CS 577 - Graphs

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Graphs

Graphs

A graph G is a pair $G = (V, E)$, where V is a set of vertices/nodes and E is a set of edges/arcs connecting a pair of vertices. That is, $E \in V \times V$.

Some Special Graphs

- Complete graph (K_4)
- Cycle (C_4)
- Path (P_4)
- Trees
- Digraph
- Directed Acyclic Graph (DAG)
- Bipartite
- Forests

Trees

Definition

- A connected graph without cycles.
- A single node may be designated as the *root* of the tree.
- Any node with degree 1 that is not the root is a *leaf*.

Properties of a tree T

1. If $|V| \geq 2$, (unrooted) T has at least 2 leaves.
2. For all nodes u and v, there exists one path between them in T.
3. $|V| = |E| + 1$ for $|V| \geq 1$.

TopHat 1

Is P_{10} a tree?

What can be represented by graphs?

- Transportation networks
- Communication networks
- Information networks
- Social networks
- Dependency networks

Connectivity

Graph Connectivity

Problem: s-t connectivity

Given a graph $G = (V, E)$, and the vertices s and t, is there a path from s to t in G?

Connected Graph

If all $(u, v) \in V \times V$ are connected, then G is connected.

Connected Components

Let $H \subset G$ be a subgraph of G. If H is connected and there are no edges between H and $G \setminus H$. Then, H is a connected component of G.

Graph Exploration/Traversal

Determining *s*-*t* Connectivity

Requires an algorithm that explores or traverses the graph by considering the edges of the graph to find all nodes connected to s.

Algorithm: Generalized Exploration

$R = \{s\}$
while $\exists$ *an edge* (u, v) *where* $u \in R$ *and* $v \notin R$ **do**
    Add v to R
end
return R

Graph Encodings and Implementation

Representations

- **Adjacency matrix**: $|V|$ by $|V|$ matrix with a 1 if nodes are adjacent.
- **Adjacency list**: For each node, list adjacent nodes.
- **Edge list**: List of all node pairs representing the edges (plus list of nodes).
- **Incidence matrix**: $|V|$ by $|E|$ matrix with a 1 if node is incident to the edge.

	Space	Find (u, v)	List of neighbours
Adjacency matrix	$O(\|V\|^2)$	$O(1)$	$O(\|V\|)$
Adjacency list	$O(\|V\| \cdot \min(\|E\|, \|V\|))$	$O(\min(\|V\|, \|E\|))$	$O(1)$
Edge list	$O(\|E\| + \|V\|)$	$O(\|E\|)$	$O(\|E\|)$
Incidence matrix	$O(\|V\|\|E\|)$	$O(\|E\|)$	$O(\|V\|\|E\|)$

Graph Exploration/Traversal

Algorithm: Generalized Exploration

$R = \{s\}$
while $\exists$ *an edge* (u, v) *where* $u \in R$ *and* $v \notin R$ **do**
  Add v to R
end
return R

TopHat 2

Which graph representation would be best suited?

Graph Exploration/Traversal

Algorithm: Generalized Exploration

$R = \{s\}$
while $\exists$ *an edge* (u, v) *where* $u \in R$ *and* $v \notin R$ **do**
  Add v to R
end
return R

Rough Running Time

- At step i: $O(|E_i| \cdot (\log |R_i| + \log |R_i|) + \log |R_i|)$, assuming R is a self-balancing BST.
- At most $|E|$ steps: $O(|E|^2 \log |V|)$

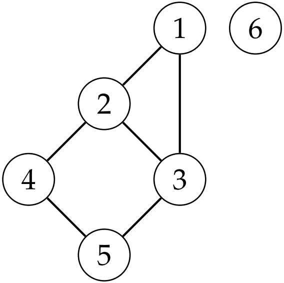
What is this algorithm lacking?

Breadth-First Search (BFS)

Process

- Also referred to as graph flooding.
- Let L_i be all the neighbours at a distance i from s.
- Starting from $i = 0$, visit all the nodes (not previously visited) in L_i. Increment i and repeat.

