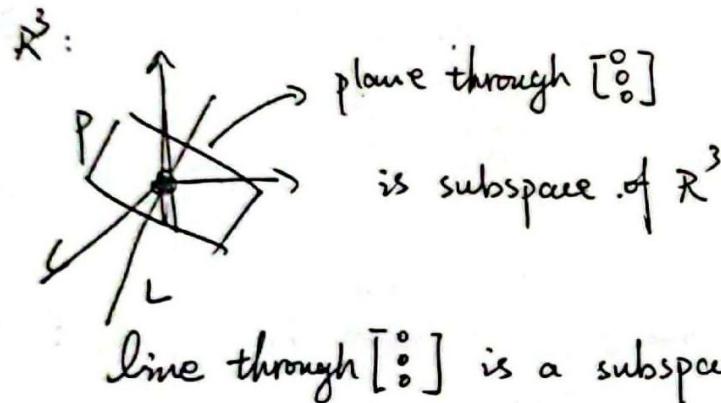


73页全篇彩色文学传世圣典♥

把前途无量的小学生们
用扶她精子
射到怀孕也没问题的吧!?

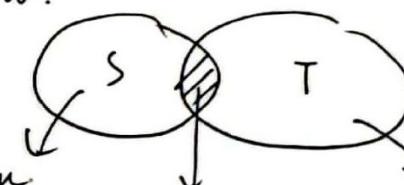
①



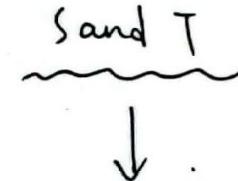
for subspace S and T.

their intersection $S \cap T$
is a subspace.

for v and w:



the linear com
of S and not in T the linear com
in T and not in
S.



$v+w$, $c v$, and $c v + d w$

it cannot add — if I use a vector in \mathbb{R}

and a vector in P. I will not get a subspace. have to do inside the
subspace.

② $P \cap L =$ all vectors in both P and L — this is a subspace

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

② column space of a matrix.

we can solve $Ax=b$ when exactly when

example: $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} = A$. a subspace in $\mathbb{R}^4 = C(A)$

b is in $C(A)$.



when b is not ~~the~~ linear com of $C(A)$

it cannot be solved (outside the $C(A)$).

= all linear com of columns.

Does $Ax=b$ have a solution for every b ? No.

because there is 4 equations but 3 unknowns.



$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

useless. pivot 1 pivot 2.

not independent.

$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ is a 2-dim subspace.

which b 's allow the system to be solved??

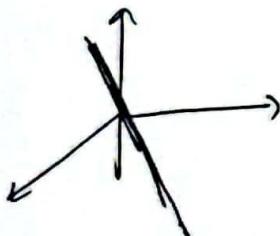
① $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ always works. ② $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

③ Nullspace.

Nullspace of $A = \text{all solutions } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
to $Ax = \vec{0}$. (in \mathbb{R}^3) .

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ -n \\ n \end{bmatrix}, (n=0,1,2,\dots)$.



$N(A)$ is line in \mathbb{R}^3 .

how can I proof it [solution to $Ax = \vec{0}$
always give a subspace]?

* If $Ax = \vec{0}$ $Aw = \vec{0}$. and $A(v+w) = Av + Aw = \vec{0}$

and $A(cv) = cAv = \vec{0}$.

if $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, all the solution

of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ cannot build as subspace.

(why) the $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not inside it.

so what how can I build a subspace?

① find some vectors and full out
the space to find a subspace.

② find the solution of $Ax = b$.
to find the subspace.

① calculation of $Ax = \vec{0}$

example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

elimination will not change it.

1. elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form.

the pivot of a matrix: (2).

= rank of matrix A

2. pivot variables.

the pivot columns: the columns with pivots.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot column

free column (we can choose any number).

when we have another x_2, x_4

$$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

special solution: the special number given to the free variable

$$\text{and } c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

contains all the linear combination of the Nullspace.

when we have R pivot vari-

and n columns, m rows matrix.

free vari = $n - R$.

free variables: ~~variables~~

$$\text{exam: } x = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} \xrightarrow{\substack{\text{back} \\ \text{sub...}}} x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} \text{means:} \\ -2 \times \text{columns 1} \\ + 1 \times \text{columns 2} \\ \text{can get } \vec{0} \end{array}$$

3. clean up the matrix: reduce the row line .

R = reduced row echelon form: ⁽¹⁾ zero above + below the pivots. ⁽²⁾ pivots = \neq

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A) \text{ in Matlab}$$

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot columns/rows .

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(more easy to back sub)}.$$

when we compose the matrix A .

rref from : $\begin{bmatrix} R & F \end{bmatrix} \xrightarrow{\text{pivot rows } r} R$.

pivot column $r \xrightarrow{n-r \text{ free col.}}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 16 \end{bmatrix} \xrightarrow{\text{elim}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 rank

nullspace matrix : columns are special solution) .

$$RN=0 \rightarrow N = \begin{bmatrix} -F \\ R \end{bmatrix} \xrightarrow{\substack{\text{②} \\ \text{④} \\ \text{①} \\ \text{⑤}}} \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

the rank of a matrix is
the same after it compose

④
Computing solution of $Ax = b$.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = b_1$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$



$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right] \xrightarrow{\text{elim1}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array} \right] \xrightarrow{\text{elim2}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right]$$

Augments matrix = $[A \ b]$

$$\boxed{0 = b_3 - b_2 - b_1}$$

Solvability condition on b

1. $Ax = b$ solvable when b in $C(A)$.

2. If a comb of rows of A gives zero rows.

then the same comb of entries of b must give zero.

① To find complete

solution $Ax = b$.

\triangle $x_{\text{particular}}$: set all free variables to 0.

solve $Ax = b$ for pivot variables.

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

\triangle . $x_{\text{nullspace}}$. $Ax_p = b$ $Ax_n = 0$

$$x_c = x_p + x_n . \quad A(x_p + x_n) = b .$$

$$x_c = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

② m × n matrix A of r. *

1. if columns of rank is full $\boxed{r=n}$ $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

$N(A) = \begin{cases} \text{zero} \\ \text{vector} \end{cases}$ solution
to $Ax=b$: $x_0 = x_p$. (0 or 1 solution).

2. full row rank $\boxed{r=m}$ $R = \begin{bmatrix} I & F \end{bmatrix}$

can solve $Ax=b$ for every b . [exists]

left with $n-r$ free variables.

