CS 577 - Greedy

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Greedy

Greedy Algorithms

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Definition from Priority Algorithms

A greedy algorithm is an algorithm that processes the input in a specified order. For each request in the input, the greedy algorithm processes it so as to minimize (resp. maximize) the objective, assuming that the request is the last request.

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For a given problem, there may be many greedy algorithms.

Is greedy Optimal?

Not always: Bin Packing Problem

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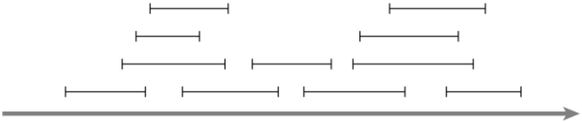
Techniques for showing that greedy is optimal:

- Always stays ahead
- Exchange argument

Stays Ahead: Interval Scheduling

Interval Scheduling



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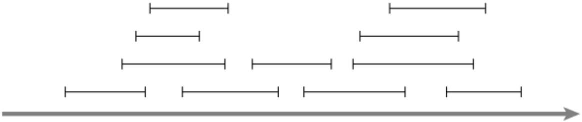

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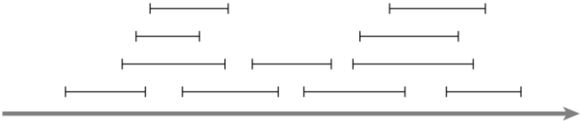

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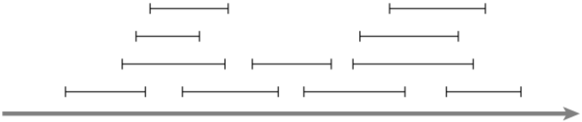

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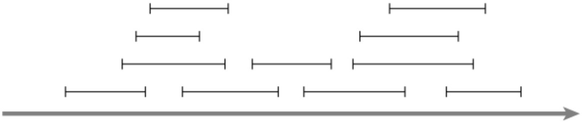

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TopHat Discussion 1: What greedy heuristic might work?

Greedy Algorithms for Interval Scheduling

Heuristic 1: Earliest First

Schedule a compatible request with the earliest start time.

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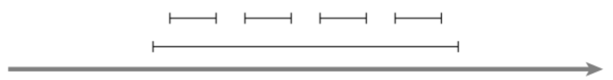

Greedy Algorithms for Interval Scheduling

Heuristic 2: Smallest Interval

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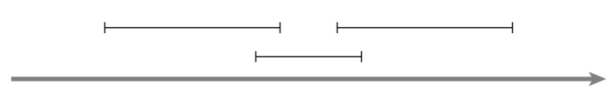

Greedy Algorithms for Interval Scheduling

Heuristic 3: Fewest Conflicts

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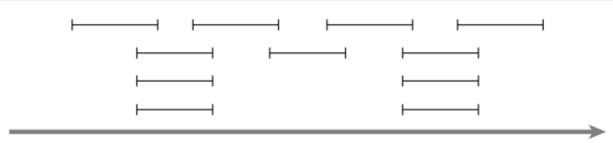

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Heuristic 4: Finish First

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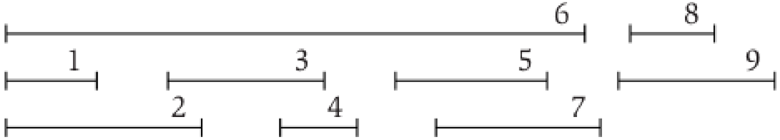
Sample Run (TopHat Q1: What is $|S|$?)



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Observation 1

Immediate from the definition of FinishFirst, *S is compatible.*

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- Hence, we can show the weaker claim of $|S| = |S^*|$ for this problem.
- Technique: "Always stays ahead"
  - At every time step i, $|S_i| \geq |S_i^*|$.

Stays Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

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For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

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- Assume true for $r - 1$.
  - By the induction hypothesis, we have that $f_{i_{r-1}} \leq f_{j_{r-1}}$.
  - The only way for S to fall behind S^* would be for FinishFirst to choose a request q with $f_q > f_{j_r}$, but this is a contradiction.

□

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The optimality of FinishFirst, essentially, follows immediately from Lemma 1.

