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A delay mathematical model for the spread and control of water borne diseases

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ABSTRACT

A non-linear SIRS mathematical model to explore the dynamics of water borne diseases like cholera is proposed and analyzed by incorporating delay in using disinfectants to control the disease. It is assumed that the only way for the spread of infection is ingestion of contaminated water by susceptibles. As the pathogens discharged by infectives reach to the aquatic environment, it is assumed that the growth rate of pathogens is proportional to the number of infectives. Further, it is assumed that disinfectants are introduced to kill pathogens with a rate proportional to the density of pathogens in the aquatic environment. The model is analyzed by using stability theory of delay differential equations. It is found that the model exhibits two equilibria, the disease free equilibrium and the endemic equilibrium. The analysis shows that under certain conditions, the cholera disease may be controlled by using disinfectants but a longer delay in their use may destabilize the system. Numerical simulation is also carried out to confirm the analytical results.

1. Introduction

Worldwide everyday more than 4500 deaths occur among children under the age of 14 due to water borne diseases like cholera, typhoid, giardiasis, shigella, hepatitis A and E, etc. The pathogens of such diseases reach in the aquatic environment through the feces or wastes of symptomatic or asymptomatic infected persons or animals and contaminate it. Drinking of this contaminated water causes water borne disease like cholera. Earlier, it was considered that cholera epidemic mainly spread due to fecal-oral transmission but after Snow's influential work (Snow, 1936), it has been found that consumption of contaminated water by the susceptibles is a main cause for the spread of cholera infection. Firstly in 1854, an Italian scientist Filippo Pacini discovered that the presence of *Vibrio cholerae* (*V. cholerae*) in this contaminated water is responsible for the spread of cholera and 30 years later his work was supported by Robert Koch in 1884 (http://www.ph.ucla.edu). When these *V. cholerae* are consumed by the human population, they accumulate in the intestine and produce an enterotoxin which results in abundant watery diarrhea, vomiting, acidosis, and leg cramps. If treatment is not promptly given, the health of an infected person deteriorates rapidly and death may occur within 1–2 days. Until late 1970s permanence of *V. cholerae* in the aquatic environment was considered for short time (i.e., for only a few hours or days) but after that it is found that *V. cholerae* may survive for a longer time (i.e., from months to years) in fresh water (Borroto, 1997). Thus a reservoir containing contaminated water in which *V. cholerae* are present is responsible for cholera epidemic. Developing countries are most affected by cholera disease due to inadequate sanitation, improper treatment of reservoirs and lack of safe water supply. For example, in 1961 the seventh cholera pandemic started in Indonesia and spread worldwide. The disease is still endemic in many parts of Africa and Asia. In 2007, WHO reported 177,963 cholera cases and 4031 deaths across 53 countries in which 99% cases occurred in African countries (World, Health, Organization, 2007). Recently, cholera has re-emerged in a western hemisphere country Haiti after a long absence of about 100 years. The disease spread rapidly in the entire Haiti, which reported 452,189 cases and 6334 deaths till 18 September 2011 since the outbreak appeared in October 2010 (PAHO, 2010).

In the past few decades, various SIS and SIRS mathematical models have been proposed and analyzed to understand the dynamics of spread of infectious diseases (Shangbing, 2007; Ghosh et al., 2000, 2004, 2005; Bailey, 1980; Hethcote, 1976; Das et al., 2005; Singh et al., 2003, 2005; Shukla et al., 2011; Ma and Li, 2009). In these models, it is assumed that disease spreads through direct contact between susceptibles and infectives as well as due to the presence of carriers in the environment. But in the case of cholera, infection is not likely to spread directly from person to person (http://www.medicinenet.com/cholera/article.htm). Therefore, direct contact of healthy person with an infected person is

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not a risk for contracting infection, whereas a healthy person may contract infection by drinking contaminated water in which pathogens *V. cholerae* are present. In the last few decades, cholera epidemic has become a major health problem for the developing countries. Hence, enormous attention is being paid on the dynamics of cholera disease (Capasso and Paveri-Fontana, 1979; Codeço, 2001; Pascual et al., 2002; Hartley et al., 2006; Jensen et al., 2006; Bertuzzo et al., 2008; Tien and Earn, 2010; Tian and Wang, 2011). For example, Capasso and Paveri-Fontana (1979), proposed and analyzed a mathematical model to describe the dynamics of 1973 cholera outbreaks of Italy. They studied the dynamics of the infected population and *V. cholerae* in their model. Later on, Codeç o (2001) generalized this model to study the role of the aquatic reservoir on the epidemic of cholera. Pascual et al. (2002) extended the Codeço's model by incorporating another equation, describing the volume of water in which the *V. cholera* live. Hartley et al. (2006) also extended Codeço's model by considering the role of hyper infectious bacteria in the spread of cholera epidemic. Also, Jensen et al. (2006) studied a mathematical model to see the role of bacteriophage on the control of cholera outbreaks. Recently, Tien and Earn (2010) proposed a mathematical model to study the dynamics of water borne diseases, like giardiasis, campylobacter, hepatitis A and E, etc. In this model, they assumed that the infection is transmitted by direct contact between susceptibles and infectives as well as through contaminated water supply containing pathogens. More recently, Tian and Wang (2011) have analyzed a cholera model by considering general incidence function for multiple transmissions and a general growth rate function for pathogens.

