Combinatorics of the geometry of Wilson loop diagrams II: Grassmann necklaces, dimensions, and denominators

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1 Introduction

This paper is the second in a two part series investigating the combinatorics and geometry underlying SYM $N = 4$ theory. This series lays out several results about the relationship between the Wilson loop diagrams and the positroid cells that give a CW complex structure to the positive Grassmanian. This paper is concerned with identifying which positroid cells correspond to which Wilson loop diagrams, and how this translates to a geometric tiling of a subspace of the positive Grassmannians.

In contrast, the first paper in this series focuses on combinatorial and matroidal properties of the Wilson loop diagrams. The main focus of that paper was to enumerate the positroid cells that could be associated to Wilson loop diagrams with a fixed number of vertices and propagators, as well as counting the number of Wilson loop diagrams in the pre-image of any positroid cell under this association.

In recent years, there has been significant interest in understanding the geometry and combinatorics underlying the field theory SYM $N = 4$ [3, 8, 12, 13, 16]. This started with the observation that BCFW diagrams, which represent the on shell interactions of this theory, correspond to plabic graphs [17]. Plabic graphs were introduced by Postnikov in [22], and provide one of the many equivalent ways to classify the positroid cells of the positive real Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$. This led to the representation of the on shell integrals associated to this theory in terms of a geometric space (the Amplituhedron) embedded in $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ [7, 8]. Moreover, integrals associated to the BCFW diagrams define a volume form on the Amplituhedron, which is useful for .... Since then, there has been significant work to understand the volume form and its physical implications [6, 13].

Meanwhile, a different body of work has studied SYM $N = 4$ theory from the point of view of Feynman integrals in twistor space [1, 2, 9, 10]. These integrals are calculated via holomorphic Wilson loops. As with the Amplituhedron, each Wilson loop represents the sum of a family of $N^k MHV$ Feynman diagrams, and adding up the corresponding sums of Feynman integrals associates a volume form to space [15]. The first author and Amat uncovered a connection between the Feynman integrals developed in this literature and positroids, defined as matroids that can be realized as an element of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ [3], while other work has shown a connection between the space parametrized by Wilson loop diagrams and the Amplituhedron [4, 12].

[margin note: we need a vague one-line description here]

[margin note: I don't know if this was supposed to say "space" or "this space", and if "this space" I don't know which space it's referring to.]

In the above work and elsewhere, it is clear that while the space parametrized by the Wilson loop diagrams is related to the Amplituhedron, they are not the same space. For instance, it is conjectured that the Amplituhedron is orientable, whereas the first author and Marcott show in [5] that the space parametrized by the Wilson loop diagrams can be seen as a non-orientable vector bundle over some submanifold of $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$. Furthermore, the cells associated to the Wilson loop diagrams are different from the cells associated to BCFW diagrams, as one sees from counting arguments in comparing the results of [16] and [4]. This is not too alarming, as one expects the space parametrized by the Wilson loop diagrams to project down to something close to the Amplituhedron [3, 12].

> can we call this a conjecture?

This paper continues this work by identifying which positroid cells are associated to which Wilson loop diagram: specifically, we give an algorithm to read off the Grassmann necklace of the associated positroid cell from the Wilson loop diagram. We also show that the positroid cells associated to Wilson loop diagrams are all $3k$ dimensional; by contrast, the BCFW cells are all $4k$ dimensional [ref]. Again, this is not too alarming, as it was shown in [5] that the actual geometric space parametrized by the Wilson loop diagrams is a non-orientable k-dimensional bundle over this $3k$ dimensional manifold in $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$.

> someone actually proved this, right?

As noted above, the sum of the Feynman integrals associated to Wilson loop diagrams gives the volume of the space; if this volume is to be finite, then any singularities appearing in individual integrals must cancel out in the overall sum. This has been verified explicitly in a few cases [4, 14], but it is hard to prove in general due to the fact that the relationship between Wilson loop diagrams and $3k$ dimensional positroid cells of $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$ is neither one-to-one nor onto. In this paper, we give a characterization of the factors appearing in the denominator of these Feynman integrals in terms of the Grassmann necklace of the associated positroid, thus providing a much simpler and more tractable link between the singularities and the diagrams.

> help?

1.1 Roadmap

In Section 2, we summarize the required background for Wilson loop diagrams (subsection 2.1), the theory of positroids in $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$ and their characterization in terms of Grassmann necklaces and Le diagrams (subsection 2.2), and the link between Wilson loop diagrams and positroids (subsection 2.3).

In Section 3 we state an algorithm for constructing the Grassmann necklace of the positroid associated to a Wilson loop diagram (Algorithm 3.6) and prove its correctness (Theorem 3.13). While it was already possible to construct the Grassmann necklace of a Wilson loop diagram via existing bijections between some of the various combinatorial objects that index the positroid cells, this process was convoluted and involved multiple steps; our algorithm allows us to track directly how each propagator in the Wilson loop diagram contributes to each term in the Grassmann necklace. Many of the technical lemmas required to prove Theorem 3.13 continue to be important in subsequent sections: in particular we have Lemma 3.4, which identifies certain configurations of propagators which must appear in any admissible Wilson loop diagram, and Lemma 3.11, which shows that the pattern of values that a propagator contributes to each term of the Grassmann necklace is both simple and predictable.

In Section 4 we examine the dimension of positroids associated to Wilson loop diagrams, and

show that the dimension is always equal to three times the number of propagators in the diagram (Theorem 4.10). While this has been proved by Marcott in <mark>[ref]</mark> using geometric methods, our proof is constructive and explicitly relates the position of the plusses in the Le diagram to the propagators in the Wilson loop diagram.

<mark>Did this get published yet?</mark>

Finally in Section 5 we characterize the denominator of the integral of a Wilson loop diagram in terms of how propagators contribute to the Grassmann necklace. This characterization is given explicitly by Algorithm 5.1 and Theorem 5.4.

2 Background

2.1 Wilson Loop diagrams

Definition 2.1. A **Wilson loop diagram** $W = (\mathcal{P}, V)$ consists of a cyclically ordered set V with one distinguished element (this plays the role of the "first" element when we need a compatible linear ordering on V), and a set $\mathcal{P} \subseteq V \times V$. The elements of V are the **vertices** of W, and elements of $\mathcal{P}$ are the **propagators** of W.

A Wilson loop diagram is depicted as a circle, with the vertices V <mark>equally</mark> spaced along the edge of the circle and listed counterclockwise in their cyclic order. The arc between two consecutive vertices is referred to as an **edge** of the diagram, and the ith edge of W is the edge from vertex i to its successor. Each propagator $(i, j) \in \mathcal{P}$ is denoted by a wavy line inside the circle joining edge i and edge j.
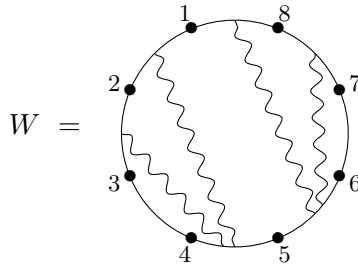

Figure 1: The Wilson loop diagram $W = \big(\{(1,4), (2,4), (5,7), (5,8)\}, [8]\big)$ which has four propagators and eight vertices.

Let $[n]$ denote the set of integers $\{1, \ldots, n\}$ with the obvious cyclic and linear orders. Our primary focus is Wilson loop diagrams with $V = [n]$, so in order to simplify the exposition we will often use $i + 1$ as shorthand for "the successor of vertex i".

Generally speaking the propagators in a Wilson loop diagram are undirected, so (i, j) and (j, i) represent the same propagator. If we need to impose a direction on a propagator $p = (i, j)$, for example in order to consider the region "inside" or "outside" of p, we write $(p, <_i)$ to indicate that p is directed from edge i to edge j, and $(p, <_j)$ for the opposite direction.

We also impose the convention that all vertices should always be interpreted in terms of the cyclic order on V; for example, if $V = [n]$ then $(n, n + 3)$ means the propagator connecting edges 3 and n. This viewpoint is beneficial for us since many of our proofs rely on restricting our attention to a region of the diagram bounded by a given propagator, or to short propagators such as $(i, i + 2)$, and these behave the same regardless of whether we cross the "first" vertex or not; however, we emphasize that this is different to the standard convention in the physics literature, where propagators are written (i, j) with $1 \leq i < j \leq n$.

Definition 2.2. Let $W = (\mathcal{P}, V)$ be a Wilson loop diagram.

1. For $p = (i, j) \in \mathcal{P}$, let $V(p) = \{i, i + 1, j, j + 1\}$ denote the set of **vertices supporting** p. For a set of propagators $P \subseteq \mathcal{P}$, define

$$V(P) = \bigcup_{p \in P} V(p)$$

   to be the vertex support of P.

2. For $U \subseteq V$, the set of **propagators supported on** U is denoted by

$$\mathrm{Prop}(U) = \{p \in \mathcal{P} \mid V(p) \cap U \neq \emptyset\}.$$

   Note that a propagator in $\mathrm{Prop}(U)$ does not need to have its entire support contained in U.

3. Vertices of W that do not support any propagators are called **non-supporting**. The set of all non-supporting vertices in W is denoted $F(\emptyset)$.

Definition 2.3. Let $W = (\mathcal{P}, V)$ be an admissible Wilson loop diagram. Then $W' = (Q, U)$ is a **subdiagram** of W if $Q \subseteq \mathcal{P}$ and $V(Q) \subseteq U \subseteq V$.

With these definitions in hand, we can impose some conditions on the density and behavior of the propagators of Wilson loop diagrams.

Definition 2.4. A Wilson loop diagram $W = (\mathcal{P}, V)$ is called **admissible** if:

1. $|V| \geq |\mathcal{P}| + 4$.

2. There does not exist a non-empty set of propagators $Q \subseteq \mathcal{P}$ such that $|V(Q)| < |Q| + 3$.

3. There does not exist a pair of propagators $\{(i, j), (k, l)\} \subseteq \mathcal{P}$ such that $i < k < j < l$.

The first condition bounds the total number of propagators in the diagram, while the second limits how densely propagators can be fitted into any part of the diagram; in particular, it prohibits propagators that connect two adjacent edges and parallel propagators that start and end on the same pair of edges. The third condition simply requires that propagators are non-crossing.

These conditions are imposed by the physical interpretation of Wilson loop diagrams (see for example [ref]). We therefore restrict our attention in this paper to admissible Wilson loop diagrams and their subdiagrams only.

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Note that a subdiagram (Q, U) of an admissible Wilson loop diagram need not be admissible itself: it inherits conditions 2 and 3 automatically, but we could have $|U| = |Q| + 3$. We call diagrams of this type **weakly admissible**; in the context of this paper, they occur only when we take an admissible diagram and delete one or more non-supporting vertices.

Due to the physical motivation behind these diagrams, we can associate a matrix $C(W)$ to each Wilson loop diagram W, which is defined elementwise by

$$C(W)_{p,q} = \begin{cases} c_{p,q} & \text{if } q \in V(p) \\ 0 & \text{if } q \notin V(p) \end{cases}. \tag{1}$$

The entries $c_{p,q}$ are non-zero real variables. The columns of $C(W)$ correspond to the vertices of W listed in their linear order, and the rows correspond to the propagators of W. $C(W)$ is therefore only defined up to rearrangement of the rows, which will not matter in this setting since we will only interpret it as parametrizing a family of elements of the $|\mathcal{P}| \times |V|$ real Grassmannian.

Example 2.5. Consider the admissible Wilson loop diagram

$$W = \big(\{(1,4), (2,4), (5,7), (5,8)\}, [8]\big),$$

from Figure 1. Ordering the propagators as listed above, we obtain the associated matrix

$$C(W) = \begin{pmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,4} & c_{1,5} & 0 & 0 & 0 \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,7} & c_{3,8} \\ c_{4,1} & 0 & 0 & 0 & c_{4,5} & c_{4,6} & 0 & c_{4,8} \end{pmatrix}.$$

The Wilson loop diagram also defines a volume form on $\Sigma(W)$, namely

$$\Omega(W) = \frac{\prod_{r=1}^{|\mathcal{P}|} \prod_{v \in V_{p_r}} \mathrm{d}c_{p_r}}{R(W)}.$$

The interested reader is referred to [ref] for more information about the volume form. Of interest to us in this paper is the denominator $R(W)$, which we define next.

Definition 2.6. Let $W = (\mathcal{P}, V)$ be an admissible Wilson loop diagram, and fix an edge e. Let $\{p_1 \ldots p_r\}$ be the propagators with one end lying on edge e, ordered so that p_1 is closest to the vertex e, p_r closest to $e + 1$, and p_i is closer to e than p_{i+1} is. Define

$$R_e = c_{p_1, e+1} \prod_{j=1}^{r-1} \big((c_{p_j, e} c_{p_{j+1}, e+1} - c_{p_{j+1}, e} c_{p_j, e+1})\big) c_{p_r, e}$$

to be the component of $R(W)$ associated to edge e. Note that if $r = 1$ (i.e. there is exactly one propagator lying on edge e) then this expression simplifies to $R_e = c_{p,e} c_{p,e+1}$. The denominator $R(W)$ is defined to be the product of all of the R_e, i.e.

$$R(W) = \prod_{e \in V} R_e.$$

Understanding the polynomial $R(W)$ (and hence the singularities of the volume form $\Omega(W)$) is important for the physics. In Section 5 we give a simple algorithm to read $R(W)$ directly from the Wilson loop diagram.

2.2 Positroids, Grassmann necklaces, and Le diagrams

Let $\left(\genfrac{}{}{0pt}{}{[n]}{k}\right)$ denote the set of all k-subsets of $[n]$.

For our purposes, a **positroid** is a matroid M (with ground set $[n]$ and bases $\mathcal{B} \subseteq \left(\genfrac{}{}{0pt}{}{[n]}{k}\right)$) which can be represented by an element of the positive Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$. In other words, there exists a full-rank $k \times n$ real matrix whose $k \times k$ minors are all nonnegative and such that the minor $[[k]|J]$ is positive if and only if $J \in \mathcal{B}$.

