CS121 Lecture 19 November 5

probabilistic experiment: tossing n independent unbiased coins

$\equiv$ choose $x \sim \{0,1\}^n$

Event

- event is the set $A \subseteq \{0,1\}^n$
- probability that A happens is $Pr[A] = \frac{|A|}{2^n}$

Q1. $n=3$. $A = \{x_0 = 1\}$, $B = \{x_0 + x_1 + x_2 \geq 2\}$, $C = \{x_0 + x_1 + x_2 = 1 \mod 2\}$

$Pr[B] = \frac{1}{2}$ $\quad Pr[A \cap B] = \frac{1}{4}$ $\quad Pr[B \cap C] = \frac{1}{8}$

$Pr[C] = \frac{1}{2}$ $\quad Pr[A \cap C] = \frac{1}{4}$

Q2. $Pr[A \cap B]$ vs $Pr[A]Pr[B]$: $>$, A and B + correl

$Pr[A \cap C]$ vs $Pr[A]Pr[C]$: $=$, A and C are independent

Operations on Events

$Pr[A$ or B happens$] = Pr[A \vee B]$

$Pr[A$ and B happen$] = Pr[A \wedge B]$

$Pr[A$ doesn't happen$] = Pr[\bar{A}] = Pr[\{0,1\}^n \setminus A] = 1 - Pr[A]$

Q3. Prove the union bound: $Pr[A \vee B] \leq \underbrace{Pr[A] + Pr[B]}$

$$\frac{|A \vee B|}{2^n} \leq \frac{|A + B|}{2^n}$$

Independence

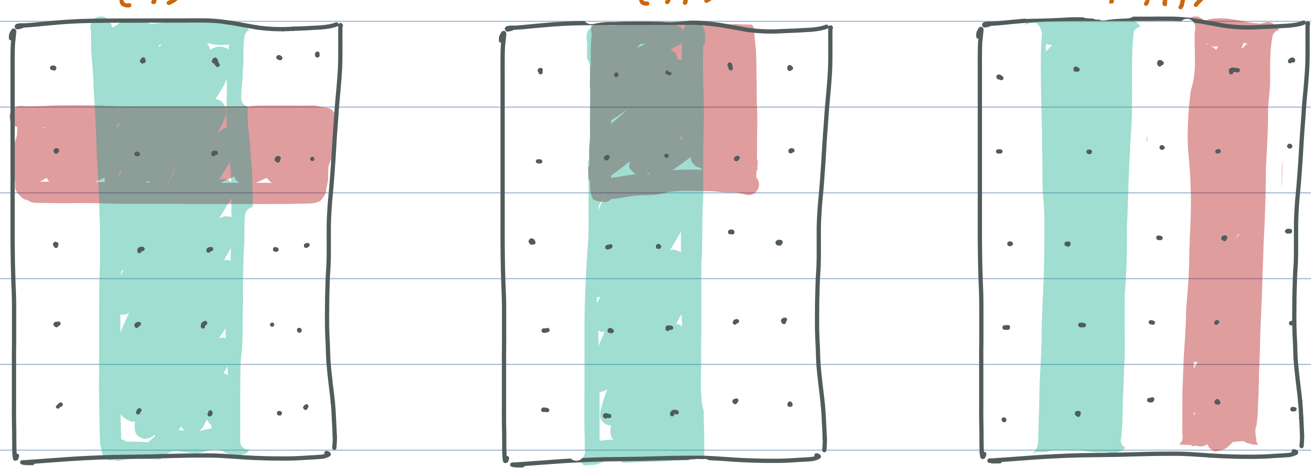
Two events are independent if $Pr[A \cap B] = Pr[A]Pr[B]$

Conditional probability:

$$\frac{Pr[A \cap B]}{Pr[A]} =: Pr[B \mid A]$$

Qn. Which pairs are independent?

