# 4 Infinite Series

## 1. INTRODUCTION

In this chapter we shall discuss the techniques of testing the *behaviour* of infinite series as regards convergence. The most important technique for series, all of whose terms are of the same sign (all positive or all negative), is to compare the given series with another suitably chosen series with known behaviour. So, first of all, comparison tests are discussed, and then some special tests for convergence are considered. *Leibnitz's test* for alternating series, *Abel's* and *Dirichlet's tests* for arbitrary term series, and *Dirichlet's* and *Riemann's* theorems on rearrangement of terms will be discussed in detail towards the end.

**1.1** A *series* is the sum of the terms of a sequence. Thus if $u_1, u_2, u_3, \ldots$ is a sequence then the sum $u_1 + u_2 + u_3 + \ldots$ of all the terms is called an *infinite series* and is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$.

Evidently we cannot just add up all the infinite number of terms of the series in the ordinary way and in fact it is not obvious that this kind of sum has any meaning. We thus start by associating with the given series, a sequence $\{S_n\}$, where $S_n$ denotes the sum of the first $n$ terms of the series. Thus,

$$S_n = u_1 + u_2 + \ldots + u_n, \quad \forall n.$$

The sequence $\{S_n\}$ is called the *sequence of partial sums* of the series and the partial sums, $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, and so on, may be regarded as approximations to the full infinite sum $\sum_{n=1}^{\infty} u_n$ of the series. If the sequence $\{S_n\}$ of partial sums converges, then the series is regarded as convergent, and $\lim S_n$ is said to be the *sum of the series*. If, however, $\{S_n\}$ does not tend to a limit, we must take it that the sum of the infinite series does not exist. We express this fact by saying that the series does not converge. In fact *an infinite series is said to converge, diverge or oscillate according as its sequence of partial sums* $\{S_n\}$ *converges, diverges or oscillates.*

### 1.2 A Necessary Condition for Convergence

**Theorem 1.** *A necessary condition for convergence of an infinite series* $\sum u_n$ *is that* $\lim_{n\to\infty} u_n = 0$.

Let $S_n = u_1 + u_2 + \ldots + u_n$ so that $\{S_n\}$ is the sequence of partial sums.

Since the series converges, therefore, the sequence $\{S_n\}$ also converges.

Consequently

$$\lim_{n\to\infty} S_n = s(\text{say})$$

Now,

$$u_n = S_n - S_{n-1},\ n > 1$$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= s - s = 0$$

Hence for a convergent series,

$$\lim u_n = 0$$

In other words, a series cannot converge if its $n$th term does not tend to zero,

**Notes:**

1. It must be clearly understood that $\lim u_n = 0$ does not prove that a series in convergent, for there exist series which do not converge even though $\lim u_n = 0$. See Example 2.
2. However, $\lim u_n \neq 0$ proves that the series does not converge, see Example 1 below.

**Example 1.** Show that the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$$

is not convergent.

■ Here

$$u_n = \frac{n}{n+1}$$

$$\therefore \qquad \lim u_n = \lim \frac{n}{n+1} = 1 \neq 0$$

Since $\lim u_n \neq 0$, therefore, the series is not convergent.

## 1.3 Cauchy's General Principle of Convergence for Series

A necessary and sufficient condition for the convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that the sequence of its partial sums $\{S_n\}$ is convergent.

Therefore, a test for convergence of infinite series may be derived from our knowledge of sequences.

**Theorem 2.** *A series $\sum u_n$ converges iff for each $\varepsilon > 0$, there exists a positive integer $m$ such that*

$$\left| u_{n+1} + u_{n+2} + \ldots + u_{n+p} \right| < \varepsilon,\ \forall\ n \geq m \text{ and } p \geq 1$$

By Cauchy's General Principle of Convergence (for sequences), the sequence $\{S_n\}$ of partial sums of $\sum u_n$ converges iff to each $\varepsilon > 0\ \exists$ a positive integer $m$, such that

$$\left| S_{n+p} - S_n \right| < \varepsilon,\ \forall\ n \geq m \text{ and } p \geq 1$$

or
$$\left|u_{n+1}+u_{n+2}+\dots+u_{n+p}\right|<\varepsilon,\ \forall\, n \geq m \text{ and } p \geq 1$$

**Example 2.** Show that the series $\Sigma\frac{1}{n}$ does not converge.

■ Suppose, if possible, the series converges.

Therefore, for any given $\varepsilon\left(\text{say}, \frac{1}{4}\right)$, $\exists$ a positive integer $m$ such that

$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+p}\right|<\varepsilon,\ \forall\, n \geq m \text{ and } p \geq 1$$

In particular, if $n = m$ and $p = m$, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+p}=\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m}$$
$$> m\cdot\frac{1}{2m}=\frac{1}{2}>\varepsilon$$

Thus, there is a contradiction. Hence, the given series does not converge.

It may also be seen that $\lim u = \lim\frac{1}{n}=0$ even though the series does not converge.

**Example 3.** If $u_n > 0$ and $\Sigma u_n$ is convergent, with the sum $S$, then prove that

$$\frac{u_n}{u_1+u_2+\dots+u_n}<\frac{2u_n}{S},$$

when $n$ is sufficiently large. Also prove that $\sum\frac{u_n}{u_1+u_2+\dots+u_n}$ is convergent.

■ Since $\Sigma u_n$ is convergent with the sum $S$, therefore, for $\varepsilon > 0\ \exists$ a positive integer $m$, such that

$$|S_n - S| < \varepsilon\ \ \forall\, n \geq m, \text{ where } S_n = u_1+u_2+\dots+u_n,$$

or
$$S-\varepsilon < S_n < S+\varepsilon,\ \forall\, n \geq m$$

In particular for $\varepsilon = \frac{1}{2}S > 0$,

$$\frac{1}{2}S < S_n < \frac{3}{2}S,\ \ \forall n \geq m$$

$\Rightarrow$
$$\frac{2}{S} > \frac{1}{S_n} > \frac{2}{3S},\ \ \forall\, n \geq m$$

$\therefore$
$$\frac{u_n}{S_n} < \frac{2u_n}{S},\ \ \forall\, n \geq m.$$

Now, $\frac{u_{n+1}}{S_{n+1}}+\frac{u_{n+2}}{S_{n+2}}+\dots+\frac{u_{n+p}}{S_{n+p}}<\frac{2}{S}\left(S_{n+p}-S_n\right),\ \forall\, n \geq m$ and $p \geq 1$.

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Since $\sum u_n$ is convergent, then for given $\varepsilon > 0$, there exists a positive integer $m_1$ such that

$$S_{n+p} - S_n < \frac{\varepsilon S}{2}, \forall n \geq m_1$$

$$\frac{u_{n+1}}{S_{n+1}} + \frac{u_{n+2}}{S_{n+2}} + \ldots + \frac{u_{n+p}}{S_{n+p}} < \frac{2}{S} \cdot \frac{\varepsilon S}{2} = \varepsilon, \quad \forall n \geq \max(m, m_1)$$

Hence by *Cauchy's General Principle of Convergence*, $\sum \frac{u_n}{u_1 + u_2 + \ldots + u_n}$ is convergent.

## 1.4 Some Preliminary Theorems

**Theorem 3.** *If $\Sigma u_n = u$, then $\Sigma c u_n = cu$, independent of $n$.*

The result follows at once from the identity

$$\sum_{r=1}^{n} c u_r = c \sum_{r=1}^{n} u_r$$

on making $n$ tend to infinity.

**Theorem 4.** If $\sum_{n=1}^{\infty} u_n = u$, *then* $\sum_{n=0}^{\infty} u_n = u + u_0$, *and* $\sum_{n=2}^{\infty} u_n = u - u_1$.

Let $S_n' = \sum_{r=0}^{n} u_r$ and $S_n = \sum_{r=1}^{n} u_r$.

Clearly

$$S_n' = u_0 + S_n$$

$\therefore$ By letting $n$ tend to infinity

$$\sum_{n=0}^{\infty} u_n = u_0 + u$$

The proof of the second part is similar.

A slight modification and extension enables us to conclude that the insertion or removal of any finite number of terms from a convergent series does not affect its convergence. Of course, the sums of the various series are related in the expected way.

It is also clear that if the series $\sum_{n=1}^{\infty} u_n$ is divergent, the changed series $\sum_{n=1}^{\infty} c u_n$, $\sum_{n=0}^{\infty} u_n$ or $\sum_{n=2}^{\infty} u_n$ is also divergent.

Hence, the behaviour of a series as regards convergence is not altered by

(*i*) *the alteration, addition or omission of a finite number of terms; or*

(*ii*) *multiplication of all the terms by a finite number other than zero.*

**Theorem 5.** ***Convergent series may be added or subtracted term by term.*** *If $\Sigma u_n = u$ and $\Sigma v_n = v$ then $\Sigma w_n = u \pm v$, where $w_n = u_n \pm v_n$, for all $n$.*

The result follows from the identity

$$\sum_{r=1}^{n} w_r = \sum_{r=1}^{n} (u_r \pm v_r) = \sum_{r=1}^{n} u_r \pm \sum_{r=1}^{n} v_r$$

by making $n$ tend to infinity.

The same proof shows that

(*i*) if any two of the three series are convergent, the third is also convergent,

(*ii*) if one of the series is divergent and another convergent then the third is necessarily divergent, but

(*iii*) if two of the series are divergent, no conclusion can be drawn about the behaviour of the third, which may converge or diverge.

**Theorem 6.** *If a series $\sum u_n$ converges to the sum u then so does any series obtained from $\sum u_n$ by grouping the terms in brackets without altering the order of the terms.*

Suppose that the series derived from $\sum u_n$ by the insertion of brackets is $\sum v_n$ and let $\sigma_r$ denote the $r$th partial sum of the series $\sum v_n$. Suppose that $\sigma_r$ contains $n_r$ terms of the given series then since the order of the terms is unaltered, $\sigma_r = S_{n_r}$.

Also as $r \to \infty, n_r \to \infty$.

Since the given series converges to $u$, the sequence $\{S_n\}$ of its partial sums also converges to $u$.

Hence, as $r \to \infty$ or $n_r \to \infty$, $s_{n_r} \to u$, and hence $\sigma_r \to u$.

**Remarks:**

1. Converse of the theorem is not always true.

   For example, the series $(1 - 1) + (1 - 1) + \ldots$ is convergent, whereas the series $1 - 1 + 1 - 1 + \ldots$, obtained by removing the brackets is not.

   Hence, in convergent series brackets may be *inserted* at will without affecting convergence but may not be removed.

   In the case of convergent positive term series, or the absolutely convergent series, however, it will be shown later that the brackets may be inserted or removed without affecting convergence.

2. The theorem may as well be proved by the following alternative argument.

   Since the given series converges to $u$, its sequence $\{S_n\}$ of partial sums will also converge to $u$. Therefore, the sequence $\{\sigma_n\}$ of partial sums of $\sum v_n$, being a subsequence of $\{S_n\}$ will also converge to $u$.

3. The theorem may be restated, "*A series obtained from a given convergent series by a grouping of terms converges to the same limit*".

   By *grouping* we simply mean the placing of brackets or associating the terms of the series without changing the order of the terms.

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## 2. POSITIVE TERM SERIES

Series with non-negative terms are the simplest and the most important type of series one comes across. The simplicity arises mainly from the fact that the sequence of its partial sums is monotonic increasing.

