CS 577 - Greedy

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Spring 2023

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Greedy

GREEDY ALGORITHMS

What is a Greedy Algorithm (GREEDY)?

- Typically, thought of as a *heuristic* that is locally optimal.
- Is GREEDY always the best? No, but a good place to start.
- This notion has yet to be fully formalized, and it often problem specific.

Definition from Priority Algorithms

A greedy algorithm is an algorithm that processes the input in a specified order. For each request in the input, the greedy algorithm processes it so as to minimize (resp. maximize) the objective, assuming that the request is the last request.

For a given problem, there may be many greedy algorithms.

Is greedy Optimal?

Not always: Bin Packing Problem

- Bins of size 1, and requests of size $(0, 1]$.
- Objective: Pack the items in the minimum number of bins.
- Greedy heuristic: First Fit Increasing (ffi)

Non-optimal example:

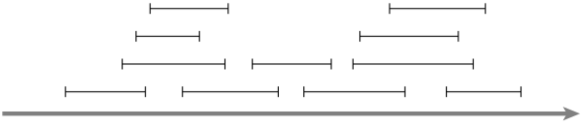
- $\sigma = \langle 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon \rangle$
- ffi: 3 bins
- opt: 2 bins

Techniques for showing that greedy is optimal:

- Always stays ahead
- Exchange argument

Stays Ahead: Interval Scheduling

Interval Scheduling



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$
- A request $r_i = (s_i, f_i)$, where s_i is the start time and f_i is the finish time.
- Objective: Produce a *compatible* schedule S that has maximum cardinality.
- Compatible schedule S: $\forall r_i, r_j \in S, f_i \leq s_j \vee f_j \leq s_i$.

TopHat Discussion 1: What greedy heuristic might work?

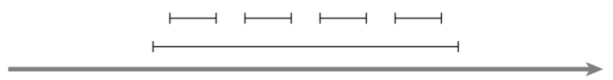
Greedy Algorithms for Interval Scheduling

Heuristic 1: Earliest First

Schedule a compatible request with the earliest start time.

Optimal?

Counter-example:



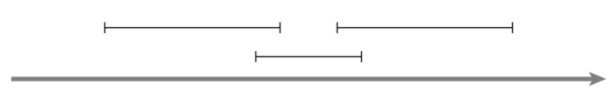
Greedy Algorithms for Interval Scheduling

Heuristic 2: Smallest Interval

Schedule a compatible request r_i with the smallest interval $(f_i - s_i)$.

Optimal?

Counter-example:



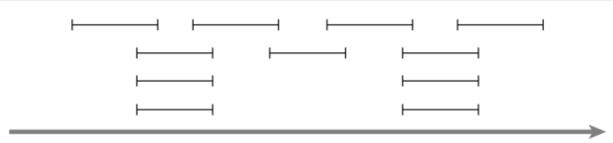
Greedy Algorithms for Interval Scheduling

Heuristic 3: Fewest Conflicts

Schedule a compatible request with the fewest remaining conflicts.

Optimal?

Counter-example:



Greedy Algorithms for Interval Scheduling

Heuristic 4: Finish First

Schedule a compatible request with the smallest finish time.

Optimal?

Counter-example? Let's try and prove it.

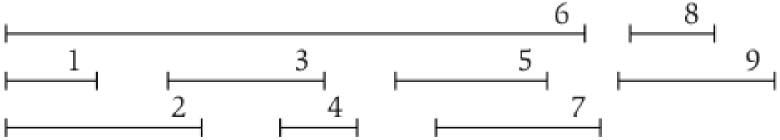
Exercise: Formalize the algorithm (pseudocode)

Heuristic 4: Finish First

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Sample Run (TopHat Q1: What is $|S|$?)



Analysis of FinishFirst

Observation 1

Immediate from the definition of FinishFirst, S is compatible.

Showing Optimality

Let S^* be an optimal solution.

- We can show the strong claim that $S = S^*$.
- Can there be multiple S^*? Yes.
- Hence, we can show the weaker claim of $|S| = |S^*|$ for this problem.
- Technique: "Always stays ahead"
  - At every time step i, $|S_i| \geq |S_i^*|$.

Stays Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

Proof.

The proof is by induction.

