Combinatorics of the geometry of Wilson loop diagrams

Susama Agarwala, Siân Fryer, and Karen Yeats

February 5, 2019

This paper studies the combinatorics of Wilson Loop Diagrams.

1 Wilson Loop diagrams

What are Wilson loop diagrams and their integrals.

Definition 1.1. A Wilson loop diagram is given by the following data: a cyclicly ordered set V, along with a choice of first vertex (labeled 1), and k pairs, called propagators, written $\{p_r = (i_r, j_r)\}_{r=1}^k$.

Generally speaking, the propagators are undirected, so $p = (i, j) = (j, i)$. In order to fix a convention, we write $p = (i, j)$ with $i + 1 < j$ relative to the first vertex. However, at times (section 3.2), we consider directed propagators, where $p = (i, j)$ denotes a particular propagator flowing in one direction, while $p = (j, i)$ denotes the *same* propagators flowing in the opposite direction.

Addendum 1

We depict this data as a circle with marked points, called vertices. The vertices are labeled by V (preserving the cyclic ordering). The arc between consecutive vertices are called edges. There are k wavy lines in the interior of the diagram, depicting the proagators, with endpoints on the edges. A propagator, $p = (i, j)$ has one endpoint on the edge between the vertex labeled i and $i + 1$ and another endpoint on the edge defined by j and $j + 1$. The condition on i_r and j_r means that the propagator does not go between adjacent edges. Let $\mathcal{P} = \{p_r\}_{r=1}^k$ be the set of propagators. Then we write

$$W = (\mathcal{P}, V) \ .$$

Note that the marked circle gives the vertices of W a cyclic ordering. The choice of first vertex gives it a compatible linear order. Both the cyclic and the linear order become the correct perspective at various points in this paper.

Often we take V to be $[n]$, the cyclically ordered set of integers, $1 \dots n$. In this case, we write $W = (\mathcal{P}, [n])$. We introduce some notation to speak of vertices supporting a propagator, and the set of propagators supported on a vertex set.

Definition 1.2. Let $W = (\mathcal{P}, [n])$.

1. For $p \in \mathcal{P}$, let $V(p) = \{i_p, i_p + 1, j_p, j_p + 1\}$ be the set of vertices supporting p. Then, for $P \subseteq \mathcal{P}$, the set $V(P) = \cup_{p \in P} V(p)$ is the vertex support of P.
2. For $V \subseteq [n]$, write $\text{Prop}(V) = \{p \in \mathcal{P} | V(p) \cap V \neq \emptyset\}$.
3. For $P \subseteq \mathcal{P}$, define $F(P) = V(P^c)^c$ to be the set of vertices in $[n]$ that do not support any propagators outside the set P.
4. The set of vertices that are not in the support of any propagotors is denoted $F(\emptyset)$. Vertices in this set are called non-supporting.

Remark 1.3. Note that a more explicit definition of $F(P)$ is

$$F(P) = \big(V(P) \setminus V(P^c)\big) \cup F(\emptyset) .$$

Furthermore, note that by construction $\text{Prop}(F(P)) \subseteq P$.

It is sometimes useful to discuss propagators in terms of the edges supporting them, rather than the vertices.

Definition 1.4. The i^{th} edge of W is the edge of the external polygon that lies between the vertices i and $i + 1$.

In this manner, the propagator $p = (i, j)$ is supported by the i^{th} and j^{th} edges.

Definition 1.5. A Wilson loop diagram is admissible if

1. $|V| \geq |\mathcal{P}| + 4$
2. There does not exists a set of propagators, $P \subseteq \mathcal{P}$ such that $|V(P)| < |P| + 3$.
3. There does not exist a pair of propagators, $p, q \subseteq \mathcal{P}$ such that $i_p < i_q < j_p < j_q$.

A Wilson loop diagram is weakly admissible if the second and third conditions hold.

The first conditions states that there are at least four more vertices than propagators in an admissible Wilson Loop Diagram. The second imposes an upper bound on how densely the propagators can be fitted in the diagram. The third ensures that ensures that no propagators cross in the interior of the diagram. In other words, a Wilson loop diagram, $(\mathcal{P}, [n])$ is admissible if and only if $n > \mathcal{P} + 4$, and has neither crossing propagators nor any pairs of propagators that start and end on the same pair of non-adjacent edges.

Note that if we take any admissible Wilson loop diagram and remove the unsupported vertices then we will obtain a weakly admissible Wilson loop diagram that may or may not be admissible itself.

In what follows, we will talk about admissible Wilson Loop diagrams and subdiagrams thereof.

Definition 1.6. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The weakly admissible diagram, W' is a subdiagram of W, written $W' \subseteq W$, if

$$W' = (P, V); \quad P \subseteq \mathcal{P}; \quad V(P) \subseteq V \subseteq [n] .$$

There is one particular type subdiagram that deserves special attention.

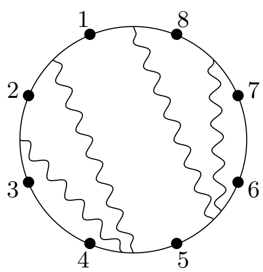
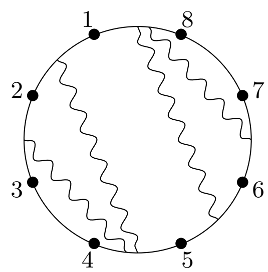
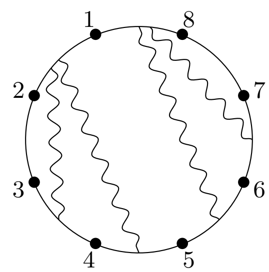
Definition 1.7. For W an admissible diagram, $(P, V(P))$ is exact if $|V(P)| = |P| + 3$.

The exact subdiagrams define an equivalence relation amongst Wilson loop diagrams.

Definition 1.8. There is an equivalence relationship on the set of admissible Wilson loops diagrams given by the transititve closure the following binary relation: $W = (\mathcal{P}, [n]) \sim W' = (\mathcal{P}', n)$ if

1. There exist two different exact subdiagrams, $(P, V(P))$ and $(P', V(P'))$ of W and W' respectively such that $V(P) = V(P')$.
2. The complementary subdiagrams are identical: $(\mathcal{P} \setminus P, V(P)^c) = (\mathcal{P}' \setminus P', V(P')^c)$.

Example 1.9. Note that since this is an equivalence relation, we may find that two Wilson loop diagrams are equivalent, even if they do not have complements of (non-trivial) exact subdiagrams in common. Consider the following three Wilson loop diagrams,

$W_1 =$  $W_2 =$  $W_3 =$  .

The diagrams $W_1 \sim W_2$ because $(\{(5,8),(5,7)\},\{5,6,7,8,1\})$ and $(\{(5,8),(7,8)\},\{5,6,7,8,1\})$ are the corresponding differing subdiagrams. Furthermore, there is the equivalence $W_2 \sim W_3$ due to the exact subdiagrams $(\{(1,4),(3,4)\},\{1,2,3,4,5\})$ and $(\{(1,4),(1,3)\},\{1,2,3,4,5\})$. This forces an equivalence between W_1 and W_3, even though one cannot partition the propagators of each into a an exact subdiagram (that may vary between the diagrams) and a complement that is fixed.

Each Wilson loop diagram, $W = (\mathcal{P}, [n])$ with $|\mathcal{P}| = k$ is associated to a $k \times n$ matrix with non-zero real variable entries, called $C(W)$:

$$C(W)_{p,q} = \begin{cases} c_{p,q} & \text{if } q \in V(p) \\ 0 & \text{if } q \notin V(p) \end{cases} . \tag{1}$$

Example 1.10. For example, ordering the propagators of W_1 from Example 1.9:

$$(1,4),\ (2,4),\ (5,7),\ (5,8)$$

we may write

$$C(W_1) = \begin{pmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,4} & c_{1,5} & 0 & 0 & 0 \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,7} & c_{3,8} \\ c_{4,1} & 0 & 0 & 0 & c_{4,5} & c_{4,6} & 0 & c_{4,8} \end{pmatrix} .$$

These $C(W)$ parametrize a subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ as shown in [?], call it $\Sigma(W)$. The Wilson loop diagrams also define a volume form on $\Sigma(W)$:

$$\Omega(W) = \frac{\prod_{r=1}^{|\mathcal{P}|} \prod_{v \in V_{p_r}} \mathrm{d}c_{p_r}}{R(W)} \ .$$

The denominator $R(W)$ is a polynomial defined by 2×2 and 1×1 minors of $C(W)$ as defined below.

Definition 1.11. For $W = (\mathcal{P}, [n])$, $R(W) = \prod_{e=1}^{n} R_e$, with R_e defined by the propagators ending on it. For any edge e of W, order the propagators incident on e as $\{p_1 \dots p_r\}$, ordered such that p_1 is closest to the vertex e, p_r closest to $e+1$, and p_i is closer to e than p_{i+1}. Then

$$R_e = c_{p_1,e+1} \prod_{j=1}^{r-1} \left((c_{p_j,e} c_{p_{j+1},e+1} - c_{p_{j+1},e} c_{p_j,e+1}) \right) c_{p_s,e} \ .$$

Note that in this notation, if $r = 1$, $R_e = c_{p,e} c_{p,e+1}$.

2 Equivalence classes of Wilson loop diagrams

In [1], Agarwala and Amat show that Wilson loop diagrams can be interpreted as positroids, a certain well behaved class of realizable matroids (this correspondence is stated precisely in Theorem 2.1 below). This opens up the study of Wilson loop diagrams to techniques from geometry and combinatorics. In particular we examine the partial result shown in [1], that shows that equivalent Wilson Loop Diagrams define the same matroid. In section 2.4, we make the relationship between Wilson Loop diagrams and matroids explicit. We count the size of each equivalence class, and enumerate the the number of said classes. We also show that two Wilson Loop diagrams map to the same matroid if and only if they are equivalent.

In subsection 2.1, we discuss a few matroidal facts about Wilson loop diagrams. In section 2.2, we show that there is a one to one correspondence between exact subdiagrams of Wilson loop diagrams and triangulated pieces of the corresponding polygon partion. In section 2.3, we prove some matroidal properties of exact subdiagrams. We show that a subdiagram of W defines a uniform matroid if and only if the subdiagram is exact (Theorem 2.21). Furthermore, this uniform matroid is the restriction of the dual matriod of W restricted to the exact subdiagram (Remark 2.20). The main result of section 2.4 shows that that two admissible Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). We obtain a formula for the number of admissible Wilson loop diagrams in each equivalence class (Corollary 2.25) as well as the number of equivalence classes. In this way, we give a comple characterization of the relationship between Wilson Loop diagrams and their associated matroids.

2.1 Wilson loop diagrams as matroids

We first give a quick summary of the matroid terminology that we will need; it is not intended as a comprehensive introduction to matroids and the interested reader is referred to [ref][find a good matroid reference].

A *matroid* $M = (E, \mathcal{B})$ consists of a finite ground set E and a non-empty family $\mathcal{B} \subseteq \mathcal{P}(E)$ whose elements satisfy the *basis exchange property*: for any distinct $B_1, B_2 \in \mathcal{B}$ and any $a \in B_1 \setminus B_2$, there exists some $b \in B_2 \setminus B_1$ such that $(B_1 \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ as well. The elements of $\mathcal{B}$ are called the *bases* of the matroid. Note that the basis exchange property immediately implies that all bases have the same size.

A subset $A \subseteq E$ is called *independent* in M if $A \subseteq B$ for some $B \in \mathcal{B}$, and *dependent* else. The *rank* $\text{rk}\,(A)$ of a subset $A \subseteq E$ is the size of the largest independent set contained in A. The rank of the matroid itself is defined to be $\text{rk}\,(E)$.

A *circuit* in M is a minimally dependent set. That is, it is a set $C \subseteq E$ such that C is dependent but $C \setminus \{e\}$ is independent for any $e \in C$. A union of circuits is called a *cycle*. On the other hand, a *flat* is a maximally dependent set, i.e. a set $F \subseteq E$ such that $\text{rk}\,(F \cup \{e\}) = \text{rk}\,(F) + 1$ for any $e \in E \setminus F$. Unsurprisingly, a *cyclic flat* is a set which is both a flat and a cycle. The set of circuits in a matroid uniquely defines that matroid, as does the set of flats; thus one could specify a matroid by listing its independent sets, bases, circuits, or flats. [ref]

Finally, we describe several important types of matroids. A matroid of rank k with a ground set of size n is called *realizable* if there exists some $A \in Gr(k, n)$ whose non-zero $k \times k$ minors are exactly those with columns indexed by elements of $\mathcal{B}$. A *positroid* is a matroid which can be realized by an element of the totally nonnegative Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Finally, a *uniform matroid* of rank r is a matroid in which any set of size $\leq r$ is independent.

Matroid theory relates to the study of Wilson loop diagrams as follows. In [1], Agarwala and Amat show that every admissible Wilson loop diagram with k propagators defines a positroid of rank k, and that the independent sets can be read directly from the diagram:

Theorem 2.1. *[1, Theorem 3.6] Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ defines a matroid $M(W)$ with ground set $[n]$. The independent sets are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| < |U|$.*

In other words, the independent sets of $M(W)$ correspond to the sets of vertices in W such that no subset supports fewer propagators than the vertices it contains.

Throughout, we take the *matroid defined by* W to be the matroid $M(W)$ of Theorem 2.1. Note that since vertices of the diagram W correspond to columns of the associated matrix $C(W)$, $M(W)$ can also be thought of as the matroid realized by $C(W)$.

Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $M(W)$ its associated matroid. Where it will not cause confusion we conflate the two objects, identifying vertices of W with elements of the ground set $[n]$ in $M(W)$.

In particular, this allows us to prove results about $M(W)$ by considering the behavior of propagators in W. We record a few elementary facts about the rank and cycles of $M(W)$ here as an example of this.

Lemma 2.2. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. Then:*

1. *The rank of a set $V \subseteq [n]$ is bounded above by $\min\{|V|, |Prop(V)|\}$, with $rk\,(V) = |V|$ if and only if V is an independent set.*

2. *If $C \subseteq [n]$ is a cycle, then rk $(C) = |Prop(C)|$.*

3. *If $[n]$ can be partitioned into at least two non-empty sets, each of which support different sets of propagators that form a partition of the propagotor set,*

$$[n] = \sqcup_i V(P_i) \quad s.t. \quad \sqcup P_i = \mathcal{P}; \quad V(P_i) \cap V(P_j) = \emptyset; \quad P_i \cap P_j = \emptyset \ ,$$

then the matroid $M(W)$ is seperable,

$$M(W) = \bigoplus_i M(P_i, V(P_i)) \ .$$

4. *$F(P)$ is a flat of $M(W)$, thus justifying the name "propagator flat".*

Proof. The first part of (1) is [1, Equation (9)] and surrounding discussion, and the second part is standard matroid theory. (2) is [1, Lemma 3.27]. (3) is a direct consequence of [1, Lemma 3.20] and the fact that $F(P_1)^c = V(P_1^c)$

To prove (4), we need to show that $F(P)$ is maximally dependent. If $F(P) = [n]$ then this is automatic, so suppose not and let $v \in [n] \setminus F(P)$. In other words, $v \in V(P^c)$ and so v supports some propagator $q \notin P$. Let $S \subseteq F(P)$ be an independent set of maximal size. Then $\text{Prop}(S) \subseteq P$ by (1), and no subset of S supports fewer propagators than the number of vertices it contains (this is the definition of an independent set in $M(W)$). Since v supports a new propagator $q \notin P$, the set $S \cup \{v\} \subseteq F(P) \cup \{v\}$ also satisfies this independence condition. Thus rk $(F(P) \cup \{v\}) =$ rk $(F(P)) + 1$, as required. □

Note that this means that the set, $F(\emptyset)$, of non-supporting vertices is the maximal subset of vertices of W of rank 0. That is, it is the unique flat of rank 0 in $M(W)$.

2.2 Polygon partitions of Wilson loop diagrams

The equivalence relation on Wilson loop diagrams is defined in terms of exact subdiagrams; thus in order to understand the equivalence, we need a way to extract and compare exact subdiagrams. We do this via the notion of a polygon partition of W.

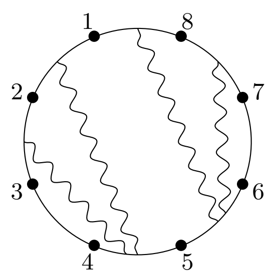
Definition 2.3. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The *polygon partition* associated to W, denoted $\tau(W)$, is defined as follows.

