inzva Algorithm Programme 2018-2019

Bundle 10

Dynamic programming - 2

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1 Bitmask DP

1.1 What is Bitmask?

Let's say that we have a set of objects. How can we represent a subset of this set? One way is using a map and mapping each object with a Boolean value indicating whether the object is picked. Another way is if the objects can be indexed by integers, we can use a Boolean array. However, this can be slow due to the operations of the map and array structures. If the size of the set is not too large (less than 64), a bitmask is much more useful and convenient.

An integer is a sequence of bits. Thus, we can use integers to represent a small set of Boolean values. We can do all the set operations using the bit operations. The bit operations are faster than the map and array operations. Therefore, the bit operations takes less time. In some problems, the time difference may be significant.

In bitmask the i_{th} bit from the right represents the i_{th} object. For example, let A = $\{1, 2, 3, 4, 5\}$, we can represent B = $\{1, 2, 4\}$ with the 11 (01011) bitmask.

1.2 Bitmask operations

- `Add the` i_{th} `object to the subset:`
  mask = mask | (1 << i), set the i_{th} bit to 1.
- `Remove the` i_{th} `object from the subset:`
  mask = mask & $\sim$ (1 << i), set the i_{th} bit to 0.
- `Check whether the` i_{th} `object is in the subset:`
  mask & (1 << i), check whether the i_{th} bit is set. If the expression is equal to 1, then the i_{th} object is in the subset. If the expression is equal to 0, then the i_{th} object is not in the subset.
- `Toggle the existence of the` i_{th} `object:`
  mask = mask ^ (1 << i), XOR the i_{th} bit with 1. This operation turns 1 to 0 and 0 to 1.
- `Count the number of objects in the subset:`
  `__builtin_popcount(mask)`, use a builtin function of GCC that counts the number of 1's in an int variable. (`__builtin_popcountll()` for long long)

1.3 Iterating over Subsets

Iterate through all the subsets of a set with size n:
$for\ (int\ x = 0;\ x < (1 << n);\ ++x)$
Iterate through all the subsets of a subset with the mask y:
$for\ (int\ x = y;\ x > 0;\ x = (y\&(x-1)))$

1.4 Task Assignment Problem

There are N people and N tasks and each task is going to be allocated to a single person. We are also given a matrix *cost* of size $N \times N$, where $cost[i][j]$ denotes how much a person is going to charge for a task. Now we need to assign each task to a person in such a way that the total cost is minimum. Note that each task is to be allocated to a single person, and each person will be allocated only one task.

Naive Approach: Try $N!$ possible assignment. Time complexity: $O(N!)$.

DP Approach: For every possible subset find new subsets can be generated from this subset and update DP array. Here, we use bitmask to represent subsets and iterate over them.
Time complexity: $O(2^N * N)$.

Note: The Hungarian Algorithm solves this problem in $O(N^3)$ time complexity.

Solution code for DP approach:

```
for (int mask = 0; mask < (1<<n); ++mask)
{
    for (int j = 0; j < n; ++j)
    {
        if((mask & (1 << j)) == 0) // jth task not assigned
        {
            dp[mask | (1 << j)] =
                min( dp[mask | (1 << j)] ,
                     dp[mask] + cost[__builtin_popcount(mask)][j]);
        }
    }
}
//after this operation our answer stored in dp[ (1<<N) - 1 ]
```

2 DP on Rooted Trees

We define functions for nodes of the trees which we calculate recursively based on children of a node. One of the states in our DP is usually a $node_i$, denoting that we are solving it for the sub-tree of $node_i$.

2.1 Problem

Given a tree T of N (1-indexed) nodes, where each $node_i$ has C_i coins attached to it. You have to choose a subset of nodes such that no two adjacent nodes (nodes connected directly by an edge) are chosen and the sum of coins attached to nodes in the chosen subset is maximized.

We define $dp1(V)$ and $dp2(V)$ as the optimal solution for when we are choosing nodes from sub-tree of node V and if we include node V in our answer or not, respectively. Our final answer is the maximum of two cases, $max(dp1(V), dp2(V))$.

