ON THE REALIZABLE TYPES OF NUMERICAL SEMIGROUPS

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Abstract. In this paper, we consider the following question: "given the multiplicity m and embedding dimension e of a numerical semigroup S, what can be said about the type t of S?" We approach this question from a poset-theoretic perspective, utilizing the notions of numerical semigroups, Apéry sets, and Kunz nilsemigroups in our proof processes. We make significant headway on this question, in the form of computationally verified realizable types of numerical semigroups, as well as generalized semigroup families proven to cover various bounds of the parameters m, e and t.

1. Introduction

A semigroup S is defined as follows: If n_1,...,n_k is a set of relatively prime positive integers, then

$$S = \langle n_1, ..., n_k\rangle = \{x_1n_1 + x_2n_2 + ... + x_kn_k : x_1, ..., x_k \in \mathbb{N}_0\}$$

is known as a numerical semigroup [1]. The elements n_1,...,n_k are referred to as the *minimal generators* of $\langle n_1, ..., n_k\rangle$ [1]. Two important semigroup properties we will look at are *multiplicity*, which is denoted $m(S)$ [3], and the *embedding dimension*, $e(S)$ [3]. The *Apéry Set* of S, with $m = m(S)$, is the set

$$Ap(S) := \{n \in S : n - m \notin S\}$$

of minimal elements of S within each equivalence class modulo m. Since S is cofinite, we are guaranteed $|Ap(S)| = m$, and that $Ap(S)$ contains exactly one element in each equivalence class modulo m. As such, we often write

$$Ap(S) = \{a_0, a_1, ..., a_{m-1}\}$$

with each $a_i \equiv i \ mod \ m$, and view the subscripts as elements of $\mathbb{Z}_m$ [3].

Recall that a *nilsemigroup* is a semigroup with a universally absorbing element ∞, called the *nil*. Let $(N, +)$ be a nilsemigroup that is finite, has an identity $0 \in \mathbb{N}$, and is *partly cancellative*: $a + b = a + c \neq \infty$ implies $b = c$ for all $b, c \in \mathbb{N}$. The

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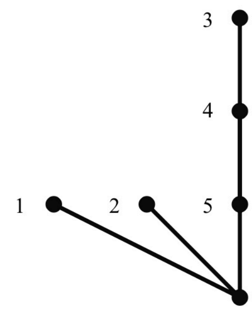

FIGURE 1. The Kunz poset P of S from Example 1.1

Kunz nilsemigroup N of a numerical semigroup S is obtained from $S/\sim$, where $\sim$ is the congruence that relates $a \sim b$ whenever $a = b$ or $a, b/ \in Ap(S)$ (the set $S \backslash Ap(S)$ comprises the nil of $S/\sim$), by replacing each non-nil element with its equivalence class in $\mathbb{Z}_m$ [3].

A finite partly cancellative nilsemigroup N may be visualized by examining the *divisibility poset* P of non-nil elements, wherein $b \preceq c$ when $c = a + b$ for some $a \in N$. If N is the Kunz nilsemigroup of a numerical semigroup S, we call P the *Kunz poset* of S [3]. Examining P allows us to visualize properties such as m, e, and t, where $t = t(S)$ is the *type*, or the number of maximal elements of P.

Example 1.1. Suppose we have

$$S = \langle 6, 7, 28, 29 \rangle.$$

Examining its Kunz poset P in Figure 1, we see the following:
P contains $m = 6$ total elements, $e = 4$ minimal generators, and $t = 3$ maximal elements.

From each semigroup, we obtain a Kunz poset, where we examine these properties. Our goal is to take a generalized Kunz poset, with varying m, e, and t, and prove there exist semigroups with all 3 desired properties. We will do this by bounding m, e, and t, and finding a generalized poset that can be adjusted to fit any m, e, and t values within the bounds. If we build a valid Apéry set of S using the related Kunz poset, we derive a semigroup S, thus proving the existence of a semigroup for the related Kunz poset.

