Falling Stars in Small Open Economies*

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Abstract

Natural rates of interest (r-star) have fallen steadily over the last half-century across the advanced world, exacerbated sharply by the Great Recession. This paper considers the implications of local asymmetric and global symmetric shocks to the long-run equilibrium interest rate on monetary policy and exchange rates in the small open economy. We derive a structural equation for the equilibrium exchange rate that is a function of a purchasing power parity component, cyclical component and expected natural rate of interest differential. To estimate the reduced form, we first implement the Kalman filter in a benchmark multi-step maximum likelihood procedure to derive natural rates of interest for seven advanced open economies. We then implement second-generation panel unit-root tests to establish mean reversion in exchange rates and thus their equilibria. This paper subsequently finds empirical support for an equilibrium real exchange rate and equilibrium real interest rate differential parity and identifies a number of theoretical implications of falling r-stars estimated across the advanced world.

Keywords: Natural rate of interest, Monetary policy, Exchange rates

JEL classification: C32, E43, E52, E58, F41

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1 Introduction

Introduced by Wicksell (1898), the natural rate of interest (r-star) is the real interest rate consistent with equilibrium output and stable inflation. It is a core indicator for policymakers, whom pursue contractionary monetary policy in reaction to short-term interest rates above equilibrium and expansionary policy in reaction to interest rates below it. The hidden nature of r-star implies it must be extracted from the data rather than being directly observed or measured. The existing literature has therefore mainly focused on its estimation and drivers. On the contrary, few have investigated the implications of trends in the natural rate of interest within an open economy setting, which is of particular importance given the growing international influence of domestic policy.

The open economy dimension of r-star first transpires in the general equilibrium model of Galí and Monacelli (2005), hereafter GM, where the equilibrium interest rate is derived as a function of both domestic and foreign factors. Empirically, the case for global determinants of r-star is also made by Holston, Laubach and Williams (2017), hereafter HLW, in which forecast error variance decomposition of the estimated natural rate of interest between major advanced economies reveal substantial interdependence that is unexplained by ordinarily closed models. Despite these indications of international drivers of the natural rate of interest, work on the impact of long-term trends in r-star on open economy fundamentals such as exchange rates is relatively scarce.

This paper investigates the implications for exchange rates and monetary policy that arise from declining natural rates of interest. We extend the benchmark small open economy model to derive a novel expression for the equilibrium exchange rate that is a function of the expected natural rate of interest differential, output gap and price level. To proceed, we estimate r-star, the output gap and its trend growth rate for several advanced (G7) economies, and use these estimates to investigate the relationship between equilibrium exchange rates and expected natural rate differentials across the advanced world. In doing so, we revive the empirically stubborn real exchange rate real interest rate differential relationship within a long-run equilibrium setting.

Our analysis yields a number of important theoretical and empirical results. As for the former, we find that if r-star is expected to fall below the foreign natural rate, exchange rates must depreciate given prices and the output gap. Second, given optimal policy, global shocks are absorbed by a general decline in r-star to maintain global demand. Third, whilst global symmetric shocks imply comovement in policy rates, local asymmetric shocks cause shifts in domestic rates that result in exchange rate adjustments to preserve the flexible price equilibrium. Fourth, optimal policy yields a forward-looking Taylor rule with a time-varying natural rate that is a function of both local and global factors. As for the latter, we prove the law of PPP holds across the G7 and that there exists a significant long-run cointegrating relationship between equilibrium exchange rates and natural rate differentials that conforms to the theoretical findings of our model.

The remainder of this paper is organised as follows. Section 2 reviews the existing literature on r-star and exchange rate determination. Section 3 outlines the main theoretical framework from which we derive an expression for the equilibrium exchange rate. Section 4 outlines the empirical specification of the state-space system used to estimate r-star. Section 5 presents and discusses the key results. Section 6 discusses the implications of our findings. Finally, Section 7 concludes.

2 Literature

This paper concerns two strands of the existing literature; the estimation of r-star and its implications for exchange rate determination. As for the former, the semi-structural approach of Laubach and Williams (2003) and Holston, Laubach and Williams (2017) is of particular importance. Their application of the Kalman filter in a multi-stage maximum likelihood procedure reveals a secular decline in r-star across major advanced economies. The authors also detect comovement in natural rates, supporting a global factor in their time variation. Despite high standard errors, general findings prove robust to a number of extensions (e.g. Clark and Kozicki, 2005; Mésonnier and Renne, 2007; Trehan and Wu, 2007; Berger and Kempa, 2014; Lewis and Vazquez-Grande, 2018; Brand and Mazelis, 2019; Krustev, 2019). Such imprecision is largely attributed to filter and parameter uncertainty. In addition, the pile-up problem (Stock, 1994) in estimates for the standard deviations of innovations arising from the maximum likelihood estimation of parameters influenced by highly persistent shifts is also of concern within the existing literature.

Responses to issues of uncertainty have ranged from alternative models to varying estimation techniques (e.g. Barsky et al., 2014; Hamilton et al., 2016; Johannsen and Mertens, 2016; Kiley, 2015; Lubik and Matthes, 2015; Pescatori and Turunen, 2016; Christensen and Rudebusch, 2017). In addition, much of the remaining literature focuses primarily on the causes of the decline in r-star unexplained by trend growth, which is largely attributed to long-term trends in demography, productivity, risk preferences, the supply of safe assets, public policy and inequality (see for instance Carvalho, Ferrero and Nechio, 2016; Gagnon et al., 2016; Gordon, 2016; Rachel and Smith, 2017; Eggertsson et al., 2019; Rachel and Summers, 2019). The estimation and drivers of r-star however, are not the focus of this research. Instead, we estimate the natural rate as an intermediate input for our analysis of exchange rates and monetary policy in the small open economy.

As for the latter, the theoretical framework for r-star often begins with Woodford (2003) and Galí (2008) in which equilibrium conditions for domestic inflation and the output gap are derived to analyse a host of monetary policy counterfactuals. This particular framework has been extended in various aspects to investigate time variation in the natural rate of interest. For instance, Barsky et al. (2014) build upon the framework of Smets and Wouters (2007) and show that deviations in the natural rate from policy rates create instability in output and increase price and wage variability. Implications for monetary policy are also captured in extensions to Laubach and Williams (2003), for instance Hamilton et al. (2016) suggest that uncertainty in estimated natural rates imply policy rules deviate less from the inflation target and output. These notable studies, among others, outline the policy implications of a time varying natural rate of interest (Orphanides and Williams, 2002; Barsky et al., 2014; Williams, 2015; 2016). However, the implications on the conduct of monetary policy in terms of exchange rates are largely omitted from this existing body of literature, partly due to the closed nature of these general equilibrium models.

In an open economy setting, real interest rate differentials are considered key to models of exchange rate determination. These are largely sorted into the monetary approach, which is divided further into the monetarist (flexible-prices) and overshooting (sticky-prices) model, and the portfolio balance approach. In particular, frictional approaches to exchange rate determination suggest that arbitrage between goods and asset market prices may cause temporary real interest rate differentials between countries due to monetary shocks and deviations in real exchange rates from their equilibria (see Dornbusch, 1976; Mussa, 1984). Furthermore, alternative empirical approaches to real exchange rates and real interest rate differentials involve optimisation models (see for instance Grilli and Roubini, 1992; Obstfeld and Rogoff, 1995) that seek to capture the impact of liquidity shocks on real interest rates, and by extension, real exchange rates.

However, the empirical evidence for this relation is mixed. For instance, Campbell and Clarida (1987) find that volatility in exchange rates is only partly explained by expected real interest rate differentials. Meese and Rogoff (1988) use standard regression and cointegration techniques to detect statistically significant relationships between real exchange rates and interest rate differentials, albeit the results are inconclusive. Edison and Pauls (1993) identify no cointegration between non-stationary exchange rates and real interest rates, nor a long-run relationship using error-correction models. Despite this, the authors present some evidence to suggest that this relationship may exist over higher-frequencies (see also Baxter, 1994). Some studies also find supporting evidence given longer horizons, non-linearity, and regime switching (see for instance Nakagawa, 2002; Kanas and Genius, 2005; Hoffman and MacDonald, 2009).

In light of such findings, our arguments may be summarised as follows. The existing literature on real exchange rate determination tends to focus largely on disequilibrium interest rates, where the former is often highly persistent due to the degree of mean reversion captured by long half-life of deviations from Purchasing Power Parity (PPP) (see Parsley and Wei, 1996; Frankel and Rose, 1996). In equilibrium however, exchange rates are assumed stationary and the law of PPP holds, yet this equilibrium is seldom translated to real interest rate differentials. We propose an alternative perspective on the real exchange rate real interest rate puzzle by considering it in equilibrium; as a relation between equilibrium exchange rates and equilibrium interest rates. We argue that whilst the relationship between real exchange rates and real interest rates is empirically inconclusive, the evidence between equilibrium real exchange rates and r-star is significant.

The open economy model of Galí and Monacelli (2005) facilitates this task; to our knowledge it has not been used to extend equilibrium exchange rates to r-star. As an exception, Clarida (2019) uses a simple variation of the model, in which an expression for the exchange rate is derived as a function of a business cycle component, purchasing-parity-power component and expected natural rate differential. However, this result is not tested empirically nor interpreted within the context of declining r-stars across the advanced world, which is a novelty in our research.

This paper builds on these strands of the existing literature in various respects. As the primary focus of this analysis are the implications of international trends in r-star, we make no adjustments to the benchmark HLW system despite some of the aforementioned limitations. As for the implications for exchange rates and monetary policy, we derive a theoretical solution from the Galí and Monacelli (2005) model that links equilibrium real exchange rates with r-star. Our contribution to the literature are the implications that follow from this result in light of declining natural rates of interest across the advanced world. We estimate a reduced version of our structural expression using second-generation panel estimation techniques before discussing the impact of secular trends in r-star on exchange rates and monetary policy in the small open economy.

3 Theory

3.1 Households

We proceed with Galí and Monacelli (2005), in which the small open economy is modelled among a continuum of infinitely many small economies that compose the global economy. In this setting the representative household wishes to solve the following maximisation problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1}$$

where β is the discount factor, N_t is the number of labour hours and C_t is a composite consumption function of both domestic and foreign consumption that is given by the following equation:

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} (C_{h,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{f,t})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{2}$$

where $\alpha \in [0,1]$ is the degree of openness and $\eta > 0$ is the substitutability between domestic and foreign goods. $C_{h,t}$ and $C_{f,t}$ denote CES functions, where the latter is itself a composite of goods imported from country i (see Appendix A for a representation of the goods structure).[1]

This maximisation problem is subject to a sequence of budget constraints that are derived using optimal conditions for both household consumption and consumer price inflation as follows:

$$P_t C_t + E_t\{Q_{t,t+1} D_{t+1}\} \leqslant D_t + W_t N_t + T_t \tag{3}$$

where P_t denotes consumer price inflation, D_{t+1} is the future payoff from a presently held portfolio and $Q_{t,t+1}$ is a stochastic discount factor for future nominal payoffs of domestic households.

Assuming this constraint holds with equality and that the utility function is given by:

$$u(C_t, N_t) \equiv \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \tag{4}$$

where σ and φ denote the elasticities of substitution for consumption and labour respectively.

First order conditions yield the intratemporal choice between consumption and labour:

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} \tag{5}$$

Assuming standard conditions for an optimising household agent within the small open economy,[2] the stochastic Euler relation may be written with a one-period stochastic discount factor as follows:

$$E_t\{Q_{t,t+1}\} = \beta E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \tag{6}$$

[1] Consumption is given by $C_{h,t} \equiv (\int_0^1 C_{h_i,t}^{\frac{\varepsilon-1}{\varepsilon}} di)^{\frac{\varepsilon}{\varepsilon-1}}$ and $C_{f,t} \equiv (\int_0^1 C_{j,t}^{\frac{\gamma-1}{\gamma}} dj)^{\frac{\gamma-1}{\gamma}}$, wherein ε is the elasticity of substitution between varieties of goods produced in j and γ is the elasticity of substitution between importing countries.

[2] In particular, $\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}}$, in which $V_{t,t+1}$ is the price of an Arrow security (a one-period security yielding one unit of domestic currency at the realisation of a future state of nature, and zero otherwise).

where $E_t\{Q_{t,t+1}\}$ is defined as the market price of a one-period portfolio yielding one unit of domestic currency in the future. Log linearisation of equations (5) and (6) yields the following:[3]

$$w_t - p_t = \sigma c_t + \varphi n_t \tag{7}$$

$$c_t = E_t\{c_{t+1}\} - \sigma^{-1}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right) \tag{8}$$

3.2 Firms

Firms produce differentiated goods with technology represented by a typical production function. In particular, a representative firm faces the following maximisation problem:

$$\max_{\bar{P}_{h,t}} \textstyle\sum_{k=0}^{\infty} \theta^k E_t\left[Q_{t,t+k}\left(\bar{P}_{h,t} Y_{i,t+k|t} - TC^n_{i,t+k|t} Y_{i,t+k|t}\right)\right] \tag{9}$$

where θ^k is the probability of being restricted by today's prices in k periods, $Y_{i,t+k|t}$ is the output in $t+k$ of a firm setting prices in t and $TC^n_{i,t+k|t}$ is the total cost, subject to the constraint:

$$Y_{i,t+k|t} = \left(\frac{\bar{P}_{h,t}}{P_{h,t+k}}\right)^{-\varepsilon} C_{t+k} \tag{10}$$

First order conditions yield a standard log-linearised optimal pricing rule consistent with the firm's optimal price-setting behaviour in the closed economy case, given by the following equation:[4]

$$\bar{p}_{h,t} = \mu + (1-\theta\beta) E_t \textstyle\sum_{k=0}^{\infty} \theta^k \beta^k [mc_{t+k|t} + p_{h,t+k}] \tag{11}$$

3.3 Identities

In the small open economy framework, various identities must be introduced. Firstly, the effective terms of trade s_t between the domestic and foreign country are defined as the difference in prices:

$$s_t = p_{f,t} - p_{h,t} \tag{12}$$

Given the law of one price, we may rewrite this identity to express the terms of trade as a function of the effective nominal exchange rate e_t, world prices p_t^*, and domestic prices $p_{h,t}$ as follows:[5]

$$s_t = e_t + p_t^* - p_{h,t} \tag{13}$$

Assuming the Euler condition holds for j, international risk is given by the following expression that links domestic consumption c_t with world consumption c_t^* and the effective terms of trade s_t:

[3] Where $i_t \equiv -\log Q$ is the short-term nominal rate of interest, $\rho \equiv -\log\beta$ is the discount rate and $\pi \equiv p_t - p_{t-1}$ is the rate of consumer price inflation. Aggregate prices used to calculate CPI inflation are given by $p_t \equiv \log P_t$.

