Project 2: On Asymptotes and Intercepts

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1 Introduction

We know that the equation that defines a hyperbola on a coordinate plane is $\frac{1}{x}$. Let's look at the graph of this equation:

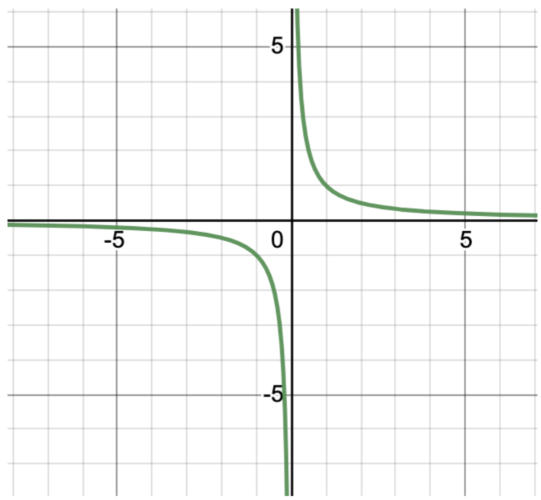

Figure 1: The graph of $\frac{1}{x}$ shown in green

This graph is a hyperbola because there is a vertical asymptote at $x = 0$. x can never equal zero, because then $\frac{1}{x}$ would be undefined. Now let's change this equation a bit, and make it $\frac{1}{x-1}$. This is graphed in Figure 2.

Looking at the graph, we can see that there is a vertical asymptote at $x = 1$. x cannot equal 1 for the same reason that in the previous equation, $\frac{1}{x}$, x could not equal 0.

Next, let's look at iterations of $\frac{1}{x-1}$. Let's refer to $\frac{1}{x-1}$ as $j(x)$. The graph of three iterations of $j(x)$ is shown in Figure 2.

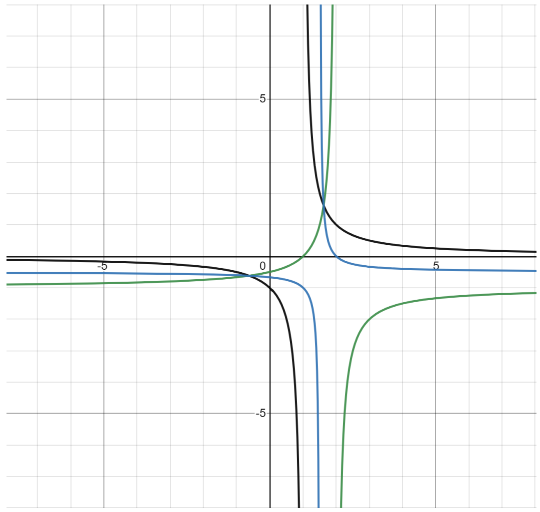

Figure 2: The graph of $j(x)$ (black) $j^2(x)$ (green) and $j^2(x)$ (blue)

Something interesting seems to be happening here. The x intercept of $j^2(x)$ seems to be the vertical asymptote of $j(x)$. If we look at Figure 2, we can see that the vertical asymptote of $j^2(x)$ seems to be the x intercept of $j^3(x)$. This bring us to an interesting pattern. Let's take our equation a step further.

Take the equation $f(x) = \frac{a}{ax-1}$. The graph of multiple iterations of $f(x)$ when $a = 1$ is shown in Figure 3.

