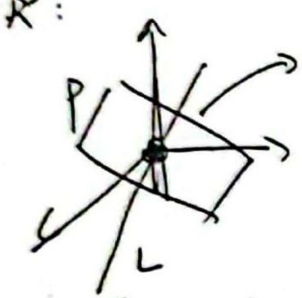
Introduction To Linear ALGEBRA

MEMEME

①

R^3:



plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is subspace of R^3

line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace.

if I have 2 subspaces P and L.

① $P \cup L$ = all vectors in P or L or both

this (is)(is not) a subspace.

it cannot add — if I use a vector in L and a vector in P. I will not get a subspace.

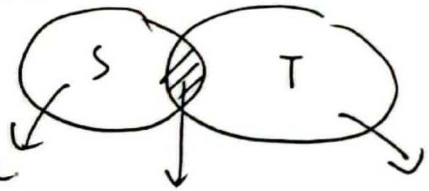
② $P \cap L$ = all vectors in both P and L — this is a subspace

$= \begin{bmatrix}0\\0\\0\end{bmatrix}$.

for subspace S and T.

their intersection $S \cap T$ is a subspace.

for v and w:



the linear com of S and not in T

the linear com in both S and T

the linear com in T and not in S.

↓

v+w. cv. and cv+dw have to do inside the subspace.

② column space of a matrix.

example: $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} = A$. a subspace in $R^4 = C(A)$

= all linear com of columns.

Does $Ax=b$ have a solution for every b? No

because there is 4 equament but 3 unknown.

$\Downarrow$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

which b's allow the system to be solve ??

① $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ always works. ② $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

I can solve $Ax=b$ when exacly when b is in $C(A)$.

$\Downarrow$

when b is not ~~all~~ the linear com of $C(A)$

it cannot be solve. (outside the $C(A)$).

$$\underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}}_{\text{useless.}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{pivot 1}} + \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\text{pivot 2.}}$$

not independtent.

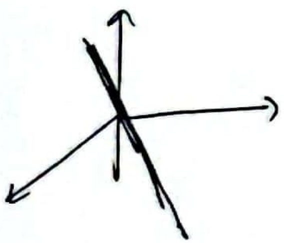
$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ is a 2-dim subspace.

③ Nullspace.

Nullspace of A = all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = \vec{0}$. (in R^3).

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. $\begin{bmatrix} n \\ n \\ -n \end{bmatrix}$. $(n = 0, 1, 2 \dots)$.



$N(A)$ is line in R^3.

how can I proof it [solution to $Ax = \vec{0}$ always give a subspace] ?

☆ If $Ax = \vec{0}$ $Aw = \vec{0}$. and $A(v+w) = Av + Aw = \vec{0}$

and $A(cv) = cAv = \vec{0}$.

if $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. all the solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ cannot build as subspace.

why the $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not inside it.

so ~~what~~ how can I build a subspace ?

① fine some vectors and full out the space to find a subspace.

② fine the solution of 等 $Ax = b$. to find the subspace.

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① calulation of $Ax = \vec{0}$

example:

elimination will not change it.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

1. elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

echelon form.

the pivot of a matrix: 2.

= rank of matrix A

2. pivot variables.

the pivot columns: the colums with pivots.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot column

free column (.we can choice any number).

free variables: ~~we can~~

exam: $x = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} \xrightarrow[\text{sub..}]{\text{back}} x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ means:

$-2\times$ colums 1

$+ 1\times$ colums 2

can get $\vec{0}$

when we have another $x_2 . x_4$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

special solution: the special number given to the free variable

and $c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

contains all the linear combination of the Nullspace.

when we have R pivot vari. and n columns, m rows matrix.

free vari = n − R.

3. clean up the matrix: reduce the row line.

R = reduced row echelon form: ① zero above + blew the pivots. ② pivots = 1

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A).$$

in Mathlad

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot colums / rows.

$$\begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\quad ① \quad ② \quad ③ \quad ④$$

(more esey to back sub).

rref from: $\begin{bmatrix} R & F \\ 0 & 0 \end{bmatrix} = R$.

→ pivot rows r.

pivot colun r   n-r free col.

when we compose the matrix A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \xrightarrow{\text{elim}} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 rank

nullspace matix: colums are speeall solution).

$$RN = 0 \longrightarrow N = \begin{bmatrix} -F \\ R \end{bmatrix}. \longrightarrow \begin{array}{c} -② \\ -④ \\ ① \\ ③ \end{array}\begin{bmatrix} \\ \\ \\ \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}.$$

the rank of a matrix is the same after it compose

Compefing solution of $Ax=b$.

$$x_1+2x_2+2x_3+2x_4=b_1$$
$$2x_1+4x_2+6x_3+8x_4=b_2$$
$$3x_1+6x_2+8x_3+10x_4=b_3.$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \xrightarrow{\text{elimi}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array}\right] \xrightarrow{\text{elim}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

Augments matrix $=[A\ b]$

$$\boxed{0=b_3-b_2-b_1}$$

Solvablity condition on b

1. $Ax=b$ solvable when b in $C(A)$.

2. If a comb of rows of A gives zero rows.

then the same comb of entries of b must give zero.

① To find ~~compet~~ complete solution $Ax=b$.

⚠1 $x_{particular}$: set all free varables to 0.

solve $Ax=b$ for pivot variables.

$$x_p=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}$$

⚠2. $x_{nullspace}$.

$Ax_p=b$ $\quad Ax_n=0$

$x_c=x_p+x_n$. $\quad A(x_p+x_n)=b$.

$$x_c=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}+c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

② m b n matrix A of r.

1. if columns of rank is full $\boxed{r = n}$ $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

$N(A) = \begin{Bmatrix} \text{zero} \\ \text{vetor} \end{Bmatrix}$ solution to $Ax = b$: $x_0 = x_p$. (0 or 1 solution).

2. full row rank $\boxed{r = m}$ $R = [\, I \ F \,]$

can solve $Ax = b$ for every b. $\boxed{\text{exists}}$

left with $n - r$ free variables.