[4] Where the term $\mu \equiv \log\frac{\varepsilon}{\varepsilon-1}$ captures the steady-state markup under flexible prices and marginal costs are given by $mc_{t+k|t} = -v + w_t - p_{h,t} - a_t$, in which the employment subsidy is captured by the term $v = ln(1-\tau)$.

[5] This assumes prices $P_{j_i,t} = \epsilon_{j,t} P^j_{j_i,t}\ \ \forall\ i,j \in [0,1]$, where $\epsilon_{j,t}$ is the bilateral nominal exchange rate and $P^j_{j_i,t}$ is the price of goods in i; it can be easily shown that $p_{f,t} = \int_0^1 (e_{j,t} + p^j_{j,t})dj = \int_0^1 e_{j,t}dj + \int_0^1 p^j_{j,t}dj = e_t + p_t^*$.

$$c_t = c_t^* + \sigma^{-1}(1-\alpha)\, s_t \tag{14}$$

Finally, we may derive an expression for consumer price inflation whose deviation from domestic inflation is proportional to the change in terms of trade given by the coefficient α as follows:

$$\pi_t = \pi_{h,t} + \alpha \Delta s_t \tag{15}$$

3.4 Clearing

Equilibrium in the goods market yields the dynamic IS curve, relating the output gap to deviations in the real interest rate from the natural rate of interest and expectations of the future output gap:

$$\tilde{y}_t = E_t\{\tilde{y}_{t+1}\} - \sigma_\alpha^{-1}(i_t - E_t\pi_{h,t+1} - r_t^n) \tag{16}$$

and the New Keynesian Phillips curve, which also nests the special case of a closed economy (for $\alpha = 0$), relating domestic inflation to deviations in output from potential and expected inflation:[6]

$$\pi_{h,t} = \beta E_t \pi_{h,t+1} + \kappa_\alpha \tilde{y}_t \tag{17}$$

The natural rate of interest may be expressed as a function of model parameters, and in particular, due to openness, the expected change in future world output, in addition to domestic productivity:[7]

$$r_t^n = \rho - \sigma_\alpha \Gamma_a (1-\rho_a) a_t + \frac{\alpha \Theta \sigma_\alpha \varphi}{\sigma_\alpha + \varphi} E_t \Delta y_{t+1}^* \tag{18}$$

3.5 Exchange

In this section we derive a general expression for the equilibrium exchange rate. In particular, we combine the terms of trade with an equation for the domestic output that arises from the IS curve:

$$e_t = p_t - p_t^* + \sigma_\alpha(\tilde{y}_t - \tilde{y}_t^*) + \sigma_\alpha(y_t^n - y_t^{n*}) \tag{19}$$

The natural rate of interest clearing the domestic (foreign) market for goods conditional on foreign (domestic) output and the domestic (foreign) natural rate of output may be written as follows:

$$r_t^n = \rho + \sigma_\alpha \alpha \Theta E_t \Delta y_{t+1}^* + \sigma_\alpha E_t \Delta y_{t+1}^n \tag{20}$$

The natural rate (r*) differential is therefore given by the expected differential between the home and foreign equilibrium output given the unconditional mean of output at home and abroad equate:

$$r_t^n - r_t^{n*} = \sigma_\alpha (E_t \Delta y_{t+1}^n - E_t \Delta y_{t+1}^{n*}) \tag{21}$$

[6] Where the slope of the New Keynesian Phillips curve is $\kappa_\alpha = \lambda(\sigma_\alpha + \varphi)$ in which $\lambda = \theta^{-1}(1-\beta\theta)(1-\theta)$.
[7] Where $\sigma_\alpha = \sigma^{-1}(1+\alpha(\omega-1))$, $\Gamma_\alpha = (\sigma_\alpha+\varphi)^{-1}(1+\varphi)$ and $\Theta = \omega - 1 = (\sigma\gamma - 1) + (1-\alpha)(\sigma\eta - 1)$.

It follows that the equilibrium exchange rate specified in equation (19) can be written as a general function of a price differential, output gap differential and expected natural rate differential:

$$e_t = p_t - p_t^* + \sigma_\alpha(\tilde{y}_t - \tilde{y}_t^*) - E_t \sum_{i=0}^{\infty} \left(r_{t+i}^n - r_{t+i}^{n*}\right) \tag{22}$$

In the special case (where $\sigma = \gamma = \eta = 1$), the natural rate of output and interest are functions of domestic productivity only.[8] The equilibrium exchange rate may finally be expressed as follows:

$$e_t = p_t + \tilde{y}_t - E_t \sum_{i=0}^{\infty} \left(r_{t+i}^n - r_{t+i}^{n*}\right) \tag{23}$$

This equation expresses the equilibrium exchange rate as a function of a purchasing power parity component in the form of prices at time t, a cyclical component in the form of the output gap at time t and an expected natural rate of interest differential at time $t + i$.

This particular result challenges the existing literature in numerous ways. Standard approaches to exchange rate determination often struggle to establish an empirical relationship between real exchange rates and real interest rate differentials. We believe the problem may lie with the framing of this theoretical relationship rather than its estimation. In particular, exchange rates are often expressed as a function of interest rate differentials within a *disequilibrium* setting. Whilst exchange rates are tested for their stationarity to validate the law of purchasing power parity, their equilibria does not extend to real interest rates. Our findings from the Galí and Monacelli (2005) small open economy model suggests that real equilibrium exchange rates are a function of expected long-run *equilibrium* real interest rate differentials, consistent with asset market equilibrium, goods market equilibrium, and long run purchasing power parity.

Given this structural relation, we proceed to empirically test the relationship between the equilibrium real exchange rate and natural rate of interest. This paper follows a reduced-form approach consistent with the existing literature. In addition, we focus on r-star as the exclusive determinant. Our parsimonious model therefore requires data on equilibrium exchange rates and equilibrium interest rates. In respect to the former, the theory of purchasing power parity implies that exchange rates are a mean-reverting stationary process. In this regard, we test the stationarity of bilateral exchange rates between each member of the G7 and exploit our multiple cross sections using second generation panel unit root tests to pass judgment on their long-run stability. To derive a series for the latter, we estimate the semi-structural natural rate of interest rate consistent with inflation and output stability using the benchmark HLW maximum likelihood system.

4 Specification

Our model specification follows that of Holston, Laubach and Williams (2017). In particular, we relax the New Keynesian model of Galí and Monacelli (2005) by initially specifying the following reduced equations of (16) and (17) that permit shocks to the output gap and inflation respectively:

[8] In particular, natural rates of interest and output are defined by $r_t^n = \rho + E_t \Delta a_{t+1}$ and $y_t^n = \Gamma_0 + a_t$.

$$\tilde{y}_t = \sum_{i=1}^{2} a_{y,i}\tilde{y}_{t-i} + \frac{a_r}{2}\sum_{i=1}^{2}\tilde{r}_{t-i} + \varepsilon_{\tilde{y},t} \tag{24}$$

$$\pi_t = b_\pi \pi_{t-1} + \frac{1-b_\pi}{3}\sum_{i=2}^{4}\pi_{t-i} + b_y\tilde{y}_{t-1} + \varepsilon_{\pi,t} \tag{25}$$

where $\varepsilon_{\tilde{y},t}$ and $\varepsilon_{\pi,t}$ denote transitory shocks to the output gap and inflation respectively, and $\tilde{r}_t$ is the real rate gap given by $r_t - r_t^*$ in which r_t^* captures persistent shocks to the relationship between r_t and $\tilde{y}_t$. The law of motion for the natural rate of interest is specified by the following equation:

$$r_t^* = g_t + z_t \tag{26}$$

where g_t is the trend growth rate of the natural rate of output and z_t is the error term capturing other factors driving r-star. In addition to these measurement equations, we specify the following transition equations of the state-space system as in Holston, Laubach and Williams (2017):

$$y_t^* = y_{t-1}^* + g_{t-1} + \varepsilon_{y*,t} \tag{27}$$

$$g_t = g_{t-1} + \varepsilon_{g,t} \tag{28}$$

$$z_t = z_{t-1} + \varepsilon_{z,t} \tag{29}$$

where (27) defines log potential output y_t^* as a random walk with drift g, which also follows a random walk process specified in equation (28). Finally, equation (29) captures the unobserved component of the natural rate, which itself is also a random walk.[9] The measurement and transition equations may be summarised as follows (see Appendix B for the full state-space representation):

$$\begin{aligned}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} 1-a_{y,1} & -a_{y,2} & 1-a_r & -a_r \\ -b_y & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ z_{t-1} \end{bmatrix} + \\ &+ \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 & a_r \\ b_y & 0 & b_\pi & 1-b_\pi & 0 \end{bmatrix}\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \pi_{t-1} \\ \pi_{t-2|4} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\tilde{y},t} + \varepsilon_{y*,t} \\ \varepsilon_{\pi,t} \end{bmatrix}\end{aligned} \tag{30}$$

[9] We assume here that the error terms in each transition equation are contemporaneously uncorrelated and normally distributed with the following first and second moments; $\mu_{y*,t} \sim (N, \sigma_{y*}^2)$, $\mu_{g,t} \sim (N, \sigma_g^2)$ and $\mu_{z,t} \sim (N, \sigma_z^2)$.

$$
\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ z_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y*,t} \\ 0 \\ \varepsilon_{g,t} \\ \varepsilon_{z,t} \end{bmatrix} \tag{31}
$$

Linearity in this system allows us to estimate the natural rate of interest, the natural rate of output and its trend growth rate using the Kalman filter.[10] However, as both real growth rates and interest rates are often influenced by highly persistent shifts, maximum likelihood estimation would likely produce estimates for the standard deviations of innovations (σ_g and σ_z) that are biased towards zero. To resolve this 'pile-up problem' (Stock, 1994), we follow LW (2003) by using the median unbiased estimator to derive estimates of ratios $\lambda_g = \sigma_g/\sigma_{y*}$ and $\lambda_z = a_r\sigma_z/\sigma_{\tilde{y}}$ that are imposed as restrictions on the remaining parameters (Stock and Watson, 1998).

As the focus of this paper concerns the implications of r-star, our modifications to the model are minimal. Instead, we implement this system to derive intermediate estimates that are subsequently used to examine the behaviour of equilibrium exchange rates and optimal monetary policy within the advanced world. In particular, we specify the lag structure in the DIS and Phillips curve, such that two lags of the output gap and the average of two lags of the real rate gap enter the former, and one lag of the output gap in addition to the first and average of the second to fourth lag of inflation enter the latter.[11] Furthermore, we calibrate the ratio of stochastic innovations for countries other than the United States in accordance with the existing literature.

Our implementation therefore proceeds as follows and in accordance with the HLW procedure. We cast equations (30)-(31) into a state-space system and first estimate the natural rate of output barring the real rate gap and assuming constant trend growth. We then derive the median unbiased estimate for the ratio between the standard deviations of innovations for the natural rate of output and its trend growth rate, which we impose as a restriction in our second estimation including the real rate gap. We derive a ratio for the component unrelated to trend growth in a similar manner, which we impose in the third stage to estimate the remaining model parameters.

We focus our analysis on seven advanced open economies that make up a significant portion of global gross domestic product and maintain a large degree of integration. In contrast to much of the literature, we isolate the three highest performing economies of the Euro Area (EA) by GDP, rather than losing them to aggregation. In addition, we also estimate the natural rate of interest for Japan, which is seldom found in the literature. Furthermore, we investigate comovement in r-stars and the implications of their secular decline for equilibrium exchange rates and optimal monetary policy between members of the G7. We thus source data on real GDP, CPI inflation and short-term interest rates between 1970:1-2019:4. Data specifics are further outlined in Appendix C.

[10] Confidence intervals for these estimates and their standard errors are calculated using a Monte Carlo procedure under constraints, which accounts for both filter and parameter uncertainty (see Hamilton (1986) for further details).

[11] The vector of unobserved states contains three lags of potential output in each stage. The second stage includes one lag of g and the third includes a second lag of g, in addition to two lags of z. We compute the conditional expectation and covariance matrix of the initial state by applying the Hodrick-Prescott filter in the maximum likelihood procedure.

5 Results

5.1 Parameter Estimates

Parameter estimates are summarised in Table 1. Time variation in trend growth and the natural rate of interest for the United States is substantial, as indicated by relative median-unbiased estimates of the innovations to σ_g and σ_z, in the ratios λ_g and λ_z respectively. Both slope parameters a_r and b_y associated with the interest rate and output gap are relatively large and statistically significant, suggesting that they are well identified. In accordance with the literature, one-sided Kalman filter estimates of r-star are imprecise with a standard error of 1.2 percentage points; yet this is the least uncertainty in estimates of the natural rate between the advanced economies.

The United Kingdom still remains unique within the G7, owing to a combination of parameter estimates. For instance, b_y is relatively high, suggesting that inflation is more sensitive to transitory shocks in the output gap; a_r is negligible, indicating a weak relationship between the real rate and output gap; and σ_π is relatively large, suggesting inflation is poorly explained by lags in inflation and the output gap. In addition, our estimates of the natural rate are also imprecise, with an average standard error of approximately 2.4 percentage points.

Estimates for Japan also reveal substantial time variation in trend growth and the natural rate of interest. Slope parameters are also large in relative magnitude and statistically significant, indicating strong identification. Standard deviations in trend growth g and the unobservable component z are also particularly high in Japan, suggesting large uncertainty in estimates of trend growth and the natural rate of interest. We also note that standard errors for the natural rate of interest in Japan are the highest in the G7 at approximately 3.73 percentage points.

As for those members within the Euro Area, our estimation of the top three largest economies by shares of real gross domestic product is informative. Germany clearly distinguishes itself from the other EA members with negligible variation in the natural rate, higher sensitivity of inflation to transitory output gap shocks and far more precise one-sided estimates of the natural rate of interest, with an average standard error of roughly 1.2 points. This is in contrast to France and Italy, whose standard errors are much higher at 3.9 and 3.8 points respectively. In addition, estimates for b_y are lower in France and Italy, indicating less sensitivity to output shocks.