2. Code results

The code for obtaining data to find realizable types of numerical semigroups can be found at [4]. We used Sage packages, Numericalsemigroup.sage from [5], and KunzPoset.sage and PlotKunzPoset.sage from [6]. Instructions for download of these packages can be found on their respective github README files. Also necessary is to download the Kunz Face Data sets derived in [7], whose files can be found on [4] labeled full_face_lattice_m3-14. The file location of this data is the same file location used for the *file_location* variable at the beginning of the Python file in [4]. The Kunz face data labeled is used to calculate possible posets, which we used to find valid, existing semigroups.

The first function, csv_creation_no_repeats(*min_elem*, *dimension* = 0, *csv_file_path* = *file_location*), creates a CSV file into the same file location as csv_file_location. It into the new folder called *csv_creations*, which is in the same folder that is holding the folder for the Kunz Face Data. This folder will hold all the CSV files for the following functions. The *min_elem* variable represents the minimum element of the numerical semigroups we are looking for. The *dimension* variable has a range from 1 to (*min_elem* - 1). (Note: this function uses multiple gigabytes of RAM as the minimal element increases above 10.) The dimension variable splits the creation of the CSV file into smaller parts to prevent the program from running out of RAM during calculations. If adequate RAM is not an issue, leave the *dimension* variable equal to 0. If necessary, assign it a value from the range given. The resulting CSV files record the minimal element, embedding dimension, type, minimal trades, and, optionally, the dimension of the Kunz face. The CSV file does not store full repeated lines of values, as we are only looking for the possible values for the previously stated m, e, t, and n.

If the data for a particular minimal element is separated by dimensions into multiple CSV files, the function merging_csvs(*min_elem*, *csv_file_path* = file_location) merges said files into one.

The function first_3_columns(*min_elem*, *csv_file_path* = *file_location*), deletes the last two columns of a CSV file from a given minimal element value. For this purpose, it deletes the minimal trades and Kunz face dimension, which are not relevant to our task. It also deletes repeated full row values. This leaves us with a list of minimal elements, embedding dimensions, and types of possible numerical semigroups. This list is what was used to create the staircase image that is featured in the next section, Figure 2. The previous three functions have been executed correctly if the function prints "CSV file '{csv_file_path}' created successfully."

The function poset_finder(*min_elem*, *embed_dim*, *Type*, *dimension* = 0, *csv_file_path* = *file_location*), creates a list of posets that can be printed in the following cell. In this cell, set the particular *min_elem*, *embed_dim*, and *type* values of interest. The last line of that cell then prints the posets using HTML. We manually sorted through

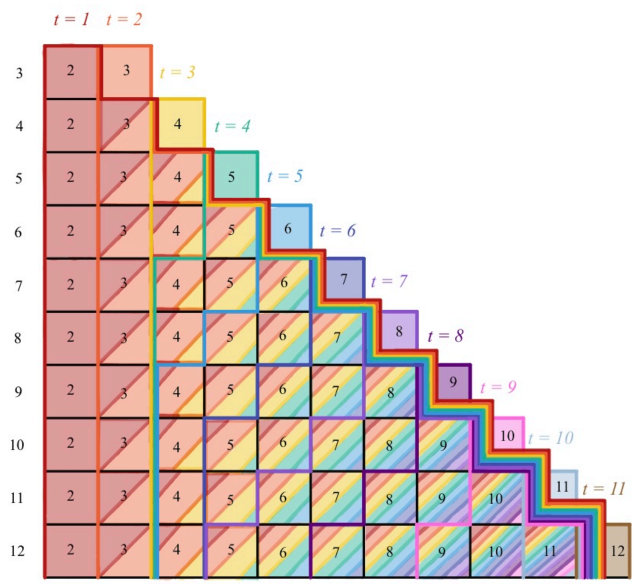

FIGURE 2. The values of $t(S) \leq 11$ attained and outlined for $m(S) \leq 12$ and $e(S) \leq 12$. A version of this image with each type demonstrated individually can be found in Figure 5.

these printed posets in order to find the patterns that led us to the creation of the generalized posets seen in 6 and 5.