3. $r=n=m$. (full rank)

$$A = \begin{bmatrix} & & \end{bmatrix} \rightarrow R = I.$$

$N(A) = \begin{cases} \text{zero} \\ \text{vector} \end{cases}$ can solve $Ax=b$ for every b .

4. $r < n$, $r < m$.

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(0 or ∞ solution).

⑤ linear independence.

① earlier that class .

② suppose A is m by n with $m \leq n$.

Then there are nonzero solution to $Ax=0$

(more unknown than equations)

Reason: There will be free variables !!

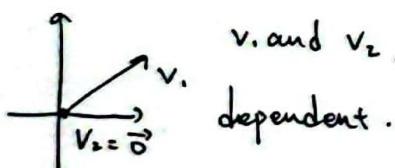
② independence .

Vectors x_1, x_2, \dots, x_n are independence if

no combination gives zero vectors (except the zero combination)

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = 0 \quad (c_1, c_2, \dots, c_n \neq 0)$$

all of them .



v₁ and v₂.
dependent. (if there is a zero vector, the
vector group most dependent) .

Repeat: when v_1, \dots, v_n are columns of A .

They are independent if nullspace of A is $\{ \begin{matrix} \text{zero} \\ \text{vector} \end{matrix} \} : \boxed{\text{rank} = n}$

They are dependent if $Ac=0$ for some nonzero c . = $\boxed{\text{rank} < n}$.

③ span a space

Vectors v_1, \dots, v_l span a space

means: The space consists of all
combination of those vectors .

Basis for a space is a sequence of

vectors . v_1, v_2, \dots, v_d

two properties .

① independent ② span a space .

examp: space is \mathbb{R}^3

one basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (standard basis)

\mathbb{R}^n : n vectors gives basis if $n \times n$
matrix with those cols is invertible

Given a space:

Every basis for the space has the

same number of vectors.

✓ Def: Dimension of the space

rank of $\underbrace{C(A)}$ = # pivot ^{columns} vectors = dimensions of $\underbrace{C(A)}$.
 $\underbrace{\text{matrix}}$ $\underbrace{\text{column space}}$.

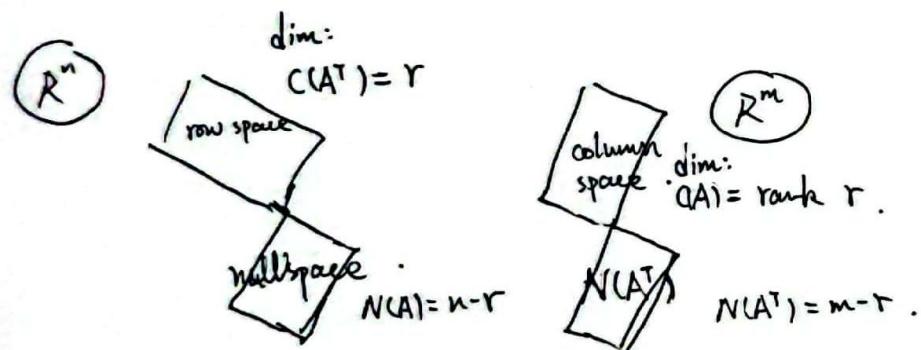
dimensions of nullspace = # free variables.

⑥ 4. subspace . (A is $m \times n$)
 column space $C(A)$ (in \mathbb{R}^m),
 nullspace $N(A)$ (in \mathbb{R}^n)
 rowspace : all combinations of rows A (in \mathbb{R}^n) .
 = all comb of columns of $A^T = C(A^T)$.
 nullspace of $A^T = N(A^T)$. (in \mathbb{R}^m) .
 = left nullspace of A .

Basis for row space is the first r of R (the third row is useless).
 reason : doing elimination is the linear combinations of rows . the solution still in the space .

② $N(A^T)$.

$$A^T y = 0 \rightarrow y^T A = 0^T .$$



① row space

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$C(R) \neq C(A)$.
 different column space
 same row space .

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = [0]$$

how to find:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{elimin}} \begin{bmatrix} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

so we have $E \cdot A = R$. [0 0 0 0]
 the row [-1 0 1] in E multiply A get

③ new vector space M

All 3×3 matrices!!

subspace of M : all upper triangular | symmetric | diagonal
matrices = D
↑ dim of D is 3.

basis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ make a diagonal matrix

①

vector space M .

$M = \text{all } 3 \times 3 \text{ matrices}$.

subspace: symmetric 3×3 .

upper triangle 3×3 .

diagonal 3×3 .

$S \cap U = \text{symm and upper triangular}$.

= diagonal 3×3 's $\dim(S \cap U) = 3$.

$S + U = \text{any element of } S + \text{any element of } U$.

= all 3×3 's $\dim(S + U) = 9$.

① Basis and dim

basis for M : (9)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

basis for S (6).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for U (6).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② rank.

We can create a matrix that rank is 1.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \quad \dim(C(A)) = \text{rank} = \dim(C(A^T))$$

$$r = 1. \quad \text{one basis of column: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for rank one matrix:

$$\text{row: } \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \ 4 \ 5]$$

$$= UV^T \quad (U \text{ and } V \text{ are column space}).$$

If we need a 5×17 matrix rank 4.

how can we make.

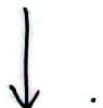
basis of S .

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

We need 4 matrix that $\begin{bmatrix} \quad \\ \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$. in this

$$N(A) + C(A^T) = 4 = n \quad C(A) + N(A^T) = 1 = m.$$

form



① Small world graph = {nodes, edges}.

M = all 5×17 matrices.

subset of rank 4 matrices. not a subspace

$2^n R^4$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad S = \text{all } v \text{ in } R^4 \text{ with } v_1 + v_2 + v_3 + v_4 = 0.$$

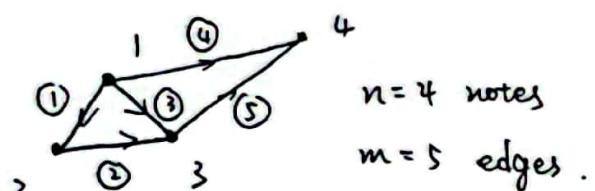
is a subspace (R^3).

