Project 3: On Asymptotes and Extrema

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1 Introduction

1.1 Background Information

Let us begin by trying the graph of $f(x) = xe^{ax}$ in Desmos for positive values of a (**Fig. 1, Fig. 2**).

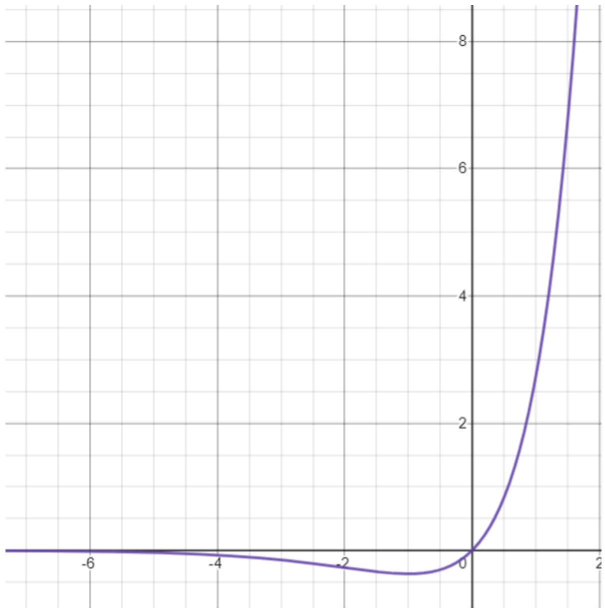

Figure 1: The graph of $f(x) = xe^{ax}$ when $a = 1$.

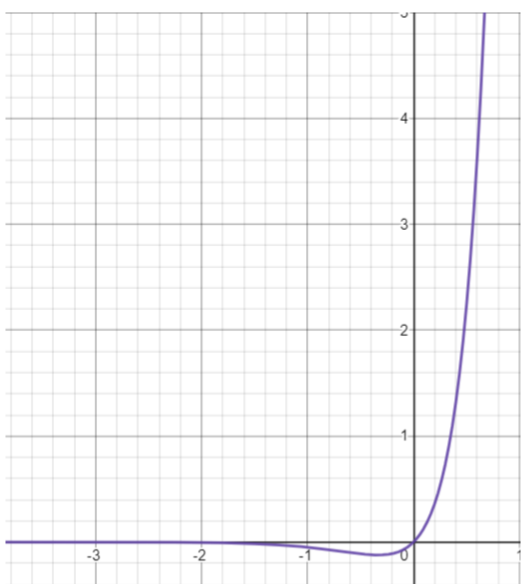

Figure 2: The graph of $f(x) = xe^{ax}$ when $a = 3$.

Here, we can see that the y-values are increasing for positive x-values. However, for negative x-values, there is a "dip" in the graph that is below the x-axis. Furthermore, as x becomes increasingly negative, the graph of $f(x) = xe^{ax}$ (for positive values of a) is inclining closer to zero.

Let us now examine the graph of $f(x) = xe^{ax}$ for negative values of a (**Fig. 3, Fig 4**).

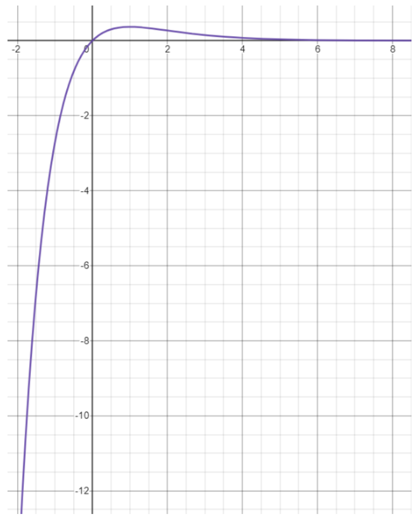

Figure 3: The graph of $f(x) = xe^{ax}$ when $a = -1$.

