This paper studies the combinatorics of Wilson Loop Diagrams.

1 Wilson Loop diagrams

What are Wilson loop diagrams and their integrals.

Definition 1.1. A Wilson loop diagram is given by the following data: a cyclicly ordered set V, and k pairs $\{p_r = (i_r, j_r)\}_{r=1}^k$ ordered such that $i_r + 1 < j_r$.

We depict this data as a convex polygon, with vertices labeled by V (preserving the cyclic ordering), and k wavy lines in the interior of the diagrams. The pair p_r defines a wavy line from the i_r^{th} edge (between i_r and $i_r + 1$) and the j_r^{th} edge. These are called propagators. The condition on i_r and j_r means that the propagator does not go between adjascent edges. Let $\mathcal{P} = \{p_r\}_{r=1}^k$ be the set of propagators. Then we write

$$W = (\mathcal{P}, V) \ .$$

Often we take V to be $[n]$, the cyclically ordered set of integers, $1 \ldots n$. In this case, we write $W = (\mathcal{P}, n)$. We introduce some notation to speak of vertices supporting a propagator, and the set of propagators supported on a vertex set.

Definition 1.2. Let $W = (\mathcal{P}, n)$.

1. For $p \in \mathcal{P}$, let $V_p = \{i_p, i_p+1, j_r, j_r+1\}$ be the set of vertices supporting p. Then, for $P \subset \mathcal{P}$, the set $V_P = \cup_{p \in P} V_p$ is the vertex support of P.
2. For $V \subset [n]$, write $\mathrm{Prop}(V) = \{p \in \mathcal{P} | V_p \cap V \neq \emptyset\}$.

Definition 1.3. A Wilson loop diagram is admissible if

1. $|V| \geq |\mathcal{P}| + 4$
2. There does not exists a set of propagators, $P \subset \mathcal{P}$ such that $|V(P)| < |P| + 3$.
3. There does not exist a pair of propagators, $p, q \subset \mathcal{P}$ such that $i_p < i_q < j_p < j_q$.

The first conditions states that there are at least four more vertices than propagators in an admissible Wilson Loop Diagram. The second imposes an upper bound on how densely the propagators can be fitted in the diagram. The third ensures that ensures that no propagators cross in the interior of the diagram. In other words, a Wilson loop diagram, $(\mathcal{P}, n)$ is admissible if and only if $n > \mathcal{P} + 4$, and has neither crossing propagators nor any pairs of propagators that start and end on the same pair of non-adjacent edges.

In what follows, we will talk about admissible Wilson Loop diagrams and subdiagrams thereof.

Definition 1.4. Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram. A subdiagram of W is defined by a a subset of propagators of W. For $P \subset \mathcal{P}$, write $W|_P = (P, V(P))$.

Note that a subdiagram of an admissible diagram need not be admissible. In particular, the condition $|V(P)| > |P| + 4$ need not hold.

There is one particular type subdiagram that deserves special attention.

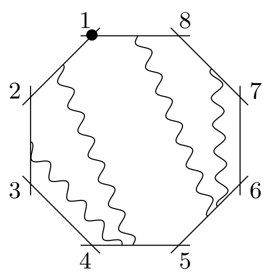
Definition 1.5. For W an admissible diagram, $(P, V(P))$ is exact if $|V(P)| = |P| + 3$.

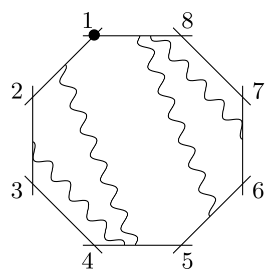
The exact subdiagrams define an equivalence relation amongst Wilson loop diagrams.

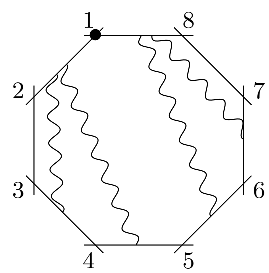
Definition 1.6. There is an equivalence relationship on the set of admissible Wilson loops diagrams: $W = (\mathcal{P}, n)$ and $W' = (\mathcal{P}', n)$ are equivalent if

1. There exist two different exact subdiagrams, $(P, V(P))$ and $(P', V(P'))$ of W and W' respectively such that $V(P) = V(P')$.
2. The complementary subdiagrams are identical: $(\mathcal{P} \setminus P, V(P)^c) = (\mathcal{P}' \setminus P', V(P')^c)$.

Example 1.7. Note that since this is an equivalence relation, we may find that two Wilson loop diagrams are equivalent, even if they do not have complements of (non-trivial) exact subdiagrams in common. Consider the following three Wilson loop diagrams,

$W_1 =$ 

$W_2 =$ 

$W_3 =$ 

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The diagrams $W_1 \equiv W_2$ because $(\{(5,8),(5,7),\{5,6,7,8,1\})$ and $(\{(5,8),(7,8),\{5,6,7,8,1\})$ are the corresponding differing subdiagrams. Furthermore, there is the equivalence $W_2 \equiv W_3$ due to the exact subdiagrams $(\{(1,4),(3,4)\},\{1,2,3,4,5\})$ and $(\{(1,4),(1,3)\},\{1,2,3,4,5\})$. This forces an equivalence between W_1 and W_3, even though one cannot partition the propagators of each into a an exact subdiagram (that may vary between the diagrams) and a complement that is fixed.

2 Wilson Loop diagrams as matroids

In [**?**], Amat and Agarwala show that admissible Wilson loop diagrams with k propagators corresponds to positroids of rank k (a matroid that can be represented by an element of $\mathbb{G}_{\mathbb{R},\geq 0}(k,n)$.)

Theorem 2.1. *[**?**]An admissible Wilson loop diagram, $W = (\mathcal{P}, n)$, defines a matroid with base set $[n]$. The independent sets are exactly those subsets $V \subseteq [n]$ such that $\nexists U \subseteq V$ satisfying $|Prop(U)| > |U|$.*

This does not, however, show that the matroids associated to each Wilson loop diagram is unique. In fact, in [?], the authors show that if two diagrams are equivalent then they define the same matroid [cite theorem here].

Next, we show that this is the only condition under which two Wilson loop diagrams define the same matroid.

Theorem 2.2. *Two Wilson loop diagrams define the same matroid if and only if the are equivalent.*

To this end, we recall some definitions and go through a series of new intermediary steps.

First we recall a few facts about matroids, with particulars about those defined by Wilson loop diagrams.

Definition 2.3. For a Wilson Loop diagram $W = (\mathcal{P}, [n])$, the associated matroid $M(W)$ can be defined by any of the following sets of subsets

1. A basis of W is any independent set, $B \subset [n]$ of size $|B| = |\mathcal{P}|$.

2. The rank of a set $|V| \subset [n]$, is bounded above by $\text{rk }(V) \leq \min\{|V|, |\text{Prop}(V)|\}$. If V is an independent set, then $\text{rk }(V) = |V|$. Furthermore, if $v \in [n]$, and $q \in \text{Prop}(v)$, then for any $V \subset [n]$ such that $q \notin \text{Prop}(V)$, $\text{rk }(V \cup v) = \text{rk }(V) + 1$. That is, adding a vertex that supports a new propagator to a set increases the rank of the set.

3. A circuit of a matroid is any set $C \subset [n]$ such that every proper subset $U \subsetneq C$ is independent. That is, $\text{rk }(C) = |C| - 1$. The union of circuits is called a cycle. If C is a circuit of an admissible Wilson loop diagram, and $e \in C$,

$$|\text{Prop}(C \setminus e))| \leq |C \setminus e| = \text{rk }(C) \leq \min\{|C|, |\text{Prop}(C \setminus e)|\} .$$

   In other words, if C is a circuit of a Wilson loop diagram, then

$$\text{rk }(C) = |C| - 1 = |\text{Prop}(C)| .$$

4. A flat, F, of a matroid is a maximally dependent set. That is, for all $e \notin F$ $\text{rk }(F \cup e) = \text{rk }(F) + 1$. If W is an admissible Wilson loop diagram, any group of propagators defines a flat, $F(P) = V(P^c)^c$, called a propagator flat. It consists of all vertices of W that do not support any propagators outside of $P \subset \mathcal{P}$. To see that this is flat, let $v \in V(P^c)$. We have that there is a $q \notin P$ such that $q \in \text{Prop}(v)$. Therefore, by the rank function, $\text{rk }(F(P) \cup v)) > \text{rk }(F(P))$.

Furthermore, in [?], the authors show that if F is a cyclic flat, then there is a $P \subset \mathcal{P}$ such that $F = F(P)$.

It is worth noting that it is not necessary to list all the flats in a matroid to defined it completely, only the cyclic flats, and those that are independent sets. Namely, if F is a dependent flat, it contains a circuit. Let $C \subset F$ be the largest cycle contained in F. Then two things are true.