= null space of $A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{\text{rank } 1}$.

$$\text{rank } 1 \quad \dim N(A) = n - r.$$

$$C(A) = R^1 \quad N(A^T) = \{0\}.$$

Graph: Notes . Edges .



$$A^T y = 0 \quad \dim N(A^T) = 5 - 3 = 2$$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(KCL)

$$y: y_1, y_2, y_3, y_4, y_5$$

Incidence matrix:

note	1	2	3	4	edge
1	-1	1	0	0	1
2	0	-1	1	0	2
3	-1	0	1	0	3
4	-1	0	0	1	4
5	0	0	-1	1	5

loop = dependent.

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x = x_1, x_2, x_3, x_4.$$

x : potentials at notes



$x_2 - x_1$: potentials differences. $\rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ means when $I = 0$ on edges.
what will happen about potentials difference.

y : currents

$$\begin{array}{l} -y_1 - y_3 - y_4 = 0 \\ y_1 - y_2 = 0 \\ y_2 + y_3 - y_5 = 0 \\ y_4 + y_5 = 0 \end{array} \quad \begin{array}{l} \text{basis for } N(A^T) \\ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \end{array}$$

not independent.

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

↑ ↑ ↑ ↑ ↑ q

pivot variable

all dependence came from loop.

so independence means graph that not loop

Tree : graph with no loop

$$\dim N(A^\top) = m-r .$$

$$\boxed{\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)} .$$

(rank = $n-1$)

$N(A)$

(Euler's Formula) .

x : potentials at nodes

$$\boxed{A^\top C A x = f}$$

KCL .

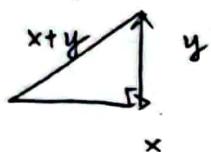
$$\downarrow e = Ax .$$

$$\uparrow A^\top y = f_{\text{in}} \quad (\text{outside currents}) .$$

e : potential difference $\xrightarrow{y = Ce}$ y : current at edges

Orthogonal vector.

for vectors:



$$x^T y = 0$$

proof: $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$$x^T x + y^T y = (x+y)^T (x+y)$$

$$= x^T x + y^T y + x^T y + x y^T$$

$$\therefore 0 = x^T y$$

[if x is zero vector, y is whatever
the x and y are orthogonal]

for subspace:

Subspace S and Subspace T are orthogonal

means: every vector in S is orthogonal
to every vector in T.

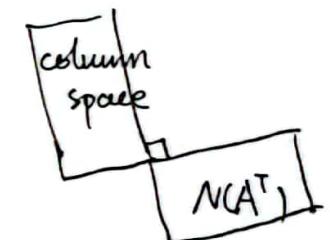
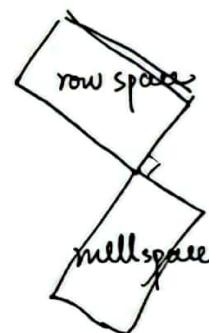
row space is orthogonal to the nullspace.

why?

$$Ax = 0 \quad \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \vdots \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$c_1(\text{row 1})^T x = 0 \quad c_2(\text{row 2})^T x = 0$$

$$c_1(\text{row } k)^T x + c_2(\text{row 2})^T x = 0.$$



row space and ~~null~~^{nullspace} are orthogonal
complements in \mathbb{R}^n .

means: Nullspace contains all vectors
 \perp rowspace.

Coming: $Ax = b$.

"solve" $Ax = b$ when there is no solution

($m \times n$: to find measurement mistakes)

$$\underbrace{A^T A}_{} x = \underbrace{A^T b}_{} \quad \text{not always invertible}.$$

$$N(A^T A) = N(A) \quad \text{rank of } A^T A = \text{rank of } A.$$

$A^T A$ is invertible exactly if

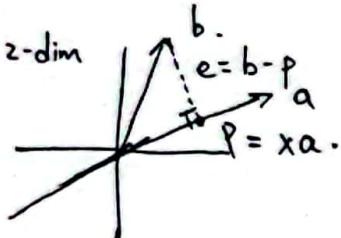
A has independent columns.

how to proof? next class.

10

.Projections

for vectors:



$$P = \alpha \frac{\alpha^T b}{\alpha^T \alpha}$$

$$P = \underbrace{Pb}_{\substack{\text{vector} \\ \text{matrix}}} \quad P = \underbrace{\frac{\alpha \alpha^T}{\alpha^T \alpha}}_{\substack{\text{matrix} \\ \text{number}}} \text{ matrix}$$

*

$C(P) = \text{line through } \alpha \quad \text{rank}(P) = 1.$

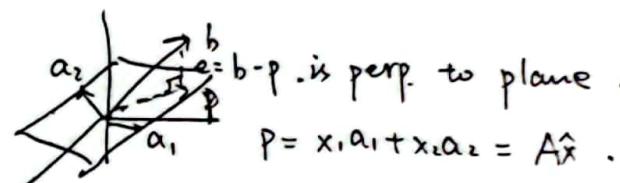
$$\boxed{P^T = P. \quad P^2 = P.}$$

why projections?

Because $Ax = b$ may have no solutions.

solve $Ax = p$ instead. p is projection of b onto the column space.

3-dim.



plane of α_1 and α_2 .

= column space of $A = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$

②

$$p = A\hat{x} \quad \text{Find } \hat{x}$$

key: $b - A\hat{x}$ is perp. to plane.

$$\overline{\alpha_1^T(b - A\hat{x}) = 0 \quad \alpha_2^T(b - A\hat{x}) = 0}.$$

$$\downarrow \quad \begin{bmatrix} \alpha_1^T \\ \alpha_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\overline{A^T(b - A\hat{x}) = 0}$$

$$\begin{array}{l} e \in N(A^T) \\ e \perp C(A) \end{array} \quad \checkmark$$

$$\begin{array}{l} A^T A \hat{x} = A^T b \\ n \times n \quad 1 \times n \end{array}$$

$$\hat{x} = (A^T A)^{-1} A^T b.$$

$$p\hat{x} = A\hat{x} = A(A^T A)^{-1} A^T b$$

$$\text{matrix } P = A(A^T A)^{-1} A^T.$$

$$\neq \cancel{A(A^{-1})^T} \cancel{A} = AA^{-1}(A^T)^{-1}A^T$$

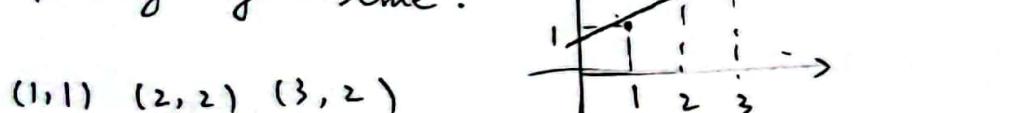
$\underbrace{\text{if not invertible if non-square.}}$

$$P^T = P \quad P^2 = P$$

$$P^2 = A(A^T A)^{-1} \cancel{A^T} \cancel{A(A^{-1})} A^T = P.$$

Least squares

Fitting by a line.



