$

(ii) $(11110.01)_2$ into a number in the decimal system.

Solve:- $(11110.01)_2 \longleftrightarrow (30.25)_{10}$

$= \left(1\times2^4+1\times2^3+1\times2^2+1\times2^1+0\times2^0 \,.\, 0\times2^{-1}+1\times2^{-2}\right)_{10}$

$= \left[16+8+4+2+0 \,.\, 0+\frac{1}{4}\right]_{10}$

$= \left[30.25\right]_{10}$

$$\boxed{\therefore (11110.01)_2 \longleftrightarrow (30.25)_{10}}$$

Ques: 5(c)) If $u = \frac{ax - by}{x^2+y^2}$; $v = \frac{(ay+bx)}{x^2+y^2}$; $w=0$

investigate the nature of motion of the liquid.

Solve:- Given: $u = \frac{ax-by}{x^2+y^2}$; $v = \frac{(ay+bx)}{x^2+y^2}$; $w = 0$ —①

from ① $\frac{\partial u}{\partial x} = \frac{a(x^2+y^2) - 2x(ax-by)}{(x^2+y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial y} = \frac{a(x^2+y^2) - 2x(ay+bx)}{(x^2+y^2)^2} = \frac{-ax^2+ay^2-2bxy}{(x^2+y^2)^2}$$

we see that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and

hence, the equation of continuity satisfied by ①

∴ ① represents a possible motion.

Moreover ① represents a 2-dimensional motion and hence vorticity components are given by —

$$\Omega_x = 0 \; ; \; \Omega_y = 0 \; ; \; \Omega_z = (\partial v/\partial x) - (\partial u/\partial y) \quad —②$$

from ① → $\frac{\partial u}{\partial y} = \frac{-b(x^2+y^2) - 2y(ax-by)}{(x^2+y^2)^2} = \frac{-bx^2+by^2-2ayx}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial x} = \frac{b(x^2+y^2) - 2x(ay+bx)}{(x^2+y^2)} = \frac{-bx^2+by^2-2axy}{(x^2+y^2)^2}$$

∴ $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

∴ from ①; $\Omega_z = 0$. Thus; $\Omega_x = \Omega_y = \Omega_z = 0$

∴ The motion is irrotational.

Ques:- 6(a)) Find the integral surface of the linear PDE

$$xp + yq = z$$

which contains the circle defined by-

$$x^2+y^2+z^2=4 \ , \ x+y+z=2$$

Solution:- The integral surface of the given PDE is generated by the integral curves of the auxillary equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \quad \text{—— (1)}$$

Integration of the first two member of Eq (1) gives

$$\log x = \log y + \log c_1$$

$$\boxed{x = yc_1} \text{ or } \boxed{c_1 = x/y} \quad \text{—— (2)}$$

Similarly, integration of last two of eq^n (1), gives.

$$\log y = \log z + \log c_2$$

$$y = zc_2 \Rightarrow \boxed{c_2 = y/z} \quad \text{—— (3)}$$

Hence, the integral surface of the given PDE is-

$$\boxed{F(c_1, c_2) = 0 \Rightarrow F\left(x/y, y/z\right) = 0} \quad \text{—— (4)}$$

If the integral surface also contains the given circle, then we have to find a relation b/w x/y and y/z.

The equation of circle is

$$x^2+y^2+z^2 = 4 \quad \text{—— (5)}$$

$$x+y+z = 2 \quad \text{—— (6)}$$

from (2) and (3), we have.

$$\boxed{y = x/c_1}, \ z = y/c_2 \Rightarrow \boxed{z = x/c_1c_2}$$

Substituting the values of y and z in eqn (5) & (6), we get.

$$x^2 + \frac{x^2}{c_1^2} + \frac{x^2}{c_1^2c_2^2} = 4$$

$$\Rightarrow x^2\left[1 + \frac{1}{c_1^2} + \frac{1}{c_1^2c_2^2}\right] = 4 \quad \text{——— (7)}$$

and.

$$x + \frac{x}{c_1} + \frac{x}{c_1c_2} = 2 \Rightarrow x\left[1 + \frac{1}{c_1} + \frac{1}{c_1c_2}\right] = 2 \quad \text{— (8)}$$

from eqn (7) & (8), we observe

$$1 + \frac{1}{c_1^2c_2^2} + \frac{1}{c_1^2} = \left[1 + \frac{1}{c_1} + \frac{1}{c_1c_2}\right]^2$$

$$\Rightarrow \quad \frac{2}{c_1} + \frac{2}{c_1c_2} + \frac{2}{c_1^2c_2} = 0$$

That is $\boxed{c_1c_2 + c_1 + 1 = 0}$ ——— (9)

Now replacing c_1 by x/y, & $c_2 = y/z$, we get

$$\Rightarrow \frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} + \frac{x}{y} + 1 = 0$$

$$\Rightarrow \frac{x}{z} + \frac{x}{y} + 1 = 0$$

$$\Rightarrow \boxed{xy + xz + yz = 0}$$

which is required integral surface.