3. FIGURE 1 - STAIRCASE IMAGE

The diagram in Figure 2 lists all values of t achieved for $m \leq 12$ and $2 \leq e \leq m$, obtained computationally using algorithms from [4]. Each row corresponds to a value of $m \leq 12$ and each boxed number in that row is the value of e achieved by some numerical semigroup with multiplicity m. Each bold colored edge demarcates the values of t that are achieved by numerical semigroups with the labeled values of e and m. If a box is shaded with a given color, we have proved the existence of a semigroup family with those values of m, e, and t within the contents of this paper.

Figure 2 puts the results of this manuscript in context. Shaded boxes indicate values of t attained by families of numerical semigroups constructed in Theorems 5.1, and 8.1, colored according to the value of t used therein. Theorem 5.1 proves the existence of numerical semigroups that posses $2 \leq e \leq m$ and $t = e - 1$, which can be seen in Figure 6. We later prove in Theorem 8.1 that if $2 \leq e < m$ and $e \geq t + 1$, then a numerical semigroup exists. This can be seen in Figures 5 and 4. In fact, we conjecture that the family obtained in Theorem 8.1 applies for all values of m, but have verified computationally up to $m = 12$.

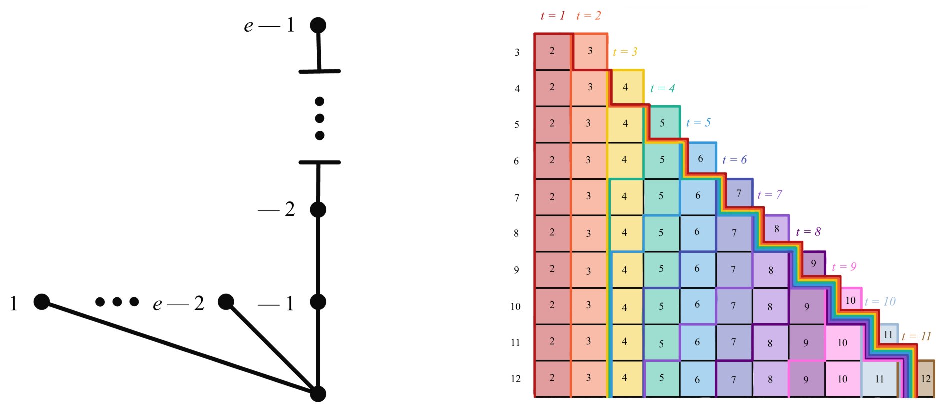

FIGURE 3. The Kunz poset of S from Theorem 5.1 (left) and the corresponding blocks it verifies in the staircase image (right).

4. First Generalized Poset

In order to prove the existence of the semigroups created in our code, we manually sorted through posets with similar structure to create a generalized poset with varying parameters. The first generalized poset we found has varying multiplicity and embedding dimension, and the type is fixed to one less than the embedding dimension. The general form of this poset is shown in Figure 6. In relationship to the staircase image in Figure 6, for any particular e value, this poset corresponds to the column such that $t = e - 1$.

We will prove the existence of this semigroup in the following section. In proving the existence of this semigroup, we first found the Apéry set. We did this by starting with the bottom of the upward "stem" of the poset and moving upwards, selecting values that were in the correct mod class. After finding the upward "stem" of the poset, we created the element $\tilde{a}$ in the semigroup to be one less than the element in the mod class $e - 1$, which is the top of the upward "stem" of this poset. By filling out the rest of the semigroup in terms of this $\tilde{a}$ element, we ensure that the sum of any two elements that are not in the upward "stem" are greater than the element in the mod class $e - 1$. The execution of this proof is seen as follows, in Theorem 5.1.

5. Verifying semigroup existence for a related Kunz poset

Theorem 5.1. *If $m \geq 3$, $2 \leq e \leq m$ and $t = e - 1$, then there exists a numerical semigroup of embedding dimension e, multiplicity m, and type t.*

Proof. Consider the numerical semigroup

$$S = \langle m,\ \tilde{a} - (e-3),\ \ldots,\ \tilde{a} - 1,\ \tilde{a},\ 2m-1 \rangle$$

with $\tilde{a} = (e-1)(2m-1) - 1$.
Its Apéry set, written as $\{0, a_1, ..., a_{m-1}\}$ with $a_i \equiv i \bmod m$, is

$$Ap(S) = \{0, \tilde{a} - (e-3), \ldots, \tilde{a}, (e-1)(2m-1), \ldots, 2(2m-1), 2m-1\}$$

In the following statements, we will prove S has the Kunz poset depicted in Figure 6 and identify whether its Apéry set elements a_i reside in the correct mod class $i \in \mathbb{Z}_m$. Let $a_0 = 0$ and $a_i = n_i$ denote the generator of S with $n_i \equiv i \bmod m$.