FinishFirst is Optimal

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By way of contradiction, assume that $|S^*| > |S|$. This implies that $m > k$. Lemma 1 shows that FinishFirst is ahead for all the k requests. That means it would be able to add the $(k+1)$-st item of S^*. As it did not, this contradicts the definition of FinishFirst. □

Implementation and Running Time

Algorithm: FinishFirst

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Overall:

$O(n \log n) + O(n) = O(n \log n)$

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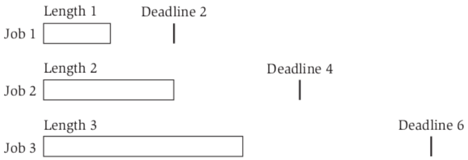
  - Objective: Minimize the number of schedules.

Exchange Argument: Minimize Max Lateness

Scheduling Problem: Minimize Lateness



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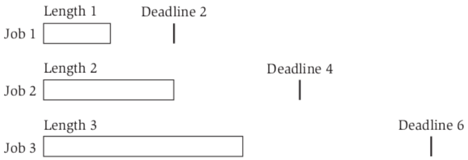

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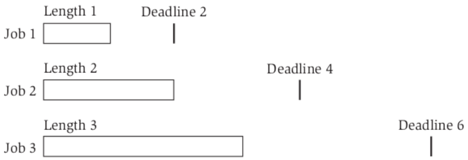

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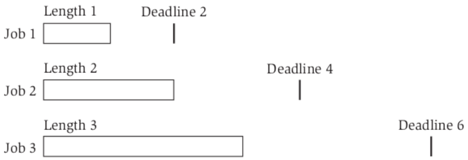

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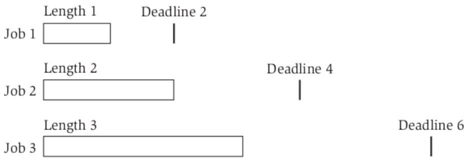

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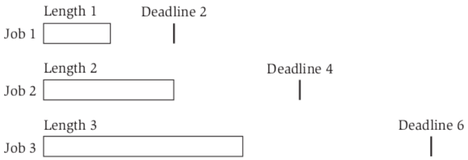

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Counter-example: Jobs (t_i, d_i): $\{(1, 100), (10, 10)\}$

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Heuristic 2: Increasing slack.

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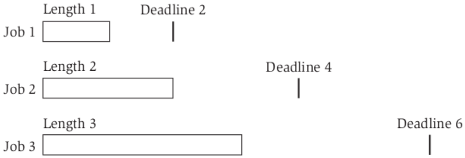
Sample Run (TopHat Q1: What is max lateness?)



Analysis of EDF

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There is an optimal schedule with no idle time.

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  - $S^* \equiv S_1 \equiv S_2 \equiv \cdots \equiv S$ for max lateness.

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A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

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Proof.

- Only vary in jobs with the same deadline.
- Jobs with same deadline must sequential.
- Ordering of jobs with same deadline won't change lateness.

□

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

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- Let $S^* := S'$ and repeat until no more inversions.

$\square$

EDF IS OPTIMAL

Corollary 6

EDF produces an optimal schedule.

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- EDF produces a schedule with no inversions and no idle time.
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Run time: Sort the jobs by deadline: $O(n \log n)$.

Shortest Path

Finding the Shortest Path

Problem Definition

We have a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ and a node s that has a path to every other node in V. For each edge e, $\ell_e \geq 0$ is the length of the edge.

- What is the shortest path from s to each other node?

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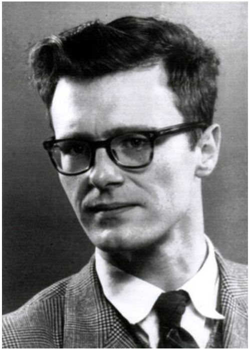

Edsger Dijkstra, 1956
Dijkstra's shortest path fame

Dijkstra's

Algorithm: *Dijkstra's*

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TopHat 3: Which technique to prove optimality?

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Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

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  - Moreover, there cannot be a shorter path to v passing through another node $y \notin S$ else y that would be added at $k + 1$.

