CS121 Lecture 14 October 22, 2019

Time heirarchy thm

- there are some problems that just don't have polynomial solutions

Today

If ISET ∈ P, 3SAT ∈ P

If MAXCUT ∈ P, 3SAT ∈ P

...

3SAT problem

Input: φ - 3CNF formula $C_0 \wedge C_1 \wedge C_2 \dots \wedge C_{m-1}$

each clause C_j = OR of 3 variables/negations

Output: $3SAT(\varphi) = 1$ iff $\exists x \in \{0,1\}^n$ satisfying φ

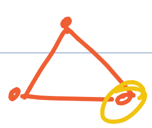
Q. What is $3SAT((x_0 \vee \bar{x}_1 \vee x_2) \wedge (\bar{x}_0 \vee x_1 \vee \bar{x}_2))$?

1 ($\exists x$: $x = 111$)

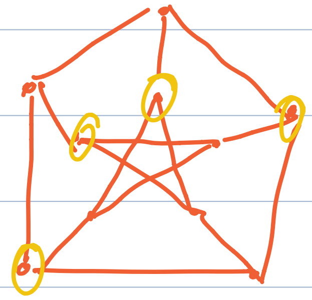
Independent Set

Input: Graph $G = (V, E)$ and number k

Output: If there are k vertices with no edges.

A.  $ISET(A, 1) = 1$ $ISET(A, 2) = 0$

B.



$ISET(B,4) = 1$

$ISET(B,5) = 0$

↳ can have max 2 inside and 2 outside

3SAT — can verify correct answer — Independent Set

3SAT — no known poly algo — Independent Set

$3SAT \leq_p ISET$

Thm. Suppose $ISET \in P$, then $3SAT \in P$.

Corollary. Suppose that $3SAT \notin P$, then $ISET \notin P$.

Proof. We will show poly-time $R: \{0,1\}^* \to \{0,1\}^*$

showing this implies Thm

$R: \{3CNF \text{ formulas}\} \to \{(\text{graphs, numbers})\}$

s.t. for every 3CNF φ, $3SAT(\varphi) = ISET(R(\varphi))$

because you're comparing polynomials

1. Assume an algo A that solves ISET in $|X|^a$ time.

2. Will show an algo B that solves 3SAT in $|X|^b$ time.

3. Intermediate Lemma: R computable in $|\varphi|^c$ time $ISET(R(\varphi)) = 3SAT(\varphi)$ $\forall \varphi$.

Define Alg $B(\varphi)$:

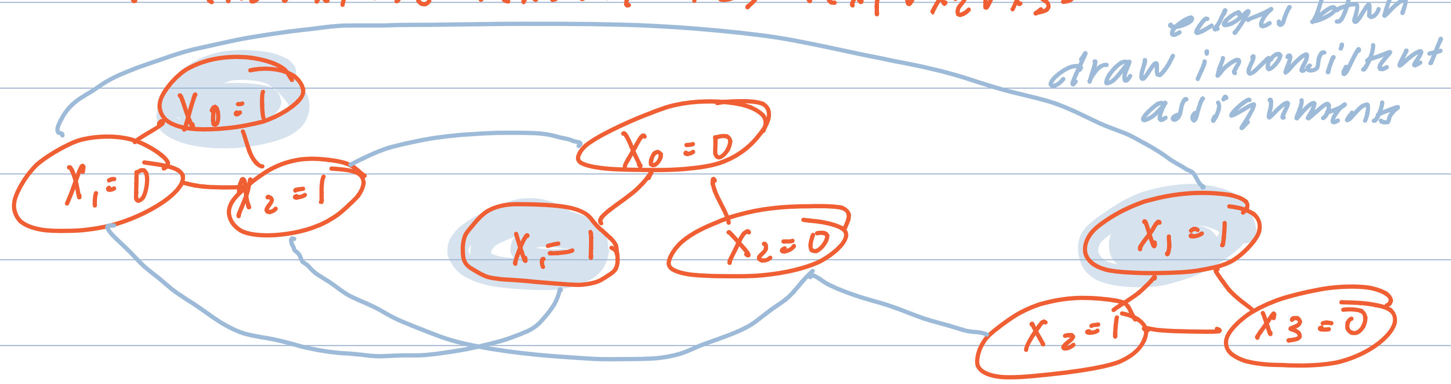
return $A(R(\varphi))$

① Running time $\leq |R(\varphi)|^a \leq (|\varphi|^c)^a$

② Actually solves: $\forall \varphi$, $B(\varphi) = 3SAT(\varphi)$

map triples to nodes $\quad ISET(R(\varphi))$

EX. $\varphi = (x_0 \vee \bar{x}_1 \vee x_2) \wedge (\bar{x}_0 \vee x_1 \vee \bar{x}_2) \wedge (x_1 \vee x_2 \vee \bar{x}_3)$