Ques: 6(b)> Reduce the equation $\frac{\partial^2 z}{\partial x^2}+2\left(\frac{\partial^2 z}{\partial x \partial y}\right)+\frac{\partial^2 z}{\partial y^2}=0$ to canonical form and hence solve it.

Solution>. given equation.

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

which can be written as -

$$r+2s+t=0 \quad \text{——} (1)$$

comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here; $R=1$, $S=2$, $T=1$

so that $S^2-4RT=0$

showing that (1) is parabolic.

The λ-quadrate equation reduces to

$$\lambda^2+2\lambda+1=0 \Rightarrow \boxed{\lambda=-1,-1} \text{(equal roots)}$$

The corresponding characteristic equation is

$$\frac{dy}{dx}-1=0 \Rightarrow dy-dx=0$$

Integrating $\Rightarrow$ $\boxed{y-x=c}$ or $\boxed{x-y=0}$

choose; $u=x-y$ and $v=x+y$ ——(2)

where we have chosen $v=x+y$, in such a manner that u and v are independent functions as verified below.

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}=1.1+1.1=2\neq 0$$

Now; $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

$$p = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad (\text{using } (2)) \quad \text{———} (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$q = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad (\text{using } (2)) \quad \text{——} (4)$$

$\therefore$ from (3) and (4);

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{——} (5)$$

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]\left[\frac{\partial z}{\partial x}\right]$$

$$\boxed{r = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)}$$

$$r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$\boxed{r = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}} \quad \text{————} (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\cdot\left[\frac{\partial z}{\partial y}\right] = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$$

$$= -\frac{\partial}{\partial u}\left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] + \frac{\partial}{\partial v}\left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$$

$$\boxed{t = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial y^2}} \quad \text{———} (7)$$

and $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$$S = \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$\boxed{S = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}} \quad \text{——— (8)}$$

Using (6), (7) and (8); in (1), the required canonical form is

$$\boxed{\frac{\partial^2 z}{\partial v^2} = 0 \quad \text{or} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0} \quad \text{——— (9)}$$

To find the Solution

Integrating (9), w.r.t v, we get

$$\frac{\partial z}{\partial v} = \phi(u) \quad \text{——— (10)}, \quad \phi \text{ is an arbitrary function}$$

Integrating (10) partially w.r.t v,

$$z = \int \phi(u)\, dv + \psi(u)$$

$$z = v\,\phi(u) + \psi(u)$$

$$\boxed{z = (x+y)\,\phi(x-y) + \psi(x-y)}$$

which is desired Solution; ϕ & ψ being arbitrary functions.

Ques: 6(c) Solve $r+s-6t = y\cos x$.

Solution:- Given equation can be re-written as -

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

$$(D^2 + DD' - 6D'^2)z = y\cos x \quad \text{—— (1)}$$

Its auxillary equation $\Rightarrow$ $m^2+m-6=0$

$$\boxed{m = 2, -3}$$

$$\boxed{\therefore C.F = \phi_1(y+2x) + \phi_2(y-3x)}$$

ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{D^2+DD'-6D'^2} y\cos x = \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$= \frac{1}{D-2D'}\int (3x+c)\cos x\, dx \quad ; \text{ where } c = y-3x$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\, dx\right]$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right] \quad \text{as } c = y-3x$$

$$= \int \left[(c'-2x)\sin x + 3\cos x\right]dx \quad ; \quad c' = y+2x$$

$$= (c'-2x)(-\cos x) - \int (-2)(-\cos x)dx + 3\sin x.$$

$$= y(-\cos x) - 2\sin x + 3\sin x \quad [\because c' = y+2x]$$

$$\boxed{P.I = \sin x - y\cos x}$$

General solution $= C.F + PI$

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x}$$

Ques: 6(d)) Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$. The edges $x=0, x=a, y=0$ are kept at temperature zero while the edge $y=b$ is kept at 100°C.

Solution:-

Temperature $u(x,y)$ in steady state in two-dimensional plate is governed by the Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——} \ (1)$$

Here; $u(0,y)=0$; $u(a,y)=0$ —— (2)

$u(x,0)=0$; $u(x,b)=100$ —— (3)

Suppose (1) has a solution of the form

$$u(x,y)=X(x)Y(y) \quad \text{——} \ (4)$$

Substituting this value of u in (1), we get

$$X''Y+XY''=0 \quad \text{or} \quad \frac{X''}{X}=\frac{-Y''}{Y} \quad \text{——} \ (5)$$

Since; x and t are independent, each side of (5) must be equal to the constant, say μ.