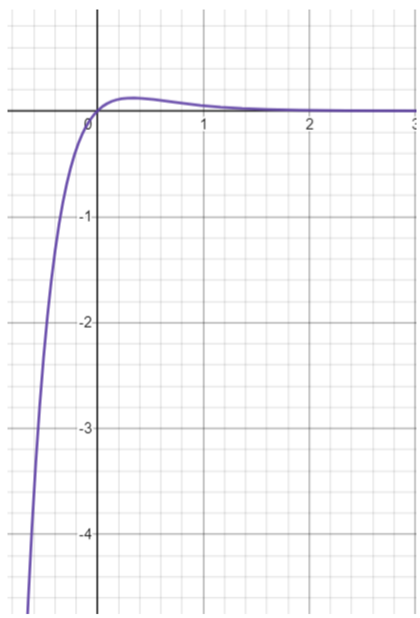

Figure 4: The graph of $f(x) = xe^{ax}$ when $a = -3$.

When a is negative, we have a similar graph; however, it seems to have been reflected across both the x- and y-axes. The "dip" is in a different location, and is now a "bump" (it goes up above the x-axis, then inclines towards the x-axis). Now, when x is becoming increasingly positive, it appears that the graph is going lower, and getting closer and closer to the x-axis.

1.2 Derived Question

Drawing on our observations in **Section 1.1**, we can ask two questions. Based on **Fig. 1 and 2**, we can ask: As x becomes increasingly negative, and a is a positive constant, does $f(x) = xe^{ax}$ have a horizontal asymptote at $y = 0$? Furthermore, based on **Fig. 3 and 4**, we can ask: As our x-value become increasingly positive, and our a-value is a negative constant, does $f(x) = xe^{ax}$ have a horizontal asymptote at $y = 0$?

For more clarity, we will be solving these two questions in different parts.

2 The Horizontal Asymptote

2.1 When a is Positive and x Approaches -∞

2.1.1 As A General Rule

To begin, we can subtract one from our x-value, since x is becoming increasingly negative. This strategy works because we are solving for a horizontal asymptote, or a horizontal line that a function approaches as x approaches ∞ or $-\infty$. In this case, we are trying to determine whether there is a horizontal asymptote at $y = 0$ when x approaches $-\infty$. Therefore, because x is so negative, subtracting one from x is less significant. Furthermore, when x is very negative, the ratio between x and $x - 1$ is approximately one, demonstrating that there is not a significant difference between x and $x - 1$.

2.1.2 Solving

Using the approach outlined above, our equation becomes $(x - 1) \cdot e^{a(x-1)}$. To better understand the components of this equation and to find its relationship to our original function, we can expand it:

$$\begin{aligned}
(x-1) \cdot e^{a(x-1)} &= (x-1) \cdot e^{(ax-a)} \\
&= (x-1) \cdot e^{ax} \cdot e^{-a} \\
&= (x-1) \cdot e^{ax} \cdot \frac{1}{e^a}
\end{aligned}$$

As x becomes increasingly negative, subtracting one from x becomes less significant, as doing so does not change the value of that term by much. Because of this, it looks like the main relationship between this equation and our original function is that this equation is being multiplied by $\frac{1}{e^a}$.

Since we restricted a to positive values, e^a will be a large number, indicating that the fraction $\frac{1}{e^a}$ is a very small value. This means that when x becomes more negative, our function is multiplied by $\frac{1}{e^a}$, and multiplying a number by a fraction over and over never makes it reach zero - it only gets very close to zero. So, zero is the horizontal asymptote for negative values of x.

To generalize the work that we have done in this section: as x becomes increasingly smaller by P amount, it is being multiplied by $\left(\frac{1}{e^a}\right)^P$, which is essentially an incredibly small fraction. We can prove this algebraically using the same steps as taken previously.

$$\begin{aligned}
(x-P) \cdot e^{a(x-P)} &= (x-P) \cdot e^{(ax-aP)} \\
&= (x-P) \cdot e^{ax} \cdot e^{-aP} \\
&= (x-P) \cdot e^{ax} \cdot \frac{1}{e^{aP}} \\
&= (x-P) \cdot e^{ax} \cdot \left(\frac{1}{e^a}\right)^P
\end{aligned}$$

When negative x is reduced by P, we can see that our original function is multiplied by $\frac{1}{e^a}$ P times, and when a number is multiplied by a fraction over and over, it never reaches zero, it only gets infinitely close.

