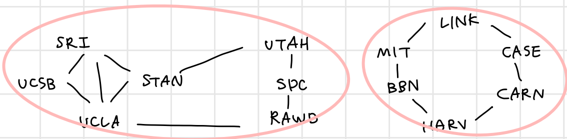
Graph Theory

- A graph consists of
  1. a set of nodes (vertices)
  2. a set of links each connecting 2 nodes (edges)

- Path: sequence of nodes, where each consecutive pair is linked
- Shortest path: Path of minimal length b/w 2 nodes (length = # of edges)
- diameter: length of longest shortest path b/w any two nodes

- Two nodes are connected if there is a path b/w them

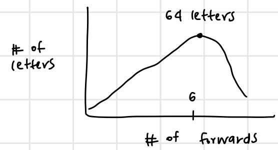


- A component is a set of nodes all pairs of which are connected with no edges in or out
  most networks have a single "giant component" with most nodes

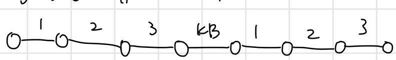
- why not two large components?

- How close are we?
  "Small world phenomenon"
  "six degrees of separation" (for any two nodes, there is a path between them of at most 6 connections)
  - more like 4-5 degrees of separation (internet)

- Stanley Milgram (1960s)
  - get letter to person in Boston
  - Forward to someone you know on a first-name basis
  - 296 people → 64 letters made it

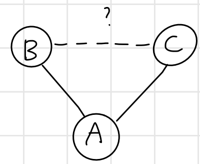


- Bacon number: Shortest path to Kevin Bacon
  - Avg. Bacon #: 2.9

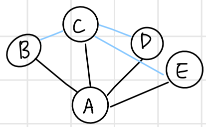


  - b/w any two nodes, 6 hops

Triadic closure & tie Strength



- triadic closure principle
  - If two people have a friend in common, more likely to be friends themselves
  - A gives opportunity for B and C to meet
  - bond over similar things

- clustering coefficient (of node A)
  - fraction of pairs of friends of A that are also friends themselves
  - 

    4 friends

    6 pairs [BC, BD] [BE, CD] [DE, CE]

  - 3 pairs are friends themselves

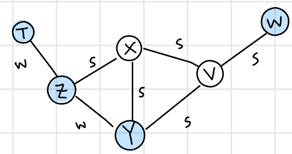
    $CC(A) = 3/6 = 1/2$

    $= \dfrac{\#\text{ triangle containing } A}{\#\text{ pairs of friends of } A}$

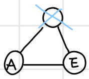
  - A has k friends
  - # pairs = "k choose 2" $= \binom{k}{2}$

    $= \dfrac{k(k-1)}{2}$

Satisfies STCP?



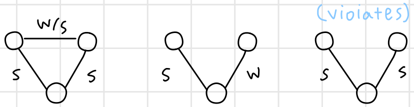
- A—E is local bridge if they have no common friends



$\longleftrightarrow$ local bridge

no triangle

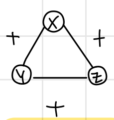
- Node A violates the Strong Triadic closure property if
  ① A has strong ties to B, C
  ② no B—C bridges



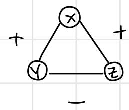
- If A (i) has at least 2 strong ties, then any local bridge A—B is weak tie
  (ii) satisfies STCP

Signed Networks and Balance

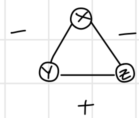
- **Structural Balance theory**
  - Assume a complete graph : edge btw every pairs of nodes
  - Each edge is + or −

(A) 

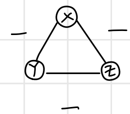
balanced

(B) 

unbalanced

(C) 

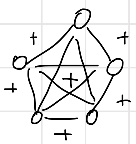
balanced

(D) 

unbalanced

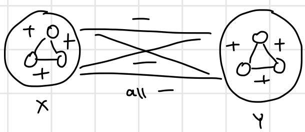
(A) All friends are good
(B) two friends of X are enemies push Y−Z to be +
(C) mutual enemy : enemy of my enemy is my friend
(D) incentive to team up

- what do balanced networks look like?