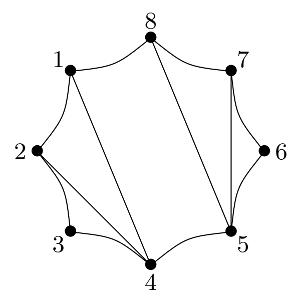
- The vertices of $\tau(W)$ correspond to the edges of W.
- Labeling the vertices of $\tau(W)$ with the edge number of Wgives a cyclic order to the vertices. Connecting consecutive vertices gives a graph theoretic cycle called the polygon of $\tau(W)$.
- Each propagator of W defines a chord edge of $\tau(W)$; specifically, a propagator $(i, j) \in \mathcal{P}$ defines a chord connecting the vertices i and j in $\tau(W)$.

Lemma 2.4. *If $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram, then $\tau(W)$ is a simple planar graph whose outer face is a cycle. It is embedded such that the vertices all lie on this infinite face*[1]*. These vertices are cyclically ordered, with a choice of first vertex giving it an additional compatible linear order.*

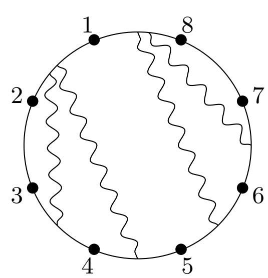
Proof. Since the vertices of $\tau(W)$ are labeled by the edges of W, which are cyclically ordered, this gives an ordering to the vertices and the outer face of $\tau(W)$ is a cycle. Since W is admissible, no pairs of propagators cross. Therefore, it is a planar embedding. Similarly, W does not admit any propagators of the form $p = (i, i+1)$; therefore there is exactly one edge connecting any two adjacent edges of $\tau(W)$. Finally, there does not exist two propagators p, q such that both p and q start at edge i and end at edge j. Therefore, no other two vertices of $\tau(W)$ can be connected by more than one edge. Finally, the embedding of $\tau(W)$ is induced from the embedding of the graph W. $\square$

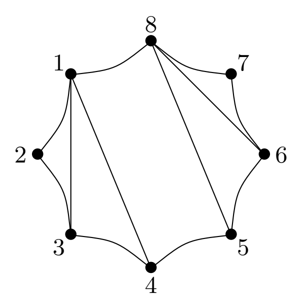
Example 2.5. In this example we return to two of the Wilson loop diagrams in Example 1.9. We can pair diagrams with their polygon partitions as follows:

$W_1 =$ 

; $\tau(W_1) =$ 

and

$W_3 =$ 

; $\tau(W_3) =$ 

.

Recall that a planar embedding of a graph is a *triangulation* if all faces, except possibly the infinite face, are triangles.

Definition 2.6. Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. A *triangulated piece* of $\tau(W)$ is a 2-connected subgraph of $\tau(W)$ which is a triangulation. We will take the convention that a subgraph consisting of a single chord edge is called a *trivial* triangulated piece. A *maximal* triangulated piece is one which is not contained in any strictly larger triangulated piece.

[1]That is, it is an *outerplanar* graph

Definition 2.7. A *decomposition* of a polygon partition $\tau(W)$ is a set of 2-connected induced subgraphs of $\tau(W)$ which partition the edges of $\tau(W)$.

Example 2.8. For the Wilson loop diagrams and polygon partitions in Example 2.5, the vertex sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ give maximal triangulated pieces for both $\tau(W_1)$ and $\tau(W_3)$. The vertex set $\{4, 5, 8, 1\}$ is not a triangulation in either polygon partition.

Lemma 2.9. *For W an admissible Wilson loop diagram, the polygon partition $\tau(W)$ has a unique decomposition into maximal triangulated pieces, and edges in the polygon of $\tau(W)$.*

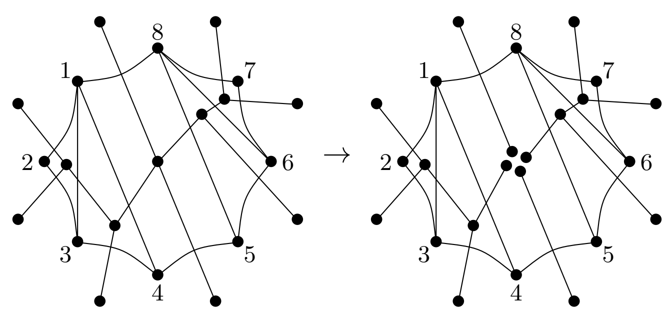
Proof. We begin by giving an algorithm for the decomposition, then prove its uniqueness. Let $W = (\mathcal{P}, [n])$, with $|\mathcal{P}| = k$.

By *splitting* a vertex v we will mean replacing v by new vertice $v_1, v_2, \dots, v_{\deg(v)}$ such that each v_i has exactly one neighbour and the union of the of the v_i is the neighbourhood[2] of v.

Let $T(W)$ be the dual graph of $\tau(W)$ with the vertex corresponding to the infinite face split. Since $\tau(W)$ is an embedded graph (with a fixed distinguished embedding) by Lemma 2.4, $T(W)$ is an uniquely defined graph.

Furthermore $T(W)$ is a tree because it is connected, has $n + k + 1$ vertices ($k + 1$ from the internal faces of $\tau(W)$ and n from the outer face) and $n+k$ edges (since $\tau(W)$ has $n+k$ edges). Additionally, since $\tau(W)$ is simple, $T(W)$ has no vertices of degree 2.

Split every vertex of $T(W)$ which has degree > 3. The connected components of $T(W)$ correspond to the decomposition of $\tau(W)$ into maximal triangulated pieces and edges originally in the polygon of $\tau(W)$. Let f be the forest thus obtained. The vertices of f either have degree 1 or 3. Trees of f with no trivalent vertices correspond to either edges in the polygon of $\tau(W)$, if they were originally leaves of $T(W)$, or to maximal trivial triangulated pieces. Splitting at all the faces that are not triangles ensures maximality of the decomposition. If the splitting were not maximal, then one could add a triangle to a connected component of the splitting, but this would imply that that splitting happened at a valence 3 vertex.



To see uniqueness, consider a different maximal decomposition of $\tau(W)$. This induces a splitting on $T(W)$, where each connected component of the new decomposition corresponds to a subtree. Call this forest f'. Since $f' \neq f$, there are two trees, t and t' in f and f' that are distinct, but

[2]The neighbourhood of a vertex is the set of adjacent vertices

share at least one edge of $T(W)$. Since f' is also maximal, t' is not a subtree of t. Therefore, the edges of t' can be found in at least two trees in the forest f. In particular, there is a vertex v in t' that corresponds to a split vertex of $T(W)$ in the original decomposition. This implies that v has valence greater than 3 in $T(W)$, and thus the corresponding face of $\tau(W)$ is not a triangle. In other words, the decomposition corresponding to f' is not a triangulation. $\square$

Corollary 2.10. *Given a maximal decomposition of $\tau(W)$, the maximal triangulated pieces are edge disjoint.*

Proof. Consider any two distinct maximal triangulated pieces of $\tau(W)$. These two pieces correspond to subtrees of $T(W)$ and intersect, at most, at a vertex in the interior of $\tau(W)$. Since the subtrees corresponding to the maximal triangulated pieces are edge disjoint, and the edges of $T(W)$ correspond to the edges of $\tau(W)$, this forces the maximal triangulated pieces to be edge disjoint as well. $\square$

We are now in a position to relate the triangulated pieces of $\tau(W)$ to exact subdiagrams of W.

Every triangulated piece t of an admissible Wilson loop diagram W corresponds to a subdiagram of W by taking the set of propagators P corresponding to edges of t and then taking the subdiagram $W_P = (P, V(P))$. Conversely, given a subdiagram $W_P = (P, V(P))$ of W we can obtain a subgraph of $\tau(W)$, called t, as follows:

- The vertex set of the subgraph t is
  - the vertices of $\tau(W)$ corresponding to edges of W defined by cyclically consecutive elements of $V(P)$ as a subset of $[n]$
- The edge set of the subgraph is
  - the edges of $\tau(W)$ corresponding to propagators of P
  - along with the outer edges of $\tau(W)$ for which both their end points are in the vertex set

Note that the subgraph t depends on how the propagators of P sit inside $[n]$, not on how they sit in W_P itself. In particular t is not the subgraph of $\tau(W)$ consisting only of edges corresponding to propagators of P.

Lemma 2.11. *Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. The triangulated pieces of $\tau(W)$ correspond to the exact subdiagrams of W using the correspondence described above.*

Proof. Let us first record a few standard facts about polygon triangulations (that is, about triangulations with all vertices on the outer face). If such a triangulation has n vertices then it has n edges on the polygon (that is, on the outer face) and $n-3$ edges which are not. No planar graph with the same vertices and the same outer face can have more edges than the triangulation, and every such simple graph with $n-3$ edges off the outer face is a triangulation.

Since W is admissible, by Lemma 2.4 $\tau(W)$ is a simple graph. Let t be a triangulated piece of the decomposition of $\tau(W)$ given in Lemma 2.9; note that t cannot be equal to $\tau(W)$ by the definition of admissible diagrams.

If t has 2 vertices then t corresponds to a propagator that connects two non-adjacent edges. Therefore, the trivial triangulation is a trivial exact subdiagram.

Now suppose that t has $m > 2$ vertices. We count how many edges of t are not on the outer face of $\tau(W)$. These are exactly the edges of t defined by propagators of W. Consider the intersection of t with the outer face of $\tau(W)$: this is a possibly disconnected subgraph of the polygon of $\tau(W)$ and this subgraph has m vertices. Call this new subgraph S, and let j be the number of connected components of S. To join the components of S into the outer face of t, t must have j edges in its outer face which are not in the outer face of $\tau(W)$. Furthermore t has $m-3$ edges not in its outer face and so also not in the outer face of $\tau(W)$. Thus there are $m-3+j$ edges of t not in the outer face of $\tau(W)$.

Each of these $m-3+j$ internal edges corresponds to a propagator in W; call this set of propagators P. Next we count the size of $V(P)$. Each of the m vertices in the outer face of t corresponds to an edge of W. These m edges define j connected components of the outer polygon of W. Thus the set $V(P)$ has $m+j$ vertices. In other words,

$$|V(P)| = m + j = |P| + 3 \ .$$

Thus the subdiagram $(P, V(P))$ defined by t is exact.

Conversely, suppose we have an exact subdiagram $(P, V(P))$ of $W = (\mathcal{P}, [n])$ supported on $|V(P)| = |P| + 3$ vertices, and let t be the subgraph of $\tau(W)$ corresponding to $(P, V(P))$.

Suppose $|P| = 1$. Let p be the element of P. The exactness condition on $(P, V(P))$ says that the four supporting vertices of p are distinct. If the support of p is four consecutive vertices, then $V(p)$ defines three consecutive boundary edges of W, so t is a single triangle, hence a triangulated piece. If the support of p is not four consecutive vertices, then the vertices which are the ends of t are separated by at least two vertices along the cycle. This implies that t is a trivial triangulated piece.

Now suppose $|P| > 1$. Let $j = |P|$, $m = |V(P)|$, and suppose that the set $V(P)$ defines c disjoint cyclic intervals of $[n]$. Then t has $m - c$ vertices. If t were a triangulation, t would have $j - c$ internal edges.

The graph t has j edges that come from propagators, and $m-2c$ edges that come from the boundary polygon of $\tau(W)$. Since t has $m - c$ vertices, it has $m - c$ external edges, of which c come from propagators. Therefore, of the j edges of t that come from propagators, $j - c$ are internal to the connected component. Therefore, t is a triangulated piece.

□

To avoid the issue of exact diagrams being subdiagrams of other exact subdiagrams (for instance, any subdiagram $(q, V(q))$, for $q \in \mathcal{P}$ is exact), we introduce the notion of maximal exact subdiagrams.

Definition 2.12. An exact subdiagram $(P, V(P))$ is a *maximal exact subdiagram* of W if there is no other exact subdiagram $(Q, V(Q))$ in W that contains $(P, V(P))$ as a strict subdiagram.

Corollary 2.13. *Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ can be uniquely decomposed into maximal exact subdiagrams. These maximal subdiagrams partition $\mathcal{P}$.*

Proof. Combining Lemmas 2.9 and 2.11 yields the unique decomposition into maximal exact subdiagrams, and Corollary 2.10 ensures that no propagator appears in more than one subdiagram in this decomposition. Since the chord edges of $\tau(W)$ correspond to the propagators of W, the decomposition of $\tau(W)$ induces a partition of $\mathcal{P}$. □

2.3 Matroidal properties of exact subdiagrams

Since Corollary 2.13 allows us to decompose any admissible Wilson loop diagram into a collection of maximal exact subdiagrams, in this section we examine the matroid properties of exact subdiagrams more closely. In this section, we show prove two results:

1. The matriod assoicated to an exact subdiagram of W can be written as a contraction of the matroid $M(W)$ by the complementary propagator flat (Theorem 2.19).
2. The matroid associated to an exact subdiagram is uniform (Theorem 2.21).

We begin by proving some useful facts about propagator flats, and flats of matroids associated to admissible Wilson loop diagrams more generally.

Lemma 2.14. *Let F be a flat in $M(W)$, and let $C \subseteq F$ be the union of all circuits contained in F. Then the following are true:*

1. *$C = F(Prop(C))$, i.e. C is a propagator flat.*
2. *$F \setminus C$ is an independent set. Furthermore, If $F \setminus C$ is an independent flat if and only if $F(\emptyset) = \emptyset$, that is W has no non-supporting vertices.*

Proof. (1) If F is an independent flat, then $C = \emptyset$ and the statement is trivially true. Now suppose that F is a dependent set, so C is non-empty.

Let $v \in C$. Clearly $\text{Prop}(v) \subseteq \text{Prop}(C)$. Since $F(P) = V(P^c)^c$, we have $F(\text{Prop}(v)) \subseteq F(\text{Prop}(C))$. Since $v \in F(\text{Prop}(v))$ by the definition of propagator flat, we have $v \in F(\text{Prop}(C))$ as required, and $C \subseteq F(\text{Prop}(C))$.

Now suppose there exists some $w \in F(\text{Prop}(C)) \setminus C$. Let B be an independent subset of C of maximal rank; we first show that $B \cup \{w\}$ is a dependent set in $M(W)$. By Lemma 2.2,

$$|\text{Prop}(C)| = \text{rk }(C) = |B| = \text{rk }(B).$$

Since $B \subseteq C$ implies that $\text{Prop}(B) \subseteq \text{Prop}(C)$, this implies that $\text{Prop}(B) = \text{Prop}(C)$. Furthermore, since $w \in F(Prop(C)$, $\text{Prop}(B \cup w) \subseteq \text{Prop}(B)$. By Lemma 2.2 $\text{rk }(B \cup w) \leq \min\{|B \cup w|, |\text{Prop}(B \cup w)|\}$, and therefore $\text{rk }(B \cup w) < |B \cup w|$. That is, $B \cup w$ is a circuit in $F(\text{Prop}(C))$. Therefore, $B \cup w \subset C$, leading to a contradiction.

For part (2), first note that $F \setminus C$ is automatically independent as it contains no circuits.

Note that since $F(\emptyset)$ has rank 0, $F(\emptyset)$ is contained in every flat of W. Therefore, if $F(\emptyset) \neq \emptyset$, then $F \setminus C$ cannot be a flat.

If $F(\emptyset) = \emptyset$, for any $e \notin F$, we certainly have rk $((F \setminus C) \cup \{e\}) =$ rk $(F \setminus C) + 1$ since F is a flat. Now let $e \in C$, and suppose that rk $((F \setminus C) \cup \{e\}) =$ rk $(F \setminus C)$. This implies that $(F \setminus C) \cup \{e\}$ is dependent, and hence contains a circuit, S. Since $F(\emptyset) = \emptyset$, this circuit must contain at least two elements but this contradicts the fact that C was the union of all circuits in F.

Thus rk $((F \setminus C) \cup \{e\}) =$ rk $(F \setminus C) + 1$ for any $e \notin F \setminus C$, and hence $F \setminus C$ is a flat. □

Corollary 2.15. *If F is a flat of a Wilson loop diagram, it can be written as the disjoint union of a cyclic propagator flat and an independent set and the set of non-supportive vertices.*

In particular, any propagator flat can be written as a union of a cyclic propagator flat, an independent set and $F(\emptyset)$ for the appropriate Wilson loop diagram.

