This paper studies the combinatorics of Wilson Loop Diagrams.

1 Wilson Loop diagrams

What are Wilson loop diagrams and their integrals.

Definition 1.1. A Wilson loop diagram is given by the following data: a cyclicly ordered set V, along with a choice of first vertex (labeled 1), and k pairs, called propagators, written $\{p_r = (i_r, j_r)\}_{r=1}^k$ with labels ordered such that $i_r + 1 < j_r$ relative to the fisrt vertex.

We depict this data as a circle with marked points, called vertices. The vertices are labeled by V (preserving the cyclic ordering). The arc between consecutive vertices are called edges. There are k wavy lines in the interior of the diagram, depicting the proagators, with endpoints on the edges. A propagator, $p = (i, j)$ has one endpoint on the edge between the vertex labeled i and $i + 1$ and another endpoint on the edge defined by j and $j + 1$. The condition on i_r and j_r means that the propagator does not go between adjacent edges. Let $\mathcal{P} = \{p_r\}_{r=1}^k$ be the set of propagators. Then we write

$$W = (\mathcal{P}, V) \ .$$

Note that the marked circle gives the vertices of W a cyclic ordering. The choice of first vertex gives it a compatible linear order. Both the cyclic and the linear order become the correct perspective at various points in this paper.

Often we take V to be $[n]$, the cyclically ordered set of integers, $1 \ldots n$. In this case, we write $W = (\mathcal{P}, [n])$. We introduce some notation to speak of vertices supporting a propagator, and the set of propagators supported on a vertex set.

Definition 1.2. Let $W = (\mathcal{P}, [n])$.

1. For $p \in \mathcal{P}$, let $V(p) = \{i_p, i_p + 1, j_p, j_p + 1\}$ be the set of vertices supporting p. Then, for $P \subseteq \mathcal{P}$, the set $V(P) = \cup_{p \in P} V(p)$ is the vertex support of P.
2. For $V \subseteq [n]$, write $\text{Prop}(V) = \{p \in \mathcal{P} | V(p) \cap V \neq \emptyset\}$.
3. For $P \subseteq \mathcal{P}$, define $F(P) = V(P^c)^c$ to be the set of vertices in $[n]$ that only support propagators in the set P.

Note that $F(\emptyset)$ is the set of vertices that are not in the support of any propagators.

did we settle on a term here? I'd like to make this a definition

It is sometimes useful to discuss propagators in terms of the edges supporting them, rather than the vertices.

Definition 1.3. The i^{th} edge of W is the edge of the external polygon that lies between the vertices i and $i + 1$.

In this manner, the propagator $p = (i, j)$ is supported by the i^{th} and j^{th} edges.

Definition 1.4. A Wilson loop diagram is admissible if

1. $|V| \geq |\mathcal{P}| + 4$
2. There does not exists a set of propagators, $P \subseteq \mathcal{P}$ such that $|V(P)| < |P| + 3$.
3. There does not exist a pair of propagators, $p, q \subseteq \mathcal{P}$ such that $i_p < i_q < j_p < j_q$.

A Wilson loop diagram is weakly admissible if the second and third conditions hold.

The first conditions states that there are at least four more vertices than propagators in an admissible Wilson Loop Diagram. The second imposes an upper bound on how densely the propagators can be fitted in the diagram. The third ensures that ensures that no propagators cross in the interior of the diagram. In other words, a Wilson loop diagram, $(\mathcal{P}, [n])$ is admissible if and only if $n > \mathcal{P} + 4$, and has neither crossing propagators nor any pairs of propagators that start and end on the same pair of non-adjacent edges.

Note that if we take any admissible Wilson loop diagram and remove the unsupported vertices then we will obtain a weakly admissible Wilson loop diagram that may or may not be admissible itself.

In what follows, we will talk about admissible Wilson Loop diagrams and subdiagrams thereof.

Definition 1.5. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The weakly admissible diagram, W' is a subdiagram of W, written $W' \subseteq W$, if

$$W' = (P, V); \quad P \subseteq \mathcal{P}; \quad V(P) \subseteq V \subseteq [n] \ .$$

There is one particular type subdiagram that deserves special attention.

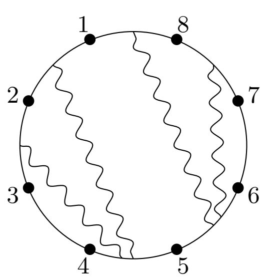
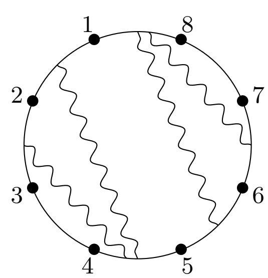
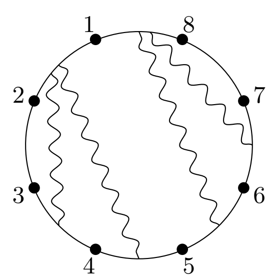
Definition 1.6. For W an admissible diagram, $(P, V(P))$ is exact if $|V(P)| = |P| + 3$.

The exact subdiagrams define an equivalence relation amongst Wilson loop diagrams.

Definition 1.7. There is an equivalence relationship on the set of admissible Wilson loops diagrams given by the transititve closure the following binary relation: $W = (\mathcal{P}, [n]) \sim W' = (\mathcal{P}', n)$ if

1. There exist two different exact subdiagrams, $(P, V(P))$ and $(P', V(P'))$ of W and W' respectively such that $V(P) = V(P')$.
2. The complementary subdiagrams are identical: $(\mathcal{P} \setminus P, V(P)^c) = (\mathcal{P}' \setminus P', V(P')^c)$.

Example 1.8. Note that since this is an equivalence relation, we may find that two Wilson loop diagrams are equivalent, even if they do not have complements of (non-trivial) exact subdiagrams in common. Consider the following three Wilson loop diagrams,

$W_1 =$  $W_2 =$  $W_3 =$  .

The diagrams $W_1 \sim W_2$ because $(\{(5,8),(5,7)\},\{5,6,7,8,1\})$ and $(\{(5,8),(7,8)\},\{5,6,7,8,1\})$ are the corresponding differing subdiagrams. Furthermore, there is the equivalence $W_2 \sim W_3$ due to the exact subdiagrams $(\{(1,4),(3,4)\},\{1,2,3,4,5\})$ and $(\{(1,4),(1,3)\},\{1,2,3,4,5\})$. This forces an equivalence between W_1 and W_3, even though one cannot partition the propagators of each into a an exact subdiagram (that may vary between the diagrams) and a complement that is fixed.

Each Wilson loop diagram, $W = (\mathcal{P}, [n])$ with $|\mathcal{P}| = k$ is associated to a $k \times n$ matrix with non-zero real variable entries, called $C(W)$:

$$C(W)_{p,q} = \begin{cases} c_{p,q} & \text{if } q \in V(p) \\ 0 & \text{if } q \notin V(p) \end{cases} . \tag{1}$$

Example 1.9. For example, ordering the propagators of W_1 from Example 1.8:

$$(1,4),\ (2,4),\ (5,7),\ (5,8)$$

we may write

$$C(W_1) = \begin{pmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,4} & c_{1,5} & 0 & 0 & 0 \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,7} & c_{3,8} \\ c_{4,1} & 0 & 0 & 0 & c_{4,5} & c_{4,6} & 0 & c_{4,8} \end{pmatrix} .$$

These $C(W)$ parametrize a subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ as shown in [**?**], call it $\Sigma(W)$. The Wilson loop diagrams also define a volume form on $\Sigma(W)$:

$$\Omega(W) = \frac{\prod_{r=1}^{|\mathcal{P}|} \prod_{v \in V_{p_r}} \mathrm{d}c_{p_r}}{R(W)} .$$

The denominator $R(W)$ is a polynomial defined by 2×2 and 1×1 minors of $C(W)$ as defined below.

Definition 1.10. For $W = (\mathcal{P}, [n])$, $R(W) = \prod_{e=1}^{n} R_e$, with R_e defined by the propagators ending on it. For any edge e of W, order the propagators incident on e as $\{p_1 \dots p_r\}$, ordered such that p_1 is closest to the vertex e, p_r closest to $e+1$, and p_i is closer to e than p_{i+1}. Then

$$R_e = c_{p_1,e+1} \prod_{j=1} (r-1) \left((c_{p_j,e} c_{p_{j+1},e+1} - c_{p_{j+1},e} c_{p_j,e+1}) \right) c_{p_s,e} .$$

Note that in this notation, if $r = 1$, $R_e = c_{p,e} c_{p,e+1}$.

2 Equivalence classes of Wilson loop diagrams

In [**?**], Agarwala and Amat show that Wilson loop diagrams can be interpreted as positroids, a certain well behaved class of realizable matroids (this correspondence is stated precisely in Theorem

2.1 below). This opens up the study of Wilson loop diagrams to techniques from geometry and combinatorics.

[outline of section, postponed until section structure is finalised]

The main results of this section are the following: we show that two admissible Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24), and we obtain a formula for the number of admissible Wilson loop diagrams in each equivalence class (Corollary 2.25). [sentence about why this matters]

2.1 Wilson loop diagrams as matroids

We first give a quick summary of the matroid terminology that we will need; it is not intended as a comprehensive introduction to matroids and the interested reader is referred to [ref][find a good matroid reference].

A *matroid* $M = (E, \mathcal{B})$ consists of a finite ground set E and a non-empty family $\mathcal{B} \subseteq \mathcal{P}(E)$ whose elements satisfy the *basis exchange property*: for any distinct $B_1, B_2 \in \mathcal{B}$ and any $a \in B_1 \setminus B_2$, there exists some $b \in B_2 \setminus B_1$ such that $(B_1 \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ as well. The elements of $\mathcal{B}$ are called the *bases* of the matroid. Note that the basis exchange property immediately implies that all bases have the same size.

A subset $A \subseteq E$ is called *independent* in M if $A \subseteq B$ for some $B \in \mathcal{B}$, and *dependent* else. The *rank* $\mathrm{rk}\ (A)$ of a subset $A \subseteq E$ is the size of the largest independent set contained in A. The rank of the matroid itself is defined to be $\mathrm{rk}\ (E)$.

A *circuit* in M is a minimally dependent set. That is, it is a set $C \subseteq E$ such that C is dependent but $C \setminus \{e\}$ is independent for any $e \in C$. A union of circuits is called a *cycle*. On the other hand, a *flat* is a maximally dependent set, i.e. a set $F \subseteq E$ such that $\mathrm{rk}\ (F \cup \{e\}) = \mathrm{rk}\ (F) + 1$ for any $e \in E \setminus F$. Unsurprisingly, a *cyclic flat* is a set which is both a flat and a cycle. The set of circuits in a matroid uniquely defines that matroid, as does the set of flats; thus one could specify a matroid by listing its independent sets, bases, circuits, or flats. [ref]

Finally, we describe several important types of matroids. A matroid of rank k with a ground set of size n is called *realizable* if there exists some $A \in Gr(k, n)$ whose non-zero $k \times k$ minors are exactly those with columns indexed by elements of $\mathcal{B}$. A *positroid* is a matroid which can be realized by an element of the totally nonnegative Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Finally, a *uniform matroid* of rank r is a matroid in which any set of size $\leq r$ is independent.

Matroid theory relates to the study of Wilson loop diagrams as follows. In [**?**], Agarwala and Amat show that every admissible Wilson loop diagram with k propagators defines a positroid of rank k, and that the independent sets can be read directly from the diagram:

Theorem 2.1. *[**?**, Theorem 3.6] Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ defines a matroid $M(W)$ with ground set $[n]$. The independent sets are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| < |U|$.*

In other words, the independent sets of $M(W)$ correspond to the sets of vertices in W such that no subset supports fewer propagators than the vertices it contains.

Throughout, we take the *matroid defined by* W to be the matroid $M(W)$ of Theorem 2.1. Note that since vertices of the diagram W correspond to columns of the associated matrix $C(W)$, $M(W)$ can also be thought of as the matroid realized by $C(W)$.

Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $M(W)$ its associated matroid. Where it will not cause confusion we conflate the two objects, identifying vertices of W with elements of the ground set $[n]$ in $M(W)$.

In particular, this allows us to prove results about $M(W)$ by considering the behavior of propagators in W. We record a few elementary facts about the rank and cycles of $M(W)$ here as an example of this.

Lemma 2.2. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. Then:*

1. *The rank of a set $V \subseteq [n]$ is bounded above by $\min\{|V|, |Prop(V)|\}$, with $rk\ (V) = |V|$ if and only if V is an independent set.*
2. *If $C \subseteq [n]$ is a cycle, then $rk\ (C) = |Prop(C)|$.*
3. *If $[n]$ can be partitioned into at least two non-empty sets, each of which support different sets of propagators that form a partition of the propagotor set,*

$$[n] = \sqcup_i V(P_i) \quad s.t. \quad \sqcup P_i = \mathcal{P}; \quad V(P_i) \cap V(P_j) = \emptyset; \quad P_i \cap P_j = \emptyset \ ,$$

then the matroid $M(W)$ is seperable,

$$M(W) = \bigoplus_i M(P_i, V(P_i)) \ .$$

Proof. The first part of (1) is [**?**, Equation (9)] and surrounding discussion, and the second part is standard matroid theory. (2) is [**?**, Lemma 3.27]. (3) is a direct consequence of [**?**, Lemma 3.20] and the fact that $F(P_1)^c = V(P_1^c)$ □

Note that this means that the set $F(\emptyset)$ is the maximal subset of vertices of W of rank 0. That is, it is the unique flat of rank 0 in $M(W)$.

I'm sprinkling comments about $F(\emptyset)$ throughout until we can come up with a better name than unsupportive vertices.

2.2 Polygon partitions of Wilson loop diagrams

The equivalence relation on Wilson loop diagrams is defined in terms of exact subdiagrams; thus in order to understand the equivalence, we need a way to extract and compare exact subdiagrams. We do this via the notion of a polygon partition of W.

Definition 2.3. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The *polygon partition* associated to W, denoted $\tau(W)$, is defined as follows.

- The vertices of $\tau(W)$ correspond to the edges of W.

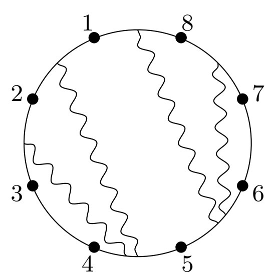
- Labeling the vertices of $\tau(W)$ with the edge number of W gives a cyclic order to the vertices. Connecting consecutive vertices gives a graph theoretic cycle called the polygon of $\tau(W)$.
- Each propagator of W defines a chord edge of $\tau(W)$; specifically, a propagator $(i, j) \in \mathcal{P}$ defines a chord connecting the vertices i and j in $\tau(W)$.

