Assignment 03

KAWIN M

19723

Qs (1) Given:

$$T^{\mu}V(s) = \mathbb{E}[r(s_0, s_1) + \gamma V(s_1) \mid s_0 = s] \quad \forall s \in S$$

To Show: T^{μ} is a γ-contraction

i.e

$$\| T^{\mu} V - T^{\mu} V' \| \leq \gamma \| V - V' \|$$

for any $V, V' \in \mathbb{R}^{|S|}$ and a scalar $\gamma \in [0,1)$

Proof:

$$\| T^{\mu} V - T^{\mu} V' \| = \| \mathbb{E}[r(s_0, s_1) + \gamma V(s_1) \mid s_0 = s] - \mathbb{E}[r(s_0, s_1) + \gamma V'(s_1) \mid s_0 = s] \|$$

$$= \| \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) [r(s_0, s_1) + \gamma V(s_1)] - \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) [r(s_0, s_1) + \gamma V'(s_1)] \|$$

$$= \| \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) [r(s_0, s_1) + \gamma V(s_1) - r(s_0, s_1) - \gamma V'(s_1)] \|$$

$$= \| \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) [\gamma (V(s_1) - V'(s_1))] \|$$

$$= \gamma \| \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) [V(s_1) - V'(s_1)] \|$$

$$\leq \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s_0 = s) \| V(s_1) - V'(s_1) \|$$

[$\because$ Using Triangle Inequality]

$$\leq \gamma \| V - V' \| \qquad \left[\because \sum_{s' \in S} \mathbb{P}(s' | s_0 = s) = 1\right]$$

$\therefore$ Hence shown that

$$\| T^{\mu} V - T^{\mu} V' \| \leq \gamma \| V - V' \|$$

$\Rightarrow T^{\mu}$ is a γ - contraction

To show: V^{μ} is a unique fixed point of T^{μ}

We know that, $\displaystyle V^{\mu}(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^{t} r(s_t, s_{t+1}) \,|\, s_0 = s\right]$

To show: $T^{\mu} V^{\mu} = V^{\mu}$

Proof:

$$T^{\mu} V^{\mu}(s) = \mathbb{E}[r(s_0, s_1) + \gamma V^{\mu}(s_1) \,|\, s_0 = s]$$

$$= \mathbb{E}\left[r(s_0, s_1) + \gamma \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, s_{t+1}) \,|\, s_1 = s'\right] \,|\, s_0 = s\right]$$

$$= \mathbb{E}\left[r(s_0, s_1) + \gamma \sum_{s'' \in S} \mathbb{P}(s'' | s_1 = s') \sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, s_{t+1}) \,|\, s_0 = s\right]$$

$$= \mathbb{E}\left[r(s_0, s_1) + \sum_{s'' \in S} \mathbb{P}(s'' | s_1 = s') \sum_{t=1}^{\infty} \gamma^{t} r(s_t, s_{t+1}) \,|\, s_0 = s\right]$$

$$= \mathbb{E}\left[\sum_{s'' \in S} \mathbb{P}(s'' | s_1 = s') \sum_{t=0}^{\infty} \gamma^{t} r(s_t, s_{t+1}) \,|\, s_0 = s\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^{t} r(s_t, s_{t+1}) \,|\, s_0 = s, s_1 = s'\right]\right]$$

Using Iterative conditioning,

$$= \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^{t} r(s_t, s_{t+1}) \,|\, s_0 = s\right]$$

$= V^{M}(s)$

Hence shown that

$$T^{M} V^{M} = V^{M}$$

i.e V^{M} is a fixed point of T^{M}

uniqueness:

Using Banach's fixed point theorem

Here, $\mathbb{R}^{d|S|}$ is a complete space with metric $\|\cdot\|$ and

T^{M} a γ-contraction with a fixed point V^{M}

From, Banach's fixed point theorem,

V^{M} is a ~~fixed~~ "unique" fixed point of T^{M}.