3. $r = n = m$. (full rank).

$A = [\quad] \rightarrow R = I$.

$N(A) = \begin{Bmatrix} \text{zero} \\ \text{vector} \end{Bmatrix}$ can solve $Ax = b$ for every b.

4. $r < n$, $r < m$.

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(0 or ∞ solution).

linear independence.

① earlier that class.

注 suppose A is m by n with $m<n$.

Then there are nonzero solution to $Ax=0$

(more unknown than equations)

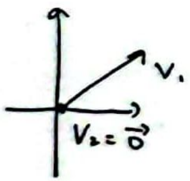
Reason: There will be free variables!!

② independance.

Vectors $x_1\ x_2 \cdots x_n$ are independence if

no combination gives zero vectors (expect the zero combination)

$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ $(\underbrace{c_1\ c_2 \cdots c_n}_{\text{all of them}} \neq 0)$.



v_1 and v_2.

dependent. (if ~~their~~ there is a zero vector, the vector group must dependent).

Repeat: when $v_1, \ldots v_n$ are columns of A.

They are independent if nullspeace of A is $\{\text{zero vector}\}$: $\boxed{\text{rank} = n}$

They are dependent if $Ac=0$ for some nonzero c. $= \boxed{\text{rank} < n}$.

③ span a speace

Vectors $v_1, \cdots v_l$ span a speace

means: The speace consists of all combination of those vectors.

Basis for a ~~speace~~ is a sequence of vectors. $v_1, v_2 \cdots v_d$

two properties.

① independent ② span a speace.

examp: speace is R^3

one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ (standard basis).

R^n: n vectors gives basis if $n \times n$ matrix with those cols is inverable

Given a space:

Every basis for the space has the same <u>number</u> of vectors.

Def: Dimention of the space

rank of $\underbrace{(A)}_{\text{matrix}}$ = # pivot ~~vectors~~ columns = dimensions of $\underbrace{C(A)}_{\text{column space}}$.

dimensions of nullspace = # free variables.

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4. subspace. (A is $m \times n$)

column space $C(A)$ (in R^m)

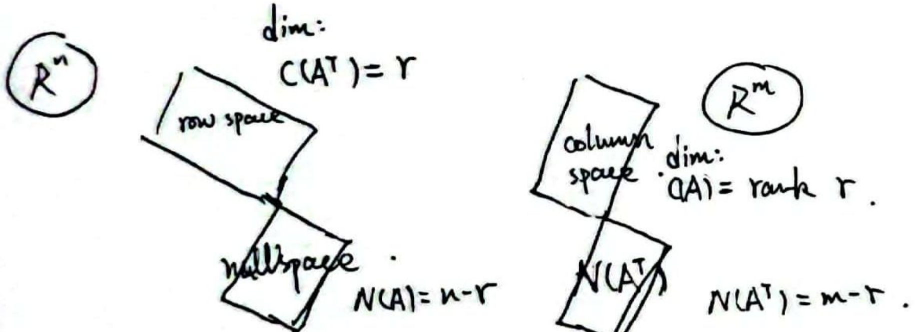
null space $N(A)$ (in R^n)

row space: all combinations of rows A (in R^n).

= all comb of columns of $A^T = C(A^T)$.

nullspace of $A^T = N(A^T)$. (in R^m).

= left nullspace of A.



① row space

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$C(R) \neq C(A)$.

different column space

same row space.

Basis for row space is the first r of R (the third row is useless).

reason: doing elimination is the linear combinations of rows. the solution still in the space.

② $N(A^T)$.

$$A^T y = 0 \longrightarrow y^T A = 0^T.$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

how to fine:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{array}{cc} A_{m \times n} & I_{m \times m} \end{array} \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{elimin}} \begin{array}{cc} R_{m \times n} & E \end{array} \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

so we have $E \cdot A = R$. [0 0 0 0]

the row [-1 0 1] in E multiply A get

③ new vector space M

All 3x3 matries!!

subspace of M : all upper trangelors | symmetrix matrixes | diagenal matrices = D

dim of D is 3.

basis : $\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ $\begin{bmatrix} 1&0&0 \\ 0&3&0 \\ 0&0&0 \end{bmatrix}$. $\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&7 \end{bmatrix}$ make a diagenal matrix

⑦ vector space M.

M = all 3x3 matrices.

subspace: symmetric 3x3.

upper triangle 3x3.

diagonal 3x3.

① Basis and dim

basis for M: (9)

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} \cdots\cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

basis for S (6).

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$$

basis for U (6).

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$S \cap U$ = symm and upper tranglar.

= diagonal 3x3's $\dim(S \cap U) = 3$.

$S + U$ = any element of S + any element of U.

= all 3x3's $\dim(S+U) = 9$.

$$\boxed{\dim S + \dim U = \dim(S \cap U) + \dim(S+U)}$$

② rank.

we can creat a matrix that rank is 1.

$$A = \begin{bmatrix}1&4&5\\2&8&10\end{bmatrix} \qquad \dim C(A) = \text{rank} = \dim C(A^T)$$

$r = 1$.

one basis of column: $\begin{bmatrix}1\\2\end{bmatrix}$

row: $[1\ 4\ 5]$

for rank one matrix:

$$A = \begin{bmatrix}1&4&5\\2&8&10\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}[1\ 4\ 5].$$

$= uv^T$ (u and v are column space).

If I need a 5×17 matrix rank 4.

how can I make.

we need 4 matrix that [ ][ ]. in this form

↓

M = all 5×17 matrices.

subset of rank 4 matrices. not a subspace

In R^4

$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ S = all v in R^4 with $v_1+v_2+v_3+v_4=0$.

is a subspace (R^3).

= null space of $A = [1\ 1\ 1\ 1]$.

rank = 1 $\dim N(A) = n-r$.

$C(A) = R^1$ $N(A^T) = \{0\}$.

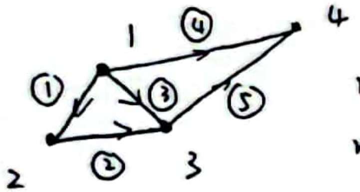
basis of S.

