CS 577 - Approximation

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Approximation Algorithms

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Dealing with hard problems

Efficient Approximation Algorithms

- Polynomial run-time
- Guaranteed (worst-case) performance.
- Bi-criteria Goal: Be as efficient as possible and as close to optimal as possible.

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Approximation Ratio

Worst-Case Approximation Analysis

- Let $\textsc{alg}(I)$ be the value of an algorithm $\textsc{alg}$ for an instance I of problem P.
- r-approximation ratio:

$$\forall I, \textsc{alg}(I) \leq r \cdot \textsc{opt}(I) + \eta$$

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- $\eta = 0$ is a strict approximation ratio.
- $\eta > 0$ is a asymptotic approximation ratio.

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Minimization vs Maximization

- Min: Approximation ratio ≥ 1.
- Max: Approximation ratio ≤ 1, or we use $c \geq 1$ where $1/c = r \leq 1$.

Heuristics

Bin Packing Problem

Definition

- Consists of:
  - an initial empty set of bins of capacity 1.
  - Finite request sequence: σ of length n.
- Each $r_i \in \sigma$ is an item with a size $s(r_i) \in (0, 1]$.
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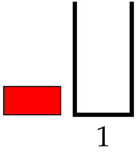

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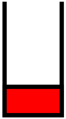

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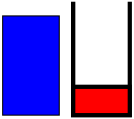

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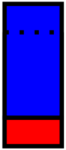

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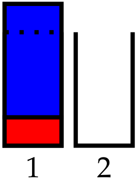

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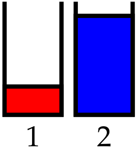

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First Fit Heuristic

First Fit (FF)

- A greedy heuristic
- Bins are opened (and ordered) sequentially.
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- Typically, approximation ratios are exact (no big-Oh).

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- For an algorithm with an approx ratio of $r \geq 1$:
  - r_ℓ is a *lower bound* if $\exists$ an instance I where $\textsc{alg}(I) \geq r_\ell \cdot \textsc{opt}(I)$.
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- A tight bound is $r_\ell = r_u$.

FF Lower Bound

Theorem 1

FF *has an approximation ratio* $\geq 4/3$.

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Proof.

Consider a sequence of n items of size $\frac{1}{3} - 2\varepsilon$, followed by $2n$ items of size $\frac{1}{3} + \varepsilon$ (assume n divisible by 3):

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- FF packing: $\frac{n}{3}$ bins for the first n items, and n bins for the last $2n$ items. Overall $\frac{4}{3}n$ bins.
- OPT packing: n bins each with 1 item of size $\frac{1}{3} - 2\varepsilon$, and 2 items of size $\frac{1}{3} + \varepsilon$.

□

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Theorem 2

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Theorem 2

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Proof.

All bins in FF are $\geq 1/2$ full (except possibly one bin). $\square$

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- If $i < k$, all bins are filled to at least $\frac{2}{3}$ (except possibly last bin).
- If $i = k$, there are at least k items of size $> \frac{1}{3}$. Since these k items can fit at most 2 to a bin, OPT $\geq \frac{2}{3}k$.

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Known Bounds

- FF has an approximation ratio of 1.7.
- FFD has an approximation ratio of 11/9.

PTAS

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For some ε:

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- Not all problems admit a PTAS.
- Some problems have inapproximability results:
  - Unless $\mathrm{NP} = \mathrm{P}$, Vertex Cover cannot be approximated within a factor of 1.3606.
  - Unless $\mathrm{NP} = \mathrm{P}$, Bin Packing cannot be strictly approximated within a factor of $3/2$.

Bin Packing Brute Force

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  - Choose M items to pack in a bin from $k+1$ different items sizes (includes 0).

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Number of Packings

- At most n bins are need, and we have R types of bins, so there are:

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feasible packings to check.

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Bin Packing (asymptotic) PTAS

Theorem 4

For any $\varepsilon > 0$*, there is an* $\textsc{alg}_\varepsilon$ *that runs in* $\mathrm{poly}(n)$ *time for which* $\textsc{alg}_\varepsilon(I) \leq (1 + 2\varepsilon)\textsc{opt}(I) + 1$ *for all* I.

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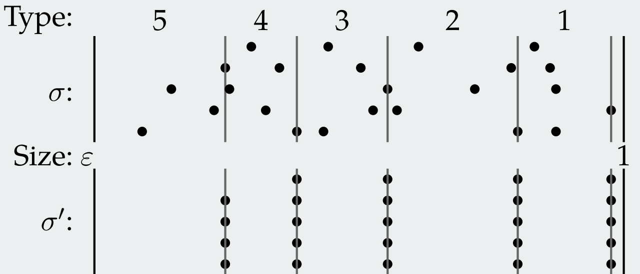
Proof (PTAS).

- Split the items of size $\geq \varepsilon$ (denoted by L) into groups containing $\lfloor n\varepsilon^2 \rfloor$ items $\implies$ $\lceil 1/\varepsilon^2 \rceil$ groups.
- Round items up to size of largest item:



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- Pack items $< \varepsilon$ using FF.

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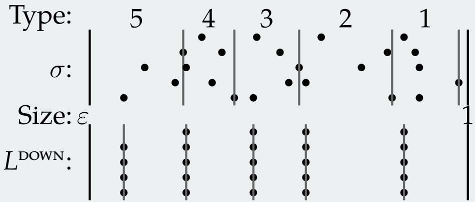
Proof (PTAS).