Let $\sum u_n$ be an infinite series of positive terms and $\{S_n\}$, the sequence of its partial sums, so that

$$S_n = u_1 + u_2 + \ldots + u_n \geq 0, \quad \forall\, n$$

$$S_n - S_{n-1} = u_n \geq 0$$

$\therefore$
$$S_n \geq S_{n-1}, \ \forall n > 1$$

$\Rightarrow$ Thus the sequence $\{S_n\}$ of partial sums of a series of positive terms is a monotonic increasing sequence.

Since a monotonic increasing sequence can either converge, or diverge to $\infty$, but cannot oscillate, therefore, there are only two possibilities for a positive term series—it may either converge or diverge to $+\infty$.

**Theorem 7.** *A positive term series converges iff the sequence of its partial sums is bounded above.*

We know that the sequence of partial sums of a positive term series is a monotonic increasing sequence and a monotonic increasing sequence converges iff it is bounded above. Therefore, it follows that a positive term series converges iff its sequence of partial sums is bounded above.

**Remarks.**

1. The sequence of partial sums of a series with negative terms can be shown to be monotonic decreasing and hence a series with negative terms converges iff the sequence of its partial sums is bounded below.
2. It may similarly be seen that a series of negative terms can either converge, or diverge to $-\infty$.
3. A series of positive terms can either converge, or diverge to $+\infty$. But a series with arbitrary terms can have five possible behaviours depending upon the behaviour of the sequence of its partial sums.

## 2.1 A Necessary Condition for Convergence of Positive Term Series

We know that for any convergent series, the $n$th term $u_n \to 0$ as $n \to \infty$, we now give another necessary condition which holds for positive term series only.

**Theorem 8. Pringsheim's Theorem.** *If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.*

We know that for a convergent series, for any $\varepsilon > 0$, a positive integer $N$ exists such that

$$\left| u_{m+1} + u_{m+2} + \ldots + u_{m+p} \right| < \frac{\varepsilon}{2}, \ \forall m \geq N, p \geq 1$$

Let us choose

$$m + p = n > 2N$$

and

$$m = \left[\frac{n}{2}\right]$$

the greatest integer not greater than $\frac{n}{2}$

$\therefore$
$$u_{m+1} + u_{m+2} + \ldots + u_n < \frac{\varepsilon}{2}$$

But $\sum u_n$ is positive monotonic decreasing.

$\therefore$
$$(n - m)\, u_n < u_{m+1} + u_{m+2} + \ldots + u_n < \frac{\varepsilon}{2}$$

or

$$\frac{1}{2}nu_n < \frac{1}{2}\varepsilon$$

i.e.,

$$nu_n < \varepsilon, \ \forall\, n \geq N$$

$\therefore$

$$\lim nu_n = 0$$

Notes:

1. The condition $u_n \to 0$ holds for all types of convergent series but a convergent series of positive monotonic decreasing terms satisfies the additional condition, $nu_n \to 0$.
2. The condition $nu_n \to 0$ is only a necessary not a sufficient condition for the convergence of the present type of series. If $nu_n$ does not tend to zero then the series $\sum u_n$ is certainly divergent, *e.g.*, the harmonic series $\sum \frac{1}{n}$ must diverge because it has positive monotonic decreasing terms and $n \cdot \frac{1}{n}$ does not tend to zero. However, $nu_n \to 0$ does not imply anything as to the possible convergence of $\sum u_n$, e.g., Abel's series $\sum \frac{1}{n \log n}$ diverges although it has positive monotonic decreasing terms, and $nu_n \to 0$.

## 2.2 Geometric Series

The positive term geometric series $1 + r + r^2 + \ldots$, converges for $r < 1$, and diverges to $+\infty$ for $r \geq 1$.

*Case I.* $0 \leq r < 1$.

Let $\{S_n\}$ be the sequence of its partial sums, so that

$$S_n = 1 + r + r^2 + \ldots + r^{n-1}$$

i.e.,

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r} \leq \frac{1}{1 - r}, \ \forall\, n,$$

so that $\{S_n\}$ is bounded above and hence the series converges.

*Case II.* When $r = 1$, $S_n = n$, so that $\{S_n\}$ is not bounded above and hence the series diverges to $+\infty$.

*Case III.* When $r > 1$, every term of $S_n$ after the first is greater than 1, so that

$$S_n > n, \ \forall\, n$$

Therefore, the sequence $\{S_n\}$ is not bounded above and consequently the given series diverges to $+\infty$.

Hence, the given series converges if $r < 1$ and diverges if $r \geq 1$.

## 2.3 A Comparison Series

As mentioned earlier, an important technique for testing the convergence of a series is to compare the given series with a suitably selected series with known behaviour. We now discuss one such series which is most frequently used for such a purpose.

**Theorem 9.** *A positive term series* $\sum \frac{1}{n^p}$ *is convergent if and only if* $p > 1$.

Let $S_n$ denote the sum of the first $n$ terms of the given series.

*Case I.* When $p > 1$.

Now

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \ldots + \frac{1}{(2^{n+1}-1)^p} < \frac{2^{n+1}-2^n}{(2^n)^p} = \frac{1}{(2^n)^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^n$$

Adding,

$$S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots + \left(\frac{1}{2^{p-1}}\right)^n$$

$$= \frac{1 - \left(\frac{1}{2^{p-1}}\right)^{n+1}}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}\left[1 - \left(\frac{1}{2^{p-1}}\right)^{n+1}\right]}{2^{p-1}-1}$$

$$< \frac{2^{p-1}}{2^{p-1}-1}, \; \forall\, n$$

We know that when $n$ is any positive integer,

$$2^{n+1} - 1 > 2^n > n$$

$\therefore$

$$S_n < S_{2^n} < S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1}-1}$$

Since for a given $p$, $\frac{2^{p-1}}{2^{p-1}-1}$ is a fixed number, therefore, the sequence $\{S_n\}$ of partial sums of the given positive term series is bounded above and hence the series converges for $p > 1$.

*Case II.* When $p \leq 1$.

We know, if $n$ is any positive integer and $p \leq 1$, then

$$n^p \leq n \Rightarrow \frac{1}{n^p} \geq \frac{1}{n}$$

$\therefore$

$$1 + \frac{1}{2^p} \geq 1 + \frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3^p} + \frac{1}{4^p} \geq \frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} \geq \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

$$\frac{1}{9^p} + \frac{1}{10^p} + \ldots + \frac{1}{16^p} \geq \frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{\left(2^{m-1}+1\right)^p} + \frac{1}{\left(2^{m-1}+2\right)^p} + \ldots + \frac{2}{\left(2^m\right)^p} \geq \frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \ldots + \frac{1}{2^m} > \frac{2^m - 2^{m^{-1}}}{2^m} = \frac{1}{2}$$

Adding,

$$S_{2^m} > \frac{m}{2}$$

We shall now show that $\{S_n\}$ is not bounded above.

If $G$ be any number, however, large, then $\exists\, m \in \mathbf{N}$ such that

$$\frac{m}{2} > G$$

Let $n > 2^m$,

$\therefore$

$$S_n > S_{2^m} > G$$

Thus, the sequence $\{S_n\}$ of partial sums of the given series is not bounded above, and hence the series diverges for $p \leq 1$.

Thus, the given series $\Sigma \frac{1}{n^p}$ converges iff $p > 1$.

## 3. COMPARISON TESTS FOR POSITIVE TERM SERIES

Two types of comparison tests shall now be discussed. In the first type, the general term of one series will be compared with the general term of the second series. In the second type, the ratio of two consecutive terms of one series will be compared to the ratio of the corresponding consecutive terms of the second series.

## 3.1 Comparison Test (*First type*)

I. *If $\sum u_n$ and $\sum v_n$ are two positive term series, and $k \neq 0$, a fixed positive real number (independent of $n$) and there exists a positive integer $m$ such that $u_n \leq kv_n$, $\forall\ n \geq m$, then*

(i) *$\sum u_n$ is convergent, if $\sum v_n$ is convergent, and*

(ii) *$\sum v_n$ is divergent, if $\sum u_n$ is divergent.*

Let $n \geq m$, $S_n = u_1 + u_2 + ... + u_n$, and $t_n = v_1 + v_2 + ... + v_n$.

Now for all $n \geq m$, we have

$$S_n - S_m = u_{m+1} + u_{m+2} + ... + u_n$$

$$\leq k\left(v_{m+1} + v_{m+2} + ... + v_n\right) = k\left(t_n - t_m\right)$$

or

$$S_n \leq kt_n + \left(S_m - kt_m\right)$$

$\Rightarrow$
$$S_n \leq kt_n + h \qquad ...(1)$$

where $h = S_m - kt_m$, is a finite quantity.

(i) If $\sum v_n$ is convergent, then the sequence $\{t_n\}$ of its partial sums is bounded above, so that $\exists$ a number $B$ such that

$$t_n \leq B,\ \forall\ n$$

So from (1), we get

$S_n \leq kB + h$, for all $n \geq m$,

$\Rightarrow$ the sequence $\{S_n\}$ is bounded above.

$\Rightarrow$ $\sum u_n$ is convergent.

(ii) If $\sum u_n$ is divergent, then the sequence $\{S_n\}$ of its partial sums is not bounded above, so that if $G$ be any number, however, large, $\exists$ a positive integer $m_0$ such that

$$S_n > G\ \ \forall\ n \geq m_0.$$

Thus from (1), $\forall\ n \geq \max\,(m, m_0)$,

$$t_n \geq \frac{1}{k}(G - h),\ k \neq 0$$

$\Rightarrow$ the sequence $\{t_n\}$ is unbounded

$\Rightarrow$ $\sum v_n$ is divergent.

II. **Limit Form.** *If $\sum u_n$ and $\sum v_n$ are two positive term series such that $\lim\limits_{n\to\infty}\left(\dfrac{u_n}{v_n}\right) = l$, where $l$ is a non-zero finite number, then the two series converge or diverge together.*

Evidently $l > 0$.

Let $\varepsilon$ be a positive number such that $l - \varepsilon > 0$.

Since $\lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = l$, therefore $\exists$ a positive integer $m$ such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon, \ \forall \, n \geq m$$

$$\Rightarrow \quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \ \forall \, n \geq m$$

$$\Rightarrow \quad (l - \varepsilon) v_n < u_n < (l + \varepsilon) v_n, \ \forall \, n \geq m \qquad \text{...(1)}$$

Now, if $\Sigma v_n$ is convergent, then from (1)

$$u_n < (l + \varepsilon) v_n, \ \forall \, n \geq m$$

so that by Test I, $\Sigma u_n$ is convergent.

Again, if $\Sigma v_n$ is divergent, then from (1)

$$u_n > (l - \varepsilon) v_n, \ \forall \, n \geq m$$

so that by Test I, $\Sigma u_n$ is divergent.

Similarly, we may show that $\Sigma v_n$ converges or diverges with $\Sigma u_n$. Hence, the two series behave alike.

