



CS 113

DISCRETE STRUCTURES

Chapter 2: The Language of Mathematics



HOMEWORK

- Section 2.1: 1-10, 25-29, 31, 33, 35, 78, 82
- Section 2.2: 4, 6, 9, 43, 45, 46, 74, 88-92, 96
- Section 2.3: 1-31 (odd)
- Section 2.4: 19-24, 29-34
- Section 2.5: 9-14, 30-31
- Section 2.6: 1-6, 8-17, 19, 23, 24
- Section 2.8: 10-15, 19-23, 26-27, 36-37, 56, 78-84

RELATIONS

- A relation is another name for a set of ordered pairs
- We need a domain set and a range set
- A sample relation is
 - Domain: $\{1, 2, 3, 4\}$
 - Range: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
 - Relation: $\{(1, 2), (2, 4), (3, 4), (4, 4), (4, 3)\}$
- Unlike a function, we can have two x-values with the same y-value
- Often, the domain and range will be the same
 - Then we say the relation is a relation on that common set

PICTURING RELATIONS

- We can use a directed graph (digraph) to picture a relation
- Let's examine the picture on p. 78 for the relation
 - $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
 - This is, of course, the less than or equal relation
- You let each value in the domain be a point
 - These points are called vertices
- You connect the points in the relation by arrows
 - The arrows go from the domain values to the range values
 - The arrows are called directed edges
- If a point is connected to itself, we call the connection a loop

PROPERTIES OF RELATIONS

- There are four key properties of relations that we need to know
- A relation can be reflexive
 - This means that (x, x) is in the relation for all x in the domain
- A relation can be symmetric
 - This means that if (x, y) is in the relation then (y, x) is also in the relation
- A relation can be transitive
 - This means that if (x, y) and (y, z) are in the relation then (x, z) is also in the relation
- A relation can be antisymmetric
 - This means that if (x, y) and (y, x) are in the relation, then $y \neq x$
 - This is the contrapositive of the book's definition

CHECKING THE DEFINITIONS

- Let's check all four properties on these relations
- In all cases, the domain and range are $\{1, 2, 3, 4\}$
- $R_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
- $R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $R_3 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)\}$
- $R_4 = \{(1, 1), (1, 2), (2, 2), (3, 3), (3, 4), (4, 4)\}$
- Can you recognize any of the relations?

CHECKING THE DEFINITIONS WITH FORMULAS

- Again, let's check all four properties on these relations
- The relations will be relations on the integers
 - Remember that this means that the domain and range are both the integers
- $R1: \leq$
- $R2: =$
- $R3: \{(x, y) \mid \text{either both } x \text{ and } y \text{ are even OR both } x \text{ and } y \text{ are odd}\}$

USING A MATRIX TO DEFINE A RELATION

- The matrix contains only 0s and 1s
- We list the elements of the domain down the rows
- We list the elements of the range across the columns
- We put a 1 at Row i , Column j whenever (i, j) is in the relation
- We fill the rest of the matrix with 0
- Let's see if we can create matrices for the relations two slides back

AN INTERESTING FACT

- **Composing relations**
 - This is like the transitive property, but applied to the two relations
- **If you have two matrices for two different relations then**
 - The product matrix is the matrix of the composition
- **Let's check this with Problem 4 on p. 96**
- **A consequence**
 - If you square a matrix of a relation, you can tell if the original relation was transitive or not



FUNCTIONS

- A function is a special kind of relation
- Like a relation, we need a domain set and a range set
- Unlike a relation, we can't have two y -values with the same x -value
- Often, the domain and range will be the same

AN EXAMPLE FUNCTION

- An example of a function is the modulus function
- You choose a base and then compute $x \bmod$ that base
- For example, suppose the base is 7, and $m(x)$ represents the modulus function
- Then $f(3) = 3$, $f(11) = 4$, $f(104) = 6$
- This function has several uses in the world of computer science
- There is a branch of math based on equations involving the modulus function
- The modulus function is useful in other branches of math too

BACK TO RELATIONS

- An equivalence relation is a relation that is reflexive, symmetric, and transitive
 - Some examples
 - Equality
 - Equality modulo an integer
- In fact, any partition of a set gives an equivalence relation
 - For example, on the set $\{1, 2, 3, 4, 5\}$, a partition is $\{1, 2\}, \{3, 4, 5\}$
 - This means that the relation is $(1, 1), (1, 2), (2, 1),$ and $(2, 2)$ and similarly for all the two-element subsets of $\{3, 4, 5\}$ along with $(3, 3), (4, 4),$ and $(5, 5)$
 - Can we check that?
- An equivalence relation breaks a set up into subsets like that

