# CHAPTER 6

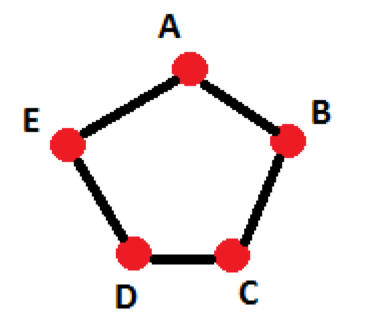
GRAPH THEORY

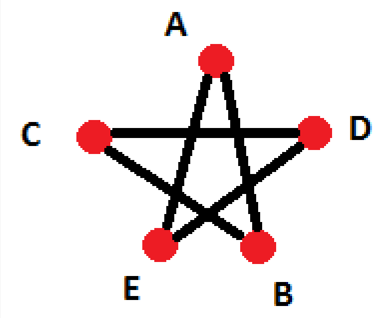
# HOMEWORK

- **Again, all homework is from the Exercises**
  - **No problems are from the Review Exercises**
- **Section 6.1, (p. 271), #5-10, 17-18, 22, 27-28, 46-48**
- **Section 6.2, (p. 281), #20-21, 28-38, 39, 41**
- **Section 6.3, (p. 296), #1-7**
- **Section 6.5, (p. 300), #1-3, 7-9, 13-14, 24-25**
- **Section 6.6, (p. 305), #1-7**
- **Section 6.7, (p. 311), #6-9, 18-24**

# DIFFERENT GRAPHS?

- From the book
  - Two people are each told to draw a graph with five vertices
    - They are told to label the vertices A, B, C, D, and E
  - They are told to add edges AB, BC, CD, DE, EA
- Here are the resulting pictures



# ISOMORPHISMS OF GRAPHS

- Are these graphs different?
- When two graphs are the same, we call them isomorphic
- This means
  - There is a 1-1 correspondence (f) between the vertices in the first graph and those in the second
  - There is a 1-1 correspondence (g) between the edges in the first graph and those in the second
  - If a vertex v in the first set is incident on an edge e, then f(v) is incident on g(e)

# ISOMORPHISMS OF GRAPHS

- The problem is this
  - Given two graphs, can we tell if they are isomorphic?
- Why does this matter?
  - If one graph has a picture that is easier to understand, it might be easier to work with
  - This gives a way to categorize graphs into categories
    - We can talk about each category separately
    - This will cover all possible graphs

# ISOMORPHIC GRAPHS AND ADJACENCY MATRICES

- It’s easy to see that isomorphic graphs have essentially the same adjacency matrices
- There is a way to rearrange the vertices in the second graph so the matrices are identical

# NON-ISOMORPHIC GRAPHS

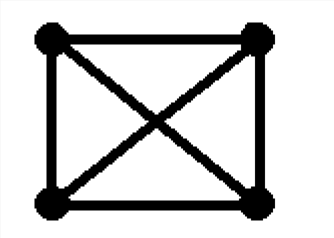
- Oftentimes, it's easier to show that two graphs are non-isomorphic
- One way to show this is if the graphs have a different number of vertices or edges
- Another way is if one graph has a vertex of degree and the other doesn't
- A characteristic of a graph that has to be the same in an isomorphic graph is called a graph invariant
- There are several other invariants

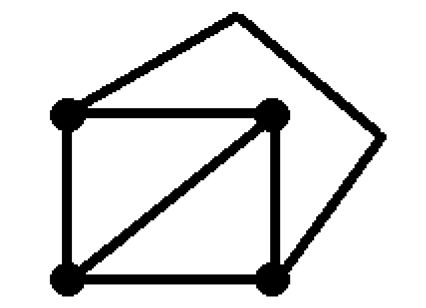
# A FUN PROBLEM

- Suppose there are three utilities
  - They are gas, water, and electricity
- Suppose there are three houses
- Can you draw a picture connecting each house to each utility without having any lines cross?

