Econ 1052 Lecture 15 Tues, March 31, 2020

~ Bayesian Games ~

	H	T
H	(-1, 1)	(1, -1)
T	(1, -1)	(-1, 1)

player 1 mixing determined by player 2 payoff (and vice versa)

version of matching pennies

↓

	H	T
H	$-1+x,\ 1+y$	$1+x,\ -1$
T	$1,\ -1+y$	$-1,\ 1$

$$x, y \overset{iid}{\sim} \begin{cases} \varepsilon & \text{w/ prob } 1/2 \\ -\varepsilon & \text{w/ prob } 1/2 \end{cases}$$

$\varepsilon > 0$

$$T_{row} = \{x_{high}, x_{low}\}$$
$$T_{column} = \{y_{high}, y_{low}\}$$

Baysian NE:

$S_{row}(x_h) = H$ $\quad$ $S_{col}(y_h) = H$

$S_{row}(x_t) = T$ $\quad$ $S_{col}(y_l) = T$

Harsany's Purification Theorem:

a generic game

All equilibria (pure or mixed) of almost any complete info game are the limits of pure strategy eq in perturbed games

⟷ $\varepsilon \to 0$, converges

↳ where all players have small independent shocks to payoffs

rules out weakly dominated strategies

Bayesian w/ continuum types

N

H

$T = (T_i)_{i \in N}$

A_i

$p \in \Delta H \times T$

u_i

$s_i : T_i \to A_i$

$s = (s_i)_{i \in N}$

Requirements:

1. symmetry

   $(s_1, \ldots, s_n)$ is symmetrical if

   $\exists \tilde{s} : T_1 \to A_1$ s.t. $\forall i : s_i(t_i) = \tilde{s}_i(t_i)$

2. monotone

   $T_i \subseteq \mathbb{R}$, $A_i \subseteq \mathbb{R}$

   and s_i is either weakly increasing or weakly decreasing

3. Linear

   $s_i(t_i) = \alpha_i t_i + \beta_i$ for constants α_i and β_i

when checking for linear eq: need to check all deviations

Application 1: Coordination Game

	Invest	Not Invest
Invest	x_1, x_2	$x_1 - 1$, 0
Not Invest	0, $x_2 - 1$	0, 0

$x_1, x_2 \overset{iid}{\sim} U[\underline{x}, \bar{x}]$

$\underline{x} < 0$, $\bar{x} > 1$

low val don't invest

high val invest ind. of other

monotone equilibria

cutoff x_i^*

$$s_i(x_i) = \begin{cases} \text{invest} & \text{if } x_i \geq x_i^* \\ \text{not} & \text{if } x_i < x_i^* \end{cases}$$

Calculate cutoff eq

Symmetric monotone eq: common x^*

$$Pr(x_j < x^*) = \frac{x^* - \underline{x}}{\bar{x} - \underline{x}}$$

Invest: $(x_i - 1)\left(\frac{x^* - \underline{x}}{\bar{x} - \underline{x}}\right) + x_i\left(\frac{\bar{x} - x^*}{\bar{x} - \underline{x}}\right)$

opponent invests

opponent doesn't invest

Don't invest: 0

Invest = Don't invest when x_i = cutoff

$$x^* = \frac{-\underline{x}}{\bar{x} - \underline{x} - 1} \in (0,1)$$

$-\underline{x} \rightarrow > 0$

$\bar{x} - \underline{x} - 1 \rightarrow > 0$

Application 2: First Price Auction

$t_1, t_2 \overset{iid}{\sim} U[0,1]$

$$u_1(b_1, b_2, t_1, t_2) = \begin{cases} t_1 - b_1 & \text{if } b_1 > b_2 \\ 0 & \text{otherwise} \end{cases}$$

