Precalculus Project Quarter 1

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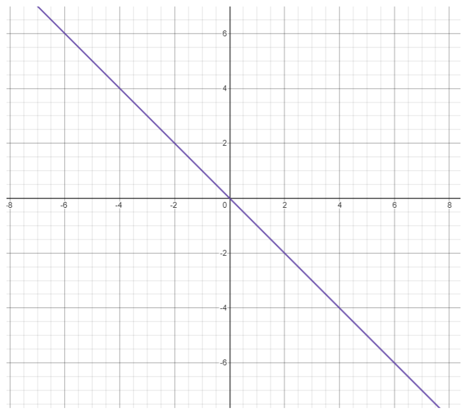
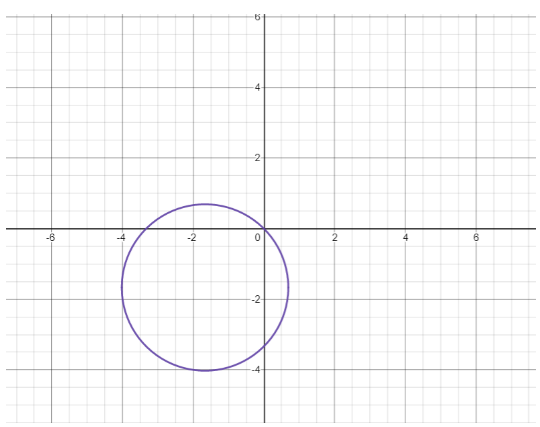
We begin by typing this equation into Desmos:

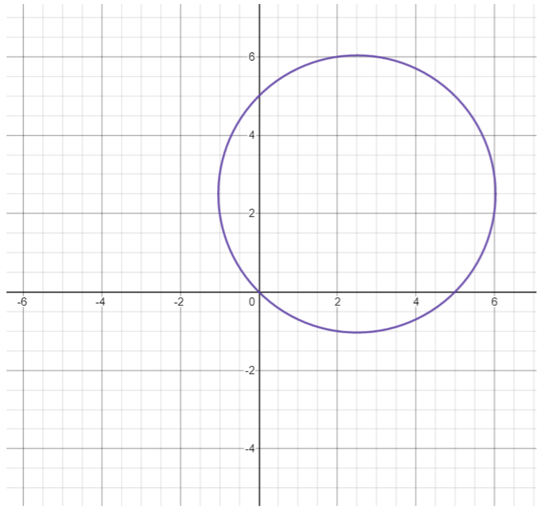
$$ax^2 + x + ay^2 + y = 0$$



When $a = 0$, the graph of the equation was a line, but when varying the a value, it seems like the graph is always a circle. This is the graph when $a = 0.3$:



This is the graph when $a = -0.2$:

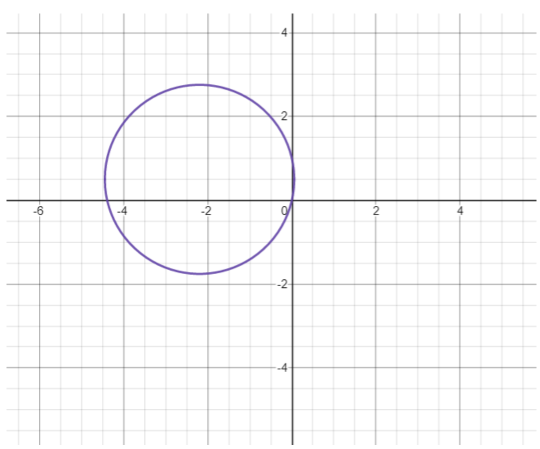


This is interesting because as negative values of a get closer to zero, the circle gets bigger and bigger until $a = 0$, where it opens into a straight line. As a begins to grow, the line closes back into a circle but it closes around the other way, and then for larger and larger values of a, the circle gets infinitely smaller.