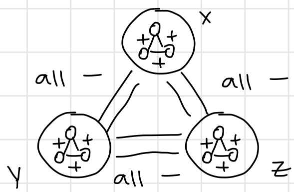
all friends

balanced



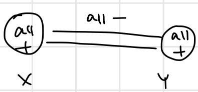
two graphs that dislike each other

balanced



unbalanced

- **Balance Theorem** : If a signed complete graph satisfies SBP, then either
  (a) all edges + or
  (b) nodes can be split into 2 sets



X, Y nonempty

Game Theory

• Prisoner's Dilemma

① \ ②	C	NC
C	-5, -5	0, -9
NC	-9, 0	-1, -1

Player ①

- If ② chooses C, then ① chooses C
- If ② chooses NC, then ① chooses C

- Two player Games
  1) players 1, 2
  2) strategies S_1, S_2
  3) payoffs $p_1(S_1, S_2)$, $P_2(S_1, S_2)$

- strategy S^* is a **best response** for 1 to S_2, if $p_1(S_1^*, S_2) \geq P_1(S_1, S_2) \; \forall S_1$
- strategy S^* is a **dominant strategy** if it's a BR to every strategy

- **Nash Equilibrium**: a pair of strategies that are BR to each other
  - multiple NE is possible

Mixed Strategies

- **Mixed Strategy NE**

		q	$1-q$
		LG	RG
p	Lk	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{4}, \frac{1}{4}$
$1-p$	Rk	$1, 0$	$\frac{1}{2}, \frac{1}{2}$

No NE in pure strategies

$Pr(L_K) = p$

prob of L_K

$Pr(L_G) = q$

prob of L_G

- $L_K \rightarrow \frac{1}{2}q + \frac{3}{4}(1-q)$
- $R_K \rightarrow 1q + \frac{1}{2}(1-q)$
- In a mixed NE, kicker and goalie, must randomize.

For K ... payoff to L_K = payoff to R_K

$\frac{1}{2}q + \frac{3}{4}(1-q) = 1q + \frac{1}{2}(1-q) \Rightarrow q^* \frac{1}{3}$

For G ... payoff to L_G = payoff to R_G

$\frac{1}{2}p + 0(1-p) = \frac{1}{4}p + \frac{1}{2}(1-p) \Rightarrow p^* \frac{2}{3}$

$\therefore$ NE is $p^* = \frac{2}{3}$, $q^* = \frac{1}{3}$

Games can have ...

1) only one pure strategy NE

ex) prisoner's Dilemma

2) Only one mixed NE

ex) penalty niches

3) Multiple pure strategy NE

ex) Hawk-Dove

4) Both pure and mixed strategy NE

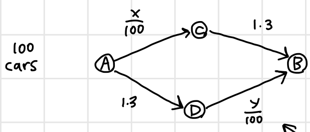
Network Traffic

	L*	R
U*	5, 5	40, 0
D	0, 30	10, 10

Only NE is (U, L)

NE not necessarily optimal

cars want to go from A to B



travel times

x = number of cars on AC

y = number of cars on DB

100 drivers start on A, go to B, independently minimize travel time

① No NE with all on ACB or all on ADB

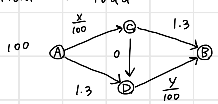
② If both paths used then both paths have the same travel time

ACB		ADB
$\frac{x}{100} + 1.3$	$=$	$1.3 + \frac{y}{100}$
x	$=$	y
x^*	$=$	$y^* = 50$

also $x + y = 100$

travel time $= 1.8$

- Add a road from C to D with 0 travel time



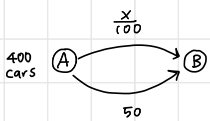
ACB : $\frac{x}{100} + 1.3$

ACDB : $\frac{x}{100} + 0 + \frac{y}{100}$

ADB : $1.3 + \frac{y}{100}$

∴ ACDB is strictly dominant travel time is 2

- Braess's paradox



x = number of cars on AB high

y = number of cars on AB low

NE: $x = 400, y = 0$

All routes that are used have the same cost

unused routes would be more expensive

Auctions

- Assume ...
  - single object
  - Buyers know own value, but not other's values
  - Values are independent + private