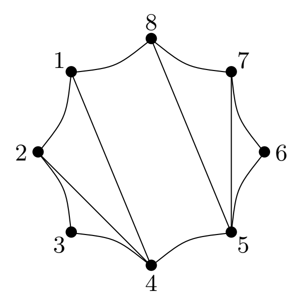
Lemma 2.4. *If $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram, then $\tau(W)$ is a simple planar graph whose outer face is a cycle. It is embedded such that the vertices all lie on this infinite face*[1]*. These vertices are cyclically ordered, with a choice of first vertex giving it an additional compatible linear order.*

Proof. Since the vertices of $\tau(W)$ are labeled by the edges of W, which are cyclically ordered, this gives an ordering to the vertices. Furthermore, since the outer circle of W is a cycle, the outer face of $\tau(W)$ is also a cycle. Since W is admissible, no pairs of propagators cross. Therefore, it is a planar embedding. Similarly, W does not admit any propagators of the form $p = (i, i+1)$; therefore there is exactly one edge connecting any two adjacent edges of $\tau(W)$. Finally, there does not exist two propagators p, q such that both p and q start at edge i and end at edge j. Therefore, no other two vertices of $\tau(W)$ can be connected by more than one edge. Finally, the embedding of $\tau(W)$ is induced from the embedding of the graph W. □

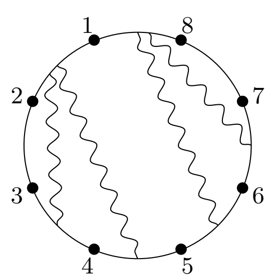
sounds awkward in the new notation

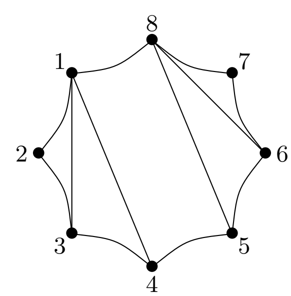
Example 2.5. In this example we return to two of the Wilson loop diagrams in Example 1.8. We can pair diagrams with their polygon partitions as follows:

$W_1 =$ 

; $\tau(W_1) =$ 

and

$W_3 =$ 

; $\tau(W_3) =$ 

.

Recall that a planar embedding of a graph is a *triangulation* if all faces, except possibly the infinite face, are triangles.

[1] That is, it is an *outerplanar* graph

Definition 2.6. Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. A *triangulated piece* of $\tau(W)$ is a 2-connected subgraph of $\tau(W)$ which is a triangulation. We will take the convention that a subgraph consisting of a single chord edge is called a *trivial* triangulated piece. A *maximal* triangulated piece is one which is not contained in any strictly larger triangulated piece.

Definition 2.7. A *decomposition* of a polygon partition $\tau(W)$ is a set of 2-connected induced subgraphs of $\tau(W)$ which partition the edges of $\tau(W)$.

Example 2.8. For the Wilson loop diagrams and polygon partitions in Example 2.5, the vertex sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ give maximal triangulated pieces for both $\tau(W_1)$ and $\tau(W_3)$. The vertex set $\{4, 5, 8, 1\}$ is not a triangulation in either polygon partition.

Lemma 2.9. *For W an admissible Wilson loop diagram, the polygon partition $\tau(W)$ has a unique decomposition into maximal triangulated pieces, and edges in the polygon of $\tau(W)$.*

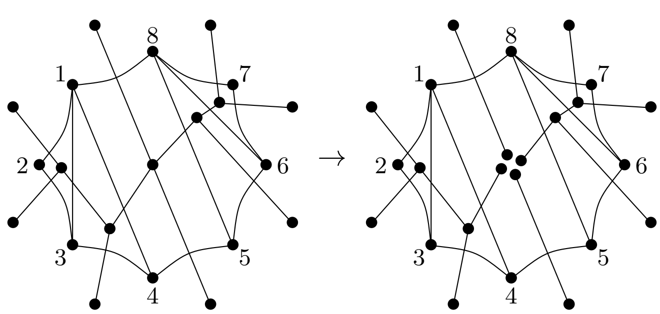
Proof. We begin by giving an algorithm for the decomposition, then prove its uniqueness. Let $W = (\mathcal{P}, [n])$, with $|\mathcal{P}| = k$.

By *splitting* a vertex v we will mean replacing v by new vertice $v_1, v_2, \ldots, v_{\deg(v)}$ such that each v_i has exactly one neighbour and the union of the of the v_i is the neighbourhood[2] of v.

Let $T(W)$ be the dual graph of $\tau(W)$ with the vertex corresponding to the infinite face split. Since $\tau(W)$ is an embedded graph (with a fixed distinguished embedding) by Lemma 2.4, $T(W)$ is an uniquely defined graph.

Furthermore $T(W)$ is a tree because it is connected, has $n + k + 1$ vertices ($k + 1$ from the internal faces of $\tau(W)$ and n from the outer face) and $n+k$ edges (since $\tau(W)$ has $n+k$ edges). Additionally, since $\tau(W)$ is simple, $T(W)$ has no vertices of degree 2.

Split every vertex of $T(W)$ which has degree > 3. The connected components of $T(W)$ correspond to the decomposition of $\tau(W)$ into maximal triangulated pieces and edges originally in the polygon of $\tau(W)$. Let f be the forest thus obtained. The vertices of f either have degree 1 or 3. Trees of f with no trivalent vertices correspond to either edges in the polygon of $\tau(W)$, if they were originally leaves of $T(W)$, or to maximal trivial triangulated pieces. Splitting at all the faces that are not triangles ensures maximality of the decomposition. If the splitting were not maximal, then one could add a triangle to a connected component of the splitting, but this would imply that that splitting happened at a valence 3 vertex.



[2] The neighbourhood of a vertex is the set of adjacent vertices

To see uniqueness, consider a different maximal decomposition of $\tau(W)$. This induces a splitting on $T(W)$, where each connected component of the new decomposition corresponds to a subtree. Call this forest f'. Since $f' \neq f$, there are two trees, t and t' in f and f' that are distinct, but share at least one edge of $T(W)$. Since f' is also maximal, t' is not a subtree of t. Therefore, the edges of t' can be found in at least two trees in the forest f. In particular, there is a vertex v in t' that corresponds to a split vertex of $T(W)$ in the original decomposition. This implies that v has valence greater than 3 in $T(W)$, and thus the corresponding face of $\tau(W)$ is not a triangle. In other words, the decomposition corresponding to f' is not a triangulation. $\square$

Corollary 2.10. *Given a maximal decomposition of $\tau(W)$, the maximal triangulated pieces are edge disjoint.*

Proof. Consider any two distinct maximal triangulated pieces of $\tau(W)$. These two pieces correspond to subtrees of $T(W)$ and intersect, at most, at a vertex in the interior of $\tau(W)$. Since the subtrees corresponding to the maximal triangulated pieces are edge disjoint, and the edges of $T(W)$ correspond to the edges of $\tau(W)$, this forces the maximal triangulated pieces to be edge disjoint as well. $\square$

We are now in a position to relate the triangulated pieces of $\tau(W)$ to exact subdiagrams of W.

Every triangulated piece t of an admissible Wilson loop diagram W corresponds to a subdiagram of W by taking the set of propagators P corresponding to edges of t and then taking the subdiagram $(P, V(P))$. Conversely, given a subdiagram $(P, V(P))$ of W we can obtain a subgraph of $\tau(W)$ as follows:

- The vertex set of the subgraph is
  - the vertices of $\tau(W)$ corresponding to edges on which propagators of P are supported
  - along with any other vertices of $\tau(W)$ for which both neighbours are as described in the previous point.
- The edge set of the subgraph is
  - the edges of $\tau(W)$ corresponding to propagators of P
  - along with the outer edges of $\tau(W)$ for which both their end points are in the vertex set

Note that the subgraph only depends on P, not the vertex set of the subdiagram, but it is not the subgraph of $\tau(W)$ consisting only of edges corresponding to propagators of P.

Lemma 2.11. *Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. The triangulated pieces of $\tau(W)$ correspond to the exact subdiagrams of W using the correspondence described above.*

Proof. Let us first record a few standard facts about polygon triangulations (that is, about triangulations with all vertices on the outer face). If such a triangulation has n vertices then it has n edges on the polygon (that is, on the outer face) and $n - 3$ edges which are not. No planar graph with the same vertices and the same outer face can have more edges than the triangulation, and every such simple graph with $n-3$ edges off the outer face is a triangulation.

Since W is admissible, by Lemma 2.4 $\tau(W)$ is a simple graph. Let t be a triangulated piece of the decomposition of $\tau(W)$ given in Lemma 2.9; note that t cannot be equal to $\tau(W)$ by the definition of admissible diagrams.

If t has 2 vertices then t corresponds to a propagator that connects two non-adjacent edges. Therefore, the trivial triangulation is a trivial exact subdiagram.

Now suppose that t has $m > 2$ vertices. We count how many edges of t are not on the outer face of $\tau(W)$. Consider the intersection of t with the outer face of $\tau(W)$: this is a possibly disconnected subgraph of the polygon of $\tau(W)$ and this subgraph has m vertices. Call this new subgraph S, and let j be the number of connected components of S. To join the components of S into the outer face of t, t must have j edges in its outer face which are not in the outer face of $\tau(W)$. Furthermore t has $m-3$ edges not in its outer face and so also not in the outer face of $\tau(W)$. Thus there are $m-3+j$ edges of t not in the outer face of $\tau(W)$.

Each of these $m-3+j$ internal edges corresponds to a propagator in W; call this set of propagators P. Next we count the size of $V(P)$. Each of the m vertices in the outer face of t corresponds to an edge of W. These m edges define j connected components of the outer polygon of W. Thus the set $V(P)$ has $m+j$ vertices. In other words,

$$|V(P)| = m + j = |P| + 3 \, .$$

Thus the subdiagram $(P, V(P))$ defined by t is exact.

Conversely, suppose we have an exact subdiagram $(P, V(P))$ of $W = (\mathcal{P}, [n])$ supported on $|V(P)| = |P| + 3$ vertices, and let t be the subgraph of $\tau(W)$ corresponding to $(P, V(P))$.

Suppose $|P| = 1$. Let p be the element of P. The exactness condition on $(P, V(P))$ says that the four supporting vertices of p are distinct. If the support of p is four consecutive vertices, then the ends of the edge corresponding to p in $\tau(W)$ have a common neighbour and so t is a single triangle, hence a triangulated piece. If the support of p is not four consecutive vertices, then the vertices which are the ends of t are separated by at least two vertices along the cycle. This implies that t is a trivial triangulated piece.

Now suppose $|P| > 1$. Let $j = |P|$, $m = |V(P)|$, and suppose that the set $V(P)$ defines c disjoint cyclic intervals of $[n]$. Then t has $m - c$ vertices. If t were a triangulation, t would have $j - c$ internal edges.

The graph t has j edges that come from propagators, and $m-2c$ edges that come from the boundary polygon of $\tau(W)$. Since t has $m - c$ vertices, it has $m - c$ external edges, of which c come from propagators. Therefore, of the j edges of t that come from propagators, $j - c$ are internal to the connected component. Therefore, t is a triangulated piece.

□

To avoid the issue of exact diagrams being subdiagrams of other exact subdiagrams (for instance, any subdiagram (q, V_q), for $q \in \mathcal{P}$ is exact), we introduce the notion of maximal exact subdiagrams.

Definition 2.12. An exact subdiagram $(P, V(P))$ is a *maximal exact subdiagram* of W if there is no other exact subdiagram $(Q, V(Q))$ in W that contains $(P, V(P))$ as a strict subdiagram.

Corollary 2.13. *Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ can be uniquely decomposed into maximal exact subdiagrams. These maximal subdiagrams partition $\mathcal{P}$.*

Proof. Combining Lemmas 2.9 and 2.11 yields the unique decomposition into maximal exact subdiagrams, and Corollary 2.10 ensures that no propagator appears in more than one subdiagram in this decomposition. Since the chord edges of $\tau(W)$ correspond to the propagators of W, the decomposition of $\tau(W)$ induces a partition of $\mathcal{P}$. □

2.3 Matroid properties of exact subdiagrams

Since Corollary 2.13 allows us to decompose any admissible Wilson loop diagram into a collection of maximal exact subdiagrams, in this section we examine the matroid properties of exact subdiagrams more closely.

Definition 2.14. Let $M = (E, \mathcal{B})$ be a matroid, and $S \subseteq E$. The *contraction* of M by S is the matroid $M/S = (E \setminus S, \mathcal{B}/S)$, where

$$\mathcal{B}/S = \{B \setminus S \bigm| |B \cap S| \text{ is maximal amonst all } B \in \mathcal{B}\}.$$

In [?], Agarwala and Amat show that certain subdiagrams of W can be realized as contractions of $M(W)$:

Lemma 2.15. *[?, Theorem 3.33] Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. If the set $V(P)^c$ has rank $|P^c|$, then the matroid defined by the subdiagram $(P, V(P))$ is equal to the contraction $M(W)/V(P)^c$.*

In Lemma 2.20 below we show that every exact subdiagram satisfies the rank condition of Lemma 2.15. In order to do this, we first examine the properties of the set by which we contract.

Definition 2.16. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. The set $F(P) := V(P^c)^c$ is called the *propagator flat* of P.

Thus in Lemma 2.15 above we would contract $M(W)$ by the propagator flat $F(P^c)$ of the *complement* of P in order to study $(P, V(P))$. The justification for this notation is given by (1) of the next lemma: the set $F(P)$ consists of vertices which *only* support propagators in P (or no propagators at all).

Lemma 2.17. *Let $F(P)$ be a propagator flat as defined above. Then*

1. *$F(\emptyset)$ is exactly the set of vertices which support no propagators, while for $P \neq \emptyset$ we have*

$$F(P) = \big(V(P) \setminus V(P^c)\big) \cup F(\emptyset).$$

updated (1) to match new definition of $F(P)$;

2. *If $Q \subseteq P$ then $F(Q) \subseteq F(P)$.*

3. *$F(P)$ is a flat of $M(W)$, thus justifying the name "propagator flat".*

Proof. The proof of (1) and (2) are routine applications of the definition and are omitted.

To prove (3), we need to show that $F(P)$ is maximally dependent. If $F(P) = [n]$ then this is automatic, so suppose not and let $v \in [n] \setminus F(P)$. In other words, $v \in V(P^c)$ and so v supports some propagator $q \notin P$. Let $S \subseteq F(P)$ be an independent set of maximal size. Then $\text{Prop}(S) \subseteq P$ by (1), and no subset of S supports fewer propagators than the number of vertices it contains (this is the definition of an independent set in $M(W)$). Since v supports a new propagator $q \notin P$, the set $S \cup \{v\} \subseteq F(P) \cup \{v\}$ also satisfies this independence condition. Thus $\text{rk}\ (F(P) \cup \{v\}) = \text{rk}\ (F(P)) + 1$, as required. □

Lemma 2.18. *Let F be a flat in $M(W)$, and let $C \subseteq F$ be the union of all circuits contained in F. Then the following are true:*

1. *$C = F(Prop(C))$, i.e. C is a propagator flat.*

2. *$F \setminus C$ is an independent flat.*

Proof. (1) If F is an independent flat, then $C = \emptyset$ and the statement is trivially true. Now suppose that F is a dependent set, so C is non-empty.

Let $v \in C$. Clearly $\text{Prop}(v) \subseteq \text{Prop}(C)$, and so by Lemma 2.17 we have $F(\text{Prop}(v)) \subseteq F(\text{Prop}(C))$. Since $v \in F(\text{Prop}(v))$ by the definition of propagator flat, we have $v \in F(\text{Prop}(C))$ as required.

Now suppose there exists some $w \in F(\text{Prop}(C)) \setminus C$. Let B be an independent subset of C of maximal rank; we first show that $B \cup \{w\}$ is a dependent set in $M(W)$. Indeed, by [**?**, Proposition 3.10], there exists some $Q \subseteq \text{Prop}(B)$ with $|Q| = |B|$ and such that B is a basis in $M(Q, V(Q))$. We therefore have

$$Q \subseteq \text{Prop}(B) \subseteq \text{Prop}(C),$$

with $|\text{Prop}(C)| = \text{rk}\ (C)$ by Lemma 2.2 and $|Q| = |B| = \text{rk}\ (C)$ by the choice of B. Thus $Q = \text{Prop}(C)$, and so B is a basis in the matroid defined by $W' := (\text{Prop}(C), V(\text{Prop}(C)))$. All that remains is to note that $w \in F(\text{Prop}(C)) \subseteq V(\text{Prop}(C))$, i.e. w is a vertex in W'; thus $B \cup \{w\}$ is dependent in W', and hence also in W. This also implies that $\text{rk}\ (F \cup w) = \text{rk}\ (F)$ since $B \subseteq F$ and so since F is a flat we must have $w \in F$.