Next we examine properties of propagator flats associated to the complement of Wilson Loop diagrams.

Lemma 2.16. *Let $W = (\mathcal{P}, [n])$ be a Wilson loop diagram, and $P \subseteq \mathcal{P}$. Then:*

1. *If $(P, V(P))$ is an exact subdiagram in W, then $F(P^c)$ is not an independent flat.*
2. *If $(P, V(P))$ is a maximal exact subdiagram in W, then $F(P^c)$ is a cyclic flat.*
3. *If $(P, V(P))$ is an exact subdiagram in W, then rk $(F(P^c)) = |P^c|$.*

Proof. First note that if $(P, V(P))$ is exact then the admissiblity of W guarantees that $F(P^c)$ is a non-empty flat. That is, $V(P) \subsetneq [n]$.

Since W is admissible, we have $n \geq |\mathcal{P}| + 4$. Rewriting this as

$$|V(P)| + |F(P^c)| \geq |P| + |P^c| + 4,$$

and combining it with the fact that $|V(P)| = |P| + 3$ (from the exactness of $(P, V(P))$), we obtain

$$|F(P^c)| > |P^c| \, . \tag{2}$$

Equation (2) is therefore saying that $F(P^c)$ supports fewer propagators than the number of vertices it contains, i.e. $F(P^c)$ is not an independent set.

For part (2), suppose that $(P, V(P))$ is a maximal exact subdiagram. By Corollary 2.15 we can decompose $F(P^c)$ as

$$V(P)^c = F(P^c) = C \sqcup S = F(\mathrm{Prop}(C)) \sqcup S, \tag{3}$$

where C is the largest cyclic flat contained in $F(P^c)$ and S is an independent set. Note that C must be non-empty since $F(P^c)$ is non-empty and dependent (by part (1)).

Since S is an independent set, $\text{rk }(S) = |S|$, and every element of S is independent of C, $\text{rk }\big(F(P^c)\big) = \text{rk }(C) + \text{rk }(S)$. By Lemma 2.2(2),

$$\text{rk }(S) = \text{rk }\big(F(P^c)\big) - \text{rk }(S) \leq |P^c| - \text{rk }(C)\ . \tag{4}$$

Let $Q^c = \text{Prop}(C)$. Then $C = F(Q^c)$ and, by Lemma 2.2(1), $\text{rk }(C) = |Q^c|$. By equation (4),

$$|P| + |S| \leq |P| + |P^c| - \text{rk }(C) = |\mathcal{P}| - |Q^c| = |Q|\ . \tag{5}$$

From equation (3),

$$|V(Q)| = V(P) \sqcup S = C^c = F(\text{Prop}(C))^c = F(Q^c)^c = V(Q)\ .$$

Combining this with equation (4) gives

$$|V(Q)| = |V(P)| + |S| = |V(P)| + 3 + |S| \leq |Q| + 3\ . \tag{6}$$

Since W is admissible, one must have $V(Q) = |Q| + 3$, that is the subdiagram $(Q, V(Q))$ is exact.

Next, note that

$$C = F(Q^c) = V(Q)^c \subset F(P^c) = V(P)^c\ ,$$

which implies that $P \subseteq Q$. That is, $(P, V(P))$ is a subdiagram of $(Q, V(Q))$. Since $(P, V(P))$ is maximally exact by hypothesis, and $(Q, V(Q))$ is exact by (6), we are forced to have $S = \emptyset$. Therefore, $F(P^c)$ is a cyclic flat.

To see (3), first note that the second point of this Lemma, combined with Lemma 2.2 that

$$\text{rk }(F(P^c)) = |\text{Prop}(F(P^c))| = |P^c|.$$

To prove this in the general case, let let $(R, V(R))$ be an exact subdiagram that is not maximal and P be the maximal exact subdiagram containing it. That is, $R \subsetneq P$.

Since $R \subset P$, $V(P) = V(R) \sqcup S$, where $S = V(P) \setminus V(R)$. Since P and R both define exact subdiagrams, we may write $|V(P)| = |V(R)| + |P \setminus R|$, where S is a vertex set of size $|P \setminus R|$ with $(P \setminus R \subset \text{Prop}(S)$. Since $|S| = |V(P) \setminus V(R)| = |P \setminus R| \leq |\text{Prop}(S)|$, by Lemma 2.2, $\text{rk }(S) = |S|$, implying that S is independent. Taking the complements, we may write

$$F(R^c) = F(P^c) \sqcup S\ .$$

Taking the rank of both sides,

$$\text{rk }(F(R^c)) = \text{rk }(F(P^c) \sqcup S) \leq \text{rk }(F(P^c)) + |S| = |P^c| + |\text{Prop}(S)| \leq |R^c|\ .$$

Combining this with (2) gives the desired result.

□

We are now ready to show that any exact subdiagram $(P, V(P)$ of W can be written as a contraction of $M(W)$ by $F(P^c)$. We begin with a definition.

Definition 2.17. Let $M = (E, \mathcal{B})$ be a matroid, and $S \subseteq E$. The *contraction* of M by S is the matroid $M/S = (E \setminus S, \mathcal{B}/S)$, where

$$\mathcal{B}/S = \{B \setminus S \mid |B \cap S| \text{ is maximal amonst all } B \in \mathcal{B}\}.$$

In [1], Agarwala and Amat show that certain subdiagrams of W can be realized as contractions of $M(W)$:

Lemma 2.18. *[1, Theorem 3.33] Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. If the set $V(P)^c$ has rank $|P^c|$, then the matroid defined by the subdiagram $(P, V(P))$ is equal to the contraction $M(W)/V(P)^c$.*

Theorem 2.19. *If $(P, V(P)$ is an exact subdiagram of of W, one may write*

$$M\big((P, V(P))\big) = M(W)/F(P^c) \; .$$

Proof. This follows from Lemma 2.17. Lemma 2.16 shows that the supports of exact subdiagrams satisfy the conditions of this lemma. □

Remark 2.20. Let M^* denote the dual of a matroid. The the dual of a contraction of a matroid is the same as the restriction of the dual matroid by the complement:

$$M/S = M^*|_{S^c} \; .$$

Then Theorem 2.19 implies that, for $(P, V(P))$ an exact subdiagram of W

$$M\big((P, V(P))\big) = M(W)/F(P^c) = M(W)^*|_{V(P)} \; .$$

Matroids coming from exact subdiagrams have an especially nice structure, namely, they are uniform. Recall from Section 2.1 that a uniform matroid of rank r is one in which all sets of size $\leq r$ are independent.

Theorem 2.21. *Let $W' := (P, V(P))$ be a subdiagram of an admissible Wilson loop diagram $W = (\mathcal{P}, [n])$. Then W' is an exact subdiagram if and only if $M(W')$ is a uniform matroid of rank $|P|$.*

Proof. It follows directly from the definitions that a matroid of rank r is uniform if and only if all circuits have rank r; we therefore focus on the circuits of $M(W')$.

We prove the following claim: W' is exact if and only if $V(P)$ contains no circuits C with rk $C < |P|$ in $(P, V(P))$. Since rk $(M(W'))$ is bounded above by $|P|$, the result follows.

Suppose $C \subseteq V(P)$ is a circuit of rank $m < |P|$; by Lemma 2.2, we know that $|\text{Prop}_{W'}(C)| = m$ as well, where the subscript to Prop specifies the diagram we are working in. Observe that $\text{Prop}_{W'}(C) \subseteq P$ by definition. The set $P \setminus \text{Prop}_{W'}(C)$ is thus nonempty, and we can consider the subdiagram $W'' := (P \setminus \text{Prop}_{W'}(C), V(P \setminus \text{Prop}_{W'}(C))$. By the density condition on subdiagrams of admissible diagrams, we have

$$|V(P \setminus \text{Prop}_{W'}(C))| \geq |P \setminus \text{Prop}_{W'}(C)| + 3.$$

It is easy to verify that $V(P \setminus \mathrm{Prop}_{W'}(C)) \subseteq V(P) \setminus C$; since $C \subseteq V(P)$ and $\mathrm{Prop}_{W'}(C) \subseteq P$, we can therefore rewrite the previous inequality as

$$|V(P)| - (m+1) \geq |V(P \setminus \mathrm{Prop}(C))| \geq |P| - m + 3.$$

Simplifying, we obtain $|V(P)| \geq |P| + 4$, i.e. $(P, V(P))$ is not an exact diagram.

Conversely, suppose that W' is not exact and for a contradiction suppose also that $M(W')$ is uniform of rank $|P|$. Take $p \in P$. Then $|V(P) \setminus V(p)| = |V(P)| - 4 \geq |P|$ by non-exactness. By uniformity there is an independent set of size $|P|$ in $V(P) \setminus V(p)$. This is impossible because the submatrix corresponding to this independent set has $|P|$ rows but the one corresponding to p is all 0 one of them is all 0 so it cannot be full rank. □

We now make a few observations about the geometry of matroids defined by exact diagrams.

In [1], the authors show that all admissible Wilson loop diagrams correspond to positroids. That is, they correspond to matroids that can be represented by elements of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$. Any positroid of rank k on n elements defines a subspace of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, namely the points which represent it. These subspaces give a CW structure on $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, with each positroid defining a cell.

Definition 2.22. Given a Wilson loop diagram $W = (\mathcal{P}, [n])$, define the positroid cell associated to a Wilson loop diagram, $\Sigma(W)$, to be the cell in the CW complex on $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ defined by the positroid $M(W)$.

With this definition in mind, we have the following corollary:

Corollary 2.23. *Let $(P, V(P))$ be an exact subdiagram of W. The matroid associated to this subdiagram corresponds to the top dimensional cell in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$.*

Proof. The unique top dimensional cell of $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ is defined by all points in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ such that all Plucker coordinates are strictly greater than 0. Since $(P, V(P))$ is an exact subdiagram, this all $|P| \times |P|$ minors are non-zero. Intersecting these with the cases with the positive Grassmannians demands that all minors be strictly positive. □

2.4 Matroids and equivalent diagrams

Now we are ready to prove the main results of this section, namely that two Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). We also count the number of equivalence classes amongst Wilon Loop diagrams with n vertices and k propagators and give a formula for the number of Wilson loop diagrams in a given equivalence class (Corollary 2.25), completing the characterization of the correspondence between Wilson Loop diagrams and matroids starting in [1].

Siân postponed reading this section until the earlier ones are fixed.

Theorem 2.24. *Let $W = (\mathcal{P}, [n])$ and $W' = (\mathcal{P}', [n])$ be two Wilson Loop diagrams. They define the same matroid if and only if $W \sim W'$.*

Proof. One direction has been proved in [1, Theorem 1.18], but we give a different proof here to be consistent with the method of this document.

Assume that W and W' are equivalent. Without loss of generality, write $W = (P \cup R, [n])$ and $W' = (P \cup R', [n])$, where $P \subset \mathcal{P} \cap \mathcal{P}'$ and $(R, V(R))$ and $(R', V(R'))$ are two maximally exact subdiagrams, with $R \neq R'$, but $V(R) = V(R')$. If this is not the case, one may always find a family of diagram, $\{W_i\}$ satisfying this condition and forming a transitive chain connecting W to W' in the equivalence class.

Let $U \subset V(R)$ be any subset of size $|U| = |R|$. Since $(R, V(R))$ defines a uniform matroid, by Lemma 2.21, the set U is independent in the subdiagram $(R, V(R))$, and thus in W. The complementary set $F(P)$ is a flat of maximal rank by lemma 2.16 (rk $(F(P)) = |P|$). Let $B \subset F(P)$ be a maximal indpendent set (rk $B = |P|$) in $F(P)$. Since $F(P)$ is a flat, adding any element of $V(R)$ to B increases the rank. Therefore, any basis of W can be written as $B \cup U$ for some B and U of the form indicated. However, since $F(P)$ is common to both W and W', and $V(R) = V(R')$, any any basis of W' can also be written $B \cup U$. Thus both matroids have the same bases sets, proving that they are the same.

For the converse, assume that the matroids associated to W and W' are the same: $M(W) = M(W') = M$. Let $\mathcal{R} = \{(P_i, V(P_i)\}_{i=1}^k$ and $\mathcal{R}' = \{(P'_i, V(P'_i)\}_{i=1}^l$ be the sets of maximally exact subdiagrams of W and W'. Write $F_i = F(P_i^c)$ and $F'_i = F(P_i'^c)$ to be the complementary cyclic flats. By Theorem 2.21 and Theorem 2.19, M/F_i and M/F'_i are uniform matroids. Since, by Theorem 2.21, any subset V of M such that $M/(V^c)$ is uniform defines an exact subdiagram of both W and W'. If $U \subset V$ are two such thsets, then the corresponding exact subdiagrams are also subsets. Therefore, $|\mathcal{R}| = |\mathcal{R}'|$, and we write $V(P_i) = V(P'_i)$, after possibly reindexing $\mathcal{R}$ and $\mathcal{R}'$.

Since the sets of propagators defining maximal exact subdiagrams partition $\mathcal{P}$, by Lemma 2.13, write $\cup_{i=1}^k P_i = \cup_{i=1}^k P'_i = \mathcal{P}$. Reorganize the vertex sets of maximal exact subdiagrams as follows:

$$\cup_{P_i \notin \mathcal{R}'} V(P_i) = V(\cup_{P_i \notin \mathcal{R}'} P_i) = F(\cup_{P_i \in \mathcal{R}'} P_i)^c \, . \tag{7}$$

The final flat may, of course, be empty.

Thus, we have partitioned the vertices of W and W' into two complementary sets. The first, $\cup_{P_i \notin \mathcal{R}'} V(P_i)$, is comprised of the union of supports of maximal exact subdiagrams whose propagators differ between W and W'. The second, $F(\cup_{P_i \in \mathcal{R}'} P_i)$, is the propagator flat of all propagators in common between W and W'.

Without loss of generality, assume that $|\mathcal{R}| = |\mathcal{R}'| = 1$. Then the two Wilson loop diagrams are equivalent. If $|\mathcal{R}| = |\mathcal{R}'| > 1$, then one may define a family of Wilson loop diagrams, W_0 to W_k defined such that $W_0 = W$ and W_i is derived from W_{i-1} by replacing the propagator set P_i with P'_i. In this manner, $W' = W_k$ and $W_i \sim W_{i+1}$, making $W \sim W'$. □

Since there is a unique way to decompose W into maximal exact subdiagrams, it is logical to ask how many diagrams there are in an equivalence class. It is a classical fact the the number of triangulations of an n-gon is the $n-2$ Catalan number, namely $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. Thus we can count the number of equivalent diagrams. [CITATION?]

Corollary 2.25. *Let W be an admissible Wilson loop diagram where the sizes of the supports of the nontrivial maximal connected exact subdiagrams are $n_1, n_2, \dots, n_j$. Then the number of admissible Wilson loop diagrams equivalent to W (including W itself) is*

$$\prod_{i=1}^{j} \frac{1}{n_i - 1} \binom{2(n_i - 2)}{n_i - 2}$$

In this section we have characterized the correspondence between Wilson loop diagrams and positroids, showing that Wilson Loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). Corollary 2.25 enumerates the fibres of the map from Wilson Loop diagrams to positroids. By counting the number of equivalence classes (Theorem **??**), we enumerate the number of positroids defined by Wilson Loop diagrams. In section **??**, Theorem **??**, we show that each Wilson Loop diagram with n vertices and k propagators define a positroid cell of dimension $3k$. Comparing the result of Theorem **??** to the number of positroid cells of a particular dimension in $\mathbb{G}_{\mathbb{R},+}(n,k)$, [CITE AND FIND FORMULA HERE], we see that *Wilson loop diagrams do not map onto all positroids of the correct dimension.*

3 Geometry of Wilson Loop diagram

Since Wilson loop diagrams correspond to positroids, it is natural to study the subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ they define.

[words about what's in this section]

Throughout this section, $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram with k propagators.

3.1 Background

[positroids]

Let $\binom{[n]}{k}$ be the set of all k-subsets of the cyclically ordered set $[n]$. For each $j \in [n]$, we can define a total order $\leq_j$ on the interval $[n]$ by

$$j <_j j+1 <_j \cdots <_j n <_j 1 \cdots <_j j-1 \,.$$

This in turn induces a total order on $\binom{[n]}{k}$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\binom{[n]}{k}$ (the **Gale order** [3]), which is defined as follows: for

$$A = \{a_1 <_j a_2 <_j \cdots <_j a_k\} \text{ and } B = \{b_1 <_j b_2 <_j \cdots <_j b_k\} \in \binom{[n]}{k},$$

we define

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\binom{[6]}{3}$ we have $\{2,5,6\} \preccurlyeq_2 \{2,6,1\}$ but $\{2,5,6\} \not\preccurlyeq_2 \{3,4,6\}$.