Lemma 2.4. *Let W be a Wilson loop diagram, and F is a flat. Let $C \subset F$ be the largest cycle contained in F. Then the following is true:*

1. *$C = F(Prop(C))$ is a cyclic flat.*
2. *$F \setminus C$ is an independent set. If W does not contain any vertices of rank 0, that is, all vertices support at least 1 propagator, then $F \setminus C$ is an independent flat.*

Proof. If F is an independent flat, then $C = \emptyset$ and the statement is trivially true.

If F is a dependent set, F contains a circuit.

To see the first point, note that rk $(C) = |\text{Prop}(C)| \geq F(\text{Prop}(C)|$. For any $v \in C$ we have that $\text{Prop}(v) \subset \text{Prop}(C)$. This implies that $v \in F(\text{Prop}(C))$. Therefore $C \subset F(\text{Prop}(C))$. This implies that $|\text{Prop}(C)| = F(\text{Prop}(C)|$. Suppose there is a $v \in F(\text{Prop}(C)) \setminus C$. Let B be an independent subset of C of maximal rank. The set $B \cup v$ is dependent and therefore contains a circuit. Therefore, $C \cup b$ is a cycle, which is a contradiction to C being the maximal cycle in F.

To see the second point, note that $F \setminus C$ is independent (as it contains no circuits). Furthermore, it is a flat. For any $e \notin F$, rk $(F \setminus C \cup e) =$ rk $(F \setminus C) + 1$, since F is a flat. For any $e \in C$, if rk $(F \setminus C \cup e) =$ rk $(F \setminus C)$, the set $F \setminus C \cup e$ is a dependent set. This occures either if rk $e = 0$, or $F \setminus C \cup e$ contains a circuit of at least 2 elements, which contradicts the hypothesis that C is the largest cycle in F. □

Therefore, in the sequel, one only needs worry about cyclic and independent flats.

Corollary 2.5. *If F is a flat of a Wilson loop diagram, it can be written as the union of a cyclic flat, and thus a propagator flat, and an independent flat.*

From this we see that any propagator flat can be written as a union of a cyclic propagator flat and an independent flat.

Given any matroid, one may restrict it to a subset of the base set. The bases of the restricted matroid come from intersecting bases of the original with the subset. It is worth noting that a the matroid defined by a subdiagram is different from the restriction of the matroid of a Wilson loop diagram to a set of vertices.

Definition 2.6. For $W = (\mathcal{P}, n)$, the restricted diagram, $W|_V$ is the matroid defined by only looking at the vertices $V \subset [n]$.

The key difference between a subdiagram and a restriction is that the propagator support function does not change in the case of restriction, while it may in the case of a subdiagram. In particular, for $v \in V$, $\text{Prop}_W(v) = \text{Prop}_{W|_V}(v)$, while $\text{Prop}_{(P,V(P))}(v) = \text{Prop}_W(v) \cap P$.

In the sequel, I conflate a Wilson loop diagram with the matroid it defines.

2.1 Exact subdiagrams and matroids

It has been proved that if two Wilson loop diagrams are equivalent, then the define the same matroid in "Matroids and Wilson Loop Diagrams". In this section, we prove the converse. Therefore, we prove several results about exact subdiagrams.

Theorem 2.7. *Let $(P, V(P))$ be a subdiagram of $W = (\mathcal{P}, [n])$. This is an exact subdiagram if and only if it defines a uniform matroid of rank $|P|$.*

Proof. A uniform matroid of rank r is one in which any set of size $\leq r$ is independent. A circuit of rank m is a set C of rank m and size $m+1$ such that every proper subset of C is independent. I.e., *a circuit is a minimal dependent set.* In other words, a matroid of rank r is uniform if and only if all circuits have rank r. If C is a circuit, then $|C| = r + 1$.

We prove the following claim: The set $V(P)$ contains no circuits, C, such that rk $C < |P|$ in $(P, V(P))$. Since the $V(P) = |P| + 3$, rk $V(P) \leq |P|$. This implies that all circuits C of $(P, V(P))$ are of maximal rank, as desired.

Consider a subdiagram $(P, V(P))$ of W. Suppose there is a circuit, $C \subset V(P)$ such that rk $C = |\text{Prop}(C)| = m < |P|$. The size of the complement of C is $|V(P) \setminus C| = |V(P)| - (m+1)$. However, since C does not contain any propagators outside of Prop(C), the set $V(P \setminus \text{Prop}(C)) \subset V(P) \setminus C$. In otherwords, there are $|P| - m$ propagators supported on (at most) $|V(P)| - m - 1$ vertices:

$$|V(P \setminus \text{Prop}(C))| \leq |V(P)| - m - 1 \ .$$

Since W is an admissible Wilson loop diagram, the subdiagram $(P, V(P))$ satisfies $|R| + 3 \leq |V(R)|$ for all $R \subset P$. This gives

$$|P| - m + 3 \leq V(P \setminus \text{Prop}(C)) \ .$$

These two inequalities are compatible if and only if

$$|P| + 3 < |V(P)| \ .$$

In other words, the matroid associated to thet subdiagram $(P, V(P))$ is not uniform if and only if $(P, V(P))$ is not exact. $\square$

In [?], the authors show that all exact Wislon loop diagrams correspond to positroids. That is, they correspond to matroids that can be represented by elements of the positive Grassmannians $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, |V|)$.

Definition 2.8. Given a Wilson loop diagram W, define the positroid cell associated to a Wislon loop diagram, $\Sigma(W)$, to be the cell in the CW complex on $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, |V|)$ defined by the realizations of W that lie in $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, |V|)$.

With this definition in mind, we have the following corrolary:

Corollary 2.9. *Let $(P, V(P))$ be an exact subdiagram of W. The matroid associated to this subdiagram corresponds to the top dimensional cell in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$.*

Proof. The unique top dimensional cell of $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ is defined by all points in $\mathbb{G}_{\mathbb{R},\geq 0}(|P|, |V(P)|)$ such that all Plucker coordinates are strictly greater than 0. Since $(P, V(P))$ is an exact subdiagram, this all $|P| \times |P|$ minors are non-zero. Intersecting these with the cases with the positive Grassmannians demands that all minors be strictly positive. $\square$

To avoid the issue of exact diagrams being subdiagrams of other exact subdiagrams (for instance, any subdiagram (q, V_q), for $q \in \mathcal{P}$ is exact), I introduce a definition.

Definition 2.10. We say $(P, V(P))$ is a maximal exact subdiagram of W if there is not other exact subdiagram $(Q, V(Q))$ containing it.

Since the matroids defined by the subdiagram $(P, V(P)$ is not the same as the matroid associated to $W_{V(P)}$, one cannot say that the $W_{V(P)}$ is a uniform matroid. Infact, for any subset $U \subset V(P)$, $\text{rk}\ _{W_{V(P)}}(U) \geq \text{rk}\ _{(P,V(P)}(U)$. This is because adding propagators to vertices can only increase the rank. In particular this implies that, for any subset $U \subset V(P)$, if $|U| \leq |P|$, then U is independent in both the restricted matroid $W_{V(P)}$, and the original, W.

In what follows, maximal exact subdiagrams play a significant role. First, we show that maximal exact subdiagrams of an admissible Wilson loop diagram are disjoint.

Lemma 2.11. *Let $W = (\mathcal{P}, n)$ be an admissible Wilson loop diagram, and $(Q, V(Q))$, $(R, V(R))$ two maximal exact subdiagrams. Then $Q \cap R = \emptyset$.*

Proof. By construction, if $T \subset Q$, then $(T, V(T))$ would be exact. Furthermore, for $p \in Q \setminus T$, $|V(T \cup p)| = |T| + 1 + 3 = |V(T)| + 1$. That is, by extending an exact subdiagram to a larger one, one extends the vertex set by exactly on vertex per proppagator. Therefore, if $T \neq \emptyset = Q \cap R$,

$$|V(Q \cup R)| = |T| + 3 + |Q \setminus T| + |R \setminus T| = |Q \cup R| + 3\ .$$

Therefore, if $(Q, V(Q))$ and $(R, V(R))$ are two exact subdiagrams such that $Q \cap R \neq \emptyset$, then $(Q \cup R, V(Q \cup R))$ is exact. □

Lemma 2.12. *Let $W = (\mathcal{P}, [n])$ be a Wilson loop diagram, and $(P, V(P))$ an exact subdiagram with $P \subsetneq \mathcal{P}$. Then the set $V(P)^c = F(P^c)$ is a propagator flat of maximal rank*

$$rk\ F(P^c) = |P^c|\ .$$

Proof. By construction, $V(P)^c$ is a propagator flat. Furthermore, it is not empty. In particular, since $(P, V(P))$ is an exact subdiagram, the vertices $V(P)$ cannot support a larger set of propagators (otherwise W would no longer be admissible). Therefore, since $P \subsetneq \mathcal{P}$, $V(P)^c = F(P^c)$ is not an empty set. It remains to check that this has maximal rank.