$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$N(A) + C(A^T) = 4 = n$ $C(A) + N(A^T) = 1 = m$.

① Small world graph = {notes, edges}.

Graph: Notes. Edges.



$n = 4$ notes

$m = 5$ edges.

Incidence matrix:

$$A = \begin{array}{c} \\ \end{array}\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

note: 1 2 3 4 (columns); edge: 1–5 (rows)

$\left.\begin{matrix}1\\2\\3\end{matrix}\right\}$ loop = dependent.

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$x = x_1, x_2\ x_3\ x_4$.

$\dim N(A) = 1 \quad \text{rank} = 3$.

one basis: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

x: potentials at notes

$\downarrow$

$x_2 - x_1$: potentials differences. $\rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ means when $I = 0$ on edges. what will happen about potentials difference.

$A^T y = 0 \qquad \dim N(A^T) = 5 - 3 = 2$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(KCL)

y: $y_1, y_2\ y_3\ y_4\ y_5$

y: currents

$$-y_1 - y_3 - y_4 = 0.$$
$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

basis for $N(A^T)$.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

not independent.

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

pivot variable

all dependence came from loop.

so independence means Graph that not loop

$\boxed{\text{Tree: graph with no loop}}$

$\dim N(A^T) = m - r$.

$$\boxed{\#\text{loops} = \#\text{edges} - (\#\text{nodes} - 1).}$$

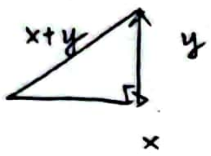
$(\text{rank} = n - \underbrace{1}_{N(A)})$ (Euler's Formular).

$$\boxed{A^T C A x = f}$$

$$\begin{tikzcd}
x:\ \text{potenials at nodes} \arrow[d, "e = Ax"] & & \text{KCL} \\
e:\ \text{potential difference} \arrow[r, "y = Ce"] & y:\ \text{current at edges} \arrow[ur, "A^T y = f \ (\text{outside currents})"'] &
\end{tikzcd}$$

Orthogonal vector.

for vectors:



$\boxed{x^Ty=0}$

proof: $|x|^2+|y|^2=|x+y|^2$

$$x^Tx+y^Ty=(x+y)^T(x+y)$$

$$=x^Tx+y^Ty+x^Ty+xy^T$$

$$\therefore\ 0=\cancel{x}x^Ty$$

[if x is zero vector, y is whatever
the x and y are orthogonal]

for subspace:

Subspace S and Subspace T are orthogonal

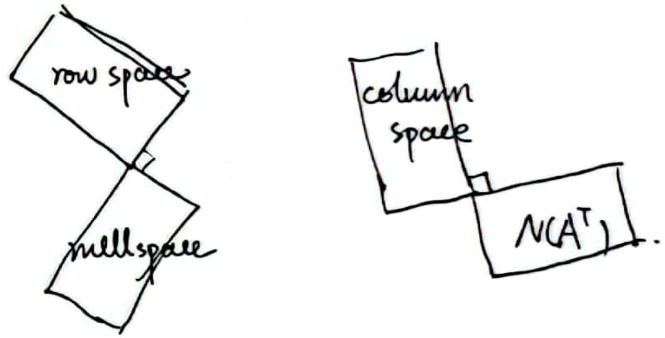
means: every vector in S is orthogonal
to ~~the~~ every vector in T.

row space is othogonal to the nullspace.

why? $Ax=0$ $\begin{bmatrix} \cancel{\text{row } 1} \\ \text{row } 2 \\ \vdots \end{bmatrix}\begin{bmatrix} x \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$.

$c_1(\text{row } 1)^Tx=0$ $\quad c_2(\text{row } 2)^Tx=0$

$c_1(\text{row } 1)^Tx+c_2(\text{row } 2)^Tx=0$.



row space and ~~row s~~ nullspace are othogonal complements in R^n.

means: Nullspace contains <u>all</u> vectors $\perp$ rowspace.

Coming: $Ax = b$.

"solve" $Ax = b$ when there is no solution

(mxn: to find measurement mistakes)

$$\underbrace{A^T A}_{\text{not always inverable}} x = A^T b$$

$N(A^T A) = N(A)$ rank of $A^T A$ = rank of A.

$A^T A$ is inverable exactly if

A has independent columns.

how to proof? next class.

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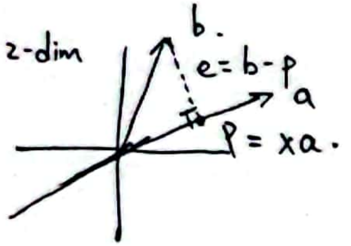
Projections

for vectors:



$a^T(b - xa) = 0$

$x a^T a = a^T b$

$x = \frac{a^T b}{a^T a}$

$p = xa$.

$$\boxed{p = a\frac{a^T b}{a^T a}}$$

$p = Pb$ — P matrix, b vector. $P = \frac{aa^T}{a^T a}$ — aa^T matrix, $a^T a$ number.

* $C(P)$ = line through a rank$(P) = 1$.

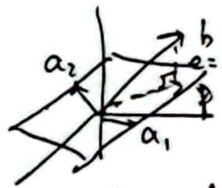
$$\boxed{P^T = P. \quad P^2 = P.}$$

why projections?

Because $Ax = b$ may have no solutions.

solve $Ax = p$ instead. p is projection of b into the column space.

3-dim.



$e = b - p$ is perp. to plane.

$p = x_1 a_1 + x_2 a_2 = A\hat{x}$.

plane of a_1 and a_2

= column space of $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

$p = A\hat{x}$ Find $\hat{x}$

key: $b - A\hat{x}$ is perp. to plane.

$a_1^T(b - A\hat{x}) = 0 \quad a_2^T(b - A\hat{x}) = 0$.

$\downarrow \quad \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$A^T(\underbrace{b - A\hat{x}}) = 0$

e in $N(A^T)$

$e \perp C(A)$ ✓

$\underbrace{A^T A}_{n\times n}\hat{x} = \underbrace{A^T b}_{1\times n}$.