Definition 3.1. A **Grassmann necklace** of type (k, n) is a sequence $(I_1, \ldots, I_n)$ of n sets $I_i \in \binom{[n]}{k}$ such that for each $i \in [n]$:

- if $i \in I_i$, then $I_{i+1} = \big(I_i \backslash \{i\}\big) \cup \{j\}$ for some $j \in [n]$.
- if $i \notin I_i$, then $I_{i+1} = I_i$.

By convention, we set $I_{n+1} = I_1$.

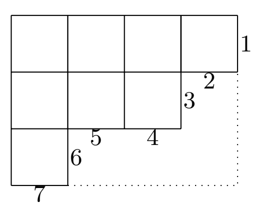
By [5, Theorem 17.1], the Grassmann necklaces of type (k, n) are in 1-1 correspondence with the positroid cells in $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. This correspondence is given explicitly in [4, Theorem 8]: if $(I_1, \ldots, I_n)$ is the Grassmann necklace associated to a positroid $M = ([n], \mathcal{B})$, then the bases of M are exactly

$$\mathcal{B} = \left\{ J \in \binom{[n]}{k} \ : \ I_i \preccurlyeq_i J \ \forall i \in [n] \right\}.$$

Definition 3.2. A **Le diagram** is a Young diagram in which every square contains either a + or a 0, subject to the rule that if a square contains a 0 then either all squares to its left (in the same row) must also contain a 0, or all squares above it (in the same column) must also contain a 0, or both.

By [ref], the set of all Le diagrams that fit within a $k \times (n-k)$ rectangle is in 1-1 correspondence with the positroid cells of $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. The dimension of a positroid cell is equal to the number of + squares in its Le diagram [ref].

The rows and columns of a Le diagram are labelled as follows: given a Le diagram fitting inside a $k \times (n-k)$ box, arrange the numbers $1, 2, \ldots, n$ along its southeast border, starting from the top-right corner. See Figure 1 for examples.



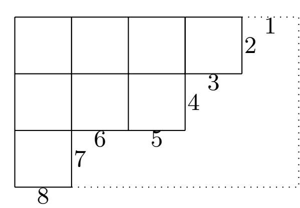


Figure 1: Row and column numbering for a Young diagram with $k = 3$, $n = 7$ (left) and $k = 3$, $n = 8$ (right). The top left box in each diagram has coordinates $(1, 7)$ (left diagram), $(2, 8)$ (right diagram).

An algorithm for constructing the Le diagram associated to a Grassmann necklace was given by Agarwala and Fryer in [ref]. Since we will make use of this algorithm in Section [ref] below, we summarise the process here.

Algorithm 3.3. *[ref] Let $(I_1, \ldots, I_n)$ be a Grassmann necklace of type (k, n). Within a $k \times (n-k)$ square, draw the Young diagram whose rows are labelled by I_1 (as per the convention above).*

For each i, $2 \leq i \leq n$:

- *Write*
$$I_1 \setminus I_i = \{a_1 > a_2 > \cdots > a_r\}, \quad I_i \setminus I_1 = \{b_1 < b_2 < \cdots < b_r\},$$
where the inequalities denote the $<_1$ order (subscripts suppressed for clarity).

- *For $1 \leq j \leq r$, place a $+$ in square (a_j, b_j) of the diagram. (We will sometimes refer to this $+$ as being* in the $a_j \to b_j$ position.*)*

After performing the above for $2 \leq i \leq n$, place a 0 in any remaining unfilled boxes.

An algorithm for constructing the Grassmann necklace of a Le diagram also exists; this was given by Oh in [ref]. A method for using the Le diagram to test whether a given k-subset is a basis for the corresponding positroid or not was given by Casteels in [ref].

3.2 Propagator configurations in admissible Wilson loop diagrams

Before we can describe the algorithm for extracting the Grassmann necklace of $M(W)$ from the Wilson loop diagram W, we require some initial results about the behavior of propagators in admissible WLD.

can we use the notation directed propagator here? to emphasize the notational change?

Definition 3.4. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $\overrightarrow{p} = (i, j) \in \mathcal{P}$ be a directional propagator and $\overleftarrow{p} = (j, i)$ be the same propagator with the opposite directionality. Call the vertices of W

$$V_{in}(\overrightarrow{p}) = \{a \in [n] | i <_i a <_i j\} \cup V(p)$$

the region of W **inside** $\overrightarrow{p}$, and

$$V_{out}(\overrightarrow{p}) = \{a \in [n] | j <_j a <_j i\} \cup V(p)\ .$$

the region **outside** $\overrightarrow{p}$. The set of propagators inside (resp. outside) $\overrightarrow{p}$ are those lying in the region inside (resp. outside) are the $q \neq p \in \mathcal{P}$ such that $V(q) \subset V_{in}(p)$ (resp. $V(q) \subset V_{out}(p)$) .

Note that under this definiton, $V_{in}(\overrightarrow{p}) = V_{out}(\overleftarrow{p})$ and $V_{out}(\overrightarrow{p}) = V_{in}(\overleftarrow{p})$. That is, the regions inside and outside of a propagator switch depending on the direction imposed on the propagators p.

Definition 3.5. Let $p = (i, j) \in cP$ be a propagator in W. Define the *length* of p to be

right now this doesn't depend on the order of i and j, but that might matter later on?

$$\ell(p) = min\big\{|V_(in)(\overrightarrow{p})|, |V_(in)(\overleftarrow{p})|\big\} - 2 = min\big\{|[i+1, j]|, |[j+1, i]|\big\}.$$

In other words, given that p partitions the vertices of W, $\ell(p)$ is the size of the smaller of the two partitions.

Remark 3.6. The following observations about propagators of short length in W are easily verified:

1. If $p = (i, i+3)$ is a propagator of length 3, then the middle vertex $i+2$ supports at most one propagator.

2. If every vertex in W supports at least one propagator, then W admits at least one propagator of length 2.

The following lemma establishes certain configurations of propagators that must exist in any admissible diagram with no non-supporting vertices (i.e. with $F(\emptyset) = \emptyset$). We make use of this result in several induction proofs below. Recall from Definition 1.5 that a *weakly admissible* diagram is one that satisfies the density and non-crossing conditions but does not require that $|V(\mathcal{P})| \geq |\mathcal{P}| + 4$.

Lemma 3.7. *Let W be a weakly admissible WLD with at least 5 vertices and in which each vertex supports at least one propagator. Then at least one of the following two things occurs.*

1. *W has a propagator of length ≤ 6 with a propagator of length 2 inside it and nothing else inside it.*

2. *There exists a pair of propagators of length 2 with the property that the first propagator is $(i, i+2)$, the second is $(j, j+2)$, no other propagator ends between vertices $i+2$ and $j+1$, and $j \in \{i+2, i+3, i+4\}$.*

Proof. Suppose first that W has a propagator of length 3, say $p = (i, i+3)$. By Remark 3.6 and the fact that every vertex of W supports at least one propagator, we have that $i+2$ supports exactly one propagator and this propagator must have length 2 by noncrossingness. This gives us an instance of configuration 1 from the statement.

Now suppose W has no propagators of length 3.

We need a bit of notation: Define the restriction of W to a region $[i, j]$ to be the subdiagram obtained by forgetting all propagators not wholly supported on vertices in the cyclic interval $[i, j]$; the vertex set remains the same as in W. Denote this diagram by $W_{[i,j]}$.

Note that if p is a propagator supported on (i, j), then we can consider two obvious restrictions of W, namely $W_{[i,j+1]}$ and $W_{[j,i+1]}$. Clearly all other propagators $q \neq p$ appear in exactly one of these subdiagrams, while p appears in both. In this case, we will say that *p bounds the regions* $[i, j+1]$ and $[j, i+1]$; the ordering of the start and end points allow us to identify one or the other region unambiguously.

With these observations in mind we can return to the proof.

We will inductively construct a sequence of pairs of propagators (p_r, q_r) satisfying: $\ell(p_r) = 2$, and p_r either forms part of configuration 1 or 2 from the statement, or there is a propagator q_r satisfying

- $\ell(q_r) \geq 4$.
- q_r bounds a region of W which contains strictly fewer vertices than that bounded by q_{r-1} and the restriction of W to this region does not contain any of the propagators $p_1, \ldots, p_r$.

Since W contains finitely many vertices, this must eventually terminate in one of the desired configurations.

Start by choosing a propagator p_1 of length 2 in W (which exists by Remark 3.6), and write (j_1, j_1+2) for its support. If it is part of one of the configurations we are looking for then we are done, so suppose otherwise. By assumption: p_1 is not in configuration 2, there are no propagators of length 3, and every vertex is supported. Therefore there must exist a propagator q_1 of length ≥ 4 with one end on edge j_1 or on edge $j_1 - 1$ or on edge $j_1 - 2$. This bounds a region $[i_1, k_1]$ (where $k_1 \in \{j_1 - 1, j_1, j_1 + 1\}$) which does not contain p_1.

Now suppose q_r exists by the induction hypothesis and bounds a region $[i_r, k_r]$ not containing any p_i for $i \leq r$. Consider the restricted diagram $W_r := W_{[i_r, k_r]}$; by the original hypotheses on W every vertex $l \in [i_r+2, k_r-2]$ (which is a non-empty interval since $\ell(q_r) \geq 4$) must have support ≥ 1, while the same is true for $l \in \{i_r, i_r+1, k_r-1, k_r\}$ because these vertices support q_r.

By Remark 3.6, W_r admits at least one propagator of length 2. W_r has no propagators of length 3 since W has none. Let p_{r+1} be a propagator of length 2 in W_r; if it forms part of configuration 1 or 2 then we are done, so assume otherwise.

Note we may replace q_r by any other propagator of length ≥ 4 in W_r which contains p_{r+1}; such a new q_r still satisfies all the hypotheses, and so, without loss of generality we may assume that q_r has minimal length among propagators of length ≥ 4 which contain p_{r+1}.

There are two cases to consider. The first case is that p_{r+1} has one end before $i_r + 3$ and the other after $k_r - 3$. Then since p_{r+1} has length 2, it must be that $[i_r+3, k_r-3]$ is at most a single edge, and so q_r has length ≤ 6. The only propagators inside q_r that do not also contain p_{r+1} must have both ends in $[i_r, i_r+3]$ or both ends in $[k_r-3, k_r]$ and hence be of length 2. Therefore, if q_r contains a propagator t of length ≤ 4, this propagator must also contain p_{r+1} which is impossible my minimality. Therefore, if q_r contains another propagator of length 2 we would have configuration 2 which we have already assumed does not occur. Consequently, q_r contains only p_{r+1} and so we have 1 which again we have assumed does not occur. Therefore the first case cannot occur.

The second case is that either at least one endpoint of p_{r+1} lies on an edge contained in the interval $[i_r+3, k_r-3]$, or both ends lie in $[i_r, i_r+3]$ or $[k_r-3, k_r]$; these situations all behave similarly. Up to symmetry we can write $p_{r+1} = (j_{r+1}, j_{r+1}+2)$ where either the edge $j_{r+1}+2$ is contained in $[i_r+3, k_r-3]$ or $j_{r+1}+2 = i_r+2$. Since p_{r+1} is not part of configuration 2 and vertex $j_{r+1}+4$ is supported (and not by q_r since $j_{r+1}+4 \leq k_r - 2$ in both situations), there is some propagator t with one end in $[j_{r+1}+3, j_{r+1}+5]$ and length ≥ 4. By noncrossingness t is contained in q_r. t does not contain p_{r+1} by minimality and so we may set $q_{r+1} = t$ to continue the induction.

The overall result then follows by induction.

□

The picture didn't apply any longer, but perhaps a new one would be nice?

Remark 3.8. In the case that all vertices of an admissible WLD W support at least two propagators, then Lemma 3.7 substantially simplifies. By Remark 3.6, W has no propagators of length 3. Configuration 1 necessarily entails vertices with support 1 as does configuration 2 unless $j = i+2$. So in the case that W has all vertices with support at least two then W must contain a pair of propagators of length 2 with the property that the first propagator is $(i, i+2)$, the second is $(i+2, i+4)$ and no other propagator ends on the edge $i+2$.

3.3 From Wilson Loop diagrams to Grassmann Necklaces

Until now, the positroid associated to a Wilson loop diagram W could only be obtained by computing the matrix $C(W)$ associated to W, listing all bases of the induced matroid $M(W)$, and constructing the Le diagram or Grassmann necklace of the positroid "by eye" from this list.

In this section, we give an algorithm for passing directly from the Wilson loop diagrams to its Grassmann necklace. This not only greatly simplifies the process above, but will also allow us to relate the behavior of the positroid $M(W)$ directly to the configuration of progators in W.

The fact that Algorithm 3.9 does construct the required Grassmann necklace is proved in Theorem 3.15.

Algorithm 3.9. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. This gives an algorithm for calculating the set I_a, for $a \in [n]$.*

1. *Fix a vertex $a \in [n]$. Set $i := a$ and $I_a = \emptyset$.*
2. *While $\mathcal{P} \neq \emptyset$, perform the following steps.*
   (a) ***Step** i **for vertex** a: If Prop$(i) \neq \emptyset$ in W, write $I_a = I_a \cup i$. Let $p \in$ Prop(p) be the clockwise most propagator supported on i. Write $W = (\mathcal{P} \setminus p, n)$.*
   (b) *If Prop$(i) = \emptyset$ do nothing.*
   (c) *Increment i by 1 and repeat from (a).*

If the algorithm assigns vertex j to propagator p from starting vertex i, we say that p *contributes* j to I_i. Notationally, we represent this by allowing the I_i symbol to represent a function as well as a set, as follows:

Definition 3.10. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. For each $i \in [n]$, define a function $I_i : \mathcal{P} \longrightarrow [n]$ by

$$I_i(p) := \text{the vertex label that } p \text{ contributes to } I_i \text{ in Algorithm 3.9,}$$

for each $p \in \mathcal{P}$.

Given a propagator p and a vertex i in its support, it will be very useful in the following to understand on what set of vertices the Grassmann necklace algorithm assigns i to p. The answer is that the set is a non-empty cyclic interval. Lemma 3.13 establishes this, but first we need a preliminary lemma which is also useful in its own right.

Lemma 3.11. *Let W be an admissible Wilson loop diagram containing at least one propagator. For any $i \in [n]$ and for any $p = (a, b)$ with $i \leq_i a <_i b$, we have $I_i(p) \neq b + 1$.*

Proof. Suppose for contradiction that we have $p = (a, b)$ with $i \leq_i a <_i b$ and $I_i(p) = b + 1$. We may choose p such that $|[a + 1, b]|$ is minimal amongst propagators with this property.

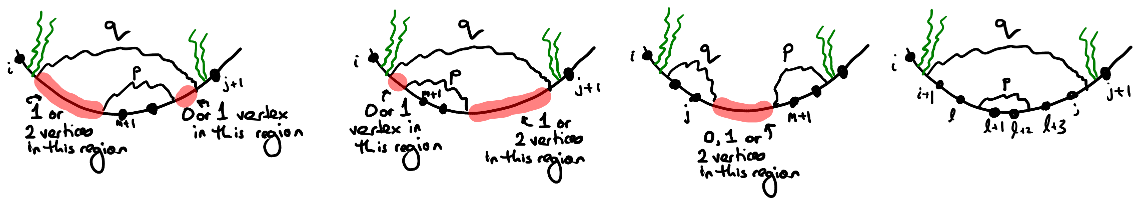


Figure 2: Four cases for admissible WLDs with no non-supporting vertices. The green half-propagators illustrate where propagators may occur, but are not required to exist; no other regions illustrated may support additional propagators.

Since $I_i(p) \neq b$, there must exist a propagator q inside of p with $I_i(q) = b$. The propagator q cannot end on the edge $(b-1, b)$, as this would contradict the minimality of p, so $q = (c, c+1, b, b+1)$ with $a <_i c <_i b$, and $I_i(q) = b$.