- For $r = 1$, the claim is true as FinishFirst first selects the request with the earliest finish time.
- Assume true for $r - 1$.
  - By the induction hypothesis, we have that $f_{i_{r-1}} \leq f_{j_{r-1}}$.
  - The only way for S to fall behind S^* would be for FinishFirst to choose a request q with $f_q > f_{j_r}$, but this is a contradiction.

□

Stays Ahead Analysis

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Lemma 1

For all i_r, j_r with $r \leq k$, we have $f_{i_r} \leq f_{j_r}$

The optimality of FinishFirst, essentially, follows immediately from Lemma 1.

FinishFirst is Optimal

- Label $S = \langle i_1, \ldots, i_k \rangle$ such that $f_{i_u} < f_{i_v}$ for $u < v$.
- Label $S^* = \langle j_1, \ldots, j_m \rangle$ such that $f_{j_u} < f_{j_v}$ for $u < v$.

Theorem 2

FinishFirst produces an optimal schedule.

Proof.

By way of contradiction, assume that $|S^*| > |S|$. This implies that $m > k$. Lemma 1 shows that FinishFirst is ahead for all the k requests. That means it would be able to add the $(k+1)$-st item of S^*. As it did not, this contradicts the definition of FinishFirst. □

Implementation and Running Time

Algorithm: FinishFirst

Let S be an initially empty set.
while *σ is not empty* **do**
  Choose $r_i \in \sigma$ with the smallest finish time (break ties arbitrarily).
  Add r_i to S.
  Remove all incompatible request in σ.
end
return S

Implementation Details

- Choose request with smallest finish time: Before processing, sort requests: $O(n \log n)$.
- Remove incompatible requests: Advance in sorted order until a request with a compatible start time.

Overall:

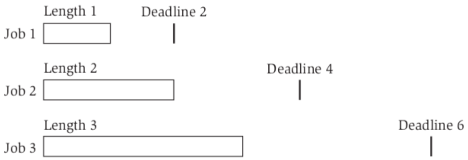
$O(n \log n) + O(n) = O(n \log n)$

Interval Extensions

- Online variant: Requests are presented in a specific order to the algorithm. At request i, the algorithm does not know n nor $r_{i+1}, \ldots, r_n$.
- Add a value to the intervals (online/offline). Now objective is to maximize the total value of scheduled intervals.
- Scheduling all intervals: Interval Colouring Problem.
  - Unlimited resources and the algorithm must produce multiple compatible schedules that cover all the requests (without duplicates between the schedules).
  - Objective: Minimize the number of schedules.

Exchange Argument: Minimize Max Lateness

Scheduling Problem: Minimize Lateness



Problem Definition

- n jobs and a single machine that can process one job at a time
- For job i:
  - t_i is the processing time, d_i is the deadline.
  - Lateness $l_i = f_i - d_i$ if finish time $f_i > d_i$; 0 otherwise.
- Objective: Build a schedule for all the jobs that minimizes the max lateness.

TopHat Discussion 2: What greedy heuristic might work?

Greedy Algorithms for Minimizing Max Lateness

Heuristic 1: Increasing processing time.

Schedule jobs by increasing t_i.

Optimal?

Counter-example: Jobs (t_i, d_i): $\{(1, 100), (10, 10)\}$

Greedy Algorithms for Minimizing Max Lateness

Heuristic 2: Increasing slack.

Schedule by increasing $d_i - t_i$.

Optimal?

Counter-example:

$$\text{Jobs } (t_i, d_i)\text{: } \{(1,2),(10,10)\}$$

Greedy Algorithms for Minimizing Max Lateness

Heuristic 3: Earliest deadline first.

Schedule by increasing d_i.

Optimal?

Counter-example? Let's try and prove it.

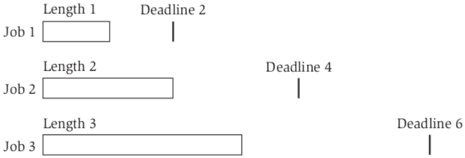
Exercise: Formalize the algorithm (pseudocode)

Heuristic 3: Earliest deadline first.

Algorithm: EDF

Let J be the set of jobs.
Let S be an initially empty list.
while *J is not empty* **do**
  Choose $j \in J$ with the smallest d_i (break ties arbitrarily).
  Append j to S.
end
return S

Sample Run (TopHat Q1: What is max lateness?)