We summarize here only the subset of positroid theory that we require in this paper; the interested reader is referred to [22] for more details. Postnikov shows in [22] that the positroids of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ are indexed by many different collections of objects, each with their own advantages and disadvantages. The two most suited to our purposes are Grassmann necklaces and Le diagrams, which we introduce below. In order to do so, we first need some preliminary definitions.

For each $j \in [n]$, we can define a total order $\leq_j$ on the interval $[n]$ by

$$j <_j j+1 <_j \cdots <_j n <_j 1 \cdots <_j j-1 \,.$$

This in turn induces a total order on $\left(\genfrac{}{}{0pt}{}{[n]}{k}\right)$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\left(\genfrac{}{}{0pt}{}{[n]}{k}\right)$ (the **Gale order** [19]), which is defined as follows: for

$$A = \{a_1 <_j a_2 <_j \cdots <_j a_k\} \text{ and } B = \{b_1 <_j b_2 <_j \cdots <_j b_k\} \in \left(\genfrac{}{}{0pt}{}{[n]}{k}\right),$$

we define

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\left(\genfrac{}{}{0pt}{}{[6]}{3}\right)$ we have $\{2,5,6\} \preccurlyeq_2 \{2,6,1\}$ but $\{2,5,6\} \not\preccurlyeq_2 \{3,4,6\}$.

Definition 2.7. A **Grassmann necklace** of type (k,n) is a sequence $(I_1,\ldots,I_n)$ of n sets $I_i \in \left(\genfrac{}{}{0pt}{}{[n]}{k}\right)$ such that for each $i \in [n]$:

- if $i \in I_i$, then $I_{i+1} = \left(I_i \backslash \{i\}\right) \cup \{j\}$ for some $j \in [n]$.

- if $i \notin I_i$, then $I_{i+1} = I_i$.

By convention, we set $I_{n+1} = I_1$.

By [22, Theorem 17.1], the Grassmann necklaces of type (k,n) are in 1-1 correspondence with the positroid cells in $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$. Each term I_i is simply the minimal (with respect to the $<_i$-lex order) basis of the positroid. A characterization of all bases of the positroid in terms of the Grassmann necklace and the Gale order was given by Oh in [20, Theorem 8]: if $(I_1,\ldots,I_n)$ is the Grassmann necklace associated to a positroid $M = ([n],\mathcal{B})$, then the bases of M are exactly

$$\mathcal{B} = \left\{ J \in \left(\genfrac{}{}{0pt}{}{[n]}{k}\right) \mid I_i \preccurlyeq_i J \ \forall i \in [n] \right\}.$$

Thus the Grassmann necklace is well suited to testing whether a given k-set is a basis for M, and for generating a list of all bases of M.

Definition 2.8. A **Le diagram** is a Young diagram in which every square contains either a + or a 0, subject to the rule that if a square contains a 0 then either all squares to its left (in the same row) must also contain a 0, or all squares above it (in the same column) must also contain a 0, or both.

By [22, Theorem 6.5], the set of all Le diagrams that fit within a $k \times (n-k)$ rectangle is in 1-1 correspondence with the positroid cells of $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$. Le diagrams are particularly useful for comparing dimensions of positroids, since the dimension of a positroid is equal to the number of + squares in its Le diagram [22, Theorem 6.5].

The rows and columns of a Le diagram are labelled as follows: given a Le diagram fitting inside a $k \times (n-k)$ box, arrange the numbers $1, 2, \ldots, n$ along its southeast border, starting from the top-right corner. See Figure 2 for examples.
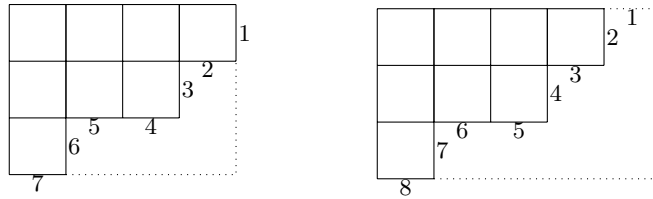

Figure 2: Row and column numbering for a Young diagram with $k = 3$, $n = 7$ (left) and $k = 3$, $n = 8$ (right). The top left box has coordinates $(1, 7)$ in the left diagram, and $(2, 8)$ in the right diagram.

Specifying a k-subset $J \subseteq [n]$ therefore uniquely determines the shape of the Le diagram, by taking the elements of J to be the row indices of the diagram.

An algorithm for constructing the Le diagram associated to a Grassmann necklace was given by Agarwala and Fryer in [18]. Since we will make use of this algorithm in Section 4 below, we summarize the process here.

Algorithm 2.9. *[18, Algorithm 2] Let $(I_1, \ldots, I_n)$ be a Grassmann necklace of type (k, n). Within a $k \times (n-k)$ square, draw the Young diagram whose rows are labelled by I_1 (as per the convention above).*

For each i, $2 \leq i \leq n$:

- *Write*
$$I_1 \setminus I_i = \{a_1 > a_2 > \cdots > a_r\}, \quad I_i \setminus I_1 = \{b_1 < b_2 < \cdots < b_r\},$$
  where the inequalities denote the $<_1$ order (subscripts suppressed for clarity).

- *For $1 \leq j \leq r$, place a + in square (a_j, b_j) of the diagram. (We will sometimes refer to this + as being in the $a_j \to b_j$ position.)*

After performing the above for $2 \leq i \leq n$, place a 0 in any remaining unfilled boxes.

An algorithm for constructing the Grassmann necklace of a Le diagram also exists; this was given by Oh in [20]. A method for using the Le diagram to test whether a given k-subset is a basis for the corresponding positroid or not was given by Casteels in [11].

2.3 Wilson loop diagrams as positroids

Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram with k propagators, and $C(W)$ the associated matrix defined in (1). Let $M(W)$ be the matroid realized by $C(W)$, i.e. the matroid with ground set $[n]$ whose independent sets are exactly the sets $V \subseteq [n]$ such that the columns of $C(W)$ indexed by V are linearly independent.

In [3], Agarwala and Amat show that $M(W)$ is also characterized by the Wilson loop diagram itself:

Theorem 2.10. *[3, Theorem 3.6] The independent sets of the matroid $M(W)$ associated to an admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ and realized by $C(W)$ are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| < |U|$.*

In other words, the independent sets of $M(W)$ correspond to sets of vertices in W in which no subset supports fewer propagators than the vertices it contains.

By studying $M(W)$ in terms of the behavior of the propagators in W, Agarwala and Amat also showed that $M(W)$ is in fact a positroid whenever W is admissible [3, Corollary 3.39]. In theory, this allows us to leverage geometric and combinatorial results about $\mathbb{G}_{\mathbb{R}, \geq 0}(k, n)$ to study Wilson loop diagrams.

However, in order to identify the positroid associated to a given Wilson loop diagram W we first have to construct $C(W)$, then make a list of all its non-zero $k \times k$ minors and extract the Le diagram or Grassmann necklace via the methods described in Section 2.2. Alternatively, one could obtain the decorated permutation of the positroid from $C(W)$ as in [ref] and then convert it into a Grassmann necklace as described in [22, Section 16].

In [4], Agarwala and Fryer apply this process in the smallest non-trivial case: admissible Wilson loop diagrams with two propagators on six vertices. Even in this simple case, we see that the mapping from admissible WLD to positroid cells is neither one-to-one nor onto, and the relationship between the original Wilson loop diagram and the resulting positroid is obscured to the point of invisibility.

Our first goal is therefore to find a better method of obtaining the positroid associated to a given Wilson loop diagram; this is the focus of the next section.

3 Wilson Loop diagrams and their Grassmann necklaces

In this section we will give an algorithm to go directly from a Wilson loop diagram to the Grassmann necklace of its positroid. This is a useful result in and of itself, as it greatly simplifies the process of identifying the positroid associated to a given WLD. It also allows us to examine how the positroid changes as we make minor changes to the Wilson loop diagram; indeed, the techniques we develop in this section to verify that our algorithm produces the correct Grassmann necklace will also be used repeatedly in later sections.

One of the key insights is that each element of the Grassmann necklace can be viewed as a function from the propagators of the diagram to the vertices in the Grassmann necklace element. This is captured in Definition 3.7, and used throughout the paper.

Throughout this section, $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram with k propagators.

3.1 Propagator configurations in admissible Wilson loop diagrams

Before we can describe the algorithm for extracting the Grassmann necklace of $M(W)$ from the Wilson loop diagram W, we require some initial results about the behavior of propagators in admissible WLD.

Definition 3.1. Let $W = (\mathcal{P}, [n])$ be a weakly admissible Wilson loop diagram, with $p \in \mathcal{P}$ supported on edges i and j. Write $(p, <_i)$ to represent the same propagator directed from i to j. Then the sets of propagators **inside** and **outside** of $(p, <_i)$ are defined to be

$$\mathcal{P}_{in}(p, <_i) = \{(k, l) \in \mathcal{P} \mid i \leq_i k <_i l \leq_i j\},$$
$$\mathcal{P}_{out}(p, <_i) = \mathcal{P} \setminus \mathcal{P}_{in}(p, <_i),$$

respectively.

Definition 3.2. Let $p = (i, j) \in \mathcal{P}$ be a propagator in W. Define the **length** of p to be

$$\ell(p) \;=\; \min\big\{|[i+1, j]|, |[j+1, i]|\big\}.$$

In other words, $\ell(p)$ is the size of the smaller of the sets of vertices on either side of the propagator p.

Remark 3.3. The following observations about configurations of propagators of short length in a weakly admissibly Wilson loop diagram W are easily verified:

1. If $p = (i, i+3)$ is a propagator of length 3, then the middle vertex $i+2$ supports at most one propagator.

2. If every vertex in W supports at least one propagator, then W admits at least one propagator of length 2.

The following lemma establishes certain configurations of propagators that must exist in any admissible diagram with no non-supporting vertices (i.e. with $F(\emptyset) = \emptyset$). We make use of this result in several induction proofs below.

Lemma 3.4. *Let W be a weakly admissible WLD with at least 5 vertices and in which each vertex supports at least one propagator. Then at least one of the following two things occurs.*

1. *W has a propagator of length ≤ 6 with a propagator of length 2 on one side it and nothing else on that side.*

2. *There exists a pair of propagators of length 2 with the property that the first propagator is $(i, i+2)$, the second is $(j, j+2)$, no other propagator ends between vertices $i+2$ and $j+1$, and $j \in \{i+2, i+3, i+4\}$.*

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Proof. Suppose first that W has a propagator of length 3, say $p = (i, i+3)$. By Remark 3.3 and the fact that every vertex of W supports at least one propagator, we have that $i+2$ supports exactly one propagator and this propagator must have length 2 by noncrossingness. This gives us an instance of configuration 1 from the statement.

Now suppose W has no propagators of length 3.

Given a propagator $p = (i, j)$ oriented $(p, <_i)$, we can construct two obvious subdiagrams of W by removing all of the propagators on one side of p and leaving the vertices intact, i.e. the subdiagrams

$$(\mathcal{P}_{in} \cup \{p\}, [n]) \quad \text{and} \quad (\mathcal{P}_{out} \cup \{p\}, [n]).$$

We will inductively construct a sequence of pairs of propagators (p_r, q_r) satisfying: $\ell(p_r) = 2$, and p_r either forms part of configuration 1 or 2 from the statement, or there is a propagator q_r satisfying

- $\ell(q_r) \geq 4$.

- $\{p_1, \ldots, p_r\}$ and $\{q_1, \ldots, q_{r-1}\}$ are all on the same side of q_r.

We can then choose the orientation of q_r so that the previous p_i and q_i are all on the outside of q_r, and By the finiteness of W this must eventually terminate in one of the desired configurations.

Start by choosing a propagator $p_1 = (j_1, j_1 + 2)$ of length 2 in W (which exists by Remark 3.3). If it is part of one of the configurations we are looking for then we are done, so suppose otherwise. By assumption: p_1 is not in configuration 2, there are no propagators of length 3, and every vertex supports at least one propagator. Therefore there must exist a propagator q_1 of length ≥ 4 with one end on edge j_1 or on edge $j_1 - 1$ or on edge $j_1 - 2$. Orient $q_1 = (i_1, k_1)$ such that $p_1 \in \mathcal{P}_{out}(q_1)$, (that is $(q_1, <_{i_1})$).

INSERT PICTURE HERE?

Now suppose $q_r = (i_r, k_r)$ exists by the induction hypothesis and is oriented $(q_r, <_{i_r})$ so that the previous p_i and q_i are on the outside. For the rest of this proof, we assume this orientation and drop the $<_{i_r}$ from the notation.

Call let $W_r := W|_{\mathcal{P}_{in} \cup q_r}$. By the original hypotheses on W every vertex in $v \in V_{in}(q_r)$ (which is a non-empty interval since $\ell(q_r) \geq 4$) must have support at least one propagator ($|\mathrm{Prop}(v)| \geq 1$), while the same is true for $V(q_r)$ because these vertices support q_r. By construction, the remaining vertices of $W|_{\mathcal{P}_{in} \cup q_r}$, i.e. $V_{out}(q_r)$, are non-supporting.

By Remark 3.3, W_r admits at least one propagator of length 2. W_r has no propagators of length 3 since W has none. Let p_{r+1} be a propagator of length 2 in W_r; if it forms part of configuration 1 or 2 then we are done, so assume otherwise.

Note we may replace q_r by any other propagator, q'_r of length ≥ 4 in W_r such that p_{r+1} and q_r are on opposite sides; such a new q'_r still satisfies all the hypotheses, and so, without loss of generality we may assume that q_r has minimal length among propagators of length ≥ 4 which have p_{r+1} on their inside.

Write $p_{r+1} = (j_{r+1}, j_{r+1} + 2)$. There are two cases to consider. The first case is that $i_r \leq j_{r+1} \leq i_r + 2$ and $k_r - 2 \leq j_{r+1} + 2 \leq k_r$ (so p_{r+1} has one end before $i_r + 3$ and the other after $k_r - 2$). Then since

note highlighting this for anyone skimming through looking for pictures to add

p_{r+1} has length 2, it must be that $(i_r + 3) + 1 \geq k_r - 2$ and so q_r has length ≤ 6. By the minimality assumption on q_r no propagator in W_r has q_r on one side and p_{r+1} on the other side. Therefore, any propagator in W_r other than p_{r+1} itself must be of the form (i, j) with $i_r \leq i, j \leq i_r + 2$ or $k_r - 2 \leq i, j \leq k_r$. That is, this propagator must be of length 2, and so we have configuration 2 which we have already assumed does not occur. Consequently, $\mathcal{P}_{in}(q_r)$ contains only p_{r+1} and so we have congifuratin 1 from the lemma statement which again we have assumed does not occur. Therefore this first case cannot occur.