We can see that $dp1(V) = C_V + \sum_{i=1}^{n} dp2(v_i)$, where n is the number of children of node V and v_i is the ith child of the V. Similarly, $dp2(V) = \sum_{i=1}^{n} max(dp1(v_i), dp2(V))$.

Complexity: $O(N)$.
Solution Code:

```
//pV is parent of V
void dfs(int V, int pV){
    //base case:
    //when dfs reaches a leaf it finds dp1 and dp2 and does not branch again.

    //for storing sums of dp1 and max(dp1, dp2) for all children of V
    int sum1=0, sum2=0;

    //traverse over all children
    for(auto v: adj[V]){
        if(v == pV) continue;
        dfs(v, V);
        sum1 += dp2[v];
        sum2 += max(dp1[v], dp2[v]);
    }

    dp1[V] = C[V] + sum1;
    dp2[V] = sum2;
}
//Nodes are 1-indexed, therefore our answer stored in dp1[1] and dp2[1]
//for the answer we take max(dp1[1],dp2[1]) after calling dfs(1,0).
```

3 DP on Directed Acyclic Graphs

As we know, nodes of a directed acyclic graph (DAG) can be sorted topologically, and DP can be implemented efficiently through this sorting.

First, we can find topological order with topological sort in $O(N)$ time complexity. Then we can find the $dp(V)$ values in topological order, where the V is a node in the DAG and $dp(V)$ is the answer for node V. Answer and implementation will differ from problem to problem.

3.1 Converting a DP Problem into a Directed Acyclic Graph

Most of DP problems can be converted into a DAG. We are going to see why that is the case.

It is obvious that while doing DP and processing a state we evaluate a state by looking all of the possible previous states; and to be able to this, all of the possible previous states must be processed before the current state. From this point of view, we can see that some states depend on the other states to be able to get processed, which gives us a DAG formation.

Note that some DP problems cannot be converted into a DAG and may require hyper-graphs. (Please refer Advanced Dynamic Programming in Semiring and Hypergraph Frameworks for more details)

Let's see a conversion on an example.

Problem:

There are N stones numbered 1,2,...,N. For each i ($1 \leq\ i\ \leq N$), the height of i_{th} stone is h_i. There is a frog who is initially on the stone 1. He will repeat the following action some number of times to reach stone N:
If the frog is currently on $stone_i$, it can jump to $stone_{i+1}$ or $stone_{i+2}$. Here, the cost of a jump from i to j is $\mid h_i - h_j \mid$. Find the minimum possible cost to reach $stone_N$.

Solution:

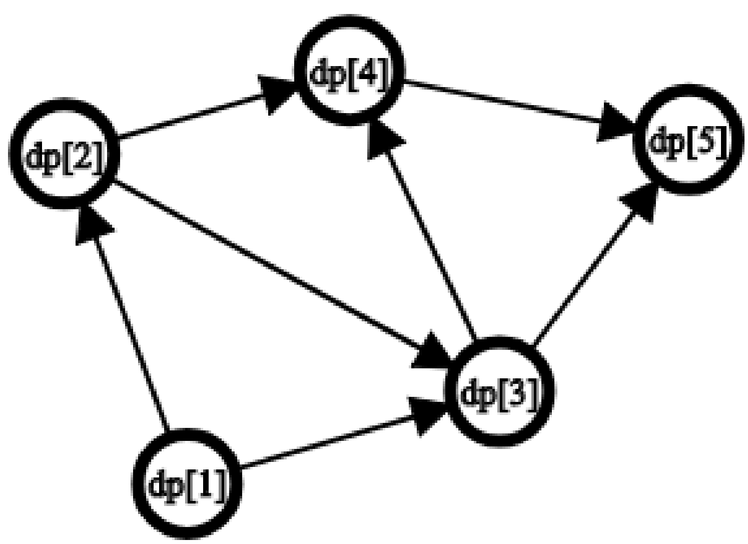
DP approach: We define $dp[i]$ as the minimum cost of the first i index. In the end, our answer will be $dp[N]$. We can denote $dp[i]$ in terms of $dp[i-1]$ and $dp[i-2]$:

$$dp[i] = min(dp[i-1] + abs(h_i - h_{i-1}),\ dp[i-2] + abs(h_i - h_{i-2}))$$

For N=5, we can see that in order to calculate $dp[5]$ we need to calculate $dp[4]$ and $dp[3]$ first and this rule applies to $dp[4]$ and $dp[3]$ as well.