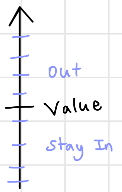
Types of Auctions

1) Ascending Bid (English) — $p \uparrow$ until only 1 bidder left
2) Descending Bid (Dutch) — $p \downarrow$ until someone accepts
3) First price — highest bid win + pay their bid
4) Second price — highest bid win + pay the 2nd highest bid

Descending Bid = First place
Ascending Bid = Second place

Ascending Bid

- Dominant strategy : to stay in until your value is recorded

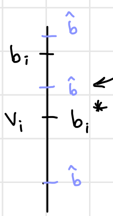


winner — bidder with highest dollar

price — second highest value

Second price

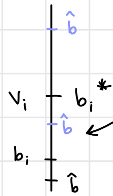
- Dominant strategy : bidding your value
  - Consider $b_i > V_i$



only matter if $b_i > \hat{b} > V_i$

win pay $\hat{b} > V_i$
(really lose)

- consider $b_i < V_i$



only matter if $V_i > \hat{b} > b_i$

Lose if bid b_i,
would won with bid $= V_i$
and paid $\hat{b} < V_i$

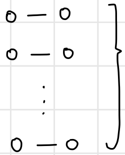
First place Auction

- Dominant strategy : clearly bid $< V_i$
  - N bidders w values drawn
    independently from uniform distribution on $[0, 1]$

$$\begin{pmatrix} \text{NE bid for} \\ \text{bidder } i \text{ with} \\ \text{value } V_i \end{pmatrix} = \left(\frac{N-1}{N}\right) V_i$$

Matching markets (intro)

- Networks Math
  - bipartite graph has nodes of two types (L and R) where every edge has one node in L and one in R
  - perfect matching is a set of n edges such that each node is in one edge



n edges

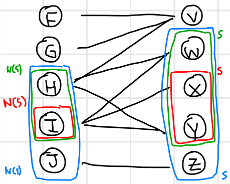
2n nodes < n Lnodes, n Rnodes

* each node appears exactly once

- S is a constricted set if N(S) has fewer nodes than S
  - S = subset of R nodes
  - N(S) = all L nodes connected to S

* If there is a constructed set, then there is no PM

- Matching theorem:
  - If a bipartite graph with no PM, then it must be CS
    - ① A PM exists – provide it
    - ② No PM – a CS is to blame



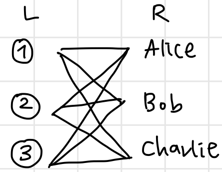
How many 오른쪽 nodes are in the largest constructed set?

(A) 2 (B) 3

(C) 4 (D) 5

Matching markets (valuation)

	Alice	Bob	charlie
Room 1	6	5	3
Room 2	3	4	3
Room 3	1	1	1



- 6 possible PM
- maximum social welfare — the sum of validation in the PM

1 — Alice
2 — Bob
3 — charlie

SW of A-1, B-2, C-3 : 6+4+4 = 14

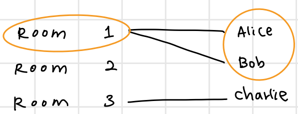
- Maximizing SW is more general than finding PM
- PMs is a special case of max SW

payoff = valuation − price

Price		Alice	Bob	charlie
3	Room 1	6	5	3
0	Room 2	3	4	3
1	Room 3	1	1	4

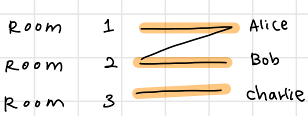
	Alice	Bob	charlie
Room 1	3	2	0
Room 2	3	4	3
Room 3	0	0	3

- preferred seller graph: buyer connects to all goods that have equal payoffs



constructed set !

→ too much demand for room 1



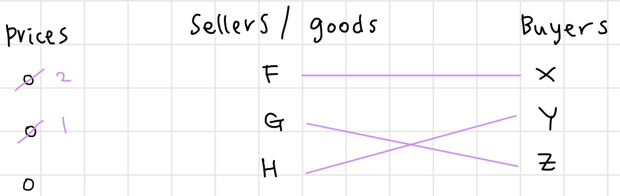
The prices are market clearing if the PSG has a PM

- If prices are MC, any resulting PM in the PSG maximizes SW (in the original valuations)
- Let M be a PM in the PSG
  - $\underbrace{\text{Total payoff}}_{\text{max}} = \underbrace{\text{sum of valuations}}_{\text{same}} - \underbrace{\text{sum of prices}}_{\text{so... it must be max}}$

Finding MCP

- How do we find market clearing price?