## 3.2 Comparison Test (*Second type*)

III. *If $\Sigma u_n$ and $\Sigma v_n$ are two positive term series and $\exists$ a positive integer m such that*

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}, \ \forall \, n \geq m,$$

*then (i) $\Sigma u_n$ is convergent, if $\Sigma v_n$ is convergent, and (ii) $\Sigma v_n$ is divergent, if $\Sigma u_n$ is divergent.*

Let $S_n = u_1 + u_2 + ... + u_n$

and $\quad t_n = v_1 + v_2 + ... + v_n$

For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdots \frac{u_{n-1}}{u_n} \geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdots \frac{v_{n-1}}{v_n} = \frac{v_m}{v_n}$$

$$\Rightarrow \quad u_n \leq \frac{u_m}{v_m} v_n$$

Since $m$ is a fixed positive integer, therefore $u_m/v_m$ is fixed number, say $k$. Thus $\forall \, n \geq m$ we have

$$u_n \leq k v_n$$

Hence by Test I, $\Sigma u_n$ converges if $\Sigma v_n$ converges and $\Sigma v_n$ diverges if $\Sigma u_n$ diverges.

Notes:

1. For practical purposes, Test II is very useful and can be easily applied.
2. For a successful application of the comparison test, we first make an estimate of the magnitude of the general term $u_n$ of the given series, and then select the auxiliary series $\sum v_n$ of such a magnitude that $\lim (u_n/v_n) = l \neq 0, \infty$, or in other words $u_n \sim v_n$. Thus for large values of $n$,
$$\sqrt{n^3+1} \sim n^{3/2}, \frac{n^r}{(1+n)^s} \sim n^{r-s}$$
$$\sin\frac{1}{n} \sim \frac{1}{n}$$
3. It will help to remember that for large $n$, $e^{an} \gg n^b \gg (\log n)^c$, where $a, b, c$ are positive numbers.

## 3.3 Solved Examples

**Example 4.** Show that the series $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ is convergent.

- We have

$$\frac{1}{2!} = \frac{1}{2}$$
$$\frac{1}{3!} < \frac{1}{2^2}$$
$$\frac{1}{4!} < \frac{1}{2^3}$$
$$\cdots \quad \cdots$$
$$\cdots \quad \cdots$$
$$\frac{1}{n!} < \frac{1}{2^{n-1}}$$
$$\therefore \quad 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

Thus each term of the given series after the second is less than the corresponding term of the convergent geometric series
$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$
Thus by Test I, the given series converges.

**Example 5.** Show that the series
$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \ldots + \frac{1}{(\log n)^p} + \ldots$$
diverges for $p > 0$.

- Since $\lim_{n\to\infty} \frac{(\log n)^p}{n} = 0,$

$$(\log n)^p < n, \ \forall\, n > 1$$

$$\frac{1}{(\log n)^p} < \frac{1}{n}, \ \forall\, n > 1$$

Let us compare the given series with the divergent series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Since each term of the given series is greater than the corresponding term of the divergent series, therefore, the given series diverges.

**Example 6.** Show that the series

$$\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \ldots \text{ is convergent}$$

- Let us denote the given series by $\sum u_n$, where

$$u_n = \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2}, \left(\sim \frac{1}{n^2}\right).$$

Let us compare it with the convergent series $\sum v_n$, where $v_n = 1/n^2$.

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2-1/n)(2)}{(2+1/n)^2 (2+2/n)^2} = \frac{1}{4}$$

Thus, the two series converge or diverge together.

Since $\sum v_n$ converges, therefore $\sum u_n$ also converges.

**Example 7.** Investigate the behaviour of the series whose $n$th term is $\sin 1/n$.

- Let $u_n = \sin 1/n$ and $v_n = 1/n$.

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$$

Therefore, the two series behave alike.

Since $\sum v_n$ diverges, therefore $\sum \sin 1/n$ also diverges.

**Example 8.** Test for convergence of the series whose $n$th term is

$$\{(n^3+1)^{1/3} - n\}$$

- Let

$$u_n = (n^3+1)^{1/3} - n$$

$$= n\left\{\left(1 + \frac{1}{n^3}\right)^{1/3} - 1\right\}$$

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$$= n\left\{\frac{1}{3n^3} + \ldots\right\} = \frac{1}{3n^2} + \ldots \left(\sim \frac{1}{n^2}\right)$$

and

$$\sum v_n = \sum \frac{1}{n^2}$$

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$$

Therefore, the two series converge and diverge together.

Since $\sum v_n$ converges, therefore, the given series also converges.

**Example 8.** Test the convergence of the series $\sum \frac{1}{n^{1+1/n}}$.

Let

$$u_n = \sum \frac{1}{n^{1+1/n}} \text{ and } v_n = \frac{1}{n}$$

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

∴

Hence, the two series $\sum u_n$ and $\sum v_n$ behave alike.

Since $\sum v_n$ is divergent, therefore $\sum \frac{1}{n^{1+1/n}}$ is also divergent.

## EXERCISE

Test the convergence of the following series:

$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \frac{7}{4\cdot 5\cdot 6} + \ldots$

$\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \ldots$

$\frac{1}{4\cdot 6} + \frac{\sqrt{3}}{6\cdot 8} + \frac{\sqrt{5}}{8\cdot 10} + \frac{\sqrt{7}}{10\cdot 12} + \ldots$

$\sum \frac{n+1}{n^p}$

$\sum \frac{1}{\sqrt{n} + \sqrt{n+1}}$

$\sum \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$ [*Hint*: Rationalize]

(i) $\sum \sin \frac{1}{n^2}$, (ii) $\sum \cos \frac{1}{n}$

8. $\Sigma \frac{1}{\sqrt{n}} \tan \frac{1}{n}$

9. $\sum_{n=1}^{\infty} e^{-n^2}$

10. $\sum_{n=1}^{\infty} \frac{5^n + 5}{3^n + 2}$

11. $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$

12. $\Sigma \left\{ \frac{\sqrt{n+1} - \sqrt{n-1}}{n} \right\}$

13. Show that the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots$ converges.

[*Hint*: $n^n > 2^n$, for $n > 2$. Compare with the convergent geometric series $\Sigma 1/2^n$]

## ANSWERS

1. Convergent, 2. Divergent, 3. Convergent, 4. Convergent, for, $p > 2$, 5. Divergent, 6. Convergent,
7. (*i*) Convergent, (*ii*) Divergent, 8. Convergent, 9. Convergent, 10. Divergent, 11. Convergent,
12. Convergent.

## 4. CAUCHY'S ROOT TEST

*If $\Sigma u_n$ is a positive term series, such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series*

(*i*) *converges, if $l < 1$,*

(*ii*) *diverges, if $l > 1$, and*

(*iii*) *the test fails to give any definite information, if $l = 1$.*

*Case I.* $l < 1$.

Let us select a positive number $\varepsilon$, such that $l + \varepsilon < 1$.

Let $l + \varepsilon = \alpha < 1$.

Since, $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore $\exists$ a positive integer $m$ such that

$$\left|(u_n)^{1/n} - l\right| < \varepsilon, \ \forall\, n \geq m$$

$$\Rightarrow \quad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon, \ \forall\, n \geq m$$

$$\Rightarrow \quad (l - \varepsilon)^n < u_n < (l + \varepsilon)^n = \alpha^n, \ \forall\, n \geq m$$

$$\Rightarrow \quad u_n < \alpha^n, \ \forall\, n \geq m.$$

But since $\sum \alpha^n$ is a convergent geometric series (common ratio $\alpha < 1$), therefore, by comparison test, the series $\sum u_n$ converges.

Case II. $l > 1$.

Let us select a positive number $\varepsilon$ such that $l - \varepsilon > 1$

Let $l - \varepsilon = \beta > 1$.

Since $\lim_{n \to \infty} (u_n)^{1/n} = l$, therefore $\exists$ a positive integer $m_1$ such that

$$l - \varepsilon < (u_n)^{1/n} < l + \varepsilon, \ \forall \ n \geq m_1$$

$$\Rightarrow \quad (l - \varepsilon)^n < u_n < (l - \varepsilon)^n, \ \forall \ n \geq m_1$$

$$\Rightarrow \quad u_n > (l - \varepsilon)^n = \beta^n, \ \forall \ n \geq m_1.$$

But since $\sum \beta^n$ is a divergent geometric series (common ratio $\beta > 1$), therefore by comparison test, the series $\sum u_n$ diverges.

**Note:** The test fails to give any definite information for $l = 1$.

Consider the two series $\sum (1/n)$ and $\sum (1/n^2)$.

$\sum (1/n)$ diverges when $\lim_{n \to \infty} (1/n)^{1/n} = 1$, and $\sum (1/n^2)$ converges when $\lim_{n \to \infty} (1/n^2)^{1/n} = 1$.

**Example 10.** Test for convergence of the series whose general term is $\left(1 + \dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

- Let $u_n = \dfrac{1}{\left(1 + 1/\sqrt{n}\right)^{n^{3/2}}}$, then

$$\lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \frac{1}{\left(1 + 1/\sqrt{n}\right)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

Hence, the series converges.

## 5. D'ALEMBERT'S RATIO TEST

*If $\sum u_n$ is a positive term series, such that $\lim_{n \to \infty} \dfrac{u_{n+1}}{u_n} = l$, then the series*

(*i*) *converges, if $l < 1$,*

(*ii*) *diverges, if $l > 1$, and*

(*iii*) *the test fails, if $l = 1$.*

*Case I.* $0 < l < 1$.

Let us select a positive number $\varepsilon$, such that $l + \varepsilon < 1$.

Let $l + \varepsilon = \alpha < 1, \alpha \neq 0$.

Since $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, therefore $\exists$ a positive integer $m$ such that

$$\left|\frac{u_{n+1}}{u_n} - l\right| < \varepsilon,\ \forall\, n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon,\ \forall\, n \geq m$$

$$\Rightarrow \qquad \frac{u_{n+1}}{u_n} < l + \varepsilon = \alpha,\ \forall\, n \geq m$$

For $n \geq m$,

$$\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdots \frac{u_n}{u_{n-1}} < \alpha^{n-m}$$

$$\Rightarrow \qquad u_n < \left(\frac{u_m}{\alpha^m}\right)\alpha^n,\ \forall\, n \geq m, \alpha < 1.$$

Since $m$ is a fixed integer, therefore $\left(\dfrac{u_m}{\alpha^m}\right)$ is a fixed finite number, say $k$.

Thus, $\forall\, n \geq m$, we have

$$u_n < k\alpha^n$$

But since $\sum \alpha^n$ is a convergent geometric series (common ratio, $\alpha < 1$), therefore by comparison test $\sum u_n$ converges.

*Case II.* $l > 1$.

Let us select a positive number $\varepsilon$, such that $l - \varepsilon > 1$.

Let $l - \varepsilon = \beta > 1$.

Since $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, therefore $\exists$ a positive integer $m_1$ such that

$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon,\ \forall\, n \geq m_1$$

$$\Rightarrow \qquad \frac{u_{n+1}}{u_n} > l - \varepsilon = \beta,\ \forall\, n \geq m_1$$

Now, for $n \geq m_1$,

$$\frac{u_n}{u_{m_1}} = \frac{u_{m_1+1}}{u_{m_1}} \cdot \frac{u_{m_1+2}}{u_{m_1+1}} \cdots \frac{u_n}{u_{n-1}} > \beta^{n-m_1}$$

$$\Rightarrow \qquad u_n > \frac{u_{m_1}}{\beta^{m_1}} \beta^n, \ \forall\, n \geq m_1$$

Since $m_1$ is a fixed integer, therefore $u_{m_1}/\beta^{m_1}$ is a fixed finite number, say $k_1$.

Thus, for $n \geq m_1$, we have

$$u_n > k_1 \beta^n$$

But $\Sigma \beta^n$ is a divergent geometric series (common ratio, $\beta > 1$), therefore by comparison test, $\Sigma u_n$ diverges.

