CHAPTER 5

RECURRENCE RELATIONS

VARIATIONS ON SOLVING RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- **We will discuss some extra problems**
- **The first group of problems will just be variations on LHRRCC**
  - **Never forget what LHRCC stands for ☺**

VARIATION 1: THE CHARACTERISTIC POLYNOMIAL HAS HIGH DEGREE

- We only solved problems with a quadratic characteristic polynomial
- That doesn't have to happen
- Here is a recurrence relation with a characteristic polynomial of higher degree
  - For example, $a_n = 2a_{n-1} + a_{n-2} – 2a_{n-3}$
    - This is solvable
  - For example, $a_n = a_{n-1} – + a_{n-2} – a_{n-3}$
    - This is s not directly solvable
    - It can be “solved” numerically

VARIATION 2: THE RECURRENCE RELATION IS NOT HOMOGENEOUS

- So now LHRRCC has change to LRCC ☺
- For this kind of problem, we use a table
  - This table is explained (in words, not as a table) on p. 244 as Problem 40
  - I have included the table here
- There is an extra step for nonhomogeneous problems that we didn't have with homogeneous problems
  - You have to find a particular solution to the nonhomogeneous problem
  - You have to find a general solution to the HOMOGENEOUS version of the problem
  - The solution is the sum of the two

THE TABLE FOR NONHOMOGENEOUS PROBLEMS

- The original equation is $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$
- The characteristic equation is then $x^2 - c_1 x - c_2 = 0$

Is 1 a root of the characteristic polynomial?	1 is not a root	1 is a single root	1 is a double root
f(n)	Format for solution to particular equation		
(constant)	D	Dn	Dn^2
Cn	$D_1 n + D_2$	$D_1 n^2 + D_2 n$	$D_1 n^3 + D_2 n^2$
Cn^2	$D_1 n^2 + D_2 n + D_3$	$D_1 n^3 + D_2 n^2 + D_3 n$	$D_1 n^4 + D_2 n^3 + D_3 n^2$

AN EXAMPLE OF A NON-HOMOGENEOUS RECURRENCE RELATION

- Let's try to solve $a_n = 6a_{n-1} - 8a_{n-2} + 3$
- The homogeneous version of the problem is $a_n = 6a_{n-1} + 8a_{n-2}$
  - The characteristic equation is?
  - The roots of the characteristic equation are?
  - The solution to the homogeneous equation is?
- Stop and solve this part of the problem

AN EXAMPLE OF A NON-HOMOGENEOUS RECURRENCE RELATION

- Let's try to solve $a_n = 6a_{n-1} – 8a_{n-2} + 3$
- The homogeneous version of the problem is $a_n = 6a_{n-1} – 8a_{n-2}$
  - The characteristic equation is $x^2 – 6x + 8 = 0$
  - The roots of the characteristic equation are 2 and 4
  - The solution to the homogeneous equation is $a_n = R(2^n) + S(4^n)$

AN EXAMPLE OF A HOMOGENEOUS RECURRENCE RELATION

- Now, we solve the nonhomogeneous problem
  - The non-homogeneous part is $f(n) = 3$
- In order to use the table, we need to know if 1 is a root of the characteristic equation
  - The answer is no, since the roots are 2, 4
- The table says we should look for a solution to have the form

  $a_n = D$
- Substituting this into the non-homogeneous equation gives

  $D = 6D – 8D + 3$
- Solving this equation gives $D = 1$

PUTTING IT ALL TOGETHER

- So the solution is

  $a_n = R(2^n) + S(4^n) + 1$
- You have to go through this entire process every time

- If initial conditions are given, you can determine R and S:
- Suppose a0 = 4 and a1 = 11
- Then we have $4 = R(2^0) + S(4^0) + 1$ and $11 = R(2^1) + S(4^1) + 1$
- Solving this system gives R = 1 and S = 2, for the solution to the original problem being

  $a_n = 2^n + 2(4^n) + 1$

A SECOND EXAMPLE

- Let's try to solve $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$
- The homogeneous version of the problem is?
- The characteristic equation is?
- The roots of the characteristic equation are?
- The solution to the homogeneous equation is?
- See if you can also determine what f(n) is