Then (5) gives

$$X''-\mu X=0 \quad \text{——} \ (6)$$

$$Y''+\mu Y=0 \quad \text{——} \ (7)$$

Using (2), (4) gives

$$X(0)Y(y)=0 \quad \text{and} \quad X(a)Y(y)=0$$

or $X(0)$ and $X(a) = 0$ ——— (8)

where we have taken $Y(y) \equiv 0$,
Since, otherwise $u \equiv 0$; which does not satisfy eq (3).
We now solve (6) under BC (8). Three cases arise:

Case I:- Let $\mu = 0$. Then solution of (6) is

$$X(x) = Ax + B \quad \text{———} \quad (9)$$

Using B.C. (8), (9) gives; $0 = B$ and $0 = Aa + B$
These give $A = B = 0$ so that $X(x) = 0$.
This leads to $u \equiv 0$ which does not satisfy (3).
So, we reject $\mu = 0$.

Case II) Let $\mu = \lambda^2$; $\lambda \neq 0$

The solution of (6) is

$$\boxed{X(x) = Ae^{x\lambda} + Be^{-x\lambda}} \quad \text{———} \quad (10)$$

Using B.C (1), (10) gives

$$\boxed{\begin{array}{l} 0 = A + B = 0 \quad \text{and} \\ 0 = e^{a\lambda} + Be^{-a\lambda} \end{array}} \quad \text{———} \quad (11)$$

Solving (11); $A = B = 0$,
So that $X(x) = 0$ and $u \equiv 0$.
So we reject $\mu = \lambda^2$

Case III) Let $\mu = -\lambda^2$, $\lambda \neq 0$, Then solution of (6) is

$$\boxed{X(x) = A\cos\lambda x + B\sin\lambda x} \quad \text{———} \quad (12).$$

Using B.C (8), (12) gives.

$$0 = A \quad \text{and} \quad 0 = A\cos\lambda a + B\sin\lambda a$$

So that $A = 0$ and $\sin\lambda a = 0$

where, we have taken $B \neq 0$, since otherwise

$X(x) \equiv 0$ and hence $u = 0$

which does not satisfy eq (3)

Now; $\sin\lambda a = 0$; gives $\lambda a = n\pi$

$$\boxed{\therefore\ \lambda = n\pi/a} \;;\; n = 1, 2, 3, \cdots \quad \text{——— (13)}$$

Hence, non-zero solutions $X_n(x)$ of (6) are given by

$$\boxed{X_n(x) = B_n \sin\frac{n\pi x}{a}} \quad \text{——— (14)}$$

Using $\mu = -\lambda^2 = -n^2\pi^2/a^2$, (7) becomes.

$$\boxed{Y'' - \frac{n^2\pi^2}{a^2}Y = 0} \quad \text{——— (15)}$$

Whose general solution is

$$\boxed{Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}} \quad \text{——— (16)}$$

Using eq (3), (4) gives

$$0 = X(x)\,Y(0) \;;\; \text{so that} \quad Y(0) = 0.$$

where, we have taken $X(x) \neq 0$. for otherwise we shall get $u = 0$, which does not satisfy (3).

But $Y(0) = 0 \Rightarrow \boxed{Y_n(y) = 0}$ ——— (17)

putting $y=0$ in (16) and using (17), we have.

$0 = C_n + D_n$; so that $\boxed{D_n = -C_n}$

Then (6), reduces to

$$\boxed{Y_n(y) = C_n(e^{n\pi y/a} - e^{-n\pi y/a}) = 2\sinh(n\pi y/a)} \quad \text{—(18)}$$

as $e^{\theta} - e^{-\theta} = 2\sinh\theta$

$$\therefore \boxed{u(x,y) = X_n(x)Y_n(y) = E_n \sin(n\pi x/a)\cdot \sinh(n\pi y/a)} \quad \text{—(19)}$$

are solutions of (1), satisfying (2) and (3).
Here; $E_n (= 2B_nC_n)$ are new arbitrary constants. In order to satisfy conditions (3), we now consider moral general solution.

i.e. $$\boxed{u(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cdot \sinh\frac{n\pi y}{a}} \quad \text{—— (20)}$$

Putting $y=b$ in (20) and using (3), we get

$$100 = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cdot \sinh\frac{n\pi b}{a}$$

Which is fourier sine series and hence E_n is given by

$$E_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a 100 \sin\frac{n\pi x}{a}\,dx = \frac{200}{a}\left[\frac{-\cos(n\pi x/a)}{(n\pi/a)}\right]_0^a$$

$$\therefore E_n = \frac{200}{n\pi}\left[1-(-1)^n\right] \operatorname{cosech}\frac{n\pi b}{a}$$

$$\therefore E_n = \begin{cases} 0 & ; \text{ if } n=2m \\ \dfrac{400 \operatorname{cosech}\{(2m-1)\pi b/a\}}{(2m-1)\pi} & ; \text{ if } n=2m-1 \end{cases}$$

with these values of E_n, (20) reduces to

$$u(x,y) = \sum_{m=1}^{\infty} E_{2m-1} \sin\frac{(2m-1)\pi x}{a} \sinh\frac{(2m-1)\pi y}{a}$$

i.e; $$\boxed{u = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\cdot\sin\frac{(2m-1)\pi x}{a}\cdot\sinh\frac{(2m-1)\pi y}{a}\,\operatorname{cosech}\frac{(2m-1)\pi b}{a}}$$

which is required solution

Ques:-7(a)) Using Gauss Seidel iterative method and the starting solution $x_1 = x_2 = x_3 = 0$ determine the solution of the following system of equations in two iterations., $10x_1 - x_2 - x_3 = 8$; $x_1 + 10x_2 + x_3 = 12$
$x_1 - x_2 + 10x_3 = 10$.