Returning our attention to $M(W)$, the dependent set $B \cup \{w\}$ must contain a circuit C', with $w \in C'$ since B was an independent set. Now $C \cup C' = C \cup \{w\} \supsetneq C$ is a cycle which, since $w \in F$, contradicts the maximality of C.

For part (2), first note that $F \setminus C$ is automatically independent as it contains no circuits.

For any $e \notin F$, we certainly have $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ since F is a flat. Now let $e \in C$, and suppose that $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C)$. This implies that $(F \setminus C) \cup \{e\}$ is dependent, and hence contains a circuit. This circuit must contain at least two elements (since all vertices support at least one propagator), but this contradicts the fact that C was the union of all circuits in F.

What if $e \in C$ supported no propagators? $\{e\}$ would then be a circuit on its own

Thus $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ for any $e \notin F \setminus C$, and hence $F \setminus C$ is a flat. □

Corollary 2.19. *If F is a flat of a Wilson loop diagram, it can be written as the disjoint union of a cyclic propagator flat and an independent flat.*

In particular, any propagator flat can be written as a union of a cyclic propagator flat and an independent flat.

Lemma 2.20. *Let $W = (\mathcal{P}, [n])$ be a Wilson loop diagram, and $P \subseteq \mathcal{P}$. Then:*

1. *If $(P, V(P))$ is an exact subdiagram in W, then $rk\ (F(P^c)) = |P^c|$.*
2. *If $(P, V(P))$ is a maximal exact subdiagram in W, then $F(P^c)$ is a cyclic flat.*

Note that $|P^c|$ is always an upper bound on the rank of $F(P^c)$ for any $P \subseteq \mathcal{P}$, since $F(P^c)$ supports at most P^c propagators. However, showing equality is far harder.

Proof. First note that if $(P, V(P))$ is exact then the admissiblity of W guarantees that $F(P^c)$ is non-empty.

Since W is admissible, we have $n \geq |\mathcal{P}| + 4$. Rewriting this as

$$|V(P)| + |F(P^c)| \geq |P| + |P^c| + 4,$$

and combining it with the fact that $|V(P)| = |P| + 3$ (from the exactness of $(P, V(P))$), we obtain

$$|F(P^c)| > |P^c| \ . \tag{2}$$

By Lemma 2.17, $F(P^c)$ is a set of vertices that supports only propagators in P^c, so in particular we have $\text{Prop}(F(P^c)) = P^c$. Equation (2) is therefore saying that $F(P^c)$ supports fewer propagators than the number of vertices it contains, i.e. $F(P^c)$ is dependent.

We address part (2) first. Suppose that $(P, V(P))$ is a maximal exact subdiagram; by Corollary 2.19 we can decompose $F(P^c)$ as

$$F(P^c) = C \sqcup S, \tag{3}$$

where C is the largest cyclic flat contained in $F(P^c)$ and S is an independent set. Note that C must be non-empty since $F(P^c)$ is non-empty and dependent. We proceed by showing that S must be empty, forcing $F(P^c)$ to be a cyclic flat.

Since $C = F(\text{Prop}(C))$ by Lemma 2.18, define $Q := \text{Prop}(C)^c$ so that we may write $C = F(Q^c) = V(Q)^c$. Note that since $C \subseteq F(P^c)$, $P \subseteq Q$. Furthermore, we have

$$|S| = \text{rk}\ (S) \leq \text{rk}\ (F(P^c)) - \text{rk}\ (C) \leq |P^c| - \text{rk}\ (C)\ , \tag{4}$$

where first equality comes from the fact that S is an independent set,and the final inequality comes from Lemma 2.2(1). Since $\text{rk}\ (C) = |\text{Prop}(C)| = |Q^c|$ by Lemma 2.2(2), we can rearrange (4) to obtain the inequality

$$|P| + |S| \leq |P| + |P^c| - \text{rk}\ (C) = |Q|\ . \tag{5}$$

Did we ever address the question of where the middle inequality comes from? Can we argue by number of propagators?

Furthermore, since $V(Q) = S \sqcup V(P)$ (by equation (3) and the definition of Q) we may write

$$|V(Q)| = |S| + |V(P)| = |S| + |P| + 3\ . \tag{6}$$

Combining this with (5) gives $|V(Q)| \leq |Q|+3$. Since W is admissible, we conclude that $(Q, V(Q))$ is exact; since $V(Q) = V(P) \sqcup S$ and $(P, V(P))$ is maximal exact, it follows that $S = \emptyset$. It now follows from (3) that $F(P^c)$ is a cyclic flat, and hence from Lemma 2.2 that

$$\text{rk }(F(P^c)) = |Prop(F(P^c))| = |P^c|.$$

Let P be a maximal exact subdiagram as above, and let $(R, V(R))$ be an exact subdiagram that is not maximal. That is, $R \subsetneq P$.

Since $R \subset P$, $V(P) = V(R) \sqcup S$. Since P and R both define exact subdiagrams, we may write $|V(P)| = |V(R)| + |P \setminus R|$, where S is a vertex set of size $|P \setminus R|$ with $(P \setminus R \subset \text{Prop}(S)$. Since $\text{rk } V(P) = \text{rk } V(R) + |P \setminus R|$, S is independent. Taking the complements, we may write

$$F(R^c) = F(P^c) \sqcup S\ .$$

Since S is an independent set of vertices of the correct size supporting the appropriate propagators, this gives $\text{rk } F(R^c) = |R^c|$.

□

In particular, any exact subdiagram $(P, V(P))$ satisfies the conditions of Lemma 2.15 and can therefore be written as a contraction of $M(W)$ by the complementary propagator flat $F(P^c)$.

Matroids coming from exact subdiagrams have an especially nice structure, as we now show. Recall from Section 2.1 that a uniform matroid of rank r is one in which all sets of size $\leq r$ are independent.

Theorem 2.21. *Let $W' := (P, V(P))$ be a subdiagram of an admissible Wilson loop diagram $W = (\mathcal{P}, [n])$. Then W' is an exact subdiagram if and only if $M(W')$ is a uniform matroid of rank $|P|$.*

Proof. It follows directly from the definitions that a matroid of rank r is uniform if and only if all circuits have rank r; we therefore focus on the circuits of $M(W')$.

We prove the following claim: W' is exact if and only if $V(P)$ contains no circuits C with $\text{rk } C < |P|$ in $(P, V(P))$. Since $\text{rk }(M(W')$ is bounded above by $|P|$, the result follows.

Suppose $C \subseteq V(P)$ is a circuit of rank $m < |P|$; by Lemma 2.2, we know that $|\text{Prop}_{W'}(C)| = m$ as well, where the subscript to Prop specifies the diagram we are working in. Observe that $\text{Prop}_{W'}(C) \subseteq P$ by definition. The set $P \setminus \text{Prop}_{W'}(C)$ is thus nonempty, and we can consider the subdiagram $W'' := (P \setminus \text{Prop}_{W'}(C), V(P \setminus \text{Prop}_{W'}(C))$. By the density condition on subdiagrams of admissible diagrams, we have

$$|V(P \setminus \text{Prop}_{W'}(C))| \geq |P \setminus \text{Prop}_{W'}(C)| + 3.$$

It is easy to verify that $V(P \setminus \text{Prop}_{W'}(C)) \subseteq V(P) \setminus C$; since $C \subseteq V(P)$ and $\text{Prop}_{W'}(C) \subseteq P$, we can therefore rewrite the previous inequality as

$$|V(P)| - (m+1) \geq |V(P \setminus \text{Prop}(C))| \geq |P| - m + 3.$$

Simplifying, we obtain $|V(P)| \geq |P| + 4$, i.e. $(P, V(P))$ is not an exact diagram.

Conversely, suppose that W' is not exact and for a contradiction suppose also that $M(W')$ is uniform of rank $|P|$. Take $p \in P$. Then $|V(P) \setminus V(p)| = |V(P)| - 4 \geq |P|$ by non-exactness. By uniformity there is an independent set of size $|P|$ in $V(P) \setminus V(p)$. This is impossible because the submatrix corresponding to this independent set has $|P|$ rows but the one corresponding to p is all 0 one of them is all 0 so it cannot be full rank. □

Before proving the main theorems of this section, we make a few observations about the geometry of matroids defined by exact diagrams.

In [**?**], the authors show that all violet Wilson loop diagrams correspond to positroids. That is, they correspond to matroids that can be represented by elements of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, |V|)$. Any positroid of rank k on n elements defines a subspace of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, namely the points which represent it. These subspaces give a CW structure on $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, with each positroid defining a cell.

Definition 2.22. Given a Wilson loop diagram $W = (\mathcal{P}, [n])$, define the positroid cell associated to a Wilson loop diagram, $\Sigma(W)$, to be the cell in the CW complex on $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ defined by the positroid $M(W)$.

With this definition in mind, we have the following corollary:

Corollary 2.23. *Let $(P, V(P))$ be an exact subdiagram of W. The matroid associated to this subdiagram corresponds to the top dimensional cell in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$.*

Proof. The unique top dimensional cell of $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ is defined by all points in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ such that all Plucker coordinates are strictly greater than 0. Since $(P, V(P))$ is an exact subdiagram, this all $|P| \times |P|$ minors are non-zero. Intersecting these with the cases with the positive Grassmannians demands that all minors be strictly positive. □

2.4 Matroids and equivalent diagrams

Now we are ready to prove the main results of this section, namely that two Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem [ref]) and a formula for the number of Wilson loop diagrams in a given equivalence class (Corollary [ref]).

Siân postponed reading this section until the earlier ones are fixed.

Theorem 2.24. *Let $W = (\mathcal{P}, [n])$ and $W' = (\mathcal{P}', [n])$ be two Wilson Loop diagrams. They define the same matroid if and only if $W \sim W'$.*

Proof. One direction has been proved in previous work, but we give a different proof here to be consistent with the method of this document.

Assume that W and W' are equivalent. Without loss of generality, write $W = (P \cup R, [n])$ and $W' = (P \cup R', [n])$, where $P \subset \mathcal{P} \cap \mathcal{P}'$ and $(R, V(R))$ and $(R', V(R'))$ are two maximally exact subdiagrams, with $R \neq R'$, but $V(R) = V(R')$. If this is not the case, one may always find a family of diagram, $\{W_i\}$ satisfying this condition and forming a transitive chain connecting W to W' in the equivalence class.

Let $U \subset V(R)$ be any subset of size $|U| = |R|$. The set U is independent in the subdiagram $(R, V(R))$, and thus in W. The complementary set $F(P)$ is a flat of maximal rank by lemma 2.20 (rk $(F(P)) = |P|$). Let $B \subset F(P)$ be a maximal indpendent set (rk $B = |P|$) in $F(P)$. Since $F(P)$ is a flat, adding any element of $V(R)$ to B increases the rank. Therefore, any basis of W can be written as $B \cup U$. However, since $F(P)$ is common to both W and W', and $V(R) = V(R')$, any any basis of W' can also be written $B \cup U$. Thus both matroids have the same bases sets, proving that they are the same.

For the converse, assume that the matroids associated to W and W' are the same: $M(W) = M(W') = M$. Let $\{(P_i, V(P_i)\}_{i=1}^{k}$ and $\{(P'_i, V(P'_i)\}_{i=1}^{l}$ be the sets of maximally exact subdiagrams of W and W'. Write $F_i = F(P_i)$ and $F'_i = F(P_i'^c)$ to be the complementary cyclic flats. By Lemma 2.21 M/F_i is a uniform matroid. Thereofore, $k = l$, and we may write $V(P_i) = V(P'_i)$.

Reorganize the vertex sets as follows:

$$\cup_{P_i \neq P'_i} V(P_i) = V(\cup_{P_i \neq P'_i} P_i) \; . \tag{7}$$

Since maximal exact subdiagrams partition $\mathcal{P}$, by Lemma 2.13, write $cup_{i=1}^{k} P_i = \cup_{i=1}^{k} P'_i = \mathcal{P}$. Then equation (7) becomes

$$\cap_{P_i \neq P'_i} (F(P_i^c))^c = F(\cup_{P_j = P'_j} P_j)^c \; .$$

These flats may, of course, be empty.

Thus, we have partitioned the vertices of W and W' into a single set that supports a union of maximal exact subdiagrams whose propagators between W and W', and the complementary propagator flats that support the propagators in common between W and W'. As mentioned above, the latter may be empty.

Define a family of Wilson loop diagrams, W_0 to W_k defined such that $W_0 = W$ and W_i is derived from W_{i-1} by replacing the propagator set P_i with P'_i. In this manner, $W' = W_k$ and $W_i \sim W_{i+1}$, making $W \sim W'$. □

Since there is a unique way to decompose W into maximal exact subdiagrams, it is logical to ask how many diagrams there are in an equivalence class. It is a classical fact the the number of triangulations of an n-gon is the $n-2$ Catalan number, namely $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. Thus we can count the number of equivalent diagrams.

Corollary 2.25. *Let W be an admissible Wilson loop diagram where the sizes of the supports of the nontrivial maximal connected exact subdiagrams are $n_1, n_2, \ldots, n_j$. Then the number of admissible Wilson loop diagrams equivalent to W (including W itself) is*

$$\prod_{i=1}^{j} \frac{1}{n_i - 1} \binom{2(n_i - 2)}{n_i - 2}$$

3 Geometry of Wilson Loop diagram

Since Wilson loop diagrams correspond to positroids, it is natural to study the subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ they define.

Talk about Grassmann Necklaces and how it defines a cell in a CW complex of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$. Also talk about how this is exactly all the non-negative matrices that represent a particular positroid.

Also talk about Le diagrams.

Throughout this section, $W = (\mathcal{P}, n)$ is an admissible Wilson loop diagram with k propagators.

3.1 Some setup results about propagators

Definition 3.1. Let p be a propagator supported on (i,j) in W. The region bounded by p and the edge of the Wilson loop diagram from i to $j+1$ is refered to as the region **inside** p, and the complement is the region **outside** p. The set of propagators inside (resp. outside) p are those lying in the region inside (resp. outside) p, excluding p itself.

Note that this depends on the order of i and j: if we view p as supported on (j,i) (for example, if we are working with the $<_j$ order on vertices) then the regions inside and outside p are swapped.

Definition 3.2. Let p be a propagator supported on (i,j). Define the *length* of p to be

right now this doesn't depend on the order of i and j, but that might matter later on?

$$\ell(p) = min\big\{|[i+1,j]|, |[j+1,i]|\big\}.$$

Remark 3.3. The following observations about propagators of short length in W are easily verified:

- If every vertex in W supports at least one propagator, then W admits at least one propagator of length 2.
- If $p = (i, i+3)$ is a propagator of length 3, then the middle vertex $i+2$ supports at most one propagator.

The following lemma establishes certain configurations of propagators that must exist in any admissible diagram with no unsupported vertices. We make use of this result in several induction proofs below.