Hence Proved

Qs (2) (a) TD(0) recursion with linear function approximation.

update rule:

$$w_{t+1} = w_t + \alpha_t[r(s_t, s_{t+1}) + \gamma \phi^T(s_{t+1}) w_t - \phi^T(s_t) w_t] \phi(s_t)$$

If $s_t = 0$ && $s_{t+1} = 0$

$$w_{t+1} = w_t + \alpha_t[r(0,0) + \gamma \phi^T(0) w_t - \phi^T(0) w_t] \phi(0)$$

$$= w_t + \alpha_t[0 + 0.8 * [1,0] w_t - [1,0] w_t] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$w_{t+1} = w_t - \alpha_t [0.2 [1,0] w_t] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Else if $s_t = 0$ && $s_{t+1} = 1$

$$w_{t+1} = w_t + \alpha_t[r(0,1) + \gamma \phi^T(1) w_t - \phi^T(0) w_t] \phi(0)$$

$$= w_t + \alpha_t[5 + 0.8 [0,1] w_t - [1,0] w_t] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$w_{t+1} = w_t + \alpha_t [5 + [-1, 0.8] w_t] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Else if $s_t = 1$ && $s_{t+1} = 0$

$$w_{t+1} = w_t + \alpha_t[r(1,0) + \gamma \phi^T(0) w_t - \phi^T(1) w_t] \phi(1)$$

$$= w_t + \alpha_t[1 + 0.8 * [1,0] w_t - [0,1] w_t] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= w_t + \alpha_t[1 + [0.8, -1] w_t] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Else [$s_t = 1$ && $s_{t+1} = 1$]

$$w_{t+1} = w_t + \alpha_t[r(1,1) + \gamma \phi^T(1) w_t - \phi^T(1) w_t] \phi(1)$$

$$= w_t + \alpha_t[3 + 0.8 * [0,1] w_t - [0,1] w_t] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w_{t+1} = w_t + \alpha_t [3 - 0.2 [0, 1] w_t] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qs (2) (b)

Limiting ODE is

$$\mathring{w}(t) = b - A w(t)$$

where,

$$b = \phi^T D r \quad \&$$

$$A = \phi^T D \phi - \gamma \phi^T D \mathbb{P} \phi$$

Here, $\phi^T = \phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbb{P} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$

$$r = \begin{bmatrix} 0 + 1 * 5 \\ \frac{1}{2} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$D = \text{diag}(d_\mu(s))$$

Here, $d_\mu^T \mathbb{P} = d_\mu^T$

$$\Rightarrow [d_\mu(0) \;\; d_\mu(1)] \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} = [d_\mu(0) \;\; d_\mu(1)]$$

$$\Rightarrow [0.5 d_\mu(1) \;\; d_\mu(0) + 0.5 d_\mu(1)] = [d_\mu(0) \;\; d_\mu(1)]$$

$\Rightarrow 0.5 d_\mu(1) = d_\mu(0)$ and

$d_\mu(0) + 0.5 d_\mu(1) = d_\mu(1)$

$$\Rightarrow d_\mu(0) = 0.5 d_\mu(1) \longrightarrow (1)$$

Also, w.k.t $d_\mu(0) + d_\mu(1) = 1 \longrightarrow (2)$

Subs ① in ②

$$0.5\, d_\mu(1) + d_\mu(1) = 1$$

$$\Rightarrow 1.5\, d_\mu(1) = 1$$

$$\Rightarrow d_\mu(1) = \frac{2}{3}$$

$$\Rightarrow d_\mu(0) = 0.5\, d_\mu(1) = \frac{1}{2} * \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow d_\mu = \begin{pmatrix} \frac{1}{3} \\ 2/3 \end{pmatrix}$$

and $D = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2/3 \end{pmatrix}$

$$b = \phi^T D \gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$A = \phi^T D \phi - \gamma \phi^T D \mathbb{P} \phi$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0.8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} - 0.8 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} - 0.8 \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} 0 & \frac{4}{15} \\ \frac{4}{15} & \frac{4}{15} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{3} & \frac{-4}{15} \\ \frac{-4}{15} & \frac{6}{15} \end{bmatrix}$$

Limiting ODE

$$\dot{\omega}(t) = b - A\omega(t)$$

$$= \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & -\frac{4}{15} \\ \frac{-4}{15} & \frac{2}{15} \end{bmatrix} \omega(t)$$