Solution:- The given equation can be rewritten as—

$$x_1 = \frac{8 + x_2 + x_3}{10} = 0.8 + 0.1x_2 + 0.1x_3$$

$$x_2 = \frac{12 - x_1 - x_3}{10} = 1.2 - 0.1x_1 - 0.1x_3$$

$$x_3 = \frac{10 - x_1 + x_2}{10} = 1 - 0.1x_1 + 0.1x_2$$

Also given that $x_1 = x_2 = x_3 = 0$

and we need to find the solution of the system of equation in two iteration.

Iteration-1

Taking; $x_1 = x_2 = x_3 = 0$

$$x_1 = 0.8 + 0.1\times0 + 0.1\times0 = 0.8$$

$$x_2 = 1.2 - 0.1\times0.8 - 0.1\times0 = 1.12.$$

$$x_3 = 1 - 0.1\times0.8 + 0.1\times1.12 = 1.032.$$

After first Iteration; $x_1 = 0.8$
$x_2 = 1.12$
$x_3 = 1.032.$

Iteration - II

$x_1 = 0.8$, $x_2 = 1.12$, $x_3 = 1.032$

$x_1 = 0.8 + 0.1 \times 1.12 + 0.1 \times 1.032 = 1.0152$

$x_2 = 1.2 - 0.1 \times 1.0152 - 0.1 \times 1.032$

$= 1.2 - 0.10152 - 0.1032 = 0.9953$

$x_3 = 1 - 0.1 \times 1.0152 + 0.1 \times 0.9953$

$= 1 - 0.10152 + 0.09953 = 0.99801.$

$\therefore$ After 2^{nd} Iteration $\Rightarrow$ $\boxed{\begin{array}{l} x_1 = 1.0152 \\ x_2 = 0.9953 \\ x_3 = 0.99801 \end{array}}$

which is required solution

Ques:- 7(b)) Using Lagrange interpolation formula Calculate the values of $f(3)$ from the following table of values of x and $f(x)$:

X	0	1	2	4	5	6
F(x)	1	14	15	5	6	19

Solution:- The given values are.

$x_0=0$; $x_1=1$, $x_2=2$, $x_3=4$, $x_4=5$, $x_5=6$

$f(0)=y_0=1$, $y_1=14$, $y_2=15$, $y_3=5$, $y_4=6$, $y_5=19$

For $x=3$; Lagrange's Interpolation formula is given by

$$f(3)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(0)+$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(1)+$$

$$\vdots$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(5).$$

$$f(3)=\frac{(2)(1)(-1)(-2)(-3)}{(-1)(-2)(-3)(-5)(-6)}\times 1+\frac{(3)(1)(-1)(-2)(-3)}{(1)(-1)(-3)(-4)(-5)}\times 14+$$

$$\frac{(3)(2)(-1)(-2)(-3)}{(2)(1)(-2)(-3)(-4)}\times 15+\frac{(3)(2)(1)(-2)(-3)}{(4)(3)(2)(-1)(-2)}\times 5+$$

$$\frac{(3)(2)(+1)(-1)(-3)}{(5)(4)(3)(1)(-1)}\times 6+\frac{(3)(2)(1)(-1)(-2)}{(6)(5)(4)(2)(1)}\times 19$$

$$f(3)=\frac{1}{20}-\frac{21}{5}+\frac{45}{4}+\frac{15}{4}-\frac{9}{5}+\frac{19}{20}$$

$$F(3)=\left[\frac{1}{20}+\frac{19}{20}\right]-\left[\frac{21}{5}+\frac{9}{5}\right]+\left[\frac{45}{4}+\frac{15}{4}\right]$$

$$F(3)=1-6+15=16-6=10$$

$$\boxed{\therefore f(3)=10}$$

Ques:- 7 (c)) Given $\frac{dy}{dx} = y - x$; where $y(0) = 2$, using the Runge-kutta fourth order method find $y(0.1)$ and $y(0.2)$. Compare the approximate solution with its exact solution $e^{0.1} = 1.10517$ $e^{0.2} = 1.2214$.