TopHat 3: This process engenders a BFS tree. Start at 1 and draw such a tree for the following.

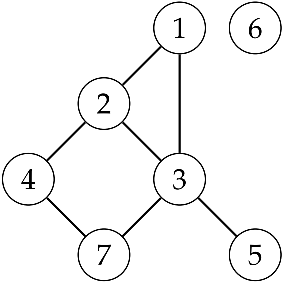


Depth-First Search (DFS)

Recursive Process starting at s

- Mark s as visited.
- For each $(s, u) \in E$ where u has not been visited, do DFS(u).

TopHat 4: This process engenders a DFS tree. Start at 1 and draw such a tree for the following.



Implementing BFS and DFS

TopHat 5

Which graph representation would be best for BFS and DFS? Why?

Implementing BFS and DFS

BFS Process

- Also referred to as graph flooding.
- Let L_i be all the neighbours at a distance i from s.
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DFS Recursive Process starting at s

- Mark s as visited.
- For each $(s, u) \in E$ where u has not been visited, do DFS(u).

Implementing BFS and DFS

Algorithm: BFS(S)

Initialize $v[u]$ = false for all nodes
Set $v[s]$ = true
Add s to tree T
Add s to queue Q
while *Q is not empty* **do**
    u = dequeue(Q)
    foreach *neighbour r of u* **do**
        if $!v[r]$ **then**
            Add (u, r) to T
            Set $v[r]$ = true
            Enqueue v
        end
    end
end
return T

Algorithm: DFS(S)

Initialize $v[u]$ = false and $p[u]$ = null for all nodes
Push s to stack S
while *S is not empty* **do**
    u = pop(S)
    if $!v[u]$ **then**
        Add $(p[u], u)$ to T
        Set $v[u]$ = true
        foreach *neighbour r of u* **do**
            Push r to stack S
            Set $p[r] = u$
        end
    end
end
return T

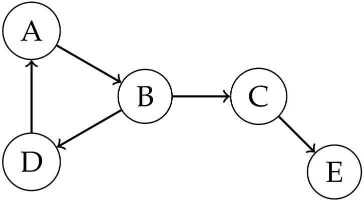
Runtime: $O(|E| + |V|)$

Strongly Connected Components

Directed Graphs

Directed Graph

- In a directed graph, the edges have a direction and are often called *arcs*.
- I.e. (u, v) is different than (v, u).



Strong Connectivity

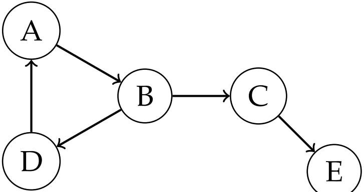
Mutually Reachable

- A pair of nodes (u, v) in a directed graph are *mutually reachable* if there is a path from u to v, and from v to u.
- Note: This property is transitive: if (u, v) and (v, w) are both mutually reachable, then u, w is mutually reachable.

Strongly Connected

A directed graph is *strongly connected* if, for every pair of nodes (u, v), u and v are mutually reachable.

G:



Strong Connectivity

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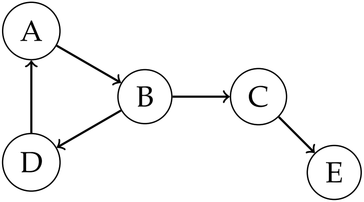
Testing for Mutually Reachable

How might we check if (u, v) is mutually reachable?

G:



Strong Connectivity

Mutually Reachable

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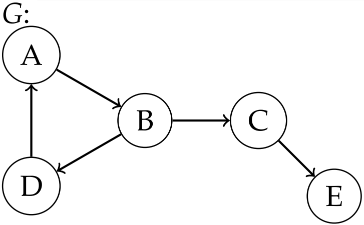
Testing for Mutually Reachable

Check if DFS/BFS from u reach v, and DFS/BFS from v reaches u.



