CHAPTER 4

COUNTING AND THE PIGEONHOLE PRINCIPLE

HOMEWORK

- Again, all homework is from the Exercises
  - No problems are from the Review Exercises
- Section 4.1 (p. 170), #5, 17-20, 28-30, 34-37, 42-46, 60, 62
- Section 4.2 (p. 182), #10-14, 25-29, 31-34, 58-62
- Section 4.4 (p. 194), #11-17, 30-33
- Section 4.5 (p. 204), #1-5, 22-26, 42-45
- Section 4.6 (p. 210), #1-3, 7-9, 15-17, 22-24
- Section 4.7 (p. 215), #1, 3-5, 10-11
- Section 4.8 (p. 219), #1-10

MORE COUNTING IDEAS

- Suppose we have a big set of items
  - For example, you have a collection of coffee cups that you want to display
  - You want to put them side by side
  - You wonder how many different ways you can arrange them
- This is called a permutation

ARRANGING THOSE COFFEE CUPS

- **Now, suppose you have only three coffee cups**
- **Let's call them A, B, and C**
- **There are 6 different ways to arrange them**
  - **The six ways are ABC, ACB, BAC, BCA, CAB, and ABA**
- **How do I know there are six different ways?**
- **Again, I could make a tree**

COUNTING ARRANGEMENTS

- In general
  - Suppose we have n items
  - We want to know how many different arrangements there are
  - Again, I want to point out that having items in different order is a different arrangement
  - These are called permutations
  - We say that order matters
- The formula for the number of different arrangements of n items is

  $n!$
- We can see this from a tree

MORE PERMUTATIONS

- Now suppose that, out of your 20 coffee cups, you want to display only 3
- Again, the order of displaying them is important
  - Let's call the cups A, B, C, …, T
  - Then the arrangement ABC is not the same as the arrangement ACB
- The number of ways of displaying 20 items in groups of 3 is (20) (19) (18)
  - I got this from a tree
- In general, the number of ways of displaying n items in groups of k where order matters is (n) (n-1) · · · (n-k+1)
- We have another way of writing that: $P(n,k) = \dfrac{n!}{(n-k)!}$

ANOTHER WAY OF ARRANGING THINGS: COMBINATIONS

- Now I will do the same thing, but with a change
- This time the order doesn't matter
- Now I want to know how many different ways I can choose three coffee cups out of 20
- The order of choosing the cups is not important
  - I just care about which three I have chosen
- This is called a combination

GETTING A FORMULA FOR COMBINATIONS

- Notice that a combination is similar to a permutation, but order doesn't matter
- And, notice that
  - For a permutation ABC, ACB, BAC, BCA, CAB, and CBA are all different
  - For a combination ABC, ACB, BAC, BCA, CAB, and CBA are all the same
- If you notice the connection between permutations and combinations, you can see that you have counted every combination exactly 3! times
  - This is 6 repeats each time
- So, in this case the formula is $\frac{20!}{17!}$, but you have to divide by 3!

A GENERAL FORMULA

- So the general formula for the number of combinations of n things taken k at a time is

$$C(n,k)=\frac{n!}{k!(n-k)!}$$

- We usually write it this way
- We can also get this formula by putting an equivalence relation on the permutations

AN EQUIVALENCE RELATION

- We can also get that formula through an equivalence relation
  - It's based on the ideas two slides back
- We start by looking at the list of permutations of 20 coffee cups taken 3 at a time
- Again, I can list them: ABC, ABD, ABE, …, RST
  - We know there are (20) (19) (18) or 6,840 items in that list
- Now I say that two items in the list are equivalent if they have the same three letters
  - I will use ~ to mean "equivalent to"
  - So ABC ~ ACB, CDT ~ TCD, etc.
- How many equivalence classes are there?
  - Notice this is the same as asking how many different combinations there are of 20 things taken 3 at a time
- This gives the formula