4
3
2
1

$$\begin{aligned}c + D &= 1 \\c + 2D &= 2 \\c + 3D &= 2\end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

11

Proj. matrix.

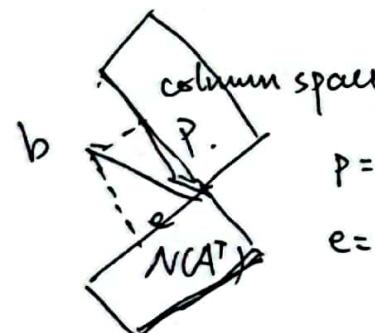
$$P = P^T b = A(A^T A)^{-1} A^T$$

if b in column space $Pb = b$

$$\text{why? } \because A\hat{x} = b \quad P\hat{x} = A(A^T A)^{-1} A^T = A\hat{x} = b$$

if $b \perp$ column space $Pb = 0$

why? $\because b \in N(A^T)$. $A^T b = 0$.



$$p = Pb \quad p + e = b.$$

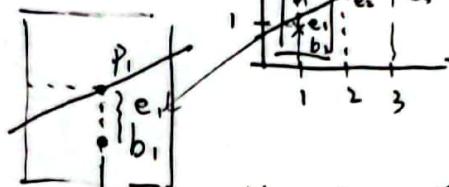
$$e = (\underbrace{I - P}_{N(A^T)}) b$$

projection onto \perp space.

examp last class

* outlier (just ignore) $\cdot c + D = 1$
 $P_1, P_3, y = c + Dt$. $c + 2D = 2$

$$c + 3D = 2 \cdot$$



$$\text{Minimize } \|A\hat{x} - b\|^2 = \|e\|^2.$$

$$\leq e_1^2 + e_2^2 + e_3^2 = (c+D-1)^2 + (c+2D-2)^2 + (c+3D-2)^2$$

$$\text{Find } \hat{x} = \begin{bmatrix} \hat{c} \\ \hat{D} \end{bmatrix}, P.$$

$$\begin{cases} A^T A \hat{x} = A^T b \\ P = A \hat{x} \end{cases} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix}$$

normal eqns

$$\begin{aligned} 3\hat{c} + 6\hat{D} &= 5 \\ 6\hat{c} + 14\hat{D} &= 11 \end{aligned} \quad \begin{bmatrix} \hat{D} = \frac{1}{2} \\ \hat{c} = \frac{2}{3} \end{bmatrix}$$

$$e_1 = \frac{1}{6} \quad e_2 = \frac{2}{6} \quad e_3 = \frac{1}{6}.$$

$$b = p + e.$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

PLS $e \perp$ column space

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$A^T A$.

$A^T A$ is invertible.

If A have independent columns

then $A^T A$ is invertible.

Reason: suppose $A^T A x = 0$. (to ~~prove~~ ^{proof} $x = 0$).

Idea: $x^T A^T A x = 0 \Rightarrow (Ax)^T (Ax) \Rightarrow Ax = 0$.

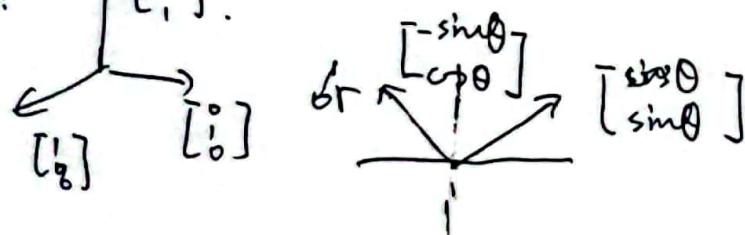
$$\Rightarrow x = 0.$$

columns definitely independent if

they are perp. unit vector.

like: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

orthonormal vector



⑪

Orthonormal Vector

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q = [q_1, \dots, q_n]$$

$$Q^T Q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \cdot \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

we call Q orthogonal matrix when

it is square
(invertible)

$$\underline{\underline{Q^T = Q^{-1}}} \quad (\text{square})$$

example :

$$\text{perm } Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I$$

$$2) Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$3) Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (\text{Adhemer}).$$

if Q has orthonormal columns

Project onto its column space.

$$P = Q(Q^T Q)^{-1} Q^T = \underline{\underline{Q}} \underline{\underline{Q^T}} = z \text{ if } Q \text{ is square.}$$

for QQ^T : ① symmetric

$$\textcircled{2} (QQ^T)^T (QQ^T) = QQ^T.$$

for $A^T A \hat{x} = A^T b$

Now A is Q .

$$Q^T Q \hat{x} = Q^T b \rightarrow \hat{x} = \underline{\underline{Q^T b}}$$

$$(\hat{x}_i = \underline{\underline{q_i^T b}}.)$$

Cram-Schmidt . orthonormality $A = QR$.

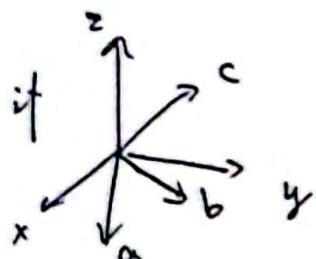
① examp: vector a, b . (independent).

$$e = B \xrightarrow{\perp} b \quad \begin{array}{l} \text{to get orthogonal} \\ \text{vectors } A, B. \end{array} \quad \begin{array}{l} \text{get . orthonormal} \\ q_1 = \frac{A}{\|A\|}, q_2 = \frac{B}{\|B\|} \end{array}$$

$$\boxed{B = b - \frac{A^T b}{A^T A} A.}$$

how to proof $A \perp B$?

$$A^T B = A^T b - A^T b \frac{A^T A}{A^T A} = 0$$



vectors a, b, c .

$$\boxed{C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B.}$$

$$q_3 = \frac{C}{\|C\|}$$

② $A = QR$

$$\boxed{[a, b] = [q_1, q_2] \begin{bmatrix} a^T q_1 & * \\ a^T q_2 & * \end{bmatrix}}$$

upper triangular

$$a \perp q_2 \quad a^T q_2 = 0.$$

③

Determinants. $\det A = |A|$.

to check the invertability

① the properties of det.