  1. Initialize prices to 0
  2. compute payoffs and make PSG
  3. PM ⇒ Done
  4. Otherwise find constructed set S — always decrease System Energy
  5. Increase price of everything in N(S) by 2
  6. m = min price, if m > 0, subtract m from all prices — net no change in system energy
  7. Go back to ②



F	G	H
~~10~~ 8	~~7~~ 5	~~4~~ 2
~~8~~ 7	~~7~~ 6	~~6~~ 5
6	9	4

- Does this process always stop?
  - If so prove that MCPs always exist
  - energy of seller = current price of item
  - energy of buyer = current max payoff
  - energy of system = energy of buyers + energy of seller

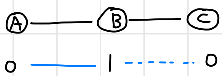
  - start at some system energy E_0 = 0 + 10 + 8 + 9 = 27
  - Everyone starts with ≥ 0 energy
  - If PM in PSG, then done
  - Otherwise, we have constructed set S
    1. increase energy of Sellers in N(S)
    2. Decrease energy of buyers in S

    ⇒ net decrease in energy

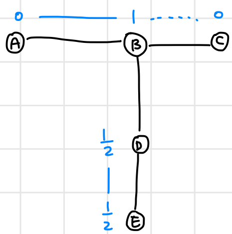
Network exchange experiments

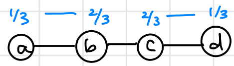
- **How does this turn out?**

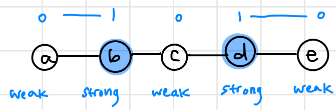
typically even split

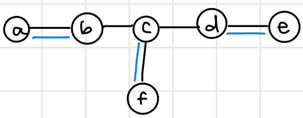


B has (almost) all power





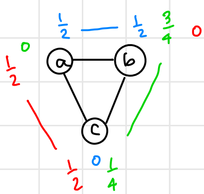




c's power up

b, d weaker

a, e stronger



No "stable" outcome

matching where each node is part of at most one edge

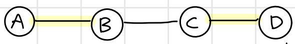
- **Extract Basic Ideas**

① Real people won't accept 0 (ultimatum Game)

② Network structure matters

Network exchange theory

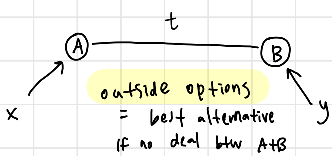
- **Outcome:**
  - matchings where each node is part of at most one edge
  - Sum of node's value on any edge in the matching is 1
  - Any node not in the matching has value 0



A	B	C	D	
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	stable
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	stable
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	unstable

  - **unstable** if any pair of nodes connected by an edge have values that sum less than 1
  - **Stable** if no instability

- **Nash Bargaining**



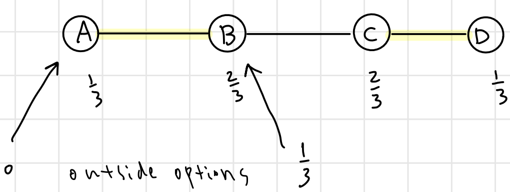
If $x+y > t$ then no deal

If $x+y < t$ then Surplus, $S = 1-(x+y)$

Nash $\Rightarrow$ Share S equally

$$\left.\begin{array}{l} A \to x + S/2 \\ B \to y + S/2 \end{array}\right]$$ **Nash Bargaining Outcome**

- Outcome is **Balanced** if for each edge in the matching, the values for the nodes on the edge are a NBO, give the values of all other nodes



$S = 1-(0+\frac{1}{3}) = \frac{2}{3}$

$$\left.\begin{array}{l} A \to 0+\frac{1}{3} = \frac{1}{3} \\ B \to \frac{1}{3}+\frac{1}{3} = \frac{2}{3} \end{array}\right]$$ NBO