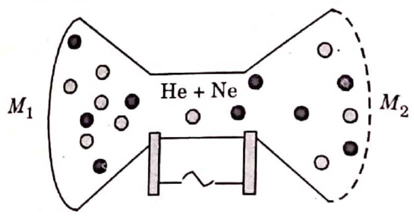


Fig. 5.23.1. Construction of He-Ne laser.

C. Working :

1. The energy level diagram is shown in Fig. 5.23.2.

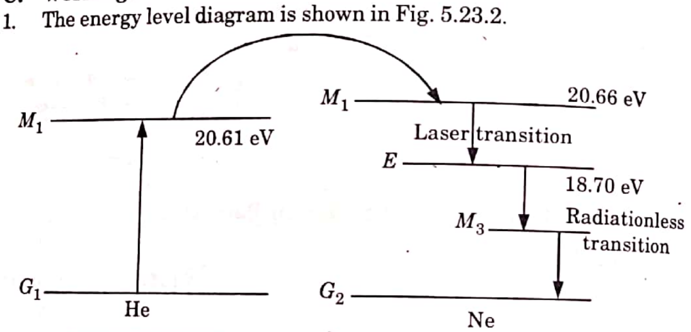


Fig. 5.23.2. Energy level diagram of He-Ne laser.

2. Pumping is achieved by using electrical discharge in the helium-neon mixture.
3. Electrons and ions in this discharge collide with He atoms raising them to a level M_1, which is meta-stable.
4. The He atoms are more readily excitable than Ne atom because they are lighter. Excitation level M_1 = 20.61 eV of He is very close to excitation level M_2 = 20.66 eV of Ne.
5. Some of the excited He atoms transfer their energy to Ne atoms of ground states by collisions between helium and neon atoms.
6. Thus, the excited He atoms return to ground state by transferring their energy to Ne atoms through collisions.
7. The kinetic energy of helium atoms provides the additional 0.05 % for exciting the neon atom.
8. This is the main pumping scheme of He-Ne system.
9. Thus, neon atoms are active centres.
10. The helium gas in the laser tube provides the pumping medium to attain the necessary population inversion for laser action. This population inversion is maintained because :
    i. The meta-stability of level E_3 ensures a ready supply of Ne atoms in level E_2.
    ii. The Ne **atoms from** level E decay rapidly to the neon ground state G_2.

11. In this way, M_2 state of Ne can become more highly populated than level E.
12. The laser transition occurs when Ne atoms fall from level M_2 to E through stimulated emission.
13. A red laser light of wavelength 6328 Å is obtained in He-Ne laser.

E. Superiority of He-Ne laser over Ruby laser :

1. He-Ne laser produces continuous laser beam while ruby laser produces light in the form of pulses.
2. He-Ne laser employs a four level pumping scheme while ruby laser employs a three level pumping scheme.

Que 5.24. **What is the advantage of four level laser systems over three level laser systems ? Describe the construction and working of ruby laser.**

AKTU 2017-18, Marks 07

Answer

A. Advantages of Four Level Laser System over Three Level Systems :

1. It is easy to achieve population inversion with four level system than with a three level system.
2. In the four level laser, the transition does not terminate at the ground state, the pumping power needed for the excitation of atoms is much lower than in a three level laser.
3. The efficiency of four level laser is much better than that of a three level laser.
4. In order to get population inversion in three level laser systems more than 50 % of the atoms in the ground level E_1 must be lifted to level E_2 while it is not necessary in case of four level lasers.
5. The threshold pump power required for population inversion in three level lasers is larger than in four level lasers.

B. Construction and Working of Ruby Laser : Refer Q. 5.22, Page 5–23A, Unit-5.

Que 5.25. **What are various applications of LASER beam ?**

Answer

1. The laser beam is used for drilling, welding and melting of hard materials like diamonds, iron, steel, etc.
2. It is used in heat treatments for hardening or annealing in metallurgy.
3. The laser beam is used in delicate surgery like cornea grafting and in the treatment of kidney stone, cancer and tumor.
4. Laser is used in holography, fibre optics and nonlinear optics.
5. During war-time, lasers are used to detect and destroy enemy missiles.
6. Now, laser-pistols, laser-rifles and laser bombs are also being made, which can be aimed at the enemy in the night.

3. When the medium is exposed to pump frequency radiation, a large number of atoms will be excited to E_3 level.
4. They do not stay at that level but rapidly undergo downward transitions to the metastable level E_2 through transitions.

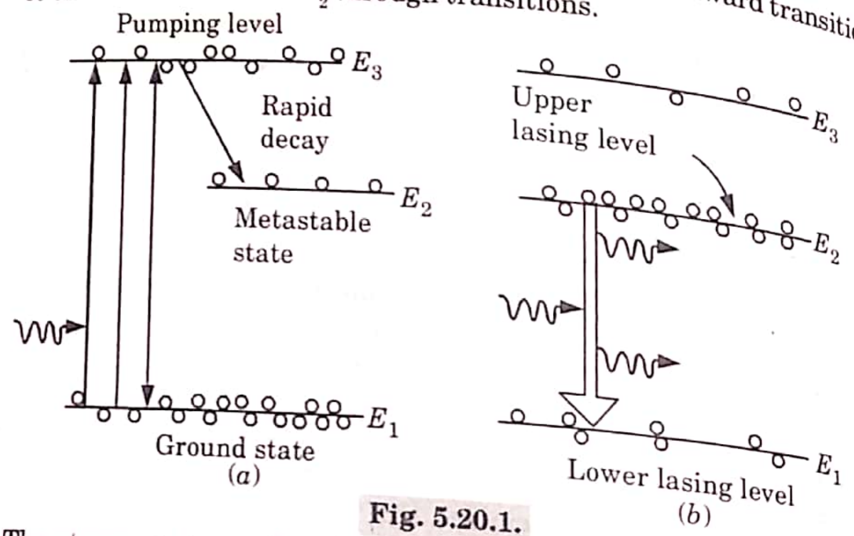


Fig. 5.20.1.

5. The atoms are trapped at this level as spontaneous transition from the level E_2 to the level E_1 is forbidden.
6. The pumping continues and after a short time there will be a large accumulation of atoms at the level E_2.
7. When more than half of the ground level atoms accumulate at E_2, the population inversion condition is achieved between the two levels E_1 and E_2.
8. Now a photon can trigger stimulated emission.

B. Four Level Pumping Scheme :

1. A typical four-level pumping scheme is shown in Fig. 5.20.2.
2. The level E_1 is the ground level, E_4 is the pumping level, E_3 is the metastable upper lasing level and E_2 is the lower lasing level. E_2, E_3 and E_4 are the excited levels.

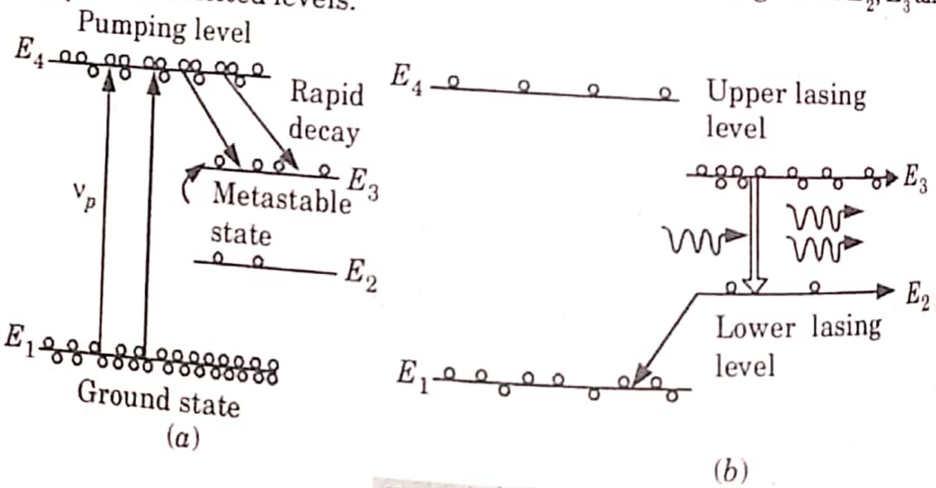


Fig. 5.20.2.

3. When light of pump frequency ν_p is incident on the lasing medium, the active centers are readily excited from the ground level to the pumping level E_4.

$$\therefore \quad \rho(\nu) = \frac{A_{21}N_2}{[B_{12}N_1 - B_{21}N_2]} \quad ...(5.18.5)$$

6. By dividing both the numerator and denominator on the right hand side of the eq. (5.18.5) with $B_{12}N_2$, we obtain,

$$\rho(\nu) = \frac{A_{21}/B_{12}}{\left[\frac{N_1}{N_2} - \frac{B_{21}}{B_{12}}\right]} \quad ...(5.18.6)$$

7. But, according to Boltzmann distribution law,

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT}$$

As $\quad E_2 - E_1 = h\nu,$

$$\therefore \quad \frac{N_2}{N_1} = e^{-h\nu/kT} \quad \text{or} \quad \frac{N_1}{N_2} = e^{h\nu/kT}$$

$$\therefore \quad \rho(\nu) = \frac{A_{21}}{B_{12}}\left[\frac{1}{e^{h\nu/kT} - B_{21}/B_{12}}\right] \quad ...(5.18.7)$$

8. To maintain thermal equilibrium, the system must release energy in the form of electromagnetic radiation.
9. It is required that the radiation be identical with black body radiation and be consistent with Planck's radiation law for any value of T.
10. According to Planck's law,

$$\rho(\nu) = \left(\frac{8\pi h\nu^3}{c^3}\right)\left[\frac{1}{e^{h\nu/kT} - 1}\right] \quad ...(5.18.8)$$

where, $\quad c$ = Velocity of light in free space.

11. Energy density $\rho(\nu)$ given by eq. (5.18.7) will be consistent with Planck's law given by eq. (5.18.8), only if

$$\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3} \quad ...(5.18.9)$$

and, $$\frac{B_{21}}{B_{12}} = 1 \quad \text{or} \quad B_{12} = B_{21} \quad ...(5.18.10)$$

12. The eq. (5.18.9) and eq. (5.18.10) are known as the Einstein's relations. The coefficients B_{12}, B_{21} and A_{21} are known as Einstein's coefficients.
13. It follows that the coefficients are related through

$$B_{12} = B_{21} = \frac{c^3}{8\pi h\nu^3} A_{21} \quad ...(5.18.11)$$

14. The relation (5.18.11) shows that the ratio of coefficients of spontaneous versus stimulated emission is proportional to the third power of frequency of the radiation. This is why it is difficult to achieve laser action in higher frequency ranges such as X-rays.

Que 5.19. **What is population inversion ?**

Answer

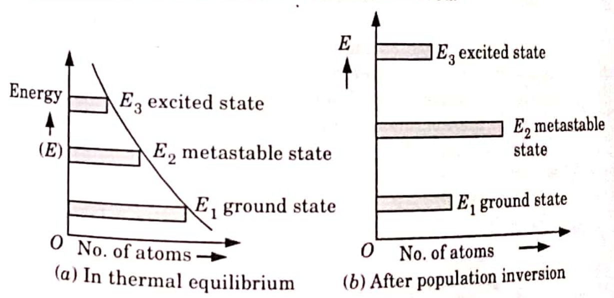
1. The phenomenon is which the number of atoms in the higher energy state becomes comparatively greater than the number of atoms in the lower energy state is known as population inversion.
2. According to Boltzmann's equation, if N_1 and N_2 are the number of atoms in the ground and excited states,

then, $$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT}$$

or $$\frac{N_2}{N_1} = e^{-\Delta E/kT}$$

where, ΔE = Energy difference between the ground state and excited state,
k = Boltzmann's constant, and
T = Absolute temperature.

3. But for atomic radiation ΔE is much greater than kT. Therefore in thermal equilibrium the population of higher state is very much smaller than the ground state *i.e.*, $N_1 > N_2$.
4. As a result the numbers of stimulated emissions are very little as compared to absorption. Therefore laser action will not take place.
5. If somehow the number of atoms in excited state are made larger than in the ground state *i.e.*, $N_2 > N_1$, the process of stimulated emission dominates and the laser action can be achieved.



(a) In thermal equilibrium (b) After population inversion

Fig. 5.19.1.

Que 5.20. **Explain the concept of 3 and 4 level laser.**

OR

Discuss the principal pumping schemes.

Answer

A. **Three Level Pumping Scheme :**
1. A typical three level pumping scheme is shown in Fig. 5.20.1.
2. The state E_1 is the ground level; E_3 is the pump level and E_2 is the metastable upper lasing level.

Answer

1. There are three conditions to achieve laser action as follows :
   i. The number of atoms in higher energy state must be greater than that in lower energy state so that the rate of emission becomes greater than the rate of absorption.
   ii. The radiation must be coherent so the probability of spontaneous emission should be negligible in comparison to the probability of stimulated emission.
   iii. The coherent beam of light must be sufficiently amplified.

Que 5.18. **What are Einstein's coefficients A and B ? Establish a relation between them.**

Answer

A. Einstein's Coefficients A and B :

1. The probability that an absorption transition occurs is given by

$$P_{12} = B_{12}\, \rho(\nu) \qquad \text{...(5.18.1)}$$

where, B_{12} = Constant of proportionality known as the Einstein's coefficient for induced absorption.

2. The probability that a spontaneous transition occurs is given by

$$(P_{21})_{spontaneous} = A_{21} \qquad \text{...(5.18.2)}$$

where, A_{21} = Constant known as the Einstein's coefficient for spontaneous emission.

3. The probability that a stimulated transition occurs is given by

$$(P_{21})_{stimulated} = B_{21}\rho(\nu) \qquad \text{...(5.18.3)}$$

where, B_{21} = Constant of proportionality known as the Einstein's coefficient for stimulated emission.

B. Relation Between Einstein's Coefficients A and B :

1. Under thermal equilibrium, the mean population N_1 and N_2 in the lower and upper energy levels respectively must remain constant.
2. This condition requires that the number of transitions from E_2 to E_1 must be equal to the number of transitions from E_1 to E_2.

   Thus,

$$\begin{pmatrix}\text{The number of atoms absorbing} \\ \text{photons per second per unit volume}\end{pmatrix} = \begin{pmatrix}\text{The number of atoms emitting} \\ \text{photons per unit volume}\end{pmatrix}$$

3. The number of atoms absorbing photons per second per unit volume

$$= B_{12}\rho(\nu)N_1$$

4. The number of atoms emitting photons per second per unit volume

$$= A_{21}N_2 + B_{21}\rho(\nu)N_2$$

5. As the number of transitions from E_1 to E_2 must equal the number of transitions from E_2 to E_1, we have

$$B_{12}\rho(\nu)N_1 = A_{21}N_2 + B_{21}\rho(\nu)N_2 \qquad \text{...(5.18.4)}$$

$$\rho(\nu)\,[B_{12}N_1 - B_{21}N_2] = A_{21}N_2$$

2. But now instead of one, two photons of energy $h\nu$ each are released.
3. When these two photons of energy $h\nu$ each are incident on another two excited state atoms, four photons of energy $h\nu$ each are released.

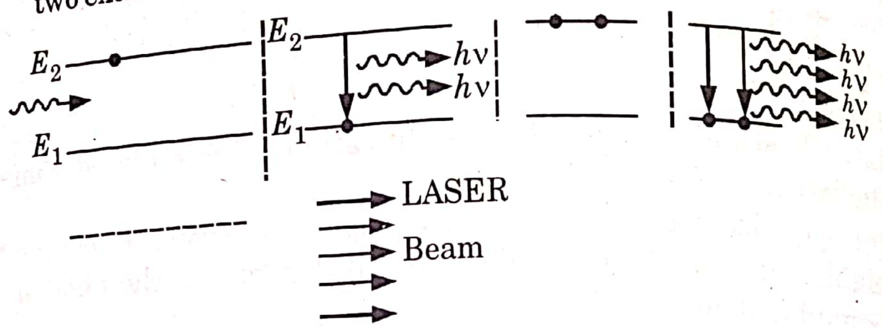


Fig. 5.15.3.

4. This process goes on continuously and as a result, a monochromatic, unidirectional beam of photon is released, which is known as stimulated emission of radiation.
5. The probability of stimulated emission of radiation is given by :

$$(P_{21})_{\text{stimulated}} = B_{21}\rho(\nu)$$

where, B_{21} = Einstein's coefficient of stimulated emission of radiation, and

$\rho(\nu)$ = Energy density.

Que 5.16. **Differentiate between spontaneous emission and stimulated emission.**

Answer

S. No.	Spontaneous Emission	Stimulated Emission
1.	It is a natural transition in which an atom is de-excited after the end of its life-time in the higher energy level.	It is an artificial transition which occurs due to de-excitation of an atom before the end of its life-time in the higher energy level.
2.	The photon emitted due to spontaneous emission can move in any direction.	The photon emitted due to stimulated emission can move only in the direction of the incident photon.
3.	The probability of spontaneous emission depends only on the properties of the two energy levels between which the transition occurs.	The probability of stimulated emission depends on the properties of the two energy levels involved in the transition as well as on the energy density of incident radiation.

Que 5.17. **Discuss necessary condition to achieve laser action.**

Que 5.15. **Explain LASER and different types of process of radiations.**

Answer

A. LASER :

1. LASER stands for "Light Amplification by Stimulated Emission of Radiation".
2. It is a device used to produce a strong, monochromatic, collimated and highly coherent beam of light and it depends on the phenomenon of "stimulated emission".

B. Processes of Radiation :

a. Absorption of Radiation :

1. When an atom is in its ground state and a photon of energy $h\nu$ is incident over it, it comes to its excited state after absorbing that photon. This process is known as absorption of radiation.

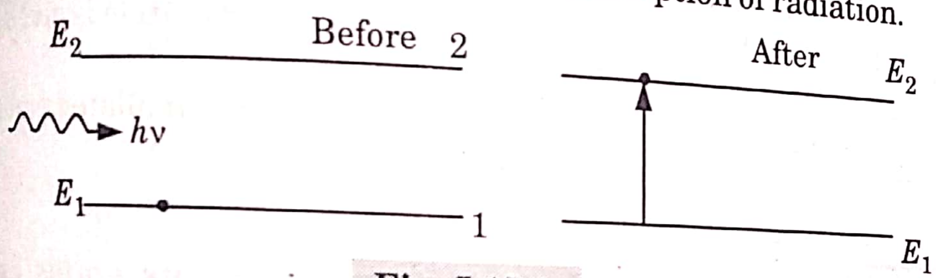


Fig. 5.15.1.

2. The probability of absorption of radiation is given by :

$$P_{12} = B_{12}\,\rho(\nu)$$

where, B_{12} = Einstein's coefficient of absorption of radiation, and

$\rho(\nu)$ = Energy density.

b. Spontaneous Emission of Radiation :

1. When an atom is in its excited state, it can remain there only for 10^{-8} sec. After that it comes to its ground state and releases a photon of energy $h\nu$. This process is called spontaneous emission of radiation.

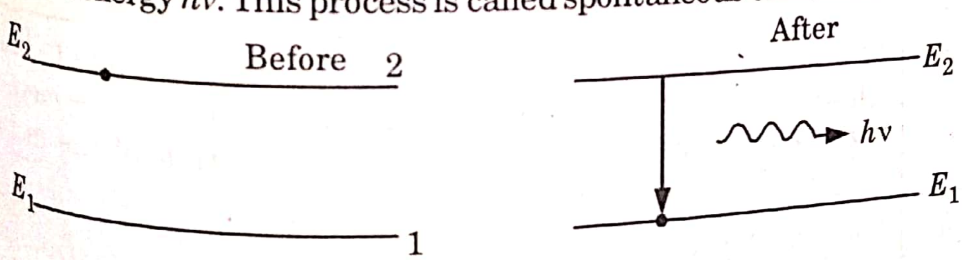


Fig. 5.15.2.

2. The probability of spontaneous emission of radiation is given by :

$$(P_{21})_{spontaneous} = A_{21}$$

where, A_{21} = Einstein's coefficient of spontaneous emission of radiation.

c. Stimulated Emission (Induced Emission) of Radiation :

1. When an atom is in its excited state and a photon of energy $h\nu$ is incident over it, atom comes to its ground state.

2. Number of guided modes,

$$N = \frac{v^2}{2} = \frac{(24.48)^2}{2} = 299.635 \approx 300$$

Que 5.9. **Describe the basic principle of communication of wave in optical fibre. A step index fibre has core refractive index 1.468, cladding refractive index 1.462. Compute the maximum radius allowed for a fibre, if it supported only one mode at a wavelength 1300 nm.**

AKTU 2015-16, Marks 10

Answer

A. Basic principle of c/0ommunication of wave in optical fibre : Refer Q. 5.2, Page 5–3A, Unit-5.

B. Numerical :

Given : $N = 1$, $\lambda = 1300$ nm, $n_1 = 1.468$, $n_2 = 1.462$
To Find : Maximum radius allowed for a fibre.

1. Number of modes supported, $N = \frac{v^2}{2}$

$$1 = \frac{v^2}{2}$$

$$v = 1.414$$

2. Let, a is radius allowed for a fibre.

$$\therefore \quad v = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

$$1.414 = \frac{2\pi \times a}{1300 \times 10^{-9}}\sqrt{(1.468)^2 - (1.462)^2}$$

$$292.56 \times 10^{-9} = a\sqrt{(1.468)^2 - (1.462)^2}$$

$$a = 2.2 \times 10^{-6}\ \text{m} = 2.2\ \mu\text{m}$$

Que 5.10. **Discuss the different types of losses in optical fibre.**

Answer

A. Absorption Losses :

1. Absorption is the most prominent factor causing the attenuation in optical fibre.
2. The absorption of light is caused by the following three different mechanisms :

i. Intrinsic Absorption :

1. It is the absorption of light by the material of the core itself.

... withstand environmental conditions.

The cost per channel is lower than that of metal counterpart.

Handling and installation cost of optical fibre system is very nominal.

Que 5.6. **Differentiate between single mode fibres and multimode fibres.**

Answer

S. No.	Single Mode Fibres	Multimode Fibres
1.	In single mode fibres there is only one path for ray propagation.	In multimode fibres, large numbers of paths are available for light ray propagation.
2.	Single mode step index fibres have less core diameter (< 10 μm) and the difference between the refractive indices of core and cladding is very small.	Multimode step index fibres have larger core diameter (50 to 200 μm) and the difference between the refractive indices of core and cladding is large.
3.	In single mode fi...	

iii. Classification of Optical Fibres Depending on the Index Profile :

a. Multimode Step Index Fibre (MMSIF) :

1. It consists of a core material surrounded by a concentric layer of cladding material with a uniform index of refraction n_2 that is only slightly less than that of core of refractive index n_1.
2. If the refractive index is plotted against the radial distance from the core, the refractive index abruptly changes at the core-cladding surface creating a step, hence the name step index.

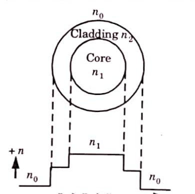


Fig. 5.4.1.

3. The name step index is due to this index profile and the term multimode is due to its feature of propagating a number of modes.
4. Its manufacturing is such that its core radius is large enough to accommodate many different rays of light or mode each entering the core at different angles.

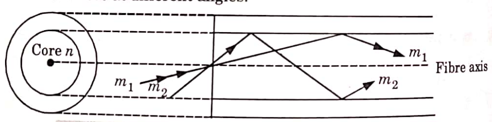


Fig. 5.4.2. Propagation in multimode step index fibre.

b. Multimode Graded Index Fibre (MMGIF) :

1. In this, the material in the core is modified so that the refractive index profile does not exhibit step index change but a parabolic refractive index profile which is maximum at the fibre axis.
2. In this fibre, index of refraction has a maximum value n_1 at the axis and lesser values falling off gradually and hence the name graded index is given to this fibre.
3. Since the light travels faster in a medium with lower refractive index, the light ray, which is farther from the fibre axis travels faster than the ray which is nearer to the axis.
4. As the refractive index is continuously changing across the fibre axis, the light ray is bent towards the fibre axis in almost sinusoidal fashion.
5. Light rays are curved towards the fibre axis by refraction.

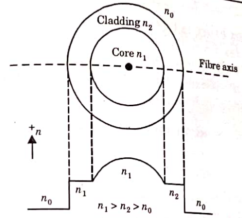


Fig. 5.4.3.

