Econ 1052: Lecture 6

Feb 18, 2020

Partner Project w/ 2 parts

$S_a = S_b = [0, \infty)$ effort on project

$U_a(S_a, S_b) = \theta\, \alpha(S_a) \beta(S_b) - S_a$

$U_b(S_a, S_b) = \theta\, \alpha(S_a) \beta(S_b) - S_b$

$\theta \geq 0$, $\alpha(\cdot)$ and $\beta(\cdot)$ increasing $\in [0,1]$

↳ probability of success

$$V_a(S_a + \Delta, S_b) - V_a(S_a, S_b) = \theta \underbrace{(\alpha(S_a + \Delta))}_{\geq 0} \underbrace{\beta(S_b)}_{\geq 0} - \Delta$$

neat and differentiable

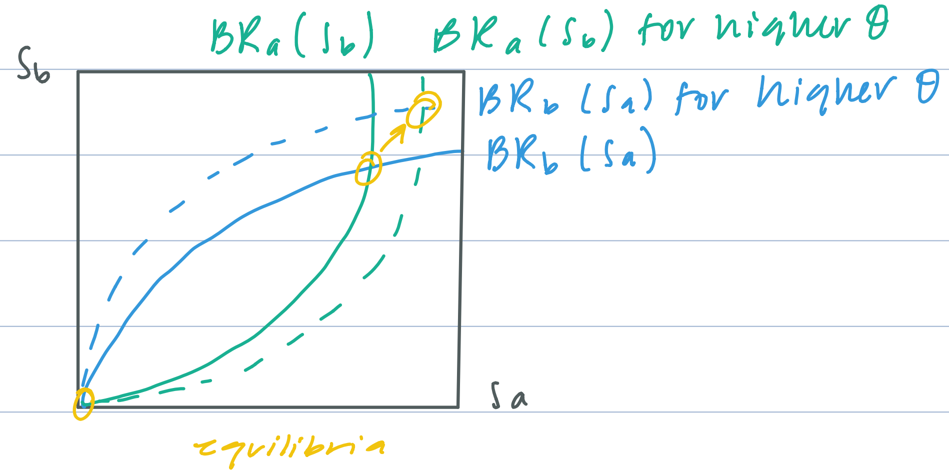
$\alpha(S_a) = S_a^{1/3}$

$\beta(S_b) = S_b^{1/3}$

no need for differentiable

$BR_a(S_b) = (\theta/3)^{3/2} S_b^{1/2}$

$BR_b(S_a) = (\theta/3)^{3/2} S_a^{1/2}$



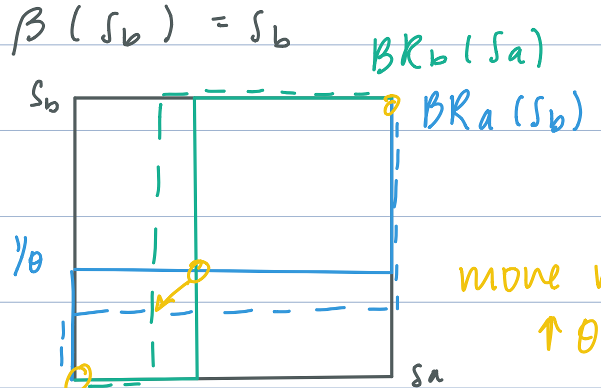
linear

$\alpha(S_a) = S_a$

$\beta(S_b) = S_b$

if opponent effort $< \theta$: 0 effort

$> \theta$: 1 effort



more valuable → internal equilibrium is smaller effort

↑ θ ↓ $1/\theta$

$/\theta$

Notation

$x \geq y$ if $\forall k \quad x^k \geq y^k$

vector vector

K elements

"join" $x \vee y \equiv (\max\{x^k, y^k\})_{k \in K}$

"meet" $x \wedge y \equiv (\min\{x^k, y^k\})_{k \in K}$

2D Functions

2 players, fixed θ

$f: \mathbb{R}^2 \to \mathbb{R}$ has increasing differences (ID) if for any $(x_1, x_2) \geq (y_1, y_2)$:

$$f(x_1, x_2) - f(y_1, x_2) \geq f(x_1, y_2) - f(y_1, y_2)$$

$$f(x_1, x_2) - f(y_1, y_2) \geq f(x_1, y_2) - f(y_1, x_2)$$

if $f(\cdot)$ twice diff $\longleftrightarrow$ $\frac{\partial^2 f}{\partial x \partial y} \geq 0$

Def. $f: \mathbb{R}^m \to \mathbb{R}$ is supermodular if $\forall x, y \in \mathbb{R}^n$:

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

Lemma: f is supermodular if for any two arguments j, k:

$\forall$ any x_{-jk} $f(\cdot, \cdot, x_{-jk})$ has increasing differences in j, k.