Finally, parameter estimates for Canada also suggest variation in the natural rate of interest and trend growth. The Canadian output gap is the most persistent among the other advanced economies as indicated by higher values for $\sum a_y$. In addition, inflation responds the least to transitory output gap variation, as evidenced by lower estimates of b_y. Estimates of r-star are highly imprecise with a standard error of roughly 3.7 percentage points. In addition, inflation is explained poorly by lags in inflation and the output gap, as evidenced by higher levels for σ_π.

Consistent with the existing literature, our results suggest that parameter estimates for the G7 indicate substantial variation in the natural rate of interest and high standard errors from parameter and filter uncertainty. In addition, we find varying degrees of identification in the measurement and transition equations of the state-space system across the advanced world. To proceed, we examine this estimated variation in r-stars before characterising their comovement by way of forecast error variance decompositions (FEVD) from a vector error correction model (VECM).

Table 1. Parameter Estimates

Parameter	US	UK	JP	DE	FR	IT	CA
λ_g	0.053	0.020	0.046	0.053	0.011	0.054	0.034
λ_z	0.036	0.018	0.053	0.003	0.022	0.029	0.039
Σa_y	0.943	0.906	0.805	0.898	0.927	0.890	0.971
a_r	-0.069	-0.006	-0.060	-0.061	-0.056	-0.040	-0.049
b_y	0.078	0.642	0.130	0.137	0.030	0.109	0.025
$\sigma_{\tilde{y}}$	0.339	0.069	0.432	0.818	0.938	0.427	0.322
σ_π	0.789	2.630	1.433	1.115	1.121	1.676	2.159
σ_{y*}	0.573	0.868	0.845	0.587	0.398	0.657	0.906
σ_g	0.121	0.069	0.152	0.124	0.018	0.142	0.123
σ_z	0.178	0.207	0.443	0.040	0.369	0.310	0.256
σ_{r*}	0.215	0.218	0.374	0.130	0.369	0.341	0.284
SE (avg)							
$r*$	1.180	2.400	3.734	1.220	3.891	3.793	3.678
g	0.400	0.414	0.520	0.429	0.507	0.471	0.538
$y*$	1.526	0.633	0.696	1.421	2.022	1.666	3.760

Note: Estimated parameters of the Holston, Laubach and Williams (2017) model. Average standard errors of the natural rate of interest, the natural rate of output and its trend growth are reported in the last three rows. Standard errors are calculated as in Hamilton (1986). σ_g is expressed here as an annualised rate.

5.2 Natural Rates of Interest

Estimates of r-star are presented in Figure 1 and indicate three phases of stagnation. In this paper we attempt to distinguish local from global shocks. To do this, we aggregate natural rates by PPP-GDP weighted averages. Firstly, G7 estimates show that between 1970-1990, the advanced natural rate of interest declines sharply from approximately 3.5-4% to 1.5-2%. Natural rates across the G7 recover on average, before declining moderately between 1990-2007 from approximately 2.5-3% to 2-2.5%. Thirdly, r-stars fall precipitously between 2008-2009 during the Great Recession, to a new normal of 0-0.5 percentage points, persisting for over a decade between 2009-2019.

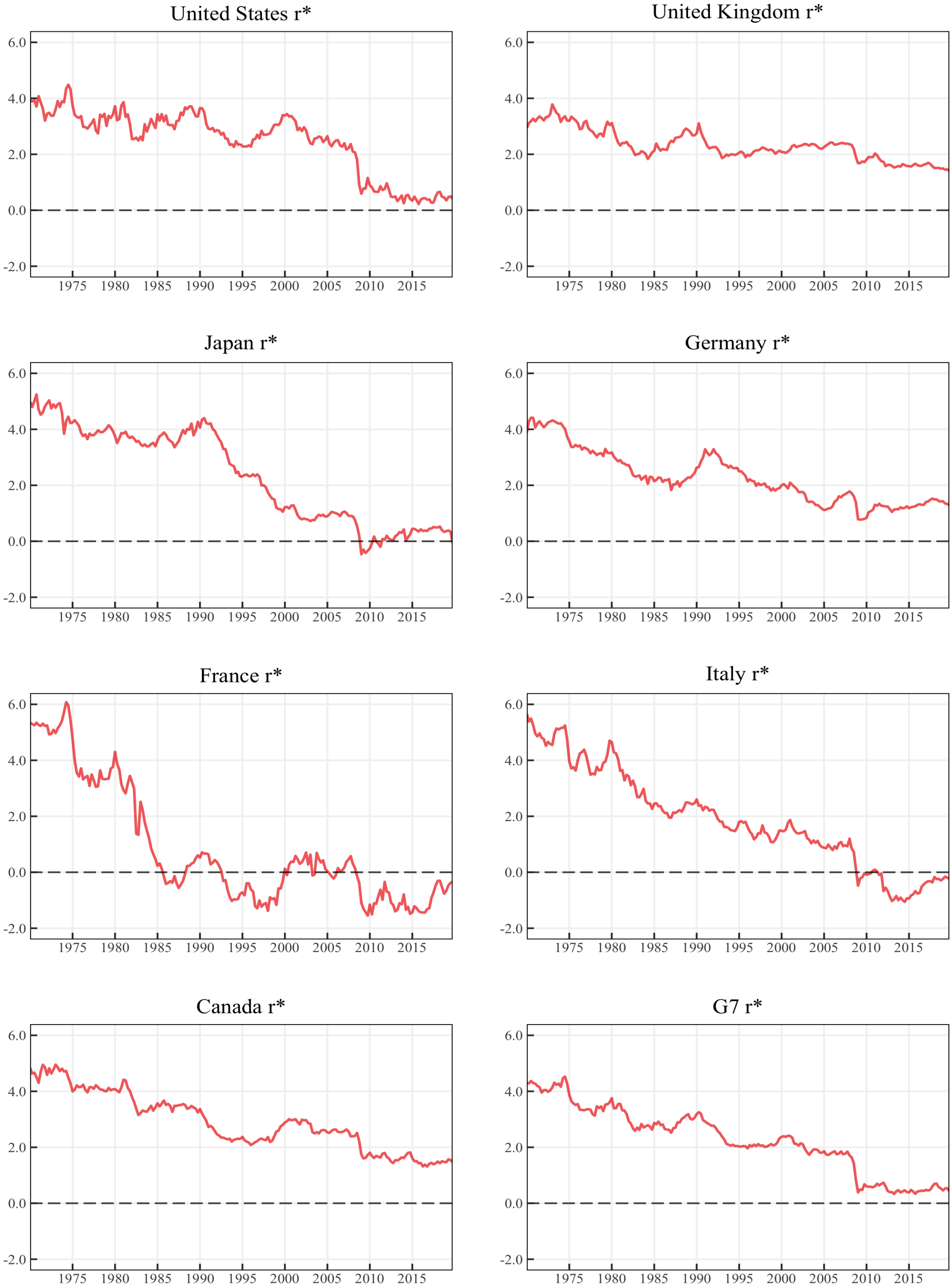

Figure 1. Advanced Natural Rates of Interest

Note: One-sided natural rate of interest (r) estimates for the advanced world (1970:1-2019:4). Aggregate G7 estimates are calculated using PPP-GDP weighted averages of the natural rate for each member state. See Appendix D for remaining one-sided estimates of the natural rate of output and its trend growth rate.*

The advanced economies all experience sharp reductions in r-star during the Great Recession, with France, Italy, and Japan among those to experience negative natural rates. This is largely explained by a similar decline in the trend growth rate, albeit for some nations such as the United States there is substantial unobserved heterogeneity. Natural rates for leading members of the Euro Area suffer further declines, chiefly due to the Euro crisis following the Great Recession, albeit with Germany relatively unscathed thereafter. From our estimates, it is also clear that variation in the natural rate after the crisis is also explained by variation in trend growth rates for most of the countries in our sample (see Appendix D for natural rate of output and trend growth rate estimates). These results prove that the financial crisis of 2008 was clearly a global symmetric shock that placed downward pressures on the natural rate across the advanced world. In addition, crises occurring around 1975 and 1990 also seem common across each of the G7 countries, as evidenced by similar downward pressures on the natural rate of interest and trend growth rates.

We note here that our analysis omits data post 2019 due to the sharp effects of the Great Lockdown that was enforced across the advanced world to suppress the coronavirus pandemic. Despite it being a clear symmetric international shock, Holston, Laubach and Williams (2020) have rightly argued that this unprecedented crisis is an extreme-tailed event that violates the Kalman filter, in which stochastic innovations are assumed Gaussian in distribution. Furthermore, transitory shocks to supply in the form of innovations in the Phillips curve are assumed serially uncorrelated, which is inconsistent with the nature of the Great Lockdown. Thus, estimates of r-star during this period are likely distorted. HLW (2020) modify their state-space system to account for this unique crisis by including a persistent albeit temporary supply shock that accounts for the effects of restrictions during the pandemic.[12] This indicator aggregates various measures of containment capturing legal restrictions, limitations to travel, and the closure of public transport. Given we are interested in the theoretical and empirical implications of time varying natural rates, we do not employ the adjusted model and restrict our analysis to the period preceding the pandemic.

5.3 Global Drivers of R-star

To confirm the international factor in natural rates of interest, we construct a vector error-correction model as in Holston, Laubach and Williams (2017), to inspect the co-movement between advanced economies. Augmented Dickey-Fuller (ADF) tests confirm non-stationarity in each natural rate of interest, motivating our VECM. The Johansen test for cointegration suggests a single cointegrating vector that unites r-star between members of the G7. Results are reported in Table 2 and suggest substantial comovement in the natural rate of interest. In addition, error correction coefficients are negative as required and indicate a slow adjustment to equilibria. Given natural rates of interest are themselves estimated with large uncertainty, caution is taken in our interpretation of these results; we focus primarily on the comomevemnt between the advanced economies. Notwithstanding, our estimates corroborate the existence of a global factor in r-star that warrants further analysis.

[12] In particular, the technical adjustment to the natural rate of output is given by $y_t^* + \zeta \frac{d_t}{100}$, where ζ is an estimated coefficient that captures the effects of the COVID-19 indicator d_t on output. The output gap is therefore expressed in terms of the adjusted natural rate of output as $100 \cdot (y_t - y_t^*) - \zeta d_t$.

Table 2. Vector Error Correction Model

$r = 1, k = 2$	US	UK	JP	DE	FR	IT	CA
ADF Test (p)	0.56	0.48	0.65	0.32	0.69	0.11	0.60
Cointegrating Vector (SE)	1 (-)	-0.45 (0.21)	0.18 (0.11)	-0.65 (0.22)	0.63 (0.08)	-0.11 (0.19)	-1.41 (0.18)
Error Correction Coefficient (SE)	-0.05 (0.04)	-0.00 (0.02)	-0.03 (0.03)	-0.00 (0.02)	-0.20 (0.05)	-0.07 (0.03)	-0.14 (0.03)
Granger Test (p)	< 0.01	0.07	0.02	0.06	0.02	0.03	0.03

Note: Estimates of a vector error correction model with one cointegrating vector (r) for the natural rate of interest (r^*) between each of the advanced economies. Standard errors are reported in parentheses.

Figure 2 presents error variance decompositions for the natural rate of interest across the advanced world over a 50-year horizon. Our results indicate substantial interdependence between members of the G7. International shocks to natural rates of interest have little influence on the natural rate of interest in the United States, capturing merely 5-10% of the variation in r-star over the long-run horizon. In the United Kingdom, the impact of foreign variation is clearly much higher, accounting for approximately 60-65% of domestic variation in the long run with the United States, Germany, and Japan accounting for the largest shares. As regards Japan, only 10-15% of domestic variation is explained by foreign variation in the natural rate of interest over a long-run horizon, with shocks to r-star in the United States accounting for the largest share.

As expected, decompositions for Germany, France, and Italy indicate strong interdependence, with variation in their natural rates accounting for a substantial portion of variation in each other. Foreign variation has a minor impact on r-star in Germany over the short run. This grows significantly over the long run to account for 75-80% of the domestic variation and is attributable to most members of the G7. In addition, roughly 50-55% of the long-run variation in France is explained by variation in the United States, Canada, and Germany. Whilst shocks to r-star account for only a small amount of the variation in Italy over the short run, it is the most interdependent over the long run, with around 90%-95% of the variation in r-star explained by shocks in remaining members of the G7. Finally, variation in r-star abroad accounts for roughly 35-40% of the variation in Canada over the long run, with the United States accounting for the largest share.

These results make a clear case for a global component in the natural rate of interest. Our result in Section 3.5 suggests that variation in natural rates between foreign and domestic economies have consequences for equilibrium exchange rates. In particular, global symmetric shocks that cause r-stars to comove have no implications for the equilibrium exchange rate between any two countries, whilst local asymmetric shocks that cause divergence in r-stars will place pressure on equilibrium exchange rates. In the next section, we attempt to characterise this relationship empirically, before discussing the implications of falling r-stars on exchange rates and optimal monetary policy.

