Econ 1052 Lecture 18 Thurs, April 9, 2020

Review

assessment (σ, b)

prob distribution over histories in information set. $\forall I, \sum_{h_i \in I} b(h_i | I) = 1$

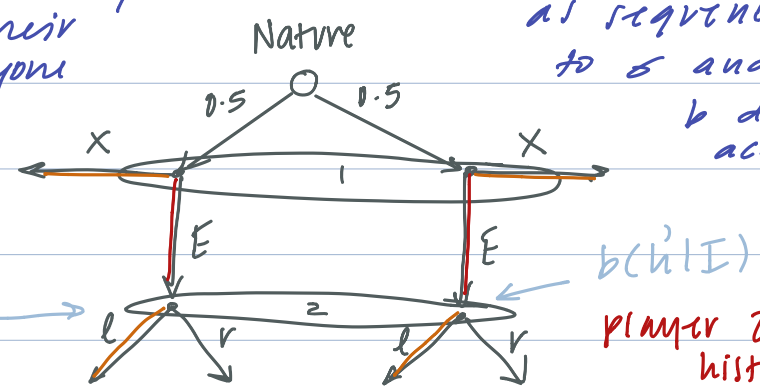
(σ, b) is a sequential equilibrium if its

sequentially rational and consistent

at each infoset, every player is BR to their beliefs given everyone else plays according to strat profile

$\hookrightarrow$ σ^m and b^m can be expressed as sequence that converges to σ and b

b derived from σ according to Bayes



player 2 must be 0.5 at each history in infoset

What does consistency mean? If infoset on path of play, b derived from Bayes

2's belief is off-path (l never played if X)

- Bayes rule doesn't tell us anything
- belief profile not consistent?

$$b(h | I) = 0.7$$
$$b(h' | I) = 0.3$$

two thinks this is informative of nature

but one doesn't know anything about nature

$$\sigma_1^m(E | I) = p^m > 0$$

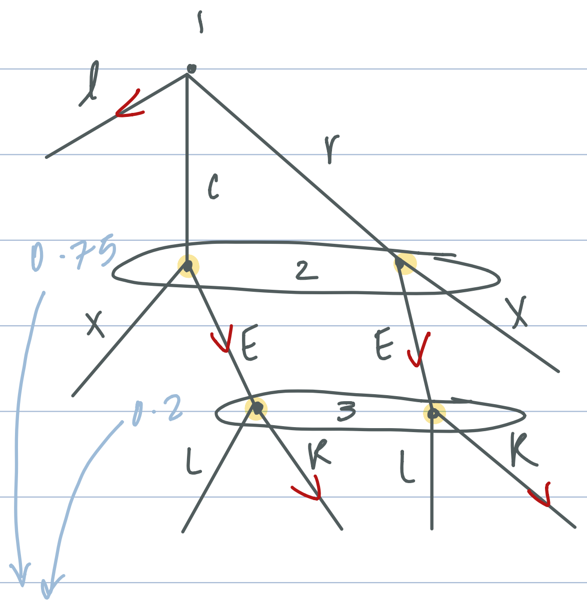
$$b^m(h | I) = b^m(h' | I) = 0.5$$

$$b(h | I) = b(h' | I) = 0.5$$

off path restrictions

* consistency doesn't define a unique set of beliefs but it does place restrictions on beliefs that other players

have



(σ, b)

σ^m fully mixed

s.t. $\sigma_m \to \sigma$

b^m is derived w/ Bayes

and $b^m \to b$

what isn't consistent for strategy in red?

- relative weights when approaching limit of 1s totally mixed strategy → 3x weight on c vs. r
- two's prob of playing E → 1 but cannot condition on 1s answer
- three | two played enter cannot learn anything about whether 1 played c or r
    - 0.75 at c → E node rather than 0.2

