Combinatorics of the geometry of Wilson loop diagrams

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This paper studies the combinatorics of Wilson Loop Diagrams.

1 Wilson Loop diagrams

What are Wilson loop diagrams and their integrals.

Definition 1.1. A Wilson loop diagram is given by the following data: a cyclicly ordered set V, along with a choice of first vertex (labeled 1), and k pairs, called propagators, written $\{p_r = (i_r, j_r)\}_{r=1}^k$.

Generally speaking, the propagators are undirected, so $p = (i, j) = (j, i)$. In order to fix a convention, we write $p = (i, j)$ with $i + 1 < j$ relative to the first vertex. However, at times (section 3.2), we consider directed propagators, where $p = (i, j)$ denotes a particular propagator flowing in one direction, while $p = (j, i)$ denotes the *same* propagators flowing in the opposite direction. Addendum 1

We depict this data as a circle with marked points, called vertices. The vertices are labeled by V (preserving the cyclic ordering). The arc between consecutive vertices are called edges. There are k wavy lines in the interior of the diagram, depicting the proagators, with endpoints on the edges. A propagator, $p = (i, j)$ has one endpoint on the edge between the vertex labeled i and $i + 1$ and another endpoint on the edge defined by j and $j + 1$. The condition on i_r and j_r means that the propagator does not go between adjacent edges. Let $\mathcal{P} = \{p_r\}_{r=1}^k$ be the set of propagators. Then we write

$$W = (\mathcal{P}, V) \ .$$

Note that the marked circle gives the vertices of W a cyclic ordering. The choice of first vertex gives it a compatible linear order. Both the cyclic and the linear order become the correct perspective at various points in this paper.

Often we take V to be $[n]$, the cyclically ordered set of integers, $1 \ldots n$. In this case, we write $W = (\mathcal{P}, [n])$. We introduce some notation to speak of vertices supporting a propagator, and the set of propagators supported on a vertex set.

Definition 1.2. Let $W = (\mathcal{P}, [n])$.

1. For $p \in \mathcal{P}$, let $V(p) = \{i_p, i_p + 1, j_p, j_p + 1\}$ be the set of vertices supporting p. Then, for $P \subseteq \mathcal{P}$, the set $V(P) = \cup_{p \in P} V(p)$ is the vertex support of P.
2. For $V \subseteq [n]$, write $\mathrm{Prop}(V) = \{p \in \mathcal{P} | V(p) \cap V \neq \emptyset\}$.
3. For $P \subseteq \mathcal{P}$, define $F(P) = V(P^c)^c$ to be the set of vertices in $[n]$ that do not support any propagators outside the set P.
4. The set of vertices that are not in the support of any propagotors is denoted $F(\emptyset)$. Vertices in this set are called non-supporting.

Remark 1.3. Note that a more explicit definition of $F(P)$ is

$$F(P) = \big(V(P) \setminus V(P^c)\big) \cup F(\emptyset) .$$

Furthermore, note that by construction $\mathrm{Prop}(F(P)) \subseteq P$.

It is sometimes useful to discuss propagators in terms of the edges supporting them, rather than the vertices.

Definition 1.4. The i^{th} edge of W is the edge of the external polygon that lies between the vertices i and $i + 1$.

In this manner, the propagator $p = (i, j)$ is supported by the i^{th} and j^{th} edges.

Definition 1.5. A Wilson loop diagram is admissible if

1. $|V| \geq |\mathcal{P}| + 4$
2. There does not exists a set of propagators, $P \subseteq \mathcal{P}$ such that $|V(P)| < |P| + 3$.
3. There does not exist a pair of propagators, $p, q \subseteq \mathcal{P}$ such that $i_p < i_q < j_p < j_q$.

A Wilson loop diagram is weakly admissible if the second and third conditions hold.

The first conditions states that there are at least four more vertices than propagators in an admissible Wilson Loop Diagram. The second imposes an upper bound on how densely the propagators can be fitted in the diagram. The third ensures that ensures that no propagators cross in the interior of the diagram. In other words, a Wilson loop diagram, $(\mathcal{P}, [n])$ is admissible if and only if $n > \mathcal{P} + 4$, and has neither crossing propagators nor any pairs of propagators that start and end on the same pair of non-adjacent edges.

Note that if we take any admissible Wilson loop diagram and remove the unsupported vertices then we will obtain a weakly admissible Wilson loop diagram that may or may not be admissible itself.

In what follows, we will talk about admissible Wilson Loop diagrams and subdiagrams thereof.

Definition 1.6. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The weakly admissible diagram, W' is a subdiagram of W, written $W' \subseteq W$, if

$$W' = (P, V); \quad P \subseteq \mathcal{P}; \quad V(P) \subseteq V \subseteq [n] .$$

There is one particular type subdiagram that deserves special attention.

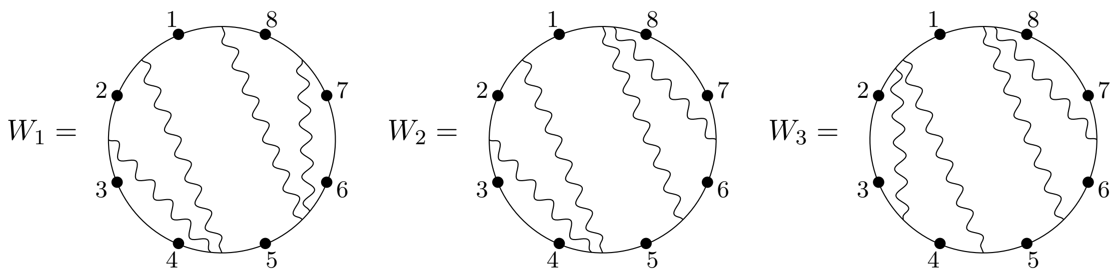
Definition 1.7. For W an admissible diagram, $(P, V(P))$ is exact if $|V(P)| = |P| + 3$.

The exact subdiagrams define an equivalence relation amongst Wilson loop diagrams.

Definition 1.8. There is an equivalence relationship on the set of admissible Wilson loops diagrams given by the transititve closure the following binary relation: $W = (\mathcal{P}, [n]) \sim W' = (\mathcal{P}', n)$ if

1. There exist two different exact subdiagrams, $(P, V(P))$ and $(P', V(P'))$ of W and W' respectively such that $V(P) = V(P')$.
2. The complementary subdiagrams are identical: $(\mathcal{P} \setminus P, V(P)^c) = (\mathcal{P}' \setminus P', V(P')^c)$.

Example 1.9. Note that since this is an equivalence relation, we may find that two Wilson loop diagrams are equivalent, even if they do not have complements of (non-trivial) exact subdiagrams in common. Consider the following three Wilson loop diagrams,



The diagrams $W_1 \sim W_2$ because $(\{(5,8),(5,7)\}, \{5,6,7,8,1\})$ and $(\{(5,8),(7,8)\}, \{5,6,7,8,1\})$ are the corresponding differing subdiagrams. Furthermore, there is the equivalence $W_2 \sim W_3$ due to the exact subdiagrams $(\{(1,4),(3,4)\}, \{1,2,3,4,5\})$ and $(\{(1,4),(1,3)\}, \{1,2,3,4,5\})$. This forces an equivalence between W_1 and W_3, even though one cannot partition the propagators of each into a an exact subdiagram (that may vary between the diagrams) and a complement that is fixed.

Each Wilson loop diagram, $W = (\mathcal{P}, [n])$ with $|\mathcal{P}| = k$ is associated to a $k \times n$ matrix with non-zero real variable entries, called $C(W)$:

$$C(W)_{p,q} = \begin{cases} c_{p,q} & \text{if } q \in V(p) \\ 0 & \text{if } q \notin V(p) \end{cases} . \tag{1}$$

Example 1.10. For example, ordering the propagators of W_1 from Example 1.9:

$$(1,4),\ (2,4),\ (5,7),\ (5,8)$$

we may write

$$C(W_1) = \begin{pmatrix} c_{1,1} & c_{1,2} & 0 & c_{1,4} & c_{1,5} & 0 & 0 & 0 \\ 0 & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3,5} & c_{3,6} & c_{3,7} & c_{3,8} \\ c_{4,1} & 0 & 0 & 0 & c_{4,5} & c_{4,6} & 0 & c_{4,8} \end{pmatrix} .$$

These $C(W)$ parametrize a subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$ as shown in [**?**], call it $\Sigma(W)$. The Wilson loop diagrams also define a volume form on $\Sigma(W)$:

$$\Omega(W) = \frac{\prod_{r=1}^{|\mathcal{P}|}\prod_{v\in V_{p_r}} \mathrm{d}c_{p_r}}{R(W)} \ .$$

The denominator $R(W)$ is a polynomial defined by 2×2 and 1×1 minors of $C(W)$ as defined below.

Definition 1.11. For $W = (\mathcal{P},[n])$, $R(W) = \prod_{e=1}^{n} R_e$, with R_e defined by the propagators ending on it. For any edge e of W, order the propagators incident on e as $\{p_1 \dots p_r\}$, ordered such that p_1 is closest to the vertex e, p_r closest to $e+1$, and p_i is closer to e than p_{i+1}. Then

$$R_e = c_{p_1,e+1}\prod_{j=1}^{r-1}\left((c_{p_j,e}c_{p_{j+1},e+1} - c_{p_{j+1},e}c_{p_j,e+1})\right)c_{p_s,e} \ .$$

Note that in this notation, if $r=1$, $R_e = c_{p,e}c_{p,e+1}$.

2 Equivalence classes of Wilson loop diagrams

In [1], Agarwala and Amat show that Wilson loop diagrams can be interpreted as positroids, a certain well behaved class of realizable matroids (this correspondence is stated precisely in Theorem 2.1 below). This opens up the study of Wilson loop diagrams to techniques from geometry and combinatorics. In particular we examine the partial result shown in [1], that shows that equivalent Wilson Loop Diagrams define the same matroid. In section 2.4, we make the relationship between Wilson Loop diagrams and matroids explicit. We count the size of each equivalence class, and enumerate the the number of said classes. We also show that two Wilson Loop diagrams map to the same matroid if and only if they are equivalent.

In subsection 2.1, we discuss a few matroidal facts about Wilson loop diagrams. In section 2.2, we show that there is a one to one correspondence between exact subdiagrams of Wilson loop diagrams and triangulated pieces of the corresponding polygon partion. In section 2.3, we prove some matroidal properties of exact subdiagrams. We show that a subdiagram of W defines a uniform matroid if and only if the subdiagram is exact (Theorem 2.21). Furthermore, this uniform matroid is the restriction of the dual matriod of W restricted to the exact subdiagram (Remark 2.20). The main result of section 2.4 shows that that two admissible Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). We obtain a formula for the number of admissible Wilson loop diagrams in each equivalence class (Corollary 2.25) as well as the number of equivalence classes. In this way, we give a comple characterization of the relationship between Wilson Loop diagrams and their associated matroids.

2.1 Wilson loop diagrams as matroids

We first give a quick summary of the matroid terminology that we will need; it is not intended as a comprehensive introduction to matroids and the interested reader is referred to [ref][find a good matroid reference].

A *matroid* $M = (E, \mathcal{B})$ consists of a finite ground set E and a non-empty family $\mathcal{B} \subseteq \mathcal{P}(E)$ whose elements satisfy the *basis exchange property*: for any distinct $B_1, B_2 \in \mathcal{B}$ and any $a \in B_1 \setminus B_2$, there exists some $b \in B_2 \setminus B_1$ such that $(B_1 \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ as well. The elements of $\mathcal{B}$ are called the *bases* of the matroid. Note that the basis exchange property immediately implies that all bases have the same size.

A subset $A \subseteq E$ is called *independent* in M if $A \subseteq B$ for some $B \in \mathcal{B}$, and *dependent* else. The *rank* $\text{rk}\ (A)$ of a subset $A \subseteq E$ is the size of the largest independent set contained in A. The rank of the matroid itself is defined to be $\text{rk}\ (E)$.

A *circuit* in M is a minimally dependent set. That is, it is a set $C \subseteq E$ such that C is dependent but $C \setminus \{e\}$ is independent for any $e \in C$. A union of circuits is called a *cycle*. On the other hand, a *flat* is a maximally dependent set, i.e. a set $F \subseteq E$ such that $\text{rk}\ (F \cup \{e\}) = \text{rk}\ (F) + 1$ for any $e \in E \setminus F$. Unsurprisingly, a *cyclic flat* is a set which is both a flat and a cycle. The set of circuits in a matroid uniquely defines that matroid, as does the set of flats; thus one could specify a matroid by listing its independent sets, bases, circuits, or flats. [ref]

Finally, we describe several important types of matroids. A matroid of rank k with a ground set of size n is called *realizable* if there exists some $A \in Gr(k, n)$ whose non-zero $k \times k$ minors are exactly those with columns indexed by elements of $\mathcal{B}$. A *positroid* is a matroid which can be realized by an element of the totally nonnegative Grassmannian $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Finally, a *uniform matroid* of rank r is a matroid in which any set of size $\leq r$ is independent.

Matroid theory relates to the study of Wilson loop diagrams as follows. In [1], Agarwala and Amat show that every admissible Wilson loop diagram with k propagators defines a positroid of rank k, and that the independent sets can be read directly from the diagram:

Theorem 2.1. *[1, Theorem 3.6] Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ defines a matroid $M(W)$ with ground set $[n]$. The independent sets are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| < |U|$.*

In other words, the independent sets of $M(W)$ correspond to the sets of vertices in W such that no subset supports fewer propagators than the vertices it contains.

Throughout, we take the *matroid defined by* W to be the matroid $M(W)$ of Theorem 2.1. Note that since vertices of the diagram W correspond to columns of the associated matrix $C(W)$, $M(W)$ can also be thought of as the matroid realized by $C(W)$.

Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $M(W)$ its associated matroid. Where it will not cause confusion we conflate the two objects, identifying vertices of W with elements of the ground set $[n]$ in $M(W)$.

In particular, this allows us to prove results about $M(W)$ by considering the behavior of propagators in W. We record a few elementary facts about the rank and cycles of $M(W)$ here as an example of this.

Lemma 2.2. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. Then:*

1. *The rank of a set $V \subseteq [n]$ is bounded above by $\min\{|V|, |Prop(V)|\}$, with $rk\ (V) = |V|$ if and only if V is an independent set.*

2. *If $C \subseteq [n]$ is a cycle, then rk $(C) = |Prop(C)|$.*

3. *If $[n]$ can be partitioned into at least two non-empty sets, each of which support different sets of propagators that form a partition of the propagotor set,*

$$[n] = \sqcup_i V(P_i) \quad s.t. \quad \sqcup P_i = \mathcal{P}; \quad V(P_i) \cap V(P_j) = \emptyset; \quad P_i \cap P_j = \emptyset \ ,$$

then the matroid $M(W)$ is seperable,

$$M(W) = \bigoplus_i M(P_i, V(P_i)) \ .$$

4. *$F(P)$ is a flat of $M(W)$, thus justifying the name "propagator flat".*

Proof. The first part of (1) is [1, Equation (9)] and surrounding discussion, and the second part is standard matroid theory. (2) is [1, Lemma 3.27]. (3) is a direct consequence of [1, Lemma 3.20] and the fact that $F(P_1)^c = V(P_1^c)$

To prove (4), we need to show that $F(P)$ is maximally dependent. If $F(P) = [n]$ then this is automatic, so suppose not and let $v \in [n] \setminus F(P)$. In other words, $v \in V(P^c)$ and so v supports some propagator $q \notin P$. Let $S \subseteq F(P)$ be an independent set of maximal size. Then $\text{Prop}(S) \subseteq P$ by (1), and no subset of S supports fewer propagators than the number of vertices it contains (this is the definition of an independent set in $M(W)$). Since v supports a new propagator $q \notin P$, the set $S \cup \{v\} \subseteq F(P) \cup \{v\}$ also satisfies this independence condition. Thus rk $(F(P) \cup \{v\}) =$ rk $(F(P)) + 1$, as required. □

Note that this means that the set, $F(\emptyset)$, of non-supporting vertices is the maximal subset of vertices of W of rank 0. That is, it is the unique flat of rank 0 in $M(W)$.

2.2 Polygon partitions of Wilson loop diagrams

The equivalence relation on Wilson loop diagrams is defined in terms of exact subdiagrams; thus in order to understand the equivalence, we need a way to extract and compare exact subdiagrams. We do this via the notion of a polygon partition of W.

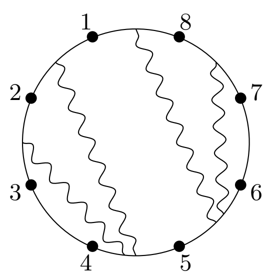
Definition 2.3. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. The *polygon partition* associated to W, denoted $\tau(W)$, is defined as follows.

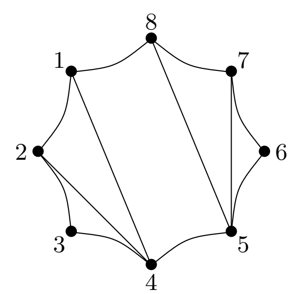
- The vertices of $\tau(W)$ correspond to the edges of W.
- Labeling the vertices of $\tau(W)$ with the edge number of Wgives a cyclic order to the vertices. Connecting consecutive vertices gives a graph theoretic cycle called the polygon of $\tau(W)$.
- Each propagator of W defines a chord edge of $\tau(W)$; specifically, a propagator $(i, j) \in \mathcal{P}$ defines a chord connecting the vertices i and j in $\tau(W)$.

Lemma 2.4. *If $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram, then $\tau(W)$ is a simple planar graph whose outer face is a cycle. It is embedded such that the vertices all lie on this infinite face*[1]*. These vertices are cyclically ordered, with a choice of first vertex giving it an additional compatible linear order.*

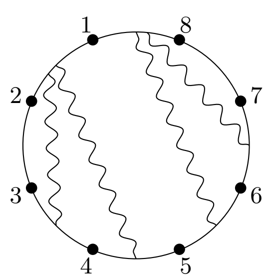
Proof. Since the vertices of $\tau(W)$ are labeled by the edges of W, which are cyclically ordered, this gives an ordering to the vertices and the outer face of $\tau(W)$ is a cycle. Since W is admissible, no pairs of propagators cross. Therefore, it is a planar embedding. Similarly, W does not admit any propagators of the form $p = (i, i+1)$; therefore there is exactly one edge connecting any two adjacent edges of $\tau(W)$. Finally, there does not exist two propagators p, q such that both p and q start at edge i and end at edge j. Therefore, no other two vertices of $\tau(W)$ can be connected by more than one edge. Finally, the embedding of $\tau(W)$ is induced from the embedding of the graph W. $\square$

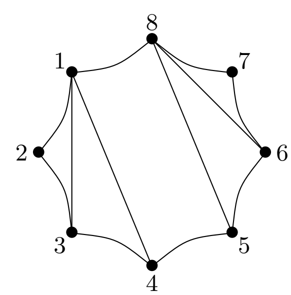
Example 2.5. In this example we return to two of the Wilson loop diagrams in Example 1.9. We can pair diagrams with their polygon partitions as follows:

$W_1 =$ 

; $\tau(W_1) =$ 

and

$W_3 =$ 

; $\tau(W_3) =$ 

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Recall that a planar embedding of a graph is a *triangulation* if all faces, except possibly the infinite face, are triangles.