**Note:** The test fails for $l = 1$ in the sense that it fails to give any definite information.

For example, consider the two series $\Sigma(1/n)$ and $\Sigma(1/n^2)$

$\Sigma \frac{1}{n}$ diverges when $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, and $\Sigma \frac{1}{n^2}$ converges when $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$.

**Remark:** Cauchy's Root Test is stronger than D' Alembert's Ratio Test and may succeed where Ratio-Test fails. For example, take the series $\Sigma u_n$, where $u_{2n-1} = 1/2^{2n-1}$ and $u_{2n} = 1/3^{2n}$, $\forall\, n$.

**Example 11.** Test for convergence of the series $\Sigma \frac{n^2 - 1}{n^2 + 1} x^n$, $x > 0$.

- Let $u_n = \frac{n^2 - 1}{n^2 + 1} x^n$.

$$\therefore \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2 + 1}{n^2 - 1} \cdot \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \cdot \frac{x^{n+1}}{x^n} = x$$

Hence by D'Alembert's Ratio Test the series converges if $x < 1$ and diverges if $x > 1$. The test fails to give any information when $x = 1$.

When $x = 1$, $u_n = \frac{n^2 - 1}{n^2 + 1}$, and $\lim\limits_{n\to\infty} u_n = 1 \neq 0$

$\Rightarrow$ The series is divergent.

Hence, the series converges if $x < 1$ and diverges if $x \geq 1$.

## 6. RAABE'S TEST

*If $\Sigma u_n$ is a positive term series, such that $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series* (*i*) *converges, if* $l > 1$,

(*ii*) *diverges, if* $l < 1$, and

(*iii*) *the test fails, if* $l = 1$.

*Case I.* $l > 1$.

Let us select a positive number $\varepsilon$, such that $l - \varepsilon > 1$.

Let $l - \varepsilon = \alpha > 1$.

Since $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, therefore $\exists$ a positive integer $m$ such that for all $n \geq m$,

$$l - \varepsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \varepsilon$$

$$\Rightarrow \qquad n\left(\frac{u_n}{u_{n+1}} - 1\right) > l - \varepsilon = \alpha$$

$$\Rightarrow \qquad nu_n - nu_{n+1} > \alpha u_{n+1}$$

$$\Rightarrow \qquad nu_n - (n+1)u_{n+1} > (\alpha - 1)u_{n+1}, \quad \alpha - 1 > 0$$

Putting $n = m, m+1, m+2, \ldots, n-1$ and adding, we get

$$mu_m - nu_n > (\alpha - 1)\left(u_{m+1} + u_{m+2} + \ldots + u_n\right)$$

$$= (\alpha - 1)(S_n - S_m), \text{ where } S_n = \sum_{r=1}^{n} u_r$$

$$\Rightarrow \qquad (\alpha - 1)(S_n - S_m) < mu_m, \quad \forall\, n \geq m$$

$$\Rightarrow \qquad S_n < S_m + \frac{m}{\alpha - 1}u_m, \quad \forall\, n \geq m,$$

Since $m$ is a fixed integer, therefore $S_m + \frac{m}{\alpha - 1}u_m$ is a fixed finite number.

Thus, the sequence $\{S_n\}$ of partial sums of the given series is bounded above and hence the series $\Sigma u_n$ is convergent.

*Case II.* $l < 1$.

Let us select a positive number $\varepsilon$, such that $l + \varepsilon < 1$.

Since $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, therefore $\exists$ a positive integer $m$ such that for all $n \geq m$,

$$l - \varepsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \varepsilon < 1$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} < \frac{n+1}{n}, \ \forall\, n \geq m$$

If $v_n = 1/n$, the series $\sum v_n$ is divergent, and

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Hence by comparison test the series $\sum u_n$ diverges.

**Notes:**

1. The test fails to give any definite information for $l = 1$. Consider the two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n(\log n)^2}$. The former is divergent, while the latter is convergent but

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$$

for both.

2. Raabe's Test is stronger that D'Alembert's Ratio Test and may succeed where Ratio-Test fails.

**Example 12.** Test for convergence of the series

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \ldots$$

■ Here

$$u_n = \frac{(1+\alpha)(2+\alpha)(3+\alpha)\ldots(n-1+\alpha)}{(1+\beta)(2+\beta)(3+\beta)\ldots(n-1+\beta)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+\alpha}{n+\beta} = 1$$

Hence, the Ratio-Test fails.

Again

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+\beta}{n+\alpha} - 1\right) = \lim_{n\to\infty} \frac{\beta-\alpha}{1+\alpha/n} = \beta - \alpha$$

Thus by Raabe's Test, the series converges if $\beta - \alpha > 1$ or $\beta > \alpha + 1$, and diverges if $\beta < \alpha$ ... The test fails for $\beta = \alpha + 1$.

But for $\beta = \alpha + 1$, the series becomes

$$\frac{\alpha}{\alpha+1} + \frac{1+\alpha}{2+\alpha} + \frac{1+\alpha}{3+\alpha} + \ldots = \sum \frac{1+\alpha}{n+\alpha}$$

which diverges, by comparison with $\sum 1/n$.

**Example 13.** Show that the series

$$\Sigma \frac{3 \cdot 6 \cdot 9 \ldots 3n}{7 \cdot 10 \cdot 13 \ldots (3n+4)} x^n, \quad x > 0$$

converges for $x \le 1$, and diverges for $x > 1$.

■ Now

$$\frac{u_n}{u_{n+1}} = x^n \frac{3 \cdot 6 \cdot 9 \ldots 3n}{7 \cdot 10 \ldots (3n+4)} \cdot \frac{7 \cdot 10 \cdot 13 \ldots (3n+4)(3n+7)}{3 \cdot 6 \cdot 9 \ldots 3n(3n+3)} \frac{1}{x^{n+1}}$$

$$= \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$\therefore \qquad \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = x$$

By Ratio Test, the series converges for $x < 1$ and diverges for $x > 1$. The test fails for $x = 1$. But for $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

$$\therefore \qquad \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n \to \infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

$\therefore$ By Raabe's Test, the series converges.

Thus, the series converges for $0 < x \le 1$ and diverges for $x > 1$.

## EXERCISE

Test the behaviour of the following series:

1. $\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \ldots$

2. $\left(\frac{1}{3}\right)^2 + \left(\frac{1 \cdot 2}{3 \cdot 5}\right)^2 + \left(\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)^2 + \ldots$

3. $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots, \; x > 0$

4. $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^5}{5\sqrt{4}} + \ldots, \; x > 0$

5. $1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \ldots$

6. $\dfrac{1^2 \cdot 2^2}{1!} + \dfrac{2^2 \cdot 3^2}{2!} + \dfrac{3^2 \cdot 4^2}{3!} + \dfrac{4^2 \cdot 5^2}{4!} + \ldots$

7. $\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \cdots$

8. $\dfrac{1}{2} + \dfrac{1}{3^2} + \dfrac{1}{2^3} + \dfrac{1}{3^4} + \ldots$

Test for convergence each of the following series whose $n$th terms are given:

9. $\dfrac{\underline{|n}}{n^n}$

10. $\dfrac{1 \cdot 2 \cdot 3 \ldots n}{7 \cdot 10 \ldots (3n+4)}$

11. $\dfrac{n^3+5}{3^n+2}$

12. $\dfrac{r^n}{n^n},\ r > 0$

13. $\dfrac{\sqrt{n}\, x^n}{\sqrt{n^2+1}},\ x > 0$

14. $\dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \dfrac{1}{n}$

15. $\dfrac{1 \cdot 3 \cdot 5 \ldots (4n-3)}{2 \cdot 4 \cdot 6 \ldots (4n-2)} \cdot \dfrac{x^{2n}}{4n}$

16. $\sqrt{\dfrac{n-1}{n^3+1}}\, x^n,\ x > 0$

17. $\dfrac{2 \cdot 4 \cdot 6 \ldots (2n+2)}{3 \cdot 5 \cdot 7 \ldots (2n+3)}\, x^{n-1},\ x > 0$

## ANSWERS

1. Converges
2. Converges
3. Converges for $x < 1$, diverges for $x \geq 1$
4. Converges for $x \leq 1$, diverges for $x > 1$
5. Converges for $|x| < 1$, diverges for $|x| > 1$; at $x = \pm 1$, converges for $p > 1$, diverges for $p \leq 1$
6. Convergent
7. Convergent
8. Convergent
9. Convergent
10. Convergent
11. Convergents
12. Convergent
13. Convergent for $x < 1$, divergent for $x \geq 1$
14. Convergent
15. Convergent for $|x| \leq 1$, divergent for $|x| > 1$
16. Convergent for $x < 1$, divergent for $x \geq 1$
17. Convergent for $x < 1$, divergent for $x \geq 1$.

## 7. LOGARITHMIC TEST

*If $\sum u_n$ is a positive term series such that,*

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l,$$

*then the series converges for $l>1$, and diverges for $l<1$.*

First, let $l>1$.

Let us select $\varepsilon>0$, such that $l-\varepsilon>1$.

Let $l-\varepsilon=\alpha>1$.

Since $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l$, therefore $\exists$ an $m$, such that

$$l-\varepsilon<n\log\frac{u_n}{u_{n+1}}<l+\varepsilon,\ \forall\ n\geq m$$

$$\Rightarrow \qquad n\log\frac{u_n}{u_{n+1}}>\alpha,\ \forall\ n\geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}}>e^{\alpha/n},\ \forall\ n\geq m$$

Now since $\{(1+1/n)^n\}$ is a monotonic increasing sequence converging to $e$, therefore

$$\left(1+\frac{1}{n}\right)^n\leq e,\ \forall\ n$$

so that we get

$$\frac{u_n}{u_{n+1}}>\left(1+\frac{1}{n}\right)^\alpha=\frac{(n+1)^\alpha}{n^\alpha}=\frac{v_n}{v_{n+1}},\ \forall\ n\geq m$$

where $v_n=1/n^\alpha$.

But since for $\alpha>1$, $\sum v_n$ converges, therefore by comparison test, $\sum u_n$ also converges.

We may similarly show that for $l<1$, the series $\sum u_n$ diverges.

**Example 14.** Test for convergence of the series

$$1+\frac{x}{1!}+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\dots,\ \text{for } x>0$$

- Ignoring the first term,

$$u_n = \frac{n^n x^n}{n!}$$

Now,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n x = ex$$

By Ratio Test, the series converges for $ex < 1$ or $x < 1/e$, and diverges for $x > 1/e$.

For $x = 1/e$,

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot e$$

Now,

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\left[1 - n\log\left(1+\frac{1}{n}\right)\right].$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right] = \frac{1}{2} < 1$$

Therefore by logarithmic test, the series diverges.

Hence the series converges for $x < 1/e$, and diverges for $x \geq 1/e$.

**Note:** Logarithmic test is generally more helpful in situations like those above, where the presence of a number li in $u_n/u_{n+1}$ makes the application of Raabe's Test difficult.

## 8. INTEGRAL TEST

*Improper Integral.* As preparatory to the introduction of Cauchy's Integral Test, it will help to remem that the infinite integral $\int_a^\infty u(x)\,dx$ is said to converge if $t(x) = \int_a^x u(x)\,dx$ tends to a finite limit as $x \to$ otherwise the integral is said to diverge.

