This paper studies the combinatorics of Wilson Loop Diagrams.

1 Wilson Loop diagrams

What are Wilson loop diagrams and their integrals.

Definition 1.1. A Wilson loop diagram is given by the following data: a cyclicly ordered set V, along with a choice of first vertex, and k pairs $\{p_r = (i_r, j_r)\}_{r=1}^k$ ordered such that $i_r + 1 < j_r$.

We depict this data as a convex polygon, with vertices labeled by V (preserving the cyclic ordering), and k wavy lines in the interior of the diagrams. We depict the choice of first vertex on V by a $\bullet$. Note that the polygon gives the verices of W a cyclic ordering, which the choice of first vertex gives it a compatible linear order. Both the cyclic and the linear order become the correct perspective at various points in this paper. The pair p_r defines a wavy line from the i_r^{th} edge (between i_r and $i_r + 1$) and the j_r^{th} edge. These are called propagators. The condition on i_r and j_r means that the propagator does not go between adjacent edges. Let $\mathcal{P} = \{p_r\}_{r=1}^k$ be the set of propagators. Then we write

$$W = (\mathcal{P}, V) \ .$$

Often we take V to be $[n]$, the cyclically ordered set of integers, $1 \dots n$. In this case, we write $W = (\mathcal{P}, n)$. We introduce some notation to speak of vertices supporting a propagator, and the set of propagators supported on a vertex set.

Definition 1.2. Let $W = (\mathcal{P}, n)$.

1. For $p \in \mathcal{P}$, let $V_p = \{i_p, i_p+1, j_p, j_p+1\}$ be the set of vertices supporting p. Then, for $P \subseteq \mathcal{P}$, the set $V_P = \cup_{p\in P} V_p$ is the vertex support of P.
2. For $V \subseteq [n]$, write $\mathrm{Prop}(V) = \{p \in \mathcal{P} | V_p \cap V \neq \emptyset\}$.
3. For $P \subseteq \mathcal{P}$, define $F(P) = V(P^c)^c$ to be the set of vertices in $[n]$ that only support propagators in the set P.

It is sometimes useful to discuss propagators in terms of the edges supporting them, rather than the vertices.

Definition 1.3. The i^{th} edge of W is the edge of the external polygon that lies between the vertices i and $i + 1$.

In this manner, the propagator $p = (i, j)$ is supported by the i^{th} and j^{th} edges.

Definition 1.4. A Wilson loop diagram is admissible if

1. $|V| \geq |\mathcal{P}| + 4$

2. There does not exists a set of propagators, $P \subset \mathcal{P}$ such that $|V(P)| < |P| + 3$.

3. There does not exist a pair of propagators, $p, q \subset \mathcal{P}$ such that $i_p < i_q < j_p < j_q$.

The first conditions states that there are at least four more vertices than propagators in an admissible Wilson Loop Diagram. The second imposes an upper bound on how densely the propagators can be fitted in the diagram. The third ensures that ensures that no propagators cross in the interior of the diagram. In other words, a Wilson loop diagram, $(\mathcal{P}, n)$ is admissible if and only if $n > \mathcal{P} + 4$, and has neither crossing propagators nor any pairs of propagators that start and end on the same pair of non-adjacent edges.

In what follows, we will talk about admissible Wilson Loop diagrams and subdiagrams thereof.

Definition 1.5. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. A subdiagram of W is defined by a a subset of propagators of W. For $P \subset \mathcal{P}$, write $W|_P = (P, V(P))$.

Note that a subdiagram of an admissible diagram need not be admissible. In particular, the condition $|V(P)| > |P| + 4$ need not hold.

Later I will also want a different kind of subobject, one with a subset of the propagators but where the vertex set contains $V(P)$ but may be larger (the most important instance being when it is $[n]$.) This kind of subobject doesn't need a particular name, nor does it need to be defined up here, but I wanted to mention it so we have it in mind.

There is one particular type subdiagram that deserves special attention.

Definition 1.6. For W an admissible diagram, $(P, V(P))$ is exact if $|V(P)| = |P| + 3$.

The exact subdiagrams define an equivalence relation amongst Wilson loop diagrams.

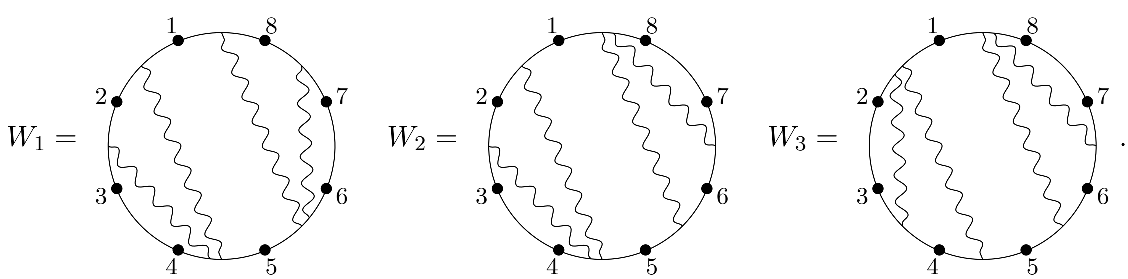
Definition 1.7. There is an equivalence relationship on the set of admissible Wilson loops diagrams generated by the following binary relation: $W = (\mathcal{P}, n) \sim W' = (\mathcal{P}', n)$ if

I feel like I need some sort of uniqueness of partition result to even be able to define this well. Or something. Can we discuss?

1. There exist two different exact subdiagrams, $(P, V(P))$ and $(P', V(P'))$ of W and W' respectively such that $V(P) = V(P')$.

2. The complementary subdiagrams are identical: $(\mathcal{P} \setminus P, V(P)^c) = (\mathcal{P}' \setminus P', V(P')^c)$.

Example 1.8. Note that since this is an equivalence relation, we may find that two Wilson loop diagrams are equivalent, even if they do not have complements of (non-trivial) exact subdiagrams in common. Consider the following three Wilson loop diagrams,



The diagrams $W_1 \sim W_2$ because $(\{(5,8),(5,7)\}, \{5,6,7,8,1\})$ and $(\{(5,8),(7,8)\}, \{5,6,7,8,1\})$ are the corresponding differing subdiagrams. Furthermore, there is the equivalence $W_2 \sim W_3$ due to the exact subdiagrams $(\{(1,4),(3,4)\},\{1,2,3,4,5\})$ and $(\{(1,4),(1,3)\},\{1,2,3,4,5\})$. This forces an equivalence between W_1 and W_3, even though one cannot partition the propagators of each into a an exact subdiagram (that may vary between the diagrams) and a complement that is fixed.

Each Wilson loop diagram, $W=(\mathcal{P},n)$ with $|\mathcal{P}|=k$ is associated to a $k\times n$ matrix with non-zero real variable entries, called $C(W)$:

$$C(W)_{p,q}=\begin{cases} c_{p,q} & \text{if } q\in V(p) \\ 0 & \text{if } q\notin V(p) \end{cases} . \tag{1}$$

Example 1.9. For example, ordering the propagators of W_1 from Example 1.8:

$$(1,4),\ (2,4),\ (5,7),\ (5,8)$$

we may write

$$C(W_1)=\begin{pmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,4} & c_{1,5} & 0 & 0 & 0 \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,7} & c_{3,8} \\ c_{4,1} & 0 & 0 & 0 & c_{4,5} & c_{4,6} & 0 & c_{4,8} \end{pmatrix} .$$

These $C(W)$ parametrize a subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ as shown in [**?**], call it $\Sigma(W)$. The Wilson loop diagrams also define a volume form on $\Sigma(W)$:

$$\Omega(W)=\frac{\prod_{r=1}^{|\mathcal{P}|}\prod_{v\in V_{p_r}}\mathrm{d}c_{p_r}}{R(W)} .$$

The denominator $R(W)$ is a polynomial defined by 2×2 and 1×1 minors of $C(W)$ as defined below.

Definition 1.10. For $W=(\mathcal{P},n)$, $R(W)=\prod_{e=1}^{n}R_e$, with R_e defined by the propagators ending on it. For any edge e of W, order the propagators incident on e as $\{p_1\dots p_r\}$, ordered such that p_1 is closest to the vertex e, p_r closest to $e+1$, and p_i is closer to e than p_{i+1}. Then

$$R_e=c_{p_1,e+1}\prod_{j=1}(r-1)\left((c_{p_j,e}c_{p_{j+1},e+1}-c_{p_{j+1},e}c_{p_j,e+1})\right)c_{p_s,e} .$$

Note that in this notation, if $r=1$, $R_e=c_{p,e}c_{p,e+1}$.

2 Equivalence classes of Wilson loop diagrams

In [**?**], Agarwala and Amat show that Wilson loop diagrams can be interpreted as positroids, a certain well behaved class of realizable matroids (this correspondence is stated precisely in Theorem 2.1 below). This opens up the study of Wilson loop diagrams to techniques from geometry and combinatorics.

[outline of section, postponed until section structure is finalised]

The main results of this section are the following: we show that two admissible Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24), and we obtain a formula for the number of admissible Wilson loop diagrams in each equivalence class (Corollary 2.25). [sentence about why this matters]

2.1 Wilson loop diagrams as matroids

We first give a quick summary of the matroid terminology that we will need; it is not intended as a comprehensive introduction to matroids and the interested reader is referred to [ref].

A *matroid* $M = (E, \mathcal{B})$ consists of a finite ground set E and a non-empty family $\mathcal{B} \subseteq \mathcal{P}(E)$ whose elements satisfy the *basis exchange property*: for any distinct $B_1, B_2 \in \mathcal{B}$ and any $a \in B_1 \setminus B_2$, there exists some $b \in B_2 \setminus B_1$ such that $(B_1 \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ as well. The elements of $\mathcal{B}$ are called the *bases* of the matroid. Note that the basis exchange property immediately implies that all bases have the same size.

A subset $A \subseteq E$ is called *independent* in M if $A \subseteq B$ for some $B \in \mathcal{B}$, and *dependent* else. The *rank* $\mathrm{rk}\ (A)$ of a subset $A \subseteq E$ is the size of the largest independent set contained in A. The rank of the matroid itself is defined to be $\mathrm{rk}\ (E)$.

A *circuit* in M is a minimally dependent set. That is, it is a set $C \subseteq E$ such that C is dependent but $C \setminus \{e\}$ is independent for any $e \in C$. A union of circuits is called a *cycle*. On the other hand, a *flat* is a maximally dependent set, i.e. a set $F \subseteq E$ such that $\mathrm{rk}\ (F \cup \{e\}) = \mathrm{rk}\ (F) + 1$ for any $e \in E \setminus F$. Unsurprisingly, a *cyclic flat* is a set which is both a flat and a cycle. The set of circuits in a matroid uniquely defines that matroid, as does the set of flats; thus one could specify a matroid by listing its independent sets, bases, circuits, or flats. [ref]

Finally, we describe several important types of matroid. A matroid of rank k with a ground set of size n is called *realizable* if there exists some $A \in Gr(k, n)$ whose non-zero $k \times k$ minors are exactly those with columns indexed by elements of $\mathcal{B}$. A *positroid* is a matroid which can be realized by an element of the totally nonnegative Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Finally, a *uniform matroid* of rank r is a matroid in which any set of size $\leq r$ is independent.

Matroid theory relates to the study of Wilson loop diagrams as follows. In [**?**], Agarwala and Amat show that every admissible Wilson loop diagram with k propagators defines a positroid of rank k, and that the independent sets can be read directly from the diagram:

Theorem 2.1. *[**?**, Theorem 3.6] Any admissible Wilson loop diagram $W = (\mathcal{P}, n)$ defines a matroid $M(W)$ with ground set $[n]$. The independent sets are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| < |U|$.*

In other words, the independent sets of $M(W)$ correspond to the sets of vertices in W such that no subset supports fewer propagators than the vertices it contains.

Throughout, we take the *matroid defined by* W to be the matroid $M(W)$ of Theorem 2.1. Note that since vertices of the diagram W correspond to columns of the associated matrix $C(W)$, $M(W)$ can also be thought of as the matroid realized by $C(W)$.

Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $M(W)$ its associated matroid. Where it will not cause confusion we conflate the two objects, identifying vertices of W with elements of the ground set $[n]$ in $M(W)$.

In particular, this allows us to prove results about $M(W)$ by considering the behavior of propagators in W. We record a few elementary facts about the rank and cycles of $M(W)$ here as an example of this.

It would be nice to have a few more; do we need any others, and/or are there any cool ones?

Lemma 2.2. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. Then:*

1. *The rank of a set $V \subseteq [n]$ is bounded above by $\min\{|V|, |Prop(V)|\}$, with $rk\ (V) = |V|$ if and only if V is an independent set.*
2. *If $C \subseteq [n]$ is a cycle, then $rk\ (C) = |Prop(C)|$.*

Proof. The first part of (1) is [**?**, Equation (9)] and surrounding discussion, and the second part is standard matroid theory. (2) is [**?**, Lemma 3.27]. □

2.2 Polygon partitions of Wilson loop diagrams

The equivalence relation on Wilson loop diagrams is defined in terms of exact subdiagrams; thus in order to understand the equivalence, we need a way to extract and compare exact subdiagrams. We do this via the notion of a polygon partition of W.

Definition 2.3. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. The *polygon partition* associated to W, denoted $\tau(W)$, is defined as follows.

- The vertices of $\tau(W)$ correspond to the edges of W.
- Labeling the vertices of $\tau(W)$ with the edge number of Wgives a cyclic order to the vertices. Connecting consecutive vertices gives a graph theoretic cycle called the polygon of $\tau(W)$.
- Each propagator of W defines a chord edge of $\tau(W)$; specifically, a propagator $(i, j) \in \mathcal{P}$ defines a chord connecting the vertices i and j in $\tau(W)$.

