Lifetime Portfolio and Consumption Choice with Defined Contribution Plans

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This version: May 30, 2025

Abstract

We examine optimal portfolio and consumption choice with Defined Contribution Plan (DCP) participation and taxation of labor and capital income. We rigorously establish the connection between the value function and the associated Hamilton-Jacobi-Bellman equation. We uncover the impact of tax sheltering, employer matching, and potential diversification effect related to the DCP on the optimal choice. We calculate the value of participation and decompose it into four components corresponding to different types of utility gains from participation. We find that suboptimal asset location significantly reduces the participation value, while suboptimal capital gains tax-timing does not.

Keywords: Defined contribution plans, Capital gains tax, Portfolio and consumption choice, Indifference valuation, Singular stochastic control.

JEL Classification: C61, G11, H24, K34.

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1 Introduction

Defined contribution plans (DCPs) are the most popular employer-sponsored retirement saving plans in the United States. According to the U.S. Department of Labor, there were 656,241 defined contribution retirement plans in the U.S. as of 2018 (560,241 were 401(k)-type plans), covering more than 100 million participants. Participants can periodically contribute a portion of their wages to pension accounts for retirement saving purpose. While pre-retirement withdrawals from the pension account are heavily penalized, investments in the pension account are eligible for favorable tax treatments such as income tax deferral and capital gains tax exemption. In addition, employers often match employees' contributions to a certain extent, which helps to further enhance the wealth growth in the employees' pension accounts.

We develop a continuous-time model for examining optimal lifetime portfolio and consumption choice when an investor participates in a DCP. By participating in the DCP, the investor can invest through two accounts: an ordinary investment account in which ordinary income is taxed upon distribution and capital gains are taxed upon realization; and a pension account in which ordinary income taxes are deferred until withdrawal time and capital gains are tax-free. In each account, the investor can allocate the wealth between a risk-free asset (bond hereafter) and a risky asset (stock hereafter). Since typical DCPs often offer a rich menu of diversified mutual funds as investment options, we interpret the stock in the pension account as a diversified stock portfolio. We allow the stock in the ordinary account to be either identical to, or different from, that in the pension account; and we use the former (the latter resp.) case to model the scenario where the investor's personal portfolio is fully diversified (under-diversified resp.). In the under-diversification (UD hereafter) case, investing with the pension account can provide the investor with a diversification benefit in addition to the tax benefit.

Our singular stochastic control model originally includes five state variables: the wealth level in the pension account; the value of the bond position in the ordinary account; the value and purchasing costs of the stock position in the ordinary account, and the calendar time. The large number of state variables makes the model difficult to solve and analyze. Fortunately, assuming the investor exhibits a constant absolute risk aversion (CARA) preference, we can reduce the number of effective state variables to three. This reduction significantly improves the tractability of the model.

On the other hand, assuming the CARA utility function gives rise to technical challenges in establishing the connection between the value function and the proper viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation; the approach proposed in Bian et al. (2021) does not apply to our case since the CARA utility function is unbounded from below. We propose a novel space-truncating approach, which enables us to prove the continuity of the value function and its coincidence with the minimum viscosity solution of the HJB equation. Thanks to this coincidence, we can numerically solve for the value function by applying a monotone finite-difference method to the HJB equation.

In the absence of capital gains tax, similar to a classic Merton problem with a CARA utility function, the investor's optimal stock exposure (in dollars) in the ordinary account is a deterministic function of time. As in classic optimal tax-timing models, the presence of capital gains tax leads to a tax-deferral region such that it is optimal not to trade the stock in the ordinary account when the stock exposure in this account lies within this region. This tax-deferral region exists because the loss from the time value of taxes, should the investor sell the stock and realize a gain, exceeds the benefit of rebalancing to a better risk exposure. The investor should trade the stock in the ordinary account only when the stock exposure is outside the tax-deferral region so the benefit of portfolio rebalancing exceeds the time value of paid taxes. In the UD case, the diversification effect can also significantly affect the investor's optimal wealth allocation. In particular, participation could alter the location of the tax-deferral region and affect the average stock exposure in the ordinary account.

When the returns of the two stocks are positively (negatively resp.) correlated, participation will decrease (increase resp.) the optimal stock exposure in the ordinary account.

In the full-diversification (FD hereafter) case, similar to Dammon et al. (2004), the optimal stock exposure in the pension account is small (but positive on average), because capital gains earned on the stock are taxed lighter than interest earned on the bond. In the UD case, nevertheless, the optimal stock exposure in the pension account can be substantial due to the diversification effect. Moreover, with a positive return correlation, the optimal stock exposures in the two accounts are negatively related: as the stock exposure in the ordinary account decreases (or increases), the optimal stock exposure in the pension account increases (or decreases) accordingly. This relationship is because stock trading in the pension account is tax-free; hence, if the investor's realized stock exposure in the ordinary account decreases (or increases) due to depreciation (or appreciation) in the stock's price, she can maintain her overall risk exposure by raising (or reducing) the stock exposure in the pension account without incurring extra tax liability. Our model also implies that when the investor's personal portfolio is underdiversified, the diversification effect can make it optimal to hold lightly-taxed stocks in the pension account. This result has implications for the well-known asset location puzzle (i.e., the phenomenon that many investors hold stocks or mutual funds in their 401(k) accounts, see, e.g., Agnew et al. (2003) and Amromin (2003)).

Through Monte Carlo simulations, we find that participation generally leads to an increase in the pre-retirement consumption level. In the absence of employer matching and diversification benefits, this can lead to a lower wealth level at retirement and thus a lower post-retirement consumption level. In the presence of employer matching and diversification benefits, by contrast, participation can lead to a higher wealth level at retirement and a higher post-retirement consumption level.

Next, we examine the investor's subjective valuation of participation. To quantify this value, we compare the indirect utility levels of two otherwise identical investors, assuming that one participates in the DCP and the other does not. We demonstrate that while the optimal investment decisions in the ordinary and pension accounts are inseparable, the dependence of the value function on the wealth levels in these two accounts is separable. This feature enables us to efficiently calculate the value of participation. Moreover, we decompose the total value of participation into four components: (i) the value of deferring taxes on the contributed labor income; (ii) the value of receiving the employer match; (iii) the value of capital gains tax exemption and dividend tax deferral in the pension account; and (iv) the value of portfolio diversification (zero in the FD case). The first and second components are expressed in a preference-free manner, while the third and fourth depend on the investor's risk preference.

In reality, investors do not necessarily trade optimally. Motivated by this, we examine the investor's subjective valuation of participation when she does not follow the optimal strategies. First, motivated by the well-documented asset location puzzle, we examine the impact of suboptimal asset location decisions. For this purpose, we focus on the FD case. In this case, the strict pecking-order rule (i.e., holding the heavily taxed assets only in the tax-deferred account) is close to optimal. Under the baseline parameterizations, adopting the strict pecking-order rule results in a loss of only 0.24% in the value of participation. In contrast, if the investor transfers all her optimal stock exposure to the pension account, she will lose about 6.72% of this value. These results indicate that it is important to adopt optimal asset location strategy to maximize the value of participation.

Second, it is empirically found that many individual investors are not sophisticated in tax-timing (see, e.g., Odean (1998) and Dhar and Zhu (2006)). Motivated by this, we examine the investor's subjective valuation of participation when she never defers capital gains taxes in the ordinary account. We find that non-deferral of capital gains taxes has only a small impact on the value, which indicates the robustness of our results with respect to the investor's capital gains tax-timing skill. Interestingly, in the UD case, the subjective value is in fact slightly higher if the investor does not defer capital gains taxes. The intuition is that participation leads to a lower optimal stock investment in the ordinary account, which reduces the cost of exhibiting tax-timing naiveté and translates into a higher value of participation.

This paper is closely related to the literature on optimal asset location and allocation decisions with both taxable and tax-deferred investments, such as Dammon et al. (2004), Shoven and Sialm (2004), Garlappi and Huang (2006), Huang (2008), Gomes et al. (2009), and Fischer and Gallmeyer (2017). Dammon et al. (2004) and Huang (2008) show that the pecking-order rule (i.e., only holding heavily taxed assets in the tax-deferred account) is optimal. However, empirical evidence shows that many investors allocate significant portions of their retirement saving portfolios to equity, violating the pecking-order rule (see, e.g., Agnew et al. (2003), Poterba and Samwick (2003), Bergstresser and Poterba (2004), Barber and Odean (2004), and Amromin (2003)), a phenomenon known as the asset location puzzle. Some recent studies have attempted to explain the asset location puzzle, including Amromin (2003) by identifying the impact of liquidity needs and labor income shocks, Shoven and Sialm (2004) by taking into consideration the tax-exempt municipal bonds and the tax-inefficient stock portfolios, Garlappi and Huang (2006) by examining borrowing and short-selling constraints, and Fischer and Gallmeyer (2017) by considering the limited use of capital losses. We contribute to this literature by showing that the prevailing under-diversification of individual investors and the potential diversification effect of the tax-deferred investment can help explain the asset location puzzle. Broadly speaking, this work is also related to the extensive literature on optimal portfolio choice with capital gains tax, including Constantinides (1983, 1984), Dammon et al. (2001), DeMiguel and Uppal (2005), Gallmeyer et al. (2006), Ben Tahar et al. (2007, 2010), Marekwica (2012), Dai et al. (2015), Fischer and Gallmeyer (2016), Fischer and Gallmeyer (2017), Cai et al. (2018), and Lei et al. (2020).

Our study also contributes to the literature on the valuation of tax-deferred investments. In a fully deterministic setting, Poterba (2004) proposes the concept of equivalent taxable wealth to measure the value of assets in the tax-deferred account. Poterba et al. (2005) calculate the certainty equivalent wealth of three specific allocation strategies in the tax-deferred account under uncertainty given non-stochastic taxable wealth over the life cycle. In a two-period model, Garlappi and Huang (2006) introduce the effective tax subsidy per dollar in the tax-deferred account, aiming to explain the asset location puzzle through the motive to smooth the volatility of tax subsidy. Huang (2008) extends the concept of equivalent taxable wealth to a multi-period setting under uncertainty, in which the risky assets in the taxable and tax-deferred account are common, their prices follow binomial processes, and the investor is allowed to borrow and short sell in the taxable account. By comparison, our indifference valuation approach allows us to address differential investment opportunities in the taxable and tax-deferred account and to incorporate short-selling constraints which are fairly binding for individual investors.

