Lab 5: Color Organ II

In this lab, we will explore cascade design, one of the most popular methods for designing active filters (filters that include active components such as op-amps).

If the output terminal of a filter circuit has a much lower (ideally zero) impedance than the input terminal of the filter circuit with which it is cascaded (whose impedance would ideally be infinite), *cascading does not change the transfer functions of the individual circuits,* and the overall transfer function of the cascade is simply the product of the transfer functions of the individual circuits. This is why buffers are so useful in filter design.

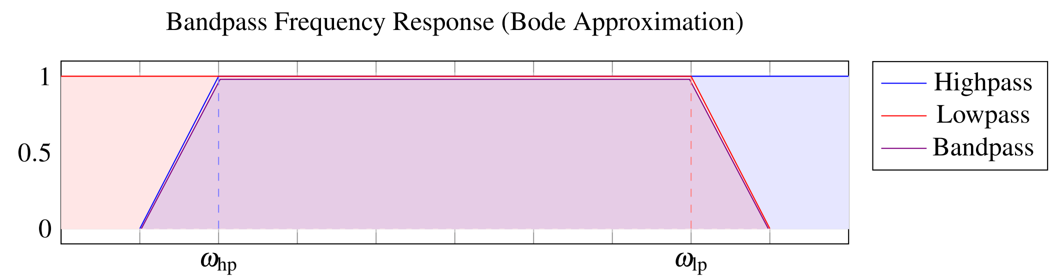
Part 1: Caught in the Midrange

In this part of the lab, you will build an active RC bandpass filter to isolate the midrange frequencies. You're familiar with first-order low- and high-pass filters already. But, can we make a first-order bandpass filter? As you probably expected, the answer is no[1]. We need to rely on our knowledge of loading and first-order filters to design an appropriate bandpass filter.

To do so, you will need to choose two cutoff frequencies ω_{hp} and ω_{lp}. Important questions to ask yourself while designing this filter include:

- Which of the two cutoff frequencies should be higher?
- What's a good amount of space to leave between your bandpass cutoffs and your bass and treble filter cutoffs?

The transfer functions of buffered cascaded filters multiply, so the frequency response of the bandpass is the multiplication of its component frequency responses:



Ponder the following while you build your filters:

- What would a pulse train (i.e., square wave with its minimum at 0) look like if you passed it through a low-pass filter? What about a high-pass filter?
- Can you think of additional applications for these filters? Try thinking about more general signals, like stock prices or images.

You are now ready to start the lab! Go to the Jupyter notebook and complete Part 1.
The remainder of this note may help you for Parts 2 and 3.

Generalizing the first-order filter

The general first-order (or **bilinear**, since it is linear in both the numerator and denominator) transfer function is as follows (recall, $s = j\omega$):

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$$

[1] See Appendix B for a discussion of phase and the third kind of first-order filter: the all-pass.

This filter has a pole at $s = -\omega_0$ and a zero at $s = -a_0/a_1$. *Think: What is the gain at $s = \infty$? What about at $s = 0$?* The gain at $s = 0$ is the DC gain, and the gain at $s = \infty$ is the high frequency gain. In this case, the high-frequency gain approaches a_1, and the DC gain is a_0/ω_0. The coefficients a_0 and a_1 determine what kind of filter we have. As an exercise, think of the relationships among the numerator coefficients that realize the different kinds of filters.

Generalizing the second-order filter

Second-order filters can be either active or passive, but as we have discussed, making the filters active allows for greater modularity and ease of design. The general second-order (or **biquadratic**) transfer function is as follows:[2]

$$T(s) = \frac{a_2 s^2 + a_1 s + a_0}{s^2 + (\omega_0/Q)s + \omega_0^2}$$

By applying the quadratic formula to the denominator, we can find the poles of this function. As with the first-order transfer function, the type of filter is determined by the numerator coefficients.

What kinds of second-order filters can we make?

Second order filters include highpass, lowpass, bandpass, bandstop, notch, and allpass. As an exercise, think of the relationships among the numerator coefficients that realize the different kinds of filters.

The natural frequency, ω_0

The natural frequency is the frequency at which the system oscillates with no damping; for example, the natural frequency of a series RLC circuit would be its frequency of oscillation with the resistance shorted (i.e., replaced with ideal wire). We will address natural frequency more in-depth later, so don't worry about it for now.

Appendix A: Impedance and Reactance

Capacitors and inductors are **reactive** components: their behavior depends on frequency. But, both components are *linear*. You are likely familiar with the concept of linearity, i.e., that a linear system (or linear transformation) must exhibit both *superposition* (the result of adding two inputs and then feeding the result through the system is the same as the result of feeding the two inputs into the system individually and then adding the outputs) and *homogeneity* (the result of multiplying the input by a constant and feeding it through the system is the same as that of feeding the input through the system and then multiplying by a constant). Linearity is a property with many important consequences, the most important of which is likely that *when a sinusoid of frequency f is fed into a linear circuit, the output will be a sinusoid at the same frequency (though amplitude and phase may change).* This property makes the frequency response a useful characteristic: the frequency response defines how the output voltage depends on the input voltage at a particular frequency, and this relationship can be fully expressed by some constant numerical value at that point.