Because x is being multiplied by $\frac{1}{e^a}$ as x becomes increasingly negative, the function will approach $y = 0$ but with never reach it. Thus, we can conclude that **$y = 0$ is indeed the horizontal asymptote for increasingly negative x-values.**

2.1.3 Analysis

We have found that as x is increasingly negative, $y = 0$ is the horizontal asymptote. This conclusion, paired with the observation that the "dip" we are studying is a minimum, suggests that the "dip" may actually be the *absolute* minimum of $f(x) = xe^{ax}$ for positive a-values.

For negative x-values, the graph of $f(x)$ continually inclines towards the x-axis without ever reaching it. In order to state that this "dip" is the absolute minimum, we need to prove that the right side of the graph is increasing for infinitely large x-values. If the y-value for increasingly positive x-values is always growing, we can conclude that the dip is the absolute minimum.

First, we can add one to our x-value, since x is approaching ∞. Doing so, our equation becomes $(x+1) \cdot e^{a(x+1)}$. To better understand the components of this equation and to find its relationship to our original function, we can expand it.

$$\begin{aligned}(x+1) \cdot e^{a(x+1)} &= (x+1) \cdot e^{(ax+a)} \\ &= (x+1) \cdot e^{ax} \cdot e^{a}\end{aligned}$$

As x becomes increasingly positive, adding one to x becomes less significant, as doing so does not change the value of that term by much. Because of this, it looks like the main relationship between this equation and our original function is that this equation is being multiplied by e^a.

Since we restricted a to positive values, e^a is a large number. This means that when x becomes more positive, $(x+1) \cdot e^{ax}$ is multiplied by e^a, and multiplying by a value that is greater than 1 over and over increases the number indefinitely.

To generalize the work that we have done in this section: as x becomes increasingly larger by P amount, it is being multiplied by $(e^a)^P$, which is essentially a very large amount. We can prove this algebraically using the same steps as taken previously.

$$\begin{aligned}(x+P) \cdot e^{a(x+P)} &= (x+P) \cdot e^{(ax+aP)} \\ &= (x+P) \cdot e^{ax} \cdot e^{aP} \\ &= (x+P) \cdot e^{ax} \cdot e^{aP} \\ &= (x+P) \cdot e^{ax} \cdot (e^a)^P\end{aligned}$$

When positive x is increased by P, we can see that our original function is multiplied by e^a P times, and when a number is multiplied a value greater than 1 over and over, it increases indefinitely.

Because as x approaches $+\infty$, the function is increasing, and as x approaches $-\infty$, our function gets increasingly closer to 0, we can conclude that **the dip in our function is the absolute minimum.**

2.2 When a is Negative and x Approaches -∞

To solve for the horizontal asymptote, we can modify the general approach provided in **section 2.1.1**, and add one to x. We are trying to determine whether there is a horizontal asymptote at $y = 0$ when x approaches ∞. Because x is so large, adding one to x does not drastically change the value of x. Additionally, when x is ∞, the ratio between ∞ and $\infty + 1$ is approximately one, which indicates that there is not a significant difference between x and $x + 1$.

Using this approach, our equation becomes $(x+1) \cdot e^{a(x+1)}$. To better understand the components of this equation and to find its relationship to our original function, we can expand it.

$$\begin{aligned}(x+1) \cdot e^{a(x+1)} &= (x+1) \cdot e^{(ax+a)} \\ &= (x+1) \cdot e^{ax} \cdot e^{a}\end{aligned}$$

Here, it seems like the main relationship between this equation and our original function is that this equation is being multiplied by e^a.

Since we restricted a to negative values, e^a will be a fraction, indicating that e^a is a very small value. This means that when x becomes more positive, it is multiplied by e^a, and multiplying by a fraction over and over never makes a number reach zero, it only gets very close. So, zero is the horizontal asymptote for positive values of x.