We claim that each Apéry set element is in the correct mod class m. The Apéry set contains elements of the form $k(2m-1)$ and $\tilde{a} + k$, as demonstrated above.
If an element a_{-k} takes the form $k(2m-1)$, where $k \in \mathbb{Z}$, we expand to find $k(2m-1) = 2mk - k \equiv -k \bmod m$. Thus, any element of the Apéry set a_{-k} with the form $k(2m-1)$ is in the correct mod class.
If an element a_i takes the form $\tilde{a} - l$, where $l \in \mathbb{Z}$, we proceed by induction. Begin at $l = 0$, found at $a_{-(e-1)}$, which we have shown is in the correct mod class. Moving one element to the left, we subtract 1 from $a_{-(e-1)}$, giving us a value for a_{-e} guaranteed to be modulo $-e$. This process is then repeated for all remaining elements. Thus, any element of the Apéry set a_i of the form $\tilde{a} - l$ is in the correct mod class.

We claim that $a_i + a_j \geq a_{i+j}$, for some $i, j \in Ap(S)$.
If $i \leq e-1$ then we immediately know $a_i + a_j \geq maxAp(S)$, where $maxAp(S) = \tilde{a} + 1 = a_{e-1}$.
If $i > e-1$, we have $a_i = p(2m-1) = a_{-p}$ and $a_j = q(2m-1) = a_{-q}$, and thus

$$a_i + a_j = (p+q)(2m-1).$$

Finally, *if* $p+q > e-1$, *then* $(p+q)(2m-1) > maxAp(S)$, *and*
if $p+q \leq e-1$, *then* $(p+q)(2m-1) = a_{i+j}$, as desired.

Lastly, we claim S has the Kunz poset depicted in Figure 6. The relations in this poset lie within the upwards "stem" on the rightmost edge of the figure. Equality exists when the Apéry set values of i and j precede the Apéry set value of $i+j$, denoted as

$$a_i + a_j = a_{i+j} \textit{ if and only if } i \prec i+j \textit{ and } j \prec i+j.$$

So, when $p+q \leq e-1$, $a_{-p} + a_{-q} = a_{-(p+q)}$. Otherwise, $a_{-p} + a_{-q} > a_{-(p+q)}$.

□

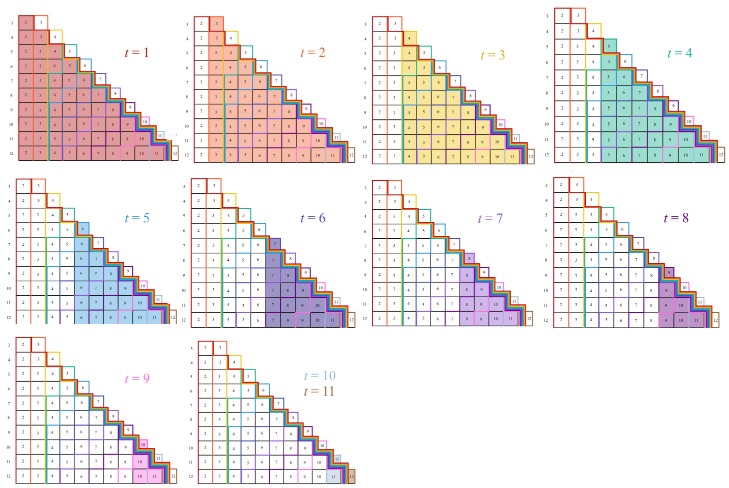

Figure 4. The values of $t(S) \leq 11$ attained and outlined for $m(S) \leq 12$ and $e(S) \leq 12$ verified by Theorem 8.1, demonstrated individually in the image by type.