The past experience reveals that the outbreak of water borne infectious diseases cause mortality of millions of people as well as expenditure of enormous amount of money in health care and disease control. It is, therefore, essential that adequate attention be paid to prevent the spread of such diseases by taking control measures into account. Government and other health care organizations formulate various strategies for preventing epidemics, like vaccination, awareness, quarantine, etc. Various vaccination models have been proposed and analyzed for the prevention and control of epidemics (Anderson and May, 1983; Shulgin et al., 1998; Kribs-Zaleta and Velasco-Hernandez, 2000; Liu et al., 2008; Naresh et al., 2008; Zhang and Teng, 2008; Li and Yang, 2011). In these models a critical vaccination rate has been derived above which the disease may be eradicated. Recently, Misra et al. (2011) have discussed a mathematical model to study the role of awareness programs driven by media on the prevalence of infectious diseases. Some quarantine mathematical models have also been proposed and analyzed (Ma et al., 2009). In view of the Haiti cholera outbreaks, some spatial studies have also been conducted to look into its dynamics (Chao et al., 2011; Tuite et al., 2011; Bertuzzo et al., 2011). These studies have provided a more promising way to explore the potential effect of various control strategies for cholera epidemic such as vaccination, provision for safe and clean drinking water, minimizing scarcity of decontaminated food, proper sanitation, good sewage system to excreta disposal, and waste water treatment. In these studies, Haiti is divided into 10 departments and it is assumed that the infection spread in a department due to the presence of pathogens in the same department as well as in the other departments. As in case of cholera epidemic, it is well known that infection spreads rapidly in human population due to the contaminated water supply. Thus the eradication of pathogen *V. cholerae* from the aquatic environment may be one of the important strategies for preventing the prevalence of cholera. This can be done through proper treatment and adequate sanitation of the water sources. For this, disinfectants may be used which either kill bacteria or inhibit their growth in the aquatic environment. Sometimes government employees (trained squads) visit to the affected areas and introduce the disinfectant in the aquatic reservoir to kill the pathogen *V. cholerae*. Since the overuse of disinfectants may be harmful to the human health (http://www.ghchealth.com/chlorine.html), it is more reasonable to use these disinfectants proportionally to the density of pathogens. For this proper sampling of water from the affected areas for the measurement of density of bacteria in the aquatic environment is needed. As there is a time lag involved in the sampling of water and introduction of disinfectants, this suggests to incorporate time delay in the modeling process. So in this paper, we also discuss about the time lag involved in the sampling and the introduction of disinfectants for controlling the growth of pathogens.

To summarize, in this paper, we consider that the cholera disease spreads only due to the ingestion of contaminated water containing pathogen *V. cholerae* by susceptibles. Further it is considered that reservoirs are contaminated due to lack of sanitation and release of sewage into the aquatic environment. It is also assumed that disinfectants are used to eradicate the pathogen *V. cholerae* from the aquatic environment. As the measured data for the density of bacteria in the aquatic environment is usually few days old, thus the rate of introduction of disinfectants at any time t is assumed to be proportional to the density of *V. cholerae* at time $(t-\tau)$ where $\tau > 0$, which appears as a time delay in the modeling process.

2. Mathematical model

Let $N(t)$ be the total human population in any region under consideration at time t which is divided into three subclasses, (i) susceptible class $X(t)$, (ii) infective class $Y(t)$, and (iii) recovered class $Z(t)$. Let $B(t)$ and $C_h(t)$ be the density of *V. cholerae* and concentration of disinfectant at time t in the aquatic environment, respectively.

In the modeling process, it is assumed that susceptibles are become infective with a rate proportional to the number of susceptibles and the density of *V. cholerae* present in the contaminated water i.e. 'βXB'. It is also assumed that infectives will recover with rate ν and move to the recovered class with a rate proportional to number of infectives i.e. 'νY', with temporary immunity against reinfection. It is also assumed that the period of this temporary immunity is $1/\delta$ and after that they join the susceptible class with a rate proportional to the number of recovered individuals i.e. 'δZ'. Also, the growth rate of density of *V. cholerae* in the aquatic environment is assumed to be proportional to the number of infectives i.e. 'μY'. Further, it is assumed that the rate of introduction of disinfectants to kill *V. cholerae* is proportional to the density of *V. cholerae* in the aquatic environment with some time lag τ i.e. '$\theta B(t-\tau)$'. The natural washout rate of disinfectant is assumed to be proportional to its concentration i.e. '$\theta_1 C_h$' and its absorption rate due to uptake by *V. cholerae* is assumed to be proportional to both the density of *V. cholerae* as well as the concentration of disinfectant i.e. '$\sigma_1 BC_h$'. Furthermore, it is assumed that water is being supplied to the houses with a rate proportional to the number of total human population i.e. '$\phi_2 N$' and thus the density of *V. cholerae* in the aquatic environment decreases with a rate proportional to this water supply and its density i.e. '$\phi_1(\phi_2 N)B$' or say 'ϕNB'. The natural decay rate of *V. cholerae* is assumed to be proportional to its density i.e. '$\mu_1 B$'. Also the eradication of *V. cholerae* due to the effect of disinfectants is assumed to be proportional to the density of *V. cholerae* as well as the concentration of disinfectant i.e. '$\sigma_2 BC_h$'. Here, it may be noted that we can write $\beta = \lambda_1(\lambda_2\phi)$, where the constant λ_2 is a fractional constant representing the consumption of supplied contaminated water by a single susceptible and the constant λ_1 is the transmission rate coefficient of susceptibles to the infective class due to consumption of contaminated water.

With the above assumptions, the mathematical formulation of the problem is as follows:

$$\dot{X}(t) = A-\beta X(t)B(t)-dX(t)+\delta Z(t),$$

$$\dot{Y}(t) = \beta X(t)B(t)-(v+\alpha+d)Y(t),$$

$$\dot{Z}(t) = vY(t)-(d+\delta)Z(t),$$

$$\dot{B}(t) = \mu Y(t)-\mu_1 B(t)-\phi N(t)B(t)-\sigma_2 B(t)C_h(t),$$

$$\dot{C_h}(t) = \theta B(t-\tau)-\theta_1 C_h(t)-\sigma_1 B(t)C_h(t), \quad (1)$$

with initial conditions $X(0)=X_0>0$, $Y(0)=Y_0\geq 0$, $Z(0)=Z_0\geq 0$, $B(\gamma)=B_0\geq 0$ for $\gamma\in[-\tau,0]$, $C_h(0)=C_{h0}\geq 0$, and $X(0)+Y(0)+Z(0)=N(0)=N_0>0$.

In the above model system (1), A is the constant immigration and birth rate of susceptible population, the constant β represents the transmission rate of susceptibles to the infective class due to consumption of contaminated water. Constants d and α represent the natural death rate and disease induced death rate of human population, respectively. The constant μ is the growth rate of density of *V. cholerae* discharged by infectives in the aquatic environment. Constants μ_1, ϕ and σ_2 represent natural decay rate of *V. cholerae*, decay rate of *V. cholerae* due to water supply to the houses and decay rate due to the introduction of disinfectant, respectively. The constant θ represents the rate of introduction of disinfectants in the aquatic environment to kill *V. cholerae*. Constants θ_1 and σ_1 represent the natural washout rate and absorption rate of disinfectants by *V. cholerae* in the aquatic environment, respectively.