Definition 2.6. Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. A *triangulated piece* of $\tau(W)$ is a 2-connected subgraph of $\tau(W)$ which is a triangulation. We will take the convention that a subgraph consisting of a single chord edge is called a *trivial* triangulated piece. A *maximal* triangulated piece is one which is not contained in any strictly larger triangulated piece.

[1]That is, it is an *outerplanar* graph

Definition 2.7. A *decomposition* of a polygon partition $\tau(W)$ is a set of 2-connected induced subgraphs of $\tau(W)$ which partition the edges of $\tau(W)$.

Example 2.8. For the Wilson loop diagrams and polygon partitions in Example 2.5, the vertex sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ give maximal triangulated pieces for both $\tau(W_1)$ and $\tau(W_3)$. The vertex set $\{4, 5, 8, 1\}$ is not a triangulation in either polygon partition.

Lemma 2.9. *For W an admissible Wilson loop diagram, the polygon partition $\tau(W)$ has a unique decomposition into maximal triangulated pieces, and edges in the polygon of $\tau(W)$.*

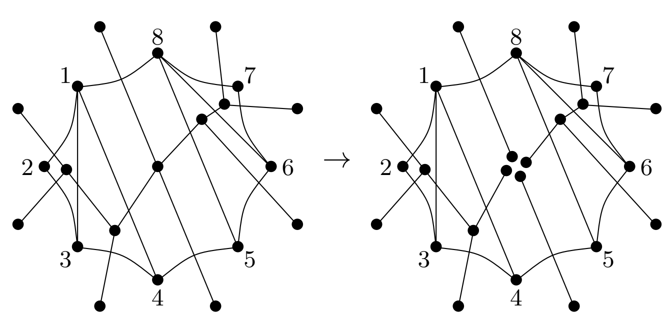
Proof. We begin by giving an algorithm for the decomposition, then prove its uniqueness. Let $W = (\mathcal{P}, [n])$, with $|\mathcal{P}| = k$.

By *splitting* a vertex v we will mean replacing v by new vertice $v_1, v_2, \ldots, v_{\deg(v)}$ such that each v_i has exactly one neighbour and the union of the of the v_i is the neighbourhood[2] of v.

Let $T(W)$ be the dual graph of $\tau(W)$ with the vertex corresponding to the infinite face split. Since $\tau(W)$ is an embedded graph (with a fixed distinguished embedding) by Lemma 2.4, $T(W)$ is an uniquely defined graph.

Furthermore $T(W)$ is a tree because it is connected, has $n + k + 1$ vertices ($k + 1$ from the internal faces of $\tau(W)$ and n from the outer face) and $n+k$ edges (since $\tau(W)$ has $n+k$ edges). Additionally, since $\tau(W)$ is simple, $T(W)$ has no vertices of degree 2.

Split every vertex of $T(W)$ which has degree > 3. The connected components of $T(W)$ correspond to the decomposition of $\tau(W)$ into maximal triangulated pieces and edges originally in the polygon of $\tau(W)$. Let f be the forest thus obtained. The vertices of f either have degree 1 or 3. Trees of f with no trivalent vertices correspond to either edges in the polygon of $\tau(W)$, if they were originally leaves of $T(W)$, or to maximal trivial triangulated pieces. Splitting at all the faces that are not triangles ensures maximality of the decomposition. If the splitting were not maximal, then one could add a triangle to a connected component of the splitting, but this would imply that that splitting happened at a valence 3 vertex.



To see uniqueness, consider a different maximal decomposition of $\tau(W)$. This induces a splitting on $T(W)$, where each connected component of the new decomposition corresponds to a subtree. Call this forest f'. Since $f' \neq f$, there are two trees, t and t' in f and f' that are distinct, but

[2]The neighbourhood of a vertex is the set of adjacent vertices

share at least one edge of $T(W)$. Since f' is also maximal, t' is not a subtree of t. Therefore, the edges of t' can be found in at least two trees in the forest f. In particular, there is a vertex v in t' that corresponds to a split vertex of $T(W)$ in the original decomposition. This implies that v has valence greater than 3 in $T(W)$, and thus the corresponding face of $\tau(W)$ is not a triangle. In other words, the decomposition corresponding to f' is not a triangulation. □

Corollary 2.10. *Given a maximal decomposition of $\tau(W)$, the maximal triangulated pieces are edge disjoint.*

Proof. Consider any two distinct maximal triangulated pieces of $\tau(W)$. These two pieces correspond to subtrees of $T(W)$ and intersect, at most, at a vertex in the interior of $\tau(W)$. Since the subtrees corresponding to the maximal triangulated pieces are edge disjoint, and the edges of $T(W)$ correspond to the edges of $\tau(W)$, this forces the maximal triangulated pieces to be edge disjoint as well. □

We are now in a position to relate the triangulated pieces of $\tau(W)$ to exact subdiagrams of W.

Every triangulated piece t of an admissible Wilson loop diagram W corresponds to a subdiagram of W by taking the set of propagators P corresponding to edges of t and then taking the subdiagram $W_P = (P, V(P))$. Conversely, given a subdiagram $W_P = (P, V(P))$ of W we can obtain a subgraph of $\tau(W)$, called t, as follows:

- The vertex set of the subgraph t is
  - the vertices of $\tau(W)$ corresponding to edges of W defined by cyclically consecutive elements of $V(P)$ as a subset of $[n]$
- The edge set of the subgraph is
  - the edges of $\tau(W)$ corresponding to propagators of P
  - along with the outer edges of $\tau(W)$ for which both their end points are in the vertex set

Note that the subgraph t depends on how the propagators of P sit inside $[n]$, not on how they sit in W_P itself. In particular t is not the subgraph of $\tau(W)$ consisting only of edges corresponding to propagators of P.

Lemma 2.11. *Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. The triangulated pieces of $\tau(W)$ correspond to the exact subdiagrams of W using the correspondence described above.*

Proof. Let us first record a few standard facts about polygon triangulations (that is, about triangulations with all vertices on the outer face). If such a triangulation has n vertices then it has n edges on the polygon (that is, on the outer face) and $n-3$ edges which are not. No planar graph with the same vertices and the same outer face can have more edges than the triangulation, and every such simple graph with $n-3$ edges off the outer face is a triangulation.

Since W is admissible, by Lemma 2.4 $\tau(W)$ is a simple graph. Let t be a triangulated piece of the decomposition of $\tau(W)$ given in Lemma 2.9; note that t cannot be equal to $\tau(W)$ by the definition of admissible diagrams.

If t has 2 vertices then t corresponds to a propagator that connects two non-adjacent edges. Therefore, the trivial triangulation is a trivial exact subdiagram.

Now suppose that t has $m > 2$ vertices. We count how many edges of t are not on the outer face of $\tau(W)$. These are exactly the edges of t defined by propagators of W. Consider the intersection of t with the outer face of $\tau(W)$: this is a possibly disconnected subgraph of the polygon of $\tau(W)$ and this subgraph has m vertices. Call this new subgraph S, and let j be the number of connected components of S. To join the components of S into the outer face of t, t must have j edges in its outer face which are not in the outer face of $\tau(W)$. Furthermore t has $m-3$ edges not in its outer face and so also not in the outer face of $\tau(W)$. Thus there are $m-3+j$ edges of t not in the outer face of $\tau(W)$.

Each of these $m-3+j$ internal edges corresponds to a propagator in W; call this set of propagators P. Next we count the size of $V(P)$. Each of the m vertices in the outer face of t corresponds to an edge of W. These m edges define j connected components of the outer polygon of W. Thus the set $V(P)$ has $m+j$ vertices. In other words,

$$|V(P)| = m + j = |P| + 3 \; .$$

Thus the subdiagram $(P, V(P))$ defined by t is exact.

Conversely, suppose we have an exact subdiagram $(P, V(P))$ of $W = (\mathcal{P}, [n])$ supported on $|V(P)| = |P| + 3$ vertices, and let t be the subgraph of $\tau(W)$ corresponding to $(P, V(P))$.

Suppose $|P| = 1$. Let p be the element of P. The exactness condition on $(P, V(P))$ says that the four supporting vertices of p are distinct. If the support of p is four consecutive vertices, then $V(p)$ defines three consecutive boundary edges of W, so t is a single triangle, hence a triangulated piece. If the support of p is not four consecutive vertices, then the vertices which are the ends of t are separated by at least two vertices along the cycle. This implies that t is a trivial triangulated piece.

Now suppose $|P| > 1$. Let $j = |P|$, $m = |V(P)|$, and suppose that the set $V(P)$ defines c disjoint cyclic intervals of $[n]$. Then t has $m - c$ vertices. If t were a triangulation, t would have $j - c$ internal edges.

The graph t has j edges that come from propagators, and $m-2c$ edges that come from the boundary polygon of $\tau(W)$. Since t has $m - c$ vertices, it has $m - c$ external edges, of which c come from propagators. Therefore, of the j edges of t that come from propagators, $j - c$ are internal to the connected component. Therefore, t is a triangulated piece.

□

To avoid the issue of exact diagrams being subdiagrams of other exact subdiagrams (for instance, any subdiagram $(q, V(q))$, for $q \in \mathcal{P}$ is exact), we introduce the notion of maximal exact subdiagrams.

Definition 2.12. An exact subdiagram $(P, V(P))$ is a *maximal exact subdiagram* of W if there is no other exact subdiagram $(Q, V(Q))$ in W that contains $(P, V(P))$ as a strict subdiagram.

Corollary 2.13. *Any admissible Wilson loop diagram $W = (\mathcal{P}, [n])$ can be uniquely decomposed into maximal exact subdiagrams. These maximal subdiagrams partition $\mathcal{P}$.*

Proof. Combining Lemmas 2.9 and 2.11 yields the unique decomposition into maximal exact subdiagrams, and Corollary 2.10 ensures that no propagator appears in more than one subdiagram in this decomposition. Since the chord edges of $\tau(W)$ correspond to the propagators of W, the decomposition of $\tau(W)$ induces a partition of $\mathcal{P}$. $\square$

2.3 Matroidal properties of exact subdiagrams

Since Corollary 2.13 allows us to decompose any admissible Wilson loop diagram into a collection of maximal exact subdiagrams, in this section we examine the matroid properties of exact subdiagrams more closely. In this section, we show prove two results:

1. The matriod assoicated to an exact subdiagram of W can be written as a contraction of the matroid $M(W)$ by the complementary propagator flat (Theorem 2.19).
2. The matroid associated to an exact subdiagram is uniform (Theorem 2.21).

We begin by proving some useful facts about propagator flats, and flats of matroids associated to admissible Wilson loop diagrams more generally.

Lemma 2.14. *Let F be a flat in $M(W)$, and let $C \subseteq F$ be the union of all circuits contained in F. Then the following are true:*

1. *$C = F(Prop(C))$, i.e. C is a propagator flat.*
2. *$F \setminus C$ is an independent set. Furthermore, If $F \setminus C$ is an independent flat if and only if $F(\emptyset) = \emptyset$, that is W has no non-supporting vertices.*

Proof. (1) If F is an independent flat, then $C = \emptyset$ and the statement is trivially true. Now suppose that F is a dependent set, so C is non-empty.

Let $v \in C$. Clearly $\text{Prop}(v) \subseteq \text{Prop}(C)$. Since $F(P) = V(P^c)^c$, we have $F(\text{Prop}(v)) \subseteq F(\text{Prop}(C))$. Since $v \in F(\text{Prop}(v))$ by the definition of propagator flat, we have $v \in F(\text{Prop}(C))$ as required, and $C \subseteq F(\text{Prop}(C))$.

Now suppose there exists some $w \in F(\text{Prop}(C)) \setminus C$. Let B be an independent subset of C of maximal rank; we first show that $B \cup \{w\}$ is a dependent set in $M(W)$. By Lemma 2.2,

$$|\text{Prop}(C)| = \text{rk }(C) = |B| = \text{rk }(B).$$

Since $B \subseteq C$ implies that $\text{Prop}(B) \subseteq \text{Prop}(C)$, this implies that $\text{Prop}(B) = \text{Prop}(C)$. Furthermore, since $w \in F(Prop(C)$, $\text{Prop}(B \cup w) \subseteq \text{Prop}(B)$. By Lemma 2.2 $\text{rk }(B \cup w) \leq \min\{|B \cup w|, |\text{Prop}(B \cup w)|\}$, and therefore $\text{rk }(B \cup w) < |B \cup w|$. That is, $B \cup w$ is a circuit in $F(\text{Prop}(C))$. Therefore, $B \cup w \subset C$, leading to a contradiction.

For part (2), first note that $F \setminus C$ is automatically independent as it contains no circuits.

Note that since $F(\emptyset)$ has rank 0, $F(\emptyset)$ is contained in every flat of W. Therefore, if $F(\emptyset) \neq \emptyset$, then $F \setminus C$ cannot be a flat.

If $F(\emptyset) = \emptyset$, for any $e \notin F$, we certainly have $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ since F is a flat. Now let $e \in C$, and suppose that $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C)$. This implies that $(F \setminus C) \cup \{e\}$ is dependent, and hence contains a circuit, S. Since $F(\emptyset) = \emptyset$, this circuit must contain at least two elements but this contradicts the fact that C was the union of all circuits in F.

Thus $\text{rk}\ ((F \setminus C) \cup \{e\}) = \text{rk}\ (F \setminus C) + 1$ for any $e \notin F \setminus C$, and hence $F \setminus C$ is a flat. $\square$

Corollary 2.15. *If F is a flat of a Wilson loop diagram, it can be written as the disjoint union of a cyclic propagator flat and an independent set and the set of non-supportive vertices.*

In particular, any propagator flat can be written as a union of a cyclic propagator flat, an independent set and $F(\emptyset)$ for the appropriate Wilson loop diagram.

Next we examine properties of propagator flats associated to the complement of Wilson Loop diagrams.

Lemma 2.16. *Let $W = (\mathcal{P}, [n])$ be a Wilson loop diagram, and $P \subseteq \mathcal{P}$. Then:*

1. *If $(P, V(P))$ is an exact subdiagram in W, then $F(P^c)$ is not an independent flat.*
2. *If $(P, V(P))$ is a maximal exact subdiagram in W, then $F(P^c)$ is a cyclic flat.*
3. *If $(P, V(P))$ is an exact subdiagram in W, then rk $(F(P^c)) = |P^c|$.*

Proof. First note that if $(P, V(P))$ is exact then the admissiblity of W guarantees that $F(P^c)$ is a non-empty flat. That is, $V(P) \subsetneq [n]$.

Since W is admissible, we have $n \geq |\mathcal{P}| + 4$. Rewriting this as

$$|V(P)| + |F(P^c)| \geq |P| + |P^c| + 4,$$

and combining it with the fact that $|V(P)| = |P| + 3$ (from the exactness of $(P, V(P))$), we obtain

$$|F(P^c)| > |P^c| \,. \tag{2}$$

Equation (2) is therefore saying that $F(P^c)$ supports fewer propagators than the number of vertices it contains, i.e. $F(P^c)$ is not an independent set.

For part (2), suppose that $(P, V(P))$ is a maximal exact subdiagram. By Corollary 2.15 we can decompose $F(P^c)$ as

$$V(P)^c = F(P^c) = C \sqcup S = F(\text{Prop}(C)) \sqcup S, \tag{3}$$

where C is the largest cyclic flat contained in $F(P^c)$ and S is an independent set. Note that C must be non-empty since $F(P^c)$ is non-empty and dependent (by part (1)).

Since S is an independent set, $\text{rk }(S) = |S|$, and every element of S is independent of C, $\text{rk }\big(F(P^c)\big) = \text{rk }(C) + \text{rk }(S)$. By Lemma 2.2(2),

$$\text{rk }(S) = \text{rk }\big(F(P^c)\big) - \text{rk }(S) \leq |P^c| - \text{rk }(C) \ . \tag{4}$$

Let $Q^c = \text{Prop}(C)$. Then $C = F(Q^c)$ and, by Lemma 2.2(1), $\text{rk }(C) = |Q^c|$. By equation (4),

$$|P| + |S| \leq |P| + |P^c| - \text{rk }(C) = |\mathcal{P}| - |Q^c| = |Q| \ . \tag{5}$$

From equation (3),

$$|V(Q)| = V(P) \sqcup S = C^c = F(\text{Prop}(C))^c = F(Q^c)^c = V(Q) \ .$$

Combining this with equation (4) gives

$$|V(Q)| = |V(P)| + |S| = |V(P)| + 3 + |S| \leq |Q| + 3 \ . \tag{6}$$

Since W is admissible, one must have $V(Q) = |Q| + 3$, that is the subdiagram $(Q, V(Q))$ is exact.

Next, note that

$$C = F(Q^c) = V(Q)^c \subset F(P^c) = V(P)^c \ ,$$

which implies that $P \subseteq Q$. That is, $(P, V(P))$ is a subdiagram of $(Q, V(Q))$. Since $(P, V(P))$ is maximally exact by hypothesis, and $(Q, V(Q))$ is exact by (6), we are forced to have $S = \emptyset$. Therefore, $F(P^c)$ is a cyclic flat.

To see (3), first note that the second point of this Lemma, combined with Lemma 2.2 that

$$\text{rk }(F(P^c)) = |\text{Prop}(F(P^c))| = |P^c|.$$

To prove this in the general case, let let $(R, V(R))$ be an exact subdiagram that is not maximal and P be the maximal exact subdiagram containing it. That is, $R \subsetneq P$.

Since $R \subset P$, $V(P) = V(R) \sqcup S$, where $S = V(P) \setminus V(R)$. Since P and R both define exact subdiagrams, we may write $|V(P)| = |V(R)| + |P \setminus R|$, where S is a vertex set of size $|P \setminus R|$ with $(P \setminus R \subset \text{Prop}(S)$. Since $|S| = |V(P) \setminus V(R)| = |P \setminus R| \leq |\text{Prop}(S)|$, by Lemma 2.2, $\text{rk }(S) = |S|$, implying that S is independent. Taking the complements, we may write

$$F(R^c) = F(P^c) \sqcup S \ .$$

Taking the rank of both sides,

$$\text{rk }(F(R^c)) = \text{rk }(F(P^c) \sqcup S) \leq \text{rk }(F(P^c)) + |S| = |P^c| + |\text{Prop}(S)| \leq |R^c| \ .$$

Combining this with (2) gives the desired result.

$\square$

We are now ready to show that any exact subdiagram $(P, V(P)$ of W can be written as a contraction of $M(W)$ by $F(P^c)$. We begin with a definition.