The second case is that one of the following three things happen (we continue to write $p_{r+1} = (j_{r+1}, j_{r+1} + 2)$)

- $i_r + 3 \leq j_{r+1} \leq k_r - 3$ or $i_r + 3 \leq j_{r+1} + 2 \leq k_r - 3$,

- $i_r = j_{r+1}$, or

- $k_r - 2 = j_{r+1}$

(Phrased more causually, this is that p_{r+1} has either at least one end on an edge bounded by the vertices in the interval $[i_r + 3, k_r - 2]$ or both ends lie in $[i_r, i_r + 3]$ or both ends lie in $[k_r - 2, k_r + 1]$.) These situations all behave similarly. By symmetry it suffices to only consider the second situation and the second possiblity of the first situation. Note that since $j_{r+1} + 4 \leq k_r - 1$ in both situations, and $j_{r+1} + 4 \in V_{in}(q_r)$, thus $j_{r+1} + 4$ is supported by a propagator in W_r other than q_r. Let t be this propagator. Since p_{r+1} is not part of configuration 2 and W_r has no propagators of length 3, we must have that the length of t is ≥ 4. If q_r and p_{r+1} were on different sides of t then this would contradict the minimal length hypothesis on q_r. Therefore t has all the previous q_i and p_i on the same side and length ≥ 4 and so we may set $q_{r+1} = t$ to continue the induction.

The overall result then follows by induction. $\qquad\square$

Remark 3.5. In the case that all vertices of an admissible WLD W support at least two propagators, then Lemma 3.4 substantially simplifies. By Remark 3.3, W has no propagators of length 3. Configuration 1 necessarily entails vertices with support 1 as does configuration 2 unless $j = i + 2$. So in the case that W has all vertices with support at least two then W must contain a pair of propagators of length 2 with the property that the first propagator is $(i, i + 2)$, the second is $(i + 2, i + 4)$ and no other propagator ends on the edge $i + 2$.

3.2 From Wilson Loop diagrams to Grassmann Necklaces

Until now, the positroid associated to a Wilson loop diagram W could only be obtained by computing the matrix $C(W)$ associated to W, listing all bases of the induced matroid $M(W)$, and constructing the Le diagram or Grassmann necklace of the positroid "by eye" from this list.

In this section, we give an algorithm for passing directly from the Wilson loop diagrams to its Grassmann necklace. This not only greatly simplifies the process above, but will also allow us to relate the behavior of the positroid $M(W)$ directly to the configuration of progators in W.

The fact that Algorithm 3.6 does construct the required Grassmann necklace is proved in Theorem 3.13. An worked example of Algorithm 3.6 is given after Definition 3.7.

Algorithm 3.6. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. This gives an algorithm for calculating the set I_i, for $i \in [n]$.*

1. *Fix a vertex $i \in [n]$. Set $a := i$ and $I_i = \emptyset$.*

2. *While $\mathcal{P} \neq \emptyset$, perform the following steps.*

   (a) ***Step*** a ***for vertex*** i*: If $Prop(a) \neq \emptyset$ in W, write $I_i = I_i \cup \{a\}$. Let $p \in Prop(a)$ be the clockwise most propagator supported on a. Write $W = (\mathcal{P} \setminus \{p\}, [n])$.*

   (b) *If $Prop(i) = \emptyset$ do nothing.*

   (c) *Increment a by 1 and repeat from (a).*

If the algorithm assigns vertex j to propagator p from starting vertex i, we say that p *contributes* j to I_i. Notationally, we represent this by allowing the I_i symbol to represent a function as well as a set, as follows:

Definition 3.7. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. For each $i \in [n]$, define a function $I_i : \mathcal{P} \longrightarrow [n]$ by

$$I_i(p) := \text{the vertex label that } p \text{ contributes to } I_i \text{ in Algorithm 3.6,}$$

for each $p \in \mathcal{P}$.

Example 3.8. Consider the admissible Wilson loop diagram
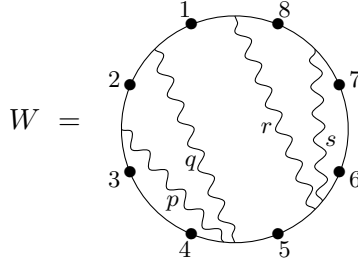
To construct I_1 we start at vertex 1, so we begin with $i = 1$, $a = 1$, and $I_1 = \emptyset$.

- $a = 1$: Since $Prop(1) = \{q, r\}$ and r is the clockwise-most of these propagators, we let $I_1 = \{1\}$ and remove propagator r from the diagram.

- $a = 2$: $Prop(2) = \{p, q\}$, with q being the clockwise-most of these propagators. So $I_1 = \{1, 2\}$ and propagator q is removed.

- $a = 3$: $Prop(3) = \{p\}$, so now $I_1 = \{1, 2, 3\}$ and propagator p is removed. The only remaining propagator in the diagram is s.

- $a = 4$: Since p and q were removed in earlier steps, we now have $Prop(4) = \emptyset$.

- $a = 5$: $Prop(5) = \{s\}$, and we have $I_1 = \{1, 2, 3, 5\}$.

There are no propagators left in the diagram, so the algorithm terminates and we have $I_1 = \{1, 2, 3, 5\}$, or 1235 for short. Viewed as a function, we have

$$I_1 : \mathcal{P} \longrightarrow [n] : \quad I_1(p) = 3, \quad I_1(q) = 2, \quad I_1(r) = 1, \quad I_1(s) = 5.$$

Applying the algorithm for all 8 starting vertices, we obtain the sets

$$1235, 2356, 3456, 4567, 5671, 6712, 7812, 8123.$$

The reader can easily verify that this sequence of k-sets satisfies the definition of a Grassmann necklace, and can (with significantly more work) also verify that this Grassmann necklace defines the positroid associated to the Wilson loop diagram W given above.

Given a propagator p and a vertex i in its support, it will be very useful in the following to understand on what set of vertices the Grassmann necklace algorithm assigns i to p. The answer is that the set is a non-empty cyclic interval. Lemma 3.11 establishes this, but first we need a preliminary lemma which is also useful in its own right.

Lemma 3.9. *Let W be an admissible Wilson loop diagram containing at least one propagator. For any $i \in [n]$ and for any $p = (a, b)$ with $i \leq_i a <_i b$, we have $I_i(p) \neq b + 1$.*

Proof. Suppose for contradiction that we have $p = (a, b)$ with $i \leq_i a <_i b$ and $I_i(p) = b + 1$. We may choose p such that $\|[a + 1, b]\|$ is minimal amongst propagators with this property.

Since $I_i(p) \neq b$, there must exist a propagator q inside of p with $I_i(q) = b$. The propagator q cannot end on the edge $(b - 1, b)$, as this would contradict the minimality of p, so $q = (c, c + 1, b, b + 1)$ with $a <_i c <_i b$, and $I_i(q) = b$.

In order for q to remain unassigned until vertex b, there must be another propagator r with an end on $(c, c + 1)$ and $I_i(r) = c + 1$; the only way this can occur is if r is outside q but inside p. Now r contributes its fourth vertex to I_i, again contradicting the minimality of p. $\square$

Corollary 3.10. *If W is an admissible Wilson loop diagram with k propagators, then Algorithm 3.6 assigns exactly k vertices to each I_i.*

Proof. It follows from the proof of Lemma 3.9 that Algorithm 3.6 can never reach the fourth vertex of a propagator's support (with respect to the starting vertex). Therefore if the algorithm starts at vertex i, it must have assigned vertices to all propagators by the time it reaches $i - 1$, ensuring that I_i contains exactly k distinct vertices. $\square$

Given a propagator p of an admissible WLD W and a vertex i in the support of p, define

$$J_p^{(W)}(i) = \{m \in [n] \mid I_m^{(W)}(p) = i\},$$

i.e. the set of indices m for which Algorithm 3.6 assigns the value i to the propagator p in W. The following lemma establishes that these sets behave in a simple and predictable manner, a fact which we will repeatedly use in subsequent proofs.
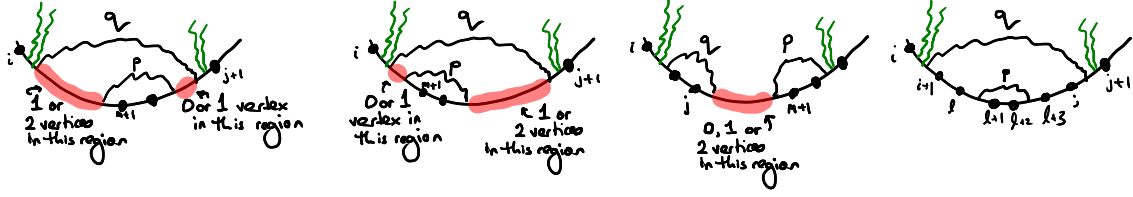
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Figure 3: Four cases for admissible WLDs with no non-supporting vertices. The green half-propagators illustrate where propagators may occur, but are not required to exist; no other regions illustrated may support additional propagators.
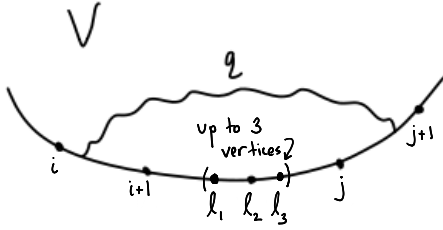

Figure 4: Diagram V is W with p removed; there are no propagators inside $q = (i,j)$, though there may be up to 4 non-supporting vertices labelled $l, l+1, \ldots$.

Lemma 3.11. *Let $p = (i,j)$ be a propagator of an admissible WLD W on n vertices. Then $J_p^{(W)}(i)$, $J_p^{(W)}(i+1)$, $J_p^{(W)}(j)$, and $J_p^{(W)}(j+1)$ are each non-empty cyclic intervals which partition $[n]$ and occur in the given cyclic order.*

Proof. We will prove the result by induction on the number of propagators. If W has one propagator then the result is immediate. Now suppose W has more than one propagator. Since non-supporting vertices have no effect on the Grassmann necklace algorithm, it suffices to prove the result for W with $F(\emptyset) = \emptyset$. Then by Lemma 3.4, W has at least one of the four situations illustrated in Figure 3.

In each of the four cases, when we remove the propagator labelled p we obtain a diagram which satisfies the statement of the theorem by the induction hypothesis, and contains a propagator $q = (i,j)$ with no other propagators inside it (although this region may or may not contain non-supporting vertices, which we will call $l, l+1, \ldots$ as necessary). Let V be the diagram obtained by removing the propagator p from W (Figure 4).

Consider the fourth case of Figure 3 first, as it is the easiest. In this case, the vertices $\{l, l+1, l+2, l+3\}$ which support p in W are non-supporting in V, so for every propagator r in V (including q) we have $J_r^{(V)} = J_r^{(W)}$; these are non-empty cyclic intervals in the correct order by the induction hypothesis. Additionally, it is clear from Figure 3 that $J_p^{(W)}(l+a) = \{l+a\}$ for $a \in \{1,2,3\}$ and $J_p^{(W)}(l) = [n] \setminus \{l+1, l+2, l+3\}$. The result therefore holds in this case.

Now we proceed to consider the first three cases of Figure 3. We first describe $J_q^{(V)}(*)$, which can

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be handled identically for all three cases.

In V there are no propagators inside q, so we see from Figure 4 that

$$l, l+1, \ldots, j \in J_q^{(V)}(j) \text{ (if } l \text{ exists) and } j+1 \in J_q^{(V)}(j+1).$$

Note that $j + 2 \notin J_q^{(V)}(j + 1)$ by Lemma 3.9, so by the induction hypothesis we must have $J_q^{(V)}(j+1) = \{j+1\}$ and $j+2 \in J_q^{(V)}(i)$. Thus there exist vertices $d, e \in [j+2, i+1]$ with $d < e$, such that

$$J_q^{(V)}(i) = [j+2, d-1], \quad J_q^{(V)}(i+1) = [d, e-1], \quad J_q^{(V)}(j) = [e, j], \quad J_q^{(V)}(j+1) = \{j+1\},$$

and all intervals are non-empty.

We now consider what happens as we move from V to each of the three remaining cases for W. We need to consider both $J_p^{(W)}$ and $J_r^{(W)}$ for $r \neq p$, since the addition of p can have a knock-on effect on later steps in the algorithm.

Left two cases: These cases behave essentially identically (except when j or $j + 1$ are not in the support of p, which can occur in the second case only; see below) so we handle the majority of the proof for these two cases simultaneously. Let $1 \leq a \leq 3$ be the number of non-supporting vertices inside q in V; so these vertices are $l, \ldots, l + a - 1$. Write $p = (m, m + 2)$ where $m \in \{i, i+1, l\}$. Note that $l, \ldots, l + a - 1$ are all in the support of p.

We first calculate $I_w^{(W)}$ for a starting vertex $w \in [n] \backslash \{l, l+1, \ldots, j, j+1\}$. Note that p has no effect on other propagators for starting vertices in this range, while the value of $I_w^{(W)}(p)$ depends on how soon q is assigned to a vertex, i.e. on the value of $I_w^{(V)}(q)$. Thus, if $w \in J_q^{(V)}(i)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{i+1, m\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(i+1)$ or $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} l & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

We also need to understand $I_w^{(W)}$ for $w \in \{l, l+1, \ldots, j, j+1\}$. For the majority of these vertices, we use the following observation: if p is the first propagator to be assigned a value by $I_w^{(W)}$, then the remainder of $I_w^{(W)}$ proceeds identically to the assignments of $I_{w+1}^{(V)}$. Thus we have for any $0 \leq b < a$

$$I_{l+b}^{(W)}(r) = \begin{cases} l+b & \text{if } r = p \\ I_{l+b+1}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Similarly, if j is in the support of p, then we have

$$I_j^{(W)}(r) = \begin{cases} j & \text{if } r = p \text{ and } j \text{ is in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is in the support of } p \end{cases}$$

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If j is not in the support of p, then we must be in the second case of Figure 3 with two vertices in the right hand region. In this case, if we start the algorithm at j we need to know whether there will be any unassigned propagators other than p when we reach vertex i, so as to know what p contributes.