To generalize the work that we have done in this section: as x becomes increasingly larger by P amount, it is being multiplied by $(e^a)^P$, which is essentially an incredibly small fraction. We can prove this algebraically using the same steps as taken previously.

$$\begin{aligned}
(x+P)\cdot e^{a(x+P)} &= (x+P)\cdot e^{(ax+aP)} \\
&= (x+P)\cdot e^{ax}\cdot e^{aP} \\
&= (x+P)\cdot e^{ax}\cdot e^{aP} \\
&= (x+P)\cdot e^{ax}\cdot (e^{a})^{P}
\end{aligned}$$

When positive x is increased by P, we can see that our original function is multiplied by e^a P times, and when a number is multiplied by a fraction over and over, it never reaches zero, it only gets infinitely close.

We can see that because x is being multiplied by e^a as x becomes increasingly positive, it will never reach zero but it will continue getting closer. Thus, we can conclude that $\boldsymbol{x = 0}$ **is indeed the horizontal asymptote for increasingly positive $\boldsymbol{x}$ values.**

2.2.1 Analysis

We have found that as x is increasingly positive, $y = 0$ is the horizontal asymptote. Furthermore, in **Fig. 3 and 4**, we can notice that the upside down "dip" (bump), we are studying is a relative maximum. Building upon this conclusion, we can ask if this bump is only a relative maximum or if it is the *absolute* maximum of $f(x) = xe^{ax}$ for negative a values. Because we already figured out that for positive values of x, the graph gets more and more inclined towards the x-axis without reaching it, we know that that side will never increase, only decrease towards zero. In order to say that this bump really is the absolute maximum, we need to prove that the left side of the graph, is decreasing for infinitely negative x values. If the y-value for increasingly negative x-values is always getting smaller or more negative, we can conclude that the bump is the absolute maximum.

To begin, we can subtract one from our x-value, since x is becoming increasingly negative. Doing so, our equation becomes $(x-1)\cdot e^{a(x-1)}$. To better understand the components of this equation and to find its relationship to our original function, we can expand it.

$$\begin{aligned}
(x-1)\cdot e^{a(x-1)} &= (x-1)\cdot e^{(ax-a)} \\
&= (x-1)\cdot e^{ax}\cdot e^{-a}
\end{aligned}$$

As x becomes increasingly negative, subtracting one from x becomes less significant, as doing so does not change the value of that term by much. Because of this, it looks like the main relationship between this equation and our original function is that this equation is being multiplied by e^{-a}.

Since we restricted a to negative values, e^{-a} is a large number. This means that when x becomes more negative, $(x-1)\cdot e^{ax}$ is being multiplied by e^{-a}, and multiplying a negative by a value that is greater than 1 over and over makes the number get more negative without ever turning around and becoming more positive.

To generalize the work that we have done in this section: as x becomes increasingly larger by P amount, it is being multiplied by $(e^{-a})^P$, which is essentially a very large amount. We can prove this algebraically using the same steps as taken previously.

$$
\begin{aligned}
(x-P)\cdot e^{a(x-P)} &= (x-P)\cdot e^{(ax-aP)} \\
&= (x-P)\cdot e^{ax}\cdot e^{-aP} \\
&= (x-P)\cdot e^{ax}\cdot e^{-aP} \\
&= (x-P)\cdot e^{ax}\cdot \left(e^{-a}\right)^{P}
\end{aligned}
$$

When negative x is decreased by P, we can see that our original function is multiplied by e^{-a} P times, and when a number is multiplied by a value greater than 1 over and over, it increases indefinitely.

As x approaches $+\infty$, the function is decreasing towards zero, and as x approaches $-\infty$, our function gets more and more negative; therefore, we can conclude that **the bump in our function is the absolute maximum.**

3 Follow-up Question

One way that we can alter this graph is to multiply the exponent ax by x, giving us the function $g(x) = xe^{ax^2}$. The resulting graph has a shape so that, when we change the value of a, it has two curved shapes in it. When a is negative, the graph looks like this:

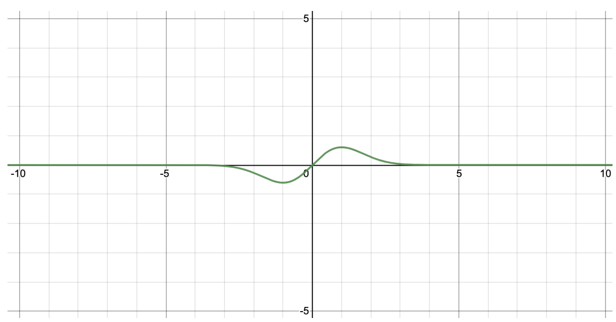

Figure 5: The graph of $g(x)$ when $a = -0.5$.