Impedance

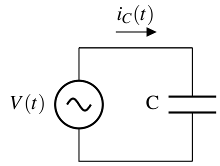
Impedance (**Z**) can be thought of as generalized resistance: it includes both *reactance* (X) and *resistance* (R). In reactive elements, voltage and current are always 90° out of phase, whereas in resistive elements, voltage and current are always in phase. Impedance is mathematically defined as follows:

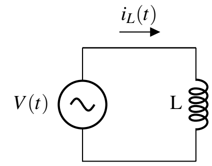
$$Z = R + jX$$

The magnitude of Z gives the ratio of amplitudes of voltage to current, and the polar angle of Z gives the phase angle between current and voltage.

By observing both the complex impedances and the differential equations governing inductive and capacitive behavior, we see that these are indeed linear components. We will intuitively derive the reactance of each by imagining each in a simple circuit in series with a sinusoidal voltage source ($V(t) = V_0 \sin(\omega t)$), as follows:

[2] There are several common ways of representing the coefficients in the general second-order transfer function, and a particularly common one is to write the coefficient of s in the denominator as 2ζ. This notation is especially frequently used with the general second-order **all-pole** transfer function, $T(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$.





Lets start with the capacitor. The current through the capacitor is given by

$$I(t) = C\frac{dV}{dt} = C\omega V_0 \cos(\omega t)$$

The current has amplitude $\omega C V_0$, and the current *leads* the voltage by 90°, since the voltage is given by a sine wave and the current by a cosine, and a cosine is just a sine shifted back by 90°; therefore, we can say the cosine is 90° ahead of the sine.

Disregarding phase (considering amplitude only), the current is

$$I = \omega C V = \frac{V}{\omega C} = \frac{V}{X_C} \implies \frac{|V|}{|I|} = \frac{1}{\omega C} = X_C$$

The analysis of an inductor follows the same pattern: imagine an inductor in the place of the capacitor in our earlier imaginary circuit, with a sinusoidal voltage source such that the current is $I(t) = I_0 \sin(\omega t)$. Using the differential equation, we find the voltage across the inductor to be

$$V(t) = L\frac{dI(t)}{dt} = L\omega I_0 \cos(\omega t)$$

Now, the voltage is a cosine while the current is a sine, so we can say the voltage leads the current by 90° in the inductor.

We have now derived the impedances and reactances for both the capacitor and the inductor:

$$Z_C = \frac{1}{j\omega C} \qquad X_C = \frac{1}{\omega C} \qquad Z_L = j\omega L \qquad X_L = \omega L$$

Why complex numbers?

Since we need to specify both magnitude and phase shift of our voltages and currents at any point in the circuit, a single number is not an adequate representation. While we could explicitly write them as $V(t) = A\sin(\omega t + \Phi)$, it is more convenient to take advantage of the geometry of complex numbers to *represent* our voltages and currents: the complex number representation keeps the magnitude and phase neatly separated so that we can just add or subtract them directly instead of having to add or subtract sinusoidal functions of time.

We will use voltage as an example to review the complex representation:

$$V_0 \cos(\omega t + \phi) \iff V_0 e^{j\phi} \text{ (polar)} = a + jb, \quad a = V_0 \cos(\phi), \quad b = V_0 \sin(\phi) \text{ (rectangular)}$$

Euler's formula: $e^{j\phi} = \cos(\phi) + j\sin(\phi)$

To find the actual voltage and current, multiply the complex representation by $e^{j\omega t}$ and take the real part, for example, $V_0 \cos(\omega t + \phi) = \Re\{e^{j\omega t}(V_0 e^{j\phi})\}$.

Appendix B: WTPh is Phase?

While magnitude is a rather intuitive concept, many people struggle to understand phase: why we care about it, how to calculate it, and how to represent it.

Calculating phase

Close Encounters of the Third Kind: The all-pass filter

We finally come to the third kind of first-order filter: the all-pass filter. This filter (ideally) does not influence magnitude as a function of frequency, but it does influence phase[3].

So why do we even define a cutoff frequency for all-pass filters?

Recall that a phase shift of 180° corresponds to an inversion of the signal: all-pass filters are frequency selective

References

Horowitz, P. and Hill, W. (2015). *The Art of Electronics*. 3rd ed. Cambridge: Cambridge University Press, ch 1.
Sedra, A. and Smith, K. (2015). *Microelectronic Circuits.* 7th ed. New York: Oxford University Press, ch 17.

[3]Common uses of all-pass filters are in equalizing delays or any system that requires phase shaping. They are commonly applied in electronic music production.