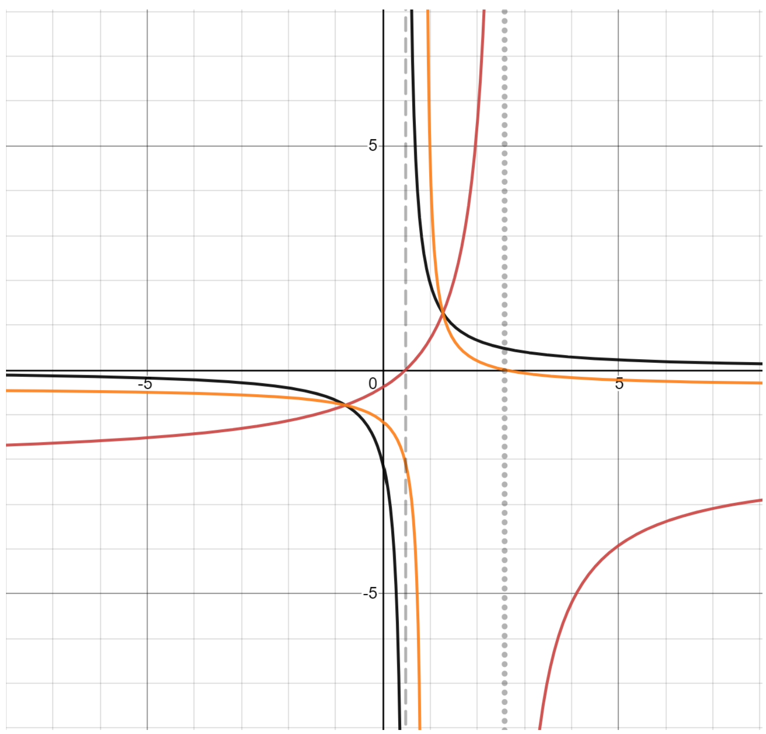

Figure 3: The graph of $f(x)$ (black), $f^2(x)$ (red), $f^3(x)$ (orange), $f(x)$'s asymptote dashed, and $f^2(x)$'s asymptote dotted

The graph of $f^2(x)$ appears to have an x intercept at the same location that $f(x)$ has a vertical asymptote. Furthermore, the x intercept of $f^3(x)$ seems to be located in the same place as the vertical asymptote of $f^2(x)$. In both these cases, the vertical asymptote of $f^n(x)$ seems to be located in the same place as the x intercept of $f^{n+1}(x)$.

This begs the question, **for a function $f(x) = \frac{a}{ax-1}$ where a is a constant which can be any real number, is the vertical asymptote of $f^n(x)$ always located in the same place as the x intercept of $f^{n+1}(x)$ where n is a natural number?**

To approach this problem, we can try checking algebraically whether the x intercept of $f^2(x)$ and the vertical asymptote of $f(x)$ are the same. We can then attempt to establish this as a pattern for all iterations of $f(x)$.

This approach will help us gain some useful knowledge, but as we will discover in section 4, there is a flaw in the core reasoning of the method - this function and its iterations do not have x-intercepts. However, approaching the question as if it has an x intercept allows us to visualize how asymptotes function in iterations, as well as help us solve a rephrased question, which we introduce in section 5.

2 Comparing the x-intercept and Vertical Asymptote of $f^2(x)$ and $f(x)$

First, we can ensure that the x intercept of $f^2(x)$ and the vertical asymptote of $f(x)$ are the same. To do this we can first find the vertical asymptote of $f(x)$. The vertical asymptote is a value that makes the equation undefined. The equation will be undefined when the denominator is 0. So, to find the vertical asymptote, we can set the denominator of $f(x)$, $ax-1$, equal to 0.

$$ax - 1 = 0$$

$$ax = 1$$
$$x = \frac{1}{a}$$

The vertical asymptote of $f(x)$ is $\frac{1}{a}$. Next, let's find the x intercept of $f^2(x)$. To do this, we can set $f^2(x)$ equal to 0. First, let's establish $f^2(x)$.

$$f^2(x) = \frac{a}{a\left(\frac{a}{ax-1}\right) - 1}$$

$$f^2(x) = \frac{a}{\frac{a^2}{ax-1} - 1}$$

$$f^2(x) = \frac{a}{\frac{a^2-ax+1}{ax-1}}$$

$$f^2(x) = \frac{a^2x - a}{a^2 - ax + 1}$$

Let's now set $f^2(x)$ equal to 0 to find the x intercept.