Definition 2.17. Let $M = (E, \mathcal{B})$ be a matroid, and $S \subseteq E$. The *contraction* of M by S is the matroid $M/S = (E \setminus S, \mathcal{B}/S)$, where

$$\mathcal{B}/S = \{B \setminus S \mid |B \cap S| \text{ is maximal amonst all } B \in \mathcal{B}\}.$$

In [1], Agarwala and Amat show that certain subdiagrams of W can be realized as contractions of $M(W)$:

Lemma 2.18. *[1, Theorem 3.33] Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and $P \subseteq \mathcal{P}$. If the set $V(P)^c$ has rank $|P^c|$, then the matroid defined by the subdiagram $(P, V(P))$ is equal to the contraction $M(W)/V(P)^c$.*

Theorem 2.19. *If $(P, V(P)$ is an exact subdiagram of of W, one may write*

$$M\big((P, V(P))\big) = M(W)/F(P^c)\ .$$

Proof. This follows from Lemma 2.17. Lemma 2.16 shows that the supports of exact subdiagrams satisfy the conditions of this lemma. □

Remark 2.20. Let M^* denote the dual of a matroid. The the dual of a contraction of a matroid is the same as the restriction of the dual matroid by the complement:

$$M/S = M^*|_{S^c}\ .$$

Then Theorem 2.19 implies that, for $(P, V(P))$ an exact subdiagram of W

$$M\big((P, V(P))\big) = M(W)/F(P^c) = M(W)^*|_{V(P)}\ .$$

Matroids coming from exact subdiagrams have an especially nice structure, namely, they are uniform. Recall from Section 2.1 that a uniform matroid of rank r is one in which all sets of size $\leq r$ are independent.

Theorem 2.21. *Let $W' := (P, V(P))$ be a subdiagram of an admissible Wilson loop diagram $W = (\mathcal{P}, [n])$. Then W' is an exact subdiagram if and only if $M(W')$ is a uniform matroid of rank $|P|$.*

Proof. It follows directly from the definitions that a matroid of rank r is uniform if and only if all circuits have rank r; we therefore focus on the circuits of $M(W')$.

We prove the following claim: W' is exact if and only if $V(P)$ contains no circuits C with $\text{rk } C < |P|$ in $(P, V(P))$. Since $\text{rk }(M(W'))$ is bounded above by $|P|$, the result follows.

Suppose $C \subseteq V(P)$ is a circuit of rank $m < |P|$; by Lemma 2.2, we know that $|\text{Prop}_{W'}(C)| = m$ as well, where the subscript to Prop specifies the diagram we are working in. Observe that $\text{Prop}_{W'}(C) \subseteq P$ by definition. The set $P \setminus \text{Prop}_{W'}(C)$ is thus nonempty, and we can consider the subdiagram $W'' := (P \setminus \text{Prop}_{W'}(C), V(P \setminus \text{Prop}_{W'}(C))$. By the density condition on subdiagrams of admissible diagrams, we have

$$|V(P \setminus \text{Prop}_{W'}(C))| \geq |P \setminus \text{Prop}_{W'}(C)| + 3.$$

It is easy to verify that $V(P \setminus \mathrm{Prop}_{W'}(C)) \subseteq V(P) \setminus C$; since $C \subseteq V(P)$ and $\mathrm{Prop}_{W'}(C) \subseteq P$, we can therefore rewrite the previous inequality as

$$|V(P)| - (m+1) \geq |V(P \setminus \mathrm{Prop}(C))| \geq |P| - m + 3.$$

Simplifying, we obtain $|V(P)| \geq |P| + 4$, i.e. $(P, V(P))$ is not an exact diagram.

Conversely, suppose that W' is not exact and for a contradiction suppose also that $M(W')$ is uniform of rank $|P|$. Take $p \in P$. Then $|V(P) \setminus V(p)| = |V(P)| - 4 \geq |P|$ by non-exactness. By uniformity there is an independent set of size $|P|$ in $V(P) \setminus V(p)$. This is impossible because the submatrix corresponding to this independent set has $|P|$ rows but the one corresponding to p is all 0 one of them is all 0 so it cannot be full rank. $\square$

We now make a few observations about the geometry of matroids defined by exact diagrams.

In [1], the authors show that all admissible Wilson loop diagrams correspond to positroids. That is, they correspond to matroids that can be represented by elements of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$. Any positroid of rank k on n elements defines a subspace of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, namely the points which represent it. These subspaces give a CW structure on $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$, with each positroid defining a cell.

Definition 2.22. Given a Wilson loop diagram $W = (\mathcal{P}, [n])$, define the positroid cell associated to a Wilson loop diagram, $\Sigma(W)$, to be the cell in the CW complex on $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ defined by the positroid $M(W)$.

With this definition in mind, we have the following corollary:

Corollary 2.23. *Let $(P, V(P))$ be an exact subdiagram of W. The matroid associated to this subdiagram corresponds to the top dimensional cell in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$.*

Proof. The unique top dimensional cell of $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ is defined by all points in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ such that all Plucker coordinates are strictly greater than 0. Since $(P, V(P))$ is an exact subdiagram, this all $|P| \times |P|$ minors are non-zero. Intersecting these with the cases with the positive Grassmannians demands that all minors be strictly positive. $\square$

2.4 Matroids and equivalent diagrams

Now we are ready to prove the main results of this section, namely that two Wilson loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). We also count the number of equivalence classes amongst Wilon Loop diagrams with n vertices and k propagators and give a formula for the number of Wilson loop diagrams in a given equivalence class (Corollary 2.25), completing the characterization of the correspondence between Wilson Loop diagrams and matroids starting in [1].

Siân postponed reading this section until the earlier ones are fixed.

Theorem 2.24. *Let $W = (\mathcal{P}, [n])$ and $W' = (\mathcal{P}', [n])$ be two Wilson Loop diagrams. They define the same matroid if and only if $W \sim W'$.*

Proof. One direction has been proved in [1, Theorem 1.18], but we give a different proof here to be consistent with the method of this document.

Assume that W and W' are equivalent. Without loss of generality, write $W = (P \cup R, [n])$ and $W' = (P \cup R', [n])$, where $P \subset \mathcal{P} \cap \mathcal{P}'$ and $(R, V(R))$ and $(R', V(R'))$ are two maximally exact subdiagrams, with $R \neq R'$, but $V(R) = V(R')$. If this is not the case, one may always find a family of diagram, $\{W_i\}$ satisfying this condition and forming a transitive chain connecting W to W' in the equivalence class.

Let $U \subset V(R)$ be any subset of size $|U| = |R|$. Since $(R, V(R))$ defines a uniform matroid, by Lemma 2.21, the set U is independent in the subdiagram $(R, V(R))$, and thus in W. The complementary set $F(P)$ is a flat of maximal rank by lemma 2.16 (rk $(F(P)) = |P|$). Let $B \subset F(P)$ be a maximal indpendent set (rk $B = |P|$) in $F(P)$. Since $F(P)$ is a flat, adding any element of $V(R)$ to B increases the rank. Therefore, any basis of W can be written as $B \cup U$ for some B and U of the form indicated. However, since $F(P)$ is common to both W and W', and $V(R) = V(R')$, any any basis of W' can also be written $B \cup U$. Thus both matroids have the same bases sets, proving that they are the same.

For the converse, assume that the matroids associated to W and W' are the same: $M(W) = M(W') = M$. Let $\mathcal{R} = \{(P_i, V(P_i)\}_{i=1}^k$ and $\mathcal{R}' = \{(P'_i, V(P'_i)\}_{i=1}^l$ be the sets of maximally exact subdiagrams of W and W'. Write $F_i = F(P_i^c)$ and $F'_i = F(P_i'^c)$ to be the complementary cyclic flats. By Theorem 2.21 and Theorem 2.19, M/F_i and M/F'_i are uniform matroids. Since, by Theorem 2.21, any subset V of M such that $M/(V^c)$ is uniform defines an exact subdiagram of both W and W'. If $U \subset V$ are two such thsets, then the corresponding exact subdiagrams are also subsets. Therefore, $|\mathcal{R}| = |\mathcal{R}'|$, and we write $V(P_i) = V(P'_i)$, after possibly reindexing $\mathcal{R}$ and $\mathcal{R}'$.

Since the sets of propagators defining maximal exact subdiagrams partition $\mathcal{P}$, by Lemma 2.13, write $\cup_{i=1}^k P_i = \cup_{i=1}^k P'_i = \mathcal{P}$. Reorganize the vertex sets of maximal exact subdiagrams as follows:

$$\cup_{P_i \notin \mathcal{R}'} V(P_i) = V(\cup_{P_i \notin \mathcal{R}'} P_i) = F(\cup_{P_i \in \mathcal{R}'} P_i)^c \; . \tag{7}$$

The final flat may, of course, be empty.

Thus, we have partitioned the vertices of W and W' into two complementary sets. The first, $\cup_{P_i \notin \mathcal{R}'} V(P_i)$, is comprised of the union of supports of maximal exact subdiagrams whose propagators differ between W and W'. The second, $F(\cup_{P_i \in \mathcal{R}'} P_i)$, is the propagator flat of all propagators in common between W and W'.

Without loss of generality, assume that $|\mathcal{R}| = |\mathcal{R}'| = 1$. Then the two Wilson loop diagrams are equivalent. If $|\mathcal{R}| = |\mathcal{R}'| > 1$, then one may define a family of Wilson loop diagrams, W_0 to W_k defined such that $W_0 = W$ and W_i is derived from W_{i-1} by replacing the propagator set P_i with P'_i. In this manner, $W' = W_k$ and $W_i \sim W_{i+1}$, making $W \sim W'$. □

Since there is a unique way to decompose W into maximal exact subdiagrams, it is logical to ask how many diagrams there are in an equivalence class. It is a classical fact the the number of triangulations of an n-gon is the $n-2$ Catalan number, namely $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. Thus we can count the number of equivalent diagrams. [CITATION?]

Corollary 2.25. *Let W be an admissible Wilson loop diagram where the sizes of the supports of the nontrivial maximal connected exact subdiagrams are $n_1, n_2, \ldots, n_j$. Then the number of admissible Wilson loop diagrams equivalent to W (including W itself) is*

$$\prod_{i=1}^{j} \frac{1}{n_i - 1} \binom{2(n_i - 2)}{n_i - 2}$$

In this section we have characterized the correspondence between Wilson loop diagrams and positroids, showing that Wilson Loop diagrams define the same matroid if and only if they are equivalent (Theorem 2.24). Corollary 2.25 enumerates the fibres of the map from Wilson Loop diagrams to positroids. By counting the number of equivalence classes (Theorem **??**), we enumerate the number of positroids defined by Wilson Loop diagrams. In section **??**, Theorem **??**, we show that each Wilson Loop diagram with n vertices and k propagators define a positroid cell of dimension $3k$. Comparing the result of Theorem **??** to the number of positroid cells of a particular dimension in $\mathbb{G}_{\mathbb{R},+}(n, k)$, [CITE AND FIND FORMULA HERE], we see that *Wilson loop diagrams do not map onto all positroids of the correct dimension.*

3 Geometry of Wilson Loop diagram

Since Wilson loop diagrams correspond to positroids, it is natural to study the subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ they define.

[words about what's in this section]

Throughout this section, $W = (\mathcal{P}, [n])$ is an admissible Wilson loop diagram with k propagators.

3.1 Background

[positroids]

Let $\binom{[n]}{k}$ be the set of all k-subsets of the cyclically ordered set $[n]$. For each $j \in [n]$, we can define a total order $\leq_j$ on the interval $[n]$ by

$$j <_j j + 1 <_j \cdots <_j n <_j 1 \cdots <_j j - 1 \, .$$

This in turn induces a total order on $\binom{[n]}{k}$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\binom{[n]}{k}$ (the **Gale order** [3]), which is defined as follows: for

$$A = \{a_1 <_j a_2 <_j \cdots <_j a_k\} \text{ and } B = \{b_1 <_j b_2 <_j \cdots <_j b_k\} \in \binom{[n]}{k},$$

we define

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\binom{[6]}{3}$ we have $\{2, 5, 6\} \preccurlyeq_2 \{2, 6, 1\}$ but $\{2, 5, 6\} \not\preccurlyeq_2 \{3, 4, 6\}$.

Definition 3.1. A **Grassmann necklace** of type (k, n) is a sequence $(I_1, \dots, I_n)$ of n sets $I_i \in \binom{[n]}{k}$ such that for each $i \in [n]$:

- if $i \in I_i$, then $I_{i+1} = \big(I_i \backslash \{i\}\big) \cup \{j\}$ for some $j \in [n]$.
- if $i \notin I_i$, then $I_{i+1} = I_i$.

By convention, we set $I_{n+1} = I_1$.

By [5, Theorem 17.1], the Grassmann necklaces of type (k, n) are in 1-1 correspondence with the positroid cells in $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. This correspondence is given explicitly in [4, Theorem 8]: if $(I_1, \dots, I_n)$ is the Grassmann necklace associated to a positroid $M = ([n], \mathcal{B})$, then the bases of M are exactly

$$\mathcal{B} = \left\{ J \in \binom{[n]}{k} \ : \ I_i \preccurlyeq_i J \ \forall i \in [n] \right\}.$$

Definition 3.2. A **Le diagram** is a Young diagram in which every square contains either a + or a 0, subject to the rule that if a square contains a 0 then either all squares to its left (in the same row) must also contain a 0, or all squares above it (in the same column) must also contain a 0, or both.

By [ref], the set of all Le diagrams that fit within a $k \times (n-k)$ rectangle is in 1-1 correspondence with the positroid cells of $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. The dimension of a positroid cell is equal to the number of + squares in its Le diagram [ref].

The rows and columns of a Le diagram are labelled as follows: given a Le diagram fitting inside a $k \times (n-k)$ box, arrange the numbers $1, 2, \dots, n$ along its southeast border, starting from the top-right corner. See Figure 1 for examples.

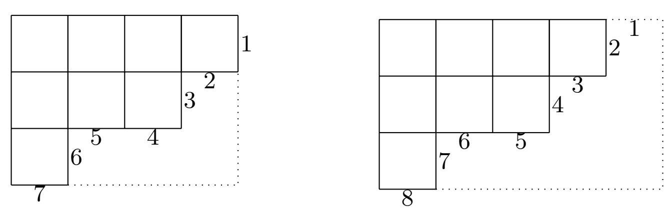


Figure 1: Row and column numbering for a Young diagram with $k = 3$, $n = 7$ (left) and $k = 3$, $n = 8$ (right). The top left box in each diagram has coordinates $(1, 7)$ (left diagram), $(2, 8)$ (right diagram).

An algorithm for constructing the Le diagram associated to a Grassmann necklace was given by Agarwala and Fryer in [ref]. Since we will make use of this algorithm in Section [ref] below, we summarise the process here.

Algorithm 3.3. *[ref] Let $(I_1, \dots, I_n)$ be a Grassmann necklace of type (k, n). Within a $k \times (n-k)$ square, draw the Young diagram whose rows are labelled by I_1 (as per the convention above).*

For each i, $2 \leq i \leq n$:

- *Write*
$$I_1 \setminus I_i = \{a_1 > a_2 > \cdots > a_r\}, \quad I_i \setminus I_1 = \{b_1 < b_2 < \cdots < b_r\},$$
where the inequalities denote the $<_1$ order (subscripts suppressed for clarity).

- *For $1 \leq j \leq r$, place a $+$ in square (a_j, b_j) of the diagram. (We will sometimes refer to this $+$ as being* in the $a_j \to b_j$ position.*)*

After performing the above for $2 \leq i \leq n$, place a 0 in any remaining unfilled boxes.

An algorithm for constructing the Grassmann necklace of a Le diagram also exists; this was given by Oh in [ref]. A method for using the Le diagram to test whether a given k-subset is a basis for the corresponding positroid or not was given by Casteels in [ref].

3.2 Propagator configurations in admissible Wilson loop diagrams

Before we can describe the algorithm for extracting the Grassmann necklace of $M(W)$ from the Wilson loop diagram W, we require some initial results about the behavior of propagators in admissible WLD.

Note that for any Wilson loop diagram, $W = (\mathcal{P}, n)$, each propagator p partions $[n]$ into three subsets of $[n]$, $\{[i+2, j-1], V(p), [j+2, i-1]\}$, the support of p and the vertices that lie on either side of p. In the sequel we refer to the latter two intervals as an interval inside and outside of p, depending on a ordering we associate to the propagator.

Definition 3.4. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, with $p = (i, j) \in \mathcal{P}$. Write $(p, <_a)$ be the same propagator with a direction induced by the linear ordering $<_a$. Then

$$V_{in}(p, <_a) = \{b \in [n] | i + 1 <_a b <_a j\}$$

the region of W **inside** p, and

$$V_{out}(p, <_a) = \{b \in [n] | j + 1 <_a b <_a i\} \ .$$

the region **outside** p.

Depending on the ordering associated to p, the vertices on the inside and outside chagne. In particular, for $p = (i, j)$, $V_{in}(p, <_i) = V_{out}(p, <_j)$ and $V_{out}(p, <_i) = V_{in}(p, <_j)$.

This in turn defines two sets of propagators, $\{\mathcal{P}_{in}(p, <_i), \mathcal{P}_{out}(p, <_i)\}$, where (for $\bullet \in \{in,\ out\}$)

$$\mathcal{P}_\bullet = \{q \in (\mathcal{P} \setminus p) | V(q) \cap V_\bullet(p) \neq \emptyset\} \ .$$

When clear, we drop the parentetical and just write $\mathcal{P}_{in}$ (resp. $\mathcal{P}_{out}$). These sets are the sets of propagators, not equal to p, whose support contains points in $V_{in}(p, <_*)$ (resp. $V_{in}(p, <_*)$), where $<_*$ refers to the appropriate ordering.If W has no propagators crossing p and no other propagator supported entirely on $V(p)$, then the set $\{\mathcal{P}_{in}, p, \mathcal{P}_{out}\}$ partitions $\mathcal{P}$. Under these circumstances, the set $\mathcal{P}_{in}$ defines the propagators with end points on one side of p, while $\mathcal{P}_{out}$ defines the propagators

Note, this means that paralell propagators are neither on the inside and outside of p. Is that okay?

with end points on the other side. In particular, for (weakly) admissible Wilson loop diagrams, the set $\{\mathcal{P}_{in}, p, \mathcal{P}_{out}\}$ is a partition for every propagators $p \in \mathcal{P}$. Recall from Definition 1.5 that a *weakly admissible* diagram is one that satisfies the density and non-crossing conditions but does not require that $|V(\mathcal{P})| \geq |\mathcal{P}| + 4$.

As with the vertex sets, which set of propagators lie inside and outside of p switch depending on the direction imposed p.

Definition 3.5. Let $p = (i, j) \in cP$ be a propagator in W. Define the *length* of p to be

right now this doesn't depend on the order of i and j, but that might matter later on?

$$\ell(p) = min\big\{|V_{in}(p)|, |V_{out}(p)|\big\} + 2 = min\big\{|[i+1, j]|, |[j+1, i]|\big\}.$$

In other words, $\ell(p)$ is the size of the smaller of the number of vertices that lie on either side of the progator p.

