CS 113
DISCRETE STRUCTURES

Chapter 2: The Language of Mathematics

HOMEWORK

- Section 2.1: 1-10, 25-29, 31, 33, 35, 78, 82
- Section 2.2: 4, 6, 9, 43, 45, 46, 74, 88-92, 96
- Section 2.3: 1-31 (odd)
- Section 2.4: 19-24, 29-34
- Section 2.5: 9-14, 30-31

RELATIONS

- A relation is another name for a set of ordered pairs
- We need a domain set and a range set
- A sample relation is
  - Domain: {1, 2, 3, 4}
  - Range: {1, 2, 3, 4, 5,6 ,7, 8, 9, 10, 11}
  - Relation: {(1,2), (2, 4), (3, 6), (4, 8), (4, 3)}
- Unlike a function, we can have two x-values with the same y-value
- Often, the domain and range will be the same
  - Then we say the relation is a relation on that common set

PICTURING RELATIONS

- **We can use a directed graph (digraph) to picture a relation**
- **Let's examine the picture on p. 78 for the relation**
  - **{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}**
    - **This is, of course, the less than or equal relation**
- **You let each value in the domain be a point**
  - **These points are called vertices**
- **You connect the points in the relation by arrows**
  - **The arrows go from the domain values to the range values**
  - **The arrows are called directed edges**
- **If a point is connected to itself, we call the connection a loop**

PROPERTIES OF RELATIONS

- There are four key properties of relations that we need to know
- A relation can be reflexive
  - This means that (x, x) is in the relation for all x in the domain
- A relation can be symmetric
  - This means that if (x, y) is in the relation then (y, x) is also in the relation
  - What does this say about the domain and range of the relation?
- A relation can be transitive
  - This means that if (x, y) and (y, z) are in the relation then (x, z) is also in the relation
- A relation can be antisymmetric
  - This means that if (x, y) and (y, x) are in the relation, then y ≠ x
  - This is the contrapositive of the book's definition

CHECKING THE DEFINITIONS

- **Let's check all four properties on these relations**
- **In all cases, the domain and range are {1, 2, 3, 4}**
- **R_1 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}**
- **R_2 = {(1, 1), (2, 2), (3, 3), (4, 4)}**
- **R_3 = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)}**
- **R_4 = {(1, 1), (1, 2), (2, 2), (3, 3), (3, 4), (4, 4)}**

- **Can you recognize any of the relations?**

CHECKING THE DEFINITIONS WITH FORMULAS

- **Again, let's check all four properties on these relations**
- **The relations will be relations on the integers**
  - **Remember that this means that the domain and range are both the integers**
- **R1: =**
- **R2: ≤**
- **R3: {(x, y) | either both x and y are even OR both x and y are odd}**

USING A MATRIX TO DEFINE A RELATION

- **The matrix contains only 0s and 1s**
- **We list the elements of the domain down the rows**
- **We list the elements of the range across the columns**
- **We put a 1 at Row i, Column j whenever (i, j) is in the relation**
- **We fill the rest of the matrix with 0**
- **Let's see if we can create matrices for the relations two slides back**

AN INTERESTING FACT

- Composing relations
  - This is like the transitive property, but applied to the two relations
- If you have two matrices for two different relations then
  - The product matrix is the matrix of the composition
- Let's check this with Problem 4 on p. 96

- A consequence
  - If you square a matrix of a relation, you can tell if the original relation was transitive or not

FUNCTIONS

- **A function is a special kind of relation**
- **Like a relation, we need a domain set and a range set**
- **Unlike a relation, we can't have two y-values with the same x-value**
- **Often, the domain and range will be the same**

AN EXAMPLE FUNCTION

- **An example of a function is the modulus function**
- **You choose a base and then compute x mod that base**
- **For example, suppose the base is 7, and m(x) represents the modulus function**
- **Then f(3) = 3, f(11) = 4, f (104) = 6**
- **This function has several uses in the world of computer science**
- **There is a branch of math based on equations involving the modulus function**
- **The modulus function is useful in other branches of math too**

BACK TO RELATIONS

- **An equivalence relation is a relation that is reflexive, symmetric, and transitive**
  - Some examples
    - Equality
    - Equality modulo an integer
- **In fact, any partition of a set gives an equivalence relation**
  - For example, on the set {1, 2, 3, 4, 5}, a partition is {1, 2}, {3, 4, 5}
    - This means that the relation is (1, 1), (1, 2), (2, 1), and (2, 2) and similarly for all the two-element subsets of {3, 4, 5} along with (3, 3), (4, 4), and (5, 5)
  - Can we check that?
- **An equivalence relation breaks a set up into subsets like that**

