Econ 1052 Lecture 20

April 16, 2020

~ Optimal Auctions Continued ~

Recap

choose	to maximize	subject to
A_i $x_i : A \to [0,1]$ $p_i : A \to \mathbb{R}$ s	$\mathbb{E}[\sum_i p_i(s(t))]$ expected revenue	• s is a BNE • participation constraint $E[t_i] \geq 0$ • feasibility sell 1

revelation mechanism — Bayes Incentive Compatability

Report type, machine will bid for you

$x_i : T \to [0,1]$ $p_i : T \to [0,1]$	reporting type truthfully $\mathbb{E}[\sum_i p_i(t)]$	• (x, p) is BIC • participation constraint • feasibility

Envelope Thm = clever way of dropping out payments

$V_i(0)$ $x_i : T \to [0,1]$	$\mathbb{E}\left[\sum_i x_i(t)\left[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right] - V_i(0)\right]$	$\bar{x}_i$ is monotone $V_i(0) \geq 0$ feasibility

relaxed problem

Envelope Thm: $V_i(t_i) = V_i(0) + \int_0^{t_i} \bar{x}_i(w)\,dw$

Ex ante expected utility

$$\mathbb{E}_{t_i}[V_i(t_i)] = V_i(0) + \int_0^1 \int_0^{t_i} \bar{x}_i(w)\,dw\, f_i(t_i)\,dt_i$$

$$= V_i(0) + \int_0^1 \bar{x}_i(t_i) \times \frac{1-F_i(t_i)}{f_i(t_i)} \underbrace{f_i(t_i)}_{\text{PDF}}\,dt_i$$

CDF

$\frac{f_i(t_i)}{1-F_i(t_i)}$: starting from 0 and gradually checking for player 1 type

conditional on not seeing player 1 type, prob of seeing player 1 type

backwards Mills Ratio

Expected utility from i $\quad \mathbb{E}_{t_i}[\bar{p}_i(t_i)] = \int_0^1 \bar{x}_i(t_i)\left[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right] f_i(t_i)\,dt_i - V_i(0)$

$= \mathbb{E}_{t_i}\left[\bar{x}_i(t_i)\left[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right]\right] - V_i(0)$

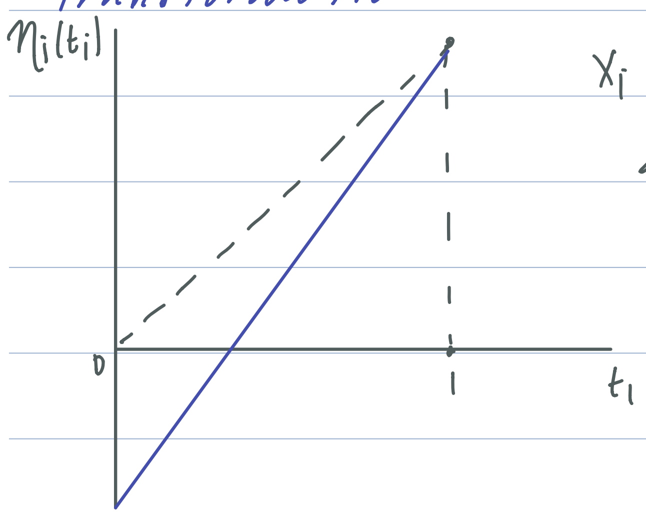
one bidder: $\mathbb{E}_{t_i}[\bar{p}_i(t_i)] = \mathbb{E}_{t_i}[\bar{x}_i(t_i)[2t_i-1]] - \cancel{V_i(0)}$

$\mathbb{E}\left[\sum_i p_i(t_i)\right] = \sum_i \mathbb{E}_t[p_i(t_i)] = \sum_i \mathbb{E}_{t_i}[\bar{p}_i(t_i)]$

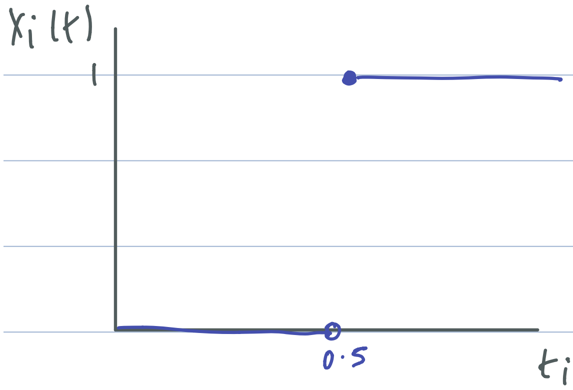
$= \sum_i \mathbb{E}_{t_i}\left[\bar{x}_i(t_i)\left[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right]\right] - V_i(0)$