6. Light rays periodically diverge and converge along the length of the fibre.

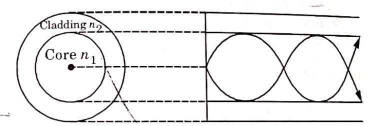


Fig. 5.4.4. Propagation in a multimode graded index fibre.

c. Single Mode Step Index Fibre (SMSIF) :

1. In this fibre, the core of a fibre is made so small that only one ray of light can enter the core and get guided by the total internal reflection hence the name single mode.
2. This will be the only ray of light or mode that can enter the core at such a shallow angle.

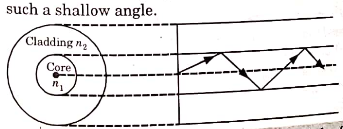


Fig. 5.4.5. Propagation in a single mode step index fibre.

3. Major advantage of this fibre is that modal dispersion is totally eliminated and because of this, such fibres are extensively used for long distance communication.
4. Different fibre designs have a specific wavelength called cut off wavelength above which it carries only one mode.
5. Single mode step index fibre has a superior transmission quality over other fibre types of the above because of the absence of modal dispersion.

Scanned by CamScanner

i.e., $$\Delta = \frac{n_1 - n_2}{n_1} \quad ...(5.3.6)$$

or $$n_1 - n_2 = \Delta\, n_1 \quad ...(5.3.7)$$

3. Eq. (5.3.5) can be written as :
$$\text{NA} = \sqrt{n_1^2 - n_2^2} = \sqrt{(n_1 - n_2)(n_1 + n_2)} \quad ...(5.3.8)$$
4. Substituting eq. (5.3.7) in eq. (5.3.8), we get
$$\text{NA} = \sqrt{\Delta\, n_1\, (n_1 + n_2)}$$
5. Since, $n_1 = n_2$; so, $n_1 + n_2 \approx 2n_1$
$$\therefore \quad \text{NA} = \sqrt{2\Delta n_1^2} = n_1\sqrt{2\Delta} \quad ...(5.3.9)$$
6. Numerical aperture can be increased by increasing 'Δ' and thus enhances the light-gathering capacity of the fibre.
7. We cannot increase Δ to a very large value because it leads to intermodal dispersion, which cause signal distortion.

Que 5.4. **Write the classification of optical fibres.**

OR

Discuss the different types of optical fibre. Why graded index fibre is better than multimode step index fibre. **AKTU 2014-15, Marks 05**

Answer

A. Classification of Optical Fibres :

i. Classification of Optical Fibre Depending on Material used :

a. Glass Fibres :

1. These fibres consist of glass as the core and also glass as the cladding.
2. These are the most widely used fibres.
3. To reduce the refractive index of cladding, impurities such as Germanium, Boron, Phosphorous or Fluoride are added to the pure glass.

b. Plastic Clad Silica or P.C.S. Fibres :

1. By replacing the cladding with a plastic coating of the refractive index lower than that of core, a plastic clad fibre is achieved.
2. Its advantage is only that the replacement of the glass cladding with plastic offers the saving in cost.
3. The limitations are :
   - i. Losses are more than the glass fibres.
   - ii. Refractive index varies with temperature.
   - iii. Fibre life is small, mainly in humid environment.

c. Plastic Fibres :

1. These fibres consist of both core and cladding of the plastic material.
2. These fibres are cheaper in comparison to the above fibres.
3. But these fibres have high losses and low bandwidth.
4. Also life of these fibres is small and refractive index varies with temperature.
5. These fibres don't need protective coating and they are more flexible.
6. Attenuation of plastic fibres is more than glass or silica fibres but even then they are frequently used for short distance computer applications.

ii. Classification of Optical Fibres Depending on Number of Modes :

a. Monomode or Single Mode Fibre :

1. In this, fibre is capable of transmitting only one mode.
2. Suppose we make the core of the fibre for any small ray of order of 2 to 8 μm, then only one ray of light can enter the core and get guided by total internal reflection.
3. Major advantage of single mode fibre is that it exhibits minimum dispersion loss and hence, the highest transmission bandwidth.
4. Only high-quality laser sources that produce a very focused beam of nearly monochromatic light can be used for single-mode operation.
5. Because of the superior transmission characteristics, such fibres are extensively used for long-distance applications.

b. Multimode Fibres :

1. In this, the fibre is capable of transmitting more than one mode, so the name multimode fibre.
2. The multimode fibre has the core diameter of the order of 50 μm *i.e.*, larger than the monomode fibre.
3. As the core radius is large enough, it accommodates many different rays of light or modes, each entering the core at different angles.
4. Since the different mode have different groups velocities, there exists considerable broadening of transmitted light pulses.
5. Hence, dispersion losses are more and bandwidth length product is small of order of 1 GHz-km.
6. These fibres are useful for moderate distances.
7. The loss of information capacity however is compensated by certain benefits of multimode fibres over monomode fibre such as :
   - i. Incoherent optical source can be used in multimode fibre due to large core diameter and large acceptance angle.
   - ii. Ease of splicing or joining.
   - iii. Lower tolerance requirements on fibre connectors.

PART-1

Introduction to Fibre Optics, Acceptance Angle, Numerical Aperture, Normalized Frequency, Classification of Fibre, Attenuation and Dispersion in Optical Fibre.

CONCEPT OUTLINE : PART-1

Optical Fibre : Optical fibre consists of a core surrounded by a cladding and a sheath. It is a thin, transparent and flexible strand. It is made up of glass or plastic. It works on the principle of total internal reflection.
Acceptance Angle : It is defined as the maximum angle that a light ray can have relative to the axis of the fibre and propagates down the fibre.
Numerical Aperture : It is a dimensionless number that characterizes the range of angles over which the fibre can accept or emit light.
Dispersion : The amplitude of the optical signal propagating in an optical fibre attenuates due to losses in fibres as well as it spreads during its propagation. Thus, the output signal received at the end becomes wider compared to the input signal. This type of distortion arises due to dispersion effect in optical fibres.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. What is optical fibre ?

Answer

1. Optical fibre is a long, thin transparent dielectric material made up of glass or plastic, which carries electromagnetic waves of optical frequencies (visible to infrared) from one end of the fibre to the other by means of multiple total internal reflection.
2. Optical fibres work as wave guides in optical communication systems.
3. An optical fibre consists of an inner cylindrical material made up of glass or plastic called core.
4. The core is surrounded by a cylindrical shell of glass or plastic called the cladding.
5. The refractive index of core (n_1) is slightly larger than the refractive index of cladding (n_2), (*i.e.*, $n_1 > n_2$).

6. The cladding is enclosed in a polyurethane jacket as shown in Fig. 5.1.1. This layer protects the fibre from the surrounding atmosphere.
7. Many fibres are grouped to form a cable. A cable may contain one to several hundred such fibres.

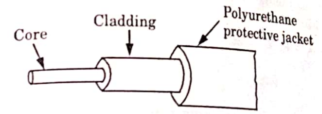


Fig. 5.1.1.

Que 5.2. Explain the principle of optical fibre.

Answer

1. The working of optical fibre is based on the principle of total internal reflection.
2. Total internal reflection is the phenomenon in which a light ray reflects completely in the first medium, when it is incident on the boundary of two different mediums.
3. When a light ray is incident on a high to low refractive index interface, then from Snell's law, Fig. 5.2.1(*a*).

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} \quad ...(5.2.1)$$

where, n_1 and n_2 = Refractive indices of denser and rarer mediums respectively.

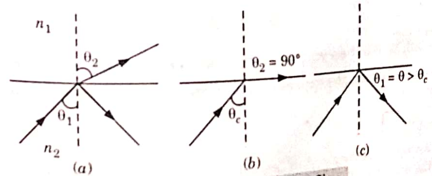


Fig. 5.2.1. Principle of optical fibre.

4. Since $n_1 > n_2$, so from eq. (5.2.1), we have

$$\frac{\sin\theta_1}{\sin\theta_2} < 1$$

i.e., $$\sin\theta_1 < \sin\theta_2$$

In this case $\lambda' = \dfrac{8037.20 + 8037.50}{2} = 8037.35$ Å

$$d\lambda' = 8037.50 - 8037.20 = 0.30$$

Resolving power, $\dfrac{\lambda'}{d\lambda'} = \dfrac{8037.35}{0.30} = 26791.17$

4. Thus, the grating will not be able to resolve the lines 8037.20 Å and 8037.50 Å in the second order because the required resolving power (26791.17) is greater than the actual resolving power (20160).

Que 4.31. **A diffraction grating used at normal incidence gives a green line (5400 Å) in a certain order n superimposed on the violet line (4050 Å) of the next higher order. If the angle of diffraction is 30°, find the value of n, also find how many lines per cm there in the grating.**

Answer

Given : $\lambda_1 = 5400$ Å $= 5400 \times 10^{-8}$ cm, $\lambda_2 = 4050 \times 10^{-8}$ cm, $\theta = 30°$, $\lambda_1 - \lambda_2 = 1350 \times 10^{-8}$ cm

To Find : i. Order of spectrum, n.

ii. Grating lines per cm.

1. The direction of principal maxima for normal incidence is given by

$$(e + d) \sin \theta = n\lambda$$

2. Let n^{th} order maxima of λ_1, coincide with $(n + 1)^{th}$ order maxima of λ_2, we have,

$$(e + d) \sin \theta = n\lambda_1 = (n + 1)\lambda_2$$

or $$n\lambda_1 = (n + 1)\lambda_2 \quad \text{or} \quad n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

$$n = \frac{4050 \times 10^{-8}}{1350 \times 10^{-8}} = 3$$

3. Now, $(e + d) \sin \theta = \dfrac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}$ or $e + d = \dfrac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2) \sin \theta}$

$$e + d = \frac{5400 \times 10^{-8} \times 4050 \times 10^{-8}}{1350 \times 10^{-8} \times \sin 30°}$$

4. Number of lines per cm,

$$N = \frac{1}{e + d} = \frac{1350 \times 10^{8}}{5400 \times 4050 \times 2} = 3086$$

☺☺☺

Que 4.27. **Find the angular separation of 5048 Å and 5016 Å wavelengths in second order spectrum obtained by a plane diffraction grating having 15000 lines per cm.**

Answer

Given : $n = 2$, $\lambda_1 = 5048$ Å $= 5048 \times 10^{-10}$ m, $\lambda_2 = 5016$ Å $= 5016 \times 10^{-10}$ m,

$$(e + d) = \frac{2.54}{15000} = 1.693 \times 10^{-4} \text{ cm} = 1.693 \times 10^{-6} \text{ m}$$

To Find : Angular separation.

Note : The value of $(e + d)$ or diffraction grating can't be 15000 lines per cm due to this value of square root comes out to be negative, hence, considering the $(e + d)$ value as 15000 lines per inch.

1. Since,
$$d\lambda = \lambda_1 - \lambda_2$$
$$= 5048 - 5016 = 32 \text{ Å} = 32 \times 10^{-10} \text{ m}$$

2.
$$\lambda = \frac{\lambda_1 + \lambda_2}{2} = \frac{5048 + 5016}{2} = 5032 \text{ Å} = 5032 \times 10^{-10} \text{ m}$$

3. Now, angular separation is given by,
$$d\theta = \frac{d\lambda}{\sqrt{\left[\left(\frac{e+d}{n}\right)^2 - \lambda^2\right]}} = \frac{32 \times 10^{-10}}{\sqrt{\left[\left(\frac{1.696 \times 10^{-6}}{2}\right)^2 - (5032 \times 10^{-10})^2\right]}}$$
$$= \frac{32 \times 10^{-10}}{\sqrt{71.65 \times 10^{-14} - 25.32 \times 10^{-14}}} = \frac{32 \times 10^{-10}}{\sqrt{10^{-14}(71.65 - 25.32)}}$$
$$= \frac{32 \times 10^{-10}}{4.63 \times 10^{-6}} = 6.91 \times 10^{-4} \text{ rad}$$

Que 4.28. **A diffraction grating used at normal incidence gives a yellow line ($\lambda = 6000$ Å) in a certain spectral order superimposed on a blue line ($\lambda = 4800$ Å) of next higher order. If the angle of diffraction is $\sin^{-1}$ (3/4), calculate the grating element.**

AKTU 2015-16, Marks 05

Answer

Given : $\lambda_1 = 6000$ Å $= 6000 \times 10^{-8}$ cm, $\lambda_2 = 4800$ Å $= 4800 \times 10^{-8}$ cm,
$\theta = \sin^{-1}(3/4) = 48.6°$

To Find : Grating element.

1.
$$\lambda_1 - \lambda_2 = (6000 - 4800) \times 10^{-8}$$

10. Therefore, for normal incidence only first order will be obtained.
11. Hence, if the width of grating element is less than twice the wavelength of light, then only first order is possible.

Que 4.25. **What is dispersive power of grating and resolving power of an optical instrument ? Explain Rayleigh's criterion of resolution.**

Answer

A. Dispersive Power of a Diffraction Grating :

1. The dispersive power of a diffraction grating is defined as the rate of change of the diffraction angle with the wavelength. It is expressed as $\frac{d\theta}{d\lambda}$.
2. For a grating, $(e + d) \sin \theta = n\lambda$

Differentiating w.r.t. λ, we get,

$$(e + d) \cos \theta \frac{d\theta}{d\lambda} = n$$

or

$$\frac{d\theta}{d\lambda} = \frac{n}{(e+d) \cos \theta}$$

B. Resolving Power of an Optical Instrument :

1. The ability of an optical instrument to produce the separate images of two objects placed very close to each other is known as resolving power.

C. Rayleigh's Criteria of Resolution :

1. According to Rayleigh's criterion, the spectral lines of equal intensity are said to be resolved, if the position of the principal maxima of one spectral line coincide with first minima of the other spectral line.

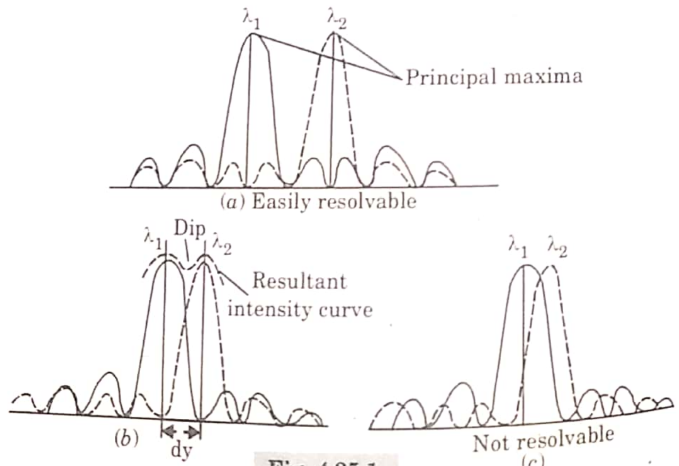


Fig. 4.25.1.

4. Hence, dispersive power, $\frac{d\theta}{d\lambda} = \frac{1}{\sqrt{\left(\frac{e+d}{n}\right)^2 - \lambda^2}}$

Que 4.24. **What do you understand by missing order spectrum ? What particular spectra would be absent if the width of transparencies and opacities of the grating are equal ? Show that only first order spectra is possible if the width of the grating element is more than wavelength of light and less than twice the wavelength of light.**

OR

What are the conditions absent spectra in the grating ?

Answer

1. Sometime for a particular angle of diffraction 'θ' satisfying the relation $(e + d) \sin\theta = n\lambda$, there is no visible spectrum obtained. This phenomenon is known as missing order spectrum.
2. We know that condition for a minima in a single slit is given by
$$e \sin\theta = m\lambda \quad \text{...(4.24.1)}$$
and the condition for the principal maxima in the n^{th} order spectrum is given by
$$(e + d) \sin\theta = n\lambda \quad \text{...(4.24.2)}$$
3. If both conditions are simultaneously satisfied, the diffracted rays from all transparencies are superimposed upon each other but the resultant intensity is zero, *i.e.* the spectrum is absent.
4. From eq. (4.24.1) and eq. (4.24.2), we have, $\frac{e+d}{e} = \frac{n}{m}$
This is the condition for absent spectra.
5. If $e = d$ then, $n = 2m$
So that 2^{nd}, 4^{th}, 6^{th}, ... order of the spectra will be missing corresponding to $m = 1, 2, 3, ...$
6. When $d = 2e$ then, $n = 3$ m
Hence, 3^{rd}, 6^{th}, 9^{th} ... spectra will be missing corresponding to $m = 1, 2, 3 ......$
7. The maximum number of spectra available with a diffraction grating in the visible region can be evaluated by using the grating equation for normal incidence as
$$(e + d) \sin\theta_n = n\lambda \quad \text{or} \quad n = \frac{(e+d)\sin\theta_n}{\lambda}$$
8. The maximum possible value of θ is 90°.
$$n_{max} = \frac{(e+d)\sin 90}{\lambda} = \frac{(e+d)}{\lambda}$$
9. If the grating element $(e + d)$ lies between λ and 2λ or grating element $(e + d) < 2\lambda$

Then, $$n_{max} < \frac{2\lambda}{\lambda} < 2$$

4. The direction of principal maxima is given by, $\sin\beta = 0$, $\beta = \pm nx$
5. We know, $$\beta = \frac{\pi}{\lambda}(e+d)\sin\theta$$

So, $$\pm n\pi = \frac{\pi}{\lambda}(e+d)\sin\theta \Rightarrow (e+d)\sin\theta = \pm n\lambda \quad ...(4.22.5)$$

6. For $n = 0$ we get, $\theta = 0$ and get zero order principal maxima. $n = 1, 2, 3, \ldots\ldots$ correspond to I^{st}, II^{nd}, III^{rd} order principal maxima.

b. Condition for Minima :

1. The intensity is minimum when, $\sin N\beta = 0$
2. But, $\sin\beta \neq 0$

Hence, $$N\beta = \pm m\pi \quad \text{or} \quad \beta = \pm\frac{m\pi}{N}$$

or, $$\frac{N\pi}{\lambda}(e+d)\sin\theta = \pm m\pi \quad \text{or} \quad N(e+d)\sin\theta = \pm m\lambda \quad ...(4.22.6)$$

where, m can take all integral values except $0, N, 2N, 3N \ldots\ldots\ldots nN$; $m = 0$ gives maxima and $m = 1, 2, 3, \ldots.. (N-1)$ give minima.

B. Secondary Maxima :

1. There are $(N-1)$ minima between two consecutive principal maxima, therefore, there are $(N-2)$ other maxima coming alternatively with the minima between two successive principal maxima.
2. Position of secondary maxima is obtained by differentiating eq. (4.22.2) w.r.t. β and equating it to zero.

$$\frac{dI}{d\beta} = A^2\frac{\sin^2\alpha}{\alpha^2}2\left[\frac{\sin N\beta}{\sin\beta}\right]\left[\frac{N\cos N\beta\sin\beta - \sin N\beta\cos\beta}{\sin^2\beta}\right] = 0$$

or, $N\cos N\beta\sin\beta - \sin N\beta\cos\beta = 0$

and, $$\tan N\beta = N\tan\beta \quad ...(4.22.7)$$

3. From Fig. 4.22.3,

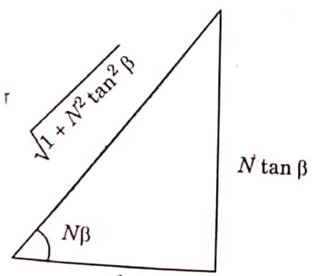


Fig. 4.22.3.

$$\sin N\beta = \frac{N\tan\beta}{\sqrt{1+N^2\tan^2\beta}}$$

Squaring both sides and dividing by $\sin^2\beta$,

$$\frac{\sin^2 N\beta}{\sin^2\beta} = \frac{N^2\tan^2\beta}{[(1+N^2\tan^2\beta)\sin^2\beta]} = \frac{N^2}{[1+(N^2-1)\sin^2\beta]}$$

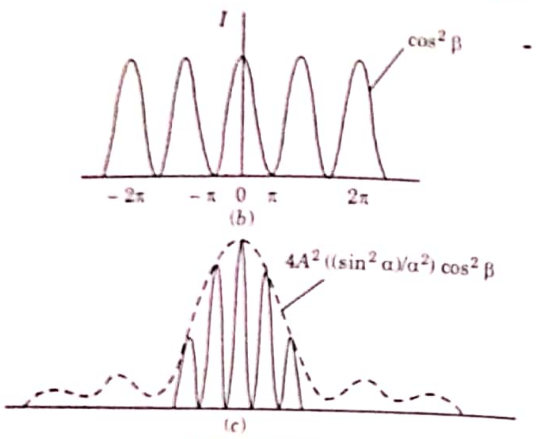


Fig. 4.21.3.

A. Conditions for Maxima :

1. If, $\cos^2 \beta = 1$ or $\beta = \pm n\pi$ [where, $n = 0, 1, 2, 3,...$]

2. But, $\beta = \dfrac{\pi}{\lambda}(e+d)\sin\theta$

$$\pm n\pi = \frac{\pi}{\lambda}(e+d)\sin\theta$$

$$(e+d)\sin\theta = \pm n\lambda \quad \text{[where, } n = 0, 1, 2, 3,...\text{]}$$

B. Conditions for Minima :

1. If, $\cos^2 \beta = 0$ or $\cos\beta = 0$

$$\beta = \pm(2n+1)\frac{\pi}{2} \quad \text{[where, } n = 0, 1, 2, 3,...\text{]}$$

2. Then, $(e+d)\sin\theta = \pm(2n+1)\dfrac{\lambda}{2}$

Que 4.22. **What do you mean by a diffraction grating ? Derive expression of Fraunhofer diffraction due to N slits.**

OR

Give the construction and theory of plane transmission grating. Explain the formation of spectra by it. **AKTU 2017-18, Marks 07**

Answer

1. The diffraction grating consists of a large number (N) of parallel slits having equal width and separated by an equal opaque space

2. It is constructed by rolling a large number of parallel and equidistant lines on a glass plate with a diamond point.

A. Explanation :

1. Let a parallel beam of monochromatic light of wavelength 'λ' be incident on 'N' slits.
2. This light diffracted at an angle θ is focused at point P on the screen by lens L_2 having same amplitude.

$$R = A\frac{\sin\alpha}{\alpha}$$

3. Let e be the width of each slit and 'd' be the opaque space between two slits, then $(e+d)$ is called grating element.
4. Path difference = $(e+d)\sin\theta$

and, phase difference,

$$2\beta = \frac{2\pi}{\lambda}(e+d)\sin\theta.$$

5. Therefore, as we pass from one vibration to another, the path goes on increasing by an amount $(e+d)\sin\theta$ and phase goes on increasing by an amount $\dfrac{2\pi}{\lambda}(e+d)\sin\theta$. Thus, phase increases in arithmetic progression.
6. Now, the resultant amplitude and intensity at point P due to N slits can be obtained by vector polygon method,

$$R' = R\frac{\sin\frac{Nd}{2}}{\sin\frac{d}{2}}$$

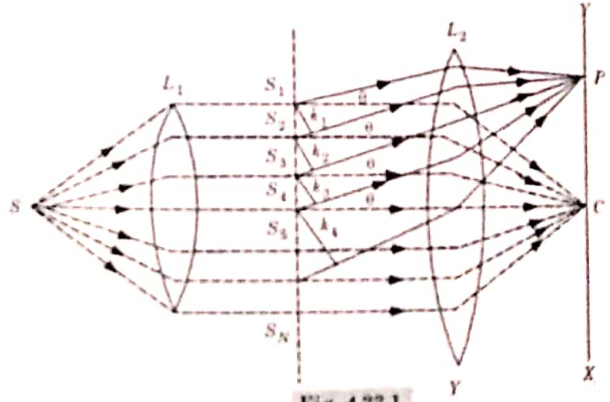


Fig. 4.22.1.

7. But, $d = 2\beta$

Que 4.21. **Discuss Fraunhofer diffraction at a double slit.**

Answer

1. Consider a parallel beam of monochromatic light having wavelength λ incident normally on two parallel slits AB and CD as shown in Fig. 4.21.1.
2. Width of each slit is 'e' and are separated by distance 'd'. Distance between S_1 and S_2 point is '$e + d$'.
3. Now each slit diffracts the light at an angle θ to incident direction.