1). $\det I = 1$. ($| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} | = 1$)

2) exchange rows: reverse sign of det (the $\det P = \begin{cases} -1 & \text{if } \text{swap} \\ 1 & \text{if } \text{not swap} \end{cases}$).

3). ($| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} | = 1$) . [for every permutation: either odd or even even]

$$\left| \begin{matrix} a+tb & b \\ c & d \end{matrix} \right| = ad - tb \cdot c \left| \begin{matrix} a & b \\ c & d \end{matrix} \right|$$

$$\left| \begin{matrix} a & a' \\ c & d \end{matrix} \right| = \left| \begin{matrix} a & b \\ c & d \end{matrix} \right| + \left| \begin{matrix} a' & b' \\ c & d \end{matrix} \right|.$$

3) linear each row.

4). 2 equal rows $\rightarrow \det = 0$.

(when exchange those rows we get same matrix).

5). Subtract $l \times \text{row } i$ from row k .

the det. doesn't change. ($\left| \begin{matrix} a & b \\ c-la & b-lb \end{matrix} \right| = \left| \begin{matrix} a & b \\ c & d \end{matrix} \right| + \left| \begin{matrix} a & b \\ -la & -lb \end{matrix} \right|$).

6). row of zeros $\rightarrow \det A = 0$. ($t=0$)

7)

$$U = \left[\begin{matrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \ddots & d_n \end{matrix} \right]$$

$\det U = d_1 \times d_2 \times \dots \times d_n$.
(product of pivots)

$$\left[\begin{matrix} d_1 & * & * \\ d_2 & \ddots & \\ \vdots & \ddots & d_n \end{matrix} \right] \xrightarrow{\text{elim}} \left[\begin{matrix} d_1 & d_2 & \dots & d_n \end{matrix} \right]$$

$$\rightarrow d_1 \left[\begin{matrix} 1 & 0 & 0 & 0 \\ d_2 & \ddots & & d_n \end{matrix} \right] \dots \rightarrow d_1 d_2 \dots \left[\begin{matrix} 1 & \dots & 1 \end{matrix} \right]$$

8) $\det A = 0$

when A is singular.

$\det A \neq 0$

when A is invertible.

$$9) \det AB = (\det A)(\det B).$$

$$\det A^{-1} = \frac{1}{\det A}. \quad \det 2A = 2^n \det A.$$

(inverable)

$$10) \cdot \det A^T = \det A. \rightarrow \text{column exchange will}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \quad \text{change the single sign.}$$

$$\text{proof: } |A^T| = |A|$$

$$|U^T||L^T| = |L||U|. \checkmark$$

$$\text{for any } \begin{vmatrix} d_1 & d_2 & \dots & d_n \\ 0 & 0 & \dots & 0 \end{vmatrix} \text{ or } \begin{vmatrix} d_1 & d_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{vmatrix}$$

↓ elim

$$\begin{vmatrix} d_1 & d_2 & 0 \\ 0 & 0 & \dots & 0 \end{vmatrix}.$$

① Formulas for A.

example: 2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= ad - bc .$$

3x3;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \dots + \dots$$

each column each row

will have a number.

$$= \alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} - \dots$$

Big formula

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} \dots a_{nn} .$$

exmp:
$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$(4, 3, 2, 1) \rightarrow +1$$

② Cofactors 3×3 . in paren

$$\det A = a_{11} (\underbrace{a_{22}a_{33} - a_{23}a_{32}}_{\text{Cofactors of } a_{11}}) + a_{12} (\dots) + a_{13} (\dots)$$

$$\begin{array}{|ccc|} \hline & a_{11} & \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \hline \end{array} + \begin{array}{|cc|} \hline & a_{12} \\ \hline a_{21} & a_{23} \\ a_{31} & a_{33} \\ \hline \end{array}$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \dots$$

Cofactor of $a_{ij} = \pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{col } j \text{ erased} \end{pmatrix} = C_{ij}$.

$i+j$ is even : +

$i+j$ is odd : -

Cofactor formulae.

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}.$$

① Formular for A^{-1}

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \underbrace{\frac{1}{\det A} C^T}_{\substack{\text{products of } n-1 \text{ entries} \\ \text{products of } n \\ \text{entries}}}$$

② we can solve the question what will happen to A^{-1} if 2 change a_{11} or (whatever).

$\left\{ \begin{array}{l} \det A \text{ will change} \\ a_{11} \cdot c_{11} \text{ will change} \end{array} \right.$

② Cramer's rule . (useless rule)

how to check: (proof $A \cdot C^T = \det A \cdot I$).

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & & \\ 0 & \ddots & & \\ 0 & & \ddots & \\ 0 & & & \det A \end{bmatrix}$$

↙ why zero?

components of x .

$$x_1 = \frac{\det B_1}{\det A} \quad x_n = \frac{\det B_n}{\det A}.$$

$$B_1 = \begin{bmatrix} b, a_1, a_2, \dots, a_n \end{bmatrix}$$

= A with column placed by b

③ examp:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

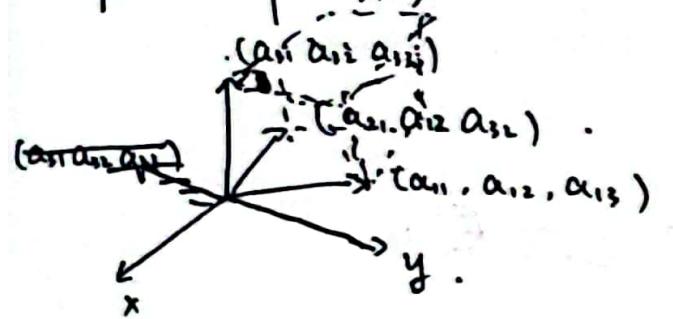
$$A_S = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix} \quad \det A_S = 0$$

$B_j = A$ with columns j replaced
by b .

③ volume

$|\det A|$ = volume of box

for examp: 3×3



check it?

① I ✓ ② ± ✓

③ 3-a.

3-b (in textbooks)

$$\text{for } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \checkmark$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix}$$

16

① Eigenvectors .

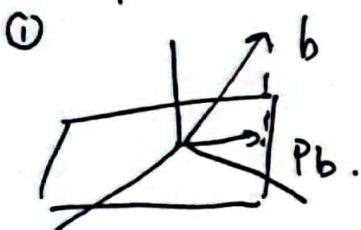
Ax - parallel to x .

$$\boxed{Ax = \lambda x}$$

eigenvector eigenvalue .

if A is singular , then $\lambda=0$ is erdue .

examp:



What x 's and λ 's for projection matrix ?