Further, if $u(x) \geq 0$ for all $x > a$, it can be shown geometrically or otherwise that the integral $\int_a^t u(x$ is a monotonic increasing function of $t$, so that the improper integral

$$\int_a^\infty u(x)\,dx, \text{ where } u(x) \geq 0, \ \forall\ x > a$$

converges iff it is bounded above, *i.e.*, $\exists$ a positive number $k$ such that

$$\int_a^t u(t)\,dt \leq k, \ \forall\ t \geq a$$

## 8.1 Cauchy's Integral Test

*If $u$ is a non-negative monotonic decreasing integrable function such that $u(n) = u_n$, for all positive integral values of $n$, then the series $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ converge or diverge together.*

As $u$ is monotonic decreasing, we have

$$u(n) \geq u(x) \geq u(n+1), \text{ whenever } n \leq x \leq n+1$$

Also, since $u$ is non-negative and integrable,

$$\int_n^{n+1} u(n)\,dx \geq \int_n^{n+1} u(x)\,dx \geq \int_n^{n+1} u(n+1)\,dx$$

$$\Rightarrow \qquad u(n) \geq \int_n^{n+1} u(x)\,dx \geq u(n+1)$$

or

$$u_n \geq \int_n^{n+1} u(x)\,dx \geq u_{n+1} \qquad \text{...(1)}$$

Let us write $S_n = u_1 + u_2 + ... + u_n$, and $I_n = \int_1^n u(x)\,dx$, and putting $n = 1, 2, ..., (n-1)$ successively, and adding, we get

$$S_n - u_n \geq I_n \geq S_n - u_1$$

$$\Rightarrow \qquad 0 < u_n \leq S_n - I_n \leq u_1 \qquad \text{...(2)}$$

Let us consider the sequence $\{(S_n - I_n)\}$.

$$\begin{aligned}(S_n - I_n) - (S_{n-1} - I_{n-1}) &= S_n - S_{n-1} - (I_n - I_{n-1}) \\ &= u_n - \int_{n-1}^{n} u(x)\,dx \\ &\leq 0 \qquad \text{[using (1)]}\end{aligned}$$

Therefore, the sequence $\{(S_n - I_n)\}$ is monotonic decreasing, bounded by 0 and $u_1$.
Hence, the sequence converges and has a limit such that

$$0 \leq \lim (S_n - I_n) \leq u_1 \qquad \text{...(3)}$$

Thus the series $\sum u_n$ converges or diverges with the integral $\int_1^{\infty} u(x)\,dx$; if convergent, the sum of the series differs from the integral by less than $u_1$; if divergent, the limit of $(S_n - I_n)$ still exists and lies between 0 and $u_1$.

**Example 15.** Show that the series $\sum(1/n^p)$ converges if $p > 1$, and diverges if $p \leq 1$.

- Let $u(x) = 1/x^p$, so that for $x \geq 1$, the function $u$ is a non-negative monotonic decreasing integrable function such that

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$$u_n = u(n) = \frac{1}{n^p}, \quad \forall\, n \in \mathbf{N}$$

By Integral Test, $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ converge or diverge together.

Let us now test the convergence of the infinite integral.

$$\because \quad \int_1^X u(x)\,dx = \int_1^X \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p}\left(X^{1-p}-1\right), & \text{if } p \neq 1 \\ \log X, & \text{if } p = 1 \end{cases}$$

$$\therefore \quad \int_1^{\infty} u(x)\,dx = \lim_{X\to\infty} \int_1^X u(x)\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } \quad p > 1 \\ \infty, & \text{if } 0 < p \leq 1 \end{cases}$$

Thus $\int_1^{\infty} u(x)\,dx$ converges if $p > 1$, and diverges if $0 < p \leq 1$.

Hence, the infinite series $\Sigma(1/n^p)$ converges if $p > 1$, and diverges if $0 < p \leq 1$.

But, when $p < 0$, the series $\Sigma(1/n^p)$ diverges for then the $n$th term $n^{-p}$ does not tend to zero as $n \to \infty$.

Hence the series $\Sigma(1/n^p)$ converges when $p > 1$, and diverges when $p \leq 1$.

**Example 16.** The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p > 0$ converges for $p > 1$ and diverges for $p \leq 1$.

■ Let $u(x) = \dfrac{1}{x(\log x)^p}$, so that for $x \geq 2$, the function $u$ is a non-negative monotonic decreasing integrable function, such that

$$u_n = u(n) = \frac{1}{n(\log n)^p}, \quad \forall\, p > 0, n \in \mathbf{N}$$

By Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} u(x)\,dx$ converge or diverge together.

Let us now test the convergence of the infinite integral.

$$\because \quad \int_2^X u(x)\,dx = \int_2^X \frac{1}{x(\log x)^p}\,dx, \quad p > 0$$

$$= \begin{cases} \dfrac{(\log X)^{1-p} - (\log 2)^{1-p}}{1-p}, & \text{if } p \neq 1, \\ \log\log X - \log\log 2, & \text{if } p = 1. \end{cases}$$

$$\therefore \int_2^\infty u(x)\,dx = \lim_{X\to\infty}\int_2^X u(x)\,dx = \begin{cases} \dfrac{(\log 2)^{1-p}}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } 0<p\le 1. \end{cases}$$

Thus $\int_2^\infty u(x)\,dx$ converges if $p>1$, and diverges if $0<p\le 1$.

Hence the series $\sum_{n=2}^{\infty}\dfrac{1}{n(\log n)^p}$, $p>0$, converges if $p>1$, and diverges if $p\le 1$.

## 9. GAUSS'S TEST

*If $\sum u_n$ is a positive terms series such that,*

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p},$$

*where $\alpha>0$, $p>1$, and $\{\gamma_n\}$ is a bounded sequence, then*

(*i*) *for $\alpha\ne 1$, $\sum u_n$ converges if $\alpha>1$, and diverges if $\alpha<1$, whatever $\beta$ may be*

(*ii*) *for $\alpha=1$, $\sum u_n$ converges if $\beta>1$, and diverges if $\beta\le 1$.*

(*i*) When $\alpha\ne 1$,

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \alpha$$

Hence by Ratio Test, the series converges if $\alpha>1$, and diverges if $\alpha<1$.

(*ii*) When $\alpha=1$,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \beta$$

Hence by Raabe's Test, the series converges if $\beta>1$, and diverges if $\beta<1$.

For $\beta=1$, we have

$$\frac{u_n}{u_{n+1}} = 1+\frac{1}{n}+\frac{\gamma_n}{n^p},\quad p>1$$

Let us compare the given series with the divergent series $\sum v_n$ where $v_n=\dfrac{1}{n\log n}$.

Now,

$$\frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} = 1+\frac{1}{n}+\frac{\gamma_n}{n^p} - \frac{(n+1)\log(n+1)}{n\log n}$$

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$$= \frac{\gamma_n}{n^p} - \frac{n+1}{n}\left[\frac{\log(n+1)}{\log n} - 1\right]$$

$$= \frac{1}{n^p}\left[\gamma_n - (n+1)\log\left(1+\frac{1}{n}\right)\cdot\frac{n^{p-1}}{\log n}\right]$$

But $\lim\limits_{n\to\infty}(n+1)\log\left(1+\frac{1}{n}\right) = \lim\limits_{n\to\infty}\left[\log\left(1+\frac{1}{n}\right)^n + \log\left(1+\frac{1}{n}\right)\right] = 1$

and $\lim\limits_{n\to\infty}\frac{n^{p-1}}{\log n} = \infty,\ p > 1$, and $\{\gamma_n\}$ is bounded, therefore, for sufficiently large values of

$\gamma_n - (n+1)\log\left(1+\frac{1}{n}\right)\cdot\frac{n^{p-1}}{\log n}$ remains negative.

Thus $\exists$ a positive integer $m$ such that

$$\gamma_n - (n+1)\log\left(1+\frac{1}{n}\right)\cdot\frac{n^{p-1}}{\log n} < 0,\quad \forall\, n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} < 0,\quad \forall\, n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}},\quad \forall\, n \geq m$$

Since $\sum v_n$ is divergent, therefore by Comparison Test, the series $\sum u_n$ is also divergent.

**Remarks:**

1. Gauss's Test is very useful and may be used after the failure of Raabe's Test or directly without recourse to other tests.
2. If

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2} + \frac{\delta}{n^3} + \ldots$$

where $\alpha, \beta, \gamma, \ldots$ are independent of $n$, then we can write

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^2}$$

where $\gamma_n = \gamma + \delta/n + \ldots$, so that $\lim \gamma_n = \gamma$, *i.e.*, $\{\gamma_n\}$ is a bounded sequence.

Thus for the application of Gauss's Test, we may expand $u_n/u_{n+1}$ in powers of $1/n$ as in (1).

**Example 17.** Test the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \ldots$$

- Now

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \ldots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \ldots (2n+1)^2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{1}{n}\right)^{-2}$$

$$= \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \ldots\right)$$

$$= 1 + \frac{1}{n} - \frac{3}{4n^2} + \ldots \text{ higher powers of } \frac{1}{n}$$

so that $\alpha = 1$ and $\beta = 1$.

Hence by Gauss's Test, the series diverges.

**Example 18.** Test for convergence of the series

$$\Sigma \frac{1^2 \cdot 3^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \ldots (2n)^2} x^{n-1}, \quad x > 0.$$

- Here

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence by Ratio Test, the series converges if $x < 1$, and diverges if $x > 1$

Now for $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4n^2 + 3n}{(2n+1)^2} = 1$$

Hence Raabe's Test fails.

Let us now apply Gauss's Test.

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2}+\ldots\right)$$

$$=1+\frac{1}{n}-\frac{1}{4n^2}+\ldots \text{ higher powers of } \frac{1}{n}$$

so that by Gauss's Test, the series diverges.

Hence, the series converges for $x < 1$, and diverges for $x \geq 1$.

**Note:** We could get the result directly by Gauss's Test, for

$$\frac{u_n}{u_{n+1}}=\frac{1}{x}\left(\frac{2n+2}{2n+1}\right)^2=\frac{1}{x}+\frac{1/x}{n}-\frac{1/4x}{n^2}+\ldots$$

where $\alpha = 1/x, \beta = 1/x$.

**Example 19.** Test for convergence of the hypergeometric series

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\,\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

for all positive values of $x$; $\alpha, \beta, \gamma$ being all positive.

- It is a positive term series.

Ignoring the first term, which does not affect the behaviour of the series, we have

$$u_n=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1\cdot 2\cdot 3\ldots\cdot n}\,\frac{\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}x^n$$

so that

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\frac{1}{x}=\frac{1}{x}$$

Hence, by Ratio Test, the series converges if $x < 1$, and diverges if $x > 1$. For $x = 1$, we have

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}=\frac{\left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)}{\left(1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)}$$

$$=\left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)\left[1-\left(\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)+\left(\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)^2+\ldots\right]$$

$$=1+\frac{1+\gamma-\alpha-\beta}{n}+\frac{(\alpha+\beta-\gamma)(\alpha+\beta-1)-\alpha\beta}{n^2}+\ldots$$

Hence by Gauss's Test the series converges if $1 + \gamma - \alpha - \beta > 1$ or $\gamma > \alpha + \beta$ and diverges if $1 + \gamma - \alpha - \beta \le 1$ or $\gamma \le \alpha + \beta$.

Thus for positive values of $\alpha, \beta, \gamma$ and $x$,

(*i*) for $x < 1$, the series converges,

(*ii*) for $x > 1$, the series diverges, and

(*iii*) for $x = 1$, the series converges if $\gamma > (\alpha + \beta)$ and diverges if $\gamma \le (\alpha + \beta)$.