This differs from the function we started with in that there are *two* indentations in the graph of $g(x)$ when a is negative. It also seems like there are two asymptotes, one on each side of the bump/dips. This is different from our original because one side is not increasing/decreasing steadily. As x grows increasingly negative, the graph appears to have a horizontal asymptote at $y = 0$; as x grows increasingly positive, the graph appears to have a horizontal asymptote at $y = 0$. So, these two indentations could be the extrema of the graph. But, we do not know for certain whether this is true, so we can ask: is the dip in the function $g(x) = xe^{ax^2}$ the absolute minimum of the graph? In this section we will only be looking at the minima, not the maxima.

3.1 Solving for the Absolute Minimum

Using this approach, our equation becomes $(x-1)\cdot e^{a(x-1)^2}$. To better understand the components of this equation and to find its relationship to our original function, we can expand it:

$$
\begin{aligned}
(x-1)\cdot e^{a(x-1)^2} &= (x-1)\cdot e^{\left(ax^2-2ax+a\right)} \\
&= (x-1)\, e^{\left(ax^2-2ax\right)}\cdot e^{a} \\
&= (x-1)\, e^{ax^2}\cdot e^{-2ax}\cdot e^{a} \\
&= (x-1)\, e^{ax^2}\cdot \frac{1}{e^{2ax}}\cdot e^{a}
\end{aligned}
$$

Here, we have our recurring $x - 1$, which we proved that is essentially x (**Section 2.1.1**). In addition, the expanded equation has two new terms: $\frac{1}{e^{2ax}}$ and e^a. As x approaches $-\infty$, the denominator of $\frac{1}{e^{2ax}}$ becomes increasingly large because a also negative. This means that $\frac{1}{e^{2ax}}$ is an increasingly small number, and approaches zero. Thus, the other terms in the equation, $(x-1)\, e^{ax^2}$ and e^a, are essentially being multiplied by zero. Using this information, we can conclude that **$y = 0$ is the horizontal asymptote for increasingly negative x-values**.

To generalize the work that we have done in this section: as x becomes increasingly smaller by P amount, it is being multiplied by $\frac{1}{e^{2ax}}$, which is an incredibly small fraction (as shown before) and essentially zero. We can prove this algebraically using the same steps as taken previously.

$$\begin{aligned}(x-P)\cdot e^{a(x-P)^2} &= (x-P)\cdot e^{\left(ax^2-2axP+aP^2\right)}\\ &= (x-P)\, e^{\left(ax^2-2axP\right)}\cdot e^{aP^2}\\ &= (x-P)\, e^{ax^2}\cdot \frac{1}{e^{2axP}}\cdot e^{aP^2}\end{aligned}$$

When negative x is reduced by P, we can see that our original function is multiplied by $\frac{1}{e^{2axP}}$. Because this is an incredibly small fraction, our function is approaching zero. Thus, we can conclude that **$y = 0$ is indeed the horizontal asymptote for increasingly negative x-values**. This can be used towards our determination that the dip we identified is the absolute minimum because we have now proved that the left side of our graph is increasing towards the x-axis and not turning around and dipping lower.

We also need to prove that for positive values of x, the graph never goes lower than our current assumed absolute minimum. If we can show that the right side of the graph (or when x is positive), never goes below the x-axis, then we know that it never dips below our current minimum.

In order to do this, we need to ensure that for any positive value of x, the graph does not dip below the x-axis; in other words, $g(x)$ is never negative. Our function is $g(x) = xe^{ax^2}$, where there are two terms being multiplied. This means that this function is only negative when one of the terms are negative. Here, one term is x and the other term is an exponential function with base e. When the base of an exponential term is positive, its output can never be negative, which means that in order for our function's output to be negative, our x-value must be the negative term. However, we have restricted our view to positive x-values only because we know the behavior of the graph for negative x-values already (solved previously). This means that there are no positive x-values that make our function negative, so it never dips below the x-axis. We can thus conclude that the left side of the graph always remains above the minimum because the minimum falls below the x-axis.