Similarly,
$dp[4]$ needs $dp[3]$ and $dp[2]$ to be calculated,
$dp[3]$ needs $dp[2]$ and $dp[1]$ to be calculated,
$dp[2]$ needs $dp[1]$ to be calculated,

From these dependencies we can construct a DAG:



3.2 DP on Directed Acyclic Graph Problem

For a given DAG with N nodes and M weighted edges, find the longest path in the DAG.

Complexity: $O(N + M)$

Solution Code:

```
// topological sort is not written here so we will take tp as it is already sorted
// note that tp is reverse topologically sorted
// vector <int> tp
// n , m and vector <pair<int,int>> adj is given.Pair denotes {node,weight}.
// flag[] denotes whether a node is processed or not.Initially all zero.
// dp[] is DP array.Initially all zero.

for (int i = 0; i < (int)tp.size(); ++i)//processing in order
{
  int curNode = tp[i];

  for (auto v : adj[curNode])//iterate through all neighbours
      if(flag[v.first])//if a neighbour is already processed
          dp[curNode] = max(dp[curNode] , dp[v.first] + v.second);

  flag[curNode] = 1;
}
//answer is max(dp[1..n])
```

4 Digit DP

Problems that require the calculation of how many numbers there are between two values (say, A and B) satisfying a particular property, can be solved using digit dynamic programming.

4.1 How to Work on Digits

While constructing our numbers recursively (from the left), we need a way to check if our number is still smaller than the given boundary number. For that, while branching, we keep a variable (named "*strict*") which limits our ability to select numbers that are bigger than the boundary number.

Let's suppose that the boundary number is A. We start filling the number from the left (most significant digit) and set *strict* to "*true*", meaning that we cannot take any number that is higher than A's corresponding digit. While branching, the values less than that digit are now going to be not strict ($strict$ = "$false$") because after that point we guarantee that the number is going to be smaller than A. For the value equal to A's corresponding digit, strictness will continue to be "*true*".

4.2 Counting Problem Example

How many numbers x are there in the range A to B, where the digit d occurs exactly k times in x? Constraints: $A, B < 10^{60}$, $k < 60$.

Brute force: $O(N \log_{10}(N))$.

Iterate over all the numbers in the range $[A, B]$ and count the number of digits equal to the d one by one for every number. This complexity is too big, hence we need a better approach.

Recursive approach: $O(3^{\log_{10} N}))$.

We can recursively fill digits of our number starting from the left. At any point of time we branch to 3 possibilities, namely picking a number that is not d and smaller than the corresponding digit of the boundary number, picking a number that is equal to the d and picking a number that is equal to the corresponding digit of the boundary number. The depth of the recursive is equal to the number of the digits in the decimal representation of the boundary number. Thus, we get a complexity of $O(3^{\log_{10} N})$. Still not enough.

Recursive with memoization: $O((\log_{10} N)^2)$.

We can represent a dp state by (*current index*, *current strictness*, *the number of* $d's$) which denotes, the number of possible configurations of the remaining digits after picking the current digit. We use $dp[\log_{10} N][2][\log_{10} N]$ array and calculate every value maximum one time, therefore the worst case is $(\log_{10} N) * 2 * (\log_{10} N)$. Thus, we get $O((\log_{10} N)^2)$ time complexity.

