CS121 Lecture 14: Modeling Efficient Computation

Define running time

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TM ⟺ RAM bisimulation

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Universal Machine

Time Hierarchy Thm

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Nonuniform complexity

$P \subseteq P_{/poly}$

Define Running Time

Def. Let $F: \{0,1\}^* \to \{0,1\}$, then the running time to solve to solve F is the time it takes to compute F on a 2019 4.8Ghz Intel Core i9 Macbook pro.

Time to compute $F(x)$ depends on:

- Hardware / architecture
- Algorithm used
- Input x

ex: mergesort is $c \cdot |x| \log |x|$

Def: Let $T: \mathbb{N} \to \mathbb{N}$ and $F: \{0,1\}^* \to \{0,1\}$. We say $F \in TIME(T(n))$, where $TIME_{TM}(T(n))$ is the set of F's $\{0,1\}^* \to \{0,1\}$, if : $\exists$ NAND-RAM program (TM TM M) P s.t. $\forall$ large enough n and $x \in \{0,1\}^n$, on input x, $\overset{M}{P}$ halts after executing $\leq T(n)$ lines and outputs $F(x)$

For TM

Why RAM?

- closest model to modern architectures
- this class only cares about polynomial vs exponential.

If $F \in TIME_{TM}(T(n))$ then $F \in TIME(10 \cdot T(n))$

If $F \in TIME(T(n))$ then $F \in TIME_{TM}(T(n)^4)$

Thm. TMs can simulate RAM

Idea. - Encode state of NAND-RAM as list (var, location, value)

- Simulate one step of RAM: scan list and update everything

Classes P and EXP

Def: $F: \{0,1\}^* \to \{0,1\}$ is in P if $\exists$ poly-time TM M that computes F

Claim: $P = \bigcup_{c=1,2,3,\ldots} TIME_{TM}(n^c) = \bigcup_{c=1,2,3\ldots} TIME(n^c)$

$\subseteq$

$\supseteq$

Def. $F:\{0,1\}^+ \to \{0,1\}$ in EXP if $\exists$ TM M of time $\leq$ $\exp(\text{poly}(n))$ computes F.

Claim. $\text{EXP} = \bigcup_{c=1,2,3\ldots} \text{TIME}_{TM}(2^{n^c}) = \bigcup_{c=1,2,3\ldots} \text{TIME}(2^{n^c})$

Time Heirarchy Theorem

Thm. $\forall^* T:\mathbb{N}\to\mathbb{N}$, $\text{TIME}(T(n)) \subsetneq \text{TIME}(T(n)\log\log n)$

- $T(n) \geq n$
- $T(n+1) \geq T(n)$ monotone
- $n \longrightarrow T(n)$ computable

more resources = more functions compute

Thm. $P \subsetneq \text{EXP}$

Proof. Let $\text{HALT}_{n^{\log n}}(M,x) = \begin{cases} 1 & M \text{ halts on } x \text{ within } \leq n^{\log n} \text{ steps} \\ 0 & \text{o/w} \end{cases}$

Q1. Prove $\text{HALT}_{n^{\log n}} \in \text{EXP}$