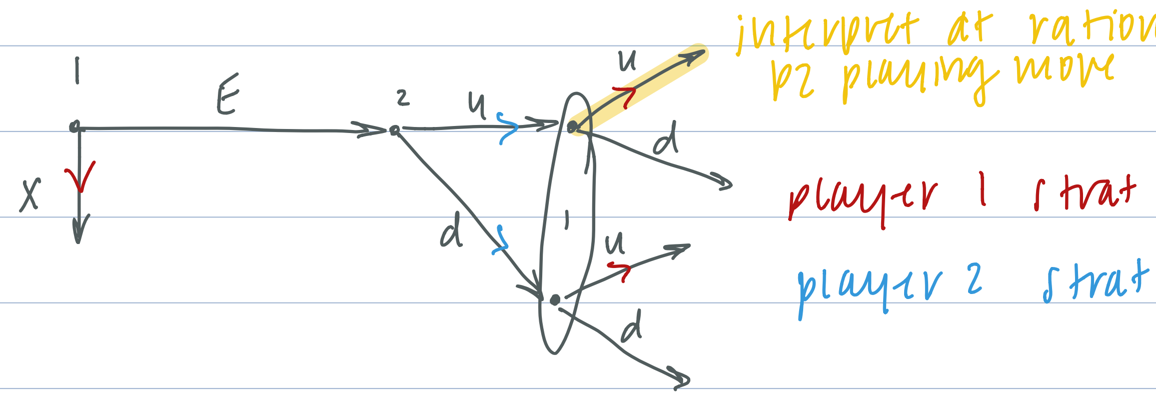
PBNE: you can only update beliefs about players type who just played

consistency:

- seq. eq.: when 1 doesn't know about Nature, two cannot derive information about Nature from one's play

- players 2 and 3 agree on relative probability (consensus) on the 0-prob event that one deviates from l.

★ if a player doesn't know something, subsequent players cannot assume anything from their move
★ if a 0-prob event happens, subsequent players reach a consensus (induced by same limited strat profile)



- player 1 plan specifies moves at her second information set (which is unreachable)
  - pins down what P2 should choose

★ choosing other player's belief about you when defining moves in non-reachable infosets

Auctions

Auctions under symmetric independent private values

N players

one object $\qquad T_i : [0,1]$

$t_i \overset{iid}{\sim} F$ $\qquad F : [0,1] \to [0,1]$

types/value

cdf strictly increasing, atomless (no jumps)

y_i = indicator = $\begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{o/w} \end{cases}$ $\qquad y \in \{0,1\}^N$

$p_i \in \mathbb{R}$ $\qquad p \in \mathbb{R}^N$

$u_i(y, p, t) = y_i t_i - p_i$

auction G

simultaneous bayesian game!

Assumptions:

- only care if you win/not
- don't care how much someone else pays
- independently distributed values

$\tilde{y}_i(b)$ whether you win w/ profile of bids b

$\tilde{p}_i(b)$ how much you pay ''

First price auction

$\tilde{y}_i(b) = 1/2$ 'tie breaking'

$\tilde{p}_i(b) = 1$ for winner, 0 for loser

Second Price

- Dominant strat to bid value

All price Auction

highest bidder wins, everyone ends up paying even if you lose

$p_i(b) = b_i$

Revenue Equivalence Thm

what type of auction to run?

Effect of format on revenue

"All efficient auctions in which 0-value bidders have 0 utility + yield the same expected revenue."

Any auction w/ efficient auction (obj goes to bidder w/ max value) → yield same $\mathbb{E}$ value for auctioneer

Generalized Envelope Thm

$\mathbb{E}[\text{revenue}] = \mathbb{E}[\sum_{\text{bidders } i \in N} \text{payment}_i]$

$\underbrace{\text{welfare}}_{\sum_i y_i t_i} - \sum_i \text{payment}_i = \sum_i \text{utility}_i$

← show the $\mathbb{E}$ is same for all auction scheme

given by assumption that efficient auction

Milgrom and Segal (2002) - Formulation of Env Thm

$f(x,t)$ → completely arbitrary

(choice variable, parameter in unit interval)

Assume: $f_2(x,t)$ exists $\forall\ x, t$

$f: X \times [0,1] \to \mathbb{R}$ ↳ $\frac{\partial f}{\partial t}$ $\exists$ for all values of t

$V(t) := \sup_{x \in X} f(x,t)$

$\mathbb{X}^*(t) = \underset{x \in \mathbb{X}}{\arg\max}\ f(x,t)$ } could be empty, one elt, multiple elts, ...