Solution Code:

```
#include <bits/stdc++.h>
using namespace std;
#define ll long long
ll A,B,d,k,dg;// dg: digit count
vector <ll> v;// digit vector
ll dp[25][2][25];
void setup(ll a)
{
    memset(dp,0,sizeof dp);
    v.clear();
    ll tmp = a;
    while(tmp)
    {
        v.push_back(tmp%10);
        tmp/=10;
    }
    dg = (ll)v.size();
    reverse(v.begin(), v.end());
}
ll rec(int idx, bool strict, int count)
{
    if(dp[idx][strict][count]) return dp[idx][strict][count];
    if(idx == dg or count > k) return (count == k);
    ll sum = 0;
    if(strict)
    {
        // all <v[idx] if d is included -1
        sum += rec(idx+1, 0, count) * (v[idx] - (d<v[idx]));
        // v[idx], if d==v[idx] send count+1
        sum += rec(idx+1, 1, count + (v[idx]==d) );
        if(d < v[idx])
            sum += rec(idx+1, 0, count+1); // d
    }
    else
    {
        sum += rec(idx+1, 0, count) * (9); // other than d (10 - 1)
        sum += rec(idx+1, 0, count+1); // d
    }
    return dp[idx][strict][count] = sum;
}
int main()
{
    cin >> A >> B >> d >> k;
    setup(B);
    ll countB = rec(0,1,0);//countB is answer of [0..B]
    setup(A-1);
    ll countA = rec(0,1,0);//countA is answer of [0..A-1]
    cout << fixed << countB - countA << endl;//difference gives us [A..B]
}
```

5 Walk Counting using Matrix Exponentiation

This method helps to count the number of *walks* with the desired *length* on a graph.

Let l be the desired length and let A and B be a node in graph G. If D is the adjacency matrix of G, then $D^l[A][B]$ is the number of *walks* from A to B with length l, where D^k denotes k_{th} power of the matrix D.[11]

5.1 Example

Directed Graph:

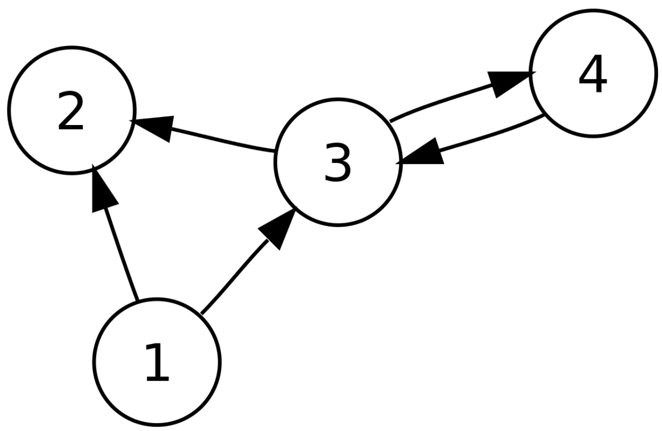


Figure 1: G, a directed graph

	1	2	3	4
1	0	1	1	0
2	0	0	0	0
3	0	1	0	1
4	0	0	1	0

Figure 2: D, adjacency matrix of G

	1	2	3	4
1	0	0	1	0
2	0	0	0	0
3	0	1	0	1
4	0	0	1	0

Figure 3: D^3, $3rd$ power of the matrix D

From the D^3 we can see that there is 4 total *walks* with 3 length.
Let S be the set of *walks* and let w be a *walk* where $w = \{n_1, n_2, ..n_k\}$ and $n_i = i_{th}$ node of the *walk*. Then,
$S = \{\{1, 3, 4, 3\}, \{3, 4, 3, 2\}, \{3, 4, 3, 4\}, \{4, 3.4.3\}\}$ and $|S| = 4$.

Using fast exponentiation on adjacency matrix we can find number of *walks* with length k in $O(N^3 \log k)$ time, where N is the number of nodes in the graph.

Note that $O(N^3)$ time complexity comes from matrix multiplication and we do $\log k$ multiplication using fast exponentiation resulting in $O(N^3 \log k)$ time complexity.