3.1.1 Analysis

We have found that as x approaches $-\infty$, our approaches the x axis from below, getting closer and closer. We have also found that as x becomes increasingly positive and approaches $+\infty$, our graph never dips below the x axis. From here, we can conclude that both sides of the graph never go below the dip present in our function meaning **the dip in $g(x) = xe^{ax^2}$ is the absolute minimum.**

4 Conclusion

By modifying the equation of this function to study the change in y-values as x-values change, we can conclude that the line $y = 0$ serves as an asymptote for $f(x)$, no matter what the value of a is. Through following the rules of exponential equations and fractions, this work has shown us that the bump seen when a is positive and, by association, the dip when a is negative are the absolute maximum and minimum of their respective graphs. Overall, this function is an excellent example of the logic of exponents and how they relate to asymptotes.

5 Further Inquiry

In this project, we examine the shape of the graph for general categories of values of a. However, when we apply a specific value to a and then change that value, we can see that the local extreme changes slightly (or significantly, depending on how large the change in a is). **(Fig. 6, 7, 8)** This leads us to wonder: what is the relationship between the distances from the function's local extreme to the x-axis as a changes at a constant rate?

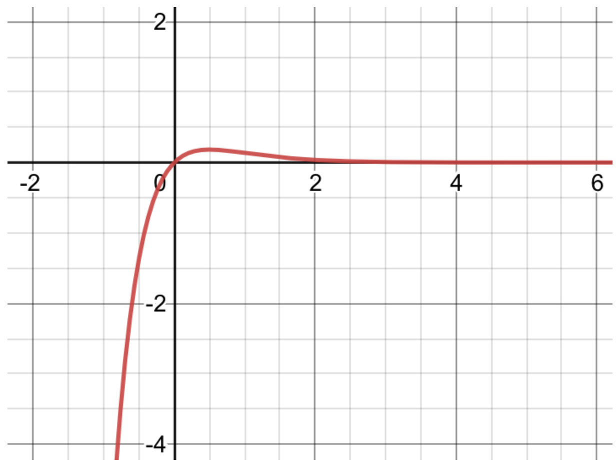

Figure 6: The graph of xe^{ax} when $a = -0.5$.

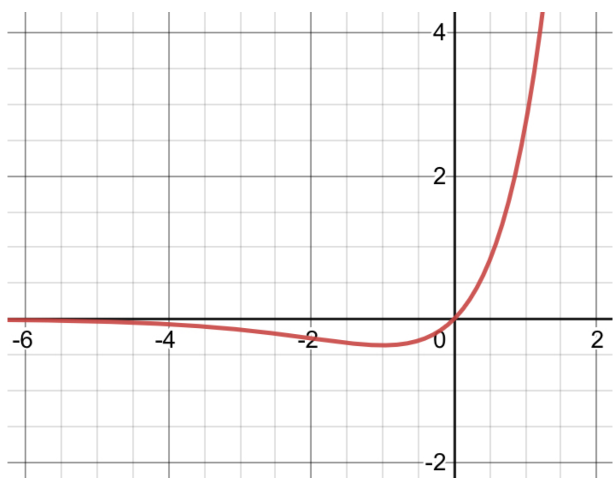

Figure 7: The graph of xe^{ax} when $a = 0.5$.

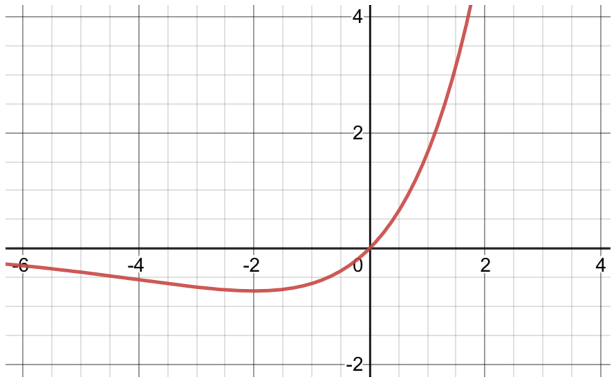

Figure 8: The graph of xe^{ax} when $a = 1$.