CS124 Lecture 19

Wed, April 8, 2020

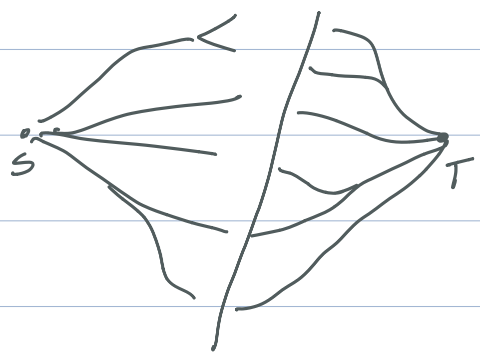
~ Flows, Duality, and Games ~

From Last Time

- Residual networks
- Augmenting path alg
- $O(E \cdot F^*)$, $O(E^2 V)$ algs
- Integer caps = Integer solutions

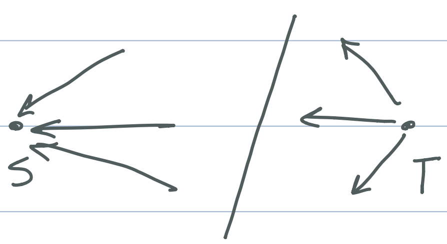
S-T Max Flow = S-T Min Cut

maximum flow ≤ minimum cut



Now: maximum flow ≥ minimum cut

max flow ≥ alg flow ≥ minimum cut ≥ max flow → max flow = min cut



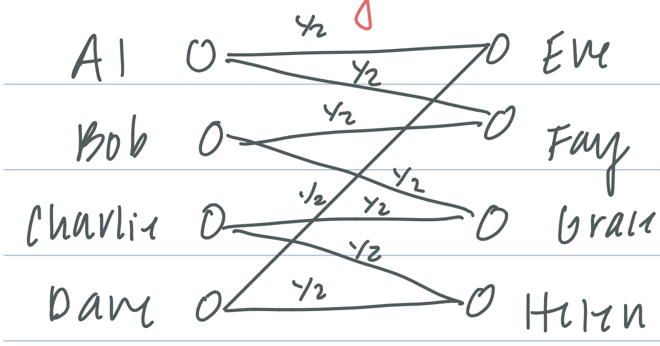
cut = all vertices connected to S in residual network

Claim: For edges crossing cut $f(e) = c(e)$

For any edge (u,v) $v \in S$, $u \notin S$, $f(e) = 0$

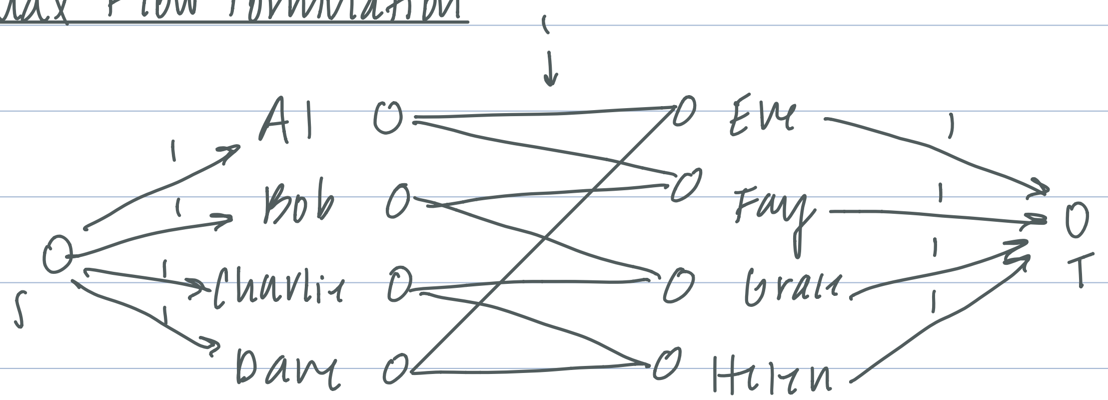
Conservation of flow: flow reaching T = capacity of the cut

Max Matching Problem



$$\left.\begin{aligned} & e_{ij} = 1, 0 \\ & \max \sum e_{ij} \\ & \sum e_{ij} \leq 1 \\ & e_{ij} \text{ connected to Al} \end{aligned}\right\} \text{LP}$$