6. Second Generalized Poset

We now wish to create a generalized poset that not only varies in multiplicity and embedding dimension, but also in type. In comparison to the poset proved in Section 5, this new poset contains a maximum element above any value of elements in the embedding dimension, allowing embedding dimension to vary while keeping type constant, if necessary. This creates the opportunity for all parameters m, e, and t to very within one generalized poset structure, as seen in Figure 5.

In the next section, we will prove the existence of semigroups that take the form of the second generalized poset in Figure 5. This correlates to the existence of semigroups reflected in Figure 4.

Due to the allowed amount of variance of the m, e, and t values, proving the existence of an Apéry set as we did in Section 5, was not easily done. In order to prove the existence of the semigroups for this second poset, we will use a different proof structure that relies on the nilsemigroup of the poset, allowing for the Apéry set elements to reside in any mod class.

7. Introduction II

In order to prove the existence of the semigroup for our second poset, we use a new method in order to avoid case-by-case issues with the mod classes for the elements within the Apéry set. Fix $m \geq 2$ and a face $F \subseteq \mathcal{C}_m$ with Kunz nilsemigroup N. We say F is *Apéry* if F° contains an Apéry point [2]. To determine the existence of Apéry points, we'll need to use notions of trades and presentations. Define a relation, or *trade* to be a linear equation relating minimal generators of S whenever two different combinations of minimal generators add to the same value [1].

Example 7.1. Suppose $S = \langle 2, 3, 5\rangle$. A minimal presentation of S [3] consisting of 2 trades is

$$\rho = \{((3,0,0),(0,2,0)),((5,0,0),(0,0,2))\}.$$

The first trade of ρ represents that 3 copies of the first element of S, or $3 \cdot 2$, is equal to two copies of the second element of S, or $2 \cdot 3$. The second trade represents that 5 copies of the first element of S, or $5 \cdot 2$, is equal to two copies of the second element of S, or $2 \cdot 5$.

Fix a Kunz nilsemigroup N and a presentation ρ of N. The *presentation lattice* [2] of N is given by

$$L_N = L_\rho = \{c_1(z_1 - z_1') + ... + c_n(z_n - z_n') : c_1, ..., c_n \in \mathbb{Z}, (z_1, z_1'), ..., (z_n, z_n') \in \rho\}$$

If ρ and ρ' are presentations of a Kunz nilsemigroup N, then $L_\rho = L_{\rho'}$ [2]. This ensures L_ρ doesn't depend on the chosen ρ of N.

We can now determine what makes F Apéry. Fix a non-degenerate face $F \subseteq \mathcal{C}_m$ with Kunz nilsemigroup N. Let $a_1, ..., a_k$ denote the minimal generators of N, and let $\alpha = (a_1, ..., a_k) \in \mathbb{Z}^e$. Then, F is Apéry if and only if $v \cdot \alpha \equiv 0 \ mod\ m$ for all $v \in Sat(L_N)$ [2].

The following proposition, taken from [2], lists specific cases where F *can not* be Apéry:

Proposition 7.2 ([2])**.** *Fix $m \geq 7$. In each of the following cases, if $F \subseteq \mathcal{C}_m$ is a non-degenerate face with $e(F) = e$ and $dim F = d$, then m is even and F is not Apéry:*
(a) $d = \frac{1}{2}m - 1$ and $e = m - 2$; or
(b) $d = 1$ and $e = m - 3$.

We will now use the following theorem from [2], which provides criteria for when F *can* be Apéry.

Theorem 7.3. [2] *Fix $m \geq 7$. There exists a face $F \subseteq \mathcal{C}_m$ with $e(F) = e$ and $dim F = d$ if:*
(a) $e = m - 1$ and $d = m - 1$;
(b) $e = m - 2$ and $\lfloor \frac{1}{2}(m - 1)\rfloor \leq d \leq m - 2$; or
(c) $2 \leq e \leq m - 3$ and $d \in [2, e]$.
Each such face F can be chosen Apéry, except where impossible by Prop 7.2.