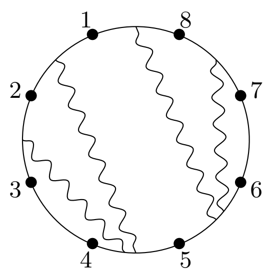
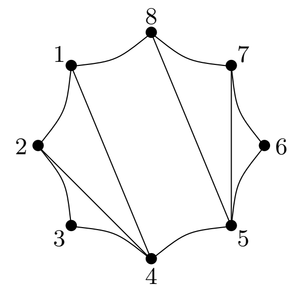
Lemma 2.4. *If $W = (\mathcal{P}, n)$ is an admissible Wilson loop diagram, then $\tau(W)$ is a simple planar graph whose outer face is a cycle. It is embedded such that the vertices all lie on this infinite face*[1]*. These vertices are cyclically ordered, with a choice of first vertex giving it an additional compatible linear order.*

Proof. Since the vertices of $\tau(W)$ are labeled by the edges of W, which are cyclically ordered, this gives an ordering to the vertices. Furthermore, since the outer polygon of W is a cycle, the outer face of $\tau(W)$ is also a cycle. Since W is admissible, no pairs of propagators cross. Therefore, it is a planar embedding. Similarly, W does not admit any propagators of the form $p = (i, i + 1)$; therefore there is exactly one edge connecting any two adjacent edges of $\tau(W)$. Finally, there does

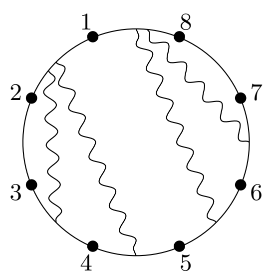
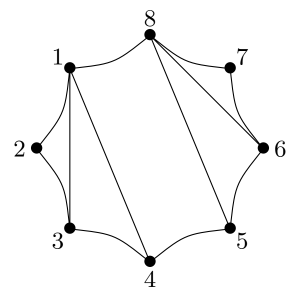
[1]That is, it is an *outerplanar* graph

not exist two propagators p, q such that both p and q start at edge i and end at edge j. Therefore, no other two vertices of $\tau(W)$ can be connected by more than one edge. Finally, the embedding of $\tau(W)$ is induced from the embedding of the graph W. $\square$

Example 2.5. In this example we return to two of the Wilson loop diagrams in Example 1.8. We can pair diagrams with their polygon partitions as follows:

$W_1 =$  ; $\tau(W_1) =$ 

and

$W_3 =$  ; $\tau(W_3) =$  .

Recall that a planar embedding of a graph is a *triangulation* if all faces, except possibly the infinite face, are triangles.

Definition 2.6. Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. A *triangulated piece* of $\tau(W)$ is a 2-connected subgraph of $\tau(W)$ which is a triangulation. We will take the convention that a subgraph consisting of a single chord edge is called a *trivial* triangulated piece. A *maximal* triangulated piece is one which is not contained in any strictly larger triangulated piece.

Definition 2.7. A *decomposition* of a polygon partition $\tau(W)$ is a set of 2-connected induced subgraphs of $\tau(W)$ which partition the edges of $\tau(W)$.

Example 2.8. For the Wilson loop diagrams and polygon partitions in Example 2.5, the vertex sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ are maximal triangulated pieces for both $\tau(W_1)$ and $\tau(W_3)$. The vertex set $\{4, 5, 8, 1\}$ is not a triangulation in either polygon partition.

do we want a picture of a dual tree here? Being lazy

Lemma 2.9. *For W an admissible Wilson loop diagram, the polygon partition $\tau(W)$ has a unique decomposition into maximal triangulated pieces, and edges in the polygon of $\tau(W)$.*

Proof. We begin by giving an algorithm for the decomposition, then prove its uniqueness. Let $W = (\mathcal{P}, n)$, with $|\mathcal{P}| = k$.

By *splitting* a vertex v we will mean replacing v by new vertice $v_1, v_2, \ldots, v_{\deg(v)}$ such that each v_i as exactly one neighbour and the union of the neighbourhoods of the v_i is the neighbourhood of v.

Let $T(W)$ be the dual graph of $\tau(W)$ with the vertex corresponding to the infinite face split. Since $\tau(W)$ is an embedded graph (with a fixed distinguished embedding) by Lemma 2.4, $T(W)$ is an uniquely defined graph.

The fixed embedding is important, because otherwise you only have a unique planar dual if the graph is 3-connected which this one isn't. I'm just noting that here so that everyone knows why I added the part about the embedding and then we can take this comment out.

Furthermore $T(W)$ is a tree because it is connected, has $n+k+1$ vertices ($k+1$ from the internal faces of $\tau(W)$ and n from the outer face) and $n+k$ edges (since $\tau(W)$ has $n+k$ edges). Additionally, since $\tau(W)$ is simple, $T(W)$ has no vertices of degree 2.

Split every vertex of $T(W)$ which has degree > 3. The connected components of $T(W)$ correspond to the decomposition of $\tau(W)$ into maximal triangulated pieces and edges originally in the polygon of $\tau(W)$. Let f be the forest thus obtained. The vertices of f either have degree 1 or 3. Trees of f with no trivalent vertices correspond to either edges in the polygon of $\tau(W)$, if they were originally leaves of $T(W)$, or to maximal trivial triangulated pieces. Splitting at all the faces that are not triangles ensures maximality of the decomposition. If the splitting were not maximal, then one could add a triangle to a connected component of the splitting, but this would imply that that splitting happened at a valence 3 vertex.

To see uniqueness, consider a different maximal decomposition of $\tau(W)$. This induces a splitting on $T(W)$, where each connected component of the new decomposition corresponds to a subtree. Call this forest f'. Since $f' \neq f$, there are two trees, t and t' in f and f' that are distinct, but share at least one edge of $T(W)$. Since f' is also maximal, t' is not a subtree of t. Therefore, the edges of t' can be found in at least two trees in the forest f. In particular, there is a vertex v in t' that corresponds to a split vertex of $T(W)$ in the original decomposition. This implies that v has valence greater than 3 in $T(W)$, and thus the corresponding face of $\tau(W)$ is not a triangle. In other words, the decomposition corresponding to f' is not a triangulation. $\square$

Corollary 2.10. *Given a maximal decomposition of $\tau(W)$, the maximal triangulated pieces are edge disjoint.*

Proof. Consider any two distinct maximal triangulated pieces of $\tau(W)$. These two pieces correspond to subtrees of $T(W)$ and intersect, at most, at a vertex in the interior of $\tau(W)$. Since the subtrees corresponding to the maximal triangulated pieces are edge disjoint, and the edges of $T(W)$ correspond to the edges of $\tau(W)$, this forces the maximal triangulated pieces to be edge disjoint as well. $\square$

We are now in a position to relate the triangulated pieces of $\tau(W)$ to exact subdiagrams of W.

Lemma 2.11. *Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. The triangulated pieces of $\tau(W)$ correspond to the exact subdiagrams of W.*

Proof. Let us first record a few standard facts about polygon triangulations (that is, about triangulations with all vertices on the outer face). If such a triangulation has n vertices then it has n edges on the polygon (that is, on the outer face) and $n-3$ edges which are not. No planar graph with the same vertices and the same outer face can have more edges than the triangulation, and every such simple graph with $n-3$ edges off the outer face is a triangulation.

Since W is admissible, by Lemma 2.4 $\tau(W)$ is a simple graph. Furthermore, t is not all of $\tau(W)$.

Let t be a triangulated piece of the decomposition of $\tau(W)$ given in Lemma 2.9; note that t cannot be equal to $\tau(W)$ by the definition of admissible diagrams.

If t has 2 vertices, then t has two connected components. Therefore, t corresponds to a propagator that connects two non-adjacent edges. Therefore, the trivial triangulation is a trivial exact subdiagram.

Now suppose that t has $m > 2$ vertices; we count how many edges of t are not on the outer face of $\tau(W)$. Consider the intersection of t with the outer face of $\tau(W)$: this is a possibly disconnected subgraph of the polygon of $\tau(W)$ with m vertices, call it S. Let k be the number of connected components of S. To join the components S must have k edges not in the outer face of $\tau(W)$. Furthermore t has $m-3$ edges not in its outer face and so also not in the outer face of $\tau(W)$. Thus there are $m-3+k$ edges of t not in the outer face of $\tau(W)$.

Each of these $m-3+k$ internal edges correspond to propagators in W, call this set P. Next we count the size of $V(P)$. Each of the m vertices in the outer face of t corresponds to an edge of W. These m edges define k connected compnents of the outer polygon of W. Thus the set $V(P)$ has $m+k$ vertices. In other words,

$$|V(P)| = m + k = |P| - 3 \ .$$

Conversely, suppose we have an exact subdiagram $(P, V(P))$ of $W = (\mathcal{P}, n)$ supported on $|V(P)| = |P| + 3$ vertices, and let t be the subgraph of $\tau(W)$ corresponding to $(P, V(P))$.

If $|P| = 1$, then $(P, V(P))$ is a trivially exact subdiagram. In this case t is an edge internal to $\tau(W)$ connecting two boundary edges, which by definition is a triangulation.

Suppose the set $V(P)$ defines c disjoint cyclic intervals of $[n]$. Then the corresponding subgraph of $\tau(W)$ has $m-c$ vertices. Call it t. If t were a triangulated component, t would have $k-c$ internal edges.

The graph t has k edges that come from propagators, and $m-2c$ edges that come from the boundary polygon of $\tau(W)$. Since t has $m-c$ vertices, it has $m-c$ external edges, of which c come from propagators. Therefore, of the k edges of t that come from propagators, $k-c$ are internal to the connected component. Therefore, this is triangulated.

$\square$

To avoid the issue of exact diagrams being subdiagrams of other exact subdiagrams (for instance, any subdiagram (q, V_q), for $q \in \mathcal{P}$ is exact), we introduce the notion of maximal exact subdiagrams.

Definition 2.12. An exact subdiagram $(P, V(P))$ is a *maximal exact subdiagram* of W if there is no other exact subdiagram $(Q, V(Q))$ in W that contains $(P, V(P))$ as a strict subdiagram.

Corollary 2.13. *Any admissible Wilson loop diagram $W = (\mathcal{P}, n)$ can be uniquely decomposed into maximal exact subdiagrams. These maximal subdiagrams partition $\mathcal{P}$.*

Proof. Combining Lemmas 2.9 and 2.11 yields the unique decomposition into maximal exact subdiagrams, and Corollary 2.10 ensures that no propagator appears in more than one subdiagram in this decomposition. Since the chord edges of $\tau(W)$ correspond to the propagators of W, the decomposition of $\tau(W)$ induces a partition of $\mathcal{P}$. $\square$

2.3 Matroid properties of exact subdiagrams

Since Corollary 2.13 allows us to decompose any admissible Wilson loop diagram into a collection of maximal exact subdiagrams, in this section we examine the matroid properties of exact subdiagrams more closely.

Definition 2.14. Let $M = (E, \mathcal{B})$ be a matroid, and $S \subseteq E$. The *contraction* of M by S is the matroid $M/S = (E \setminus S, \mathcal{B}/S)$, where

$$\mathcal{B}/S = \{B \setminus S \mid |B \cap S| \text{ is maximal amonst all } B \in \mathcal{B}\}.$$

In [**?**], Agarwala and Amat show that certain subdiagrams of W can be realized as contractions of $M(W)$:

Lemma 2.15. *[**?**, Theorem 3.33] Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. If the set $V(P)^c$ has rank $|P^c|$, then the matroid defined by the subdiagram $(P, V(P))$ is equal to the contraction $M(W)/V(P)^c$.*

In Lemma 2.20 below we show that every exact subdiagram satisfies the rank condition of Lemma 2.15. In order to do this, we first examine the properties of the set by which we contract.

Definition 2.16. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. The set $F(P) := V(P^c)^c$ is called the *propagator flat* of P.

Thus in Lemma 2.15 above we would contract $M(W)$ by the propagator flat $F(P^c)$ of the *complement* of P in order to study $(P, V(P))$. The justification for this notation is given by (1) of the next lemma: the set $F(P)$ consists of exactly those vertices which *only* support propagators in P.

Lemma 2.17. *Let $F(P)$ be a propagator flat as defined above. Then*

1. *$F(P) = V(P) \setminus V(P^c)$.*
2. *If $Q \subseteq P$ then $F(Q) \subseteq F(P)$.*
3. *$F(P)$ is a flat of $M(W)$, thus justifying the name "propagator flat".*

Proof. The proof of (1) and (2) are routine applications of the definition and are omitted.

To prove (3), we need to show that $F(P)$ is maximally dependent. If $F(P) = [n]$ then this is automatic, so suppose not and let $v \in [n] \setminus F(P)$. In other words, $v \in V(P^c)$ and so v supports some propagator $q \notin P$. Let $S \subseteq F(P)$ be an independent set of maximal size. Then $\mathrm{Prop}(S) \subseteq P$ by (1), and no subset of S supports fewer propagators than the number of vertices it contains (this is the definition of an independent set in $M(W)$). Since v supports a new propagator $q \notin P$, the set $S \cup \{v\} \subseteq F(P) \cup \{v\}$ also satisfies this independence condition. Thus $\mathrm{rk}\ (F(P) \cup \{v\}) = \mathrm{rk}\ (F(P)) + 1$, as required. $\square$

Lemma 2.18. *Let F be a flat in $M(W)$, and let $C \subseteq F$ be the union of all circuits contained in F. Then the following are true:*

1. *$C = F(Prop(C))$, i.e. C is a propagator flat.*
2. *$F \setminus C$ is an independent ~~set. Further, if W does not contain any vertices of rank 0, i.e. all vertices support at least 1 propagator, then $F \setminus C$ is an independent~~ flat.*

F and C both contain all vertices of rank 0 by definition

Proof. (1) If F is an independent flat, then $C = \emptyset$ and the statement is trivially true. Now suppose that F is a dependent set, so C is non-empty.

Let $v \in C$. Clearly $\text{Prop}(v) \subseteq \text{Prop}(C)$, and so by Lemma 2.17 we have $F(\text{Prop}(v)) \subseteq F(\text{Prop}(C))$. Since $v \in F(\text{Prop}(v))$ by the definition of propagator flat, we have $v \in F(\text{Prop}(C))$ as required.

Now suppose there exists some $w \in F(\text{Prop}(C)) \setminus C$. Let B be an independent subset of C of maximal rank; we first show that $B \cup \{w\}$ is a dependent set in $M(W)$. Indeed, by [**?**, Proposition 3.10], there exists some $Q \subseteq \text{Prop}(B)$ with $|Q| = |B|$ and such that B is a basis in $M(Q, V(Q))$. We therefore have

$$Q \subseteq \text{Prop}(B) \subseteq \text{Prop}(C),$$

with $|\text{Prop}(C)| = \text{rk}\ (C)$ by Lemma 2.2 and $|Q| = |B| = \text{rk}\ (C)$ by the choice of B. Thus $Q = \text{Prop}(C)$, and so B is a basis in the matroid defined by $W' := (\text{Prop}(C), V(\text{Prop}(C)))$. All that remains is to note that $w \in F(\text{Prop}(C)) \subseteq V(\text{Prop}(C))$, i.e. w is a vertex in W'; thus $B \cup \{w\}$ is dependent in W', and hence also in W. This also implies that $\text{rk}\ (F \cup w) = \text{rk}\ (F)$ since $B \subseteq F$ and so since F is a flat we must have $w \in F$.

Returning our attention to $M(W)$, the dependent set $B \cup \{w\}$ must contain a circuit C', with $w \in C'$ since B was an independent set. Now $C \cup C' = C \cup \{w\} \supsetneq C$ is a cycle which, since $w \in F$, contradicts the maximality of C.