The utility indifference approach is commonly applied in valuation problems with incomplete markets. Hodges and Neuberger (1989) propose this approach to solve the option pricing problem with transaction costs. In the area of derivatives pricing, this approach is further developed by a series of studies, including Davis et al. (1993), Barles and Soner (1998), Constantinides and Zariphopoulou (1999, 2001), Zariphopoulou (2001), Henderson (2002), Musiela and Zariphopoulou (2004a,b,c), Wu and Dai (2009), and Henderson and Liang (2014, 2016). Kahl et al. (2003) use the indifference principle to examine the valuation of equity-based compensation schemes with lockup restrictions. Sorensen et al. (2014) apply this approach to study the valuation of private equity that may involve long-term liquidity restriction and management fees. This paper contributes to this strand of literature by extending the approach to the valuation of the DCP participation option. Unlike prior studies, our model focuses on an important market friction–capital gains tax. The presence of capital gains tax renders the asset market incomplete, and the opportunity to invest with a tax-deferred pension account provides a way to mitigate the impact of such market friction.

The rest of this paper is organized as follows. In Sect. 2, we propose our theoretical framework.

In Sect. 3, we present some theoretical results. In Sect. 4 and 5, we conduct a numerical analysis of the model's quantitative implications. In Sect. 6, we discuss the impact of introducing additional penalty for the distribution at retirement. We conclude in Sect. 7. Proofs, technical details, and some additional results are relegated to the Appendix.

2 The Model

2.1 Economic Setting

Throughout this paper, we work with a complete probability space $(\Omega, \mathscr{F}, P)$, on which a standard two-dimensional Brownian motion $\{B_t\}_{t\geq 0} = \{(B_{1t}, B_{2t})\}_{t\geq 0}$ is defined. The Brownian motion generates a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. We consider the portfolio and consumption choice of an investor who earns labor income until the retirement time T_1. We assume the investor lives until a deterministic time T_2 with $T_1 < T_2 < \infty$. In addition, when being employed, she may participate in a defined contribution plan (DCP) sponsored by her employer.

With the participation of the DCP, the investor holds two investment accounts: one is an ordinary account, the investments in which are taxed in ordinary ways; the other is a pension account associated with the DCP, where the investments are tax-sheltered. In reality, early withdrawals from the pension account are often subject to a severe penalty (for example, the IRS stipulates that withdrawals prior to age 59.5 may be taxed with an extra rate of 10%). Accordingly, we assume the wealth in the pension account is locked up until the retirement time T_1.

In both accounts, the investor can allocate her wealth to a risk-free asset ("bond" hereafter), which offers a pre-tax interest rate of $r \geq 0$, and a risky asset ("stock" hereafter). Since most 401(k) contributions are invested in mutual funds, we let the stock in the pension account, denoted as S_2, represent a diversified stock portfolio. The stock in the ordinary account, denoted as S_1, can be either identical to or different from S_2; the former (the latter resp.) case is used to model the scenario in which the investor holds a diversified (under-diversified resp.) personal portfolio. We assume the ex-dividend returns of these two stocks have the following dynamics:

$$\begin{aligned}\frac{dS_{1t}}{S_{1t}} =& \mu_1 dt + \sigma_1 dB_{1t},\\ \frac{dS_{2t}}{S_{2t}} =& \mu_2 dt + \sigma_2(\rho dB_{1t} + \sqrt{1-\rho^2} dB_{2t}),\end{aligned}$$

where μ_j and σ_j are the constant expected return and volatility of S_j, respectively, for $j = 1,2$; and $\rho \in (-1,1]$ is the correlation between the two stocks' returns. We assume S_j continuously pays dividends at a rate of $q_j \geq 0$ for $j = 1,2$. When $\rho = 1$, we require $\mu_1 = \mu_2$, $q_1 = q_2$, and $\sigma_1 = \sigma_2$ so that the two stocks are identical.

We assume that the investor's labor income is a deterministic function of time, denoted as $L(t)$ for $t \in [0, T_1]$. We do not specify the functional form of $L(t)$ at this moment since our analysis applies to general labor income function.

Participants of DCPs are allowed to periodically contribute a proportion of their wages to the pension accounts. We assume that the contribution amount is also a deterministic function of time, denoted by $L_C(t)$. We further assume that the employer matches the investor's contribution at a constant rate $\alpha \geq 0$. Hence, the total cash injection rate to the pension account is $(1+\alpha)L_C(t)$.

Remark 2.1. $L_C(t)$ is not a control variable in our model. In practice, the amount of contribution is capped above; it is not difficult to show that in our model, it is optimal to contribute the maximum amount allowed.

In the ordinary account, any dividend distributions from the stock, interest offered by the bond, and labor income are taxed immediately at rates τ_d, τ_i, and $\tau_L \in [0,1)$, respectively. Those in the pension account, however, are tax-deferred. Any realized capital gains in the ordinary account are taxed at a rate of $\tau_g \in [0,1)$, while those in the pension account are tax-exempted. Moreover, we assume that any capital losses in the ordinary account are fully rebatable at the same rate τ_g.

Throughout this study, we assume that both stocks have positive risk-adjusted returns, that is,

$$a_1 := \frac{\mu_1 + q_1(1-\tau_d) - r(1-\tau_i)}{\sigma_1} > 0 \text{ and } a_2 := \frac{\mu_2 + q_2 - r}{\sigma_2} > 0.$$

This ensures that short selling is not optimal when the investor invests in only one of the two stocks. Moreover, we assume

$$\max\{a_1 - \rho a_2, a_2 - \rho a_1\} > 0$$

to ensure the uniqueness of optimal consumption-investment strategy in a capital gains tax-free market.

In the ordinary account, we denote by $\{x_t\}_{t\in[0,T_2]}$ the dollar amount invested in the bond, $\{y_t\}_{t\in[0,T_2]}$ the dollar amount invested in the stock S_1, and $\{k_t\}_{t\in[0,T_2]}$ the purchasing cost of the current stock position. The net wealth level in the ordinary account is

$$f(x_t, y_t, k_t) = x_t + (1-\tau_g)y_t + \tau_g k_t. \tag{2.1}$$

The trading process of the stock in the ordinary account can be characterized by two non-decreasing, right-continuous with left limits, and $\{\mathscr{F}_t\}$-adapted processes $\{I_t\}_{t\in[0,T_2]}$ and $\{D_t\}_{t\in[0,T_2]}$ with $I_{0-} = D_{0-} = 0$. More specifically, $dI_t \geq 0$ is the dollar amount of newly purchased stock shares, and $dD_t \in [0,1]$ is the fraction of the taxable stock position sold. In the pension account, we denote as $\{z_t\}_{t\in[0,T_2]}$ the wealth level and $\{\xi_t\}_{t\in[0,T_2]}$ the dollar amount invested in the stock S_2. The investor's consumption rate is denoted as $\{C_t\}_{t\in[0,T_2]}$, which is an $\{\mathscr{F}_t\}$-adapted process satisfying $\int_0^{T_2} |C_t| dt < +\infty$ almost surely.

During the pre-retirement period (i.e., $t \in [0, T_1)$), we have the following dynamics:

$$\begin{cases} dx_t = r(1-\tau_i)x_{t-}dt - C_t dt - dI_t + f(0, y_{t-}, k_{t-})dD_t \\ \qquad + (1-\tau_L)\big(L(t) - L_C(t)\big)dt + q_1(1-\tau_d)y_{t-}dt, \\ dy_t = \mu_1 y_{t-}dt + \sigma_1 y_{t-}dB_{1t} + dI_t - y_{t-}dD_t, \\ dk_t = dI_t - k_{t-}dD_t, \\ dz_t = \big(rz_t + \xi_t(\mu_2 + q_2 - r)\big)dt + \xi_t\sigma_2(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}) \\ \qquad + (1+\alpha)L_C(t)dt. \end{cases} \tag{2.2}$$

To specify the dynamics during the post-retirement period (i.e., $t \in [T_1, T_2]$), we need to specify how the wealth in the pension account is distributed after retirement. In practice, lump-sum distribution and annuity distribution are both popular. For ease of exposition, we assume a lump-sum distribution at the retirement time T_1. Since such a distribution is taxed as ordinary income (at rate τ_L), we have

$$x_{T_1} = x_{T_1-} + (1-\tau_L)z_{T_1-}. \tag{2.3}$$

Then, we have the following wealth dynamics for $t \in [T_1, T_2)$:

$$\begin{cases} dx_t = r(1-\tau_i)x_{t-}dt - C_t dt - dI_t + f(0, y_{t-}, k_{t-})dD_t + q_1(1-\tau_d)y_{t-}dt, \\ dy_t = \mu_1 y_{t-}dt + \sigma_1 y_{t-}dB_{1t} + dI_t - y_{t-}dD_t, \\ dk_t = dI_t - k_{t-}dD_t, \end{cases} \tag{2.4}$$

with $x_{T_2} = x_{T_2-}$, $y_{T_2} = y_{T_2-}$, and $k_{T_2} = k_{T_2-}$. For notational convenience, we set $\xi_t \equiv 0$ and $z_t \equiv 0$ for $t \in [T_1, T_2]$.

2.2 Discussions of the Model

2.2.1 Deterministic Death Time

In our baseline model, we assume the investor lives up to a deterministic death time T_2. Since in our model, the wealth in the pension account is assumed to be distributed at the retirement time T_1 as a lump-sum payment, the impacts of DCP participation on the investor's portfolio and consumption choice primarily focus on the pre-retirement period. Thus, the main limitation of the deterministic death time assumption is the exclusion of the possibility of dying before retirement, in which case the investor is unable to utilize the distributed pensions. Nevertheless, we argue that this assumption is unlikely consequential. On the one hand, according to the data published by the World Bank in 2022, 77% (86% resp.) of the males (females resp.) in the U.S. can survive to age 65 (see https://data.worldbank.org/indicator/SP.DYN.TO65.MA.ZS?locations=US). In addition, even if participants die before retirement, their family members are often eligible for the benefits offered by the DCP. These facts suggest that the impact of ignoring the pre-retirement death is likely small. On the other hand, since our model has a relatively large number of state variables, simplifying the model through dimension reduction is crucial for achieving numerical tractability. With a random death time, the homogeneity property (Proposition 3.5) would no longer hold, and dimension reduction will become infeasible. This will significantly increase the difficulty of theoretically analyzing and numerically solving the model.

