CS113 – Solutions to Review for Final Exam

1. Create a truth table for the expression (p ∧ q) ∨ (q ∧ r)

Solution:

p	q	r	p ∧ q	q ∧ r	(p ∧ q) ∨ (q ∧ r)
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

2. Write the inverse of: If the sum of two even numbers is even, then the sum of two odd numbers is even.

Solution: It the sum of two even numbers is not even, then the sum of two odd number is not odd.

3. Using induction, prove that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}\text{, for n=1, 2, 3, …}$$

You must follow the outline I gave in class. Points will be lost if you don't.

Solution:

Base Step: (n=1)

The left side is $\frac{1}{2}$.

The right side is $1-\frac{1}{2}$, which is also $\frac{1}{2}$

Since these are equal, the base step is proven.

Inductive Step:

Assume that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Show that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n+1}}=1-\frac{1}{2^{n+1}}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \qquad +\frac{1}{2^{n+1}}$

$=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$ because the term before $\frac{1}{2^{n+1}}$ is $\frac{1}{2^n}$

$= \quad 1-\frac{1}{2^n} \qquad +\frac{1}{2^{n+1}}$ by the induction hypothesis

$=1-\frac{2}{2^{n+1}}+\frac{1}{2^{n+1}}$ getting a common denominator of 2^{n+1}

$=1-\frac{1}{2^{n+1}}$ combining the fractions.

You must include all the parts in red; omitting them will result in a grade of at most 1 point, Also, if you write the equation after "Show that" and verify it, your grade will be at most 2 points. These are critical parts of an inductive proof. It's imperative that you get them correct.

4. Suppose X and Y are two sets.. Prove that $X' \cap Y' \subseteq (X \cup Y)'$ using the properties given below

Rules:

1. $x \in A \cap B$ -> $x \in A$ and $x \in B$
2. $x \in A \cup B$ -> $x \in A$ or $x \in B$
3. $A \subseteq B$ -> ($x \in A$ -> $x \in B$)
4. $A = B$ -> $A \subseteq B$ and $B \subseteq A$
5. $x \in A'$ -> $x \notin A$ (Here, A' means the complement of A)

Solution: First, number the rules. This makes it easier to talk about them in the proof. Then, to get a sense of what I need to do, I look at what I need to show. I need to show that (by 3) if $t \in X' \cap Y'$, then $t \in (X \cup Y)'$.

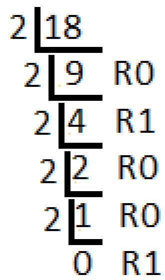
So, let $t \in X' \cap Y'$. Then, by 1, $t \in X'$, and $t \in Y'$.
By 5 applied twice, $t \notin X$ and $t \notin Y$.
Then, since $X \cup Y$ is all the things in X or Y, and t isn't in either of X or Y, then $t \notin X \cup Y$.
By 5, this means $t \in (X \cup Y)'$.

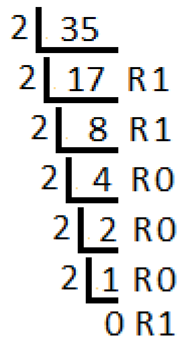
5. What is the power set of {1, 2, 3, 4}?

Solution: First, notice that there are 4 elements in the given set. That set has $2^4 = 16$ subsets. We need to list them. The power set of {1, 2, 3, 4}, written P({1, 2, 3, 4}) =
$\{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$
Notice that this is not just a list; it's a set.

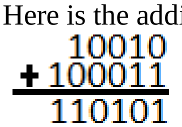
6. Convert the numbers 18 and 35 to base 2. Then add them. Then convert the answer back to base 10 to verify that you performed the various operations correctly.

Solution:
These are the conversions to binary.





So, (reading from bottom to top) $18_{10} = 10010_2$, and $35_{10} = 100011_2$.
Here is the addition



Converting the sum (110101) back to decimal gives 53. That's correct, since 18 + 35 = 53.

7. Convert the numbers as indicated.
   a.) $16_{16} = ?_{10}$
   b.) $25F_{16} = ?_2$

Solution:
Part a. 16_{16} means $1 \text{x} 16^1 + 6 \text{x} 16^0 = 16 + 6 = 22_{10}$.
Part a. Use the table below. (It will be provided on the exam.)
$25F_{16}$ is 001001011111_2.

Hex	Binary		Hex	Binary		Hex	Binary		Hex	Binary
0	0000		4	0100		8	1000		C	1100
1	0001		5	0101		9	1001		D	1101
2	0010		6	0110		A	1010		E	1110
3	0011		7	0111		B	1011		F	1111

8. Suppose aRb if $5 \mid (b - a)$. The domain for the relation is the integers. Show that this is an equivalence relation. Then determine the equivalence classes.
Solution: To show something is an equivalence relation, you have to show it's reflexive, symmetric, and transitive.
To show it reflexive, I have to show aRa for all integers a.
aRa means $5 \mid (a - a)$. $a-a = 0$. Show $5 \mid 0$.