Remark 3.6. The following observations about configurations of propagators of short length in an (weakly) admissibly Wilson loop diagram, W, are easily verified:

1. If $p = (i, i+3)$ is a propagator of length 3, then the middle vertex $i+2$ supports at most one propagator.
2. If every vertex in W supports at least one propagator, then W admits at least one propagator of length 2.

The following lemma establishes certain configurations of propagators that must exist in any admissible diagram with no non-supporting vertices (i.e. with $F(\emptyset) = \emptyset$). We make use of this result in several induction proofs below.

Lemma 3.7. *Let W be a weakly admissible WLD with at least 5 vertices and in which each vertex supports at least one propagator. Then at least one of the following two things occurs.*

1. *W has a propagator of length ≤ 6 with a propagator of length 2 on one side it and nothing else on that side.*
2. *There exists a pair of propagators of length 2 with the property that the first propagator is $(i, i+2)$, the second is $(j, j+2)$, no other propagator ends between vertices $i+2$ and $j+1$, and $j \in \{i+2, i+3, i+4\}$.*

Proof. Suppose first that W has a propagator of length 3, say $p = (i, i+3)$. By Remark 3.6 and the fact that every vertex of W supports at least one propagator, we have that $i+2$ supports exactly one propagator and this propagator must have length 2 by noncrossingness. This gives us an instance of configuration 1 from the statement.

Now suppose W has no propagators of length 3.

We need a bit of notation: Let $W = (\mathcal{P}, n)$ a weakly admissible Wilson loop diagram, and $Q \subseteq \mathcal{P}$. Define the restriction $W|_Q$ to be the diagram W with the propagators not in Q removed. That is ,

$$W|_Q = (Q, n).$$

Given a propagator $p = (i, j)$ oriented $(p, <_i)$, we can consider two obvious restrictions of W, namely

$$W|_{\mathcal{P}_{in} \cup p} \quad \text{and} \quad W|_{\mathcal{P}_{out} \cup p},$$

where the set of non-supportive vertices of $W|_{\mathcal{P}_{in} \cup p}$ is V_{out} and the set of non-supportive vertices of $W|_{\mathcal{P}_{out} \cup p}$ is V_{in}. That is, these restrictions contain the propagators on the inside and outside of $(p, <_i)$ respectively, as well as p itself. [3]

With these observations in mind we can return to the proof.

We will inductively construct a sequence of pairs of propagators (p_r, q_r) satisfying: $\ell(p_r) = 2$, and p_r either forms part of configuration 1 or 2 from the statement, or there is a propagator q_r satisfying

- $\ell(q_r) \geq 4$.
- $\{p_1, \ldots, p_r\}$ and $\{q_1, \ldots, q_{r-1}\}$ are all on the same side of q_r.

Note that we will be interested in the orientation of q_r so that the previous p_i and q_i are on the outside of q_r. By the finiteness of W this must eventually terminate in one of the desired configurations.

Start by choosing a propagator $p_1 = (j_1, j_1 + 2)$ of length 2 in W (which exists by Remark 3.6). If it is part of one of the configurations we are looking for then we are done, so suppose otherwise. By assumption: p_1 is not in configuration 2, there are no propagators of length 3, and every vertex supports at least one propagator. Therefore there must exist a propagator q_1 of length ≥ 4 with one end on edge j_1 or on edge $j_1 - 1$ or on edge $j_1 - 2$. Orient $q_1 = (i_1, k_1)$ such that $p_1 \in \mathcal{P}_{out}(q_1)$, (that is $(q_1, <_{i_1})$).

INSERT PICTURE HERE?

Now suppose $q_r = (i_r, k_r)$ exists by the induction hypothesis and is oriented $(q_r, <_{i_r})$ so that the previous p_i and q_i are on the outside. For the rest of this proof, we assume this orientation and drop the $<_{i_r}$ from the notation.

Call let $W_r := W|_{\mathcal{P}_{in} \cup q_r}$. By the original hypotheses on W every vertex in $v \in V_{in}(q_r)$ (which is a non-empty interval since $\ell(q_r) \geq 4$) must have support at least one propagator ($|\mathrm{Prop}(v)| \geq 1$), while the same is true for $V(q_r)$ because these vertices support q_r. By construction, the remaining vertices of $W|_{\mathcal{P}_{in} \cup q_r}$, i.e. $V_{out}(q_r)$, are non-supporting.

By Remark 3.6, W_r admits at least one propagator of length 2. W_r has no propagators of length 3 since W has none. Let p_{r+1} be a propagator of length 2 in W_r; if it forms part of configuration 1 or 2 then we are done, so assume otherwise.

Note we may replace q_r by any other propagator, q_r' of length ≥ 4 in W_r such that p_{r+1} and q_r are on opposite sides; such a new q_r' still satisfies all the hypotheses, and so, without loss of generality

[3] If we denote $V \subset [n]$ and $Q = \mathrm{Prop}(V^c)$, then we can write

$$M(W|_{Q^c}) = M(W)/F(Q) \oplus M_{\emptyset, n-|i-j+1|}$$

where $M_{\emptyset,k}$ is the matroid of rank 0 on a set of k. That is $M_{\emptyset, n-|i-j+1|} = M((\emptyset, n - |i - j + 1|))$.

we may assume that q_r has minimal length among propagators of length ≥ 4 which have p_{r+1} on their inside.

Write $p_{r+1} = (j_{r+1}, j_{r+1}+2)$. There are two cases to consider. The first case is that $i_r \leq j_{r+1} \leq i_r+2$ and $k_r - 2 \leq j_{r+1} + 2 \leq k_r$ (so p_{r+1} has one end before $i_r + 3$ and the other after $k_r - 2$). Then since p_{r+1} has length 2, it must be that $(i_r + 3) + 1 \geq k_r - 2$ and so q_r has length ≤ 6. By the minimality assumption on q_r no propagator in W_r has q_r on one side and p_{r+1} on the other side. Therefore, any propagator in W_r other than p_{r+1} itself must be of the form (i, j) with $i_r \leq i, j \leq i_r + 2$ or $k_r - 2 \leq i, j \leq k_r$. That is, this propagator must be of length 2, and so we have configuration 2 which we have already assumed does not occur. Consequently, $\mathcal{P}_{in}(q_r)$ contains only p_{r+1} and so we have congifuratin 1 from the lemma statement which again we have assumed does not occur. Therefore this first case cannot occur.

The second case is that one of the following three things happen (we continue to write $p_{r+1} = (j_{r+1}, j_{r+1} + 2)$)

- $i_r + 3 \leq j_{r+1} \leq k_r - 3$ or $i_r + 3 \leq j_{r+1} + 2 \leq k_r - 3$,
- $i_r = j_{r+1}$, or
- $k_r - 2 = j_{r+1}$

(Phrased more causually, this is that p_{r+1} has either at least one end on an edge bounded by the vertices in the interval $[i_r + 3, k_r - 2]$ or both ends lie in $[i_r, i_r + 3]$ or both ends lie in $[k_r - 2, k_r + 1]$.) These situations all behave similarly. By symmetry it suffices to only consider the second situation and the second possiblity of the first situation. Note that since $j_{r+1} + 4 \leq k_r - 1$ in both situations, and $j_{r+1} + 4 \in V_{in}(q_r)$, thus $j_{r+1} + 4$ is supported by a propagator in W_r other than q_r. Let t be this propagator. Since p_{r+1} is not part of configuration 2 and W_r has no propagators of length 3, we must have that the length of t is ≥ 4. If q_r and p_{r+1} were on different sides of t then this would contradict the minimal length hypothesis on q_r. Therefore t has all the previous q_i and p_i on the same side and length ≥ 4 and so we may set $q_{r+1} = t$ to continue the induction.

The overall result then follows by induction. $\square$

Remark 3.8. In the case that all vertices of an admissible WLD W support at least two propagators, then Lemma 3.7 substantially simplifies. By Remark 3.6, W has no propagators of length 3. Configuration 1 necessarily entails vertices with support 1 as does configuration 2 unless $j = i + 2$. So in the case that W has all vertices with support at least two then W must contain a pair of propagators of length 2 with the property that the first propagator is $(i, i + 2)$, the second is $(i + 2, i + 4)$ and no other propagator ends on the edge $i + 2$.

3.3 From Wilson Loop diagrams to Grassmann Necklaces

Until now, the positroid associated to a Wilson loop diagram W could only be obtained by computing the matrix $C(W)$ associated to W, listing all bases of the induced matroid $M(W)$, and constructing the Le diagram or Grassmann necklace of the positroid "by eye" from this list.

In this section, we give an algorithm for passing directly from the Wilson loop diagrams to its Grassmann necklace. This not only greatly simplifies the process above, but will also allow us to relate the behavior of the positroid $M(W)$ directly to the configuration of progators in W.

The fact that Algorithm 3.9 does construct the required Grassmann necklace is proved in Theorem 3.15.

Algorithm 3.9. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. This gives an algorithm for calculating the set I_a, for $a \in [n]$.*

1. *Fix a vertex $a \in [n]$. Set $i := a$ and $I_a = \emptyset$.*
2. *While $\mathcal{P} \neq \emptyset$, perform the following steps.*
   (a) ***Step** i **for vertex** a: If Prop(i) $\neq \emptyset$ in W, write $I_a = I_a \cup i$. Let $p \in$ Prop(p) be the clockwise most propagator supported on i. Write $W = (\mathcal{P} \setminus p, n)$.*
   (b) *If Prop(i) $= \emptyset$ do nothing.*
   (c) *Increment i by 1 and repeat from (a).*

If the algorithm assigns vertex j to propagator p from starting vertex i, we say that p *contributes* j to I_i. Notationally, we represent this by allowing the I_i symbol to represent a function as well as a set, as follows:

Definition 3.10. Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram. For each $i \in [n]$, define a function $I_i : \mathcal{P} \longrightarrow [n]$ by

$$I_i(p) := \text{the vertex label that } p \text{ contributes to } I_i \text{ in Algorithm 3.9,}$$

for each $p \in \mathcal{P}$.

Given a propagator p and a vertex i in its support, it will be very useful in the following to understand on what set of vertices the Grassmann necklace algorithm assigns i to p. The answer is that the set is a non-empty cyclic interval. Lemma 3.13 establishes this, but first we need a preliminary lemma which is also useful in its own right.

Lemma 3.11. *Let W be an admissible Wilson loop diagram containing at least one propagator. For any $i \in [n]$ and for any $p = (a, b)$ with $i \leq_i a <_i b$, we have $I_i(p) \neq b + 1$.*

Proof. Suppose for contradiction that we have $p = (a, b)$ with $i \leq_i a <_i b$ and $I_i(p) = b + 1$. We may choose p such that $|[a + 1, b]|$ is minimal amongst propagators with this property.

Since $I_i(p) \neq b$, there must exist a propagator q inside of p with $I_i(q) = b$. The propagator q cannot end on the edge $(b - 1, b)$, as this would contradict the minimality of p, so $q = (c, c + 1, b, b + 1)$ with $a <_i c <_i b$, and $I_i(q) = b$.

In order for q to remain unassigned until vertex b, there must be another propagator r with an end on $(c, c + 1)$ and $I_i(r) = c + 1$; the only way this can occur is if r is outside q but inside p. Now r contributes its fourth vertex to I_i, again contradicting the minimality of p. ∎

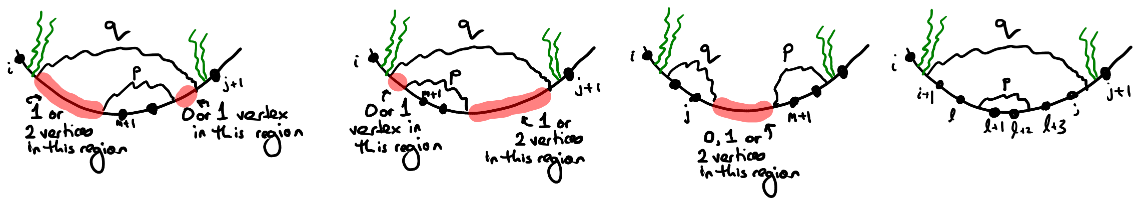


Figure 2: Four cases for admissible WLDs with no non-supporting vertices. The green half-propagators illustrate where propagators may occur, but are not required to exist; no other regions illustrated may support additional propagators.

Corollary 3.12. *If W is an admissible Wilson loop diagram with k propagators, then Algorithm 3.9 assigns exactly k vertices to each I_i.*

Proof. It follows from the proof of Lemma 3.11 that Algorithm 3.9 can never reach the fourth vertex of a propagator's support (with respect to the starting vertex). Therefore if the algorithm starts at vertex i, it must have assigned vertices to all propagators by the time it reaches $i-1$, ensuring that I_i contains exactly k distinct vertices. □

Given a propagator p of an admissible WLD W and a vertex i in the support of p, define

$$J_p^{(W)}(i) = \{m \in [n] \ : \ I_m^{(W)}(p) = i\},$$

i.e. the set of indices m for which Algorithm 3.9 assigns the value i to the propagator p in W. The following lemma establishes that these sets behave in a simple and predictable manner, a fact which we will repeatedly use in subsequent proofs.

Lemma 3.13. *Let $p = (i,j)$ be a propagator of an admissible WLD W on n vertices. Then $J_p^{(W)}(i)$, $J_p^{(W)}(i+1)$, $J_p^{(W)}(j)$, and $J_p^{(W)}(j+1)$ are each non-empty cyclic intervals which partition $[n]$ and occur in the given cyclic order.*

Proof. We will prove the result by induction on the number of propagators. If W has one propagator then the result is immediate. Now suppose W has more than one propagator. Since non-supporting vertices have no effect on the Grassmann necklace algorithm, it suffices to prove the result for W with $F(\emptyset) = \emptyset$. Then by Lemma 3.7, W has at least one of the four situations illustrated in Figure 2.

In each of the four cases, when we remove the propagator labelled p we obtain a diagram which satisfies the statement of the theorem by the induction hypothesis, and contains a propagator $q = (i,j)$ with no other propagators inside it (although this region may or may not contain non-supporting vertices, which we will call $l, l+1, \ldots$ as necessary). Let V be the diagram obtained by removing the propagator p from W (Figure 3).

Consider the fourth case of Figure 2 first, as it is the easiest. In this case, the vertices $\{l, l+1, l+2, l+3\}$ which support p in W are non-supporting in V, so for every propagator r in V (including q) we have $J_r^{(V)} = J_r^{(W)}$; these are non-empty cyclic intervals in the correct order by the induction

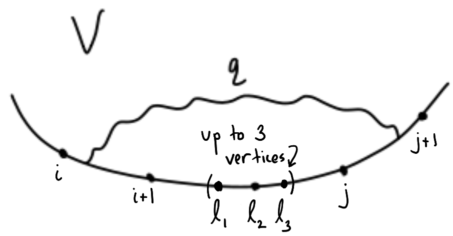


Figure 3: Diagram V is W with p removed; there are no propagators inside $q = (i, j)$, though there may be up to 4 non-supporting vertices labelled $l, l+1, \ldots$.

hypothesis. Additionally, it is clear from Figure 2 that $J_p^{(W)}(l+a) = \{l+a\}$ for $a \in \{1,2,3\}$ and $J_p^{(W)}(l) = [n]\backslash\{l+1, l+2, l+3\}$. The result therefore holds in this case.

Now we proceed to consider the first three cases of Figure 2. We first describe $J_q^{(V)}(*)$, which can be handled identically for all three cases.

In V there are no propagators inside q, so we see from Figure 3 that

$$l, l+1, \ldots, j \in J_q^{(V)}(j) \text{ (if } l \text{ exists) and } j+1 \in J_q^{(V)}(j+1).$$

Note that $j+2 \notin J_q^{(V)}(j+1)$ by Lemma 3.11, so by the induction hypothesis we must have $J_q^{(V)}(j+1) = \{j+1\}$ and $j+2 \in J_q^{(V)}(i)$. Thus there exist vertices $d, e \in [j+2, i+1]$ with $d < e$, such that

$$J_q^{(V)}(i) = [j+2, d-1], \quad J_q^{(V)}(i+1) = [d, e-1], \quad J_q^{(V)}(j) = [e, j], \quad J_q^{(V)}(j+1) = \{j+1\},$$

and all intervals are non-empty.

We now consider what happens as we move from V to each of the three remaining cases for W. We need to consider both $J_p^{(W)}$ and $J_r^{(W)}$ for $r \neq p$, since the addition of p can have a knock-on effect on later steps in the algorithm.

Left two cases: These cases behave essentially identically (except when j or $j+1$ are not in the support of p, which can occur in the second case only; see below) so we handle the majority of the proof for these two cases simultaneously. Let $1 \leq a \leq 3$ be the number of non-supporting vertices inside q in V; so these vertices are $l, \ldots, l+a-1$. Write $p = (m, m+2)$ where $m \in \{i, i+1, l\}$. Note that $l, \ldots, l+a-1$ are all in the support of p.

We first calculate $I_w^{(W)}$ for a starting vertex $w \in [n]\backslash\{l, l+1, \ldots, j, j+1\}$. Note that p has no effect on other propagators for starting vertices in this range, while the value of $I_w^{(W)}(p)$ depends on how soon q is assigned to a vertex, i.e. on the value of $I_w^{(V)}(q)$. Thus, if $w \in J_q^{(V)}(i)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{i+1, m\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(i+1)$ or $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} l & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

We also need to understand $I_w^{(W)}$ for $w \in \{l, l+1, \dots, j, j+1\}$. For the majority of these vertices, we use the following observation: if p is the first propagator to be assigned a value by $I_w^{(W)}$, then the remainder of $I_w^{(W)}$ proceeds identically to the assignments of $I_{w+1}^{(V)}$. Thus we have for any $0 \leq b < a$

$$I_{l+b}^{(W)}(r) = \begin{cases} l+b & \text{if } r = p \\ I_{l+b+1}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Similarly, if j is in the support of p, then we have

$$I_j^{(W)}(r) = \begin{cases} j & \text{if } r = p \text{ and } j \text{ is in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is in the support of } p \end{cases}$$

If j is not in the support of p, then we must be in the second case of Figure 2 with two vertices in the right hand region. In this case, if we start the algorithm at j we need to know whether there will be any unassigned propagators other than p when we reach vertex i, so as to know what p contributes.