Since we know that $X(t)+Y(t)+Z(t)=N(t)$, the above model system (1) becomes

$$\dot{Y}(t) = \beta(N(t)-Y(t)-Z(t))B(t)-(v+\alpha+d)Y(t),$$

$$\dot{Z}(t) = vY(t)-(d+\delta)Z(t),$$

$$\dot{N}(t) = A-dN(t)-\alpha Y(t),$$

$$\dot{B}(t) = \mu Y(t)-\mu_1 B(t)-\phi N(t)B(t)-\sigma_2 B(t)C_h(t),$$

$$\dot{C_h}(t) = \theta B(t-\tau)-\theta_1 C_h(t)-\sigma_1 B(t)C_h(t). \quad (2)$$

In the following, we analyze the model system (2) using stability theory of differential equations. For the solutions of model system (2), the region of attraction (Shukla et al., 2011) is given by the set

$$\Omega = \left\{(Y,Z,N,B,C_h): 0\leq Y,Z\leq N\leq \frac{A}{d}, 0\leq B\leq B_R, 0\leq C_h\leq \frac{\theta B_R}{\theta_1}\right\}, \quad (3)$$

where $B_R=\mu A/\mu_1 d$, and it attracts all solutions initiating inside the interior of positive the orthant.

3. Equilibrium analysis

The model system (2) has two non-negative equilibria, which are as follows:

(i) The disease free equilibrium $E_0(0,0,A/d,0,0)$ exists without any condition.
(ii) The endemic equilibrium E^* (Y^*, Z^*, N^*, B^*, C_h^*) exists, provided

$$R_0=\frac{\beta\mu A}{d(v+\alpha+d)(\mu_1+\phi A/d)}>1. \quad (4)$$

R_0 is known as the *basic reproduction number* for the model system (2). This denotes the average number of secondary infection that one infected individual generates indirectly through transmission of pathogens in his whole infectious period in a totally susceptible population. To show this, it is easy to note that in the expression of R_0, the constant μ is the density of pathogens in the aquatic environment discharged by a single infective per unit time. They will stay in the aquatic environment for time $1/(\mu_1+\phi A/d)$. Since there are A/d susceptibles in a totally susceptible population and they become infected due to ingestion of contaminated water with a rate β, then $\{\beta A/d\}\{\mu/(\mu_1+\phi A/d)\}$ susceptibles get infected per unit of time due to the presence of pathogens in the aquatic environment. As the total infectious period for infectives is $1/(v+\alpha+d)$, the average number of secondary infection due to a single infected individual in his/her whole infectious period is $\{\beta A/d\}\{\mu/(\mu_1+\phi A/d)\}\{1/(v+\alpha+d)\}$.

Remark 1. From the above condition (4), it may be noted that if the transmission rate of susceptibles to the infective class 'β', rate of immigration of susceptible population 'A', and the growth rate of pathogens in the aquatic environment 'μ', are sufficiently small or the recovery rate of infective population 'v' is large such that condition (4) is not satisfied then the endemic equilibrium E^* will not exist and the disease free equilibrium E_0 will be the only equilibrium of the model system (2).

3.1. Existence of equilibria

In the following, we discuss the existence of equilibrium points. The existence of E_0 is trivial, hence omitted. The endemic equilibrium E^* may be obtained by solving the following algebraic equations ($Y\neq 0$):

$$\beta(N-Y-Z)B-(v+\alpha+d)Y=0, \quad (5)$$

$$vY-(d+\delta)Z=0, \quad (6)$$

$$A-dN-\alpha Y=0, \quad (7)$$

$$\mu Y-\mu_1 B-\phi NB-\sigma_2 BC_h=0, \quad (8)$$

$$\theta B-\theta_1 C_h-\sigma_1 BC_h=0. \quad (9)$$

Now using Eqs. (6) and (7) in Eq. (5), we get

$$Y=\frac{\beta AB}{\left(\frac{\beta((\alpha+d)(d+\delta)+vd)}{(d+\delta)}B+d(v+\alpha+d)\right)}. \quad (10)$$

Again using Eqs. (7) and (9) in Eq. (8), we get

$$B=\frac{\mu Y}{\mu_1+\frac{\phi}{d}(A-\alpha Y)+\frac{\sigma_2\theta B}{(\theta_1+\sigma_1 B)}}. \quad (11)$$

Finally, from Eqs. (10) and (11) we get a quadratic equation in B, which is as follows:

$$PB^2+QB-R=0, \quad (12)$$

where

$$P=\left(\mu_1+\frac{\phi A}{d}\right)\frac{\beta\sigma_1(d+\delta+v)}{(d+\delta)}+\frac{\mu\beta\sigma_1\alpha}{d}+\frac{\beta\theta\sigma_2(\alpha+d)((d+\delta)+vd)}{d(d+\delta)},$$

$$Q=\left(\mu_1+\frac{\phi A}{d}\right)\left\{\left(\frac{\beta\{(\alpha+d)(d+\delta)+vd\}}{d(d+\delta)}\right)\theta_1+\sigma_1\right\}-\left(\frac{\phi\alpha\theta_1}{d}+\mu\sigma_1\right)\frac{\beta A}{d}+\sigma_2\theta(v+\alpha+d)$$

and $R=\theta_1\{\mu\beta A/d-(\mu_1+\phi A/d)(v+\alpha+d)\}$.

From Eq. (12) it is clear that we have a unique positive root say B^* provided $R>0$ (i.e. condition (4) is satisfied) and the value

of B^* is

$$B^* = (-Q + \sqrt{Q^2 + 4PR})/2P.$$

Now using this value of B^* in Eqs. (9) and (10), we get the positive values of C_h^* and Y^*, respectively. Finally, using this value of Y^* in Eqs. (6) and (7), we get positive values of Z^* and N^*, respectively. Thus the endemic equilibrium $E^*(Y^*, Z^*, N^*, B^*, C_h^*)$ exists provided condition (4) is satisfied.