Analysis of EDF

Observation 2

There is an optimal schedule with no idle time.

Showing Optimality

Let S^* be an optimal solution.

- Is it sufficient to show that $|S| = |S^*|$? No.
- Can there be multiple S^*? Yes.
- We need to show either $S = S^*$, or $S \equiv S^*$ for max lateness.
- Technique: "Exchange Argument"
  - Start with an optimal solution S^* and transform it over a series of steps to something equivalent to S while maintaining optimality.
  - $S^* \equiv S_1 \equiv S_2 \equiv \cdots \equiv S$ for max lateness.

Exchange Argument Analysis

Definition 3

A schedule A has an *inversion* if the are jobs i and j with i scheduled before j and $d_j < d_i$.

Lemma 4

All schedules with no inversions and no idle time have the same lateness.

Proof.

- Only vary in jobs with the same deadline.
- Jobs with same deadline must sequential.
- Ordering of jobs with same deadline won't change lateness.

□

Analysis of EDF

Theorem 5

There is an optimal schedule that has no inversions and no idle time.

Proof.

- If S^* has an inversion, then there is a pair of jobs i and j such that j is scheduled immediately after i and has $d_j < d_i$.
- We will swap i and j to create a new schedule S'. Note that S' has one less inversion than S^*.
- We need to show that S' has the same max lateness as S^*:
  - Swapping i and j means that l'_j (lateness in S') is less than that in S^*.
  - Lateness of i may increase, but: $l'_i = f'_i - d_i = f^*_j - d_i \leq f^*_j - d_j = l^*_j$.
- Let $S^* := S'$ and repeat until no more inversions.

$\square$

EDF is Optimal

Corollary 6

EDF produces an optimal schedule.

Proof.

- EDF produces a schedule with no inversions and no idle time.
- From Theorem 5, there is an optimal schedule with no inversions and no idle time.
- Lemma 4 shows that these two schedules have the same max lateness.

$\square$

Run time: Sort the jobs by deadline: $O(n \log n)$.

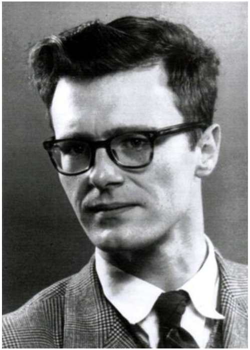
Shortest Path

Finding the Shortest Path

Problem Definition

We have a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ and a node s that has a path to every other node in V. For each edge e, $\ell_e \geq 0$ is the length of the edge.

- What is the shortest path from s to each other node?



Edsger Dijkstra, 1956
Dijkstra's shortest path fame

Dijkstra's

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
    Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest

$$d'(v) = \min_{e=(u,v):u \in S} \{d(u) + \ell_e\}$$

    Append v to S and define $d(v) = d'(v)$.
end

How is it greedy?
TopHat 3: Which technique to prove optimality?

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- For $|S| = 1$, the claim follows trivially as $S = \{s\}$.
- By the induction hypothesis, for $|S| = k$, P_u is the shortest $s - u$ path for all $u \in S$.

Correctness of Dijkstra's

Theorem 7

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

Proof.

By induction on the size of S.

- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.
  - Since P_u is a shortest path to u, P_v is the shortest path to v when considering only the nodes of S.
  - Moreover, there cannot be a shorter path to v passing through another node $y \notin S$ else y that would be added at $k + 1$.

$\square$

Dijkstra's Observations

Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

- Negative edge weights, where does it fail?
- TopHat 4: It is graph exploration, what kind of exploration?
    - Weighted (continuous) BFS

Implementation and Run Time of Dijkstra's

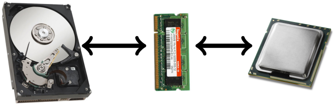
Algorithm: *Dijkstra's*

Let S be the set of explored nodes.
For each $u \in S$, we store a distance value $d(u)$.
Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
  Choose $v \notin S$ with at least one incoming edge originating from a node in S with the smallest $d'(v) = \min_{e=(u,v):u \in S}\{d(u) + \ell_e\}$
  Append v to S and define $d(v) = d'(v)$.
end

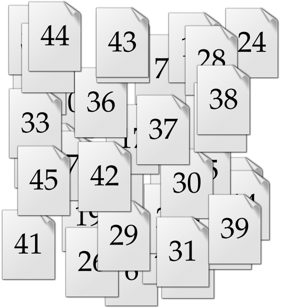
- TopHat 5: Number of iterations of the loop? $n - 1$
- Key Operations:
  - Finding the min: Easy in $O(m)$
- Overall: $O(mn)$
- How can we get $O(m \log n)$?