EQUIVALENCE RELATIONS AND PARTITIONS

- An equivalence relation creates a partition
 - The subsets in the partition are the items that are related to each other
 - These subsets are called equivalence classes
- Let's determine if the relations below are equivalence equations on the integers
 - If they are, let's also find the equivalence classes
 - $\{(x,y) \mid \text{Both } x, y \text{ are even or both are odd}\}$
 - For a negative to be even, use: n is even if $n = 2k$ where k is an integer
 - $\{(x,y) \mid 5 \text{ divides into } y-x\}$
 - $\{(x,y) \mid y = 3x\}$
 - $\{(x,y) \mid 1/x = 1/y\}$

MORE EQUIVALENCE RELATIONS AND PARTITIONS

- Let's also check if these are equivalence relations on the integers \times the integers
 - Again, if it is, we want to know the equivalence classes
 - $\{(x,y), (z,t) \mid xy=zt\}$
 - $\{(x,y), (z,t) \mid 2 \text{ divides into } x-z \text{ and } 3 \text{ divides into } y-t\}$
- A potential equivalence relation on a clock
 - x is equivalent to y if $x = y$ or x is directly across from y on a clock
 - Is this an equivalence relation?
 - Again, if it is, we want to know the equivalence classes

PARTIAL ORDERS

- A partial order is a relation that is reflexive, antisymmetric, and transitive
 - $\{(x,y) \mid x \leq y\}$ is an example
 - In fact, this is the model partial order
- The relation $\{(S,T) \mid S \subseteq T\}$ is a partial order on the set $P(\{1, 2, 3, 4,5\})$
- One interesting thing about a partial order is that not every two elements of the domain are comparable
- If every two elements of the domain are comparable, the order is called a total order

THE INVERSE OF A RELATION

- Every relation has an inverse
- You just “reverse” every pair in the relation
- So, for example the relation $\{(1, 2), (3, 4)\}$ is a relation on $\{1, 2, 3, 4\}$
- Its inverse is $\{(2, 1), (4, 3)\}$

APPLICATION 1-HASH FUNCTIONS

- A hash function is a way to find data
- The domain values of a hash function are called keys
- A common example of a hash function is to
 - Start with a data value
 - Return an index into an array or a disk file
- For example, suppose you are creating a phone list for the company you work for
 - The domain values will be people's names
 - The range values will be an index into the array that stores the numbers

HASH FUNCTOINS

- One popular way to design a hash function is to use the mod function
- This is done in two parts
 - Part 1: Use some function to assign an “index” to the key
 - Part 2: Calculate that answer mod the size of the array
- Finding a good hash function is tough
- The challenge is to find one that assigns elements to the array without having two elements hash to the same position

APPLICATION 2-PSEUDORANDOM NUMBERS

- Last semester, we used random numbers
- We had the computer generate random numbers
- The numbers we got were not really random
 - They were generated by a formula
- The formula is $x_{i+1} = kx_i \bmod n$
 - n is typically a prime number or a power of a prime
 - There are also special ways to choose k
- Since these numbers are not actually random, they are called pseudorandom

“ONTO” FUNCTIONS

- A function is called onto (or surjective or a surjection) if it uses up all its range elements
- For example, $f(x) = x^2$
 - The domain for the example function will be the real numbers
 - If the range is the real numbers, the function is NOT onto
 - If the range is the non-negative real numbers, the function is onto

ONE-TO-ONE FUNCTIONS

- A function is called 1-1 (injective or an injection) if two different domain elements go to two different range elements
- Using the same example as before
 - The range for the example function will be the real numbers
 - If the domain is the real numbers, the function is NOT 1-1
 - If the domain is the non-negative real numbers, the function is 1-1

ONE-TO-ONE AND ONTO FUNCTIONS

- One reason this is important is that it allows us to tell if two sets have the same number of elements
- Matching two sets based on the number of elements is called a one-to-one correspondence
- A set has four elements because you can find a one-to-one correspondence between the set and $\{1, 2, 3, 4\}$
- Can you find a one-to-one correspondence between the even numbers and the integers?
- How about the fractions and the integers?
- There are other uses
- They are all similar
- A function that is 1-1 and onto is called a bijection

ANOTHER VALUE OF 1-1 AND ONTO FUNCTIONS

- A function that is 1-1 and onto has an inverse
 - If a function is not 1-1 or is not onto, it does not have an inverse
- The inverse of a function is another function that undoes the original function
- It is written $f^{-1}(x)$
- Let's look at some examples



QUESTIONS

- Any questions?