CS 577 # Assignment 3: Greedy Algorithms Fall 2022

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Name: _____ Wisc id: _____

Greedy Algorithms

1. In one or two sentences, describe what a greedy algorithm is. Your definition should be informal, something you could share with a non computer scientist.

Solution: A "greedy" algorithm only makes decisions which yield the highest immediate reward, but does not take into account how the decision will affect future states and their rewards. This is essentially treating every decision the algorithm makes as if that decision is the only one affecting the final result.

2. There are many different problems all described as "scheduling" problems. In the following questions, pay attention to the details of the problem setup, as they will change each time!

   (a) Let each job have a start time, an end time, and a value. We want to schedule as much value of non-conflicting jobs as possible. Use a counterexample to show that Earliest Finish First (the greedy algorithm we used for jobs with all equal value) does NOT work in this case.

Solution: Two jobs: The first job runs from time 0-2 and is worth 1. The second job runs from time 1-3 and is worth 2. Earliest Finish First selects the first job and scores 1, but the optimum is the second job and total value 2.

   (b) *Kleinberg, Jon. Algorithm Design (p. 191, q. 7)* Now let each job consist of two durations. A job i must be preprocessed for p_i time on a supercomputer, and then finished for f_i time on a standard PC. There are enough PCs available to run all jobs at the same time, but there is only one supercomputer (which can only run a single job at a time). The completion time of a schedule is defined as the earliest time when all jobs are done running on both the supercomputer and the PCs. Give a polynomial time algorithm that finds a schedule with the earliest completion time possible.

Solution: Longest Finish Time First

We sort all jobs by their finish times (ignoring preprocessing time) in $O(N \log N)$ time. Then we run jobs through the supercomputer in descending finish times. We assign each job to a standard PC for finishing as soon as it is done preprocessing.

(c) Prove the correctness and efficiency of your algorithm from part (c).

Solution: LFTF runs in $O(N \log N)$ time as described above as this is the time required to sort, and selecting jobs from the sorted list can be done in $O(N)$ time.

For correctness, suppose some optimal algorithm OPT chooses job x immediately before job y at some point, but $f_x \leq f_y$. (This is an inversion from LFTF.) We claim that exchanging jobs x and y results in an optimal schedule.

First, no other job's completion time is affected, because x, y has the same preprocessing duration as y, x. Because we preprocess y earlier as a result of the exchange, only job x may finish later. If we call the end of the combined preprocessing time p, then job x completes at $p + f_x$ after the exchange. But job y completed at $p + f_y$ before the exchange, and $f_x \leq f_y$. Thus, no job may increase the overall completion time and we may exchange all jobs into LFTF order while maintaining optimality.

3. *Kleinberg, Jon. Algorithm Design (p. 190, q. 5)*

(a) Consider a long road with houses scattered along it. We want to place cell phone towers along the road so that every house is within four miles of at least one tower. Give an efficient algorithm that achieves this goal using the minimum possible number of towers.

Solution: Note: We assume that the road is straight.

Proceed along the road from one end. Every time we reach a house not already in range of a tower, proceed four miles farther and plant a tower there.

(b) Prove the correctness of your algorithm.

Solution: We'll use a "greedy stays ahead" approach. First, we show by induction that the ith tower the greedy algorithm places is at least as far along the road as the ith tower any optimal algorithm places. Let g_i be the position of the ith tower placed by the greedy algorithm, and o_j be the position of the jth tower placed by the optimal algorithm. We have $g_1 < g_2 < \ldots < g_m$ and $o_1 < o_2 < \ldots < o_n$, where the greedy algorithm places m towers and the optimal algorithm places n. We would like to show $g_i \geq o_i$ for all $1 \leq i \leq n$.

For the base case, we have $g_1 \geq o_1$ because both algorithms must cover the first house, and the greedy algorithm places the first tower as far along the road as possible while still covering the first house.

Now assume $g_{k-1} \geq o_{k-1}$ for some $k \leq n-1$. Suppose the greedy algorithm puts tower k 4 miles down the road from house h at position x_h, so $g_k = x_h + 4$. Since the greedy algorithm only would have done this if h was not already in range of the tower greedy places at g_{k-1}, we have $x_h - g_{k-1} > 4$. Then $x_h - o_{k-1} \geq x_h - g_{k-1} > 4$, so h is also not in range of the tower at o_{k-1}. Thus the optimal algorithm must also place tower k to cover house h, so $o_k \leq x_h + 4 = g_k$. This completes the proof by induction that $g_i \geq o_i$ for all $1 \leq i \leq n$.

