Mathematical Induction

A technique for verifying a formula based on 0, 1, 2, 3, … (You can actually start at any integer, even negative ones!)

It has two parts

A starting point (Called the base step)

The inductive step

For induction, I won't ask you to define or explain it; I will ask you to use it

There are two ways we have used it

You can use it by itself to verify a formula is true; or

You can use it to verify that a guess at a closed form of a recurrence relation is correct

Steps for math induction:

1. Prove the base step. Label it "Base Step"
2. Prove the inductive step. Label it "Inductive step". Then:

A.) Write "Assume _____".

Fill in the blank with the formula for n. Write this assumption down!

B.) Write "Show _____"

Fill in the blank with the same formula, but change n to n+1.

C.) Verify the formula in B. You need to manipulate the formula in B in order to use the formula you wrote for A.

Keep this in mind as you look at the examples. If you omit the phrases Base Step", "Inductive Step", "Assume _____", and/or "Show _____", you will lose points.
This is emphasized in the comments in red in the examples that follow.

Here is a proof that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$, for n =1, 2, 3, …by induction.

Proof: (Induction)

Base step: (n=1)

The left side is 1

The right side is $\frac{1(1+1)}{2}=\frac{1(2)}{2}=\frac{2}{2}=1$.

Since the left side equals the right side, the base case is proven.

You need to have almost exactly what I have in the base step. Specifically,
You need to check the left side of the equation.
You also need to check the right side of the equation.
You then need to state that they are equal so that you have proved the base case.

Inductive step:

Assume $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

(This assumption is called the induction hypothesis. It's not so important that you know its name. I just use the name so I can refer to it later.)

To prove: $1+2+3+\cdots+n+1=\frac{(n+1)(n+1+1)}{2}$.

Note: That fraction simplifies to $\frac{(n+1)(n+2)}{2}$

Again, here you need to have the parts that I have above. That is, you need to say, "Assume …".
You also need to say,
"To prove: …".

$1+2+3+\cdots+n+1$

$=1+2+3+\cdots++n+(n+1)$ because n is the integer immediately before n+1

$=\frac{n(n+1)}{2}+n+1$ by the inductive hypothesis

$=\frac{n^2+n}{2}+\frac{2(n+1)}{2}$ distributing in the fraction and getting a common denominator

$=\frac{n^2+n}{2}+\frac{2n+2}{2}$ by the inductive hypothesis

$=\frac{n^2+3n+2}{2}$ combining the fractions

$=\frac{(n+1)(n+2)}{2}$ factoring the numerator. This is what we need to show.

At the end, be sure to summarize with something like, "This is what we need to show."

Example 1: Verify that $1+3+5+...+(2n-1)=n^2$, for n = 1, 2, 3, …

Proof (Induction)

Base step (n=1)

Substitute n = 1 into the left side and simplify. This gives $1+3+5+...+(2\bullet 1-1)=1+3+5+...+1$. Of course, this is just 1—there is nothing in the middle.

The right side says we should get 1^2, which is 1.

Since these two answers are the same, we have verified the formula for n = 1.

Inductive step:

Assume $1+3+5+...+(2n-1)=n^2$

To prove: $1+3+5+...+(2(n+1)-1)=(n+1)^2$. Cleaning up both the left and right sides gives $1+3+5+...+(2n+1)=n^2+2n+1$. This is what we need to show.

Remember, we need to make $1+3+5+...+2n-1$ appear in the formula on the left side of the equal sign. This looks like the start of the left side. Let's continue the left side after this, checking on what we get. The next number after $2n-1$ is $(2n-1)+2$, since you add 2 to get from one odd number to the next. Then $(2n-1)+2$ simplifies to $2n+1$. Notice that $2n+1$ is the last term in out series. Now we have

$$\begin{aligned}&1+3+5+...+(2n+1)\\=&1+3+5+...+(2n-1)+(2n+1)\\=&\qquad n^2 \qquad\qquad +(2n+1)\end{aligned}$$

, which is exactly what we want. So we have verified the formula.