In order for q to remain unassigned until vertex b, there must be another propagator r with an end on $(c, c+1)$ and $I_i(r) = c+1$; the only way this can occur is if r is outside q but inside p. Now r contributes its fourth vertex to I_i, again contradicting the minimality of p. □

Corollary 3.12. *If W is an admissible Wilson loop diagram with k propagators, then Algorithm 3.9 assigns exactly k vertices to each I_i.*

Proof. It follows from the proof of Lemma 3.11 that Algorithm 3.9 can never reach the fourth vertex of a propagator's support (with respect to the starting vertex). Therefore if the algorithm starts at vertex i, it must have assigned vertices to all propagators by the time it reaches $i-1$, ensuring that I_i contains exactly k distinct vertices. □

Given a propagator p of an admissible WLD W and a vertex i in the support of p, define

$$J_p^{(W)}(i) = \{m \in [n] \;:\; I_m^{(W)}(p) = i\},$$

i.e. the set of indices m for which Algorithm 3.9 assigns the value i to the propagator p in W. The following lemma establishes that these sets behave in a simple and predictable manner, a fact which we will repeatedly use in subsequent proofs.

Lemma 3.13. *Let $p = (i, j)$ be a propagator of an admissible WLD W on n vertices. Then $J_p^{(W)}(i)$, $J_p^{(W)}(i+1)$, $J_p^{(W)}(j)$, and $J_p^{(W)}(j+1)$ are each non-empty cyclic intervals which partition $[n]$ and occur in the given cyclic order.*

Proof. We will prove the result by induction on the number of propagators. If W has one propagator then the result is immediate. Now suppose W has more than one propagator. Since non-supporting vertices have no effect on the Grassmann necklace algorithm, it suffices to prove the result for W with $F(\emptyset) = \emptyset$. Then by Lemma 3.7, W has at least one of the four situations illustrated in Figure 2.

In each of the four cases, when we remove the propagator labelled p we obtain a diagram which satisfies the statement of the theorem by the induction hypothesis, and contains a propagator

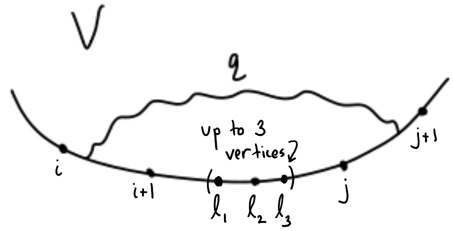


Figure 3: Diagram V is W with p removed; there are no propagators inside $q = (i, j)$, though there may be up to 4 non-supporting vertices labelled $l, l + 1, \ldots$.

$q = (i, j)$ with no other propagators inside it (although this region may or may not contain non-supporting vertices, which we will call $l, l + 1, \ldots$ as necessary). Let V be the diagram obtained by removing the propagator p from W (Figure 3).

Consider the fourth case of Figure 2 first, as it is the easiest. In this case, the vertices $\{l, l + 1, l + 2, l + 3\}$ which support p in W are non-supporting in V, so for every propagator r in V (including q) we have $J_r^{(V)} = J_r^{(W)}$; these are non-empty cyclic intervals in the correct order by the induction hypothesis. Additionally, it is clear from Figure 2 that $J_p^{(W)}(l + a) = \{l + a\}$ for $a \in \{1, 2, 3\}$ and $J_p^{(W)}(l) = [n]\backslash\{l + 1, l + 2, l + 3\}$. The result therefore holds in this case.

Now we proceed to consider the first three cases of Figure 2. We first describe $J_q^{(V)}(*)$, which can be handled identically for all three cases.

In V there are no propagators inside q, so we see from Figure 3 that

$$l, l + 1, \ldots, j \in J_q^{(V)}(j) \text{ (if } l \text{ exists) and } j + 1 \in J_q^{(V)}(j + 1).$$

Note that $j + 2 \notin J_q^{(V)}(j + 1)$ by Lemma 3.11, so by the induction hypothesis we must have $J_q^{(V)}(j + 1) = \{j + 1\}$ and $j + 2 \in J_q^{(V)}(i)$. Thus there exist vertices $d, e \in [j + 2, i + 1]$ with $d < e$, such that

$$J_q^{(V)}(i) = [j + 2, d - 1], \quad J_q^{(V)}(i + 1) = [d, e - 1], \quad J_q^{(V)}(j) = [e, j], \quad J_q^{(V)}(j + 1) = \{j + 1\},$$

and all intervals are non-empty.

We now consider what happens as we move from V to each of the three remaining cases for W. We need to consider both $J_p^{(W)}$ and $J_r^{(W)}$ for $r \neq p$, since the addition of p can have a knock-on effect on later steps in the algorithm.

Left two cases: These cases behave essentially identically (except when j or $j + 1$ are not in the support of p, which can occur in the second case only; see below) so we handle the majority of the proof for these two cases simultaneously. Let $1 \leq a \leq 3$ be the number of non-supporting vertices inside q in V; so these vertices are $l, \ldots, l + a - 1$. Write $p = (m, m + 2)$ where $m \in \{i, i + 1, l\}$. Note that $l, \ldots, l + a - 1$ are all in the support of p.

We first calculate $I_w^{(W)}$ for a starting vertex $w \in [n]\backslash\{l, l + 1, \ldots, j, j + 1\}$. Note that p has no effect on other propagators for starting vertices in this range, while the value of $I_w^{(W)}(p)$ depends on how soon q is assigned to a vertex, i.e. on the value of $I_w^{(V)}(q)$. Thus, if $w \in J_q^{(V)}(i)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{i+1, m\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(i+1)$ or $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} l & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

We also need to understand $I_w^{(W)}$ for $w \in \{l, l+1, \ldots, j, j+1\}$. For the majority of these vertices, we use the following observation: if p is the first propagator to be assigned a value by $I_w^{(W)}$, then the remainder of $I_w^{(W)}$ proceeds identically to the assignments of $I_{w+1}^{(V)}$. Thus we have for any $0 \leq b < a$

$$I_{l+b}^{(W)}(r) = \begin{cases} l+b & \text{if } r = p \\ I_{l+b+1}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Similarly, if j is in the support of p, then we have

$$I_j^{(W)}(r) = \begin{cases} j & \text{if } r = p \text{ and } j \text{ is in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is in the support of } p \end{cases}$$

If j is not in the support of p, then we must be in the second case of Figure 2 with two vertices in the right hand region. In this case, if we start the algorithm at j we need to know whether there will be any unassigned propagators other than p when we reach vertex i, so as to know what p contributes.

Consider the WLD X formed from V by moving the second end of q to the edge $j-1$ instead of j. X is still admissible since we have not decreased the support of any set of propagators, so the induction hypothesis applies to it as well. Note that $J_j^{(V)}(r) = J_j^{(X)}(r)$ for all $r \neq p$ and $J_j^{(X)}(r) = J_{j+1}^{(X)}(r)$ for $r \neq q, p$. Additionally $I_{j+1}^{(X)}(q) = i$ by the induction hypothesis applied to X, and so if we start at $j+1$ and assign propagators to vertices according to the algorithm, when we reach vertex i in X all propagators other than q must have been assigned. Therefore if we start at j in W, we first assign q to j then proceed to assign as in X starting at $j+1$, and hence when we get to i the only remaining unassigned propagator is p. Therefore

$$I_j^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j \text{ is not in the support of } p \\ I_j^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is not in the support of } p \end{cases}$$

Finally, we consider what happens when we start the algorithm at vertex $j+1$. If $j+1$ is in the support of p then we can argue as above to get

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = p \text{ and } j+1 \text{ is in the support of } p \\ I_{j+2}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is in the support of } p. \end{cases}$$

Now suppose $j+1$ is not in the support of p. If we start at $j+1$ we need to know whether there are any unassigned propagators supported on edge i when we reach vertex i. We already know that $J_q^{(V)}(j+1) = \{j+1\}$; in particular this means that q contributes i in $I_{j+2}^{(V)}$. However the construction of $I_{j+1}^{(V)}$ first associates q to $j+1$ and then proceeds identically to $I_{j+2}^{(V)}$. In particular if i was assigned in $I_{j+1}^{(V)}$, then it would not be available to assign to q in $I_{j+2}^{(V)}$ as all other propagators supported at i in V come before q.

Therefore p is the only potentially unassigned propagator on edge i when we reach vertex i in W, and

$$I_{j+1}^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j+1 \text{ is not in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is not in the support of } p \end{cases}$$

We can now describe the intervals $J_r^{(W)}(*)$ for the first two cases of Figure 2. For $r \neq p$ the intervals are clearly still cyclic and appear in the correct order, and we can assemble the intervals for the $J_p^{(W)}(*)$ as follows.

- If $m = l$ then either $a = 2$ (so $l+1 = m+1$, $j = m+2$, and $j+1 = m+3$) or $a = 3$ (so $l+1 = m+1$, $l+2 = m+2$, $j = m+3$, and $j+1$ is not in the support of p), and in both cases

  $$J_p^{(W)}(m) = [m+4, m], \qquad J_p^{(W)}(m+1) = \{m+1\},$$
  $$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are nonempty and otherwise as required.

- If $m = i+1$ then checking each of the three different possibilities for a we likewise get

  $$J_p^{(W)}(m) = [m+4, d-1], \qquad J_p^{(W)}(m+1) = [d, m+1],$$
  $$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are nonempty and otherwise as required.

- If $m = i$ then $a = 1$ or $a = 2$, in the former case $l = m+2$, $j = m+3$ and $j+1$ is not in the support of p so

  $$J_p^{(W)}(m) = \{m+4\}, \qquad J_p^{(W)}(m+1) = [m+5, d-1],$$
  $$J_p^{(W)}(m+2) = [d, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  while in the latter $l = m+2$, $l+1 = m+3$, and j and $j+1$ are not in the support of p so

  $$J_p^{(W)}(m) = [m+4, j+1], \qquad J_p^{(W)}(m+1) = [j+2, d-1],$$
  $$J_p^{(W)}(m+2) = [d, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are again as required.

Third case: In this case there are no non-supporting vertices $l, l+1, \ldots$ inside q. Again write $p = (m, m+2)$ where $m \in \{j, j+1, j+2\}$. We proceed as in the previous cases, by computing $I_w^{(W)}$ for vertices w in roughly increasing order of difficulty.

For $w \in [n]\backslash\{j+1, m, m+1, m+2, m+3\}$: if $w \in J_q^{(V)}(i)$ or $w \in J_q^{(V)}(i+1)$ then

$$I_w^{(W)}(r) = \begin{cases} m & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{m, j+1\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Finally, for $j+1$ and the vertices in the support of p, we have

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = q \\ j+2 & \text{if } r = p \\ I_{j+3}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$
$$I_m^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } q \text{ not supported on } m \\ m+1 & \text{if } r = p \text{ and } q \text{ supported on } m \\ I_m^{(V)}(r) & \text{if } r \neq p \end{cases}$$
$$I_{m+1}^{(W)}(r) = \begin{cases} m+1 & \text{if } r = p \text{ and } q \text{ not supported on } m+1 \\ m+2 & \text{if } r = p \text{ and } q \text{ supported on } m+1 \\ I_{m+1}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$
$$I_{m+2}^{(W)}(r) = \begin{cases} m+2 & \text{if } r = p \\ I_{m+3}^{(V)}(r) & \text{if } r \neq p \end{cases}$$
$$I_{m+3}^{(W)}(r) = \begin{cases} m+3 & \text{if } r = p \\ I_{m+4}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Note that $I_{m+2}^{(V)}(r) = I_{m+3}^{(V)}(r)$ for all propagators r in V, and that if $j+1 \notin \{m, m+1, m+2, m+3\}$ then $j+2$ and $j+3$ are non-supporting vertices in V, so in that case $I_{j+2}^{(V)}(r) = I_{j+3}^{(V)}(r) = I_{j+4}^{(V)}(r)$ for r in V.

Therefore, once again we can see that the $J_r^{(W)}(*)$ are cyclic for all $r \neq p$ in W. Assembling the intervals for p we have:

- if $m = j$ then

$$J_p^{(W)}(m) = [m+4, e-1], \qquad J_p^{(W)}(m+1) = [e, m],$$
$$J_p^{(W)}(m+2) = [m+1, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j+1$ then

$$J_p^{(W)}(m) = [m+4, m-1], \qquad J_p^{(W)}(m+1) = [m, m+1],$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j + 2$ then

$$J_p^{(W)}(m) = [m+4, m], \qquad J_p^{(W)}(m+1) = \{m+1\},$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\}.$$

The result now follows by induction. □

We need one more lemma before we can prove that Algorithm 3.9 does in fact give the Grassmann necklace of the positroid associated to W.

Lemma 3.14. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and let $M(W)$ be its associated matroid. A subset $J \subseteq [n]$ is an independent set of $M(W)$ if and only if there exists an injective set map $f : J \to \mathcal{P}$ with the property that for each $j \in J$ we have $j \in V(f(j))$.*

One of the most important uses of this lemma is for bases. The lemma says that a subset J of $[n]$ is a basis of $M(W)$ if and only if there is a set bijection between J and $\mathcal{P}$ with the property that for each $j \in J$ the propagator associated to j under the bijection is supported on vertex j.

Proof. Because the nonzero entries of $C(W)$ are independent indeterminants, J is an independent set is if and only if there is some choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns.

Each entry in $C(W)$ identifies a propagator by the row of the entry and a vertex by the column of the entry. The entry is nonzero if and only if the propagator is supported on that vertex.

Consequently, a choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns is equivalent to an assignment of the propagators of J to supporting vertices so that no two are assigned to the same vertex. Such an assignment of the propagators of J to supporting vertices is exactly a map f as described in the statment, hence proving the result. □

Theorem 3.15. *The sequence of k-subsets $(I_1, \ldots, I_n)$ obtained by applying Algorithm 3.9 to all vertices of an admissible diagram W is exactly the Grassmann necklace of $M(W)$.*

Proof. For each $i \in [n]$, let I_i be the set of vertices assigned to the propagators of W by Algorithm 3.9 with starting vertex i. By Lemma 3.12, we know that $|I_i| = k$ for each $i \in [n]$. By [ref], the sequence $(I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in [n]$.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_{i \backslash \{i\}}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.9 at vertex i and vertex $i+1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i; thus one end of p must lie on either edge $i-1$ or edge i. In both cases let b denote the edge supporting the other end of p.

Case I: Suppose p has one end on edge $i-1$. Then p is not supported on $i+1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i+1$ up to $b-1$, that is $I_{i+1} \cap [i+1, b-1] = I_i \cap [i+1, b-1]$. Furthermore, by Lemma 3.13, when building I_{i+1} it must happen that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when the algorithm reaches vertex b there cannot be any unassigned propagators remaining that are before p. This is also true in when the algorithm constructing I_i reaches b, since the same propagators have been assigned beforehand. Finally, we also note that $k \geq_i b+1$.

Let W' be the diagram obtained from W by removing both p and all propagators inside of p (recall Definition 3.4).

By the above observations, if we commence Algorithm 3.9 in W' from vertex b, then we are in the same situation with respect to unassigned propagators as if we began at i in W and proceeded to b following the algorithm; the propagators we assigned in the latter case are exactly the ones removed to build W'. Similarly, starting at $i+1$ in W and moving to $b+1$ leaves us in the same situation with respect to unassigned propagators as beginning at $b+1$ in W' would. This gives the equations

$$I_i^{(W)} \cap [b, i-1] = I_b^{(W')}$$
$$I_{i+1}^{(W)} \cap [b+1, i-1] = I_{b+1}^{(W')}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{(W')} \setminus \{b\}$ and $k \notin I_{b+1}^{(W')}$, contradicting the minimality of (W, i, k).

Case II: Suppose p has one end on edge i. Note that by assumption we have $I_i(p) = i$; this means that p must be the first propagator lying on edge i, and hence we must have $I_{i+1}(p) = i+1$ as well. Observe also that $k \geq_i i+2$ since $i+1 \in I_{i+1}$.

Let W' be the diagram obtained from W by removing only the propagator p. Then we have

$$I_i^{(W)} \backslash \{i\} = I_{i+1}^{(W')}$$
$$I_{i+1}^{(W)} \backslash \{i+1\} = I_{i+2}^{(W')}$$

since in both cases the algorithm in W' proceeds identically to that in W after assigning p. Since $k \neq i+1$, we have $k \in I_{i+1}^{(W')} \setminus \{i+1\}$ but $k \notin I_{i+2}^{(W')}$, contradicting the minimiality of (W, i, k).