Notice that $n = |V(P)| + |F(P^c)| \geq |P| + |P^c| + 4$, and $|V(P)| = |P| + 3$. Therefore

$$|F(P^c)| > |P^c|\ .$$

In other words, $F(P^c)$ is a dependent flat. By corollary 2.5 $F(P^c)$ is the union of a cyclic flat and an independent flat. We claim that it is a cyclic flat. Therefore, rk $(F(P^c)) = |P^c|$.

Suppose $F(P^c) = C \cup F$, where C is the largest cyclic flat contained in $F(P^c)$ and F is a non-empty independent set. Since $C = F(\text{Prop}(C))$, we can write $V(Q)^c = F(\text{Prop}(C))$, with

$$Q = (\text{Prop}(C))^c\ . \tag{1}$$

Furthermore,

$$\text{rk }(F) = |F| \leq \text{rk }(F(P^c)) - \text{rk }(C) \leq |P^c| - \text{rk }(C)\ . \tag{2}$$

The first equality comes from the fact that F is an independent set. Combining (1) and (2) gives

$$|P| + |F| \leq |P| + |P^c| - \text{rk }(C) = |Q|\ . \tag{3}$$

Furthermore, since $V(Q) = F \cup V(P)$, we may write

$$|V(Q)| = |F| + |V(P)| = |F| + |P| + 3\ . \tag{4}$$

Combining this with (3) gives $|V(Q)| - 3 \leq |Q|$. That is, either W was not admissible, or $(P, V(P))$ was not a maximal exact subdiagram. In otherwords, F cannot be non-empty, proving that $F(P^c)$ is a cyclic flat, and therefore of maximal rank. □

While one still cannot equate a subdiagram with a restriction, this lemma allows one to write a maximal exact subdiagram as a contraction. First, recall a result from [**?**] about contracted diagrams and matroids.

Lemma 2.13. *For $W = (\mathcal{P}, [n])$, write the contracted diagram $W/P = (P^c, V(P)^c)$. If rk $F(P) = |P|$, we may identify*

$$M(W/P) = M(W)/F(P)\ .$$

In other words, a maximally exact subdiagram $(P, V(P))$ can be written as a contraction by the complementary propagator flat:

$$(P, V(P)) = W/F(P^c)\ .$$

Now I am ready to prove the main theorem of this section.

Theorem 2.14. *Let $W = (\mathcal{P}, [n])$ and $W' = (\mathcal{P}', [n])$ be two Wilson Loop diagrams. They define the same matroid if and only if $W \sim W'$.*

Proof. One direction has been proved in previous work, but I give a different proof here to be consistent with the method of this document.

Assume that W and W' are equivalent. Without loss of generality, write $W = (P \cup R, [n])$ and $W' = (P \cup R', [n])$, where $P \subset \mathcal{P} \cap \mathcal{P}'$ and $(R, V(R))$ and $(R', V(R'))$ are two maximally exact subdiagrams, with $R \neq R'$, but $V(R) = V(R')$. If this is not the case, one may always find a family of diagram, $\{W_i\}$ satisfying this condition and forming a transitive chain connecting W to W' in the equivalence class.

Let $U \subset V(R)$ be any subset of size $|U| = |R|$. The set U is independent in the subdiagram $(R, V(R))$, and thus in W. The complementary set $F(P)$ is a flat of maximal rank by lemma 2.12 (rk $(F(P)) = |P|$). Let $B \subset F(P)$ be a maximal indpendent set (rk $B = |P|$) in $F(P)$. Since $F(P)$ is a flat, adding any element of $V(R)$ to B increases the rank. Therefore, any basis of W can be written as $B \cup U$. However, since $F(P)$ is common to both W and W', and $V(R) = V(R')$, any any basis of W' can also be written $B \cup U$. Thus both matroids have the same bases sets, proving that they are the same.

For the converse, assume that the matroids associated to W and W' are the same: $M(W) = M(W') = M$. Let $\{(P_i, V(P_i)\}_{i=1}^k$ and $\{(P_i', V(P_i')\}_{i=1}^l$ be the sets of maximally exact subdiagrams of W and W'. Write $F_i = F(P_i)$ and $F_i' = F(P_i'^c)$ to be the complementary cyclic flats. By Lemma **??** M/F_i is a uniform matroid. Thereofore, $k = l$, and we may write $V(P_i) = V(P_i')$.

Reorganize the vertex sets as follows:

$$\cup_{P_i \neq P_i'} V(P_i) = V(\cup_{P_i \neq P_i'} P_i) \ . \tag{5}$$

Since maximal exact subdiagrams are disjoint, and single propapagators may define maximal subdiagrams, $\cup_{i=1}^k P_i = \cup_{i=1}^k P_i' = \mathcal{P}$. Then equation (5) becomes

$$\cap_{P_i \neq P_i'} (F(P_i^c))^c = F(\cup_{P_j = P_j'} P_j)^c \ .$$

These flats may, of course, be empty. Is this paragraph necessary?

Define a family of Wilson loop diagrams, W_0 to W_k defined such that $W_0 = W$ and W_i is derived from W_{i-1} by replacing the propagator set P_i with P_i'. In this manner, $W' = W_k$ and $W_i \sim W_{i+1}$, making $W \sim W'$. □

2.2 Counting exact subdiagrams

Since equivalent Wislon loop diagrams are the only ones that define the same matroid as each other, it is useful to enumerate the number of equivalent diagrams any Wilson loop diagram has.

Definition 2.15. Let W be an admissible Wilson loop diagram. The *polygon partition* associated to W, denotes $\tau(W)$ is defined as follows.

- The vertices of $\tau(W)$ correspond to the edges of W.
- The polygon of $\tau(W)$ is the graph theoretic cycle made of the vertices of $\tau(W)$ and where two vertices are adjacent if the corresponding edges of W meet at a vertex of W.
- There is a chord edge of $\tau(W)$ for each propagator of W. Given a propagator fo W, this edge of $\tau(W)$ joins the two vertices of $\tau(W)$ corresponding to the edges of W where the two ends of the propagator lie.

For an admissible W no propagators cross and so no chord edges of $\tau(W)$ cross. Thus, in graph theoretic language, we can view $\tau(W)$ as a planar embedding of a graph with no cut vertices, where all vertices lie on the infinite face, along with a distinguished start vertex and direction around the infinite face. From this viewpoint the polygon of $tau(W)$ is the boundary of the infinite face.

draw some examples

A planar embedding of a graph is a *triangulation* if all faces, except possibly the infinite, face are triangles.

Definition 2.16. Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. A *triangulated piece* of $\tau(W)$ is a 2-connected subgraph of $\tau(W)$ which is a triangulation and which is maximal with respect to that property. We will take the convention that a subgraph consisting of two vertices joined by an edge which is in no triangle is also a triangulated piece, called a *trivial* triangulated piece.

A decomposition of a polygon partition $\tau(W)$ is a set of 2-connected induced subgraphs of $\tau(W)$ which partition the edges of $\tau(W)$.

Lemma 2.17. *Every polygon partition $\tau(W)$ has a unique decomposition into triangulated pieces.*

draw an example

Proof. First consider the weaker notion of decompositions of $\tau(W)$ into 2-connected pieces which are either triangulations (but not necessarily maximal) or subgraphs consisting of two vertices joined at an edge. At least one such decomposition must exist as we can simply take each edge as defining it's own subgraph.

Now suppose we have two distinct such decompositions. Since they are distinct there must be some edge e of $\tau(W)$ for which the subgraph containing this edge are differs between the decompositions. Let T_1 and T_2 be these two subgraphs. The graph $T_1 \cup T_2$ defined by taking the union of the vertex sets of T_1 and T_2 along with the union of the edge sets of T_1 and T_2 is still a subgraph of $\tau(W)$. Furthermore, since all vertices of $\tau(W)$ lie on the external face, and T_1 and T_2 are 2-connected, then T_1 and T_2 can share at most two vertices and these must be on the external faces of T_1 and of T_2. Therefore $T_1 \cup T_2$ is the result of gluing two triangulations along edges on each of their external faces and hence is also a triangulation. Combining all subgraphs which share edges we obtain a decomposition containing both original decompositions.

Therefore the set of such decompositions of $\tau(W)$ is not just a poset under containment but is a finite directed set and thus has a unique maximal element.

Also, the subgraphs making up this maximal element are themselves maximal as if not then taking the union with the decomposition formed from a single triangulation which properly contains one of the triangulations of the maximal element along with the remaining edges as individual subgraphs, would give a decompositions containing the purported maximal one. Furthermore, no non-maximal decomposition can be a decomposition into maximal triangulations because since it is properly contained in the maximal element some subgraph of this decomposition must be properly contained in some subgraph of the maximal decomposisiton and hence not be a maximal triangulation.

Thus the maximal decomposition is the desired unique decomposition into (maximal by definition) triangulated pieces. $\square$

That previous proof is rather belaboured, so it will probably want to be cut down, but I'm not sure which parts will be obvious to our audience and which will not be.