Consider the WLD X formed from V by moving the second end of q to the edge $j-1$ instead of j. X is still admissible since we have not decreased the support of any set of propagators, so the induction hypothesis applies to it as well. Note that $J_j^{(V)}(r) = J_j^{(X)}(r)$ for all $r \neq p$ and $J_j^{(X)}(r) = J_{j+1}^{(X)}(r)$ for $r \neq q, p$. Additionally $I_{j+1}^{(X)}(q) = i$ by the induction hypothesis applied to X, and so if we start at $j+1$ and assign propagators to vertices according to the algorithm, when we reach vertex i in X all propagators other than q must have been assigned. Therefore if we start at j in W, we first assign q to j then proceed to assign as in X starting at $j+1$, and hence when we get to i the only remaining unassigned propagator is p. Therefore

$$I_j^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j \text{ is not in the support of } p \\ I_j^{(V)}(r) & \text{if } r \neq p \text{ and } j \text{ is not in the support of } p \end{cases}$$

Finally, we consider what happens when we start the algorithm at vertex $j+1$. If $j+1$ is in the support of p then we can argue as above to get

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = p \text{ and } j+1 \text{ is in the support of } p \\ I_{j+2}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is in the support of } p. \end{cases}$$

Now suppose $j+1$ is not in the support of p. If we start at $j+1$ we need to know whether there are any unassigned propagators supported on edge i when we reach vertex i. We already know that $J_q^{(V)}(j+1) = \{j+1\}$; in particular this means that q contributes i in $I_{j+2}^{(V)}$. However the construction of $I_{j+1}^{(V)}$ first associates q to $j+1$ and then proceeds identically to $I_{j+2}^{(V)}$. In particular if i was assigned in $I_{j+1}^{(V)}$, then it would not be available to assign to q in $I_{j+2}^{(V)}$ as all other propagators supported at i in V come before q.

Therefore p is the only potentially unassigned propagator on edge i when we reach vertex i in W, and

$$I_{j+1}^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } j+1 \text{ is not in the support of } p \\ I_{j+1}^{(V)}(r) & \text{if } r \neq p \text{ and } j+1 \text{ is not in the support of } p \end{cases}$$

We can now describe the intervals $J_r^{(W)}(*)$ for the first two cases of Figure 2. For $r \neq p$ the intervals are clearly still cyclic and appear in the correct order, and we can assemble the intervals for the $J_p^{(W)}(*)$ as follows.

- If $m = l$ then either $a = 2$ (so $l+1 = m+1$, $j = m+2$, and $j+1 = m+3$) or $a = 3$ (so $l+1 = m+1$, $l+2 = m+2$, $j = m+3$, and $j+1$ is not in the support of p), and in both cases

$$J_p^{(W)}(m) = [m+4, m], \qquad J_p^{(W)}(m+1) = \{m+1\},$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are nonempty and otherwise as required.

- If $m = i+1$ then checking each of the three different possibilities for a we likewise get

$$J_p^{(W)}(m) = [m+4, d-1], \qquad J_p^{(W)}(m+1) = [d, m+1],$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are nonempty and otherwise as required.

- If $m = i$ then $a = 1$ or $a = 2$, in the former case $l = m+2$, $j = m+3$ and $j+1$ is not in the support of p so

$$J_p^{(W)}(m) = \{m+4\}, \qquad J_p^{(W)}(m+1) = [m+5, d-1],$$
$$J_p^{(W)}(m+2) = [d, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  while in the latter $l = m+2$, $l+1 = m+3$, and j and $j+1$ are not in the support of p so

$$J_p^{(W)}(m) = [m+4, j+1], \qquad J_p^{(W)}(m+1) = [j+2, d-1],$$
$$J_p^{(W)}(m+2) = [d, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

  which are again as required.

Third case: In this case there are no non-supporting vertices $l, l+1, \ldots$ inside q. Again write $p = (m, m+2)$ where $m \in \{j, j+1, j+2\}$. We proceed as in the previous cases, by computing $I_w^{(W)}$ for vertices w in roughly increasing order of difficulty.

For $w \in [n]\backslash\{j+1, m, m+1, m+2, m+3\}$: if $w \in J_q^{(V)}(i)$ or $w \in J_q^{(V)}(i+1)$ then

$$I_w^{(W)}(r) = \begin{cases} m & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

while if $w \in J_q^{(V)}(j)$ then

$$I_w^{(W)}(r) = \begin{cases} \max\{m, j+1\} & \text{if } r = p \\ I_w^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Finally, for $j+1$ and the vertices in the support of p, we have

$$I_{j+1}^{(W)}(r) = \begin{cases} j+1 & \text{if } r = q \\ j+2 & \text{if } r = p \\ I_{j+3}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_m^{(W)}(r) = \begin{cases} m & \text{if } r = p \text{ and } q \text{ not supported on } m \\ m+1 & \text{if } r = p \text{ and } q \text{ supported on } m \\ I_m^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+1}^{(W)}(r) = \begin{cases} m+1 & \text{if } r = p \text{ and } q \text{ not supported on } m+1 \\ m+2 & \text{if } r = p \text{ and } q \text{ supported on } m+1 \\ I_{m+1}^{(V)}(r) & \text{if } r \neq p, q \end{cases}$$

$$I_{m+2}^{(W)}(r) = \begin{cases} m+2 & \text{if } r = p \\ I_{m+3}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

$$I_{m+3}^{(W)}(r) = \begin{cases} m+3 & \text{if } r = p \\ I_{m+4}^{(V)}(r) & \text{if } r \neq p \end{cases}$$

Note that $I_{m+2}^{(V)}(r) = I_{m+3}^{(V)}(r)$ for all propagators r in V, and that if $j+1 \notin \{m, m+1, m+2, m+3\}$ then $j+2$ and $j+3$ are non-supporting vertices in V, so in that case $I_{j+2}^{(V)}(r) = I_{j+3}^{(V)}(r) = I_{j+4}^{(V)}(r)$ for r in V.

Therefore, once again we can see that the $J_r^{(W)}(*)$ are cyclic for all $r \neq p$ in W. Assembling the intervals for p we have:

- if $m = j$ then

$$J_p^{(W)}(m) = [m+4, e-1], \qquad J_p^{(W)}(m+1) = [e, m],$$
$$J_p^{(W)}(m+2) = [m+1, m+2], \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j+1$ then

$$J_p^{(W)}(m) = [m+4, m-1], \qquad J_p^{(W)}(m+1) = [m, m+1],$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\},$$

- if $m = j+2$ then

$$J_p^{(W)}(m) = [m+4, m], \qquad J_p^{(W)}(m+1) = \{m+1\},$$
$$J_p^{(W)}(m+2) = \{m+2\}, \qquad J_p^{(W)}(m+3) = \{m+3\}.$$

The result now follows by induction. $\square$

We need one more lemma before we can prove that Algorithm 3.9 does in fact give the Grassmann necklace of the positroid associated to W.

Lemma 3.14. *Let $W = (\mathcal{P}, [n])$ be an admissible Wilson loop diagram and let $M(W)$ be its associated matroid. A subset $J \subseteq [n]$ is an independent set of $M(W)$ if and only if there exists an injective set map $f : J \to \mathcal{P}$ with the property that for each $j \in J$ we have $j \in V(f(j))$.*

One of the most important uses of this lemma is for bases. The lemma says that a subset J of $[n]$ is a basis of $M(W)$ if and only if there is a set bijection between J and $\mathcal{P}$ with the property that for each $j \in J$ the propagator associated to j under the bijection is supported on vertex j.

Proof. Because the nonzero entries of $C(W)$ are independent indeterminants, J is an independent set is if and only if there is some choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns.

Each entry in $C(W)$ identifies a propagator by the row of the entry and a vertex by the column of the entry. The entry is nonzero if and only if the propagator is supported on that vertex.

Consequently, a choice of $|J|$ nonzero entries of $C(W)$ one in each row associated to an element of J and each in different columns is equivalent to an assignment of the propagators of J to supporting vertices so that no two are assigned to the same vertex. Such an assignment of the propagators of J to supporting vertices is exactly a map f as described in the statment, hence proving the result. $\square$

Theorem 3.15. *The sequence of k-subsets $(I_1, \ldots, I_n)$ obtained by applying Algorithm 3.9 to all vertices of an admissible diagram W is exactly the Grassmann necklace of $M(W)$.*

Proof. For each $i \in [n]$, let I_i be the set of vertices assigned to the propagators of W by Algorithm 3.9 with starting vertex i. By Lemma 3.12, we know that $|I_i| = k$ for each $i \in [n]$. By [ref], the sequence $(I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in [n]$.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_{i \backslash \{i\}}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.9 at vertex i and vertex $i+1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i; thus one end of p must lie on either edge $i-1$ or edge i. In both cases let b denote the edge supporting the other end of p.

Case I: Suppose p has one end on edge $i-1$. Then p is not supported on $i+1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i+1$ up to $b-1$, that is $I_{i+1} \cap [i+1, b-1] = I_i \cap [i+1, b-1]$. Furthermore, by Lemma 3.13, when building I_{i+1} it must happen that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when the algorithm reaches vertex b there cannot be any unassigned propagators remaining that are before p. This is also true in when the algorithm constructing I_i reaches b, since the same propagators have been assigned beforehand. Finally, we also note that $k \geq_i b+1$.

Let W' be the diagram obtained from W by removing both p and all propagators inside of p (recall Definition 3.4).

By the above observations, if we commence Algorithm 3.9 in W' from vertex b, then we are in the same situation with respect to unassigned propagators as if we began at i in W and proceeded to b following the algorithm; the propagators we assigned in the latter case are exactly the ones removed to build W'. Similarly, starting at $i+1$ in W and moving to $b+1$ leaves us in the same situation with respect to unassigned propagators as beginning at $b+1$ in W' would. This gives the equations

$$I_i^{(W)} \cap [b, i-1] = I_b^{(W')}$$
$$I_{i+1}^{(W)} \cap [b+1, i-1] = I_{b+1}^{(W')}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{(W')} \setminus \{b\}$ and $k \notin I_{b+1}^{(W')}$, contradicting the minimality of (W, i, k).

Case II: Suppose p has one end on edge i. Note that by assumption we have $I_i(p) = i$; this means that p must be the first propagator lying on edge i, and hence we must have $I_{i+1}(p) = i+1$ as well. Observe also that $k \geq_i i+2$ since $i+1 \in I_{i+1}$.

Let W' be the diagram obtained from W by removing only the propagator p. Then we have

$$I_i^{(W)} \setminus \{i\} = I_{i+1}^{(W')}$$
$$I_{i+1}^{(W)} \setminus \{i+1\} = I_{i+2}^{(W')}$$

since in both cases the algorithm in W' proceeds identically to that in W after assigning p. Since $k \neq i+1$, we have $k \in I_{i+1}^{(W')} \setminus \{i+1\}$ but $k \notin I_{i+2}^{(W')}$, contradicting the minimiality of (W, i, k).

We have shown that $(I_1, \ldots, I_n)$ is a Grassmann necklace; it remains to check that this Grassmann necklace corresponds to the positroid $M(W)$. We need to show that:

- For each $i \in [n]$, I_i is a basis for $M(W)$.
- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $M(W)$.

Siân note to self: check whether it's lex smaller or Gale smaller

The algorithm is pairing each $j \in I_i$ with a unique propagator supported on that vertex so by Lemma 3.14 I_i is a basis for $M(W)$.

Suppose we have a k-set J such that J is a basis for $M(W)$ and yet is lexicographically less than I_i with respect to $<_i$. By Lemma 3.14 there is a set bijection between J and the propagators of W such that for each $j \in J$, the propagator associated to j is supported on vertex j. Choose one such bijection. For a propagator p of W write $J(p)$ for the associated j according to this bijection.

Since J is lexicographically smaller than I_i, the $<_i$-smallest element of the symmetric difference of J and I_i is some $j_0 \in J$, $j_0 \notin I_i$. Let p_1 be the propagator such that $J(p_1) = j_0$. Since $j_0 \notin I_i$

but p_1 is supported at j_0, then p_1 must have been assigned to an earlier vertex by I_i, i.e. we have $I_i(p_1) <_i j_0$. Let $j_1 = I_i(p_1)$.

However, j_0 is the $<_i$-smallest element in the symmetric difference of J and I_i, so in particular there is some propagator p_2 such that $J(p_2) = I_i(p_1)$. Now, let $j_2 = I_i(p_2)$ and note that $j_2 \neq J(p_2)$ (since $I_i(p_1) = J(p_2)$), so either $I_i(p_2) \notin J$ or there is a propagator p_3 such that $J(p_3) = I_i(p_2)$. Continuing likewise we get a list of vertices j_k and propagators p_k such that $J(p_k) = I_i(p_{k-1}) = j_{k-1}$.

We claim that the propagators and vertices in this list are distinct. The claim is proved by induction. The definition of j_0 gives that j_0 and j_1 are distinct and that p_1 and p_2 are distinct. Now, suppose we have p_k and j_k with $I_i(p_k) = j_k$ and with $j_k \in J$. Then p_{k+1} is the propagator associated to j_k by the fixed bijection we have chosed for J. Since j_k is distinct from the previous j_j, p_{k+1} is distinct from the previous p_j. Then $j_{k+1} = I_i(p_{k+1})$, but I_i is also a bijection, so since p_{k+1} is distinct from the other p_j we have that j_{k+1} is distinct from the other j_j for $1 \leq j \leq k$. Additionally $j_{k+1} \neq j_0$ since $j_0 \notin I_i$ but $j_{k+1} \in I_i$. This proves the claim.

By finiteness the list must end, and the only way it can end is with some p_k such that $j_k = I_i(p_k) \notin J$, so in particular $j_k >_i j_0$.

Next we will prove that $j_k <_i j_0$ for all $k > 0$. This gives a contradiction to the exisitence of J and completes the proof of the theorem. This proof will also be by induction, however, for the induction to go through nicely we will need to prove the following slightly stronger statement: for all $k > 0$ in the list

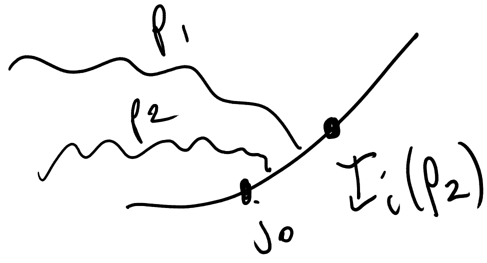
- $j_k <_i j_0$ and
- either
  - $I_i(p_k) <_i J(p_k)$, or
  - there exists an $ell < k$ such that $I_i(p_\ell) <_i J(p_\ell)$, those two vertices are supported by distinct ends of p_ℓ and p_k is on the side of p_l which runs from $I_i(p_\ell)$ to $J(p_\ell)$.

We already observed that $j_1 <_i j_0$ and $I_i(p_1) <_k J(p_1)$ so we have the base case for the induction. Suppose we have the inductive hypothesis for all indices less than $k > 1$ and that p_k exists.

By definition $J(p_k) = I_i(p_{k-1}) = j_{k-1}$, so p_{k-1} and p_k are both supported at j_{k-1}, but $I_i(p_{k-1}) = j_{k-1}$. Thus either p_{k-1} appears before p_k in the neighbourhood of j_{k-1} or p_k was taken earlier by I_i so $I_i(p_k) <_i I_i(p_{k-1})$. In the latter case we are done as $j_k = I_i(p_k) <_i j_{k-1} <_i j_0$ which is what we want. So we now assume that p_{k-1} appears before p_k in the neighbourhood of j_{k-1}.

If $I_i(p_{k-1}) <_i J(p_{k-1})$ and these are vertices supported by distinct ends of p_{k-1} then p_k coming after p_{k-1} around j_{k-1} implies that p_k is on the $I_i(p_{k-1})$ to $J(p_{k-1})$ side of p_{k-1}. Therefore $j_k = I_i(p_k) \leq_i J(P_{k-1}) + 1$. If $k-1 \neq 1$ then this gives $j_k \leq_i I_i(p_{k-2}) + 1 <_i j_0$ so $j_k \leq_i j_0$, but

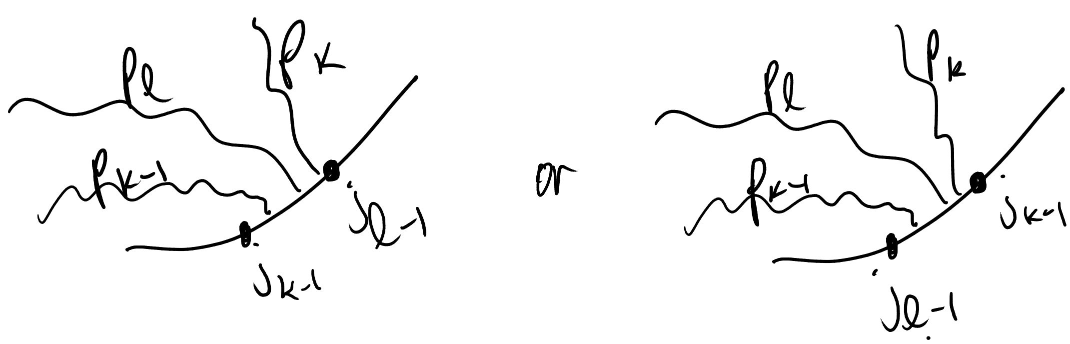
$j_0 \notin I_i$ so $j_k <_i j_0$. If $k-1=1$ then either $j_k <_i j_0$ or we have the configuration



which is impossible as p_2 is supported at j_0 but not taken in I_i until after j_0 even though $j_0 \notin I_i$. Taking all of this paragraph together we are done unless $I_i(p_{k-1}) \not<_i J(p_{k-1})$ so now assume this.

By the induction hypothesis, $I_i(p_{k-1}) \not<_i J(p_{k-1})$ implies that there exists and $\ell < k-1$ such that p_{k-1} is on the side of p_ℓ that goes from $I_i(p_\ell)$ to $J(p_\ell)$. If j_{k-1} is not in the support of the $J(p_\ell)$ end of p_ℓ then p_{k-1} before p_k around j_{k-1} implies that p_k is on the same side of p_ℓ as p_{k-1} is. Then the same arguments as above with ℓ in place of $k-1$ give $j_k \leq_i J(P_\ell)+1$ implying $j_k <_i j_0$ which is everything we need.

It remains to consider when j_{k-1} is in the support of the $J(p_\ell)$ end of p_ℓ, call this case $(*)$. Since $J(p_\ell) = j_{\ell-1}$ and the j_i are distinct this means that we have one of the following two configurations



If $p_\ell = p_1$ then $j_{\ell-1} = j_0$ and then as $j_0 \notin I_i$ but p_k is supported at j_0 we get $I_i(p_k) <_i j_0$ and also $I_i(p_k) <_i j_{k-1} = J(p_k)$ as the only vertices not yet taken for $I_i(p_k)$ must be before both labelled vertices in the illustration.

If $p_\ell \neq p_1$ then we argue as above with $\ell - 1$ in place of $k-1$: in particular $J(p_\ell) = I_i(p_{\ell-1})$ but p_k is supported at $p_{\ell-1}$, so either $I_i(p_k) <_i I_i(p_{\ell-1})$ in which case as before we get desired conclusion, or p_k comes before $p_{\ell-1}$ at $j_{\ell-1}$. Then again argue as before getting a new $p_{\ell'}$ in place of p_ℓ, except that then case $(*)$ must occur simultaneously for both $p_{\ell'}$ and p_ℓ with respect to p_k. This can only occur if $j_{\ell-1}, j_{k-1}, j_{\ell'-1}$ are consecutive. Iterating the argument one more time we finally cannot have case $(*)$ as then all neighbouring vertices of j_{k-1} are used up.

This completes the induction and thence the proof of the theorem.