To verify if it has a unique globally asymptotically stable equilibrium point

To show: A^{-1} exist

Proof: $|A| = \frac{1}{3} * \frac{2}{5} - \frac{4}{15} * \frac{4}{15}$

$$= \frac{2}{15} - \frac{16}{15^2} = \frac{30-16}{15^2}$$

$$= \frac{14}{15^2} \neq 0$$

$$\therefore |A| \neq 0$$

$$\Rightarrow A^{-1} \text{ exists}$$

Hence verified

The unique equilibrium point

$$w_* = A^{-1} b$$

Here, $A^{-1} = \begin{bmatrix} \frac{45}{7} & \frac{30}{7} \\ \frac{30}{7} & \frac{75}{14} \end{bmatrix}$

$$w_* = \begin{bmatrix} \frac{45}{7} & \frac{30}{7} \\ \frac{30}{7} & \frac{75}{14} \end{bmatrix} \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$w_* = \begin{bmatrix} \frac{115}{7} \\ \frac{100}{7} \end{bmatrix}$$

Qs 3 a

Convex Optimization

Differentiating w.r.t w and setting to 0

$$\Rightarrow 2\delta \|w_t - w\| \cdot \frac{(w - w_t)}{\|w_t - w\|} + \sum_{m=0}^{t} 2\Big[\phi^T(s_m)w - \phi^T(s_m)w_t$$

$$- \sum_{k=m}^{t} (\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\Big] \phi^T(s_m) = 0$$

$$\Rightarrow 2\delta w + 2\sum_{m=0}^{t} \phi^T(s_m) \underset{1\times d}{w} \underset{d\times 1}{\phi(s_m)} = 2\delta w_t + 2\sum_{m=0}^{t}\Big[\phi^T(s_m) w_t$$

$$+ \sum_{k=m}^{t} (\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\Big]\phi(s_m)$$

cancelling out 2 on all terms and rearranging $\phi^T(s_m) w \phi(s_m)$ to $\phi(s_m)\phi^T(s_m) w$ [$\because \phi^T(s_m) w$ is a scalar]

$$\Rightarrow \delta w + \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m) w = \delta w_t + \sum_{m=0}^{t}\Big[\phi^T(s_m) w_t$$

$$+ \sum_{k=m}^{t} (\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\Big]\phi(s_m)$$

LHS:

$$\Big[\delta \cdot I + \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m)\Big] w$$

$$\Rightarrow w = \frac{\delta w_t}{[\delta \cdot I + \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m)]} + \frac{\sum_{m=0}^{t} \phi^T(s_m) w_t \phi(s_m)}{[\delta \cdot I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)]}$$

$$+\frac{\sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\,\phi(s_m)}{\left[\delta . I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)\right]}$$

$$\Rightarrow w = \frac{\left[\delta I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)\right] w_t}{\left[\delta I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)\right]} + \frac{1}{\left[\delta I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)\right]}\sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\,\phi(s_m)$$

$$\therefore \boxed{B = \delta I + \sum_{m=0}^{t}\phi(s_m)\phi^T(s_m)}$$

$$\Rightarrow w = w_t + B^{-1}\sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m} d_t(s_k, s_{k+1})\,\phi(s_m)$$

$$= w_t + B^{-1}\sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m}\left[r(s_k, s_{k+1}) + (\gamma\phi(s_{k+1}) - \phi(s_k))^T w_t\right]\phi(s_m)$$

[Expanding $d_t(s_k, s_{k+1})$]

Separating out terms with w_t

$$= w_t + B^{-1}\left[\sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m}\left[(\gamma\phi(s_{k+1}) - \phi(s_k))^T w_t\right]\phi(s_m) + \sum_{m=0}^{t}\sum_{k=m}^{t}(\gamma\lambda)^{k-m}\, r(s_k, s_{k+1})\,\phi(s_m)\right]$$

$$\therefore \boxed{A = \sum_{m=0}^{t}\phi(s_m)\sum_{k=m}^{t}(\gamma\lambda)^{k-m}(\gamma\phi(s_{k+1}) - \phi(s_k))^T}$$

$$b = \sum_{m=0}^{t} \sum_{k=m}^{t} (Y\lambda)^{k-m} \gamma(s_k, s_{k+1}) \phi(s_m)$$

$$= \sum_{m=0}^{t} \phi(s_m) \sum_{k=m}^{t} (Y\lambda)^{k-m} \gamma(s_k, s_{k+1}) \cancel{\phi}$$