2.2.2 Timing of Taxing Contributions

In our baseline specification, we assume contributions to the pension account are taxed at the time of withdrawal. This rule applies to regular IRA plans. In the case of Roth-type retirement plans, the income taxes are paid at the time of contribution. To apply our model to this case, one only needs to change equations (2.2) and (2.3) to

$$
\begin{aligned}
dz_t = & \big(rz_t + \xi_t(\mu_2 + q_2 - r)\big)dt + \xi_t\sigma_2(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}) \\
& + (1-\tau_L)(1+\alpha)L_C(t)dt,
\end{aligned}
$$

and $x_{T_1} = x_{T_1-} + z_{T_1-}$ respectively.

2.2.3 Tax Consequences of Lump-sum Distribution

In practice, individuals may move into higher tax brackets if they make large withdrawals from pension accounts, resulting in increased marginal income tax rates. Consequently, the lump-sum distribution assumed in our model may also be subject to such penalties. In our baseline model, we ignore this potential effect, as is common in the literature (e.g., Dammon et al. (2004)). In Sect. 6, we discuss the impacts of incorporating this effect.

2.3 The Investor's Problem

In this study, we assume the investor exhibits constant absolute risk aversion (CARA) preference:

$$
U(v) = -e^{-\gamma v},
$$

where $\gamma > 0$ is her absolute risk aversion coefficient. For any consumption-investment strategy $\Theta = (C, I, D, \xi)$ and $t \in [0, T_2]$, we denote as $\{(x_s, y_s, k_s, z_s)\}_{s\in[t,T_2]}$ the future values of state variables with $(x_t, y_t, k_t, z_t) = (x, y, k, z)$ by following strategy Θ from time t on. As a convention, we set $z_t = 0$ for $t \in [T_1, T_2]$. Because individual investors rarely short sell stocks, we assume

$$
y_s \geq 0, k_s \geq 0, \xi_s \geq 0, \tag{2.5}
$$

for $s \in [t, T_2]$. A strategy Θ is admissible if the constraint (2.5) is satisfied. We denote as $\mathscr{A}_t(x,y,k,z)$ ($\mathscr{A}_t(x,y,k)$ resp.) the set of all admissible strategies for state (x,y,k,z) ((x,y,k) resp.) starting from $t \in [0,T_1)$ ($t \in [T_1,T_2]$ resp.).

Remark 2.2. We do not need the conventional constraint (7) in Lo et al. (2004) or (5) in Liu (2004) to prevent the investor from engaging in Ponzi schemes since the investor's bequest incentive can already prevent her from unlimited consumption.

The investor's objective is to maximize the expected utility derived from life-time consumption plus bequest, that is,

$$\max_{\Theta \in \mathscr{A}_0(x,y,k,0)} E_0^{x,y,k,0}\left[\omega \int_0^{T_2} e^{-\beta s} U(C_s)ds + (1-\omega)e^{-\beta T_2} U\big(G(x_{T_2}, y_{T_2}, k_{T_2}; \iota)\big)\right], \tag{2.6}$$

where β is the time discounting factor; $E_t^{x,y,k,z}$ is the expectation conditional on $(x_t, y_t, k_t, z_t) = (x,y,k,z)$; $\omega \in (0,1)$ measures the importance of life-time consumption relative to bequest;

$$G(x_{T_2}, y_{T_2}, k_{T_2}; \iota) = f(x_{T_2}, y_{T_2}, k_{T_2}) + \iota \tau_g (y_{T_2} - k_{T_2})^+$$

is the net wealth level at the death time; and $\iota \in \{0,1\}$ with $\iota = 1$ if the taxes on inherited wealth were waived and $\iota = 0$ if otherwise.

For $(x,y,k,z) \in \mathbb{R}^4$ with $y \geq 0$ and $k \geq 0$, denote as $V(x,y,k,z,t)$ the value function during the pre-retirement period $t \in [0,T_1)$:

$$\begin{aligned} V(x,y,k,z,t) := \max_{\Theta \in \mathscr{A}_t(x,y,k,z)} E_t^{x,y,k,z}\Bigg[&\omega \int_t^{T_2} e^{-\beta(s-t)} U(C_s)ds \\ &+ (1-\omega)e^{-\beta(T_2-t)} U\big(G(x_{T_2}, y_{T_2}, k_{T_2}; \iota)\big)\Bigg] \end{aligned} \tag{2.7}$$

and $V^R(x,y,k,t)$ the value function during the post-retirement period $t \in [T_1, T_2]$:

$$V^R(x,y,k,t) := \max_{\Theta \in \mathscr{A}_t(x,y,k)} E_t^{x,y,k} \left[\omega \int_t^{T_2} e^{-\beta(s-t)} U(C_s) ds \right.$$
$$\left. + (1-\omega) e^{-\beta(T_2-t)} U\big(G(x_{T_2}, y_{T_2}, k_{T_2}; \iota)\big) \right]. \tag{2.8}$$

Remark 2.3. In subsequent analyses, the value functions in a capital gains tax-free market (i.e., with $\tau_g = 0$) will be utilized. When the capital gains tax is absent, we can use the wealth level in the ordinary account (i.e., $w_t := x_t + y_t$) as a combined state variable and eliminate the state variable k_t. We denote $\bar{V}(w,z,t)$ ($\bar{V}^R(w,t)$ resp.) the corresponding value function before (after resp.) retirement. Their formal definitions are provided in (A.1) and (A.2) in Appendix A.

3 Theoretical Analysis

In this section, we present some analytical results derived from our model.

3.1 No Capital Gains Tax Case

We begin by presenting the solution to the case without capital gains taxes. The details and proof of this solution can be found in Appendix A. For later use, let

$$\begin{aligned} p_1(t) &= \int_t^{T_2} e^{r(1-\tau_i)(T_2-s)} ds = \frac{e^{r(1-\tau_i)(T_2-t)} - 1}{r(1-\tau_i)}, \\ p(t) &= \frac{e^{r(1-\tau_i)(T_2-t)}}{p_1(t)+1}, \qquad (3.1) \\ g(t) &= \frac{(1-\tau_L) e^{r(1-\tau_i)(T_2-T_1)+r(T_1-t)}}{p_1(t)+1} \qquad (3.2) \end{aligned}$$

for $t \in [0, T_2]$; and let

$$f_1(t) = (1 - \tau_L) \int_{t \wedge T_1}^{T_1} (L(s) - L_C(s)) e^{-r(1-\tau_i)(s-t)} ds \tag{3.3}$$

be the time-t value of future labor income (post-tax and post-contribution) in the ordinary account, and

$$f_2(t) = (1 + \alpha) \int_{t \wedge T_1}^{T_1} L_C(s) e^{-r(s-t)} ds \tag{3.4}$$

be the time-t value of future contributions (pre-tax) in the pension account. Then, we have the following result:

Proposition 3.1. *Assume $|\rho| < 1$. In the absence of capital gains taxes, the optimal consumption-investment strategies in the pre-retirement period $[0, T_1)$ are*

$$C_t^* = p(t)\big(w_t + f_1(t)\big) + g(t)\big(z_t + f_2(t)\big) + \bar{h}(t) - \frac{1}{\gamma} \log \frac{p(t)}{\omega}, \tag{3.5}$$

$$y_t^* = \begin{cases} \frac{\max\{a_1 - \rho a_2, 0\}}{(1-\rho^2)\gamma\sigma_1 p(t)} & \text{if } a_2 - \rho a_1 > 0, \\ \frac{a_1}{\gamma\sigma_1 p(t)} & \text{if } a_2 - \rho a_1 \leq 0, \end{cases} \tag{3.6}$$

$$\xi_t^* = \begin{cases} \frac{\max\{a_2 - \rho a_1, 0\}}{(1-\rho^2)\gamma\sigma_2 g(t)} & \text{if } a_1 - \rho a_2 > 0, \\ \frac{a_2}{\gamma\sigma_2 g(t)} & \text{if } a_1 - \rho a_2 \leq 0, \end{cases} \tag{3.7}$$

and those in the post-retirement period $[T_1, T_2]$ are

$$C_t^* = p(t) w_t + \bar{h}(t) - \frac{1}{\gamma} \log \frac{p(t)}{\omega}, \tag{3.8}$$

$$y_t^* = \frac{a_1}{\gamma\sigma_1 p(t)}. \tag{3.9}$$

Here, $w_t = x_t + y_t$ is the total wealth level in the ordinary account, $\bar{h}(t)$ is given by (A.3) for $t \in [0, T_2]$ in Appendix A. Moreover, the value functions in the capital gains tax-free market (i.e., $\bar{V}(w,z,t)$ and $\bar{V}^R(w,t)$) take strictly negative values.

Remark 3.2. Expressions (3.5) and (3.8) indicate that, in the absence of capital gains taxes, the optimal consumption rate is an affine function of the wealth levels in each account held by the investor. Expressions (3.6), (3.7), and (3.9) indicate that the optimal dollar exposure to the stock in each account is a deterministic function of time.

Remark 3.3. Expressions (3.6) and (3.9) indicate that in the absence of capital gains taxes, DCP participation can affect the optimal stock exposure in the ordinary account. In fact, by extending t to the entire period of $[0, T_2]$, (3.9) is also a nonparticipating investor's optimal stock exposure in the ordinary account, which may differ from (3.6). Later, we will show that this effect carries over to the case with capital gains tax.

3.2 Capital Gains Tax Case

We next turn to the case with capital gains taxes.

3.2.1 Properties of the Value Functions

We first present some properties of the value functions defined by (2.7) and (2.8). The next proposition indicates that the value functions in the capital gains tax-free market, with a certain adjustment in parameter values, are the upper bounds of the value functions V and V^R, respectively.

Proposition 3.4. *Given $(x, y, k, z) \in \mathbb{R}^4$ with $y \geq 0$ and $k \geq 0$.*

(1) Assume $\iota = 0$ and let $\tilde{q}_1 = q_1/(1-\tau_g)$. Then, we have

$$\begin{aligned} V(x,y,k,z,t) &\le \bar{V}\big(x+(1-\tau_g)y+\tau_g k, z, t; \mu_1, \tilde{q}_1\big), && t \in [0,T_1), \\ V^R(x,y,k,t) &\le \bar{V}^R\big(x+(1-\tau_g)y+\tau_g k, t; \mu_1, \tilde{q}_1\big), && t \in [T_1,T_2]. \end{aligned}$$

(2) Assume $\iota = 1$. Then, we have

$$\begin{aligned} V(x,y,k,z,t) &\le \bar{V}\big(x+y+\tau_g(k-y)^+, z, t; \mu_1, q_1\big), && t \in [0,T_1), \\ V^R(x,y,k,t) &\le \bar{V}^R\big(x+y+\tau_g(k-y)^+, t; \mu_1, q_1\big), && t \in [T_1,T_2]. \end{aligned}$$

Here, the notations $\bar{V}(w,z,t;\mu_1,q_1)$ and $\bar{V}^R(w,t;\mu_1,q_1)$ indicate the dependency of $\bar{V}$ and $\bar{V}^R$ on the parameters μ_1 and q_1.