Strong Connectivity

Mutually Reachable

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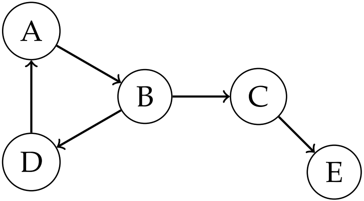
- Note: This property is transitive: if (u, v) and (v, w) are both mutually reachable, then u, w is mutually reachable.

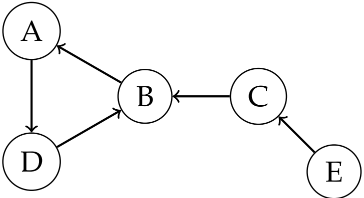
Testing for Mutually Reachable

Check if DFS/BFS from u in G reaches v, and DFS/BFS from u in G^{REV} reaches v.

G:



G^{REV}:



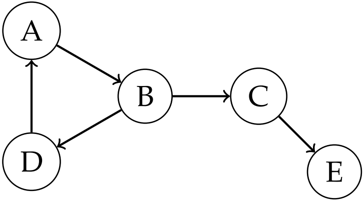
Strongly Connected Components

Strongly Connected Component (SCC)

A maximal strongly connected subgraph.

TopHat 6: How many SCC in G? 3

G:



Strongly Connected Components

Problem

Find the SCCs in a digraph G.

Kosaraju's Algorithm

1. Populate a stack S with a DFS on G.
2. Build G^{REV} for G, and set all nodes to unvisited.
3. While S is not empty:
    1. Pop node v from S.
    2. If v is unvisited, run DFS on G^{REV} from v to extract an SCC.

TopHat 7: What is the time complexity of Kosaraju's Algorithm? $O(|E| + |V|)$

Topological Ordering

Directed Graphs

Directed Graph

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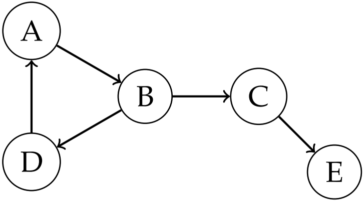
- I.e. (u, v) is different than (v, u).



Directed Graphs

Directed Graph

- In a directed graph, the edges have a direction and are often called *arcs*.
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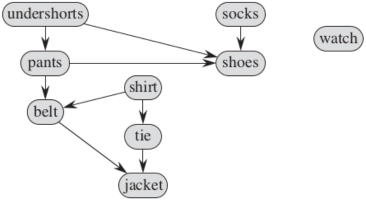
Directed Acyclic Graph (DAG)

- A directed graph with no directed cycles.
- Precedence relationships.

Getting dressed:

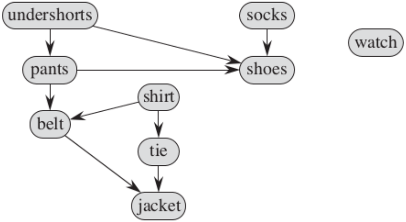


Topological Ordering

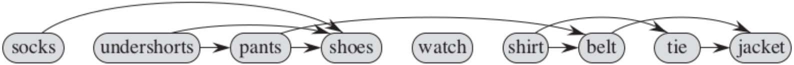
Definition

An ordering of the nodes of a DAG which respected the precedence relations.

Getting dressed DAG:



Topological ordering:



DAGs and Topological Ordering

Observation 1

If G has a topological ordering, then G is a DAG.

Key Property

In every DAG G, there is a node v with no incoming edges.

Proof (Exercise)

- By way of contradiction, assume all nodes in G have an incoming edge.
- Pick an arbitrary node u and follow the incoming node back to v. Since all nodes have an incoming edge, when can repeat this infinitely.
- After visiting $|V| + 1$ nodes, by the Pigeon Hole principle, we have visited some node w twice $\implies$ G contains a cycle.

DAGs and Topological Ordering

Observation 1

If G has a topological ordering, then G is a DAG.

Key Property

In every DAG G, there is a node v with no incoming edges.

- The Key Property allows us to show that all DAGs have a topological ordering.
- Prove it by induction.
- Does the inductive proof imply an algorithm to build a topological ordering from a DAG? If so, what is it?