$b_i \in [0,1]$

symmetric linear eq

$s_i(t_i) = \alpha + \beta t_i$ for α, β constants

$$\forall t_i: s_i(t_i) \in \arg\max_{b_i} \underbrace{P(b_i > b_{-i})}_{P(b_i > \alpha + \beta t_{-i}) = \frac{b_i - \alpha}{\beta}} [t_i - b_i]$$

looking for $> t_{-i}$

FOC: $(t_i - b_i) - (b_i - \alpha) = 0$

$\rightarrow b_i = \frac{\alpha}{2} + \frac{t_i}{2} \rightarrow = \alpha + \beta t_i$

$s_i(t_i) = \frac{\alpha}{2} + \frac{t_i}{2}$ $\qquad \alpha = 0, \beta = 1/2$

New Version

2 bidders

$t_i \overset{iid}{\sim} U[0,1]$

1st price auction

BNE $s_i(t_i) = t_i/2$

2nd price auction

BNE

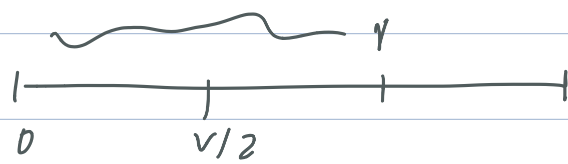
dominant strategy

$s_i(t_i) = t_i$

$b_2 = t_2 \sim U[0,1]$

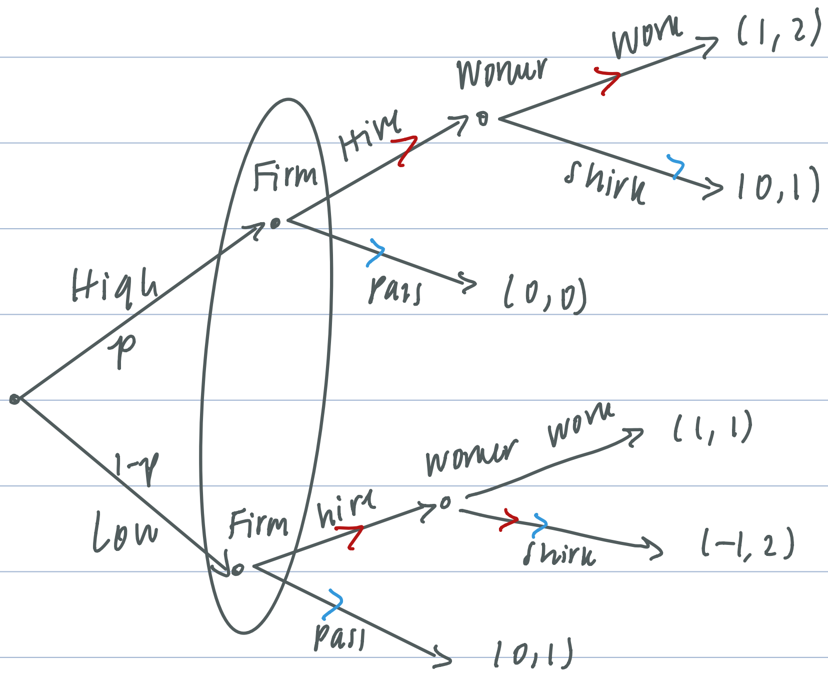
$v = b_1$

$\mathbb{E}[b_2 \mid b_2 < \overset{v}{\cancel{b_1}}] = v/2$



$$\int_0^1 P(v > b_2)\, \mathbb{E}[b_2 \mid b_2 < v]\, dv$$