For part (2), first note that $F \setminus C$ is automatically independent as it contains no circuits. Now suppose that all vertices in W have rank ≥ 1.

can get rid of?

For any $e \notin F$, we certainly have $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ since F is a flat. Now let $e \in C$, and suppose that $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C)$. This implies that $(F \setminus C) \cup \{e\}$ is dependent, and hence contains a circuit. This circuit must contain at least two elements (since all vertices support at least one propagator), but this contradicts the fact that C was the union of all circuits in F.

Thus $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ for any $e \notin F \setminus C$, and hence $F \setminus C$ is a flat. $\square$

Corollary 2.19. *If F is a flat of a Wilson loop diagram, it can be written as the disjoint union of a cyclic propagator flat and an independent flat.*

In particular, any propagator flat can be written as a union of a cyclic propagator flat and an independent flat.

Lemma 2.20. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and $P \subsetneq \mathcal{P}$. Then:*

should hold if $P = \mathcal{P}$.

1. *If $(P, V(P))$ is an exact subdiagram in W, then $\text{rk}\ (F(P^c)) = |P^c|$.*

2. *If* $(P, V(P))$ *is a maximal exact subdiagram in* W, *then* $F(P^c)$ *is a cyclic flat.*

Note that by definition, since $F(P^c)$ has at $|P^c|$ propagators in it, this is an upper bound on the rank. However, showing equality on the first part is harder.

Proof. We first verify that $F(P^c) = V(P)^c$ is non-empty and dependent for any exact subdiagram $(P, V(P))$. The definition of exactness means that the vertices $V(P)$ cannot support a larger set of propagators (otherwise W would no longer be admissible). Therefore, since $P \subsetneq \mathcal{P}$, there must be at least one propagator supported on $V(P)^c$, i.e. it cannot be empty.

Since W is admissible, we have $n \geq |\mathcal{P}| + 4$. Rewriting this as

$$|V(P)| + |F(P^c)| \geq |P| + |P^c| + 4,$$

and combining it with the fact that $|V(P)| = |P| + 3$ (from the exactness of $(P, V(P))$), we obtain

$$|F(P^c)| > |P^c| \,. \tag{2}$$

By Lemma 2.17, $F(P^c)$ is the set of vertices that *only* supports propagators in P^c, so in particular we have $\mathrm{Prop}(F(P^c)) = P^c$. Equation (2) is therefore saying that $F(P^c)$ supports fewer propagators than the number of vertices it contains, i.e. $F(P^c)$ is dependent.

We address part (2) first. Suppose that $(P, V(P))$ is a maximal exact subdiagram; by Corollary 2.19 we can decompose $F(P^c)$ as

$$F(P^c) = C \sqcup S, \tag{3}$$

where C is the largest cyclic flat contained in $F(P^c)$ and S is an independent set. Note that C must be non-empty since $F(P^c)$ is dependent. We proceed by showing that S must be empty, forcing $F(P^c)$ to be a cyclic flat.

Does this fix confusion later?

Since $C = F(\mathrm{Prop}(C))$ by Lemma 2.18, define $Q := \mathrm{Prop}(C)^c$ so that we may write $C = F(Q^c) = V(Q)^c$. Note that since $C \subseteq F(P^c)$, $P \subseteq Q$. Furthermore, we have

$$\mathrm{rk}\ (S) = |S| \leq \mathrm{rk}\ (F(P^c)) - \mathrm{rk}\ (C) \leq |P^c| - \mathrm{rk}\ (C) \,, \tag{4}$$

where first equality comes from the fact that S is an independent set, and the final inequality comes from Lemma 2.2(1). Since $\mathrm{rk}\ (C) = |\mathrm{Prop}(C)| = |Q^c|$ by Lemma 2.2(2), we can rearrange (4) to obtain the inequality

$$|P| + |S| \leq |P| + |P^c| - \mathrm{rk}\ (C) = |Q| \,. \tag{5}$$

Furthermore, since $V(Q) = S \sqcup V(P)$ (by equation [ref] and the definition of Q) we may write

$$|V(Q)| = |S| + |V(P)| = |S| + |P| + 3 \,. \tag{6}$$

Combining this with (5) gives $|V(Q)| - 3 \leq |Q|$. That is, either W was not admissible, or $(P, V(P))$ was not a maximal exact subdiagram. In other words, S cannot be non-empty, proving that $F(P^c)$ is a cyclic flat, and therefore of maximal rank.

This only follows if $P \subseteq Q$, which I can't see? We don't ever seem to use the fact that S is empty/nonem

REPLACE BELOW WITH: Let P be a maximal exact subdiagram as above, and let $(R, V(R))$ be an exact subdiagram that is not maximal. That is, $R \subsetneq P$.

Let $(Q, V(Q))$ be an exact subdiagram that is not maximal, with $Q \subset P$ defining the maximal exact subdiagram above .

POSSIBLY REPLACE BELOW WITH: Since $R \subset P$, $V(P) = V(R) \sqcup S$. Since P and R both define exact subdiagrams, we may write $|V(P)| = |V(R)| + |P \setminus R|$, where S is a vertex set of size $|P \setminus R|$ with $(P \setminus R \subset \text{Prop}(S)$. Since rk $V(P) =$ rk $V(R) + |P \setminus R|$, S is independent. Taking the complements, we may write

$$F(R^c) = F(P^c) \sqcup S\ .$$

Since S is an independent set of vertices of the correct size supporting the appropriate propagators, this gives rk $F(R^c) = |R^c|$.

If I've understood this right, we would need to show that any exact subdiagram appears as $\text{Prop}(C)^c$ for some maximal ex-

Using the fact that P and Q are both exact diagrams, we have $|V(P)| = |V(Q)| + |P \setminus Q|$. Then $V(Q)^c = F(P^c) \cup S$, where S is an independent set of size $|P \setminus Q|$ and with $(P \setminus Q) \subset \text{Prop}(S)$. Therefore, rk $(F(Q^c)) = |Q^c|$. $\square$

I'm lost here.

why? I

better if I fix typo?

confusion

In particular, any exact subdiagram $(P, V(P))$ satisfies the conditions of Lemma 2.15 and can therefore be written as a contraction of $M(W)$ by the complementary propagator flat $F(P^c)$.

Matroids coming from exact subdiagrams have an especially nice structure, as we now show. Recall from Section 2.1 that a uniform matroid of rank r is one in which all sets of size $\leq r$ are independent.

Theorem 2.21. *Let $W' := (P, V(P))$ be a subdiagram of an admissible Wilson loop diagram $W = (\mathcal{P}, n)$. Then W' is an exact subdiagram if and only if $M(W')$ is a uniform matroid of rank $|P|$.*

Proof. It follows directly from the definitions that a matroid of rank r is uniform if and only if all circuits have rank r; we therefore focus on the circuits of $M(W')$.

We prove the following claim: W' is exact if and only if $V(P)$ contains no circuits C with rk $C < |P|$ in $(P, V(P))$. Since rk $(M(W')$ is bounded above by $|P|$, the result follows.

Suppose $C \subseteq V(P)$ is a circuit of rank $m < |P|$; by Lemma 2.2, we know that $|\text{Prop}_{W'}(C)| = m$ as well, where the subscript to Prop specifies the diagram we are working in. Observe that $\text{Prop}_{W'}(C) \subseteq P$ by definition. The set $P \setminus \text{Prop}_{W'}(C)$ is thus nonempty, and we can consider the subdiagram $W'' := (P \setminus \text{Prop}_{W'}(C), V(P \setminus \text{Prop}_{W'}(C))$. By the density condition on subdiagrams of admissible diagrams, we have

$$|V(P \setminus \text{Prop}_{W'}(C))| \geq |P \setminus \text{Prop}_{W'}(C)| + 3.$$

It is easy to verify that $V(P \setminus \text{Prop}_{W'}(C)) \subseteq V(P) \setminus C$; since $C \subseteq V(P)$ and $\text{Prop}_{W'}(C) \subseteq P$, we can therefore rewrite the previous inequality as

$$|V(P)| - (m+1) \geq |V(P \setminus \text{Prop}(C))| \geq |P| - m + 3.$$

Simplifying, we obtain $|V(P)| \geq |P| + 4$, i.e. $(P, V(P))$ is not an exact diagram.

Conversely, suppose that W' is not exact and for a contradiction suppose also that $M(W')$ is uniform of rank $|P|$. Take $p \in P$. Then $|V(P) \setminus V(p)| = |V(P)| - 4 \geq |P|$ by non-exactness. By uniformity there is an independent set of size $|P|$ in $V(P) \setminus V(p)$. This is impossible because the submatrix corresponding to this independent set has $|P|$ rows but the one corresponding to p is all 0 one of them is all 0 so it cannot be full rank. □

Before proving the main theorems of this section, we make a few observations about the geometry of matroids defined by exact diagrams.

In [**?**], the authors show that all exact Wilson loop diagrams correspond to positroids. That is, they correspond to matroids that can be represented by elements of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|,|V|)$.

Check: positroids or totally positive? Why specify exactness?

Definition 2.22. Given a Wilson loop diagram W, define the positroid cell associated to a Wilson loop diagram, $\Sigma(W)$, to be the cell in the CW complex on $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|,|V|)$ defined by the realizations of W that lie in $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|,|V|)$.

we haven't specified what this means

With this definition in mind, we have the following corollary:

Corollary 2.23. *Let $(P, V(P))$ be an exact subdiagram of W. The matroid associated to this subdiagram corresponds to the top dimensional cell in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|,|V(P)|)$.*

Proof. The unique top dimensional cell of $\mathbb{G}_{\mathbb{R},\geq 0}(|P|,|V(P)|)$ is defined by all points in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|,|V(P)|)$ such that all Plucker coordinates are strictly greater than 0. Since $(P, V(P))$ is an exact subdiagram, this all $|P| \times |P|$ minors are non-zero. Intersecting these with the cases with the positive Grassmannians demands that all minors be strictly positive. □

2.4 Matroids and equivalent diagrams

Now we are ready to prove the main results of this section, namely that two Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem [ref]) and a formula for the number of Wilson loop diagrams in a given equivalence class (Corollary [ref]).

Siân postponed reading this section until the earlier ones are fixed.

Theorem 2.24. *Let $W = (\mathcal{P}, [n])$ and $W' = (\mathcal{P}', [n])$ be two Wilson Loop diagrams. They define the same matroid if and only if $W \sim W'$.*

Proof. One direction has been proved in previous work, but we give a different proof here to be consistent with the method of this document.

Assume that W and W' are equivalent. Without loss of generality, write $W = (P \cup R, [n])$ and $W' = (P \cup R', [n])$, where $P \subset \mathcal{P} \cap \mathcal{P}'$ and $(R, V(R))$ and $(R', V(R'))$ are two maximally exact subdiagrams, with $R \neq R'$, but $V(R) = V(R')$. If this is not the case, one may always find a family of diagram, $\{W_i\}$ satisfying this condition and forming a transitive chain connecting W to W' in the equivalence class.

Let $U \subset V(R)$ be any subset of size $|U| = |R|$. The set U is independent in the subdiagram $(R, V(R))$, and thus in W. The complementary set $F(P)$ is a flat of maximal rank by lemma 2.20 (rk $(F(P)) = |P|$). Let $B \subset F(P)$ be a maximal indpendent set (rk $B = |P|$) in $F(P)$. Since $F(P)$ is a flat, adding any element of $V(R)$ to B increases the rank. Therefore, any basis of W can be written as $B \cup U$. However, since $F(P)$ is common to both W and W', and $V(R) = V(R')$, any any basis of W' can also be written $B \cup U$. Thus both matroids have the same bases sets, proving that they are the same.

For the converse, assume that the matroids associated to W and W' are the same: $M(W) = M(W') = M$. Let $\{(P_i, V(P_i)\}_{i=1}^k$ and $\{(P'_i, V(P'_i)\}_{i=1}^l$ be the sets of maximally exact subdiagrams of W and W'. Write $F_i = F(P_i)$ and $F'_i = F(P_i'^c)$ to be the complementary cyclic flats. By Lemma 2.21 M/F_i is a uniform matroid. Thereofore, $k = l$, and we may write $V(P_i) = V(P'_i)$.

Reorganize the vertex sets as follows:

$$\cup_{P_i \neq P'_i} V(P_i) = V(\cup_{P_i \neq P'_i} P_i) \ . \tag{7}$$

Since maximal exact subdiagrams partition $\mathcal{P}$, by Lemma 2.13, write $cup_{i=1}^k P_i = \cup_{i=1}^k P'_i = \mathcal{P}$. Then equation (7) becomes

$$\cap_{P_i \neq P'_i} (F(P_i^c))^c = F(\cup_{P_j = P'_j} P_j)^c \ .$$

These flats may, of course, be empty.

Thus, we have partitioned the vertices of W and W' into a single set that supports a union of maximal exact subdiagrams whose propagators between W and W', and the complementary propagator flats that support the propagators in common between W and W'. As mentioned above, the latter may be empty.

Define a family of Wilson loop diagrams, W_0 to W_k defined such that $W_0 = W$ and W_i is derived from W_{i-1} by replacing the propagator set P_i with P'_i. In this manner, $W' = W_k$ and $W_i \sim W_{i+1}$, making $W \sim W'$. □

Since there is a unique way to decompose W into maximal exact subdiagrams, it is logical to ask how many diagrams there are in an equivalence class. It is a classical fact the the number of triangulations of an n-gon is the $n-2$ Catalan number, namely $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. Thus we can count the number of equivalent diagrams.

Corollary 2.25. *Let W be an admissible Wilson loop diagram where the sizes of the supports of the nontrivial maximal connected exact subdiagrams are $n_1, n_2, \ldots, n_j$. Then the number of admissible Wilson loop diagrams equivalent to W (including W itself) is*

$$\prod_{i=1}^{j} \frac{1}{n_i - 1}\binom{2(n_i-2)}{n_i - 2}$$

3 Geometry of Wilson Loop diagram

Since Wilson loop diagrams correspond to positroids, it is natural to understand the subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ they define.

Talk about Grassmann Necklaces and how it defines a cell in a CW complex of $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Also talk about how this is exactly all the non-negative matrices that represent a particular positroid.

Also talk about Le diagrams.

3.1 From Wilson Loop diagrams to Grassmann Necklaces

Here, we give an algorithm for passing from Wilson loop diagrams to Grassmann Necklaces.