$$f^2(x) = \frac{a^2x - a}{a^2 - ax + 1} = 0$$

$$a^2x - a = 0$$

$$a^2x = a$$

$$x = \frac{a}{a^2}$$

$$x = \frac{1}{a}$$

Both the x intercept and the vertical asymptote are the same. However, to be sure we can also test our answer by substituting $x = \frac{1}{a}$ into $f^2(x)$. If we do this, we should get 0, as the x intercept has a y value of 0. Substituting $x = \frac{1}{a}$ into $f^2(x)$:

$$f^2(x) = \frac{a^2\left(\frac{1}{a}\right) - a}{a^2 - a\left(\frac{1}{a}\right) + 1}$$

$$f^2(x) = \frac{a - a}{a^2 - 1 + 1}$$

$$f^2(x) = \frac{0}{a^2}$$

$$f^2(x) = 0$$

Thus, $f^2(x) = 0$. This shows that the vertical asymptote of $f(x)$ is indeed the x-intercept of $f^2(x)$. Now, we must prove this true for all iterations of $f(x)$.

3 Does This Logic Work for All Iterations $f^n(x)$?

In order to show that this holds true for all iterations of $f(x)$, where an iteration is $f^n(x)$ and n is a natural number, we cannot simply solve for the vertical asymptote and x-iteration of $f^n(x)$ and $f^{n+1}(x)$, respectively.

Solving for the vertical asymptote and x-intercept of $f^n(x)$ and $f^{n+1}(x)$ respectively, for every natural number n, would entail that we algebraically solve for the vertical asymptote and x-intercept of *every* iteration of $f(x)$, because our question asks, "is the vertical asymptote of $f^n(x)$ ***always*** located in the same place as the x intercept of $f^{n+1}(x)$ where n is a natural number?"

Thus, since it is not possible to calculate all of the wanted values, the vertical asymptote and x-intercept for all, iterations of $f(x)$, we cannot appropriately address or solve the question that we are actually asking, using this method.

Therefore, we must find another method to find out if the vertical asymptote of $f^n(x)$ is ***always*** located in the same place as the x intercept of $f^{n+1}(x)$ where n is a natural number.

If not manually, the next most plausible method to establish a pattern for all iterations of $f(x)$ is algebraically. What does this mean, though? We must try to find an algebraic pattern in the values of the vertical asymptotes of the iterations of $f(x)$ and the x-intercepts of $f(x)$ for a given n.

To embark on this new solution, we use brute force—solving for the values of the asymptote and x-intercept of the first few iterations of $f(x)$, where a is still unknown, and searching for a relationship that depends on n.

We've already determined that the vertical asymptote of $f(x)$ is $x = \frac{1}{a}$ and that the x-intercept of $f^2(x)$ is $\frac{1}{a}$, so we can start with the vertical asymptote of $f^2(x)$. First, let's find $f^2(x)$.

$$f(x) = \frac{a}{ax - 1}$$

$$f^2(x) = \frac{a}{a\left(\frac{a}{ax-1}\right) - 1}$$

$$f^2(x) = \frac{a}{\frac{a^2}{ax-1} - 1}$$

$$f^2(x) = \frac{a}{\frac{a^2}{ax-1} - \frac{ax-1}{ax-1}}$$

$$f^2(x) = \frac{a}{\frac{a^2-ax+1}{ax-1}}$$

$$f^2(x) = \frac{a\left(ax-1\right)}{a^2-ax+1}$$

$$f^2(x) = \frac{a^2x-a}{a^2-ax+1}$$

Now that we have the fully simplified version of $f^2(x)$, we can solve for its vertical asymptote. To do this, we set the denominator equal to 0, to find out what value of x would make the function undefined. We get the following.

$$a^2 - ax + 1 = 0$$

$$a^2 - ax = -1$$

$$ax - a^2 = 1$$

$$a(x-a) = 1$$

$$\frac{a(x-a)}{a} = \frac{1}{a}$$

$$x - a = \frac{1}{a}$$

$$x = \frac{1}{a} + a$$

$$x = \frac{a^2+1}{a}$$

Thus, the vertical asymptote of $f^2(x)$ is $x = \dfrac{a^2+1}{a}$.