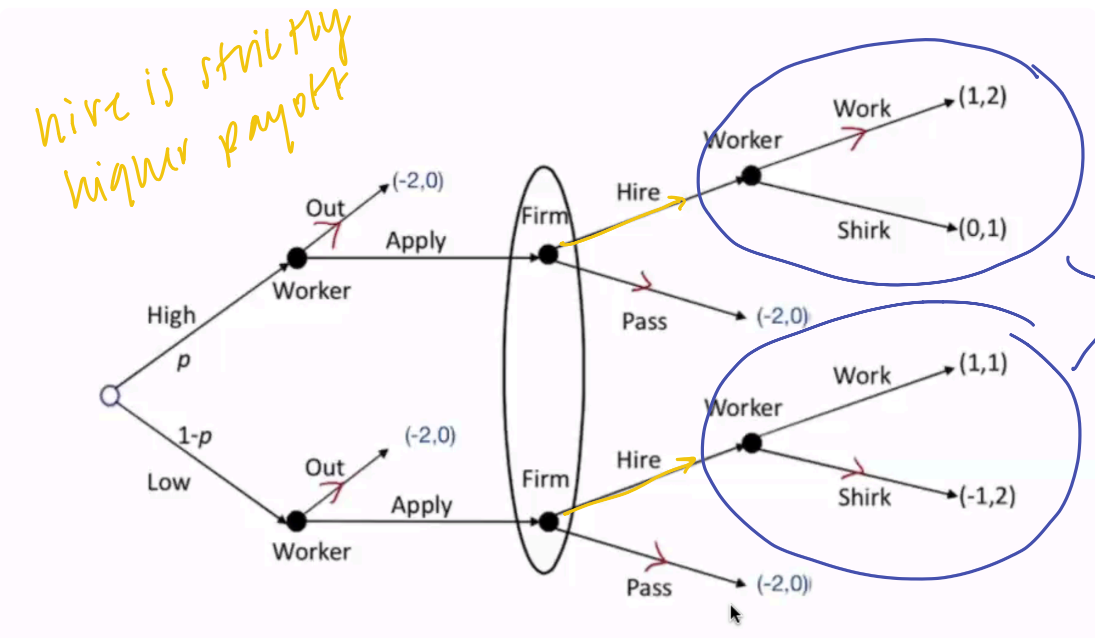
$$= \int_0^1 \frac{v^2}{2}\, dv$$



for high p, what we expect

not SPNE!

Bayesian NE

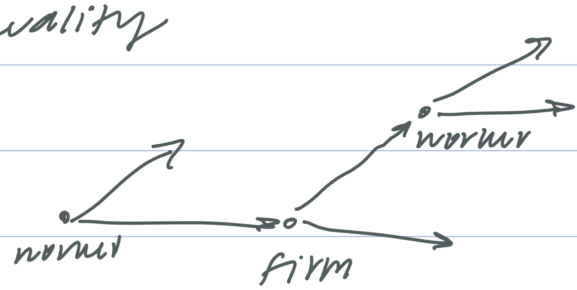


proper subgames

An 'unreasonable' SPNE

desparate Firm, worker applies knowing high or low quality

Firm decides w/o knowing quality



$b_{worker}(t_w^{high} | h)$

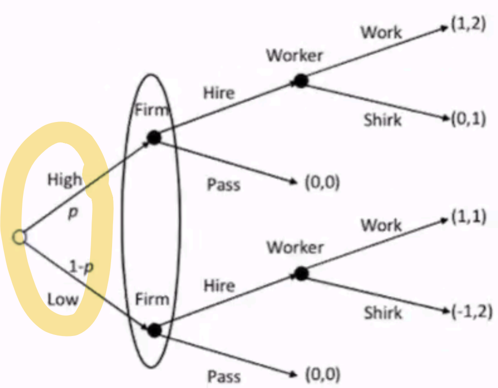
A *dynamic Bayesian game* (with publicly observed actions) consists of

- a set of players $N = \{1, \ldots, n\}$,
- a set of type profiles $T = \{T_1, \ldots, T_n\}$,
- a probability distribution p_i on T_i for each player i,
- a game tree (in which all past actions are known),
- an assignment of each non-terminal history to a player (including Nature), and
- a payoff function $u_i : T \times Z \to \mathbb{R}$ for each player where Z is the set of terminal nodes.

independence!!