□

In an arbitrary Grassmann necklace, it is possible for an index i to appear in no terms of the Grassmann necklace (a *loop*) or in all terms of the necklace (a *coloop*). Using Theorem 3.15, a characterization of the loops and coloops of the Grassmann necklace associated to a Wilson loop diagram follows easily.

Corollary 3.16. *Grassmann necklaces coming from admissible Wilson loop diagrams have no coloops. A vertex j is a loop if and only if j supports no propagators.*

Proof. For any $i \in [n]$, $i-1$ is maximal with respect to the $<_i$ order. Therefore there can be no propagator p with $I_i(p) = i-1$ by Lemma 3.11, i.e. $i-1 \notin I_i$. Thus the Grassmann necklace admis no coloops.

If $j \in [n]$ is a loop then $j \notin I_j$, which can only happen if there are no propagators supported on vertex j. Conversely, if j supports no propagators, then Algorithm 3.9 never assigns a propagator to j and hence $j \notin I_i$ for all $i \in [n]$. □

3.4 Dimension of the Wilson Loop cells

Our next goal is to show that the dimension of the positroid cell defined by a Wilson loop diagram $(\mathcal{P}, [n])$ has dimension $3|\mathcal{P}|$. Marcott in ***cite this*** has a different proof which is geometric and more elegant, but it is not easy to track the effect of a particular propagator. Our approach is much more explicit.

Recall that the dimension of a positroid cell is equal to the number of plusses in the associated Le diagram [5, Theorem 6.5]. By combining Algorithms 3.9 and 3.3 (converting from WLD to Grassmann necklace to Le diagram) we explicitly describe the effect of adding another propagator to a Wilson loop diagram in terms of the plusses of the associated Le diagrams, and hence give a recursive proof of the $3|\mathcal{P}|$-dimensionality of the cells.

We start with several lemmas, of roughly increasing degree of technicality.

Lemma 3.17. *Let W be an admissible Wilson loop diagram with k propagators, and with a vertex i that supports no propagators. Let V be W with vertex i removed. Then the Le diagram of W is obtained from the Le diagram of V by inserting an extra column containing all 0s in position i (i.e. such that the new column has the label i).*

Proof. By Algorithm 3.9 the Grassmann necklace of W is obtained from the Grassmann necklace of V by duplicating the ith element of the Grassmann necklace of V (shifting indices as appropriate), and incrementing all indices greater than i in each Grassmann necklace element. Formally, if $I_1^V, \dots, I_{n-1}^V$ and $I_1^W, \dots, I_n^W$ are the Grassmann necklaces of V and W respectively then

$$I_j^W = \begin{cases} \{\ell \in I_j^V : \ell < i\} \cup \{\ell + 1 \in I_j^V : \ell \geq i\} & \text{if } j \leq i \\ \{\ell \in I_{j-1}^V : \ell < i\} \cup \{\ell + 1 \in I_{j-1}^V : \ell \geq i\} & \text{if } j > i. \end{cases}$$

By Lemma 3.16 we know that $i \notin I_1^W$, and so i must label a horizontal edge on the boundary of the Le diagram of W, i.e. it must be a column label. The shapes of the Le diagram of V and W are the same except for the insertion of this column since I_1^* is the same for V and W except for the incrementation of the indices $\geq i$ in the transition from the necklace for V to the necklace for W. □

Lemma 3.18. *If two Wilson loop diagrams differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Proof. By [5, Proposition 17.10], the dimension of a positroid (and hence the number of plusses in its Le diagram) is $k(n-k) - A(\pi_W)$, where $A(\pi_W)$ denotes the number of alignments of the decorated permutation π_W of the positroid associated to W. (See [5, Figure 17.1] and preceding discussion.)

It can easily be seen from [5, Section 17] and Algorithm 3.9 that dihedral transformations of a Wilson loop diagram W correspond to dihedral transformations of the chord diagram representation of π_W. Since the number of alignments in a chord diagram is preserved under dihedral transformations, the result follows. □

Lemma 3.19. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators. Then there is some dihedral transformation W' of W such that W' has a propagator p with the following properties.*

- *$p = (i, n-1)$ for some i, and p has no propagators inside it (that is $i+2, \dots, n-2$ do not support any propagators in W').*
- *Either the edge i in W' only supports p or the edge i in W' supports exactly one other propagator $q = (j, i)$ with no other propagators inside q.*

Proof. Remove all vertices of W which do not support any propagators to get a weakly admissible Wilson loop diagram V. Lemma 3.7 applied to V gives a length 2 propagator p in V for which either no other propagator is supported on one of the supporting edges of p or there is a second length 2 propagator which is the only other propagator supported on one of the supporting edges of p. (Figure 2 shows the possible configurations arising from Lemma 3.7, and the reader can easily check that in each case p must be in one of the two situations described above.)

We can now make a dihedral transformation of V to obtain a diagram satisfying the statement of the lemma with p and q both length 2. Restoring the vertices which do not support any propagators, we obtain a dihedral transformation W' of W as desired (with potentially longer lengths for p and q). □

Combining Lemmas 3.17, 3.18, and 3.19, it therefore suffices to study the Le diagrams of weakly admissible Wilson loop diagrams admitting one of the configurations described in Lemma 3.19 with propagators p and q (if q exists) both of length 2. See Figure 4 for an illustration of the two possibilities.

The next few lemmas describe how diagrams of this type are related to the corresponding diagram with propagator p removed, first in terms of the Grassmann necklaces and then in terms of the

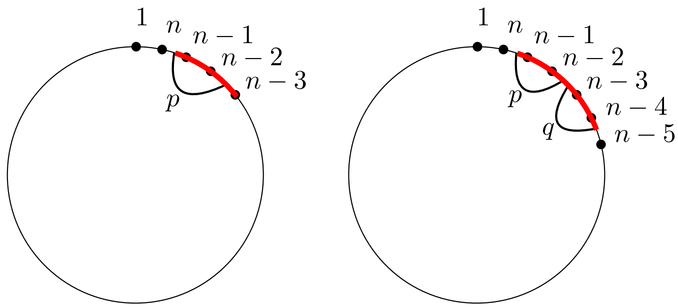


Figure 4: The two cases for W and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

Le diagrams. These technical lemmas will form the backbone of the inductive step in the main dimensionality argument.

Lemma 3.20. *Let W be an admissible Wilson loop diagram with $n \geq 1$ propagators, and suppose that W admits one of the configurations described in Lemma 3.19, with p and q (if q exists) both of length 2. Let V be W with p removed. Then*

$$\begin{aligned}
I_1^W &= I_1^V \cup \{n-3\} \\
I_n^W &= I_1^V \cup \{n\} \\
I_{n-1}^W &= I_n^V \cup \{n-1\} \\
I_{n-2}^W &= \begin{cases} I_{n-2}^V \cup \{n-2\} & \textit{if } n-2 \notin I_{n-2}^V \\ I_{n-2}^V \cup \{n-1\} & \textit{if } n-2 \in I_{n-2}^V,\ n-1 \notin I_{n-2}^V \\ (I_n^V - \{n-5\}) \cup \{n-1, n-2\} & \textit{if } n-1, n-2 \in I_{n-2}^V \end{cases} \\
I_k^W &= \begin{cases} I_k^V \cup \{n-3\} & \textit{if } n-3 \notin I_k^V \\ I_k^V \cup \{n-2\} & \textit{if } n-3 \in I_k^V \end{cases} \\
&\quad \textit{for } 1 < k < n-2
\end{aligned}$$

Proof. The two possible cases for W are illustrated in Figure 4; all references to the “left hand case” or “right hand case” below refer to the diagrams in this figure.

We first consider I_1^W, i.e. the set obtained by applying the Grassmann necklace algorithm to W with starting vertex 1. Note that $n-3 \notin I_1^V$: for the right hand case this is clear from the diagram (since Algorithm 3.9 would assign q no later than vertex $n-4$), while for the left hand case it follows from Lemma 3.11. Therefore when we start at vertex 1 and apply the Grassmann necklace algorithm to W, p is the only remaining unassigned propagator when we reach vertex $n-3$, and so we have $I_1^W(p) = n-3$ and $I_1^W = I_1^V \cup \{n-3\}$.

Next consider I_n^W. From the figure we see that in both cases we have $I_n^W(p) = n$. We are now at vertex 1, and the unassigned propagators are exactly those that appear in V. Therefore the

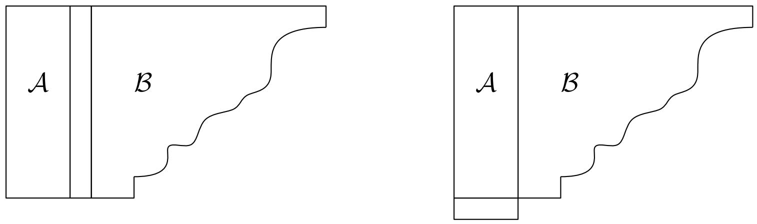


Figure 5: Le diagrams for V (left) and W (right).

algorithm continues as in I_1^V, i.e. we have $I_n^W = I_1^V \cup \{n\}$. By a similar argument, we must have $I_{n-1}^W = I_n^V \cup \{n-1\}$.

Now consider I_{n-2}^W. If $n-2 \notin I_{n-2}^V$ (i.e. $n-2$ supports no propagators in V), then in W the algorithm assigns p to $n-2$ and this does not affect the rest of the construction of I_{n-2}^V, so we obtain $I_{n-2}^W = I_{n-2}^V \cup \{n-2\}$ as above.

On the other hand, if $n-2 \in I_{n-2}^V$ then we must be in the right hand case of Figure 4 and the algorithm assigns q to vertex $n-2$ in both V and W (since q is always the clockwise-most propagator supported on vertex $n-2$; see Figure 4). If $n-1 \notin I_{n-2}^V$, then in W the algorithm assigns p to $n-1$ and then proceeds identically to I_{n-2}^V for the remainder of its steps; thus $I_{n-2}^W = I_{n-2}^V \cup \{n-1\}$ in this case.

Finally, if $n-2, n-1 \in I_{n-2}^V$ then in W the algorithm assigns q to $n-2$ and p to $n-1$ as above, but this is different to what occured in I_{n-2}^V so we cannot use the same argument as above. We are now at vertex n and only propagators p and q have been assigned; thus we are proceeding as in the construction of I_n^V but without propagator q. By Lemma 3.13 we know that q was assigned to $n-5$ by I_n^V, and from the diagram we see that the only way this could occur is if all other propagators in V had already been assigned when we reached vertex $n-5$; thus q was the final propagator to be assigned in I_n^V. Therefore $I_{n-2}^W = (I_n^V \setminus \{n-5\}) \cup \{n-1, n-2\}$ in this case. This completes all cases for I_{n-2}^W.

The arguments for I_k^W $(1 < k < n-2)$ proceed analagously to those of I_{n-2}^W, with one simplification: we cannot have both $n-3$ and $n-2$ in I_k^V since q is the only propagator that could be assigned to either of them, and it cannot be assigned to both.

This covers all cases and hence completes the proof. □

When doing the final formatting, try to ensure that Figure 5 and Lemma 3.21 are on the same page

Lemma 3.21. *Let V and W be as in Lemma 3.20. The shape of the Le diagram of V can be built from left to right of the following blocks: a rectangle which is 3 columns wide, one more column of the same height, and a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of W can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of V, and the same partition shape as in V.*

Proof. See Figure 5 for an illustration of the shapes described in the statement of the lemma.

Recall that I_1 determines the shape of the Le diagram. By Lemma 3.20 we know that $n, n-1, n-2, n-3 \notin I_1^V$ (which yields the leftmost four columns of the diagram for V) and that

$I_1^W = I_1^V \cup \{n-3\}$. This implies that the right hand boundary of the shape of V is the same as the right hand boundary of the shape of W except that W has one additional row of 3 boxes while V has an additional column in the $n-3$ position; that is, an extra column fourth from the left. $\square$

As illustrated in Figure 5, the pieces of the Le diagrams of V and W will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of each diagram are identical, and that the relationship between the plusses in the $\mathcal{A}$ regions can be described explicitly.

We do this by applying Algorithm 3.3, which constructs the Le diagram associated to a Grassmann necklace, to the Grassmann necklaces of V and W. As described in Section [ref], if Algorithm 3.3 places a + in the box with row index i and column index j, we say that this plus is in the $i \to j$ position, and refer to it as the plus defined by "the (hook) path from i to j". Note that the collection of paths contributed by a single Grassmann necklace term must be non-crossing.

Finally, we also note that when we speak of a plus in the Le diagram of V being the same as in W or vice versa, we mean that the position of the plus in $\mathcal{A}$ or $\mathcal{B}$ is the same; because of the column insertion the absolute indices may differ.

Lemma 3.22. *Let V and W be as in Lemma 3.20. Then I_n^W and I_{n-1}^W together yield the same plusses as I_n^V did, along with two extra plusses which appear in the leftmost two boxes of the bottom row of the Le diagram of W.*

Proof. By Lemma 3.20 we have $I_n^W = I_1^V \cup \{n\}$ and $I_1^W = I_1^V \cup \{n-3\}$. Thus

$$I_1^W \setminus I_n^W = \{n-3\}, \qquad I_n^W \setminus I_1^W = \{n\},$$

and so by Algorithm 3.3 we have a plus in the $(n-3) \to n$ position, i.e. in the leftmost box of the bottom row.

Also by Lemma 3.20 we have $I_{n-1}^W = I_n^V \cup \{n-1\}$. Recall from the proof of Lemma 3.20 that $n-3, n-2, n-1, n \notin I_1^V$; therefore $I_1^W \setminus I_{n-1}^W = (I_1^V \setminus I_n^V) \cup \{n-3\}$, and $n-3$ is maximal in this set. Similarly,

$$I_{n-1}^W \setminus I_1^W = (I_n^V \setminus I_1^V) \cup \{n-1\} \subseteq \{n-1, n\},$$

where the final inclusion follows from the definition of Grassmann necklace. In particular, $n-1$ is minimal in this set, so Algorithm 3.3 yields a plus in the $(n-3) \to (n-1)$ position (i.e. the second box in the bottom row of the Le diagram of W) along with any plusses yielded by I_n^V. $\square$

Lemma 3.23. *Let V and W be as in Lemma 3.20, and suppose that $n-2 \notin I_{n-2}^V$. Then I_{n-2}^W contributes all of the same plusses as $I_{n-1}^V = I_{n-2}^V$, along with a new $(n-3) \to (n-2)$ plus.*

Proof. If $n-2 \notin I_{n-2}^V$ then $I_{n-1}^V = I_{n-2}^V$ by definition, and by Lemma 3.20 we have $I_{n-2}^W = I_{n-1}^V \cup \{n-2\}$. Note that $n-3 \notin I_{n-2}^V$ by Lemma 3.11. Therefore the paths controlling the plusses contributed by I_{n-2}^W are exactly the paths for I_{n-1}^V along with the $(n-3) \to (n-2)$ path. This gives the statement of the lemma. $\square$

Lemma 3.24. *Let V and W be as in Lemma 3.20, and suppose that $n-2, n-1 \in I_{n-2}^V$.*

Then I_{n-2}^W and I_{n-3}^W contribute the following plusses to the Le diagram of W

- *An* $(n-3) \to (n-2)$ *plus and an* $(n-5) \to (n-1)$ *plus.*
- *All of the* I^V_{n-1} *plusses.*
- *In the Le diagram of* V*,* I^V_{n-2} *contributes an* $n-5 \to n-2$ *plus and no other term in the Grassmann necklace of* V *gives a plus in this column.* I^W_{n-2} *does not contribute this* $+$ *but yields an* $n-5 \to n-1$ *plus instead.*
- *All other plusses of* I^V_{n-2}*.*
- *In the Le diagram of* V*,* I^V_{n-3} *gives a plus in the* $n-3$ *column. This* $+$ *is shifted over into the* $n-2$ *column in* W*.*
- *All other plusses of* I^V_{n-3}*.*

Furthermore, no element of the Gramann necklace of V *gives an* $n-5 \to n-1$ *plus.*

Proof. By Lemma 3.20 $I^W_{n-2} = (I^V_n - \{n-5\}) \cup \{n-1, n-2\}$. Also, by the location of q in the WLD, $n-2 \notin I^V_{n-3}$ and and $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Gramann necklace of V looks like

$$I^V_{n-3} \underset{\substack{n-3 \text{ out} \\ n-2 \text{ in}}}{\to} I^V_{n-2} \underset{\substack{n-2 \text{ out} \\ n-5 \text{ in}}}{\to} I^V_{n-1} \underset{\substack{n-1 \text{ out} \\ \text{something in}}}{\to} I^V_n \underset{\substack{n \text{ out} \\ \text{something in}}}{\to} I^V_1 \tag{8}$$

where the first "something" is either n or an element of I^V_1 and the second "something" is an element of I^V_1. Additionally all elements not explicitly mentioned must be in I^V_1 as they remain unchanged through this portion of the necklace.

Using this information now determine the symmetric difference of I^V_{n-2} and I^V_1: $n-1, n-2$ and possibly n are in I^V_{n-2} but not in I^V_1. $n-5$ is in I^V_1 as are at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$. This means that the plusses in the Le diagram of V coming from I^V_{n-2} are as in the first part of Figure 8. Stepping to I^V_{n-1} simply removes the $n-5 \to n-2$ path, see the second part of Figure 8.

Stepping to I^V_n, $n-1$ is taken out and either n is put in if it was not there before, or one of a or b is put in and hence no longer available as a right end for a path. This gives two possible configurations illustrated in the bottom two parts of Figure 8.

Now we know that $I^W_{n-2} = (I^V_n - \{n-5\}) \cup \{n-1, n-2\}$ so the paths for building plusses from I^W_{n-2} go from the set $\{n-5, n-3\}$ along with whichever of a and b is not in I^V_n to $\{n-2, n-1, n\}$. This means that we get plusses as in Figure 9 where the left and right cases correspond to the left and right cases in the bottom parts of Figure 8

This proves the first item of statement of the lemma.

Now consider I^V_{n-3}. By (9) I^V_{n-3} contributes the same plusses as $I^V_{n_2}$ except that it contributes an $n-5 \to n-3$ plus in place of the $n-5 \to n-2$ plus. Also, we have $n-3 \in I^V_{n-3}$ be the location of q and so $I^W_k = I^V_k \cup \{n-2\}$. Thus the paths for I^W_{n-3} are the same as for I^V_{n-3} except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (9) shows that $n-2$ only appears in I^V_{n-2} among the necklace elements of V.