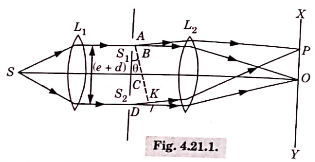


Fig. 4.21.1.

5. From the theory of diffraction due to single slit we know that, resultant amplitude is

$$R = A\frac{\sin\alpha}{\alpha} \quad \text{and} \quad \alpha = \frac{\pi}{\lambda} e \sin\theta$$

6. Let S_1 and S_2 are two coherent sources, each sending wavelets of amplitude $A\frac{\sin\alpha}{\alpha}$ in the direction of θ.
7. Therefore, the resultant amplitude due to interference of these two waves at point P can be calculated as :
   i. Draw perpendiculars S_1K on S_2K.
   ii. Path difference, $S_2K = (e + d)\sin\theta$ ...(4.21.1)
   and, phase difference,
   $$\delta = \frac{2\pi}{\lambda}(e + d)\sin\theta \qquad ...(4.21.2)$$
   iii. If R' is resultant amplitude at point P then, according to Fig. 4.21.2.
   $$OB^2 = OA^2 + AB^2 + 2AB.OA\cos\delta$$
   $$R'^2 = R^2 + R^2 + 2RR\cos\delta$$
   $$= 2R^2 + 2R^2\cos\delta$$
   $$R'^2 = 2R^2(1 + \cos\delta) = 2R^2\left(2\cos^2\frac{\delta}{2}\right)$$

$$R'^2 = 4R^2\cos^2\frac{\delta}{2}$$

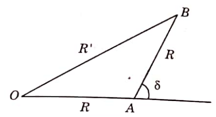


Fig. 4.21.2.

iv. But, $\quad R = A\frac{\sin\alpha}{\alpha}$

then, $\quad R'^2 = 4\frac{A^2\sin^2\alpha}{\alpha^2}\cos^2\frac{\delta}{2}$

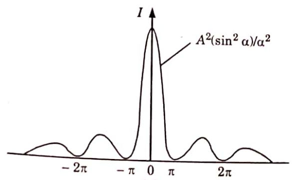
v. Let, $\quad \beta = \frac{\delta}{2} = \frac{\pi}{\lambda}(e + d)\sin\theta$

then, $\quad R'^2 = 4A^2\frac{\sin^2\alpha}{\alpha^2}.\cos^2\beta$

8. Now, by definition,

$$I = R'^2 = \frac{4A^2\sin^2\alpha.\cos^2\beta}{\alpha^2}$$

9. Hence, the resultant intensity depends upon following two factors :
   i. $\frac{A^2\sin^2\alpha}{\alpha^2}$ due to diffraction, and
   ii. $\cos^2\beta$, due to interference.



(a)

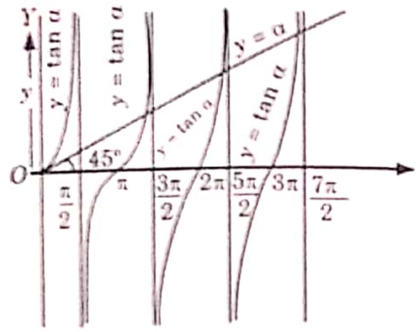


Fig. 4.19.2.

6. At $\alpha = 0$, the position of principal maxima,

$$I = I_o \left(\frac{\sin\alpha}{\alpha}\right)^2 = I_o$$

7. At $\alpha = \frac{3\pi}{2}$, the intensity of first secondary maxima,

$$I_1 = I_o \left(\frac{\sin\alpha}{\alpha}\right)^2 \Rightarrow I_o \left(\frac{\sin\frac{3\pi}{2}}{\frac{3\pi}{2}}\right)^2 = \frac{4I_o}{9\pi^2}$$

8. At $\alpha = \frac{5\pi}{2}$, the intensity of second secondary maxima,

$$I_2 = I_o \left(\frac{\sin\frac{5\pi}{2}}{\frac{5\pi}{2}}\right)^2 = \frac{4I_o}{25\pi^2}$$

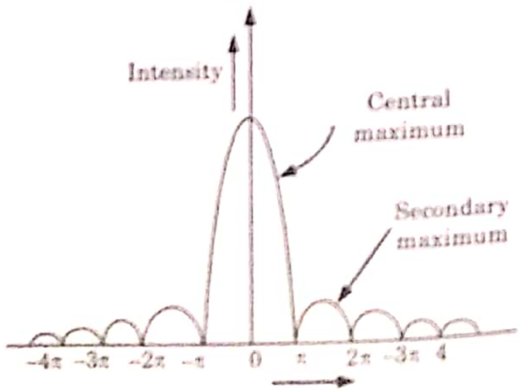


Fig. 4.19.3.

9. At $\alpha = \frac{7\pi}{2}$, $\quad I_3 = \frac{4I_o}{49\pi^2}$

6. To find intensity at P let us draw normal AK.
7. Path difference of rays meeting at P is

$$BK = e \sin\theta$$

and, phase difference $= \left(\frac{2\pi}{\lambda}\right) e \sin\theta$

8. Let AB be divided into large number of equal parts. The secondary waves originating from these parts will be of equal amplitude 'a' (say).
9. Then phase difference between two successive waves will be

$$\delta = \frac{1}{n}\left(\frac{2\pi}{\lambda}\right) e \sin\theta$$

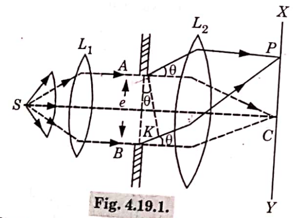


Fig. 4.19.1.

10. Now, according to n simple harmonic waves,

$$R = \frac{a \sin\frac{n\delta}{2}}{\sin\delta/2} = \frac{a \sin\left(\frac{\pi e \sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi e \sin\theta}{n\lambda}\right)}$$

11. Let $n\delta = 2\alpha$

and, $\alpha = \frac{\pi}{\lambda} e \sin\theta$

Then, $R = \frac{a \sin\alpha}{\sin\left(\frac{\alpha}{n}\right)}$ $\left[\because \frac{\alpha}{n} \text{ is very small. So, } \sin\frac{\alpha}{n} = \frac{\alpha}{n}\right]$

or $R = na \frac{\sin\alpha}{\alpha} \Rightarrow R = \frac{A_o \sin\alpha}{\alpha}$ $[\because na = A_o]$

12. Intensity at point 'P',

$$I = R^2 = \frac{A_o^2 \sin^2\alpha}{\alpha^2} \quad ...(4.19.1)$$

a. Position of Maxima :

1. $\frac{\sin\alpha}{\alpha} = 1$ when $\alpha \to 0$

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = \lim_{\alpha \to 0} \frac{1}{\alpha}\left(\alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - ...\right) = 1$$

2. $$I = I_o(1)^2 \Rightarrow I = I_o$$

3. $$\alpha = \frac{\pi}{\lambda} e \sin\theta$$

$$\frac{\pi}{\lambda} e \sin\theta = 0 \Rightarrow \sin\theta = 0$$

$$\theta = 0$$

4. Point C is central maxima or principal maxima.

b. Position of Minima :

1. If $\frac{\sin\alpha}{\alpha} = 0$

$\Rightarrow$ $\sin\alpha = 0$

$\alpha \neq 0$

$\alpha = \pm m\pi$ [where, m = 1, 2, 3....]

2. Hence, $\frac{e\pi \sin\theta}{\lambda} = \pm m\pi$

$e \sin\theta = \pm (m\lambda)$ [where, m = 1, 2, 3.....] ...(4.19.2)

3. Eq. (4.19.2) gives the direction of first, second, third .... minima and this equation is called diffraction equation.

c. Secondary Maxima :

1. The condition of secondary maxima may be obtained by differentiating eq. (4.19.1) w.r.t. α and equating it to zero.

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\left[A_o^2\left(\frac{\sin\alpha}{\alpha}\right)^2\right] = A_o^2 2\frac{\sin\alpha}{\alpha}\left[\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right]$$

2. Either $\frac{\sin\alpha}{\alpha} = 0 \Rightarrow \sin\alpha = 0$

or $\alpha\cos\alpha - \sin\alpha = 0$

$$\alpha = \tan\alpha$$

3. $\sin\alpha = 0$, gives position of principal minima and position of secondary maxima is given by

$$\alpha = \tan\alpha \quad ...(4.19.3)$$

4. Eq. (4.19.3) can be solved graphically by plotting the curves

$$y = \alpha \quad \text{and} \quad y = \tan\alpha$$

5. According to curves, the point of intersection of these two curves gives the value of α satisfying the equation $\alpha = \tan\alpha$. These points correspond to the value of

$$\alpha = 0, \pm\frac{3\pi}{2}, \frac{\pm 5\pi}{2}, \pm\frac{7\pi}{2} ....$$

$$\therefore \quad R = \frac{0.50 \times 0.50}{4 \times 10 \times 6.0 \times 10^{-5}} = 104.17 \text{ cm}$$

2. If t is the thickness of the film corresponding to a ring of D_n diameter, then,

$$2t = \frac{D_n^2}{4R} \text{ or } t = \frac{D_n^2}{8R} = \frac{0.50 \times 0.50}{8 \times 104.17} = 2.99 \times 10^{-4} \text{ cm}$$

PART-2

Fraunhofer Diffraction at Single Slit and at Double Slit, Absent Spectra, Diffraction Grating, Spectra with Grating, Dispersive Power, Rayleigh's Criterion of Resolution, Resolving Power of Grating.

CONCEPT OUTLINE : PART-2

Diffraction : Diffraction of light is a phenomenon of bending of light and spreading out towards the geometrical shadow when passed through an obstruction.

Rayleigh's Criteria : The spectral lines of equal intensity are said to be resolved, if the position of the principal maxima of one spectral line coincide with first minima of the other spectral line.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.18. **What is meant by diffraction of light ? Write name of the two classes of diffraction and explain it.**

Answer

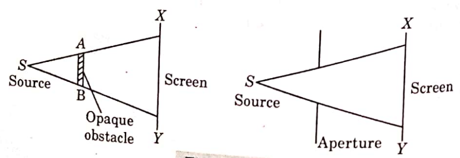


Fig. 4.18.1.

1. The departure of light path from true rectilinear path or the bending of light around corners of an obstacle is called diffraction of light.

2. When light passes through small apertures or by the side of small obstacles it does not follow rectilinear path strictly, but bends at round corners of the obstacles.
3. Diffraction phenomena can be divided into two types :

a. **Fresnel Diffraction :**
   1. In Fresnel diffraction, the source of light and the screen, both are placed at finite distance from the diffraction element (obstacle or apertures) in which incident wavefront is either spherical or cylindrical and no lens are used.

b. **Fraunhofer Diffraction :**
   1. In Fraunhofer diffraction, source of light and the screen both are placed at infinite distance from diffraction element in which incident wavefront is often plane.
   2. Here convex lenses are used for focusing diffracted light.

Que 4.19. **Describe Fraunhofer diffraction due to a single slit and show that relative intensities of successive maxima are nearly**

$$1 : \frac{1}{22} : \frac{1}{61} : \frac{1}{121} \ .....$$

OR

Discuss the phenomenon of diffraction at a single slit and show that the relative intensities of successive maxima are nearly

$$1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2} \ ......$$

AKTU 2014-15, Marks 05

OR

Explain the diffraction pattern obtained with diffraction at single slit. By what fraction the intensity of second maximum reduced from principal maximum ? **AKTU 2013-14, Marks 05**

Answer

A. **Fraunhofer Diffraction due to a Single Slit :**

1. The light from a monochromatic source S is converted into parallel beam of light by convex lens L_1.
2. Now this beam is incident normally on a slit AB of width 'e'.
3. Now according to Huygen's wave theory, every point in AB sends out secondary waves which are superimposed to give diffraction pattern on screen XY.
4. In this diffraction pattern, a central bright band is obtained because the rays from AB reach at C in same phase and here the intensity is maximum.
5. The rays which are directed through an angle θ are focused at point 'P'.

4. On subtracting eq. (4.15.1) from eq. (4.15.2),

$$D^2_{n+p} - D_n^{\,2} = \frac{4p\lambda R}{\mu} \quad \text{(For liquid)} \qquad ...(4.15.3)$$

5. For air, $\mu = 1$

$$D^2_{n+p} - D_n^{\,2} = 4p\lambda R \qquad ...(4.15.4)$$

6. On dividing eq. (4.15.4) by eq. (4.15.3),

$$\mu = \frac{(D^2_{n+p} - D^2_n)_{\text{air}}}{(D^2_{n+p} - D^2_n)_{\text{liquid}}}$$

Que 4.16. **Newton's rings are observed by keeping a spherical surface of 100 cm radius on a plane glass plate. If the diameter of the 15th bright ring is 0.590 cm and the diameter of the 5th ring is 0.336 cm, what is the wavelength of light used ?**

AKTU 2014-15, Marks 05

Answer

Given : $D_{15} = 0.590$ cm, $D_5 = 0.336$ cm, $p = 15 - 5 = 10$, $R = 100$ cm.

To Find : Wavelength of light.

b. For Dark Rings :

1. Since, we know, $2t = n\lambda$

   Substituting the value of 't' from eq. (4.12.1),

$$2\,\frac{r^2}{2R} = n\lambda \text{ or } r^2 = n\lambda R$$

2. If radius of n^{th} dark ring is r_n.

   Then, $r_n^{\,2} = n\lambda R$

3. D_n is diameter of n^{th} dark ring,

$$r_n = D_n / 2$$

then, $$\left(\frac{D_n}{2}\right)^2 = n\lambda R \quad \text{or} \quad D_n^{\,2} = 4n\lambda R$$

$$D_n = \sqrt{4n\lambda R}$$

4. Let, $$K = \sqrt{4\lambda R}$$

   Therefore, $$D_n = K\sqrt{n} \text{ or } D_n \propto \sqrt{n}$$

5. Diameter of dark ring is proportional to the square root of natural number.

Que 4.13. **Explain the formation of Newton's ring ? If in a Newton's ring experiment, the air in the interspaces is replaced by a liquid of refractive index 1.33, in what proportion would the diameter of the rings changed ?**

AKTU 2015-16, Marks 10

Answer

A. Formation of Newton's Ring : Refer Q. 4.11, Page 4–16A, Unit-4.

B. Numerical :

Given : μ = 1.33 (refractive index of liquid)

To Find : Proportion of change in diameter.

1. $$\frac{\text{Diameter of a ring in liquid film}}{\text{Diameter of the same ring in air film}} = \frac{1}{\sqrt{\mu}}$$

$$= \frac{1}{\sqrt{1.33}} = 0.867$$

2. So, the diameter of rings decreased by the portion of 0.867 of natural diameter.

Que 4.14. **Show that the diameter D_n of the n^{th} Newton's ring, when two surface of radius R_1 and R_2 are placed in contact is given by the relation : $\frac{1}{R_1} \pm \frac{1}{R_2} = \frac{4n\lambda}{D_n^{\,2}}$.**

Answer

Newton's Rings formed by two Curved Surfaces :

a. Case I :

1. When a planoconvex lens of radius of curvature R_1 is placed on the planoconcave lens of radius R_2.
2. Let at point A the thickness of air film is 't' and n^{th} dark ring is passing through A and its radius is r_n.
3. According to Fig. 4.14.1.

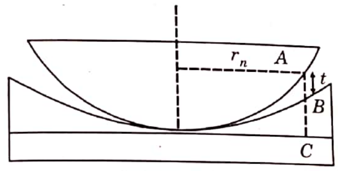


Fig. 4.14.1.

$$t = t_1 - t_2 \qquad [\because AC = t_1 \text{ and } BC = t_2]$$

$$= \frac{r_n^2}{2R_1} - \frac{r_n^2}{2R_2} \text{ or } t = \frac{r_n^{\,2}}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or $$2t = r_n^{\,2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad ...(4.14.1)$$

4. For dark ring,

$$2\mu t = n\lambda$$

$$2t = n\lambda \qquad [\because \mu = 1 \text{ for air}] \qquad ...(4.14.2)$$

5. From eq. (4.14.1) and eq. (4.14.2),

$$r_n^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = n\lambda$$

or $$\frac{1}{R_1} - \frac{1}{R_2} = \frac{n\lambda}{r_n^2}$$

6. But, $$D_n = 2r_n$$

$$\therefore \quad \frac{1}{R_1} - \frac{1}{R_2} = \frac{4n\lambda}{D_n^2}$$

b. Case II :

1. Let both the lenses are planoconvex and their curved surface is in contact.
2. Let 't' is thickness of air film at point A and R_1 and R_2 are radius of curvature of lenses respectively as shown in Fig 4.14.2.

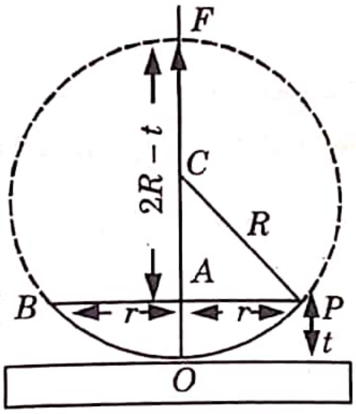


Fig. 4.12.1.

3. In actual, R is quite large and t is very small. So, t^2 is neglected.
Hence,
$$r^2 = 2Rt$$
$$t = \frac{r^2}{2R} \quad \text{...(4.12.1)}$$

a. For Bright Rings :

1. Since, we know that,
$$2t = (2n-1)\frac{\lambda}{2}$$
On putting the value of t from eq. (4.12.1),
$$2\frac{r^2}{2R} = (2n-1)\frac{\lambda}{2} \quad \text{or} \quad r^2 = (2n-1)\frac{\lambda R}{2}$$
2. If radius of n^{th} bright ring is r_n,
Then, $$r_n^2 = \frac{(2n-1)\lambda R}{2} \quad \text{...(4.12.2)}$$
3. D_n is diameter of n^{th} bright ring,
$$\therefore \quad r_n^2 = \left(\frac{D_n}{2}\right)^2$$
Now, from eq. (4.12.2),
$$\left(\frac{D_n}{2}\right)^2 = \frac{(2n-1)\lambda R}{2}$$
or $$D_n^2 = 2(2n-1)\lambda R$$
or $$D_n = \sqrt{2(2n-1)\lambda R}$$
4. Let, $$K = \sqrt{2\lambda R}$$
$$D_n = K\sqrt{2n-1} \quad \text{[where, } n = 0, 1, 2, 3... \text{]}$$
$$D_n \propto \sqrt{2n-1}$$
5. Diameter of bright rings is proportional to the square root of odd natural

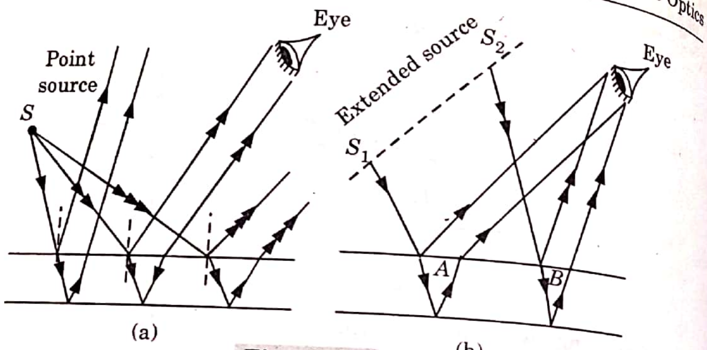


Fig. 4.10.1.

Que 4.11. **What are Newton's rings ? Explain with diagram.**

Answer

1. When a plano-convex lens of large radius of curvature is placed on a plane glass plate with convex surface in contact, a thin film between the lower surface of the lens and the upper surface of glass plate is formed.
2. The thickness of this film is very small (or zero) at the contact point and gradually increases from contact point to outward.
3. When a monochromatic light falls on the film, we get dark and bright concentric circular fringes having uniform thickness.
4. These rings are first investigated by Newton and are called Newton's rings.

A. Experimental Arrangement :

1. According to Fig. 4.11.1, S is a monochromatic source of light placed at the focus of lens L_1.
2. A horizontal beam of light fall on the glass plate G placed at 45° to the incident beam.
3. This beam is partly reflected from glass plate G.
4. This reflected beam fall normally on the lens L, placed on glass plate E.
5. Hence, the interference occurs between the rays reflected from the upper and lower surface of the film.
6. Interference rings are seen with the help of lower power microscope M.
7. The fringes are circular because the air film is symmetrical about the point of contact with lens and glass plate.

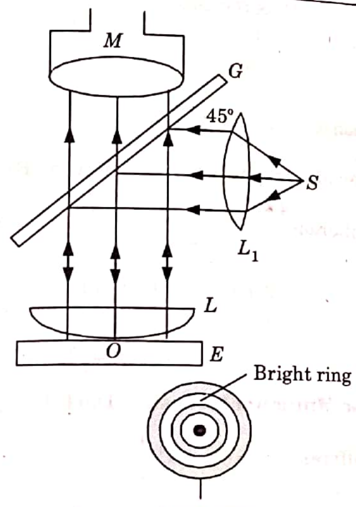


Fig. 4.11.1.

B. Explanation :

1. According to Fig. 4.11.2, rays (1) and (2) are reflected interfering rays corresponding to incident ray SP.

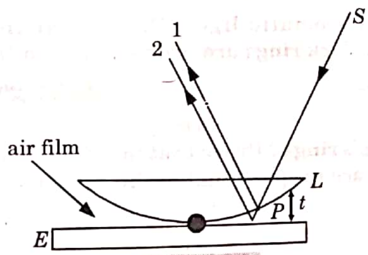


Fig. 4.11.2.

2. Now the effective path difference between (1) and (2) rays is given as :

$$\Delta = 2\mu t \cos r + \frac{\lambda}{2}$$

where, μ = Refractive index, and

t = thickness.

3. For normal incidence, $r = 0$, therefore

$$\cos \theta = 1$$

4. Hence, $$\Delta = 2\mu t + \frac{\lambda}{2}$$

1. Since, the condition for dark band is

$$2\mu t \cos r = n\lambda \quad \text{...(4.8.1)}$$

2. If n and $(n + 1)$ are the orders for dark bands for wavelengths λ_1 and λ_2 respectively, then,

$$2\mu t \cos r = n\lambda_1 \quad \text{...(4.8.2)}$$

and $$2\mu t \cos r = (n + 1)\lambda_2$$

or $$2\mu t \cos r = n\lambda_1 = (n + 1)\lambda_2 \quad \text{...(4.8.3)}$$

or $$n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

3. On putting the value of n in eq. (4.8.2),

$$2\mu t \cos r = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \text{ or } t = \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)2\mu \cos r}$$

4. But, $$\cos r = \sqrt{1 - \sin^2 r} = \sqrt{1 - \left(\frac{\sin i}{\mu}\right)^2} \quad \left(\because \mu = \frac{\sin i}{\sin r}\right)$$

$$\cos r = \sqrt{1 - \left(\frac{4/5}{4/3}\right)^2} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

5. Now, $$t = \frac{6.1 \times 10^{-5} \times 6.0 \times 10^{-5}}{(6.1 \times 10^{-5} - 6.0 \times 10^{-5}) \times 2 \times \frac{4}{3} \times \frac{4}{5}}$$

$$= 0.0017 \text{ cm.}$$

Que 4.9. **Two plane glass surfaces in contact along one edge are separated at the opposite edge by a thin wire. If 20 interference fringes are observed between these edges, in sodium light of wavelength, $\lambda = 5890$ Å of normal incidence, find the diameter of the wire.**

Answer

Que 4.6. **A man whose eyes are 150 cm above the oil film on water surface observes greenish colour at a distance of 100 cm from his feet. Find the thickness of the film.**

(μ_{oil} = 1.4, μ_{water} = 1.33, λ_{green} = 5000 Å)

Answer

Given : μ_{oil} = 1.4, μ_{water} = 1.33, λ_{green} = 5000 Å

To Find : Thickness of the film.