- Approximation ratio:
  - if items $< \varepsilon$ fit in $\textsc{alg}_\varepsilon(L)$ bins:

$$\textsc{alg}_\varepsilon(I) = \textsc{alg}_\varepsilon(L) \leq \textsc{alg}_\varepsilon(L^{\textsc{down}}) + n\varepsilon^2$$



  - Let S'' be optimal packing of $L^{\textsc{down}}$ ($\textsc{alg}_\varepsilon(L^{\textsc{down}}) \leq \textsc{opt}(L)$).
  - We can pack σ using S'':
    (1) Group 1 items in bins by themselves ($\leq n\varepsilon^2$ bins at most).
    (2) For $\lceil 1/\varepsilon^2 \rceil \geq i > 1$, pack the items of group i in the spots for the items of group $i-1$ in S''.

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$$\begin{aligned}\textsc{alg}_\varepsilon(I) = \textsc{alg}_\varepsilon(L) &\le \textsc{alg}_\varepsilon(L^{\textsc{down}}) + n\varepsilon^2 \\ &\le \textsc{opt}(L) + \varepsilon\textsc{opt}(L) \\ &= (1+\varepsilon)\textsc{opt}(L)\end{aligned}$$

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Proof (PTAS).

- Approximation ratio:
  - if items $< \varepsilon$ open new bins:
    - All bins (except possibly last one) are $(1 - \varepsilon)$ full:

$$\begin{aligned} \textsc{opt}(I) &\geq (\textsc{alg}_\varepsilon(I) - 1)(1 - \varepsilon) \\ \iff \textsc{alg}_\varepsilon(I) &\leq (1 + 2\varepsilon)\textsc{opt}(I) + 1 \end{aligned}$$

$\square$

Linear Programming

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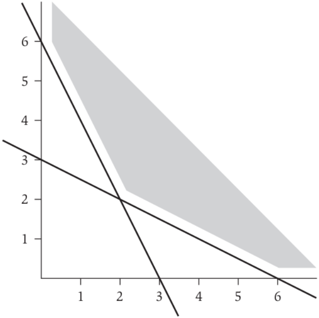

Basic Linear Program

$$
\begin{array}{rlcl}
\min & 1.5x_1 + x_2 & & \\
\text{s.t.} & x_1 + 2x_2 & \geq & 6 \\
& 2x_1 + x_2 & \geq & 6 \\
& x_1 \geq 0, x_2 \geq 0 & &
\end{array}
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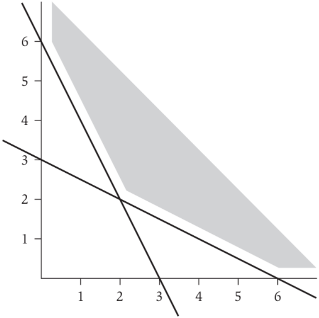

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TH1: What is the solution vector (x_1, x_2)?

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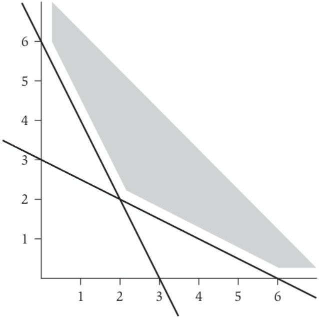

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General Linear Program

- Optimize a linear object function subject to linear constraints.

$$\begin{array}{rllll} \text{optimize}_x & c^T x & & & \\ \text{subject to} & Ax & \leq & b & \\ & x_j & \geq & 0 & \forall j \end{array}$$

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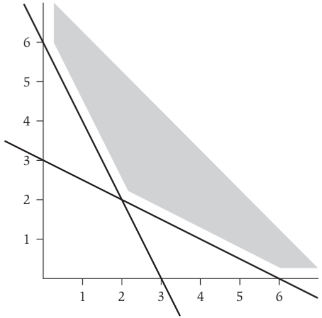

Basic Linear Program

$$c^T = \begin{bmatrix} 1.5 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -6 \\ -6 \end{bmatrix}$$

General Linear Program

- Optimize a linear object function subject to linear constraints.

$$\begin{array}{rlll} \text{optimize}_x & c^T x & & \\ \text{subject to} & Ax & \leq & b \\ & x_j & \geq & 0 \qquad \forall j \end{array}$$

Linear Programming (LP)

Some Notes

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Vertex Cover

ILP

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\min\limits_{x} & \sum_{i \in V} w_i x_i & & \\
\text{s.t.} & x_i + x_j \geq & 1 & \forall (i,j) \in E \\
& x_i \in & \{0,1\} & \forall i \in V
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Weighted Vertex Cover

- A graph $G = (V, E)$.
- For each $v_i : w_i \geq 0$.
- Goal: Find the minimum weighted vertex cover.

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What does a solution mean?

- Say $x_i = 0.42$ in a solution to the LP:
  - TH2: What is the minimum value for x_j if $(i,j) \in E$?

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  - $x_j \geq 0.58$ if $(i,j) \in E$

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Rounding Fractional Solution

Some methods:

- Fixed rounding rule
- Treat it as probability and do a probabilistic rounding.

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$$w(S) = \sum_{i \in S} w_i \leq 2 \sum_{i \in S} w_i x_i \leq 2 \sum_i w_i x_i \leq 2w(S^*)$$

Appendix

References

Image Sources I

https://brand.wisc.edu/web/logos/