CS 124 Lecture 21 April 15, 2020

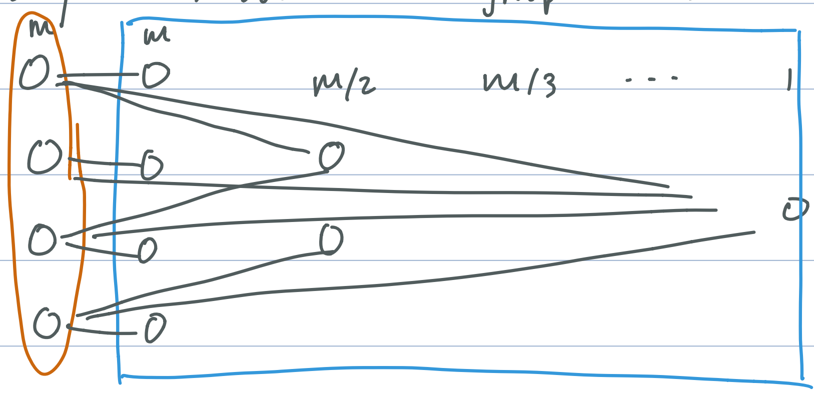
~ Approximation Algorithms ~

multiplicative factor of optimal

Vertex Cover

Greedy vertex cover on layered graph w/ nodes connected to base

$m = k!$



vertex cover of size m

degree m

greedy alg chooses and removes edges since it has highest degree and take all vertices but first layer

m log m vertices

Good Algo

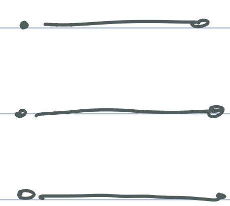
Take any edge

Put both vertices in cover

Remove adj edges

+ continue

within a factor of 2 of OPT



matching: collection of disjoint edges

optimal VC $\geq$ max matching

$\geq$ alg matching

alg finds VC = 2 $\times$ alg matching

OPT VC $\geq$ 1/2 VC found by alg

factor of 2 from optimal

Maximum Cut

Given a graph $G = (V, E)$

Give a cut V_1, V_2

$(V_1 \cup V_2 = V, \quad V_1 \cap V_2 = \emptyset)$

so that the total # of edges crossing cut is maximized

can be extrapolated to weighted edges

maxcut = NPC

Randomized Approx

Expected value of algs output and say what factor of opt it is.

- Flip a coin for each vertex

$P(\text{edge crosses cut}) = 1/2$

$E[\text{size of cut}] = |E|/2$

max cut $\leq |E|$, so within factor of 2

Deterministic Alg

Local search

Alg.

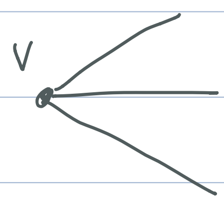
Split into S_1 and S_2

while $\exists$ vertex for which switching increases the cut

switch

⟶ Each move increases size of cut, cut size bounded so alg will stop.

Claim: Within factor of 2 from optimal



For some vertex, at least half must be across the cut (if more than half are on same side you should move the vertex across)

$\therefore$ half the total edges cross the cut:

$$C = \frac{1}{2} \times \left(\sum_{v \in V_1} |(v,u) \in E : u \in V_2| + \sum_{v \in V_2} |(v,u) \in E : u \in V_1| \right)$$

$$\geq 1/2 \left(\sum_{v \in V_1} 1/2 \cdot \delta(v) + \sum_{v \in V_2} 1/2 \cdot \delta(v) \right)$$

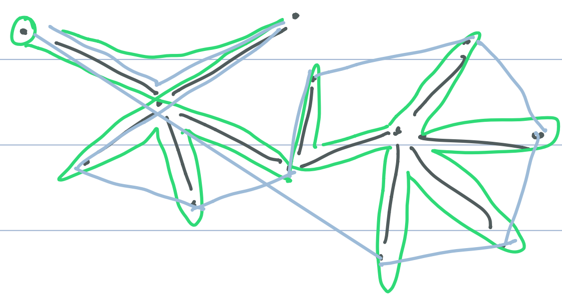
$$= 1/4 \left(\sum_v \delta(u) \right) = 1/4 \times (2|E|) = 1/2 |E| \quad \blacksquare$$

Euclidean TSP

2D points in space

(alg works w/ higher D)

starting vertex DFS (visits more than once)



short-circuit

Alg.

1. Find an MST
2. Walk (DFS) over MST to find pseudo-tour
3. Short circuit repeats to get a tour

   triangle inequality via straight line

length of tour $\leq$ length of pseudo tour $= 2 \cdot$ Length of MST

length of MST $\leq$ optimal tour

since every tour contains an MST

3/2 - approximation exists as well

complicated DP to get $(1+\varepsilon)$-approx w/ runtime $O(n^{1/\varepsilon})$ for any $\varepsilon \geq 0$

NPC → Integer LP

turn into LP to solve

MAX-SAT

(max 3SAT for simplicity)

maximize # of satisfyable clauses

Randomized

- random solution

  Pr(clause is satisfied) = 7/8

$E[\#\text{ satisfied clauses}] = 7/8 \times |\#\text{clauses}|$

k variable clause : $1 - 2^{-k}$

<u>LP</u>

$(x_2 \vee \bar{x}_4 \vee x_6 \vee \bar{x}_8)$

0 if false, 1 if true

$\downarrow$

$y_2 + (1-y_4) + y_6 + (1-y_8) \geq z_j$ max $\sum_j z_j$

$y_i \in \{0,1\}$ versus $0 \leq y_i \leq 1$

$z_j \in \{0,1\}$ $0 \leq z_i \leq 1$

} possibility of non 0/1 values

rounding may disconnect y and z values

* relaxation allows more solutions

Treat y_i as probability and calculate z_i accordingly

Claim. Provable guarantee

max $\sum z_j \geq OPT$

k variables $(x_1 \vee x_2 \vee \cdots \vee x_k)$

$y_1 + y_2 + \cdots + y_k \geq z_j$ $z_j = \beta$

Prob[z-th clause is satisfied] is $\geq c\beta$ for some const c.

Expected value after randomized rounding $\geq c \cdot \sum z_j$

$\geq c \cdot OPT$

worst case: $y_1 + y_2 + \cdots + y_k = \beta$

$1 - \prod_i (1-y_i)$ = prob clause rounds to true

minimized when $y_i = \beta/k$

$1 - (1-\beta/k)^k \geq (1 - 1/e)\beta$

$\uparrow$

constant c

best global approx is random btwn 3SAT and LP