Mathematics Derivation in detail

1. Rotation matrix – Convention y-x'

"P" and "B" denote the coordinate of the moving platform and the body frame respectively.

"I" – Intermittent coordinate

$$^BR_P = \,^BR_I(\theta,y)\,^IR_P(\phi,x') = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

The location of the SPS upper joint expressed in the moving platform

$$^Pb_l = \begin{bmatrix} -r \\ -a \\ 0 \end{bmatrix}, \qquad ^Pb_l = \begin{bmatrix} -r \\ a \\ 0 \end{bmatrix}$$

Represent the location in the body frame.

$$^Bb_l = \,^BR_P\,^Pb_l = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}\begin{bmatrix} -r \\ -a \\ 0 \end{bmatrix} = \begin{bmatrix} -rc\theta - as\theta s\phi \\ -ac\phi \\ rs\theta - ac\theta s\phi \end{bmatrix}$$

$$^Bb_r = \,^BR_P\,^Pb_r = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}\begin{bmatrix} -r \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} -rc\theta + as\theta s\phi \\ ac\phi \\ rs\theta + ac\theta s\phi \end{bmatrix}$$

2. Inverse Kinematics and Forward Kinematics

The position of the SPS lower joint expressed in the body frame

$$^Ba_l = \begin{bmatrix} -r \\ -b \\ h_0 \end{bmatrix}, \qquad ^Ba_l = \begin{bmatrix} -r \\ b \\ -h_0 \end{bmatrix}$$

The length of the SPS link must be equal to the distance between the lower joint and upper one.

$$d_l^2 = \left(^Bb_l - \,^Ba_l\right)^T\left(^Bb_l - \,^Ba_l\right) = (-rc\theta - as\theta s\phi + r)^2 + (-ac\phi + b)^2 + (rs\theta - ac\theta s\phi - h_0)^2$$

$$d_r^2 = \left(^Bb_r - \,^Ba_r\right)^T\left(^Bb_r - \,^Ba_r\right) = (-rc\theta + as\theta s\phi + r)^2 + (ac\phi - b)^2 + (rs\theta + ac\theta s\phi - h_0)^2$$

Given ϕ, θ, you can get the link length. (Inverse Kinematics solution)

Given d_l, d_r, find out ϕ, θ. (Forward Kinematics)

Rearrange the equations

$$f_1(\phi, \theta) = 2r^2 + a^2 + b^2 + h_0^2 - d_l^2 - 2r^2 c\theta - 2ars\theta s\phi - 2abc\phi + 2rh_0 s\theta - 2ah_0 c\theta s\phi = 0$$

$$f_2(\phi, \theta) = 2r^2 + a^2 + b^2 + h_0^2 - d_r^2 - 2r^2 c\theta + 2ars\theta s\phi - 2abc\phi + 2rh_0 s\theta + 2ah_0 c\theta s\phi = 0$$

Newton Raphson Method for multivariable.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{x = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}} \begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} - \begin{bmatrix} f_1(x_{1,k}, x_{2,k}) \\ f_2(x_{1,k}, x_{2,k}) \end{bmatrix}$$

$$J(x_k) \begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} = f(x_k)$$

$$\begin{bmatrix} x_{1,k+1} - x_{1,k} \\ x_{2,k+1} - x_{2,k} \end{bmatrix} = J^{-1}(x_k) f(x_k)$$

$$x_{k+1} = x_k + J^{-1}(x_k) f(x_k)$$

Iterate the procedure until f meets tolerance you defined.

Jacobian

$$J = \begin{bmatrix} \dfrac{\partial f_1(\phi, \theta)}{\partial \phi} & \dfrac{\partial f_1(\phi, \theta)}{\partial \theta} \\ \dfrac{\partial f_2(\phi, \theta)}{\partial \theta} & \dfrac{\partial f_2(\phi, \theta)}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} -2ars\theta c\phi + 2abs\phi - 2ah_0 c\theta c\phi & 2r^2 s\theta - 2arc\theta s\phi + 2rh_0 c\theta + 2ah_0 s\theta s\phi \\ 2ars\theta c\phi + 2abs\phi + 2ah_0 c\theta c\phi & 2r^2 s\theta + 2arc\theta s\phi + 2rh_0 c\theta - 2ah_0 s\theta s\phi \end{bmatrix}$$