Solution :- Given ; $x = 0$, $y(0) = 2$

$$\frac{dy}{dx} = f(x, y) = y - x \quad , h = 0.1$$

To find ; $y(0.1) = ?$, $y(0.2) = ?$

For $y(0.1) = ?$

$y(0) = 2$, $x_0 = 0$, $h = 0.1$

$$y(0.1) = y(0) + k.$$

$$K = \frac{1}{6}\left(K_1 + K_4 + 2(K_2 + K_3)\right).$$

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}K_1)$$

$$K_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}K_2)$$

$$K_4 = hf(x_0 + h, y_0 + K_3).$$

$$\therefore K_1 = 0.1 f(0, 2) = 0.1 \times 2 = 0.2$$

$$K_2 = 0.1 f(0.05, 2.1) = 0.1 \times (2.05) = 0.205.$$

$$K_3 = 0.1 f(0.05, 2.1025) = 0.1 \times (2.0525) = 0.20525$$

$$K_4 = 0.1 f(0.1, 2.20525) = 0.1(2.10525) = 0.210525.$$

$$K = \frac{1}{6}\left[0.2 + 0.210525 + 2(0.205 + 0.20525)\right]$$

$$K = \frac{1}{6}\left[0.410525 + 0.8205\right] = 0.20517.$$

$$y(0.1) = y(0) + K = 2 + 0.20517 = 2.20517.$$

For $y(0.2)$:- $y(0.1) = 2.20517,\ h = 0.1,\ x = 0.1$

$$K_1 = 0.1 f(0.1, 2.20517) = 0.1 \times 2.10517$$
$$= 0.210517$$

$$K_2 = 0.1 \cdot f(0.15, 2.3104) = 0.1 \times 2.1604$$
$$= 0.21604.$$

$$K_3 = 0.1 f(0.15, 2.3132) = 0.1 \times 2.163$$
$$= 0.2163$$

$$K_4 = 0.1 (0.2, 2.42147) = 0.1 \times 2.22147$$
$$= 0.22215.$$

$$K = \frac{1}{6}\left[0.210517 + 0.22215 + 2(0.21604 + 0.2163)\right]$$

$$K = \frac{1}{6}(1.2974) = 0.2162.$$

$$y(0.2) = y(0.1) + K = 2.20517 + 0.2162 = 2.42137.$$

$$\therefore \boxed{y(0.1) = 2.20517 \quad ; \quad y(0.2) = 2.42137}$$

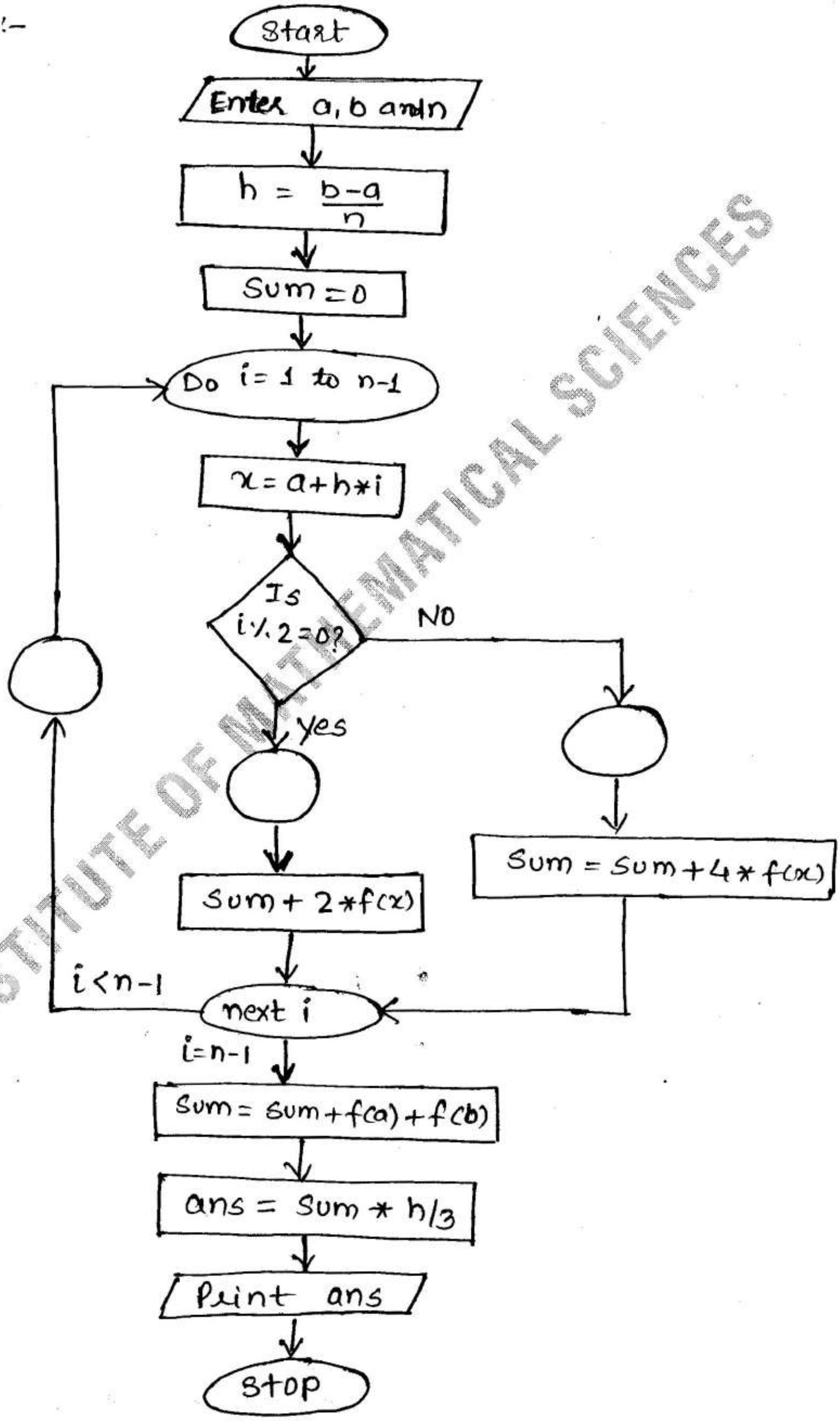
$$\text{Error} = \left|y(0.1) - e^{0.1}\right| = |2.20517 - 1.10517|$$
$$= \underline{\underline{1.1}}$$