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Paging

Paging Problem

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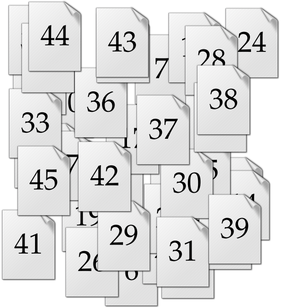

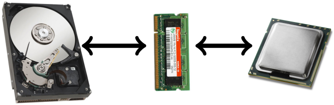

Cache:

Requests:

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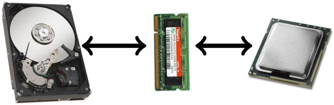

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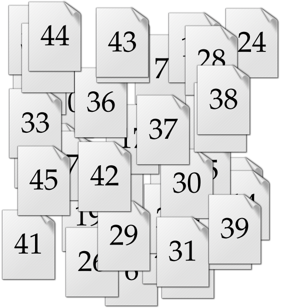

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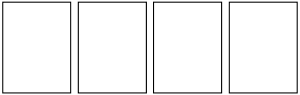

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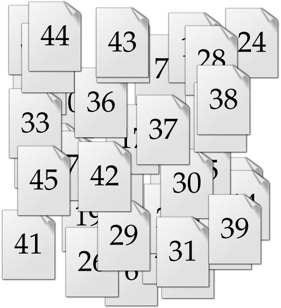

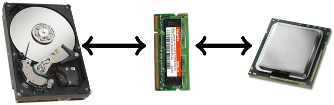

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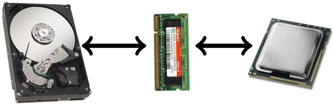

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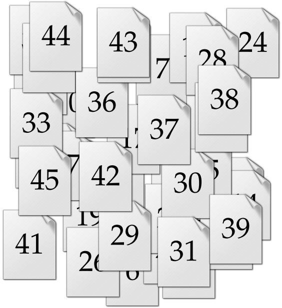

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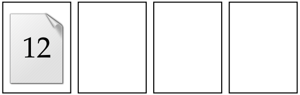

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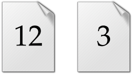

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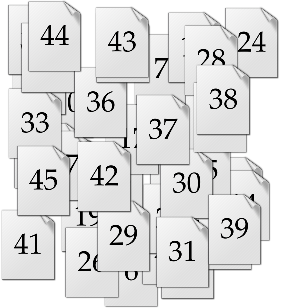

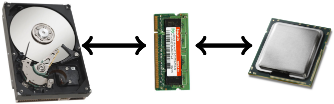

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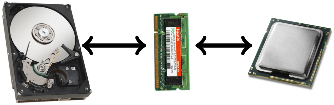

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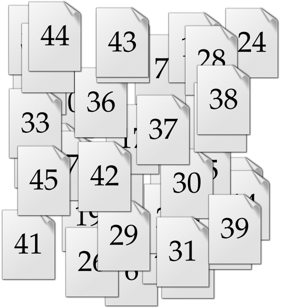

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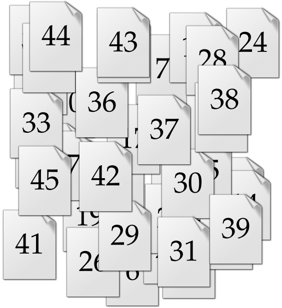

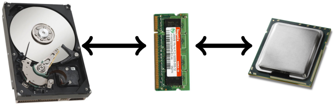

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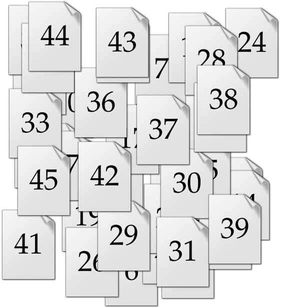

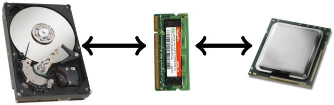

Cache: 12 | 3 | 33 |

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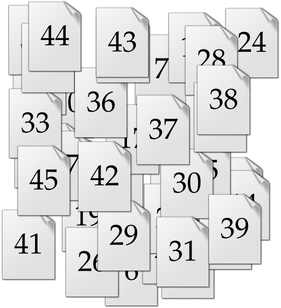

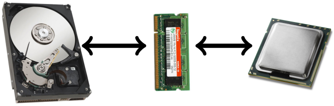

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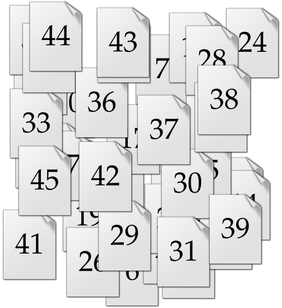