We have shown that $(I_1, \ldots, I_n)$ is a Grassmann necklace; it remains to check that this Grassmann necklace corresponds to the positroid $M(W)$. We need to show that:

- For each $i \in [n]$, I_i is a basis for $M(W)$.
- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $M(W)$.

Siân note to self: check whether it's lex smaller or Gale smaller

The algorithm is pairing each $j \in I_i$ with a unique propagator supported on that vertex so by Lemma 3.14 I_i is a basis for $M(W)$.

Suppose we have a k-set J such that J is a basis for $M(W)$ and yet is lexicographically less than I_i with respect to $<_i$. By Lemma 3.14 there is a set bijection between J and the propagators of W

such that for each $j \in J$, the propagator associated to j is supported on vertex j. Choose one such bijection. For a propagator p of W write $J(p)$ for the associated j according to this bijection.

Since J is lexicographically smaller than I_i, the $<_i$-smallest element of the symmetric difference of J and I_i is some $j_0 \in J$, $j_0 \notin I_i$. Let p be the propagator such that $J(p) = j_0$. Since $j_0 \notin I_i$ but p is supported at j_0, then p must have been assigned to an earlier vertex by I_i, i.e. we have $I_i(p) <_i j_0$. Thus the propagator p has the following property, which we call property A:

$$I_i(p) <_i J(p) \quad \text{and} \quad I_i \cap [i, I_i(p)] = J \cap [i, I_i(p)]. \tag{A}$$

From the previous paragraph we conclude that if we ever had a J which is lexicographically less than I_i and yet a basis of $M(W)$, then there exists a propagator p which has property A.

Finally, we will prove that whenever there is a propagator p which has property A then there is another propagator r which also has property A and for which $I_i(r) <_i I_i(p)$. This would lead to an infinte regress, contradicting the finiteness of I_i; thus there could be no such J, which will complete the proof of the proposition.

So suppose p has property A. The same vertices in $[i, I_i(p)]$ are in the image of the assignment from J and the image of the assignment from I_i so the same number of propagators are assigned to vertices in this interval by each assignment. However p is one of the propagators assigned to a vertex in this interval in I_i but not in J. Thus there exists a propagator q for which $J(q) \in [i, I_i(p)]$ but $I_i(q) >_i I_i(p)$. Therefore $J(q) <_i I_i(q)$ and $I_i \cap [i, J(q)] = J \cap [i, J(q)]$. This is analogous to property A but with the roles of J and I_i switched. Thus by the same argument we must have a propagator r for which $I_i(r) \in [i, J(q)]$ but $J(r) >_i J(q)$. It follows that r also has property A and that

$$I_i(r) \leq_i J(q) \leq_i I_i(p)$$

as required. $\square$

We need $I_i(r) <_i I_i(p)$ but I'm having trouble seeing it?

In an arbitrary Grassmann necklace, it is possible for an index i to appear in no terms of the Grassmann necklace (a *loop*) or in all terms of the necklace (a *coloop*). Using Theorem 3.15, a characterization of the loops and coloops of the Grassmann necklace associated to a Wilson loop diagram follows easily.

Corollary 3.16. *Grassmann necklaces coming from admissible Wilson loop diagrams have no coloops. A vertex j is a loop if and only if j supports no propagators.*

Proof. For any $i \in [n]$, $i - 1$ is maximal with respect to the $<_i$ order. Therefore there can be no propagator p with $I_i(p) = i - 1$ by Lemma 3.11, i.e. $i - 1 \notin I_i$. Thus the Grassmann necklace admis no coloops.

If $j \in [n]$ is a loop then $j \notin I_j$, which can only happen if there are no propagators supported on vertex j. Conversely, if j supports no propagators, then Algorithm 3.9 never assigns a propagator to j and hence $j \notin I_i$ for all $i \in [n]$. $\square$

3.4 Dimension of the Wilson Loop cells

Our next goal is to show that the dimension of the positroid cell defined by a Wilson loop diagram $(\mathcal{P}, [n])$ has dimension $3|\mathcal{P}|$. Marcott in ***cite this*** has a different proof which is geometric and more elegant, but it is not easy to track the effect of a particular propagator. Our approach is much more explicit.

Recall that the dimension of a positroid cell is equal to the number of plusses in the associated Le diagram [5, Theorem 6.5]. By combining Algorithms 3.9 and 3.3 (converting from WLD to Grassmann necklace to Le diagram) we explicitly describe the effect of adding another propagator to a Wilson loop diagram in terms of the plusses of the associated Le diagrams, and hence give a recursive proof of the $3|\mathcal{P}|$-dimensionality of the cells.

We start with several lemmas, of roughly increasing degree of technicality.

Lemma 3.17. *Let W be an admissible Wilson loop diagram with k propagators, and with a vertex i which supports no propagators. Let V be W with vertex i removed. Then the Le diagram of W is obtained from the Le diagram of V by inserting an extra column containing all 0s in position i (i.e. such that the new column has the label i).*

Proof. By Algorithm 3.9 the Grassmann necklace of W is obtained from the Grassmann necklace of V by duplicating the ith element of the Grassmann necklace of V (shifting indices as appropriate), and incrementing all indices greater than i in each Grassmann necklace element. Formally, if $I_1^{(V)}, \ldots, I_{n-1}^{(V)}$ and $I_1^{(W)}, \ldots, I_n^{(W)}$ are the Grassmann necklaces of V and W respectively then

$$I_j^{(W)} = \begin{cases} \{\ell \in I_j^{(V)} : \ell < i\} \cup \{\ell + 1 \in I_j^{(V)} : \ell \geq i\} & \text{if } j \leq i \\ \{\ell \in I_{j-1}^{(V)} : \ell < i\} \cup \{\ell + 1 \in I_{j-1}^{(V)} : \ell \geq i\} & \text{if } j > i. \end{cases}$$

By Lemma 3.16 we know that $i \notin I_w^W$, and so i must label a horizontal edge on the boundary of the Le diagram of W, i.e. it must be a column label. The shapes of the Le diagram of V and W are the same except for the insertion of this column since I_1^* is the same for V and W except for the incrementation of the indices $\geq i$ in the transition from the necklace for V to the necklace for W. $\square$

Lemma 3.18. *If two Wilson loop diagrams differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Proof. By [5, Proposition 17.10], the dimension of a positroid (and hence the number of plusses in its Le diagram) is $k(n-k) - A(\pi_W)$, where $A(\pi_W)$ denotes the number of alignments (see [5, Figure 17.1] and preceding discussion) of the decorated permutation π_W of the positroid associated to W.

It can easily be seen from [5, Section 17] and Algorithm 3.9 that dihedral transformations of a Wilson loop diagram W correspond to dihedral transformations of the chord diagram representation of π_W. Since the number of alignments in a chord diagram is preserved under dihedral transformations, the result follows. $\square$

Lemma 3.19. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators. Then there is some dihedral transformation W' of W such that there is a propagator p of W' with the following properties.*

- *$p = (i, n-1)$ with no propagators inside it (that is $i+2, \ldots, n-2$ do not support any propagators in W').*
- *Either the edge i in W' only supports p or the edge i in W' supports exactly one other propagator $q = (j, i)$ with no other propagators inside q.*

Proof. Remove all vertices of W which do not support any propagators to get a weakly admissible Wilson loop diagram V. Lemma 3.7 applied to V gives a length 2 propagator p in V for which either no other propagator is supported on one of the supporting edges of p or there is a second length 2 propagator which is the only other propagator supported on one of the supporting edges of p. Figure 2 shows the possible cases, and the reader can easily check that in each case p must be in one of the two situations described above.

By Lemma 3.18 there is a dihedral transformation of V which yields a diagram satisfying the statement of the lemma with p and q both length 2. Restoring the vertices which do not support any propagators, we obtain a dihedral transformation W' of W as desired (with potentially longer lengths for p and q). $\square$

Combining Lemmas 3.17, 3.18, and 3.19, it therefore suffices to study the Le diagrams of weakly admissible Wilson loop diagrams admitting one of the configurations described in Lemma 3.19 with propagators p and q (if q exists) both of length 2. See Figure 4 for an illustration of the two possibilities.

The next few lemmas describe how diagrams of this type are related to the corresponding diagram with propagator p removed, first in terms of the Grassmann necklaces and then in terms of the Le diagrams. These technical lemmas will form the backbone of the inductive step in the main dimensionality argument.

Lemma 3.20. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators, and suppose that W admits one of the configurations described in Lemma 3.19, with p and q (if q exists) both of length 2. Let V be W with p removed. Then*

$$\begin{aligned}
I_1^W &= I_1^V \cup \{n-3\} \\
I_n^W &= I_1^V \cup \{n\} \\
I_{n-1}^W &= I_n^V \cup \{n-1\} \\
I_{n-2}^W &= \begin{cases} I_{n-2}^V \cup \{n-2\} & \textit{if } n-2 \notin I_{n-2}^V \\ I_{n-2}^V \cup \{n-1\} & \textit{if } n-2 \in I_{n-2}^V,\ n-1 \notin I_{n-2}^V \\ (I_n^V - \{n-5\}) \cup \{n-1, n-2\} & \textit{if } n-1, n-2 \in I_{n-2}^V \end{cases} \\
I_k^W &= \begin{cases} I_k^V \cup \{n-3\} & \textit{if } n-3 \notin I_k^V \\ I_k^V \cup \{n-2\} & \textit{if } n-3 \in I_k^V \end{cases} \\
&\quad \textit{for } 1 < k < n-2
\end{aligned}$$

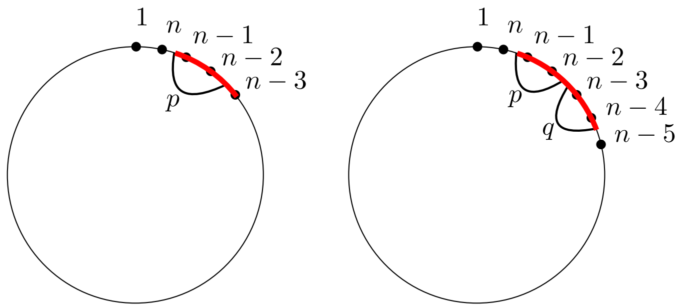


Figure 4: The different possibilities for W and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

Proof. The two possible situations are illustrated in Figure 4. The arguments proceed in a similar manner to Lemma 3.13.

We first consider $I_1^{(W)}$. Note that $n-3 \notin I_1^{(V)}$: for the right hand case this is clear from the diagram (since Algorithm 3.9 would assign q no later than vertex $n-4$), while for the left hand case it follows from Lemma 3.11. Therefore when we start at vertex 1 and apply the Grassmann necklace algorithm to W, p is the only unassigned propagator when we reach vertex $n-3$ and we have $I_1^{(W)}(p) = n-3$.

Next, consider $I_n^{(W)}$, i.e. we apply the Grassmann necklace algorithm with starting vertex n. In both cases we have $I_n^{(W)}(p) = n$. We are now at vertex 1, and the unassigned propagators are exactly those that appear in V. Therefore the algorithm continues as in $I_1^{(V)}$, i.e. we have $I_n^W = I_1^V \cup \{n\}$. Similarly, we must have $I_{n-1}^W = I_n^V \cup \{n-1\}$.

Now consider I_{n-2}^W. If $n-2 \notin I_{n-2}^V$ (i.e. $n-2$ supports no propagators in V), then in W the algorithm assigns p to $n-2$ and this does not affect the rest of the construction of I_{n-2}^V, so we obtain $I_{n-2}^W = I_{n-2}^V \cup \{n-2\}$ as above.

□

Lemma 3.21. *Let V and W be as in Lemma 3.20. The shape of the Le diagram of V can be built from left to right of the following blocks: a rectangle with 3 columns, one more column of the same length, a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of W can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of V, and the same partition shape as in V.*

Proof. See Figure 5 for an illustration of the shapes described in the statement of the lemma.

Recall that I_1 determines the shape of the Le diagram. By Lemma 3.20, we know that $I_1^W = I_1^V \cup \{n-3\}$, and that $n, n-1, n-2, n-3 \notin I_1^{(V)}$. This implies that the right hand boundary of the shape of V is the same as the right hand boundary of the shape of W except that W has one additional row of 3 boxes while V has an additional column in the $n-3$ position; that is, an extra column fourth from the left. □

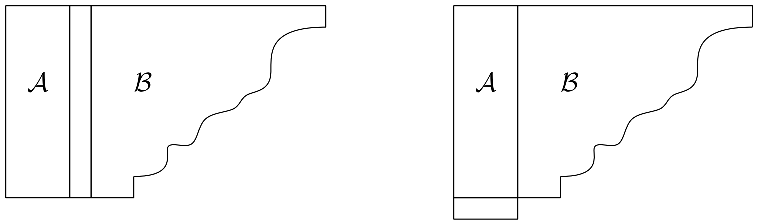


Figure 5: Le diagrams for V (left) and W (right).

The pieces of the Le diagrams of V and W will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows, as in Figure 5. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of each diagram are identical, and that the plusses in the $\mathcal{A}$ parts are very closely related.

Note that when we speak of a plus in the Le diagram of V being the same as in W or vice versa, we mean that the position of the plus in $\mathcal{A}$ or $\mathcal{B}$ is the same; because of the column insertion the absolute indices may differ.

Lemma 3.22. *Let V and W be as in Lemma 3.20. Then I_n^W and I_{n-1}^W together yield the same plusses as I_n^V did, along with extra plusses in the leftmost two boxes of the bottom row of the Le diagram of W.*

Proof. By Lemma 3.20 we have $I_n^W = I_1^V \cup \{n\}$ and $I_1^{(W)} = I_1^{(V)} \cup \{n-3\}$. Thus

$$I_1^{(W)} \setminus I_n^{(W)} = \{n-3\}, \qquad\qquad I_n^{(W)} \setminus I_1^{(W)} = \{n\},$$

and so by Algorithm 3.3 we have a plus in the $(n-3) \to n$ position, i.e. in the leftmost box of the bottom row.

Also by Lemma 3.20 we have $I_{n-1}^W = I_n^V \cup \{n-1\}$. Recall from the proof of Lemma 3.20 that $n-3, n-2, n-1, n \notin I_1^V$; therefore $I_1^W \setminus I_{n-1}^W = (I_1^V \setminus I_n^V) \cup \{n-3\}$, and $n-3$ is maximal in this set. Similarly,

$$I_{n-1}^W \setminus I_1^W = (I_n^V \setminus I_1^V) \cup \{n-1\} \subseteq \{n-1, n\},$$

where the final inclusion follows from the definition of Grassmann necklace. In particular, $n-1$ is minimal in this set, so Algorithm 3.3 yields a plus in the $(n-3) \to (n-1)$ position (i.e. the second box in the bottom row of the Le diagram of W) along with any plusses yielded by I_n^V.

□

Lemma 3.23. *Let V and W be as in Lemma 3.20, with $n-2 \notin I_{n-2}^V$. Then I_{n-2}^W contributes an $(n-3) \to (n-2)$ plus, and all of the $I_{n-1}^V = I_{n-2}^V$ plusses.*

Proof. If $n-2 \notin I_{n-2}^V$ then $I_{n-1}^V = I_{n-2}^V$ and by Lemma 3.20 we have $I_{n-2}^W = I_{n-1}^V \cup \{n-2\}$. Note that $n-3 \notin I_{n-2}^V$ by Lemma 3.11. Therefore the paths for I_{n-2}^W are the paths for I_{n-1}^V along with the $(n-3) \to (n-2)$ path. This gives the statement of the lemma.

□

Lemma 3.24. *Let V and W be as in Lemma 3.20, and suppose that $n-2, n-1 \in I^V_{n-2}$. Then I^W_{n-2} and I^W_{n-3} contribute the following plusses to the Le diagram of W*

- *An $(n-3) \to (n-2)$ plus and an $(n-5) \to (n-1)$ plus.*
- *All of the I^V_{n-1} plusses.*
- *In the Le diagram of V, I^V_{n-2} contributes an $n-5 \to n-2$ plus and no other term in the Grassmann necklace of V gives a plus in this column. I^W_{n-2} does not contribute this $+$ but yields an $n-5 \to n-1$ plus instead.*
- *All other plusses of I^V_{n-2}.*
- *In the Le diagram of V, I^V_{n-3} gives a plus in the $n-3$ column. This $+$ is shifted over into the $n-2$ column in W.*
- *All other plusses of I^V_{n-3}.*

Furthermore, no element of the Gramann necklace of V gives an $n-5 \to n-1$ plus.