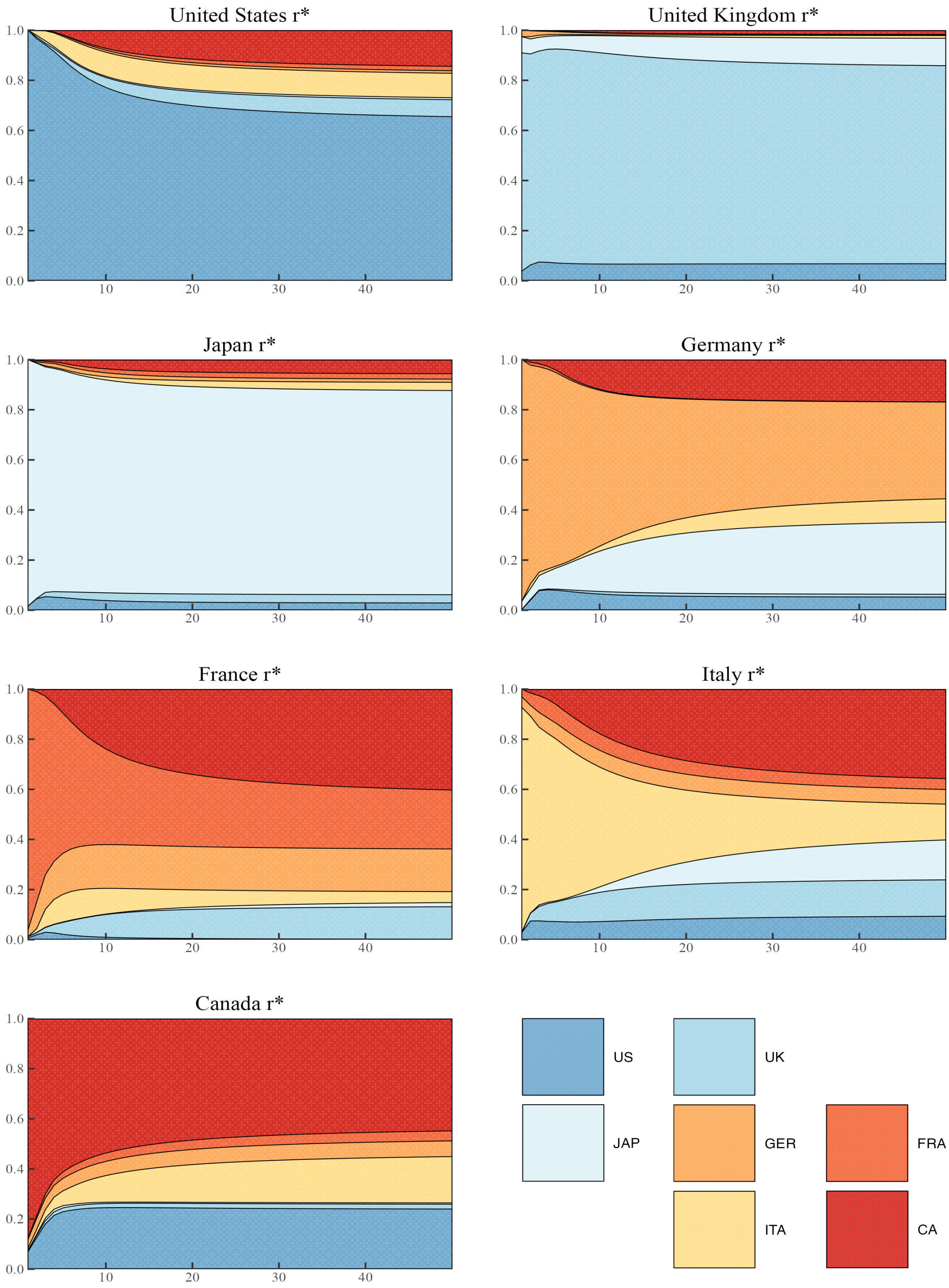

Figure 2. Forecast Error Variance Decompositions

Note: Forecast error variance decomposition of the natural rate of interest (r^*) across the advanced world (50 period horizon). Results extracted from a vector error correction model with one cointegrating vector.

5.4 Equilibrium Exchange Rates

To establish equilibria in real exchange rates, we test their stationarity, and by extension, the theory of purchasing power parity (PPP). Given we are also interested in cross-country r-star differentials, we test their stationarity alongside real exchange rates. Our sample ranges from 1965:1 to 2019:4 and involves data on nominal exchange rates sourced from the OECD Main Economic Indicators (MEI) database. This paper uses data sourced earlier on the consumer price index to compute real exchange rates. Individual Augmented Dickey-Fuller tests are reported in Appendix E. Results are inconclusive and suggest less than half of the bilateral exchange rates are stationary at the 10% significance level, with almost none at the 5%. In contrast, individual unit root tests detect strong nonstationarity in all differentials between members of the G7.

Given the lack of equilibria in real exchange rates established by individual unit root tests, we proceed to exploit the rich panel structure of our data and test the stationarity of exchange rates across the advanced world. In this regard, significant developments in the literature have rendered standard first-generation approaches (see Levin, Lin and Chu, 2002; Im, Pesaran and Shin, 2003) to exchange rates and purchasing power parity inappropriate in the presence of cross-sectional dependence. As comovement in the open economy is inescapable, it is important to test for such dependence and adjust our analysis accordingly. We therefore implement the Pesaran (2015) CD test for cross-sectional dependence and estimate the exponent of dependence à la Bailey, Kapetanios and Pesaran (2016) to determine its strength across our sample.

These results are reported in Table 3. The CD test strongly rejects the null hypothesis of weak cross-sectional dependence for both real exchange rates and real natural rate differentials in each member state. As for our estimates of the exponent, Chudik et al. (2011) propose four categories of dependence; weak ($\alpha = 0$), semi-weak ($0 < \alpha < 0.5$), semi-strong ($0.5 \leqslant \alpha < 1$), and strong ($\alpha = 1$). Under weak dependence, the aggregate effect of the common factors remains invariant in the limit of the cross-sections. Our estimates for the exponent α using the bias-adjusted estimator of Bailey, Kapetanios and Pesaran (2016) clearly suggest strong presence of dependence, in which the aggregate effect of the common factors increases in strength in the limit of the cross-sections. These results highlight the need for second-generation approaches that account for cross-sectional dependence in the estimation of panel data models.

To proceed, we use the cross-sectionally augmented Im, Pesaran and Shin (2003) test to examine the stationarity of real exchange rates and real natural rate differentials across the G7. Results are presented in Table 4. The CIPS test finds strong evidence against the null hypothesis of a unit root in real exchange rates across all panels. This is also largely robust to the inclusion of a trend, barring France and Italy. As for r-star differentials, our findings are less conclusive, and in many instances, not robust to the inclusion of a time trend. It is clear that nonstationarity prevails across a number of panels, and given individual ADF tests confirm unit roots across all differentials, it is reasonable to conclude at the very least mixed orders of integration. We thus focus our subsequent analysis on second-generation panel cointegration tests and proceed to test the long and short-run relationship between real equilibrium exchange rates and real equilibrium interest rate differentials in a cross-sectionally dependent panel autoregressive distributed lag model.

Table 3. Cross-sectional Dependence Tests

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
	cd	exp	cd	exp
United States	40.05***	0.90***	19.79***	1.00***
United Kingdom	18.14***	0.90***	19.98***	1.00***
Japan	52.82***	1.00***	40.39***	1.00***
Germany	17.75***	0.90***	10.74***	1.00***
France	5.72***	0.75*	50.47***	1.00***
Italy	13.80***	0.90***	27.21***	1.00***
Canada	33.42***	0.90***	4.26***	1.00***

*Note: Cross-sectional dependence (CD) test statistics (Pesaran, 2015) and the estimated exponent of cross-sectional dependence (Bailey, Kapetanios and Pesaran, 2016). Each member is fixed as the foreign economy in a panel of real exchange rates and r-star differentials. */**/*** denote significance at 10%, 5% and 1%.*

Table 4. Panel Unit Root Tests

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
	const	+ trend	const	+ trend
United States	-2.90***	-3.08***	-1.73	-3.47***
United Kingdom	-2.66***	-2.90***	-2.73***	-2.72
Japan	-3.33***	-3.28***	-2.85***	-3.28***
Germany	-3.10***	-3.26***	-2.07	-3.02**
France	-2.44**	-2.72	-2.36**	-3.83***
Italy	-2.62***	-2.60	-2.56***	-2.38
Canada	-3.01***	-3.06***	-1.51	-2.63

*Note: Test statistics associated with the cross-sectionally augmented Im, Pesaran and Shin (2003) (CIPS) unit root test including a constant, with and without a trend. Each member is fixed as the foreign economy in a panel of real exchange rates and r-star differentials. */**/*** denote significance at 10%, 5% and 1%.*

5.5 The REEREI Relation

We investigate the long-run real equilibrium exchange rate-real equilibrium interest rate (REEREI) relationship using multiple panel cointegration tests that accommodate cross-sectional dependence (Westerlund, 2007). These second-generation tests rely upon structural dynamics and do not suffer from the large losses of power in residual cointegration tests arising due to the failure of imposed common-factor restrictions. The initial group-mean tests (G_τ and G_α) test the alternative that one unit of the panel is cointegrated, whilst the latter panel-based tests (P_τ and P_α) test if the panel is entirely cointegrated. Our results for all four test statistics are presented in Table 5.

Table 5. Panel Cointegration Tests

1965:1-2019:4	G_τ	G_α	P_τ	P_α
United States	-2.66***	-11.70**	-6.92***	-13.01***
United Kingdom	-2.99***	-15.23***	-7.17***	-14.58***
Japan	-2.38*	-9.45	-5.84**	-8.39**
Germany	-2.81***	-12.89***	-6.87***	-11.79***
France	-3.23***	-17.41***	-7.49***	-15.64***
Italy	-2.65***	-12.49***	-6.55***	-12.71***
Canada	-2.67***	-13.12***	-6.73***	-13.67***

*Note: Group and panel test statistics associated with the Westerlund (2007) tests for panel cointegration including a constant. Lag length is selected by the AIC. Each member is fixed as the foreign economy in a panel of real exchange rates and r-star differentials. */**/*** denote significance at 10%, 5% and 1%.*

Westerlund (2007) tests strongly reject the null hypothesis of a lack of cointegration in favour of both alternatives across the G7. Japan is the only exception to this in respect to group-mean tests, albeit we continue to reject the null associated with both panel-based tests for cointegration. Given the clear presence of a long-run relationship between equilibrium exchange rates and natural rates of interest, we proceed to estimate an appropriate model that accounts for cross-sectional dependence. As our unit root tests suggest mixed orders of integration, we construct and estimate a novel panel error correction representation of a cross-sectionally augmented autoregressive distributed lag model using the dynamic common correlated effects estimator.[13]

To outline this approach, consider the following theoretical dynamic ARDL(1,1) panel model of the relationship between a response variable $y_{i,t}$, and an observed explanatory variable $x_{i,t}$ with unobserved common factors and heterogeneous coefficients:

$$y_{i,t} = \mu_i + \lambda_i y_{i,t-1} + \alpha_{0,i} x_{i,t} + \beta_{1,i} x_{i,t-1} + u_{i,t} \tag{32}$$

$$u_{i,t} = \sum_{l=1}^{m} \zeta_{y,i,t} f_{t,l} + e_{i,t} \tag{33}$$

$$x_{i,t} = \sum_{l=1}^{m} \zeta_{x,i,t} f_{t,l} + v_{i,t} \tag{34}$$

where μ_i is the unit-specific fixed effect and $v_{i,t}$ is the cross-section unit-specific error. m denotes the number of unobserved common factors $f_{t,l}$ in the explanatory variable. Loading coefficients $\zeta_{y,i,t}$ and $\zeta_{x,i,t}$ are heterogeneous across units and heterogeneous coefficients are functions of random deviations with zero mean independent of the error term and common factors.

[13] Refer to Chudik et al. (2016) for the original implementation and related applications to cross-sectional augmented distributed lag models. To our knowledge, such approaches to panel cross-sectional dependence have not been widely applied in the real exchange rate real interest rate literature, which is a novelty in our research.

As Chudik and Pesaran (2015) have outlined, equation (32) may be estimated consistently by an estimator that approximates the common factors with the cross-sectional averages, yielding the following cross-sectionally augmented ARDL specification:[14]

$$y_{i,t} = \mu_i + \lambda_i y_{i,t-1} + \alpha_{0,i} x_{i,t} + \beta_{1,i} x_{i,t-1} + \sum_{l=0}^{p} \boldsymbol{\gamma}'_{i,l} \bar{\mathbf{z}}_{t-l} + e_{i,t} \tag{35}$$

where $\boldsymbol{\gamma}_{i,l} = (\gamma_{y,i,l}, \gamma_{x,i,l})$ are the estimated coefficients of the cross-sectional averages of the dependent and independent variables given by $\bar{\mathbf{z}}_{t-l} = (\bar{y}_t, \bar{x}_t)'$. This paper estimates the augmented model by the mean group estimator (Pesaran, 2006; Chudik and Pesaran, 2019), otherwise known as the common correlated effects mean group estimator (CCE-MG). In the steady state, the long-run effect of the explanatory variable is given by:

$$\theta_i = \frac{\alpha_{0,i} + \beta_{1,i}}{1 - \lambda_i} \tag{36}$$

Finally, the cross-sectionally augmented ARDL model given by (35) may be reparameterised into the following error correction model (ECM) that is a function of the short and long-run dynamics in addition to an error correction component:

$$\Delta y_{i,t} = \mu_i - \phi_i \big(y_{i,t-1} - \theta_{1,i} x_{i,t} \big) - \beta_{1,i} \Delta x_{i,t} + \sum_{l=0}^{p} \boldsymbol{\gamma}'_{i,l} \bar{\mathbf{z}}_{t-l} + e_{i,t} \tag{37}$$

where $\phi_i = (1 - \lambda_i)$ is the error correction coefficient (speed of adjustment) associated with the error correction term, $\alpha_{0,i}$ is the short-run effect and θ_i is the long-run effect defined in (36). Cross-sectional averages clearly prevail in this ECM representation of the ARDL model and estimates of the long-run effect are numerically identical.

Model estimates are presented in Table 6. Our results show a statistically significant negative relationship between real equilibrium exchange rates and real equilibrium interest rate differentials across most members of the G7. The United Kingdom and Italy are distinguished by large long-run coefficients, in which a marginal rise in the differential between domestic and foreign natural rates of interest is associated with a fall in the real equilibrium exchange rate by approximately 23 and 25 percentage points respectively. The United States exhibits a similar long-run relationship, with a marginal rise in the differential leading to an 10 percentage point reduction in the exchange rate. Germany is the only member that exhibits a positive relationship. We caveat our results once more by noting that our estimates rely on r-stars that are themselves generated regressors from an estimated state-space model. Our objective here is to purely characterise any comovement. In this regard, we find that the direction of our coefficients are largely in conformity with our theoretical result for exchange rate determination. Finally, we note here that error correction coefficients are negative and statistically significant, suggesting a gradual speed of adjustment toward equilibria.

[14] We follow Chudik and Pesaran (2015) by including the floor of $\sqrt[3]{T}$ lags of the cross-sectional averages in our specification. As our panel for each member of the G7 consists of 6 groups each with 216 observations, this implies that $p = \sqrt[3]{216} = 6$ lags are to be included in the augmented autoregressive distributed lag model.