$$\text{Error} = \left|y(0.2) - e^{0.2}\right| = |2.42137 - 1.2214|$$
$$\simeq \underline{\underline{1.2}}$$

Note:- If $y(0) = 1$; then it will be negligible error from exact solution

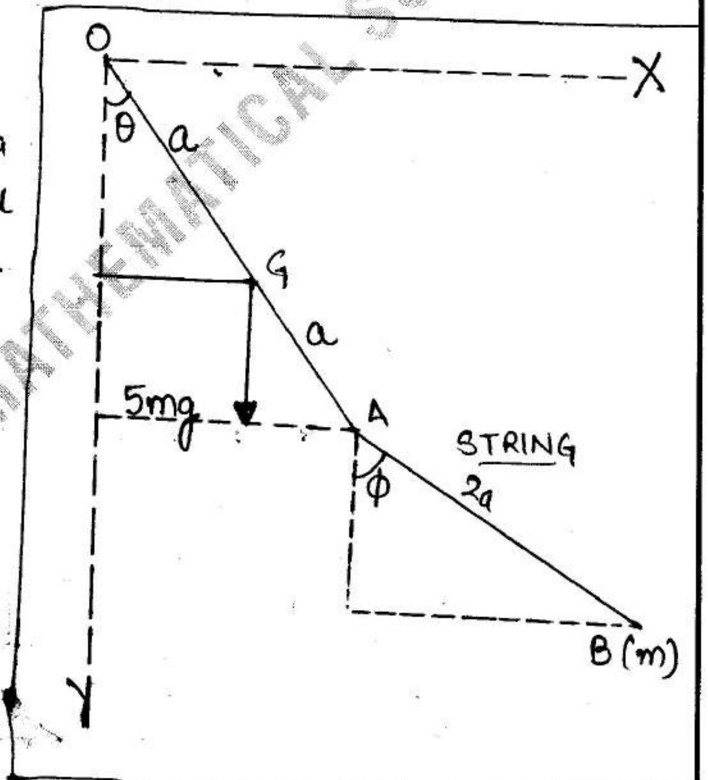
Ques:- 7(d)) Draw a flow chart for Simpson's one third rule?

Solution:-



Ques: 8(a)) A uniform rod, of mass 5m and length 2a, turns freely about one end which is fixed, to its other extremity is attached one end of a light string, of length 2a, which carries at its other end a particle of mass 'm' show that the periods of small oscillations in a verticle plane are same as those of simple pendulum of lengths $2a/3$ and $20a/7$.

Solution:-



Let, OA be the rod of mass 5m and length 2a turning about the fixed end O. AB the string of length 2a and m the mass attached at the end B.

At time 't', let the rod and the string makes an angles θ and ϕ to the vertical respectively.

Referred to O as origin, horizontal and vertical lines OX and OY as axes, the co-ordinates of OG. 'G' of the rod and that of the end 'B' are given by

$$x_G = a\sin\theta \ ; \quad y_G = a\cos\theta \ ; \ x_B = 2a(\sin\theta + \sin\phi)$$

$$y_B = 2a(\cos\theta + \cos\phi)$$

$\therefore$ If v_G and v_B are the velocities of 'G' and 'm' at B, then

$$V_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (a\cos\theta\,\dot{\theta})^2 + (-a\sin\theta\,\dot{\theta})^2 = a^2\dot{\theta}^2$$

$$V_B^2 = \dot{x}_B^2 + \dot{y}_B^2 = [2a(\cos\theta\cdot\dot{\theta} + \cos\phi\,\dot{\phi})]^2 + [2a(-\sin\theta\,\dot{\theta} - \sin\phi\,\dot{\phi})]^2$$

$$= 4a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)]$$

$$= 4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

[$\because$ θ and ϕ are small]