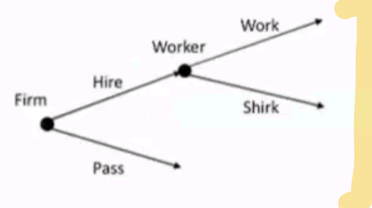
dropped (H)

extensive-form game



dynamic Bayesian game.

- the set of players is $N = \{\text{Firm, Worker}\}$;
- the set of types is $T_F = \{t_F\}$ for the firm and $T_W = \{H, L\}$ for the worker, where t_F is a dummy type, and h and l stand for high-ability and low-ability, respectively;
- the probability of H is $p_W(H) = p$, and the probability of l is $p_W(L) = 1 - p$;
- the "game tree" and the assignment of players are as follows



- the payoffs are as given in Figure 15.1, as a function of types.

dynamic Bayesian Game

strategy

$s_i(t_i, h) \in A(h)$ & perfect information game!

$\sigma_i(t_i, h) \in \Delta A(h)$

belief system

$b_i : H \to \Delta T_i$ $\quad b(t_1, \dots t_n \mid h)$

$(b) = (b_i)_{i \in N}$ $\quad = b_1(t_1 \mid h) \cdot b_2(t_2 \mid h) \dots$

$b_i(t_i^L \mid h) =$ belief of player called to play at h of the probability that i has type t_i^L.

assessment

strat profile

(σ, b) belief system

Definition 15.1. An *assessment* is a pair (σ, b) of a strategy profile σ and a belief system b.

Natural to require:

After i plays a at h:

$$b_j(t_j \mid \underbrace{h, a}_{\text{next history}}) = b_j(t_j \mid h) \qquad (\text{for all } j \neq i,\ t_j \in T_j). \tag{15.1}$$

i's choice at h is not informative of j's type

reasonable if t_i, t_j not independent?

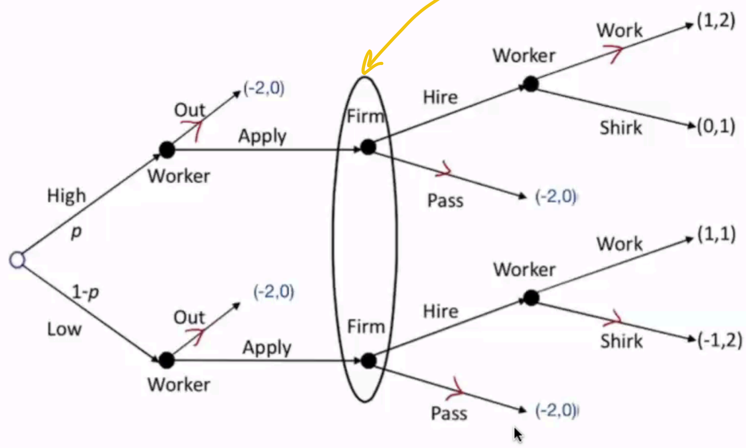
$$b_i(t_i \mid h, a) = \frac{\sigma_i(a \mid t_i, h)\, b_i(t_i \mid h)}{\Pr(a \mid \sigma_i, h)} \qquad \text{if } \Pr(a \mid \sigma_i, h) > 0. \tag{15.3}$$

Definition 15.2. An assessment (σ, b) is said to be *sequentially rational* if at each history h, the player i who is to move at h maximizes her expected utility

1. given her type t_i and her beliefs $b(\cdot \mid h)$ about the other players' types at history h, and
2. given that the players will play according to σ in the continuation game.

Definition 15.3. An assessment (σ, b) is said to be a *perfect Bayesian Nash equilibrium* (henceforth PBNE) if it is sequentially rational and satisfies (15.1) and (15.3) throughout.

at this history, the firm can profitably deviate



An 'unreasonable' SPNE

not a PBNE

Hiring +1

Passing -2