$\hat{x} = (A^TA)^{-1}A^Tb.$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb.$

maxtrix $P = A(A^TA)^{-1}A^T.$

$\neq \cancel{A(A^T)^{-1}A^{-1}A^T}\ AA^{-1}(A^T)^{-1}A^T.$

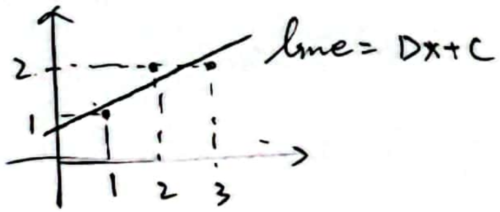
not inverable if no- square.

$P^T = P \qquad P^2 = P$

$P^2 = A(A^TA)^{-1}\cancel{A^TA(A^TA)^{-1}}A^T = P.$

Least squares

Fitting by a line.



(1,1) (2,2) (3,2)

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

⑪

Proj. matrix.

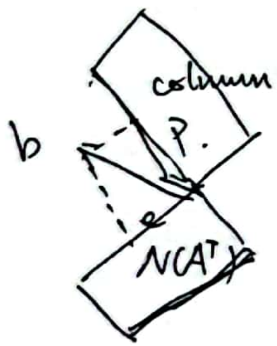
$p = Pb = A(A^TA)^{-1}A^T$

if b in column space $Pb = b$

why? $\because Ax = b$ $\quad Pb = A(A^TA)^{-1}A^T = Ax = b$

if $b \perp$ column space $Pb = 0$

why? $\because b$ in $N(A^T)$. $\quad A^Tb = 0$.

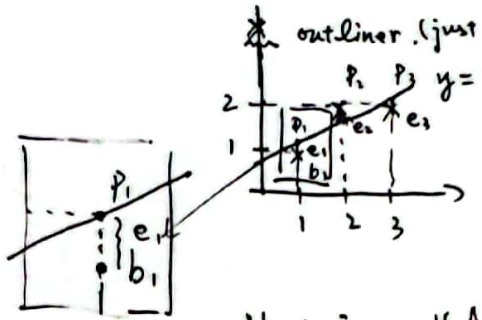


$p = Pb \qquad p + e = b.$

$e = \underset{\sim\sim\sim}{(I-P)}b$

pojection onto $\perp$ space.

examp last class



$C + D = 1$

$C + 2D = 2$

$C + 3D = 2.$

$Ax = b.$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Minimize $\|Ax - b\|^2 = \|e\|^2$.

$\leq e_1^2 + e_2^2 + e_3^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p.

$$\begin{cases} A^TA\hat{x} = A^Tb \\ p = A\hat{x} \end{cases}$$

normal eqns

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \left|\begin{matrix} 1 \\ 2 \\ 2 \end{matrix}\right. = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \left|\begin{matrix} 5 \\ 11 \end{matrix}\right.$$

$\Rightarrow 3C + 6D = 5$

$6C + 14D = 11.$

$$\boxed{\begin{matrix} D = \frac{1}{2} \\ C = \frac{2}{3} \end{matrix}}$$

$e_1 = 1/6 \quad e_2 = -2/6 \quad e_3 = 1/6.$

$b = p + e.$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

$p \perp e$ $\quad \because e \perp$ column space $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

A^TA .

A^TA is inverable.

若 A have independent columns

then A^TA is inverable.

reason: suppose $A^TAx = 0$. (to ~~prove~~ proof $x = 0$).

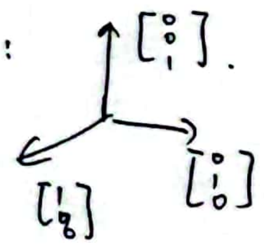
idea: $x^TA^TAx = 0 = (Ax)^T(Ax) \Rightarrow Ax = 0$.

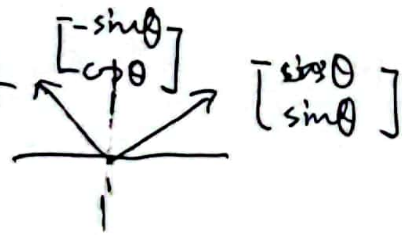
$\Rightarrow x = 0$.

columns definitely independent if

they are perp. unit vector.

orthonormal vector.

like: 

or 

⑰

Othonormal Vector

$$q_i^T q_j = \begin{cases} 0 & i \neq j. \\ 1 & i = j. \end{cases}$$

$$Q = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix} = I$$

we call Q othogonal matrix when it is square (inverable)

$$\underbrace{Q^T = Q^{-1}}_{\text{(square)}}$$

example :

1) perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$Q^T Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I.$$

2) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3) $Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ (Adhemer).

if Q has othonormal colums

Project onto its column space.

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T = I \text{ if } Q \text{ is square}.$$

for QQ^T: ① symmetrix

② $(QQ^T)^T(QQ^T) = QQ^T$.

for $A^TA\hat{x} = A^Tb$

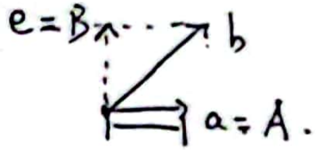
Now A is Q.

$$Q^TQ\hat{x} = Q^Tb. \longrightarrow \hat{x} = Q^Tb.$$

$$(\hat{x}_i = q_i^T b.).$$

Gram-Schmidt. orthonormality $A = QR$.

① examp: vector a, b. (independent).