Table 6. Panel ARDL-ECM Estimates

1965:1-2019:4	US	UK	JP	DE	FR	IT	CA
Short-run est.	-0.00	-0.02	-0.02	0.00	-0.00	-0.01	0.00
coefficient (SE)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Long-run est.	-0.10	-0.28	-0.01	0.05	-0.02	-0.25	-0.03
coefficient (SE)	(0.08)	(0.15)	(0.08)	(0.04)	(0.05)	(0.10)	(0.03)
Error-correction	-0.06	-0.05	-0.06	-0.05	-0.06	-0.05	-0.05
coefficient (SE)	(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)

Note: Estimated coefficients of the panel error correction representation of a cross-sectionally augmented autoregressive distributed lag model by the (dynamic) common correlated effects (mean group) estimator à la Chudik et al. (2016). 6 lags of the cs averages are included. Standard errors are reported in parentheses.

6 Implications

6.1 Exchange Rates

Widespread secular decline in natural rates of interest across the advanced world have significant implications for exchange rates between member states and collectively with the rest of the world. Equation (23) suggests that if the natural rate in i is expected to fall below that of the natural rate in j, the exchange rate must depreciate relative to domestic prices, given the output gap. Country-specific implications of time varying neutral rates due to domestic shocks are therefore captured by an inverse relationship between deviations from foreign natural rates and the equilibrium exchange rate. This result implies that global international crises, such as the Great Recession, place minimal pressure on exchange rates over the long-run across the advanced world. This is due to the fact that symmetric variation in the domestic and foreign natural rate of interest do not generate differentials that force exchange rate adjustments. Natural rates of interest therefore vary bilaterally to insulate the impact of external volatility on domestic equilibrium exchange rates.

We note here that whilst unobserved international heterogeneity ought to be controlled for, it is difficult to identify and isolate such phenomena across our panel of advanced economies. Existing research suggests that these shocks have also declined in magnitude over the last half-century (see for instance Stock and Watson, 2005; Kose et al., 2012). Fushing et al. (2010) propose three such episodes arising in our sample; the OPEC crisis of 1973, the dot-com crisis of 2001, and the global financial crisis of 2008. However, their symmetry is contested within the literature and it is unclear if the comovement in r-stars completely suppressed exchange rate pressures during such periods. We also acknowledge that the equilibrium exchange rate may be consistent with factors other than those related to the asset market, which, but for parsimony, we limit our analysis to. In this regard, it may be beneficial to incorporate other factors arising from the goods market that are thought to contribute to exchange rate determination. We leave such an endeavor to future research.

Finally, as natural rates of interest across the advanced world fall, exchange rate adjustments due to country-specific shocks may appear to be beggar-thy-neighbour and therefore imply a race to the bottom. However, these adjustments are in fact to balance trade and maintain equilibrium in the goods market. In addition, exchange rate depreciation due to expected country-specific shocks to r-star coincide with expected increases in domestic productivity and output, which necessitates greater levels of global demand to equilibrate the market for goods at home and abroad.

6.2 Monetary Policy

Galí and Monacelli (2005) show that under the assumptions for which domestic inflation targeting is optimal, the flexible equilibrium is a divine coincidence. To close this model, we minimise the following welfare loss function in the presence of nominal rigidity at an efficient steady state:

$$\max \mathbb{W} = -\tfrac{1}{2} E_0 \textstyle\sum_{t=0}^{\infty} \beta^t \left[\pi_{h,t}^2 + \alpha \tilde{y}_t^2\right] \tag{38}$$

which is subject to two constraints; the Dynamic Investment Savings curve, as specified in (16), and the New Keynesian Phillips curve as specified in (17) plus an auto regressive component:[15]

$$\text{s.t} \quad \tilde{y}_t = E_t \tilde{y}_{t+1} - \sigma_\alpha^{-1}(i_t - E_t \pi_{h,t+1} - r_t^n) \tag{39}$$

$$\text{s.t} \quad \pi_{h,t} = \beta E_t \pi_{h,t+1} + \kappa_\alpha \tilde{y}_t + u_t \tag{40}$$

Solving for optimal monetary policy using the Euler equation yields a forward looking Taylor rule with a time varying natural rate that consists of a global and country-specific component (refer to Appendix A.1 for proofs). Optimal policy in an open economy is therefore given by the system:[16]

$$i_t = r_t^n + \left(1 + \frac{\sigma_\alpha \kappa_\alpha (1 - \rho_u)}{\alpha \rho_u}\right) E_t \pi_{h,t+1} \tag{41}$$

$$e_t = p_{t-1} + \left(1 - \tfrac{\kappa_\alpha}{\alpha}\right) \delta u_t - E_t \textstyle\sum_{i=0}^{\infty} \left(r_{t+i}^n - r_{t+i}^{n*}\right) \tag{42}$$

Time varying natural rates in small open economies also have significant monetary policy implications. In this model, policy rates set by the central bank are a function of local and global factors. Declines in the domestic natural rate due to country-specific factors place pressure on local policy rates set by central banks. In addition, the global component in policy rates imply that symmetric shocks to natural rates in the advanced world are associated with positive cross-country correlated shifts in policy rates. This was particularly evident during the Great Recession, when policy rates across the advanced world exhibited a high degree of correlation in their decline. Shocks to r-star that are not country-specific therefore imply that any downward policy rate adjustments are merely reactionary rather than an active result of game theoretical coordination.

[15] Where the error component is given by $u_t = \rho_u u_{t-1} + \varepsilon_t^u$

[16] Where we define the parameter $\delta = \alpha^{-1}(\kappa_\alpha^2 + \alpha(1 - \beta\rho_u))$

Our result also implies that optimal monetary policy differentiates cost push shocks and natural rate of interest shocks in terms of their impact on market equilibrium. As regards the former, cost push shocks imply that policymakers must trade between inflation and output stability. In reaction to greater inflation expectations, monetary policymakers must increase nominal rates even further so as to force growth below the flexible equilibrium. As for the latter, shocks to r-star are reflected proportionally in policy rates, resulting in exchange rate variation. Therefore, shocks to the natural rate of interest under optimal monetary policy imply policy rate and exchange rate adjustments that are sufficient to preserve the flexible equilibrium in the small open economy.

As for welfare, time varying natural rates of interest imply optimal monetary policy is efficient under discretion given particular assumptions. In the unique case for which intertemporal elasticity of substitution is unity, the Nash equilibrium is a forward looking Taylor rule with a time varying equilibrium interest rate that maximises international welfare. However, this is not consistent with the general, and more probable case, that the elasticity of substitution is not unity ($\sigma \neq 1$). In this instance, Clarida (2019) uses the model of Clarida, Galí and Gertler (2002) to prove that optimal policy is still a forward looking Taylor rule, albeit with game theoretical gains to cooperation.

7 Conclusion

This paper estimates the natural rate of interest for seven advanced economies; the United States, the United Kingdom, Japan, Germany, France, Italy, and Canada, using the Laubach and Williams (2003) and Holston, Laubach and Williams (2017) model. Our results corroborate trends identified within the existing literature and highlight a new normal that has persisted for over a decade after the Great Recession. In addition, we identify significant comovement in equilibrium interest rates across the advanced world, particularly between members of the Euro Area.

We argue that within the small open economy, variation in r-stars is an important determinant of equilibrium exchange rates, for which we derive a novel structural expression using the general equilibrium framework of Galí and Monacelli (2005), that is a function of deviations in the natural rate between the domestic and foreign country. We find empirical support for this relationship and show that a marginal rise in natural rate of interest differentials do indeed place negative pressures on equilibrium exchange rates. This result calls into question sticky-price approaches to exchange rate determination within a disequilibrium setting and suggests that the real exchange rate and real interest rate differential parity exists only under equilibrium conditions.

The implications of this result are extensive, particularly within the context of declining natural rates of interest across the advanced G7 economies. As for equilibrium exchange rates, asymmetric variation in r-star due to country-specific shocks result in exchange rate adjustments to maintain equilibria. In contrast, symmetric variation due to global shocks are absorbed by joint variation in r-star, which insulates aggregate demand from exchange rate volatility. As for monetary policy in the small open economy, our study shows that nominal policy rates are a function of both local and global factors. In particular, global shocks result in positive correlation between policy rates across the advanced world by reaction as opposed to game theoretical coordination. Finally, time-varying natural rates of interest also imply monetary policy is efficient and maximises global welfare.

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APPENDIX

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This document provides all supplementary material for the paper "*Falling Stars in Small Open Economies*". Appendix A extends the Galí and Monacelli (2005) small open economy framework to the equilibrium exchange rate. Appendix B outlines the Holston, Laubach and Williams (2017) maximum likelihood procedure used to estimate the natural rate of interest (r-star). Appendix C details the sources of data for all G7 members. Appendix D presents all remaining estimates of the trend growth rate and output gap. Finally, Appendix E reports individual Augmented Dickey-Fuller tests on real exchange rates and natural rate of interest differentials.

A Equilibrium Exchange Rate

A.1 Households

Representative households maximise the following:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

$$\text{where} \quad C_t = \left[(1-\alpha)^{\frac{1}{\eta}} (C_{h,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{f,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

C_t is a composite consumption function, $\alpha \in [0,1]$ measures the degree of openness and $\eta > 0$ measures the degree substitutability between domestic and foreign goods. The consumption of domestic goods, foreign goods and imported goods are given by the following functions respectively:

$$C_{h,t} \equiv \left(\int_0^1 C_{h_i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad C_{f,t} \equiv \left(\int_0^1 C_{j,t}^{\frac{\gamma-1}{\gamma}} dj\right)^{\frac{\gamma-1}{\gamma}}, \quad C_{j,t} \equiv \left(\int_0^1 C_{j_i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $i \in [0,1]$ and $\varepsilon > 1$ is the elasticity of substitution between variables produced in any given country (including home) and γ represents the elasticity of substitution between importing countries. Maximisation of the objective function is subject to a sequence of budget constraints:

$$\int_0^1 P_{h_i,t} C_{h_i,t} di + \int_0^1 \int_0^1 P_{j_i,t} C_{j_i,t} di dj + E_t\{Q_{t,t+1} D_{t+1}\} \leqslant D_t + W_t N_t + T_t$$

where the domestic price on goods i is denoted by $P_{h_i,t}$ and the price of goods i imported from country j is denoted by $P_{j_i,t}$. The future payoff from a portfolio held to maturity is denoted by D_{t+1}. Finally, the stochastic discount factor for future nominal payoffs is denoted by $Q_{t,t+1}$.

To proceed, let consumption expenditures be $\int_0^1 P_{h_i,t} C_{h_i,t} di = Z_{h,t}$ such that:

$$\max_{C_{h_i,t}} \left(\int_0^1 C_{h_i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{s.t.} \quad \int_0^1 P_{h_i,t} C_{h_i,t} di \leqslant Z_t$$

Solving, taking the parity between goods and substituting into our constraint yields:

$$C_{j_i,t} = \frac{Z_t P_{j_i,t}^{-\varepsilon}}{\int_0^1 P_{h_i,t}^{1-\varepsilon} di} \quad \text{and} \quad C_{h_i,t} = \frac{Z_t P_{h_i,t}^{-\varepsilon}}{\int_0^1 P_{h_i,t}^{1-\varepsilon} di}$$

Inserting into $C_{h,t}$ and equating to unity:

$$C_{h,t} = \left[\int_0^1 \left(\frac{Z_t P_{h_i,t}^{\varepsilon}}{\int_0^1 P_{h_i,t}^{1-\varepsilon} di}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = Z_t \left(\int_0^1 P_{h_i,t}^{1-\varepsilon} di\right)^{\frac{1}{\varepsilon-1}} = 1$$

where if $P_{h,t} \equiv Z_{ht}|_{C_{h,t}=1}$, then $P_{h,t} = \left(\int_0^1 P_{h_i,t}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$ and optimal consumption is:

$$C_{h_i,t} = \left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon} \frac{Z_{h,t}}{P_{h,t}}$$

Inserting into $C_{h,t}$ and rearranging:

$$C_{h,t} = \left(\int_0^1 \left[\left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon} \frac{Z_{h,t}}{P_{h,t}}\right]^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \Rightarrow Z_{h,t} = \int_0^1 P_{h_i,t} C_{h_i,t} di$$

Inserting into $C_{h_i,t}$ to yield:

$$C_{h_i,t} = \left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon} C_{h,t}$$

In a similar fashion, we have that the aggregated price index for imported goods and the optimal consumption vector of goods i imported from country j may be expressed as follows:

$$P_{j,t} = \left(\int_0^1 P_{j_i,t}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}, \quad C_{j_i,t} = \left(\frac{P_{j_i,t}}{P_{j,t}}\right)^{-\varepsilon} C_{j,t}$$

In addition, the aggregated price index for all imported foreign goods and the relevant optimal consumption vector or import basket from country j may be expressed as follows:

$$P_{f,t} = \left(\int_0^1 P_{j,t}^{1-\gamma} di\right)^{\frac{1}{1-\gamma}}, \quad C_{j,t} = \left(\frac{P_{j,t}}{P_{f,t}}\right)^{-\gamma} C_{f,t}$$

Finally, the optimal distribution of domestic and international goods is:

$$C_{h,t} = (1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t \text{ and } C_{f,t} = \alpha\left(\frac{P_{f,t}}{P_t}\right)^{-\eta} C_t$$

where the consumer price index is defined as:

$$P_t = \left[(1-\alpha)P_{h,t}^{1-\eta} + \alpha P_{f,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

Total consumption expenditure may be written as:

$$P_t C_t = P_{h,t} C_{h,t} + P_{f,t} C_{f,t}$$

The budget constraint can now be rewritten as:

$$\int_0^1 P_{h_i,t} C_{h_i,t} di + \int_0^1 \int_0^1 P_{j_i,t} C_{j_i,t} di dj + E_t\{Q_{t,t+1} D_{t+1}\}$$

$$\Rightarrow \quad P_tC_t + E_t\{Q_{t,t+1}D_{t+1}\} \leqslant D_t + W_tN_t + T_t$$

Households maximise the following utility function:

$$u(C_t, N_t) \equiv \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

Assuming our budget constraint holds with equality:

$$P_tC_t = D_t + W_tN_t + T_t - E_t\{Q_{t,t+1}D_{t+1}\}$$

The intertemporal problem may therefore be written as:

$$\max_{D_{t+1}\forall t} u(C_t, N_t) + \beta E_t\{u(C_{t+1}, N_{t+1})\} \ \text{ s.t. } \ P_tC_t = D_t + W_tN_t + T_t - E_t \int V_{t,t+1}D_{t+1}d\tau$$

$$\text{and } \ \text{ s.t. } \ E_tP_{t+1}C_{t+1} = E_t\left(\int \xi_{t,t+1}D_{t+1}d\tau + W_{t+1}N_{t+1} + T_{t+1} - V_{t,t+2}D_{t+2}\right)$$

where $E_t \int V_{t,t+1}D_{t+1}d\tau$ is the price of the one-period portfolio yielding a random payoff D_{t+1} integrated over all states of nature. $V_{t,t+1}$ is the price of an Arrow security. $\xi_{t,t+1}$ is the probability that a given future state is realised. The price and stochastic discount factor may be written as:

$$E_t\frac{V_{t,t+1}}{\xi_{tt,t+1}}D_{t+1}, \quad Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{tt,t+1}}$$