Let $\binom{[n]}{k}$ be the set of all k-subsets of the cyclically ordered set $[n]$. For each $j \in [n]$, we can define a total order $\leq_j$ on the interval $[n]$ by

$$j <_j j+1 <_j \cdots <_j n <_j 1 \cdots <_j j-1 \ .$$

This in turn induces a total order on $\binom{[n]}{k}$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\binom{[n]}{k}$ (the *Gale order*), which is defined as follows: if $A = [a_1 <_j a_2 <_j \cdots <_j a_k]$, $B = [b_1 <_j b_2 <_j \cdots <_j b_k] \in \binom{[n]}{k}$, then

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\binom{[6]}{3}$ we have $[2,5,6] \preccurlyeq_2 [2,6,1]$ but $[2,5,6] \not\preccurlyeq_2 [3,4,6]$.

Definition 3.1. A Grassmann necklace of type (k,n) is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of n elements $I_i \in \binom{[n]}{k}$, such that

- if $i \in I_i$, then $I_{i+1} = \big(I_i \backslash \{i\}\big) \cup \{j\}$ for some $j \in [\![1, n]\!]$.
- if $i \notin I_i$, then $I_{i+1} = I_i$.

By [ref], the Grassmann necklaces of type (k,n) are in 1-1 correspondence with the positroid cells in $Gr(k,n)^{tnn}$. Further, if $\mathcal{I}$ is the Grassmann necklace associated to a cell $\mathcal{P}$, then the bases of $\mathcal{P}$ can be computed using the Gale order $\preccurlyeq_i$ for each $i \in [n]$:

$$\mathcal{B}(\mathcal{P}) = \left\{ J \in \binom{[n]}{k} \ : \ I_i \preccurlyeq_i J \ \forall i \in [n] \right\}.$$

We now describe an algorithm that, when applied to an admissible Wilson loop diagram, produces exactly the Grassmann necklace of the corresponding positroid.

Algorithm 3.2. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. This gives an algorithm for calculating the set I_a, for $a \in [n]$.*

1. *Fix a vertex $a \in [n]$. Set $i := a$ and $I_a = \emptyset$.*
2. *While $\mathcal{P} \neq \emptyset$, perform the following steps.*
   - (a) ***Step** i **for vertex** a: If $Prop(i) \neq \emptyset$ in W, write $I_a = I_a \cup i$. Let $p \in Prop(p)$ be the clockwise most propagator supported on i. Write $W = (\mathcal{P} \setminus p, n)$.*
   - (b) *If $Prop(i) = \emptyset$ do nothing.*
   - (c) *Increment i by 1 and repeat from (a).*

If the propagator p is removed at vertex i according to the algorithm for I_a, we say that p *contributes* i to I_a.

Next we prove that this algorithm actually gives a Grassmann Necklace as claimed. We begin with some useful definitions and lemmas

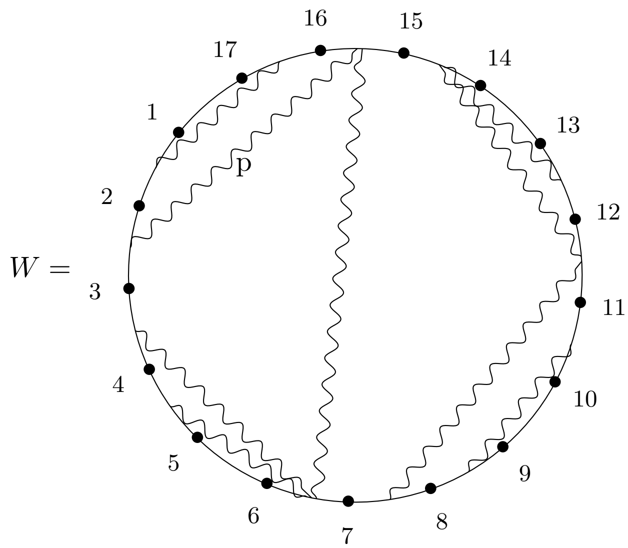
Definition 3.3. Let $W = (\mathcal{P}, n)$ be an admissible diagram and $a \in [n]$ a vertex. For $p = (i, j) \in \mathcal{P}$ an a propagator of W and $v \in \{i, j\}$ a vertex of W, let $P_v(p)$ or $P_{v+1}(p)$ to be two *protecting sets of* p *at* v *and* $v+1$ be a set of propagators defined as follows:

1. Work under the linear order on $[n]$, $<_{v+1}$, every propagator is now written as $p = (i_p, j_p)$ such that $i_p <_{v+1} j_p$.
2. If there is a propagator $s_1 \in \mathcal{P}$ such that $s_1 = (i_{s_1}, v)$ or $s_1 = (i_{s_1}, v-1)$, then $s_1 \in P_v(p)$.
3. If s_1 as above does not exists, then $P_v(p) = \emptyset$. In this case, $P_w(p) = \emptyset$. When $P_v(p) \neq \emptyset$, if there exists a propagator $t_1 \in \mathcal{P}$ defined such that $t_1 = (i_{t_1}, v)$ and $s_1 = (i_{s_1}, v)$ with $i_p <_{v+1} i_{t_1} <_{v+1} i_{s_1}$, $P_w(p) = t_1$.
4. If $s_1 = (i_{s_1}, v)$ then
$$P_v(p) = \begin{cases} \{s_1\} \cup P_{i_{s_1}}(s_1) \cup P_{i_{s_1}+1}(s_1) & \text{if } P_{i_{s_1}+1}(s_1) \neq \emptyset \\ s_1 & \text{else} \end{cases}.$$
If $s_1 = (i_{s_1}, v-1)$ then
$$P_v(p) = \begin{cases} \emptyset & \text{if } P_{v-1}(s_1) = \emptyset \\ \{s_1\} \cup P_{v-1}(s_1) & \text{else} \end{cases}$$
.

The protective sets $P_\cdot(p)$ are designed to prevent p from contributing the vertex indicated in the subscript at certain points in the Grassmann Necklace, I_a. Write $P_v(p) = \{s_1, \ldots s_r, t_2, \ldots t_r\}$ with the $t_x = P_{i_{s_{x-1}}+1}(s_{x-1})$ and $s_x \in P_{i_{s_{x-1}}}(s_{x-1})$ for $x \geq 2$. We place the restrictions on the vertex a as follows:

1. P_v prevents p from contributing to I_a if
$$i_{s_r} <_{v+1} a - 1 <_{v+1} v$$
2. $P_{v+1}(p)$ prevents p from contributing to I_a only if $P_v(p)$ prevents p from contributing v to I_a.

Example 3.4. Consider the admissible Wilson loop diagram



Consider the two sets of propagators $Q = \{(3,6),(4,6),(6,15)\}$ and $R = \{(7,11),(8,10),(11,14),(12,14)\}$. The propagator $p = (2,15)$ is protected at $a = 5$ from contributing 15 to I_5 by the set $P_{15} = Q$. There is no set preventing p from contributing 16 to I_{15}. At $a = 9$, the set $P_{15} = \{(6,15)\} \subset Q$. We may just as easily right $P_{15} = Q$, though the other two propagators in the set prove unnecessary to keep p from contributing 15 to I_9. At $a = 14$, both P_{15} and P_{16} exists, with $P_{15} = \{(11,14),(12,14)\} \subset R$ and $P_{16} = \{(6,15)\} \subset Q$. We may just as easily right $P_{15} = R$ and $P_{16} = R$, though the other two propagators in each set are unnecessary.

Note that for $a = 1$, the propagator p contributes 2 to I_1. However, in this case, we say that neither P_{15} or P_{16} exist.

Lemma 3.5. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $p = (i,j) \in \mathcal{P}$, and $a \in [n]$ be a vertex. For $v \in \{i,j\}$, if $P_v(p) \neq \emptyset$, write $P_v(p) = \{s_1 \dots s_r, t_2, \dots t_r\}$ as above. Then p does not contribute v to I_a if $i_{s_r} + 1 <_{v+1} a <_{v+1} v+1$. If both $P_v(p)$ and $P_{v+1}(p)$ are non empty, then p also does not contribute $v+1$ to I_a.*

Proof. Note that if $s_1 \in P_v(p)$ exists at step v of Algorithm 3.2, then p cannot contribute p to I_a. Further, if $P_{v+1}(p) \neq \emptyset$, p cannot contribute $v+1$ to I_a either.

Next we show that if $P_v(p)$ is as above, with $i_{s_r} + 1 <_{v+1} a$, then the propagator s_1 exists at step v of Algorithm 3.2. Let x be the largest integer that such that $a <_{v_1} j_x + 1$. Then s_x cannot contribute to I_a before vertex v. By construction $P_{i_{s_x}+1}(s_{x-1}) \subset P_v(p)$. Therefore, s_{x-1} does not contribute $i_{s_x} + 1$ to I_a. Thus we conclude that s_1 does not contribute i_{s_1} or $i_{s_1} + 1$ to I_a. In other words, s_1 has not been removed from the diagram at vertex v.

□

Thus we say that $P_v(p)$ protects p at v from a, for $a \in [i_{s_r} + 2, v]$. There is an easy corollary to this, describing when such a protecting set cannot exist.

Corollary 3.6. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $p = (i, j) \in \mathcal{P}$. Picking a $v \in \{i, j\}$ write $p = (i_p, j_p)$ with $i_p <_{v+1} j_p$. If $a \in [j_p + 1, i_p + 1]$ then there can be no protecting set P_v or P_{v+1} protecting p at v or $v + 1$ from a.*

Proof. Since W is admissible, it cannot have crossing propagators. Therefore neither P_v cannot have the requisite propagator s_r with $i_{s_r} + 1 <_{v+1} a$. $\square$

The converse, however requires an extra condition.

Lemma 3.7. *For $W = (\mathcal{P}, n)$, $p = (i, j)$, and $v \in \{i, j\}$ and $a \in [n]$ a vertex of W, if v (resp. $v + 1$) is the first vertex of V_p in the $<_a$ order such that P_v (resp. neither P_v nor P_{v+1}) does not exist, then p contributes v (resp. $v + 1$) to I_a.*

Proof. Write $p = (i_p, j_p)$ with $i_p <_{v+1} j_p$.

If p doesn't contribute v (or $v + 1$) to I_a, then either p contributes a different supporting vertex u earlier in the $<_a$ order or there is a propagator s_1 at step v of the algorithm 3.2 preventing p from contributing v to I_a, (or there are a pair of propagators s_1 and t_1 at step v preventing p from contributing $v + 1$ at I_a.)

If there are propagators $s_1, t_1 \in \mathcal{P}$ that are still in the diagram at step v, (write them $s_1 = (i_{s_1}, j_{s_1})$, and $t_1 = (i_{t_1}, j_{t_1})$) with $j_{s_1} \in \{v, v - 1\}$, and $i_p \leq_{v+1} i_{t_1} <_v i_{s_1} <_v j_{s_1} \leq_v j_{t_1} = v$) then s_1 or t_1 did not contribute i_{s_1} or $i_{s_1} + 1$ (resp. i_{t_1} or $i_{t_1} + 1$) to I_a. Then one of the following three situations hold:

1. $i_{s_1} + 1 <_{v+1} a$ and $j_{s_1} = v$: In this case, s_1 cannot be removed until v.
2. $i_{s_1} + 1 <_{v+1} a$ and $j_{s_1} = v - 1$: in this case there exists a propagator $s_2 = (i_{s_2}, j_{s_2})$ with $i_{s_1} \leq_{v+1} i_{s_2} <_{v+1} j_{s_2}$ and $j_{s_2} \in \{j_{s_1} - 1, j_{s_1}\}$.
3. $i_{s_1} + 1 \not<_{v+1} a$ and $j_{s_1} = v$: and there must be two other propagators $s_2 = (i_{s_2}, j_{s_2})$ and $t_2 = (i_{t_2}, j_{t_2})$, with $j_{s_2} \in \{i_{s_1}, i_{s_1} - 1\}$ and $i_{s_1} \leq_{v+1} i_{t_2} <_{v+1} i_{s_2} <_{v+1} j_{s_2} \leq_{v+1} j_{t_2} = j_{s_1}$

However, now one must ensure that the newly identified propagators s_2 and t_3 do not contribute to I_a before i_{s_1}. One does this by a similar construction to above. In this manner, we begin to build the sets $P_v(p)$ and $P_{v+1}(p)$. By finiteness and admissiblity of the Wilson loop diagram, both these sets are finite; one cannot continue to add propagators s_i, t_i ad infinitum.

We claim that this implies that at some point there is a terminal propagator, $s_m \in P_v(p)$ such that $i_{s_m} + 1 <_{v+1} a$, and

$$\text{either } = \begin{cases} j_{s_m} = i_{s_{m-1}} \\ j_{s_m} = j_{s_{m-1}} \end{cases} .$$

We show that all other possibilities lead to a contradiction. For the remainder of this proof, indicate by p_k some propagator identified at an earlier stage of this construction. That is, for $0 < k < m$, $p_k \in \{s_k, t_k\}$. Futher define $p_0 = p$.

If $j_{s_m} \in \{i_{s_{m-1}} - 1, j_{s_{m-1}} - 1\}$, then, this cannot be the final propagator added to $P_v(p)$, as one would need a means of preventing s_m from contributing $i_{s_{m-1}} - 1$ or $j_{s_{m-1}} - 1$ to I_a.

If $i_{s_m} + 1 \not<_{v+1} a$, then there are two.

$j_{s_m} = i_{s_{m-1}}$ **and** $i_{s_m} \in \{i_{p_k} + 2, i_{p_k} + 1\}$ for some $k < m$. By admissibility, there is not room to place both the propagators s_{m+1}, and t_{m+1}. Thus s_m is removed at the step i_{s_m} or $i_{s_m} + 1$. Therefore, s_1 contributes at either i_{s_1} or $i_{s_1} + 1$. This leads to a contradiction.

$j_{s_m} = i_{s_{m-1}}$ **and** $i_{s_m} = i_{p_k} + 2$ for some $k < m$. This requires either crossing propagators or propagators starting and ending at the same edge, which violates admissibility.

□

correct but clear as mud? Help?

We use these results to prove some properties about the Algorithm 3.2.

Proposition 3.8. *If $W = (\mathcal{P}, n)$ is an admissible Wilson loop diagram with $|\mathcal{P}| = k$, then Algorithm 3.2 puts exactly k distinct labels on each edge.*

Proof. It is enough to show that the algorithm terminates in at most n steps for each $a \in [n]$. In other words, that all k propagators of W are removed by the time the algorithm has cycled through all the vertices, starting at a. Suppose this doesn't happen, i.e. there is a propagator $p = (i, j) \in \mathcal{P}$ which survives the first n steps of the algorithm for some starting vertex a. Without loss of generality, assume $a \leq_a i <_a j$. Otherwise, change the roles of i and j.

Since p is not removed by step $j + 1$, by Lemma 3.5 there are 4 protecting sets, P_i, P_{i+1}, P_j , and P_{j+1} respectively, allowing p to remain after n steps.