Consider the WLD X formed from V by moving the second end of q to the edge $j - 1$ instead of j. X is still admissible since we have not decreased the support of any set of propagators, so the induction hypothesis applies to it as well. Note that $J_j^{(V)}(r) = J_j^{(X)}(r)$ for all $r \neq p$ and $J_j^{(X)}(r) = J_{j+1}^{(X)}(r)$ for $r \neq q, p$. Additionally $I_{j+1}^{(X)}(q) = i$ by the induction hypothesis applied to X, and so if we start at $j + 1$ and assign propagators to vertices according to the algorithm, when we reach vertex i in X all propagators other than q must have been assigned. Therefore if we start at j in W, we first assign q to j then proceed to assign as in X starting at $j + 1$, and hence when we get to i the only remaining unassigned propagator is p. Therefore

$$
I_j^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j \text{ is not in the support of } p \\ I_j^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is not in the support of } p \end{cases}
$$

Finally, we consider what happens when we start the algorithm at vertex $j + 1$. If $j + 1$ is in the support of p then we can argue as above to get

$$
I_{j+1}^{(W)}(r) = \begin{cases} j + 1 & \text{if } r = p \text{ and } j + 1 \text{ is in the support of } p \\ I_{j+2}^{(V)}(r) & \text{if } r \neq p \text{ and } j + 1 \text{ is in the support of } p. \end{cases}
$$

Now suppose $j + 1$ is not in the support of p. If we start at $j + 1$ we need to know whether there are any unassigned propagators supported on edge i when we reach vertex i. We already know that $J_q^{(V)}(j + 1) = \{j + 1\}$; in particular this means that q contributes i in $I_{j+2}^{(V)}$. However the construction of $I_{j+1}^{(V)}$ first associates q to $j + 1$ and then proceeds identically to $I_{j+2}^{(V)}$. In particular if i was assigned in $I_{j+1}^{(V)}$, then it would not be available to assign to q in $I_{j+2}^{(V)}$ as all other propagators supported at i in V come before q.

Therefore p is the only potentially unassigned propagator on edge i when we reach vertex i in W, and

$$
I_{j+1}^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j + 1 \text{ is not in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j + 1 \text{ is not in the support of } p \end{cases}
$$

We can now describe the intervals $J_r^{(W)}(*)$ for the first two cases of Figure 3. For $r \neq p$ the intervals are clearly still cyclic and appear in the correct order, and we can assemble the intervals for the $J_p^{(W)}(*)$ as follows.

- If $m = l$ then either $a = 2$ (so $l + 1 = m + 1$, $j = m + 2$, and $j + 1 = m + 3$) or $a = 3$ (so $l + 1 = m + 1$, $l + 2 = m + 2$, $j = m + 3$, and $j + 1$ is not in the support of p), and in both cases

$$
J_p^{(W)}(m) = [m + 4, m], \qquad J_p^{(W)}(m + 1) = \{m + 1\},
$$
$$
J_p^{(W)}(m + 2) = \{m + 2\}, \qquad J_p^{(W)}(m + 3) = \{m + 3\},
$$

which are nonempty and otherwise as required.

- If $m = i + 1$ then checking each of the three different possibilities for a we likewise get

$$J_p^{(W)}(m) = [m + 4, d - 1], \qquad J_p^{(W)}(m + 1) = [d, m + 1],$$
$$J_p^{(W)}(m + 2) = \{m + 2\}, \qquad J_p^{(W)}(m + 3) = \{m + 3\},$$

which are nonempty and otherwise as required.

- If $m = i$ then $a = 1$ or $a = 2$, in the former case $l = m + 2$, $j = m + 3$ and $j + 1$ is not in the support of p so

$$J_p^{(W)}(m) = \{m + 4\}, \qquad J_p^{(W)}(m + 1) = [m + 5, d - 1],$$
$$J_p^{(W)}(m + 2) = [d, m + 2], \qquad J_p^{(W)}(m + 3) = \{m + 3\},$$

while in the latter $l = m + 2$, $l + 1 = m + 3$, and j and $j + 1$ are not in the support of p so

$$J_p^{(W)}(m) = [m + 4, j + 1], \qquad J_p^{(W)}(m + 1) = [j + 2, d - 1],$$
$$J_p^{(W)}(m + 2) = [d, m + 2], \qquad J_p^{(W)}(m + 3) = \{m + 3\},$$

which are again as required.

Third case: In this case there are no non-supporting vertices $l, l + 1, \ldots$ inside q. Again write $p = (m, m + 2)$ where $m \in \{j, j + 1, j + 2\}$. We proceed as in the previous cases, by computing $I_w^{(W)}$ for vertices w in roughly increasing order of difficulty.

For $w \in [n] \backslash \{j + 1, m, m + 1, m + 2, m + 3\}$: if $w \in J_q^{(V)}(i)$ or $w \in J_q^{(V)}(i + 1)$ then

$$I_w^{(W)}(r) = \begin{cases} m & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{m, j + 1\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Finally, for $j + 1$ and the vertices in the support of p, we have

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = q \\ j+2 & \text{if } r = p \\ I_{j+3}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_{m}^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } q \text{ not supported on } m \\ m+1 & \text{if } r = p \text{ and } q \text{ supported on } m \\ I_{m}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+1}^{(W)}(r) = \begin{cases} m+1 & \text{if } r = p \text{ and } q \text{ not supported on } m+1 \\ m+2 & \text{if } r = p \text{ and } q \text{ supported on } m+1 \\ I_{m+1}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_{m+2}^{(W)}(r) = \begin{cases} m+2 & \text{if } r = p \\ I_{m+3}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+3}^{(W)}(r) = \begin{cases} m+3 & \text{if } r = p \\ I_{m+4}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Note that $I_{m+2}^{(V)}(r) = I_{m+3}^{(V)}(r)$ for all propagators r in V, and that if $j+1 \notin \{m, m+1, m+2, m+3\}$ then $j+2$ and $j+3$ are non-supporting vertices in V, so in that case $I_{j+2}^{(V)}(r) = I_{j+3}^{(V)}(r) = I_{j+4}^{(V)}(r)$ for r in V.

Therefore, once again we can see that the $J_r^{(W)}(*)$ are cyclic for all $r \neq p$ in W. Assembling the intervals for p we have:

- if $m = j$ then

$$J_p^{(W)}(m) = [m+4, e-1], \qquad J_p^{(W)}(m+1) = [e, m],$$
$$J_p^{(W)}(m+2) = [m+1, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j+1$ then

$$J_p^{(W)}(m) = [m+4, m-1], \qquad J_p^{(W)}(m+1) = [m, m+1],$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j+2$ then

$$J_p^{(W)}(m) = [m+4, m], \qquad J_p^{(W)}(m+1) = \{m+1\},$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\}.$$

The result now follows by induction. $\qquad\square$

We need one more lemma before we can prove that Algorithm 3.6 does in fact give the Grassmann necklace of the positroid associated to W.

Lemma 3.12. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and let $M(W)$ be its associated matroid. A subset $J \subseteq [n]$ is an independent set of $M(W)$ if and only if there exists an injective set map $f : J \to \mathcal{P}$ with the property that for each $j \in J$ we have $j \in V(f(j))$.*

One of the most important uses of this lemma is for bases. The lemma says that a subset J of $[n]$ is a basis of $M(W)$ if and only if there is a set bijection between J and $\mathcal{P}$ with the property that for each $j \in J$ the propagator associated to j under the bijection is supported on vertex j.

Proof. Because the nonzero entries of $C(W)$ are independent indeterminants, J is an independent set is if and only if there is some choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns.

Each entry in $C(W)$ identifies a propagator by the row of the entry and a vertex by the column of the entry. The entry is nonzero if and only if the propagator is supported on that vertex.

Consequently, a choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns is equivalent to an assignment of the propagators of J to supporting vertices so that no two are assigned to the same vertex. Such an assignment of the propagators of J to supporting vertices is exactly a map f as described in the statment, hence proving the result. $\qquad\square$

Theorem 3.13. *The sequence of k-subsets $(I_1, \ldots, I_n)$ obtained by applying Algorithm 3.6 to all vertices of an admissible diagram W is exactly the Grassmann necklace of $M(W)$.*

Proof. For each $i \in [n]$, let I_i be the set of vertices assigned to the propagators of W by Algorithm 3.6 with starting vertex i. By Lemma 3.10, we know that $|I_i| = k$ for each $i \in [n]$. It is easy to verify that the sequence $(I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in [n]$; this is the characterization we will use in this proof.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_i \setminus \{i\}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.6 at vertex i and vertex $i + 1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i; thus one end of p must lie on either edge $i - 1$ or edge i. In both cases let b denote the edge supporting the other end of p.

Case I: Suppose p has one end on edge $i - 1$. Then p is not supported on $i + 1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i + 1$ up to $b - 1$, that is $I_{i+1} \cap [i+1, b-1] = I_i \cap [i+1, b-1]$. Furthermore, by Lemma 3.11, when building I_{i+1} it must happen that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when the algorithm reaches vertex b there cannot be any unassigned propagators remaining that are before p. This is also true in when the algorithm constructing I_i reaches b, since the same propagators have been assigned beforehand. Finally, we also note that $k \geq_i b + 1$.

Let W' be the diagram obtained from W by removing both p and all propagators inside of p (recall Definition 3.1).

By the above observations, if we commence Algorithm 3.6 in W' from vertex b, then we are in the same situation with respect to unassigned propagators as if we began at i in W and proceeded to b following the algorithm; the propagators we assigned in the latter case are exactly the ones removed to build W'. Similarly, starting at $i + 1$ in W and moving to $b + 1$ leaves us in the same situation with respect to unassigned propagators as beginning at $b + 1$ in W' would. This gives the equations

$$I_i^{(W)} \cap [b, i - 1] = I_b^{(W')}$$
$$I_{i+1}^{(W)} \cap [b + 1, i - 1] = I_{b+1}^{(W')}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{(W')} \setminus \{b\}$ and $k \notin I_{b+1}^{(W')}$, contradicting the minimality of (W, i, k).

Case II: Suppose p has one end on edge i. Note that by assumption we have $I_i(p) = i$; this means that p must be the first propagator lying on edge i, and hence we must have $I_{i+1}(p) = i + 1$ as well. Observe also that $k \geq_i i + 2$ since $i + 1 \in I_{i+1}$.

Let W' be the diagram obtained from W by removing only the propagator p. Then we have

$$I_i^{(W)} \setminus \{i\} = I_{i+1}^{(W')}$$
$$I_{i+1}^{(W)} \setminus \{i + 1\} = I_{i+2}^{(W')}$$

since in both cases the algorithm in W' proceeds identically to that in W after assigning p. Since $k \neq i + 1$, we have $k \in I_{i+1}^{(W')} \setminus \{i + 1\}$ but $k \notin I_{i+2}^{(W')}$, contradicting the minimiality of (W, i, k).

We have shown that $(I_1, \ldots, I_n)$ is a Grassmann necklace; it remains to check that this Grassmann necklace corresponds to the positroid $M(W)$. We need to show that:

- For each $i \in [n]$, I_i is a basis for $M(W)$.

- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $M(W)$.

The algorithm is pairing each $j \in I_i$ with a unique propagator supported on that vertex so by Lemma 3.12 it immediately follows that I_i is a basis for $M(W)$.

Suppose we have a k-set J such that J is a basis for $M(W)$ and yet is lexicographically less than I_i with respect to $<_i$. By Lemma 3.12 there is a set bijection between J and the propagators of W such that for each $j \in J$, the propagator associated to j is supported on vertex j. Choose one such bijection. For a propagator p of W write $J(p)$ for the associated j according to this bijection.

Since J is lexicographically smaller than I_i, the $<_i$-smallest element of the symmetric difference of J and I_i is some $j_0 \in J$, $j_0 \notin I_i$. Let p_1 be the propagator such that $J(p_1) = j_0$. Since $j_0 \notin I_i$ but p_1 is supported at j_0, then p_1 must have been assigned to an earlier vertex by I_i, i.e. we have $I_i(p_1) <_i j_0$. Let $j_1 = I_i(p_1)$.

However, j_0 is the $<_i$-smallest element in the symmetric difference of J and I_i, so we must have $j_1 \in J$ as well; in particular, there is some propagator p_2 such that $J(p_2) = I_i(p_1)$. Let $j_2 = I_i(p_2)$ and note that $j_2 \neq J(p_2)$ (since $I_i(p_1) = J(p_2)$), so either $I_i(p_2) \notin J$ or there is a propagator p_3 such that $J(p_3) = I_i(p_2)$. Continue this process for as many steps as possible (i.e. for as long as $I_i(p_*) \in J$); this produces a list of vertices j_k and propagators p_k such that

$$J(p_k) = I_i(p_{k-1}) = j_{k-1}.$$

We claim that the propagators and vertices in this list are distinct; this will show that the process must eventually terminate. The claim is proved by induction. The definition of j_0 gives that j_0 and j_1 are distinct and that p_1 and p_2 are distinct. Now, suppose we have p_k and j_k with $I_i(p_k) = j_k$ and with $j_k \in J$. Then there must exist a propagator p_{k+1} with $J(p_{k+1}) = j_k$. Since $J : \mathcal{P} \longrightarrow [n]$ is injective and j_k is not equal to any of the previous vertices $j_0, \ldots, j_{k-1}$ by assumption, p_{k+1} must be distinct from the previous p_j. Define $j_{k+1} = I_i(p_{k+1})$, but since I_i is also a bijection and p_{k+1} is distinct from the earlier p_j, it follows that j_{k+1} is distinct from the other j_j for $1 \leq j \leq k$. Additionally $j_{k+1} \neq j_0$ since $j_0 \notin I_i$ but $j_{k+1} \in I_i$. This proves the claim.