To show it symmetric, I have to show If aRb, then bRa.
aRb means $5 \mid (b - a)$. Then $b-a = 5k$ for some integer k. (I need to show $5 \mid (a-b)$
$a-b = -(b-a) = -5k = 5(-k)$ So $5 \mid (a-b)$.

Transitive. I have to show that, if aRb and bRc, then aRc.
aRb means that $5 \mid (b-a)$
bRc means that $5 \mid (c-b)$ I need to show that $c-a = 5n$. So aRc.

For the equivalence classes. Start with one value in the domain (which, in this case, is the integers.) Find everything equivalent to it. Then to find another equivalence class, start with an omitted value. Find everything equivalent to it. Repeat until you have used up all the values in the domain. Here, there are five equivalence classes. They are

$\{0, 5, 10, \ldots\} \cup \{-5, -10, -15, \ldots\}$
$\{1, 6, 11, \ldots\} \cup \{-4, -9, -14, \ldots\}$
$\{2, 7, 12, \ldots\} \cup \{-3, -8, -13, \ldots\}$
$\{3, 8, 13, \ldots\} \cup \{-2, -7, -12, \ldots\}$
$\{4, 9, 14, \ldots\} \cup \{-1, -6, -11, \ldots\}$

9. What is a bijection? What can it be used for?

Solution: A bijection is a function that is one-to-one (an injection) and onto (a surjection). It has two main purposes. The first is that if a function is a bijection, it has an inverse. The second is that a bijection is also called a one-to-one correspondence between two sets. It is a way to tell if two sets have the same cardinality.

10. A committee of five people is to be formed from 10 women and eight men. The committee should have at least one man and one woman. How many different committees are there?

Solution: Using the complement idea, it will be a lot easier to work with the complement. The complement of a committee with "at least one man and one woman" is a committee that is all men or a committee that is all women. (Otherwise, you have to have some men and some women.) The total number of committees that can be formed is $\binom{18}{5} = 8568$. The number of committees that are composed entirely of women is $\binom{10}{5} = 252$. The number of committees that are composed entirely of men is $\binom{8}{5} = 56$. So, the number of committees that have at least one man and one woman is 8568 – 252 – 56, which is 8260.

11. An event is rolling two dice to decide how many squares a piece should move in a game. You want to avoid the square 5 spaces ahead. What is the probability that a random roll avoids the bad square?

Solution: I will solve this problem using complement ideas. If you roll two dice, there are 36 possibilities. (They are all listed in the Chapter 4 PowerPoint.) That is the sample space, S. The event we are interested in is E = {Ways to get a 5}. There are four ways to get a 5. (You can find these by looking through that list.) Then, the probability of getting a 5 is $P(E) = \frac{N(E)}{N(S)} = \frac{4}{36} = \frac{1}{9}$. Make sure you write it this way. So, by the complement theorem, $P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{9} = \frac{8}{9}$. You could also get this result by counting the non-5s directly.

12. Are the two events rolling an even number on a die and rolling a number less than 4 on a die independent?

Solution:
Before starting, let's write down the sample space S: S = {1, 2, 3, 4, 5, 6}. Let's call the event of getting an even number E: E = {2, 4, 6}. Let's call the event of getting a number less than 4 F: F = {1, 2, 3}. (Notice that E and F are subsets of S.) Now, there are two ways to check if the events are independent.
Method 1: Two events E and F are independent if P(E|F) = P(E).

$$P\left(E \mid F\right) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{N(E \cap F)}{N(S)}}{\frac{N(F)}{N(S)}} = \frac{1}{3}, \text{ and } P(E) = \frac{N(E)}{N(S)} = \frac{3}{6} = \frac{1}{2}..$$ Since P(E|F) is not equal to P(E), the events are not independent.

Method 2: Two events E and F are independent if P(E) P(F) = P(E ∩ F). Here P(E) = 1/2. P(F) =1/2. E ∩ F = {2}, and P(E ∩ F) = 1/6. Since 1/2 is not equal to 1/6, the events are not independent.

13. Solve the recurrence relation

$a_0 = 3$

$a_1 = 2$

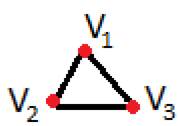
$a_n = 2a_{n-1} - 4a_{n-2}, n \geq 2$

Solution:
This is a linear, homogeneous recurrence relation with constant coefficients. The characteristic equation is $x^2 - 3x - 4 = 0$. The roots are 4, -1. The solution before considering the initial values is $a_n = R(4^n) + S(-1)^n$. Substituting the values of a_0 and a_1 gives the system

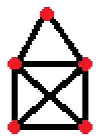
$$\begin{cases} 3 = a_0 = R(4^0) + S(-1)^0 \\ 2 = a_1 = R(4^1) + S(-1)^1 \end{cases}$$, which simplifies to $$\begin{cases} R + S = 3 \\ 4R - S = 2 \end{cases}$$, which has the solution R=1, S=2. The solution to the original recurrence relation is $a_n = (4^n) + 2(-1)^n$.