## EXERCISE

Test the convergence of the series:

1. $1 + \frac{2x}{2!} + \frac{3^2 \cdot x^2}{3!} + \frac{4^3 \cdot x^3}{4!} + \ldots$, for $x > 0$

2. $\frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots$, for $x > 0$

3. $\frac{a + x}{1!} + \frac{(a + 2x)^2}{2!} + \frac{(a + 3x)^3}{3!} + \ldots$, for $a, x > 0$.

4. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$

5. Apply Cauchy's Integral Test to test the convergence of the series.

   (*i*) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$, (*ii*) $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

6. Test the convergence of $\sum_{n=3}^{\infty} \frac{1}{n \log n \, (\log \log n)^p}$, $p > 0$

   Test for convergence the series whose $n$th term is

7. $\frac{(n!)^2}{(2n)!} x^n$, $x > 0$,

8. $\frac{n!}{(n + 1)^n} x^n$, $x > 0$.

9. Prove that $1 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} + \frac{\alpha(\alpha + 1)(\alpha + 2)}{\beta(\beta + 1)(\beta + 2)} + \ldots$, where $\alpha$ and $\beta$ are positive, converges if $\beta > \alpha + 1$, and diverges if $\beta \le \alpha + 1$.

## ANSWERS

1. Converges if $x < 1/e$, diverges if $x \geq 1/e$.
2. Converges if $|x| \leq 1$, diverges if $|x| > 1$.
3. Converges for $x < 1/e$, diverges for $x \geq 1/e$.
4. Diverges
5. (*i*) Convergent (*ii*) Convergent.
6. Converges for $p > 1$, diverges for $p \leq 1$.
7. Converges for $x < 4$, diverges for $x \geq 4$.
8. Converges for $0 < x < e$, diverges for $x \geq e$.

## 10. SERIES WITH ARBITRARY TERMS

So far we have considered series with positive terms only. We shall now discuss series with terms having any sign whatsoever.

### 10.1 Alternating Series

A series whose terms are alternatively positive and negative, *e.g.*,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

is called an *alternating series*.

**Leibnitz Test.** *If the alternating series*

$$u_1-u_2+u_3-u_4+\ldots,(u_n>0,\ \forall\ n)$$

*is such that*

(*i*) $u_{n+1}\leq u_n,\ \forall\ n$, *and*

(*ii*) $\lim\limits_{n\to\infty} u_n=0$,

*then the series converges.*

Let $S_n=u_1-u_2+u_3-u_4+\ldots+(-1)^{n-1}u_n$

Now for all $n$,

$$S_{2n+2}-S_{2n}=u_{2n+1}-u_{2n+2}\geq 0$$

$\Rightarrow$ $$S_{2n+2}\geq S_{2n}$$

$\Rightarrow$ $\{S_{2n}\}$ is a monotonic increasing sequence.

Again

$$\begin{aligned}S_{2n}&=u_1-u_2+u_3-\ldots+u_{2n-1}-u_{2n}\\&=u_1-(u_2-u_3)-(u_4-u_5)-\ldots-(u_{2n-2}-u_{2n-1})-u_{2n}\end{aligned}$$

But since $u_{n+1}\leq u_n$, for all $n$, therefore, each bracket on the right is positive and hence

$$S_{2n}<u_1,\ \forall\ n$$

Thus, the monotonic increasing sequence $\{S_{2n}\}$ is bounded above and is consequently convergent.

Let $\lim\limits_{n\to\infty} S_{2n} = S$.

We shall now show that the sequence $\{S_{2n+1}\}$ also converges to the same limit $S$.

Now

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

$$\therefore \quad \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

But by condition (*ii*),

$$\lim_{n\to\infty} u_{2n+1} = 0$$

$$\therefore \quad \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = S$$

Thus, the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit $S$. We shall now show that the sequence $\{S_n\}$ also converges to $S$.

Let $\varepsilon > 0$ be given.

Since the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to $S$, therefore $\exists$ positive integers $m_1$, $m_2$, respectively, such that

$$|S_{2n} - S| < \varepsilon, \ \forall\, n \geq m_1 \qquad \text{...(1)}$$

and

$$|S_{2n+1} - S| < \varepsilon, \ \forall\, n \geq m_2 \qquad \text{...(2)}$$

Thus from (1) and (2), we have

$$|S_n - S| < \varepsilon, \ \forall\, n \geq \max\,(2m_1, 2m_2 + 1)$$

$\Rightarrow$ $\{S_n\}$ converges to $S$

$\Rightarrow$ The series $\sum(-1)^{n-1} u_n$ converges.

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**Example 20.** Show that the series $\dfrac{1}{1^p} - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \ldots$ converges for $p > 0$.

- Let $u_n = 1/n^p$.
  Here

$$u_{n+1} \leq u_n, \ \forall\, n$$

and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$$

Hence by Leibnitz test, the alternating series $\sum \dfrac{(-1)^{n-1}}{n^p}$ converges.

## 10.2 Absolute Convergence

A series $\Sigma u_n$ is said to be *absolutely convergent* if the series $\Sigma u_n$ obtained on taking every term of given series with a positive sign is convergent, *i.e.*, if the series $\Sigma|u_n|$ is convergent.

**Conditional Convergence.** A series which is convergent but is not absolutely convergent is calle *conditionally convergent* series.

### ILLUSTRATIONS

1. The series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\dots$$

is absolutely convergent because the series

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\dots$$

obtained on taking every term of the given series with a positive sign, is convergent.

2. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

is convergent by Leibnitz test, but the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$$

obtained on taking every term with a positive sign is divergent. Thus, the given series is condition convergent.

**Theorem 10.** *Every absolutely convergent series is convergent.*

Let $\Sigma u_n$ be absolutely convergent, so that $\Sigma|u_n|$ is convergent.

Hence, for any $\varepsilon>0$, by Cauchy's General Principle of convergence, $\exists$ a positive integer $m$ such th

$$|u_{n+1}|+|u_{n+2}|+\dots+|u_{n+p}|<\varepsilon, \quad \forall\, n\ge m \wedge p\ge 1$$

Also for all $n$ and $p>1$,

$$|u_{n+1}+u_{n+2}+\dots+u_{n+p}|\le|u_{n+1}|+|u_{n+2}|+\dots+|u_{n+p}|<\varepsilon, \quad \forall\, n\ge m\wedge p\ge 1.$$

Hence, by Cauchy's General Principle of convergence the series $\Sigma u_n$ converges.

**Aliter.** Let $\Sigma u_n$ be absolutely convergent, so that $\Sigma|u_n|$ is convergent.

Let $a_n=\begin{cases}u_n, & \text{if } u_n\ge 0\\ 0, & \text{if } u_n<0\end{cases}$ and $b_n=\begin{cases}-u_n, & \text{if } u_n<0\\ 0, & \text{if } u_n\ge 0.\end{cases}$

Then clearly,

$$a_n\ge 0,\, b_n\ge 0,$$

$$u_n = a_n - b_n \qquad \text{...(2)}$$

and

$$|u_n| = a_n + b_n \qquad \text{...(3)}$$

From (1) and (3), it follows that

$$a_n \le |u_n|, \; b_n \le |u_n|$$

Since $\Sigma|u_n|$ is convergent, therefore, by Comparison Test, both $\Sigma a_n$ and $\Sigma b_n$ are convergent.

Hence, by Theorem 5, $\Sigma(a_n - b_n)$ is convergent.

Hence from (2), it follows that $\Sigma u_n$ is convergent.

**Remarks:**

1. The divergence of $\Sigma|u_n|$ does not imply the divergence of $\Sigma u_n$.

   For example, if $u_n = \frac{(-1)^{n-1}}{n}$, we have seen above that $\Sigma|u_n|$ is divergent, whereas $\Sigma u_n$ is convergent.

2. The very great significance of the concept of Absolute Convergence is that the convergence of absolutely convergent series is much more easy to recognise than that of conditionally convergent series—usually by comparison with series of positive terms. In fact all the tests for positive term series become available for the purpose. But this significance becomes more visible in the discussion of rearrangement of series—so much so that we may operate on absolutely convergent series, precisely as we operate on sums of a finite number of terms, whereas in the case of conditionally convergent series this in general is not possible.

**Example 21.** Show that for any fixed value of $x$, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

- Let $u_n = \frac{\sin nx}{n^2}$, so that $|u_n| = \frac{|\sin nx|}{n^2}$

  Now $\left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2}$, $\forall\, n$ and $\Sigma \frac{1}{n^2}$ converges.

  Hence, by Comparison Test, the series $\Sigma\left|\frac{\sin nx}{n^2}\right|$ converges.

  Since every absolutely convergent series is convergent, therefore $\Sigma \frac{\sin nx}{n^2}$ is convergent.



**Example 22.** Show that the series

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

converges absolutely for all values of $x$.

- Let $u_n = \frac{x^n}{n!}$.

Now, $\lim_{n\to\infty} \dfrac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \dfrac{n+1}{|x|} \to \infty$ except when $x = 0$.

So by Ratio Test, the series converges absolutely for all $x$ except possibly zero.

But for $x = 0$ the series evidently converges absolutely.

Hence, the series converges absolutely for all values of $x$.

Note: Since for a convergent series $\sum u_n$, $\lim_{n\to\infty} u_n = 0$,

$$\therefore \quad \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

A useful result.

**Ex.** If $b > 0$, then show that the series

$$x + \frac{a-b}{\lfloor 2} x^2 + \frac{(a-b)(a-2b)}{\lfloor 3} x^3 + \ldots \text{ converges absolutely for } |x| < b^{-1}.$$

**Example 23.** Show that

$$\lim_{n\to\infty} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n = 0,$$

where $|x| < 1$ and $m$ is any real number.

■ Consider the series $\sum u_n$, where

$$u_n = \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n$$

Now

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left|\frac{n}{m-n}\right| \cdot \frac{1}{|x|}$$

$$= \lim_{n\to\infty} \left|\frac{1}{\frac{m}{n} - 1}\right| \cdot \frac{1}{|x|} = \frac{1}{|x|}$$

Hence, the series $\sum u_n$ converges absolutely for $|x| < 1$

$\Rightarrow$ The series $\sum u_n$ converges for $|x| < 1$

$\Rightarrow$ $\lim_{n\to\infty} u_n = 0$, for $|x| < 1$

*i.e.,*
$$\lim_{n\to\infty} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n = 0, \text{ if } |x| < 1$$

Note: The results of Examples 22 and 23 are very useful.

Also see Examples 12 and 13 of Ch. 3.

## EXERCISE

1. Show that the following series are convergent:

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$

(iii) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots$

(iv) $\frac{\log 2}{2^2}-\frac{\log 3}{3^2}+\frac{\log 4}{4^2}-\ldots$

2. Prove that the following series are absolutely convergent:

(i) $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$

(ii) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

(iii) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$

3. Show that $\lim\limits_{n\to\infty}\frac{n^r}{x^n}=0$, if $x>1$.

4. Show that the following series are conditionally convergent:

(i) $\Sigma\frac{(-1)^{n+1}}{\sqrt{n}}$

(ii) $\Sigma\frac{(-1)^{n+1}}{3n-2}$

5. Show that the series $\Sigma\frac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p>1$, but conditionally convergent for $0<p\le 1$.