Proof. By Lemma 3.20 $I^W_{n-2} = (I^V_n - \{n-5\}) \cup \{n-1, n-2\}$. Also, by the location of q in the WLD, $n-2 \notin I^V_{n-3}$ and and $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Gramann necklace of V looks like

$$I^V_{n-3} \underset{\substack{n-3\text{ out}\\ n-2\text{ in}}}{\rightarrow} I^V_{n-2} \underset{\substack{n-2\text{ out}\\ n-5\text{ in}}}{\rightarrow} I^V_{n-1} \underset{\substack{n-1\text{ out}\\ \text{something in}}}{\rightarrow} I^V_n \underset{\substack{n\text{ out}\\ \text{something in}}}{\rightarrow} I^V_1 \tag{8}$$

where the first "something" is either n or an element of I^V_1 and the second "something" is an element of I^V_1. Additionally all elements not explicitly mentioned must be in I^V_1 as they remain unchanged through this portion of the necklace.

Using this information now determine the symmetric difference of I^V_{n-2} and I^V_1: $n-1, n-2$ and possibly n are in I^V_{n-2} but not in I^V_1. $n-5$ is in I^V_1 as are at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$. This means that the plusses in the Le diagram of V coming from I^V_{n-2} are as in the first part of Figure 6. Stepping to I^V_{n-1} simply removes the $n-5 \to n-2$ path, see the second part of Figure 6.

Stepping to I^V_n, $n-1$ is taken out and either n is put in if it was not there before, or one of a or b is put in and hence no longer available as a right end for a path. This gives two possible configurations illustrated in the bottom two parts of Figure 6.

Now we know that $I^W_{n-2} = (I^V_n - \{n-5\}) \cup \{n-1, n-2\}$ so the paths for building plusses from I^W_{n-2} go from the set $\{n-5, n-3\}$ along with whichever of a and b is not in I^V_n to $\{n-2, n-1, n\}$. This means that we get plusses as in Figure 7 where the left and right cases correspond to the left and right cases in the bottom parts of Figure 6

This proves the first item of statement of the lemma.

Now consider I^V_{n-3}. By (8) I^V_{n-3} contributes the same plusses as $I^V_{n_2}$ except that it contributes an $n-5 \to n-3$ plus in place of the $n-5 \to n-2$ plus. Also, we have $n-3 \in I^V_{n-3}$ be the location

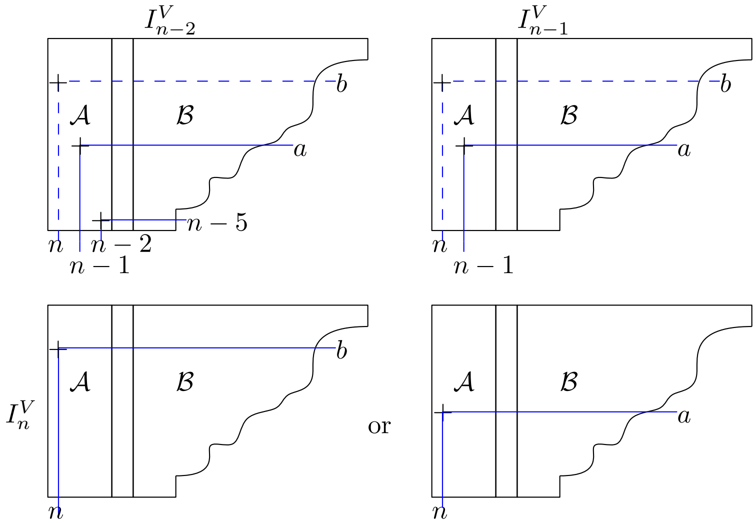


Figure 6: Plusses coming from I_{n-2}^V (top left), I_{n-1}^V (top right) and I_n^V bottom when $n-1, n-2 \in I_{n-1}^V$. The blue lines are the non-intersecting paths. The dashed blue lines may or may not appear, but if one appears then they both do.

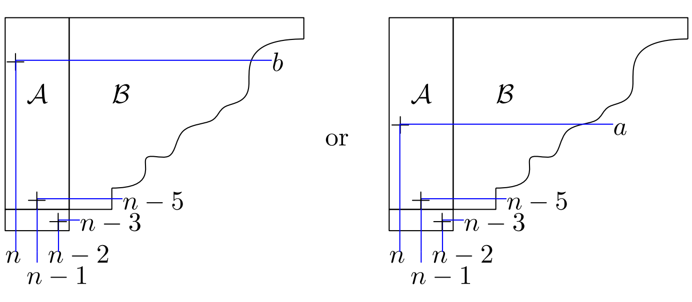


Figure 7: Plusses coming from I_{n-2}^W.

of q and so $I_k^W = I_k^V \cup \{n-2\}$. Thus the paths for I_{n-3}^W are the same as for I_{n-3}^V except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (8) shows that $n-2$ only appears in I_{n-2}^V among the necklace elements of V.

Also note that I_{n-3}^V, I_{n-2}^V, and I_{n-1}^V share their plusses outside of the $n-3$ and $n-2$ columns. This proves the remaining statements of the lemma except the furthermore.

Finally, suppose there were a $n-5 \to n-1$ plus in the Le diagram of V. By the algorithm, it would have to come when $n-5 \notin I_j^V$. By (8) this means that it would have to come from I_{n-4}^V, I_{n-3}^V, or I_{n-2}^V. The analysis above shows it does not come from I_{n-3}^V or I_{n-2}^V. Now, $n-4 \in I_{n-4}^V$ by the location of q and so I_{n-4}^V must give an $n-5 \to n-4$ plus and so cannot give an $n-5 \to n-1$ plus.

□

Lemma 3.25. *Let V and W be as in Lemma 3.20 and take $1 < k < n-2$. Suppose that if $n-2 \in I_{n-2}^V$ then also $n-1 \in I_{n-2}^V$. Then I_k^W gives the same plusses as I_k^V except that if there is a plus in the $n-3$ column for I_k^V then this plus is shifted into the $n-2$ column and no plus was already in that location.*

Proof. If $n-3 \notin I_k^V$ then by Lemma 3.20 $I_k^W = I_k^V \cup \{n-3\}$. Then since $n-3$ is the largest element of I_1^W this transformation leaves the disjoint paths unchanged and so the plusses carry over from V to W directly.

If $n-3 \in I_k^V$ then by Lemma 3.20 $I_k^W = I_k^V \cup \{n-2\}$. If $n-2$ supports no propagators in V then certainly no pluses appear in the $n-2$ column of the Le diagram of V. If $n-2$ supports at least one propagator in V then by hypothesis so is $n-1$ and so we satisfy the hypotheses of Lemma 3.24. Thus the only necklace element of V containing $n-2$ is I_{n-2}^V and this particular plus is not contributed to the Le diagram of D by I_{n-2}^W.

From I_k^V there is a path from some vertical edge to the bottom edge $n-3$. In I_k^W, $n-3$ is a vertical edge with no path and instead there must be a path to $n-2$. By the previous paragraph no other path can end in $n-2$, so shifting the path that did go to $n-3$ to go to $n-2$ while leaving the others the same maintains non-crossingness and so must be the paths for I_k^W. Thus the plus in the $n-3$ column for C is shifted into the $n-2$ column, where there was no plus before, and no other plusses are changed. □

Theorem 3.26. *The number of plusses in the Le diagram of an admissible Wilson loop diagram is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a Wilson loop diagram W with one propagator supported on vertices $i < j < k < \ell$ has Le diagram a single row with $|W| - i$ boxes. Labelling them from left to right by $|W|, \ldots, |W| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider Wilson loop diagrams with $k > 1$ propagators. By Lemma 3.17 it suffices to prove the result for weakly admissible Wilson loop diagrams with k propagators and no vertices that do not support at least one propagator. By Lemma 3.18 it suffices to prove the result for at least one

Wilson loop diagram from each dihedral orbit. Take a weakly admissible Wilson loop diagram W with k propagators and no vertices which do not support at least one propagator. Make a dihedral transformation of W if necessary so that W has a propagator p with the properties in Lemma 3.19 relative to W. If $n-1$ supports only p but $n-2$ supports at least one other propagator, then flip W on the line perpendicular to the edge from $n-2$ to $n-1$. This will be our W for the rest of the proof.

Let V be W with p removed. Note that if $n-2$ supports at least one propagator in V then so is $n-1$ by the end of the previous paragraph and so if $n-2$ supports at least one propagator in V then the hypotheses of Lemma 3.24 are satisfied.

From Lemma 3.21 we know how the shapes of the Le diagrams of V and W relate; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 3.22, 3.23, and 3.24 tell us that the three boxes of the bottom row of the Le diagram of W each have a plus. Lemmas 3.22, 3.23, 3.24, and 3.25 show that there is a bijection between the plusses of the Le diagram of V and the plusses of the Le diagram of W that are not in the bottom row. This bijection can be described as follows.

- Plusses from $\mathcal{B}$ for V maintain their positions in $\mathcal{B}$ for W.
- Plusses from the first two columns (the n and the $n-1$ columns) of $\mathcal{A}$ for V maintain their positions in $\mathcal{A}$ for W.
- If there is a plus in the $n-2$ column of $\mathcal{A}$ in V then Lemma 3.24 applies, so there is exactly one such plus. This plus maps to the $n-5 \to n-1$ plus for W.
- The plusses in the $n-3$ column for V shift over to the third column (the $n-2$ column) of $\mathcal{A}$ in W.

This map is clearly reversible and hence bijective except possibly for the $n-5 \to n-1$ plus for W. If the Le diagram of W has an $n-5 \to n-1$ plus then look at the Le diagram for V. If the Le diagram for V has a plus in the $n-2$ column then Lemma 3.24 applies and so there is no $n-5 \to n-1$ plus in the Le diagram of V and the $n-5 \to n-1$ plus of W can be uniquely mapped to the plus in the $n-2$ column of the Le diagram of V. If the Le diagram for V has no plus in the $n-2$ column, then leave the $n-5 \to n-1$ plus where it is in moving back to V. This reverses the map.

From all of this we get that the number of plusses in the Le diagram for W is three more than the number of plusses in the Le diagram for V. Applying induction completes the proof. $\square$

4 Poles of Wilson Loop Integrals

The results of Section 3 allow us to relate the position of propagators in a Wilson loop diagram W to minors of $C(W)$, which we use in this section to understand the denominator of the integral $I(W)$ associated to a Wilson loop diagram (see Definition 1.11).

The main result of this section is Theorem 4.4, which expresses the denominator $R(W)$ in terms of the Grassmann necklace of W. This simplifies the computation of $R(W)$ and allows us to directly relate the poles of the integral to the combinatorics of the diagram.

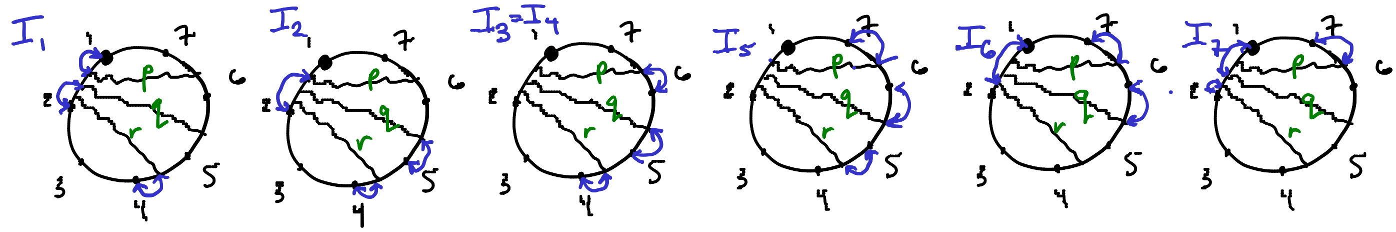


Figure 8: Example WLD for illustrating Algorithm 4.1 and bijections between propagators and vertices for each Grassmann necklace element.

We first give an algorithm which extracts the required minors from the Grassmann necklace.

Algorithm 4.1. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and let $C(W)$ be the matrix of W as defined in* (1) *(see Section 1).*

- *For each $i \in [n]$, we construct a factor r_i as follows:*
  - *Let $S_i = \{p \in \mathcal{P} \mid I_{i-1}(p) \neq I_i(p)\}$. (By convention, set $I_{-1} = I_n$.)*
  - *Write Δ_{I_i} for the determinant of the $k \times k$ minor of $C(W)$ with columns indexed by I_i.*
  - *Let r_i be Δ_{I_i} with all variables from rows associated to $p \notin S_i$ set to* 1.
- *Define $R = \prod_{i=1}^{n} r_i$.*

As we take each Δ_{I_i} to simply be the determinant of a particular submatrix of $C(W)$, the sign of each Δ_{I_i} is well-defined. The goal is to show that R is equal to the denominator of the Wilson loop diagram as defined in Definition 1.11.

Example 4.2. Consider the Wilson loop diagram in Figure 8. Assigning propagators p, q, s to rows $1, 2, 3$ respectively, we obtain the matrix

$$C(W) = \begin{bmatrix} a & b & 0 & 0 & 0 & c & d \\ e & f & 0 & 0 & g & h & 0 \\ i & j & 0 & k & l & 0 & 0 \end{bmatrix}$$

The Grassmann necklace of this diagram is

$$I_1 = \{1, 2, 4\}, I_2 = \{2, 4, 5\}, I_3 = \{4, 5, 6\}, I_4 = \{4, 5, 6\},$$
$$I_5 = \{5, 6, 7\}, I_6 = \{6, 7, 1\}, I_7 = \{7, 1, 2\}.$$

Figure 8 indicates the pairings between propagators and vertices for each $i \in [1, 7]$.

From I_1 to I_2, the propagators p and q change which vertex they are assigned to but r is assigned to vertex 4 in both, so $S_2 = \{p, q\}$. Then

$$\Delta_{I_2} = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ j & k & l \end{bmatrix} = kgb, \qquad r_2 = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ 1 & 1 & 1 \end{bmatrix} = gb.$$

where the 1s in the third row of the second matrix correspond to the fact that $I_1(s) = I_2(s)$. Continuing likewise, we get $r_3 = c$, $r_4 = 1$ (since $I_4 = I_3$), $r_5 = lhd$, and $r_6 = i$.

At I_7 the situation is more complicated: we have $S_7 = \{q, s\}$, so we find that $\Delta_{I_7} = d(ej - fi)$ and $r_7 = ej - fi$. This quadratic factor corresponds to the fact that q and s share an edge *and* contribute both endpoints of that edge to I_7; see Proposition 4.3 below.

Finally, we have $r_1 = (af - be)k$. Putting everything together, we obtain

$$R = (af - be)kgbclhdi(ej - fi)$$

which is squarefree and contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$. If one were to construct the denominator $R(W)$ associated to this Wilson loop diagram as per Definition 1.11, we would find that (up to integer multiples) we have $R(W) = R$.

Proposition 4.3. *With notation as in Algorithm 4.1 we have the following:*

1. *Each Δ_{I_i} is homogeneous, as is each r_i.*
2. *Each Δ_{I_i} splits into linear and quadratic factors. All linear factors of Δ_{I_i} are single variables and all irreducible quadratic factors are 2×2 determinants of single variables.*
3. *Quadratic factors in Δ_{I_i} arise precisely when propagators p and q are supported on a common edge (a, b) with $I_i(p) = a$ and $I_i(q) = b$.*
4. *r_i divides Δ_{I_i}.*
5. *The ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.*

Proof. 1. The nonzero entries of $C(W)$ are independent indeterminates and so every $i \times i$ minor of $C(W)$ is either homogeneous of degree i or is 0. Thus each Δ_{I_i} is homogeneous. Furthermore, each row contributes one factor to each term in the expansion of Δ_{I_i} so the result of setting the variables from a subset of rows to 1 is still homogeneous. Thus each r_i is homogeneous.

2. Using the expression for the determinant as a sum over permutations we see that Δ_{I_i} is a sum over bijections between I_i and $\mathcal{P}$. The nonzero terms in this sum are precisely those bijections such that each propagator is associated to one of its supporting vertices in I_i, since only those locations in $C(W)$ are nonzero. Since the nonzero entries of $C(W)$ are independent there can be no cancellation between terms in this expansion.

   Suppose Δ_{I_i} has an irreducible factor f. Let $\mathcal{P}'$ be the set of propagators which contribute a variable to f and let J be the set of vertices which contribute a variable to f.