1. The condition for maxima,

$$2\mu t \cos r = (2n-1)\frac{\lambda}{2} \quad \text{[where, } n = 1, 2, 3....]$$

or
$$t = \frac{(2n-1)\lambda}{4\mu \cos r}$$

2. From Fig. 4.6.1,

$$\tan i = \frac{100}{150} = \frac{2}{3}$$

$$\sin i = \frac{2}{\sqrt{13}}$$

3. Since,
$$\mu = \frac{\sin i}{\sin r}$$

∴
$$\sin r = \frac{\sin i}{\mu} = \frac{\frac{2}{\sqrt{13}}}{1.4} = 0.3962$$

and
$$\cos r = \sqrt{1-\sin^2 r} = \sqrt{[1-(0.3962)^2]} = 0.9182$$

4. Therefore,
$$t = \frac{(2n-1)\lambda}{4\mu \cos r} = \frac{(2n-1)5\times 10^{-7}}{4\times 1.4\times 0.9182}$$
$$= (2n-1) \times 9.724 \times 10^{-8} \text{ m}$$

5. Putting n = 1, 2, 3...... value of t is calculated.

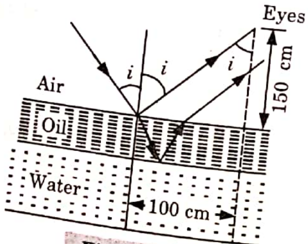


Fig. 4.6.1.

Que 4.7. **Light of wavelength 5893 Å is reflected at nearly normal incidence from a soap film of μ = 1.42. What is the least thickness of this film that will appear :**

a. dark

b. bright

Answer

Given : λ = 5893 Å , μ = 1.42

To Find : Least thickness of this film that will appear :

i. Dark

ii. Bright

i. Least Thickness of Dark Film :

1. Since, the condition for the dark film in reflected system is

$$2\mu t \cos r = n\lambda$$

2. For normal incidence, r = 0 and cos r = 1

∴
$$2\mu t = n\lambda \quad \text{or} \quad t = n\lambda/2\mu$$

3. For least thickness of the film, n = 1

$$t = \frac{\lambda}{2\mu}$$

$$t = \frac{5893\times 10^{-8}}{2\times 1.42} = 2.075\times 10^{-5} \text{ cm}$$

ii. Least Thickness of Bright Film :

1. The condition for bright film

$$2\mu t \cos r = (2n-1)\frac{\lambda}{2}$$

2. For normal incidence, r = 0 and cos r = 1

∴
$$2\mu t = (2n-1)\frac{\lambda}{2}$$

3. For least thickness, n = 1

$$2\mu t = (2\times 1-1)\frac{\lambda}{2} \text{ or } 2\mu t = \frac{\lambda}{2}$$

and
$$t = \frac{\lambda}{4\mu} = \frac{5893\times 10^{-8}}{4\times 1.42} = 1.0375\times 10^{-5} \text{ cm}$$

Que 4.8. **White light is incident on a soap film at an angle $\sin^{-1}\frac{4}{5}$ and the reflected light is observed with spectroscope. It is foun that two consecutive dark bands correspond to wavelengt 6.1×10^{-5} cm and 6.0×10^{-5} cm. If the μ of the film be 4/3, calculate it thickness.**

3. Similarly, if x_{n+1} is the distance of $(n + 1)^{th}$ dark fringe, then

$$2\mu\, x_{n+1} \tan\theta = (n + 1)\lambda \quad ...(4.4.4)$$

4. Subtracting eq. (4.4.3) from eq. (4.4.4), we get,

$$2\mu\,(x_{n+1} - x_n) \tan\theta = \lambda$$

or

$$x_{n+1} - x_n = \frac{\lambda}{2\mu \tan\theta} \quad ...(4.4.5)$$

5. For very small value of θ, $\tan\theta \approx \theta$

$\therefore$ Fringe width, $$\omega = x_{n+1} - x_n = \frac{\lambda}{2\mu\,\theta} \quad ...(4.4.6)$$

where, θ is measured in radian.

6. Similarly, we can obtain same formula for the fringe width of bright fringes, that is the fringe width of bright fringe is expressed as,

$$\omega = \frac{\lambda}{2\mu\theta} \quad ...(4.4.7)$$

7. It is clear from eq. (4.4.6) and eq. (4.4.7) that for a given wedge angle θ, the fringe width of dark or bright fringes is constant (as λ and μ is constant). It means that the interference fringes are equidistant from one another.

Que 4.5. **White light falls normally on a film of soapy water whose thickness is 1.5×10^{-5} cm and refractive index is 1.33. Which wavelength in the visible region will be reflected most strongly ?**

Answer

b. Condition for Minima :

1. $$\Delta = (2n + 1)\frac{\lambda}{2}$$

where, $n = 0, 1, 2, 3...$

2. Then, $$2\mu t \cos r + \frac{\lambda}{2} = (2n + 1)\frac{\lambda}{2}$$

$$2\mu t \cos r = n\lambda \quad ...(4.3.4)$$

B. Interference in a Thin Film by Transmitted Light :

1. From Fig. 4.3.1, the path difference between two transmitted rays, CP and FQ,

$$\Delta = \text{Path } CDF \text{ in film} - \text{path } CR \text{ in air}$$

$$\Delta = \mu\,(CD + DF) - CR \quad ...(4.3.5)$$

2. Now in ΔCDM, $\cos r = \dfrac{DM}{CD} \Rightarrow CD = \dfrac{t}{\cos r}$ and $CM = t \tan r$

3. In ΔDMF, $\cos r = \dfrac{DM}{DF} \Rightarrow DF = \dfrac{t}{\cos r}$ and $MF = t \tan r$

4. In ΔCRF, $\sin i = \dfrac{CR}{CF} \Rightarrow CR = CF \sin i$

or $CR = (CM + MF)\sin i$

$CR = 2t \tan r \sin i$

5. On putting the value of CD, DF and CR in eq. (4.3.5),

$$\Delta = \frac{2\mu t}{\cos r} - 2t \tan r \sin i$$

$$= \frac{2\mu t}{\cos r} - 2t\frac{\sin r}{\cos r}\mu \sin r \qquad \left[\because \mu = \frac{\sin i}{\sin r}\right]$$

$$\Delta = \frac{2\mu t}{\cos r}[1 - \sin^2 r]$$

$$\Delta = 2\mu t \cos r \quad ...(4.3.6)$$

a. Condition for Maxima :

1. If $$\Delta = 2n\frac{\lambda}{2}$$ [where, $n = 0, 1, 2, 3...$]

2. Then, $$2\mu t \cos r = 2n\frac{\lambda}{2}$$

or $$2\mu t \cos r = n\lambda \quad ...(4.3.7)$$

b. Condition for Minima :

1. If $$\Delta = (2n + 1)\frac{\lambda}{2}$$ [where, $n = 0, 1, 2, 3...$]

2. Then, $$2\mu t \cos r = (2n + 1)\frac{\lambda}{2} \quad ...(4.3.8)$$

B. Condition for Minimum Intensity (Destructive Interference, I_{min}) :

1. If $\cos \delta = -1$ i.e., $\delta = (2n + 1)\pi$
   where, $n = 0, 1, 2, 3...$
2. $I_{min} = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$
3. Hence, $I_{min} < I_1 + I_2$
4. $$\text{Path difference} = \frac{\lambda}{2\pi} \times \text{Phase difference}$$
   $$= \frac{\lambda}{2\pi} \times (2n + 1)\pi = (2n + 1)\frac{\lambda}{2}$$
   = odd multiple of $\lambda/2$.

Que 4.3. **Discuss the interference in thin film due to reflected light. What happens when film is excess thin ?**

AKTU 2013-14, Marks 05

OR

Explain the phenomenon of interference in thin films due to reflected light and transmitted light.

Answer

1. Consider a parallel sided transparent thin film of thickness t and refractive index $\mu > 1$.
2. Let SA a monochromatic light of wavelength λ be incident on the upper surface of the film at an angle i. This ray gets partially reflected along AB and partially refracted along AC direction.
3. Now at point C it again gets reflected along CD and transmitted along DE.

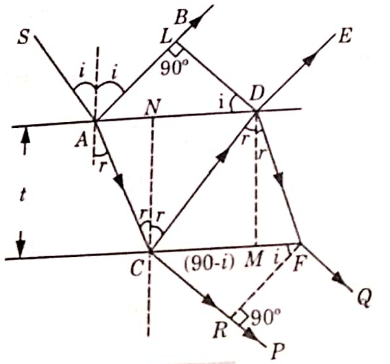


Fig. 4.3.1.

A. Interference in a Thin Film by Reflected Light :

1. According to Fig. 4.3.2, the path difference between AB and DE rays,
   $$\Delta = \text{path } ACD \text{ in film} - \text{path } AL \text{ in air}$$
   $$\Delta = \mu\,(AC + CD) - AL \qquad ...(4.3.1)$$
2. Now in ΔANC and ΔNCD,
   $$\cos r = \frac{CN}{AC} = \frac{CN}{CD}$$
   $$AC = CD = \frac{t}{\cos r}$$
3. Now in ΔALD,
   $$\sin i = \frac{AL}{AD} \Rightarrow AL = AD \sin i$$
   $$AL = (AN + ND) \sin i$$
4. But, from ΔANC and ΔNCD,
   $$AN = t \tan r \quad \text{and} \quad ND = t \tan r$$
   So, $$AL = 2t \tan r \sin i$$
5. Putting the values of AC, CD, and AL in eq. (4.3.1),
   $$\Delta = \mu\left(\frac{2t}{\cos r}\right) - 2t \tan r . \sin i$$
   $$= \frac{2\mu t}{\cos r} - 2t\frac{\sin r}{\cos r}.\sin i$$
   $$\therefore \quad \Delta = \frac{2\mu t}{\cos r}[1 - \sin^2 r] \qquad \left[\because \mu = \frac{\sin i}{\sin r}\right]$$
   $$\Delta = 2\mu t \cos r$$
6. Since, the ray AB is reflected at the surface of a denser medium. Therefore, it undergoes a phase change of π or path difference of $\frac{\lambda}{2}$. The effective path difference between AB and DE is
   $$\Delta = 2\mu t \cos r + \frac{\lambda}{2} \qquad ...(4.3.2)$$

a. Condition for Maxima :

1. If $\Delta = 2n\frac{\lambda}{2}$
   where, $n = 0, 1, 2, 3...$
2. Then, $$2\mu t \cos r + \frac{\lambda}{2} = 2n\frac{\lambda}{2}$$
   $$2\mu t \cos r = (2n - 1)\frac{\lambda}{2} \qquad ...(4.3.3)$$

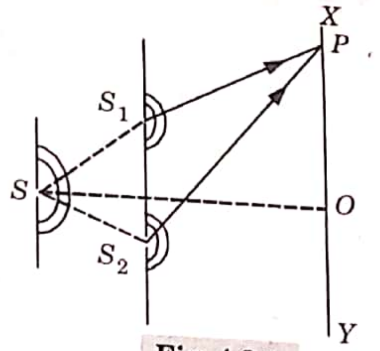


Fig. 4.2.1.

3. According to principle of superposition, the resultant displacement at point P is

$$y = y_1 + y_2$$
$$y = a_1 \sin \omega t + a_2 \sin (\omega t + \delta)$$
$$= a_1 \sin \omega t + a_2 [\sin \omega t \cos \delta + \cos \omega t \sin \delta]$$

or $$y = \sin \omega t\,(a_1 + a_2 \cos \delta) + (a_2 \sin \delta) \cos \omega t \quad ...(4.2.3)$$

4. Let us take,

$$A \cos \phi = a_1 + a_2 \cos \delta \quad ...(4.2.4)$$
$$A \sin \phi = a_2 \sin \delta \quad ...(4.2.5)$$

Then, eq. (4.2.3) becomes,

$$y = A \cos \phi \sin \omega t + A \sin \phi \cos \omega t$$

or $$y = A \sin (\omega t + \phi) \quad ...(4.2.6)$$

5. Since, eq. (4.2.6) the resultant wave equation having amplitude A this can be obtained by squaring eq. (4.2.4) and eq. (4.2.5) and adding,

$$A^2 = a_1^2 + a_2^2 \cos^2 \delta + 2a_1a_2 \cos \delta + a_2^2 \sin^2 \delta$$
$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \delta$$

6. By the definition, intensity is directly proportional to the square of amplitude,

$$I \propto A^2$$

or $$I = KA^2 \quad [K = 1, \text{ in arbitrary unit}]$$

$\therefore$ $$I = A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \delta$$

A. Condition for Maximum Intensity (Constructive Interference, I_{max}) :

1. If $\cos \delta = 1$ *i.e.*, $\delta = 2n\pi$

   where, $n = 0, 1, 2, 3,...$

2. $$I_{max} = a_1^2 + a_2^2 + 2a_1a_2$$
   $$I_{max} = (a_1 + a_2)^2$$

3. So, $$I_{max} > I_1 + I_2$$

4. $$\text{Path difference} = \frac{\lambda}{2\pi} \times \text{Phase difference}$$
   $$= \frac{\lambda}{2\pi} \times 2n\pi = 2n\frac{\lambda}{2}$$
   $$= \text{even multiple of } \lambda/2.$$

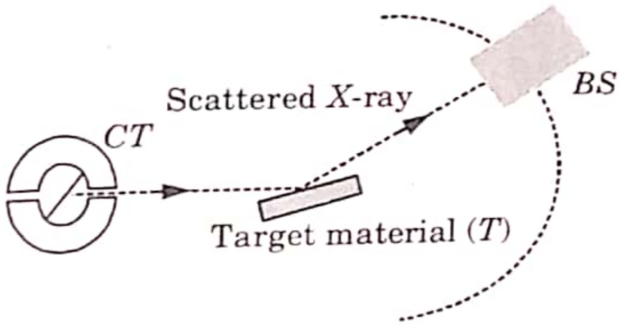
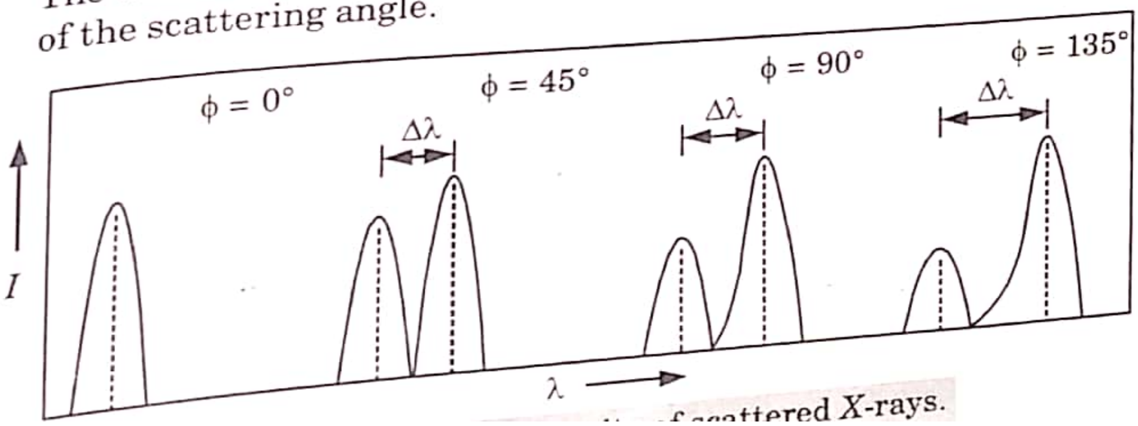


Fig. 3.24.1. Experimental verification of Compton effect.

2. Monochromatic X-rays of wavelength λ from a Coolidge tube CT are allowed to fall on a target material T such as a small block of carbon.
3. The scattered X-rays of wavelength λ' are received by a Bragg's spectrometer BS, which can move along the arc of a circle.
4. The wavelength of the scattered X-rays is measured for different values of the scattering angle.



...scattered X-rays.

$$P = \frac{2}{L}\int_{x_1}^{x_2} \frac{1}{2}\left(1 - \cos\frac{2\pi nx}{L}\right) dx = \frac{1}{L}\left[x - \frac{L}{2\pi n}\sin\frac{2n\pi x}{L}\right]_{x_1}^{x_2}$$

3. Since, $x_1 = 0.45\,L$ and $x_2 = 0.55\,L$, for ground state, $n = 1$

$$P = \frac{1}{L}\left[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{0.45L}^{0.55L}$$

$$= \frac{1}{L}\left[\left(0.55L - \frac{L}{2\pi}\sin 1.1\pi\right) - \left(0.45L - \frac{L}{2\pi}\sin 0.9\pi\right)\right]$$

$$= \left[\left(0.55 - \frac{1}{2\pi}\sin 198°\right) - \left(0.45 - \frac{1}{2\pi}\sin 162°\right)\right]$$

$$P = 0.198362 = 19.8\ \%$$

Que 3.23. **Discuss Compton effect and derive an expression for Compton shift.**

OR

Derive an expression for Compton shift showing dependency on angle of scattering.

Answer

1. When a monochromatic beam of high frequency radiation is scattered by a substance, the scattered radiation contain two components-one having a lower frequency or greater wavelength and the other having the same frequency or wavelength.
2. The radiation of unchanged frequency in the scattered beam is known as 'unmodified radiation' while the radiation of lower frequency or slightly higher wavelength is called as 'modified radiation'.
3. This phenomenon is known as 'Compton effect'.
4. Let a photon of energy $h\nu$ collides with an electron at rest.
5. During the collision it gives a fraction of energy to the free electron. The electron gains kinetic energy and recoil as shown in Fig. 3.23.1.

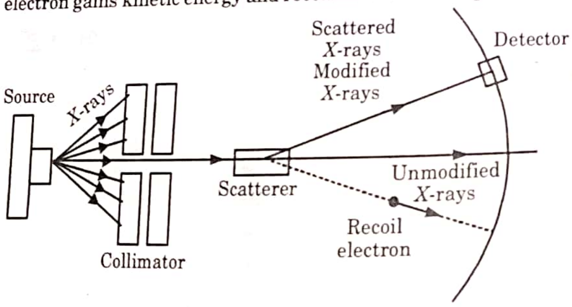
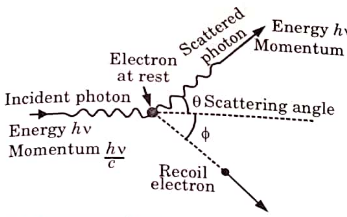
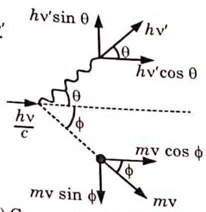


Fig. 3.23.1. Compton effect.



(*a*) Geometry of Compton scattering



(*b*) Components of momentum before and after collision

Fig. 3.23.2.

a. Before collision :
   i. Energy of incident photon = $h\nu$
   ii. Momentum of incident photon = $\frac{h\nu}{c}$
   iii. Rest energy of electron = m_0c^2
   iv. Momentum of rest electron = 0
b. After collision :
   i. Energy of scattered photon = $h\nu'$
   ii. Momentum of scattered photon = $\frac{h\nu'}{c}$
   iii. Energy of electron = mc^2
   iv. Momentum of recoil electron = mv

6. According to the principle of conservation of energy,
$$h\nu + m_0c^2 = h\nu' + mc^2 \quad ...(3.23.1)$$
7. Again using the principle of conservation of momentum along and perpendicular to the direction of incident, we get,
Momentum before collision = Momentum after collision
$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\theta + mv\cos\phi \quad ...(3.23.2)$$
$$0 + 0 = \frac{h\nu'}{c}\sin\theta - mv\sin\phi \quad ...(3.23.3)$$
8. From eq. (3.23.2), we get,
$$mvc\cos\phi = h\nu - h\nu'\cos\theta \quad ...(3.23.4)$$
9. From eq. (3.23.3), we get,
$$mvc\sin\phi = h\nu'\sin\theta \quad ...(3.23.5)$$
10. Squaring eq. (3.23.4) and eq. (3.23.5) and then adding, we get,
$$m^2v^2c^2 = (h\nu - h\nu'\cos\theta)^2 + (h\nu'\sin\theta)^2$$
$$= h^2\nu^2 - 2h^2\nu\nu'\cos\theta + h^2\nu'^2\cos^2\theta + h^2\nu'^2\sin^2\theta$$
$$= h^2[\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta] \quad ...(3.23.6)$$
11. From eq. (3.23.1), we get,
$$mc^2 = h(\nu - \nu') + m_0c^2$$

Given : $\psi = ax$.

To Find : i. Probability of particle can be found between $x = 0.35$ to $x = 0.45$.

ii. Expectation value.

1. The probability of finding the particle between x_1 and x_2 when it is in n^{th} state is,

$$P = \int_{x_1}^{x_2} |\psi_n|^2 \, dx$$

2. Here, $x_1 = 0.35$ and $x_2 = 0.45$

Therefore, $P = \int_{0.35}^{0.45} (ax)^2 dx = a^2 \int_{0.35}^{0.45} x^2 dx$

$$P = \frac{a^2}{3}\left[x^3\right]_{0.35}^{0.45} = \frac{a^2}{3} [(0.45)^3 - (0.35)^3]$$

$$= \frac{a^2}{3} [0.091125 - 0.042875] = 0.0161\, a^2$$

3. The expectation value of the position of particle is given by

$$< x > = \int_{-\infty}^{\infty} x |\psi_n(x)|^2 \, dx$$

4. Since, the particle is confined in a box having its limit $x = 0$ to $x = 1$ then,

$$< x > = \int_0^1 x.(ax)^2 \, dx = a^2 \int_0^1 x^3 dx$$

$$< x > = \frac{a^2}{4} = 0.25\, a^2$$

Que 3.22. **Determine the probabilities of finding a particle trapped in a box of length L in the region from $0.45\,L$ to $0.55\,L$ for the ground state.** **AKTU 2017-18**

Answer

2. Now applying normalization condition to find constant A,

$$\int_0^L |\psi_n(x)|^2\, dx = 1$$

$$\int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\frac{A^2}{2}\int_0^L \left(1 - \cos\frac{2n\pi}{L}x\right) dx = 1$$

$$\frac{A^2}{2}\left[x - \frac{\sin\frac{2n\pi x}{L}}{\frac{2n\pi}{L}}\right]_0^L = 1$$

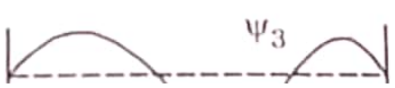
$$\frac{A^2}{2} L = 1$$

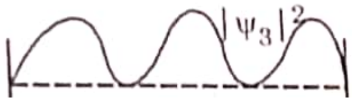
$$A = \sqrt{\frac{2}{L}}$$

3. So, eq. (3.18.1) becomes, $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$.

This is normalization function.

$n = 3$





6. If we consider the scattering of only a single photon by a crystal or the passage of only a single photon through a narrow slit, then it is impossible to observe the usual pattern of intensity variation or diffraction.
7. In this situation, we can only say that the probability of photon striking the screen is highest at places where the wave theory predicts a maximum and lowest at places where the wave theory predicts a minimum.
8. Eq. (3.14.3) shows that $< E^2 >$ is a measure of the probability of photon crossing unit area per second at the point under consideration. Hence, in one dimension, $<E^2>$ is a measure of the probability per unit length of finding the photon at the position x at time t.

Que 3.15. **The wave function of a particle confined to a box of length L is**

$$\psi(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} \qquad 0 < x < L$$

and $\psi(x) = 0$ everywhere else.
Calculate probability of finding the particle in region $0 < x < \frac{L}{2}$.

Answer

1. The probability of finding the particle in interval dx at distance x is
$$p(x)dx = |\psi|^2 dx = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) dx$$
2. The probability in region $0 < x < \frac{L}{2}$ is
$$P = \int_0^{L/2} p(x)dx = \int_0^{L/2} \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) dx$$
$$= \frac{1}{L} \int_0^{L/2} \left(1 - \cos\frac{2\pi x}{L}\right) dx$$
$$= \frac{1}{L} \cdot \frac{L}{2} = \frac{1}{2}$$

Que 3.16. **Discuss the stationary state solutions in brief.**

Answer

1. A state of the system in which probability distribution function $\psi\psi^*$ is independent of time is called stationary state of the system.
where, ψ = wave function, and
ψ^* = complex conjugate of wave function.
2. Let the probability distribution function $\psi\psi^*$ for a system in the state is given by the wave function

11. From eq. (3.13.8) and eq. (3.13.9), we get

$$\nabla^2\psi = \frac{-4\pi^2 2m[E-V]\psi}{h^2}$$

$$\therefore \quad \nabla^2\psi + \frac{2m[E-V]\psi}{\hbar^2} = 0 \qquad \left[\text{where, } \hbar = \frac{h}{2\pi}\right] \qquad ...(3.13.10)$$

This is required time-independent Schrodinger wave equation.