Inserting our constraints, the unconstrained problem becomes:

$$\max_{D_{t+1}} u\left(\frac{1}{P_t}\left(D_t + W_tN_t + T_t - E_t\int V_{t,t+1}D_{t+1}d\tau\right), N_t\right)$$

$$+\beta E_t u\left(\frac{1}{P_{t+1}}\left(\xi_{t,t+1}D_{t+1} + W_{t+1}N_{t+1} + T_{t+1} - E_tV_{t,t+2}D_{t+2}\right), N_{t+1}\right)$$

First-order conditions:

$$E_t\left(-u_{c,t}\frac{V_{t,t+1}}{P_t} + u_{c,t+1}\frac{\xi_{t,t+1}}{P_{t+1}}\right) = 0$$

$$\Rightarrow \quad Q_{t,t+1} = E_t\left(\frac{u_{c,t+1}}{u_{c,t}}\frac{P_t}{P_{t+1}}\right)$$

Taking expectations:

$$E_t\{Q_{t,t+1}\} = \beta E_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}$$

Defining $\rho \equiv -\ln\beta$ and $i_t \equiv -\ln Q_t$, we may write:

$$1 = E_t\{e^{(i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - \rho)}\}$$

In steady state $i_t = \rho + \pi + \sigma\gamma$, Taylor expansion yields the Euler:

$$c_t = E_t\{c_{t+1}\} - \sigma^{-1}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right)$$

Log-linearised first-order conditions for labour-leisure yield:

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t} \quad \Rightarrow \quad w_t - p_t = \sigma c_t + \varphi n_t$$

A.2 Terms of Trade and Inflation

Bilateral and thus effective terms of trade with country j are:

$$S_t = \frac{P_{f,t}}{P_{h,t}} = \left(\int_0^1 S_{j,t}^{1-\gamma} dj\right)^{\frac{1}{1-\gamma}}$$

First-order approximation around the steady state (with $S_{j,t} = S_j = 1, \quad \forall j \in [1, 0]$):

$$S_t \approx \left(\int_0^1 S_j^{1-\gamma} dj\right)^{\frac{1}{1-\gamma}} + \frac{1}{1-\gamma}\left(\int_0^1 (1-\gamma) S_j^{-\gamma}(S_{j,t} - S_j) dj\right) = 1 + \int_0^1 (S_{j,t} - 1) dj$$

$$\Rightarrow \quad s_t = \ln S_t = p_{f,t} - p_{h,t} \approx \textstyle\int_0^1 s_{j,t} dj \quad \text{when} \quad \gamma = 1$$

First-order approximation of CPI around the steady state (with $P_h = P_f = P$):

$$P_t \approx \left[(1-\alpha)P^{1-\eta} + \alpha P^{1-\eta}\right]^{\frac{1}{1-\eta}} + \frac{1}{1-\eta}\left[(1-\alpha)P^{1-\eta} + \alpha P^{1-\eta}\right]^{\frac{1}{1-\eta}-1}$$

$$\cdot \left[(1-\alpha)(1-\eta)P^{-\eta}(P_{h,t} - P) + \alpha(1-\eta)P^{-\eta}(P_{f,t} - P)\right]$$

$$\Rightarrow \quad p_t \approx (1-\alpha)p_{h,t} + \alpha p_{f,t} = p_{h,t} + \alpha s_t \quad \text{when} \quad \eta = 1$$

Domestic inflation $\pi_{h,t} \equiv p_{h,t+1} - p_{h,t}$ may therefore be written as:

$$\Rightarrow \quad \pi_t = p_t - p_{t-1} = (p_{h,t} + \alpha s_t) - (P_{h,t-1} + \alpha s_{t-1}) = \pi_{h,t} + \alpha \Delta s_t$$

A.3 Real Exchange Rate

Given the law of one price $P_{j_i,t} = \epsilon_{j,t} P_{j_i,t}^j$ holds, aggregation yields:

$$P_{j,t} = \left(\int_0^1 P_{j_i,t}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} = \left[\int_0^1 (\epsilon_{j,t} P_{j_i,t}^j)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$$

$$= \left(\epsilon_{j,t}^{1-\varepsilon} \int_0^1 P_{j_i,t}^{j1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}} = \epsilon_{j,t}\left(\int_0^1 P_{j_i,t}^{j1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}} = \epsilon_{j,t} P_{j,t}^j$$

where $P^j_{j,t} = \left(\int_0^1 P^{j\,1-\varepsilon}_{j_i,t} di\right)^{\frac{1}{1-\varepsilon}}$ is the aggregate price level in country j.

Substituting $P_{j,t}$ into the aggregate price equation for all imported goods $P_{f,t}$:

$$P_{f,t} = \left(\int_0^1 P_{j,t}^{1-\gamma} di\right)^{\frac{1}{1-\gamma}} = \left(\int_0^1 \left(\epsilon_{j,t} P^j_{j,t}\right)^{1-\gamma} dj\right)^{\frac{1}{1-\gamma}}$$

Log-linearising around a symmetric steady state yields:

$$p_{f,t} = \int_0^1 (e_{j,t} + p^j_{j,t}) dj = \int_0^1 e_{j,t} dj + \int_0^1 p^j_{j,t} dj = e_t + p^*_t$$

where $e_t = \int_0^1 e_{j,t} dj$ is the log effective nominal exchange rate, $p^j_{j,t} = \int_0^1 p^j_{j,t} dj$ is the log domestic price index and $p^*_t = \int_0^1 p^j_{j,t} dj$ is the log foreign price index. Inserting into terms of trade yields:

$$s_t = p_{f,t} - p_{h,t} = e_t + p^*_t - p_{h,t}$$

The bilateral exchange rate with country j is defined as the ratio of CPI in domestic currency:

$$\mathcal{Q}_{j,t} = \frac{\epsilon_{j,t} P^j_{j,t}}{P_t}$$

Given the log effective real exchange rate may be defined as $q_t \equiv \int_0^1 q_{j,t} dj$, we may write:

$$q_t = \int_0^1 (e_{j,t} + p^j_{j,t} - p_t) dj = e_t + p^*_t - p_t = s_t + p_{h,t} - (p_{h,t} + \alpha s_t)$$

$$\Rightarrow \quad q_t = (1-\alpha) s_t \quad \text{for} \quad \eta \neq 1$$

A.4 International Risk

Assuming complete international markets, the Euler condition holds across country j:

$$1 = \beta E_t \left\{ Q^{-1}_{t,t+1} \left(\frac{C^j_{t+1}}{C^j_t} \right)^{-\sigma} \frac{P_{j,t}}{P_{j,t+1}} \right\} = \beta E_t \left\{ Q^{-1}_{t,t+1} \left(\frac{C^j_{t+1}}{C^j_t} \right)^{-\sigma} \frac{\epsilon_{j,t} P^j_{j,t}}{\epsilon_{j,t+1} P^j_{j,t+1}} \right\}$$

Deriving the relative international choice as ratio of the domestic to international condition:

$$1 = \frac{\beta E_t \left\{ Q^{-1}_{t,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}}{\beta E_t \left\{ Q^{-1}_{t,t+1} \left(\frac{C^j_{t+1}}{C^j_t} \right)^{-\sigma} \frac{\epsilon_{j,t} P^j_{j,t}}{\epsilon_{j,t+1} P^j_{j,t+1}} \right\}} = E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \left(\frac{C^j_{t+1}}{C^j_t} \right)^{\sigma} \frac{\epsilon_{j,t+1} P^j_{j,t+1}}{\epsilon_{j,t} P^j_{j,t}} \right\}$$

$$\Rightarrow \quad C_t = E_t \left\{ \left(\frac{C^j_{t+1}}{C^j_t} \right)^{\sigma} C_t^{j-\sigma} \frac{\mathcal{Q}_{j,t+1}}{\mathcal{Q}_{j,t}} \right\}^{-\frac{1}{\sigma}} = E_t \left\{ \frac{C_{t+1}}{C^j_{t+1} \mathcal{Q}^{\frac{1}{\sigma}}_{j,t+1}} \right\} C^j_t \mathcal{Q}^{\frac{1}{\sigma}}_{j,t}$$

$$\Rightarrow \quad C_t = \vartheta_j C_t^j \mathcal{Q}_{j,t}^{\frac{1}{\sigma}} \quad \text{where} \quad \vartheta_t \equiv E_t \left\{ \frac{C_{t+1}}{C_{t+1}^j \mathcal{Q}_{j,t+1}^{\frac{1}{\sigma}}} \right\} \quad \text{with} \quad \vartheta_j = \vartheta = 1 \forall j$$

Given world consumption is simply $c_t^* \equiv \int_0^1 c_t^j dj$, we may integrate over the log of consumption:

$$c_t = \int_0^1 \left(c_t^j + \frac{1}{\sigma} q_{j,t} \right) dj = c_t^* + \frac{1}{\sigma} q_t = c_t^* + \left(\frac{1-\alpha}{\sigma} \right) s_t$$

A.5 Firms and Technology

With production function $Y_{j,t} = A_t N_{j,t}$ and an optimal price setting rule $P_t = \frac{\varepsilon}{\varepsilon-1} \frac{W_t}{MPN_t}$, optimal subsidies $\tau = \frac{1}{\varepsilon}$ imply $P_t = \frac{W_t}{MPN_t}$. Market clearing in the analogous closed market yields $y_t = a_t + (1-\alpha) n_t$, which given constant returns can be written as $y_t = a_t - n_t$ and $\text{mpn}_t^n = a_t$, thus we have marginal costs $\text{mc}_t^r = -v + w_t - p_{h,t} - a_t$, where the subsidy $v \equiv -\ln(1-\tau)$.

The firm's maximisation problem may be expressed as:

$$\max_{\bar{P}_t} \sum_{k=0}^{\infty} \theta^k E_t \left[Q_{t,t+k} \left(\bar{P}_t Y_{i,t+k|t} - TC_{i,t+k|t}^n Y_{i,t+k|t} \right) \right] \quad \text{s.t.} \quad Y_{i,t+k|t} = \left(\frac{\bar{P}_t}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}$$

where $Y_{i,t+k|t}$ is the output, $TC_{i,t+k|t}^n$ is the total cost and $\bar{P}_t Y_{i,t+k|t} - TC_{i,t+k|t}^n Y_{i,t+k|t}$ is the undiscounted profits. We may unconstrain the problem by inserting the constraint:

$$\max_{\bar{P}_t} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \left(\bar{P}_t \left(\frac{\bar{P}_t}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - TC_{i,t+k|t}^n \left(\frac{\bar{P}_t}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right]$$

Taking first order conditions:

$$\Rightarrow \quad \sum_{k=0}^{\infty} \theta^k E_t \left(Q_{t,t+k} Y_{t+k|t} \bar{P}_t \right) = \frac{\varepsilon}{\varepsilon - 1} \sum_{k=0}^{\infty} \theta^k E_t \left(Q_{t,t+k} Y_{t+k|t} MC_{t+k|t}^n \right)$$

Substituting in for $Q_{t,t+k}$ and $Y_{t+k|t}$, and solving for the optimal price:

$$\Rightarrow \quad \bar{P}_t = \frac{\varepsilon}{\varepsilon - 1} \frac{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon} MC_{t+k|t}^r}{E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1}}$$

In foresight we divide by P_{t-1}:

$$\Rightarrow \quad \frac{\bar{P}_t}{P_{t-1}} E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon-1} = \frac{1}{P_{t-1}} \frac{\varepsilon}{\varepsilon - 1} E_t \sum_{k=0}^{\infty} \theta^k \beta^k C_{t+k}^{1-\sigma} P_{t+k}^{\varepsilon} MC_{t+k|t}^r$$

Expansion of the LHS around the steady state and factorising:

$$\Rightarrow \quad C^{1-\sigma} P^{\varepsilon-1} E_t \sum_{k=0}^{\infty} \theta^k \beta^k [1 + \bar{p}_t - p_{t-1} + (\varepsilon - 1)(p_{t+k} - p) + (1-\sigma)(c_{t+k} - c)]$$

Expansion of the RHS around the steady state and factorising:

$$\Rightarrow \quad C^{1-\sigma}p^{\varepsilon-1}E_t\sum_{k=0}^{\infty}\theta^k\beta^k[1-p_{t-1}+\varepsilon p_{t+k}-(\varepsilon-1)p+(1-\sigma)(c_{t+k}-c)+(mc^r_{t+k|t}-mc^r)]$$

Equating and solving for $\bar{p}_t$:

$$\Rightarrow \quad \bar{p}_t = \mu + (1-\theta\beta)E_t\sum_{k=0}^{\infty}\theta^k\beta^k[mc^r_{t+k|t}+p_{t+k}] \;\; \text{with} \;\; \mu = ln\left(\frac{\varepsilon}{\varepsilon-1}\right)$$

$$\text{where} \quad -mc^r = -\ln MC^r = \ln\frac{\varepsilon}{\varepsilon-1} = ln\left(1+\frac{\varepsilon-(\varepsilon-1)}{(\varepsilon-1)}\right) \approx \frac{\varepsilon}{\varepsilon-1}-1 = \mu$$

A.6 Uncovered interest rate parity

The budget constraint may be written as a function of domestic and foreign bonds:

$$P_tC_t + Q_{t,t+1}B_{t+1} + Q^*_{t,t+1}\epsilon_{t+1}B_{t+1} \leqslant B_t + \epsilon_t B^*_t + W_tN_t + T_t$$

The ratio of optimal conditions between assets implies:

$$\Rightarrow \quad \frac{Q^*_{t,t+1}}{Q_{t,t+1}} = E_t\left\{\frac{\epsilon_{t+1}}{\epsilon_t}\right\}$$