Lemma 2.18. *Let W be an admissible Wilson loop diagram and $\tau(W)$ its polygon partition. The nontrivial triangulated pieces of $\tau(W)$ correspond to the maximal connected exact subdiagrams of W.*

Proof. Let us first record a few standard facts about polygon triangulations (that is about triangulations with all vertices on the outer face). If such a triangulation has n vertices then it has n edges on the polygon (that is on the outer face) and $n-3$ edges which are not. No planar graph with the same vertices and the same outer face can have more edges than the triangulation and every such simple graph with $n-3$ edges off the outer face is a triangulation.

Since W is admissible, $\tau(W)$ has no double edges and so is a simple graph. Let t be a nontrivial triangulated piece of $\tau(W)$ and suppose t has n vertices. We want to count how many edges of t are not on the outer face of $\tau(W)$. Consider the intersection of t with the outer face of $\tau(W)$. This is a subgraph with n vertices of the polygon of $\tau(W)$, call it S. Let k be the number of connected components of S. Also S is a spanning subgraph of the outer face of t and S has n edges. To join the components S must have k edges not in the outer face of $\tau(W)$. Furthermore t has $n-3$ edges not in its outer face and so also not in the outer face of $\tau(W)$. Thus there are $n-3+k$ edges of t not in the outer face of $\tau(W)$.

These edges correspond to propagators in W, so the open subdiagram of W corresponding to $\tau(W)$ has $n-3+k$ propagators. Next let's count the vertices of this subdiagram. Each of the n vertices in the outer face of t corresponds to an edge of the open subdiagram and the edges of this open subdiagram form k connected compnents of the outer polygon. Thus the open subdiagram has $n+k$ vertices. So we see that the open subdiagram corresponding to t has exactly three more vertices than propagators. Propagators can never be supported at higher density than this by admissibility and so the support of the propagators of the subdiagram is all the vertices of the subdiagram and hence the subdiagram is exact. We have not used the maximality of the triangulation in the above and so any triangulated subgraph corresponds to an exact subdiagram.

Conversely if we have an exact subdiagram of W with n edges forming k connected components of the outer polygon then the corresponding subgraph of $\tau(W)$ has n vertices and $n-k+(n-3+k) = 2n-3$ edges, of which n are in the outer face and hence is a triangulation.

The statement of the lemma then follows form the fact that inclusion is preserved under τ and so maximality also corresponds under τ. □

Definition 2.19. Say two admissible Wilson loop diagrams W_1 and W_2 are *triangulation-equivalent* if there is a bijection α between the set of triangulated pieces in the decomposition of $\tau(W_1)$ and the set of triangulated pieces in the decomposition of $\tau(W_2)$, where t and $\alpha(t)$ have the same vertex set for all triangulated pieces t of $\tau(W_1)$.

draw an example

Theorem 2.20. *Two admissible Wilson loop diagrams have the same matroid iff they are triangulation equivalent*

Again do I want matroid or positroid in this theorem?

Proof. By Lemma 2.18 two admissible Wilson loop diagrams are triangulation equivalent iff their decompositions into maximal connected exact subdiagrams have matching supports, that is if they are equivalent. Theorem **??** then gives the result. □

It is a classical fact the the number of triangulations of an n-gon is the $n-2$ Catalan number, namely $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. Thus we can count the number of equivalent diagrams.

Corollary 2.21. *Let W be an admissible Wilson loop diagram where the sizes of the supports of the maximal connected exact subdiagrams are $n_1, n_2, \ldots, n_j$. Then the number of admissible Wilson loop diagrams equivalent to W (including W itself) is*

$$\prod_{i=1}^{j} \frac{1}{n_i - 1}\binom{2(n_i-2)}{n_i-2}$$

3 Geometry of Wilson Loop diagram

Since Wilson loop diagrams correspond to positroids, it is natural to understand the subspace of $\mathbb{G}_{\mathbb{R},\geq 0}(|\mathcal{P}|, n)$ they define.

Talk about Grassmann Necklaces and how it defines a cell in a CW complex of $\mathbb{G}_{\mathbb{R},\geq 0}(k, n)$. Also talk about how this is exactly all the non-negative matrices that represent a particular positroid.

Also talk about Le diagrams.

3.1 From Wilson Loop diagrams to Grassmann Necklaces

Here, we give an algorithm for passing from Wilson loop diagrams to Grassmann Necklaces.

Let $[\![1, n]\!]$ denote the set of integers $\{1, 2, \ldots, n\}$, and $\binom{[n]}{k}$ the set of all k-subsets of $[\![1, n]\!]$. For each $j \in [\![1, n]\!]$, we can define a total order $\leq_j$ on the interval $[\![1, n]\!]$ by

$$j <_j j+1 <_j \cdots <_j n <_j 1 \cdots <_j j-1.$$

This in turn induces a total order on $\binom{[n]}{k}$, namely the lexicographic order with respect to $<_j$. It also induces a separate partial order $\preccurlyeq_j$ on $\binom{[n]}{k}$ (the *Gale order*), which is defined as follows: if $A = [a_1 <_j a_2 <_j \cdots <_j a_k]$, $B = [b_1 <_j b_2 <_j \cdots <_j b_k] \in \binom{[n]}{k}$, then

$$A \preccurlyeq_j B \text{ if and only if } a_r \leq_j b_r \text{ for all } 1 \leq r \leq k.$$

For example, in $\binom{[6]}{3}$ we have $[2, 5, 6] \preccurlyeq_2 [2, 6, 1]$ but $[2, 5, 6] \not\preccurlyeq_2 [3, 4, 6]$.

Definition 3.1. A Grassmann necklace of type (k, n) is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of n elements $I_i \in \binom{[n]}{k}$, such that

- if $i \in I_i$, then $I_{i+1} = \big(I_i \backslash \{i\}\big) \cup \{j\}$ for some $j \in [\![1, n]\!]$.
- if $i \notin I_i$, then $I_{i+1} = I_i$.

By [ref], the Grassmann necklaces of type (k, n) are in 1-1 correspondence with the positroid cells in $Gr(k,n)^{tnn}$. Further, if $\mathcal{I}$ is the Grassmann necklace associated to a cell $\mathcal{P}$, then the bases of $\mathcal{P}$ can be computed using the Gale order $\preccurlyeq_i$ for each $i \in [\![1, n]\!]$:

$$\mathcal{B}(\mathcal{P}) = \left\{ J \in \binom{[n]}{k} \ : \ I_i \preccurlyeq_i J \ \forall i \in [\![1, n]\!] \right\}.$$

We now describe an algorithm that, when applied to an admissible Wilson loop diagram, produces exactly the Grassmann necklace of the corresponding positroid.

Algorithm 3.2. *Let W be a Wilson loop diagram with n vertices. Note that whenever the algorithm below refers to the "rightmost" propagator supported on a vertex i, we mean the rightmost from the point of view of standing in the middle of the diagram and looking directly at i.*

1. *Fix a vertex $s \in [\![1, n]\!]$, and set $i := s$.*
   - (a) ***Step** i **for vertex** s: If there is at least one propagator in the current diagram supported on i, add the label i to the edge $(s, s+1)$ and delete the rightmost propagator supported on i; otherwise do nothing.*
   - (b) *If there are no propagators left in the diagram at this point, go to (2) below.*
   - (c) *Otherwise, increment i by 1 and repeat from (a).*
2. *Replace all propagators in the diagram, and repeat from (1) for a new vertex s. Continue until the algorithm has been applied to all vertices.*

Proposition 3.3. *If W is an admissible Wilson loop diagram, then Algorithm 3.2 puts exactly k distinct labels on each edge.*

Proof. It is enough to show that the algorithm terminates in at most n steps for each vertex. Suppose this doesn't happen, i.e. there is a propagator P which survives the first n steps of the algorithm for some starting vertex s. We will show that this always gives a contradiction in an admissible diagram by constructing an inductive sequence of propagators on strictly decreasing regions of the diagram.

Rotating and cyclically renumbering if necessary, we can assume that $s = 1$ and that P is supported on $(a, a+1, b, b+1)$ with $1 \leq a < b$. Since P is not removed at step $b+1$, there is another propagator p_1 to its right which is also supported on $b+1$: see Figure 1.

The propagator p_1 forms the base step for the following induction:

Suppose that p_r is a propagator supported on $(i, i+1, m, m+1)$ and removed at step $m+1$ of the algorithm; we will show that there must also be a propagator p_{r+1} supported on $(i', i'+1, m', m'+1)$ with $i \leq i'$ and $m \geq m'$, which is removed at step $m'+1$. Since we can't have both $i' = i$ and $m' = m$, p_{r+1} bounds a region containing strictly fewer vertices than p_r.

Notice that if we restrict our attention to the region bounded by p_r and the edges of the diagram from i to $m+1$, then propagators outside of this region can affect the algorithm only at vertices i, $i+1$. (The starting vertex 1 lies outside this region, by assumption.)