Lemma 3.4. *Let W be a weakly admissible WLD with at least 5 vertices and in which each vertex supports at least one propagator. Then at least one of the following two things occurs.*

1. *W has a propagator of length ≤ 6 with a propagator of length 2 inside it and nothing else inside it.*
2. *There exists a pair of propagators of length 2 with the property that the first propagator is $(i, i+2)$, the second is $(j, j+2)$, no other propagator ends between vertices $i+2$ and $j+1$, and $j \in \{i+2, i+3, i+4\}$.*

Proof. Suppose first that W has a propagator of length 3, $p = (i, i+3)$. By Remark 3.3 along with the fact that every vertex of W supports at least one propagator, we have that $i+2$ supports exactly one propagator and this propagator must have length 2 by noncrossingness. This gives us an instance of configuration 1 from the statement.

Now suppose W has no propagators of length 3.

We need a bit of notation: Define the restriction of W to a region $[i, j]$ to be the subdiagram obtained by forgetting all propagators not wholly supported on vertices in the cyclic interval $[i, j]$; the vertex set remains the same as in W. Denote this diagram by $W_{[i,j]}$.

Note that if p is a propagator supported on (i, j), then we can consider two obvious restrictions of W, namely $W_{[i,j+1]}$ and $W_{[j,i+1]}$. Clearly all other propagators $q \neq p$ appear in exactly one of these subdiagrams, while p appears in both. In this case, we will say that p *bounds the regions* $[i, j + 1]$ and $[j, i + 1]$; the ordering of the start and end points allow us to identify one or the other region unambiguously.

With these observations in mind we can return to the proof.

We will inductively construct a sequence of pairs of propagators (p_r, q_r) satisfying: $\ell(p_r) = 2$, and p_r either forms part of configuration 1 or 2 from the statement, or there is a propagator q_r satisfying

- $\ell(q_r) \geq 4$.
- q_r bounds a region of W which contains strictly fewer vertices than that bounded by q_{r-1} and the restriction of W to this region does not contain any of the propagators $p_1, \ldots, p_r$.

Since W contains finitely many vertices, this must eventually terminate in one of the desired configurations.

Start by choosing a propagator p_1 of length 2 in W (which exists by Remark 3.3), and write $(j_1, j_1 + 2)$ for its support. If it is part of one of the configurations we are looking for then we are done, so suppose otherwise: By assumption p_1 is not in configuration 2, there are no propagators of length 3, and every vertex is supported. Therefore there must exist a propagator q_1 of length ≥ 4 with one end on edge j_1 or on edge $j_1 - 1$ or on edge $j_1 - 2$. This bounds a region $[i_1, k_1]$ (where $k_1 \in \{j_1 - 1, j_1, j_1 + 1\}$) which does not contain p_1.

Now suppose q_r exists by the induction hypothesis and bounds a region $[i_r, k_r]$ not containing any p_i for $i \leq r$. Consider the restricted diagram $W_r := W_{[i_r,k_r]}$; by the original hypotheses on W every vertex $l \in [i_r + 2, k_r - 2]$ (which is a non-empty interval since $\ell(q_r) \geq 4$) must have support ≥ 1, while the same is true for $l \in \{i_r, i_r + 1, k_r - 1, k_r\}$ because of q_r.

By Remark 3.3, W_r admits at least one propagator of length 2. W_r has no propagators of length 3 since W has none. Let p_{r+1} be a propagator of length 2 in W_r; if it forms part of configuration 1 or 2 then we are done, so assume otherwise.

Note we may replace q_r by any other propagator of length ≥ 4 in W_r which contains p_{r+1}; such a new q_r still satisfies all the hypotheses, and so, without loss of generality we may assume that q_r has minimal length among propagators of length ≥ 4 which contain p_{r+1}.

There are two cases to consider. The first case is that p_{r+1} has one end before $i_r + 3$ and the other after $k_r - 3$. Then since p_{r+1} has length 2, it must be that $[i_r + 3, k_r - 3]$ is at most a single edge, and so q_r has length ≤ 6. The only propagators inside q_r that do not also contain p_{r+1} must have both ends in $[i_r, i_r + 3]$ or both ends in $[k_r - 3, k_r]$ and hence be of length 2. Therefore, if q_r contains a propagator t of length ≤ 4, this propagator must also contain p_{r+1} which is impossible my minimality. Therefore, if q_r contains another propagator of length 2 we would have configuration 2 which we have already assumed does not occur. Consequently, q_r contains only p_{r+1} and so we have 1 which again we have assumed does not occur. Therefore the first case cannot occur.

The second case is that either at least one endpoint of p_{r+1} lies on an edge contained in the interval $[i_r + 3, k_r - 3]$, or both ends lie in $[i_r, i_r + 3]$ or $[k_r - 3, k_r]$; these situations all behave similarly. Up to symmetry we can write $p_{r+1} = (j_{r+1}, j_{r+1} + 2)$ where either the edge $j_{r+1} + 2$ is contained in $[i_r + 3, k_r - 3]$ or $j_{r+1} + 2 = i_r + 2$. Since p_{r+1} is not part of configuration 2 and vertex $j_{r+1} + 4$ is supported (and not by q_r since $j_{r+1} + 4 \leq k_r - 2$ in both situations), there is some propagator t with one end in $[j_{r+1} + 3, j_{r+1} + 5]$ and length ≥ 4. By noncrossingness t is contained in q_r. t does not contain p_{r+1} by minimality and so we may set $q_{r+1} = t$ to continue the induction.

The overall result, then follows by induction.

□

The picture didn't apply any longer, but perhaps a new one would be nice?

Also, the way you had it set out, you never needed a minimality argument; this was essentially beacause in your third picture, you could choose the side which didn't contain p_{r+1}. The same argument could work here, but I was finding it was getting a bit heavy to explain it that way, and the minimality approach seemed like it was lighter. See what you think

Remark 3.5. In the case that all vertices of an admissible WLD W support at least two propagators, then Lemma 3.4 substantially simplifies. By Remark 3.3, W has no propagators of length 3. Configuration 1 necessarily entails vertices with support 1 as does configuration 2 unless $j = i + 2$. So in the case that W has all vertices with support at least two then W must contain a pair of propagators of length 2 with the property that the first propagator is $(i, i + 2)$, the second is $(i + 2, i + 4)$ and no other propagator ends on the edge $i + 2$.

3.2 From Wilson Loop diagrams to Grassmann Necklaces

Here, we give an algorithm for passing from Wilson loop diagrams to Grassmann Necklaces.

Let $\binom{[n]}{k}$ be the set of all k-subsets of the cyclically ordered set $[n]$. For each $j \in [n]$, we can define a total order $\leq_j$ on the interval $[n]$ by

$$j <_j j + 1 <_j \cdots <_j n <_j 1 \cdots <_j j - 1 \,.$$

This in turn induces a total order on $\binom{[n]}{k}$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\binom{[n]}{k}$ (the *Gale order*), which is defined as follows: if $A = [a_1 <_j a_2 <_j \cdots <_j a_k]$, $B = [b_1 <_j b_2 <_j \cdots <_j b_k] \in \binom{[n]}{k}$, then

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\binom{[6]}{3}$ we have $[2, 5, 6] \preccurlyeq_2 [2, 6, 1]$ but $[2, 5, 6] \not\preccurlyeq_2 [3, 4, 6]$.

Definition 3.6. A Grassmann necklace of type (k, n) is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of n elements $I_i \in \binom{[n]}{k}$, such that

- if $i \in I_i$, then $I_{i+1} = \big(I_i \backslash \{i\}\big) \cup \{j\}$ for some $j \in [\![1, n]\!]$.
- if $i \notin I_i$, then $I_{i+1} = I_i$.

By [ref], the Grassmann necklaces of type (k,n) are in 1-1 correspondence with the positroid cells in $Gr(k,n)^{tnn}$. Further, if $\mathcal{I}$ is the Grassmann necklace associated to a cell $\mathcal{P}$, then the bases of $\mathcal{P}$ can be computed using the Gale order $\preccurlyeq_i$ for each $i \in [n]$:

$$\mathcal{B}(\mathcal{P}) = \left\{ J \in \binom{[n]}{k} \;:\; I_i \preccurlyeq_i J \; \forall i \in [n] \right\}.$$

We now describe an algorithm that, when applied to an admissible Wilson loop diagram, produces exactly the Grassmann necklace of the corresponding positroid.

Algorithm 3.7. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. This gives an algorithm for calculating the set I_a, for $a \in [n]$.*

1. *Fix a vertex $a \in [n]$. Set $i := a$ and $I_a = \emptyset$.*
2. *While $\mathcal{P} \neq \emptyset$, perform the following steps.*
   (a) ***Step** i **for vertex** a: If Prop$(i) \neq \emptyset$ in W, write $I_a = I_a \cup i$. Let $p \in$ Prop(p) be the clockwise most propagator supported on i. Write $W = (\mathcal{P} \setminus p, n)$.*
   (b) *If Prop$(i) = \emptyset$ do nothing.*
   (c) *Increment i by 1 and repeat from (a).*

If the propagator p is removed at vertex i according to the algorithm for I_a, we say that p *contributes* i to I_a. Notationally, we represent this by allowing the I_i symbol to represent a function as well as a set, as follows:

Definition 3.8. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. For each $a \in [n]$, define a function $I_a : \mathcal{P} \longrightarrow [n]$ by

$$I_a(p) := \text{the vertex label that } p \text{ contributes to } I_a \text{ in Algorithm 3.7},$$

for each $p \in \mathcal{P}$.

Lemma 3.9. *Let W be an admissible Wilson loop diagram containing at least one propagator. For any $i \in [n]$ and for any $p = (a,b)$ with $i \leq_i a <_i b$, we have $I_i(p) \neq b+1$.*

Proof. Suppose for contradiction that we have $p = (a,b)$ with $i \leq_i a <_i b$ and $I_i(p) = b+1$. We may choose p such that $|[\![a+1,b]\!]|$ is minimal amongst propagators with this property.

Since $I_i(p) \neq b$, there must exist a propagator q inside of p with $I_i(q) = b$. The propagator q cannot end on the edge $(b-1,b)$, as this would contradict the minimality of p, so $q = (c, c+1, b, b+1)$ with $a <_i c <_i b$, and $I_i(q) = b$.

In order for q to remain unassigned until vertex b, there must be another propagator r with an end on $(c, c+1)$ and $I_i(r) = c+1$; the only way this can occur is if r is outside q but inside p. Now r contributes its fourth vertex to I_i, again contradicting the minimality of p. ☐

Corollary 3.10. *If W is an admissible Wilson loop diagram with k propagators, then Algorithm 3.7 assigns exactly k vertices to each I_i.*

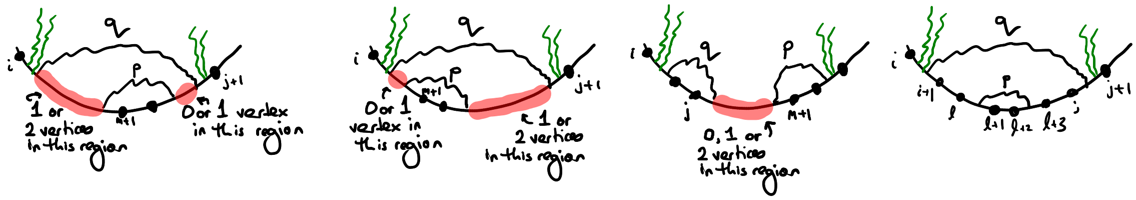


Figure 1: Four cases for admissible WLDs with no unsupported vertices. The green half-propagators illustrate where propagators may occur, but are not required to exist; no other regions illustrated may support additional propagators.

Proof. It follows from the proof of Lemma 3.9 that Algorithm 3.7 can never reach the fourth vertex of a propagator's support (with respect to the starting vertex). Therefore if the algorithm starts at vertex i, it must have assigned vertices to all propagators by the time it reaches $i-1$, ensuring that I_i contains exactly k distinct vertices. □

Given a propagator p and a vertex i in its support, it will be very useful in the following to understand on what set of vertices the Grassmann necklace algorithm assigns i to p. The answer is that the set is a non-empty cyclic interval. Lemma 3.11 establishes this, but first we need a preliminary lemma which is also useful in its own right.

Given a propagator p of an admissible WLD W and a vertex i in the support of p we will use the notation $J_p^{(W)}(i)$ for the set of indices m where $I_m(p) = l$ where the I_m are the sets calculated by Algorithm 3.7, and using the notation of Definition 3.8.

Lemma 3.11. *Let $p = (i, j)$ be a propagator of an admissible WLD W on n vertices. Then $J_p^{(W)}(i)$, $J_p^{(W)}(i+1)$, $J_p^{(W)}(j)$, and $J_p^{(W)}(j+1)$ are each non-empty cyclic intervals of $\{1, \ldots, n\}$ which partition $\{1, \ldots, n\}$ and occur in the given cyclic order.*

Proof. We will prove the result by induction on the number of propagators. If W has one propagator then the result is immediate. Now suppose W has more than one propagator. Since unsupported vertices have no effect on the Grassmann necklace algorithm, it suffices to prove the result for W with no unsupported vertices. Then by Lemma 3.4 W has at least one of the four situations illustrated in figure 1.

In each of the four cases, when we remove the propagator labelled p we obtain a diagram which satisfies the statement of the theorem by the induction hypothesis, and contains a propagator $q = (i, j)$ with no other propagators inside it (although this region may or may not contain unsupported vertices which we will call $l, l+1, \ldots$ as necessary). Call this diagram V (figure 2).

Consider the fourth case of Figure 1 first, as it is the easiest. In this case the support of p is unsupported in V so for every propagator r in V (including q) $J_r^{(V)} = J_r^{(W)}$. Additionally, $J_p^{(W)}(l+a) = \{l+a\}$ for $a \in \{1, 2, 3\}$ and $J_p^{(W)}(l) = [n] \backslash \{l+1, l+2, l+3\}$ which proves the result in this case.

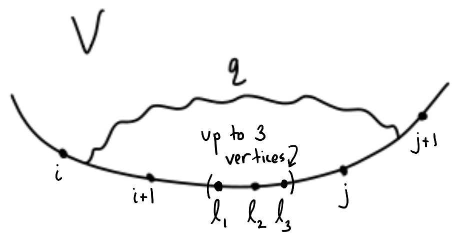


Figure 2: Diagram V is W with p removed; there are no propagators inside $q = (i, j)$, though there may be up to 4 unsupported vertices labelled $l, l+1, \ldots$.

Now we proceed to consider the first three cases of Figure 1. We first describe $J_q^{(V)}(*)$: since there are no propagators inside q, we see from the diagram that

$$l, l+1, \ldots, j \in J_q^{(V)}(j) \text{ (if } l \text{ exists) and } j+1 \in J_q^{(V)}(j+1).$$

Note that $j+2 \notin J_q^{(V)}(j+1)$ by Lemma 3.9, so $J_q^{(V)}(j+1) = \{j+1\}$ and $j+2 \in J_q^{(V)}(i)$ by the induction hypothesis. Thus there exist vertices $d, e \in [j+2, i+1]$ with $d < e$, such that

$$J_q^{(V)}(i) = [j+2, d-1], \quad J_q^{(V)}(i+1) = [d, e-1], \quad J_q^{(V)}(j) = [e, j], \quad J_q^{(V)}(j+1) = \{j+1\},$$

and all intervals are non-empty.

We now examine what happens in each of the three cases.