6. Show that the following series are absolutely convergent:

(i) $\Sigma(-1)^{n-1}\left\{\frac{1}{n^2}+\frac{1}{(n+1)^2}\right\}$

(ii) $\Sigma(-1)^{n-1}\left\{\frac{1}{n^{5/2}}+\frac{1}{(n+1)^{5/2}}\right\}$

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(vi) $\Sigma(-1)^n \dfrac{n+2}{2^n+5}$

7. Show that the series $\Sigma\left(\dfrac{1}{n}+\dfrac{(-1)^{n-1}}{\sqrt{n}}\right)$ is divergent.

8. Show that the series $1-\dfrac{1}{3\cdot 4}+\dfrac{1}{5\cdot 4^2}-\dfrac{1}{7\cdot 4^3}+\ldots$ converges.

9. Use Cauchy's Integral Test to show that the following series converge:

(i) $\sum_{n=0}^{\infty}\left(\dfrac{1+n}{1+n^2}\right)^2$

(ii) $\sum_{n=2}^{\infty}\dfrac{1}{\sqrt{n}}\log\dfrac{n+1}{n-1}$

10. Show that the following series are absolutely convergent:

(i) $\Sigma \dfrac{\sin n\alpha}{n^2}$

(ii) $\Sigma(-1)^{n+1}\dfrac{n^3}{2^n}$

(iii) $\dfrac{1}{2}-\dfrac{1}{2}\cdot\dfrac{1}{2^2}+\dfrac{1}{3}\cdot\dfrac{1}{2^3}-\dfrac{1}{4}\cdot\dfrac{1}{2^4}+\ldots$

11. Show that the following series are conditionally convergent:

(i) $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\log(n+1)}$

(ii) $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n-\log n}$

12. Establish the divergence of the series $2-\dfrac{3}{2}+\dfrac{4}{3}-\dfrac{5}{4}+\ldots$

13. Show that if $\Sigma a_n^2$ and $\Sigma b_n^2$ are convergent infinite series, then $\Sigma a_n b_n$ is an absolutely convergent series.

14. Show that if the series $\Sigma a_n$ is absolutely convergent, then the series $\Sigma\dfrac{n+1}{n}a_n$ is also absolutely convergent.

## 10.3 Tests for Series of Arbitrary Terms

We now consider arbitrary term series which are convergent (but not necessarily absolutely) and obtain tests for their convergence. We first prove an important lemma, due to Abel.

**Lemma.** *If $b_n$ is a positive, monotonic decreasing function and if $A_n$ is bounded, then the series $\Sigma A_n(b_n-b_{n+1})$ is absolutely convergent.*

Since $A_n$ is bounded, therefore $\exists$ a positive number $k$ such that

$$|A_n| \le k, \ \forall n$$

Thus

$$\sum_{n=1}^{m} \left| A_n\left(b_n - b_{n+1}\right) \right| = \sum_{n=1}^{m} |A_n| \left(b_n - b_{n+1}\right) \ \left[\because \ b_n - b_{n+1} \ge 0\right]$$

$$\le k \sum_{n=1}^{m} \left(b_n - b_{n+1}\right) \ = k\left(b_1 - b_{m+1}\right) < kb_1.$$

Thus the sequence of partial sums of the positive term series $\sum \left| A_n(b_n - b_{n+1}) \right|$ is bounded above by $kb_1$, so that the series $\sum \left| A_n(b_n - b_{n+1}) \right|$ converges, *i.e.*, the series $\sum A_n(b_n - b_{n+1})$ converges absolutely.

**Note:** The lemma may be restated as follows:

*If $\{b_n\}$ is positive, monotonic decreasing sequence and if $\{A_n\}$ is a bounded sequence, then the series $\sum A_n(b_n - b_{n+1})$ is absolutely convergent.*

## 10.4 Abel's Test

*If $b_n$ is a positive monotonic decreasing function and if $\sum u_n$ is a convergent series, then the series $\sum u_n b_n$ is also convergent.*

Let $v_n = u_n b_n$ and $S_n = \sum_{r=1}^{n} u_r, V_n = \sum_{r=1}^{n} v_r$ be $n$th partial sums. Then

$$V_n = u_1 b_1 + u_2 b_2 + \ldots + u_n b_n$$

$$= S_1 b_1 + (S_2 - S_1)b_2 + \ldots + \left(S_n - S_{n-1}\right)b_n$$

$$= S_1\left(b_1 - b_2\right) + S_2\left(b_2 - b_3\right) + \ldots + S_{n-1}\left(b_{n-1} - b_n\right) + S_n b_n$$

$$= \sum_{r=1}^{n-1} S_r\left(b_r - b_{r+1}\right) + S_n b_n \qquad \ldots(1)$$

Since the series $\sum u_n$ is convergent, therefore the sequence $\{S_n\}$ is also convergent and hence bounded. Also $b_n$ is a positive and monotonic decreasing function. Therefore, by the above Lemma, the series $\sum S_n(b_n - b_{n+1})$ and hence the partial sum $\sum_{r=1}^{n-1} S_r(b_r - b_{r+1})$ tend to a finite limits as $n \to \infty$.

Also, since $\{b_n\}$ is monotonic decreasing and bounded below by zero, therefore $\{b_n\}$ is convergent and so $b_n$ tends to a finite limit as $n \to \infty$. Hence, $S_n b_n$ tends to finite a limit as $n \to \infty$.

Using the above results we find from (1) that $V_n$ tends to a finite limit as $n \to \infty$, *i.e.*, the sequence $\{V_n\}$ of partial sums of $\sum v_n$ converges. Consequently the series $\sum v_n$ or $\sum u_n b_n$ converges.

**Corollary.** *A convergent series $\sum u_n$ (which need not converge absolutely) remains convergent if its terms are each multiplied by a factor $a_n$, provided that the sequence $\{a_n\}$ is bounded and monotonic.*

Under the given conditions, $\{a_n\}$ converges to a limit $a$, say. Let us write $b_n = a - a_n$, when $\{a_n\}$ is an increasing sequence, and $b_n = a_n - a$ when $\{a_n\}$ is decreasing. Then it is clear that the sequence $\{b_n\}$ monotonically decreases to the limit zero. With this function $b_n$, we deduce as above that the series $\sum u_n b_n$ converges.

Also, since $\sum u_n$ and hence $\sum a u_n$ converges, the convergence of $\sum u_n a_n$ follows.

## 10.5 Dirichlet's Test

*If $b_n$ is a positive, monotonic decreasing function with limit zero, and if, for the series $\sum u_n$, the sequence $\{S_n\}$ of partial sums is bounded, then the series $\sum u_n b_n$ is convergent.*

Using the notation of § 10.4 we get as before

$$V_n = \sum_{r=1}^{n-1} S_r (b_r - b_{r+1}) + S_n b_n$$

Since $S_n$ is bounded and $b_n$ is positive and monotonic decreasing, therefore, by the above Lemma $\sum_{r=1}^{n-1} S_r (b_r - b_{r+1})$ tends to a finite limit as $n \to \infty$.

Also since $b_n \to 0$ as $n \to \infty$ and since $S_n$ is bounded, therefore $S_n b_n \to 0$ as $n \to \infty$.

Using the above results, we find from (1) that $V_n$ tends to a finite limit as $n \to \infty$ and hence the series $\sum v_n (= \sum u_n b_n)$ converges.

The case $u_n = (-1)^{n-1}$ of the above theorem is of considerable importance.

**Corollary.** *Leibnitz test is a particular case of Dirichlet's test.*

Since the sequence of partial sums of the series $\sum (-1)^{n-1}$ is bounded (for, $S_n = 0$, if $n$ is even and $S_n = 1$, if $n$ is odd) therefore by taking $u_n = (-1)^{n-1}$, $\sum u_n b_n$ reduces to $b_1 - b_2 + b_3 - b_4 + \dots$. Thus we obtain, "If $b_n$ is positive and monotonic decreasing to the limit zero, then the alternating series $b_1 - b_2 + b_3 - b_4 + \dots$ is convergent."

**Example 24.** Show that the series $0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \dots$ converges.

- The given series can be considered to have arisen as a result of multiplication of the terms of series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$$

by the terms of the sequence

$$0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4} \dots$$

Since the series (1) is convergent and the sequence (2) is monotonic and bounded, therefore by A Test, the given series converges.

**Example 25.** Test the convergence of the series

$$\sum \frac{(n^3+1)^{1/3}-n}{\log n}$$

■ Let $u_n = \{(n^3+1)^{1/3} - n\}$, and $b_n = \dfrac{1}{\log n}$.

Then the given series can be written as $\sum b_n u_n$.

Since $\sum u_n$ converges and $\{b_n\}$ is a positive monotonic decreasing sequence, therefore, by Abel's Test, the given series converges.

**Example 26.** Show that the series $1 - \dfrac{1}{3 \cdot 2^2} + \dfrac{1}{5 \cdot 3^2} - \dfrac{1}{7 \cdot 4^2} + \ldots$ is convergent.

■ Let $\sum u_n = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$, and $b_n = \dfrac{1}{2n-1}$.

Then the given series can be written as $\sum b_n u_n$.

Since $\sum u_n$ converges and $b_n$ is positive and monotonic decreasing, therefore by Abel's Test, the given series converges.

## 11. REARRANGEMENT OF TERMS

It is a well known fact that a finite sum keeps the same value, no matter how the terms of the sum are arranged. This property, however, is by no means universally true for infinite series. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

converges to a sum, say $S$. On rearranging the terms so that each positive term is followed by two negative terms, the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$$

converges to the sum $\dfrac{1}{2}S$.

Another rearrangement gives the series (with sign changed)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2^8} - 1 + \frac{1}{2^8+2} + \ldots + \frac{1}{2^{16}} - \frac{1}{3} + \ldots$$

which is divergent.

Consequently, the *rearrangement* (or equally well *dearrangement*) or changing the order of the terms in a series may not only alter the sum of the series but may change its nature all together. So we naturally ask *under what conditions may we rearrange the terms of the series without altering its value*?

A series $\sum v_n$ is said to arise from a series $\sum u_n$ by a *rearrangement* of terms, if there exists a one-to-one correspondence between the terms of the two series, so that every term in the first series occupies a perfectly definite place in the second series, and conversely. Thus, corresponding to any number of terms (say $n$) in the first series, we can find a number $p$ such that $p$ terms of the second series contain all the $n$ terms (and some others) of the first; and conversely.

**11.1 Theorem 11. Dirichlet's theorem.** *A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series.*

We shall prove the theorem in two parts, first for the positive term series, and then for series of arbitrary terms.

(*i*) Let $\sum u_n$ be a given series of positive terms which converges to a sum, say $S$. Let $\sum v_n$ be the rearranged series. Let $S_m$ and $\sigma_m$ respectively denote the partial sums of the series $\sum u_n$ and $\sum v_n$.

As $\sigma_m$ consists of $m$ terms of the series $\sum v_n$, we can find a number $p$ such that $p$ terms of the series $\sum u_n$ contain all the $m$ terms (and some more) of the former and since we are dealing with positive term series, therefore $\sigma_m \leq S_p \leq S$.

Thus the sequence $\{\sigma_m\}$ of partial sums of the series $\sum v_n$ of positive terms is bounded above by $S$. Therefore, the sequence $\{\sigma_m\}$ and consequently the series $\sum v_n$ converges to a limit, say $\sigma$, where $\sigma \leq S$.

Considering $\sum u_n$ as rearrangement of $\sum v_n$, we can similarly prove that $S \leq \sigma$.