(i) (ii) (iii)



✓ X X

	(i)	(ii)	(iii)
A	2/5	2/5	1/5
B	1/5	6/25	1/5
$A \cap B$	2/25	4/25	0

Disjoint =! independent

More than 2 events

3 events A, B, C are independent if every pair A, C

A, B and B, C are independent and

$Pr[A \cap B \cap C] = Pr[A]Pr[B]Pr[C]$

Random Variables

- Assign a number to every outcome of the coins

$X : \{0,1\}^* \to \mathbb{R}$

example: $X(x) = x_0 + x_1 + x_2$

Expectation

Average value of X: $\mathbb{E}(X) = \sum_{x \in \{0,1\}^n} \frac{X(x)}{2^n} = \sum_{v \in \mathbb{R}} v \cdot \Pr[X = v]$

Q5. What is $\mathbb{E}(X)$?

v	$\Pr[X=v]$
0	1/8
1	3/8
2	3/8
3	1/8

$0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$

Linearity of Expectation

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Independent Random Variables

X, Y are independent r.v.s if $\{X = u\}$ and $\{Y = v\}$ are independent $\forall u, v$

$\Pr[\bigwedge_{i \in S} X_i = v_i] = \prod_{i \in S} \Pr[X_i = v_i]$

Q6. Let $x \sim \{0,1\}^n$. Let $X_0 = x_0$, $X_1 = x_1$, ... $X_{n-1} = x_{n-1}$

Let $Y_0 = Y_1 = \dots = Y_{k-1} = x_0$

Let $X = X_0 + X_1 + \dots + X_{n-1}$, $Y = Y_0 + Y_1 + \dots Y_{k-1}$

Are $X_0 \dots X_{n-1}$ independent? Yes, they are iid.
Are $Y_0 \dots Y_{n-1}$ independent? No, fixed value.

Compute $\mathbb{E}[X]$:
$= \mathbb{E}[X_0] + \mathbb{E}[X_1] + \dots + \mathbb{E}[X_{n-1}] = n/2$

Compute $\mathbb{E}[Y]$:
$= \mathbb{E}[Y_0] + \mathbb{E}[Y_1] + \dots + \mathbb{E}[Y_{k-1}] = k/2$

For $n = 100$, estimate $\Pr[Y \in (0.4k, 0.6k)]$
$= 0$ (either 0 or 100, each w/ prob 1/2)

$\Pr[X \in (0.4n, 0.6n)]$
= most of the time = 96%

Concentration Bounds

If X is the sum of n independent random variables
— bell curve line.

$\Pr[X \notin (0.99, 1.01)\mathbb{E}[X]] < \exp(-\delta \cdot n)$

Chernoff Bounds:

Let $X_0, \dots X_{n-1}$ iid rv's w/ $X_i \in [0,1]$. Then $X = X_0 + X_1 + \dots + X_{n-1}$ and $\mu = \mathbb{E}[X]$ for every $\epsilon > 0$,

$\Pr[|X - \mu| > \epsilon\mu] < 2 \cdot \exp(-\epsilon^2 \cdot n/4)$

Q7. Suppose average age in neighborhood is 20. Prove at most 1/4 of residents are 80 or older.

proof by contradiction:

1/4 are 80 , 3/4 have average age x

$$\frac{1}{4} \cdot 80 + \frac{3}{4} \cdot x = 20$$

$$80 + 3x = 80$$

$x = 0$ $\longrightarrow$ if more than 1/4, negative age

Markov's Inequality

Let X be non-neg rv and $\mu = \mathbb{E}[X]$. Then for every $k \geq 1$, $\Pr[X \geq k\mu] \leq 1/k$

Variance and Chebychev

If X is r.v. w/ $\mu = \mathbb{E}[X]$ then $\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

Chebychev:

For every rv X: $\Pr[X \geq k$ deviations from $\mu] \leq \frac{1}{k^2}$

(proof: Markov on $Y = (X-\mu)^2$)

Compare w/ $X = \sum X_i$ iid or other "bell-curve" rvs where $\Pr[X \geq k$ deviations from $\mu] \leq \exp(-k^2)$