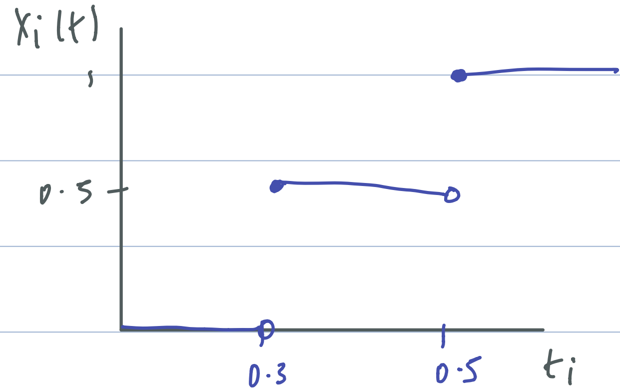
$= \mathbb{E}_t\left[\sum_i x_i(t)\underbrace{\left[t_i - \frac{1-F_i(t_i)}{f_i(t_i)}\right]}_{\eta_i(t_i)\text{ – virtual value}}\right] - V_i(0)$

transformation



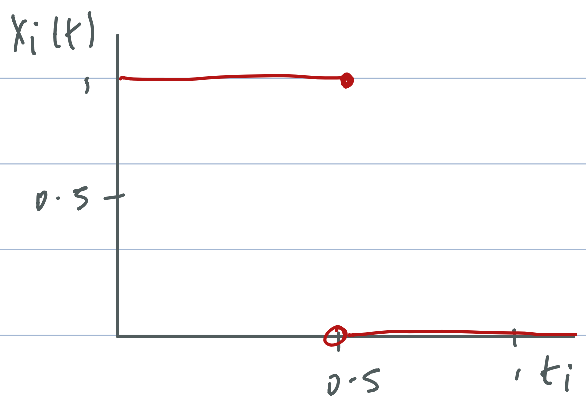
x_i that maximizes distorted values also maximizes expected payments allocation function





$$p_i(t_i) = \begin{cases} 0.5 & t_i \geq 0.5 \\ 0.15 & t_i \in [0.3, 0.5] \\ 0 & o/w \end{cases}$$

un-supported allocation rule



$$\text{utility}_i = x_i(\hat{t}_i)t_i - p_i(\hat{t}_i)$$

$t_i < t_i'$ and $x_i(t_i) > x_i'(t_i')$

low doesn't want to imitate high

$$x_i(t_i)t_i - p_i(t_i) \geq x_i(t_i')t_i - p_i(t_i')$$

$$p_i(t_i') - p_i(t_i) \geq t_i(x_i(t_i') - x_i(t_i))$$

contradiction

high doesn't want to imitate low

$$x_i(t_i')t_i' - p_i(t_i') \geq x_i(t_i)t_i' - p_i(t_i)$$

$$t_i'(x_i(t_i') - x_i(t_i)) \geq p_i(t_i') - p_i(t_i)$$

monotonicity: $\bar{x}_i(t_i)$ must be weakly increasing in t_i

$$\eta_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

F is uniform $\rightarrow$ $\eta_i(t_i) = 2t_i - 1$

then: ① All players report $\hat{t}_i$

② award object to $\arg\max_i \eta_i(\hat{t}_i)$

③ if $\max_i \eta_i(\hat{t}_i) > 0$:
charge winner $\inf t_i' \mid \eta_i(t_i') \geq \eta_j(\hat{t}_j)$ for all j.
withold otherwise

losers pay 0

$\to V_i(0) = 0$

+ similar to second-price auction

maximizes relaxed problem $\Rightarrow$ maximizes strict problem

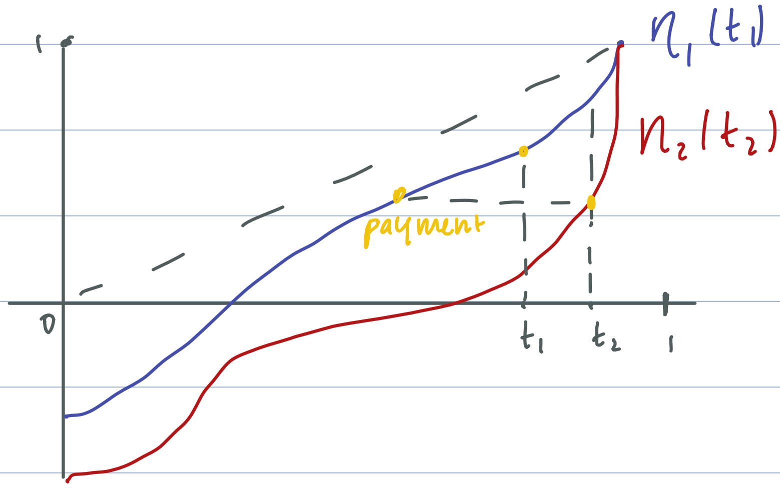
F_i uniform

$\forall i : \eta_i(t_i) = 2t_i - 1$

$t_1 = 0.1$
$t_2 = 0.2$
no positive virtual values

$\eta_2(t_2) > \eta_1(t_1) \iff t_2 > t_1$

award object to no one if both values $< \frac{1}{2}$.
to highest bidder if either value $> \frac{1}{2}$
charge w/ inner $\max\{t_{-i}, \frac{1}{2}\}$



Thm. If η_i is strictly increasing for all i, then the scoring auction maximizes expected revenue in the class of all games satisfying

1. BIC
2. participation constraints
3. feasibility

this model is unrealistic because real auctions:

- effort to participate
- opportunity cost of not participating in a diff auction

$$\eta_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

reserve when virtual value function crosses 0