We can now solve for the x-intercept and vertical asymptote of $f^3(x)$. Let's first establish a simplified version of $f^3(x)$.

$$f(x) = \frac{a}{ax-1}$$

$$f^2(x) = \frac{a^2x-a}{a^2-ax+1}$$

$$f^3(x) = \frac{a}{a\left(\frac{a^2x-a}{a^2-ax+1}\right) - 1}$$

$$f^3(x) = \frac{a}{\frac{a^3x-a^2}{a^2-ax+1} - 1}$$

$$f^3(x) = \frac{a}{\frac{a^3x-a^2}{a^2-ax+1} - \frac{a^2-ax+1}{a^2-ax+1}}$$

$$f^3(x) = \frac{a}{\frac{a^3x-2a^2+ax-1}{a^2-ax+1}}$$

$$f^3(x) = \frac{a\left(a^2 - ax + 1\right)}{a^3x - 2a^2 + ax - 1}$$

$$f^3(x) = \frac{a^3 - a^2x + a}{a^3x - 2a^2 + ax - 1}$$

Now that we have the fully simplified version of $f^3(x)$, we can solve for its x-intercept. To do this, we set the function equal to 0. We get the following.

$$\frac{a^3 - a^2x + a}{a^3x - 2a^2 + ax - 1} = 0$$

$$a^3 - a^2x + a = 0$$

$$a(a^2 - ax + 1) = 0$$

$$a^2 - ax + 1 = 0$$

$$a^2 - ax = -1$$

$$ax - a^2 = 1$$

$$a(x - a) = 1$$

$$\frac{a(x - a)}{a} = \frac{1}{a}$$

$$x - a = \frac{1}{a}$$

$$x = \frac{1}{a} + a$$

$$x = \frac{a^2 + 1}{a}$$

Thus, the x-intercept of $f^3(x)$ is $x = \dfrac{a^2 + 1}{a}$.

We now solve for the vertical asymptote of $f^3(x)$. To do this, we set the denominator equal to 0, to find out what value of x would not work in the denominator. We get

$$a^3x - 2a^2 + ax - 1 = 0$$

$$a^3x - 2a^2 + ax = 1$$

$$a^2x - 2a + x = \frac{1}{a}$$

$$a^2x + x = \frac{1}{a} + 2a$$

$$x(a^2 + 1) = \frac{1 + 2a^2}{a}$$

$$x = \frac{2a^2 + 1}{a^3 + a}$$

Thus, the vertical asymptote of $f^3(x)$ is $x = \frac{2a^2+1}{a^3+a}$.

Now we have two short lists, one of the x-intercepts of the first three iterations of $f(x)$ (although $f(x)$ itself has no x-intercept), and the vertical asymptotes of the first three iterations of $f(x)$:

$$x\text{-intercepts (with increasing } n\text{):}\ \left(\text{N/A}, x=\frac{1}{a}, x=\frac{a^2+1}{a}\right)$$

$$\text{vertical asymptotes (with increasing } n\text{):}\ \left(x=\frac{1}{a}, x=\frac{a^2+1}{a}, x=\frac{2a^2+1}{a^3+a}\right)$$

Strictly from observation, although our hypothesis still stands with the calculated iterations, the x-intercepts and vertical asymptotes of the first three iterations of $f(x)$ seem to share little to no relation with each other and with n.

Calculating these values for further iterations won't help either, in that a new pattern won't emerge from them. To show this, we calculate the values for iterations $n = 4$ and $n = 5$ respectively.