By finiteness the list must end, and the only way it can end is with some p_k such that $j_k = I_i(p_k) \notin J$. In particular, this implies that $j_k >_i j_0$.

Next we will prove that $j_k <_i j_0$ for all $k > 0$; this gives a contradiction to the existence of J and completes the proof of the theorem. This proof will also be by induction; however, for the induction to go through nicely we will need to prove the following slightly stronger statement: for all $k > 0$ in the list

- $j_k <_i j_0$ and

- either

    - $I_i(p_k) <_i J(p_k)$, or
    - there exists an $\ell < k$ such that $I_i(p_\ell) <_i J(p_\ell)$, those two vertices support distinct ends of p_ℓ, and p_k is inside p_l when p_l is directed from $I_i(p_\ell)$ to $J(p_\ell)$.

We observed above that $j_1 <_i j_0$ and $I_i(p_1) <_i J(p_1)$ so we have the base case for the induction. Now fix $k > 1$ and suppose that the inductive hypothesis holds for all indices less than k and that p_k exists.

By definition $J(p_k) = I_i(p_{k-1}) = j_{k-1}$, so p_{k-1} and p_k are both supported at vertex j_{k-1}.

Since I_i always chooses the earliest unassigned propagator supported at a vertex, it follows that either p_{k-1} appears before p_k in the neighbourhood of j_{k-1}, or that p_k was assigned earlier than p_{k-1} by the algorithm for I_i; note that the ordering $p_1, p_2, \ldots, p_k$ need not match the order in which Algorithm 3.6 assigns propagators to vertices.

In the latter case we are done, since p_k being assigned earlier than p_{k-1} by I_i means exactly that $I_i(p_k) <_i I_i(p_{k-1})$. Recalling that $I_i(p_{k-1}) = J(p_k) = j_{k-1}$, we obtain $I_i(p_k) <_i J(p_k)$ and $j_k = I_i(p_k) <_i j_{k-1} <_i j_0$, as required.

Now assume that p_{k-1} appears before p_k in the neighbourhood of j_{k-1}.

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If $I_i(p_{k-1}) <_i J(p_{k-1})$ and these are vertices supported by distinct ends of p_{k-1} then p_k coming after p_{k-1} around j_{k-1} implies that p_k is on the $I_i(p_{k-1})$ to $J(p_{k-1})$ side of p_{k-1}. Therefore $j_k = I_i(p_k) \leq_i J(P_{k-1}) + 1$. If $k - 1 \neq 1$ then this gives $j_k \leq_i I_i(p_{k-2}) + 1 <_i j_0$ so $j_k \leq_i j_0$, but $j_0 \notin I_i$ so $j_k <_i j_0$. If $k - 1 = 1$ then either $j_k <_i j_0$ or we have the configuration
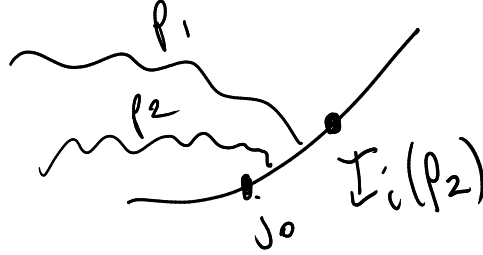
which is impossible as p_2 is supported at j_0 but not taken in I_i until after j_0 even though $j_0 \notin I_i$. Taking all of this paragraph together we are done unless $I_i(p_{k-1}) \not<_i J(p_{k-1})$ so now assume this.

By the induction hypothesis, $I_i(p_{k-1}) \not<_i J(p_{k-1})$ implies that there exists and $\ell < k - 1$ such that p_{k-1} is on the side of p_ℓ that goes from $I_i(p_\ell)$ to $J(p_\ell)$. If j_{k-1} is not in the support of the $J(p_\ell)$ end of p_ℓ then p_{k-1} before p_k around j_{k-1} implies that p_k is on the same side of p_ℓ as p_{k-1} is. Then the same arguments as above with ℓ in place of $k - 1$ give $j_k \leq_i J(P_\ell) + 1$ implying $j_k <_i j_0$ which is everything we need.

It remains to consider when j_{k-1} is in the support of the $J(p_\ell)$ end of p_ℓ, call this case $(*)$. Since $J(p_\ell) = j_{\ell-1}$ and the j_i are distinct this means that we have one of the following two configurations
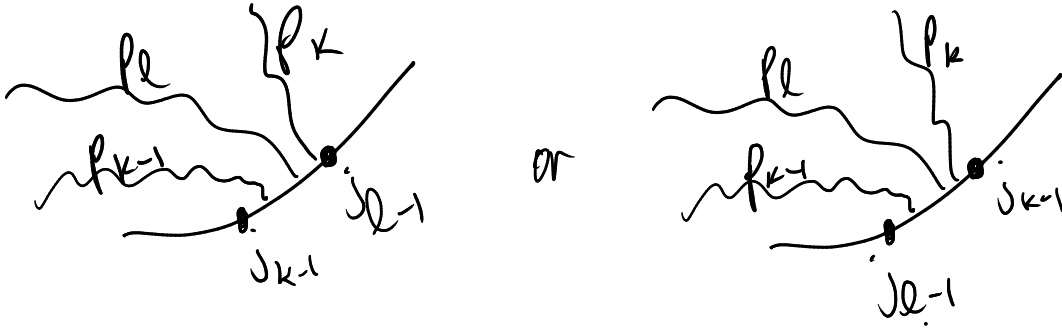
If $p_\ell = p_1$ then $j_{\ell-1} = j_0$ and then as $j_0 \notin I_i$ but p_k is supported at j_0 we get $I_i(p_k) <_i j_0$ and also $I_i(p_k) <_i j_{k-1} = J(p_k)$ as the only vertices not yet taken for $I_i(p_k)$ must be before both labelled vertices in the illustration.

If $p_\ell \neq p_1$ then we argue as above with $\ell - 1$ in place of $k - 1$: in particular $J(p_\ell) = I_i(p_{\ell-1})$ but p_k is supported at $p_{\ell-1}$, so either $I_i(p_k) <_i I_i(p_{\ell-1})$ in which case as before we get desired conclusion,

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or p_k comes before $p_{\ell-1}$ at $j_{\ell-1}$. Then again argue as before getting a new $p_{\ell'}$ in place of p_ℓ, except that then case $(*)$ must occur simultaneously for both $p_{\ell'}$ and p_ℓ with respect to p_k. This can only occur if $j_{\ell-1}, j_{k-1}, j_{\ell'-1}$ are consecutive. Iterating the argument one more time we finally cannot have case $(*)$ as then all neighbouring vertices of j_{k-1} are used up.

This completes the induction and thence the proof of the theorem.

$\square$

In an arbitrary Grassmann necklace, it is possible for an index i to appear in no terms of the Grassmann necklace (a *loop*) or in all terms of the necklace (a *coloop*). Using Theorem 3.13, a characterization of the loops and coloops of the Grassmann necklace associated to a Wilson loop diagram follows easily.

Corollary 3.14. *Grassmann necklaces coming from admissible Wilson loop diagrams have no coloops. A vertex j is a loop if and only if j supports no propagators.*

Proof. For any $i \in [n]$, $i-1$ is maximal with respect to the $<_i$ order. Therefore there can be no propagator p with $I_i(p) = i-1$ by Lemma 3.9, i.e. $i-1 \notin I_i$. Thus the Grassmann necklace admis no coloops.

If $j \in [n]$ is a loop then $j \notin I_j$, which can only happen if there are no propagators supported on vertex j. Conversely, if j supports no propagators, then Algorithm 3.6 never assigns a propagator to j and hence $j \notin I_i$ for all $i \in [n]$. $\square$

4 Dimension of the Wilson Loop cells

Our next goal is to show that the dimension of the positroid cell defined by a Wilson loop diagram $(\mathcal{P}, [n])$ has dimension $3|\mathcal{P}|$. Marcott in ***cite this*** has a different proof which is geometric and more elegant, but it is not easy to track the effect of a particular propagator. Our approach is much more explicit.

Recall that the dimension of a positroid cell is equal to the number of plusses in the associated Le diagram [22, Theorem 6.5]. By combining Algorithms 3.6 and 2.9 (converting from WLD to Grassmann necklace to Le diagram) we explicitly describe the effect of adding another propagator to a Wilson loop diagram in terms of the plusses of the associated Le diagrams, and hence give a recursive proof of the $3|\mathcal{P}|$-dimensionality of the cells.

We start with several lemmas, of roughly increasing degree of technicality.

Lemma 4.1. *Let W be an admissible Wilson loop diagram with k propagators, and with a vertex i that supports no propagators. Let V be W with vertex i removed. Then the Le diagram of W is obtained from the Le diagram of V by inserting an extra column containing all 0s in position i (i.e. such that the new column has the label i).*

Proof. By Algorithm 3.6 the Grassmann necklace of W is obtained from the Grassmann necklace of V by duplicating the ith element of the Grassmann necklace of V (shifting indices as appropriate),

and incrementing all indices greater than i in each Grassmann necklace element. Formally, if $I_1^V, \ldots, I_{n-1}^V$ and $I_1^W, \ldots, I_n^W$ are the Grassmann necklaces of V and W respectively then

$$I_j^W = \begin{cases} \{\ell \in I_j^V : \ell < i\} \cup \{\ell + 1 \in I_j^V : \ell \geq i\} & \text{if } j \leq i \\ \{\ell \in I_{j-1}^V : \ell < i\} \cup \{\ell + 1 \in I_{j-1}^V : \ell \geq i\} & \text{if } j > i. \end{cases}$$

By Lemma 3.14 we know that $i \notin I_1^W$, and so i must label a horizontal edge on the boundary of the Le diagram of W, i.e. it must be a column label. The shapes of the Le diagram of V and W are the same except for the insertion of this column since I_1^* is the same for V and W except for the incrementation of the indices $\geq i$ in the transition from the necklace for V to the necklace for W. $\qquad\square$

Lemma 4.2. *If two Wilson loop diagrams differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Proof. By [22, Proposition 17.10], the dimension of a positroid (and hence the number of plusses in its Le diagram) is $k(n - k) - A(\pi_W)$, where $A(\pi_W)$ denotes the number of alignments of the decorated permutation π_W of the positroid associated to W. (See [22, Figure 17.1] and preceding discussion.)

It can easily be seen from [22, Section 17] and Algorithm 3.6 that dihedral transformations of a Wilson loop diagram W correspond to dihedral transformations of the chord diagram representation of π_W. Since the number of alignments in a chord diagram is preserved under dihedral transformations, the result follows. $\qquad\square$

Lemma 4.3. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators. Then there is some dihedral transformation W' of W such that W' has a propagator p with the following properties.*

- *$p = (i, n - 1)$ for some i, and p has no propagators inside it (that is $i + 2, \ldots, n - 2$ do not support any propagators in W').*

- *Either the edge i in W' only supports p or the edge i in W' supports exactly one other propagator $q = (j, i)$ with no other propagators inside q.*

Proof. Remove all vertices of W which do not support any propagators to get a weakly admissible Wilson loop diagram V. Lemma 3.4 applied to V gives a length 2 propagator p in V for which either no other propagator is supported on one of the supporting edges of p or there is a second length 2 propagator which is the only other propagator supported on one of the supporting edges of p. (Figure 3 shows the possible configurations arising from Lemma 3.4, and the reader can easily check that in each case p must be in one of the two situations described above.)

We can now make a dihedral transformation of V to obtain a diagram satisfying the statement of the lemma with p and q both length 2. Restoring the vertices which do not support any propagators, we obtain a dihedral transformation W' of W as desired (with potentially longer lengths for p and q). $\qquad\square$
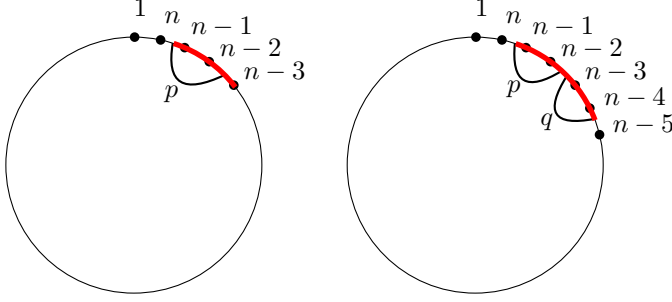
Figure 5: The two cases for W and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

Combining Lemmas 4.1, 4.2, and 4.3, it therefore suffices to study the Le diagrams of weakly admissible Wilson loop diagrams admitting one of the configurations described in Lemma 4.3 with propagators p and q (if q exists) both of length 2. See Figure 5 for an illustration of the two possibilities.

The next few lemmas describe how diagrams of this type are related to the corresponding diagram with propagator p removed, first in terms of the Grassmann necklaces and then in terms of the Le diagrams. These technical lemmas will form the backbone of the inductive step in the main dimensionality argument.

Lemma 4.4. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators, and suppose that W admits one of the configurations described in Lemma 4.3, with p and q (if q exists) both of length 2. Let V be W with p removed. Then*

$$I_1^W = I_1^V \cup \{n-3\}$$
$$I_n^W = I_1^V \cup \{n\}$$
$$I_{n-1}^W = I_n^V \cup \{n-1\}$$
$$I_{n-2}^W = \begin{cases} I_{n-2}^V \cup \{n-2\} & \text{if } n-2 \notin I_{n-2}^V \\ I_{n-2}^V \cup \{n-1\} & \text{if } n-2 \in I_{n-2}^V, \ n-1 \notin I_{n-2}^V \\ (I_n^V - \{n-5\}) \cup \{n-1, n-2\} & \text{if } n-1, n-2 \in I_{n-2}^V \end{cases}$$
$$I_k^W = \begin{cases} I_k^V \cup \{n-3\} & \text{if } n-3 \notin I_k^V \\ I_k^V \cup \{n-2\} & \text{if } n-3 \in I_k^V \end{cases}$$
$$\text{for } 1 < k < n-2$$

Proof. The two possible cases for W are illustrated in Figure 5; all references to the "left hand case" or "right hand case" below refer to the diagrams in this figure.