NOTATION

- For C(n,k), the book starts by saying what n is
- Then they call a combination of k items a k-combination
  - They do the same for permutations
- Many people (and calculators) use ${}_nC_k$ for C(n, k)
- Another common way of writing C (n, k) is $\binom{n}{k}$
  - It looks like a fraction, but without the "fraction bar"
- We also read that as "n choose k"
- We only have one alternate way of writing P (n, k), which is ${}_nP_k$

THOSE FORMULAS

- Memorize those two formulas
- The number of permutations of n things taken k at a time is
  - Order matters

$$P(n,k) = \frac{n!}{(n-k)!}$$

- The number of combinations of n things taken k at a time is
  - Order doesn't matter

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

SOME PROBLEMS

- These problems are from p. 182
- #7, 8, 9, 10, 12
  - These problems are just applications of the formulas

COMBINATORIAL IDENTITIES

- **The numbers C(n,k) are called combinatorial coefficients**
- **There are many good formulas based on the combinatorial coefficients**
- **These are called combinatorial identities**
- **Let's visit a place where we can find some of them**

PASCAL'S TRIANGLE

- There is a triangle that contains these numbers
- It's called Pascal's Triangle
- Here is a picture of the top five rows of the triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

BUILDING THE TRIANGLE

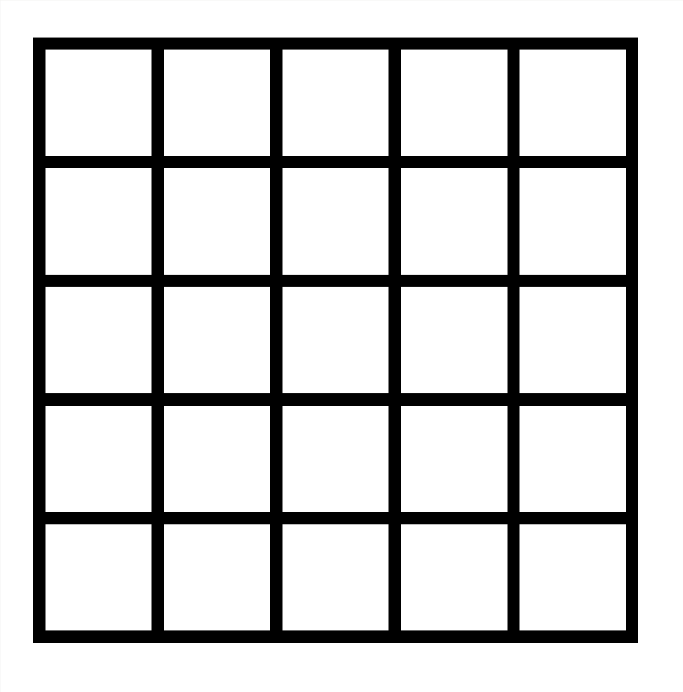
- How do you build the triangle?
  - Start with a triangle of 1s
  - For each new row
    - Put 1s on the outside
    - Each new entry in the row is the sum of the two elements diagonally above it
      - This is actually a formula that appears in the table!
- Take a minute to see if you can find any patterns or formulas in the triangle

CHECKING THE TRIANGLE

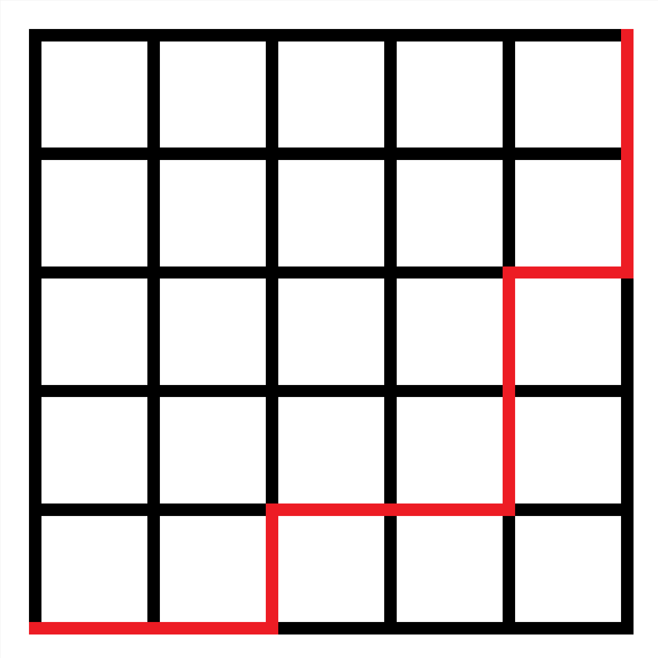
- The importance of the triangle is that it gives the binomial coefficients
- The n^{th} row contains the binomial coefficients C(n,i)
  - The first row is Row 0, corresponding to n=0, or C(0,?)
  - The first column in a row is Column 0, corresponding to C(?,0)
- There are many useful formulas hidden in Pascal's triangle, as well as others we can identify
- We will discuss these later in the chapter