Changing the order of summation to calculate A_t and b_t iteratively

$$A_t = \sum_{k=0}^{t} \sum_{m=0}^{k} (Y\lambda)^{k-m} \phi(s_m) \left(Y\phi(s_{k+1}) - \phi(s_k)\right)^T.$$

$$b_t = \sum_{k=0}^{t} \sum_{m=0}^{k} (Y\lambda)^{k-m} \phi(s_m) \gamma(s_k, s_{k+1})$$

Iterative scheme for computing

① $$B_{t+1} = \delta I + \sum_{m=0}^{t+1} \phi(s_m) \phi^T(s_m)$$

$$= \delta I + \sum_{m=0}^{t} \phi(s_m) \phi^T(s_m) + \phi(s_{t+1}) \phi^T(s_{t+1})$$

$$B_{t+1} = B_t + \phi(s_{t+1}) \phi^T(s_{t+1})$$

② $$A_{t+1} = \sum_{k=0}^{t+1} \sum_{m=0}^{k} (Y\lambda)^{k-m} \phi(s_m) \left(Y\phi(s_{k+1}) - \phi(s_k)\right)^T$$

$$= \sum_{k=0}^{t} \sum_{m=0}^{k} (Y\lambda)^{k-m} \phi(s_m) \left(Y\phi(s_{k+1}) - \phi(s_k)\right)^T$$

$$+ \sum_{m=0}^{t+1} (Y\lambda)^{t+1-m} \phi(s_{t+1}) \left(Y\phi(s_{t+2}) - \phi(s_{t+1})\right)^T$$

$$A_{t+1} = A_t + \sum_{m=0}^{t+1} (\gamma\lambda)^{t+1-m} \phi(s_m)\left(\gamma\phi(s_{t+2}) - \phi(s_{t+1})\right)^T$$

③ $$b_{t+1} = \sum_{k=0}^{t+1} \sum_{m=0}^{k} (\gamma\lambda)^{k-m} \phi(s_m)\, r(s_k, s_{k+1})$$

$$= \sum_{k=0}^{t} \sum_{m=0}^{k} (\gamma\lambda)^{k-m} \phi(s_m)\, r(s_k, s_{k+1}) + \sum_{m=0}^{t+1} (\gamma\lambda)^{t+1-m} \phi(s_m)\, r(s_{t+1}, s_{t+2})$$

$$b_{t+1} = b_t + \sum_{m=0}^{t+1} (\gamma\lambda)^{t+1-m} \phi(s_m)\, r(s_{t+1}, s_{t+2})$$

Qs (3)(b)

For the inverse of B_t to exist:

$$|B_t| \neq 0$$

$$\Rightarrow \left| \delta I + \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m) \right| \neq 0$$

Rearranging

$$\Rightarrow \left| \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m) - (-\delta) I \right| \neq 0$$

Let, $\sum_{m=0}^{t} \phi(s_m)\phi^T(s_m)$ be a matrix C, then

$$\Rightarrow |C - (-\delta) I| \neq 0$$

$\Rightarrow (-\delta)$ is not a Eigen value of matrix C.

That is, for the inverse of B_t to exist:

$$\delta \neq \left(-\text{Eigen value of } \sum_{m=0}^{t} \phi(s_m)\phi^T(s_m)\right)$$

Qs (3)(c)

when $\lambda = 0$, the optimization problem is same as the TD(0) algorithm with Linear Function Approximation.