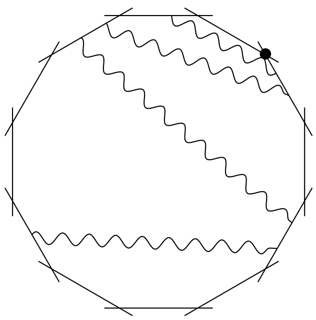

Figure 1: Base step: if P survives n steps of the algorithm, there must be another propagator p_1 to its right on edge $(b, b+1)$.

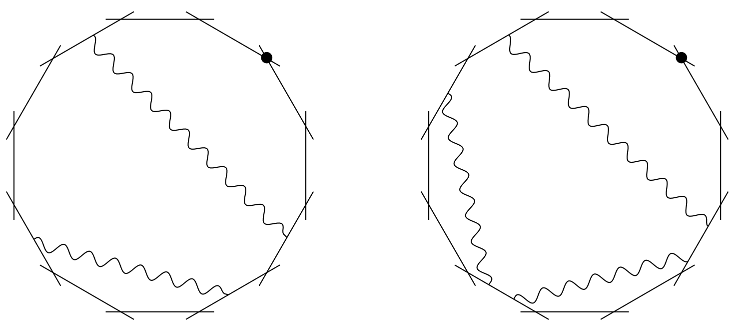

Figure 2: Inductive step: if propagator q is removed at step m, then it lies on either edge $(m-1, m)$ (Case I) or $(m, m+1)$ (Case II).

Given a propagator p_r supported on $(i, i+1, m, m+1)$ and removed at step $m+1$, we must also have a propagator q to its right which is removed at step m ("protecting" p_r at step m). This splits into two cases:

Case I: If q is supported on $(k, k+1, m-1, m)$ for some $k \geq i$, then $p_{r+1} := q$ satisfies the conditions of the induction step.

Case II: Suppose q is supported on $(k, k+1, m, m+1)$ for some k; since the diagram is admissible, we have $k > i$. In particular, anything that happens at vertex $k+1$ is unaffected by propagators outside the region bounded by p_r, since $k+1 > i+1$. But q must survive until step m, so there must be another propagator q' to the right of q, which protects q at step $k+1$. The admissibility of W implies that q' is supported on $(j, j+1, k, k+1)$ for some $j \geq i$, and we can take $p_{r+1} := q'$.

By induction, we will eventually find a propagator on a region with 3 or fewer vertices, which is inadmissible. $\square$

To prove that this algorithm gives the Grassmann necklace of the diagram we need a lemma.

Lemma 3.4. *Given a WLD and a propagator p, each covered vertex of p is contributed to a non-empty cyclic interval in the Graßmann necklace.*

this lemma should be rephrased to be more in your language, it was just copied from the rank section, also it needs a cite or a proof

Proposition 3.5. *The sequence of k-subsets obtained by applying Algorithm 3.2 to an admissible diagram W is exactly the Grassmann necklace of the positroid associated to W.*

I think the statement should be clearer about what this sequence is, in the way that the first line of the proof currently is.

Proof. Let I_i be the set of labels on edge $(i, i+1)$; the sequence $\mathcal{I} = (I_1, \ldots, I_n)$ is a Grassmann necklace if and only if $I_{i+1} \supseteq I_i \backslash \{i\}$ for all $i \in \{1, \ldots, n\}$.

Suppose for a contradition that there exists an admissible diagram for which there exists an i with $k \in I_{i\backslash\{i\}}$ and $k \notin I_{i+1}$. Fix n. Let the triple (W, i, k) be such a counterexample on n vertices which is minimal with respect to the number of propagators.

If $i \notin I_i$, then there are no propagators supported on i at all. In this case it is clear that applying Algorithm 3.2 at vertex i and vertex $i+1$ produces exactly the same result, i.e. $I_{i+1} = I_i$, and so (W, i, k) is not a counterexample at all.

Now suppose that $i \in I_i$. Let p be the propagator which contributes i to I_i. Either p has one end on the edge $(i-1, i)$ or it has one end on the edge $(i, i+1)$. In both cases let $(b, b+1)$ be the edge with the other end of p.

case I: Suppose p has one end on $(i-1, i)$. Then p is not supported on $i+1$, so in building I_{i+1} we will take the same propagators as in the construction of I_i from vertices $i+1$ up to $b-1$, that is $I_{i+1} \cap [\![i+1, b-1]\!] = I_i \cap [\![i+1, b-1]\!]$. Furthermore, by Lemma 3.4, in building I_{i+1}, it must be that p is taken at vertex b, as otherwise b would never be contributed by p. Consequently, in building I_{i+1}, when at vertex b no propagator still remaining is before p. This is also true in building I_i when at b since the same propagators have been taken beforehand. Additionally $k \geq_i b+1$.

Let W' be the diagram obtained from W by removing p and all propagators under it in the sense of supported between $i+1$ and $b-1$.

By the above observations when we are in W' at b then we are in the same situation with respect to the remaining propagators as if we began at i in W and moved to b following the algorithm; the propagators we took in the latter case are exactly the ones removed to build W'. Similarly starting at $i+1$ in W and moving to $b+1$ leaves us in the same situations with respect to the remaining propagators as beginning at $b+1$ in W'. This gives the equations

$$I_i^W \cap [\![b, i-1]\!] = I_b^{W'}$$
$$I_{i+1}^W \cap [\![b+1, i-1]\!] = I_{b+1}^{W'}$$

where the diagram is indicated in the superscript. Thus we have $k \in I_b^{W'} \backslash \{b\}$ and $k \notin I_{b+1}^{W'}$ contradicting the minimality of (W, i, k).

case II: Suppose p has one end on $(i, i+1)$. Note that by definition p is the propagator contributing i to I_i and so it must be the first propagator in $(i, i+1)$ and hence p contributes $i+1$ to I_{i+1}. Observe that $k \geq_i i+2$ since $i+1 \in I_{i+1}$.

Let W' be the diagram obtained from W just by removing p. Then

$$I_i^W \backslash \{i\} = I_{i+1}^{W'}$$
$$I_{i+1}^W \backslash \{i+1\} = I_{i+2}^{W'}$$

since in both cases in W' we are simply taking the remaining propagators in the same way as we would have in W after taking p for the previous vertex. Since $k \neq i+1$, we have $k \in I_{i+1}^{W'} \backslash \{i+1\}$ but $k \notin I_{i+2}^{W'}$ contradicting the minimiality of (W, i, k)

We have shown that $\mathcal{I}$ is a Grassmann necklace; it remains to check that this Grassmann necklace defines the positroid $\mathcal{P}_W$ associated to W. We need to show that:

I tried to make the first part of this proof a bit tidier by using contradiction. I will do the same for the second part (what is currently below) and also finish the proof, when I have a chance

- For each $i \in [\![1, n]\!]$, I_i is a basis for $\mathcal{P}_W$.
- If J is lexicographically smaller than I_i with respect to $<_i$, then J is not a basis for $\mathcal{P}_W$.

Recall from [ref] that a k-subset $J \in \binom{[n]}{k}$ is a basis for $\mathcal{P}_W$ if and only if it has no subset $U \subseteq J$ such that $|U| > |Prop(U)|$. It is immediately clear from the construction that I_i is a basis for each i: the algorithm is effectively pairing each $j \in I_i$ with a unique propagator supported on that vertex, so $|Prop(U)| \geq U$ for all $U \subseteq I_i$.

If $\exists l \in J$ with $Prop(l) = \emptyset$ then J is clearly not a basis, so suppose otherwise. Write

$$I_i = [i_1 <_i i_2 <_i \cdots <_i i_r <_i \cdots <_i i_k], \quad J = [i_1 <_i i_2 <_i \cdots <_i j_r <_i \cdots <_i j_k],$$

so that I_i and J differ for the first time in the rth position. Then

$$r - 1 \geq |Prop(\{i_1, \ldots, i_{r-1}\})| = |Prop(\{i_1, \ldots, i_{r+1}, j_r\})|,$$

If $|Prop(\{i_1, \ldots, i_{r-1}|\}| = r - 1$ then we are done; otherwise

□

3.2 Dimension of the Wilson Loop cells

In [**?**], Agarwala and Fryer give an algorithm for passing from a Grassmann Necklace to a Le diagram. We use this algorithm here to pass from a Wilson loop diagram to its associated Le diagram. In this manner, we show that the positroid cell defined by a Wilson loop diagram has dimension $3|\mathcal{P}|$. Maybe say something about Amplituhedra having $4k$ dimension, but this is not quite what we have, since we are ignoring one column.