   The first claim is that the minor of $C(W)$ associated to $\mathcal{P}'$ and J is precisely f.

   Proof of claim: By the structure of determinants we know that $\Delta_{I_i} = fg$, where g involves only variables associated to propagators not in $\mathcal{P}'$ and associated to vertices not in J.

   Expanding out fg yields a signed sum of monomials. In each of these monomials, f contributes those variables associated both to a propagator in $\mathcal{P}'$ and to a vertex in J, and g contributes those variables associated both to a propagator not in $\mathcal{P}'$ and to a vertex not in J, and no other variables appear.

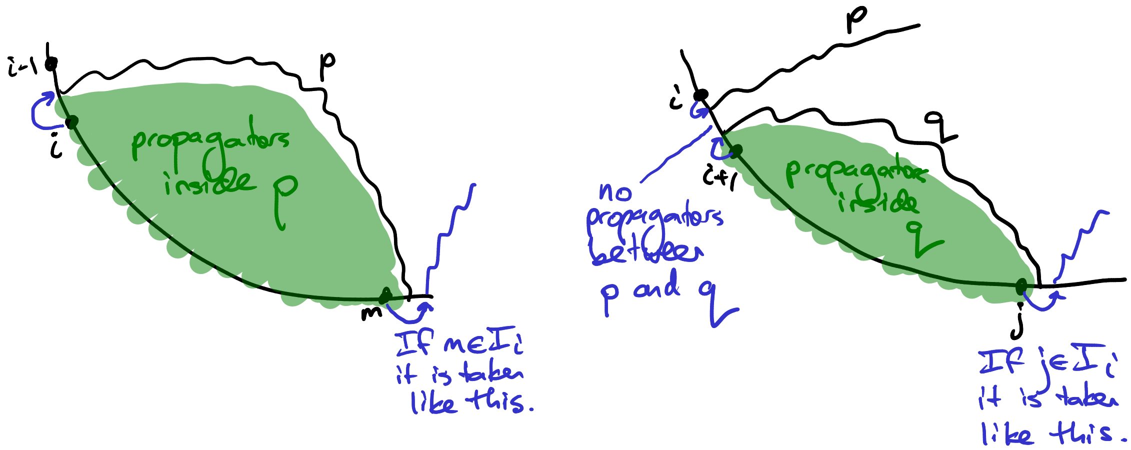


Figure 9: The two cases in the proof that no factors of Δ_{I_i} have degree 3 or more.

Since there is no cancellation between terms, this means that the full expansion over permutations of Δ_{I_i} contains no other nonzero terms and hence no other variables. Therefore Δ_{I_i} is equal to the determinant of the matrix obtained by taking the submatrix of $C(W)$ with columns indexed by I_i and setting any variables not appearing in Δ_{I_i} to 0. This new matrix is, up to permutations of rows and columns, a block matrix with one block for $\mathcal{P}'$ and J and the other block for the complements. Thus its determinant, and hence also Δ_{I_i}, is the product of the minors for these two blocks. By considering which variables appear, these two factors must also be f and g, and so in particular f is the minor of $C(W)$ associated to $\mathcal{P}'$ and J. This proves the claim.

A consequence of this claim is that every linear factor of Δ_{I_i} is a 1×1 minor of $C(W)$, hence is a single variable, and every irreducible quadratic factor of Δ_{I_i} is a 2×2 minor of $C(W)$, hence is a 2×2 determinant of single variables.

All that remains is to prove that Δ_{I_i} has no irreducible factors of degree 3 or more. Suppose for a contradiction that f is a factor of Δ_{I_i} of degree ≥ 3. Note that by removing the propagators which come before those contributing to f and changing i to be the first vertex which contributes to f, we obtain a different admissible diagram for which f still divides Δ_{I_i} but also $i \in I_i$ and i contributes to f. Showing that this different admissible diagram gives a contradiction is sufficient, and so we may assume that $i \in I_i$ and i contributes to f. Finally, we can suppose that W is minimal in number of propagators with the above occuring.

Let p be the propagator such that $I_i(p) = i$. There are two cases to consider, depending on which edge p is supported on. These are illustrated in Figure 9

Case 1: Suppose p has one end on the edge $(i-1, i)$. Thus p is supported on $(i-1, i, m, m+1)$ for some $m >_i i$, and $I_{i+1}(p) = m$ by Lemma 3.13.

Let S be the set of propagators inside p along with p itself. I_i and I_{i+1} can only differ once p contributes to I_{i+1}, so $I_i(q) = I_{i+1}(q)$ for each $q \in S \backslash \{p\}$. Thus if a propagator contributes m in I_i then it must lie outside p.

If neither m nor $m+1$ appear in I_i then by Corollary 3.16 $V(p) \cap I_i = \{i\}$, and so the row of p in the matrix of Δ_{I_i} has only one nonzero entry; hence Δ_{I_i} has a linear factor contributed by p and i, which is a contradiction. So we must have at least one of m and $m+1$ in I_i. However, all propagators in S are mapped by the function $I_i(\cdot)$ to vertices strictly before m, so the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the $|S| \times |S|$ matrix indexed by the propagators in S and the vertices in $I_i(S)$. No other propagators can be supported on these vertices since all other propagators are outside of p, and p is the first propagator supported at i; this explains the zero block. Therefore $\Delta_{I_i} = \det A \det C$, and both factors are nontrivial since at least one of m and $m+1$ appear in I_i. If we remove the propagator outside of p that contributes m or $m+1$, we get a smaller diagram for which $\Delta_{I_i} = \det A$. This contradicts the minimality of our choices unless $\det A$ is quadratic, which in turn contradicts our assumption that i and p contribute to an irreducible factor f of degree at least 3.

Case 2: Suppose p has one end on the edge $(i, i+1)$. If no other propagators are supported on i then the column of $C(W)$ corresponding to vertex i has only one nonzero entry in it, and so Δ_{I_i} has a linear factor contributed by p and i; as above, this is a contradiction. Thus we can take q to be the propagator such that $I_i(q) = i+1$. We know that q has one end on the edge $(i, i+1)$ and is adjacent to p on that edge in the counterclockwise direction (see Figure 9). Write $(i, i+1, j, j+1)$ for the support of q. The situation for q is very similar to case 1: in particular, we have $I_{i+1}(q) = j$ by Lemma 3.13 and so if $j \in I_i$ then the propagator which contributes j is outside of q.

Similarly to Case 1, let S be the set of propagators inside q along with p and q themselves. Then all propagators in S are mapped by $I_i(\cdot)$ to vertices strictly before j and no other propagators are supported on vertices strictly before j. Thus the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the submatrix indexed by the propagators in S and the vertices in $I_i(S)$. Again two things can now happen. If some vertex j or larger (with respect to $>_i$) belongs to I_i then B and C are at least one column wide, and so the block form of the matrix gives a nontrivial factorization of Δ_{I_i}. This yields a contradiction as in Case 1: either W contains unnecessary propagators which contradicts our minimality assumption, or $\det A$ is quadratic which contradicts the assumption that p and i contribute to f, an irreducible factor of degree at least 3.

On the other hand, if no vertex $\geq_i j$ is in I_i then $\Delta_{I_i} = \det A$. Looking in more detail into A, note that the only vertices in the support of p and q which belong to I_i are i and $i+1$, and hence

$$A = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}$$

where D is the 2×2 matrix indexed by the propagators p and q and the vertices i and $i+1$. Thus p and i contribute to a quadratic factor of Δ_{I_i}, once again contradicting our assumptions.

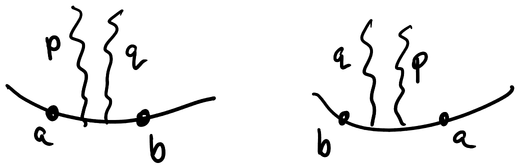


Figure 10: The situations giving a quadratic factor with variables appearing in r_i.

All cases have now been covered and so Δ_{I_i} has only irreducible factors of degree 2 or less.

3. Suppose propagators p and q are supported on a common edge (a, b), with $I_i(p) = a$ and $I_i(q) = b$. Let $x_{p,a}, x_{p,b}, x_{q,a}, x_{q,b}$ be the associated variables in $C(W)$. For any fixed bijection σ from $\mathcal{P} - \{p, q\}$ to $I_i - \{a, b\}$ for which each propagator is supported on its image under the bijection, we can extend σ to a bijection of all propagators with I_i in two ways: either $p \mapsto a$ and $q \mapsto b$ or $p \mapsto b$ and $q \mapsto a$. The sum of the contributions of all these bijections to Δ_{I_i} is therefore the product of $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ with the minor coming from $\mathcal{P} - \{p, q\}$ and $I_i - \{a, b\}$. Since there is no cancellation of terms in the expansion of Δ_{I_i}, if any other terms appear then they will cause a factor which is not in the form described in the previous part. Therefore no such terms exist and $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ is a factor of Δ_{I_i}.

   Now let f be a quadratic factor of Δ_{I_i}. By part (2) we know that f is a 2×2 minor coming from two propagators, call them p and q, and two vertices, call them $a <_i b$. It remains to show that a and b are adjacent. From this we can conclude that p and q each have one end on (a, b), as any other way for both p and q to be supported on two consecutive vertices would contradict noncrossing or the density requirement of admissibility.

   As in the proof of part (2), make a new admissible diagram by removing the propagators which come before f and set $i = a$. The cases in the proof of part (2) show how Δ_{I_i} factors: in particular the vertices supporting the other end of p either do not appear in I_i, or they contribute to a different factor of Δ_{I_i} than p and a do. By assumption b contributes to the same factor as a. Therefore (a, b) is an edge.

4. Consider $p \in S_i$, and note that Δ_{I_i} is homogeneous linear in the variables of the row corresponding to p. By part (2), either exactly one variable in the row corresponding to p appears in Δ_{I_i} and this variable is a factor of Δ_{I_i}, or exactly two variables from the row corresponding to p appear in Δ_{I_i} and they appear as part of a quadratic factor. In the first case let the variable be x, then x is a factor of both r_i and Δ_{I_i} and is the only variable from this row in either polynomial.

   Now suppose two variables from the row p appear in a quadratic factor f. By part (3), there is another propagator q and an edge (a, b) such that f is the 2×2 minor coming from p, q and a, b, with $I_i(p) = a$, $I_i(q) = b$. There are two situations which can occur, both illustrated in Figure 10; we show that in both cases it follows that $q \in S_i$ as well.

   In both cases, since $I_{i-1}(p) \neq a$ by assumption it follows from Lemma 3.13 that $I_{i-1}(p) <_{i-1} a$ and no other vertex supporting p lies between $I_{i-1}(p)$ and a. In the case that $b <_i a$ and q is

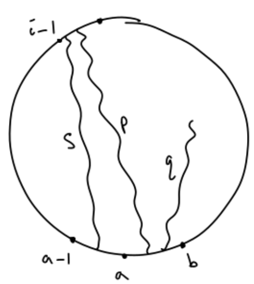


Figure 11: In order to obtain $I_{i-1}(s) = a$, propagators s and p must each have an end on the edge $(i-2, i-1)$.

taken before p in I_i, this means that $I_{i-1}(p) = b$ and so $I_{i-1}(q) \neq b$. Thus $q \in S_i$ and so f is a factor of r_i.

Now consider the case where $a <_i b$, and suppose for contradiction that $q \notin S_i$, i.e. that $I_{i-1}(q) = b$. Since $I_{i-1}(p) \neq a$, there must be some other propagator s with $I_{i-1}(s) = a$ (else I_{i-1} assigns q to a). This propagator cannot lie on edge (a, b) since by Lemma 3.13 we must have $I_i(s) = a$ or b, contradicting the fact that $I_i(p) = a$ and $I_i(q) = b$; thus s has an end on $(a-1, a)$ and is inside p from the point of view of $i-1$.

Say s is supported on $(j, j+1, a-1, a)$ and p is supported on $(k, k+1, a, b)$ with $i-1 \leq_{i-1} k+1 \leq_{i-1} j+1$. But by Lemma 3.11, if $I_{i-1}(s) = a$ then a cannot be maximal in the support of s with respect to $<_{i-1}$; thus we must have $i-1 = j+1$, and we are in the situation in figure 11.

Since p changed its association from I_{i-1} to I_i, we have $I_{i-1}(p) = i-1$ by Lemma 3.13. From figure 11 it follows that I_{i-1} assigns p to $i-1$ and then proceeds identically to I_i for all vertices inside p, implying that $I_{i-1}(s) = I_i(s)$. Since $I_{i-1}(s) = a$ and $I_i(s) \neq a$, this is a contradiction.

Thus $q \in S_i$ after all, and so again f is a factor of r_i as required.

5. If W has zero propagators then all $I_i = \emptyset$ and both R and $\prod_{i=1}^{n} \Delta_{I_i}$ are equal to 1, so the result holds in this case. Now assume W has at least one propagator.

   First we show that every factor of $\prod_{i=1}^{n} \Delta_{I_i}$ divides R. Take an irreducible factor f of $\prod_{i=1}^{n} \Delta_{I_i}$. There exists some i such that $f | \Delta_{I_i}$ but $f \nmid \Delta_{I_{i-1}}$, since otherwise the variables corresponding to the propagators contributing to f which do not themselves appear in f could never appear, contradicting Lemma 3.13. If f is a linear factor, say from associating propagator p to vertex a, then $I_i(p) = a$ and $I_{i-1}(p) \neq a$ so this factor appears in r_i. If f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively, then again we cannot have both $I_{i-1}(p) = a$ and $I_{i-1}(q) = b$, else f divides Δ_{i-1}. However, by the proof of part (4), if one of p, q belongs to S_i then the other does as well. Thus f divides r_i.

   Next we need to show that R is squarefree. Suppose $f^2 | R$. If f is a linear factor, say from associating propagator p to vertex a, then there must be two distinct points in the Grassmann necklace algorithm where p changes from not being associated to vertex a to being associated to vertex a. This contradicts Lemma 3.13. Now suppose f is a quadratic factor, say from propagators p and q supported on the edge (a,b) with p before q on the edge. In this case it is not possible for any I_i to associate p to b and q to a. Furthermore, we know by part (4) that p changes from not being associated to a to being associated to a if and only if q changes from not being associated to b to being associated to b. Thus $f^2|R$ implies that twice in the Grassmann necklace p must change from not being associated to vertex a to being associated to vertex a. This is again a contradiction, and so R is squarefree.

Taking everything together we have that $R|\prod_{i=1}^n \Delta_{I_i}$, R contains all factors of $\prod_{i=1}^n \Delta_{I_i}$ and R is squarefree. Therefore the ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^n \Delta_{I_i}$.

□

Theorem 4.4. *Given any admissible Wilson loop diagram W, let $\{I_1, \dots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral, $R(W)$ (see Definition 1.11), is a $\mathbb{Q}$-multiple of the radical of $\prod_{i=1}^n \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i. Equivalently, $R(W)$ is a $\mathbb{Q}$-multiple of the R of Algorithm 4.1.*

Proof. The equivalence of the final statement of the theorem is due to Proposition 4.3. It remains to prove that $R(W)$ is an integer multiple of the R of Algorithm 4.1.

To this end, first note that $R(W)$ and R both have total degree $4|\mathcal{P}|$; the degree of $R(W)$ is immediate from the definition while that of R follows from Lemma 3.13. By Proposition 4.3 every factor of R is either a single variable or a quadratic factor coming from two propagators supported on a common edge. The factors of each R_e making up $R(W)$ in the notation of Definition 1.11 are all of this form and hence every factor of R divides $R(W)$. Finally, since R is squarefree, this implies that $R(W)$ is a $\mathbb{Q}$-multiple of R. □

explain why this result was interesting

Is there any point in working out the $\mathbb{Q}$-coefficient, or otherwise discussing the field?

References

[1] S. Agarwala, E. M. Amat. Wilson loop diagrams and positroids. *Commun. Math. Phys.* 2016.

[2] S. Agarwala, S. Fryer. An algorithm to construct the Le diagram associated to a Grassmann necklace `arXiv:1803.01726`

[3] D. Gale. Optimal assignments in an ordered set: An application of matroid theory. *J. Combinatorial Theory*, 4: 176 - 180 (1968).

[4] S. Oh. Positroids and Schubert matroids. *J. Combin. Theory Ser. A*, 118(8):2426–2435, (2011)

[5] A. Postnikov. Total positivity, Grassmannians, and networks. `arXiv:math/0609764`.