CME Code Documentation and Formulation

We wish to use finite element method to solve the problem described by the following differential equation and boundary conditions

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

Formulation of Linear elements:

$$\left(\frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

$$f_1^e = -\frac{1}{h_e}\left[\frac{x_b}{3}\left(x_b^3 - x_a^3\right) - \frac{1}{4}\left(x_b^4 - x_a^4\right)\right]$$

$$f_2^e = -\frac{1}{h_e}\left[\frac{1}{4}\left(x_b^4 - x_a^4\right) - \frac{x_a}{3}\left(x_b^3 - x_a^3\right)\right]$$

Formulation of Quadratic elements:

$$\left(\frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix}$$

$$f_1^e = -\frac{h_e}{60}\left(-h_e^2 + 10x_a^2\right)$$

$$f_2^e = -\frac{h_e}{15}\left(3h_e^2 + 10x_a^2 + 10x_a^2 h_e\right)$$

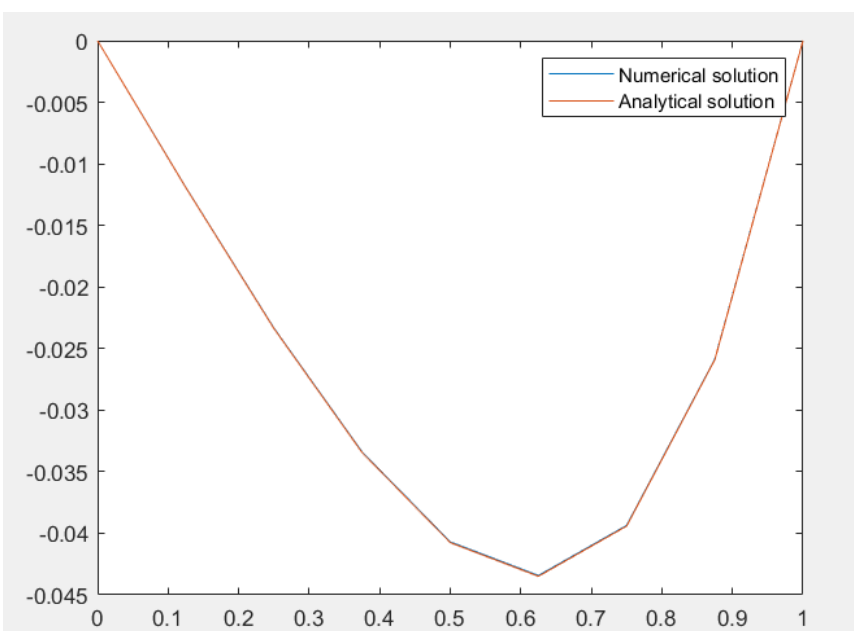
$$f_3^e = -\frac{h_e}{60}\left(9h_e^2 + 20x_a^2 + 20x_a h_e\right)$$

The coefficient matrix and the force are both calculated using the above formulations for both linear and quadratic elements given a = 1, c = -1, f(x) = x^2

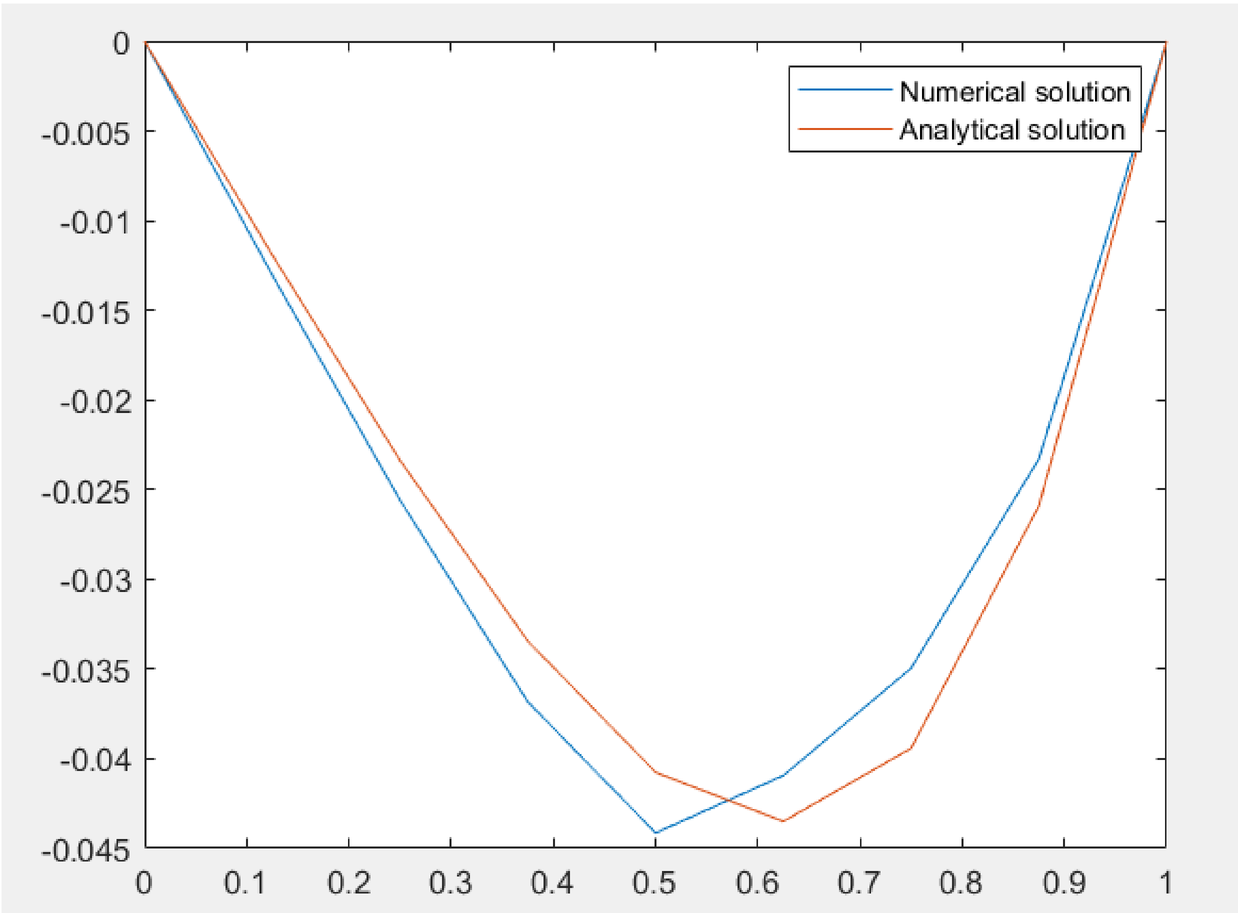
The K and the F matrices are found out using the above formulation and matrix multiplication is applied as shown below to get the results.