When I speak of a *WLD*, I mean one which satisfies your density hypothesis and other standard hypotheses. When I speak of a propagator *contributing* a vertex to an element of the Graßmann[1] necklace I mean that according to the rule you guys have to build the Graßmann necklace from the diagram by starting at a vertex and taking the clockwise-most covering propagator which hasn't already been taken, when a propagator is taken then it is contributing that vertex. Also, I will be using your algorithm to convert the Graßmann necklace into a Le diagram by non-intersecting paths.

[1] I suppose it is a bit of an affectation to use an eszett when writing in English, and actually the eszett is kind of ugly in this font, but since I've started to I'll stick with it for now but you certainly shouldn't feel obliged to do it too.

We need four lemmas from stuff you guys have already figured out. The first one is a corollary of your lemma characterizing intervals contributed by each covered vertex of a propagator.

The first lemma is Lemma 3.4 and I moved it up to a previous section.

The second lemma is the fact that we understand uncovered vertices. You probably see many ways to prove this from things you already know. One would be to say that from your algorithm the column of the uncovered vertex simply plays no role.

Lemma 3.6. *Let D be a WLD with an uncovered vertex i. Let C be D with i removed. Then the Le diagram of D is the Le diagram of C with an extra column of all $0s$ inserted in the $|D| - i + 1th$ position from the left.*

The third lemma is Sian's result.

Lemma 3.7. *Let D be a WLD with all vertices covered by at least two propagators. Then There exist two propagators in D with the following properties.*

- *The first propagator goes from the edge i, $i + 1$ to the edge $i + 2$, $i + 3$.*
- *The second propagator goes from the edge $i + 2$, $i + 3$ to the edge $i + 4$, $i + 5$*
- *No other propagator is on the edge $i + 2$, $i + 3$.*

Actually the result I need is not exactly Sian's result and what I will actually use is her proof. Thus I will use notation and definitions (including length of a propagator and the specific p_i and q_i construction from the proof) from her note without further discussion. The result I will actually use will be that in place of the hypothesis that D has all vertices covered by at least two propagators I have that all vertices are covered by at least 1 propagator and there are no propagators of length 3.

The fourth lemma is that we can rotate or reflect without changing the number of plusses. The best way to see this is probably geometrically so I leave that up to you.

Lemma 3.8. *If two WLDs differ by a dihedral transformation then their Le diagrams have same number of plusses.*

Now we're ready to go.

3.2.1 Changes in Graßmann necklaces for nice configurations

Given a propagator p in a WLD D, note that p divides remaining propagators of D into two sets depending on which side of p they live.

Lemma 3.9. *Let D be a WLD with $n \geq 1$ propagators. Then there is some dihedral transformation D' of D such that there is a propagator p with the following properties.*

- *p goes between the edge i, $i + 1$ and the edge $n - 1$, n in D'.*

- *$i+2, \ldots, n-2$ are not covered by any propagators in D' (which is trivially true if $\{i+1, \ldots, n-2\} = \varnothing$).*
- *Either $i+1$ in D' is only covered by p, or $i+1$ is covered by exactly one other propagator q.*
- *If we are in the second case above then q goes between the edge j, $j+1$ and the edge i, $i+1$ and $j+2, \ldots, i-1$ are not covered by any propagators.*

Proof. Since $n \geq 1$ the dual tree of D has at least two vertices and so has at least two leaves. Each edge going to a leaf of the dual tree corresponds to a propagator which has no other propagators on one side of it, so there is at least one such propagator in D. There are two cases to consider.

First suppose there is a propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let this propagator be p. Rotate and reflect D as necessary so that the end of p which does not share its edge comes first, then the side of p with no other propagators, and then the other end of p which is on edge $n-1$, n. Call this WLD D'. It is a straightforward check that the properties in the statement are satisfied with $i+1$ only covered by p.

Next suppose there is no propagator which has no other propagators on one side of it and for which one of its ends is on an edge with no other propagators. Let C be D with all uncovered vertices removed. Suppose C contains two propagators one of length 2 and one of length 3 with the length 2 propagator on the small side of the length 3 propagator. Then we are in the case already considered because we can rotate and reflect C so that these two propagators both have one end on the edge $|C|-1$,$|C|$, and have their other ends on the edge $|C|-2$, $|C|-3$ and the edge $|C|-3$, $|C|-4$, so taking the length 2 edge as p we satisfy the properties of the statement, and this remains true in D. Thus we now assume this does not occur, so in particular every propagator of length 3 in C has no other propagators on its small side. But then the middle vertex on this small side is uncovered contradicting the construction of C. Thus C has no propagators of size 3.

The claim then is that in C there must exist two propagators as in Lemma 3.7. The proof of the claim follows by Sian's proof. We do not have the hypothesis that all vertices are covered by at least two propagators, but this is only used in the proof of Lemma 3.7 to force certain propagators (the q_i) to have length at least 4, so our this case we instead use that C has no propagators of size 3.

Now rotate and flip C so that the two propagators as in Lemma 3.7 cover $\{|C|-5, \ldots, |C|\}$ and let p be the propagator covering $\{|C|-3, \ldots, |C|\}$ and let q be the other one. Return to D with a corresponding rotation and flip so that vertex n in D corresponds to vertex $|C|$ in C and direction in the polygon is preserved. Let this dihedral transformation of D be D'. Then the propagator p satisfies the conditions of the statement with propagator q. $\square$

Given a WLD D, I_i^D denotes the ith Graßmann necklace element corresponding to D.

Lemma 3.10. *Let D be a WLD with $n \geq 1$ propagators and let p be a propagator of D so that the properties of Lemma 3.9 are satisfied for p in D (that is D is already appropriately transformed).*

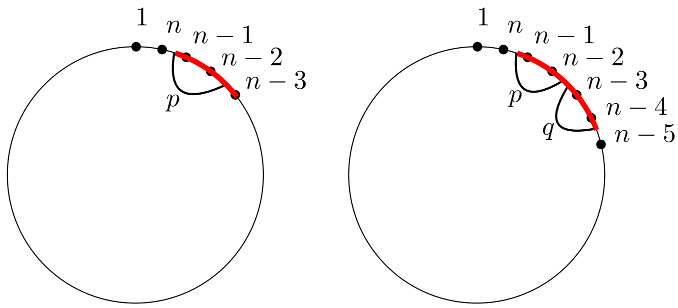

Figure 3: The different possibilities for D and p. No other propagators can end in the fat red sections. Other segments may have additional propagators ending in them.

Furthermore suppose that that p has length 2, as does q if we are in the case of Lemma 3.9 involving q. Let C be D with p removed but no change in the vertices. Then

$$\begin{aligned}
I_1^D &= I_1^C \cup \{n-3\}\\
I_n^D &= I_1^C \cup \{n\}\\
I_{n-1}^D &= I_n^C \cup \{n-1\}\\
I_{n-2}^D &= \begin{cases} I_{n-2}^C \cup \{n-2\} & \text{if } n-2 \notin I_{n-2}^C\\ I_{n-2}^C \cup \{n-1\} & \text{if } n-2 \in I_{n-2}^C,\ n-1 \notin I_{n-2}^C\\ (I_n^C - \{n-5\}) \cup \{n-1, n-2\} & \text{if } n-1, n-2 \in I_{n-2}^C \end{cases}\\
I_k^D &= \begin{cases} I_k^C \cup \{n-3\} & \text{if } n-3 \notin I_k^C\\ I_k^C \cup \{n-2\} & \text{if } n-3 \in I_k^C \end{cases}\\
&\text{for } 1 < k < n-2
\end{aligned}$$

Proof. The two possible situations are illustrated in Figure 3.

If $n-3 \in I_1^C$ then $n-3 \notin I_1^D$, but then p never contributes $n-3$ contradicting Lemma 3.4. Thus $n-3 \notin I_1^C$ and so in building the Graßmann necklace when we get to vertex $n-3$ any other covering propagators of C have already been taken and so we can now take p. Therefore $I_1^D = I_1^C \cup \{n-3\}$.

In constructing I_n^D, first for vertex n we take propagator p. Then we are at vertex 1 and precisely the propagators of C remain. Thus the rest of the construction will give I_1^C. Therefore $I_n^D = I_1^C \cup \{n\}$. Essentially the same reasoning gives $I_{n-1}^D = I_n^C \cup \{n-1\}$.