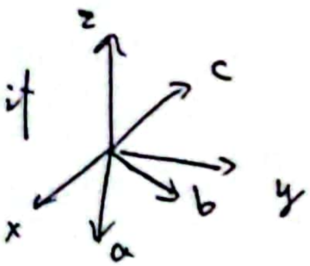
$\longrightarrow$ to get orthogonal vectors $A \cdot B$. $\longrightarrow$ get. orthonormal $q_1 = \frac{A}{|A|}$ $q_2 = \frac{B}{|B|}$

$$\boxed{B = b - \frac{A^T b}{A^T A} A.}$$

how to proof $A \perp B$?

$$A^T B = A^T b - A^T b \frac{A^T A}{A^T A} = 0$$

if



vectors a, b, c.

$$\boxed{C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B.}$$

$q_3 = \frac{C}{|C|}$

② $A = QR$

$$[a_1\ b_1] = [q_1 \cdot q_2] \begin{bmatrix} a_1^T q_1 & * \\ a_1^T q_2 & * \end{bmatrix}$$

$\rightarrow$ upper triangular

$a_1 \perp q_2$ $a_1^T q_2 = 0$.

13

Determinants. $\det A = |A|$.

to check the inverablity

① the properties of det.

1). $\det I = 1$. ($\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$)

2) exchange rows: reverse sign of det (the det $P = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$).

($\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$). [for every permutation: either odd or ~~even~~ even]

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = \cancel{ad - bc} \cdot t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

3) linear each row.

4). 2 equal row $\rightarrow \det = 0$.

(when exchange those rows we get same matrix).

5). Substract $l \times$ row i from row k.

the det. doesn't change. ($\begin{vmatrix} a & b \\ c-la & b-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$).

6). row of zeros $\rightarrow \det A = 0$. ($t=0$)

7)

$$U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & & d_n \end{bmatrix}$$

$\det U = d_1 \times d_2 \times \cdots d_n$. (product of pivots)

$$\begin{bmatrix} d_1 & * & * \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix} \xrightarrow{\text{elim}} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix}$$

$$\rightarrow d_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \cdots\cdots \rightarrow d_1 d_2 \cdots \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

8) $\det A = 0$

when A is singluar.

$\det A \neq 0$

when A is inverable.

9) det AB = (det A)(det B).

det $A^{-1} = 1/\det A$. (inverable)   det $2A = 2^n \det A$.

10). det A^T = det A. $\longrightarrow$ column exchange will change the single sign.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

prof: $|A^T| = |A|$

$|U^T||L^T| = |L||U|$ ✓

for any $\begin{vmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{vmatrix}$ or $\begin{vmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{vmatrix}$

$\downarrow$ elim

$\begin{vmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{vmatrix}$.

13

① Formula for A.

examp: 2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0}$$

$$= ad - bc.$$

3×3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \cdots + \cdots$$

each column each row will have a number.

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \cdots\cdots$$

Big formula

$$\det A = \underbrace{\sum}_{n!\ \text{terms}} \pm a_{1\alpha} a_{2\beta} \cdots a_{n\omega}.$$

n

$n-1$

$(\alpha, \beta, \cdots \omega)$ = permutation of $(1, 2, \cdots n)$.

examp:

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

det = 0

$(4, 3, 2, 1) \rightarrow +1$

$(3, 2, 1, 4) \rightarrow -1$

② Cofactors 3×3. in parens

$$\det A = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{\text{Cofactors of } a_{11}}$$

$$a_{12}(\cdots\cdots)$$

$$a_{13}(\quad)$$

$$\begin{vmatrix} \cancel{a_{11}} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & \cancel{a_{12}} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix}$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \cdots$$

$$\text{Cofactor of } a_{ij} = \underset{\sim}{\pm} \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{col } j \end{pmatrix} \text{erased} = C_{ij}.$$

$\downarrow$

$i+j$ is even : $+$

$i+j$ is odd : $-$

Cofactor formular.

$$\det A = a_{11}C_{11} + a_{12}C_{21} + \cdots + a_{1n}C_{1n}.$$

15

① Formular for A^{-1}

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\underbrace{\det A}_{\text{products of } n \text{ entries}}} \underbrace{C^T}_{}$$ products of $n-1$ entries

how to check: (proof $A \cdot C^T = \det A \cdot I$).

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ 0 & \ddots & \\ 0 & & \\ 0 & & \det A \end{bmatrix}$$

why zero?

③ examp:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A_s = \begin{bmatrix} a & b \\ a & -b \end{bmatrix} \quad \det A_s = 0$$

④ we can solve the question what will happen to A^{-1} if I change a_{11} or (whatever).

$$\begin{cases} \det A \text{ will change} \\ a_{11} \cdot C_{11} \text{ will change} \end{cases}$$

② Cramer's rule. (useless rule)

$$Ax = b$$

$$x = A^{-1}b = \frac{1}{\det A} C^T b \,..$$

componments of x.

$$x_1 = \frac{\det B_1}{\det A} \qquad x_n = \frac{\det B_n}{\det A}.$$

$$B_1 = \begin{bmatrix} b & a_1 & a_2 \cdots a_n \end{bmatrix}$$

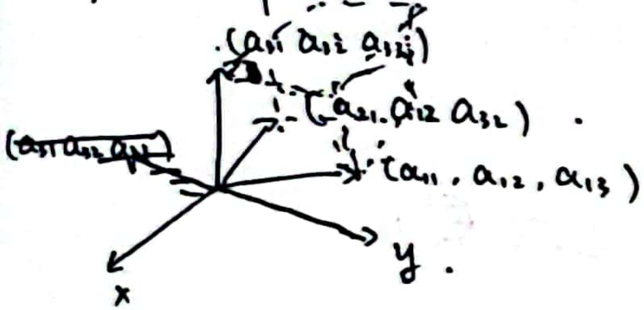
= A with column placed by b

B_j = A with columns j replaced by b.

③ volume

$|\det A|$ = volume of box

for examp: 3×3



check it?

① I ✓ ② ± ✓

③ 3-a. 3-b (in textbooks).

for $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix}$ ✓ $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix}$

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① Eigenvectors.

Ax - prarallel to x.

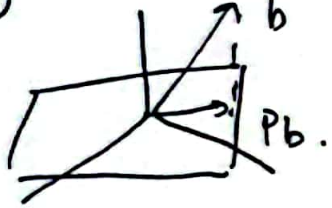
$$\boxed{\underset{\sim}{A x} = \underset{\sim}{\lambda} x}$$

eigenvector eigenvalue.

if A is sigular, then $\lambda = 0$ is erduce.

examp:

①



What x's and λ's for projection matrix?