Left two cases: In both of these cases, p has limited effect on the Grassmann necklace of W due to the presence of at least one previously unsupported vertex l. Let $1 \leq a \leq 3$ be the number of unsupported vertices inside q in V; so these vertices are $l, \ldots, l+a-1$. Write $p = (m, m+2)$ where $m \in \{i, i+1, l\}$. Note that $l, \ldots, l+a-1$ are all in the support of p. We first calculate $I_w^{(W)}$ for a starting vertex $w \in [n] \backslash \{l, l+1, \ldots, j, j+1\}$: if $w \in J_q^{(V)}(i)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{i+1, m\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(i+1)$ or $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} l & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

For the majority of the remaining vertices, we use the following observation: if p is the first propagator to contribute to a Grassmann necklace term $I_w^{(W)}$, then the remainder of $I_w^{(W)}$ proceeds identically to the assignments of $I_{w+1}^{(V)}$. Thus in both cases, we have for any $0 \leq b < a$

$$I_{l+b}^{(W)}(r) = \begin{cases} l+b & \text{if } r = p \\ I_{l+b}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

If j is in the support of p which occurs in all cases except the second case with two vertices in the right hand region then

$$I_j^{(W)}(r) = \begin{cases} j & \text{if } r = p \text{ and } j \text{ is in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is in the support of } p \\ I_j^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is not in the support of } p \end{cases}$$

If j is not in the support of p, then we must be in the second case with two vertices in the right hand region. In this case if we start at j we need to know whether propagators other than p will remain around i and $i+1$ when we reach vertex i in the Grassmann necklace algorithm so as to know what p contributes. Consider the WLD X formed from V by moving the second end of q to the edge $j-1$ instead of j. X is still admissible since we have not decreased the support of any set of propagators, so the induction hypothesis applies to it as well. Note that $J_j^{(V)}(r) = J_j^{(X)}(r)$ for all $r \neq p$ and $J_j^{(X)}(r) = J_{j+1}^{(X)}(r)$ for $r \neq q, p$. Additionally $I_{j+1}^{(X)}(q) = i$ by the induction hypothesis applied to X, and so starting at $j+1$ and assigning propagators to vertices according to the Grassmann necklace algorithm, when we get to the vertex i in X all propagators other than q must have been assigned. Starting then at j in W we first assign q to j, then proceed to assign as in X starting at $j+1$ and then when we get to i the only propagator left to assign is j. Therefore

$$I_j^{(W)}(p) = m \text{ if } j \text{ is not in the support of } p$$

For the left two cases of Figure 1 it remains to consider the starting vertex $j+1$. If $j+1$ is in the support of p then we can argue as above to get

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = p \text{ and } j+1 \text{ is in the support of } p \\ I_{j+2}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is in the support of } p. \end{cases}$$

Now suppose $j+1$ is not in the support of p. If we start at $j+1$ we need to know whether any other propagators remain on edge i when we reach vertex i. We already know that $J_q^{(V)}(j+1) = \{j+1\}$; in particular this means that q contributes i in $I_{j+2}^{(V)}$. However the construction of $I_{j+1}^{(V)}$ first associates q to $j+1$ and then proceeds identically to $I_{j+2}^{(V)}$. In particular if i was assigned in $I_{j+1}^{(V)}$, then it would not be available to assign to q in $I_{j+2}^{(V)}$ as all other propagators supported at i in V come before q. Therefore p is the only potentially unassigned propagator on edge i when we reach vertex i in W, and

$$I_{j+1}^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j+1 \text{ is not in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is not in the support of } p \end{cases}$$

Taking all the possibilities for the left two cases of Figure 1, the intervals $J_r^{(W)}(*)$ are still cyclic for all $r \neq p$, and we can assemble the intervals for the $J_p^{(W)}(*)$ as follows.

- If $m = l$ then either $a = 2$ (so $l+1 = m+1$, $j = m+2$, and $j+1 = m+3$) or $a = 3$ (so $l+1 = m+1$, $l+2 = m+2$, $j = m+3$, and $j+1$ is not in the support of p), and in both cases

  $J_p^{(W)}(m) = [m+4, m],\ J_p^{(W)}(m+1) = \{m+1\},\ J_p^{(W)}(m+2) = \{m+2\},\ J_p^{(W)}(m+3) = \{m+3\},$

which are nonempty and otherwise as required.

- If $m = i+1$ then checking each of the three different possibilities for a we likewise get

  $J_p^{(W)}(m) = [m{+}4, d{-}1],\ J_p^{(W)}(m{+}1) = [d, m{+}1],\ J_p^{(W)}(m{+}2) = \{m{+}2\},\ J_p^{(W)}(m{+}3) = \{m{+}3\},$

  which are nonempty and otherwise as required.

- If $m = i$ then $a = 1$ or $a = 2$, in the former case $l = m+2$, $j = m+3$ and $j+1$ is not in the support of p so

  $J_p^{(W)}(m) = \{m{+}3\},\ J_p^{(W)}(m{+}1) = [m{+}4, d{-}1],\ J_p^{(W)}(m{+}2) = [d, m{+}1],\ J_p^{(W)}(m{+}3) = \{m{+}2\},$

  while in the latter $l = m+2$, $l+1 = m+3$, and j and $j+1$ are not in the support of p so

  $J_p^{(W)}(m) = [m{+}4, j{+}1],\ J_p^{(W)}(m{+}1) = [j{+}2, d{-}1],\ J_p^{(W)}(m{+}2) = [d, m{+}1],\ J_p^{(W)}(m{+}3) = \{m{+}3\},$

  which are again as required.

Third case: In this case there are no unsupported vertices $l, \ldots$ inside q. Again write $p = (m, m{+}2)$ where $m \in \{j, j{+}1, j{+}2\}$. For $w \in [n] \backslash \{j{+}1, m, m{+}1, m{+}2, m{+}3\}$: if $w \in J_q^{(V)}(i)$ or $w \in J_q^{(V)}(i{+}1)$ then

$$I_w^{(W)}(r) = \begin{cases} m & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{m, j+1\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Finally, for $j+1$ and the vertices in the support of p, we have

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = q \\ j+2 & \text{if } r = p \\ I_{j+3}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_m^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } q \text{ not supported on } m \\ m+1 & \text{if } r = p \text{ and } q \text{ supported on } m \\ I_m^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+1}^{(W)}(r) = \begin{cases} m+1 & \text{if } r = p \text{ and } q \text{ not supported on } m+1 \\ m+2 & \text{if } r = p \text{ and } q \text{ supported on } m+1 \\ I_{m+1}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_{m+2}^{(W)}(r) = \begin{cases} m+2 & \text{if } r = p \\ I_{m+3}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+3}^{(W)}(r) = \begin{cases} m+3 & \text{if } r = p \\ I_{m+4}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Note that $I^{(V)}_{m+2}(r) = I^{(V)}_{m+3}(r)$ for all propagators r in V, and that if $j+1 \notin \{m, m+1, m+2, m+3\}$ then $j+2$ and $j+3$ are unsupported in V so in that case $I^{(V)}_{j+2}(r) = I^{(V)}_{j+3}(r) = I^{(V)}_{j+4}(r)$ for r in V. Therefore, once again we can see that the $J^{(W)}_r(*)$ are cyclic for all r in V. Assembling the intervals for p we get, if $m = j$ then

$$J^{(W)}_p(m) = [m+4, e-1],\ J^{(W)}_p(m+1) = [e, m],\ J^{(W)}_p(m+2) = [m+1, m+2],\ J^{(W)}_p(m+3) = \{m+3\},$$

if $m = j+1$ then

$$J^{(W)}_p(m) = [m+4, m-1],\ J^{(W)}_p(m+1) = [m, m+1],\ J^{(W)}_p(m+2) = \{m+2\},\ J^{(W)}_p(m+3) = \{m+3\},$$

and if $m = j+2$ then

$$J^{(W)}_p(m) = [m+4, m],\ J^{(W)}_p(m+1) = \{m+1\},\ J^{(W)}_p(m+2) = \{m+2\},\ J^{(W)}_p(m+3) = \{m+3\}.$$

The result now follows by induction.

□

We need one more lemma before we can prove that Algorithm 3.7 does in fact give the Grassmann necklace of the positroid associated to W.

Lemma 3.12. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and let $M(W)$ be its associated matroid. A subset $J \subseteq [n]$ is an independent set of $M(W)$ if and only if there exists an injective set map $f : J \to \mathcal{P}$ with the property that for each $j \in J$ we have $j \in V(f(j))$.*

One of the most important uses of this lemma is for bases. The lemma says that a subset J of $[n]$ is a basis of $M(W)$ if and only if there is a set bijection between J and $\mathcal{P}$ with the property that for each $j \in J$ the propagator associated to j under the bijection is supported on vertex j.

Proof. Because the nonzero entries of $C(W)$ are independent indeterminants, J is an independent set is if and only if there is some choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns.

Each entry in $C(W)$ identifies a propagator by the row of the entry and a vertex by the column of the entry. The entry is nonzero if and only if the propagator is supported on that vertex.

Consequently, a choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns is equivalent to an assignment of the propagators of J to supporting vertices so that no two are assigned to the same vertex. Such an assignment of the propagators of J to supporting vertices is exactly a map f as described in the statment, hence proving the result. □

Proposition 3.13. *The sequence of k-subsets obtained by applying Algorithm 3.7 to an admissible diagram W is exactly the Grassmann necklace of the positroid associated to W; specifically if we let I_i be the set of labels on edge $(i, i+1)$ then the sequence $\mathcal{I} = (I_1, \ldots, I_n)$ is the Grassmann necklace of the positroid associated to W.*

Proof. Let I_i be the set of labels on edge $(i, i+1)$; the sequence $\mathcal{I} = (I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in \{1, \ldots, n\}$.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_{i\backslash\{i\}}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.7 at vertex i and vertex $i+1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i. Either p has one end on the edge $(i-1, i)$ or it has one end on the edge $(i, i+1)$. In both cases let $(b, b+1)$ be the edge with the other end of p.

case I: Suppose p has one end on $(i-1, i)$. Then p is not supported on $i+1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i+1$ up to $b-1$, that is $I_{i+1} \cap [\![i+1, b-1]\!] = I_i \cap [\![i+1, b-1]\!]$. Furthermore, by Lemma 3.11, in building I_{i+1}, it must be that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when at vertex b no propagator still remaining is before p. This is also true in building I_i when at b since the same propagators have been taken beforehand. Additionally $k \geq_i b+1$.

Let W' be the diagram obtained from W by removing p and all propagators under it in the sense of supported between $i+1$ and $b-1$.

By the above observations when we are in W' at b then we are in the same situation with respect to the remaining propagators as if we began at i in W and moved to b following the algorithm; the propagators we took in the latter case are exactly the ones removed to build W'. Similarly starting at $i+1$ in W and moving to $b+1$ leaves us in the same situations with respect to the remaining propagators as beginning at $b+1$ in W'. This gives the equations

$$I_i^W \cap [\![b, i-1]\!] = I_b^{W'}$$
$$I_{i+1}^W \cap [\![b+1, i-1]\!] = I_{b+1}^{W'}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{W'} \backslash \{b\}$ and $k \notin I_{b+1}^{W'}$ contradicting the minimality of (W, i, k).

case II: Suppose p has one end on $(i, i+1)$. Note that by definition p is the propagator contributing i to I_i and so it must be the first propagator in $(i, i+1)$ and hence p contributes $i+1$ to I_{i+1}. Observe that $k \geq_i i+2$ since $i+1 \in I_{i+1}$.

Let W' be the diagram obtained from W just by removing p. Then

$$I_i^W \backslash \{i\} = I_{i+1}^{W'}$$
$$I_{i+1}^W \backslash \{i+1\} = I_{i+2}^{W'}$$

since in both cases in W' we are simply taking the remaining propagators in the same way as we would have in W after taking p for the previous vertex. Since $k \neq i+1$, we have $k \in I_{i+1}^{W'} \backslash \{i+1\}$ but $k \notin I_{i+2}^{W'}$ contradicting the minimiality of (W, i, k)

We have shown that $\mathcal{I}$ is a Grassmann necklace; it remains to check that this Grassmann necklace defines the positroid $\mathcal{P}_W$ associated to W. We need to show that:

- For each $i \in [\![1, n]\!]$, I_i is a basis for $\mathcal{P}_W$.
- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $\mathcal{P}_W$.

The algorithm is pairing each $j \in I_i$ with a unique propagator supported on that vertex so by Lemma 3.12 I_i is a basis for $\mathcal{P}_W$.

Suppose we have J such that J is a basis and yet is lexicographically less than I_i with respect to $<_i$. By Lemma 3.12 there is a set bijection between J and the propagators of W such that the propagator associated to j is supported on vertex j. Choose one such bijection. For a propagator p of W write $J(p)$ for the associated j according to this bijection. Similarly write $I_i(p)$ for the vertex assigned to p by the algorithm. Since J is lexicographically smaller than I_i, the $<_i$-smallest element of the symmetric difference of J and I_i is some $j_0 \in J$, $j_0 \notin I_i$.

Let p be the propagator such that $J(p) = j_0$. Since $j_0 \notin I_i$ but p is supported at j_0, then $I_i(p) <_i j_0$. Thus the propagator p has the following property: $I_i(p) <_i J(p)$ and $I_i \cap [\![i, I_i(p)]\!] = J \cap [\![i, I_i(p)]\!]$. Call this property A.

From the previous paragraph we conclude that if we ever had a J which is lexicographically less than I_i and yet a basis of $\mathcal{P}_W$ then there exists a propagator p which has property A.

Finally, we will prove that whenever there is a propagator p which has property A then there is another propagator r which also has property A and for which $I_i(r) <_i I_i(p)$. Since I_i is finite this would lead to infinte regress and hence is impossible and thus there can be no such J which will complete the proof of the proposition.

So suppose p has property A. The same vertices in $[\![i, I_i(p)]\!]$ are in the image of the assignment from J and the image of the assignment from I_i so the same number of propagators are assigned to vertices in this interval by each assignment. However p is one of the propagators assigned to a vertex in this interval in I_i but not in J. Thus there exists a propagator q for which $J(q) \in [\![i, I_i(p)]\!]$ but $I_i(q) >_i I_i(p)$. Therefore $J(q) <_i I_i(q)$ and $I_i \cap [\![i, J(q)]\!] = J \cap [\![i, J(q)]\!]$. This is analogous to property A but with the roles of J and I_i switched. Thus by the same argument we must have a propagator r for which $I_i(r) \in [\![i, J(q)]\!]$ but $J(r) >_i J(q)$. Therefore r has property A which is what we wanted to prove.

□

In an arbitrary Grassmann necklace, it is possible for an index i to appear in no terms of the Grassmann necklace (a *loop*) or in all terms of the necklace (a *coloop*). Clearly any vertex of W that supports no propagators will yield a loop [ref][check: is this iff?], but with Theorem [ref][cyclic intervals thm] in hand we can now easily verify that admissible Wilson loop diagrams do not admit coloops.

Corollary 3.14. *Grassmann necklaces coming from admissible Wilson loop diagrams have no coloops, that is no indices a such that $a \in I_i$ for all i.*

Proof. For any $a \in [n]$, $a-1$ is maximal with respect to the $<_a$ order. Therefore there is no propagator p with $I_a(p) = a-1$ by Lemma 3.9, i.e. $a-1 \notin I_a$. $\square$

3.3 Dimension of the Wilson Loop cells

Agarwala and Marcott show that the dimension of the positroid cell defined by a Wilson loop diagram $(\mathcal{P}, [n])$ has dimension $3|\mathcal{P}|$. Their proof is geometric and elegant, but it is not easy to track the effect of a particular propagator.

The dimension of a positroid cell is the same as the number of plusses in the associated Le diagram ***cite something***, and so with Algorithm 3.7 along with the algorithm for passing from a Grassmann necklace to a Le diagram (reversing Oh's algorithm) described by some of us in [**?**], we can, at the cost of a bit of messy grunt work, give a different recursive proof of the $3|\mathcal{P}|$-dimensionality that explicitly describes the changes in the positions of the plusses in the Le diagram.