Now consider I_{n-2}^D. If $n-2 \notin I_{n-2}^C$ then at vertex $n-2$ we take p and this does not affect the rest of the construction of I_{n-2}^C, so $I_{n-2}^D = I_{n-2}^C \cup \{n-2\}$. An analogous argument takes care of the first case for I_k^D. Suppose $n-2 \in I_{n-2}^C$ but $n-1 \notin I_{n-2}^C$. Then at vertex $n-2$ we take the same propagator in C as in D (in particular not p) because p is the counterclockwisemost propagator covering $n-2$ and so the last propagator the algorithm would choose at this vertex, and by hypothesis $n-2$ is covered in C. Howver, $n-1$ is untaken in I_{n-2}^C so at this vertex we will take p in I_{n-2}^D. Following this, now that p is out of the way without bumping any other propagators, the construction continues as in I_{n-2}^C. Therefore $I_{n-2}^D = I_{n-2}^C \cup \{n-1\}$. An analogous argument

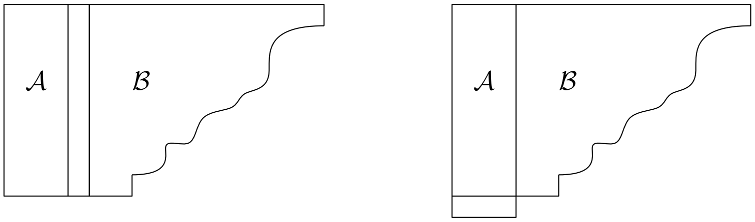

Figure 4: Le diagrams for C (left) and D (right).

takes care of the case when we have $1 < k < n-2$ and $n-3 \in I_k^C$ but $n-2 \notin I_k^C$. Furthermore, either $n-2$ is uncovered in C or only q covers $n-2$ in C and q is also the only propagator covering $n-3$. Thus it is not possible for $n-3$ and $n-2$ to both be in I_k^C. This means that all the cases for I_k^D are now proved.

The remaining case is when $n-1, n-2 \in I_{n-2}^C$ for the construction of I_{n-2}^D. We must then have a propagator q as in the right hand side of Figure 3. In the construction of I_{n-2}^D, at vertex $n-2$ we take propagator q, as in I_{n-2}^C. Then at vertex $n-1$ we take propagator p which is different from what occurs in I_{n-2}^C. Next we are at vertex n and propagators p and q have been taken. Thus we are proceeding like I_n^C but without propagator q. Fortunately we can determine explicitely how propagator q contributes to I_n^C. By Lemma 3.4 propagator q contributes $n-5$ to I_n^C, and the only way this can occur is if all other propagators of C were already taken by the time we got to vertex $n-5$. Therefore $I_{n-2}^D = (I_n^C - \{n-5\}) \cup \{n-1, n-2\}$. This covers all cases and hence completes the proof. □

3.2.2 The number of plusses from Graßmann necklaces in nice configurations

Lemma 3.11. *Let C and D be as in Lemma 3.10. The shape of the Le diagram of C can be built from left to right of the following blocks: a rectangle with 3 columns, one more column of the same length, a partition shape with at most as many rows as the rectangle. The shape of the Le diagram of D can be built from left to right of the following blocks: a rectangle with 3 columns and one more row than the first rectangle of C, the same partition shape as in C.*

Proof. I_1 determines the shape of the Le diagram. By Lemma 3.10, $I_1^D = I_1^C \cup \{n-3\}$. This implies that the right hand boundary of the shape of C is the same as the right hand boundary of the shape of D except that D has one additional row of 3 boxes while C has an additional column in the $n-3$ position, that is a new column fourth from the left. □

The shapes of the Le diagrams of C and D are illustrated in Figure 4. The pieces of the Le diagrams will be called $\mathcal{A}$ and $\mathcal{B}$ in what follows, as in the figure. Over the course of the next few lemmas we will prove that the plusses in the $\mathcal{B}$ parts of the Le diagrams of C and D are identical and the plusses in the $\mathcal{A}$ parts are very closely related. When we speak of a plus in the Le diagram of D being the same as in D or vice versa, we mean that the plus' position in $\mathcal{A}$ or $\mathcal{B}$ is the same. Because of the column insertion the absolute indices may differ.

Suppose we are following the Graßmann necklace to Le diagram algorithm, and we put a plus in a box because of a path from vertical boundary edge i to bottom boundary edge j. Then say this plus is in the $i \to j$ position.

Lemma 3.12. *Let C and D be as in Lemma 3.10. The I_n^D and I_{n-1}^D elements of the Graßmann necklace of D give all the same plusses as I_n^C along with plusses in the leftmost two boxes of the bottom row of the Le diagram of D.*

Proof. By Lemma 3.10 $I_n^D = I_1^C \cup \{n\}$, so by the Graßmann necklace to Le diagram algorithm the only plus this builds in the Le diagram of D is the one in the $n-3 \to n$ position, that is in the leftmost box of the bottom row.

Also by Lemma 3.10 $I_{n-1}^D = I_n^C \cup \{n-1\}$. Additionally $n-3 \notin I_n^C$ since if it were then $n-3$ would also be in I_1^C and hence propagator p could not contribute $n-3$ to I_1^D in contradiction to Lemma 3.4. Similarly $n-1, n-2 \notin I_n^C$. Thus the paths putting the plusses in from I_n^C lie completely in $\mathcal{B}$ or take some vertical boundary edge $> n-3$ to n. Now view these paths in the Le diagram of D and note that the path $n-3 \to n-1$ is compatible, and so these paths together build the plusses that I_{n-1}^D contributes. That is, we get all the plusses from I_n^C along with a $n-3 \to n-1$ plus, that is a plus in the second to the right box of the bottom row. $\square$

Lemma 3.13. *Let C and D be as in Lemma 3.10 with $n-2 \notin I_{n-2}^C$.*

The I_{n-2}^D element of the Graßmann necklace of D gives an $n-3 \to n-2$ plus and all the $I_{n-1}^C = I_{n-2}^C$ plusses.

Proof. If $n-2 \notin I_{n-2}^C$ then $I_{n-1}^C = I_{n-2}^C$ and by Lemma 3.10 $I_{n-2}^D = I_{n-1}^C \cup \{n-2\}$. Note that $n-3 \notin I_{n-2}^C$ by Lemma 3.4. Therefore the paths for I_{n-2}^D are the paths for I_{n-1}^C along with the $n-3 \to n-2$ path. This gives the statement of the lemma.

$\square$

Lemma 3.14. *Let C and D be as in Lemma 3.10 with $n-2, n-1 \in I_{n-2}^C$.*

The I_{n-2}^D and I_{n-3}^D elements of the Graßmann necklace of D gives the following plusses:

- *An $n-3 \to n-2$ plus and an $n-5 \to n-1$ plus.*
- *All the I_{n-1}^C plusses.*
- *I_{n-2}^C gives an $n-5 \to n-2$ plus and no other term in the Graßmann necklace of C gives a plus in this column. This $+$ does not appear in D from I_{n-2}^D but an $n-5 \to n-1$ plus does instead.*
- *All other plusses of I_{n-2}^C.*
- *I_{n-3}^C gives a plus in the $n-3$ column. This $+$ is shifted over into the $n-2$ column in D.*
- *All other plusses of I_{n-3}^C.*

Furthermore, no element of the Graßmann necklace of C gives an $n-5 \to n-1$ plus.

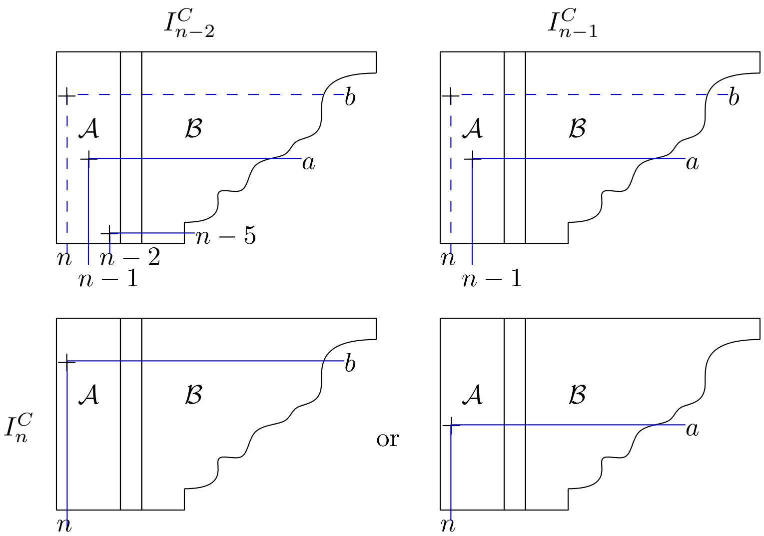

Figure 5: Plusses coming from I^C_{n-2} (top left), I^C_{n-1} (top right) and I^C_n bottom when $n-1, n-2 \in I^C_{n-1}$. The blue lines are the non-intersecting paths. The dashed blue lines may or may not appear, but if one appears then they both do.

Proof. By Lemma 3.10 $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$. Also, by the location of q in the WLD, $n-2 \notin I^C_{n-3}$ and and $n-5$ is the index of the lowest vertical edge in $\mathcal{B}$. Thus this section of the Graßmann necklace of C looks like

$$I^C_{n-3} \underset{\substack{n-3 \text{ out}\\ n-2 \text{ in}}}{\rightarrow} I^C_{n-2} \underset{\substack{n-2 \text{ out}\\ n-5 \text{ in}}}{\rightarrow} I^C_{n-1} \underset{\substack{n-1 \text{ out}\\ \text{something in}}}{\rightarrow} I^C_n \underset{\substack{n \text{ out}\\ \text{something in}}}{\rightarrow} I^C_1 \tag{6}$$

where the first "something" is either n or an element of I^C_1 and the second "something" is an element of I^C_1. Additionally all elements not explicitly mentioned must be in I^C_1 as they remain unchanged through this portion of the necklace.