Paging

Paging Problem



$\mathcal{U}$:



Cache: 12 3 33 14

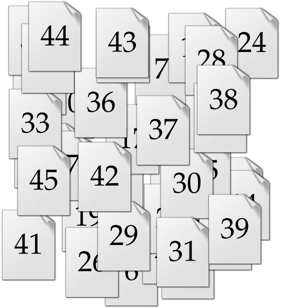
Requests: 12 3 33 14 12 20

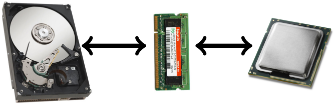
Definition

- $\mathcal{U}$: universe of pages ($|\mathcal{U}| > k$).
- Cache of size k.
- Requests are to the pages of $\mathcal{U}$.
- Goal: Minimize the number of page faults (requests to pages not in the cache).

Paging Problem

$\mathcal{U}$:





Cache: 12 3 33 14

Requests: 12 3 33 14 12 20

Eviction Strategies

- When designing an algorithm, we are picking an eviction strategy.
- In the offline version, the algorithm knows the request sequence. What might be a good eviction strategy?

Offline Greedy Algorithm

Farthest-in-Future (FF)

Evict the page whose next request is the furthest into the future.

Small Run:

- $\mathcal{U} = \{a, b, c\}$
- $k = 2$
- $\sigma = \langle a, b, c, b, c, a, b \rangle$
- TopHat 6: How many faults in small run?

TopHat 7: Which strategy to prove optimality?

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- If on request $j+1$, S behaves as S_{FF}. Then define S' as S and the claim follows.
- Otherwise, say S evicts u and S_{FF} evicts v. We will build S' by following S_{FF} for the first $j+1$ requests. Note that the number of faults are the same for S and S' up to $j+1$, and the caches match except for u and v.

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

Proof.

- From $j+2$ onward, S' follows S until either:
  1. S evicts v. In this case, S' evicts u.
  2. S evicts $g \neq v$ to bring u into the cache. In this case, S' evicts g and brings in v.
  - Note: Since S_{FF} evicts v at $j+1$, u must be requested before v.
- In either case, both S and S' have a page fault, and afterwards their cache match. □

Proving FF Optimality

Exchange Argument

Theorem 8

Let S be a schedule for the n request that make the same eviction decisions as S_{FF} for the first j items. Then, there is a schedule S' that makes the same eviction requests as S_{FF} for the first $j+1$ items with no more faults than S.

How do we get optimality of S_{FF} from Theorem 8?

By induction: We begin with the optimal schedule S^* and inductively apply Theorem 8 for $j = 1, 2, 3, \ldots, n$, which after the n iterations, produces S_{FF}.

MST

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

Observation 3

Let $T = (V, F)$ be a minimum-cost solution to the problem described above. Then, T is a tree.

Proof.

- By the definition of the problem, T must be connected.
- By way of contradiction, assume that T has a cycle C. Remove any edge from C resulting in a graph T'. T' is still connect and has a cost less than T.

□

Algorithm Design

TopHat Discussion 3: What greedy heuristic might work?

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

Assume Distinct Weights

WLOG (without loss of generality)

Theorem 9

(HW Q2) If all edge weights in a connected graph are distinct, then G has a unique MST.

Observation 4

All we need is a consistent tie-breaker when $c_{e_1} = c_{e_2}$ for some pair of edges. I.e. based on the labels of the vertices of $e_1 \cup e_2$.

Assumption: all edge weights are distinct.

Analyzing MST Heuristics

Lemma 10

Let $S \subset V$ be an non-empty proper subset of the nodes, and let $e = (v, w)$ be the minimum cost edge connecting S and $V \setminus S$. Then, every MST contains e.

Proof.