12. For free particle, $V = 0$

$$\therefore \quad \nabla^2\psi + \frac{2m}{\hbar^2}E\psi = 0$$

B. Time Dependent Schrodinger Wave Equation :

1. We know that wave function is $\psi = \psi_o e^{-i\omega t}$
2. On differentiating w.r.t. time, we get,

$$\frac{\partial\psi}{\partial t} = -i\omega\,\psi_o e^{-i\omega t}$$

or
$$\frac{\partial\psi}{\partial t} = -i\,(2\pi\nu)\,\psi \qquad ...(3.13.11)$$

3. But
$$E = h\nu \Rightarrow \nu = \frac{E}{h}$$

4. So, eq. (3.13.11) becomes,

$$\frac{\partial\psi}{\partial t} = -i2\pi\left(\frac{E}{h}\right)\psi$$

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}E\psi \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

or
$$E\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

or
$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad ...(3.13.12)$$

5. Now time independent Schrodinger wave equation is,

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}[E\psi - V\psi] = 0$$

6. Using eq. (3.13.12), we get,

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left[i\hbar\frac{\partial\psi}{\partial t} - V\psi\right] = 0$$

$$\nabla^2\psi - \frac{2m}{\hbar^2}V\psi = -\frac{2m}{\hbar^2}i\hbar\frac{\partial\psi}{\partial t}$$

$$\left(\nabla^2 - \frac{2m}{\hbar^2}V\right)\psi = -\frac{2m}{\hbar^2}i\hbar\frac{\partial\psi}{\partial t}$$

or
$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This is required time dependent Schrodinger wave equation.

7. $-\dfrac{\hbar^2}{2m}\nabla^2 + V = H \rightarrow$ is known as Hamiltonian operator.

and, $\quad i\hbar\dfrac{\partial\psi}{\partial t} = E\psi \rightarrow$ energy operator.

Then, $\quad H\psi = E\psi$

Que 3.14. **Discuss the Born's interpretation of wave function.**

Answer

1. Max Born interpreted the relation between the wave function $\psi(x,$ and the location of the particle by drawing an analogy between t intensity of light or photon beam and the intensity of electron beam.
2. Consider a beam of light (EM-wave) incident normally on a screen. T magnitude of electric field vector $\vec{E}$ of the beam is given by

$$E = E_0 \sin(kx - \omega t)$$

where, $\quad E_0$ = Amplitude of the electric field.

3. For an EM-wave, the intensity I at a point due to a monochromatic bea of frequency ν is given by

$$I = c\varepsilon_0 \langle E^2 \rangle \qquad ...(3.14$$

Here, $\langle E^2 \rangle$ = Average of the square of the instantaneo magnitudes of the electric field vector of t wave over a complete cycle,

c = Velocity of light in free space, and

ε_0 = Electric permittivity of free space.

4. The intensity may also be interpreted as the number N of photons ea of energy $h\nu$ crossing unit area in unit time at the point und consideration normal to the direction of the photons.

Thus,
$$I = N h\nu \qquad ...(3.14$$

5. Comparing eq. (3.14.1) and eq. (3.14.2), we get,

$$N h\nu = c\varepsilon_0 \langle E^2 \rangle$$

or
$$N = \frac{c\varepsilon_0}{h\nu}\langle E^2 \rangle$$

or
$$N \propto \langle E^2 \rangle \qquad ...(3.1$$

This relation is valid only when a large number of photons are invol *i.e.*, the beam has the large intensity.

$$\lambda = \frac{h}{mv}$$

$$\lambda = \frac{6.63 \times 10^{-34}}{1.67 \times 10^{-27} \times 1.5 \times 10^{7}} = 2.646 \times 10^{-14} \text{ m}$$

[$\because$ $m = 1.67 \times 10^{-27}$ kg and $h = 6.63 \times 10^{-34}$ J-s]

$= 2.646 \times 10^{-4}$ Å

PART-2

Time Dependent and Time Independent Schrodinger's Wave Equation, Born Interpretation of Wave Function, Solution to Stationary State, Schrodinger Wave Equation for One-Dimensional Particle in a Box, Compton Effect.

CONCEPT OUTLINE : PART-2

Wave Function and its Significance :

The wave function ψ is described as mathematical function whose variation builds up matter waves. $|\psi|^2$ defines the probability density of finding the particle within the given confined limits. ψ is defined as probability amplitude and $|\psi|^2$ is defined as probability density.

Schrodinger's Wave Equation :

This wave equation is a fundamental equation in quantum mechanics and describes the variation of wave function ψ in space and time.

Compton Effect : The phenomenon in which the wavelength of the incident X-rays increases and hence the energy decreases due to scattering from an atom is known as Compton effect.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.13. **What is Schrodinger wave equation ? Derive time independent and time dependent Schrodinger wave equations.**

AKTU 2015-16, 2016-17

Answer

1. Schrodinger's equation, which is the fundamental equation of quantum mechanics, is a wave equation in the variable ψ.

12. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2mqV}}$$

Que 3.9. **Derive an expression for de-Broglie wavelength of helium atom having energy at temperature T K.**

Answer

1. According to kinetic theory of gases, the average kinetic energy (E_k) of the material particle is given as

$$E_k = \frac{3}{2} KT$$

2. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \frac{h}{\sqrt{2m \times \frac{3KT}{2}}} = \frac{h}{\sqrt{3mKT}}$$

where, $K = 1.38 \times 10^{-23}$ J/K, and

T = temperature (K).

Que 3.10. **The kinetic energy of an electron is 4.55×10^{-25} J. Calculate the velocity, momentum and wavelength of the electron.**

Answer

Given : $E_k = 4.55 \times 10^{-25}$ J

To Find : i. Velocity
ii. Momentum
iii. Wavelength of electron.

1. If m_o is the rest mass of electron, v is the velocity of the electron, then its kinetic energy (E_k) is given by

$$E_k = \frac{1}{2} m_o v^2$$

$$v = \sqrt{\frac{2E_k}{m_o}} = \sqrt{\frac{2 \times 4.55 \times 10^{-25}}{9.1 \times 10^{-31}}} = 1 \times 10^3 \text{ m/s}$$

$[\because \; m_o = 9.1 \times 10^{-3} \text{ kg}]$

2. Momentum of electron,

$$p = m_o v = 9.1 \times 10^{-31} \times 10^3 = 9.1 \times 10^{-28} \text{ kg m/s}$$

3. Wavelength of electron,

$$\lambda = h/p = (6.63 \times 10^{-34})/(9.1 \times 10^{-28}) = 7.29 \times 10^{-7} \text{ m}$$

Que 3.11. **Find the de-Broglie wavelength of neutron of energy 12.8 MeV (given that $h = 6.625 \times 10^{-34}$J–s, mass of neutron $(m_n) = 1.675 \times 10^{-27}$ kg and 1 eV = 1.6×10^{-19} Joule).**

Answer

Given : E_k = 12.8 MeV, $h = 6.625 \times 10^{-34}$ J-s, $m_n = 1.675 \times 10^{-27}$ kg, 1 eV = 1.6×10^{-19} J.
To Find : de-Broglie wavelength.

1. Rest mass energy of neutron is given as

$$m_o c^2 = 1.675 \times 10^{-27} \times (3 \times 10^8)^2$$
$$= 1.5075 \times 10^{-10} \text{ J}$$
$$= \frac{1.507 \times 10^{-10}}{1.6 \times 10^{-19}} = 942.18 \text{ MeV}$$

2. The given energy 12.8 MeV is very less compared to the rest mass energy of neutron, therefore relativistic consideration in this case is not applicable.

3. Now de-Broglie wavelength of the neutron is given as

$$\lambda = \frac{h}{\sqrt{2mE_k}}$$

$$E_k = 12.8 \times 10^6 \times (1.6 \times 10^{-19}) \text{ J}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 1.675 \times 10^{-27} \times 12.8 \times 10^6 \times 1.6 \times 10^{-19}}}$$
$$= \frac{6.625 \times 10^{-34}}{8.28 \times 10^{-20}}$$
$$= 8 \times 10^{-15} \text{ m}$$
$$= 8 \times 10^{-5} \text{ Å}$$

Que 3.12. **Calculate the de-Broglie's wavelength associated with a proton moving with a velocity equal to $\frac{1}{20}$ th of light velocity.**

Answer

Given : v = (1/20) c = $\frac{1}{20} \times 3 \times 10^8$ m/s = 1.5×10^7 m/s
To Find : de-Broglie wavelength

1. Formula for de-Broglie's wavelength :

4. From eq. (3.8.1) and eq. (3.8.2),

$$mc^2 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{mc^2}$$

$$\lambda = \frac{h}{mc} \Rightarrow \lambda = \frac{h}{p} \qquad [\because \; mc = p]$$

5. In place of photon, we take material particle having mass 'm' moving with velocity 'v'. The momentum,

$$p = m\text{v}$$

6. The wavelength of wave associated with particle is,

$$\lambda = \frac{h}{m\text{v}} = \frac{h}{p}$$

This is de-Broglie's wavelength.

7. If E_k is kinetic energy of material particle of mass 'm' moving with velocity 'v' then,

$$E_k = \frac{1}{2} m\text{v}^2$$

$$E_k = \frac{m^2 \text{v}^2}{2m}$$

or $$E_k = \frac{(m\text{v})^2}{2m} = \frac{p^2}{2m}$$

or $$p = \sqrt{2mE_k}$$

8. The de-Broglie's wavelength, $\lambda = \dfrac{h}{\sqrt{2mE_k}}$

9. According to kinetic theory of gases, the average kinetic energy (E_k) of the material particle is given as

$$E_k = \frac{3}{2} KT$$

10. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2m \times \dfrac{3KT}{2}}} = \frac{h}{\sqrt{3mKT}}$$

where, $K = 1.38 \times 10^{-23}$ J/K

T = temperature (K).

11. Suppose material particle is accelerated by potential difference of V volt then,

$$E_k = qV$$

where, q = charge of particle.

a. Wien's Law from Planck's Radiation Law :

1. For shorter wavelengths λT will be small and hence $e^{hc/\lambda kT} >> 1$
2. Hence, for small values of λT Planck's formula reduces to

$$u_\lambda \, d\lambda = \frac{8\pi\, hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT}} = 8\pi\, hc\, \lambda^{-5} e^{-hc/\lambda kT} d\lambda$$

or $\quad u_\lambda \, d\lambda = A\lambda^{-5} e^{-a/\lambda T} d\lambda \quad$ ...(3.5.3)

where, $\quad A$ = Constant (= $8\pi\, hc$), and

a = Constant (= hc/k).

3. Eq. (3.5.3) is Wien's law.
4. This result shows that at shorter wavelengths Planck's law approaches Wien's law and hence at shorter wavelengths Planck's law and Wien's law agrees (Fig. 3.5.1).

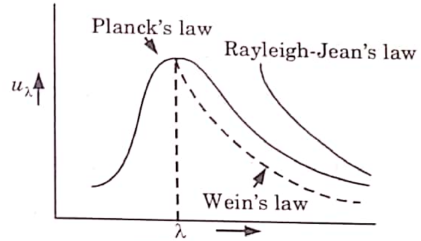


Fig. 3.5.1.

b. Rayleigh-Jean's Law :

1. For longer wavelengths $e^{hc/\lambda kT}$ is small and can be expanded as follows :

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda\, kT} \approx \frac{hc}{\lambda\, kT}$$

2. Hence, for longer wavelengths Planck's formula reduces to

$$u_\lambda d\lambda = \frac{8\pi\, hc}{\lambda^5} \frac{\lambda kT}{hc} d\lambda$$

or $\quad u_\lambda d\lambda = \dfrac{8\pi\, kT}{\lambda^4} d\lambda \quad$ ...(3.5.4)

3. Eq. (3.5.4) shows that for longer wavelengths Planck's law approaches to Rayleigh-Jean's law and thus at longer wavelengths Planck's law and Rayleigh-Jean's law agree (Fig. 3.5.1).
4. Thus, it is concluded that the Planck's radiation law successfully explained the entire shape of the curves giving the energy distribution in black body radiation.

Que 3.6. **Discuss the wave particle duality.**

Answer

1. The depth of penetration is defined as the depth in which the strength of electric field associated with the electromagnetic wave reduces to $1/e$ times of its initial value.
2. Depth of penetration or skin depth is a very important parameter in describing conductor behaviour in electromagnetic field and in radio communication.

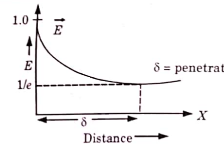


Fig. 2.19.1.

3. The reciprocal of attenuation constant is called skin depth or depth of penetration *i.e.*,

$$\delta = \frac{1}{\text{attenuation constant}}$$

4. For good conductors, penetration depth decreases with increase in frequency and is given as

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

5. For poor conductors, skin depth is independent of frequency.

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

Que 2.20. **Show that for a poor conductor, the skin depth can be expressed as**

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

Answer

1. We know that for a conducting medium, the propagation constant can be expressed as

$$K = \alpha + i\beta$$

Here, $$\alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

and $$\beta = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$

and skin depth is given by

$$\delta = \frac{1}{\alpha}$$

2. For a poor conductor $\sigma << \varepsilon\omega$, so we can approximate the first term in square root bracket of right hand side of expression of α using the binomial theorem as

$$\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} = \left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{\frac{1}{2}} = 1+\frac{\sigma^2}{2\varepsilon^2\omega^2} - \frac{\sigma^4}{8\,\varepsilon^4\omega^4} + \ldots = 1 + \frac{\sigma^2}{2\varepsilon^2\omega^2}$$

i.e., $$\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 = \frac{\sigma^2}{2\,\varepsilon^2\,\omega^2}$$

3. The expression of α therefore reduces to

$$\alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\frac{\sigma^2}{2\,\varepsilon^2\omega^2}\right]^{1/2}$$

or $$\alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\,\frac{\sigma}{\sqrt{2}\,\varepsilon\omega} \quad \text{or} \quad \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

4. Hence, the skin depth for a poor conductor can be expressed as

$$\delta = \frac{1}{\alpha} = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

Que 2.21. **For silver, $\mu = \mu_o$ and $\sigma = 3 \times 10^7$ mhos/m. Calculate the skin depth at 10^8 Hz frequency. Given, $\mu_o = 4\pi \times 10^7$ N/A^2.**

Answer

Given : $\mu = \mu_0 = 4\pi \times 10^{-7}$ N/m^2, $\sigma = 3 \times 10^7$ mhos/m, $f = 10^8$ Hz
To Find : Skin depth.

1. Since silver is a good conductor, therefore, the skin depth is given by,

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{(2\pi f)\,\mu_0\sigma}}$$

$$\delta = \sqrt{\frac{2}{(2\pi \times 10^8) \times 4\pi \times 10^{-7} \times 3 \times 10^7}}$$

$$\delta = 9.19 \times 10^{-6}\ \text{m}$$

☺☺☺

2. According to Einstein's mass-energy relation

Energy, $U = mc^2$ or $m = \frac{U}{c^2}$

$\therefore$ $\vec{p} = \frac{U}{c^2}\vec{v}$ ...(2.17.2)

3. The energy density in plane electromagnetic wave in free space is given by

$u = \varepsilon_0 E^2$ ...(2.17.3)

where E is the magnitude of electric field.

4. Thus, the momentum density or momentum per unit volume associated with an electromagnetic wave is

$\vec{p} = \frac{u}{c^2}\vec{v}$ ...(2.17.4)

5. If the electromagnetic waves are propagating along X-axis, then

$\vec{v} = c\hat{i}$

$\therefore$ $\vec{p} = \frac{u}{c}\hat{i}$ ...(2.17.5)

6. The Poynting vector

$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

$\therefore$ $\vec{S} = \frac{E^2}{\mu_0 c}\hat{i}$ $\left[\because \vec{E} \times \vec{B} = \frac{E^2}{c}\hat{i}\right]$ ...(2.17.6)

7. Substituting the value of E^2 from eq. (2.17.3) in eq. (2.17.6), we get

$\vec{S} = \frac{u}{\varepsilon_0 c \mu_0}\hat{i} = uc\,\hat{i}$ $\left(\because c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$

or $u\hat{i} = \frac{\vec{S}}{c}$ ...(2.17.7)

8. Putting the value of $u\hat{i}$ from eq. (2.17.7) in eq. (2.17.5), we get

$\vec{p} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2}(\vec{E} \times \vec{B})$

or $\vec{p} = \varepsilon_0(\vec{E} \times \vec{B})$ $\left(\because \frac{1}{\mu_0 c^2} = \varepsilon_0\right)$ ...(2.17.8)

9. Eq. (2.17.8) represents momentum per unit volume for an electromagnetic wave. The value of this momentum is,

$p = \frac{u}{c}$ or $u = pc$

i.e., Energy density = wave momentum × wave velocity.

Que 2.18. **Define radiation pressure. Derive the relation between radiation pressure and energy density.**

Answer

A. Radiation Pressure :

1. When electromagnetic wave strikes a surface, its momentum changes. The rate of change of momentum is equal to the force. This force acting on the unit area of the surface exerts a pressure, called radiation pressure.

B. Relation between Radiation Pressure and Energy Density :

1. Let a plane electromagnetic wave incident normally on a perfectly absorbing surface of area A for a time t.
2. If energy U is absorbed during this time, the momentum p delivered to the surface is given, according to Maxwell's prediction, by

$$p = \frac{U}{c}$$

3. If S is the energy passing per unit area per unit time, then

$$U = SA\,t$$

$\therefore$ $$p = \frac{S\,At}{c}$$

where S is the magnitude of Poynting vector.

4. But $\frac{S}{c} = u$ (energy density)

$\therefore$ $$p = uAt$$

5. From Newton's law average force F acting on the surface is equal to the average rate at which momentum is delivered to the surface. Therefore,

$$F = \frac{p}{t} = uA$$

6. The radiation pressure p_{rad} exerted on the surface.

$$p_{rad} = \frac{F}{A} = u$$

Hence, the radiation pressure exerted by a normally incident plane electromagnetic wave on a perfect absorber is equal to the energy density in the wave.

7. For a perfect reflector or for a perfectly reflecting surface, the radiation after reflection has a momentum equal in magnitude but opposite in direction to the incident radiation. The momentum imparted to the surface will therefore be twice as on perfect absorber. That is,

$$p_{rad} = 2u$$

Que 2.19. **Discuss depth of penetration or skin depth.**

Answer

1. When the electromagnetic waves travel in free space, the electrostatic energy density u_e and magnetostatic energy density u_m are given by

$$u_e = \frac{1}{2} \varepsilon_0 E^2$$

$$u_m = \frac{1}{2} \mu_0 H^2$$

2. Total energy density of EM-wave is given by

$$u = u_e + u_m = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 H^2$$

3. But for a plane electromagnetic wave in free space

$$\frac{E}{\mu} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \text{ or } H = \sqrt{\frac{\varepsilon_0}{\mu_0}} E$$

4. So,

$$u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 \cdot \frac{\varepsilon_0}{\mu_0} E^2$$

$$u = \varepsilon_0 E^2$$

This is the total electromagnetic energy density.

$$\left.\begin{aligned} \text{div } \vec{D} &= \rho \\ \text{div } \vec{B} &= 0 \\ \text{curl } \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{curl } \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\} \quad ...(2.14.1)$$

2. For the propagation of EM-waves in free space (or vacuum), we have

$$\sigma = 0, \rho = 0, \varepsilon_r = 1, \mu_r = 1$$

i.e., $\vec{J} = \sigma\vec{E} = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0$ and $\mu = \mu_0 \mu_r = \mu_0$

3. Thus, the Maxwell's relations for the propagation of EM-wave in free space (or vacuum) are given by

$$\left.\begin{aligned} \text{div } \vec{D} &= 0 \\ \text{div } \vec{B} &= 0 \\ \text{curl } \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{curl } \vec{H} &= \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\} \quad ...(2.14.2)$$

4. For free space, $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$, so we have

$$\left.\begin{aligned} \text{div } \vec{E} &= 0 \\ \text{div } \vec{H} &= 0 \\ \text{curl } \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\ \text{curl } \vec{H} &= \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \quad ...(2.14.3)$$

5. Taking curl on both sides of the third relation in eq. (2.14.3), we have

$$\text{curl (curl } \vec{E}) = \text{curl}\left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right)$$

$$\text{grad (div } \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \text{ curl}\left(\frac{\partial \vec{H}}{\partial t}\right)$$

$$[\because \text{ grad (div } \vec{A}) - \nabla^2 \vec{A} = \text{curl (curl } \vec{A})]$$

$$-\nabla^2 \vec{E} = -\mu_0 \left(\frac{\partial}{\partial t}\right)(\text{curl } \vec{H}) \quad [\because \text{ div } \vec{E} = 0]$$

$$-\nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \quad \left[\because \text{ curl } \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$$

$$-\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 E}{\partial t^2}\right) \quad ...(2.14.4)$$

6. Now taking curl on both sides of fourth relation in eq. (2.14.3), we can show that

$$\nabla^2 \vec{H} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2}\right) \quad ...(2.14.5)$$

7. Comparing eq. (2.14.4) and eq. (2.14.5) with the general wave eq.,

$$\nabla^2 \vec{\psi} = \frac{1}{v^2}\left(\frac{\partial^2 \psi}{\partial t^2}\right), \text{ v is the wave velocity}$$

we get $\qquad \varepsilon_0 \mu_0 = \dfrac{1}{v^2}$

i.e., $\qquad v = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} = \dfrac{1}{\sqrt{(4\pi \varepsilon_0)\left(\dfrac{\mu_0}{4\pi}\right)}}$

$$v = \frac{1}{\sqrt{\left(\dfrac{1}{9 \times 10^9}\right)(10^{-7})}} = \frac{1}{\sqrt{\left(\dfrac{1}{9 \times 10^{16}}\right)}}$$

$(\because \; 1/4\pi\varepsilon_0 = 9 \times 10^9 \text{ N-m}^2/\text{C}^2$ and $\mu_0/4\pi = 10^{-7}$ Wb/A-m)

$$v = \sqrt{9 \times 10^{16}} = 3 \times 10^8 \text{ m/s} = c \text{ (speed of light)}$$

This shows that EM-wave travel with the speed of light in free space (or vacuum).

Que 2.15. **Show that *E*, *H* and direction of propagation form a set of orthogonal vectors.**

Answer

1. Considering *EM*-waves as plane waves, the electric and magnetic vectors $\vec{E}$ and $\vec{H}$ can be expressed as

$$\vec{E} = \vec{E}_0 \exp [i(\vec{K} \cdot \vec{r} - \omega t)]$$

$$\vec{H} = \vec{H}_0 \exp [i(\vec{K} \cdot \vec{r} - \omega t)]$$

2. These two equations represent the plane-wave solutions of Maxwell's relations for the propagation of *EM*-waves in free space (or vacuum). In this condition, the del operator (∇) and partial time derivative operator $\left(\dfrac{\partial}{\partial t}\right)$ can be expressed as

12. Also, the EM-energy density is given by

$$u_{em} = \frac{\mu_0}{2} H^2 + \frac{\varepsilon_0}{2} E^2 \quad ...(2.10.10)$$

13. Using the eq. (2.10.9) and eq. (2.10.10) in eq. (2.10.8), we obtain

$$\frac{d}{dt}\int_V u_M \, dV = \frac{-d}{dt}\int_V u_{em} \, dV - \int_V \text{div}\,(\vec{S})\, dV$$

or $$\int_V \text{div}\,(\vec{S})\, dV = -\frac{d}{dt}\int_V (u_M + u_{em})\, dV$$

or $$\int_V \text{div}\,(\vec{S})\, dV = \int_V \frac{\partial}{\partial t}(u_M + u_{em})\, dV$$

or $$\text{div}\,\vec{S} = -\frac{\partial}{\partial t}(u_M + u_{em})$$

C. Physical Significance :

The Poynting vector $\vec{S}$ describes the flow of energy in the same way as the current density vector $\vec{J}$ describes the flow of charge. Since the equation of continuity expresses the conservation of charge, the Poynting theorem represents the conservation of energy.