Using this information now determine the symmetric difference of I^C_{n-2} and I^C_1: $n-1, n-2$ and possibly n are in I^C_{n-2} but not in I^C_1. $n-5$ is in I^C_1 as are at least one and at most two other elements. If there is one such element call it a. If there are two call them a and b with $a > b$. This means that the plusses in the Le diagram of C coming from I^C_{n-2} are as in the first part of Figure 5. Stepping to I^C_{n-1} simply removes the $n-5 \to n-2$ path, see the second part of Figure 5.

Stepping to I^C_n, $n-1$ is taken out and either n is put in if it was not there before, or one of a or b is put in and hence no longer available as a right end for a path. This gives two possible configurations illustrated in the bottom two parts of Figure 5.

Now we know that $I^D_{n-2} = (I^C_n - \{n-5\}) \cup \{n-1, n-2\}$ so the paths for building plusses from I^D_{n-2} go from the set $\{n-5, n-3\}$ along with whichever of a and b is not in I^C_n to $\{n-2, n-1, n\}$. This means that we get plusses as in Figure 6 where the left and right cases correspond to the left and right cases in the bottom parts of Figure 5

This proves the first item of statement of the lemma.

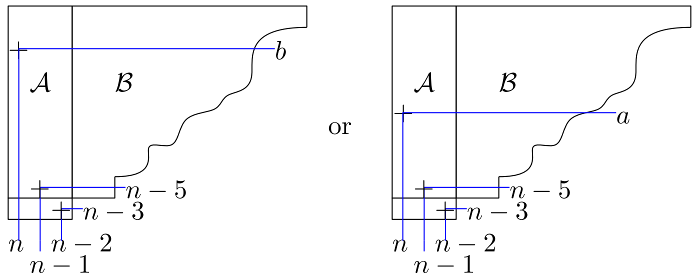

Figure 6: Plusses coming from I^D_{n-2}.

Now consider I^C_{n-3}. By (6) I^C_{n-3} contributes the same plusses as I^C_{n2} except that it contributes an $n-5 \to n-3$ plus in place of the $n-5 \to n-2$ plus. Also, we have $n-3 \in I^C_{n-3}$ be the location of q and so $I^D_k = I^C_k \cup \{n-2\}$. Thus the paths for I^D_{n-3} are the same as for I^C_{n-3} except that the path that did go to $n-3$ now goes to $n-2$. This cannot conflict with another path since (6) shows that $n-2$ only appears in I^C_{n-2} among the necklace elements of C.

Also note that I^C_{n-3}, I^C_{n-2}, and I^C_{n-1} share their plusses outside of the $n-3$ and $n-2$ columns. This proves the remaining statements of the lemma except the furthermore.

Finally, suppose there were a $n-5 \to n-1$ plus in the Le diagram of C. By the algorithm, it would have to come when $n-5 \notin I^C_j$. By (6) this means that it would have to come from I^C_{n-4}, I^C_{n-3}, or I^C_{n-2}. The analysis above shows it does not come from I^C_{n-3} or I^C_{n-2}. Now, $n-4 \in I^C_{n-4}$ by the location of q and so I^C_{n-4} must give an $n-5 \to n-4$ plus and so cannot give an $n-5 \to n-1$ plus.

□

Lemma 3.15. *Let C and D be as in Lemma 3.10 and take $1 < k < n-2$. Suppose that if $n-2 \in I^C_{n-2}$ then also $n-1 \in I^C_{n-2}$. The I^D_k element of the Graßmann necklace of D gives the same plusses as I^C_k except that if there is a plus in the $n-3$ column for I^C_k then this plus is shifted into the $n-2$ column and no plus was already in that location.*

Proof. If $n-3 \notin I^C_k$ then by Lemma 3.10 $I^D_k = I^C_k \cup \{n-3\}$. Then since $n-3$ is the largest element of I^D_1 this transformation leaves the disjoint paths unchanged and so the plusses carry over from C to D directly.

If $n-3 \in I^C_k$ then by Lemma 3.10 $I^D_k = I^C_k \cup \{n-2\}$. If $n-2$ is not covered in C then certainly no pluses appear in the $n-2$ column of the Le diagram of C. If $n-2$ is covered in C then by hypothesis so is $n-1$ and so we satisfy the hypotheses of Lemma 3.14. Thus the only necklace element of C containing $n-2$ is I^C_{n-2} and this particular plus is not contributed to the Le diagram of D by I^D_{n-2}.

From I^C_k there is a path from some vertical edge to the bottom edge $n-3$. In I^D_k, $n-3$ is a vertical edge with no path and instead there must be a path to $n-2$. By the previous paragraph no other path can end in $n-2$, so shifting the path that did go to $n-3$ to go to $n-2$ while leaving the others the same maintains non-crossingness and so must be the paths for I^D_k. Thus the plus in the

$n-3$ column for C is shifted into the $n-2$ column, where there was no plus before, and no other plusses are changed. $\square$

Theorem 3.16. *The number of plusses in the Le diagram of a WLD is three times the number of propagators.*

Proof. The proof is by induction on the number of propagators.

First note that a WLD D with one propagator covering vertices $i < j < k < \ell$ has Le diagram a single row with $|D| - i$ boxes. Labelling them from left to right by $|D|, \dots, |D| - i + 1$, by the algorithm there are plusses in the j, k, and ℓ positions.

Now consider WLDs with $k > 1$ propagators. By Lemma 3.6 it suffices to prove the result for WLDs with k propagators and no uncovered vertices. By Lemma 3.8 it suffices to prove the result for at least one WLD from each dihedral orbit. Take a WLD diagram D with k propagators. Make a dihedral transformation of D if necessary so that D has a propagator p with the properties in Lemma 3.9 relative to D. If $n-1$ is only covered by p but $n-2$ is covered by at least one other propagator, then flip D on the line perpendicular to the edge from $n-2$ to $n-1$. This will be our D for the rest of the proof.

Let C be D with p removed but no change in the vertices. Note that if $n-2$ is covered in C then so is $n-1$ by the end of the previous paragraph and so if $n-2$ is covered in C then the hypotheses of Lemma 3.14 are satisfied.

From Lemma 3.11 we know how the shapes of the Le diagrams of C and D relate; let $\mathcal{A}$ and $\mathcal{B}$ be as described after that lemma. Lemmas 3.12, 3.13, and 3.14 tell us that the three boxes of the bottom row of the Le diagram of D each have a plus. Lemmas 3.12, 3.13, 3.14, and 3.15 show that there is a bijection between the plusses of the Le diagram of C and the plusses of the Le diagram of D that are not in the bottom row which can be described as follows.

- Plusses from $\mathcal{B}$ for C maintain their positions in $\mathcal{B}$ for D.
- Plusses from the first two columns (the n and the $n-1$ columns) of $\mathcal{A}$ for C maintain their positions in $\mathcal{A}$ for D.
- If there is a plus in the $n-2$ column of $\mathcal{A}$ in then Lemma 3.14 applies, so there is exactly one such plus. This plus maps to the $n-5 \to n-1$ plus for D.
- The plusses in the $n-3$ column for C shift over to the third column (the $n-2$ column) of $\mathcal{A}$ in D.

This map is clearly reversible and hence bijective except possibly for the $n-5 \to n-1$ plus for D. If the Le diagram of D has an $n-5 \to n-1$ plus then look at the Le diagram for C. If the Le diagram for C has a plus in the $n-2$ column then Lemma 3.14 applies and so there is no $n-5 \to n-1$ plus in the Le diagram of C and the $n-5 \to n-1$ plus of D can be uniquely mapped to the plus in the $n-2$ column of the Le diagram of C. If the Le diagram for C has no plus in the $n-2$ column, then leave the $n-5 \to n-1$ plus where it is in moving back to C. This reverses the map.

From all of this we get that the number of plusses in the Le diagram for D is three more than the number of plusses in the Le diagram for C. Applying induction completes the proof. $\square$

Maybe proved that inadmissible diagrams, while they correspond to matroids, can't be mapped to the positroids of the correct dimension?

4 Poles of Wilson Loop Integrals

I'd like to end this paper with a proof of a conjecture (that I think I need for a subsequent paper, and this seems the natural place to put it. This is as follows:

Conjecture 4.1. *Given any admissibly Wilson loop diagram W, let $GN(W) = \{I_1, \ldots I_n\}$ be the associated Grasmann necklace. Then the denominator of the integral $I(W)$ is the find correct word for this s the minimal polynomial of $\prod_{i=1}^{n} \Delta_{I_i}$, where Δ_{I_i} is the determinant of the $k \times k$ minor indicated by I_i.*

i