14. Draw K_3.

Solution:
K_3 is the complete graph on 3 vertices. It's a triangle.



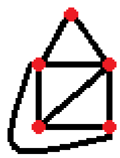
15. Can you find an Euler cycle in the graph pictured?

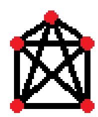


Solution: Check the degree of every vertex. If all degrees are even, there is an Euler cycle. If you find even one vertex with an odd degree, stop; there is no Euler cycle. If all vertices have even degree, then try to actually find an Euler cycle.

In this case, the two bottom vertices each have degree 3. Because of that, there is no Euler cycle.

16. State Euler's formula and verify it for each of the graphs below.





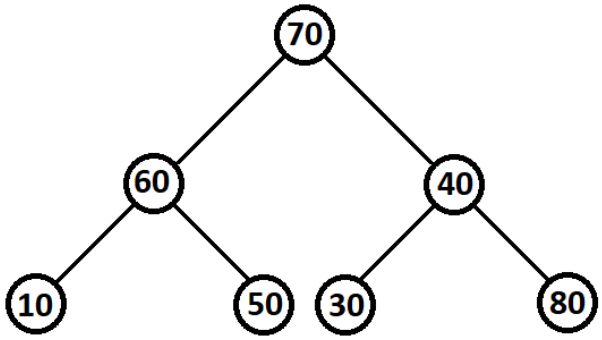
Solution: First, Euler's formula only works for a connected planar graph. (It has wider application, but not to non-planar graphs.) The formula is $F = E - V + 2$. V is the number of vertices. E is the number of edges. F is the number of faces. (Don't forget to count the "outside" of the graph as one face.) Just check that the formula holds.

For the graph on the left: $V = 5$, $E = 8$, and $F = 5$. According to the formula, 5 should equal $8 - 5 + 2$, and it does.
For the graph on the right: The graph is not planar. (This is because it's K_5, which we know is not planar.) Since it's not planar, Euler's formula doesn't hold.

17. For the tree below, identify the root, a leaf, two nodes that are siblings, two nodes that are a parent and a child. Also, tell the height of the tree.

Solution: See the book for the vocabulary.

18. Show a pre-order traversal of the tree pictured.



Solution: (I also give an inorder and post-order traversal.
Pre-order: 70,60,10,50,40,30,80
Inorder: 10,60,50,70,30,40,80
Post-order: 10,50,60,30,80,40,70

19. Give one example or applications of trees.

Solution: There are several. See PowerPoint 7-1 or the textbook.

20. Know the four equivalent conditions of Theorem 7.2.3. Also know what it means to say "the following are equivalent" in a theorem.

Solution:
Theorem 7.2.3 says that the four conditions listed are equivalent for a graph T with n vertices. The conditions are
T is a tree
T is connected and acyclic
T is connected and has n-1 edges
T is acyclic and has n-1 edges

Notice that you need to say that T is a graph with n vertices before listing the four conditions of the theorem..

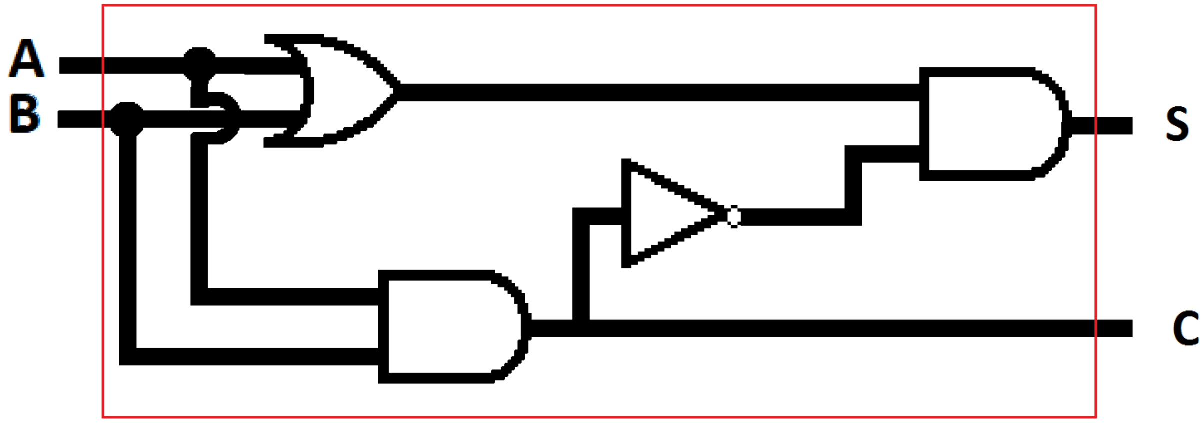
To say "the following are equivalent" means the statements all have the same truth value" Either all are true or all are false.

21. Explain what a spanning tree for a graph is, and which graphs have spanning trees.

Solution: A spanning tree for a graph is a subgraph that contains all vertices of the graph. Every connected graph has a spanning tree.

22. Draw the circuit for a half adder

Solution:



Note: If you use other gates, you need to explain how they work