Log linearising yields the uncovered interest rate parity condition:

$$\Rightarrow \quad i_t = i^*_t + E_t\Delta e_{t+1}$$

Using terms of trade, and solving the resulting stochastic difference equation:

$$\Rightarrow \quad s_t = E_t\sum_{k=0}^{\infty}[(i^*_{t+k}-\pi^*_{t+k})-(i_{t+k}-\pi_{h,t+k})]$$

A.7 Equilibrium (Demand)

Inserting $C_{h,t}$ into $C_{h_i,t}$ yields:

$$C_{h,t} = (1-\alpha)\left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon}\left(\frac{P_{h,t}}{P_t}\right)^{-\eta}C_t$$

Note that:

$$C^j_{h_i,t} = \left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon}C^j_{h,t}, \quad C^j_{h,t} = \left(\frac{P_{h,t}}{\varepsilon_{j,t}P^j_{f,t}}\right)^{-\gamma}C^j_{f,t}, \quad C^j_{f,t} = \alpha\left(\frac{P^j_{f,t}}{P^j_t}\right)^{-\eta}C^j_t$$

Combining to derive the demand for domestic goods i in country j:

$$C^j_{h_i,t} = \alpha \left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon} \left(\frac{P_{h,t}}{\varepsilon_{j,t} P^j_{f,t}}\right)^{-\gamma} \left(\frac{P^j_{f,t}}{P^j_t}\right)^{-\eta} C^j_t$$

Inserting into the clearing condition $Y_{i,t} = C_{h,t} + \int_0^1 C^j_{h_i,t} di$ yields:

$$\Rightarrow \quad Y_{i,t} = \left(\frac{P_{h_i,t}}{P_{h,t}}\right)^{-\varepsilon} \left[(1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{h,t}}{\epsilon_{j,t} P^j_{f,t}}\right)^{-\gamma} \left(\frac{P^j_{f,t}}{P^j_t}\right)^{-\eta} C^j_t dj\right]$$

Aggregating given that $Y_t \equiv \left(\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$ and simplifying:

$$\Rightarrow Y_t = (1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{h,t}}{\epsilon_{j,t} P^j_{f,t}}\right)^{-\gamma} \left(\frac{P^j_{f,t}}{P^j_t}\right)^{-\eta} C^j_t dj$$

Factorising yields:

$$\Rightarrow \quad Y_t = \left(\frac{P_{h,t}}{P_t}\right)^{-\eta} \left[(1-\alpha)C_t + \alpha \int_0^1 \left(\frac{\epsilon_{j,t} P^j_{f,t}}{P_{h,t}}\right)^{\gamma-\eta} \mathcal{Q}^\eta_{j,t} C^j_t dj\right]$$

Given $S^j_t \equiv \frac{\epsilon_{j,t} P^j_{f,t}}{P_{j,t}}$, $S_{j,t} \equiv \frac{P_{j,t}}{P_{h,t}}$ and $C_t = C^j_t \mathcal{Q}^{\frac{1}{\sigma}}_{i,t}$ we may write:

$$\Rightarrow \quad Y_t = \left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t \left[(1-\alpha) + \alpha \int_0^1 (S^j_t S_{j,t})^{\gamma-\eta} \mathcal{Q}^{\eta - \frac{1}{\sigma}}_{j,t} dj\right]$$

Log linearising around the steady state, recalling $q_t = (1-\alpha)s_t$:

$$\Rightarrow \quad y_t = c_t + \frac{\alpha\omega}{\sigma} s_t \quad \text{where} \quad \omega \equiv \sigma\gamma + (1-\alpha)(\sigma\eta - 1) > 0$$

Aggregating across countries:

$$y^j_t = \int_0^1 \left(c^j_t + \frac{\alpha\omega}{\sigma} s^j_t\right) dj = \int_0^1 c^j_t dj \equiv c^*_t$$

Given $c_t = \int_0^1 \left(c^j_t + \frac{1}{\sigma} q_{j,t}\right) dj = c^*_t + \frac{1}{\sigma} q_t = c^*_t + \left(\frac{1-\alpha}{\sigma}\right) s_t$ we have that:

$$\Rightarrow y_t = y^*_t + \frac{1}{\sigma_\alpha} s_t \quad \text{where} \quad \sigma_\alpha = \frac{\sigma}{1 + \alpha(\omega - 1)} > 0$$

Solving for c_t and combining with the Euler equation to derive the IS relation:

$$\Rightarrow \quad y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - \rho) - \frac{\alpha\omega}{\sigma} E_t \Delta s_{t+1}$$

Given $\pi_t = \pi_{h,t} + \alpha \Delta s_t$ we may write:

$$\Rightarrow \quad y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{h,t+1} - \rho) - \frac{\alpha\Theta}{\sigma} E_t \Delta s_{t+1}$$

where $\Theta \equiv \omega - 1 = (\sigma\gamma - 1) + (1-\alpha)(\sigma\eta - 1) > 0$.

Substituting in for s_t and simplifying:

$$\Rightarrow \quad y_t = E_t y_{t+1} - \frac{1}{\sigma_\alpha}(i_t - E_t \pi_{h,t+1} - \rho) + \alpha\Theta E_t \Delta y^*_{t+1}$$

A.8 Equilibrium (Trade)

Net exports may be defined as:

$$nx_t \equiv \frac{1}{Y}\left(Y_t - \frac{P_t}{P_{h,t}} C_t\right)$$

First order approximation around the steady state yields:

$$nx_t \approx y_t - c_t - \alpha s_t$$

Given $y_t = c_t + \frac{\alpha\omega}{\sigma} s_t$, we may write:

$$\Rightarrow \quad nx_t = \frac{\alpha\omega}{\sigma} s_t - \alpha s_t = \alpha\left(\frac{\omega}{\sigma} - 1\right) s_t$$

$$\text{where} \quad \sigma = \eta = \gamma = 1 \quad \text{and} \quad nx_t = 0 \quad \forall t$$

A.9 Equilibrium (Supply)

Market clearing in the labour market implies:

$$\Rightarrow \quad N_t = \int_0^1 \frac{Y_{i,t}}{A_t} di = \int_0^1 \frac{Y_t}{A_t}\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} di = \frac{Y_t}{A_t}\int_0^1 \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} di$$

Given $w_t - p_t = \sigma c_t + \varphi n_t$ and $p_t = p_{h,t} + \alpha s_t$, we may write marginal costs derived earlier as $mc_t^r = -v + \sigma c_t + \varphi n_t + \alpha s_t - a_t$. Furthermore, knowing $c_t = c_t^* + \frac{1-\alpha}{\sigma} s_t$ and $n_t = y_t - a_t$ we may write this as $mc_t^r = -v + \sigma y_t^* + \varphi y_t + s_t - (1-\varphi)a_t$. Finally, recall that $y_t = y_t^* + \frac{1}{\sigma_\alpha} s_t$ and that $mc^r = -\mu$ under flexible prices, implying $mc_t^r = -\mu = -v + (\sigma_\alpha + \varphi) y_t^n + (\sigma - \sigma_\alpha) y_t^* - (1+\varphi) a_t$.

Given $\sigma_\alpha \equiv \frac{\sigma}{1+\alpha\Theta}$ we may solve for the natural rate of output:

$$\Rightarrow \quad y_t^n = \Gamma_0 + \Gamma_a a_t + \Gamma_* y_t^*$$

$$\text{where} \quad \Gamma_0 \equiv \frac{v - \mu}{\sigma_\alpha + \varphi}, \quad \Gamma_a \equiv \frac{1+\varphi}{\sigma_\alpha + \varphi} > 0, \quad \text{and} \quad \Gamma_* \equiv \frac{\alpha\Theta\sigma_\alpha}{\sigma_\alpha + \varphi}$$

A.10 New Keynesian Phillips Curve

Given $\widehat{mc}_t^r \equiv mc_t^r - mc^r$, we may write marginal costs as:

$$\widehat{mc}_t^r = (\sigma_\alpha + \varphi)\tilde{y}_t$$

and substitute this into our inflation function:

$$\Rightarrow \quad \pi_{h,t} = \beta E_t \pi_{h,t+1} + \kappa_\alpha \tilde{y}_t$$

$$\text{where} \quad \kappa_\alpha = \lambda(\sigma_\alpha + \varphi) \quad \text{and} \quad \lambda = \theta^{-1}(1-\beta\theta)(1-\theta)$$

A.11 Dynamic IS Curve

Given $r_t \equiv i_t - E_t\pi_{h,t+1}$ we may write output as:

$$\Rightarrow \quad y_t = E_t y_{t+1} - \sigma_\alpha^{-1}(r_t - \rho) + \alpha\Theta E_t \Delta y_{t+1}^*$$

Note that this holds for output, and that $\tilde{y}_t \equiv y_t - y_t^n$ implies:

$$\Rightarrow \quad \tilde{y}_t = E_t \tilde{y}_{t+1} - \sigma_\alpha^{-1}(i_t - E_t\pi_{h,t+1} - r_t^n)$$

A.12 Natural Rate of Interest

Note that from the dynamic IS curve, the following is implied:

$$\Rightarrow \quad E_t \Delta \tilde{y}_{t+1} = \sigma_\alpha^{-1}(i_t - E_t\pi_{h,t+1} - r_t^n)$$

Solving for the natural rate given this is analogously true for y_t and y_t^n yields:

$$\Rightarrow \quad r_t^n = \rho - \sigma_\alpha \Gamma_a (1-\rho_a) a_t + \frac{\alpha\Theta\sigma_\alpha\varphi}{\sigma_\alpha + \varphi} E_t \Delta y_{t+1}^*$$

A.13 Equilibrium Exchange Rate

Given $s_t = e_t + p_t^* - p_t$ and $s_t = \sigma_\alpha(y_t - y_t^*)$, we may write the exchange rate as:

$$\Rightarrow \quad e_t = p_t - p_t^* + \sigma_\alpha(\tilde{y}_t - \tilde{y}_t^*) + \sigma_\alpha(y_t^n - y_t^{n*})$$

The domestic natural rate may be expressed as:

$$r_t^n = \rho + \sigma_\alpha \alpha \Theta E_t \Delta y_{t+1}^* + \sigma_\alpha E_t \Delta y_{t+1}^n$$

Given the unconditional mean of output equate, we may write:

$$r_t^n - r_t^{n*} = \sigma_\alpha (E_t \Delta y_{t+1}^n - E_t \Delta y_{t+1}^{n*})$$

The equilibrium exchange rate may therefore be expressed as:

$$e_t = p_t - p_t^* + \sigma_\alpha(\tilde{y}_t - \tilde{y}_t^*) - E_t \sum_{i=0}^{\infty} \left(r_{t+i}^n - r_{t+i}^{n*}\right)$$

Alternatively, given $r_t^n + \sigma_\alpha E_t \Delta \tilde{y}_{t+1} = i_t - E_t \pi_{h,t+1}$, substitute directly:

$$s_t = -E_t \sum_{k=0}^{\infty} [(r_{t+k}^n - r_{t+k}^{n*}) + \sigma_\alpha(\Delta \tilde{y}_{t+k} - \Delta \tilde{y}_{t+k}^*)]$$

Note that if $\sigma = \gamma = \eta = 1$ the natural rate is independent of global output growth:

$$r_t^n = \rho + E_t \Delta y_{t+1}^n \quad \text{as} \quad \Theta \equiv \omega - 1 = (\sigma\gamma - 1) + (1 - \alpha)(\sigma\eta - 1) = 0$$

This implies the unique case (for $\sigma = \gamma = \eta = 1$):

$$e_t = p_t + \tilde{y}_t - E_t \sum_{i=0}^{\infty} \left(r_{t+i}^n - r_{t+i}^{n*}\right)$$

A.14 Optimal Monetary Policy

Under the assumptions for which DIT is optimal, the flexible equilibrium is $\pi_{h,t} = \tilde{y}_t = 0$. Optimal monetary policy derived from minimising welfare loss yields a divine coincidence.

Consider the standard welfare function in the presence of rigidity at the efficient steady state. Monetary policymakers optimise the objective function (subject to constraints) in discretion:

$$\max \mathbb{W} = -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[\pi_{h,t}^2 + \alpha \tilde{y}_t^2\right]$$

$$-\lambda_{1,t}[\tilde{y}_t - E_t \tilde{y}_{t+1} + \sigma_\alpha^{-1}(i_t - E_t \pi_{h,t+1} - r_t^n)]$$

$$-\lambda_{2,t}[\pi_{h,t} - \beta E_t \pi_{h,t+1} - \kappa_\alpha \tilde{y}_t - u_t]$$

First order conditions and solving simultaneously yields:

$$\Rightarrow \quad \tilde{y}_t = -\frac{\kappa_\alpha}{\alpha} \pi_{h,t}$$

Inserting into the Phillips curve:

$$\Rightarrow \quad \pi_{h,t} = \frac{\alpha}{\alpha + \kappa_\alpha^2} \beta E_t \pi_{h,t+1} + \frac{\alpha}{\alpha + \kappa_\alpha^2} u_t$$

Iterating forward with lag operators and simplifying:

$$\Rightarrow \quad \pi_{h,t} = \zeta u_t \quad \text{where} \quad \zeta = \frac{\alpha}{\kappa_\alpha^2 + \alpha(1 - \beta\rho_u)}$$

Inserting into the output gap and the Euler condition:

$$\Rightarrow \quad i_t = r_t^n + \left(\sigma_\alpha \frac{\kappa_\alpha}{\alpha}(1-\rho_u) + \rho_u\right)\zeta u_t$$

The final system is:

$$i_t = r_t^n + \left(1 + \frac{\sigma_\alpha \kappa_\alpha (1-\rho_u)}{\alpha \rho_u}\right) E_t \pi_{t+1}$$
$$e_t = p_{t-1} + \left(1 - \tfrac{\kappa_\alpha}{\alpha}\right)\zeta u_t - E_t \textstyle\sum_{i=0}^{\infty}\left(r_{t+i}^n - r_{t+i}^{n*}\right)$$

where the latter follows from the fact that:

$$\zeta u_t - \frac{\kappa_\alpha}{\alpha}\zeta u_t = \pi_t + \tilde{y}_t$$

B Natural Rate of Interest

B.1 Stage I Specification:

$\mathbf{y}_t = \left[y_t, \pi_t\right]' \quad \mathbf{x}_t = \left[y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}\right]' \quad \xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*\right]'$

$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix} \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 \\ b_y & 0 & b_\pi & 1-b_\pi \end{bmatrix}$

$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y*}^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Vector estimated by maximum likelihood:

$\theta_1 = \left[a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}\right]$

B.2 Stage II Specification:

$\mathbf{y}_t = \left[y_t, \pi_t\right]' \quad \mathbf{x}_t = \left[y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1\right]' \quad \xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*\right]'$

$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & a_g \\ 0 & -b_y & 0 & 0 \end{bmatrix} \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 \end{bmatrix}$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} \sigma_{y*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y*})^2 \end{bmatrix}$$

Vector estimated by maximum likelihood:

$$\theta_2 = \left[a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*} \right]$$

B.3 Stage III Specification:

$$\mathbf{y}_t = \left[y_t, \pi_t \right]' \quad \mathbf{x}_t = \left[y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4} \right]'$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2} \right]'$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ b_y & 0 & 0 & 0 & b_\pi & 1 - b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} (1+\lambda_g^2)\sigma_{y*}^2 & 0 & 0 & (\lambda_g \sigma_{y*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\lambda_g \sigma_{y*})^2 & 0 & 0 & (\lambda_g \sigma_{y*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\frac{\lambda_z \sigma_{\tilde{y}}}{a_r})^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Vector estimated by maximum likelihood:

$$\theta_3 = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*} \right]$$

C Data

C.1 United States

Real GDP and core PCE is sourced from the Bureau of Economic Analysis (BEA). We define the short-term interest rate as the federal funds rate, taken from the Board of Governors of the Federal Reserve. As in Holston, Laubach and Williams (2017), we use the Federal Reserve Bank of New York's discount rate prior to 1965, sourced directly from the IMF International Financial Statistics Yearbooks (IFS). We collect data from the St. Louis Federal Reserve Economic Database (FRED).