By exchange argument:

- Let T be a spanning tree that does not contain e.
- Let $e' = (v', w')$, where e' **is in** $P_{v,w} \in T$**,** $v' \in S$**, and** $w' \in V \setminus S$.
- Let $T' = T \setminus e' \cup e$.
- T' is connected as e is a $P_{v,w} \in T'$.
- Since $c_e < c_{e'}$, cost of T' is less than T.

$\square$

Kruskal's Algorithm is Optimal

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Theorem 11

Kruskal's Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be the edge added at any step i.
- Since e does not create a cycle, $v \in S$ and $w \notin S$ (WLOG).
- As c_e is the minimum cost edge, the claim follows from Lemma 10.

□

Prim's Algorithm is Optimal

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Theorem 12

Prim's Algorithm produces an MST.

Proof.

- Immediate from Lemma 10.
- That is, Prim's algorithm does exactly what Lemma 10 describes.

□

Reverse-Delete is Optimal

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

How should we prove that it produces an MST?

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Proof.

- Let T be a spanning tree that does contain e.
- Let S and $V \setminus S$ be the nodes of the connected components after removing e from T.
- Let e' be an edge in C that connects S and $V \setminus S$.
- Let $T' = T \setminus e \cup e'$.
- T' is connected as e' reconnects S and $V \setminus S$.
- Since $c_e > c_{e'}$, cost of T' is less than T.

$\square$

Reverse-Delete is Optimal

Lemma 13

Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

Theorem 14

Reverse-Delete Algorithm produces an MST.

Proof.

- Let $e = (v, w)$ be an edge removed at any step i.
- By definition e, belongs to a cycle C.
- As c_e is the maximum cost edge of C, the claim follows from Lemma 13.

□

Implementing Prim's Algorithm

Prim's (1957) Algorithm

- Initialize a node set S with an arbitrary node s.
- Keep the least expensive edge as long as it does not create a cycle.

Key Operations

- Retrieve the minimum valued edge between S and $V \setminus S$.
- Prim's and Dijkstra's have nearly identical implementations (but different minimizers)!

Priority Queue (min-heap)

- `ExtractMin` ($O(1)$): $n - 1$ times.
- `ChangeKey` ($O(\log(n))$): m times.

Overall: $O(m \log(n))$

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: ($O(m \log m)$ and, since $m \leq n^2$, $O(m \log n)$).
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure

- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Union-Find / Disjoint-Set

Key Operations

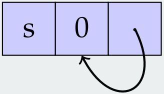
- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

Basic Container

`node`	`rank`	`parent`

Initializing Data Structure for Kruskal's

For each node s, create a singleton set. That is each container has rank 0 and points to itself.



Union-Find Operations

`Find(`x`)`: $O(\log n)$

- If x`.parent` points to x, return x.
- Else `Find(`x`.parent)`
- $O(\log n)$ requires balanced trees.
- $O(\alpha(n))$ with *path compression*.

`Union(`x`,`y`)`: $O(1)$

- (WLOG) x`.rank` $\geq y$`.rank`:
  y`.parent` $= x$
- If x`.rank` $= y$`.rank`:
  x`.rank` $:= x$`.rank + 1`
- By using rank, we maintain balanced sets if we start with balanced sets.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
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Key Operations

- Sorting the edges: $(O(m \log m)$ and, since $m \leq n^2$, $O(m \log n))$.
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure
TH: How many Find and Unions?

- `Find(x)`: Finds the set containing x.
- `Union(x,y)`: Joins two sets x and y.

Implementing Kruskal's Algorithm

Kruskal's (1956) Algorithm

- Sort edges by cost from lowest to highest.
- Insert edges unless insertion would create a cycle.

Key Operations

- Sorting the edges: $(O(m \log m)$ and, since $m \leq n^2$, $O(m \log n))$.
- Maintain sets of connected components that we merge.
- Initialize one set per node: $O(n)$.

Union-Find Data Structure

- `Find(x)`: $2m$ times $O(\log n)$ (can be $O(\alpha(n)))$.
- `Union(x,y)`: $n - 1$ times $O(1)$.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

Dijktra's

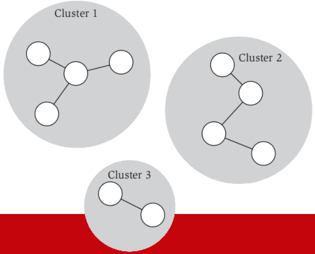
- Traverses a graph starting from some node s.
- Builds a tree T.
- All s to u paths in T are the shortest such path in G.