We first consider I_1^W, i.e. the set obtained by applying the Grassmann necklace algorithm to W with starting vertex 1. Note that $n-3 \notin I_1^V$: for the right hand case this is clear from the diagram (since Algorithm 3.6 would assign q no later than vertex $n-4$), while for the left hand case it
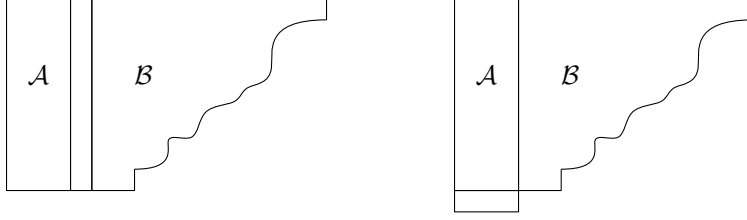
Figure 6: Le diagrams for V (left) and W (right).

follows from Lemma 3.9. Therefore when we start at vertex 1 and apply the Grassmann necklace algorithm to W, p is the only remaining unassigned propagator when we reach vertex $n-3$, and so we have $I_1^W(p) = n-3$ and $I_1^W = I_1^V \cup \{n-3\}$.

Next consider I_n^W. From the figure we see that in both cases we have $I_n^W(p) = n$. We are now at vertex 1, and the unassigned propagators are exactly those that appear in V. Therefore the algorithm continues as in I_1^V, i.e. we have $I_n^W = I_1^V \cup \{n\}$. By a similar argument, we must have $I_{n-1}^W = I_n^V \cup \{n-1\}$.

Now consider I_{n-2}^W. If $n-2 \notin I_{n-2}^V$ (i.e. $n-2$ supports no propagators in V), then $I_{n-2}^W(p) = n-2$ and this assignment of p does not affect the rest of the construction of I_{n-2}^V, so we obtain $I_{n-2}^W = I_{n-2}^V \cup \{n-2\}$ as above.

On the other hand, if $n-2 \in I_{n-2}^V$ then we must be in the right hand case of Figure 5 and $I_{n-2}^V(q) = I_{n-2}^W(q) = n-2$ (since q is always the clockwise-most propagator supported on vertex $n-2$; see Figure 5). If $n-1 \notin I_{n-2}^V$, then $I_{n-2}^W(p) = n-1$ and then the algorithm proceeds identically to I_{n-2}^V for the remainder of its steps; thus $I_{n-2}^W = I_{n-2}^V \cup \{n-1\}$ in this case.

Finally, if $n-2, n-1 \in I_{n-2}^V$ then $I_{n-2}^W(q) = n-2$ and $I_{n-2}^W(p) = n-1$ as above, but this is different to what occured in I_{n-2}^V so we cannot use the same argument as above. We are now at vertex n and only propagators p and q have been assigned; thus we are proceeding as in the construction of I_n^V but without propagator q. By Lemma 3.11 we know that $I_n^V(q) = n-5$, and from the diagram we see that the only way this could occur is if all other propagators in V had already been assigned when we reached vertex $n-5$; thus q was the final propagator to be assigned in I_n^V. Therefore $I_{n-2}^W = (I_n^V \setminus \{n-5\}) \cup \{n-1, n-2\}$ in this case. This completes all cases for I_{n-2}^W.

The arguments for I_k^W ($1 < k < n-2$) proceed analagously to those of I_{n-2}^W, with one simplification: we cannot have both $n-3$ and $n-2$ in I_k^V since q is the only propagator that could be assigned to either of them, and it cannot be assigned to both.

This covers all cases and hence completes the proof. $\qquad\square$

Lemma 4.5. *Let V and W be as in Lemma 4.4. The shape of the Le diagram of V can be built from left to right of the following blocks: a rectangle which is 3 columns wide, one more column of the same height, and a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of W can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of V, and the same partition shape as in V.*

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Proof. See Figure 6 for an illustration of the shapes described in the statement of the lemma.

Recall that I_1 determines the shape of the Le diagram. By Lemma 4.4 we know that $n, n-1, n-2, n-3 \notin I_1^V$ (which yields the leftmost four columns of the diagram for V) and that $I_1^W = I_1^V \cup \{n-3\}$. This implies that the right hand boundary of the shape of V is the same as the right hand boundary of the shape of W except that W has one additional row of 3 boxes while V has an additional column in the $n-3$ position; that is, an extra column fourth from the left. □

As illustrated in Figure 6, the pieces of the Le diagrams of V and W will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of each diagram are identical, and that the relationship between the plusses in the $\mathcal{A}$ regions can be described explicitly.

We do this by applying Algorithm 2.9, which constructs the Le diagram associated to a Grassmann necklace, to the Grassmann necklaces of V and W. As described in Section 2.2, if Algorithm 2.9 places a $+$ in the box with row index i and column index j, we say that this plus is in the $i \to j$ position, and refer to it as the plus defined by "the (hook) path from i to j". Note that the collection of paths contributed by a single Grassmann necklace term must be non-crossing.

Finally, we also note that when we speak of a plus in the Le diagram of V being the same as in W or vice versa, we mean that the position of the plus in $\mathcal{A}$ or $\mathcal{B}$ is the same; because of the column insertion the absolute indices may differ.

Lemma 4.6. *Let V and W be as in Lemma 4.4. Then I_n^W and I_{n-1}^W together yield the same plusses as I_n^V did, along with two extra plusses which appear in the leftmost two boxes of the bottom row of the Le diagram of W.*

Proof. By Lemma 4.4 we have $I_n^W = I_1^V \cup \{n\}$ and $I_1^W = I_1^V \cup \{n-3\}$. Thus

$$I_1^W \setminus I_n^W = \{n-3\}, \qquad I_n^W \setminus I_1^W = \{n\},$$

and so by Algorithm 2.9 we have a plus in the $(n-3) \to n$ position, i.e. in the leftmost box of the bottom row.

Also by Lemma 4.4 we have $I_{n-1}^W = I_n^V \cup \{n-1\}$. Recall from the proof of Lemma 4.4 that $n-3, n-2, n-1, n \notin I_1^V$; therefore $I_1^W \setminus I_{n-1}^W = (I_1^V \setminus I_n^V) \cup \{n-3\}$, and $n-3$ is maximal in this set. Similarly,

$$I_{n-1}^W \setminus I_1^W = (I_n^V \setminus I_1^V) \cup \{n-1\} \subseteq \{n-1, n\},$$

where the final inclusion follows from the definition of Grassmann necklace. In particular, $n-1$ is minimal in this set, so Algorithm 2.9 yields a plus in the $(n-3) \to (n-1)$ position (i.e. the second box in the bottom row of the Le diagram of W) along with any plusses yielded by I_n^V. □

Lemma 4.7. *Let V and W be as in Lemma 4.4, and suppose that $n-2 \notin I_{n-2}^V$. Then I_{n-2}^W contributes all of the same plusses as $I_{n-1}^V = I_{n-2}^V$, along with a new $(n-3) \to (n-2)$ plus.*

Proof. If $n-2 \notin I_{n-2}^V$ then $I_{n-1}^V = I_{n-2}^V$ by definition, and by Lemma 4.4 we have $I_{n-2}^W = I_{n-1}^V \cup \{n-2\}$. Note that $n-3 \notin I_{n-2}^V$ by Lemma 3.9. Therefore the paths controlling the plusses contributed by I_{n-2}^W are exactly the paths for I_{n-1}^V along with the $(n-3) \to (n-2)$ path. This gives the statement of the lemma. □

Lemma 4.8. *Let V and W be as in Lemma 4.4, and suppose that $n-2, n-1 \in I^V_{n-2}$. Then:*

1. *I^W_{n-2} contributes an $(n-3) \to (n-2)$ plus and an $(n-5) \to (n-1)$ plus, along with any plusses contributed by I^V_n.*

2. *I^W_{n-3} contributes the same plusses as I^V_{n-3}, except that the plus in the $n-3$ column (which necessarily exists) is shifted one square left into the $n-2$ column.*

3. *In the Le diagram of V, I^V_{n-2} contributes an $(n-5) \to (n-2)$ plus and no other term in the Grassmann necklace of V gives a plus in this column. I^W_{n-2} does not contribute this $+$ but yields the $(n-5) \to (n-1)$ plus from point (1) instead.*

4. *All other plusses from I^V_{n-2} and all plusses from I^V_{n-1} were already contributed by I^V_{n-3}, so they yield no new information.*

5. *No element of the Grassmann necklace of V contributes a $(n-5) \to (n-1)$ plus.*

Proof. By Lemma 4.4 we have $I^W_{n-2} = (I^V_n \setminus \{n-5\}) \cup \{n-1, n-2\}$ and $I^W_1 = I^V_1 \cup \{n-3\}$. We are necessarily in the case where W admits a propagator q (the right hand case of Figure 5) and we can see from the location of q that $n-2 \notin I^V_{n-3}$ and that $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Also note that by Lemma 3.11 and Corollary 3.14 we must have $n \in I^V_n$. Thus this section of the Gramann necklace of V looks like

$$I^V_{n-3} \xrightarrow[\substack{n-3 \text{ out} \\ n-2 \text{ in}}]{} I^V_{n-2} \xrightarrow[\substack{n-2 \text{ out} \\ n-5 \text{ in}}]{} I^V_{n-1} \xrightarrow[\substack{n-1 \text{ out} \\ \text{something in}}]{} I^V_n \xrightarrow[\substack{n \text{ out} \\ \text{something in}}]{} I^V_1 \qquad (2)$$

where the first "something" is either n or an element of I^V_1, and the second "something" is an element of I^V_1.

Using this information we can determine the symmetric difference of I^V_{n-2} and I^V_1. Note that $I^V_{n-2} \setminus I^V_1$ and $I^V_1 \setminus I^V_{n-2}$ each have size ≤ 3 by the definition of a Grassmann necklace. We have $n-1, n-2 \in I^V_{n-2} \setminus I^V_1$ for certain, and the third element (if it exists) must be n. On the other hand we have $n-5 \in I^V_1 \setminus I^V_{n-2}$, along with at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$.

This means that the plusses in the Le diagram of V coming from I^V_{n-2} are as in the first part of Figure 7. Stepping to I^V_{n-1} simply removes the $(n-5) \to (n-2)$ path (second part of Figure 7), so I^V_{n-1} contributes no new information about the Le diagram of V.

With this information in hand, we can now tackle the points in the statement of the lemma. We know that

$$I^W_{n-2} = (I^V_n - \{n-5\}) \cup \{n-1, n-2\}, \quad n-5 \in I^W_1, \quad n-3 \notin I^W_{n-2},$$

and so the paths for building plusses from I^W_{n-2} start at

$$I^W_1 \setminus I^W_{n-2} = \{n-3, n-5\} \cup (I^V_1 \setminus I^V_n)$$

and end at

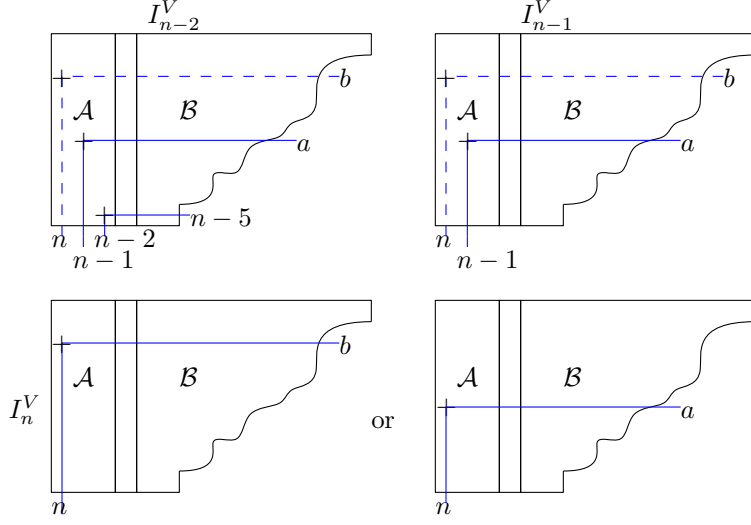$$I^W_{n-2} \setminus I^W_1 = \{n-2, n-1\} \cup (I^V_n \setminus I^V_1).$$

Figure 7: Plusses coming from I_{n-2}^V (left) and I_{n-1}^V (right), in the case where $n-1, n-2 \in I_{n-2}^V$. The blue lines are the non-intersecting paths defining the position of the plusses. The dashed blue lines may or may not appear, but if one appears then they both do.
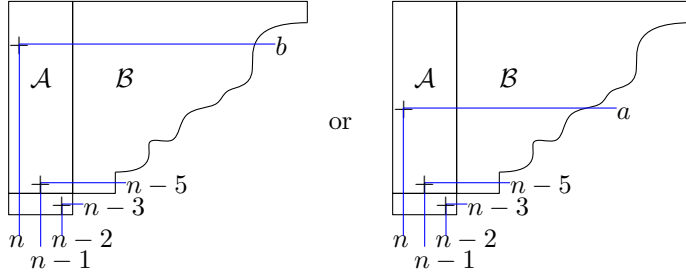

Figure 8: Plusses coming from I_{n-2}^W.

This means that I_{n-2}^W contributes plusses as in Figure 8, while I_n^V contributes as in the bottom line of Figure 7, which proves point (1) and most of point (3). From equation (2) we see that $n-2$ appears only in I_{n-2}^V, so no other Grassmann necklace term for V can contribute a plus in the $n-2$ column; this completes the proof of item (3).

Now consider I_{n-3}^V. By (2) I_{n-3}^V contributes almost the same plusses as I_{n-2}^V: the only difference is that it contributes an $(n-5) \to (n-3)$ plus in place of the $(n-5) \to (n-2)$ plus. Since we must have $n-3 \in I_{n-3}^V$ (because the propagator q exists in V), it follows from Lemma 4.4 that $I_{n-3}^W = I_{n-3}^V \cup \{n-2\}$. Thus the paths for I_{n-3}^W are the same as those for I_{n-3}^V except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (2) shows that $n-2$ only appears in I_{n-2}^V among the necklace elements of V.

This proves point (2), and combined with the observation above that I_{n-1}^V contributes no new plusses compared to I_{n-2}^V, we also obtain item (4).