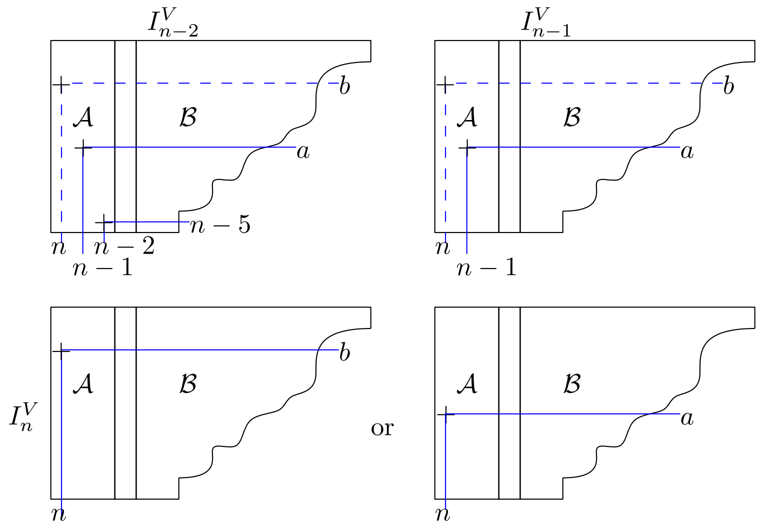


Figure 6: Plusses coming from I_{n-2}^V (top left), I_{n-1}^V (top right) and I_n^V bottom when $n-1, n-2 \in I_{n-1}^V$. The blue lines are the non-intersecting paths. The dashed blue lines may or may not appear, but if one appears then they both do.

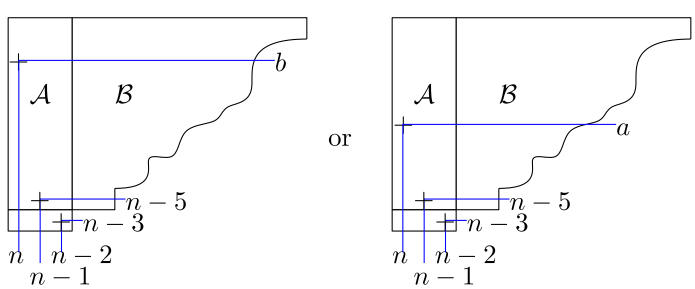


Figure 7: Plusses coming from I_{n-2}^W.

Also note that I^V_{n-3}, I^V_{n-2}, and I^V_{n-1} share their plusses outside of the $n-3$ and $n-2$ columns. This proves the remaining statements of the lemma except the furthermore.

Finally, suppose there were a $n-5 \to n-1$ plus in the Le diagram of V. By the algorithm, it would have to come when $n-5 \notin I^V_j$. By (9) this means that it would have to come from I^V_{n-4}, I^V_{n-3}, or I^V_{n-2}. The analysis above shows it does not come from I^V_{n-3} or I^V_{n-2}. Now, $n-4 \in I^V_{n-4}$ by the location of q and so I^V_{n-4} must give an $n-5 \to n-4$ plus and so cannot give an $n-5 \to n-1$ plus.

□

This is a re-organised version of the previous lemma, keeping both in place for now for comparison.

Lemma 3.25. *Let V and W be as in Lemma 3.20, and suppose that $n-2, n-1 \in I^V_{n-2}$. Then:*

1. *I^W_{n-2} contributes an $(n-3) \to (n-2)$ plus and an $(n-5) \to (n-1)$ plus, along with any plusses contributed by I^V_n.*
2. *I^W_{n-3} contributes the same plusses as I^V_{n-3}, except that the plus in the $n-3$ column is shifted one square left into the $n-2$ column.*
3. *In the Le diagram of V, I^V_{n-2} contributes an $(n-5) \to (n-2)$ plus and no other term in the Grassmann necklace of V gives a plus in this column. I^W_{n-2} does not contribute this + but yields the $(n-5) \to (n-1)$ plus from point (1) instead.*
4. *All other plusses from I^V_{n-2} and all plusses from I^V_{n-1} were already contributed by I^V_{n-3}, so they yield no new information.*
5. *No element of the Grassmann necklace of V contributes a $(n-5) \to (n-1)$ plus.*

Proof. By Lemma 3.20 we have $I^W_{n-2} = (I^V_n \setminus \{n-5\}) \cup \{n-1, n-2\}$ and $I^W_1 = I^V_1 \cup \{n-3\}$. We are necessarily in the case where W admits a propagator q (the right hand case of Figure 4) and we can see from the location of q that $n-2 \notin I^V_{n-3}$ and that $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Gramann necklace of V looks like

$$I^V_{n-3} \xrightarrow[\substack{n-2 \text{ in}}]{} I^V_{n-2} \xrightarrow[\substack{n-2 \text{ out} \\ n-5 \text{ in}}]{} I^V_{n-1} \xrightarrow[\substack{n-1 \text{ out} \\ \text{something in}}]{} I^V_n \xrightarrow[\substack{n \text{ out} \\ \text{something in}}]{\text{no change, } or} I^V_1 \tag{9}$$

where the first "something" is either n or an element of I^V_1, and the second "something" (if it exists) is an element of I^V_1. If $n \notin I^V_n$ then $I^V_n = I^V_1$.

Using this information we can determine the symmetric difference of I^V_{n-2} and I^V_1, which has size ≤ 3 by the definition of a Grassmann necklace. We have $n-1, n-2 \in I^V_{n-2} \setminus I^V_1$ for certain, and the third element (if it exists) must be n. On the other hand we have $n-5 \in I^V_1 \setminus I^V_{n-2}$, along with at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$.

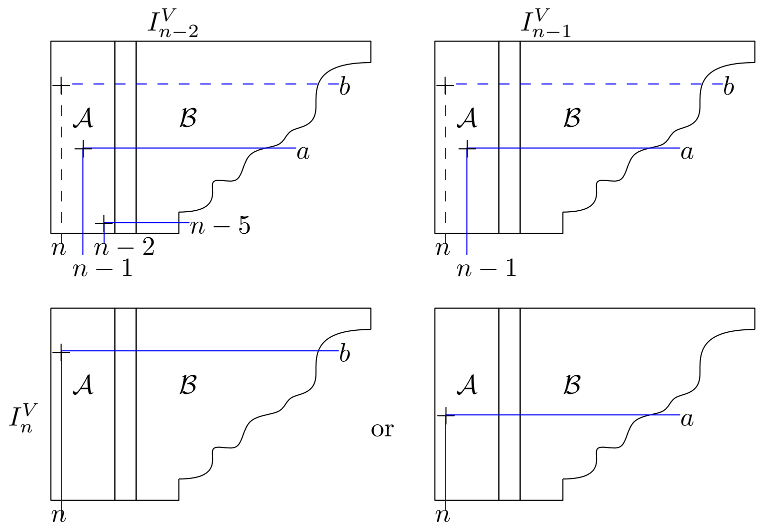


Figure 8: Plusses coming from I_{n-2}^V (left) and I_{n-1}^V (right), in the case where $n-1, n-2 \in I_{n-2}^V$. The blue lines are the non-intersecting paths defining the position of the plusses. The dashed blue lines may or may not appear, but if one appears then they both do.

This means that the plusses in the Le diagram of V coming from I_{n-2}^V are as in the first part of Figure 8. Stepping to I_{n-1}^V simply removes the $(n-5) \to (n-2)$ path (second part of Figure 8), so I_{n-1}^V contributes no new information about the Le diagram of V.

Remove the I_n^V diagrams

With this information in hand, we can now tackle the points in the statement of the lemma. We know that

$$I_{n-2}^W = (I_n^V - \{n-5\}) \cup \{n-1, n-2\}, \quad n-5 \in I_1^W, \quad n-3 \notin I_{n-2}^W,$$

and so the paths for building plusses from I_{n-2}^W start at

$$I_1^W \setminus I_{n-2}^W = \{n-3, n-5\} \cup (I_1^V \setminus I_n^V)$$

and end at

$$I_{n-2}^W \setminus I_1^W = \{n-2, n-1\} \cup (I_n^V \setminus I_1^V).$$

This means that I_{n-2}^W contributes plusses as in Figure 9, which proves point (1) and most of point (3). From equation (9) we see that $n-2$ appears only in I_{n-2}^V, so no other Grassmann necklace term for V can contribute a plus in the $n-2$ column; this completes the proof of item (3).

Modify this figure to include the case where there's no plus for column n.

Now consider I_{n-3}^V. By (9) I_{n-3}^V contributes almost the same plusses as I_{n-2}^V: the only difference is that it contributes an $(n-5) \to (n-3)$ plus in place of the $(n-5) \to (n-2)$ plus. Since we must have $n-3 \in I_{n-3}^V$ (because the propagator q exists in V), it follows from Lemma 3.20 that $I_{n-3}^W = I_{n-3}^V \cup \{n-2\}$. Thus the paths for I_{n-3}^W are the same as those for I_{n-3}^V except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (9) shows that $n-2$ only appears in I_{n-2}^V among the necklace elements of V.

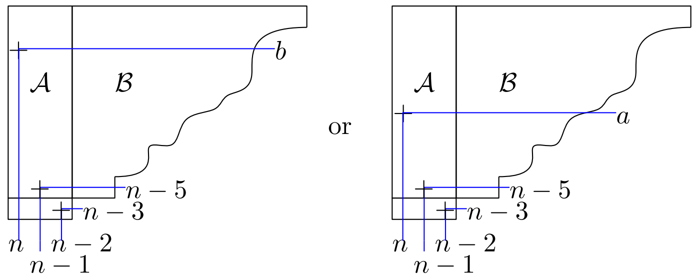


Figure 9: Plusses coming from I^W_{n-2}.

This proves point (2), and combined with the observation above that I^V_{n-1} contributes no new plusses compared to I^V_{n-2}, we also obtain item (4).

Finally, if the Le diagram of V did admit a $(n-5) \to (n-1)$ plus, it could only have been contributed by a necklace term that doesn't contain $n-5$. By equation (9) the only terms that could have this property are I^V_{n-2}, I^V_{n-3}, and I^V_{n-4}, and the analysis above shows that I^V_{n-2} and I^V_{n-3} do not contribute a plus in this position.

Recall that $n-5$ is the largest row index in the Le diagram of V, and $n-5 \in I^V_1 \setminus I^V_{n-4}$ so I^V_{n-4} does contribute a path starting at $n-5$. Since $n-4 \in I^V_{n-4}$ by the location of the propagator q, we must also have a path ending at $n-4$. Since the paths cannot cross, this implies that I^V_{n-4} must contribute a $(n-5) \to (n-4)$ plus, and hence cannot contribute a $(n-5) \to (n-1)$ plus. This proves point (5) and completes the proof of the lemma. $\square$

Lemma 3.26. *Let V and W be as in Lemma 3.20, and suppose that if $n-2 \in I^V_{n-2}$ then $n-1 \in I^V_{n-2}$ also. Then for each k in the range $1 < k < n-2$, I^W_k contributes the same plusses as I^V_k, except that if I^V_k contributed a plus in the $n-3$ column of the Le diagram of V then this plus is shifted one square left in the Le diagram of W, and no plus was already in that location in the Le diagram of W.*

Proof. Recall that $I^W_1 = I^V_1 \cup \{n-3\}$ and that $n-2, n-1, n \notin I^W_1$.

If $n-3 \notin I^V_k$ then by Lemma 3.20 we have $I^W_k = I^V_k \cup \{n-3\}$. Then since $n-3$ is the largest element of I^W_1 this transformation leaves the disjoint paths unchanged and so the plusses carry over from V to W directly.

If $n-3 \in I^V_k$ then I^V_k must contribute a plus in the $n-3$ column of the Le diagram of V, and by Lemma 3.20 we have $I^W_k = I^V_k \cup \{n-2\}$. If $n-2$ supports no propagators in V then certainly no pluses appear in the $n-2$ column of the Le diagram of V. If $n-2$ supports at least one propagator in V then $n-2 \in I^V_{n-2}$ and so by hypothesis $n-1 \in I^V_{n-2}$ as well. By Lemma 3.24, the only necklace element of V containing $n-2$ is I^V_{n-2} and the corresponding plus is not contributed to the Le diagram of W by I^W_{n-2}.

Since $n-3 \in I^V_k \setminus I^V_1$, we must have a path from some vertical edge i to the bottom edge $n-3$ in the Le diagram of V. In the Le diagram of W, the index $n-3$ labels a vertical edge with no path starting at it (since $n-3$ belongs to both I^W_1 and I^W_k), and there must be a path leading to $n-2$ since $n-2 \in I^W_k \setminus I^W_1$. By the previous paragraph no other path from I^V_k could end at $n-2$, and since the paths cannot cross we conclude that the $i \to (n-3)$ path in the Le diagram of V must become a $i \to (n-2)$ path in the Le diagram of W. All other paths are unchanged.

Thus the plus contributed by I_k^V in the $n-3$ column of the Le diagram of V is shifted into the $n-2$ column in the Le diagram for W, where there was no plus before, and no other plusses are changed. $\square$

Theorem 3.27. *The number of plusses in the Le diagram of an admissible Wilson loop diagram is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a Wilson loop diagram W with one propagator supported on vertices $i < j < k < \ell$ has Le diagram a single row with $|W| - i$ boxes. Labelling the columns from left to right by $|W|, \dots, |W| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider Wilson loop diagrams with $k > 1$ propagators. By Lemma 3.17 it suffices to prove the result for weakly admissible Wilson loop diagrams with k propagators and no non-supporting vertices. By Lemma 3.18 it suffices to prove the result for at least one Wilson loop diagram from each dihedral orbit. Take a weakly admissible Wilson loop diagram W with k propagators and no non-supporting vertices. Make a dihedral transformation of W if necessary so that W has a propagator p with the properties in Lemma 3.19.

We make one further simplification: if our W is in Case 2 of Figure 4 and $n-1$ supports only p but $n-2$ supports at least one other propagator, then flip W on the line perpendicular to the edge from $n-2$ to $n-1$ to obtain a diagram with the configuration of Case 1. This eliminates the possibility that we could have $n-2 \in I_{n-2}^W$ but $n-1 \notin I_{n-2}^W$.

This diagram will be our W for the remainder of the proof. Let V be W with p removed.

From Lemma 3.21 we know how the shapes of the Le diagrams of V and W are related; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 3.22, 3.23, and 3.24 tell us that the three boxes of the bottom row of the Le diagram of W each have a plus. Lemmas 3.22 through 3.26 show that there is a bijection between the plusses of the Le diagram of V and the plusses of the Le diagram of W that are not in the bottom row. This bijection can be described as follows:

- All plusses from $\mathcal{B}$ for V maintain their positions in $\mathcal{B}$ for W.
- All plusses from the leftmost two columns (the n and the $n-1$ columns) of $\mathcal{A}$ for V maintain their positions in $\mathcal{A}$ for W.
- If there is a plus in the $n-2$ column of $\mathcal{A}$ in V then Lemma 3.24 applies, so there is exactly one such plus. This plus is mapped to the $(n-5) \to (n-1)$ plus for W.
- All plusses in the $n-3$ column for V are shifted one square to the left in $\mathcal{A}$ for W, i.e. into the $n-2$ column.

Note that this map is reversible and hence bijective. Indeed, the only possible ambiguity is at the $(n-5) \to (n-1)$ plus in W (if it exists), which could have come from either of the first or third bullet points. However, if the third bullet point applies then by Lemma 3.24 there is no $(n-5) \to (n-1)$ plus in V, i.e. the first bullet point does not apply.

Therefore the Le diagram of W contains $3(k-1)$ plusses in bijection with the plusses from the Le diagram of V and 3 new plusses in the bottom row, yielding $3k$ in total. Applying induction completes the proof. □

4 Poles of Wilson Loop Integrals

The results of Section 3 allow us to relate the position of propagators in a Wilson loop diagram W to minors of $C(W)$, which we use in this section to understand the denominator of the integral $I(W)$ associated to a Wilson loop diagram (see Definition 1.11).

The main result of this section is Theorem 4.4, which expresses the denominator $R(W)$ in terms of the Grassmann necklace of W. This simplifies the computation of $R(W)$ and allows us to directly relate the poles of the integral to the combinatorics of the diagram.

We first give an algorithm which extracts the required minors from the Grassmann necklace.

Algorithm 4.1. *Let $W = (\mathcal{P}, n)$ be a Wilson loop diagram, and let $C(W)$ be the matrix of W as defined in* (1) *(see Section 1).*

- *For each $i \in [n]$, we construct a factor r_i as follows:*
  - *Let $S_i = \{p \in \mathcal{P} \mid I_{i-1}(p) \neq I_i(p)\}$. (By convention, set $I_{-1} = I_n$.)*
  - *Let r_i be the determinant of the $|S_i| \times |S_i|$ minor of $C(W)$ with rows indexed by S_i and columns indexed by $I_i(S_i)$.*
- *Define $R = \prod_{i=1}^{n} r_i$.*

As we discuss the algorithm it will also be useful to have the notation Δ_{I_i} for the determinant of the $k \times k$ minor of $C(W)$ with columned indexed by I_i.

As we take each r_i to simply be the determinant of a particular submatrix of $C(W)$, the sign of each r_i is well-defined. The goal is to show that R is equal to the denominator of the Wilson loop diagram as defined in Definition 1.11.

Example 4.2. Consider the Wilson loop diagram in Figure 10. Assigning propagators p, q, s to rows $1, 2, 3$ respectively, we obtain the matrix

$$C(W) = \begin{bmatrix} a & b & 0 & 0 & 0 & c & d \\ e & f & 0 & 0 & g & h & 0 \\ i & j & 0 & k & l & 0 & 0 \end{bmatrix}$$

The Grassmann necklace of this diagram is

$$I_1 = \{1, 2, 4\}, I_2 = \{2, 4, 5\}, I_3 = \{4, 5, 6\}, I_4 = \{4, 5, 6\},$$
$$I_5 = \{5, 6, 7\}, I_6 = \{6, 7, 1\}, I_7 = \{7, 1, 2\}.$$

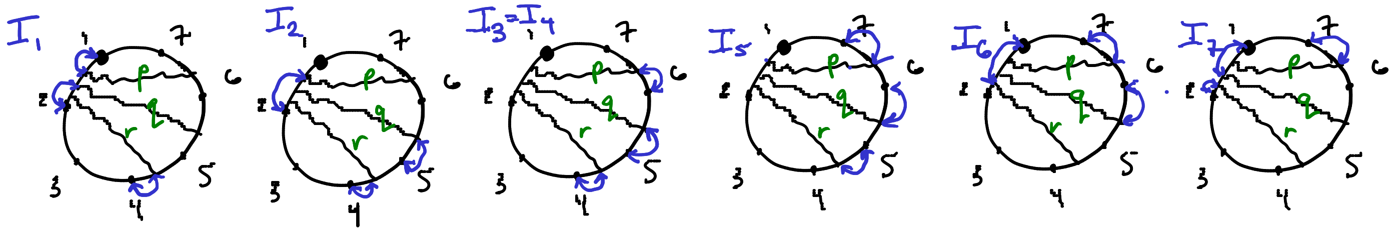


Figure 10: Example WLD for illustrating Algorithm 4.1 and bijections between propagators and vertices for each Grassmann necklace element.

Figure 10 indicates the pairings between propagators and vertices for each $i \in [1, 7]$.