Any x the plane: $Px = x$, $\lambda = 1$

Any $x \perp$ plane: $Px = 0$, $\lambda = 0$

② permutation.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{cases} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \lambda = 1. \quad Ax = x \\ x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \lambda = -1 \quad Ax = -x. \end{cases}$

Fact:

sum of λ's $= a_{11} + a_{22} + \cdots + a_{nn}$.

how to solve $Ax = \lambda x$.

Rewrite: $\underbrace{(A - \lambda I)}_{\text{singluar}} x = 0$.

$\det(A - \lambda I) = 0$. Find lamda First.

examp: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$= \lambda^2 - 6\lambda + 8$

$\lambda_1 = 2 \quad \lambda_2 = 4$.

$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

thinking: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{+3I} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases} \longrightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = 2. \end{cases}$

if $Ax = \lambda x$ then $(A + 3I)x = (\lambda + 3)x$.

Not so great.

If $Ax = \lambda x$, B has eigenvalues α

$Ax_1 = \lambda x$, $Bx = \alpha x$

$(A+B)x = (\lambda + \alpha)x$ $\times$

examp:

① 90° rotation $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

trace $= 0 + 0 = \lambda_1 + \lambda_2$

det $= 1 = \lambda_1 \lambda_2$.

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda_1 = i \quad \lambda_2 = -i$.

② $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\det(A - \lambda I) = (3-\lambda)(3-\lambda)$

$\lambda_1 = 3 \quad \lambda_2 = 3$.

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$. $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_2 = \Big[$ No second independent

17

1 Diagonalize the matrix

Suppose n independent eigenvectors of A.

Put them into columns of S.

diagonal eigenvalue matrix Λ

$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = S\Lambda$$

$$AS = S\Lambda$$

$$\boxed{S^{-1}AS = \Lambda \qquad A = S\Lambda S^{-1}}$$

if $Ax = \lambda x$

$$\boxed{A^2x = \lambda A x = \lambda^2 x. \qquad A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}}$$

A is sure to have n independent eigenvectors (and be digonaliable) if all λ's are different (no repeated λ's)

Theorm

$A^k \to 0$ as $k \to \infty$

if all eigenvalue $|\lambda| < 1$

repeated eigenvalues // may or may not have n indepent

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$\lambda = 2, 2 \qquad A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cannot be diagonalized.

② Equation. $u_{k+1} = Au_k$.

Start with given vector u_0

$$u_1 = Au_0, \quad u_2 = A^2 u_0$$

$$\boxed{u_k = A^k u_0}$$

To really solve:

$$u_0 = c_1 x_1 + \dots + c_n x_n = Sc$$

$$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n$$

$$= \Lambda^{100} Sc = u_{100}.$$

examp:

Fibonacci: 0, 1, 1, 2, 3, 5 ..... $\boxed{F_{100} = ?}$

$$F_{k+2} = F_{k+1} + F_k.$$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \qquad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix}$$

$$\begin{cases} \lambda_1 = \frac{1}{2}(1+\sqrt{5}) & x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \\ \lambda_2 = \frac{1}{2}(1-\sqrt{5}) & x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \end{cases}$$

$$u_0 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_{100} = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2.$$

19

Differential Equations $\frac{du}{dt} = Au$.

① examp: $\frac{du_1}{dt} = -u_1 + 2u_2$ and $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{du_2}{dt} = u_1 - 2u_1$

$\longrightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\lambda = 0, -3$. $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

check: $\frac{du}{dt} = Au$. Plug in $e^{\lambda_1 t} x_1 = \lambda_1 e^{\lambda_1 t} x_1$

$= A e^{\lambda t} x_1$

At $t = 0$.

$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}^{S} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution $c_1 = c_2 = \frac{1}{3}$

Steady state

$u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① Stability $u(t) \to 0$

need $e^{\lambda t} \to 0$ = Real part $\lambda < 0$

② Steady state.

$\lambda_1 = 0$ and other Real part $\lambda < 0$

③ Blow up if any Real part $\lambda > 0$.

In 2x2 matrix stability: $\lambda_1 < 0$ and $\lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; trace: $a + d = \lambda_1 + \lambda_2 < 0$

det $= \lambda_1 \lambda_2 > 0$.

② for $Au = \frac{du}{dt}$

Set $u = Sv$.

$S \frac{dv}{dt} = \cancel{S} ASv$

$$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\downarrow$$

$$\cancel{\frac{dv_1}{dt} = A} \quad v(t) = e^{\Lambda t}v(0)$$

$$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$$

③ Matrix exponential e^{At}

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \cdots + \frac{1}{n!}(At)^n + \cdots$$

$$= \sum_{0}^{\infty} \frac{x^n}{n!}$$

how to check $Se^{\Lambda t}S^{-1} = e^{At}$

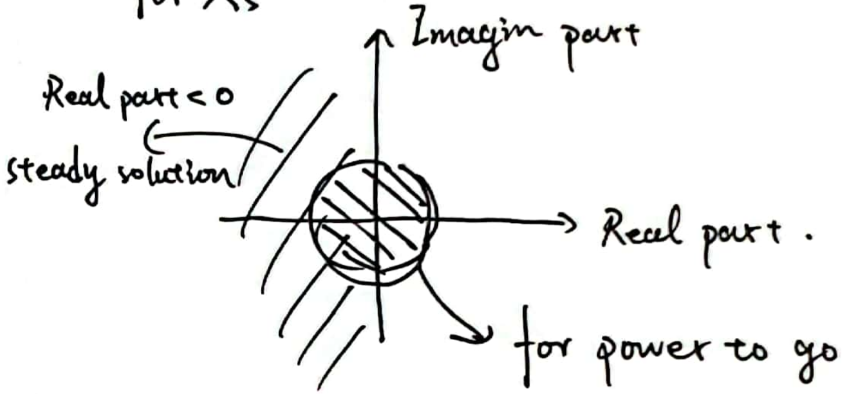
$$e^{At} = \overset{SIS^{-1}}{I} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} + \cdots + \cdots$$

$$= Se^{\Lambda t}S \quad \text{(if A can be diagonal)}$$

what is $e^{\Lambda t}$?