Finally, if the Le diagram of V did admit a $(n-5) \to (n-1)$ plus, it could only have been contributed by a necklace term that doesn't contain $n-5$. By equation (2) the only terms that

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could have this property are I_{n-2}^V, I_{n-3}^V, and I_{n-4}^V, and the analysis above shows that I_{n-2}^V and I_{n-3}^V do not contribute a plus in this position.

Recall that $n-5$ is the largest row index in the Le diagram of V, and $n-5 \in I_1^V \setminus I_{n-4}^V$ so I_{n-4}^V does contribute a path starting at $n-5$. Since $n-4 \in I_{n-4}^V$ by the location of the propagator q, we must also have a path ending at $n-4$. Since the paths cannot cross, this implies that I_{n-4}^V must contribute a $(n-5) \to (n-4)$ plus, and hence cannot contribute a $(n-5) \to (n-1)$ plus. This proves point (5) and completes the proof of the lemma. $\square$

Lemma 4.9. *Let V and W be as in Lemma 4.4, and suppose that if $n-2 \in I_{n-2}^V$ then $n-1 \in I_{n-2}^V$ also. Then for each k in the range $1 < k < n-2$, I_k^W contributes the same plusses as I_k^V, except that if I_k^V contributed a plus in the $n-3$ column of the Le diagram of V then this plus is shifted one square left in the Le diagram of W, and no plus was already in that location in the Le diagram of W.*

Proof. Recall that $I_1^W = I_1^V \cup \{n-3\}$ and that $n-2, n-1, n \notin I_1^W$.

If $n-3 \notin I_k^V$ then by Lemma 4.4 we have $I_k^W = I_k^V \cup \{n-3\}$. Then since $n-3$ is the largest element of I_1^W this transformation leaves the disjoint paths unchanged and so the plusses carry over from V to W directly.

If $n-3 \in I_k^V$ then I_k^V must contribute a plus in the $n-3$ column of the Le diagram of V, and by Lemma 4.4 we have $I_k^W = I_k^V \cup \{n-2\}$. If $n-2$ supports no propagators in V then certainly no pluses appear in the $n-2$ column of the Le diagram of V. If $n-2$ supports at least one propagator in V then $n-2 \in I_{n-2}^V$ and so by hypothesis $n-1 \in I_{n-2}^V$ as well. By Lemma 4.8, the only necklace element of V containing $n-2$ is I_{n-2}^V and the corresponding plus is not contributed to the Le diagram of W by I_{n-2}^W.

Since $n-3 \in I_k^V \setminus I_1^V$, we must have a path from some vertical edge i to the bottom edge $n-3$ in the Le diagram of V. In the Le diagram of W, the index $n-3$ labels a vertical edge with no path starting at it (since $n-3$ belongs to both I_1^W and I_k^W), and there must be a path leading to $n-2$ since $n-2 \in I_k^W \setminus I_1^W$. By the previous paragraph no other path from I_k^V could end at $n-2$, and since the paths cannot cross we conclude that the $i \to (n-3)$ path in the Le diagram of V must become a $i \to (n-2)$ path in the Le diagram of W. All other paths are unchanged.

Thus the plus contributed by I_k^V in the $n-3$ column of the Le diagram of V is shifted into the $n-2$ column in the Le diagram for W, where there was no plus before, and no other plusses are changed. $\square$

Theorem 4.10. *The number of plusses in the Le diagram of an admissible Wilson loop diagram is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a Wilson loop diagram W with one propagator supported on vertices $i < j < k < \ell$ has Le diagram a single row with $|W| - i$ boxes. Labelling the columns from left to right by $|W|, \ldots, |W| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider Wilson loop diagrams with $k > 1$ propagators. By Lemma 4.1 it suffices to prove the result for weakly admissible Wilson loop diagrams with k propagators and no non-supporting

vertices. By Lemma 4.2 it suffices to prove the result for at least one Wilson loop diagram from each dihedral orbit. Take a weakly admissible Wilson loop diagram W with k propagators and no non-supporting vertices. Make a dihedral transformation of W if necessary so that W has a propagator p with the properties in Lemma 4.3.

We make one further simplification: if our W is in Case 2 of Figure 5 and $n - 1$ supports only p but $n - 2$ supports at least one other propagator, then flip W on the line perpendicular to the edge from $n - 2$ to $n - 1$ to obtain a diagram with the configuration of Case 1. This eliminates the possibility that we could have $n - 2 \in I_{n-2}^W$ but $n - 1 \notin I_{n-2}^W$.

This diagram will be our W for the remainder of the proof. Let V be W with p removed.

From Lemma 4.5 we know how the shapes of the Le diagrams of V and W are related; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 4.6, 4.7, and 4.8 tell us that the three boxes of the bottom row of the Le diagram of W each have a plus. Lemmas 4.6 through 4.9 show that there is a bijection between the plusses of the Le diagram of V and the plusses of the Le diagram of W that are not in the bottom row. This bijection can be described as follows:

- All plusses from $\mathcal{B}$ for V maintain their positions in $\mathcal{B}$ for W.

- All plusses from the leftmost two columns (the n and the $n - 1$ columns) of $\mathcal{A}$ for V maintain their positions in $\mathcal{A}$ for W.

- If there is a plus in the $n - 2$ column of $\mathcal{A}$ in V then Lemma 4.8 applies, so there is exactly one such plus. This plus is mapped to the $(n - 5) \to (n - 1)$ plus for W.

- All plusses in the $n - 3$ column for V are shifted one square to the left in $\mathcal{A}$ for W, i.e. into the $n - 2$ column.

Note that this map is reversible and hence bijective. Indeed, the only possible ambiguity is at the $(n - 5) \to (n - 1)$ plus in W (if it exists), which could have come from either of the first or third bullet points. However, if the third bullet point applies then by Lemma 4.8 there is no $(n - 5) \to (n - 1)$ plus in V, i.e. the first bullet point does not apply.

Therefore the Le diagram of W contains $3(k - 1)$ plusses in bijection with the plusses from the Le diagram of V and 3 new plusses in the bottom row, yielding $3k$ in total. Applying induction completes the proof. □

5 Poles of Wilson Loop Integrals

The results of Section 3 allow us to relate the position of propagators in a Wilson loop diagram W to minors of $C(W)$, which we use in this section to understand the denominator of the integral $I(W)$ associated to a Wilson loop diagram (see Definition 2.6).

The main result of this section is Theorem 5.4, which expresses the denominator $R(W)$ in terms of the Grassmann necklace of W. This simplifies the computation of $R(W)$ and allows us to directly relate the poles of the integral to the combinatorics of the diagram.

We first give an algorithm which extracts the required minors from the Grassmann necklace.
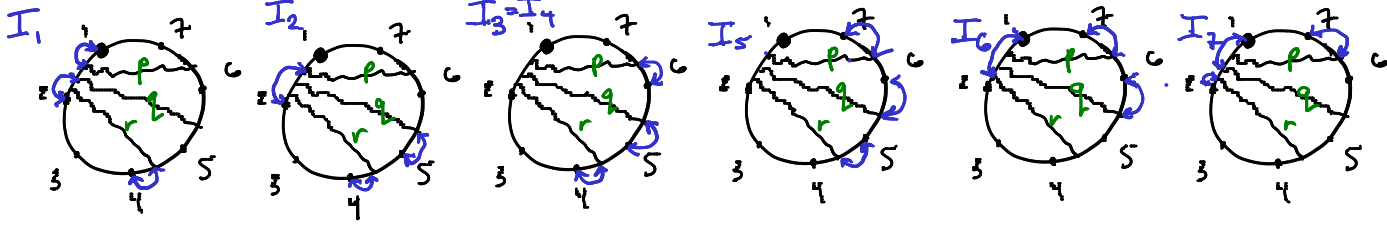
Figure 9: Example WLD for illustrating Algorithm 5.1 and bijections between propagators and vertices for each Grassmann necklace element.

Algorithm 5.1. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and let $C(W)$ be the matrix of W as defined in (1) (see Section 2.1).*

- *For each $i \in [n]$, we construct a factor r_i as follows:*
  - *Let $S_i = \{p \in \mathcal{P} \mid I_{i-1}(p) \neq I_i(p)\}$. (By convention, set $I_{-1} = I_n$.)*
  - *Let r_i be the determinant of the $|S_i| \times |S_i|$ minor of $C(W)$ with rows indexed by S_i and columns indexed by $I_i(S_i)$.*

- *Define $R = \prod_{i=1}^{n} r_i$.*

As we discuss the algorithm it will also be useful to have the notation Δ_{I_i} for the determinant of the $k \times k$ minor of $C(W)$ with columned indexed by I_i.

As we take each r_i to simply be the determinant of a particular submatrix of $C(W)$, the sign of each r_i is well-defined. The goal is to show that R is equal to the denominator of the Wilson loop diagram as defined in Definition 2.6.

Example 5.2. Consider the Wilson loop diagram in Figure 9. Assigning propagators p, q, s to rows $1, 2, 3$ respectively, we obtain the matrix

$$C(W) = \begin{bmatrix} a & b & 0 & 0 & 0 & c & d \\ e & f & 0 & 0 & g & h & 0 \\ i & j & 0 & k & l & 0 & 0 \end{bmatrix}$$

The Grassmann necklace of this diagram is

$$I_1 = \{1, 2, 4\}, I_2 = \{2, 4, 5\}, I_3 = \{4, 5, 6\}, I_4 = \{4, 5, 6\},$$
$$I_5 = \{5, 6, 7\}, I_6 = \{6, 7, 1\}, I_7 = \{7, 1, 2\}.$$

Figure 9 indicates the pairings between propagators and vertices for each $i \in [1, 7]$.

From I_1 to I_2, the propagators p and q change which vertex they are assigned to but r is assigned to vertex 4 in both, so $S_2 = \{p, q\}$. Then

$$\Delta_{I_2} = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ j & k & l \end{bmatrix} = kgb, \qquad r_2 = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ 1 & 1 & 1 \end{bmatrix} = gb.$$

where the 1s in the third row of the second matrix correspond to the fact that $I_1(s) = I_2(s)$. Continuing likewise, we get $r_3 = c$, $r_4 = 1$ (since $I_4 = I_3$), $r_5 = lhd$, and $r_6 = i$.

At I_7 the situation is more complicated: we have $S_7 = \{q, s\}$, so we find that $\Delta_{I_7} = d(ej - fi)$ and $r_7 = ej - fi$. This quadratic factor corresponds to the fact that q and s share an edge *and* contribute both endpoints of that edge to I_7; see Proposition 5.3 below.

Finally, we have $r_1 = (af - be)k$. Putting everything together, we obtain

$$R = (af - be)kgbclhdi(ej - fi)$$

which is squarefree and contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$. If one were to construct the denominator $R(W)$ associated to this Wilson loop diagram as per Definition 2.6, we would find that we have $R(W) = R$.

Proposition 5.3. *With notation as in Algorithm 5.1 we have the following:*

1. *Each Δ_{I_i} is homogeneous, as is each r_i.*

2. *Each Δ_{I_i} splits into linear and quadratic factors. All linear factors of Δ_{I_i} are single variables and all irreducible quadratic factors are 2×2 determinants of single variables.*

3. *Quadratic factors in Δ_{I_i} arise precisely when propagators p and q are supported on a common edge (a, b) with $I_i(p) = a$ and $I_i(q) = b$.*

4. *r_i divides Δ_{I_i}.*

5. *The ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.*

Proof.   1. The nonzero entries of $C(W)$ are independent indeterminates and so every $i \times i$ minor of $C(W)$ is either homogeneous of degree i or is 0. Thus each Δ_{I_i} and each r_i is homogeneous.

2. Using the expression for the determinant as a sum over permutations we see that Δ_{I_i} is a sum over bijections between I_i and $\mathcal{P}$. The nonzero terms in this sum are precisely those bijections such that each propagator is associated to one of its supporting vertices in I_i, since only those locations in $C(W)$ are nonzero. Since the nonzero entries of $C(W)$ are independent there can be no cancellation between terms in this expansion.

Suppose Δ_{I_i} has an irreducible factor f. Let $\mathcal{P}'$ be the set of propagators which contribute a variable to f and let J be the set of vertices which contribute a variable to f.

The first claim is that the minor of $C(W)$ associated to $\mathcal{P}'$ and J is precisely f.

Proof of claim: By the structure of determinants we know that $\Delta_{I_i} = fg$, where g involves only variables associated to propagators not in $\mathcal{P}'$ and associated to vertices not in J.

Expanding out fg yields a signed sum of monomials. In each of these monomials, f contributes those variables associated both to a propagator in $\mathcal{P}'$ and to a vertex in J, and g contributes those variables associated both to a propagator not in $\mathcal{P}'$ and to a vertex not in J, and no other variables appear.

Since there is no cancellation between terms, this means that the full expansion over permutations of Δ_{I_i} contains no other nonzero terms and hence no other variables. Therefore Δ_{I_i}
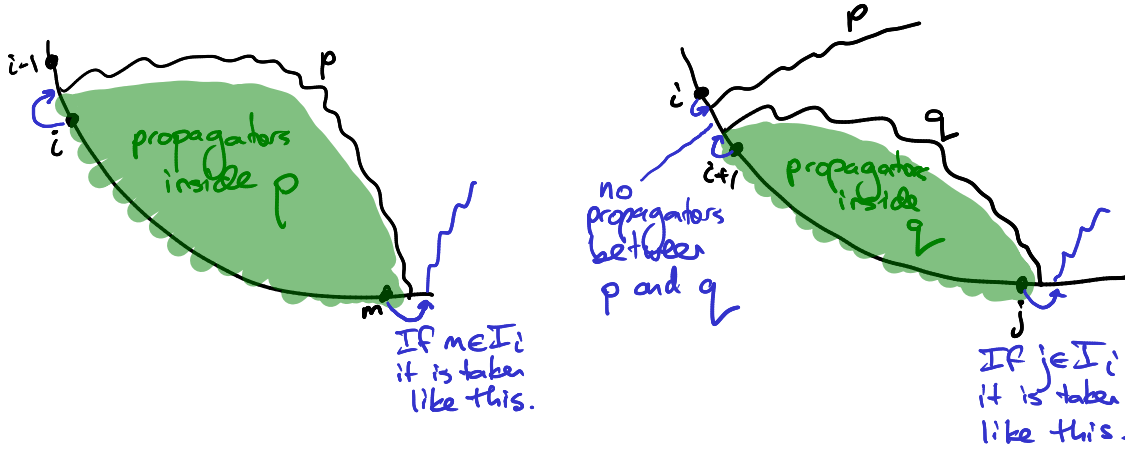
Figure 10: The two cases in the proof that no factors of Δ_{I_i} have degree 3 or more.

is equal to the determinant of the matrix obtained by taking the submatrix of $C(W)$ with columns indexed by I_i and setting any variables not appearing in Δ_{I_i} to 0. This new matrix is, up to permutations of rows and columns, a block matrix with one block for $\mathcal{P}'$ and J and the other block for the complements. Thus its determinant, and hence also Δ_{I_i}, is the product of the minors for these two blocks. By considering which variables appear, these two factors must also be f and g, and so in particular f is the minor of $C(W)$ associated to $\mathcal{P}'$ and J. This proves the claim.