$x = 1101$

polynomial time algorithm

1. G = graph w/ $3m$ vertices we will name $(j,1)$, $(j,2)$, and $(j,3)$ for $j \in [m]$
2. for every $j \in [m]$, add edges $(j,1)-(j,2)$, $(j,2)-(j,3)$, $(j,3)-(j,1)$
3. for every pair of clauses C_j and $C_{j'}$, if literals (j,a) and (j',b) conflict, add $(j,a)-(j',b)$ edge
4. return (G, m)

Claim 1. Completeness : If $3SAT(\varphi)=1$, $ISET(G,m)=1$

Pf. Assume x satisfies φ. Then for every C_j, there is a literal (j,a) satisfied. Add (j,a) to set S. ∎

Claim 2. Soundness : If $ISET(G,m)=1$ then $3SAT(\varphi)=1$

Pf: Assume S independent set of size m in G.

Q. Show that S contains exactly one vertex per triangle.

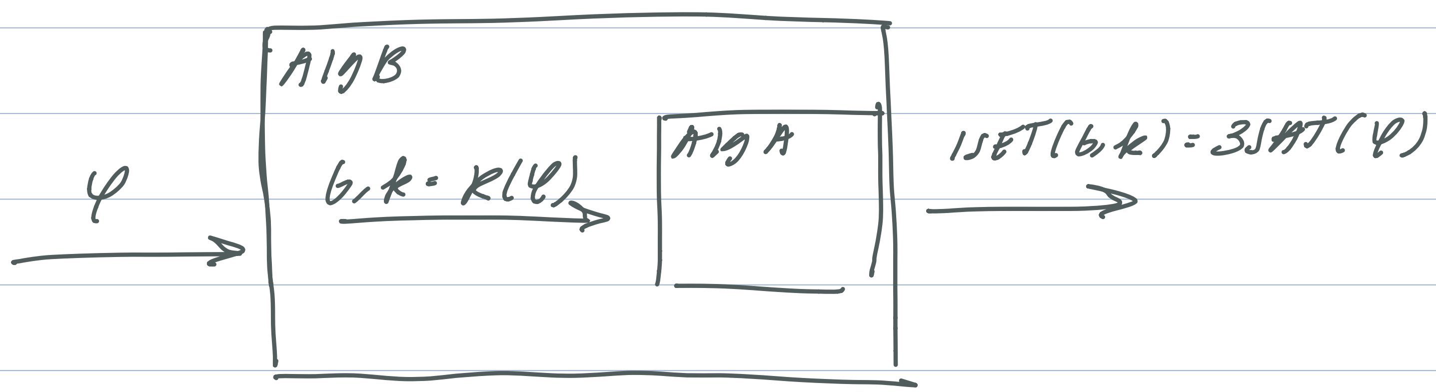
two vertices → means there is an edge.

0 vertices → m vertices for m-1 triangles
(by pigeonhole, one triangle will have 2)

Set $x_i^* = 1$ if S contains vertex tagged "$x_i = 1$", o/w $x_i = 0$.
For every clause C_j there is a vertex in S tagged "$x_i = b$"
We claim $x_i^* = b$.
- if $b = 1$: by definition
- if $b = 0$: S can't contain vertex tagged "$x_i = 1$" since its independent. $\blacksquare$

Polynomial-time Reduction

We showed: Poly time A for ISET → Poly time B for 3SAT



Def. Let $F, G: \{0,1\}^* \to \{0,1\}$. We say $F \leq_p G$ if $\exists$ poly-time $R: \{0,1\}^* \to \{0,1\}$ s.t. $\forall_{x \in \{0,1\}^*}\ F(x) = G(R(x))$

Exercise in book:
If $F \leq_p G$ and $G \leq_p H$, then $F \leq_p H$.