Proof. See Appendix B.1. □

Proposition 3.4 along with the strict negativity of the functions $\bar{V}$ and $\bar{V}^R$ ensures that the value functions in the presence of capital gains taxes (i.e., V and V^R) also take strictly negative values, which is necessary for defining the value of participation later.

The following proposition characterizes the separation property of the value functions, which allows us to reduce the problem's dimensionality by two.

Proposition 3.5. *There exists a function $\tilde{h} : [0,+\infty) \times [0,+\infty) \times [0,T_2] \to \mathbb{R}$ such that for $(x,y,k,z) \in \mathbb{R}^4$ with $y \ge 0$ and $k \ge 0$,*

$$V(x,y,k,z,t) = -e^{-\gamma(p(t)(x+(1-\tau_g)y+\tau_g k+f_1(t))+g(t)(z+f_2(t))+\tilde{h}(y,k,t))}, \quad t \in [0,T_1), \tag{3.10}$$

$$V^R(x,y,k,t) = -e^{-\gamma(p(t)(x+(1-\tau_g)y+\tau_g k)+\tilde{h}(y,k,t))}, \quad t \in [T_1,T_2], \tag{3.11}$$

where the functions $p(t)$, $g(t)$, $f_1(t)$, and $f_2(t)$ are defined in (3.1), (3.2), (3.3), and (3.4), respec-

tively.

The functional form of the value functions in the above proposition follows from Lemma B.1 in Appendix B.2. From the strict negativity of the value functions V and V^R, we can infer the finiteness of the function $\tilde{h}$.

3.2.2 Value of Participation

To examine the value of participating in the DCP at the initial time, we first establish the following result, which suggests that in our baseline model participation is always beneficial to the investor:

Proposition 3.6. *For $(x,y,k) \in \mathbb{R}^3$ with $y \geq 0$ and $k \geq 0$, it holds that*

$$V(x,y,k,0,0) \geq V_0(x,y,k,0),$$

where $V_0(x,y,k,t)$ is the value function of an otherwise identical but non-participating investor.

Proof. See Appendix B.3. □

Proposition 3.6 allows us to define the value of participation as the amount of extra cash which makes a non-participating investor derive the same utility level as a participating investor, as follows:

Definition 3.7. For $(x,y,k) \in \mathbb{R}^3$ with $y \geq 0$ and $k \geq 0$, the value of participation, denoted as $\delta := \delta(x,y,k)$, is the unique solution to the following equation:

$$V(x,y,k,0,0) = V_0(x+\delta,y,k,0). \tag{3.12}$$

Remark 3.8. Similar to (3.10), it can be shown that

$$V_0(x,y,k,t) = -e^{-\gamma(p(t)(x+(1-\tau_g)y+\tau_g k+\hat{f}_1(t))+h_0(y,b,t))} \tag{3.13}$$

for some function h_0, where

$$\hat{f}_1(t) = (1-\tau_L)\int_{t\wedge T_1}^{T_1} L(s)e^{-r(1-\tau_i)(s-t)}ds$$

is the time-t value of the future labor income after tax. (3.13) directly implies the continuity of V_0 in x. The existence and uniqueness of the solution to equation (3.12) then follow from Proposition 3.6, $\lim_{x\to+\infty} V_0(x,y,k,t) = 0$, and the monotonicity and continuity of V_0 in x.

3.2.3 Characterization of Value Functions

In this section, we establish the connection between the value functions and the Hamilton-Jacobi-Bellman (HJB) equations associated with the optimal control problem. For simplicity, we present the analysis for the period $t \in [T_1, T_2]$; the analysis for $t \in [0, T_1]$ is analogous.

Fixing an arbitrary positive number $\Lambda > 0$, we first consider the control problem (2.6) with a compulsory sell condition on $y_t^2 + k_t^2 > \Lambda^2$. Put differently, we consider the case, where the investor is forced to sell the stock in her ordinary account whenever $y_t^2 + k_t^2 > \Lambda^2$ so that her after-sale position lies on the boundary $y_t^2 + k_t^2 = \Lambda^2$. We denote the corresponding optimal value function as $V^R(x,y,k,t;\Lambda)$. Due to a homogeneity property analogous to Proposition 3.5 but for the function $V^R(x,y,k,t;\Lambda)$, we set

$$\tilde{h}(y,k,t;\Lambda) := -\frac{\log(-V^R(x,y,k,t;\Lambda))}{\gamma} - p(t)f(x,y,k)$$

in the domain $\mathscr{D}_\Lambda := \{(y,k,t) \in \mathbb{R}^3 : y,k \geq 0, y^2+k^2 \leq \Lambda^2, T_1 \leq t \leq T_2\}$. Then, the HJB equation associated with $\tilde{h}(y,k,t;\Lambda)$ turns out to be (cf. e.g., Bian et al. (2021))

$$\max\{\tilde{h}_t + \mathscr{L}_0\tilde{h}, \mathscr{B}_0\tilde{h}, \mathscr{S}_0\tilde{h}\} = 0, \qquad \forall y,k > 0, y^2+k^2 < \Lambda^2, T_1 \leq t < T_2, \tag{3.14}$$

with the boundary conditions

$$\mathscr{S}_0\tilde{h} = 0, \qquad \text{when } (y,k,t) \in \Gamma'_\Lambda, \tag{3.15}$$

$$\max\{\tilde{h}_t + \mathscr{L}_0\tilde{h}, \mathscr{B}_0\tilde{h}, \mathscr{S}_0\tilde{h}\} \geq 0, \qquad \text{when } yk = 0, 0 \leq y,k < \Lambda, \tag{3.16}$$

and terminal condition

$$\tilde{h}(y,k,T_2;\Lambda) = -\frac{1}{\gamma}\log(1-\omega), \qquad \forall y,k \geq 0, y^2+k^2 \leq \Lambda^2, \tag{3.17}$$

where $\Gamma'_\Lambda := \{(y,k,t) \in \partial\mathscr{D}_\Lambda : y^2+k^2 = \Lambda^2\}$, and the operators are

$$\begin{aligned}
\mathscr{L}_0\tilde{h} :=& - p(t)\tilde{h} + \big(\mu_1 - \gamma\sigma_1^2(1-\tau_g)p(t)y\big)y\tilde{h}_y + \frac{1}{2}\sigma_1^2 y^2(\tilde{h}_{yy} - \gamma\tilde{h}_y^2) \\
&+ \big((\mu_1 - r)(1-\tau_g)y - r\tau_g k\big)p(t) - \frac{\gamma}{2}\sigma_1^2 p(t)^2(1-\tau_g)^2 y^2 \\
&+ \frac{p(t)}{\gamma}\left(\log\frac{p(t)}{\omega} - 1\right) + \frac{\beta}{\gamma}, \\
\mathscr{B}_0\tilde{h} :=& \tilde{h}_y + \tilde{h}_k, \\
\mathscr{S}_0\tilde{h} :=& - y\tilde{h}_y - k\tilde{h}_k.
\end{aligned}$$

We have the following result:

Theorem 3.9. *1. $\tilde{h}(y,k,t;\Lambda)$ is the minimal one among all the viscosity solutions to (3.14), which are locally uniformly Lipschitz-continuous in (y,k) and satisfy the boundary conditions (3.15) and (3.16), and the terminal condition (3.17).*

2. For any $t \in [T_1,T_2]$ and $(x,y,k) \in \mathbb{R}^3$ with $y,k \geq 0$, we have

$$\lim_{\Lambda\to\infty} V^R(x,y,k,t;\Lambda) = V^R(x,y,k,t).$$

Proof. See Appendix C. □

Remark 3.10. Bian et al. (2021) assume constant relative risk aversion (CRRA) utility functions $U(x) = \frac{x^\gamma}{\gamma}$ with $\gamma \in (-\infty, 1)$. By approximating the singular controls with regular controls, they establish the result that for $\gamma \in (0, 1)$, the value function is the smallest one of the infinitely many viscosity solutions to the associated HJB equation. However, their approach does not apply to our case, as it heavily relies on the finite lower bound of the utility function. Our approach involves truncating the solvency region to make both the value and the purchasing cost of the stock position remain bounded. This allows us to employ estimates on the wealth process to control the impact of the unboundedness of the utility function on the value function, which eventually enables us to establish the desired connection between the regular control and singular control value functions in our model.

3.2.4 Characterizations of Optimal Strategies

Next, we characterize the optimal consumption-investment strategy implied by our model. Thanks to Proposition 3.5, we can reduce the spatial dimensionality of the investor's optimization problem from four to two, which substantially improves the numerical tractability. Furthermore, we make the following transformation:

$$b = \frac{k}{y}, h(y, b, t) = \tilde{h}(y, k, t), \tag{3.18}$$

where $\tilde{h}$ is defined in Proposition 3.5, and b is the basis-price ratio of the stock in the ordinary account. For the sake of convenience, we refer to the function h as the reduced value function with DCP participation. The buy condition $\mathscr{B}_0\tilde{h} = 0$ and the sell condition $\mathscr{S}_0\tilde{h} = 0$ become $\mathscr{B}h = 0$

and $\mathscr{S}h = 0$, respectively, where

$$\mathscr{B}h = yh_y + (1-b)h_b, \mathscr{S}h = -h_y. \tag{3.19}$$

In what follows, we characterize the optimal consumption-investment strategy in the y-b plane.

The optimal consumption rate for $t \in [0, T_2)$ is

$$C_t^* = p(t)\big(f(x_t, y_t, b_t y_t) + f_1(t)\big) + g(t)\big(z_t + f_2(t)\big) + h(y_t, b_t, t) - \frac{1}{\gamma}\log\frac{p(t)}{\omega}. \tag{3.20}$$

Here, the function f is the net wealth level in the ordinary account, as defined by Equation (2.1).

The optimal dollar exposure to the stock S_2 in the pension account for $t \in [0, T_1)$ is

$$\xi_t^* = \left(\frac{a_2 + \gamma\rho\sigma_1\big(b_t h_b(y_t, b_t, t) - y_t h_y(y_t, b_t, t) - (1-\tau_g)p(t)y_t\big)}{\gamma\sigma_2 g(t)}\right)^+. \tag{3.21}$$

Remark 3.11. Equation (3.21) implies that the states in the ordinary account (i.e., y and b) have an impact on the optimal risky investment in the pension account. Therefore, the optimal trading strategies in the ordinary account and the pension accounts are inseparable.