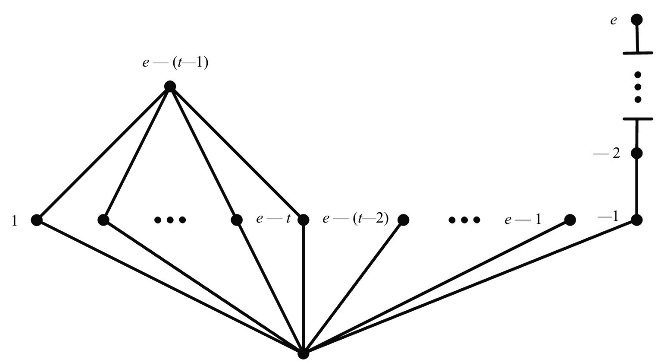

Figure 5. The General Kunz Poset from Theorem 8.1. This verifies the blocks in Figure 4.

Figure 6. The Kunz poset of S from Theorem 5.1 (left) and the corresponding blocks it verifies in the staircase image (right).

Beginning with our second generalized Kunz poset, we use the mod class of each Apéry element to ensure each $a_k \in Ap(S)$ satisfies $a_i + a_j \geq a_{i+j}$. Once we determine the different equalities contained within the poset, we create a presentation of trades, and thus satisfy the previous theorem to prove there exists a semigroup for the poset in Figure 5.

8. Proving Semigroup Existence for second generalized poset

Theorem 8.1. *If $m \geq 3$, $2 < e \leq m$ and $e \geq t + 1$, then there exists a numerical semigroup of embedding dimension e, multiplicity m, and type t.*

In the following statements, we will prove that the generalized poset in Figure 5 has an Apéry face F with a Kunz nilsemigroup N and a saturated lattice L_N. Let $a_1, ..., a_k$ ($k = 1, ..., m - 1$ modulo m) denote the minimal generators of N, as follows:

$$
\begin{aligned}
a_1 = a_2 = \cdots = a_{e-t} &= \frac{4(m-e)+2}{2} = 2(m-e)+1 \\
a_{e-(t-1)} &= 4(m-e)+2 \\
a_{e-(t-2)} = a_{e-(t-3)} = \cdots = a_{e-1} &= 4(m-e)+1 \\
a_e &= 4(m-e) \\
a_{e+1} &= 4(m-(e+1)) \\
a_{e+2} &= 4(m-(e+2)) \\
&\vdots \\
a_{-1} &= 4
\end{aligned}
$$

Note: between a_{-1} and a_e, values of $a_k = 4(m-k)$.

We claim that $a_i + a_j \geq a_{i+j}$. With out loss of generality, if $e-(t-1) \leq i \leq e$, then we know $a_i + a_j \geq \max\{N\}$, as the smallest a_j can be is 4, where $\max\{N\} = 4(m-e)+2$.

If $1 \leq i \leq e-t$ and $1 \leq j \leq e-t$, then $a_i + a_j = \max\{N\} \geq a_{i+j}$.

If $1 \leq i \leq e-t$ and $e \leq j \leq -1$, then $1 \leq (i+j) \leq e-t$, in which case $a_i + a_j \geq a_{i+j}$ as $a_i = a_{i+j}$ or a_{i+j} is closer to -1 than a_j. So, $a_j \geq a_{i+j}$. In this case, $i+j$ cannot lie between $e-(t-1)$ and $e-1$ because $(e+(e-t)) \bmod m < (e-t) \bmod m$ and $((e-t)-(e-m) \bmod m > e \bmod m$, $(e-m) \bmod m = e \bmod m$.

If $e \leq i \leq -1$ and $e \leq j \leq -1$ then either $i+j \geq e$ which means $a_i + a_j = a_{i+j}$. If $i+j < e$ then $a_i + a_j > a_e$ and $a_e = \max\{N\} - 2$ which is a smaller difference than $a_{-1} = 4$, our smallest element. Thus $a_i + a_j \geq \max\{N\}$.