Example 2: Verify that $2+4+8+...+2^n = 2^{n+1}-2$, for n=1, 2, 3, …
Proof: (Induction)

Base step: (n = 1)
The left side is $2+4+8+...+2^1$. This is just 2. Again, there is no middle.
The right side is $2^{1+1}-2$, which is 2. Since those two agree, we have verified the formula for n = 1.

Inductive step:
Assume $2+4+8+...+2^n = 2^{n+1}-2$.
To prove: $2+4+8+...+2^{n+1} = 2^{n+1+1}-2$. The right side of this is $2^{n+2}-2$.
Notice that the left side of the original equation is $2^1+2^2+2^3+...+2^n$.
Here is the process:

$$
\begin{aligned}
& 2+4+8+...+2^{n+1} \\
&= 2+4+8+...+2^n+2^{n+1} \\
&= \quad 2^{n+1}-2 \quad +2^{n+1} \\
&= 2^{n+1}+2^{n+1}-2 \\
&= 2\bullet 2^{n+1}-2 \\
&= 2^{n+2}-2,
\end{aligned}
$$

which is what we wanted. We have verified the formula.

Example 3: Verify that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$, for n = 1, 2, 3, …

Notice that the left side is $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^n}$

Proof: (Induction)

Base step (n = 1)

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^1}$ is $\frac{1}{2}+...+\frac{1}{2^1}$, which is really just $\frac{1}{2}$.

The right side is $1-\frac{1}{2^1}=1-\frac{1}{2}=\frac{1}{2}$. Since these agree, we have verified the formula for n = 1.

Inductive step;

Assume: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$

To prove:: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^{n+1}}=1-\frac{1}{2^{n+1}}$

Here is the process.

$$
\begin{aligned}
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+ \quad ... \quad +\frac{1}{2^{n+1}}\\
&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^n}+\frac{1}{2^{n+1}}\\
&= \qquad 1-\frac{1}{2^n} \qquad +\frac{1}{2^{n+1}}
\end{aligned}
$$

Simplify this.

$1-\frac{1}{2^n}+\frac{1}{2^{n+1}}=1-\frac{2}{2^{n+1}}+\frac{1}{2^{n+1}}=1-\frac{1}{2^{n+1}}$. That's what we needed to show. We verified the formula.

I m including one other example, just to show an example of when n doesn't start at 1.

Example 4: Verify that $n + 6 < 2n$, for $n = 7, 8, 9, \ldots$
(Notice that this isn't true for $n = 1, 2, 3, \ldots, 6$. But you can see that it's true for $n > 6$, where it's always true. So, I'm ignoring the obvious fact that you can solve the inequality in 0.01 second. You don't need induction to prove it.)

Proof: (Induction)
Base step: ($n = 7$. Yes, that's 7. It's not 1!)
The left side is $7 + 6$, which is 13.
The right side is $2(7)$, which is 14.
Since $13 < 14$, the base step is proved.

Inductive step;
Assume: $n + 6 < 2n$
To prove:: $(n+1) + 6 < 2(n+1)$
Here is the process.

$(n + 1) + 6$	
$= (n+6) + 1$	by rearranging (the commutative, associative properties and definition of addition)
$< 2n + 1$	by the inductive hypothesis
$< 2n+2$	because $1 < 2$
$= 2(n + 1)$	by factoring

So, $n + 6 < 2n$, for $n = 7, 8, 9, \ldots$

If you thought this was too easy, try to prove that $2n + 4 < 2^n$, for $n = 4, 5, 6, \ldots$

One more comment. In the proof for the inductive step, you must start with one side of the equation/inequality and arrive at the other. It is not okay to work on both sides to get to a common middle.

Note: In these examples, I spaced out the formulas. You should do this too! The point of that is to show how one formula relates to the formula that came before.