Any x the plane : $Px=x$, $\lambda=1$

Any $x \perp$ plane : $Px=0$, $\lambda=0$

② permutation .

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1 \quad Ax = x \\ x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1 \quad Ax = -x \end{array} \right.$$

Fact:

sum of λ 's = $a_{11} + a_{22} + \dots + a_{nn}$.

how to solve $Ax=\lambda x$.

Rewrite : $(A - \lambda I)x = 0$:

singular .

$\det(A - \lambda I) = 0$. Find lambda First .

examp:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = 4 \quad = \lambda^2 - 6\lambda + 8$$

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{thinking: } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{+3Z} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array} \right. \rightarrow \left\{ \begin{array}{l} \lambda_1 = 4 \\ \lambda_2 = 2 \end{array} \right.$$

if $Ax = \lambda x$ then $(A+3I)x = (\lambda+3)x$.

Not so great.

if $Ax = \lambda x$, B has eigenvalues α

$$Ax = \lambda x, Bx = \alpha x$$

$$(A+B)x = (\lambda + \alpha)x \quad \times$$

② $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \det(A - \lambda I) = (3-\lambda)(3-\lambda)$

$$\lambda_1 = 3 \quad \lambda_2 = 3$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}x = 0 \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} \text{No second} \\ \text{independent,} \end{bmatrix}$$

examp: ~~Q~~.

① 90° rotation $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\text{trace} = 0+0 = \lambda_1 + \lambda_2$$

$$\det = 1 = \lambda_1 \lambda_2$$

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\underline{\lambda_1 = i} \quad \underline{\lambda_2 = -i}$$

① Diagonalize the matrix

Suppose n independent eigenvectors of A .

Put them into columns of S . diagonal eigenvalue
matrix Λ

$$AS = A \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda$$

$$AS = S\Lambda$$

$$S^{-1}AS = \Lambda \quad A = S\Lambda S^{-1}$$

if $Ax = \lambda x$

$$A^2x = \lambda Ax = \lambda^2 x. \quad A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}.$$

A is sure to have n independent eigenvectors (and be diagonalizable)

if all λ 's are different
(no repeated λ 's)

Theorem

$$A^K \rightarrow 0 \text{ as } K \rightarrow \infty$$

if all eigenvalues $|\lambda| < 1$

~~repeated~~

repeated eigenvalues // may or may not have n independent

examp:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$\lambda = 2, 2 \quad A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cannot be diagonalized.

④ Equation . $u_{k+1} = Au_k$.

Start with given vector u_0

$$u_1 = Au_0, \quad u_2 = A^2u_0$$

$$\boxed{u_k = A^k u_0}$$

To really solve:

$$u_0 = c_1 x_1 + \dots + c_n x_n = S_C$$

$$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n$$

$$= \Lambda^{100} S_C = u_{100}.$$

$$\text{Fibonacci: } 0, 1, 1, 2, 3, 5, \dots, \boxed{F_{100} = ?}$$

$$F_{k+2} = F_{k+1} + F_k.$$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \quad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix}$$

$$\begin{cases} \lambda_1 = \frac{1}{2}(1+\sqrt{5}) & x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \\ \lambda_2 = \frac{1}{2}(1-\sqrt{5}) & x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \end{cases}$$

$$u_0 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_{100} = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2,$$

⑨ Differential Equations $\frac{du}{dt} = Au$.

Steady state
 $u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① example: $\frac{du_1}{dt} = -u_1 + 2u_2$ and $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{du_2}{dt} = u_1 - 2u_2$

$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\lambda = 0, -3$. $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: $u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$

check: $\frac{du}{dt} = Au$. Plug in $e^{\lambda_1 t} x_1 = \lambda_1 e^{\lambda_1 t} x_1$
 $= A e^{\lambda_1 t} x_1$

At $t=0$.

$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ⑤

Solution $C_1 = C_2 = \frac{1}{3}$

① Stability $u(t) \rightarrow 0$
need $e^{\lambda t} \rightarrow 0$ = Real part $\lambda < 0$

② Steady state.
 $\lambda_1 = 0$ and other Real part $\lambda < 0$

③ Blow up if any Real part $\lambda > 0$.

In 2×2 matrix stability: $\lambda_1 < 0$ and $\lambda_2 < 0$
 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; trace: $a+d = \lambda_1 + \lambda_2 < 0$
 $\det = \lambda_1 \lambda_2 > 0$.

② for $Au = \frac{du}{dt}$
Set $u = Sv$.

$S \frac{dv}{dt} = \frac{d}{dt} ASv$

$$\frac{dv}{dt} = S^{-1} A S v = \Lambda v$$

↓

$$\frac{dv}{dt} = \Lambda \quad v(t) = e^{\Lambda t} v(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{A t} u(0)$$

③ Matrix exponential e^{At}

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \dots + \frac{1}{n!}(At)^n + \dots$$

$$= \sum_0^{\infty} \frac{x^n}{n!}$$

how to check $S e^{\Lambda t} S^{-1} = e^{At}$

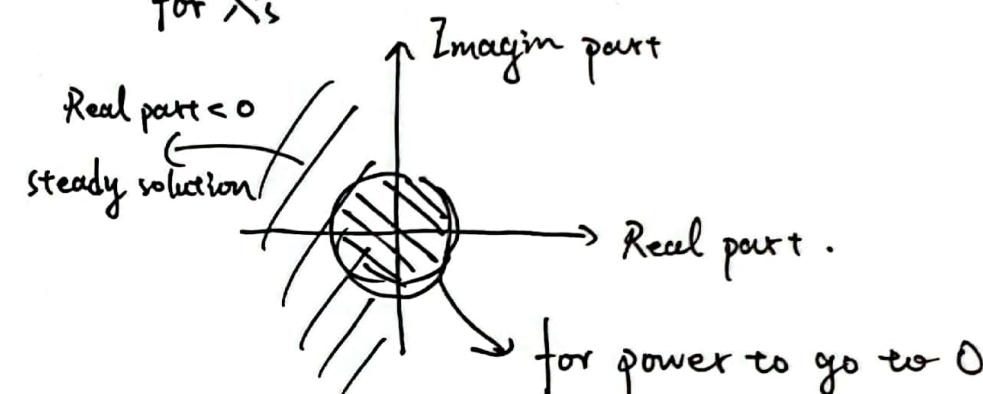
$$e^{At} = \underbrace{(I)}_{\rightarrow SIS^{-1}} + S \Lambda S^{-1} t + \frac{S \Lambda^2 S^{-1} t^2}{2!} + \dots$$

$$= S e^{\Lambda t} S \quad (\text{if } A \text{ can be diagonal})$$

what is $e^{\Lambda t}$?