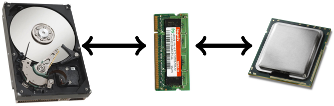

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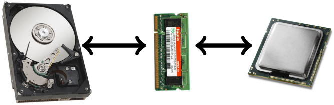

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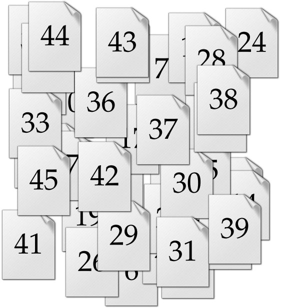

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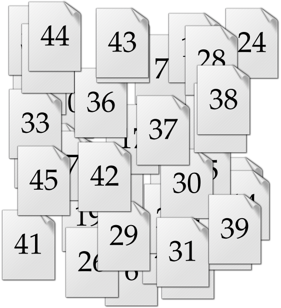

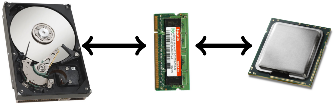

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Eviction Strategies

- When designing an algorithm, we are picking an eviction strategy.

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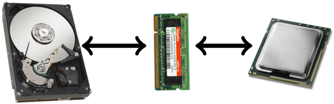

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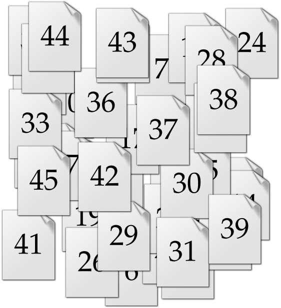

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Eviction Strategies

- When designing an algorithm, we are picking an eviction strategy.
- In the offline version, the algorithm knows the request sequence. What might be a good eviction strategy?

Offline Greedy Algorithm

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

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Small Run:

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TopHat 7: Which strategy to prove optimality?

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proving FF Optimality

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- If on request $j+1$, S behaves as S_{FF}. Then define S' as S and the claim follows.
- Otherwise, say S evicts u and S_{FF} evicts v. We will build S' by following S_{FF} for the first $j+1$ requests. Note that the number of faults are the same for S and S' up to $j+1$, and the caches match except for u and v.

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- From $j+2$ onward, S' follows S until either:
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- In either case, both S and S' have a page fault, and afterwards their cache match. □

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How do we get optimality of S_{FF} from Theorem 8?

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How do we get optimality of S_{FF} from Theorem 8?

By induction: We begin with the optimal schedule S^* and inductively apply Theorem 8 for $j = 1, 2, 3, \ldots, n$, which after the n iterations, produces S_{FF}.

MST

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

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Proof.

- By the definition of the problem, T must be connected.
- By way of contradiction, assume that T has a cycle C. Remove any edge from C resulting in a graph T'. T' is still connect and has a cost less than T.

$\square$

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

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Jarník's (1929), Kruskal's (1956), Prim's (1957), Loberman and Weinberger (1957), Dijkstra's (1958) Algorithm

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WLOG (without loss of generality)

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All we need is a consistent tie-breaker when $c_{e_1} = c_{e_2}$ for some pair of edges. I.e. based on the labels of the vertices of $e_1 \cup e_2$.

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Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

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- That is, Prim's algorithm does exactly what Lemma 10 describes.

□

Reverse-Delete is Optimal

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
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How should we prove that it produces an MST?

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Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

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- As c_e is the maximum cost edge of C, the claim follows from Lemma 13.

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- Retrieve the minimum valued edge between S and $V \setminus S$.

Implementing Prim's Algorithm

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Priority Queue (min-heap)

- `ExtractMin` ($O(1)$): $n-1$ times.
- `ChangeKey` ($O(\log(n))$): m times.

Overall: $O(m \log(n))$

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Union-Find Data Structure

- `Find(x)`: Finds the set containing x. $(O(\log n)$ can be $O(\alpha(n)))$
- `Union(x,y)`: Joins two sets x and y. $(O(1))$

Union-Find / Disjoint-Set

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`node`	`rank`	`parent`

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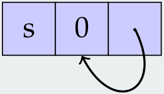
Basic Container

`node`	`rank`	`parent`

Initializing Data Structure for Kruskal's

For each node s, create a singleton set. That is each container has rank 0 and points to itself.



