Topology Notes for 2-24-2010

Important Lemma (from last time):

Let (X, FX), (Y, FY) and (A, FA) be topological spaces.

Let

Then, h is continuous if and only if f and g are continuous.

Proof: See notes from 2/22.

( Suppose f and g are continuous. We prove h is continuous by use of the theorem that if the preimages under a function of all basis elements are open, then the function is continuous.

Let

f and g are continuous, so .

We claim that

Since is the intersection of two open sets, it must be open. Therefore, h is continuous.

Relating Products to Subspaces:

“Small Fact:”

Let (X, FX), (Y, FY) be topological spaces with subspaces (A, FA), (B, FB) respectively.

Let FS denote the subspace topology on and let be the product topology on . Then .

Commentary: This means that we can either look at as a subspace of , then induce the topology, or induce the topologies on A and B, then cross. Essentially, taking the cross product and creating subspaces “commute.”

Proof: We start by reviewing which sets are open under both topologies.

Let . Then,

(where it is understood that )

Now let . Then,

(where it is understood that )

We claim that:

Therefore, .

Products and Quotient Spaces:

Taking products and quotients does not “commute” in the same way that taking products and subspaces does. We begin to demonstrate a counterexample to this notion.

Consider with the usual topology, with x~y if x=y or . This looks kind of like a “sideways infinite flower.” If an set open in contains a (“the”) natural number, then its preimage must contain all natural numbers, so it must contain an open interval about “the” natural number in every direction (all infinity of them).

Let be the projection map.

Let be the identity, where both ’s have the usual topology.

i is a quotient map where ~ is =.

Another way of seeing that i is a quotient map is that:

where the second

Now, consider by

Claim: is not a quotient map. We want to show that the product topology on is not the same as the quotient topology

To be continued…