However, since the cyclic interval $[i, j] \subseteq [a, j]$, P_j and P_{j+1} cannot be protecting sets by Corollary 3.6. □

Lemma 3.9. *For every $v \in V_p$, Let $A_v \subset [n]$ be the set of vertices where p contributes v to I_a. Then A_v is a non-empty cyclic interval.*

Proof. Fix $p = (i, j)$. By symmetry, we need only consider the vertices j, $j + 1$.

Let $k_1 \ldots k_r$ be the set of propagators of the form $k_x = (i_{k_x}, j)$, with $i <_{j+1} k_1 <_{j+1} \ldots k_r <_{j+1} j$. Write k_{r+1} to be the counterclockwise most propagator with an end on the edge $j - 1$, if it exists. If $r \geq 2$ or $r + 1 = 2$, then both $P_j(p)$ and $P_{j+1}(p)$ exists for some subset of $a \in [n]$. If $r = 1$ or $r + 1 = 2$ then only $P_j(p)$ exists. Otherwise, $P_j(p) = \emptyset$.

Consider a protecting set $P_j(p) \subset \mathcal{P}$ such that $k_1 \in P_j(p)$, if it exists. Write it

$$P_j(p) = \{s_1 \ldots s_r, t_2, \ldots t_r\} .$$

Define $b_1 = i_{s_r} + 1$. By construction, $P_j(p)$ prevents p from contributing j to I_a for all $a \in [b_1 + 1, j]$. We see that, if $r > 0$, for all $a \in [i + 2, b_1]$, p contributes j to I_a. (Note that by construction, one cannot have $i_{s_r} + 1 <_{j+1} i + 2$.) This is because in this range, j is the first vertex of p encountered by Algorithm **??**. Furthermore, by Lemma 3.7, and definition 3.3 if $i_{s_r} + 1 \not<_{j+1} i + 2$ P_1 does not

verify

protect p at j for a in this range. If $r = 0$ then p contributes j for all $a \in [i+2, j]$. In this case, we say $b_1 = j$.

Now suppose r is such that both $P_j(p)$ and $P_{j+1}(p)$ exists for some a. This time $k_2 \subset P_j(p)$. Define $b_2 = i_{s_r} + 1$, where now we use s_r to indicate the last propagator in this new protecting set. By construction, $P_{j+1}(p)$ prevents p from contributing $j+1$ to I_a for all $a \in [b_2+2, j+1]$. Therefore, if both protecting sets exist, for all $a \in [b_1+1, b_2]$ p contributes $j+1$ to I_a. This is exactly the range in which there is exactly one valid protecting set. Therefore, in this range, $j+1$ is the first vertex of p without a protective set. If there is no a such that both $P_j(p)$ and $P_{j+1}(p)$ do not hold, then p contributes $j+1$ for all $a \in [b_1+1, j+1]$.

Finally, for $r > 1$, and $j_{k_2} = j$,$a \in [b2+1, j+1]$, p contributes i to I_a. Again, for all a in this range, both sets $P_j(p)$ and $P_{j+1}(p)$ protect p at j and $j+1$ from a. Furthermore, by Corrollary **??**, there can be no protective sets of i or $i+1$ in this range.

By switching the roles of i and j, we see that for every element of V_p, there is a non-empty cyclic interval such that p contributes v to all I_a for in said interval. □

Proposition 3.10. *The sequence of k-subsets obtained by applying Algorithm 3.2 to an admissible diagram W is exactly the Grassmann necklace of the positroid associated to W; specifically if we let I_i be the set of labels on edge $(i, i+1)$ then the sequence $\mathcal{I} = (I_1, \ldots, I_n)$ is the Grassmann necklace of the positroid associated to W.*

Proof. Let I_i be the set of labels on edge $(i, i+1)$; the sequence $\mathcal{I} = (I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in \{1, \ldots, n\}$.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_{i\backslash\{i\}}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.2 at vertex i and vertex $i+1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i. Either p has one end on the edge $(i-1, i)$ or it has one end on the edge $(i, i+1)$. In both cases let $(b, b+1)$ be the edge with the other end of p.

case I: Suppose p has one end on $(i-1, i)$. Then p is not supported on $i+1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i+1$ up to $b-1$, that is $I_{i+1} \cap [\![i+1, b-1]\!] = I_i \cap [\![i+1, b-1]\!]$. Furthermore, by Lemma **??**, in building I_{i+1}, it must be that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when at vertex b no propagator still remaining is before p. This is also true in building I_i when at b since the same propagators have been taken beforehand. Additionally $k \geq_i b+1$.

Let W' be the diagram obtained from W by removing p and all propagators under it in the sense of supported between $i+1$ and $b-1$.

By the above observations when we are in W' at b then we are in the same situation with respect to the remaining propagators as if we began at i in W and moved to b following the algorithm; the propagators we took in the latter case are exactly the ones removed to build W'. Similarly starting at $i+1$ in W and moving to $b+1$ leaves us in the same situations with respect to the remaining propagators as beginning at $b+1$ in W'. This gives the equations

$$I_i^W \cap [\![b, i-1]\!] = I_b^{W'}$$
$$I_{i+1}^W \cap [\![b+1, i-1]\!] = I_{b+1}^{W'}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{W'} \backslash \{b\}$ and $k \notin I_{b+1}^{W'}$ contradicting the minimality of (W, i, k).

case II: Suppose p has one end on $(i, i+1)$. Note that by definition p is the propagator contributing i to I_i and so it must be the first propagator in $(i, i+1)$ and hence p contributes $i+1$ to I_{i+1}. Observe that $k \geq_i i+2$ since $i+1 \in I_{i+1}$.

Let W' be the diagram obtained from W just by removing p. Then

$$I_i^W \backslash \{i\} = I_{i+1}^{W'}$$
$$I_{i+1}^W \backslash \{i+1\} = I_{i+2}^{W'}$$

since in both cases in W' we are simply taking the remaining propagators in the same way as we would have in W after taking p for the previous vertex. Since $k \neq i+1$, we have $k \in I_{i+1}^{W'} \backslash \{i+1\}$ but $k \notin I_{i+2}^{W'}$ contradicting the minimiality of (W, i, k)

We have shown that $\mathcal{I}$ is a Grassmann necklace; it remains to check that this Grassmann necklace defines the positroid $\mathcal{P}_W$ associated to W. We need to show that:

- For each $i \in [\![1, n]\!]$, I_i is a basis for $\mathcal{P}_W$.
- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $\mathcal{P}_W$.

The algorithm is pairing each $j \in I_i$ with a unique propagator supported on that vertex so by Lemma **??** I_i is a basis for $\mathcal{P}_W$.

Suppose we have J such that J is a basis and yet is lexicographically less than I_i with respect to $<_i$. By Lemma **??** there is a set bijection between J and the propagators of W such that the propagator associated to j is supported on vertex j. Choose one such bijection. For a propagator p of W write $J(p)$ for the associated j according to this bijection. Similarly write $I_i(p)$ for the vertex assigned to p by the algorithm. Since J is lexicographically smaller than I_i, the $<_i$-smallest element of the symmetric difference of J and I_i is some $j_0 \in J$, $j_0 \notin I_i$.

Let p be the propagator such that $J(p) = j_0$. Since $j_0 \notin I_i$ but p is supported at j_0, then $I_i(p) <_i j_0$. Thus the propagator p has the following property: $I_i(p) <_i J(p)$ and $I_i \cap [\![i, I_i(p)]\!] = J \cap [\![i, I_i(p)]\!]$. Call this property A.

From the previous paragraph we conclude that if we ever had a J which is lexicographically less than I_i and yet a basis of $\mathcal{P}_W$ then there exists a propagator p which has property A.

Finally, we will prove that whenever there is a propagator p which has property A then there is another propagator r which also has property A and for which $I_i(r) <_i I_i(p)$. Since I_i is finite this would lead to infinte regress and hence is impossible and thus there can be no such J which will complete the proof of the proposition.

So suppose p has property A. The same vertices in $[\![i, I_i(p)]\!]$ are in the image of the assignment from J and the image of the assignment from I_i so the same number of propagators are assigned to vertices in this interval by each assignment. However p is one of the propagators assigned to a vertex in this interval in I_i but not in J. Thus there exists a propagator q for which $J(q) \in [\![i, I_i(p)]\!]$ but $I_i(q) >_i I_i(p)$. Therefore $J(q) <_i I_i(q)$ and $I_i \cap [\![i, J(q)]\!] = J \cap [\![i, J(q)]\!]$. This is analogous to property A but with the roles of J and I_i switched. Thus by the same argument we must have a propagator r for which $I_i(r) \in [\![i, J(q)]\!]$ but $J(r) >_i J(q)$. Therefore r has property A which is what we wanted to prove.

□

3.2 Dimension of the Wilson Loop cells

In [**?**], Agarwala and Fryer give an algorithm for passing from a Grassmann Necklace to a Le diagram. We use this algorithm here to pass from a Wilson loop diagram to its associated Le diagram. In this manner, we show that the positroid cell defined by a Wilson loop diagram has dimension $3|\mathcal{P}|$. Maybe say something about Amplituhedra having $4k$ dimension, but this is not quite what we have, since we are ignoring one column.

When I speak of a *WLD*, I mean one which satisfies your density hypothesis and other standard hypotheses. When I speak of a propagator *contributing* a vertex to an element of the Graßmann[2] necklace I mean that according to the rule you guys have to build the Graßmann necklace from the diagram by starting at a vertex and taking the clockwise-most covering propagator which hasn't already been taken, when a propagator is taken then it is contributing that vertex. Also, I will be using your algorithm to convert the Graßmann necklace into a Le diagram by non-intersecting paths.

We need four lemmas from stuff you guys have already figured out. The first one is a corollary of your lemma characterizing intervals contributed by each covered vertex of a propagator.

The first lemma is Lemma **??** and I moved it up to a previous section.

The second lemma is the fact that we understand uncovered vertices. You probably see many ways to prove this from things you already know. One would be to say that from your algorithm the column of the uncovered vertex simply plays no role.

Lemma 3.11. *Let D be a WLD with an uncovered vertex i. Let C be D with i removed. Then the Le diagram of D is the Le diagram of C with an extra column of all 0s inserted in the $|D| - i + 1$th position from the left.*

The third lemma is Sian's result.

[2] I suppose it is a bit of an affectation to use an eszett when writing in English, and actually the eszett is kind of ugly in this font, but since I've started to I'll stick with it for now but you certainly shouldn't feel obliged to do it too.

Lemma 3.12. *Let D be a WLD with all vertices covered by at least two propagators. Then There exist two propagators in D with the following properties.*

- *The first propagator goes from the edge i, $i+1$ to the edge $i+2$, $i+3$.*
- *The second propagator goes from the edge $i+2$, $i+3$ to the edge $i+4$, $i+5$*
- *No other propagator is on the edge $i+2$, $i+3$.*

Actually the result I need is not exactly Sian's result and what I will actually use is her proof. Thus I will use notation and definitions (including length of a propagator and the specific p_i and q_i construction from the proof) from her note without further discussion. The result I will actually use will be that in place of the hypothesis that D has all vertices covered by at least two propagators I have that all vertices are covered by at least 1 propagator and there are no propagators of length 3.

The fourth lemma is that we can rotate or reflect without changing the number of plusses. The best way to see this is probably geometrically so I leave that up to you.

Lemma 3.13. *If two WLDs differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Now we're ready to go.

3.2.1 Changes in Graßmann necklaces for nice configurations

Given a propagator p in a WLD D, note that p divides remaining propagators of D into two sets depending on which side of p they live.

Lemma 3.14. *Let D be a WLD with $n \geq 1$ propagators. Then there is some dihedral transformation D' of D such that there is a propagator p with the following properties.*

- *p goes between the edge i, $i+1$ and the edge $n-1$, n in D'.*
- *$i+2, \ldots, n-2$ are not covered by any propagators in D' (which is trivially true if $\{i+1, \ldots, n-2\} = \varnothing$).*
- *Either $i+1$ in D' is only covered by p, or $i+1$ is covered by exactly one other propagator q.*
- *If we are in the second case above then q goes between the edge j, $j+1$ and the edge i, $i+1$ and $j+2, \ldots, i-1$ are not covered by any propagators.*

Proof. Since $n \geq 1$ the dual tree of D has at least two vertices and so has at least two leaves. Each edge going to a leaf of the dual tree corresponds to a propagator which has no other propagators on one side of it, so there is at least one such propagator in D. There are two cases to consider.

First suppose there is a propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let this propagator be p. Rotate and reflect D as necessary so that the end of p which does not share its edge comes first, then the side of p with no other propagators, and then the other end of p which is on edge $n-1$, n. Call this WLD D'. It is a straightforward check that the properties in the statement are satisfied with $i+1$ only covered by p.

Next suppose there is no propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let C be D with all uncovered vertices removed. Suppose C contains two propagators one of length 2 and one of length 3 with the length 2 propagator on the small side of the length 3 propagator. Then we are in the case already considered because we can rotate and reflect C so that these two propagators both have one end on the edge $|C|-1$,$|C|$, and have their other ends on the edge $|C|-2$, $|C|-3$ and the edge $|C|-3$, $|C|-4$, so taking the length 2 edge as p we satisfy the properties of the statement, and this remains true in D. Thus we now assume this does not occur, so in particular every propagator of length 3 in C has no other propagators on its small side. But then the middle vertex on this small side is uncovered contradicting the construction of C. Thus C has no propagators of size 3.

The claim then is that in C there must exist two propagators as in Lemma 3.12. The proof of the claim follows by Sian's proof. We do not have the hypothesis that all vertices are covered by at least two propagators, but this is only used in the proof of Lemma 3.12 to force certain propagators (the q_i) to have length at least 4, so our this case we instead use that C has no propagators of size 3.

Now rotate and flip C so that the two propagators as in Lemma 3.12 cover $\{|C|-5,\ldots,|C|\}$ and let p be the propagator covering $\{|C|-3,\ldots,|C|\}$ and let q be the other one. Return to D with a corresponding rotation and flip so that vertex n in D corresponds to vertex $|C|$ in C and direction in the polygon is preserved. Let this dihedral transformation of D be D'. Then the propagator p satisfies the conditions of the statement with propagator q. $\square$

Given a WLD D, I_i^D denotes the ith Graßmann necklace element corresponding to D.

