Mathematics Derivation in detail

1. Rotation matrix – Convention y-x′

“P” and “B” denote the coordinate of the moving platform and the body frame respectively.

“I” – Intermittent coordinate

$$ {}^{B}R_{P} = {}^{B}R_{I}(\theta, y)\ {}^{I}R_{P}(\phi, x') = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\theta \end{bmatrix} = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} $$

The location of the SPS upper joint expressed in the moving platform

$$ {}^{P}b_{l} = \begin{bmatrix} -r \\ -a \\ 0 \end{bmatrix}, \qquad {}^{P}b_{l} = \begin{bmatrix} -r \\ a \\ 0 \end{bmatrix} $$

Represent the location in the body frame.

$$ {}^{B}b_{l} = {}^{B}R_{P}\,{}^{P}b_{l} = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} -r \\ -a \\ 0 \end{bmatrix} = \begin{bmatrix} -rc\theta - as\theta s\phi \\ -ac\phi \\ rs\theta - ac\theta s\phi \end{bmatrix} $$

$$ {}^{B}b_{r} = {}^{B}R_{P}\,{}^{P}b_{r} = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} -r \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} -rc\theta + as\theta s\phi \\ ac\phi \\ rs\theta + ac\theta s\phi \end{bmatrix} $$

2. Inverse Kinematics and Forward Kinematics

The position of the SPS lower joint expressed in the body frame

$$ {}^{B}a_{l} = \begin{bmatrix} -r \\ -b \\ h_0 \end{bmatrix}, \qquad {}^{B}a_{l} = \begin{bmatrix} -r \\ b \\ -h_0 \end{bmatrix} $$

The length of the SPS link must be equal to the distance between the lower joint and upper one.

$$ d_l^2 = \left({}^{B}b_l - {}^{B}a_l\right)^T \left({}^{B}b_l - {}^{B}a_l\right) = (-rc\theta - as\theta s\phi + r)^2 + (-ac\phi + b)^2 + (rs\theta - ac\theta s\phi - h_0)^2 $$

$$ d_r^2 = \left({}^{B}b_r - {}^{B}a_r\right)^T \left({}^{B}b_r - {}^{B}a_r\right) = (-rc\theta + as\theta s\phi + r)^2 + (ac\phi - b)^2 + (rs\theta + ac\theta s\phi - h_0)^2 $$

Given ϕ, θ, you can get the link length. (Inverse Kinematics solution)

Given d_l, d_r, find out ϕ, θ. (Forward Kinematics)

Rearrange the equations

$$f_1(\phi,\theta) = 2r^2 + a^2 + b^2 + h_0^2 - d_l^2 - 2r^2 c\theta - 2ars\theta s\phi - 2abc\phi + 2rh_0 s\theta - 2ah_0 c\theta s\phi = 0$$

$$f_2(\phi,\theta) = 2r^2 + a^2 + b^2 + h_0^2 - d_r^2 - 2r^2 c\theta + 2ars\theta s\phi - 2abc\phi + 2rh_0 s\theta + 2ah_0 c\theta s\phi = 0$$

Newton Raphson Method for multivariable.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}} \begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} + \begin{bmatrix} f_1(x_{1,k}, x_{2,k}) \\ f_2(x_{1,k}, x_{2,k}) \end{bmatrix}$$

$$\boldsymbol{J}(\boldsymbol{x_k}) \begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} = -\boldsymbol{f}(\boldsymbol{x_k})$$

$$\begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} = -\boldsymbol{J}^{-\boldsymbol{1}}(\boldsymbol{x_k})\boldsymbol{f}(\boldsymbol{x_k})$$

$$\boldsymbol{x_{k+1}} = \boldsymbol{x_k} - \boldsymbol{J}^{-\boldsymbol{1}}(\boldsymbol{x_k})\boldsymbol{f}(\boldsymbol{x_k})$$

Iterate the procedure until $\boldsymbol{f}$ meets tolerance you defined.

Jacobian

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1(\phi,\theta)}{\partial \phi} & \frac{\partial f_1(\phi,\theta)}{\partial \theta} \\ \frac{\partial f_2(\phi,\theta)}{\partial \theta} & \frac{\partial f_2(\phi,\theta)}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} -2ars\theta c\phi + 2abs\phi - 2ah_0 c\theta c\phi & 2r^2 s\theta - 2arc\theta s\phi + 2rh_0 c\theta + 2ah_0 s\theta s\phi \\ 2ars\theta c\phi + 2abs\phi + 2ah_0 c\theta c\phi & 2r^2 s\theta + 2arc\theta s\phi + 2rh_0 c\theta - 2ah_0 s\theta s\phi \end{bmatrix}$$