MST Algorithms

- Explores a graph G edges.
- Builds a tree T.
- T is minimum cost to connect all nodes in G.

Clustering

k-Clustering



Maximizing Spacing Problem

- A universe $\mathcal{U} := \{p_1, \ldots, p_n\}$ of n objects.
- Distance function $d : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ such that, for all $p_i, p_j \in \mathcal{U}$:
    - $d(p_i, p_i) = 0$
    - $d(p_i, p_j) > 0$
    - $d(p_i, p_j) = d(p_j, p_i)$
- Objective: Partition $\mathcal{U}$ into k non-empty groups $\mathcal{C} := C_1, \ldots, C_k$ with maximum spacing:

$$\text{maximize} \min_{C_i, C_j \in \mathcal{C}} \min_{u \in C_i, v \in C_j} d(u, v)$$

Algorithm Design

TopHat Discussion 4: What greedy approach might work?

Algorithm Design

Algorithm

- Build an MST.
- Remove $k-1$ largest edges.

k-Clusters at max spacing?

- Start with a tree, remove $k-1$ edges: We get a forest of k trees.
- By definition largest edges are removed so max spacing.

TopHat Q10: Which MST algorithm?

Kruskal's ($O(m \log n)$ which is $O(n^2 \log n)$ for clustering):

- Merge sets from lowest to most expensive edges.
- Stop when we have k sets.

Prefix Codes

Binary Encoding

Fixed-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^k$.
  $\gamma(S) := \{000, 001, 010, 011, 100\}$.
- Ex. ASCII
- TopHat Q11: Decode 000010.

Variable-Width Encoding

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{0, 1, 10, 01, 11\}$.
- TopHat Q12: How many ways to decode 0010?

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.
- 0010 invalid sequence
- TopHat 13: Decode 1101.

Unique Variable-Width Encodings

Prefix Codes

Encoding of S such that no encoding of a symbol in S is a prefix of another.

- Set of symbols $S := \{a, b, c, d, e\}$.
- Encoding function $\gamma : S \to \{0, 1\}^*$.
  $\gamma(S) := \{11, 01, 001, 000, 100\}$.

Easy Decoding

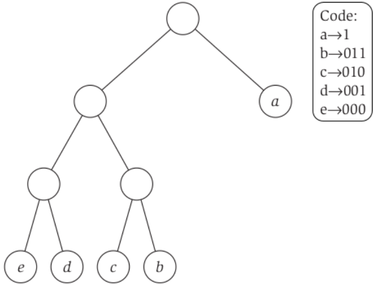
Scan left to right, once an encoding is matched, output symbol.

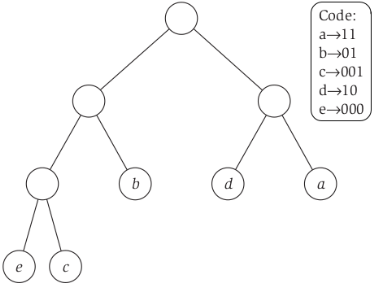
Optimal Prefix Codes

- For a set of symbols S, let f_x denote the frequency of x in the text to be encoded.
- Average bits $\textsc{abl}(\gamma) := \sum_{x \in S} f_x \cdot |\gamma(x)|$.
- Goal: Find γ that minimizes $\textsc{abl}$.

Algorithm Design

Prefix Binary Trees





Optimal Prefix Tree is Full

Theorem 15

The binary tree corresponding to the optimal prefix code is full.

Proof.

By exchange argument:

- Let T be an optimal prefix tree with a node u with one child v.
- Let T' be T with u replaced with v.
- Distance to v decreases by 1 in T', a contradiction.

□

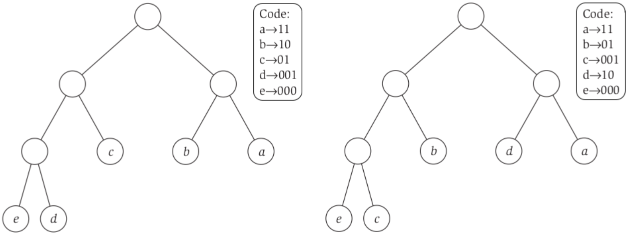
TOP-DOWN APPROACH

Algorithm

- Split S into two sets such that the sets frequency are $1/2$ the total frequency.
- Recurse on new sets until singletons.

$$f_a = .32, f_b = .25, f_c = .2, f_d = .18, f_e = .05$$

$\textsc{abl}(\textsc{opt}) = 2.23$ $\textsc{abl}(\text{TopDown}) = 2.25$



What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Proof.