EQUIVALENCE RELATIONS AND PARTITIONS

- An equivalence relation creates a partition
  - The subsets in the partition are the items that are related to each other
  - These subsets are called equivalence classes
- Let's determine if the relations below are equivalence equations on the integers
  - If they are, let's also find the equivalence classes
  - {(x,y)| Both x, y are even or both are odd}
    - For a negative to be even, use: n is even if n = 2k where k is an integer
  - {(x,y)| 5 divides into x-y}
  - {(x,y)| y = 3x}
  - {(x,y)| 1/x = 1/y}

MORE EQUIVALENCE RELATIONS AND PARTITIONS

- **Let's also check if these are equivalence relations on the integers x the integers**
  - **Again, if it is, we want to know the equivalence classes**
  - **{(x,y), (z,t)| xy=zt}**
  - **{(x,y), (z,t)| 2 divides into x-z and 3 divides into y-t}**
- **A potential equivalence relation on a clock**
  - **Is this an equivalence relation?**
  - **Again, if it is, we want to know the equivalence classes**
  - **x is equivalent to y if x = y or x is directly across from y on a clock**

PARTIAL ORDERS

- A partial order is a relation that is reflexive, antisymmetric, and transitive
  - {(x,y)|x<=y} is an example
  - In fact, this is the model partial order
- The relation $\{(S,T) \mid S \subseteq T\}$ is a partial order on the set P({1, 2, 3, 4,5})
- One interesting thing about a partial order is that not every two elements of the domain are comparable
- If every two elements of the domain are comparable, the order is called a total order

THE INVERSE OF A RELATION

- **Every relation has an inverse**
- **You just "reverse" every pair in the relation**
- **So, for example the relation {(1, 2), (3, 4)} is a relation on {1, 2, 3, 4}**
- **Its inverse is {(2, 1), (4, 3)}**

APPLICATION 1-HASH FUNCTIONS

- A hash function is a way to find data
- The domain values of a hash function are called keys
- A common example of a hash function is to
  - Start with a data value
  - Return an index into an array or a disk file
- For example, suppose you are creating a phone list for the company you work for
  - The domain values will be people's names
  - The range values will be an index into the array that stores the numbers

HASH FUNCTOINS

- One popular way to design a hash function is to use the mod function
- This is done in two parts
  - Part 1: Use some function to assign an “index” to the key
  - Part 2: Calculate that answer mod the size of the array
- Finding a good hash function is tough
- The challenge is to find one that assigns elements to the array without having two elements hash to the same position

APPLICATION 2-PSEUDORANDOM NUMBERS

- **Last semester, we used random numbers**
- **We had the computer generate random numbers**
- **The numbers we got were not really random**
  - **They were generated by a formula**
- **The formula is $x_{i+1} = kx_i \bmod n$**
  - **n is typically a prime number or a power of a prime**
  - **There are also special ways to choose k**
- **Since these numbers are not actually random, they are called pseudorandom**

"ONTO" FUNCTIONS

- **A function is called onto (or surjective or a surjection) if it uses up all its range elements**
- **For example, $f(x) = x^2$**
  - **The domain for the example function will be the real numbers**
  - **If the range is the real numbers, the function is NOT onto**
  - **If the range is the non-negative real numbers, the function is onto**

ONE-TO-ONE FUNCTIONS

- **A function is called 1-1 if two different domain elements go to two different range elements**
- **Using the same example as before**
  - **The range for the example function will be the real numbers**
  - **If the domain is the real numbers, the function is NOT 1-1**
  - **If the domain is the non-negative real numbers, the function is 1-1**

ONE-TO-ONE AND ONTO

- One reason this is important is that it allows us to tell if two sets have the same number of elements
- Matching two sets based on the number of elements is called a one-to-one correspondence
- A set has four elements because you can find a one-to-one correspondence between the set and {1, 2, 3, 4}
- Can you find a one-to-one correspondence between the even numbers and the integers?
- How about the fractions and the integers?
- There are other uses
- They are all similar
- A function that is 1-1 and onto is called a bijection

ANOTHER VALUE FROM 1-1 AND ONTO FUNCTIONS

- **A function that is 1-1 and onto has an inverse**
  - **If a function is not 1-1 or is not ontio, it has no inverse**
- **That is a function that "undoes" another function**
- **It is written as $\boldsymbol{f^{-1}(x)}$**
- **Let's find some inverses**

QUESTIONS

- Any questions?