Next, we describe the optimal trading strategy in the ordinary account. Because capital losses are assumed to be fully rebatable, when $b > 1$ (i.e., with capital losses in the ordinary account), wash-sale is optimal (see Appendix B.4). For $0 \le b \le 1$ (i.e., with capital gains in the ordinary account), we can define (and identify from the numerical solution) the sell region (SR), the buy region (BR), and the no-trade region (NR) as follows:

$$\begin{aligned}
SR =& \{(y,b,t) : 0 \le b \le 1, \mathscr{S}h = 0\}, \\
BR =& \{(y,b,t) : 0 \le b \le 1, \mathscr{B}h = 0\}, \\
NR =& \{(y,b,t) : 0 \le b \le 1, \mathscr{B}h < 0 < -\mathscr{S}h\}.
\end{aligned}$$

It is optimal to sell (buy resp.) the stock when the investor's position lies in SR (BR resp.) and to take no action in NR. We conjecture and numerically verify later that there exist two functions $y^{\pm}(b,t)$ such that

$$
\begin{aligned}
SR =& \{(y,b,t) : 0 \le b \le 1, y > y^{+}(b,t)\}, \\
BR =& \{(y,b,t) : 0 \le b \le 1, y < y^{-}(b,t)\}, \\
NR =& \{(y,b,t) : 0 \le b \le 1, y^{-}(b,t) \le y \le y^{+}(b,t)\}.
\end{aligned}
$$

4 Numerical Analysis

In this section, we present a numerical analysis of the model's main outcomes. The numerical results for the optimal consumption-investment policies are obtained by applying the penalty method in conjunction with the finite-difference method Dai and Zhong (2010) to the HJB equations associated with the function $h(y,b,t)$. Meanwhile, the time evolution patterns of consumption and tax bills are calculated by using Monte Carlo simulations. Further details can be found in Appendix D.

4.1 Parameter Choices

We set the pre-tax risk-free rate to $r = 0.04$. Since the stock in the pension account (i.e., S_2) is interpreted as a diversified stock portfolio, we set its dividend rate, ex-dividend expected return, and volatility to $q_2 = 0.03$, $\mu_2 = 0.09$, and $\sigma_2 = 0.2$, respectively. Regarding the stock in the ordinary account, we consider two cases. In the "full-diversification" (FD for short) case, we assume the investment in the ordinary account is also diversified, and set the stock therein to be identical to that in the pension account (i.e., $q_1 = q_2$, $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$, and $\rho = 1$). In the "under-diversification" (UD for short) case, we assume that the stock in the ordinary account differs from that in the pension account, and set its return volatility to a higher level of $\sigma_1 = 0.3$ and its correlation with the stock in the pension account to $\rho = 0.5$; the other parameter values remain the same as in the FD case.

We assume the investor enters the labor market at age 20, retires at age 65 and will decease at age 78. Since the initial time $t = 0$ corresponds to age 20, we set the retirement time $T_1 = 65 - 20 = 45$ and the time of death $T_2 = 78 - 20 = 58$. We set the tax rates for labor income, dividends, and interests to $\tau_L = \tau_d = \tau_i = 0.3$, and the capital gains tax rate to $\tau_g = 0.225$. These tax rates approximate those faced by a representative middle income investor in the US. We set the investor's absolute risk aversion coefficient to $\gamma = 0.001$ and her time discounting factor to $\beta = 0.01$. The utility weighting parameter is set to $\omega = 0.9$.

Mincer (1958, 1974) propose a labor income function of $L(t) = L_0 e^{m_0 \tilde{s}_t + m_1 X_t + m_2 X_t^2}$, where L_0 represents the labor income of an individual without education and work experience, $\tilde{s}_t$ denotes the total years of education at time t, X_t is the total years of working experience at time t, and m_0, m_1, and m_2 are three constants. This specification captures the hump-shaped pattern exhibited by the average individual's lifetime income. We map the Mincer equation to our model by holding $\tilde{s}_t$ constant for the investor; this constant then reflects the impact of an average duration of education on the investor's labor income. Thus, we can specify the labor income function in our model as $L(t) = e^{l_0 t^2 + l_1 t + l_2}$, where l_0, l_1, and l_2 are constants. To estimate these constants, we obtain the earnings data from the US Bureau of Labor Statistics (https://www.bls.gov/news.release/wkyeng.t01.htm). Specifically, the dataset contains information on average weekly earnings for different (age, gender) combinations. For each age bracket, we first take the average of the weekly earnings of females and males, then multiply the average value by 52 to obtain the average annual earnings. Next, we approximate the annual earnings at ages 20, 40,and 60 using the estimated annual earnings for age brackets 16–24, 35–44, and 55–64, respectively. This yields annual earnings of \$25,948 at age 20, \$47,996 at age 40, and \$49,764 at

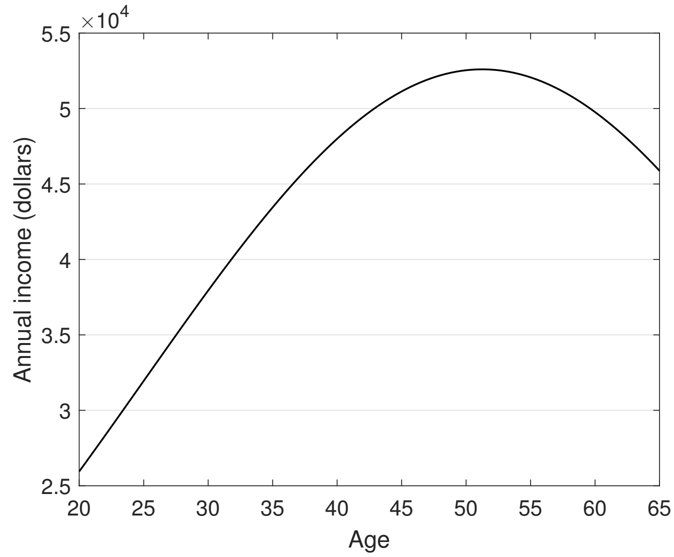


Figure 4.1: Labor Income Process
This figure shows the labor income function used in our numerical analysis. Specifically, $L(t) = e^{l_0 t^2 + l_1 t + l_2}$ with $l_0 = -0.0007$, $l_1 = 0.0452$, and $l_2 = 10.1638$, where $t = Age - 20$.

age 60. The estimated constants are $l_0 = -0.0007$, $l_1 = 0.0452$, and $l_2 = 10.1638$. We plot the corresponding labor income process in Figure 4.1.

We set the investor's pension contribution rate to 6% of the labor income, i.e., $L_C(t) = 0.06L(t)$. In the base case, consistent with typical 401(k) plans (see, e.g., Papke and Poterba (1995)), we set the employer matching ratio to $\alpha = 0.5$. For ease of reference, we summarize the default parameter values in Table 4.1.

4.2 Optimal Investment and Consumption Strategies

In this subsection, we examine the optimal investment and consumption policies implied by our model. We highlight that the optimal investment strategies in both the ordinary and the pension account are independent of labor income, investor's contribution, and employer's matching to the pension account, whereas the optimal consumption strategy is not.

Table 4.1: Default Parameter Values

This table summarizes our baseline parameter values.

Parameter	Symbol	Baseline value
Time-to-retirement	T_1	45
Time-to-death	T_2	58
Absolute risk-aversion coefficient	γ	0.001
Subjective discounting factor	β	0.01
Utility weighting parameter	ω	0.9
Tax rates for ordinary income	$\tau_L = \tau_i = \tau_d$	0.3
Tax rate for capital gains	τ_g	0.225
Labor income function	$L(t)$	$= e^{-0.0007t^2+0.0452t+10.1638}$
Contribution rate (including employer matching)	$(1+\alpha)L_C(t)$	$= (1+0.5)\times 0.06\times L(t)$
Pre-tax interest rate	r	0.04
Stock 2 (in the pension account)		
Pre-tax dividend yield	q_2	0.03
Expected return	μ_2	0.09
Return volatility	σ_2	0.2
Stock 1 (in the ordinary account)		
Pre-tax dividend yield	q_1	0.03 (FD and UD Case)
Expected return	μ_1	0.09 (FD and UD Case)
Return volatility	σ_1	0.2 (FD Case)
		0.3 (UD Case)
Correlation with stock 1	ρ	1.0 (FD Case)
		0.5 (UD Case)

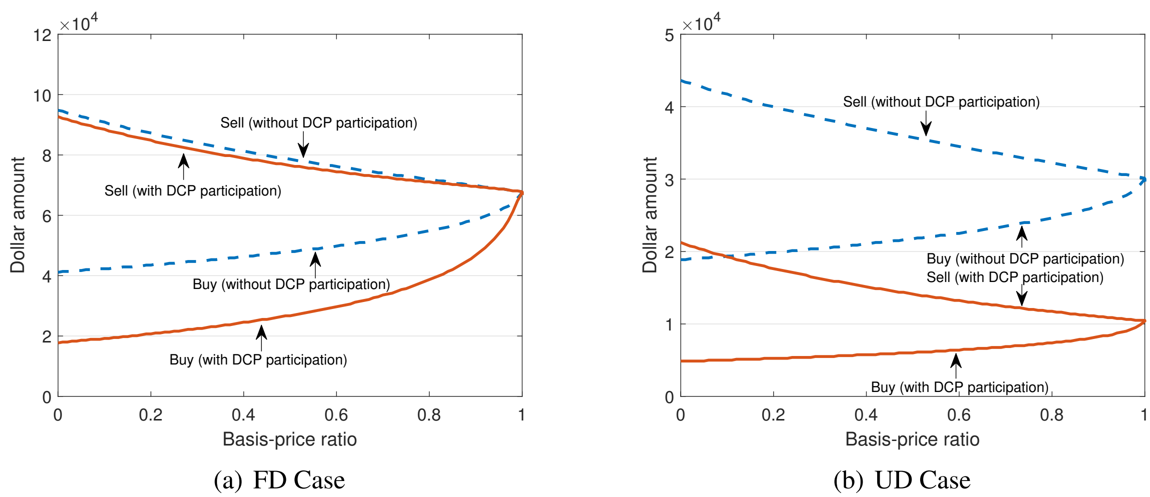


Figure 4.2: Optimal Investment Policies in the Ordinary Account
This figure shows the investor's optimal trading boundaries in the ordinary account at age 30.