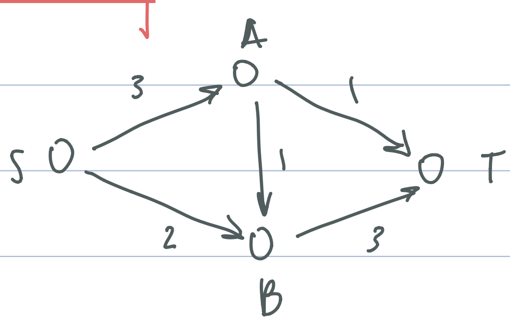
Max Flow Formulation



(int) max flow = matching

int max flow = max matching

Duality



$$\max \; f_{SA} + f_{SB}$$

$$f_{SA} \leq 3$$

$$f_{SB} \leq 2$$

$$f_{AB} \leq 1$$

$$f_{AT} \leq 1$$

$$f_{BT} \leq 3$$

$$f_{xy} \geq 0 \;\; \forall x, y$$

$$f_{SA} - f_{AB} - f_{AT} = 0$$

$$f_{SB} + f_{AB} - f_{BT} = 0$$

y_{SA} = 1 if they cross the cut
0 o/w

u_A = 1 if A is in the cut w/ S
0 o/w

$$\min \; 3y_{SA} + 2y_{SB} + y_{AB} + y_{AT} + 3y_{BT}$$

$$y_{SA} + u_A \geq 1 \qquad \text{all } y \geq 0$$
$$y_{SB} + u_B \geq 1$$
$$y_{AB} - u_A + u_B \geq 0$$
$$y_{AT} - u_A \geq 0$$
$$y_{BT} - u_B \geq 0$$

exclude $y_{AB} = u_B = 0$, $u_A = 1$

transpose coeff matrix
max → min
constrained/unconstrained
$\leq$ → $=$ constraints

☆ If LP has a bounded solution, so does its dual and the solutions match. EX: max flow = min cut.

2 Player Matrix Games

Rock - Paper - Scissors

Column Player

$$\begin{array}{cc} & \begin{array}{ccc} R & P & S \end{array} \\ \text{Row Player} \begin{array}{c} R \\ P \\ S \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

Strategies

pure and mixed

↓ pure: same move given situation

mixed ↳ choosing randomly

row player!

$$\begin{array}{c} x_1 \\ x_2 \end{array} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

max z

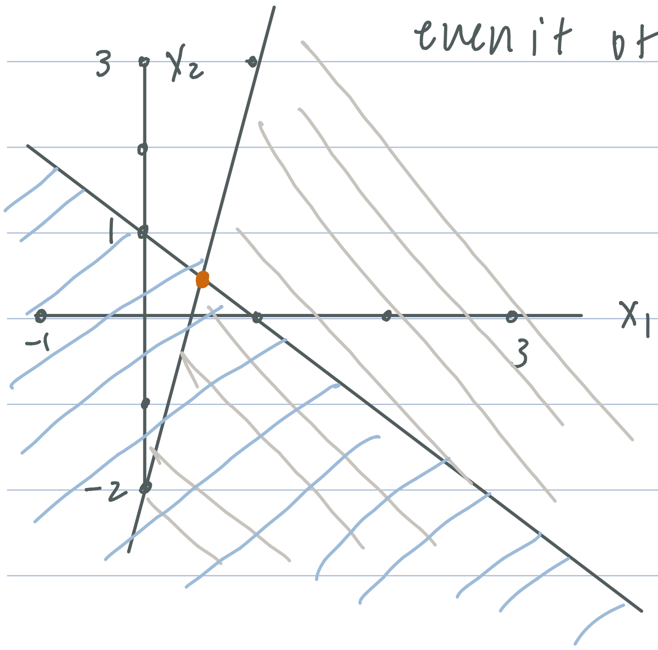
$x_1 + x_2 = 1$

$x_1 \geq 0, \; x_2 \geq 0$

$z \leq 3x_1 - 2x_2$

$z \leq -x_1 + x_2$

optimal strat: choose mixed strat that maximizes payoff even if other player knows the strategy



$3x_1 - 2x_2 = -x_1 + x_2 = \boxed{1/7}$

$4x_1 = 3x_2$

$x_1 = \frac{3}{7} \quad x_2 = \frac{4}{7}$

row player expected value guarantee $\geq \frac{1}{7}$

column player

DUEL LPs!

min w

$w \geq 3y_1 - y_2$

$w \geq -2y_1 + y_2$

$y_1 + y_2 = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$\longrightarrow$

$3y_1 - y_2 = -2y_1 + y_2 = \boxed{1/7}$

$y_1 = 2/7, \; y_2 = 5/7$

column player expected value guarantee $\leq 1/7$

★ value of game to row player $= 1/7$