Que 2.11. **A 500 watt lamp radiates power uniformly in all directions. Calculate the electric and magnetic field intensities at 1 m distance from the lamp.**

Answer

Given : Energy of the lamp = 500 watt

To Find : i. Electric field intensity at 1 m distance from lamp.

ii. Magnetic field intensity at 1 m distance from lamp.

1. Area illuminated = $4\pi r^2 = 4\pi (1)^2 = 4\pi \text{ m}^2$
2. Therefore, Energy radiated per unit area per second = $\frac{500}{4\pi}$
3. Hence, from Poynting theorem

$$|\vec{S}| = |\vec{E} \times \vec{H}| = EH = \frac{500}{4\pi} \quad ...(2.11.1)$$

and $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \quad ...(2.11.2)$$

4. Multiplying eq. (2.11.1) and eq. (2.11.2), we get

$$E^2 = \frac{500}{4\pi} \times 377 = 15000.36$$

or $$E = 122.475 \text{ V/m}$$

and $$H = \frac{E}{377} = 0.325 \text{ A/m}$$

Que 2.12. **Calculate the magnitude of Poynting vector at the surface of the sun. Given that power radiated by the sun = 3.8×10^{26} Watts and radius of the sun = 7×10^8 m.**

Answer

Given : Power radiated by the sun, Power = 3.8×10^{26} Watts

Radius of the sun, $r = 7 \times 10^8$ m

To Find : Magnitude of Poynting vector at the surface of the sun.

1. The Poynting vector is given by :

$$S = \frac{\text{Power}}{4\pi r^2} = \frac{3.8 \times 10^{26}}{4\pi(7 \times 10^8)^2} = \frac{3.8 \times 10^{26}}{1.96\pi \times 10^{18}}$$
$$= 6.171 \times 10^7 \text{ W/m}^2$$

Que 2.13. **Define EM waves. State a few properties of electromagnetic waves.**

Answer

1. EM waves are coupled electric and magnetic oscillations that move with speed of light and exhibit typical wave behaviour.
2. The properties of electromagnetic waves are as follows :
   i. In free space or vacuum, the EM wave travel with speed of light.
   ii. The electrostatic energy density is equal to the magnetic energy density.
   iii. These waves carry both energy and momentum, which can be delivered to a surface.
   iv. EM waves are transverse in nature.
   v. Electromagnetic waves of different frequencies can exist.

Que 2.14. **Derive electromagnetic wave equation in free space.**

OR

Using Maxwell's equations, derive electromagnetic wave equations in vacuum and prove that wave propagate with speed of light.

Answer

1. Maxwell's relations are given by,

$$\overline{E}.\text{curl}\,\overline{H} = \overline{E}\cdot\left[J + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right]$$

or $$\overline{E}.\text{curl}\,\overline{H} = \overline{E}\cdot\overline{J} + \varepsilon_0\,\overline{E}\cdot\frac{\partial \overline{E}}{\partial t}$$

or $$\overline{E}\cdot\overline{J} = \overline{E}\cdot\text{curl}\,\overline{H} - \varepsilon_0\,\overline{E}\cdot\frac{\partial \overline{E}}{\partial t}$$

or $$\overline{E}\cdot\overline{J} = \overline{H}\,.\,\text{curl}\,\overline{E} - \text{div}\,(\overline{E}\times\overline{H}) - \varepsilon_0\,\overline{E}\cdot\frac{\partial \overline{E}}{\partial t} \quad ...(2.10.5)$$

$$[\because\ \text{div}\,(\vec{A}\times\vec{B}) = \vec{B}.\,\text{curl}\,\vec{A} - \vec{A}\,.\,\text{curl}\,\vec{B}]$$

8. Maxwell's third relation is

$$\text{curl}\ \vec{E} = -\frac{\partial \overline{B}}{\partial t}$$

or $$\text{curl}\ \vec{E} = -\mu_0\frac{\partial \overline{H}}{\partial t} \quad ...(2.10.6)$$

9. Using eq. (2.10.6) in eq. (2.10.5), we have

$$\overline{E}\,.\overline{J} = \overline{H}\left(-\mu_0\frac{\partial \overline{H}}{\partial t}\right) - \text{div}\,(\overline{E}\times\overline{H}) - \varepsilon_0\,\overline{E}\cdot\frac{\partial \overline{E}}{\partial t}$$

or $$\overline{E}\,.\overline{J} = -\mu_0\overline{H}\frac{\partial \overline{H}}{\partial t} - \varepsilon_0\overline{E}\cdot\frac{\partial \overline{E}}{\partial t} - \text{div}\,(\overline{E}\times\overline{H})$$

or $$\overline{E}\,.\overline{J} = -\frac{\partial}{\partial t}\left(\frac{\mu_0}{2}H^2 + \frac{\varepsilon_0}{2}E^2\right) - \text{div}\,(\overline{E}\times\overline{H})$$

or $$\overline{E}\,.\overline{J} = -\frac{\partial}{\partial t}\left(\frac{\mu_0}{2}H^2 + \frac{\varepsilon_0}{2}E^2\right) - \text{div}\,(\vec{S}) \quad ...(2.10.7)$$

where $\vec{S} = (\vec{E}\times\vec{H})$ is the Poynting vector.

10 Using eq. (2.10.7) in eq. (2.10.3), we have

$$\frac{dW}{dt} = \int_V \left[-\frac{\partial}{\partial t}\left(\frac{\mu_0}{2}H^2 + \frac{\varepsilon_0}{2}E^2\right) - \text{div}\,(\vec{S})\right]dV$$

or $$\frac{dW}{dt} = \frac{-d}{dt}\int_V\left(\frac{\mu_0}{2}H^2 + \frac{\varepsilon_0}{2}E^2\right)dV - \int_V \text{div}\,(\vec{S})\,dV \quad ...(2.10.8)$$

11. Since, dW/dt represents the power density that is transferred into EM-field and increases the mechanical energy (kinetic, potential or whatever) of the charges, therefore, if u_M denotes the mechanical energy, then we have

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_M\,dV \quad ...(2.10.9)$$

PART-2

Energy in an Electromagnetic Field, Poynting Vector and Poynting Theorem, Plane Electromagnetic Waves in Vacuum and their Transverse Nature, Relation Between Electric and Magnetic Fields of an Electromagnetic Waves.

CONCEPT OUTLINE : PART-2

Poynting Vector : The magnitude and the direction of flow of energy per unit area per unit time in an EM-wave travelling in free space (or vacuum) can be expressed by a vector known as Poynting vector $\vec{S}$.

Electromagnetic Waves : Electromagnetic waves are coupled electric and magnetic oscillations that move with speed of light and exhibit typical wave behaviour.

Energy Density of Electromagnetic Wave : The total electromagnetic energy density

$$u = \varepsilon_0 E^2$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.9. **What is Poynting vector ?**

Answer

A. Poynting Vector :

1. The magnitude and the direction of flow of energy per unit area per unit time in an EM- wave travelling in free space (or vacuum) can be expressed by a vector known as Poynting vector $\vec{S}$.

B. Expression for Poynting Vector :

1. Let us consider a small volume element $dV = A\,dx$, where A is its cross sectional area and dx its length along X-axis.
2. If u is the EM-energy density of the EM-waves, then

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 \quad ...(2.9.1)$$

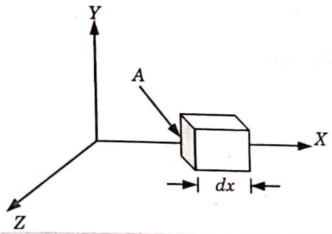


Fig. 2.9.1. Illustration of concept of poynting vector.

3. Hence, the energy associated with volume element dV is given by

$$U = u\,dV = \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) A\,dx \quad ...(2.9.2)$$

4. The relation between the magnitudes of field vectors $\vec{E}$ and $\vec{H}$ is

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 C = \frac{1}{\varepsilon_0 C} \quad ...(2.9.3)$$

5. Using eq. (2.9.3) and eq. (2.9.2), we have

$$U = \left(\frac{1}{2}\varepsilon_0 E \frac{H}{\varepsilon_0 C} + \frac{1}{2}\mu_0 H \frac{E}{\mu_0 C}\right) A\,dx$$

or $$U = \left(\frac{1}{2C} + \frac{1}{2C}\right) EHA\,dx$$

or $$U = \frac{EHA\,dx}{(dx/dt)} \qquad \left[\because\ C = \frac{dx}{dt}\right]$$

or $$U = EHAdt \quad ...(2.9.4)$$

6. Hence, by definition, the EM-energy passing through per unit area per unit time with EM-wave, *i.e.*, magnitude of Poynting vector is

$$S = \frac{U}{Adt} = \frac{EHA\,dt}{A\,dt} = EH$$

7. In vector form, this expression can be expressed as

$$\vec{S} = \vec{E} \times \vec{H}$$

Que 2.10. **Derive Poynting theorem and explain its physical significance.**

Que 2.7. **Explain the concept of displacement current and show how it led the modification of Ampere's law.**

Answer

1. Let us consider an electric circuit consisting of a battery B, resistance R, key K and a capacitor C in series as shown in Fig. 2.7.1.

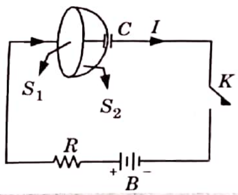


Fig. 2.7.1. Electric circuit showing the concept of displacement current.

2. When we close the circuit by pressing the key K, the charging of the capacitor starts.
3. Consider a circular surface S_1 and a semispherical surface S_2 such that both are bound by the same closed path. During charging, there is no actual flow of charge between the plates of the capacitor.
4. Hence, the current flows through surface S_1 but not through S_2.
5. Applying Ampere's law for the surface S_1, we have
$$\oint \vec{B}.d\vec{l} = \mu_0 I \quad ...(2.7.1)$$
Applying Ampere's law for the surface S_2, we have
$$\oint \vec{B}.d\vec{l} = 0 \quad ...(2.7.2)$$
6. Since the results of eq. (2.7.1) and eq. (2.7.2) contradict each other, these equations cannot be corrected.
7. Maxwell tried to improve the contradiction between the two equations by adding an additional term $\mu_0 I_D$ on the right-hand side of eq. (2.7.1).
8. Thus, the modified Amperes law can be expressed as
$$\oint \vec{B}.d\vec{l} = \mu_0 (I + I_D) \quad ...(2.7.3)$$
where I_D is known as displacement current.
9. The displacement current is given by
$$I_D = A\, J_D \quad ...(2.7.4)$$
where A is the area of the plates of capacitor and J_D is the displacement current density.
10. According to Maxwell's fourth relation, the displacement current density is given by

$$J_D = \frac{\partial D}{\partial t} \quad ...(2.7.5)$$

11. From eq. (2.7.4) and eq. (2.7.5), we have
$$I_D = A\frac{\partial D}{\partial t}$$
or
$$I_D = A\frac{\partial(\varepsilon_0 E)}{\partial t} \quad [\because D = \varepsilon_0 E]$$
or
$$I_D = A\varepsilon_0 \frac{\partial(E)}{\partial t} \quad ...(2.7.6)$$
12. If σ is the surface charge density of the plates of capacitor and q is the charge on each plate, then we know that
$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0} \quad [\because \sigma = (q/A)]$$
13. Using this relation in eq. (2.7.6), we get
$$I_D = A\varepsilon_0 \frac{\partial}{\partial t}\left(\frac{q}{A\varepsilon_0}\right)$$
i.e.,
$$I_D = \frac{\partial q}{\partial t} = I$$
14. This shows that the displacement current in the space between the plates of capacitor during charging is equal to the conduction current.
15. Thus, the concept of displacement current needs a modification to Ampere's law.

Que 2.8. **Explain the characteristics of displacement current.**

Answer

1. The displacement current is the current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not linked with the motion of charges.
2. The magnitude of displacement current is equal to the rate of change of magnitude of electric displacement vector, *i.e.*, $J_D = (\partial D/\partial t)$.
3. It serves the purpose to make the total current continuous across the discontinuity in conduction current. As an example, a battery charging a capacitor produces a closed current loop in terms of total current $I_{total} = I + I_D$.
4. The displacement current in a good conductor is negligible compared to the conduction current at any frequency less than the optical frequencies ($\approx 10^{15}$ Hz).

5. In free space, surface current density, $\vec{J} = 0$

So, $$\oint_C \vec{H}.d\vec{l} = \int_S \vec{J}_D\, d\vec{s}$$

or $$\oint_C \vec{H}.d\vec{l} = \int_S \frac{\partial \vec{D}}{\partial t}.d\vec{s}$$

Que 2.5. **Given the physical significance of Maxwell's equations.**

Answer

A. Physical Significance of Maxwell's First Equation :

1. It signifies that the total flux of electric displacement through a closed surface enclosing a volume is equal to the net charge $q \left(= \int_V \rho\, dV \right)$ contained within that volume.

B. Physical Significance of Maxwell's Second Equation :

1. It signifies that the net outward flux of magnetic induction through a surface enclosing a volume is equal to zero.
2. This shows the non-existence of monopoles in nature.

C. Physical Significance of Maxwell's Third Equation :

1. It signifies that the emf $\left(= \oint_C \vec{E}.d\vec{l} \right)$ induced around a closed path is equal to the negative rate of change of magnetic flux $\left(= -\int_S \vec{B}.d\vec{s} \right)$ linked with that closed path.

D. Physical Significance of Maxwell's Fourth Equation :

1. It signifies that the mmf $\left(= \oint_C \vec{H}.d\vec{l} \right)$ around a closed path is equal to the sum of the conduction current and displacement current linked with that closed path.

Que 2.6. **Using Maxwell equation Curl $B = \mu_0 \left[J + \frac{\partial D}{\partial t} \right]$, prove that div $D = \rho$.**

Answer

1. The Maxwell's equation is given by :
$$\text{Curl } \vec{B} = \mu_0 \left\{ \vec{J} + \frac{\partial \vec{D}}{\partial t} \right\}$$
But we know, $$\vec{B} = \mu_0 \vec{H}$$
So, $$\text{Curl } H = \left\{ \vec{J} + \frac{\partial \vec{D}}{\partial t} \right\} \quad \text{...(2.6.1)}$$
2. Taking the divergence of eq. (2.6.1), we obtain
$$\text{div(Curl } H) = \text{div} \left\{ \vec{J} + \frac{\partial \vec{D}}{\partial t} \right\} = 0$$
3. Since the div of any vector quantity is zero *i.e.*,
$$\text{div (Curl } H) = 0$$
Therefore, $$\text{div} \left\{ \vec{J} + \frac{\partial \vec{D}}{\partial t} \right\} = 0$$
$$\text{div } \vec{J} + \text{div} \left\{ \frac{\partial \vec{D}}{\partial t} \right\} = 0$$
$$\text{div } \vec{J} + \left\{ \frac{\partial\, \text{div } \vec{D}}{\partial t} \right\} = 0 \quad \text{...(2.6.2)}$$
4. From continuity equation, we have $\text{div } \vec{J} + \frac{\partial \rho}{\partial t} = 0$

i.e., $$\text{div } \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{...(2.6.3)}$$
5. Using eq. (2.6.3) in eq. (2.6.2), we obtain
$$-\frac{\partial \rho}{\partial t} + \frac{\partial\, \text{div } \vec{D}}{\partial t} = 0$$
$$\frac{\partial \rho}{\partial t} = \frac{\partial\, \text{div } \vec{D}}{\partial t}$$
$$\text{div } \vec{D} = \rho$$

This is the Maxwell's fourth relation or equation.

11. From this relation, it is clear that the displacement current density relates the electric field vector $\vec{E}$ as ($\vec{D} = \varepsilon \vec{E}$) to the magnetic field vector $\vec{H}$.

12. In free space, surface current density $\vec{J} = 0$

So,
$$\text{curl}\,(\vec{H}) = \frac{\partial \vec{D}}{\partial t}$$

Que 2.4. **Derive the Maxwell's equation in integral form.**

Answer

A. Maxwell's First Relation in Integral Form :

1. The Maxwell's first relation is
$$\text{div}\,\vec{D} = \rho \qquad \text{...(2.4.1)}$$
2. Integrating this over an arbitrary volume V bounded by a closed surface S, we have
$$\int_V (\text{div}\,\vec{D})\,dV = \int_V \rho\,dV \qquad \text{...(2.4.2)}$$
3. Using Gauss' divergence theorem on the left hand side of eq. (2.4.2), we get
$$\int_S \vec{D}.d\vec{s} = \int_V \rho\,dV \qquad \left(\because \int_S \vec{A}.d\vec{s} = \int_V \text{div}\,\vec{A}\,dV\right)$$
where S is the surface which bounds the volume V.
4. In free space, volume charge density, $\rho = 0$.

So,
$$\int_S \vec{D}.d\vec{s} = 0$$
This is the integral form of the Maxwell's first relation.

B. Maxwell's Second Relation in Integral Form :

1. The Maxwell's second relation is
$$\text{div}\,\vec{B} = 0 \qquad \text{...(2.4.3)}$$
2. Integrating this over an arbitrary volume V bounded by a closed surface S, we have
$$\int_V \text{div}\,\vec{B}\,dV = 0 \qquad \text{...(2.4.4)}$$
3. Using Gauss' divergence theorem on the left hand side of eq. (2.4.4), we get

$$\int_S \vec{B}\,d\vec{s} = 0 \qquad \left(\because \int_S \vec{A}.d\vec{s} = \int_V \text{div}\,\vec{A}\,dV\right) \qquad \text{...(2.4.5)}$$
Where S is the surface which bounds the volume V.
4. This is the integral form of Maxwell's second relation.

C. Maxwell's Third Relation in Integral Form :

1. The third Maxwell's relation is
$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{...(2.4.6)}$$
2. Integrating this over an arbitrary surface S bounded by a closed loop C, we have
$$\int_S \text{curl}\,\vec{E}.d\vec{s} = -\int_S \frac{\partial}{\partial t}(\vec{B}.d\vec{s}) \qquad \text{...(2.4.7)}$$
3. Using Stoke's curl theorem on left hand side of eq. (2.4.7), we get
$$\int_C \vec{E}.d\vec{l} = -\int_S \frac{\partial}{\partial t}(\vec{B}.d\vec{s}) \qquad \left(\because \int_C \vec{A}.d\vec{l} = \int_S \text{curl}\,\vec{A}\cdot d\vec{s}\right)$$
i.e.,
$$\int_C \vec{E}.d\vec{l} = \frac{-d}{dt}\left(\int_S \vec{B}.d\vec{s}\right)$$
Here C is the closed loop which bounds surface S.
4. This is the integral form of Maxwell's third relation.

D. Maxwell's Fourth Relation in Integral Form :

1. The Maxwell's fourth relation is
$$\text{curl}\,(\vec{H}) = J + \frac{\partial \vec{D}}{\partial t} \qquad \text{...(2.4.8)}$$
2. Integrating this over an arbitrary surface S bounded by a closed loop C, we have
$$\int_S \text{curl}\,\vec{H}.d\vec{s} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s} \qquad \text{...(2.4.9)}$$
3. Using Stoke's curl theorem on left hand side of eq. (2.4.9), we get
$$\int_C \vec{H}.d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s} \qquad \left(\because \int_C \vec{A}.d\vec{l} = \int_S \text{curl}\,\vec{A}\cdot d\vec{s}\right)$$
or
$$\int_C \vec{H}.d\vec{l} = \int_S (\vec{J} + \vec{J}_D)\cdot d\vec{s}$$
Here, C is the closed loop which bounds surface S.
4. This is the integral form of Maxwell's fourth relation.

$$\int_S \left[\text{curl}\,\vec{E}\,.\,d\vec{s} + \frac{\partial}{\partial t}(\vec{B}\,.\,d\vec{s})\right] = 0$$

$$\int_S \left[\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right]\cdot(d\vec{s}) = 0 \qquad ...(2.3.16)$$

7. Since, S is an arbitrary surface, eq. (2.3.16) holds only if its integrand is zero, *i.e.*,

$$\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

or
$$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is the Maxwell's third relation.

D. Derivation of Maxwell's Fourth Relation :

1. According to Ampere's circuit law :

"The line-integral of magnetic induction vector $\vec{B}$ around a closed current carrying loop is equal to μ_0 times the current flowing in the loop".

2. Thus,
$$\int_C \vec{B}\,.\,d\vec{l} = \mu_0 I \qquad ...(2.3.17)$$

3. Since, $\vec{B} = \mu_0 \vec{H}$, we have
$$\int_C \mu_0\,\vec{H}\,.\,d\vec{l} = \mu_0 I$$

or
$$\int_C \vec{H}\,.\,d\vec{l} = I \qquad ...(2.3.18)$$

4. Let us consider a small surface elements $d\vec{s}$ of the surface S bounded by the closed loop C. If $\vec{J}$ be the surface current density of the loop, then the current flowing in the closed loop can be expressed as
$$I = \int_S \vec{J}\,.\,d\vec{s} \qquad ...(2.3.19)$$

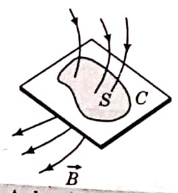


Fig. 2.3.4. A closed current-carrying loop.

5. Substituting the value of I from eq. (2.3.19) in eq. (2.3.18), we have
$$\int_C \vec{H}\,.\,d\vec{l} = \int_S \vec{J}\,.\,d\vec{s} \qquad ...(2.3.20)$$

6. Using Stokes curl theorem on left hand side of eq. (2.3.20), we get
$$\int_S \text{curl}\,\vec{H}\,.\,d\vec{s} = \int_S \vec{J}\,.\,d\vec{s} \qquad \left(\because \int_C \vec{A}\,.\,d\vec{l} = \int_S \text{curl}\,\vec{A}\cdot d\vec{s}\right)$$

or
$$\text{curl}\,\vec{H} = \vec{J} \qquad ...(2.3.21)$$

7. Using eq. (2.3.21) in the equation of continuity, we get
$$\text{div}\,(\text{curl}\,\vec{H}\,) + \frac{\partial \rho}{\partial t} = 0 \qquad \left(\because \text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0\right)$$

i.e.,
$$\frac{\partial \rho}{\partial t} = 0 \qquad [\because \text{div}\,(\text{curl}\,\vec{H}\,) = 0] \;...(2.3.22)$$

8. Eq. (2.3.22) is applicable for the steady-state conditions in which the charge density is not changing with time. This shows that for time varying fields, Ampere's law should be modified. For this, Maxwell suggested that eq. (2.3.21) should be modified as follows :
$$\text{curl}\,\vec{H} = \vec{J} + \vec{J}_D \qquad ...(2.3.23)$$

where $\vec{J}_D$ is known as the displacement current density.

9. Taking divergence on both sides of eq. (2.3.23), we get
$$\text{div}\,(\text{curl}\,\vec{H}\,) = \text{div}\,(\vec{J} + \vec{J}_D)$$

or
$$0 = \text{div}\,(\vec{J} + \vec{J}_D) \qquad [\because \text{div}\,(\text{curl}\,\vec{H}\,) = 0\,]$$

or
$$\text{div}\,\vec{J} = -\,\text{div}\,\vec{J}_D$$

or
$$\frac{\partial \rho}{\partial t} = \text{div}\,\vec{J}_D \qquad \left[\because \text{div}\,(\vec{J}) + \frac{\partial \rho}{\partial t} = 0\right]$$

or
$$\frac{\partial (\text{div}\,\vec{D})}{\partial t} = \text{div}\,\vec{J}_D$$

or
$$\text{div}\left(\frac{\partial \vec{D}}{\partial t}\right) = \text{div}\,\vec{J}_D$$

or
$$\frac{\partial \vec{D}}{\partial t} = \vec{J}_D \qquad ...(2.3.24)$$

10. From eq. (2.3.23) and eq. (2.3.24), we get
$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\text{div } \vec{B} = 0$$

This is the Maxwell's second relation or equation.

6. This relation states that there are no magnetic monopoles in the world.

C. Derivation of Maxwell's Third Relation :

1. According to Faraday's law of electromagnetic induction :
"The induced emf (electromotive force) produced in a current carrying coil is equal to the negative time-rate of magnetic flux Φ_M associated with the coil".