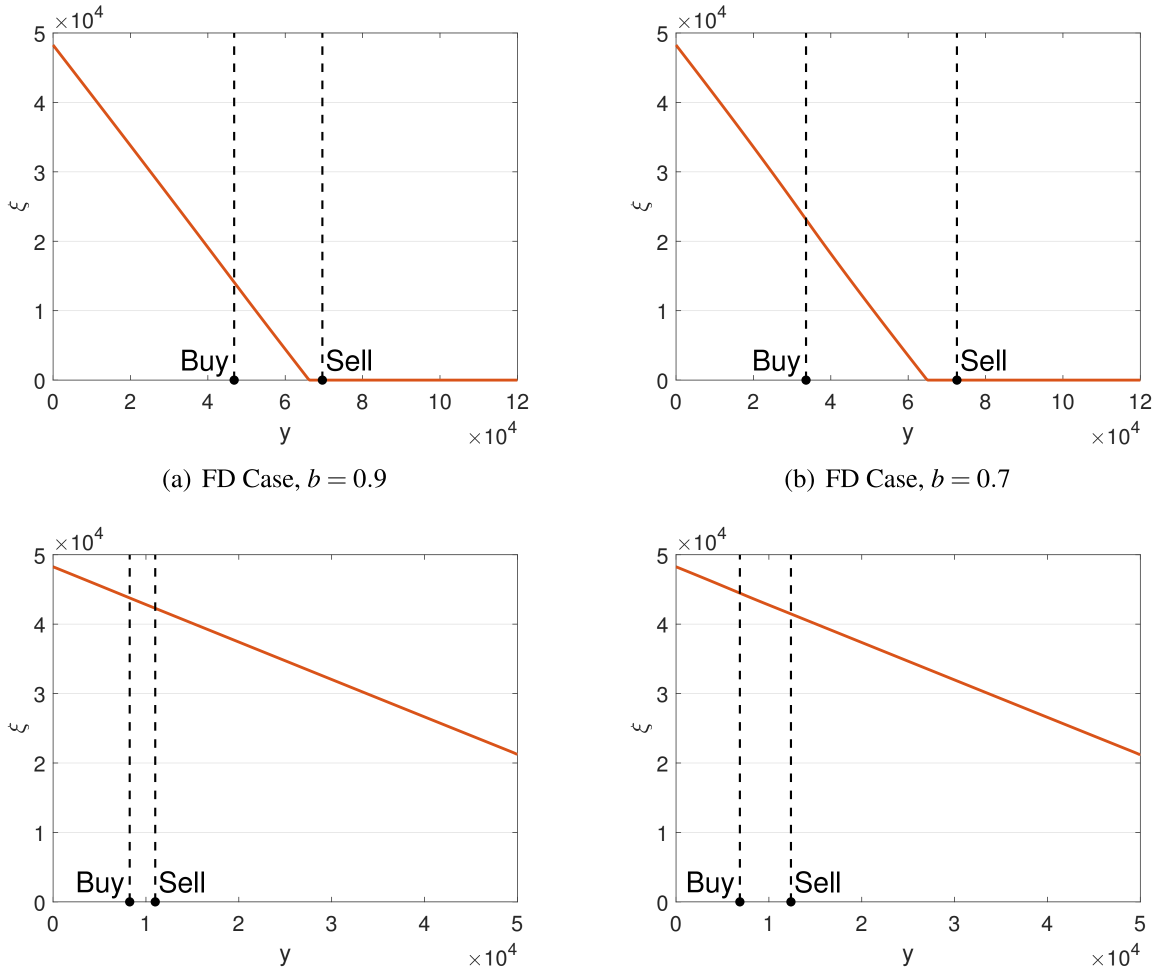
4.2.1 Optimal Investment Policies: Ordinary Account

In Figure 4.2, we show the optimal investment strategy in the ordinary account. Due to the presence of capital gains tax, the investor has an incentive to defer capital gains realizations. This incentive is revealed by the existence of a no-trade region, which is delimited by a buy boundary and a sell boundary. Specifically, when the investor has capital gains, it is optimal to sell shares only when the dollar value of the stock position exceeds the sell boundary and to buy shares only when the dollar value drops below the buy boundary. When the dollar exposure lies in the no-trade region, it is optimal to stay inactive because the benefit from rebalancing risk exposure (by realizing some capital gains) is insufficient to compensate for the loss on the time value of taxes.

When the investor has capital losses in her ordinary account, realizing them immediately and then rebuilding stock position through purchasing (i.e., wash-sale) is optimal since it is assumed that these losses are fully rebatable (see Proposition B.2 in Appendix B.4).

Note that the solid (dashed resp.) lines in Figure 4.2 show the optimal trading boundaries when the agent invests with (without resp.) the pension account. In the FD case as shown in Figure 4.2(a), if the investor invests using the pension account, both the Buy and the Sell boundaries move downward, indicating lower stock allocation in the ordinary account. This is because in this case, it can be optimal to put some stock investments in the pension account to enjoy the more favorable tax treatment therein. In addition, compared to the Sell boundary, the Buy boundary shifts downward more significantly. The intuition is that if the investor's total stock exposure drops because of declining stock price, it could be optimal to increase stock allocation in the pension account rather than in the ordinary account, because it allows the investor to enjoy greater tax benefits in the future as the stock price rebounds.

Figure 4.2(b) illustrates the optimal investment policy in the ordinary account obtained in the UD case. In this scenario, the optimal stock exposure in the ordinary account is significantly influenced by DCP participation. Because we assume a positive (but imperfect) correlation between



(a) FD Case, $b = 0.9$

(b) FD Case, $b = 0.7$

(c) UD Case, $b = 0.9$

(d) UD Case, $b = 0.7$

Figure 4.3: Optimal Investment Policies in the Pension Account
This figure shows, for two basis–price ratios in the ordinary account, the investor's optimal stock exposure in the pension account at age 30.

the two stocks' returns in the UD case, diversifying the portfolio with the stock in the pension account results in a lower optimal stock exposure in the ordinary account. Consequently, both the buy boundary and the sell boundary shift downward significantly.

4.2.2 Optimal Investment Policies: Pension Account

In Figure 4.3, we show the optimal dollar exposure to the stock in the pension account, at age 30, against the dollar exposure to the stock in the ordinary account (i.e., y). We show the result for two basis–price ratios (i.e., b): 0.9 and 0.7, respectively. The optimal trading boundaries in the ordinary account are also marked in the figures.

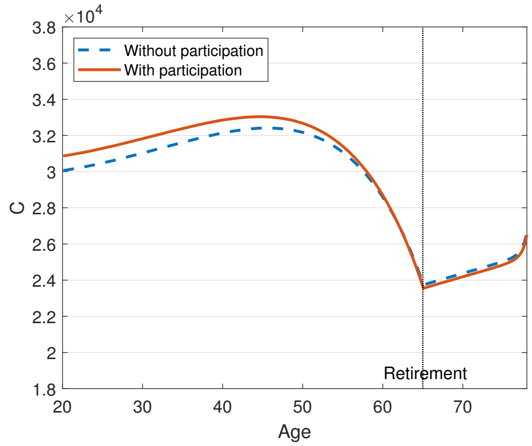
Figure 4.3 has several implications. First, as shown in Figures 4.3(a) and 4.3(b), it can be optimal to hold some stocks in the pension account even in the FD case. This is because holding stocks in the pension account allows the investor to enjoy the favorable tax treatment on both stock dividends and capital gains. Therefore, it can be optimal to violate the well-known pecking-order location rule. Nonetheless, in the FD case, the optimal stock exposure in the pension account is small relative to that in the ordinary account, because the investor has a strong incentive to allocate bonds in the pension account due to heavier taxes on interest earned in the ordinary account. By comparison, Figures 4.3(c) and 4.3(d) show that in the UD case, the stock position in the pension account can be more sizable due to the diversification effect.

Second, Figure 4.3 indicates a negative relation between the optimal stock exposures in the ordinary and in the pension account. This pattern appears because trading the stock in the pension account is tax-free; hence, if the investor's stock exposure in the ordinary account decreases (increases resp.) due to depreciation (appreciation resp.) in stock price, she can adjust her overall stock exposure by increasing (decreasing resp.) stock exposure in the pension account without assuming tax liability. In this case, the tax benefit dominates the cost of unbalanced stock exposures in the two accounts.

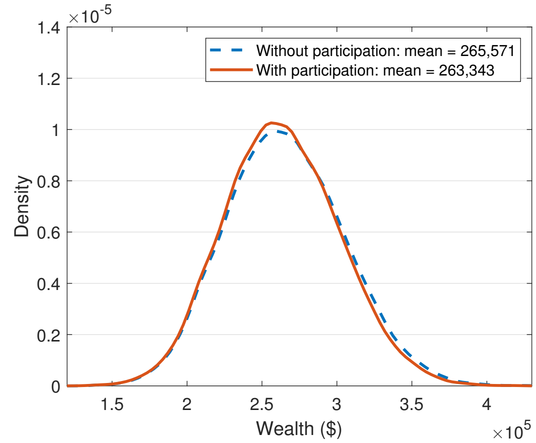
4.2.3 Optimal Consumption Policies

The investor's optimal consumption rate depends on her capital gains status, risk exposure, wealth level in the two accounts, labor income, and contributions (including employer matching) to the pension account. In Figure 4.4, we show the time evolution of the average consumption rate, and the distribution of the total wealth at retirement, calculated by performing Monte Carlo simulations.