# PLANAR GRAPHS

- The answer is "No"
- This is not a planar graph
- A graph is called a planar graph if it can be drawn in the plane without any edges crossing
- There can be many pictures of a graph
  - It's planar if at least one picture can be drawn without edges crossing
- Here is an example



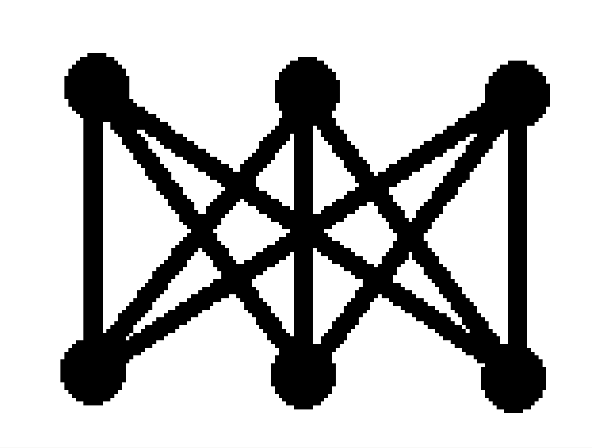


# REASONS FOR PLANAR GRAPHS

- Why do we care if a graph is planar or not?
- If you are an engineer designing a printed circuit board, it's much easier to design in the plane
- If you are designing a train system, you might prefer that tracks not cross
  - This simplifies the train schedules

# $K_{3,3}$

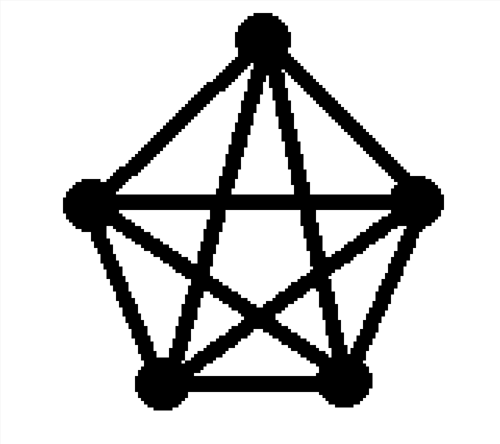
- The graph with the utilities and houses is called $K_{3,3}$
- It is a complete bipartite graph
- Here is a picture of it



- It is not planar

# ANOTHER NONPLANAR GRAPH

- **Here is another nonplanar graph**
- **This is $K_5$**

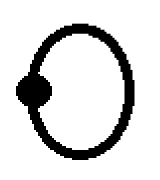


# VOCABULARY FOR PLANAR GRAPHS

- We know what vertices and edges are
- A face is a region of the plane surrounded by a simple cycle
- We also call the "outside" of the graph a face
  - Notice that the word face only makes sense for a planar graph
- In 1752 Euler proved a very important formula

  $f = e - v + 2$
- This is true for any connected planar graph
  - Euler's formula holds for a lot more than just graphs!

# A PROOF OF EULER'S FORMULA

- It's a proof by induction on the number of edges
- It's true for a connected planar graph with only one edge
- G looks like one of  or 
- For the graph on the left

  f = 1, v = 2, e = 1
- For the graph on the right

  f = 2, v = 1, e = 1
- In both cases, Euler's formula holds

# PROOF OF EULER’S FORMULA, CONTINUED

- We assume Euler’s formula holds for a graph with fewer than e edges
- We need to show it holds for a graph with e edges
  - Also assume the graph has f faces and v vertices
- I will break the proof into two cases
  - Case I: G has no cycles
  - Case II: G has a cycle

# PROOF OF EULER'S FORMULA, CONTINUED-CASE 1

- This is Case 1: G has no cycles
- Start at any vertex
- Follow an edge to another vertex
- Continue following edges, until you can't
  - This has to happen or the graph contains a cycle
- You have a arrived at a vertex of degree 1
- Delete this vertex and its edge
- By induction, Euler's formula holds for this smaller graph
  - The formula here is $f = (e-1) - (v-1) + 2$
- Cleaning this up with algebra shows Euler's formula holds for the original graph

# PROOF OF EULER'S FORMULA, CONTINUED-CASE 2

- This is Case 2: G has a cycle
- Choose any edge in the cycle
- Notice that this edge is part of a boundary for two faces
- Delete that edge
  - Don't delete any vertices
- Notice that we lost one face and one edge
- By the inductive hypothesis, $(f-1) = (e-1) + v + 2$
- Again, cleaning this up with algebra gives the formula

# MORE ABOUT NONPLANAR GRAPHS

- A question that we have is
  - How can we determine if a graph is nonplanar?
- We need some new vocabulary to answer the question
- The first is series reduction
  - A series reduction removes a vertex of degree 2 and replaces its two edges with a new edge
- If you change a graph into a new graph by series reductions, we say that the second graph can be obtained by series reductions on the first
- We will agree that any graph can be obtained by a series reduction on itself

# NONPLANAR GRAPHS

- One more word:  homeomorphic
- Two graphs are homeomorphic if they can both be reduced to the same graph using series reductions

- The importance of this vocabulary is Kuratowski's Theorem (6.7.7)
  - A graph is planar iff it does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$
- Look at Example 6.7.8 on p. 309

# HOMEWORK

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