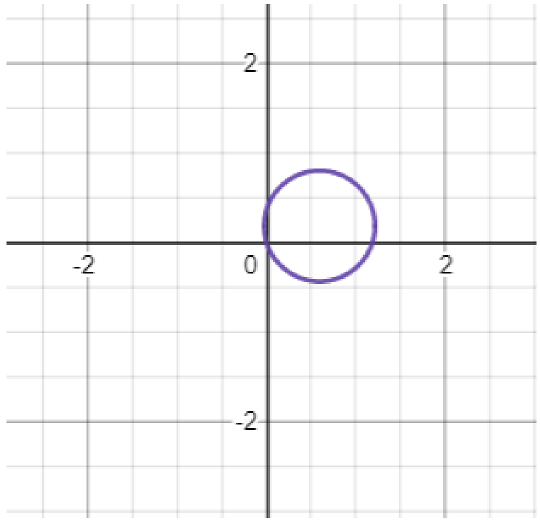
Our initial wondering is *for every non-zero value of a, is $ax^2 + x + ay^2 + y = 0$ a circle?* But we can see that because it is very close to the standard form of a circle, the answer will probably be relatively straightforward, and we would get more information by adding another variable.

We now consider the graph of $ax^2 + bx + ay^2 + y = 0$

This is the graph when $a = -1$ and $b = -4.4$:



This is the graph when $a = -2.7$ and $b = 3.2$:



We can observe the same thing in this graph as before, where as a ad b values change, a circle gets bigger and bigger before opening into a line, and then closing around the other side.

We will instead ask the same question but with this new equation.

For every non-zero value of a and any value of b, is $ax^2 + bx + ay^2 + y = 0$ a circle?

Solving for the Solution

We can begin by putting our equation into the standard form of a circle, $(x - h)^2 + (y - k)^2 = r^2$. We can do this using the completing the square method.

$$ax^2 + bx + ay^2 + y = 0$$

$$x^2 + \frac{b}{a}x + y^2 + \frac{1}{a}y = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \left(y + \frac{1}{2a}\right)^2 - \frac{1}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{1}{2a}\right)^2 = \frac{b^2 + 1}{4a^2}$$

We know that the value of the radius squared is $\frac{b^2+1}{4a^2}$, so the radius is the square root of this value. In order for our equation to represent the graph of a circle, this expression must be nonzero and non-negative. We know b^2 will always be a positive number or zero for any real value of b and so will a^2 for any real value of a. So, there is no imaginary outcome for the square root because neither the numerator nor the denominator will ever be negative for this expression.

We also need to make sure that the radius is never 0, because that would give us a point and not a circle. We can check this by setting $\frac{b^2+1}{4a^2}$ equal to 0 and solving for b.

$$\frac{b^2+1}{4a^2} = 0$$

$$b^2 + 1 = 0$$

$$b^2 = -1$$

We can see that because $b^2 = -1$, there is no real value for b that satisfies the equation and makes the expression equal to 0. This means that when we take the square root, we will get a positive number as the radius.

We can keep our constraint where a must be non-zero because as we can see from our expression, if $a = 0$, that would make the denominator of $\frac{b^2+1}{4a^2}$ equal to zero. That would mean division by zero, which is impossible. We also know that a must be non zero because at one point when we were solving to put our equation in standard form, we divided everything by a. If $a = 0$, this would not have worked because division by zero is invalid.

The reason that our equation $\left(x + \frac{b}{2a}\right)^2 + \left(y + \frac{1}{2a}\right)^2 = \frac{b^2+1}{4a^2}$ can still have the variables a and b in the center and radius is because they represent constant values, whereas x and y cannot be in the radius or center because their values vary.

From this information, we can see that the radius will always be a positive number for all values of a and b excluding the value of $a = 0$.

It's hard for us to know what varying the values of a and b do to our graph because both a and b are in our radius and center values.

But we observed previously in our graph that when the a value increased and got farther away from zero, our circles got smaller. This means that the radius was getting smaller. We can look at our current expression for the radius, $\sqrt{\frac{b^2+1}{4a^2}}$, and we can see that because a is in the denominator, the larger a is, the smaller our circle will be.

We also noticed that when the a value decreased and got farther away from zero, our circles got smaller. This means that the radius was getting smaller. We can again see that because a is in the denominator, and it is being squared, the smaller the a value, the smaller our circle will be.

This shows that our original observation about the a value stays true in our formula.

We can conclude that for every non-zero value of a ***and every value of*** b ***in*** $ax^2+bx+ay^2+y = 0$***, the graph is a circle.***

Further Inquiry

1. What happens to our graph when $ax^2 + bx + ay^2 + y = c$?
2. What happens to our graph when $ax^2 + bx + ay^2 + cy = 0$?
3. How does our graph change when $acx^2 + bcx + ay^2 + y = 0$?