Remark 2. From Eqs. (10) and (11), it is easy to show that $dY^*/d\mu$, $dY^*/d\theta_1$ are positive whereas $dY^*/d\theta$, $dY^*/d\sigma_2$ are negative. This implies that as the rate of discharge of *V. cholerae* by the infectives 'μ' and natural washout rate of disinfectant 'θ_1' increase, the equilibrium level of number of infectives increases. Further, it decreases as the rate of introduction of disinfectant 'θ' and the decay rate of pathogens due to effect of disinfectant 'σ_2' increase.

4. Stability analysis

In this section, we study the local stability of the disease free equilibrium E_0 and the endemic equilibrium E^* without delay as well as with delay. We also look for the possibility of the Hopf-bifurcation by taking time delay 'τ' as a bifurcation parameter.

4.1. Local stability analysis without delay (i.e. $\tau = 0$)

Here we show the local stability behavior of equilibrium E_0 and E^* without delay by finding the eigenvalues of the corresponding Jacobian matrices obtained for model system (2).

The Jacobian matrix for the model system (2) is as follows:

$$M = \begin{pmatrix} -a_{11} & -\beta B & \beta B & a_{14} & 0 \\ \nu & -(d+\delta) & 0 & 0 & 0 \\ -\alpha & 0 & -d & 0 & 0 \\ \mu & 0 & -\phi B & -a_{44} & -\sigma_2 B \\ 0 & 0 & 0 & a_{54} & -a_{55} \end{pmatrix},$$

where $a_{11} = \beta B + (\nu + \alpha + d)$, $a_{14} = \beta(N - Y - Z)$, $a_{44} = \mu_1 + \phi N + \sigma_2 C_h$, $a_{54} = \theta - \sigma_1 C_h$ and $a_{55} = \theta_1 + \sigma_1 B$.

Let M_0 and M^* be the Jacobian matrix evaluated at the equilibrium E_0 and E^* respectively.

From the matrix M_0, we may easily note that its three eigenvalues are $-(d+\delta)$, $-d$, and $-\theta_1$ which are clearly negative, whereas other two eigenvalues are given by the roots of the following quadratic equation:

$$\psi^2 + P'\psi - Q' = 0, \tag{13}$$

where $P' = \{(\mu_1 + \phi A/d) + (\nu + \alpha + d)\}$, and $Q' = \{\beta A\mu/d - (\nu + \alpha + d)(\mu_1 + \phi A/d)\}$.

It is easy to see that if condition (4) is satisfied then Eq. (13) has one negative and one positive root. Thus the equilibrium E_0 is unstable whenever condition (4) is satisfied, which leads to the existence of the endemic equilibrium E^*.

Now we study the local stability behavior of the endemic equilibrium E^* by using Routh–Hurwitz criterion. The characteristic equation for the matrix M^* is given as

$$\Phi^5 + p_1\Phi^4 + p_2\Phi^3 + p_3\Phi^2 + p_4\Phi + p_5 = 0, \tag{14}$$

where

$$p_1 = a_{11}^* + (d+\delta) + d + a_{44}^* + a_{55}^*,$$

$$p_2 = a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^* + (a_{44}^* + a_{55}^*)(a_{11}^* + (d+\delta) + d) + (a_{11}^* d + (d+\delta)d + a_{11}^*(d+\delta)) + \beta B^* \nu - \beta B^* \alpha + a_{14}^* \mu,$$

$$p_3 = (a_{11}^* + (d+\delta) + d)(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) + (a_{44}^* + a_{55}^*)(a_{11}^* d + (d+\delta)d + a_{11}^*(d+\delta)) + a_{11}^*(d+\delta)d + \beta B^* \nu d + \beta B^* \nu(a_{44}^* + a_{55}^*) - \beta B^* \alpha((d+\delta) + a_{44}^* + a_{55}^*) + a_{14}^*(\alpha\phi B^* + \mu d + \mu((d+\delta) + a_{55}^*)),$$

$$p_4 = (a_{11}^* d + (d+\delta)d + a_{11}^*(d+\delta))(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) + a_{11}^*(d+\delta)d(a_{44}^* + a_{55}^*) + \beta B^* \nu d(a_{44}^* + a_{55}^*) + \beta B^* \nu(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) - \beta B^* \alpha(d+\delta)(a_{44}^* + a_{55}^*) - \beta B^* \alpha(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) + a_{14}^*((d+\delta) + a_{55}^*)(\alpha\phi B^* + \mu d) + a_{14}^*(d+\delta)a_{55}^* \mu,$$

$$p_5 = a_{11}^*(d+\delta)d(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) + \beta B^* \nu d(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) - \beta B^* \alpha(d+\delta)(a_{44}^* a_{55}^* + \sigma_2 B^* a_{54}^*) + a_{14}^*(d+\delta)a_{55}^*(\alpha\phi B^* + \mu d).$$

Here $a_{11}^* = \beta B^* + (\nu + \alpha + d)$, $a_{14}^* = \beta(N^* - Y^*)$, $a_{44}^* = \mu_1 + \phi N^* + \sigma_2 C_h^*$, $a_{54}^* = \theta - \sigma_1 C_h^* = \theta_1 C_h^*/B^*$ and $a_{55}^* = \theta_1 + \sigma_1 B^*$.

Now it is easy to see that $p_1 > 0$. Thus for the local stability of the endemic equilibrium $E^*(Y^*, Z^*, N^*, C^*, C_h^*)$ of model system (2) without delay, we have the following result.

Theorem 1. *The endemic equilibrium $E^*(Y^*, Z^*, N^*, B^*, C_h^*)$ is locally asymptotically stable for $\tau = 0$ iff the following conditions are satisfied*:

$$p_5 > 0,\ p_1p_2 - p_3 > 0,\ p_3(p_1p_2 - p_3) - p_1(p_1p_4 - p_5) > 0 \text{ and } p_4(p_3(p_1p_2 - p_3) - p_1(p_1p_4 - p_5)) - p_5(p_2(p_1p_2 - p_3) - (p_1p_4 - p_5)) > 0.$$

where p_1, p_2, p_3, p_4, and p_5 are defined as above.