We need a number of lemmas, of roughly increasing degree of technicality.

Lemma 3.15. *Let W be an admissible Wilson loop diagram with k propagators and with a vertex i which supports no propagators. Let V be W with i removed. Then the Le diagram of W is the Le diagram of V with an extra column of all 0s inserted so as to be labelled i on its bottom in the Oh labelling of the border of the shape of the Le diagram*

I'm not sure how much I should assume here in terms of what the reader already knows

Proof. By Algorithm 3.7 the Grassmann necklace of W is obtained from the Grassmann necklace of V by duplicating the ith element of the Grassmann necklace of V (shifting indices as appropriate), and incrementing all indices greater than i in each Grassmann necklace element. Formally, if $I_1^{(V)}, \dots, I_{n-1}^{(V)}$ and $I_1^{(W)}, \dots, I_n^{(W)}$ are the Grassmann necklaces of V and W respectively then

$$I_j^{(W)} = \begin{cases} \{\ell \in I_j^{(V)} : \ell < i\} \cup \{\ell + 1 \in I_j^{(V)} : \ell \geq i\} & \text{if } j \leq i \\ \{\ell \in I_{j-1}^{(V)} : \ell < i\} \cup \{\ell + 1 \in I_{j-1}^{(V)} : \ell \geq i\} & \text{if } j > i. \end{cases}$$

Since i does not appear in the Grassmann necklace of W, in particular it is not in the first element of the necklace, and so in the Oh labelling of the boundary of the partition shape for the Le diagram (inside the rectangle $(k, n-k)$), i labels a horizontal edge or equivalently the bottom of a column of boxes. The shapes of the Le diagrams of V and W are the same except for the insertion of this column since the first Grassmann necklace elements are the same except for the incrementation of the indices $\geq i$ in the transition from the neccklace for V to the necklace for W.

By the algorithm of [**?**], since i never appears in the Grassmann necklace of W, the column labelled i is filled with all 0s. Otherwise the algorithm of [**?**] behaves identically on V and W with the plusses for the incremented indices shifted one column to the left, thus skipping over column i in the Le diagram for W. $\square$

Lemma 3.16. *If two Wilson loop diagrams differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Can I just cite some-

Karen got to here

one of you provide a quick geometric proof. It would be a horrible mess to do by hand combinatorially

Given a propagator p in a WLD D, note that p divides remaining propagators of D into two sets depending on which side of p they live.

Lemma 3.17. *Let D be a WLD with $n \geq 1$ propagators. Then there is some dihedral transformation D' of D such that there is a propagator p with the following properties.*

- *p goes between the edge i, $i+1$ and the edge $n-1$, n in D'.*
- *$i+2, \ldots, n-2$ are not covered by any propagators in D' (which is trivially true if $\{i+1, \ldots, n-2\} = \varnothing$).*
- *Either $i+1$ in D' is only covered by p, or $i+1$ is covered by exactly one other propagator q.*
- *If we are in the second case above then q goes between the edge j, $j+1$ and the edge i, $i+1$ and $j+2, \ldots, i-1$ are not covered by any propagators.*

Proof. Since $n \geq 1$ the dual tree of D has at least two vertices and so has at least two leaves. Each edge going to a leaf of the dual tree corresponds to a propagator which has no other propagators on one side of it, so there is at least one such propagator in D. There are two cases to consider.

First suppose there is a propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let this propagator be p. Rotate and reflect D as necessary so that the end of p which does not share its edge comes first, then the side of p with no other propagators, and then the other end of p which is on edge $n-1$, n. Call this WLD D'. It is a straightforward check that the properties in the statement are satisfied with $i+1$ only covered by p.

Next suppose there is no propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let C be D with all uncovered vertices removed. Suppose C contains two propagators one of length 2 and one of length 3 with the length 2 propagator on the small side of the length 3 propagator. Then we are in the case already considered because we can rotate and reflect C so that these two propagators both have one end on the edge $|C|-1$,$|C|$, and have their other ends on the edge $|C|-2$, $|C|-3$ and the edge $|C|-3$, $|C|-4$, so taking the length 2 edge as p we satisfy the properties of the statement, and this remains true in D. Thus we now assume this does not occur, so in particular every propagator of length 3 in C has no other propagators on its small side. But then the middle vertex on this small side is uncovered contradicting the construction of C. Thus C has no propagators of size 3.

The claim then is that in C there must exist two propagators as in Lemma 3.4. The proof of the claim follows by Sian's proof. We do not have the hypothesis that all vertices are covered by at least two propagators, but this is only used in the proof of Lemma 3.4 to force certain propagators (the q_i) to have length at least 4, so our this case we instead use that C has no propagators of size 3.

Now rotate and flip C so that the two propagators as in Lemma 3.4 cover $\{|C|-5, \ldots, |C|\}$ and let p be the propagator covering $\{|C|-3, \ldots, |C|\}$ and let q be the other one. Return to D with a

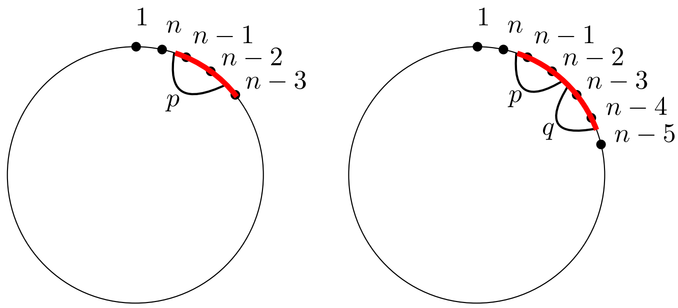


Figure 3: The different possibilities for D and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

corresponding rotation and flip so that vertex n in D corresponds to vertex $|C|$ in C and direction in the polygon is preserved. Let this dihedral transformation of D be D'. Then the propagator p satisfies the conditions of the statement with propagator q. $\square$

Given a WLD D, I_i^D denotes the ith Graßmann necklace element corresponding to D.

Lemma 3.18. *Let D be a WLD with $n \geq 1$ propagators and let p be a propagator of D so that the properties of Lemma 3.17 are satisfied for p in D (that is D is already appropriately transformed). Furthermore suppose that that p has length 2, as does q if we are in the case of Lemma 3.17 involving q. Let C be D with p removed but no change in the vertices. Then*

$$
\begin{aligned}
I_1^D &= I_1^C \cup \{n-3\} \\
I_n^D &= I_1^C \cup \{n\} \\
I_{n-1}^D &= I_n^C \cup \{n-1\} \\
I_{n-2}^D &= \begin{cases} I_{n-2}^C \cup \{n-2\} & \text{if } n-2 \notin I_{n-2}^C \\ I_{n-2}^C \cup \{n-1\} & \text{if } n-2 \in I_{n-2}^C,\ n-1 \notin I_{n-2}^C \\ (I_n^C - \{n-5\}) \cup \{n-1, n-2\} & \text{if } n-1, n-2 \in I_{n-2}^C \end{cases} \\
I_k^D &= \begin{cases} I_k^C \cup \{n-3\} & \text{if } n-3 \notin I_k^C \\ I_k^C \cup \{n-2\} & \text{if } n-3 \in I_k^C \end{cases} \\
&\text{for } 1 < k < n-2
\end{aligned}
$$

Proof. The two possible situations are illustrated in Figure 3.

If $n-3 \in I_1^C$ then $n-3 \notin I_1^D$, but then p never contributes $n-3$ contradicting Lemma 3.11. Thus $n-3 \notin I_1^C$ and so in building the Graßmann necklace when we get to vertex $n-3$ any other covering propagators of C have already been taken and so we can now take p. Therefore $I_1^D = I_1^C \cup \{n-3\}$.

In constructing I_n^D, first for vertex n we take propagator p. Then we are at vertex 1 and precisely the propagators of C remain. Thus the rest of the construction will give I_1^C. Therefore $I_n^D = I_1^C \cup \{n\}$. Essentially the same reasoning gives $I_{n-1}^D = I_n^C \cup \{n-1\}$.

Now consider I^D_{n-2}. If $n-2 \notin I^C_{n-2}$ then at vertex $n-2$ we take p and this does not affect the rest of the construction of I^C_{n-2}, so $I^D_{n-2} = I^C_{n-2} \cup \{n-2\}$. An analogous argument takes care of the first case for I^D_k. Suppose $n-2 \in I^C_{n-2}$ but $n-1 \notin I^C_{n-2}$. Then at vertex $n-2$ we take the same propagator in C as in D (in particular not p) because p is the counterclockwisemost propagator covering $n-2$ and so the last propagator the algorithm would choose at this vertex, and by hypothesis $n-2$ is covered in C. Howver, $n-1$ is untaken in I^C_{n-2} so at this vertex we will take p in I^D_{n-2}. Following this, now that p is out of the way without bumping any other propagators, the construction continues as in I^C_{n-2}. Therefore $I^D_{n-2} = I^C_{n-2} \cup \{n-1\}$. An analogous argument takes care of the case when we have $1 < k < n-2$ and $n-3 \in I^C_k$ but $n-2 \notin I^C_k$. Furthermore, either $n-2$ is uncovered in C or only q covers $n-2$ in C and q is also the only propagator covering $n-3$. Thus it is not possible for $n-3$ and $n-2$ to both be in I^C_k. This means that all the cases for I^D_k are now proved.

The remaining case is when $n-1, n-2 \in I^C_{n-2}$ for the construction of I^D_{n-2}. We must then have a propagator q as in the right hand side of Figure 3. In the construction of I^D_{n-2}, at vertex $n-2$ we take propagator q, as in I^C_{n-2}. Then at vertex $n-1$ we take propagator p which is different from what occurs in I^C_{n-2}. Next we are at vertex n and propagators p and q have been taken. Thus we are proceeding like I^C_n but without propagator q. Fortunately we can determine explicitely how propagator q contributes to I^C_n. By Lemma 3.11 propagator q contributes $n-5$ to I^C_n, and the only way this can occur is if all other propagators of C were already taken by the time we got to vertex $n-5$. Therefore $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$. This covers all cases and hence completes the proof. □

3.3.1 The number of plusses from Graßmann necklaces in nice configurations

Lemma 3.19. *Let C and D be as in Lemma 3.18. The shape of the Le diagram of C can be built from left to right of the following blocks: a rectangle with 3 columns, one more column of the same length, a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of D can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of C, the same partition shape as in C.*

Proof. I_1 determines the shape of the Le diagram. By Lemma 3.18, $I^D_1 = I^C_1 \cup \{n-3\}$. This implies that the right hand boundary of the shape of C is the same as the right hand boundary of the shape of D except that D has one additional row of 3 boxes while C has an additional column in the $n-3$ position, that is a new column fourth from the left. □

The shapes of the Le diagrams of C and D are illustrated in Figure 4. The pieces of the Le diagrams will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows, as in the figure. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of the Le diagrams of C and D are identical and the plusses in the $\mathcal{A}$ parts are very closely related. When we speak of a plus in the Le diagram of D being the same as in D or vice versa, we mean that the plus' position in $\mathcal{A}$ or $\mathcal{B}$ is the same. Because of the column insertion the absolute indices may differ.

Suppose we are following the Graßmann necklace to Le diagram algorithm, and we put a plus in a box because of a path from vertical boundary edge i to bottom boundary edge j. Then say this plus is in the $i \to j$ position.

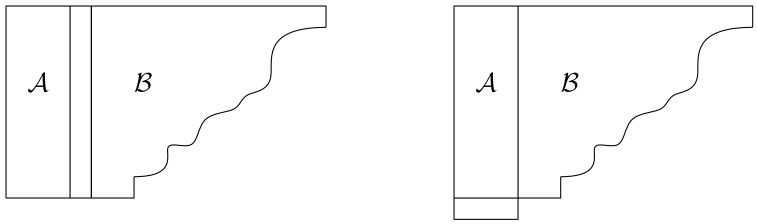


Figure 4: Le diagrams for C (left) and D (right).

Lemma 3.20. *Let C and D be as in Lemma 3.18. The I_n^D and I_{n-1}^D elements of the Graßmann necklace of D give all the same plusses as I_n^C along with plusses in the leftmost two boxes of the bottom row of the Le diagram of D.*

Proof. By Lemma 3.18 $I_n^D = I_1^C \cup \{n\}$, so by the Graßmann necklace to Le diagram algorithm the only plus this builds in the Le diagram of D is the one in the $n-3 \to n$ position, that is in the leftmost box of the bottom row.

Also by Lemma 3.18 $I_{n-1}^D = I_n^C \cup \{n-1\}$. Additionally $n-3 \notin I_n^C$ since if it were then $n-3$ would also be in I_1^C and hence propagator p could not contribute $n-3$ to I_1^D in contradiction to Lemma 3.11. Similarly $n-1, n-2 \notin I_n^C$. Thus the paths putting the plusses in from I_n^C lie completely in $\mathcal{B}$ or take some vertical boundary edge $> n-3$ to n. Now view these paths in the Le diagram of D and note that the path $n-3 \to n-1$ is compatible, and so these paths together build the plusses that I_{n-1}^D contributes. That is, we get all the plusses from I_n^C along with a $n-3 \to n-1$ plus, that is a plus in the second to the right box of the bottom row. □

Lemma 3.21. *Let C and D be as in Lemma 3.18 with $n-2 \notin I_{n-2}^C$.*

The I_{n-2}^D element of the Graßmann necklace of D gives an $n-3 \to n-2$ plus and all the $I_{n-1}^C = I_{n-2}^C$ plusses.

Proof. If $n-2 \notin I_{n-2}^C$ then $I_{n-1}^C = I_{n-2}^C$ and by Lemma 3.18 $I_{n-2}^D = I_{n-1}^C \cup \{n-2\}$. Note that $n-3 \notin I_{n-2}^C$ by Lemma 3.11. Therefore the paths for I_{n-2}^D are the paths for I_{n-1}^C along with the $n-3 \to n-2$ path. This gives the statement of the lemma.

□

Lemma 3.22. *Let C and D be as in Lemma 3.18 with $n-2, n-1 \in I_{n-2}^C$.*

The I_{n-2}^D and I_{n-3}^D elements of the Graßmann necklace of D gives the following plusses:

- *An $n-3 \to n-2$ plus and an $n-5 \to n-1$ plus.*
- *All the I_{n-1}^C plusses.*
- *I_{n-2}^C gives an $n-5 \to n-2$ plus and no other term in the Graßmann necklace of C gives a plus in this column. This $+$ does not appear in D from I_{n-2}^D but an $n-5 \to n-1$ plus does instead.*
- *All other plusses of I_{n-2}^C.*

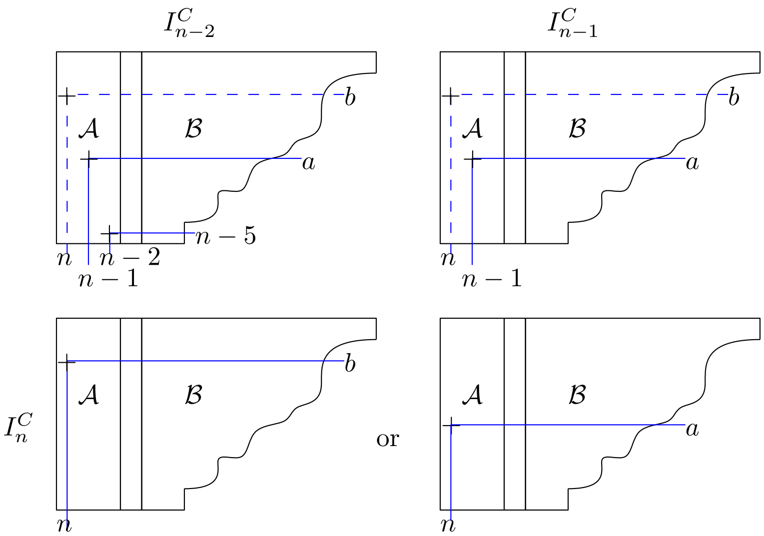


Figure 5: Plusses coming from I^C_{n-2} (top left), I^C_{n-1} (top right) and I^C_n bottom when $n-1, n-2 \in I^C_{n-1}$. The blue lines are the non-intersecting paths. The dashed blue lines may or may not appear, but if one appears then they both do.