A consequence of this claim is that every linear factor of Δ_{I_i} is a 1×1 minor of $C(W)$, hence is a single variable, and every irreducible quadratic factor of Δ_{I_i} is a 2×2 minor of $C(W)$, hence is a 2×2 determinant of single variables.

All that remains is to prove that Δ_{I_i} has no irreducible factors of degree 3 or more. Suppose for a contradiction that f is a factor of Δ_{I_i} of degree ≥ 3. Note that by removing the propagators which come before those contributing to f and changing i to be the first vertex which contributes to f, we obtain a different admissible diagram for which f still divides Δ_{I_i} but also $i \in I_i$ and i contributes to f. Showing that this different admissible diagram gives a contradiction is sufficient, and so we may assume that $i \in I_i$ and i contributes to f. Finally, we can suppose that W is minimal in number of propagators with the above occuring.

Let p be the propagator such that $I_i(p) = i$. There are two cases to consider, depending on which edge p is supported on. These are illustrated in Figure 10

Case 1: Suppose p has one end on the edge $(i-1, i)$. Thus p is supported on $(i-1, i, m, m+1)$ for some $m >_i i$, and $I_{i+1}(p) = m$ by Lemma 3.11.

Let S be the set of propagators inside p along with p itself. I_i and I_{i+1} can only differ once p contributes to I_{i+1}, so $I_i(q) = I_{i+1}(q)$ for each $q \in S \backslash \{p\}$. Thus if a propagator contributes m in I_i then it must lie outside p.

If neither m nor $m+1$ appear in I_i then by Corollary 3.14 $V(p) \cap I_i = \{i\}$, and so the row of p in the matrix of Δ_{I_i} has only one nonzero entry; hence Δ_{I_i} has a linear factor contributed

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by p and i, which is a contradiction. So we must have at least one of m and $m+1$ in I_i. However, all propagators in S are mapped by the function $I_i(\cdot)$ to vertices strictly before m, so the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the $|S| \times |S|$ matrix indexed by the propagators in S and the vertices in $I_i(S)$. No other propagators can be supported on these vertices since all other propagators are outside of p, and p is the first propagator supported at i; this explains the zero block. Therefore $\Delta_{I_i} = \det A \det C$, and both factors are nontrivial since at least one of m and $m+1$ appear in I_i. If we remove the propagator outside of p that contributes m or $m+1$, we get a smaller diagram for which $\Delta_{I_i} = \det A$. This contradicts the minimality of our choices unless $\det A$ is quadratic, which in turn contradicts our assumption that i and p contribute to an irreducible factor f of degree at least 3.

Case 2: Suppose p has one end on the edge $(i, i+1)$. If no other propagators are supported on i then the column of $C(W)$ corresponding to vertex i has only one nonzero entry in it, and so Δ_{I_i} has a linear factor contributed by p and i; as above, this is a contradiction. Thus we can take q to be the propagator such that $I_i(q) = i+1$. We know that q has one end on the edge $(i, i+1)$ and is adjacent to p on that edge in the counterclockwise direction (see Figure 10). Write $(i, i+1, j, j+1)$ for the support of q. The situation for q is very similar to case 1: in particular, we have $I_{i+1}(q) = j$ by Lemma 3.11 and so if $j \in I_i$ then the propagator which contributes j is outside of q.

Similarly to Case 1, let S be the set of propagators inside q along with p and q themselves. Then all propagators in S are mapped by $I_i(\cdot)$ to vertices strictly before j and no other propagators are supported on vertices strictly before j. Thus the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the submatrix indexed by the propagators in S and the vertices in $I_i(S)$. Again two things can now happen. If some vertex j or larger (with respect to $>_i$) belongs to I_i then B and C are at least one column wide, and so the block form of the matrix gives a nontrivial factorization of Δ_{I_i}. This yields a contradiction as in Case 1: either W contains unnecessary propagators which contradicts our minimality assumption, or $\det A$ is quadratic which contradicts the assumption that p and i contribute to f, an irreducible factor of degree at least 3.

On the other hand, if no vertex $\geq_i j$ is in I_i then $\Delta_{I_i} = \det A$. Looking in more detail into A, note that the only vertices in the support of p and q which belong to I_i are i and $i+1$, and hence

$$A = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}$$

where D is the 2×2 matrix indexed by the propagators p and q and the vertices i and $i+1$. Thus p and i contribute to a quadratic factor of Δ_{I_i}, once again contradicting our assumptions.

All cases have now been covered and so Δ_{I_i} has only irreducible factors of degree 2 or less.
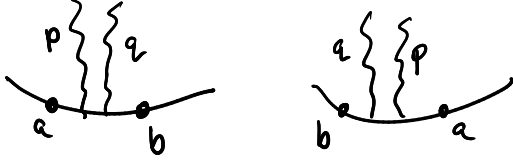
Figure 11: The situations giving a quadratic factor with variables appearing in r_i.

3. Suppose propagators p and q are supported on a common edge (a, b), with $I_i(p) = a$ and $I_i(q) = b$. Let $x_{p,a}, x_{p,b}, x_{q,a}, x_{q,b}$ be the associated variables in $C(W)$. For any fixed bijection σ from $\mathcal{P} - \{p, q\}$ to $I_i - \{a, b\}$ for which each propagator is supported on its image under the bijection, we can extend σ to a bijection of all propagators with I_i in two ways: either $p \mapsto a$ and $q \mapsto b$ or $p \mapsto b$ and $q \mapsto a$. The sum of the contributions of all these bijections to Δ_{I_i} is therefore the product of $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ with the minor coming from $\mathcal{P} - \{p, q\}$ and $I_i - \{a, b\}$. Since there is no cancellation of terms in the expansion of Δ_{I_i}, if any other terms appear then they will cause a factor which is not in the form described in the previous part. Therefore no such terms exist and $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ is a factor of Δ_{I_i}.

   Now let f be a quadratic factor of Δ_{I_i}. By part (2) we know that f is a 2×2 minor coming from two propagators, call them p and q, and two vertices, call them $a <_i b$. It remains to show that a and b are adjacent. From this we can conclude that p and q each have one end on (a, b), as any other way for both p and q to be supported on two consecutive vertices would contradict noncrossing or the density requirement of admissibility.

   As in the proof of part (2), make a new admissible diagram by removing the propagators which come before f and set $i = a$. The cases in the proof of part (2) show how Δ_{I_i} factors: in particular the vertices supporting the other end of p either do not appear in I_i, or they contribute to a different factor of Δ_{I_i} than p and a do. By assumption b contributes to the same factor as a. Therefore (a, b) is an edge.

4. Consider $p \in S_i$, and note that Δ_{I_i} is homogeneous linear in the variables of the row corresponding to p. By part (2), either exactly one variable in the row corresponding to p appears in Δ_{I_i} and this variable is a factor of Δ_{I_i}, or exactly two variables from the row corresponding to p appear in Δ_{I_i} and they appear as part of a quadratic factor. In the first case let the variable be x. Then x is a factor of Δ_{I_i} and so in particular the monomial in Δ_{I_i} corresponding to the bijection between propagators and vertices of I_i associates the column of x to p. Thus x also appears in r_i and since the matrix for r_i is a minor of the matrix for Δ_{I_i} and every term in Δ_{I_i} involves x, we also have that every term in r_i involves x so x is a factor of both r_i and Δ_{I_i} and is the only variable from this row in either polynomial.

   Now suppose two variables from the row p appear in a quadratic factor f. By part (3), there is another propagator q and an edge (a, b) such that f is the 2×2 minor coming from p, q and a, b, with $I_i(p) = a$, $I_i(q) = b$. There are two situations which can occur, both illustrated in Figure 11; we show that in both cases it follows that $q \in S_i$ as well.
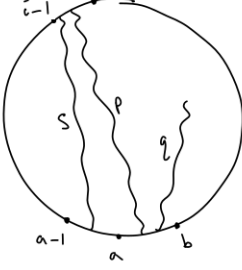
Figure 12: In order to obtain $I_{i-1}(s) = a$, propagators s and p must each have an end on the edge $(i-2, i-1)$.

In both cases, since $I_{i-1}(p) \neq a$ by assumption it follows from Lemma 3.11 that $I_{i-1}(p) <_{i-1} a$ and no other vertex supporting p lies between $I_{i-1}(p)$ and a. In the case that $b <_i a$ and q is taken before p in I_i, this means that $I_{i-1}(p) = b$ and so $I_{i-1}(q) \neq b$. Thus $q \in S_i$ and so f is a factor of r_i.

Now consider the case where $a <_i b$, and suppose for contradiction that $q \notin S_i$, i.e. that $I_{i-1}(q) = b$. Since $I_{i-1}(p) \neq a$, there must be some other propagator s with $I_{i-1}(s) = a$ (else I_{i-1} assigns q to a). This propagator cannot lie on edge (a, b) since by Lemma 3.11 we must have $I_i(s) = a$ or b, contradicting the fact that $I_i(p) = a$ and $I_i(q) = b$; thus s has an end on $(a-1, a)$ and is inside p from the point of view of $i-1$.

Say s is supported on $(j, j+1, a-1, a)$ and p is supported on $(k, k+1, a, b)$ with $i-1 \leq_{i-1} k+1 \leq_{i-1} j+1$. But by Lemma 3.9, if $I_{i-1}(s) = a$ then a cannot be maximal in the support of s with respect to $<_{i-1}$; thus we must have $i-1 = j+1$, and we are in the situation in figure 12.

Since p changed its association from I_{i-1} to I_i, we have $I_{i-1}(p) = i-1$ by Lemma 3.11. From figure 12 it follows that I_{i-1} assigns p to $i-1$ and then proceeds identically to I_i for all vertices inside p, implying that $I_{i-1}(s) = I_i(s)$. Since $I_{i-1}(s) = a$ and $I_i(s) \neq a$, this is a contradiction.

Thus $q \in S_i$ after all, and so f is a factor of r_i as required.

5. If W has zero propagators then all $I_i = \emptyset$ and both R and $\prod_{i=1}^n \Delta_{I_i}$ are equal to 1, so the result holds in this case. Now assume W has at least one propagator.

First we show that every factor of $\prod_{i=1}^n \Delta_{I_i}$ divides R. Take an irreducible factor f of $\prod_{i=1}^n \Delta_{I_i}$. There exists some i such that $f | \Delta_{I_i}$ but $f \nmid \Delta_{I_{i-1}}$, since otherwise the variables corresponding to the propagators contributing to f which do not themselves appear in f could never appear, contradicting Lemma 3.11. If f is a linear factor, say from associating propagator p to vertex a, then $I_i(p) = a$ and $I_{i-1}(p) \neq a$ so this factor appears in r_i. If f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively, then again we cannot have both $I_{i-1}(p) = a$ and $I_{i-1}(q) = b$, else f divides Δ_{i-1}. However, by the proof of part (4), if one of p, q belongs to S_i then the other does as well. Thus f divides r_i.

Next we need to show that R is squarefree. Suppose $f^2 | R$. If f is a linear factor, say from associating propagator p to vertex a, then there must be two distinct points in the Grassmann

necklace algorithm where p changes from not being associated to vertex a to being associated to vertex a. This contradicts Lemma 3.11. Now suppose f is a quadratic factor, say from propagators p and q supported on the edge (a, b) with p before q on the edge. In this case it is not possible for any I_i to associate p to b and q to a. Furthermore, we know by part (4) that p changes from not being associated to a to being associated to a if and only if q changes from not being associated to b to being associated to b. Thus $f^2|R$ implies that twice in the Grassmann necklace p must change from not being associated to vertex a to being associated to vertex a. This is again a contradiction, and so R is squarefree.

Taking everything together we have that $R|\prod_{i=1}^n \Delta_{I_i}$, R contains all factors of $\prod_{i=1}^n \Delta_{I_i}$ and R is squarefree. Therefore the ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^n \Delta_{I_i}$.

$\square$

Theorem 5.4. *Given any admissible Wilson loop diagram W, let $\{I_1, \ldots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral, $R(W)$ (see Definition 2.6), is the R of Algorithm 5.1, which is also, up to scalar multiple, the radical of $\prod_{i=1}^n \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i.*

Proof. The equivalence up to scalar multiple of R and the radical of $\prod_{i=1}^n \Delta_{I_i}$ is due to Proposition 5.3. It remains to prove that $R(W)$ is the R of Algorithm 5.1.

To this end, first note that $R(W)$ and R both have total degree $4|\mathcal{P}|$; the degree of $R(W)$ is immediate from the definition while that of R follows from Lemma 3.11. By Proposition 5.3 every factor of R is either a single variable or a quadratic factor coming from two propagators supported on a common edge. The factors of each R_e making up $R(W)$ in the notation of Definition 2.6 are all of this form and hence every factor of R divides $R(W)$. Finally, since R is squarefree, this implies that $R(W)$ is a scalar multiple of R.

Finally then we need to check the scalar. By Definition 2.6 each linear factor appears with coefficient 1 and each 2×2 determinant factor appears with the same sign as the determinant of the corresponding minor in $C(W)$. Therefore $R = R(W)$. $\square$

explain why this result was interesting

> Note that there is still a mention of scalar multiple in this theorem because it seems to me that the def of radical is only up to scalar multiple, but the part relating $R(W)$ to R is exact with no scalar multiples.

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