4.2. Local stability analysis with delay (i.e. $\tau \neq 0$)

In this section we analyze the model system (2) with delay. We also derive the stability conditions for the equilibrium E^* as well as the conditions for Hopf-bifurcation.

Now linearizing the system (2) about the endemic equilibrium E^* $(Y^*, Z^*, N^*, B^*, C_h^*)$, by using the following transformations: $Y = Y^* + y$, $Z = Z^* + z$, $N = N^* + n$, $B = B^* + b$, $C_h = C_h^* + c_h$, we have

$$\dot{u} = Cu(t) + Du(t - \tau), \tag{15}$$

where $u(t) = [y, z, n, b, c_h]^T$,

$$C = \begin{pmatrix} -a_{11}^* & -\beta B^* & \beta B^* & a_{14}^* & 0 \\ \nu & -(d+\delta) & 0 & 0 & 0 \\ -\alpha & 0 & -d & 0 & 0 \\ \mu & 0 & -\phi B^* & -a_{44}^* & -\sigma_2 B^* \\ 0 & 0 & 0 & -\sigma_1 C_h^* & -a_{55}^* \end{pmatrix} \text{ and}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \end{pmatrix}.$$

Here y, z, n, b and c_h are small perturbations around the equilibrium E^*.

The characteristic equation for the linearized system (15) is given by the following equation:

$$P(\psi) + Q(\psi)e^{-\psi\tau} = 0, \tag{16}$$

where $P(\psi) = \psi^5 + A_1\psi^4 + A_2\psi^3 + A_3\psi^2 + A_4\psi + A_5$ and $Q(\psi) = B_1\psi^3 + B_2\psi^2 + B_3\psi + B_4$.

In the above expressions of $P(\psi)$ and $Q(\psi)$, A_i's $(i = 1, 2, 3, 4, 5)$ and B_j's $(j = 1, 2, 3, 4)$ are given as follows:

$$A_1 = a_{11}^* + (d+\delta) + d + a_{44}^* + a_{55}^*,$$

$$A_2 = a_{44}^* a_{55}^* - \sigma_2 B^* \sigma_1 C_h^* + (a_{44}^* + a_{55}^*)(a_{11}^* + (d+\delta) + d)$$

$$+(a_{11}^*d+(d+\delta)d+a_{11}^*(d+\delta))+\beta B^*v-\beta B^*\alpha+a_{14}^*\mu,$$

$$A_3=(a_{11}^*+(d+\delta)+d)(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)$$
$$+(a_{44}^*+a_{55}^*)(a_{11}^*d+(d+\delta)d+a_{11}^*(d+\delta))+a_{11}^*(d+\delta)d+\beta B^*vd$$
$$+\beta B^*v(a_{44}^*+a_{55}^*)-\beta B^*\alpha((d+\delta)+a_{44}^*+a_{55}^*)$$
$$+a_{14}^*(\alpha\phi B^*+\mu d+\mu((d+\delta)+a_{55}^*)),$$

$$A_4=(a_{11}^*d+(d+\delta)d+a_{11}^*(d+\delta))(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)$$
$$+a_{11}^*(d+\delta)d(a_{44}^*+a_{55}^*)+\beta B^*vd(a_{44}^*+a_{55}^*)$$
$$+\beta B^*v(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)-\beta B^*\alpha(d+\delta)(a_{44}^*+a_{55}^*)$$
$$-\beta B^*\alpha(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)+a_{14}^*((d+\delta)+a_{55}^*)(\alpha\phi B^*+\mu d)$$
$$+a_{14}^*(d+\delta)a_{55}^*\mu,$$

$$A_5=a_{11}^*(d+\delta)d(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)+\beta B^*vd(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)$$
$$-\beta B^*\alpha(d+\delta)(a_{44}^*a_{55}^*-\sigma_2B^*\sigma_1C_h^*)+a_{14}^*(d+\delta)a_{55}^*(\alpha\phi B^*+\mu d),$$

$$B_1=\theta\sigma_2B^*,$$

$$B_2=\theta\sigma_2B^*(a_{11}^*+(d+\delta)+d),$$

$$B_3=\theta\sigma_2B^*(a_{11}^*d+(d+\delta)d+a_{11}^*(d+\delta)+\beta B^*v-\beta B^*\alpha),$$

$$B_4=\theta\sigma_2B^*(a_{11}^*(d+\delta)d+\beta B^*v\alpha-\beta B^*\alpha(d+\delta)).$$

To show Hopf-bifurcation, we must have a pair of purely imaginary roots of characteristic Eq. (16). For this substituting $\psi=i\omega$ ($\omega>0$) into Eq. (16) and separating real and imaginary parts, we get the following transcendental equations:

$$A_1\omega^4-A_3\omega^2+A_5=(B_2\omega^2-B_4)\cos(\omega\tau)-(B_3\omega-B_1\omega^3)\sin(\omega\tau), \tag{17}$$

and

$$\omega^5-A_2\omega^3+A_4\omega=-(B_3\omega-B_1\omega^3)\cos(\omega\tau)-(B_2\omega^2-B_4)\sin(\omega\tau). \tag{18}$$

Now squaring and adding Eqs. (17) and (18), we get the following equation in ω:

$$(A_1\omega^4-A_3\omega^2+A_5)^2+(\omega^5-A_2\omega^3+A_4\omega)^2$$
$$=(B_2\omega^2-B_4)^2+(B_3\omega-B_1\omega^3)^2. \tag{19}$$

Substituting $\omega^2=\eta$ in above Eq. (19), we get

$$h(\eta)=\eta^5+D_1\eta^4+D_2\eta^3+D_3\eta^2+D_4\eta+D_5=0, \tag{20}$$

where $D_1=A_1^2-2A_2$, $D_2=A_2^2+2A_4-2A_1A_3-B_1^2$, $D_3=A_3^2+2A_1A_5+2B_1B_3-B_2^2-2A_2A_4$, $D_4=A_4^2-2A_3A_5+2B_2B_4-B_3^2$ and $D_5=A_5^2-B_4^2$.

Now if the coefficients in $h(\eta)$ satisfy the conditions of the Routh–Hurwitz criterion, then Eq. (20) will not have any positive real root, thus we may not get any positive value of ω which satisfies the transcendental equations (17) and (18). In this case the result may be written in the form of following theorem.