- *I^C_{n-3} gives a plus in the $n-3$ column. This + is shifted over into the $n-2$ column in D.*
- *All other plusses of I^C_{n-3}.*

Furthermore, no element of the Graßmann necklace of C gives an $n-5 \to n-1$ plus.

Proof. By Lemma 3.18 $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$. Also, by the location of q in the WLD, $n-2 \notin I^C_{n-3}$ and and $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Graßmann necklace of C looks like

$$I^C_{n-3} \underset{\substack{n-3 \text{ out} \\ n-2 \text{ in}}}{\rightarrow} I^C_{n-2} \underset{\substack{n-2 \text{ out} \\ n-5 \text{ in}}}{\rightarrow} I^C_{n-1} \underset{\substack{n-1 \text{ out} \\ \text{something in}}}{\rightarrow} I^C_n \underset{\substack{n \text{ out} \\ \text{something in}}}{\rightarrow} I^C_1 \tag{8}$$

where the first "something" is either n or an element of I^C_1 and the second "something" is an element of I^C_1. Additionally all elements not explicitly mentioned must be in I^C_1 as they remain unchanged through this portion of the necklace.

Using this information now determine the symmetric difference of I^C_{n-2} and I^C_1: $n-1, n-2$ and possibly n are in I^C_{n-2} but not in I^C_1. $n-5$ is in I^C_1 as are at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$. This means that the plusses in the Le diagram of C coming from I^C_{n-2} are as in the first part of Figure 5. Stepping to I^C_{n-1} simply removes the $n-5 \to n-2$ path, see the second part of Figure 5.

Stepping to I^C_n, $n-1$ is taken out and either n is put in if it was not there before, or one of a or b is put in and hence no longer available as a right end for a path. This gives two possible configurations illustrated in the bottom two parts of Figure 5.

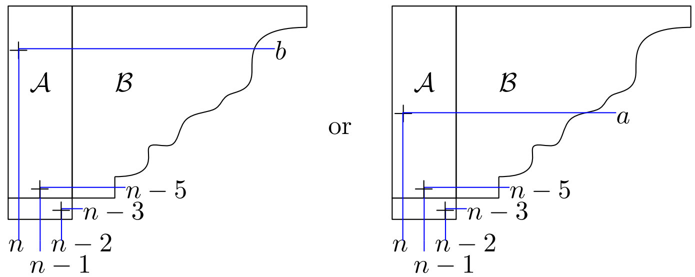


Figure 6: Plusses coming from I^D_{n-2}.

Now we know that $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$ so the paths for building plusses from I^D_{n-2} go from the set $\{n-5, n-3\}$ along with whichever of a and b is not in I^C_n to $\{n-2, n-1, n\}$. This means that we get plusses as in Figure 6 where the left and right cases correspond to the left and right cases in the bottom parts of Figure 5

This proves the first item of statement of the lemma.

Now consider I^C_{n-3}. By (8) I^C_{n-3} contributes the same plusses as $I^C_{n_2}$ except that it contributes an $n-5 \to n-3$ plus in place of the $n-5 \to n-2$ plus. Also, we have $n-3 \in I^C_{n-3}$ be the location of q and so $I^D_k = I^C_k \cup \{n-2\}$. Thus the paths for I^D_{n-3} are the same as for I^C_{n-3} except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (8) shows that $n-2$ only appears in I^C_{n-2} among the necklace elements of C.

Also note that I^C_{n-3}, I^C_{n-2}, and I^C_{n-1} share their plusses outside of the $n-3$ and $n-2$ columns. This proves the remaining statements of the lemma except the furthermore.

Finally, suppose there were a $n-5 \to n-1$ plus in the Le diagram of C. By the algorithm, it would have to come when $n-5 \notin I^C_j$. By (8) this means that it would have to come from I^C_{n-4}, I^C_{n-3}, or I^C_{n-2}. The analysis above shows it does not come from I^C_{n-3} or I^C_{n-2}. Now, $n-4 \in I^C_{n-4}$ by the location of q and so I^C_{n-4} must give an $n-5 \to n-4$ plus and so cannot give an $n-5 \to n-1$ plus.

□

Lemma 3.23. *Let C and D be as in Lemma 3.18 and take $1 < k < n-2$. Suppose that if $n-2 \in I^C_{n-2}$ then also $n-1 \in I^C_{n-2}$. The I^D_k element of the Graßmann necklace of D gives the same plusses as I^C_k except that if there is a plus in the $n-3$ column for I^C_k then this plus is shifted into the $n-2$ column and no plus was already in that location.*

Proof. If $n-3 \notin I^C_k$ then by Lemma 3.18 $I^D_k = I^C_k \cup \{n-3\}$. Then since $n-3$ is the largest element of I^D_1 this transformation leaves the disjoint paths unchanged and so the plusses carry over from C to D directly.

If $n-3 \in I^C_k$ then by Lemma 3.18 $I^D_k = I^C_k \cup \{n-2\}$. If $n-2$ is not covered in C then certainly no pluses appear in the $n-2$ column of the Le diagram of C. If $n-2$ is covered in C then by hypothesis so is $n-1$ and so we satisfy the hypotheses of Lemma 3.22. Thus the only necklace element of C containing $n-2$ is I^C_{n-2} and this particular plus is not contributed to the Le diagram of D by I^D_{n-2}.

From I_k^C there is a path from some vertical edge to the bottom edge $n-3$. In I_k^D, $n-3$ is a vertical edge with no path and instead there must be a path to $n-2$. By the previous paragraph no other path can end in $n-2$, so shifting the path that did go to $n-3$ to go to $n-2$ while leaving the others the same maintains non-crossingness and so must be the paths for I_k^D. Thus the plus in the $n-3$ column for C is shifted into the $n-2$ column, where there was no plus before, and no other plusses are changed. $\square$

Theorem 3.24. *The number of plusses in the Le diagram of a WLD is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a WLD D with one propagator covering vertices $i < j < k < \ell$ has Le diagram a single row with $|D| - i$ boxes. Labelling them from left to right by $|D|, \ldots, |D| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider WLDs with $k > 1$ propagators. By Lemma 3.15 it suffices to prove the result for WLDs with k propagators and no uncovered vertices. By Lemma 3.16 it suffices to prove the result for at least one WLD from each dihedral orbit. Take a WLD diagram D with k propagators. Make a dihedral transformation of D if necessary so that D has a propagator p with the properties in Lemma 3.17 relative to D. If $n-1$ is only covered by p but $n-2$ is covered by at least one other propagator, then flip D on the line perpendicular to the edge from $n-2$ to $n-1$. This will be our D for the rest of the proof.

Let C be D with p removed but no change in the vertices. Note that if $n-2$ is covered in C then so is $n-1$ by the end of the previous paragraph and so if $n-2$ is covered in C then the hypotheses of Lemma 3.22 are satisfied.

From Lemma 3.19 we know how the shapes of the Le diagrams of C and D relate; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 3.20, 3.21, and 3.22 tell us that the three boxes of the bottom row of the Le diagram of D each have a plus. Lemmas 3.20, 3.21, 3.22, and 3.23 show that there is a bijection between the plusses of the Le diagram of C and the plusses of the Le diagram of D that are not in the bottom row which can be described as follows.

- Plusses from $\mathcal{B}$ for C maintain their positions in $\mathcal{B}$ for D.
- Plusses from the first two columns (the n and the $n-1$ columns) of $\mathcal{A}$ for C maintain their positions in $\mathcal{A}$ for D.
- If there is a plus in the $n-2$ column of $\mathcal{A}$ in then Lemma 3.22 applies, so there is exactly one such plus. This plus maps to the $n-5 \to n-1$ plus for D.
- The plusses in the $n-3$ column for C shift over to the third column (the $n-2$ column) of $\mathcal{A}$ in D.

This map is clearly reversible and hence bijective except possibly for the $n-5 \to n-1$ plus for D. If the Le diagram of D has an $n-5 \to n-1$ plus then look at the Le diagram for C. If the Le diagram for C has a plus in the $n-2$ column then Lemma 3.22 applies and so there is no

$n-5 \to n-1$ plus in the Le diagram of C and the $n-5 \to n-1$ plus of D can be uniquely mapped to the plus in the $n-2$ column of the Le diagram of C. If the Le diagram for C has no plus in the $n-2$ column, then leave the $n-5 \to n-1$ plus where it is in moving back to C. This reverses the map.

From all of this we get that the number of plusses in the Le diagram for D is three more than the number of plusses in the Le diagram for C. Applying induction completes the proof. □

Maybe proved that inadmissible diagrams, while they correspond to matroids, can't be mapped to the positroids of the correct dimension?

4 Poles of Wilson Loop Integrals

The results of Section 3 allow us to relate the position of propagators in a Wilson loop diagram W to minors of $C(W)$, which we use in this section to understand the denominator of the integral $I(W)$ associated to a Wilson loop diagram (see Definition 1.10).

The main result of this section is Theorem 4.4, which expresses the denominator $R(W)$ in terms of the Grassmann necklace of W. This simplifies the computation of $R(W)$ and allows us to directly relate the poles of the integral to the combinatorics of the diagram.

We first give an algorithm which extracts the required minors from the Grassmann necklace.

Algorithm 4.1. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and let $C(W)$ be the matrix of W as defined in* (1) *(see Section 1).*

- *For each $i \in [\![1, n]\!]$, we construct a factor r_i as follows:*
  - *Let $S_i = \{p \in \mathcal{P} \mid I_{i-1}(p) \neq I_i(p)\}$. (By convention, set $I_{-1} = I_n$.)*
  - *Write Δ_{I_i} for the determinant of the $k \times k$ minor of $C(W)$ with columns indexed by I_i.*
  - *Let r_i be Δ_{I_i} with all variables from rows associated to $p \notin S_i$ set to 1.*
- *Define $R = \prod_{i=1}^{n} r_i$.*

What are we doing about the sign of the determinants?

Example 4.2. Consider the Wilson loop diagram in Figure 7. Assigning propagators p, q, s to rows $1, 2, 3$ respectively, we obtain the matrix

$$C(W) = \begin{bmatrix} a & b & 0 & 0 & 0 & c & d \\ e & f & 0 & 0 & g & h & 0 \\ i & j & 0 & k & l & 0 & 0 \end{bmatrix}$$

I've named the propagators p, q, s; could whoever codes up the figure please add that?

The Grassmann necklace of this diagram is

$$I_1 = \{1,2,4\}, I_2 = \{2,4,5\}, I_3 = \{4,5,6\}, I_4 = \{4,5,6\},$$
$$I_5 = \{5,6,7\}, I_6 = \{6,7,1\}, I_7 = \{7,1,2\}.$$

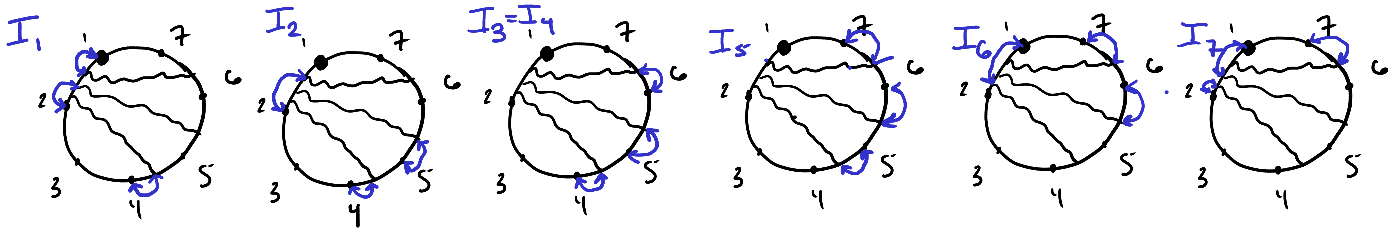


Figure 7: Example WLD for illustrating Algorithm 4.1 and bijections between propagators and vertices for each Grassmann necklace element.

Figure 7 also indicates the pairings between propagators and vertices for each $i \in [\![1,7]\!]$.

From I_1 to I_2, the propagators p and q change which vertex they are assigned to but r is assigned to vertex 4 in both, so $S_2 = \{p, q\}$. Then

$$\Delta_{I_2} = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ j & k & l \end{bmatrix} = kgb, \qquad r_2 = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ 1 & 1 & 1 \end{bmatrix} = gb.$$

where the 1s in the third row of the second matrix correspond to the fact that $I_1(s) = I_2(s)$. Continuing likewise, we get $r_3 = c$, $r_4 = 1$ (since $I_4 = I_3$), $r_5 = lhd$, and $r_6 = i$.

At I_7 the situation is more complicated: we have $S_7 = \{q, s\}$, so we find that $\Delta_{I_7} = d(ej - fi)$ and $r_7 = ej - fi$. This quadratic factor corresponds to the fact that q and s share an edge *and* contribute both endpoints of that edge to I_7; see Proposition 4.3 below.

Finally, we have $r_1 = (af - be)k$. Putting everything together, we obtain

$$R = (af - be)kgbclhdi(ej - fi)$$

which is squarefree and contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$. If one were to construct the denominator $R(W)$ associated to this Wilson loop diagram as per Definition 1.10, we would find that (up to integer multiples) we have $R(W) = R$.

Proposition 4.3. *With notation as in Algorithm 4.1 we have the following:*

1. *Each Δ_{I_i} is homogeneous, as is each r_i.*
2. *Each Δ_{I_i} splits into linear and quadratic factors. All linear factors of Δ_{I_i} are single variables and all irreducible quadratic factors are 2×2 determinants of single variables.*
3. *Quadratic factors in r_i arise precisely when propagators p and q are supported on a common edge (a, b) with $I_i(p) = a$ and $I_i(q) = b$.*
4. *r_i divides Δ_{I_i}.*
5. *The ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.*

Proof. 1. The nonzero entries of $C(W)$ are independent indeterminates and so every $i\times i$ minor of $C(W)$ is either homogeneous of degree i or is 0. Thus each Δ_{I_i} is homogeneous. Furthermore, each row contributes one factor to each term in the expansion of Δ_{I_i} so the result of setting the variables from a subset of rows to 1 is still homogeneous. Thus each r_i is homogeneous.

2. Using the expression for the determinant as a sum over permutations we see that Δ_{I_i} is a sum over bijections between I_i and $\mathcal{P}$. The nonzero terms in this sum are precisely those bijections such that each propagator is associated to one of its supporting vertices in I_i, since only those locations in $C(W)$ are nonzero. Since the nonzero entries of $C(W)$ are independent there can be no cancellation between terms in this expansion.

   Suppose Δ_{I_i} has an irreducible factor f. Let $\mathcal{P}'$ be the set of propagators which contribute a variable to f and let J be the set of vertices which contribute a variable to f.

   The first claim is that the minor of $C(W)$ associated to $\mathcal{P}'$ and J is precisely f.