The algebra will not be detailed here, but please know that we use a similar method as above to find the values of the x-intercepts and the vertical asymptotes, setting the function equal to 0 and setting the denominator of the function equal to 0, respectively.

Finding the x-intercept of $f^4(x)$ gives us that

$$x=\frac{2a^2+1}{a^3+a}$$

Finding the vertical asymptote gives us that

$$x=\frac{1+3a^2+a^4}{2a^3+a}$$

Finding the x-intercept of $f^5(x)$ gives us that

$$x=\frac{1+3a^2+a^4}{2a^3+a}$$

Finding the vertical asymptote gives us that

$$x=\frac{1+4a^2+3a^4}{a^5+3a^3+a}$$

As is clear from our newer results, our hypothesis still stands, but we get no new information about the *specific* values' relationship with n, which would help a great deal in fully proving our hypothesis true.

We seem to have met a dead end here—there seems to be no direct relationship between the iteration number, n, of f and the values that iteration of the function produces, the x-intercepts and the vertical asymptotes, which means we cannot expect to brute force this problem.

This means that we should search for another way to establish this pattern. So, let us attempt a different path. By using the following method, given that the pattern holds true for one iteration of f, we can guarantee that it works for all iterations n. The approach that we spent so much time on allows us to think of more innovative way to solve this problem, and this new approach would in theory work much better for our solution.

We have already established that the vertical asymptote of $f(x)$ is the same as the x-intercept of $f^2(x)$, we've algebraically proven that both have a value of $\frac{1}{a}$. Thus, we have proven that our hypothesis holds true for iterations $n = 1$ and $n = 2$.

We can then expand this to all values of n by using a new variable, which we will call p. p will be defined as the most recent iteration of $f^n(x)$, or $p = f^{n-1}(x)$. This means we can algebraically represent the value that creates the vertical asymptote of $f^n(x)$, and manipulate it. We can then redefine our functions $f^n(x)$ and $f^{n+1}(x)$ using this variable p:

$$f^n(x) = f(f^{n-1}(x)) = f(p)$$

$$f^{n+1}(x) = f(f(f^{n-1}(x))) = f^2(p)$$

Since $f(p)$ is the same as $f^n(x)$, we can then plug p into $f(x)$ to get a general equation for $f^n(x)$:

$$f^n(x) = f(p) = \frac{1}{ap - 1}$$

We can then solve this equation to find where $f^n(x)$ has an asymptote, in the same way as we did for just $f(x)$:

$$ap - 1 = 0$$

$$ap = 1$$

$$p = \frac{1}{a}$$

So we know that whenever $p = \frac{1}{a}$, the function $f^n(x)$ will have an asymptote. We already have established that $f^2(\frac{1}{a}) = 0$, and so therefore our function $f^2(p)$ will be equal to zero at the same p that makes $f(p)$ have a vertical asymptote. That is to say, at $p = \frac{1}{a}$, $f^2(p) = 0$, and $f(p)$ is undefined. Essentially, this means that at the position where $f(p)$ (which we defined as $f^n(x)$) has an asymptote, the function $f^2(p)$ (which we defined as $f^{n+1}(x)$) will have a zero, or an x-intercept.

So, we've proven our assertion true, and the vertical asymptote of $f^n(x)$ is always located in the same place (meaning x-coordinate) as the x intercept of $f^{n+1}(x)$ where n is a natural number.

4 A Flaw in Our Current Approach

If we take $f^2(x)$ in its original form before simplification, $f^2(x) = \frac{a}{a(\frac{a}{ax-1})-1}$ and substitute in the asymptote of our previous iteration, $x = \frac{1}{a}$, the supposed x intercept, it should be equal to 0.

$$\begin{aligned} f^2(x) &= \frac{a}{a\left(\frac{a}{ax-1}\right) - 1} \\ &= \frac{\frac{1}{a}}{\frac{1}{a}\left(\frac{a}{\frac{1}{a}(a)-1}\right) - 1} \\ &= \frac{\frac{1}{a}}{\frac{1}{a}\left(\frac{a}{0}\right) - 1} \end{aligned}$$

We can see here that during our simplification, the denominator of one of the fractions becomes 0, so this function is actually **undefined** for the value $x = \frac{1}{a}$. Therefore, our calculation above is wrong on a fundamental level (although it did give us an important insight), and $x = \frac{1}{a}$ is not the x intercept, meaning the vertical asymptote of our function $f^n(x)$ is not the x intercept of $f^{n+1}(x)$.