If $f_y < f_z$, exchange the labelling of y and z. Since depth$(u) <$ depth(v), ABL(T^*) must decrease with the new labelling. □

What if we knew the optimal tree?

Let T^* be the optimal (unlabelled) prefix tree.

Lemma 16

Let u, v be leaves of $T*$ such that $depth(u) < depth(v)$, where u is labelled with y and v is labelled with z. Then, $f_y \geq f_z$.

Labelling T*

- Order symbols by increasing frequency.
- Assign them to leaves of T^* by decreasing depth.

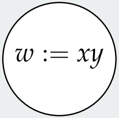
Observation 5

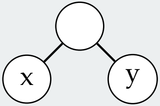
In T^*, the lowest frequency letters are siblings.

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- (1) Bottom-up by lowest frequency:
  - Let x and y be the lowest frequency symbols.
  - Set $S := S \setminus \{x, y\} \cup \{w := xy\}$ and $f_w = f_x + f_y$.
  - Repeat until $|S| = 1$.
- (2) Generate the tree:
  - $T :=$ root with element from S.
  - Replace 

 with 

  - Repeat until leaves of T are original symbols.

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Proof.

$$\begin{aligned}
\textsc{abl}(T) &= \sum_{x \in S} f_x \cdot \mathrm{depth}(x) \\
&= f_y \cdot \mathrm{depth}(y) + f_z \cdot \mathrm{depth}(z) + \sum_{x \in S; x \notin \{y,z\}} f_x \cdot \mathrm{depth}(x) \\
&= f_w + f_w \cdot \mathrm{depth}(w) + \sum_{x \in S \setminus \{y,z\}} f_x \cdot \mathrm{depth}(x) \\
&= f_w + \textsc{abl}(T')
\end{aligned}$$

□

Huffman Codes are Optimal

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\textsc{abl}(T') = \textsc{abl}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

- Base case $|S| = 2$
- Inductive step: We have T. By way of contradiction, assume $\textsc{abl}(Z) \leq \textsc{abl}(T)$.

HUFFMAN CODES ARE OPTIMAL

Lemma 17

Let T' be the tree at the $(k-1)$-st step, and let T be the tree at the k-th step. $\text{ABL}(T') = \text{ABL}(T) - f_w$, where w is the symbol replaced in the k-th step by y and z.

Theorem 18

Huffman Algorithm is optimal.

Proof.

By induction:

- We observed that y and z are siblings. Hence:

$$\begin{aligned} & \text{ABL}(Z) < \text{ABL}(T) \\ \iff\ & \text{ABL}(Z') + f_w < \text{ABL}(T') + f_w, \text{ by Lemma 17} \\ \iff\ & \text{ABL}(Z') < \text{ABL}(T'), \text{ a contradiction.} \end{aligned}$$

□

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- (1) Bottom-up by lowest frequency:
  - Let x and y be the lowest frequency symbols.
  - Set $S := S \setminus \{x, y\} \cup \{w := xy\}$ and $f_w = f_x + f_y$.
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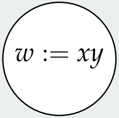
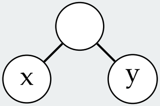
  - Replace 

with 

  - Repeat until leaves of T are original symbols.

Runtime: $|S| - 1$ recursions with find min over $|S_i|$ elements

Bottom-Up Approach

Huffman Code

Huffman's Algorithm

- (1) Bottom-up by lowest frequency:
  - Let x and y be the lowest frequency symbols.
  - Set $S := S \setminus \{x, y\} \cup \{w := xy\}$ and $f_w = f_x + f_y$.
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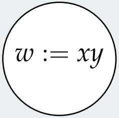
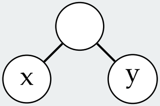
  - Replace 

with 

  - Repeat until leaves of T are original symbols.

Runtime: $O(|S|^2)$

what about $O(|S| \log |S|)$? Priority Queue (min-heap)