   Proof of claim: By the structure of determinants we know that $\Delta_{I_i} = fg$, where g involves only variables associated to propagators not in $\mathcal{P}'$ and associated to vertices not in J.

   Expanding out fg yields a signed sum of monomials. In each of these monomials, f contributes those variables associated both to a propagator in $\mathcal{P}'$ and to a vertex in J, and g contributes those variables associated both to a propagator not in $\mathcal{P}'$ and to a vertex not in J, and no other variables appear.

   Since there is no cancellation between terms, this means that Δ_{I_i} itself contains no other variables. Therefore Δ_{I_i} is equal to the determinant of the matrix obtained by taking the submatrix of $C(W)$ with columns indexed by I_i and setting any variables not appearing in Δ_{I_i} to 0. This new matrix is, up to permutations of rows and columns, a block matrix with one block for $\mathcal{P}'$ and J and the other block for the complements. Thus its determinant, and hence also Δ_{I_i}, is the product of the minors for these two blocks. By considering which variables appear, these two factors must also be f and g, and so in particular f is the minor of $C(W)$ associated to $\mathcal{P}'$ and J.

   A consequence of this claim is that every linear factor of Δ_{I_i} is a 1×1 minor of $C(W)$, hence is a single variable, and every irreducible quadratic factor of Δ_{I_i} is a 2×2 minor of $C(W)$, hence is a 2×2 determinant of single variables.

   All that remains is to prove that Δ_{I_i} has no irreducible factors of degree 3 or more. Suppose for a contradiction that f is a factor of Δ_{I_i} of degree ≥ 3. Note that by removing the propagators which come before those contributing to f and changing i to be the first vertex which contributes to f, we obtain a different admissible diagram for which f still divides Δ_{I_i} but also $i \in I_i$ and i contributes to f. Showing that this different admissible diagram gives a contradiction is sufficient, and so we may assume that $i \in I_i$ and i contributes to f. Finally, we can suppose that W is minimal in number of propagators with the above occuring.

   would it be neater to note at the beginning that removing uncovered vertices has no effect so we assume there are no redundant vertices, and hence $i \in I_i$ for each i?

   Let p be the propagator such that $I_i(p) = i$. There are two cases to consider, depending on which edge p is supported on. These are illustrated in Figure 8

   Case 1: Suppose p has one end on the edge $(i-1, i)$. Thus p is supported on $(i-1, i, m, m+1)$ for some $m >_i i$, and $I_{i+1}(p) = m$ by Lemma 3.11.

   Let S be the set of propagators inside p along with p itself. I_i and I_{i+1} can only differ once p contributes to I_{i+1}, so $I_i(q) = I_{i+1}(q)$ for each $q \in S\backslash\{p\}$. Thus if a propagator contributes

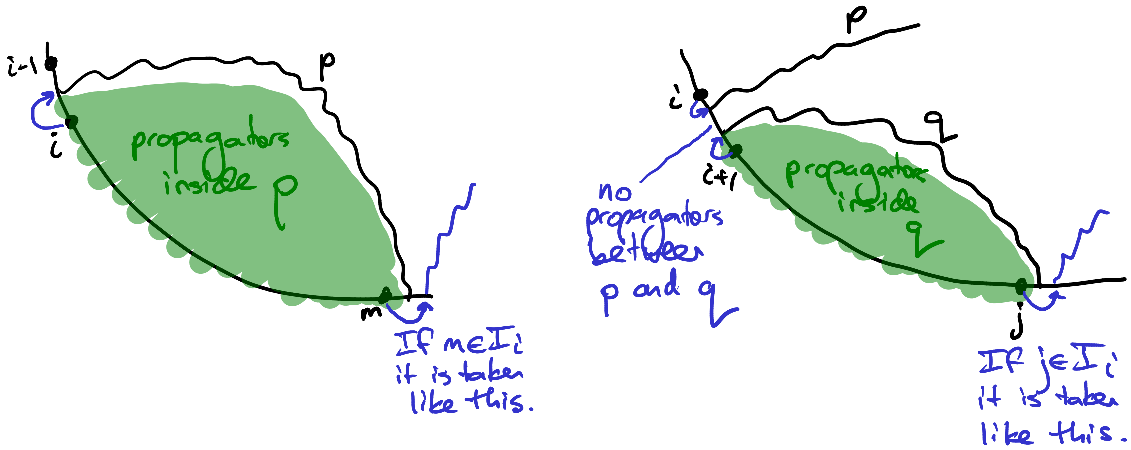


Figure 8: The two cases in the proof that no factors of Δ_{I_i} have degree 3 or more.

m in I_i then it must lie outside p.

If neither m nor $m+1$ appear in I_i then $V(p) \cap I_i = \{i\}$, and so the row of p in the matrix of Δ_{I_i} has only one nonzero entry; hence Δ_{I_i} has a linear factor contributed by p and i, which is a contradiction. So we must have at least one of m and $m+1$ in I_i. However, all propagators in S are mapped by the function $I_i(\cdot)$ to vertices strictly before m, so the matrix giving Δ_{I_i} has the form

check: is it obvious that $i-1 \notin I_i$ or are we using the no coloops result?

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the $|S| \times |S|$ matrix indexed by the propagators in S and the vertices in $I_i(S)$. No other propagators can be supported on these vertices since all other propagators are outside of p, and p is the first propagator supported at i; this explains the zero block. Therefore $\Delta_{I_i} = \det A \det C$, and both factors are nontrivial since at least one of m and $m+1$ appear in I_i. If we remove the propagator outside of p that contributes m or $m+1$, we get a smaller diagram for which $\Delta_{I_i} = \det A$. This contradicts the minimality of our choices unless $\det A$ is quadratic, which in turn contradicts our assumption that i and p contribute to an irreducible factor f of degree at least 3.

Case 2: Suppose p has one end on the edge $(i, i+1)$. If no other propagators are supported on i then the column of $C(W)$ corresponding to vertex i has only one nonzero entry in it, and so Δ_{I_i} has a linear factor contributed by p and i; as above, this is a contradiction. Thus we can take q to be the propagator such that $I_i(q) = i+1$. We know that q has one end on the edge $(i, i+1)$ and is adjacent to p on that edge in the counterclockwise direction (see Figure 8). Write $(i, i+1, j, j+1)$ for the support of q. The situation for q is very similar to case 1: in particular, we have $I_{i+1}(q) = j$ by Lemma 3.11 and so if $j \in I_i$ then the propagator which contributes j is outside of q.

Similarly to Case 1, let S be the set of propagators inside q along with p and q themselves. Then all propagators in S are mapped by $I_i(\cdot)$ to vertices strictly before j and no other propagators are supported on vertices strictly before j. Thus the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the submatrix indexed by the propagators in S and the vertices in $I_i(S)$. Again two things can now happen. If some vertex j or larger (with respect to $>_i$) belongs to I_i then B and C are at least one column wide, and so the block form of the matrix gives a nontrivial factorization of Δ_{I_i}. This yields a contradiction as in Case 1: either W contains unnecessary propagators which contradicts our minimality assumption, or $\det A$ is quadratic which contradicts the assumption that p and i contribute to f, an irreducible factor of degree at least 3.

On the other hand, if no vertex $\geq_i j$ is in I_i then $\Delta_{I_i} = \det A$. Looking in more detail into A, note that the only vertices in the support of p and q which belong to I_i are i and $i+1$, and hence

$$A = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}$$

where D is the 2×2 matrix indexed by the propagators p and q and the vertices i and $i+1$. Thus p and i contribute to a quadratic factor of Δ_{I_i}, once again contradicting our assumptions.

All cases have now been covered and so Δ_{I_i} has only irreducible factors of degree 2 or less.

3. Suppose propagators p and q are supported on a common edge (a, b), with $I_i(p) = a$ and $I_i(q) = b$. Let $x_{p,a}, x_{p,b}, x_{q,a}, x_{q,b}$ be the associated variables in $C(W)$. For any fixed bijection σ from $\mathcal{P} - \{p, q\}$ to $I_i - \{a, b\}$ for which each propagator is supported on its image under the bijection, we can extend σ to a bijection of all propagators with I_i in two ways: either $p \mapsto a$ and $q \mapsto b$ or $p \mapsto b$ and $q \mapsto a$. The sum of the contributions of all these bijections to Δ_{I_i} is therefore the product of $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ with the minor coming from the remaining propagators and $I_i - \{a, b\}$. Since there is no cancellation of terms in the expansion of Δ_{I_i}, if any other terms appear then they will cause a factor which is not in the form described in the previous part. Therefore no such terms exist and $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ is a factor of Δ_{I_i}.

   Now let f be a quadratic factor of Δ_{I_i}. By part (2) we know that f is a 2×2 minor coming from two propagators, call them p and q, and two vertices, call them $a <_i b$. It remains to show that a and b are adjacent. From this we can conclude that p and q each have one end on (a, b), as any other way for both p and q to be supported on two consecutive vertices would contradict noncrossing or the density requirement of admissibility.

   As in the proof of part (2), make a new admissible diagram by removing the propagators which come before f and set $i = a$. The cases in the proof of part (2) show how Δ_{I_i} factors: in particular the vertices supporting the other end of p either do not appear in I_i, or they contribute to a different factor of Δ_{I_i} than p and a do. By assumption b contributes to the same factor as a. Therefore (a, b) is an edge.

4. Consider $p \in S_i$, and note that Δ_{I_i} is homogeneous linear in the variables of the row corresponding to p. By part (2), either exactly one variable in the row corresponding to p appears in Δ_{I_i} and this variable is a factor of Δ_{I_i}, or exactly two variables from the row corresponding to p appear in Δ_{I_i} and they appear as part of a quadratic factor. In the first case: let the

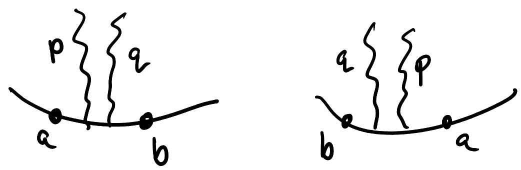


Figure 9: The situations giving a quadratic factor with variables appearing in r_i.

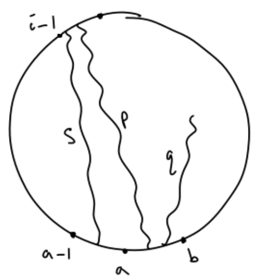


Figure 10: In order to obtain $I_{i-1}(s) = a$, propagators s and p must each have an end on the edge $(i-2, i-1)$.

variable be x, then x is a factor of both r_i and Δ_{I_i} and is the only variable from this row in either polynomial.

Now suppose two variables from the row p appear in a quadratic factor f. By part (3), there is another propagator q and an edge (a, b) such that f is the 2×2 minor coming from p, q and a, b, with $I_i(p) = a$, $I_i(q) = b$. There are two situations which can occur, both illustrated in Figure 9; we show that in both cases it follows that $q \in S_i$ as well.

In both cases, since $I_{i-1}(p) \neq a$ by assumption it follows from Lemma 3.11 that $I_{i-1}(p) <_{i-1} a$ and no other vertex supporting p lies between $I_{i-1}(p)$ and a. In the case that $b <_i a$ and q is taken before p in I_i, this means that $I_{i-1}(p) = b$ and so $I_{i-1}(q) \neq b$. Thus $q \in S_i$ and so f is a factor of r_i.

Now consider the case where $a <_i b$, and suppose for contradiction that $q \notin S_i$, i.e. that $I_{i-1}(q) = b$. Since $I_{i-1}(p) \neq a$, there must be some other propagator s with $I_{i-1}(s) = a$ (else I_{i-1} assigns q to a). This propagator cannot lie on edge (a, b) since by Lemma 3.11 we must have $I_i(s) = a$ or b, contradicting the fact that $I_i(p) = a$ and $I_i(q) = b$; thus s has an end on $(a-1, a)$ and is inside p from the point of view of $i-1$.

Say s is supported on $(j, j+1, a-1, a)$ and p is supported on $(k, k+1, a, b)$ with $i-1 \leq_{i-1} k+1 \leq_{i-1} j+1$. But by Lemma 3.9, if $I_{i-1}(s) = a$ then a cannot be maximal in the support of s with respect to $<_{i-1}$; thus we must have $i-1 = j+1$, and we are in the situation in figure 10.

Since p changed its association from I_{i-1} to I_i, we have $I_{i-1}(p) = i-1$ by Lemma 3.11.

From figure 10 it follows that I_{i-1} assigns p to $i-1$ and then proceeds identically to I_i for all vertices inside p, implying that $I_{i-1}(s) = I_i(s)$. Since $I_{i-1}(s) = a$ and $I_i(s) \neq a$, this is a contradiction.

Thus $q \in S_i$ after all, and so again f is a factor of r_i as required.

5. If W has zero propagators then all $I_i = \emptyset$ and both R and $\prod_{i=1}^n \Delta_{I_i}$ are equal to 1, so the result holds in this case. Now assume W has at least one propagator.

   First we show that every factor of $\prod_{i=1}^n \Delta_{I_i}$ divides R. Take an irreducible factor f of $\prod_{i=1}^n \Delta_{I_i}$. There exists some i such that $f|\Delta_{I_i}$ but $f\nmid\Delta_{I_{i-1}}$, since otherwise the variables corresponding to the propagators contributing to f which do not themselves appear in f could never appear, contradicting Lemma 3.11. If f is a linear factor, say from associating propagator p to vertex a, then $I_i(p) = a$ and $I_{i-1}(p) \neq a$ so this factor appears in r_i. If f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively, then again we cannot have both $I_{i-1}(p) = a$ and $I_{i-1}(q) = b$, else f divides Δ_{i-1}. However, by the proof of part (4), if one of p, q belongs to S_i then the other does as well. Thus f divides r_i.

   Next we need to show that R is squarefree. Suppose $f^2|R$. If f is a linear factor, say from associating propagator p to vertex a, then there must be two distinct points in the Grassmann necklace algorithm where p changes from not being associated to vertex a to being associated to vertex a. This contradicts Lemma 3.11. Now suppose f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively. Since p comes before q on edge (a, b)it is not possible for any I_i to associate p to b and q to a. Furthermore, we know by part (4) that p changes from not being associated to a to being associated to a if and only if q changes from not being associated to b to being associated to b. Thus $f^2|R$ implies that twice in the Grassmann necklace p must change from not being associated to vertex a to being associated to vertex a. This is again a contradiction, and so R is squarefree.

   I can't figure out why this happens; what am I missing?

   Taking everything together we have that $R|\prod_{i=1}^n \Delta_{I_i}$, R contains all factors of $\prod_{i=1}^n \Delta_{I_i}$ and R is squarefree. Therefore the ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^n \Delta_{I_i}$.

□

Theorem 4.4. *Given any admissible Wilson loop diagram W, let $GN(W) = \{I_1, \ldots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral, $R(W)$ (see Definition 1.10), is an integer multiple of the radical of $\prod_{i=1}^n \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i.*

Proof. In view of Proposition 4.3 it remains to prove that the R of Algorithm 4.1 is $R(W)$, the denominator of the integral $I(W)$.

To this end, first note that $R(W)$ and R both have total degree $4|\mathcal{P}|$; the degree of $R(W)$ is immediate from the definition while that of R follows from Lemma 3.11. By Proposition 4.3 every factor of R is either a single variable or a quadratic factor coming from two propagators supported on a common edge and hence every factor of R divides $R(W)$. Finally, since R is squarefree and monic, this implies that $R(W)$ is an integer multiple of R. □

Is this obvious? Or does it follow from a previous result?

explain why this result was interesting