ANOTHER EXAMPLE OF COUNTING: PATHS THROUGH A GRID

- Suppose you want to walk through a square grid
  - You start at the lower left
  - You go to the upper right
  - You can only go up and to the right
- We want to know how many different paths there are
- To have a concrete example, we will look at a 5x5 grid



COUNTING THE PATHS



- Let's simplify the problem
- For a path, we will create a list that describes the path
  - U means go up, R means go right
  - For example, in the grid pictured here, the path shown is
    - R, R, U, R, R, U, U, R, U, U
- The list for any path has 2n blanks, where n is the size of one side of the grid
- To find all the paths, we notice:
  - If we place the Rs in the blanks first, then the Us have to go in the leftover spots
  - And half of the blanks will be filled with Rs, half with Us
- So there are C(2n,n) paths

GOOD AND BAD PATHS

- We will call a path good if it doesn't cross the diagonal
  - The path on the previous slide is a good path
- If a path crosses the diagonal, we call it a bad path
- How many good paths are there?
- Call B_n the number of bad paths and G_n the number of good paths
  - Notice that $B_n + G_n = C(2n, n)$
- We will count the bad paths
- We will use this to get the number of good paths

COUNTING THE BAD PATHS

- We will find a 1-1 correspondence between the bad paths in an n x n grid and the paths in an (n-1) x (n+1) grid
- Notice that a bad path crosses the diagonal only by going up
- In the path, we find the U that crosses the diagonal
- We flip all positions to the right of that: U changes to R, R changes to U
- For example, the bad path R, U, R, U, **U**, R, R, R, U, U
- converts to R, U, R, U, **U**, U, U, U, R, R
- There are C(2n,n-1) paths in this new grid
  - We changed some Us to Rs and some Rs to Us
  - Specifically, there is one fewer R in the changed path

UNDERSTANDING THE MAP

- **A bad path has this form**

Original Path	Some Us and Rs	U	Some Us and Rs
	Equal numbers of Us and Rs	One single U	One more R than Us
New Path	Equal numbers of Us and Rs	One single U	One more U than Rs

- **We see that we have lost one R and gained one U**
- **So this procedure maps from an nxn grid to an (n-1)x(n+1) grid**

CHECKING THE 1-1 CORRESPONDENCE

- It's easy to see that this map is onto
  - Take any path in the (n-1)x(n+1) grid
    - These paths all have n-1 Rs, and n+1 Us
  - Find the first extra U
  - Flip all the Rs and Us after this one
  - You get an nxn path
- To see that it's 1-1
  - There is only one way to back up, and it's the way described above

CALCULATING G_n

- So, $G_n = C(2n,n) – B_n$
  $= C(2n,n) – C(2n,n-1)$
  $= \cdot\cdot\cdot = C(2n,n) / (n+1)$
- The derivation is straightforward using what C(?.?) means and some algebra
- The algebra is shown on p.181 in the text
- $\dfrac{C(2n,n)}{n+1}$ is called a Catalan number
- The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132, 429

HOMEWORK

- **You should now be able to complete the homework from Section 4.2**
- **Section 4.2 (p. 182): 10-14, 25-29, 31-34, 58-62**