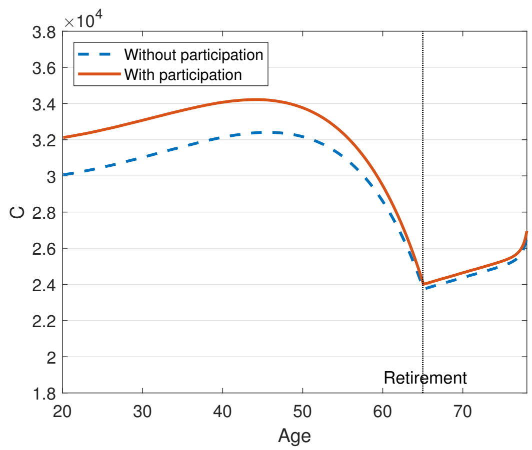
We can make some general observations from Figure 4.4. First, the investor's consumption rate



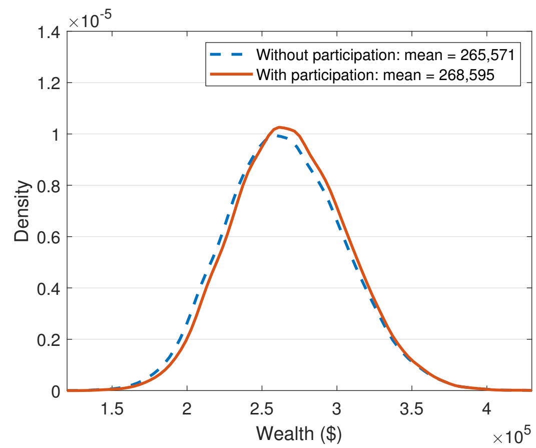
(a) Consumption Rate, FD Case without Matching



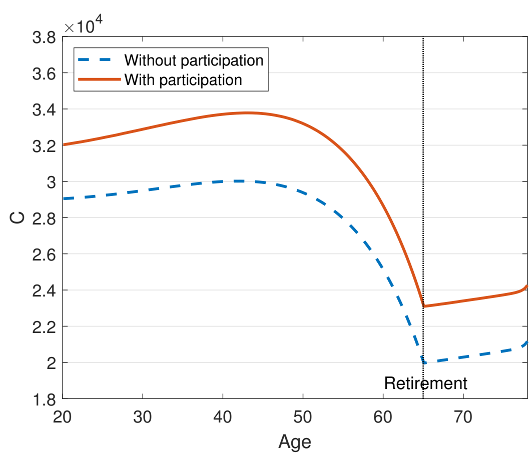
(b) Retirement Wealth, FD Case without Matching



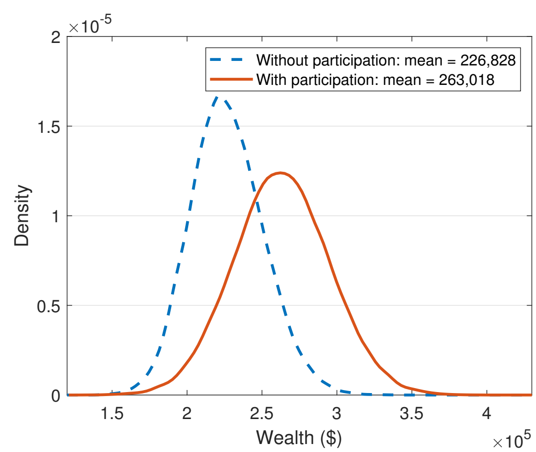
(c) Consumption Rate, FD Case with Matching



(d) Retirement Wealth, FD Case with Matching



(e) Consumption Rate, UD Case with Matching



(f) Retirement Wealth, UD Case with Matching

Figure 4.4: Life-time Consumption Path and Retirement Wealth
The left subfigures show the time evolution of the investor's average consumption rate, and the right subfigures show the distribution of the investor's wealth level at retirement. We set $x_{0-} = y_{0-} = k_{0-} = z_{0-} = 0$ when generating the results.

decreases when approaching the retirement time, which is consistent with empirical patterns of the individuals' life-cycle consumption profiles (see, e.g., Carroll and Summers (1991), Attanasio and Weber (2010), and Pagel (2017)). This is because prior to retirement, the investor significantly lowers future income expectation and decreases consumption as a response. Second, participation leads to a higher pre-retirement consumption level. Intuitively, during the pre-retirement period, the participating investor raises her consumption level because she anticipates faster wealth growth due to the tax benefits offered in the pension account.

Figures 4.4(a) and 4.4(b) illustrate that in the FD case without employer matching, participation may result in a lower wealth level at retirement and a reduced post-retirement consumption level. This is due to the high pre-retirement consumption level chosen by a participating investor. In contrast, Figures 4.4(c) and 4.4(d) show that with employer matching, the participating investor's retirement wealth level and post-retirement consumption level can exceed those of the nonparticipating investor.

Next, we consider the diversification effect in the UD case. Figures 4.4(e) and 4.4(f) indicate that, compared to the FD case, in the UD case, the participating investor can achieve much higher retirement wealth and life-time consumption levels than the non-participating investor, due to the additional diversification benefit gained from participation.

4.2.4 Life-Cycle Pattern of Tax Bills

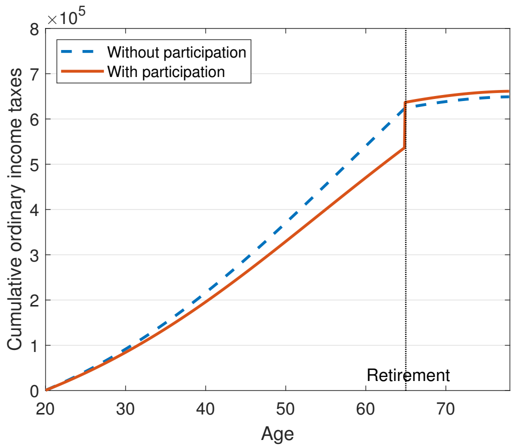
In this subsection, we examine the implications of participation for the investor's life-time tax liability.

In Figure 4.5, we show the investor's cumulative ordinary income tax bills. Before retirement, a participating investor incurs less ordinary income taxes, partly because the taxes on the contributed labor income are deferred and partly because of the reduction in bond investment, and thus the interest taxes, due to raised consumption. If we incorporate the income taxes incurred after retirement, then a participating investor eventually pays more income taxes (in nominal terms). The reason is that the participating investor will hold a significant amount of wealth in the pension account, which are taxed at the time of distribution.

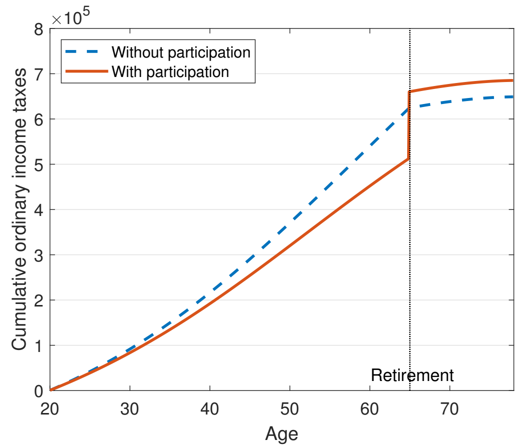
Compared to Figure 4.5(a), Figure 4.5(b) illustrates that with employer matching, fewer taxes are incurred before retirement, but more are incurred after retirement. The intuition is as follows. With a higher employer matching, before retirement, the investor increases her consumption level, leading to a reduction in her bond position and taxes on interest. Upon retirement, the investor has a higher wealth level in the pension account, leading to more ordinary income taxes paid upon distribution.

Figure 4.5(c) demonstrates a more prominent change in the ordinary tax bill at retirement in the UD case compared to in the FD case. This is because in the UD case, the stock exposure in the pension account is higher, leading to a higher wealth level in the pension account at retirement and heavier income taxes paid upon distribution.

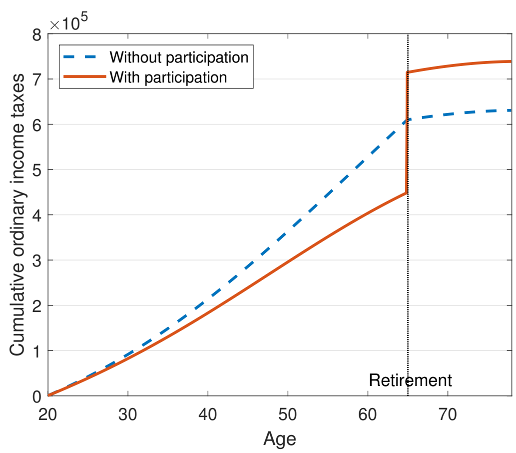
In Figure 4.6, we plot the investor's cumulative capital gains taxes incurred over the life time. Note that changes in the employer matching ratio do not affect the optimal investment policies and the capital gains taxes incurred by the investor; as a result, we do not consider this factor here. Figure 4.6(a) shows that in the FD case, a participating investor incurs fewer capital gains taxes during both the pre- and post-retirement period. This occurs because the investor reduces stock investment in her ordinary account when also investing in the pension account, thereby decreasing the amount of capital gains taxes. Nonetheless, the capital gains tax bills only decrease slightly because the equity allocation in the pension account is small. Figure 4.6(b) reveals that in the UD case, the contrast is more pronounced compared to the FD case, due to the lower stock exposure in the ordinary account.



(a) FD Case without Matching



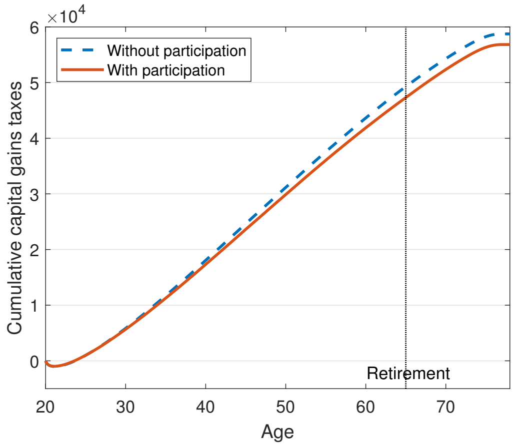
(b) FD Case with Matching



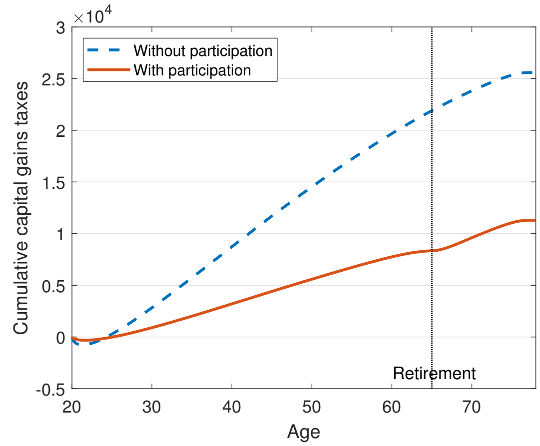
(c) UD Case with Matching

Figure 4.5: Ordinary Income Tax Bills

This figure shows the cumulative ordinary income tax bills incurred by the investor over life-cycle. We set $x_{0-} = y_{0-} = k_{0-} = z_{0-} = 0$ when generating the results.



(a) FD Case



(b) UD Case

Figure 4.6: Capital Gains Tax Bills
This figure shows the cumulative capital gains tax bills incurred by the investor over life-cycle. We set $x_{0-} = y_{0-} = k_{0-} = z_{0-} = 0$ when generating the results.