Lemma 3.15. *Let D be a WLD with $n \geq 1$ propagators and let p be a propagator of D so that the properties of Lemma 3.14 are satisfied for p in D (that is D is already appropriately transformed). Furthermore suppose that that p has length 2, as does q if we are in the case of Lemma 3.14 involving q. Let C be D with p removed but no change in the vertices. Then*

$$
\begin{aligned}
I_1^D &= I_1^C \cup \{n-3\} \\
I_n^D &= I_1^C \cup \{n\} \\
I_{n-1}^D &= I_n^C \cup \{n-1\} \\
I_{n-2}^D &= \begin{cases} I_{n-2}^C \cup \{n-2\} & \text{if } n-2 \notin I_{n-2}^C \\ I_{n-2}^C \cup \{n-1\} & \text{if } n-2 \in I_{n-2}^C,\ n-1 \notin I_{n-2}^C \\ (I_n^C - \{n-5\}) \cup \{n-1, n-2\} & \text{if } n-1, n-2 \in I_{n-2}^C \end{cases} \\
I_k^D &= \begin{cases} I_k^C \cup \{n-3\} & \text{if } n-3 \notin I_k^C \\ I_k^C \cup \{n-2\} & \text{if } n-3 \in I_k^C \end{cases} \\
&\text{for } 1 < k < n-2
\end{aligned}
$$

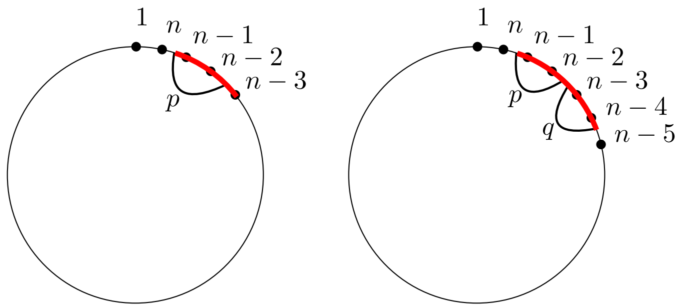


Figure 1: The different possibilities for D and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

Proof. The two possible situations are illustrated in Figure 1.

If $n-3 \in I_1^C$ then $n-3 \notin I_1^D$, but then p never contributes $n-3$ contradicting Lemma **??**. Thus $n-3 \notin I_1^C$ and so in building the Graßmann necklace when we get to vertex $n-3$ any other covering propagators of C have already been taken and so we can now take p. Therefore $I_1^D = I_1^C \cup \{n-3\}$.

In constructing I_n^D, first for vertex n we take propagator p. Then we are at vertex 1 and precisely the propagators of C remain. Thus the rest of the construction will give I_1^C. Therefore $I_n^D = I_1^C \cup \{n\}$. Essentially the same reasoning gives $I_{n-1}^D = I_n^C \cup \{n-1\}$.

Now consider I_{n-2}^D. If $n-2 \notin I_{n-2}^C$ then at vertex $n-2$ we take p and this does not affect the rest of the construction of I_{n-2}^C, so $I_{n-2}^D = I_{n-2}^C \cup \{n-2\}$. An analogous argument takes care of the first case for I_k^D. Suppose $n-2 \in I_{n-2}^C$ but $n-1 \notin I_{n-2}^C$. Then at vertex $n-2$ we take the same propagator in C as in D (in particular not p) because p is the counterclockwisemost propagator covering $n-2$ and so the last propagator the algorithm would choose at this vertex, and by hypothesis $n-2$ is covered in C. Howver, $n-1$ is untaken in I_{n-2}^C so at this vertex we will take p in I_{n-2}^D. Following this, now that p is out of the way without bumping any other propagators, the construction continues as in I_{n-2}^C. Therefore $I_{n-2}^D = I_{n-2}^C \cup \{n-1\}$. An analogous argument takes care of the case when we have $1 < k < n-2$ and $n-3 \in I_k^C$ but $n-2 \notin I_k^C$. Furthermore, either $n-2$ is uncovered in C or only q covers $n-2$ in C and q is also the only propagator covering $n-3$. Thus it is not possible for $n-3$ and $n-2$ to both be in I_k^C. This means that all the cases for I_k^D are now proved.

The remaining case is when $n-1, n-2 \in I_{n-2}^C$ for the construction of I_{n-2}^D. We must then have a propagator q as in the right hand side of Figure 1. In the construction of I_{n-2}^D, at vertex $n-2$ we take propagator q, as in I_{n-2}^C. Then at vertex $n-1$ we take propagator p which is different from what occurs in I_{n-2}^C. Next we are at vertex n and propagators p and q have been taken. Thus we are proceeding like I_n^C but without propagator q. Fortunately we can determine explicitely how propagator q contributes to I_n^C. By Lemma **??** propagator q contributes $n-5$ to I_n^C, and the only way this can occur is if all other propagators of C were already taken by the time we got to vertex $n-5$. Therefore $I_{n-2}^D = (I_n^C - \{n-5\}) \cup \{n-1, n-2\}$. This covers all cases and hence completes the proof. $\square$

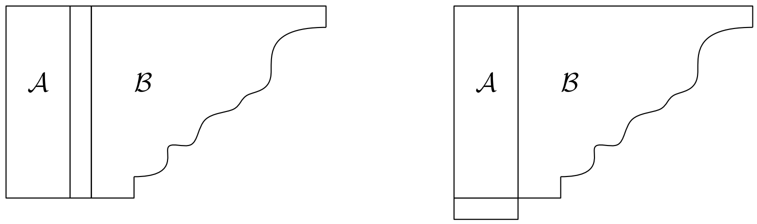


Figure 2: Le diagrams for C (left) and D (right).

3.2.2 The number of plusses from Graßmann necklaces in nice configurations

Lemma 3.16. *Let C and D be as in Lemma 3.15. The shape of the Le diagram of C can be built from left to right of the following blocks: a rectangle with 3 columns, one more column of the same length, a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of D can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of C, the same partition shape as in C.*

Proof. I_1 determines the shape of the Le diagram. By Lemma 3.15, $I_1^D = I_1^C \cup \{n-3\}$. This implies that the right hand boundary of the shape of C is the same as the right hand boundary of the shape of D except that D has one additional row of 3 boxes while C has an additional column in the $n-3$ position, that is a new column fourth from the left. $\square$

The shapes of the Le diagrams of C and D are illustrated in Figure 2. The pieces of the Le diagrams will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows, as in the figure. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of the Le diagrams of C and D are identical and the plusses in the $\mathcal{A}$ parts are very closely related. When we speak of a plus in the Le diagram of D being the same as in D or vice versa, we mean that the plus' position in $\mathcal{A}$ or $\mathcal{B}$ is the same. Because of the column insertion the absolute indices may differ.

Suppose we are following the Graßmann necklace to Le diagram algorithm, and we put a plus in a box because of a path from vertical boundary edge i to bottom boundary edge j. Then say this plus is in the $i \to j$ position.

Lemma 3.17. *Let C and D be as in Lemma 3.15. The I_n^D and I_{n-1}^D elements of the Graßmann necklace of D give all the same plusses as I_n^C along with plusses in the leftmost two boxes of the bottom row of the Le diagram of D.*

Proof. By Lemma 3.15 $I_n^D = I_1^C \cup \{n\}$, so by the Graßmann necklace to Le diagram algorithm the only plus this builds in the Le diagram of D is the one in the $n-3 \to n$ position, that is in the leftmost box of the bottom row.

Also by Lemma 3.15 $I_{n-1}^D = I_n^C \cup \{n-1\}$. Additionally $n-3 \notin I_n^C$ since if it were then $n-3$ would also be in I_1^C and hence propagator p could not contribute $n-3$ to I_1^D in contradiction to Lemma **??**. Similarly $n-1, n-2 \notin I_n^C$. Thus the paths putting the plusses in from I_n^C lie completely in $\mathcal{B}$ or take some vertical boundary edge $> n-3$ to n. Now view these paths in the Le diagram of D and note that the path $n-3 \to n-1$ is compatible, and so these paths together build the plusses that I^D_{n-1} contributes. That is, we get all the plusses from I^C_n along with a $n-3 \to n-1$ plus, that is a plus in the second to the right box of the bottom row. $\square$

Lemma 3.18. *Let C and D be as in Lemma 3.15 with $n-2 \notin I^C_{n-2}$.*

The I^D_{n-2} element of the Graßmann necklace of D gives an $n-3 \to n-2$ plus and all the $I^C_{n-1} = I^C_{n-2}$ plusses.

Proof. If $n-2 \notin I^C_{n-2}$ then $I^C_{n-1} = I^C_{n-2}$ and by Lemma 3.15 $I^D_{n-2} = I^C_{n-1} \cup \{n-2\}$. Note that $n-3 \notin I^C_{n-2}$ by Lemma **??**. Therefore the paths for I^D_{n-2} are the paths for I^C_{n-1} along with the $n-3 \to n-2$ path. This gives the statement of the lemma.

$\square$

Lemma 3.19. *Let C and D be as in Lemma 3.15 with $n-2, n-1 \in I^C_{n-2}$.*

The I^D_{n-2} and I^D_{n-3} elements of the Graßmann necklace of D gives the following plusses:

- *An $n-3 \to n-2$ plus and an $n-5 \to n-1$ plus.*
- *All the I^C_{n-1} plusses.*
- *I^C_{n-2} gives an $n-5 \to n-2$ plus and no other term in the Graßmann necklace of C gives a plus in this column. This $+$ does not appear in D from I^D_{n-2} but an $n-5 \to n-1$ plus does instead.*
- *All other plusses of I^C_{n-2}.*
- *I^C_{n-3} gives a plus in the $n-3$ column. This $+$ is shifted over into the $n-2$ column in D.*
- *All other plusses of I^C_{n-3}.*

Furthermore, no element of the Graßmann necklace of C gives an $n-5 \to n-1$ plus.

Proof. By Lemma 3.15 $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$. Also, by the location of q in the WLD, $n-2 \notin I^C_{n-3}$ and and $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Graßmann necklace of C looks like

$$I^C_{n-3} \underset{\substack{n-3 \text{ out} \\ n-2 \text{ in}}}{\to} I^C_{n-2} \underset{\substack{n-2 \text{ out} \\ n-5 \text{ in}}}{\to} I^C_{n-1} \underset{\substack{n-1 \text{ out} \\ \text{something in}}}{\to} I^C_n \underset{\substack{n \text{ out} \\ \text{something in}}}{\to} I^C_1 \tag{8}$$

where the first "something" is either n or an element of I^C_1 and the second "something" is an element of I^C_1. Additionally all elements not explicitly mentioned must be in I^C_1 as they remain unchanged through this portion of the necklace.

Using this information now determine the symmetric difference of I^C_{n-2} and I^C_1: $n-1, n-2$ and possibly n are in I^C_{n-2} but not in I^C_1. $n-5$ is in I^C_1 as are at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$. This means that the plusses in the Le diagram of C coming from I^C_{n-2} are as in the first part of Figure 3. Stepping to I^C_{n-1} simply removes the $n-5 \to n-2$ path, see the second part of Figure 3.

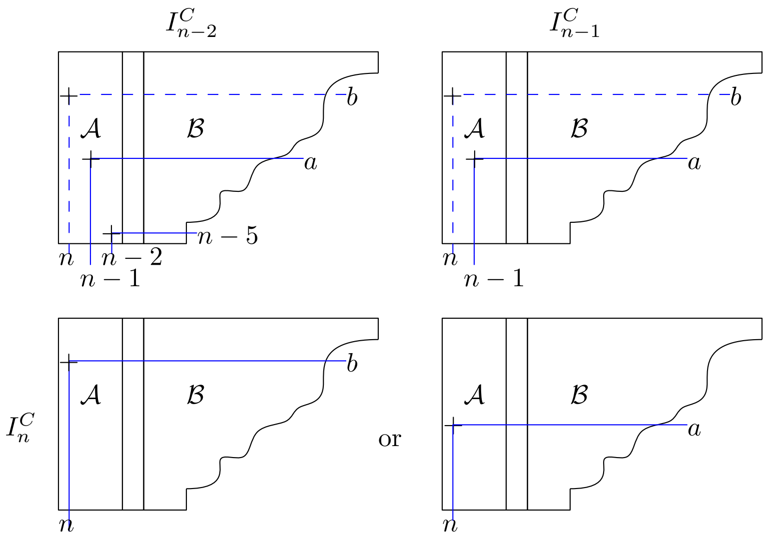


Figure 3: Plusses coming from I^C_{n-2} (top left), I^C_{n-1} (top right) and I^C_n bottom when $n-1, n-2 \in I^C_{n-1}$. The blue lines are the non-intersecting paths. The dashed blue lines may or may not appear, but if one appears then they both do.

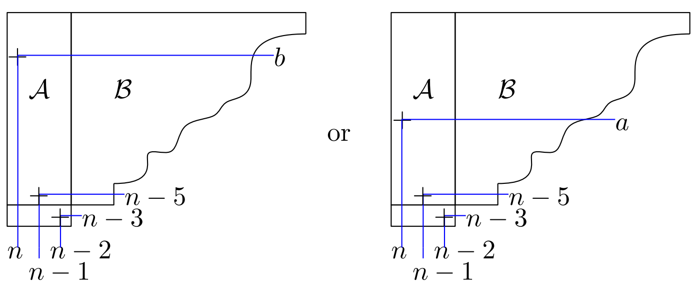


Figure 4: Plusses coming from I^D_{n-2}.

Stepping to I^C_n, $n-1$ is taken out and either n is put in if it was not there before, or one of a or b is put in and hence no longer available as a right end for a path. This gives two possible configurations illustrated in the bottom two parts of Figure 3.

Now we know that $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$ so the paths for building plusses from I^D_{n-2} go from the set $\{n-5, n-3\}$ along with whichever of a and b is not in I^C_n to $\{n-2, n-1, n\}$. This means that we get plusses as in Figure 4 where the left and right cases correspond to the left and right cases in the bottom parts of Figure 3

This proves the first item of statement of the lemma.

Now consider I^C_{n-3}. By (8) I^C_{n-3} contributes the same plusses as I^C_{n2} except that it contributes an $n-5 \to n-3$ plus in place of the $n-5 \to n-2$ plus. Also, we have $n-3 \in I^C_{n-3}$ be the location of q and so $I^D_k = I^C_k \cup \{n-2\}$. Thus the paths for I^D_{n-3} are the same as for I^C_{n-3} except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (8) shows that $n-2$ only appears in I^C_{n-2} among the necklace elements of C.

Also note that I^C_{n-3}, I^C_{n-2}, and I^C_{n-1} share their plusses outside of the $n-3$ and $n-2$ columns. This proves the remaining statements of the lemma except the furthermore.

Finally, suppose there were a $n-5 \to n-1$ plus in the Le diagram of C. By the algorithm, it would have to come when $n-5 \notin I^C_j$. By (8) this means that it would have to come from I^C_{n-4}, I^C_{n-3}, or I^C_{n-2}. The analysis above shows it does not come from I^C_{n-3} or I^C_{n-2}. Now, $n-4 \in I^C_{n-4}$ by the location of q and so I^C_{n-4} must give an $n-5 \to n-4$ plus and so cannot give an $n-5 \to n-1$ plus.