Theorem 2. *If the coefficients in $h(\eta)$ (i.e. D_i's) satisfy the conditions of the Routh–Hurwitz criterion, then the endemic equilibrium E^* of model system (2) is asymptotically stable for all delay $\tau>0$ provided it is stable in the absence of delay.*

Assuming contrary that the values of D_i's, ($i=1, 2, 3, 4, 5$) in (20) do not satisfy the Routh–Hurwitz criterion. In this case a simple assumption for the existence of a positive root of equation (20) is $D_5<0$, which gives

$$A_5^2-B_4^2<0. \tag{21}$$

Now if condition (21) holds, then Eq. (20) has a positive root η_0 and thus Eq. (18) has a pair of purely imaginary roots $\pm i\omega_0$. From transcendental equations (17) and (18) we may obtain

$$\tan\omega\tau=N_1/N_2, \tag{22}$$

where $N_1=(A_1\omega_0^4-A_3\omega_0^2+A_5)(B_1\omega_0^3-B_3\omega_0)+(\omega_0^5-A_2\omega_0^3+A_4\omega_0)(B_4-B_2\omega_0^2)$, and $N_2=(A_1\omega_0^4-A_3\omega_0^2+A_5)(B_2\omega_0^2-B_4)-(\omega_0^5-A_2\omega_0^3+A_4\omega_0)(B_3\omega_0-B_1\omega_0^3)$.

Now τ_k corresponding to this positive value of ω_0 is given as follows:

$$\tau_k=k\pi/\omega_0+(1/\omega_0)tan^{-1}(N_1/N_2), \tag{23}$$

where $k=0,1,2,3,\ldots$

By using Butler's lemma (Freedman and Rao, 1983), we can say that the endemic equilibrium of model system (2) remains stable for $\tau<\tau_0$.

Now we investigate whether there is a phenomenon of Hopf-bifurcation as τ increases through τ_0. For this we need the following lemma.

Lemma 1. *The following transversality condition is satisfied*:

$$\text{sign}\{d(Re(\psi))/d\tau\}_{\tau=\tau_0}>0. \tag{24}$$

Proof. Differentiating Eq. (16) with respect to τ, we get

$$(d\psi/d\tau)^{-1}=(P_1+P_2e^{-\psi\tau})/\psi P_3e^{-\psi\tau}-\tau/\psi$$
$$=P_1/\psi P_3e^{-\psi\tau}+P_2/\psi P_3-\tau/\psi, \tag{25}$$

where $P_1=(5\psi^4+4A_1\psi^3+3A_2\psi^2+2A_3\psi+A_4)$, $P_2=3B_1\psi^2+2B_2\psi+B_3$, and $P_3=(B_1\psi^3+B_2\psi^2+B_3\psi+B_4)$. □

Now using Eqs. (16), (17) and (18) in above Eq. (25) and after a little algebraic manipulation, we get

$$[d(Re(\psi))/d\tau]^{-1}_{\psi=i\omega_0}=[Re(d\psi/d\tau)^{-1}]_{\psi=i\omega_0}$$
$$=(5\eta_0^4+4D_1\eta_0^3+3D_2\eta_0^2+2D_3\eta_0+D_4)/\wedge$$
$$=h'(\eta_0)/\wedge. \tag{26}$$

where $\wedge=(A_1\omega_0^4-A_3\omega_0^2+A_5)^2+(\omega_0^5-A_2\omega_0^3+A_4\omega_0)^2=(B_2\omega_0^2-B_4)^2+(B_3\omega_0-B_1\omega_0^3)^2$.

Here it may be noted that $h'(\eta_0)>0$ if the condition (21) is satisfied. This proves the Lemma 1. Thus we have the following result:

Theorem 3. *If the condition (21) is satisfied then the stable endemic equilibrium E^* of model system (2) remains stable for $\tau<\tau_0$ and becomes unstable for $\tau>\tau_0$. The condition for Hopf-bifurcation is also satisfied yielding the required periodic solution from the endemic equilibrium E^* as τ passes through τ_0, i.e. Hopf-bifurcation occurs at $\tau=\tau_0$* (Gopalsamy, 1992).

The above theorem implies that if delay in introduction of disinfectants is small (less than τ_0), the disease may be controlled. However in the case of longer delay (greater than τ_0), all variables start showing oscillatory behavior i.e., sometimes the number of infectives will be high and sometimes low and it may be difficult to make the prediction regarding the size of epidemic. Thus the implementation of control measures to prevent the disease will be difficult.

5. Numerical simulation

In this section, we perform a numerical simulation of model system (2) to confirm our analytical results. We choose a set of parameter values in model system (2) according to Table 1. Some of these parameter values have been taken from (Codeço, 2001; Hartley et al., 2006; Bertuzzo et al., 2008).

For the above set of parameter values, the condition (4) for existence of endemic equilibrium E^* is satisfied and the basic reproduction number R_0 is 5.6. The endemic equilibrium E^* for this data is $Y^*=201$ persons, $Z^*=38{,}337$ persons, $N^*=179{,}872$ persons, $B^*=2920$ cells/l, $C_h^*=1.35$ mg-chlorine/l.

For $\tau = 0$, the eigenvalues of the variational matrix corresponding to the equilibrium E^* for the model (2) are $-2.02236+1.10247\ i$, $-2.02236-1.10247\ i$, -0.050665, -0.00218, and -0.00006. It may be noted here that three eigenvalues of the variational matrix M^* are negative whereas its other two eigenvalues are with negative real part. Hence, the endemic equilibrium E^* is locally asymptotically stable in the absence of delay.

We have shown the effect of various parameters on the incidence of disease in Fig. 1. From this figure it is easy to note that the size of the infected class increases as the discharge rate of *V. cholerae* by infectives in the aquatic environment or/and the transfer rate of *V. cholerae* due to water supply increases. It is also clear that this size decreases as the introduction rate of disinfectants or/and the effect of disinfectants on *V. cholerae* increases. This shows that by the eradication of *V. cholerae* using disinfectants we can control the prevalence of cholera epidemic.