It 'T' be the K.E. and W the work function of the system, then

T = K.E of the rod + K.E of the particle

$$T = \left[\frac{1}{2}\cdot 5m\cdot\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}\cdot 5m\cdot V_G^2\right] + \frac{1}{2}mV_B^2$$

$$T = \frac{1}{2}\cdot 5m\left(\frac{1}{3}a^2\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \frac{1}{2}m\,4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

or $$\boxed{T = ma^2\left(\frac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\right)}$$

and $$W = 5mg\cdot y_G + mg\,y_B + C$$

$$W = 5mga\cos\theta + mg\cdot 2a(\cos\theta + \cos\phi) + C$$

$$W = 7mga\cos\theta + mg\,2a\cdot\cos\phi + C$$

$$\boxed{W = mga(7\cos\theta + 2\cos\phi)}$$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $$\frac{d}{dt}\left[ma^2\left(\frac{32}{3}\dot{\theta} + 4\dot{\phi}\right)\right] - 0 = -7mga\sin\theta = -7mga\,\theta$$

[$\because$ θ is small]

or: $$\boxed{32\ddot{\theta} + 12\ddot{\phi} = -21c\theta}$$; taking $g/a = c$ —— (1)

And Lagrange's ϕ-equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\phi}}\right)-\frac{\partial T}{\partial \phi}=\frac{\partial W}{\partial \phi}$$

i.e $\frac{d}{dt}\left[ma^2(4\dot{\phi}+4\dot{\theta})\right]-0=-2mga\sin\phi=-2mga\phi$ $[\because \phi \approx \text{small}]$

or $\boxed{2\ddot{\theta}+2\ddot{\phi}=-c\phi}$ —— (2)

Equation (1) and (2) can be written as.

$$(32D^2+21c)\theta+12D^2\phi=0 \text{ and } 2D^2\theta+(2D^2+c)\phi=0$$

Eliminating ϕ between these two equations, we get

$$[(2D^2+c)(32D^2+21c)-24D^4]\theta=0$$

or $\boxed{(40D^4+74D^2c+21c^2)\theta=0}$ —— (3)

let, the solution of (3) be given by.

$$\theta=A\cos(pt+B)$$

$\therefore$ $D^2\theta=-p^2\theta$ and $D^4\theta=p^4\theta$

Substituting in (3), we get

$$(40p^4-74cp^2+21c^2)\theta=0$$

or $(2p^2-3c)(20p^2-7c)=0$ $\quad\because \theta\neq 0$

$\therefore$ $p_1^2=\frac{3c}{2}=\frac{3g}{2a}$ and.

$p_2^2=\frac{7c}{20}=\frac{7g}{20a}$

Hence, the lengths of simple equivalent pendulum are.

$$\boxed{\frac{g}{p_1^2} \text{ and } \frac{g}{p_2^2} \quad \text{i.e. } \frac{2a}{3} \text{ and } \frac{20a}{7}}$$

Ques:- 8(b)> Use Hamilton's equations to find the cartesian equations of motion of a particle moving in three dimensions in a force field of potential V.

Solution:- Let (x, y, z) be the co-ordinates of a particles moving in three dimensions, at time t.

$$\boxed{\therefore \text{ K.E} \Rightarrow T = \frac{1}{2} m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}$$

Since; V is the potential energy,

$$\therefore L = T - V = \frac{1}{2} m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V$$

Here x, y, z are the generalised co-ordinates

$$\therefore P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ ; \ P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ ; \ P_z = \frac{\partial L}{\partial \dot{z}} m\dot{z} \quad \text{—(1)}$$

Since, L does not contain 't' explicitely, therefore.

$$H = T + V = \frac{1}{2} m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + V$$

$$\text{Or} \Rightarrow H = \frac{1}{2m}(P_x^2+P_y^2+P_z^2) + V \quad \text{— (using (1))}$$

Hence, the six Hamilton's equations are, (note that V is a function of x, y, z and t)

$$\dot{P}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \quad \text{— } (H_1)$$

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m} \quad \text{— } (H_2)$$

$$\dot{P}_y = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y} \quad \text{— } (H_3)$$

$$\dot{y} = \frac{\partial H}{\partial P_y} = \frac{P_y}{m} \quad \text{— } (H_4)$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z} \quad \text{——} \; (H_5)$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \text{——} \; (H_6)$$

from (H_1) and (H_2), we have

$$\boxed{m\ddot{x} = \dot{p}_x = -\frac{\partial V}{\partial x}} \quad \text{——} \; (2)$$

Similarly from (H_3), (H_4) and (H_5), (H_6), we get

$$\boxed{m\ddot{y} = \dot{p}_y = -\frac{\partial V}{\partial y}} \quad \text{——} \; (3)$$

$$\boxed{m\ddot{z} = \dot{p}_z = -\frac{\partial V}{\partial z}} \quad \text{——} \; (4)$$