We can see why this does not work in this circumstance specifically, but can we generalize it a bit more? Is the vertical asymptote of any function the x intercept of the next iteration? Our answer is no, and here is why: The vertical asymptote of any function is, in simplest terms, an impossible output. This means that whatever is inputted into the function, the value of the vertical asymptote can never be outputted. We know that the output of one iteration is the input of the next iteration of that function. So, if something can never be outputted from one iteration, it can never be inputted into the next iteration. Based on this, the vertical asymptote of one iteration can never be the x intercept of the next iteration.

Now that we have proved this and explained why it is the case, then why did our original solution seem to work in the first place? What went wrong in our process and our check step? Let's start with our process. When solving this, during the simplification process, we disregarded values that made the expressions in the denominators equal to 0 by just simplifying them. For example, we made this mistake when simplifying $f^2(x)$.

$$f^2(x) = \frac{a}{\frac{a^2-ax+1}{ax-1}}$$

$$f^2(x) = \frac{a^2x - a}{a^2 - ax + 1}$$

Here, by moving the denominator of the value in the denominator to the numerator, we eliminate an discontinuities that could be formed from that expression. The same thing happened with our check step. We need to make sure that if at any point in our process, there is an expression in any denominator, that it cannot equal 0. However, this was not an approach done in vain. We can use the logic we employed previously to solve a rephrased and refined question.

But we have learned an important principle from our first attempt. Our goal was first to find a general equation that would show whether or not $f^n(x)$'s asymptote intersects $f^{n+1}(x)$'s x-intercept, all in terms of n. But due to the lack on an x-intercept, we cannot simply solve for said equation algebraically. We will have to rephrase the question in a way that deals with the principles of iteration and asymptotes.

5 Rephrasing the Question

Now that we have found out that there wasn't an x-intercept at the place we were referring to in the function, we need rephrase it in order to make conclusions about the functions that are true.

We can use the information about how we are wrong to come up with this new question. As discussed earlier, due to the principles of iteration, the asymptote we see in our function will cause the next iteration of it to have a hole located on the same x-value as the asymptote on the x-axis. So, it seems reasonable to focus on holes instead of x-intercepts in our new question.

We can pose the following question: "Is it true that an iteration $f^n(x)$ of $f(x) = \frac{a}{ax-1}$ has a vertical asymptote at the x value where the iteration $f^{n+1}(x)$ has a removable discontinuity on the x axis?"

6 The Same Logic Could Work for the Rephrased Question

We can solve this new question in a way resembling the previous question. We can solve for the vertical asymptote of a function $f^n(x)$, and simply plug it into the rule for $f^{n+1}(x)$ and see if it is defined. If it isn't, then we know there will always be a hole in the next iteration where $f^n(x)$ has a vertical asymptote. Since we have already done most of this math, we know that the vertical asymptote of $f(x)$ is $\frac{1}{a}$, and that the function rule for $f^2(x)$ is $\frac{a}{a\left(\frac{a}{ax-1}\right)-1}$. Plugging in $x = \frac{1}{a}$, we can see that $f^2(x)$ is indeed always undefined at $x = \frac{1}{a}$:

$$f^2\left(\frac{1}{a}\right) = \frac{a}{a\left(\frac{a}{a\frac{1}{a}-1}\right)-1}$$

$$f^2\left(\frac{1}{a}\right) = \frac{a}{a\left(\frac{a}{1-1}\right)-1}$$

$$f^2\left(\frac{1}{a}\right) = \frac{a}{a\left(\frac{a}{0}\right)-1}$$

Since we are dividing by zero, this value cannot exist, and the function is undefined.