C.2 United Kingdom

Real GDP is taken from the Office for National Statistics (ONS). We use core CPI as the primary measure of inflation, albeit prior to 1970 we use CPI due to data scarcity. This data is taken from the OECD Main Economic Indicators (MEI) database. The short-term interest rate is the Bank of England's Official Bank Rate published on the Bank of England Statistical Interactive Database.

C.3 Japan

Real GDP is computed using data on gross domestic product. We use core CPI (all items non-food and non-energy) as the primary measure of inflation and the short-term nominal (immediate) bank rate as the interest rate. All data is sourced from leading indicators in the OECD MEI database.

C.4 Germany, France, Italy

Real GDP is sourced from the Eurostat database for all countries. We use core CPI as our measure of inflation and source this data from the OECD MEI database for all countries. Short-term nominal interest rates are 3-month rates for all countries, which converge once the European Central Bank is established across the Euro Area. This data is also sourced from the OECD MEI database.

C.5 Canada

Real GDP is sourced from the International Monetary Fund's (IMF) International Financial Statistics (IFS) database. Core CPI is sourced from the Bank of Canada to derive the series for inflation, albeit prior to the second quarter of 2001 we use CPI containing all items. The short-term nominal interest rate is the end-period (monthly) overnight bank rate transformed to a quarterly frequency. Data for inflation and interest rates are sourced from Statistique Canada (Canada Statistics).

C.6 Exchange Rates

We source data on $n!/r!(n-r)! = 21$ unique bilateral exchange rates for all seven countries in our sample. Rates for Germany, France and Italy converge with the establishment of the Euro. All data is sourced from the OECD MEI database. Although prior data was available, our sample ranges from 1965:1-2019:4 to remain consistent with data used to estimate the natural rate of interest.

D Model Estimates

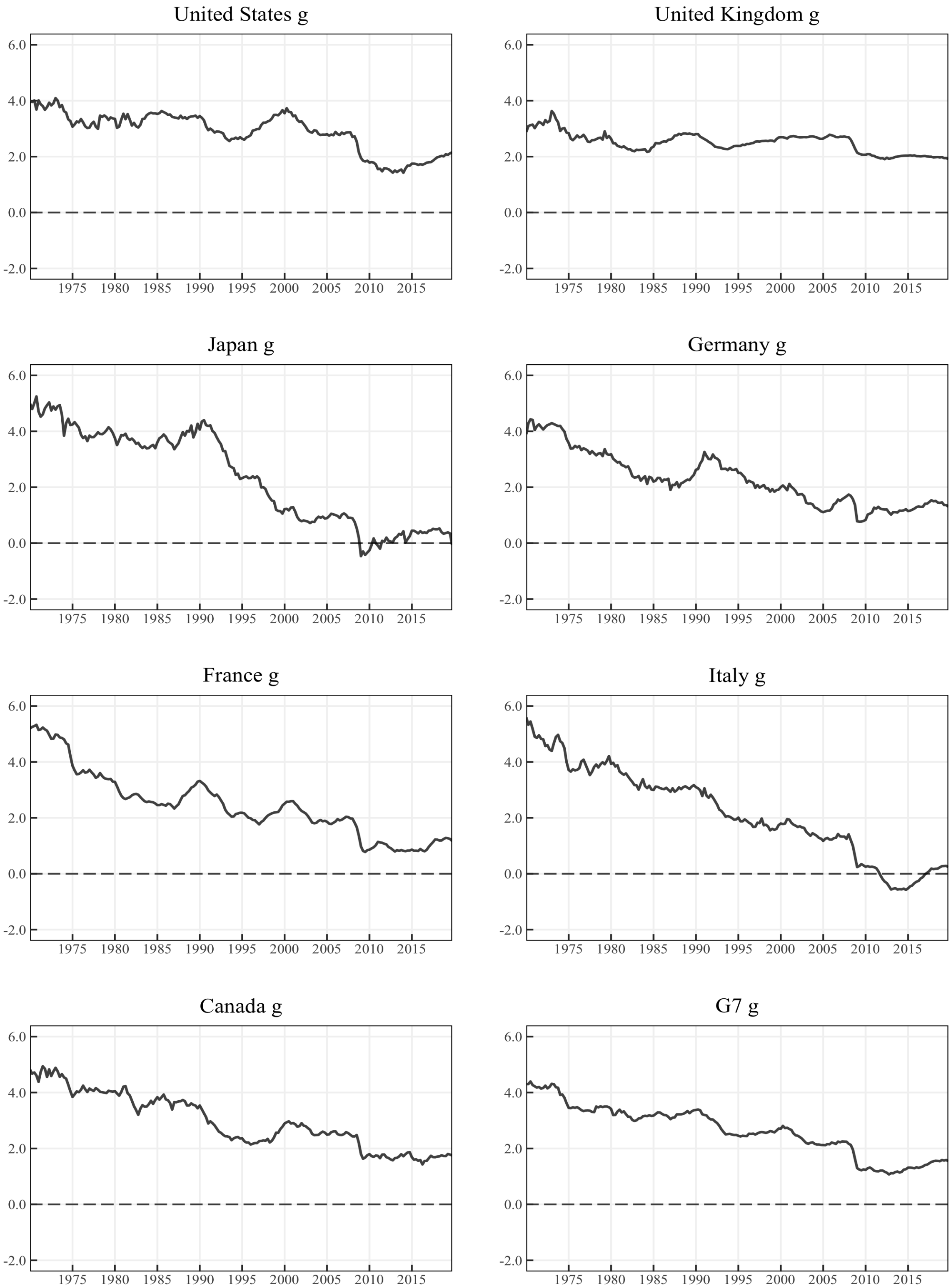

Figure D.1. Advanced Trend Growth Rates

Note: G7 trend growth rates (g), 1970:1-2019:4

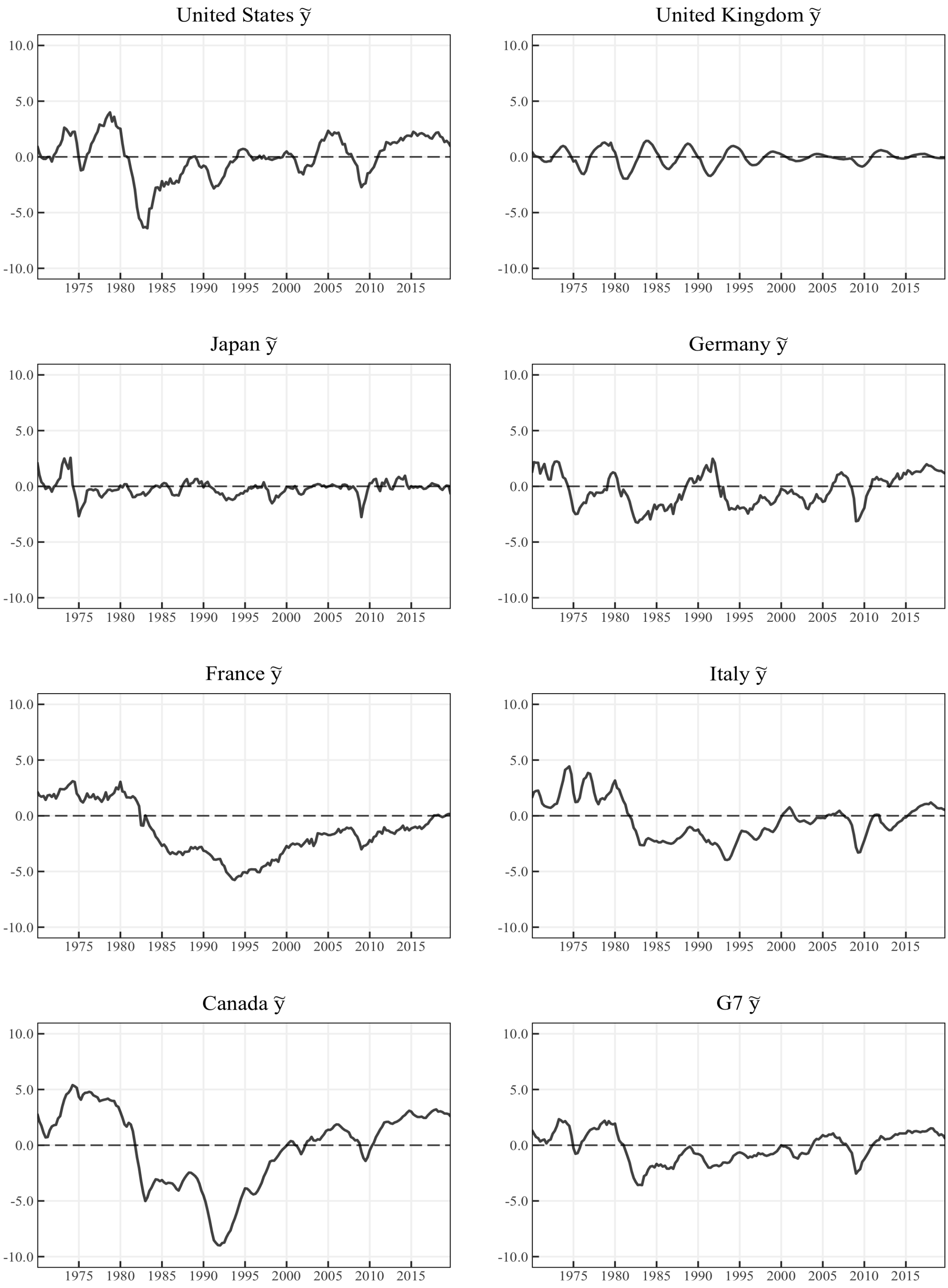

Figure D.2. Advanced Output Gaps

Note: G7 output gaps (ỹ), 1970:1-2019:4

E Augmented Dickey-Fuller Tests

Table E.1. Augmented Dickey-Fuller Tests: United States

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
United States	const	+ trend	const	+ trend
United Kingdom	0.12	0.14	0.57	0.15
Japan	0.17	0.69	0.65	0.08
Germany	0.09	0.24	0.38	0.62
France	0.06	0.15	0.53	0.94
Italy	0.12	0.14	0.57	0.38
Canada	0.08	0.23	0.28	0.61

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.2. Augmented Dickey-Fuller Tests: United Kingdom

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
United Kingdom	const	+ trend	const	+ trend
United States	0.12	0.14	0.57	0.15
Japan	0.03	0.08	0.45	0.25
Germany	0.02	0.10	0.40	0.27
France	0.03	0.08	0.46	0.67
Italy	0.16	0.42	0.81	0.01
Canada	0.12	0.23	0.73	0.02

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.3. Augmented Dickey-Fuller Tests, Japan

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
Japan	const	+ trend	const	+ trend
United States	0.17	0.69	0.65	0.08
United Kingdom	0.05	0.09	0.45	0.25
Germany	0.21	0.65	0.49	0.27
France	0.15	0.42	0.49	0.69
Italy	0.10	0.30	0.52	0.01
Canada	0.24	0.71	0.80	0.04

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.4. Augmented Dickey-Fuller Tests, Germany

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
Germany	const	+ trend	const	+ trend
United States	0.09	0.24	0.38	0.62
United Kingdom	0.02	0.10	0.40	0.27
Japan	0.25	0.68	0.49	0.27
France	0.02	0.05	0.56	0.89
Italy	0.03	0.11	0.45	0.08
Canada	0.05	0.14	0.31	0.51

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.5. Augmented Dickey-Fuller Tests, France

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
France	const	+ trend	const	+ trend
United States	0.06	0.15	0.53	0.94
United Kingdom	0.03	0.08	0.46	0.67
Japan	0.17	0.61	0.49	0.69
Germany	0.02	0.05	0.56	0.89
Italy	0.34	0.36	0.33	0.75
Canada	0.02	0.06	0.35	0.76

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.6. Augmented Dickey-Fuller Tests, Italy

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
Italy	const	+ trend	const	+ trend
United States	0.12	0.14	0.57	0.38
United Kingdom	0.16	0.42	0.81	0.01
Japan	0.13	0.51	0.52	0.01
Germany	0.03	0.11	0.45	0.08
France	0.34	0.36	0.33	0.75
Canada	0.04	0.03	0.05	0.00

Note: p-values associated with the ADF test including a constant and both a constant and time trend.

Table E.7. Augmented Dickey-Fuller Tests, Canada

1965:1-2019:4	Real Exchange Rate (q)		Real Natural Rate (∂^n)	
Canada	const	+ trend	const	+ trend
United States	0.08	0.23	0.28	0.61
United Kingdom	0.12	0.23	0.73	0.02
Japan	0.21	0.66	0.80	0.04
Germany	0.05	0.14	0.31	0.51
France	0.02	0.06	0.35	0.76
Italy	0.04	0.03	0.05	0.00

Note: p-values associated with the ADF test including a constant and both a constant and time trend.