Hence, $\sigma = S$, *i.e.*, the two series converge to the same sum.

(*ii*) Let $\sum u_n$ be an absolutely convergent series of arbitrary terms and $\sum v_n$, the rearrangement of $\sum u_n$.

Let

$$a_n = \begin{cases} u_n, & \text{if } u_n \geq 0 \\ 0, & \text{if } u_n < 0 \end{cases}$$

$$b_n = \begin{cases} -u_n, & \text{if } u_n < 0 \\ 0, & \text{if } u_n \geq 0 \end{cases}$$

$$a_n' = \begin{cases} v_n, & \text{if } v_n \geq 0 \\ 0, & \text{if } v_n < 0 \end{cases}$$

$$b_n' = \begin{cases} -v_n, & \text{if } v_n < 0 \\ 0, & \text{if } v_n \geq 0 \end{cases}$$

Thus clearly $a_n, b_n, a_n', b_n'$ are non-negative, and

$$\left.\begin{aligned} u_n = a_n - b_n, |u_n| = a_n + b_n \\ v_n = a_n' - b_n', |v_n| = a_n' + b_n' \end{aligned}\right\} \quad \ldots(1)$$

$$\Rightarrow \quad \left.\begin{aligned} a_n = \frac{1}{2}(|u_n| + u_n) \\ b_n = \frac{1}{2}(|u_n| - u_n) \end{aligned}\right\} \quad \ldots(2)$$

and

$$\left.\begin{aligned} a'_n &= \frac{1}{2}(|v_n| + v_n) \\ b'_n &= \frac{1}{2}(|v_n| - v_n) \end{aligned}\right\} \qquad ...(3)$$

Since $\Sigma u_n$ is absolutely convergent, therefore from (2), $\Sigma a_n$ and $\Sigma b_n$ are also convergent. (Ref. Theorem 5). Again, since $\Sigma a_n$ and $\Sigma b_n$ are convergent series of non-negative terms, $\Sigma a'_n$ and $\Sigma b'_n$ are respectively their rearrangements, therefore by what has been proved above, $\Sigma a'_n$ and $\Sigma b'_n$ are also convergent. Also if $a, b, a', b'$, denote the sum of the series $\Sigma a_n, \Sigma b_n, \Sigma a'_n, \Sigma b'_n$ respectively, then $a = a'$ and $b = b'$.

From (1), it follows at once that $\Sigma v_n$ and $\Sigma |v_n|$ are convergent and

$$\sum_{n=1}^{\infty} v_n = a' - b' = a - b = \sum_{n=1}^{\infty} u_n$$

$$\sum_{n=1}^{\infty} |v_n| = a' + b' = a + b = \sum_{n=1}^{\infty} |u_n|$$

Hence, the rearranged series $\Sigma v_n$ converges absolutely to the same sum as $\Sigma u_n$.

**Remarks:**

1. For a positive term series the theorem may be stated as follows:

   *A series of positive terms, if convergent, has a sum independent of the order of its terms, but if divergent, it remains divergent however its terms are rearranged.*

   For the *divergent* case, one may argue as follows:

   If $\Sigma u_n$ is divergent, $\Sigma v_n$ cannot converge; for the foregoing argument shows that if $\Sigma v_n$ converges, $\Sigma u_n$ (regarded as a rearrangement of $\Sigma v_n$) must also converge. Consequently $\Sigma v_n$ is divergent.

2. The above theorem (along with Theorem 5) shows that the brackets may be inserted or removed, or terms be picked and placed at random without changing the behaviour of a positive term series or an absolutely convergent series. In fact they behave exactly like finite sums.

3. An absolutely convergent series, because it remains convergent, with unaltered sum under every rearrangement of terms, is also called *unconditionally convergent*.

**11.2** We shall now prove a theorem which though of no practical importance, is of considerable theoretical interest.

**Theorem 12.** ***Riemann's theorem.*** *By an appropriate rearrangement of terms, a conditionally convergent series $\Sigma u_n$ can be made*

(*i*) *to converge to any number $\sigma$*, or

(*ii*) *to diverge to $+\infty$, or*

(*iii*) *to diverge to $-\infty$, or*

(*iv*) *to oscillate finitely, or*

(*v*) *to oscillate infinitely.*

Let

$$a_n = \begin{cases} u_n, & \text{if } u_n \geq 0 \\ 0, & \text{if } u_n < 0 \end{cases} \quad \text{and} \quad b_n = \begin{cases} -u_n, & \text{if } u_n < 0 \\ 0, & \text{if } u_n \geq 0. \end{cases}$$

Then clearly $a_n, b_n$ are non-negative, and

$$u_n = a_n - b_n, \ |u_n| = a_n + b_n$$

Since $\sum u_n$ is conditionally convergent, therefore $\sum |u_n|$ diverges and hence from (1) at least one of the series $\sum a_n, \sum b_n$ diverges.

Again, since $\sum u_n$ converges, therefore it follows from (1) that the two series $\sum a_n, \sum b_n$ either both converge or both diverge (being non-negative term series, they cannot oscillate). Thus we infer that $\sum a_n$ and $\sum b_n$ both diverge.

Also $a_n \to 0, b_n \to 0$ as $n \to \infty$ $(\because\ u_n \to 0)$.

(*i*) We shall first show that a rearrangement $\sum v_n$, of $\sum u_n$ can be found which converges to a number, $\sigma$.

Let $n_1$ be the least number of terms of $\sum a_n$, whose sum

$$a_1 + a_2 + a_3 + \ldots + a_{n_1} > \sigma$$

Let $m_1$ be the least number of terms of $\sum b_n$, such that

$$a_1 + a_2 + a_3 + \ldots + a_{n_1} - b_1 - b_2 - \ldots - b_{m_1} < \sigma$$

Again, let $n_2$ be the least number of the next (terms following $a_{n_1}$) of $\sum a_n$, such that

$$a_1 + a_2 + \ldots + a_{n_1} - b_1 - b_2 - \ldots - b_{m_1} + a_{n_1+1} + a_{n_1+2} + \ldots + a_{n_1+n_2} > \sigma$$

Let $m_2$ be the least number of the next terms of $\sum b_n$ such that when $(-b_{m_1+1} - b_{m_1+2} - \ldots - b_{m_1+m_2})$ is added to the above sum, makes it less than $\sigma$. The process may be continued indefinitely. The process indicated above is always possible because of the divergence of the two series $\sum a_n, \sum b_n$.

Let $\sum v_n$ be the rearranged series and $\{\sigma_n\}$ its sequence of partial sums.

Clearly

$$\sigma_{n_1} > \sigma, \sigma_{n_1+m_1} < \sigma, \sigma_{n_1+m_1+n_2} > \sigma, \ldots$$

Therefore, it can be easily shown that the sequence $\{\sigma_n\}$ converges to $\sigma$.

$\Rightarrow$ The rearrangement $\sum v_n$ converges to $\sigma$.

(*ii*) We shall now show that a suitable rearrangement of $\sum u_n$ can be found which diverges to $+\infty$.

Let us consider the rearrangement

$$a_1 + a_2 + \ldots + a_m - b_1 + a_{m_1+1} + a_{m_1+2} + \ldots + a_{m_2} - b_2 + a_{m_2+1} + \ldots,$$

in which a group of positive terms is followed by a single negative term.

This is certainly a rearrangement of $\sum u_n$ and let us denote it by $\sum v_n$ and its partial sum by

Now, since the series $\sum a_n$ is divergent, its partial sums are therefore unbounded.

Let us first choose $m_1$ so large that

$$a_1 + a_2 + a_3 + \ldots + a_{m_1} > 1 + b_1$$

then $m_2 > m_1$ so large that

$$a_1 + a_2 + \ldots + a_{m_1} + a_{m_1+1} + \ldots + a_{m_2} > 2 + b_1 + b_2$$

and generally, $m_n > m_{n-1}$, so large that

$$a_1 + a_2 + \ldots + a_{m_n} > n + b_1 + b_2 + \ldots + b_n,$$

where $n = 1, 2, 3, \ldots$

Now, since each of the partial sum $S_{m_1+1}, S_{m_2+2} \ldots$ of $\sum v_n$ whose last term is a negative term '$-b_n$' is greater than $n$ ($n = 1, 2, 3, \ldots$), therefore these partial sums are unbounded above and consequently the series $\sum v_n$ diverges to $+\infty$.

(*iii*) By considering the rearrangement

$$-b_1 - b_2 - \ldots - b_{m_1} + a_1 - b_{m_1+1} - b_{m_1+2} - \ldots - b_{m_2} + a_2 - b_{m_2+1} - \ldots$$

it can be shown, as before, that the rearrangement diverges to $-\infty$.

Other cases may similarly be proved by considering suitable rearrangements of the given series.

**Remark:** As proved earlier the absolutely convergent series remain convergent, with unaltered sum, without any condition on rearrangement of terms but it is not so in the case of non-absolutely convergent series (*i.e.*, series which converge but not absolutely). Such series change their behaviour by change in the order of the terms. This is precisely the reason why such series are called *conditionally convergent*.

**Example 27.** Criticise the following paradox.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

■ The given series

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$$

$$= 0$$

Hence, the series converges to zero.

The given series is conditionally convergent and hence can be made to converge to any limit by a rearrangement of the terms.

## EXERCISE

1. Assuming the convergence of $\sum u_n$, show that the following series are convergent:

(*i*) $\sum \frac{u_n}{n}$, (*ii*) $\sum_{n=2}^{\infty} \frac{u_n}{\log n}$, (*iii*) $\sum \frac{u_n}{\log_a n}$.

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(iv) $\sum\left(\frac{n+1}{n}\right)u_n$, (v) $\sum n^{1/n}u_n$, (vi) $\sum\left(1+\frac{1}{n}\right)^n u_n$.

2. Examine the convergence of the series:

(i) $\sum\frac{\cos n\theta}{n^\alpha}$, (ii) $\sum\frac{\sin n\theta}{n^\alpha}$

[*Hint*:

$$S_n=\sum_{r=1}^{n}\cos r\theta=\frac{\cos[(n-1)\theta/2]\cdot\sin(n\theta/2)}{\sin(\theta/2)},\ \theta\neq 0,\ 2k\pi$$

$$S_n=\sum_{r=1}^{n}\sin r\theta=\frac{\sin[(n-1)\theta/2]\cdot\sin(n\theta/2)}{\sin(\theta/2)},\ \theta\neq 0,\ 2k\pi.$$

Thus $S_n$ is a bounded function, *i.e.*, the series $\sum\cos n\theta$ or $\sum\sin n\theta$ is such that its partial sums are bounded when $\theta$ is neither zero nor a multiple of $2\pi$. By Dirichlet's test, the series converges for $\alpha>0$. For $\alpha\leq 0$ since the $n$th term does not tend to zero, both the series diverge.

For $\theta=0$ or $2k\pi$, the series $\sum[(\cos n\theta)/n^\alpha]$ reduces to $\sum(1/n^\alpha)$ which is convergent for $\alpha>1$, the series $\sum[(\sin n\theta)/n^\alpha]$ reduces to a series of zeros which is evidently convergent for all values of $\alpha$.]

3. Show that the series

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)\frac{\sin n\theta}{n}$$

converges, absolutely for $\theta=k\pi$, $k$ is any integer, and conditionally for all other real values of $\theta$.

4. Examine the convergence of the series:

(i) $\sum\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right\}\cos n\theta$,

(ii) $\sum\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right\}\sin^2 n\theta$.