From I_1 to I_2, the propagators p and q change which vertex they are assigned to but r is assigned to vertex 4 in both, so $S_2 = \{p, q\}$. Then

$$\Delta_{I_2} = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ j & k & l \end{bmatrix} = kgb, \qquad r_2 = \det \begin{bmatrix} b & 0 & 0 \\ f & 0 & g \\ 1 & 1 & 1 \end{bmatrix} = gb.$$

where the 1s in the third row of the second matrix correspond to the fact that $I_1(s) = I_2(s)$. Continuing likewise, we get $r_3 = c$, $r_4 = 1$ (since $I_4 = I_3$), $r_5 = lhd$, and $r_6 = i$.

At I_7 the situation is more complicated: we have $S_7 = \{q, s\}$, so we find that $\Delta_{I_7} = d(ej - fi)$ and $r_7 = ej - fi$. This quadratic factor corresponds to the fact that q and s share an edge *and* contribute both endpoints of that edge to I_7; see Proposition 4.3 below.

Finally, we have $r_1 = (af - be)k$. Putting everything together, we obtain

$$R = (af - be)kgbclhdi(ej - fi)$$

which is squarefree and contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$. If one were to construct the denominator $R(W)$ associated to this Wilson loop diagram as per Definition 1.11, we would find that we have $R(W) = R$.

Proposition 4.3. *With notation as in Algorithm 4.1 we have the following:*

1. *Each Δ_{I_i} is homogeneous, as is each r_i.*
2. *Each Δ_{I_i} splits into linear and quadratic factors. All linear factors of Δ_{I_i} are single variables and all irreducible quadratic factors are 2×2 determinants of single variables.*
3. *Quadratic factors in Δ_{I_i} arise precisely when propagators p and q are supported on a common edge (a, b) with $I_i(p) = a$ and $I_i(q) = b$.*
4. *r_i divides Δ_{I_i}.*
5. *The ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.*

Proof. 1. The nonzero entries of $C(W)$ are independent indeterminates and so every $i \times i$ minor of $C(W)$ is either homogeneous of degree i or is 0. Thus each Δ_{I_i} and each r_i is homogeneous.

2. Using the expression for the determinant as a sum over permutations we see that Δ_{I_i} is a sum over bijections between I_i and $\mathcal{P}$. The nonzero terms in this sum are precisely those bijections such that each propagator is associated to one of its supporting vertices in I_i, since only those locations in $C(W)$ are nonzero. Since the nonzero entries of $C(W)$ are independent there can be no cancellation between terms in this expansion.

   Suppose Δ_{I_i} has an irreducible factor f. Let $\mathcal{P}'$ be the set of propagators which contribute a variable to f and let J be the set of vertices which contribute a variable to f.

   The first claim is that the minor of $C(W)$ associated to $\mathcal{P}'$ and J is precisely f.

   Proof of claim: By the structure of determinants we know that $\Delta_{I_i} = fg$, where g involves only variables associated to propagators not in $\mathcal{P}'$ and associated to vertices not in J.

   Expanding out fg yields a signed sum of monomials. In each of these monomials, f contributes those variables associated both to a propagator in $\mathcal{P}'$ and to a vertex in J, and g contributes those variables associated both to a propagator not in $\mathcal{P}'$ and to a vertex not in J, and no other variables appear.

   Since there is no cancellation between terms, this means that the full expansion over permutations of Δ_{I_i} contains no other nonzero terms and hence no other variables. Therefore Δ_{I_i} is equal to the determinant of the matrix obtained by taking the submatrix of $C(W)$ with columns indexed by I_i and setting any variables not appearing in Δ_{I_i} to 0. This new matrix is, up to permutations of rows and columns, a block matrix with one block for $\mathcal{P}'$ and J and the other block for the complements. Thus its determinant, and hence also Δ_{I_i}, is the product of the minors for these two blocks. By considering which variables appear, these two factors must also be f and g, and so in particular f is the minor of $C(W)$ associated to $\mathcal{P}'$ and J. This proves the claim.

   A consequence of this claim is that every linear factor of Δ_{I_i} is a 1×1 minor of $C(W)$, hence is a single variable, and every irreducible quadratic factor of Δ_{I_i} is a 2×2 minor of $C(W)$, hence is a 2×2 determinant of single variables.

   All that remains is to prove that Δ_{I_i} has no irreducible factors of degree 3 or more. Suppose for a contradiction that f is a factor of Δ_{I_i} of degree ≥ 3. Note that by removing the propagators which come before those contributing to f and changing i to be the first vertex which contributes to f, we obtain a different admissible diagram for which f still divides Δ_{I_i} but also $i \in I_i$ and i contributes to f. Showing that this different admissible diagram gives a contradiction is sufficient, and so we may assume that $i \in I_i$ and i contributes to f. Finally, we can suppose that W is minimal in number of propagators with the above occuring.

   Let p be the propagator such that $I_i(p) = i$. There are two cases to consider, depending on which edge p is supported on. These are illustrated in Figure 11

   Case 1: Suppose p has one end on the edge $(i-1, i)$. Thus p is supported on $(i-1, i, m, m+1)$ for some $m >_i i$, and $I_{i+1}(p) = m$ by Lemma 3.13.

   Let S be the set of propagators inside p along with p itself. I_i and I_{i+1} can only differ once p contributes to I_{i+1}, so $I_i(q) = I_{i+1}(q)$ for each $q \in S \backslash \{p\}$. Thus if a propagator contributes m in I_i then it must lie outside p.

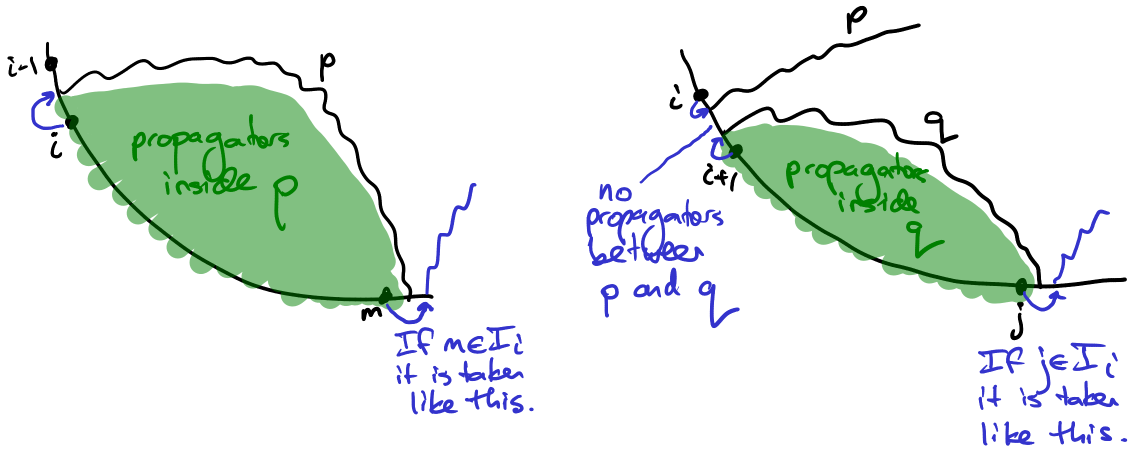


Figure 11: The two cases in the proof that no factors of Δ_{I_i} have degree 3 or more.

If neither m nor $m+1$ appear in I_i then by Corollary 3.16 $V(p) \cap I_i = \{i\}$, and so the row of p in the matrix of Δ_{I_i} has only one nonzero entry; hence Δ_{I_i} has a linear factor contributed by p and i, which is a contradiction. So we must have at least one of m and $m+1$ in I_i. However, all propagators in S are mapped by the function $I_i(\cdot)$ to vertices strictly before m, so the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the $|S| \times |S|$ matrix indexed by the propagators in S and the vertices in $I_i(S)$. No other propagators can be supported on these vertices since all other propagators are outside of p, and p is the first propagator supported at i; this explains the zero block. Therefore $\Delta_{I_i} = \det A \det C$, and both factors are nontrivial since at least one of m and $m+1$ appear in I_i. If we remove the propagator outside of p that contributes m or $m+1$, we get a smaller diagram for which $\Delta_{I_i} = \det A$. This contradicts the minimality of our choices unless $\det A$ is quadratic, which in turn contradicts our assumption that i and p contribute to an irreducible factor f of degree at least 3.

Case 2: Suppose p has one end on the edge $(i, i+1)$. If no other propagators are supported on i then the column of $C(W)$ corresponding to vertex i has only one nonzero entry in it, and so Δ_{I_i} has a linear factor contributed by p and i; as above, this is a contradiction. Thus we can take q to be the propagator such that $I_i(q) = i+1$. We know that q has one end on the edge $(i, i+1)$ and is adjacent to p on that edge in the counterclockwise direction (see Figure 11). Write $(i, i+1, j, j+1)$ for the support of q. The situation for q is very similar to case 1: in particular, we have $I_{i+1}(q) = j$ by Lemma 3.13 and so if $j \in I_i$ then the propagator which contributes j is outside of q.

Similarly to Case 1, let S be the set of propagators inside q along with p and q themselves. Then all propagators in S are mapped by $I_i(\cdot)$ to vertices strictly before j and no other propagators are supported on vertices strictly before j. Thus the matrix giving Δ_{I_i} has the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is the submatrix indexed by the propagators in S and the vertices in $I_i(S)$. Again two things can now happen. If some vertex j or larger (with respect to $>_i$) belongs to I_i then B and C are at least one column wide, and so the block form of the matrix gives a nontrivial factorization of Δ_{I_i}. This yields a contradiction as in Case 1: either W contains unnecessary propagators which contradicts our minimality assumption, or $\det A$ is quadratic which contradicts the assumption that p and i contribute to f, an irreducible factor of degree at least 3.

On the other hand, if no vertex $\geq_i j$ is in I_i then $\Delta_{I_i} = \det A$. Looking in more detail into A, note that the only vertices in the support of p and q which belong to I_i are i and $i+1$, and hence

$$A = \begin{bmatrix} D & 0 \\ E & F \end{bmatrix}$$

where D is the 2×2 matrix indexed by the propagators p and q and the vertices i and $i+1$. Thus p and i contribute to a quadratic factor of Δ_{I_i}, once again contradicting our assumptions.

All cases have now been covered and so Δ_{I_i} has only irreducible factors of degree 2 or less.

3. Suppose propagators p and q are supported on a common edge (a, b), with $I_i(p) = a$ and $I_i(q) = b$. Let $x_{p,a}, x_{p,b}, x_{q,a}, x_{q,b}$ be the associated variables in $C(W)$. For any fixed bijection σ from $\mathcal{P} - \{p, q\}$ to $I_i - \{a, b\}$ for which each propagator is supported on its image under the bijection, we can extend σ to a bijection of all propagators with I_i in two ways: either $p \mapsto a$ and $q \mapsto b$ or $p \mapsto b$ and $q \mapsto a$. The sum of the contributions of all these bijections to Δ_{I_i} is therefore the product of $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ with the minor coming from $\mathcal{P} - \{p, q\}$ and $I_i - \{a, b\}$. Since there is no cancellation of terms in the expansion of Δ_{I_i}, if any other terms appear then they will cause a factor which is not in the form described in the previous part. Therefore no such terms exist and $x_{p,a}x_{q,b} - x_{p,b}x_{q,a}$ is a factor of Δ_{I_i}.

   Now let f be a quadratic factor of Δ_{I_i}. By part (2) we know that f is a 2×2 minor coming from two propagators, call them p and q, and two vertices, call them $a <_i b$. It remains to show that a and b are adjacent. From this we can conclude that p and q each have one end on (a, b), as any other way for both p and q to be supported on two consecutive vertices would contradict noncrossing or the density requirement of admissibility.

   As in the proof of part (2), make a new admissible diagram by removing the propagators which come before f and set $i = a$. The cases in the proof of part (2) show how Δ_{I_i} factors: in particular the vertices supporting the other end of p either do not appear in I_i, or they contribute to a different factor of Δ_{I_i} than p and a do. By assumption b contributes to the same factor as a. Therefore (a, b) is an edge.

4. Consider $p \in S_i$, and note that Δ_{I_i} is homogeneous linear in the variables of the row corresponding to p. By part (2), either exactly one variable in the row corresponding to p appears in Δ_{I_i} and this variable is a factor of Δ_{I_i}, or exactly two variables from the row corresponding

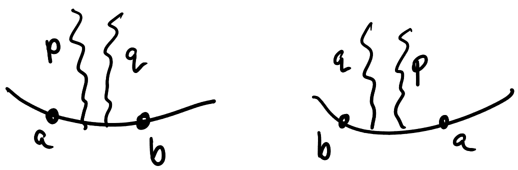


Figure 12: The situations giving a quadratic factor with variables appearing in r_i.

to p appear in Δ_{I_i} and they appear as part of a quadratic factor. In the first case let the variable be x. Then x is a factor of Δ_{I_i} and so in particular the monomial in Δ_{I_i} corresponding to the bijection between propagators and vertices of I_i associates the column of x to p. Thus x also appears in r_i and since the matrix for r_i is a minor of the matrix for Δ_{I_i} and every term in Δ_{I_i} involves x, we also have that every term in r_i involves x so x is a factor of both r_i and Δ_{I_i} and is the only variable from this row in either polynomial.

Now suppose two variables from the row p appear in a quadratic factor f. By part (3), there is another propagator q and an edge (a, b) such that f is the 2×2 minor coming from p, q and a, b, with $I_i(p) = a$, $I_i(q) = b$. There are two situations which can occur, both illustrated in Figure 12; we show that in both cases it follows that $q \in S_i$ as well.

In both cases, since $I_{i-1}(p) \neq a$ by assumption it follows from Lemma 3.13 that $I_{i-1}(p) <_{i-1} a$ and no other vertex supporting p lies between $I_{i-1}(p)$ and a. In the case that $b <_i a$ and q is taken before p in I_i, this means that $I_{i-1}(p) = b$ and so $I_{i-1}(q) \neq b$. Thus $q \in S_i$ and so f is a factor of r_i.

Now consider the case where $a <_i b$, and suppose for contradiction that $q \notin S_i$, i.e. that $I_{i-1}(q) = b$. Since $I_{i-1}(p) \neq a$, there must be some other propagator s with $I_{i-1}(s) = a$ (else I_{i-1} assigns q to a). This propagator cannot lie on edge (a, b) since by Lemma 3.13 we must have $I_i(s) = a$ or b, contradicting the fact that $I_i(p) = a$ and $I_i(q) = b$; thus s has an end on $(a - 1, a)$ and is inside p from the point of view of $i - 1$.

Say s is supported on $(j, j + 1, a - 1, a)$ and p is supported on $(k, k + 1, a, b)$ with $i - 1 \leq_{i-1} k + 1 \leq_{i-1} j + 1$. But by Lemma 3.11, if $I_{i-1}(s) = a$ then a cannot be maximal in the support of s with respect to $<_{i-1}$; thus we must have $i - 1 = j + 1$, and we are in the situation in figure 13.

Since p changed its association from I_{i-1} to I_i, we have $I_{i-1}(p) = i - 1$ by Lemma 3.13. From figure 13 it follows that I_{i-1} assigns p to $i - 1$ and then proceeds identically to I_i for all vertices inside p, implying that $I_{i-1}(s) = I_i(s)$. Since $I_{i-1}(s) = a$ and $I_i(s) \neq a$, this is a contradiction.

Thus $q \in S_i$ after all, and so f is a factor of r_i as required.

5. If W has zero propagators then all $I_i = \emptyset$ and both R and $\prod_{i=1}^n \Delta_{I_i}$ are equal to 1, so the result holds in this case. Now assume W has at least one propagator.

   First we show that every factor of $\prod_{i=1}^n \Delta_{I_i}$ divides R. Take an irreducible factor f of

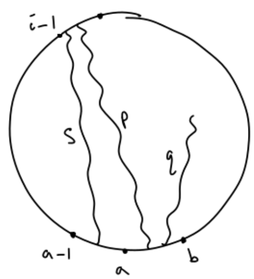


Figure 13: In order to obtain $I_{i-1}(s) = a$, propagators s and p must each have an end on the edge $(i-2, i-1)$.

$\prod_{i=1}^{n} \Delta_{I_i}$. There exists some i such that $f | \Delta_{I_i}$ but $f \nmid \Delta_{I_{i-1}}$, since otherwise the variables corresponding to the propagators contributing to f which do not themselves appear in f could never appear, contradicting Lemma 3.13. If f is a linear factor, say from associating propagator p to vertex a, then $I_i(p) = a$ and $I_{i-1}(p) \neq a$ so this factor appears in r_i. If f is a quadratic factor, say from associating propagators p and q to vertices a and b respectively, then again we cannot have both $I_{i-1}(p) = a$ and $I_{i-1}(q) = b$, else f divides Δ_{i-1}. However, by the proof of part (4), if one of p, q belongs to S_i then the other does as well. Thus f divides r_i.

Next we need to show that R is squarefree. Suppose $f^2 | R$. If f is a linear factor, say from associating propagator p to vertex a, then there must be two distinct points in the Grassmann necklace algorithm where p changes from not being associated to vertex a to being associated to vertex a. This contradicts Lemma 3.13. Now suppose f is a quadratic factor, say from propagators p and q supported on the edge (a, b) with p before q on the edge. In this case it is not possible for any I_i to associate p to b and q to a. Furthermore, we know by part (4) that p changes from not being associated to a to being associated to a if and only if q changes from not being associated to b to being associated to b. Thus $f^2 | R$ implies that twice in the Grassmann necklace p must change from not being associated to vertex a to being associated to vertex a. This is again a contradiction, and so R is squarefree.

Taking everything together we have that $R | \prod_{i=1}^{n} \Delta_{I_i}$, R contains all factors of $\prod_{i=1}^{n} \Delta_{I_i}$ and R is squarefree. Therefore the ideal generated by R is the radical of the ideal generated by $\prod_{i=1}^{n} \Delta_{I_i}$.

□

Theorem 4.4. *Given any admissible Wilson loop diagram W, let $\{I_1, \ldots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral, $R(W)$ (see Definition 1.11), is the R of Algorithm 4.1, which is also, up to scalar multiple, the radical of $\prod_{i=1}^{n} \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i.*

Proof. The equivalence up to scalar multiple of R and the radical of $\prod_{i=1}^{n} \Delta_{I_i}$ is due to Proposition 4.3. It remains to prove that $R(W)$ is the R of Algorithm 4.1.

Note that there is still a mention of scalar multiple in this theorem because it seems to me that the def of radical is only up to scalar multiple, but the

To this end, first note that $R(W)$ and R both have total degree $4|\mathcal{P}|$; the degree of $R(W)$ is immediate from the definition while that of R follows from Lemma 3.13. By Proposition 4.3 every factor of R is either a single variable or a quadratic factor coming from two propagators supported on a common edge. The factors of each R_e making up $R(W)$ in the notation of Definition 1.11 are all of this form and hence every factor of R divides $R(W)$. Finally, since R is squarefree, this implies that $R(W)$ is a scalar multiple of R.

Finally then we need to check the scalar. By Definition 1.11 each linear factor appears with coefficient 1 and each 2×2 deteriminant factor appears with the same sign as the determinant of the corresponding minor in $C(W)$. Therefore $R = R(W)$. □

explain why this result was interesting

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