Kaysey Nguyen
Normal, Gamma, and Beta Distributions

First is the normal distribution and it is the "most widely used continuous probability distribution" (Wackerly, et al. 178). It is a bell-shaped curve and is symmetric around the mean (Frost, "Normal Distribution"). The distribution has a density function shown below from page 178, 'Definition 4.8' (Wackerly, et al).:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}$$

It is for random variables if the standard deviation is greater than zero and if the mean is between negative and positive infinity (Wackerly, et al. 178). Seeing the density function, other than the random variable y, the other two parameters are the standard deviation and the mean. The mean "defines the location of the peak for the bell curve" while the standard deviation "defines the width" of the distribution (Frost, "Normal Distribution").

There is also the empirical rule for the distribution and it is the mean plus or minus the standard deviation and is the percentage of the data contained in that standard deviation. The rule is in one standard deviation, 68% is contained, two has 95%, and the third has 99.7% (Frost, "Normal Distribution"). Along with the empirical rule, there is also z-score which is "the distance from the mean of a normal distribution expressed in units of standard deviation" (Wackerly, et al. 180). In other words it lets you know how far a datapoint lies from the mean for a given population or observation, but z-scores for a normal distribution is given in a table so depending on the height and area, the z-score will mostly be the same. The formula for z-score is below (Wackerly, et al. 180):

$$z = \frac{y-\mu}{\sigma}$$

Next is the gamma distribution which is similar to the exponential distribution (Kim) as it is a "continuous probability distribution that models right-skewed data" (Frost, "Gamma Distribution"). Both are used to predict time of an event, however exponential can model "only the time until the next event" while gamma can model "the elapsed time between various numbers of events" (Frost, "Gamma Distribution"). To put it more simply, exponential "predicts the wait time until the *very first* event" while gamma is used "[t]o predict the wait time until future events" (Kim).

The gamma function, which is different from the gamma probability density function (pdf), is an improper integral and can be found on page 185 of the *Mathematical Statistics with Applications* by Wackerly and others under 'Definition 4.9'.

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

In this formula, alpha, α, has to be greater than 0. Alpha has to be greater than 0 because if it is, the gamma distribution would be a two-parameter one instead of a three-parameter one. The gamma pdf can be found on the same page and has alpha, beta, β, the threshold as the parameters.

$$f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

Not shown is 0 for elsewhere on the function, being 0 or a negative. Here both alpha and beta have to be greater than 0 and it is where y is greater than 0 and less than infinity. The threshold parameter can be any number as it "defines the smallest value in a gamma distribution.

Some analysts refer to this parameter as the location" (Frost, "Gamma Distribution") and the values or the upper bound has to be greater than the threshold. If the threshold is set to 0, the gamma distribution will be a two-parameter one. The alpha parameter is called the "shape parameter" while the beta parameter is called the "scale parameter" (Wackerly, et al. 185). The shape parameter "specifies the number of events you are modeling" while the scale parameter "represents the mean time between events" (Frost, "Gamma Distribution").

Frost includes another parameter called the 'Rate Parameter' represented by lambda, λ. It can be used in place of the scale parameter, β. It is also the "mean rate of occurrence during one unit of time in the Poisson distribution" (Frost, "Gamma Distribution"). Below is a formula with lambda and the gamma function in the denominator from Aerin Kim's article on gamma distribution. In the article, alpha was the letter k so I replaced k with α because it is the number of events. t is also time units and I replaced it with y to match the pdf equation and e is the euler number.

$$\frac{\lambda^{\alpha} y^{\alpha-1} e^{-\lambda y}}{\Gamma(\alpha)}$$

Compared to the pdf the β^{α} becomes λ because it is actually $\frac{1}{\beta}$ and is moved to the numerator. Along with the pdf, there is also the expected mean and expected variance which can be found on page 186 (Wackerly, et al.).

$$\mu = E(Y) = \alpha\beta \ and \ \sigma^2 = V(Y) = \alpha\beta^2$$

As stated before, the distribution can be used in time wait models, but there are more examples such as being used in "reliability (failure) modeling [and] service time modeling (Queuing theory)" (Kim). Some life events the distribution is used for are in "predict[ing] rainfall, the reliability of mechanical tools and machines, or any applications that only have positive results" (Metwalli).

Lastly, beta distribution is "a two-parameter density function defined over the closed interval $0 \leq y \leq 1$. It is often used as a model for proportions…" (Wackerly, et al. 194) The two parameters like in gamma distribution are alpha, α, and beta, β. The distribution is also used as a "prior distribution for binomial proportions in Bayesian analysis" (Weisstein). The function is shown below with 0 for elsewhere (Wackerly, et al. 194).

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)}$$

Compared to the gamma distribution where α and β are shape and scale parameters, beta distributions use α and β as both shape parameters and they both must be positive (Frost, "Beta Distribution"). In the denominator, results in the gamma function with more gamma stuff (Wackerly, et al. 194):

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Given the complicated functions with α and β, the expected mean and variance are quite simple as only α and β are needed and they are as followed:

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \ and \ \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

To end off, the applications of the beta distribution along with being used as a model for proportions and the Bayesian analysis, it can be used for "the *Rule of Succession* (a famous example being Pierre-Simon Laplace's treatment of the sunrise problem), and Task duration modeling… [and] project/planning control systems" (Glen).

Works Cited

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