CHAPTER 4

COUNTING AND THE PIGEONHOLE PRINCIPLE

HOMEWORK

- **Homework**
  - **Section 4.4 (p. 194) #11-17, (No: 30-33)**
  - **Section 4.5 (p. 204) #1-5, 22-26, (No : 42-45)**
  - **Section 4.6 (p. 210) #1-3, 7-9, 15-17, (No : 22-24)**
  - **Section 4.7 (p. 219) #1, 3-5, 10-11**

WORDS THAT CAN BE MADE FROM THE LETTERS IN THE WORD FREMONT

- By "words" here I mean sequences of letters
- The sequences may not be actual English words
  - For example, I will consider FTMNR to be a word
- The question is
  - How many 7-letter words can we make from "Fremont"
- There are 7 letters, so the number of words (permutations!) is 7!

WORDS THAT CAN BE MADE FROM THE LETTERS IN THE WORDS APPLE PIE

- Let's pretend that apple pie is one big word applepie
- The number of letters there is 8
- Let me number the letters

1	2	3	4	5	6	7	8
A	P	P	L	E	P	I	E

- I will indicate words by listing the letters
- For example, 2-7-5 is pie
- What is 3-7-4-8?

COUNTING THE WORDS

- How many different 8-letter words can we make?
- It would not be fair to say that the number is 8!
- For example, all of 1,2,3,4,5,6,7,8 and 1,6,3,4,8,2,7,5 and 1,2,6,4,5,3,7,8 spell applepie
- You have counted applepie three times!
  - The word applepie appears even more than three times!
- So we have overcounted

COUNTING THE WORDS SYSTEMATICALLY

- First, let's draw eight slots for the letters: __ __ __ __ __ __ __ __
- Next, let's focus on the letters that appear multiple times
  - There are 3 P's and 2 E's
- So, focusing on the P's we have C(8,3) ways of putting the P's into the blanks
- Once those have been placed, there are 5 spaces left
- This means that there are C(5,2) ways of placing the E's
- Then, for the remaining letters, we have 3! ways of doing that

CALCULATING THE ANSWER

- So the calculation for the answer is

$$C(8,3)C(5,2)\bullet 3! = \frac{8!}{3!\cancel{5!}}\frac{\cancel{5!}}{2!\cancel{3!}}\frac{\cancel{3!}}{1} = \frac{8!}{3!2!}$$

GENERALIZING THE FORMULA

- Suppose S is a set containing

  n_1 identical items of type 1

  n_2 identical items of type 2

  .

  .

  .

  n_t identical items of type t
- Suppose n is the number of items in S
- Then the number of orderings of S is $$\frac{n!}{n_1!n_2!\cdots n_t!}$$

MORE GENERALIZED COMBINATIONS

- Now, suppose we have a set containing t elements
- We want to choose k-element subsets with repetition
- We want to know how many there are
- Then the number of combinations is C (k+t-1, k)

- Proof:
  - Suppose the set is $\{a_1, a_2, ..., a_t\}$
  - Create a row with k+t-1 slots
- Put k x's and t-1 bars in the slots; x represents something; a bar represents a divider
- X's up to first bar are a_1s, etc.
- There are C(k+t-1) choices for the bars

COMBINATORIAL IDENTITIES

- New terminology: C(n,k) is a binomial coefficient
- Identity 1:
  - C(n+1,k) = C(n,k+1) + C(n,k)
- Identity 2: (The Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^k b^{n-k}$$

- Identity 3:
  - The sum of Row n in Pascal's triangle is 2^n , or in symbols $\sum_{k=0}^{k=n} C(n,k) = 2^n$

- Let's try p. 215, #22

THE PIGEONHOLE PRINCIPLE-VERSION 1

- Suppose we have k+1 pigeons
- Suppose we also have k holes to hold the pigeons
  - Each hole can hold only one pigeon
- We put pigeon treats in the holes
- The pigeons fly to the holes
- One pigeon will be left out
- We usually say this another way
- If we have k+1 pigeons and k pigeonholes to put them into, then
  - Some hole has at least 2 pigeons
- This still works if we have more than k+1 pigeons

THE PIGEONHOLE PRINCIPLE-VERSION 2

- **Now suppose we have a function from a set X to a set Y**
- **We also assume that both X and Y are finite sets and that X is "bigger" than Y**
  - **This means that |X| > |Y|**
- **Then we can say that f(a) = f(b) for some different a and b in X**

THE PIGEONHOLE PRINCIPLE-VERSION 3

- **Now, the setting is the same as in Version 2:**
- **f is a function from X to Y**
- **Both X and Y are finite sets**
- **$|X| > |Y|$**

- **Then, if k is the ceiling of $|X| / |Y|$**
- **There are at least k values, $x_1, x_2, \ldots, x_k$ with $f(x_1) = f(x_2) = \ldots = f(x_k)$**

AN APPLICATION

- This is essentially Example 4.8.3 on p. 217
- Suppose a school has 200 different computer courses they offer
  - For example
    - Introduction to C++
    - Introduction to Java
    - Discrete Structures
- The courses are numbered 101, 102, 103, …, 300 all in a row
- We choose 101 different courses
- Then at least two courses have consecutive numbers

THE PROOF

- **Let's name the chosen courses $c_1, c_2, c_3, \ldots, c_{101}$**
- **Let's create another list $c_1, c_2, c_3, \ldots, c_{101}, c_1+1, c_2+1, c_3+1, \ldots, c_{101}+1$**
- **Let's create a function from the second set to the original list of courses**
  - **Treat both lists as sets**
- **Now, there are 202 elements in the second (domain) set**
- **Also, there are 200 elements in the original course list (range set)**
- **The function is $f(x) = x$**
  - **Notice that this makes sense, as both domain and range are in {101, 102, 103, …, 300}**
- **By the pigeonhole principle, there are two domain elements that map to the same range element**

FINDING THE COURSES

- **This means there are two items in the second set that are equal**
- **Now, no two c_i can be equal**
- **Also, no two c_{i+1} can be equal**
- **So, one c_i must equal a c_{j+1}**
- **Of course, this means the two consecutive course are c_i, c_{i+1}**