Topkis Monotonicity Theorem

Conditions

Agent chose $x \in X$ nonempty, closed, bounded

$$X = X_1 \times X_2 \times X_3 \times \cdots X_m \subseteq \mathbb{R}^m$$

parameters $\theta \in \Theta$ nonempty

$$\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \subseteq \mathbb{R}^n$$

to maximize $u: X \times \Theta \to \mathbb{R}$

continuous, supermodular

Theorem

For problems satisfying the LHS conditions, the largest and smallest maximizers:

$$\overline{B}(\theta) = \max \{x \in X \mid u(x,\theta) \geq u(y,\theta) \text{ for all } y \in X\}$$

$$\& \quad \underline{B}(\theta) = \min \{x \in X \mid u(x,\theta) \geq u(y,\theta) \text{ for all } y \in X\}$$

exist and are weakly increasing in θ.

If $x \in \overline{B}(\theta)$ then $\forall y \in B(\theta)$, $x \geq y$.

Take $\theta > \theta'$: want to show $\overline{B}(\theta) > \overline{B}(\theta')$.

want to show $x = \overline{B}(\theta) \vee \overline{B}(\theta')$

$y = \overline{B}(\theta) \wedge \overline{B}(\theta')$

suppose not $\overline{B}(\theta) \geq \overline{B}(\theta') \longrightarrow x > \overline{B}(\theta)$

$$0 > u(\underset{\text{join}}{x},\theta) - u(\overline{B}(\theta),\theta) \underset{\text{supermod}}{\geq} u(x,\theta') - u(\overline{B}(\theta),\theta')$$

$$\geq u(\overline{B}(\theta'),\theta') - u(\underset{\text{meet}}{y},\theta') \overset{\text{supermod}}{\geq} u(\overline{B}(\theta'),\theta') - u(y,\theta') \geq 0$$

$\overline{B}(\theta')$ is optimal at 0

Implications

$u_a(s_a, s_b, \theta) = \theta\, \alpha(s_a) B_b(s_b) - s_a$

$BR_n(S_b, \theta)$

goods G

$X \in \{0,1\}^G$

$u(X, p) = v(X) - p \cdot x$ supermodular in $(X, -p)$

supermodular

Supermodular Game Properties

<u>Normal-Form Game</u>

G is supermodular if $S_i = S_{i1} \times S_{i2} \times S_{i3} \times \cdots \subseteq \mathbb{R}^{m_i}$

$\bar{S}_i$ largest strategy

$\underline{S}_i$ smallest strategy

$\bar{S}$ largest strategy profile

$\underline{S}$ smallest strategy profile

$v_i : S \to \mathbb{R}$ is supermodular and continuous

<u>Bertrand Game</u>

- Bertrand games have strategic complements
- Linear Bertrand w/ product differentiation

$$Q_i(p) = A - \alpha_i p_i + \sum_{j \neq i} B_j p_j$$

$$u_i(p) = (p_i - c_i) Q_i(p)$$

$$\frac{\partial^2 u_i}{\partial p_i \partial p_j} = B_j \geq 0$$

Linear Carnot

3-player carnot is not supermodular

$P(q) = A - q_1 - q_2$

$V_i(q_i) = q_i P(q) - c_i(q_i)$

$$\frac{\partial^2 u}{\partial q_1 \partial q_2} = -1 \not\geq 0$$

1 choose q_1

2 chooses $V_2 = -q_2$

$\bar{B}_i(s_{-i}) \equiv$ i's largest BR to s_{-i}

$\underline{B}_i(s_{-i}) \equiv$ i's smallest BR to s_{-i}

$\bar{B}(s) = (\bar{B}_i(s_i))_{i \in N}$

$\bar{B}(\bar{B}(\bar{s}))$

highest and take highest BR

lowest and take lowest BR

$\bar{B}^k(\bar{s}) \qquad \underline{B}^k(\underline{s})$

Thm. For any supermodular game,

$$\bar{s}^* = \lim_{k \to \infty} \bar{B}^k(\bar{s}) \quad \text{and}$$

$$\underline{s}^* = \lim_{k \to \infty} \underline{B}^k(\underline{s})$$

are Nash Equilibria and $\forall$ rationalizable s,

$\underline{s}^* \leq s \leq \bar{s}^*$.

As you change payoff params, how do equilibria change?

Family of supermodular games G^θ where $\theta \in \mathbb{R}^m$,

$u_i(s, \theta)$ is supermodular.

Then $\bar{s}^+(\theta)$ and $\underline{s}^+(\theta)$ are weakly increasing in θ.

If $\theta \geq \theta'$ then $\bar{s}(\theta) \geq \bar{s}(\theta')$