Thus,
$$e = -\frac{d}{dt}(\Phi_M) \quad \text{...(2.3.11)}$$

2. If $\vec{E}$ is the strength of the electric field corresponding to the induced emf e, then the induced emf can be expressed as line-integral of $\vec{E}$ around the coil Fig. 2.3.3, *i.e.*,
$$e = \int_C \vec{E} \cdot d\vec{l} \quad \text{...(2.3.12)}$$

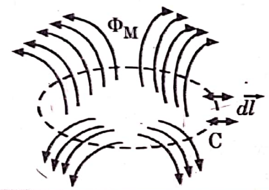


Fig. 2.3.3. Induced emf in a current carrying coil.

3. Comparing eq. (2.3.11) and eq. (2.3.12), we have
$$\int_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}(\Phi_M) \quad \text{...(2.3.13)}$$

4. The magnetic flux can be expressed as
$$\Phi_M = \int_S \vec{B} \cdot d\vec{s} \quad \text{...(2.3.14)}$$

5. Substituting the value of Φ_M in eq. (2.3.13), we get
$$\int_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_C \vec{B} \cdot d\vec{s}\right)$$

or
$$\int_C \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial}{\partial t}(\vec{B} \cdot d\vec{s}) \quad \text{...(2.3.15)}$$

6. Using the Stokes curl theorem on the left hand side of eq. (2.3.15), we get
$$\int_S \text{curl } \vec{E} \cdot d\vec{s} = -\int_S \frac{\partial}{\partial t}(\vec{B} \cdot d\vec{s}) \qquad \left(\because \int_C \vec{A} \cdot d\vec{l} = \int_S \text{curl } \vec{A} \cdot d\vec{s}\right)$$

Answer

1. The expression for relativistic K.E. is

$$K = (m - m_0)c^2 = \left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\right)c^2 = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

2. Expanding using binomial theorem,

$$K = m_0c^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots - 1\right]$$

3. Since, $v << c$ *i.e.*, $v/c << 1$, so, higher terms may be neglected.

Thus, $K = m_0c^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right] \approx \frac{1}{2}m_0 v^2$ .... (Classical K.E.)

4. Therefore, if $v << c$ then relativistic K.E. will convert into classical K.E.

Que 1.30. **Calculate the workdone to increase speed of an electron of rest energy 0.5 MeV from 0.6c to 0.8c.** **AKTU 2014-15, Marks 05**

Answer

Given : Initial velocity = 0.6c, Final velocity = 0.8c, m_oc^2 = 0.5 MeV.
To Find : Amount of workdone.

1. K.E. of electron, $K = m - m_oc^2 = \left[\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} - m_o\right]c^2$

$$K = m_oc^2\left[\left\{1-\left(\frac{v}{c}\right)^2\right\}^{-\frac{1}{2}} - 1\right]$$

2. Now initial kinetic energy,

$$K_1 = m_oc^2\left[\left\{1-\left(\frac{0.6c}{c}\right)^2\right\}^{-\frac{1}{2}} - 1\right] = 0.25\, m_oc^2$$

$\Rightarrow$ $K_1 = 0.25 \times 0.5 \times 10^6$ eV $= 1.25 \times 10^5$ eV

3. Final K.E., $K_2 = m_oc^2\left[\left\{1-\left(\frac{0.8c}{c}\right)^2\right\}^{-\frac{1}{2}} - 1\right]$

$K_2 = 0.666 \times 0.5 \times 10^6$ eV $= 3.33 \times 10^5$ eV

Amount of work $= K_2 - K_1$
$= 3.33 \times 10^5 - 1.25 \times 10^5 = 2.08 \times 10^5$ eV

$= 2.08 \times 10^5 \times 1.6 \times 10^{-19}$ J
$= 3.328 \times 10^{-14}$ J [$\because$ 1 eV $= 1.6 \times 10^{-19}$ J]

Que 1.31. **A charged particle shows an acceleration of 4.2×10^{12} cm/s² under an electric field at low speed. Compute the acceleration of the particle under the same field when the speed has reached a value 2.88×10^{10} cm/s. The speed of light is 3×10^{10} cm/s.**

Answer

Given : $a_0 = 4.2 \times 10^{12}$ cm/s², $v = 2.88 \times 10^{10}$ cm/s, $c = 3 \times 10^{10}$ cm/s.
To Find : Acceleration of particle.

1. At low speed $v << c$, the effective mass $m = m_o$
2. Acceleration at low speed,

$$a_0 = \frac{F}{m_o} = 4.2 \times 10^{12} \text{ cm/ s}^2$$

3. Now, $m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o}{\sqrt{1-\left(\frac{2.88\times 10^{10}}{3\times 10^{10}}\right)^2}} = \frac{m_o}{0.28}$

4. Now, acceleration at high speed,

$$a = \frac{F}{m} = \frac{F}{m_o/0.28} = \frac{0.28F}{m_o}$$

$$a = 0.28 \times 4.2 \times 10^{12} = 1.176 \times 10^{12} \text{ cm/s}^2$$

Que 1.32. **If the kinetic energy of a body is twice its rest mass energy, find its velocity.** **AKTU 2015-16, Marks 05**

Answer

Given : K.E. $= 2\, m_oc^2$
To Find : Velocity of body.

1. Kinetic energy, K.E. = 2 × rest mass energy

$$\frac{1}{2}mv^2 = 2 \times m_oc^2$$

2. We know that, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\Rightarrow \quad P^2 = 2Km_0 + \frac{K^2}{c^2}$$

$$\Rightarrow \quad P = \sqrt{\frac{K^2}{c^2} + 2m_0 K}$$

Que 1.26. **Show that the relativistic form of Newton's second law, when $\bar{F}$ is parallel to $\bar{v}$ is**

$$\bar{F} = m_o \frac{d\bar{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}$$

Answer

1. Newton's second law,

$$\bar{F} = \frac{d\bar{P}}{dt} = \frac{d}{dt}(m\bar{v})$$

2. But $m = \dfrac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

So, $$\bar{F} = \frac{d}{dt}\left[\frac{m_o \bar{v}}{\sqrt{1 - \dfrac{v^2}{c^2}}}\right]$$

$$\bar{F} = m_o\left[\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}\frac{d\bar{v}}{dt} + \frac{\bar{v}\left(-\dfrac{1}{2}\right)\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}\left(\dfrac{-2\bar{v}}{c^2}\right)}{\left(1 - \dfrac{v^2}{c^2}\right)}\frac{d\bar{v}}{dt}\right]$$

$$= m_o\left[\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}\frac{d\bar{v}}{dt} + \frac{\dfrac{v^2}{c^2}}{\left(1 - \dfrac{v^2}{c^2}\right)^{3/2}}\frac{d\bar{v}}{dt}\right]$$

$$\bar{F} = m_o\frac{d\bar{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = m_o\frac{d\bar{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}$$

Que 1.27. **The mass of a moving electron is 11 times its rest mass. Find its kinetic energy and momentum.** **AKTU 2011-12, Marks 05**

Answer

Given : $m = 11\, m_0$

To Find : i. Kinetic energy.
ii. Momentum

1. Since, $m = \dfrac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

So $$11\, m_o = \frac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{11} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{121}$$

$$\frac{v}{c} = 0.995$$

$$v = 0.995 \times 3 \times 10^8 = 2.985 \times 10^8 \text{ m/s}$$

2. Kinetic Energy, K.E. $= (m - m_0)\, c^2$
$= (11\, m_0 - m_0)\, c^2 = 10\, m_0 c^2$
$= 10 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2 = 8.19 \times 10^{-13}$ J

3. Momentum, $P = mv = 11\, m_0 v = 11 \times 9.1 \times 10^{-31} \times 2.985 \times 10^8$
$P = 2.987 \times 10^{-21}$ N-s

Que 1.28. **The total energy of a moving meson is exactly twice its rest energy. Find the speed of meson.** **AKTU 2012-13, Marks 05**

Answer

Given : $E = 2E_o$

To Find : Speed of meson.

1. As given
$$E = 2E_0$$
$$mc^2 = 2m_o c^2$$
or
$$m = 2m_o$$

2. Since, $$m = \frac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$2m_0 = \frac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

$$2 = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \Rightarrow v = 0.866\, c$$

$$= 0.866 \times 3 \times 10^8 = 2.59 \times 10^8 \text{ m/s}$$

Que 1.29. **Show that the relativistic K.E. will convert into classical K.E. if v << c.**

$$\Rightarrow \quad F = \frac{m\,d\text{v}}{dt} + \text{v}\frac{dm}{dt}$$

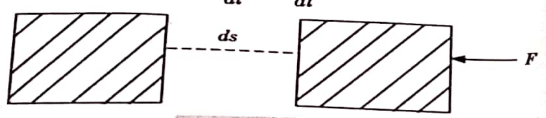


Fig. 1.24.1.

5. Multiplying '*ds*' on both sides, we get

$$\Rightarrow \quad F.ds = m\frac{ds}{dt}.d\text{v} + \text{v}\frac{ds}{dt}.dm$$

6. From eq. (1.24.1), $dK = m\,\text{v}\,d\text{v} + \text{v}^2\,dm$ ...(1.24.2) $\left(\because \frac{ds}{dt} = \text{v}\right)$

7. But we know that, $m = \dfrac{m_o}{\sqrt{1-\dfrac{\text{v}^2}{c^2}}}$ (m_0 = rest mass of particle)

On differentiating, we get

$$dm = m_0\left(-\frac{1}{2}\right)\left(1-\frac{\text{v}^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2\text{v}}{c^2}\right)d\text{v} = \frac{m_0\,\text{v}\,d\text{v}}{c^2\left(1-\dfrac{\text{v}^2}{c^2}\right)^{\frac{3}{2}}}$$

$$dm = \frac{m\,\text{v}\,d\text{v}}{c^2\left(1-\dfrac{\text{v}^2}{c^2}\right)} \qquad \left(\because \quad m = \frac{m_0}{\sqrt{1-\dfrac{\text{v}^2}{c^2}}}\right)$$

$$dm\,(c^2 - \text{v}^2) = m\text{v}\,d\text{v} \qquad ...(1.24.3)$$

8. Now putting the value of eq. (1.24.3) in eq. (1.24.2), we get

$$dK = (c^2 - \text{v}^2)\,dm + \text{v}^2\,dm = c^2\,dm$$

If the change in kinetic energy of a particle be K when it's mass changes from rest mass m_0 to relativistic mass m, then

$$\int_0^K dK = \int_{m_0}^{m} c^2\,dm$$

$$K = c^2(m - m_0) = c^2(\Delta m)$$

Total energy of particle,

$$E = \text{Relativistic K.E.} + \text{Rest mass energy}$$

$$E = (m - m_0)c^2 + m_0c^2 = mc^2$$

This is Einstein's mass energy-relation, which states mass energy equivalence.

Evidence of its Validity :

In nuclear reaction such as fission and fusion. These reactions take place in nuclear reactor and during the explosion of atom bomb. The cause of production of energy in stars and some other processes becomes known today only due to the discovery of this important mass energy relation.

2. In process of annihilation of matter, an electron and a positron give up all its mass into two photons. The entire mass is converted into energy. This verifies mass-energy relation.

Que 1.25. **Derive the relation**

a. $E^2 = P^2c^2 + m_0^2c^4$, **and**

b. $P = \sqrt{\dfrac{K^2}{c^2} + 2m_0K}$ **where, K is kinetic energy.**

Answer

a. **Derivation for $E^2 = P^2c^2 + m_0^2c^4$:**

1. Total energy of a particle is, $E = mc^2$ ...(1.25.1)

2. The relativistic mass, $m = \dfrac{m_0}{\sqrt{1-\dfrac{\text{v}^2}{c^2}}}$ ...(1.25.2)

3. Putting the value of eq. (1.25.2) in eq. (1.25.1),

$$E = \frac{m_0c^2}{\sqrt{1-\dfrac{\text{v}^2}{c^2}}} = \frac{m_0c^2}{\sqrt{1-\dfrac{m^2\text{v}^2}{m^2c^2}}} = \frac{m_0c^2}{\sqrt{1-\dfrac{m^2\text{v}^2c^2}{m^2c^4}}}$$

$$\Rightarrow \quad E = \frac{m_0c^2}{\sqrt{1-\dfrac{P^2c^2}{m^2c^4}}}$$

$$\Rightarrow \quad E^2 = \frac{m_0^2c^4}{1-\dfrac{P^2c^2}{m^2c^4}} = \frac{m_0^2c^4}{1-\dfrac{P^2c^2}{E^2}} \qquad [\because \quad E = mc^2]$$

$$\Rightarrow \quad E^2\left(1-\frac{P^2c^2}{E^2}\right) = m_0^2c^4$$

$$\Rightarrow \quad E^2 - P^2c^2 = m_0^2c^4$$

$$\Rightarrow \quad E^2 = P^2c^2 + m_0^2c^4$$

b. **Derivation for $P = \sqrt{\dfrac{K^2}{c^2} + 2m_0K}$:**

1. Total energy, E = relativistic kinetic energy + rest mass energy

$$\Rightarrow \quad E = K + m_0c^2$$

$$\Rightarrow \quad K = E - m_0c^2$$

$$\Rightarrow \quad K = \sqrt{m_0^2c^4 + P^2c^2} - m_0c^2 \quad \left(\because E = \sqrt{m_0^2c^4 + P^2c^2}\right)$$

$$\Rightarrow \quad K + m_0c^2 = \sqrt{m_0^2c^4 + P^2c^2} \qquad ...(1.25.3)$$

2. On squaring both side of eq. (1.25.3), we get

$$\Rightarrow \quad K^2 + m_0^2c^4 + 2Km_0c^2 = m_0^2c^4 + P^2c^2$$

$$\Rightarrow \quad K^2 + 2Km_0c^2 = P^2c^2$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1+\frac{u'v}{c^2} = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}} \quad \text{...(1.23.5)}$$

7. Similarly, we can take eq. (1.23.2) and proceed in the same manner,

$$1-\frac{u'v}{c^2} = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}} \quad \text{...(1.23.6)}$$

8. Putting eq. (1.23.5) and eq. (1.23.6) in eq. (1.23.4),

$$\frac{m_1}{m_2} = \sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}}$$

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_o = \text{constant}$$

9. If body B is at rest in stationary frame s that is $u_2 = 0$ before collision and $m_2 = m_0$ in frame s.
10. As bodies A and B are identical and have same mass in s'. So, $m_1 = m$ (relativistic mass) for $u_1 = v$.

Therefore, $$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}$$

Que 1.24. **Derive Einstein mass energy relation $E = mc^2$ and discuss**

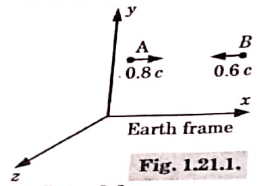


Fig. 1.21.1.

2. Velocity of earth w.r.t. B, $v = 0.6\,c$
3. Velocity of A w.r.t. earth, $u' = 0.8\,c$
4. Velocity of A w.r.t. B,

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{0.8c + 0.6c}{1 + \frac{(0.8c)(0.6c)}{c^2}}$$

$$\Rightarrow \quad u = \frac{1.4c}{1.48} = 0.946c$$

5. Velocity of earth w.r.t. A,
$$v = -0.8\,c$$
6. Velocity of B w.r.t. earth,
$$u' = -0.6\,c$$
7. Velocity of B w.r.t. A,

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{(-0.6c) + (-0.8c)}{1 + \frac{(-0.8)(-0.6)c^2}{c^2}}$$

$$u = -0.946\,c$$

8. Negative sign indicates that velocity of B w.r.t. A is towards left.

Que 1.22. **Show that no particle can attain a velocity larger than velocity of light.**

Answer

1. Let $v_x' = c$ and $v = c$
2. we know,

$$v_x = \frac{v_x' + v}{1 + \frac{v_x' v}{c^2}}$$

$$v_x = \frac{c + c}{1 + \frac{c^2}{c^2}} = \frac{2c}{2} = c$$

PART-3

Variation of Mass with Velocity, Einstein's Mass Energy Relation, Relativistic Relation between Energy and Momentum, Massless Particle.

CONCEPT OUTLINE : PART-3

Variation of Mass with Velocity : Mass is a function of the velocity of the body. It increases with increasing velocity represented by the relation :

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Mass Energy Equivalence : The variation of mass with velocity has modified the idea of energy, so that, a relationship can be established between mass and energy.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.23. **Deduce expression for variation of mass with velocity.**

OR

Show that the relativistic invariance of the law of conservation of momentum leads to the concept of variation of mass with velocity.

AKTU 2015-16, Marks 10

Answer

1. Suppose s and s' are two frame of references in which s' is moving with a constant velocity 'v' w.r.t. observer o.
2. Two identical bodies A and B having same mass m are moving with velocity u' but in opposite direction in s' frame.
3. After some time both collides and stick together and momentarily comes to rest in s' frame.
4. Now from velocity addition theorem,

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{...(1.23.1)}$$

$$u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \quad \text{...(1.23.2)}$$

Where, u_1 and u_2 = velocity of A and B in s frame before collision, and

m_1 and m_2 = their masses in s frame.

5. From the law of conservation of linear momentum,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,v \quad \text{...(1.23.3)}$$

$$u_y = \frac{u_y'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c^2}u_x'} \quad ...(1.20.6)$$

9. Similarly, $$u_z = \frac{u_z'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c^2}u_x'} \quad ...(1.20.7)$$

10. If motion of object is only in x direction then $u = u_x$ and equation becomes

$$u = \frac{u'+v}{1+\frac{vu'}{c^2}}$$

B. Consistency with Einstein's Second Postulate :

Case–1 : If $u' = c$, then

$$u = \frac{c+v}{1+\frac{v}{c^2}c} \Rightarrow u = \frac{(c+v).c}{(c+v)} = c$$

Case–2 : If $v = c$ then

$$u = \frac{u'+c}{1+\frac{c.u'}{c^2}} = c$$

Case–3 : If $v = c$ and $u' = c$ then

$$u = \frac{c+c}{1+\frac{c.c}{c^2}} = c$$

1. So the velocity of any object cannot be greater than 'c', whatever be the velocity of moving frame or velocity of object in that frame.
2. Therefore, the relativistic velocity addition theorem is consistent with the Einstein's second postulate of special theory of relativity.

Que 1.21. **Rocket *A* travels towards the right and rocket *B* travels towards the left with velocity 0.8c and 0.6c respectively relative to the earth. What is the velocity of rocket :**

a. *A*, measured from *B*, and

b. *B*, measured from *A* ?

Answer

Answer

Given : $t_o = 2.5 \times 10^{-8}$ s, v $= 2.4 \times 10^8$ m/s

To Find :
i. Mean life time.
ii. Distance travelled in one mean life time.
iii. Distance travelled without relativistic effect.

1. Mean life time,
$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.5\times10^{-8}}{\sqrt{1-\left(\frac{2.4\times10^8}{3\times10^8}\right)^2}} = \frac{2.5\times10^{-8}}{\sqrt{1-(0.8)^2}}$$
$$= \frac{2.5\times10^{-8}}{0.6} = 4.17 \times 10^{-8} \text{ sec}$$
2. The distance travelled = $(2.4 \times 10^8) \times (4.166 \times 10^{-8}) = 10$ m .
3. The distance travelled without relativistic effect
$$= (2.4 \times 10^8) \times (2.5 \times 10^{-8}) = 6 \text{ m.}$$

Que 1.18. **The half life of a particular particle as measured in the laboratory comes out to be 4.0×10^{-8} sec, when its speed is 0.8 c and 3.0×10^{-8} sec, when its speed is 0.6 c. Explain this.**

Answer

1. The time interval in motion is given by
$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{(where, } t_0 = \text{proper time interval)}$$
2. The proper half life of the given particle is
$$t_0 = t\sqrt{1-\frac{v^2}{c^2}}$$
3. In the first case $t = 4.0 \times 10^{-8}$ sec and v = 0.8 c
$$t_0 = 4.0 \times 10^{-8}\sqrt{1-\left(\frac{0.8\,c}{c}\right)^2} = 2.4 \times 10^{-8} \text{ sec}$$
4. As proper half life is independent of velocity, therefore half life of the particle when speed is 0.6 c must be given by
$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.4\times10^{-8}}{\sqrt{1-\left(\frac{0.6\,c}{c}\right)^2}} = \frac{2.4\times10^{-8}}{0.8} = 3 \times 10^{-8} \text{ sec}$$
which is actual observation.
5. Thus the variation of half life of given particle is due to relativistic time dilation.

Que 1.19. **At what speed should a clock be moved so that it may appear to lose 1 minute in each hour ?**

Answer

Given : Time loss = 1 min
To Find : Speed of clock.

1. Since, rest clock takes 60 minutes for T time interval.
∴ Rest clock takes 1 minute for T/60 time interval.
2. Now, moving clock takes 59 minutes for same T time interval.
∴ Moving clock takes 1 minute for T/59 time interval.
3. Here, $t_0 = \frac{T}{60}$ and $t = \frac{T}{59}$
4. From time dilation formula, $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$
$$\frac{T}{59} = \frac{\frac{T}{60}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{59}{60}$$
$$v = 5.45 \times 10^7 \text{ m/s}$$

Que 1.20. **Deduce the relativistic velocity addition theorem. Show that it is consistent with Einstein's second postulate.**

AKTU 2014-15, Marks 05

AKTU 2017-18, Marks 07

Answer

A. Relativistic Velocity Addition Theorem :
1. Let s and s' be two frame of references in which s' is moving with a constant velocity v in the x direction w.r.t. frame s.
2. Let P be a point having coordinate (x, y, z, t) and (x', y', z', t') in frame s and s' at any instant of time.
3. In these two frames the components of the velocities of that particle along x, y and z axis will be given by
$$u_x = \frac{dx}{dt} \text{ and } u_x' = \frac{dx'}{dt'}$$
$$u_y = \frac{dy}{dt} \text{ and } u_y' = \frac{dy'}{dt'}$$
$$u_z = \frac{dz}{dt} \text{ and } u_z' = \frac{dz'}{dt'}$$
4. Now using the Lorentz transformation equation,
$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Que 1.15. How much time does a metre stick moving at 0.1 c relative to an observer take to pass the observer ? The metre stick is parallel to its motion.

Answer

Given : $L_0 = 1$ m, $v = 0.1\,c$

To Find : Time of stick to pass observer.

1. Since, $$L = L_0\sqrt{1-\frac{v^2}{c^2}} = L_0\sqrt{1-0.01} = 1 \times \sqrt{0.99} = 0.994 \text{ m}$$
2. $$\text{Time} = \frac{L}{v} = \frac{0.994}{0.1\times 3\times 10^8} = 3.31 \times 10^{-8} \text{ sec}$$

Que 1.16. What is time dilation ? Find out its equation using Lorentz transformation and give an example to show that time dilation is a real effect.

Answer

A. Time Dilation :

1. In the special theory of relativity, the moving clock is found to run slower than a clock at rest does. This effect is known as time dilation.
2. Suppose s and s' are two frames of references. Frame s' is moving with constant velocity v in the positive x direction w.r.t. frame s.
3. If (t_1', t_2') be the times of occurrence of two events measured by the clock in frame s' and t_o be the corresponding time interval, then we have
$$t_o = t'_2 - t'_1 \quad \text{...(1.16.1)}$$
4. If (t_1, t_2) be the times of occurrence of the same events measured by the another clock in the stationary frame s and t be the corresponding time interval, then we have
$$t = t_2 - t_1$$
5. Using Lorentz transformation equation,
$$t = \frac{t'_2 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t'_1 + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$t = \frac{t'_2 + \frac{vx'}{c^2} - t'_1 - \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{...(1.16.2)}$$

6. From eq. (1.16.1) and eq. (1.16.2), we get, $t > t_o$
So the relativistic interval of time is more than proper interval of time.

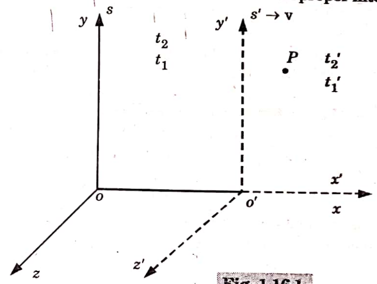


Fig. 1.16.1.