If X, Y, Z are the external forces parallel to the axes, at (x, y, z), then

$$-\frac{\partial V}{\partial x} = X, \quad -\frac{\partial V}{\partial y} = Y, \quad -\frac{\partial V}{\partial z} = Z$$

Hence, from (2), (3), (4) the equations of motion of a particle moving in three dimensions are.

$$\boxed{m\ddot{x} = X \; ; \; m\ddot{y} = Y \; ; \; m\ddot{z} = Z}$$

Which are required solution

Ques: 8(c) Test whether the motion specified by

$$q = \frac{k^2(x\hat{j} - y\hat{i})}{x^2+y^2} \quad (k = \text{Constant}),$$

is possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Solution:-

Let $q = u\hat{i} + v\hat{j} + w\hat{k}$. Then here

$$u = -\frac{k^2 y}{x^2+y^2}, \quad v = \frac{k^2 x}{x^2+y^2}; \quad w = 0 \quad \text{—— (1)}$$

The equation of continuity for an incompressible fluid is –

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{—— (2)}$$

from (1) $\Rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{2k^2 xy}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial y} = -\frac{2k^2 xy}{(x^2+y^2)^2}; \quad \frac{\partial w}{\partial z} = 0$$

Hence; (2) is satisfied and so the motion specified by given 'q' is possible.

The equation of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{-k^2y/x^2+y^2} = \frac{dy}{k^2x/x^2+y^2} = \frac{dz}{0} \quad \text{—— (3)}$$

Taking the last fraction.

$$dz = 0 \quad \text{so that} \quad \boxed{z = c_1} \quad \text{——— (4)}$$

Taking the first two fractions in (3) and simplifying we get

$$\frac{dx}{-y} = \frac{dy}{x} \Rightarrow 2x\,dx + 2y\,dy = 0$$

Integrating $\Rightarrow$ $\boxed{x^2+y^2 = c_2}$ c_2 being an arbitrary constant. ——— (5)

(4) and (5) together give the streamlines.

Clearly, the streamlines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

Again: $$\text{curl } q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-k^2 y}{x^2+y^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix}$$

$$\text{curl } q = k^2\left\{\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right\}\hat{k} = 0$$

Hence, the flow is of the potential kind and we can find velocity potential $\phi(x,y,z)$, such that $q = -\nabla\phi$. Thus, we have.

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \quad \text{——— (6)}$$

$$\frac{\partial\phi}{\partial y} = -v = \frac{-k^2 x}{x^2+y^2} \quad \text{——— (7)}$$

$$\frac{\partial\phi}{\partial z} = -w = 0 \quad \text{——— (8)}$$

Equation (8), shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x,y)$.

Integrating (6),

$$\boxed{\phi(x,y) = k^2 \tan^{-1}(x/y) + f(y)} \quad \text{——— (9)}$$

where $f(y)$ is an arbitrary function

From (9) $$\boxed{\frac{\partial \phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2}} \quad \text{——— (10)}$$

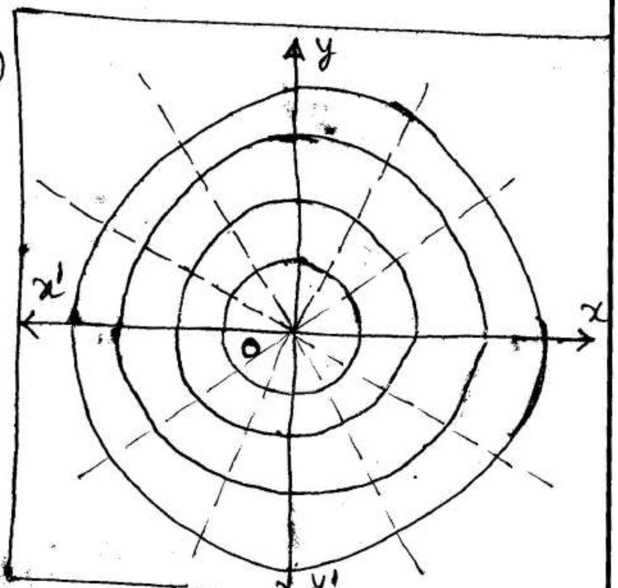
Comparing (7) and (10), we have

$f'(y) = 0$; so that

$f(y) = \text{constant}$

Since the constant can be omitted while writting velocity potential, the required velocity potential can be taken as

$$\phi(x,y) = k^2 \tan^{-1}(x/y) \text{ — (11)}$$



The equipotentials are given by -

$k^2 \tan^{-1}(x/y) = \text{constant}$

$= k^2 \tan^{-1} c$

or $\boxed{x = cy}$,

c being a constant

which are planes through the z-axis. They are intersected by the streamlines as shown in figure. Dotted lines represents equipotentials and ordinary lines represents streamlines.