CHAPTER 5

RECURRENCE RELATIONS

HOMEWORK

- Again, all homework is from the Exercises
  - No problems are from the Review Exercises
- Section 5.1, (p. 232), #4-8, 18-20, 37-40
- Section 5.2, (p. 244), #14-18, 34-36

EXTRA POINTS ON MONDAY'S TEST

- If you find a math typo on these slides, I will give 1 point back on the Chapter 4 exam

SOLVING A RECURRENCE RELATION

- Solving a recurrence relation means to put it into closed form
- Here is an example

  $a_1 = 3$

  $a_{n+1} = a_n + 2, n >= 1$
- We write out several terms

  $a_1 = 3$

  $a_2 = a_1 + 2 = 3 + 2$

  $a_3 = a_2 + 2 = 3 + 2 + 2$
- Continuing, we see that

  $a_n = 3+2(n-1)$
- This process is called iteration

PROBLEMS TO TRY

- Try to solve #2 on p. 244 by iteration. Assume $a_0 = 1$
- How about $a_0 = 3$, $a_n = 2a_{n-1} + n$?

LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- We will learn a technique for solving these problems
- Linear means?
- Homogeneous means?
- Constant coefficients means?
- Initial conditions means?
- Let's call these LHRRCC for short
  - Don't forget what those letters stand for!

- So, our recurrence relations will look like

  $a_n = c_1 a_{n-1} + c_2 a_{n-2}, + c_3 a_{n-3} + \ldots + c_k a_{n-k}$
- with initial conditions

  $a_0 = d_0, a_1 = d_1, a_2 = d_2, \ldots, a_{k-1} = d_{k-1}$

SOLVING ANOTHER RECURRENCE RELATION

- Let's solve this recurrence relation using iteration

  $a_1 = 3$

  $a_{n+1} = 2a_n$, n >= 1

- Like before, we write out several terms

  $a_1 = 3$

  $a_2 = 2a_1 = 2(3)$

  $a_3 = 2a_2 = 2(2(3))$

- Continuing, we see that

  $a_n = 3(2^{n-1})$

A PATTERN

- We now know how to solve recurrence relations like

  $a_0 = d$

  $a_n = ca_{n-1}$, $n >= 1$

- The solution is

  $a_n = dc^n$

BUILDING ON THAT PROCESS

- Let's solve #15 on p. 244

  $a_0 = 1$

  $a_1 = 0$

  $a_n = 6a_{n-1} – 8a_{n-2}$, $n >= 2$
- Following our previous problem, we guess that the solution has the form

  $a_n = r^n$
- Then

  $a_{n-1} = r^{n-1}$

  $a_{n-2} = r^{n-2}$
- The recurrence equation becomes

  $r^n = 6r^{n-1} – 8r^{n-2}$

CONTINUING ON TO A SOLUTION

- The recurrence equation from the previous slide is

  $r^n = 6r^{n-1} - 8r^{n-2}$

- Subtracting $6r^{n-1} - 8r^{n-2}$ from both sides and factoring out r^{n-2} on the right side gives

  $r^{n-2}(r^2 - 6r + 8) = 0$

- Factoring the quadratic on the right gives

  $r^{n-2}(r-4)(r-2) = 0$

- Since r isn't 0, $r = 2$ or $r = 4$
- This gives us two solutions

  $a_n = 2^n$ and $a_n = 4^n$

AN INTERESTING IDEA

- On the previous slide we found these two solutions

  $a_n = 2^n$ and $a_n = 4^n$

- It turns out that

  $a_n = R(2^n)$ or $a_n = R(4^n)$

- will also work, for <u>any</u> choice of R!
  - You need to check this!
- In addition

  $a_n = R(2^n) + S(4^n)$

- will also work, for <u>any</u> choice of R and S!
  - You need to check this too

FINDING R AND S

- We use the initial conditions

  $a_0 = 1$

  $a_1 = 0$
- to find R and S as shown on the next slide

- If we have no initial conditions, we stop here and write the solution as

  $a_n = R(2^n) + S(4^n)$
- Remember, any R and S will work
- We call this a general solution

THE FULL SOLUTION TO THE ORIGINAL PROBLEM

- We have

  $a_n = R(2^n) + S(4^n)$

  $a_0 = 1$

  $a_1 = 0$
- $a_0 = 1$ gives $R(2^0) + S(4^0) = 1$, or $R + S = 1$
- $a_1 = 0$ gives $R(2^1) + S(4^1) = 0$, or $2R + 4S = 0$
- Solving this system, we get $R = 2$, $S = -1$
- The full solution is then

  $a_n = 2(2^n) - (4^n)$
- You could rewrite this as

  $a_n = 2^{n+1} - 4^n$

SOLVING LHRRCC-A SUMMARY

- Start with the recurrence relation

  $a_n = c_1a_{n-1} + c_2a_{n-2}, + c_3a_{n-3} + \ldots + c_ka_{n-k}$

  $a_0 = d_0, a_1 = d_1, a_2 = d_2, \ldots, a_{k-1} = d_{k-1}$
- Assume the answer has the format

  $a_n = r^n$
- Find the characteristic polynomial for the recurrence relation
  - This is the quadratic polynomial we found while solving our problem
- Solve the polynomial for its roots, A, B
- The solutions are then

  $a_n = R(A^n) + S(B^n)$
- Use the initial conditions to find R and S
- Substitute in R and S and write the full solution

THE FIBONACCI SEQUENCE

- Here's an interesting idea:
- We know about the Fibonacci sequence.
  - The Fibonacci sequence is a special case of a Lucas sequence.
- Call the n^{th} term F_n
- Many people know the solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

- Let's see if we can find that

A PRACTICE PROBLEM

- Let's try p. 244, #16

A WRINKLE

- Let's try to solve

  $a_n = 4a_{n-1} – 4a_{n-2}$
- The characteristic polynomial is

  $x^2 – 4x + 4 = 0$
- The roots are 2 and 2
- The solution is then

  $a_n = R(2^n) + S(2^n)$
- This is really

  $a_n = k(2^n)$

ANOTHER SOLUTION

- It turns out that

  $a_n = n(2^n)$

- is another solution
  - You should check this
- And, just like before, the full solution is

  $a_n = Rn(2^n) + S(2^n)$

REPEATED ROOTS

- This happened because the polynomial x2 – 4x +4 has two identical roots
  - We call the root a repeated root or a multiple root
- Any time that happens, we insert n as a factor in one of the terms in the solution

- Let's try some practice problems
- p. 244, #22

HOMEWORK

- We now can solve the problems in Section 5.2
- As a reminder, all homework is from the Exercises
  - No problems are from the Review Exercises
- Section 5.2, (p. 244), #14-18, 34-36