For the above set of parameter values, condition (21) for the existence of a pair of purely imaginary roots of characteristic Eq. (16) is also satisfied. By using Eq. (23), we have computed numerical value of τ_0, which is 5.6 days. Now applying Theorem 3, it is clear that the endemic equilibrium E^* of model system (2) is stable for $\tau < 5.6$ days and unstable for $\tau > 5.6$ days. These results have been shown in Figs. 2 and 3, for $\tau = 5$ days and $\tau = 6$ days respectively. From Fig. 2, it may be noted that all the variables are approaching to their equilibrium states, showing the stability of endemic equilibrium E^* for $\tau < \tau_0$. From Fig. 3, we may note that all the variables are showing oscillatory behavior, thus the endemic equilibrium E^* is unstable for $\tau > \tau_0$. The biological meaning of the above discussion is that if the measured data for the density of *V. cholerae* is older than 5.6 days then the number of infectives will oscillate and in this case it will be difficult to predict the size and severity of cholera epidemic. If one wants to predict the size of epidemic then the measured density of *V. cholerae* in the aquatic environment should not be older than 5.6 days for the above parameter values.

We have also calculated the variation of the critical value of time delay (τ_0) with respect to the introduction rate of disinfectant (θ), which is shown in Table 2. This variation shows that delay in using disinfectants results in the introduction of higher amount of disinfectants to inhibit the growth of *V. cholerae*.

Table 1
Parameter values in model system (2).

Parameter	Value	Unit
A	10	*person day*$^{-1}$
β	0.0000001	*l (cells day)*$^{-1}$
ν	0.2	*day*$^{-1}$
α	0.005	*day*$^{-1}$
d	0.00005	*day*$^{-1}$
δ	0.001	*day*$^{-1}$
μ	25	*cells (l person day)*$^{-1}$
μ_1	0.03	*day*$^{-1}$
ϕ	0.000002	*(person day)*$^{-1}$
σ_2	0.99	*l (mg day)*$^{-1}$
θ	0.001	*mg-chlorine (cell day)*$^{-1}$
θ_1	1	*day*$^{-1}$
σ_1	0.0004	*l (cell day)*$^{-1}$

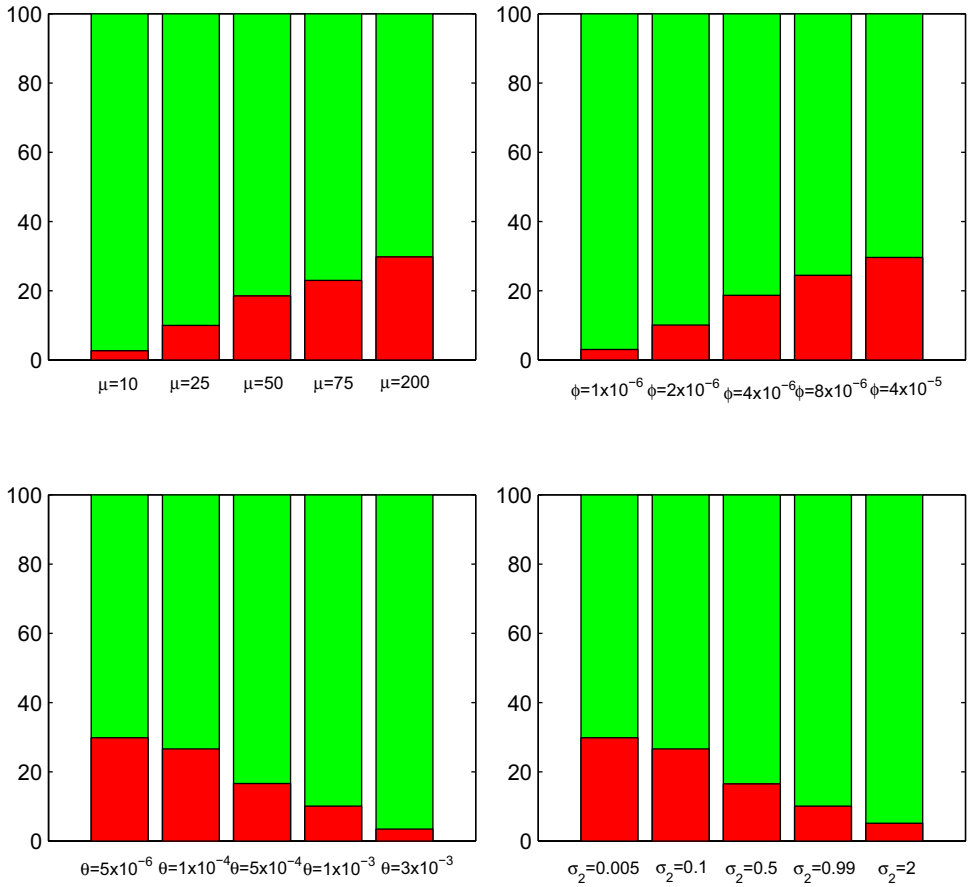

Fig. 1. Change in percentage of infectives for different values of μ, ϕ, θ and σ_2. In this figure red and green portion represents the percentage of Y and $X/100$ respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