Now use another induction to show that our algorithm always uses the minimum number of towers required to cover houses $1 \ldots k$ for any given k. (This is our inductive hypothesis.)

Base case: Any approach must use at least one tower to cover one house, and our algorithm uses exactly one.

Inductive step: Assume the hypothesis holds on $1 \ldots k$, so the greedy and optimal solutions both use i towers to cover houses $1 \ldots k$. Let house $k+1$ be at position x_{k+1}. Either the greedy algorithm uses an extra tower to cover house $k+1$ or it doesn't. If it doesn't, then the greedy solution on $1 \ldots k$ is still optimal on $1 \ldots k+1$. If it does use an extra tower, then the ith tower it placed didn't cover house $k+1$, so we have $x_{k+1} - o_i \geq x_{k+1} - g_i > 4$, so the optimal algorithm must also use an extra tower to cover house $k+1$.

4. *Kleinberg, Jon. Algorithm Design (p. 197, q. 18)* Your friends are planning to drive north from Madison to the town of Superior, Wisconsin over winter break. They have drawn a directed graph with nodes representing potential stops and edges representing the roads between them.

   They have also found a weather forecasting site that can accurately predict how long it will take to traverse one of the edges on their graph, given the starting time t. This is important because some of the roads on their graph are affected strongly by the seasons and by extreme weather. It's guaranteed that it never takes negative time to traverse an edge, and that you can never arrive earlier by starting later.

   (a) Design an algorithm your friends can use to plot the quickest route. You may assume that they start at time $t = 0$, and that the predictions made by the weather forecasting site are accurate.

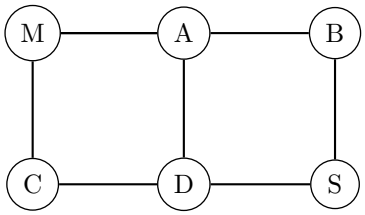
Solution: Run Dijkstra's shortest paths algorithm, starting at Madison. Whenever we add node x to our shortest paths tree with shortest path $M \to x$ of length t, we query the forecasting site for the lengths of all edges outgoing from x at time t. Edge "lengths" change as a function of time, but since we can never reach the next node earlier by departing later, the key property behind Dijkstra's algorithm still holds: if an unvisited node x can be reached earlier than any other unvisited node, the shortest path through x does not go through any other unvisited nodes, since these can only be reached after we could reach x.

To make recovering the path easier, each entry in the shortest path tree record includes both the shortest path distance to x and also the node y such that (y, x) was the last edge used.

(b) Demonstrate how your algorithm works using a small example with 6 nodes. Your demonstration should include any data structures you maintain during the execution of your algorithm and any queries you make to the weather forecasting site. For example, if your algorithm maintains a "current path" that grows from (M)adison to (S)uperior, you might show something like the following table:

Path	Total time
M	0
M,A	2
M,A,E	5
M,A,E,F	6
M,A,E	5
M,A,E,H	10
M,A,E,H,S	13

Solution:



Shortest paths:

Node	Distance	Predecessor	Priority queue (edges out of the shortest paths tree)
M	0		MA0:4,MC0:6
A	4	M	MC0:6,AD4:6,AB4:11
C	6	M	AD4:6,AB4:11
D	6	A	AB4:11,DS6:11
B	11	A	DS6:11
S	11	D	

Reconstructed path: M,A,D,S

Note: Priority queue holds queries of the form M to A at time t: t+edge time.

Coding Question

5. Implement the optimal algorithm for interval scheduling (for a definition of the problem, see the Greedy slides on Canvas) in either C, C++, C#, Java, or Python. Be efficient and implement it in $O(n \log n)$ time, where n is the number of jobs.

   The input will start with an positive integer, giving the number of instances that follow. For each instance, there will be a positive integer, giving the number of jobs. For each job, there will be a pair of positive integers i and j, where $i < j$, and i is the start time, and j is the end time.

   A sample input is the following:

   ```
   2
   1
   1 4
   3
   1 2
   3 4
   2 6
   ```

   The sample input has two instances. The first instance has one job to schedule with a start time of 1 and an end time of 4. The second instance has 3 jobs.

   For each instance, your program should output the number of intervals scheduled on a separate line. Each output line should be terminated by a newline. The correct output to the sample input would be:

   ```
   1
   2
   ```