The presentation of trades can be broken into two cases, as follows:
If $e-(t-1)$ is even, the equalities in N are:

$$
\begin{aligned}
a_1 + a_{e-t} &= a_{e-(t-1)} \\
a_2 + a_{(e-t)-1} &= a_{e-(t-1)} \\
&\vdots \\
2 \cdot a_{(e-t)/2} &= a_{e-(t-1)}
\end{aligned}
$$

and the possible trades that exist in N, written here as equalities, are given below.

$$
\begin{aligned}
a_1 + a_{e-t} &= a_2 + a_{(e-t)-1} \\
a_1 + a_{e-t} &= a_3 + a_{(e-t)-2} \\
&\vdots \\
a_1 + a_{e-t} &= 2 \cdot a_{(e-t)/2}
\end{aligned}
$$

These trades are then translated into a trade matrix:

$$\begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & \cdots & 0 & \cdots & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & \cdots & 0 & \cdots & -1 & 0 & 0 & 1 \\
\vdots & & & & \ddots & & \iddots & & & & \vdots \\
1 & 0 & \cdots & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & -2 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}$$

This trade matrix is equivalent to the trade matrix for poset b in Figure 4 of [2] when $k+1$ is even.

If $e-(t-1)$ is odd, the equalities in N are:

$$\begin{aligned}
a_1 + a_{e-t} &= a_{e-(t-1)} \\
a_2 + a_{(e-t)-1} &= a_{e-(t-1)} \\
&\vdots \\
a_{(e-t)/2} + a_{((e-t)/2)+1} &= a_{e-(t-1)}
\end{aligned}$$

and the possible trades that exist in N, written here as equalities, are given below.

$$\begin{aligned}
a_1 + a_{e-t} &= a_2 + a_{(e-t)-1} \\
a_1 + a_{e-t} &= a_3 + a_{(e-t)-2} \\
&\vdots \\
a_1 + a_{e-t} &= 2 \cdot a_{(e-t)/2} + a_{((e-t)/2)+1}
\end{aligned}$$

These trades are then translated into a trade matrix:

$$\begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & \cdots & 0 & 0 & \cdots & -1 & 0 & 0 & 1 \\
\vdots & & & & \ddots & & & & \iddots & & & \vdots \\
1 & 0 & 0 & \cdots & -1 & 0 & 0 & -1 & \cdots & 0 & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}$$

This trade matrix is equivalent to the trade matrix for poset b in Figure 4 of [2] when $k+1$ is odd.

Thus, we have shown that the trades in N create a trade matrix that is equivalent to a trade matrix from [2], which has already been proven to be saturated in Theorem 4.7 of [2]. Thus, we prove the existence of semigroups for the poset in Figure 5 and their corresponding boxes on the diagram in Figure 4. Figures 6 and 4 combine to create

Figure 2, which encapsulates and summarizes the findings of the research, reflecting the proven existence of semigroups derived in Theorems 5.1 and 8.1.

References

[1] S. Chapman, P. García-Sánchez, and C. O'Neill *Distances Between Factorizations in the Chicken McNugget Monoid*, 1–6

[2] L. Borevitz, T. Gomes, J. Ma, H. Niergarth, C. O'Neill, D. Pocklington, R. Stolk, J. Wang, and S. Xue *On Faces of the Kunz Cone and the Numerical Semigroups Within Them*, 7–8, 15–17.

[3] C. Elmacioglu, K. Hilmer, C. O'Neill, M. Okandan, and H. Park-Kaufmann *On The Cardinality of Minimal Presentations of Numerical Semigroups*, 1, 3–7

[4] K. Andrus, N. Larsen, A. Maclennan, C. Vicars *Data Creation for Finding Possible Dimensions of Existing Numerical Semigroups* (Sage software) `https://github.com/KNAC-Semigroup-Research/Semigroup-Research-Functions-KNAC`

[5] C. O'Neill, *numsgps-sage* (Sage software), `https://github.com/coneill-math/numsgps-sage`.

[6] T. Gomes, C. O'Neill, C. Preuss, and E. Torres Davila, *KunzPoset* (Sage software), `https://github.com/coneill-math/kunzpolyhedron`.

[7] W. Bruns, P. García-Sánchez, C. O'Neill, D.Wilburn *Wilf's conjecture in fixed multiplicity* (Sage software), `https://doi.org/10.1142/S021819672050023X`.

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