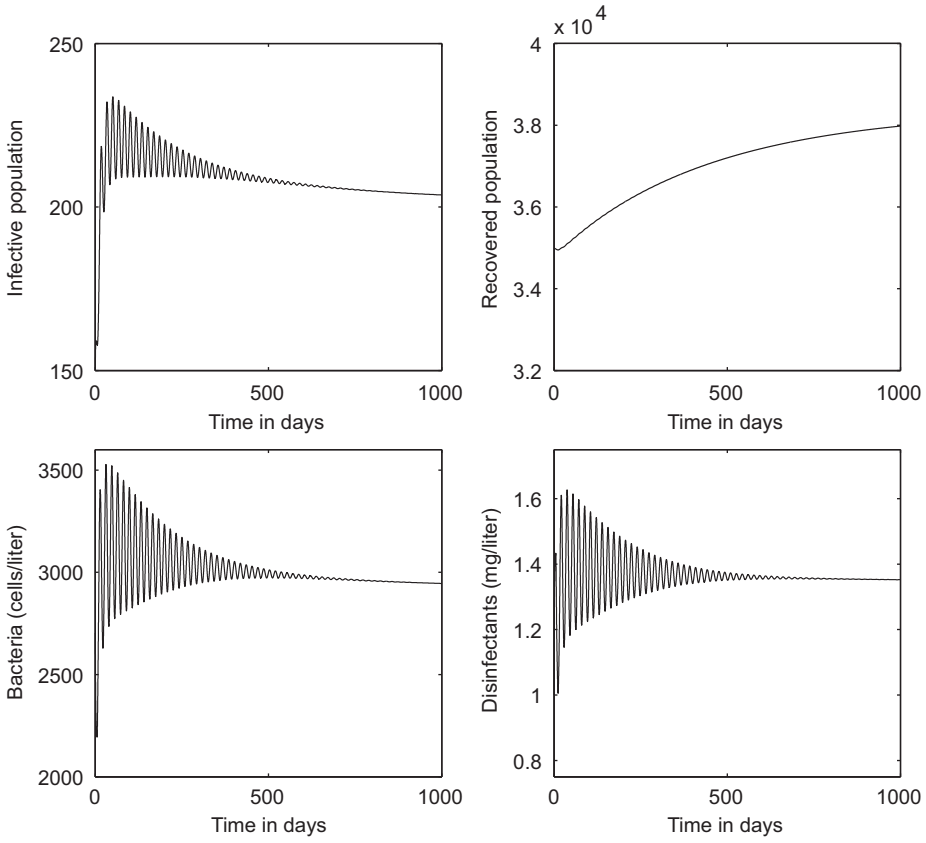

Fig. 2. Variation of Y, Z, B and C_h with respect to time t for $\tau = 5$ days.

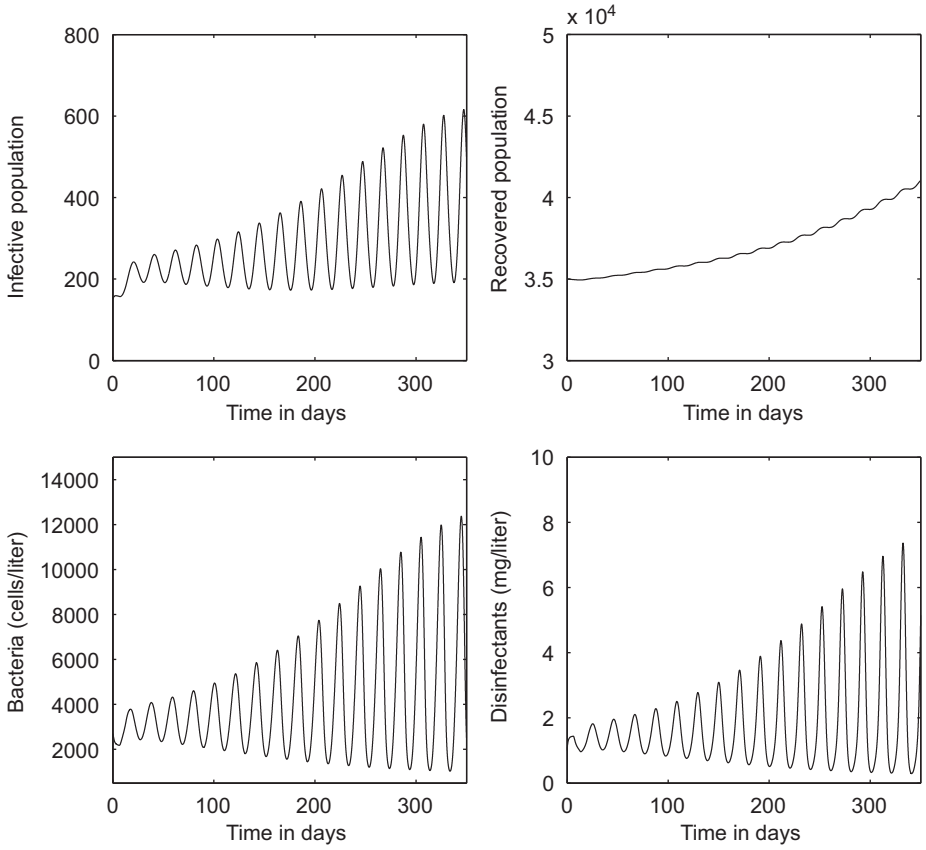

Fig. 3. Variation of Y, Z, B and C_h with respect to time t for $\tau = 6$ days.

Table 2
Variation of τ_0 for different values of θ.

θ ((mg-chlorine)/(cell.day)$^{-1}$)	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
τ_0 days	5.60	6.46	7.13	7.56	7.86	8.07	8.22	8.35	8.45	8.53

6. Conclusion

In this paper, a non-linear SIRS delay epidemic model for the control of cholera disease, using disinfectant to inhibit the growth of pathogen *V. cholerae*, has been proposed and analyzed. It is found that the model has two non-negative equilibria, (i) the disease free equilibrium E_0 and (ii) the endemic equilibrium E^*. The disease free equilibrium E_0 exists without any condition whereas the endemic equilibrium E^* exists provided $R_0 > 1$. Through the analysis of the model it has been found that in the absence of delay, the disease free equilibrium E_0 is unstable whenever the endemic equilibrium E^* exists i.e., $R_0 > 1$, however the endemic equilibrium is locally stable under some conditions stated in Theorem 1 Further it has been found that the stable endemic equilibrium E^* remains stable for all $\tau > 0$ under certain conditions given in Theorem 2. It is also shown that the endemic equilibrium enters a Hopf-bifurcation as τ crosses some critical value τ_0. This critical value of τ_0 has been obtained analytically and is given by Eq. (23).

The model analysis shows that by eradicating *V. cholerae* from the aquatic reservoirs using disinfectants, we can control the disease provided the time lag between the sampling of water for the measurement of density of *V. cholerae* and introduction of disinfectant is small (less than τ_0). If this time lag between sampling and introduction of disinfectant is large (more than τ_0), the disease may not be controlled. In the latter case, the disease will persist in the human population and the prediction about the size and severity of the disease will be difficult. Furthermore, our simulation analysis shows that the higher amount of introducing disinfectants will increase this time lag. But a large amount of disinfectants is hazardous to human's health. Therefore, a quick response to control the *V. cholerae* density in the aquatic environment using disinfectants is required, so that the epidemic can be controlled without any adverse effect of these disinfectants on the human's health.

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