Qs (4) Given: $\mathbb{E}\|x_0\|^2 < \infty$

To show: $\mathbb{E}\|M_t\|^2 < \infty \quad \forall t \geq 1$

Claim: $\mathbb{E}\|x_t\|^2 < \infty$

Proof using Induction:

Base case: $t = 0$

$\mathbb{E}\|x_0\|^2 < \infty$ [Given]

Induction Hypothesis:

Let $\mathbb{E}\|x_k\|^2 < \infty$ be true for any $t = k \in \mathbb{N}$

To Prove:

$\mathbb{E}\|x_{k+1}\|^2 < \infty$

Proof:

$$\mathbb{E}\|x_{k+1}\|^2 = \mathbb{E}\|x_k + \alpha_t[r(s_t, s_{t+1}) + \gamma\phi^T(s_{t+1})\cancel{\omega_t}\,x_k - \phi^T(s_t)\underbrace{\cancel{\omega_t}}_{x_k}]\phi(s_t)\|^2$$

$$= \mathbb{E}\Big[\|x_k\|^2 + \|\alpha_t r(s_t, s_{t+1})\phi(s_t)\|^2 + \|\alpha_t(\gamma\phi^T(s_{t+1}) - \phi^T(s_t))x_k\phi(s_t)\|^2\Big]$$

$$= \mathbb{E}\|x_k\|^2 + \mathbb{E}\|\alpha_t r(s_t, s_{t+1})\phi(s_t)\|^2 + \mathbb{E}\|\alpha_t(\gamma\phi^T(s_{t+1}) - \phi^T(s_t))x_k\phi(s_t)\|^2$$

$$= \mathbb{E}\|x_k\|^2 + \mathbb{E}\|(\alpha_t r(s_t, s_{t+1})\phi(s_t))\|^2 + \alpha_t^2\,\mathbb{E}\|(\gamma\phi^T(s_{t+1}) - \phi^T(s_t))x_k\phi(s_t)\|^2$$

Using cauchy-schwarz Inequality

$$\leq \mathbb{E}\|x_k\|^2 + (\alpha_t \gamma(s_t, s_{t+1}))^2 \mathbb{E}\|\phi(s_t)\|^2 + \alpha_t^2 \mathbb{E}\|\gamma\phi^T(s_{t+1}) - \phi^T(s_t)\|^2 \mathbb{E}\|x_k\|^2 \mathbb{E}\|\phi(s_t)\|^2$$

considering each term,

$\mathbb{E}\|x_k\|^2 < \infty$ [From Induction Hypothesis]

$(\alpha_t \gamma(s_t, s_{t+1}))^2 \mathbb{E}\|\phi(s_t)\|^2 < \infty$ [constant]

$\alpha_t^2 \mathbb{E}\|\gamma\phi^T(s_{t+1}) - \phi^T(s_t)\|^2 \mathbb{E}\|\phi(s_t)\|^2 \mathbb{E}\|x_k\|^2 < \infty$ [constant and Induction Hypothesis]

$$\Rightarrow \mathbb{E}\|x_{k+1}\|^2 \leq \mathbb{E}\|x_k\|^2 + (\alpha_t \gamma(s_t, s_{t+1}))^2 \mathbb{E}\|\phi(s_t)\|^2 + \alpha_t^2 \mathbb{E}\|\gamma\phi^T(s_{t+1}) - \phi^T(s_t)\|^2 \mathbb{E}\|x_k\|^2 \mathbb{E}\|\phi(s_t)\|^2 < \infty$$

$$\Rightarrow \mathbb{E}\|x_{k+1}\|^2 < \infty$$

Hence Proved that $\mathbb{E}\|x_t\|^2 < \infty$

To Prove: $\mathbb{E}\|M_t\|^2 < \infty \quad \forall t$

we know that,

$$\mathbb{E}\|M_{t+1}\|^2 \leq K_M[1 + \mathbb{E}\|\cancel{w}_t \, x\|^2]$$

$$\Rightarrow \mathbb{E}\|M_t\|^2 \le K_M[1 + \mathbb{E}\|\cancel{\omega}x_{t-1}\|^2]$$

$$\le K_M + K_M \mathbb{E}\|\cancel{\omega}x_{t-1}\|^2$$

Here, $K_M < \infty$ and

$K_M \mathbb{E}\|x_{t-1}\|^2 < \infty$ [From claim]

$\therefore \mathbb{E}\|M_t\|^2 < \infty$

Hence Proved