Union-Find Operations

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- (WLOG) x`.rank` $\geq y$`.rank`:
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- If x`.rank` $= y$`.rank`:
  x`.rank` $:= x$`.rank + 1`
- By using rank, we maintain balanced sets if we start with balanced sets.

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Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
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Key Operations

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Union-Find Data Structure
TH: How many Find and Unions?

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Union-Find Data Structure

- `Find(x)`: $2m$ times $O(\log n)$ (can be $O(\alpha(n))$).
- `Union(x,y)`: $n - 1$ times $O(1)$.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

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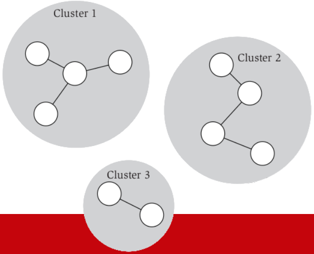
MST Algorithms

- Explores a graph G edges.
- Builds a tree T.
- T is minimum cost to connect all nodes in G.

Clustering

k-Clustering



Maximizing Spacing Problem

- A universe $\mathcal{U} := \{p_1, \ldots, p_n\}$ of n objects.
- Distance function $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ such that, for all $p_i, p_j \in \mathcal{U}$:
  - $d(p_i, p_i) = 0$
  - $d(p_i, p_j) > 0$
  - $d(p_i, p_j) = d(p_j, p_i)$
- Objective: Partition $\mathcal{U}$ into k non-empty groups $\mathcal{C} := C_1, \ldots, C_k$ with maximum spacing:

$$\text{maximize} \min_{C_i, C_j \in \mathcal{C}} \min_{u \in C_i, v \in C_j} d(u, v)$$

Algorithm Design

TopHat Discussion 4: What greedy approach might work?

Algorithm Design

Algorithm

- Build an MST.
- Remove $k - 1$ largest edges.

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- Start with a tree, remove $k-1$ edges: We get a forest of k trees.
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TopHat Q10: Which MST algorithm?

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TopHat Q10: Which MST algorithm?

Kruskal's ($O(m \log n)$ which is $O(n^2 \log n)$ for clustering):

- Merge sets from lowest to most expensive edges.
- Stop when we have k sets.

Prefix Codes

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII

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Variable-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
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  $\gamma(S) := \{0, 1, 10, 01, 11\}$.

Binary Encoding

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- Set of symbols $S := \{a, b, c, d, e\}$.
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- TopHat Q12: How many ways to decode 0010?

Unique Variable-Width Encodings

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Encoding of S such that no encoding of a symbol in S is a prefix of another.

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- TopHat 13: Decode 1101.

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Scan left to right, once an encoding is matched, output symbol.

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Easy Decoding

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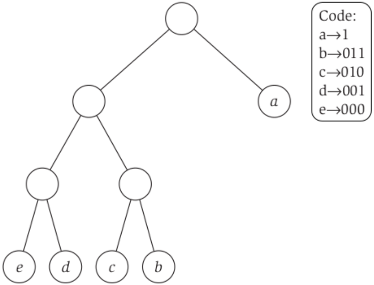
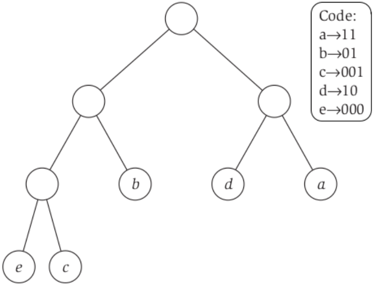
Optimal Prefix Codes

- For a set of symbols S, let f_x denote the frequency of x in the text to be encoded.
- Average bits $\textsc{abl}(\gamma) := \sum_{x \in S} f_x \cdot |\gamma(x)|$.
- Goal: Find γ that minimizes $\textsc{abl}$.

Algorithm Design

Prefix Binary Trees





Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

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By exchange argument:

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- Let T' be T with u replaced with v.
- Distance to v decreases by 1 in T', a contradiction.

□

Top-Down Approach

Algorithm

- Split S into two sets such that the sets frequency are $1/2$ the total frequency.
- Recurse on new sets until singletons.