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_2 t} \end{bmatrix}$$

for λ's



for power to go to 0

$|\lambda| < 0$.

examp: $y'' + by' + ky = 0$

$$y' = y'$$

1 2nd order equat

→ 2×1 order system

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y' \\ y \end{bmatrix}}_{u}$$

⑫

MarKov matrix

①

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

two propecties

① All entries ≥ 0 .

② All columns add to 1 .

the key points about MarKow

1. $\lambda = 1$ is an eigonvalue

2. All other $|\lambda_i| < 1$.

$$u_K = A^K u_0 = \underbrace{c_1 \lambda_1^K}_{=1} x_1 + \underbrace{c_2 \lambda_2^K}_{|\ |<0} x_2 \cdots \rightarrow \underset{\geq 0}{\text{steady}} \text{ state}$$

how to proof "$\lambda = 1$ is an eigonvalue" ?

$$A - I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

$\xRightarrow{r_1 + r_2 + r_3}$ to get [0 0 0] row

∴ the rows are dependent , this is a sigular matrix .

$\lambda = 1$ is one of the eigonvalue

✳ : eigonvalue of A and A^T are the same .

$\det(A - \lambda I) = 0 \rightarrow \det(A^T - \lambda I) = 0$

② [illegible] $u_{k+1} = Au_k$ A is Markov.

the flow of people $A \Leftrightarrow B$.

$$\begin{bmatrix} C_A \\ C_B \end{bmatrix}_{(t=k+1)} = \begin{bmatrix} \underbrace{0.9}_{} & 0.2 \\ \underbrace{0.1}_{} & \underbrace{0.8}_{} \end{bmatrix} \begin{bmatrix} C_A \\ C_B \end{bmatrix}_{(t=k)}$$

stay in A　→ B→A.

A→B　stay in B

$$\begin{bmatrix} C_A \\ C_B \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

eigonvalue: $\lambda_1 = 1$ $\lambda_2 = 0.7$

eigonvector: $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$c_1 = 1000/3$　$c_2 = 2000/3$.

③ projections with othonormal basis.

$q_1 . q_2 \cdots q_n$

Any $v = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$.

$$\boxed{q_1^T v = x_1} \underbrace{q_1^T q_1}_{=1} + 0 \cdots\cdots + 0.$$

Fourier series.

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots\cdots$

for vectors

how to get a_1

$\int f(x) \cos x \, dx = a_1 \int \cos^2 x \, dx$

$v^T w = v_1 w_1 + \cdots + v_n w_n$

Fuctions $f^T g = \int_0^{2\pi} f(x) g(x) dx$.

20

Symmatric matrix.

① $A = A^T$

① the eigonvalue are $\boxed{\text{real}}$.

② the eigonvectors are $\boxed{\text{perpendicular}}$

for matrix A

usually case:

$$A = S\Lambda S^{-1}$$

symmetric case:

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$$

I: Why real eigonvalues?

$$Ax = \lambda x \xRightarrow{\text{always}} A\bar{x} = \bar{\lambda}\bar{x} \xRightarrow{\text{trans}} \bar{x}^T A = \bar{x}^T \underbrace{\bar{\lambda}}_{\text{number}}$$

$$\bar{x}^T A x = \lambda \bar{x}^T x \qquad \bar{x}^T A \cdot x = \bar{x}^T \bar{\lambda} x$$

$$\therefore \lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$$

$$\therefore \boxed{\lambda = \bar{\lambda}}$$

if x is complex $\bar{x}^T x = [\bar{x}_1, \bar{x}_2 \cdots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$(\text{length})^2$

$= \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n$

$\because (a+bi)(a-bi) = a^2 + b^2$

Good matrix $\begin{cases} \text{Real } \lambda\text{'s} \\ \text{Perpendicular } x\text{'s} \end{cases}$

$A = \bar{A}^T$ (for complex)

② For $A = A^T$

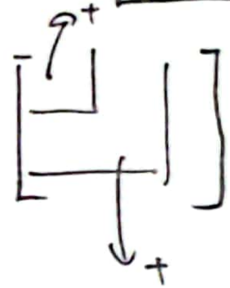
$$A = Q\Lambda Q^T$$

$$= \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} - q_1^T - \\ \vdots \\ - q_n^T - \end{bmatrix}$$

$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots$

Every symmetris matrix is a combination of perpendicular projection matrices

all sub-determines $\boxed{\text{postive}}$

$=$ 

③ for symmetric matrix

Sign of pivot same as sign of λ's

postive pivots = # postive λ's

④ Postive definite symmetric matrix.

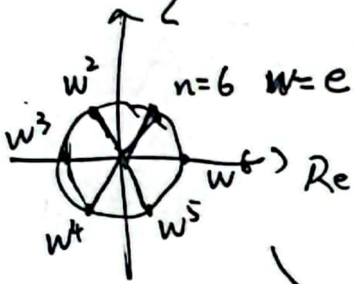
all the eigvalue are $\boxed{\text{posive}}$

all pivots are $\boxed{\text{postive}}$

21 Fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} \qquad (F_n)_{ij} = w^{ij}.$$

$$w^n = 1 \qquad w = e^{i2\pi/n}$$



$$(W_{12})^2 = W_6$$

$$F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$ cols othrogonal.

othronomal.

$$F_4^H F_4 = I \qquad F_4^{-1} = F_4^H$$

?
$$\underbrace{\begin{bmatrix} F_{64} \end{bmatrix}}_{\text{calculate for } (64)^2 \text{ times}} = \underbrace{\begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}\overset{P}{\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}}_{\text{calculate for } [2(32)^2 + 32] \text{ times}}$$

F_{32}

$$= [\ ]\begin{bmatrix} F_{16} & \\ & \end{bmatrix}[\ ]$$

$$D = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & w^{31} \end{bmatrix}$$

$$6 \times 32$$

$$= \frac{64}{2}\log_2 64.$$

For n square

calculate for $\boxed{\frac{1}{2} n \log_2 n}$ times

Complex Vectors.

for $z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ in C^n

① length of z

$z^T z$ not good

if ($z_1 = 1$ and $z_2 = i \longrightarrow z^T z = 0$)

$\bar{z}^T z$ is good

(we need $|z|^2$ not z^2)

$\bar{z}_i z = |z_i|^2$

we call $\bar{z}^T$ to z^H (Hermitian)

inner product turns to

$\bar{y}^T x = y^H x$

for symmetric matrix

$A^T = A$ not good if A is complex.