7. Therefore a moving clock appears to go slow (*i.e.*, take more time to complete a rotation compared to a rest clock).
8. If $v = c$, then, $t = \infty$
9. If v is very less than c then $t = t_o$

B. Time Dilation is a Real Effect :

1. The cosmic ray particles called μ-meson are created at high altitude about 10 km above earth's atmosphere and are projected towards earth surface with velocity 2.994×10^8 m/s.
2. It decays into e^+ (Positron) with an average life time of about 2×10^{-6} sec.
3. Therefore, in its life time μ-meson can travel a distance,
$$d = vt = 2.994 \times 10^8 \times 2 \times 10^{-6} = 600 \text{ m}$$
4. But it is found at earth surface also. It is possible because of time dilation effect.
$$t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times 10^{-6}}{\sqrt{1-\left(\frac{2.994\times 10^8}{3\times 10^8}\right)^2}} \quad [\because t_o = 2 \times 10^{-6} \text{ sec.}]$$
$$= 3.17 \times 10^{-5} \text{ sec}$$
5. In this time a μ-meson can travel a distance
$$d = 2.994 \times 10^8 \times 3.17 \times 10^{-5} = 9490.98 \text{ m} \approx 10 \text{ km}$$
6. This shows that time dilation is a real effect.

Que 1.17. The proper mean life time of + μ meson is 2.5×10^{-8} sec. Calculate :

1. **Mean life time of + μ meson travelling with the velocity 2.4×10^8 m/s.**
2. **The distance travelled by this + μ meson during one mean life time.**
3. **The distance travelled without relativistic effect.**

4. Length of moving rod,

$$L = \sqrt{L_x^2 + L_y^2} = \left[(0.7L_0)^2 + \left(\frac{L_0}{2}\right)^2\right]^{1/2} = 0.86L_0$$

5. Percentage contraction in length $= \dfrac{L_0 - L}{L_0} \times 100$

$$= \frac{L_0 - 0.86L_0}{L_0} \times 100 = 14\ \%$$

Que 1.14. **What will be the apparent length of rod of length 5 m and inclined at an angle 60° to horizontal. This rod is moving with a speed of 3×10^7 m/s.**

Answer

Given : $L_0 = 5$ m, inclination angle of rod = 60°, v = 3×10^7 m/s
To Find : Apparent length.

1. Since, frame of reference is moving along x-direction. So length of rod appears to change in x-direction only.
2. So, new length in x-direction,

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = 5\cos 60^\circ \sqrt{1 - \left(\frac{3 \times 10^7}{3 \times 10^8}\right)^2}$$

$$L_x = 2.487 \text{ m}$$

3. New apparent length of rod,

$$= \sqrt{L_x^2 + (5\sin 60^\circ)^2}$$

$$= \sqrt{(2.487)^2 + (5\sin 60^\circ)^2} = 4.99 \text{ m}$$

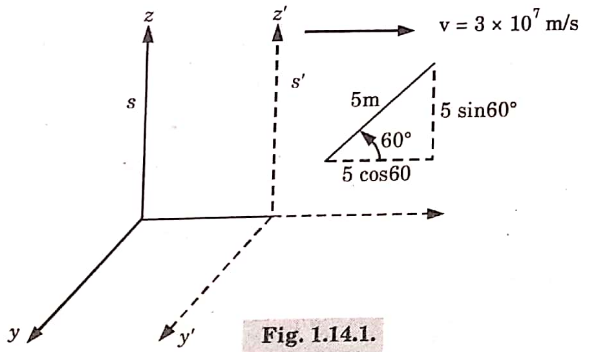


Fig. 1.14.1.

$\Rightarrow \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$

3. Hence $x^2 + y^2 + z^2 = c^2 t^2$ is invariant under Lorentz transformation.

Que 1.10. **As measured by O a bulb goes off at $x = 100$ km, $y = 10$ km, $z = 1$ km and $t = 5 \times 10^{-4}$ sec. What are the coordinates x', y', z' and t' of this event as determined by a second observer O' moving relative to O at $-0.8\,c$ along the common x-x' axis ?**

Answer

Given : $x = 100$ km, $y = 10$ km, $z = 1$ km
$v = -0.8 \times 3 \times 10^5$ km/s, $t = 5 \times 10^{-4}$ s
To Find : Coordinates x', y', z' and t'.

1. Since,
$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$x' = \frac{100 - (-0.8 \times 3 \times 10^5) \times 5 \times 10^{-4}}{\sqrt{1-(0.8)^2}} = 366.66 \text{ km}$$
$[\because \; c = 3 \times 10^5 \text{ km/s}]$

2.
$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{5 \times 10^{-4} - \frac{(-0.8 \times 3 \times 10^5) \times 100}{(3 \times 10^5)^2}}{\sqrt{1-(0.8)^2}}$$
$$= 12.77 \times 10^{-4} \text{ s}$$

3.
$$y' = y = 10 \text{ km}$$
$$z' = z = 1 \text{ km}$$

Que 1.11. **Show that a moving circle will appeared to be an ellipse if it is seen from a frame which is at rest.**

Answer

1. The equation of circle is $x^2 + y^2 = a^2$.
2. Putting the values from Lorentz transformation,
$$\left[\frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}\right]^2 + y'^2 = a^2$$
$$\frac{x'^2 + v^2 t'^2 + 2x'.vt'}{\left(1 - \frac{v^2}{c^2}\right)} + y'^2 = a^2$$
$$x'^2 + v^2t'^2 + 2x'\,vt' + y'^2 - y'^2\frac{v^2}{c^2} = a^2 - \frac{a^2 v^2}{c^2}$$

$$x'^2 + y'^2\left(1 - \frac{v^2}{c^2}\right) = a^2\left(1 - \frac{v^2}{c^2}\right) - v^2 t'^2 - 2x'vt'$$
$$\frac{x'^2}{\left(1-\frac{v^2}{c^2}\right)} + \frac{y'^2}{1} = a^2 - \frac{v^2 t'^2 + 2x'vt'}{\left(1 - \frac{v^2}{c^2}\right)} \quad \text{...(1.11.1)}$$

3. Let,
$$P^2 = 1 - \frac{v^2}{c^2} \quad \text{and} \quad Q^2 = a^2 - \frac{v^2t'^2 + 2x'vt'}{\left(1-\frac{v^2}{c^2}\right)}$$

Hence, eq. (1.11.1) becomes,
$$\frac{x'^2}{P^2} + \frac{y'^2}{1^2} = Q^2$$
$$\frac{x'^2}{(QP)^2} + \frac{y'^2}{Q^2} = 1 \quad \text{...(1.11.2)}$$

4. Eq. (1.11.2) represents an equation of ellipse.

Que 1.12. **What is length contraction ? Find out its equation using Lorentz transformation.**

Answer

1. The appeared decrease in the length of a body in the direction of motion is called length contraction.
2. Let us consider two frame of reference s and s' in which frame s' is moving with velocity v along x-axis.
3. A rod of length 'L_0' is moving horizontally in frame s'.

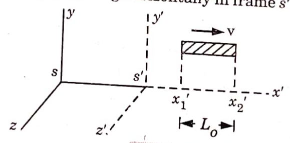


Fig. 1.12.1.

3. According to Lorentz transformation,
$$x_1' = \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$x_2' = \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Scanned by CamScanner

Answer

A. Lorentz Transformation Equations from Einstein's Postulates :
Refer Q. 1.7, Page 1–9A, Unit-1.

B. Condition at which Lorentz Transformations Reduce to Galilean Transformations :

1. Lorentz transformation equation,

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ...(1.8.1)$$

2. At low velocities means $v <<< c$

Thus, $1 - \frac{v^2}{c^2} = 1$

So eq. (1.8.1) reduces to, $x = x' + vt'$ ...(1.8.2)

Eq. (1.8.2) is a Galilean transformation equation.

3. It means at low velocities, the Lorentz transformation reduces to Galilean transformation.

5. Putting x' in eq. (1.7.2),

$$x = k\,[k\,(x - vt) + vt'] = [k^2\,(x - vt) + v\,kt']$$
$$x = k^2x - k^2vt + kvt'$$
$$kvt' = (1 - k^2)x + k^2vt$$
$$t' = \frac{(1-k^2)x + k^2vt}{kv} = \frac{(1-k^2)x}{kv} + kt \qquad ...(1.7.3)$$

6. According to second postulate of special theory of relativity, speed of light is a constant quantity.

In frame s, $x = ct$ ...(1.7.4)

In frame s', $x' = ct'$ ...(1.7.5)

7. Putting the value of x' and t' from eq. (1.7.1) and eq. (1.7.3) in eq. (1.7.5),

$$k\,(x - vt) = \frac{cx}{vk}(1-k^2) + ckt$$
$$x\left[k - \frac{c}{vk}(1-k^2)\right] = [ck + vk]\,t$$
$$x = \frac{[ck + vk]t}{\left[k - \frac{c}{vk}(1-k^2)\right]} \qquad ...(1.7.6)$$

9. On comparing eq. (1.7.6) with eq. (1.7.4),

$$c = \frac{(ck + vk)}{\left[k - \frac{c}{vk}(1-k^2)\right]}$$
$$ck + vk = ck - \frac{c^2}{vk}(1-k^2)$$
$$vk = -\frac{c^2}{vk}(1-k^2)$$
$$v^2\,k^2 = -c^2 + c^2k^2$$
$$k^2\,(v^2 - c^2) + c^2 = 0$$
$$k^2\,(c^2 - v^2) = c^2$$
$$\frac{k^2}{c^2}[c^2 - v^2] = 1$$
$$k^2\left[1 - \frac{v^2}{c^2}\right] = 1$$
$$k^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

or
$$k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

10. Putting the value of k in eq. (1.7.1),

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}[x - vt]$$ (First Lorentz transformation equation)

7. Expanding binomially, $T_2 = \frac{2d}{c}\left[1+\frac{v^2}{2c^2}\right]$ ...(1.4.2)

[Neglecting higher power term]

8. So, time difference,

$$\Delta t = T_1 - T_2 = \frac{2d}{c}\left[1+\frac{v^2}{c^2}\right] - \frac{2d}{c}\left[1+\frac{v^2}{2c^2}\right]$$

$$= \frac{2d}{c}\left[1+\frac{v^2}{c^2}-1-\frac{v^2}{2c^2}\right] = \frac{2d}{c}\left[\frac{v^2}{2c^2}\right] = \frac{dv^2}{c^3} \quad ...(1.4.3)$$

9. Now, the apparatus is rotated by 90° so that the position of mirror M_1 and M_2 gets interchange. So time taken from P to M_1 is,

$$T_1' = \frac{2d}{c}\left[1+\frac{v^2}{2c^2}\right]$$

and time taken from P to M_2 is,

$$T_2' = \frac{2d}{c}\left[1+\frac{v^2}{c^2}\right]$$

10. Time difference, $\Delta t' = T_1' - T_2'$

$$= \frac{2d}{c}\left[1+\frac{v^2}{2c^2}-\frac{v^2}{c^2}-1\right] = -\frac{2dv^2}{2c^3} = -\frac{dv^2}{c^3} \quad ...(1.4.4)$$

11. So, total time difference,

$$\Delta T = \Delta t - \Delta t' = \frac{dv^2}{c^3} - \left(-\frac{dv^2}{c^3}\right) = \frac{2dv^2}{c^3}$$

12. Since, path difference = speed of light × ΔT

$$\Delta = c \times \Delta T = c \times \frac{2dv^2}{c^3} = \frac{2dv^2}{c^2} \quad ...(1.4.5)$$

13. If λ is the wavelength of light used then the path difference in terms of the number of fringes is given by,

$$n = \Delta / \lambda = \frac{2dv^2}{c^2\lambda}$$

15. Taking, $d = 11$ m

Velocity of earth, $v = 3 \times 10^4$ m/s

Velocity of light, $c = 3 \times 10^8$ m/s

$\lambda = 6000$ Å [1 Å = 10^{-10} m]

$$n = \frac{2 \times 11 \times 9 \times 10^8}{9 \times 10^{16} \times 6000 \times 10^{-10}} = \frac{22 \times 10^2}{6000} = \frac{22}{60} = \frac{11}{30}$$

$$n = 0.36$$

16. If such type of medium like 'ether' exists in the atmosphere, there must be a fringe shift of 0.36.

17. Michelson–Morley performed that experiment several times in different situations, in different weather conditions but no fringe shift was obtained hence Huygens concept of 'ether drag' is wrong. This is known as negative result.

Que 1.5. **What will be the expected fringe shift on the basis of stationary ether hypothesis in Michelson-Morley experiment ? If the effective length of each part is 8 m and wavelength used is 8000 Å ?**

Answer

Given : $d = 8$ m, $\lambda = 8000 \times 10^{-10}$ m

To Find : Fringe shift.

1. We know that fringe shift is given by

$$n = \frac{2dv^2}{c^2\lambda} = \frac{2 \times 8 \times (3 \times 10^4)^2}{(3 \times 10^8)^2 \times 8 \times 10^{-7}} = 0.2$$

[∵ $v = 3 \times 10^4$ m/s and $c = 3 \times 10^8$ m/s]

Que 1.6. **State Einstein's postulates of special theory of relativity. Explain why Galilean relativity failed to explain actual results of Michelson-Morley experiment.**

AKTU 2013-14, Marks 05

Answer

A. **Einstein's Postulates :** There are two postulates of the special theory of relativity proposed by Einstein :

i. **Postulate 1 :**

The laws of physics are the same in all inertial frames of reference moving with a constant velocity with respect to one another.

Explanation :

1. If the laws of physics had different forms for observers in different frames in relative motion, one could determine from these differences which objects are stationary in space and which are moving.
2. As there is no universal frame of reference, therefore this distinction between objects cannot be made. Hence universal frame of reference is absent.

ii. **Postulate 2 :**

The speed of light in free space has the same value in all inertial frames of reference. This speed is 2.998 × 10^8 m/s.

Explanation :

1. This postulate is directly followed from the result of Michelson-Morley experiment.

B. **Reason for Failure Galilean transformation to Explain Actual Results of Michelson Morley Experiment :**

1. In Galilean transformations the speed of light was not taken to be constant in all inertial frames.
2. These equations were based on absolute time and absolute space.
3. The above two assumptions contradict the Einstein postulates.

Similarly, $\dfrac{du_y'}{dt'} = \dfrac{du_y}{dt}$ and $\dfrac{du_z'}{dt'} = \dfrac{du_z}{dt}$

11. Since,

$$\frac{du_x'}{dt'} = a_x' \; ; \; \frac{du_y'}{dt'} = a_y' \; ; \; \frac{du_z'}{dt'} = a_z'$$

$$\frac{du_x}{dt} = a_x \; ; \; \frac{du_y}{dt} = a_y \; ; \; \frac{du_z}{dt} = a_z$$

12. Then we get

$$a_x' = a_x \quad \text{...(1.2.9)}$$

$$a_y' = a_y \quad \text{...(1.2.10)}$$

$$a_z' = a_z \quad \text{...(1.2.11)}$$

or writing these equations collectively, $\vec{a}' = \vec{a}$

13. The measured components of acceleration of a particle are independent of the uniform relative velocity of the reference frames.
14. In other words, acceleration remains invariant when passing from one inertial frame to another that is in uniform relative translational motion.

Que 1.3. **Show that the distance between points is invariant under Galilean transformations.**

Answer

1. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinate of two points P and Q in rest frame s. Then the distance between them will be

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. Now the distance between them measured in moving frame s' is

$$d' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}$$

3. Using Galilean transformation

$$x_2' = x_2 - v_x t, \; y_2' = y_2 - v_y t \text{ and } z_2' = z_2 - v_z t$$

$$x_1' = x_1 - v_x t, \; y_1' = y_1 - v_y t \text{ and } z_1' = z_1 - v_z t$$

4. Hence

$$d' = \sqrt{[(x_2 - v_x t) - (x_1 - v_x t)]^2 + [(y_2 - v_y t) - (y_1 - v_y t)]^2 + [(z_2 - v_z t) - (z_1 - v_z t)]^2}$$

$$d' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\therefore \quad d' = d$$

Que 1.4. **Discuss the objective and outcome of Michelson-Morley experiment.**

Answer

A. Objective of Michelson-Morley Experiment :

1. The main objective of conducting the Michelson–Morley experiment was to confirm the existence of stationary ether.
2. According to Morley if there exist some imaginary medium like 'ether' in the earth atmosphere, there should be some time difference between relative motion of body with respect to earth and against the motion of earth.

3. Due to this time difference there exist some path difference and if such path difference occurs, Huygens concept is correct and if it does not occur then Huygens concept is wrong.

B. Michelson-Morley Experiment :

1. In Michelson-Morley experiment there is a semi-silvered glass-plate P and two plane mirrors M_1 and M_2 which are mutually perpendicular and equidistant from plate P.
2. There is a monochromatic light source in front of glass plate P.
3. The whole arrangement is fixed on a wooden stand and that wooden stand is dipped in a mercury pond. So it becomes easy to rotate.
4. Let v be the speed of imaginary medium (ether) w.r.t. earth and c is the velocity of light, so time taken to move the light ray from plate P to M_2 and reflected back,

$$T_1 = \frac{d}{c+v} + \frac{d}{c-v} = d\left[\frac{2c}{(c^2 - v^2)}\right]$$

$$T_1 = \frac{2dc}{c^2\left[1 - \dfrac{v^2}{c^2}\right]} = \frac{2d}{c\left[1 - \dfrac{v^2}{c^2}\right]}$$

5. Expanding binomially,

$$T_1 = \frac{2d}{c}\left[1 + \frac{v^2}{c^2}\right] \quad \text{...(1.4.1)}$$

[Neglecting higher power term]

6. Time taken to move a light ray from plate P to M_1 and to reflect back,

$$T_2 = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{2d}{c}\left[1 - \frac{v^2}{c^2}\right]^{-\frac{1}{2}}$$

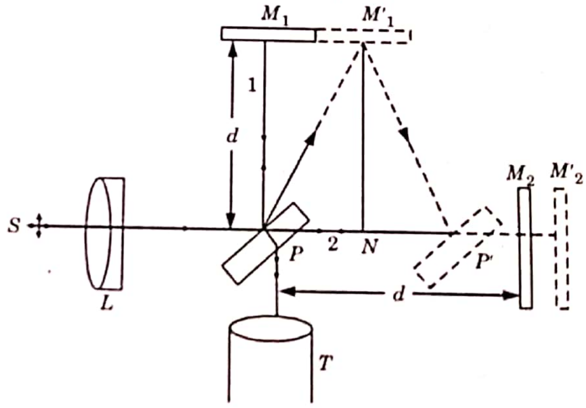


Fig. 1.4.1. The Michelson–Morley experiment.

4. Suppose P be a point in the space.
5. Now from Fig. 1.1.1,

$$\left.\begin{aligned} x &= x' + vt && \text{...(1.1.1)} \\ y &= y' && \text{...(1.1.2)} \\ z &= z' && \text{...(1.1.3)} \\ t &= t' && \text{...(1.1.4)} \end{aligned}\right\} \text{No relative motion}$$

6. Eq. (1.1.1) to eq. (1.1.4) are position and time transformation equations in s and s' frame.
7. Differentiating eq. (1.1.1) w.r.t. t on both sides,

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{v\,dt'}{dt} \quad \text{...(1.1.5)}$$

$$\frac{dx}{dt} = \frac{dx'}{dt'} + \frac{v\,dt'}{dt'} \quad (\because\ t = t' \quad \therefore\ dt = dt')$$

$$\Rightarrow \quad u_x = u'_x + v \quad \text{...(1.1.6)}$$

8. Differentiating eq. (1.1.2) w.r.t. t,

$$\frac{dy}{dt} = \frac{dy'}{dt} \quad \text{...(1.1.7)}$$

$$\frac{dy}{dt} = \frac{dy'}{dt'} \quad (\because\ t = t')$$

$$\Rightarrow \quad u_y = u'_y \quad \text{...(1.1.8)}$$

9. Similarly, $u_z = u'_z$ ...(1.1.9)
10. Now differentiating eq. (1.1.6) w.r.t. t, we get,

$$\frac{du_x}{dt} = \frac{du'_x}{dt} + \frac{dv}{dt}$$

$$\frac{du_x}{dt} = \frac{du'_x}{dt} \quad (\because\ v = \text{constant})$$

$$\frac{du_x}{dt} = \frac{du'_x}{dt'} \quad (\because\ t = t')$$

$$\Rightarrow \quad a_x = a'_x \quad \text{...(1.1.10)}$$

11. Similarly on differentiating eq. (1.1.8) and eq. (1.1.9), we get

$$a_y = a'_y \quad \text{...(1.1.11)}$$

$$a_z = a'_z \quad \text{...(1.1.12)}$$

12. Eq. (1.1.10), eq. (1.1.11) and eq. (1.1.12) shows that the acceleration is invariant in both frames.
13. So a frame of reference moving with constant velocity is an inertial frame.

Que 1.2. **Derive the Galilean transformation equations and show that its acceleration components are invariant.**

AKTU 2015-16, Marks 05

Answer

1. Suppose we are in an inertial frame of reference s and the coordinates of some event that occurs at the time t are x, y, z as shown in Fig. 1.2.1.

2. An observer located in a different inertial frame s' which is moving with respect to s at the constant velocity $\vec{v}$, will find that the same event occurs at time t' and has the position coordinates x', y' and z'.

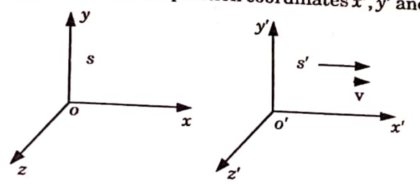


Fig. 1.2.1. Galilean transformation.

3. Assume that $\vec{v}$ is in positive x direction.
4. When origins of s and s' coincide, measurements in the x direction made in s is greater than those of s' by $\vec{v}t$ (distance).
5. Hence,

$$x' = x - vt \quad \text{...(1.2.1)}$$

$$y' = y \quad \text{...(1.2.2)}$$

$$z' = z \quad \text{...(1.2.3)}$$

$$t' = t \quad \text{...(1.2.4)}$$

These set of equations are known as Galilean transformations.

6. Differentiating eq. (1.2.1) with respect to t, we get

$$\frac{dx'}{dt} = \frac{dx}{dt} - v\left.\frac{dt}{dt}\right\} \text{no relative motion}$$

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \quad (\because t = t' \quad \therefore\ dt' = dt)$$

7. Similarly $\dfrac{dy'}{dt'} = \dfrac{dy}{dt}$

and $\dfrac{dz'}{dt'} = \dfrac{dz}{dt}$

8. Since, $dx'/dt' = u_x'$, the x-component of the velocity measured in s', and $dx/dt = u_x$, etc., then,

$$u_x' = u_x - v \quad \text{...(1.2.5)}$$

$$u_y' = u_y \quad \text{...(1.2.6)}$$

$$u_z' = u_z \quad \text{...(1.2.7)}$$

9. Eq. (1.2.5), eq. (1.2.6) and eq. (1.2.7) can be written collectively in the vector form as

$$\vec{u}' = \vec{u} - \vec{v} \quad \text{...(1.2.8)}$$

10. To obtain the acceleration transformation, we differentiate the eq. (1.2.5), eq. (1.2.6) and eq. (1.2.7) with respect to time such that

$$\frac{du'_x}{dt'} = \frac{d}{dt}(u_x - v) = \frac{du_x}{dt}$$

CONCEPT OUTLINE : PART-1

Frame of Reference : It is that coordinate system which is used to identify the position or motion of an object.

Types of Frame of Reference :

a. Inertial frame of reference, and
b. Non-inertial frame of reference.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Show that the frame of reference moving with constant velocity v is an inertial frame of reference.**

Answer

1. Let s is a frame of reference which is in rest to an observer and s' is another frame of reference moving with constant velocity v in the positive x direction with respect to the same observer.
2. o and o' are origin of frame s and s' respectively.

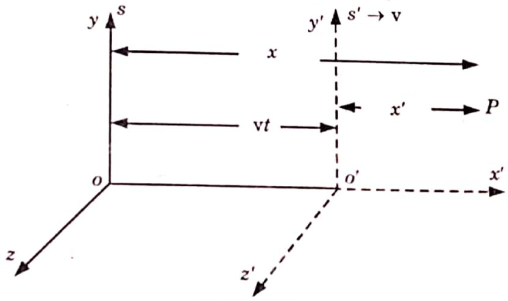


Fig. 1.1.1.

3. Initially o and o' coincide with each other at time $t = t' = 0$, where t and t' = time measured in s and s' frames respectively.

CONTENTS

KAS 101/201 : PHYSICS