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & 0 \\ 0 & \cdots & \lambda_n \end{bmatrix} \rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & 0 \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

for λ 's



$$\text{examp: } y'' + by' + ky = 0$$

$$y' = y'$$

1 2nd order equat
→ 2x1 order system

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

A u

⑪

Markov matrix

①

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

two properties

① All entries ≥ 0 .

② All columns add to 1.

the key points about Markov

1. $\lambda=1$ is an eigenvalue

2. All other $|\lambda| < 1$.

how to proof " $\lambda=1$ is an eigenvalue"?

$$A - I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

$\underline{r_1+r_2+r_3}$ to get $[0 \ 0 \ 0]$ row

\therefore the rows are dependent, this is a singular matrix.

$\lambda=1$ is one of the eigenvalue

*: eigenvalue of A and A^T

are the same.

$$\det(A - \lambda I) = 0 \rightarrow \det(A^T - \lambda I) = 0$$

$$u_k = A^k u_0 = \underbrace{c_1 \lambda_1^k}_{=1} x_1 + \underbrace{c_2 \lambda_2^k}_{<0} x_2, \dots \rightarrow \begin{array}{l} \text{steady state} \\ \geq 0 \end{array}$$

② $u_{k+1} = A u_k$ A is Markov.

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the flow of people $A \Leftrightarrow B$.

$$\begin{bmatrix} C_A \\ C_B \end{bmatrix}_{(t=k+1)} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \end{bmatrix}_{(t=k)}$$

↑ stay in A ↓ A \rightarrow B ↑ B \rightarrow A ↓ stay in B

$$\begin{bmatrix} C_A \\ C_B \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

eigenvalue: $\lambda_1 = 1$ $\lambda_2 = 0.7$

eigenvector: $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$c_1 = 1000/3 \quad c_2 = 2000/3$$

③ projections with orthonormal basis.

$$q_1, q_2, \dots, q_n$$

$$\text{Any } v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$$

$$\boxed{q_i^T v = x_1 q_i^T q_1 + 0 \dots + 0} = 1$$

Fourier series.

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

for vectors

how to get a_i .

$$V^T W = v_1 w_1 + \dots + v_n w_n$$

$$\int f(x) \cos x dx = a_1 \int \cos^2 x dx$$

Functions $f^T g = \int_0^{2\pi} f(x) g(x) dx$.

② Symmetric matrix.

①

$$A = A^T$$

① the eigenvalues are real.

② the eigenvectors are perpendicular

for matrix A

Usually case:

$$A = S \Lambda S^{-1}$$

Symmetric case:

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T$$

I:

Why real eigenvalues?

$$Ax = \lambda x \xrightarrow{\text{always}} A\bar{x} = \bar{\lambda}\bar{x} \xrightarrow{\text{trans}} \bar{x}^T A = \bar{x}^T \bar{\lambda} \xrightarrow{\text{number}}$$

$$\bar{x}^T A x = \lambda \bar{x}^T x$$

$$\therefore \cancel{\lambda} \bar{x}^T x = \bar{\lambda} \bar{x}^T x$$
$$\therefore \boxed{\lambda = \bar{\lambda}}$$

if \bar{x} is complex $\bar{x}^T \bar{x} = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(length)² $\therefore \sum_{i=1}^n \bar{x}_i x_i = \sum_{i=1}^n (\bar{a}+bi)(a-bi) = a^2+b^2$

Cool matrix { Real λ 's
Perpendicular x 's
 $A = \bar{A}^T$ (for complex)

② For $A = A^T$

$$A = Q \Lambda Q^T$$

$$= \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$\bar{x}^T A \cdot x = \bar{x}^T \bar{\lambda} x$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots$$

Every symmetric matrix is
a combination of perpendicular
projection matrices

all λ determines positive
sub- = $\begin{bmatrix} q^+ \\ - \\ \vdash \end{bmatrix}$

③ for symmetric matrix

Sign of pivot same as sign of λ 's

positive pivots = # positive λ 's

④ Positive definite symmetric matrix .

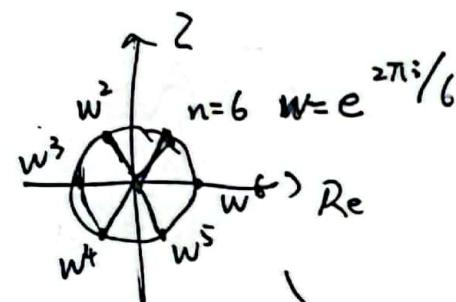
all the eigenvalues are positive

all pivots are positive

④ Fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} \quad (F_n)_{ij} = w^{ij}.$$

$$w^n = 1 \quad w = e^{i \frac{2\pi}{n}}$$



$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

cols orthogonal
otherwise orthonormal.

$$F_4^H F_4 = I$$

$$F_4^{-1} = F_4^H$$

⑤ $\begin{bmatrix} F_{64} \end{bmatrix} = \underbrace{\begin{bmatrix} I & D \\ I & -D \end{bmatrix}}_{\text{calculate}} \underbrace{\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}}_{\text{calculate for } 2(32)^2 + 32 \text{ times}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}^P}_{\text{for } (64)^2 \text{ times}}$

$$F_{32} \leftarrow \begin{bmatrix} \dots \\ \dots \\ F_{16} \\ \dots \\ \dots \end{bmatrix} \quad D = \begin{bmatrix} 1 & \dots & \dots & w^{31} \end{bmatrix}$$

$\downarrow \quad 6 \times 32$

$$= \frac{64}{2} \log_2 64.$$

For n square
calculate for $\boxed{\frac{1}{2} n \log_2 n}$ times

Complex Vectors .

for $z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ in C^n

① Length of z

$z^T z$ not good

if ($z_1=1$ and $z_2=i \rightarrow z^T z=0$)

$\bar{z}^T z$ is good

(we need $|z|^2$ not z^2)

$$\bar{z}_i z = |z_i|^2$$

we call \bar{z}^T to z^H (Hermitian)

inner product turns to

$$\bar{y}^T x = y^H x$$

for symmetric matrix

$A^T = A$ not good if A is complex .

$$\bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} \quad \textcircled{*} = A^H \quad (\text{Hermitian matrix})$$

$$A^H = A$$

Perpendicular .

$$q_1, q_2, \dots, q_n$$

$$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q = [q_1, \dots, q_n]$$

$$\underbrace{Q^H Q}_{\sim} = I$$

unitary matrix .