□

Lemma 3.20. *Let C and D be as in Lemma 3.15 and take $1 < k < n-2$. Suppose that if $n-2 \in I^C_{n-2}$ then also $n-1 \in I^C_{n-2}$. The I^D_k element of the Graßmann necklace of D gives the same plusses as I^C_k except that if there is a plus in the $n-3$ column for I^C_k then this plus is shifted into the $n-2$ column and no plus was already in that location.*

Proof. If $n-3 \notin I^C_k$ then by Lemma 3.15 $I^D_k = I^C_k \cup \{n-3\}$. Then since $n-3$ is the largest element of I^D_1 this transformation leaves the disjoint paths unchanged and so the plusses carry over from C to D directly.

If $n-3 \in I^C_k$ then by Lemma 3.15 $I^D_k = I^C_k \cup \{n-2\}$. If $n-2$ is not covered in C then certainly no pluses appear in the $n-2$ column of the Le diagram of C. If $n-2$ is covered in C then by hypothesis so is $n-1$ and so we satisfy the hypotheses of Lemma 3.19. Thus the only necklace element of C containing $n-2$ is I^C_{n-2} and this particular plus is not contributed to the Le diagram of D by I^D_{n-2}.

From I^C_k there is a path from some vertical edge to the bottom edge $n-3$. In I^D_k, $n-3$ is a vertical edge with no path and instead there must be a path to $n-2$. By the previous paragraph no other path can end in $n-2$, so shifting the path that did go to $n-3$ to go to $n-2$ while leaving the others the same maintains non-crossingness and so must be the paths for I^D_k. Thus the plus in the $n-3$ column for C is shifted into the $n-2$ column, where there was no plus before, and no other plusses are changed. □

Theorem 3.21. *The number of plusses in the Le diagram of a WLD is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a WLD D with one propagator covering vertices $i < j < k < \ell$ has Le diagram a single row with $|D| - i$ boxes. Labelling them from left to right by $|D|, \dots, |D| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider WLDs with $k > 1$ propagators. By Lemma 3.11 it suffices to prove the result for WLDs with k propagators and no uncovered vertices. By Lemma 3.13 it suffices to prove the result for at least one WLD from each dihedral orbit. Take a WLD diagram D with k propagators. Make a dihedral transformation of D if necessary so that D has a propagator p with the properties in Lemma 3.14 relative to D. If $n-1$ is only covered by p but $n-2$ is covered by at least one other propagator, then flip D on the line perpendicular to the edge from $n-2$ to $n-1$. This will be our D for the rest of the proof.

Let C be D with p removed but no change in the vertices. Note that if $n-2$ is covered in C then so is $n-1$ by the end of the previous paragraph and so if $n-2$ is covered in C then the hypotheses of Lemma 3.19 are satisfied.

From Lemma 3.16 we know how the shapes of the Le diagrams of C and D relate; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 3.17, 3.18, and 3.19 tell us that the three boxes of the bottom row of the Le diagram of D each have a plus. Lemmas 3.17, 3.18, 3.19, and 3.20 show that there is a bijection between the plusses of the Le diagram of C and the plusses of the Le diagram of D that are not in the bottom row which can be described as follows.

- Plusses from $\mathcal{B}$ for C maintain their positions in $\mathcal{B}$ for D.
- Plusses from the first two columns (the n and the $n-1$ columns) of $\mathcal{A}$ for C maintain their positions in $\mathcal{A}$ for D.
- If there is a plus in the $n-2$ column of $\mathcal{A}$ in then Lemma 3.19 applies, so there is exactly one such plus. This plus maps to the $n-5 \to n-1$ plus for D.
- The plusses in the $n-3$ column for C shift over to the third column (the $n-2$ column) of $\mathcal{A}$ in D.

This map is clearly reversible and hence bijective except possibly for the $n-5 \to n-1$ plus for D. If the Le diagram of D has an $n-5 \to n-1$ plus then look at the Le diagram for C. If the Le diagram for C has a plus in the $n-2$ column then Lemma 3.19 applies and so there is no $n-5 \to n-1$ plus in the Le diagram of C and the $n-5 \to n-1$ plus of D can be uniquely mapped to the plus in the $n-2$ column of the Le diagram of C. If the Le diagram for C has no plus in the $n-2$ column, then leave the $n-5 \to n-1$ plus where it is in moving back to C. This reverses the map.

From all of this we get that the number of plusses in the Le diagram for D is three more than the number of plusses in the Le diagram for C. Applying induction completes the proof. □

Maybe proved that inadmissible diagrams, while they correspond to matroids, can't be mapped to the positroids of the correct dimension?

4 Poles of Wilson Loop Integrals

4.1 Some definitions and results that should probably be moved to section 3 but I don't know if anyone's editing that right now

It will often be useful to refer to the vertex label contributed by a propagator to a Grassmann necklace term. Notationally, we represent this by allowing the I_i symbol to represent a function as well as a set, as follows:

Definition 4.1. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. For each $i \in [\![1, n]\!]$, define a function $I_i : \mathcal{P} \longrightarrow [\![1, n]\!]$ by

$$I_i(p) := \text{the vertex label that } p \text{ contributes to } I_i \text{ in Algorithm 3.2,}$$

for each $p \in \mathcal{P}$.

Definition 4.2. Let p be a propagator supported on (i, j). The region bounded by p and the edge of the Wilson loop diagram from i to $j+1$ is refered to as the region **inside** p, and the complement is the region **outside** p. The set of propagators inside (resp. outside) p are those lying in the region inside (resp. outside) p, excluding p itself.

Depending where this ends up, possibly include an example here.

Note that this depends on the order of i and j: if we view p as supported on (j, i) then the regions inside and outside p are swapped.

Lemma 4.3. *Let W be an admissible Wilson loop diagram containing at least one propagator. For any $i \in [n]$ and for any $p = (a, b)$ with $i \leq_i a <_i b$, we have $I_i(p) \neq b + 1$.*

Proof. Suppose for contradiction that we have $p = (a, b)$ with $i \leq_i a <_i b$ and $I_i(p) = b + 1$. We may choose p such that $|[\![a + 1, b]\!]|$ is minimal amongst propagators with this property.

Since $I_i(p) \neq b$, there must exist a propagator q inside of p with $I_i(q) = b$. The propagator q cannot end on the edge $(b - 1, b)$, as this would contradict the minimality of p, so $q = (c, c + 1, b, b + 1)$ with $a <_i c <_i b$, and $I_i(q) = b$.

In order for q to remain unassigned until vertex b, there must be another propagator r with an end on $(c, c + 1)$ and $I_i(r) = c + 1$; the only way this can occur is if r is outside q but inside p. Now r contributes its fourth vertex to I_i, again contradicting the minimality of p. □

Corollary 4.4. *If W is an admissible Wilson loop diagram with k propagators, then Algorithm 3.2 assigns exactly k vertices to each I_i.*

Proof. It follows from the proof of Lemma 4.3 that Algorithm 3.2 can never reach the fourth vertex of a propagator's support (with respect to the starting vertex). Therefore if the algorithm starts at vertex i, it must have assigned vertices to all propagators by the time it reaches $i - 1$, ensuring that I_i contains exactly k distinct vertices. □

Corollary 4.5. *Grassmann necklaces coming from admissible Wilson loop diagrams have no coloops, that is no indices a such that $a \in I_i$ for all i.*

Proof. For any $a \in [n]$, $a - 1$ is maximal with respect to the $<_a$ order. Therefore there is no propagator p with $I_a(p) = a - 1$ by Lemma 4.3, i.e. $a - 1 \notin I_a$. □

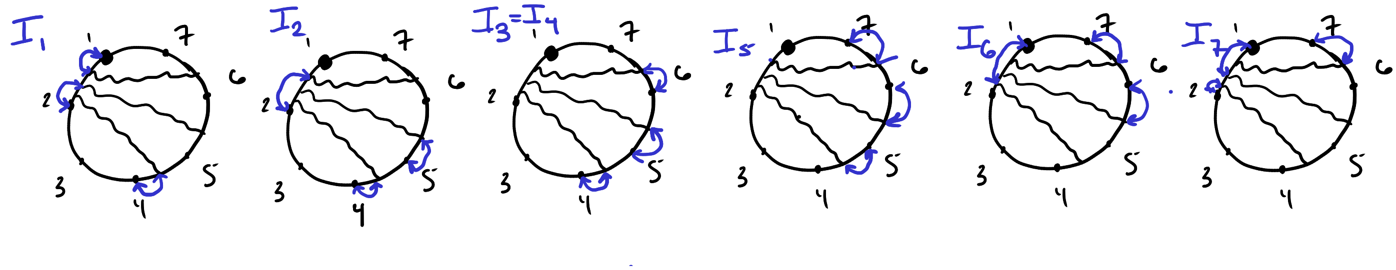


Figure 5: Example WLD for illustrating Algorithm 4.6 and bijections between propagators and vertices for each Grassmann necklace element.

4.2 actual beginning of section 4; this subsection header can go away once the results above have found their place

The results of Section 3 allow us to relate the position of propagators in a Wilson loop diagram W to minors of $C(W)$, which we use in this section to understand the denominator of the integral $I(W)$ associated to a Wilson loop diagram (see Definition 1.10).

The main result of this section is Theorem 4.9, which expresses the denominator $R(W)$ in terms of the Grassmann necklace of W. This simplifies the computation of $R(W)$ and allows us to directly relate the poles of the integral to the combinatorics of the diagram.

We first give an algorithm which extracts the required minors from the Grassmann necklace.

Algorithm 4.6. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and let $C(W)$ be the matrix of W as defined in* (1) *(see Section 1).*

- *For each $i \in [\![1, n]\!]$, we construct a factor r_i as follows:*
  - *Let $S_i = \{p \in \mathcal{P} \mid I_{i-1}(p) \neq I_i(p)\}$. (By convention, set $I_{-1} = I_n$.)*
  - *Write Δ_{I_i} for the determinant of the $k \times k$ minor of $C(W)$ with columns indexed by I_i.*
  - *Let r_i be Δ_{I_i} with all variables from rows associated to $p \notin S_i$ set to 1.*
- *Define $R = \prod_{i=1}^{n} r_i$.*

What are we doing about the sign of the determinants?

Example 4.7. Consider the Wilson loop diagram in Figure 5. Assigning propagators p, q, s to rows $1, 2, 3$ respectively, we obtain the matrix

I've named the propagators p, q, s; could whoever codes up the figure please add that?

$$C(W) = \begin{bmatrix} a & b & 0 & 0 & 0 & c & d \\ e & f & 0 & 0 & g & h & 0 \\ i & j & 0 & k & l & 0 & 0 \end{bmatrix}$$

The Grassmann necklace of this diagram is

$$I_1 = \{1,2,4\}, I_2 = \{2,4,5\}, I_3 = \{4,5,6\}, I_4 = \{4,5,6\},$$
$$I_5 = \{5,6,7\}, I_6 = \{6,7,1\}, I_7 = \{7,1,2\}.$$

Figure 5 also indicates the pairings between propagators and vertices for each $i \in [\![1,7]\!]$.

From I_1 to I_2, the propagators p and q change which vertex they are assigned to but r is assigned to vertex 4 in both, so $S_2 = \{p,q\}$. Then

$$\Delta_{I_2} = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ j & k & l \end{bmatrix} = kgb, \qquad r_2 = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ 1 & 1 & 1 \end{bmatrix} = gb.$$

where the 1s in the third row of the second matrix correspond to the fact that $I_1(s) = I_2(s)$. Continuing likewise, we get $r_3 = c$, $r_4 = 1$ (since $I_4 = I_3$), $r_5 = lhd$, and $r_6 = i$.

At I_7 the situation is more complicated: we have $S_7 = \{q,s\}$, so we find that $\Delta_{I_7} = d(ej - fi)$ and $r_7 = ej - fi$. This quadratic factor corresponds to the fact that q and s share an edge *and* contribute both endpoints of that edge to I_7; see Proposition 4.8 below.

Finally, we have $r_1 = (af - be)k$. Putting everything together, we obtain

$$R = (af - be)kgbclhdi(ej - fi)$$

which is squarefree and contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$. If one were to construct the denominator $R(W)$ associated to this Wilson loop diagram as per Definition 1.10, we would find that (up to integer multiples) we have $R(W) = R$.

Proposition 4.8. *With notation as in Algorithm 4.6 we have the following:*

1. *Each Δ_{I_i} is homogeneous, as is each r_i.*
2. *Each Δ_{I_i} splits into linear and quadratic factors. All linear factors of Δ_{I_i} are single variables and all irreducible quadratic factors are 2×2 determinants of single variables.*
3. *Quadratic factors in r_i arise precisely when propagators p and q are supported on a common edge (a,b) with $I_i(p) = a$ and $I_i(q) = b$.*
4. *r_i divides Δ_{I_i}.*
5. *The ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.*

Proof. 1. The nonzero entries of $C(W)$ are independent indeterminates and so every $i \times i$ minor of $C(W)$ is either homogeneous of degree i or is 0. Thus each Δ_{I_i} is homogeneous. Furthermore, each row contributes one factor to each term in the expansion of Δ_{I_i} so the result of setting the variables from a subset of rows to 1 is still homogeneous. Thus each r_i is homogeneous.

2. Using the expression for the determinant as a sum over permutations we see that Δ_{I_i} is a sum over bijections between I_i and $\mathcal{P}$. The nonzero terms in this sum are precisely those bijections such that each propagator is associated to one of its supporting vertices in I_i, since only those locations in $C(W)$ are nonzero. Since the nonzero entries of $C(W)$ are independent there can be no cancellation between terms in this expansion.

   Suppose Δ_{I_i} has an irreducible factor f. Let $\mathcal{P}'$ be the set of propagators which contribute a variable to f and let J be the set of vertices which contribute a variable to f.

   The first claim is that the minor of $C(W)$ associated to $\mathcal{P}'$ and J is precisely f.

   Proof of claim: By the structure of determinants we know that $\Delta_{I_i} = fg$, where g involves only variables associated to propagators not in $\mathcal{P}'$ and associated to vertices not in J.

   Expanding out fg yields a signed sum of monomials. In each of these monomials, f contributes those variables associated both to a propagator in $\mathcal{P}'$ and to a vertex in J, and g contributes those variables associated both to a propagator not in $\mathcal{P}'$ and to a vertex not in J, and no other variables appear.