6 Tree Child-Sibling Notation

In this method, we change the structure of the tree. In the standard tree, each parent node is connected to all of its children. Here, instead of having each node store the pointers to all of its children, a node will store a pointer to just one of its children. Apart from this, the node will also store a pointer to its immediate right sibling.[1]

In this notation we can see that every node at most has 2 children (left(first-child), right(first-sibling)), therefore this notation represents a binary tree.

This notation is also called LCRS (left child-right sibling).

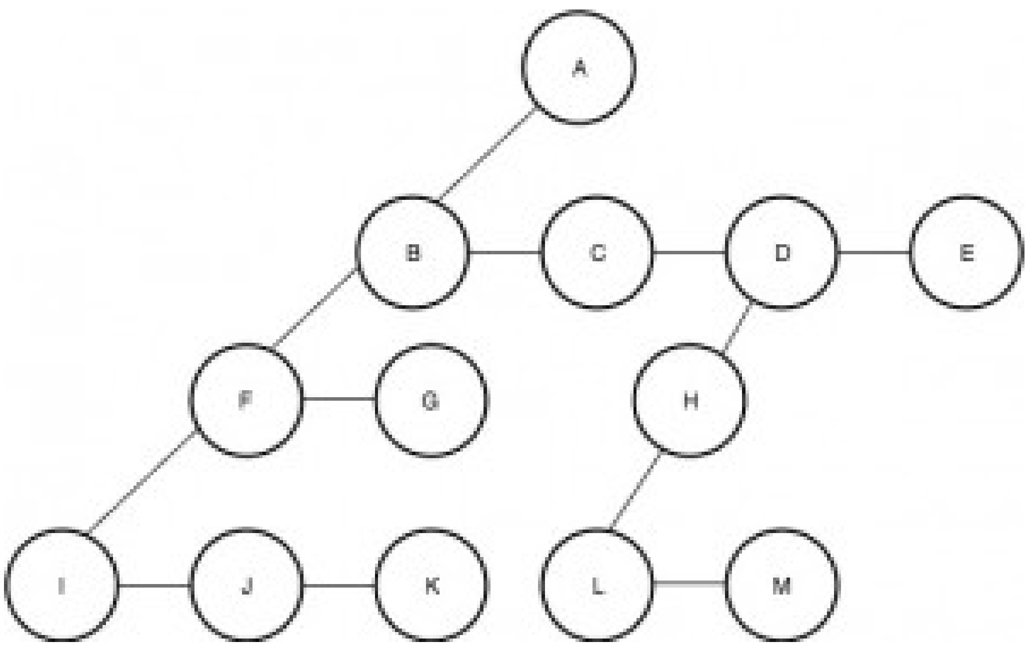


Figure 4: a tree notated with child-sibling notation [2]

6.1 Why You Would Use the LCRS Notation

The main reason for using the LCRS notation is to save memory. In the LCRS notation we do use less memory than the standard notation.

When you might use the LCRS Notation:

- `Memory is extremely scarce`
- `Random access to a node's children is not required`

Possible cases:

1. `If you needed to store a staggeringly huge multi-way tree in main memory.`

   For example, the phylogenetic tree.

2. `In specialized data structures in which the tree structure is being used in very specific ways.`

   For example, the heap data structure. The main operations used on the Heap data structure are:

   - `Remove the root of a tree and process each of its children,`
   - `Join two trees together by making one tree a child of the other.`

   These two operations can be done efficiently on an LCRS structure. Therefore, using the LCRS structure is convenient while working on a heap data structure. [6]

References

[1] LCRS article on Wikipedia

[2] Link to the Figure 3

[3] Bitmask Tutorial on HackerEarth

[4] NOI IOI training week-5

[5] DP on Graphs MIT

[6] LCRS possible uses Stackoverflow

[7] DP on Tree on Codeforces

[8] Digit DP on Codeforces

[9] Digit DP on HackerRank

[10] Graph Drawing/Editing

[11] Walk Counting on Sciencedirect