We can generalize this by utilizing a previous strategy, where we set $p = f^{n-1}(x)$, and substitute p for x in both functions. A final important fact is that $f(p)$ is essentially the same as $f^n(x)$, and $f^2(p)$ is the same as $f^{n+1}(x)$, as was previously established. So, we know that when $x = \frac{1}{a}$, there is an asymptote of $f(x)$. So, setting $p = \frac{1}{a}$, we then know that $f^n(x)$ will have an asymptote. It will be very difficult to locate where this asymptote is, but what is important is that there is indeed an asymptote, and that it happens given a specific value of $f^{n-1}(x)$.

Then, we will plug $p = \frac{1}{a}$ into $f^2(p)$. As we saw above, this function is completely undefined, as $f^2\left(\frac{1}{a}\right)$ divides by zero. Since we know that $f^2(p)$ is $f^{n+1}(x)$, $f^{n+1}(x)$ must also be undefined at this value of p. Thus, we know that whenever $f^{n-1}(x)$ creates a vertical asymptote in $f^n(x)$, it must also create a hole in $f^{n+1}(x)$ at that same position.

We know that this removable discontinuity will always be on the x-axis as when we simplify $f^2(x)$ it produces an output of 0 when $x = \frac{1}{a}$ is plugged into it:

$$f^2\left(\frac{1}{a}\right) = \frac{a^2\left(\frac{1}{a}\right)-a}{a^2-a\left(\frac{1}{a}\right)+1}$$

$$f^2\left(\frac{1}{a}\right) = \frac{a-a}{a^2-1+1}$$

$$f^2\left(\frac{1}{a}\right) = \frac{0}{a^2}$$

$$f^2\left(\frac{1}{a}\right) = 0$$

Since $f^2\left(\frac{1}{a}\right)$ would be equal to zero if the point existed and was on the graph, we know that the hole that $f^2(p)$ has at $p = \frac{1}{a}$ (and therefore at the vertical intercept of $f(p)$) will always be at $y = 0$, or on the x-axis.

7 Conclusion

From our work in the previous section, we were able to conclude that there is always a hole in $f^{n+1}(x)$ wherever $f^n(x)$ has a vertical asymptote. This means that the answer to our revised question: "Is it true that an iteration $f^n(x)$ of $f(x) = \frac{a}{ax-1}$ has a vertical asymptote at the x value where the iteration $f^{n+1}(x)$ has a removable discontinuity?" is $\boxed{\textbf{yes}}$. As we've found, this statement is always true for all values of a and for any given iteration $f^n(x)$. Proving this statement as true gives us more information about the nature of iterations. More importantly, it helps prove something that is essential to understanding the domain and range of iterations. Since the x value that creates an asymptote makes the function $f^n(x)$ undefined, it doesn't exist on the said functions domain. Therefore, it cannot exist on the domain of f^{n+1}, as it would be undefined at that location. This general fact about functions is very important to keep in mind when it comes to iterations of any kind, given that it allows us to understand how a domain of a function changes as $f(x)$ is repeatedly iterated.

8 Further Inquiry

In this project, we focused on specifically the function $f(x) = \frac{1}{ax-1}$ and its positive iterations ($f^2(x)$, $f^3(x)$, etc...). Two possible avenues of further inquiry can be drawn from this question. One would be exploring a similar function, which would be $f(x) = \frac{1}{ax^b-1}$, and seeing if it behaves in the same way described throughout the rest of this paper. The other avenue for inquiry would be to explore the inverse and negative iterations of $f(x)$ ($f^{-1}(x)$, $f^{-2}(x)$, $f^{-3}(x)$, etc.), and seeing if it also follows the rules that we were able to establish from this paper.