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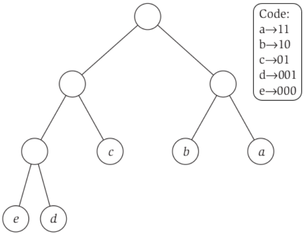
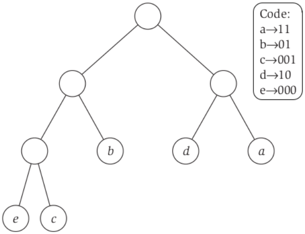
$$f_a = .32, f_b = .25, f_c = .2, f_d = .18, f_e = .05$$

$\textsc{abl}(\textsc{opt}) = 2.23$ $\textsc{abl}(\text{TopDown}) = 2.25$





What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

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Proof.

If $f_y < f_z$, exchange the labelling of y and z. Since $\text{depth}(u) < \text{depth}(v)$, $\textsc{abl}(T^*)$ must decrease with the new labelling. $\square$

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- Order symbols by increasing frequency.
- Assign them to leaves of T^* by decreasing depth.

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Observation 5

In T^*, the lowest frequency letters are siblings.

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

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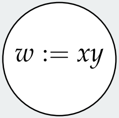
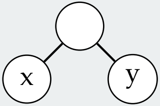
  - Replace 

with 

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Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

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Proof.

$$
\begin{aligned}
\textsc{abl}(T) &= \sum_{x \in S} f_x \cdot \mathrm{depth}(x) \\
&= f_y \cdot \mathrm{depth}(y) + f_z \cdot \mathrm{depth}(z) + \sum_{x \in S; x \notin \{y,z\}} f_x \cdot \mathrm{depth}(x) \\
&= f_w + f_w \cdot \mathrm{depth}(w) + \sum_{x \in S \setminus \{y,z\}} f_x \cdot \mathrm{depth}(x) \\
&= f_w + \textsc{abl}(T')
\end{aligned}
$$

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By induction:

- Base case $|S| = 2$

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- Inductive step: We have T. By way of contradiction, assume $\textsc{abl}(Z) \leq \textsc{abl}(T)$.

HUFFMAN CODES ARE OPTIMAL

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- We observed that y and z are siblings. Hence:

$$\begin{aligned} & \textsc{abl}(Z) < \textsc{abl}(T) \\ \iff \ & \textsc{abl}(Z') + f_w < \textsc{abl}(T') + f_w, \text{ by Lemma 17} \\ \iff \ & \textsc{abl}(Z') < \textsc{abl}(T'), \text{ a contradiction.} \end{aligned}$$

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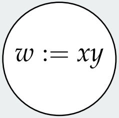
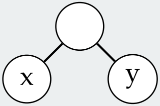
  - Replace 

with 

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Runtime:

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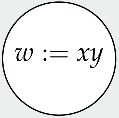
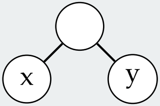
  - Replace 

with 

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Runtime: $|S| - 1$ recursions with find min over $|S_i|$ elements

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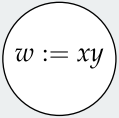
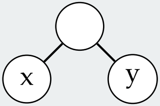
  - Replace 

with 

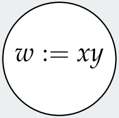
  - Repeat until leaves of T are original symbols.

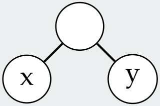
Runtime: $O(|S|^2)$

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- (1) Bottom-up by lowest frequency:
  - Let x and y be the lowest frequency symbols.
  - Set $S := S \setminus \{x, y\} \cup \{w := xy\}$ and $f_w = f_x + f_y$.
  - Repeat until $|S| = 1$.
- (2) Generate the tree:
  - $T :=$ root with element from S.
  - Replace 

with 

  - Repeat until leaves of T are original symbols.

Runtime: $O(|S|^2)$

what about $O(|S| \log |S|)$?

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- (1) Bottom-up by lowest frequency:
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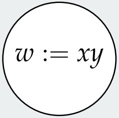
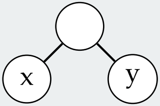
  - Replace 

with 

  - Repeat until leaves of T are original symbols.

Runtime: $O(|S|^2)$

what about $O(|S| \log |S|)$? Priority Queue (min-heap)

Appendix

References

Image Sources I

https://www.cse.unsw.edu.au/~cs1521/17s2/lecs/notices/slide068.html

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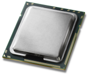

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Image Sources II

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