$\bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} = A^H$

(Hermitian matrix)

$A^H = A$

Perpendicular.

$q_1\ q_2 \cdots q_n$

$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j. \\ 1 & i = j \end{cases}$

$Q = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix}$

$Q^H Q = I.$

$\longrightarrow$ unitary matrix.

22

Positive Definite Matrix (symmetris)

how to test ? for $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0 \quad \lambda_2 > 0$

② $a > 0 \quad ac - b^2 > 0$

③ pivots $a > 0 \quad \frac{ac-b^2}{a} > 0$

④ $*$: $x^TAx > 0$ except $x = 0$

examp:

$$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \xrightarrow{x^TAx} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

semi-definite

$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 20 \end{cases}$ pivots : 2 .

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2 \text{ (quadratie from)}$$

$$ax^2 + 2bxy + cy^2 > 0 \text{ ?}$$

for

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{x^TAx} 2x^2 + 12xy + 20y^2$$

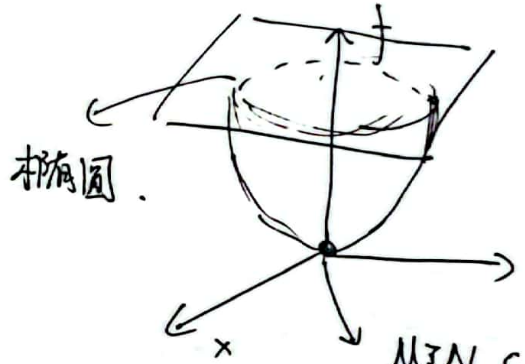
$\begin{cases} \det = 4 > 0 \\ \text{trace} = 22 > 0 \end{cases} \longrightarrow \lambda_1, \lambda_2 > 0$.

$$f(x,y) = 2x^2 + 12xy + 20y^2 = \underbrace{2(x+3y)^2 + 2y^2}_{U}$$

pivots 1 · times · pivots 2

$$\begin{array}{ccc} A & & U \\ \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} & \longrightarrow & \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} \end{array}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$



MIN $\begin{cases} \text{1st derivative} = 0 \\ \text{2nd derivative} > 0 . \end{cases}$

For 2nd derivatives ~~fo~~ in Matrix.

or bring it to $x^T A x$ to see $\underline{C > 0}$?

~~(by bring it to $x^T A x$)~~

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \xrightarrow{\text{must positive}} \text{has MIN (in } f(x,y)) \longrightarrow \text{for } \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

definite

$ax^2 + 2bxy + cy^2$ can become ~~$(x+by)^2 + cy^2$~~ $A(x+By)^2 + Cy^2$

$f_{xx} = 2a \quad f_{yx} = 2b$

$f_{xy} = 2b \quad f_{yy} = 2c \longrightarrow C > 0$?

for 3x3 example.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

dets $2, 3, 4$

pivots $2, \ 3/2, \ 4/3$

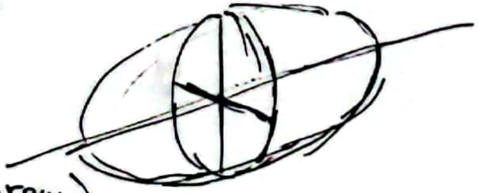
pivots × pivots = dets

eigenvalues: $2-\sqrt{2}, \ 2+\sqrt{2}, \ 2$

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0.$$

if $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 = 1$ what will happen about his chape?

$A = Q \Lambda Q^T$ (~~p~~ axis theorem)



eigenvector: the driections of principal axes

eigenvalue: the lengths of axes

positive definite means.

$x^TAx > 0$

- If A and B is pos def.

(A+B) is pos def ($x^T(A+B)x > 0$)

- Now A m×n.

A^TA is pos def ? YES

square . symmetric

$x^TA^TAx = (Ax)^T(Ax) = |Ax|^2 \geq 0 \longrightarrow > 0$

when rank is $\underline{n}$.

Similar Matrix.

$A_{(n\times n)}$ and $B_{(n\times n)}$ is similar

means: for some inversable M.

$B = M^{-1}AM$.

examp: A is similar to Λ

$S^{-1}AS = \Lambda$

if M is $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ A is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$M^{-1}AM = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix} = B$

$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

what is the same between A, B and Λ?

THEY HAVE the SAME EIGENVALE

how to check?

$Ax = \lambda x$. $(B = M^{-1}AM)$.

$\downarrow$

$\underbrace{M^{-1}AM}_{B}M^{-1}x = \lambda M^{-1}x \longrightarrow BM^{-1}x = \lambda M^{-1}x$.

eigenvectors of B is

M^{-1}(eigenvectors of A).

For Bad Case. $\lambda_1 = \lambda_2$.

suppose $\lambda_1 = \lambda_2 = 4$.

- one small family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ because for any $M \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, the family only itself.

- big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, it cannot diagonalized.

Jordan form.

nearly "diagonalize".

more member of family:

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$ trace = 8
det = 16.

for $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]_A$ and $\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]_B$

$\lambda = 0, 0, 0, 0$ $\quad N(A) = 2 \longrightarrow$ missing 2

but they are not similar (the block size is diff).

Jordan block.

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

the same num.

Every square A is similar to a Jordan matrix J.

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_n} \end{bmatrix}$$

blocks = # eigenvectors.

① Good case: $J = \Lambda$.