   Since there is no cancellation between terms, this means that Δ_{I_i} itself contains no other variables. Therefore Δ_{I_i} is equal to the determinant of the matrix obtained by taking the submatrix of $C(W)$ with columns indexed by I_i and setting any variables not appearing in Δ_{I_i} to 0. This new matrix is, up to permutations of rows and columns, a block matrix with one block for $\mathcal{P}'$ and J and the other block for the complements. Thus its determinant, and hence also Δ_{I_i}, is the product of the minors for these two blocks. By considering which variables appear, these two factors must also be f and g, and so in particular f is the minor of $C(W)$ associated to $\mathcal{P}'$ and J.

   A consequence of this claim is that every linear factor of Δ_{I_i} is a 1×1 minor of $C(W)$, hence is a single variable, and every irreducible quadratic factor of Δ_{I_i} is a 2×2 minor of $C(W)$, hence is a 2×2 determinant of single variables.

   All that remains is to prove that Δ_{I_i} has no irreducible factors of degree 3 or more. Suppose for a contradiction that f is a factor of Δ_{I_i} of degree ≥ 3. Note that by removing the propagators which come before those contributing to f and changing i to be the first vertex which contributes to f, we obtain a different admissible diagram for which f still divides Δ_{I_i} but also $i \in I_i$ and i contributes to f. Showing that this different admissible diagram gives a contradiction is sufficient, and so we may assume that $i \in I_i$ and i contributes to f. Finally, we can suppose that W is minimal in number of propagators with the above occuring.

   would it be neater to note at the beginning that removing uncovered vertices has no effect so we assume there are no redundant vertices, and hence $i \in I_i$ for each i?

   Let p be the propagator such that $I_i(p) = i$. There are two cases to consider, depending on which edge p is supported on. These are illustrated in Figure 6

   Case 1: Suppose p has one end on the edge $(i-1, i)$. Thus p is supported on $(i-1, i, m, m+1)$ for some $m >_i i$, and $I_{i+1}(p) = m$ by Lemma **??**.

   Let S be the set of propagators inside p along with p itself. I_i and I_{i+1} can only differ once p contributes to I_{i+1}, so $I_i(q) = I_{i+1}(q)$ for each $q \in S \backslash \{p\}$. Thus if a propagator contributes m in I_i then it must lie outside p.

   If neither m nor $m+1$ appear in I_i then $V(p) \cap I_i = \{i\}$, and so the row of p in the matrix of Δ_{I_i} has only one nonzero entry; hence Δ_{I_i} has a linear factor contributed by p and i, which is a contradiction. So we must have at least one of m and $m+1$ in I_i. However, all propagators

   check: is it obvious that $i-1 \notin I_i$ or are we using the no coloops result?

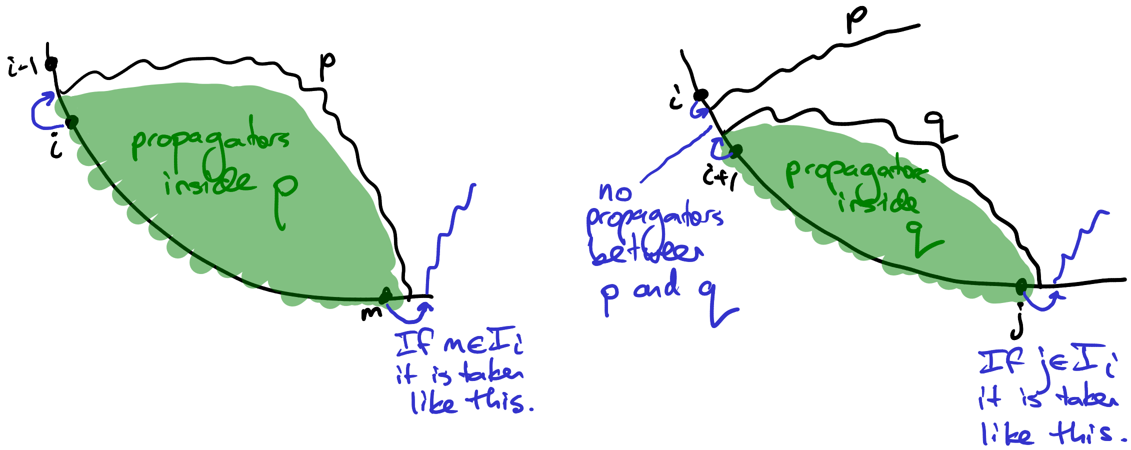


Figure 6: The two cases in the proof that no factors of Δ_{I_i} have degree 3 or more.

in S are mapped by the function $I_i(\cdot)$ to vertices strictly before m, so the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the $|S| \times |S|$ matrix indexed by the propagators in S and the vertices in $I_i(S)$. No other propagators can be supported on these vertices since all other propagators are outside of p, and p is the first propagator supported at i; this explains the zero block. Therefore $\Delta_{I_i} = \det A \det C$, and both factors are nontrivial since at least one of m and $m+1$ appear in I_i. If we remove the propagator outside of p that contributes m or $m+1$, we get a smaller diagram for which $\Delta_{I_i} = \det A$. This contradicts the minimality of our choices unless $\det A$ is quadratic, which in turn contradicts our assumption that i and p contribute to an irreducible factor f of degree at least 3.

Case 2: Suppose p has one end on the edge $(i, i+1)$. If no other propagators are supported on i then the column of $C(W)$ corresponding to vertex i has only one nonzero entry in it, and so Δ_{I_i} has a linear factor contributed by p and i; as above, this is a contradiction. Thus we can take q to be the propagator such that $I_i(q) = i+1$. We know that q has one end on the edge $(i, i+1)$ and is adjacent to p on that edge in the counterclockwise direction (see Figure 6). Write $(i, i+1, j, j+1)$ for the support of q. The situation for q is very similar to case 1: in particular, we have $I_{i+1}(q) = j$ by Lemma **??** and so if $j \in I_i$ then the propagator which contributes j is outside of q.

Similarly to Case 1, let S be the set of propagators inside q along with p and q themselves. Then all propagators in S are mapped by $I_i(\cdot)$ to vertices strictly before j and no other propagators are supported on vertices strictly before j. Thus the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the submatrix indexed by the propagators in S and the vertices in $I_i(S)$. Again two things can now happen. If some vertex j or larger (with respect to $>_i$) belongs to I_i then B and C are at least one column wide, and so the block form of the matrix gives a nontrivial factorization of Δ_{I_i}. This yields a contradiction as in Case 1: either W contains unnecessary propagators which contradicts our minimality assumption, or $\det A$ is quadratic which contradicts the assumption that p and i contribute to f, an irreducible factor of degree at least 3.

On the other hand, if no vertex $\geq_i j$ is in I_i then $\Delta_{I_i} = \det A$. Looking in more detail into A, note that the only vertices in the support of p and q which belong to I_i are i and $i+1$, and hence

$$A = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}$$

where D is the 2×2 matrix indexed by the propagators p and q and the vertices i and $i+1$. Thus p and i contribute to a quadratic factor of Δ_{I_i}, once again contradicting our assumptions.

All cases have now been covered and so Δ_{I_i} has only irreducible factors of degree 2 or less.

3. Suppose propagators p and q are supported on a common edge (a, b), with $I_i(p) = a$ and $I_i(q) = b$. Let $x_{p,a}, x_{p,b}, x_{q,a}, x_{q,b}$ be the associated variables in $C(W)$. For any fixed bijection σ from $\mathcal{P} - \{p, q\}$ to $I_i - \{a, b\}$ for which each propagator is supported on its image under the bijection, we can extend σ to a bijection of all propagators with I_i in two ways: either $p \mapsto a$ and $q \mapsto b$ or $p \mapsto b$ and $q \mapsto a$. The sum of the contributions of all these bijections to Δ_{I_i} is therefore the product of $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ with the minor coming from the remaining propagators and $I_i - \{a, b\}$. Since there is no cancellation of terms in the expansion of Δ_{I_i}, if any other terms appear then they will cause a factor which is not in the form described in the previous part. Therefore no such terms exist and $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ is a factor of Δ_{I_i}.

   Now let f be a quadratic factor of Δ_{I_i}. By part (2) we know that f is a 2×2 minor coming from two propagators, call them p and q, and two vertices, call them $a <_i b$. It remains to show that a and b are adjacent. From this we can conclude that p and q each have one end on (a, b), as any other way for both p and q to be supported on two consecutive vertices would contradict noncrossing or the density requirement of admissibility.

   As in the proof of part (2), make a new admissible diagram by removing the propagators which come before f and set $i = a$. The cases in the proof of part (2) show how Δ_{I_i} factors: in particular the vertices supporting the other end of p either do not appear in I_i, or they contribute to a different factor of Δ_{I_i} than p and a do. By assumption b contributes to the same factor as a. Therefore (a, b) is an edge.

4. Consider $p \in S_i$, and note that Δ_{I_i} is homogeneous linear in the variables of the row corresponding to p. By part (2), either exactly one variable in the row corresponding to p appears in Δ_{I_i} and this variable is a factor of Δ_{I_i}, or exactly two variables from the row corresponding to p appear in Δ_{I_i} and they appear as part of a quadratic factor. In the first case: let the variable be x, then x is a factor of both r_i and Δ_{I_i} and is the only variable from this row in either polynomial.

   Now suppose two variables from the row p appear in a quadratic factor f. By part (3), there is another propagator q and an edge (a, b) such that f is the 2×2 minor coming from p, q

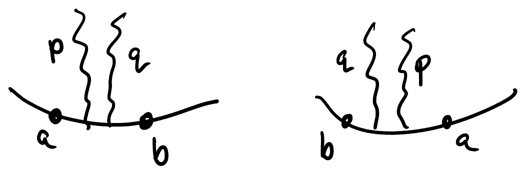


Figure 7: The situations giving a quadratic factor with variables appearing in r_i.

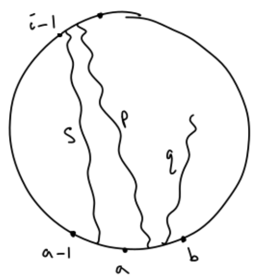


Figure 8: In order to obtain $I_{i-1}(s) = a$, propagators s and p must each have an end on the edge $(i-2, i-1)$.

and a, b, with $I_i(p) = a$, $I_i(q) = b$. There are two situations which can occur, both illustrated in Figure 7; we show that in both cases it follows that $q \in S_i$ as well.

In both cases, since $I_{i-1}(p) \neq a$ by assumption it follows from Lemma **??** that $I_{i-1}(p) <_{i-1} a$ and no other vertex supporting p lies between $I_{i-1}(p)$ and a. In the case that $b <_i a$ and q is taken before p in I_i, this means that $I_{i-1}(p) = b$ and so $I_{i-1}(q) \neq b$. Thus $q \in S_i$ and so f is a factor of r_i.

Now consider the case where $a <_i b$, and suppose for contradiction that $q \notin S_i$, i.e. that $I_{i-1}(q) = b$. Since $I_{i-1}(p) \neq a$, there must be some other propagator s with $I_{i-1}(s) = a$ (else I_{i-1} assigns q to a). This propagator cannot lie on edge (a, b) since by Lemma **??** we must have $I_i(s) = a$ or b, contradicting the fact that $I_i(p) = a$ and $I_i(q) = b$; thus s has an end on $(a-1, a)$ and is inside p from the point of view of $i-1$.

Say s is supported on $(j, j+1, a-1, a)$ and p is supported on $(k, k+1, a, b)$ with $i-1 \leq_{i-1} k+1 \leq_{i-1} j+1$. But by Lemma 4.3, if $I_{i-1}(s) = a$ then a cannot be maximal in the support of s with respect to $<_{i-1}$; thus we must have $i-1 = j+1$, and we are in the situation in figure 8.

Since p changed its association from I_{i-1} to I_i, we have $I_{i-1}(p) = i-1$ by Lemma **??**. From figure 8 it follows that I_{i-1} assigns p to $i-1$ and then proceeds identically to I_i for all vertices inside p, implying that $I_{i-1}(s) = I_i(s)$. Since $I_{i-1}(s) = a$ and $I_i(s) \neq a$, this is a contradiction.

Thus $q \in S_i$ after all, and so again f is a factor of r_i as required.

5. If W has zero propagators then all $I_i = \emptyset$ and both R and $\prod_{i=1}^{n} \Delta_{I_i}$ are equal to 1, so the

result holds in this case. Now assume W has at least one propagator.

First we show that every factor of $\prod_{i=1}^{n} \Delta_{I_i}$ divides R. Take an irreducible factor f of $\prod_{i=1}^{n} \Delta_{I_i}$. There exists some i such that $f|\Delta_{I_i}$ but $f\nmid\Delta_{I_{i-1}}$, since otherwise the variables corresponding to the propagators contributing to f which do not themselves appear in f could never appear, contradicting Lemma **??**. If f is a linear factor, say from associating propagator p to vertex a, then $I_i(p) = a$ and $I_{i-1}(p) \neq a$ so this factor appears in r_i. If f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively, then again we cannot have both $I_{i-1}(p) = a$ and $I_{i-1}(q) = b$, else f divides Δ_{i-1}. However, by the proof of part (4), if one of p, q belongs to S_i then the other does as well. Thus f divides r_i.

Next we need to show that R is squarefree. Suppose $f^2|R$. If f is a linear factor, say from associating propagator p to vertex a, then there must be two distinct points in the Grassmann necklace algorithm where p changes from not being associated to vertex a to being associated to vertex a. This contradicts Lemma **??**. Now suppose f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively. Since p comes before q on edge (a, b) it is not possible for any I_i to associate p to b and q to a. Furthermore, we know by part (4) that p changes from not being associated to a to being associated to a if and only if q changes from not being associated to b to being associated to b. Thus $f^2|R$ implies that twice in the Grassmann necklace p must change from not being associated to vertex a to being associated to vertex a. This is again a contradiction, and so R is squarefree.

I can't figure out why this happens; what am I missing?

Taking everything together we have that $R| \prod_{i=1}^{n} \Delta_{I_i}$, R contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$ and R is squarefree. Therefore the ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.

□

Theorem 4.9. *Given any admissible Wilson loop diagram W, let $GN(W) = \{I_1, \dots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral, $R(W)$ (see Definition 1.10), is an integer multiple of the radical of $\prod_{i=1}^{n} \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i.*

Proof. In view of Proposition 4.8 it remains to prove that the R of Algorithm 4.6 is $R(W)$, the denominator of the integral $I(W)$.

To this end, first note that $R(W)$ and R both have total degree $4|\mathcal{P}|$; the degree of $R(W)$ is immediate from the definition while that of R follows from Lemma **??**. By Proposition 4.8 every factor of R is either a single variable or a quadratic factor coming from two propagators supported on a common edge and hence every factor of R divides $R(W)$. Finally, since R is squarefree and monic, this implies that $R(W)$ is an integer multiple of R. □

Is this obvious? Or does it follow from a previous result?

explain why this result was interesting