- Stop and solve this part of the problem

THE ANSWERS

- The equation is $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$
- The homogeneous version of the problem is $a_n = 2a_{n-1} + 8a_{n-2}$
  - The characteristic equation is $x^2 – 2x – 8 = 0$
  - The roots of the characteristic equation are –2, 4
  - The solution to the homogeneous equation is $a_n = R(-2)^n + S(4^n)$

    $f(n) = 81n^2$

CONTINUING ON

- Since 1 is not a root of the characteristic equation, the particular solution has the form

  $a_n = D_1n^2 + D_2n + D_3$

- Notice that

  $a_{n-1} = D_1(n-1)^2 + D_2(n-1) + D_3$

  $a_{n-2} = D_1(n-2)^2 + D_2(n-2) + D_3$

- The recurrence relation $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$ becomes

  $D_1n^2 + D_2n + D_3 = 2[D_1(n-1)^2 + D_2(n-1) + D_3] + 8[D_1(n-2)^2 + D_2(n-2) + D_3] + 81n^2$

FINDING THE CONSTANTS-PART 1

- We have

  $D_1n^2 + D_2n + D_3 = 2[D_1(n-1)^2 + D_2(n-1) + D_3] + 8[D_1(n-2)^2 + D_2(n-2) + D_3] + 81n^2$

- Expanding powers of n gives

  $D_1n^2 + D_2n + D_3 = 2D_1(n^2-2n+1) + 2D_1D_2(n-1) + 2D_1D_3 + 8D_1(n^2-4n+4) + D_2(n-2) + D_3 + 81n^2$

- Distributing and grouping based on powers of n gives

  $(-9D_1 - 81)n^2 + (36D_1 - 9D_2)n + (34D_1 + 18D_2 - 9D_3) = 0$

- Since this must be true for all n, we have three separate equations

  $(-9D_1 - 81) = 0$

  $(36D_1 - 9D_2) = 0$

  $(34D_1 + 18D_2 - 9D_3) = 0$

FINISHING THE PROBLEM

- Solving the first equation gives $D_1 = -9$
- Substituting that into the second equation and solving gives $D_2 = -36$
- Substituting both D_1 and D_2 into the third equation and solving gives $D_3 = -38$
- So the particular solution is $a_n = -9n^2 - 36n - 38$
- The solution to the original recurrence relation is

  $a_n = R(-2)^n + S(4^n) - 9n^2 - 36n - 38$

A PROBLEM TO TRY

- $a_n = 4a_{n-1} – 3a_{n-2} + 20$
- The homogeneous version of the problem is?
- The characteristic equation is?
- The roots of the characteristic equation are?
- The solution to the homogeneous equation is?
- See if you can also determine what f(n) is

- Stop and solve this part of the problem

FINISHING THE PROBLEM

- You can see that 1 is a root of the characteristic equation
- This means the particular solution has the form $a_n = Dn$
- Then

  $a_{n-1} = D(n-1) = Dn - D$ and $a_{n-2} = D(n-2) = Dn - 2D$
- We need to solve $a_n = 4a_{n-1} - 3a_{n-2} + 20$

  $Dn = 4Dn - 4D - 3Dn + 6D + 20$
- Continuing to solve gives the particular solution of

  $a_n = -10n$
- For the full solution, find the general solution, and add $-10n$

THE HALTING PROBLEM

- There is one leftover idea from this chapter
- This is the Halting Problem
- Can you write a program that
  - takes as its input another program and that program's input, and
  - determines if that program will run infinitely or stop?

THE HALTING DECIDER

- Suppose you did have a program that could decide
- Let's call it the Halting Decider.

- I will create another program
- Let's call it the Halting Undecider
  - This program also takes as its input a program
  - This program does the opposite of its input.
    - If the input program stops, the new program runs forever, (like an infinite loop)
    - If the input program runs forever, the new program stops.

HALTING OR NOT?

- Now, we take the Undecider, use it as input to itself, and give this to the Halting Decider
- Now, either a program stops or it doesn't
  - We know this because the Halting Decider can tell us this
- Let's look at the new Undecider program
  - If it stops, it runs forever
  - If it runs forever, it stops
- So, there is no Halting Decider, because it can't tell us if this program halts or not
- We call the halting problem "undecidable"
  - This means that it's impossible to decide if the problem has a solution or not

PRACTICE PROBLEMS

- p. 244, #41-47