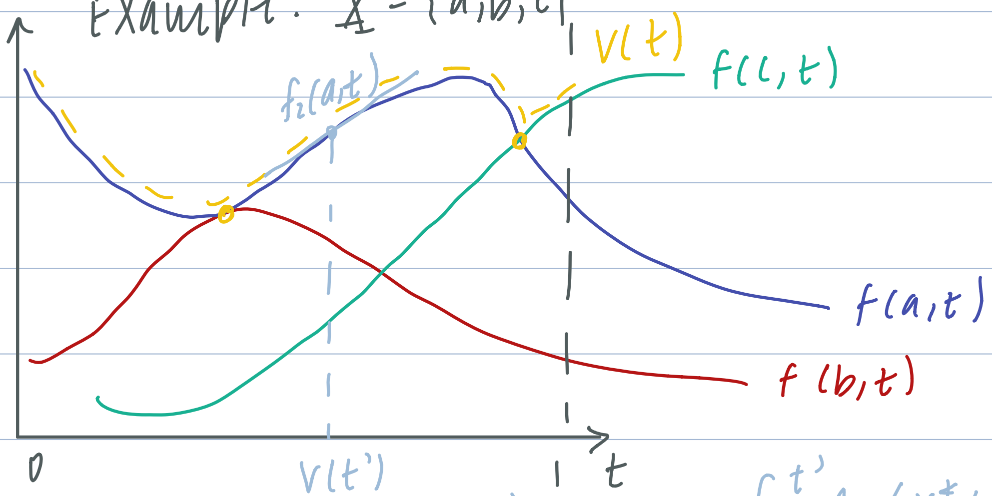
Thm. Let $t \in (0,1)$ and $x \in \mathbb{X}^*(t)$.

If $f_2(x,t)$ and $V'(t)$ both exist, then

$$V'(t) = f_2(x,t)$$

Example: $\mathbb{X} = \{a,b,c\}$

$V(t)$ = upper envelope



$V(t') = V(0) + \int_0^{t'} f_2(x^*(s), s)\, ds$

optimizer unique at t $\longrightarrow$ $V(t)$ differentiable

$V(t)$ differentiable $\longrightarrow$ $\forall\, x \in \mathbb{X}^*(t)$

$$V'(t) = f_2(x,t)$$

Proof. Take any $x \in \mathbb{X}^*(t)$

① $V(t) - f(x,t) = 0$ by definition

② $\forall\, \hat{t}$: $V(\hat{t}) - f(x, \hat{t}) \geq 0$ since x constant, do at least as well

re-optimize

keep x the same

$$\therefore\ t \in \underset{\hat{t}}{\arg\min}\ V(\hat{t}) - f(x,\hat{t})$$

First order condition must hold:

$$V'(\hat{t}) - f_2(x, \hat{t}) = 0 \quad \text{at} \quad t = \hat{t}$$

Thm. Let $f : X \times [0,1] \to \mathbb{R}$

Suppose that:

1. $\forall t : X^*(t) \neq \emptyset$ max is nonempty
2. $\forall x, t : f_2(x,t)$ exists derivative wrt param $\exists$
3.* For some $B < \infty$, $\forall x, t$, $|f_2(x,t)| \leq B$

then for any selection $x^*(\cdot)$ from $X^*(\cdot)$ and all $t \in [0,1]$, $\underbrace{V(t) = V(0) + \int_0^t f_2(x^*(s), s)\, ds}$

* can be weakened

can integrate up using optimal choice

Thm. Let b, b' be any auction s.t.

1. $\exists$ an efficient (highest bidder always wins) BNE
2. 0-value players have 0 utility

then b and b' have same $\exists$ revenue.

* will pin utility for each player

Let s be an efficient BNE of b.

$$\tilde{u}(\hat{t}, t) = t_i \underbrace{F(\hat{t}_i)^{|N|-1}} - \mathbb{E}_{t_{-i}}[p_i(s_i(\hat{t}_i), s_{-i}(t_{-i}))]$$

E utility of type t_i playing type $s_i(\hat{t}_i)$

→ because win if everyone else has type $\hat{t}_i$ >

☆ s is a BNE → $\max_{\hat{t}_i} \tilde{u}_i(\hat{t}_i, t_i) = \tilde{u}(t_i, t_i) := V(t_i)$

$\tilde{u}_2(\hat{t}_i, t_i) = F(\hat{t}_i)^{|N|-1}$

$t_i^*(s)$ is a maximizer

By envelope thm, $\underbrace{V(t_i)}_{\text{interim } \mathbb{E} \text{ utility of type } t_i} = V(0) + \int_0^{t_i} u_2(t_i^*(r), r)\, dr$

$= V(0) + \int_0^{t_i} u_2(r, r)$

since BNE

$= \underbrace{V(0)}_{=0 \text{ since 0-value types have 0 utility}} + \underbrace{\int_0^{t_i} F(r)^{|N|-1} dr}_{\text{no dependence on strategy}}$

$\mathbb{E}[\text{payment}] = \underbrace{\mathbb{E}[\text{welfare}]}_{\text{equal in } G \text{ and } G'} - \sum_i \underbrace{\mathbb{E}[\text{utility}]}_{= \mathbb{E}_t\left[\int_0^{t_i} F(r)^{|N|-1} dr\right]}$

equal in G and G'

∎