22

Positive Definite Matrix (symmetric)

for $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{x^T A x} 2x_1^2 + 12x_1x_2 + 20x_2^2$

how to test? for $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$\textcircled{1} \quad \lambda_1 > 0 \quad \lambda_2 > 0$$

$$\textcircled{2} \quad a > 0 \quad ac - b^2 > 0$$

$$\textcircled{3} \quad \text{pivots } a > 0 \quad \frac{ac - b^2}{a} > 0$$

$$\textcircled{4} \quad \cancel{\textcircled{*}} : \underline{x^T A x > 0 \text{ except } x=0}$$

example:

$$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \xrightarrow{x^T A x} [x_1 \ x_2] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

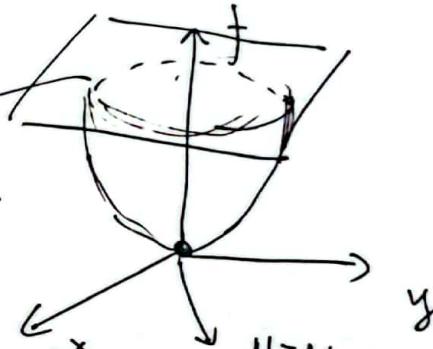
semi-definite

$$\left\{ \begin{array}{l} \lambda_1 = 0 \quad \text{pivots: 2.} \\ \lambda_2 = 20 \end{array} \right. = 2x_1^2 + 12x_1x_2 + 18x_2^2 \quad (\text{quadratic form})$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$ax^2 + 2bxy + cy^2 > 0 ? \quad \text{No}$$

极值图.



pivots 1 times pivots 2

$$\begin{bmatrix} A \\ \vdots \\ 2 & 6 \\ \vdots \\ 6 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} U \\ \vdots \\ 2 & 6 \\ \vdots \\ 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \text{1st derivative} = 0 \\ \text{2nd derivative} > 0. \end{array} \right.$$

For 2nd derivatives ~~in~~ in Matrix .

or bring it to $x^T A x$ to see $C > 0$?

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \xrightarrow[\text{definite}]{\substack{\text{must positive} \\ (\text{by bringing it to } x^T A x)}} \text{has MIN (in } f(x,y)) \rightarrow \text{for } \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

for 3×3 example .

$$a^2x^2 + 2bxy + cy^2 \text{ can become } \cancel{(x+By)^2} + Cy^2$$

$$\begin{cases} f_{xx} = 2a & f_{yy} = 2c \\ f_{xy} = 2b & f_{yx} = 2b \end{cases} \quad A(x+By)^2 + Cy^2 \rightarrow C > 0 ?$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{dets } 2, 3, 4$$

$$\text{pivots } 2, \frac{3}{2}, \frac{4}{3}$$

$$\text{pivots} \times \text{pivots} = \text{dets}$$

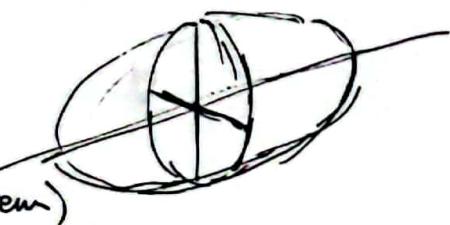
$$\text{eigenvalues: } 2-\sqrt{2}, 2+\sqrt{2}, 2$$

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0.$$

if $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 = 1$ what will happen about his shape ?

$$A = Q \Lambda Q^T$$

(axis theorem)



eigenvector: the directions of principal axes
eigenvalue: the lengths of axes

29

positive definite means .

$$x^T A x > 0$$

- If A and B is pos def.

$(A+B)$ is pos def ($x^T(A+B)x > 0$)

- Now A $n \times n$.

$\underbrace{A^T A}$ is pos def ? YES
square . symmetrie

$$x^T A^T A x = (Ax)^T (Ax) = |Ax|^2 \geq 0 \rightarrow \underbrace{\geq 0}_{\text{when rank is } n} .$$

Similar Matrix .

$A_{(n \times n)}$ and $B_{(n \times n)}$ is similar

means: for some invertible M .

$$B = M^{-1} A M$$

examp: A is similar to Λ

$$S^{-1} A S = \Lambda$$

$$\text{if } M \text{ is } \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \text{ } A \text{ is } \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M^{-1} A M = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix} = B$$

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

what is the same between A , B and Λ ?

THEY HAVE the SAME EIGENVALUE

how to check ?

$$Ax = \lambda x . \quad (B = M^{-1} A M)$$

↓

$$\frac{M^{-1} A M \cdot M^{-1} x}{B} = \lambda M^{-1} x \rightarrow \underbrace{BM^{-1} x}_{\text{eigenvectors of } B} = \underbrace{\lambda M^{-1} x}_{\text{eigenvectors of } A} .$$

eigenvectors of B is

M^{-1} (eigenvectors of A).

For Bad Case. $\lambda_1 = \lambda_2$.

suppose $\lambda_1 = \lambda_2 = 4$.

- one family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ because for small

any $M \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, the family only itself.

- big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, it cannot diagonalized.
↑
nearly "diagonalize" -
Jordan form.

more member of family:

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \quad \text{trace} = 8$$

$$\det = 16.$$

for $\begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} A$ and $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} B$

$$\lambda = 0, 0, 0, 0 \quad NCA = 2 \rightarrow \text{missing } 2$$

but they are not similar (the block size is diff).
Jordan block.

$$J_i = \begin{bmatrix} \lambda_i & & & 0 \\ & \lambda_i & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}$$

↓ the same num.

Every square A is similar to a Jordan matrix J .

$$J = \begin{bmatrix} [J_1] & & & \\ & [J_2] & & \\ & & \ddots & \\ & & & [J_n] \end{bmatrix} \quad \# \text{ blocks} = \# \text{ eigenvectors.}$$

① Good case: $J = \lambda$.