$$
\begin{bmatrix}
K_{11}^1 & K_{12}^1 & & & & \\
K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & & \mathbf{0} & \\
 & K_{21}^2 & K_{22}^2 + K_{11}^3 & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \mathbf{0} & & & K_{22}^{N-1} + K_{11}^N & K_{12}^N \\
 & & & & K_{21}^N & K_{22}^N
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_N \\ U_{N+1} \end{Bmatrix}
= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^{N-1} + f_1^N \\ f_2^N \end{Bmatrix}
+ \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \vdots \\ Q_2^{N-1} + Q_1^N \\ Q_2^N \end{Bmatrix}
$$

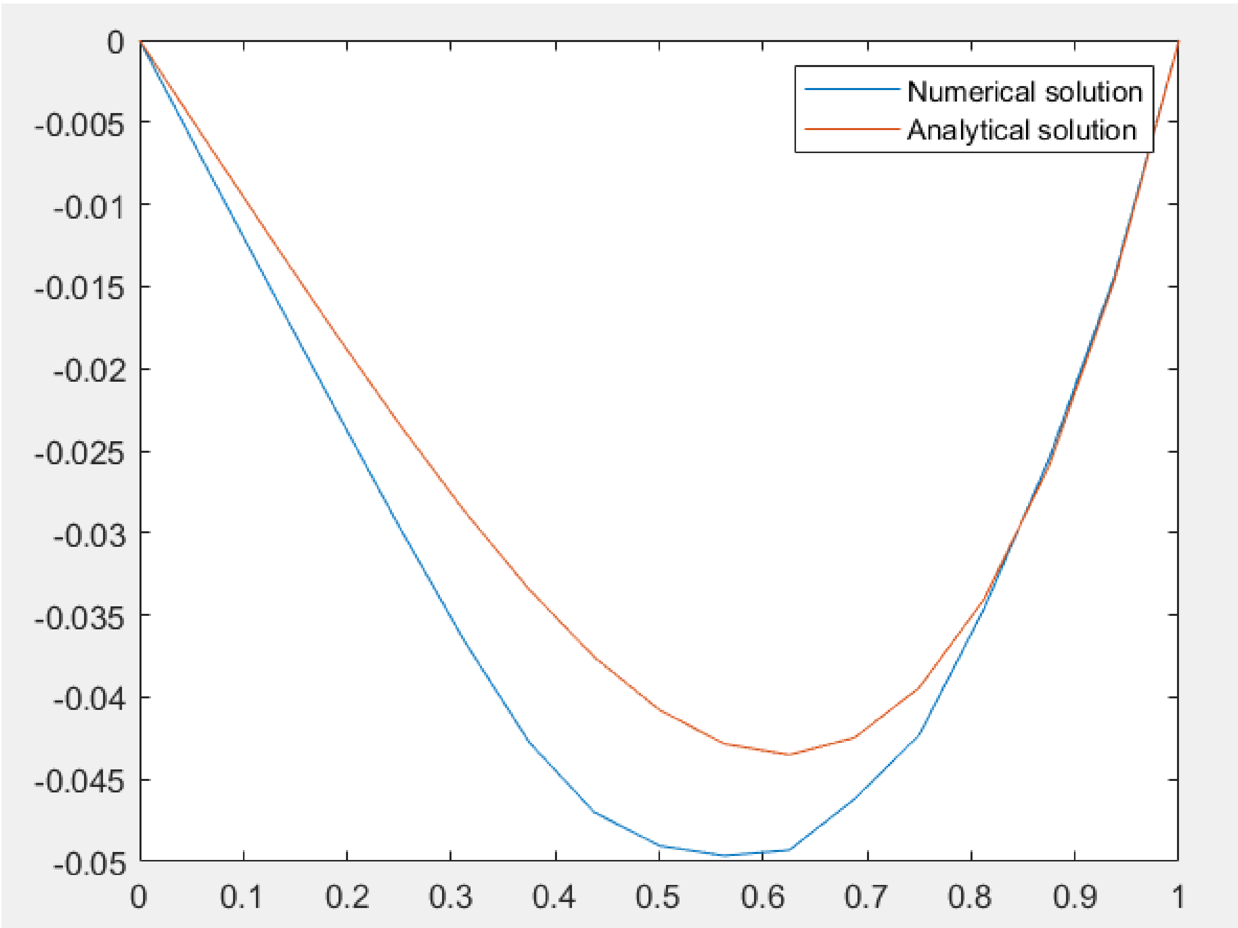
Results for 8 linear elements:



Results for 4 quadratic elements:



Results for 8 quadratic elements:



Hence the results for both linear and quadratic elements are plotted above.