4.3 The Effect of Return Correlation

In reality, most stocks' returns are positively correlated with the stock market return; hence, we set a positive correlation parameter in the UD case. Nevertheless, it is worth discussing the impact of setting a negative correlation parameter. In this case, we find that (1) DCP participation can lead to a higher stock exposure in the ordinary account, and a participating investor may pay more capital gains taxes as a result; and (2) the optimal stock exposures in the two accounts can be positively related. We do not present these results to save space; they are available upon request.

5 Value of Participation

In this section, we perform a quantitative analysis of the economic value of participating in the DCP.

5.1 A Decomposition Formula

Due to equations (3.10) and (3.13), the value of participation defined in (3.12) can be expressed as follows:

$$\begin{aligned}\delta(x,y,k) =& \frac{h^f(y,b,0)-h_0^f(y,b,0)}{p(0)} \\ &+\left(\frac{h(y,b,0)-h_0(y,b,0)}{p(0)}-\frac{h^f(y,b,0)-h_0^f(y,b,0)}{p(0)}\right) \\ &+\left((1-\tau_L)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds-(1-\tau_L)\int_0^{T_1} L_C(s)e^{-r(1-\tau_i)s}ds\right) \\ &+\alpha(1-\tau_L)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds \\ :=& \delta_1+\delta_2+\delta_3+\delta_4,\end{aligned} \tag{5.1}$$

where $h^f(y,b,t)$ or $h_0^f(y,b,t)$ is the reduced value function with or without DCP participation, respectively, for the corresponding FD case (i.e., setting the stock in the ordinary account identical to that in the pension account).

The terms in Equation (5.1) have the following interpretations. The first term

$$\delta_1 = \frac{h^f(y,b,0)-h_0^f(y,b,0)}{p(0)} \tag{5.2}$$

measures the utility gains related to stock investment in the pension account, excluding the diversification benefit. These utility gains include the benefit of dividend tax deferral and capital gains tax exemption. The second term

$$\delta_2 = \frac{h(y,b,0)-h_0(y,b,0)}{p(0)}-\frac{h^f(y,b,0)-h_0^f(y,b,0)}{p(0)} \tag{5.3}$$

measures the diversification benefit. The third term

$$\delta_3 = (1-\tau_L)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds - (1-\tau_L)\int_0^{T_1} L_C(s)e^{-r(1-\tau_i)s}ds$$

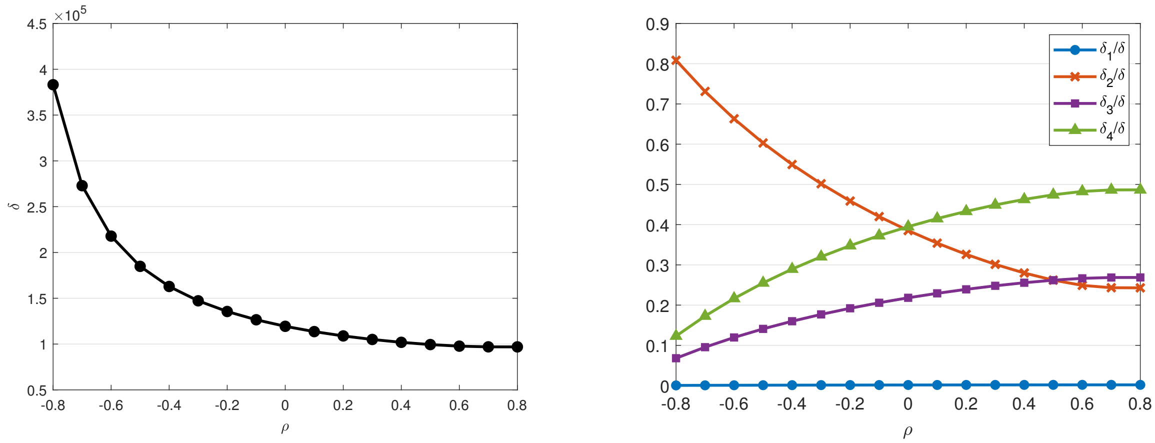
represents the value of deferring interest taxes on the investor's contribution. To understand this, note that the first part, $(1-\tau_L)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds$, measures the present value of the positive wealth shock at retirement time resulting exclusively from the investor's contribution (i.e., without considering possible capital gains or losses incurred in the pension account and employer matching); and the second part, $(1-\tau_L)\int_0^{T_1} L_C(s)e^{-r(1-\tau_i)s}ds$, is the present value of contributions if these contributions are not contributed to the pension account. Here, the factor $e^{r\tau_i T_1}$ reveals the value of deferring taxes on interest to the retirement time T_1.

The last term

$$\delta_4 = \alpha(1-\tau_L)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds$$

is the present value of employer matching.

In Figure 5.1, we show the value of participation against the correlation coefficient of the two stocks' returns, setting other parameter values as those in the UD case and the matching ratio to $\alpha = 0.5$. In the left subfigure, we plot the total value of participation (i.e., δ). The result suggests that participating in the DCP can indeed generate substantial benefits to the investor. In particular, the value of the participation option is \$96,927 when the return correlation equals 0.8, and it increases to \$119,378 or \$383,143 when the return correlation is 0 or -0.8, respectively. In the right subfigure, we show the relative importance of each contributing component (i.e., the ratios δ_1/δ, δ_2/δ, δ_3/δ, and δ_4/δ, respectively). In the reported case, the employer matching contributes 47.42%, the income tax deferral contributes 26.19%, the tax-saving on stock investment contributes a small share of 0.18%, and the diversification benefit contributes 26.21% of the total value. As



(a) Total Value of Participation

(b) Weight of Each Component

Figure 5.1: Value of Participation
This figure shows the toal value of participation (left subfigure) and its decomposition (right subfigure) in the UD case with a employer matching ratio of $\alpha = 0.5$.

the return correlation increases, the value of the stock investment in the pension account decreases due to the decreasing diversification benefit.

Remark 5.1. We find that changing the investor's risk aversion coefficient γ, discounting factor β, or utility weighting parameter ω has little impact on the value of participation. In our model, we assume full distribution of the pension wealth at the retirement time for tractability. In reality, participants can strategically withdraw funds from the pension account after retirement, which allows them to further exploit the benefit from participating. If this feature is incorporated, the value of participation should further increase.

5.2 Impact of Suboptimal Trading on the Value of Participation

In the previous section, the value of participation is calculated under the assumption that the investor adopts the optimal trading strategy in both the ordinary and the pension account. In reality, investors do not necessarily trade optimally. Motivated by this, in this section, we examine the investor's subjective valuation of participation when she does not follow the optimal strategies.

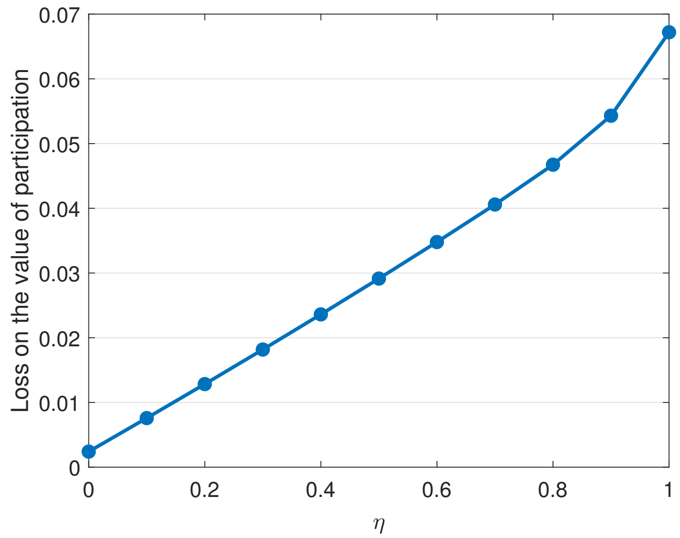


Figure 5.2: Suboptimal Equity Location and Loss on the Value
This figure shows the loss on the value of participation, as a fraction of the plan's full value, when the investor adopts suboptimal asset location rules, for the FD case with $\alpha = 0.5$. When generating the results, we assume the investor invests a dollar amount of $\eta y^*(t)$ in the stock in her pension account, where $y^*(t)$ is her optimal dollar exposure to the stock in the absence of capital gains tax.

It is empirically found that some investors over-invest in tax-light assets such as stocks in their pension accounts, which is tax-inefficient (see, e.g., Amromin (2003)). Motivated by this, we examine the impact of the investor's asset location decision on the option value. We consider the FD case, since in this case, similar to the results reported in Dammon et al. (2004), the optimal stock allocation in the pension account is small and a strict pecking-order rule is close to be optimal.

To examine the impact of adopting a suboptimal asset location strategy, we consider the following scenarios. Denote the investor's optimal stock exposure (i.e., the Merton line) in the ordinary account in the absence of capital gains tax as $y^*(t)$. Assume the investor suboptimally maintains a dollar exposure of $\eta y^*(t)$ in her pension account with some $\eta \in [0,1]$. Taking this investment strategy in the pension account as given, the investor chooses her best investment strategy in the ordinary account and consumption strategy.

In Figure 5.2, we plot the loss on the option value against the value of η. The case with $\eta = 0$ corresponds to the strict pecking-order strategy, which incurs a slight loss on the option value

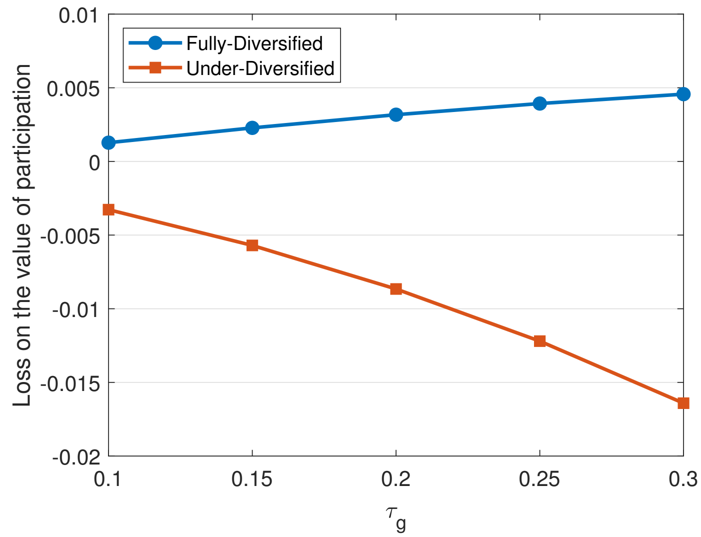


Figure 5.3: Tax-timing Skill and Value of Participation
This figure plots the percentage loss on the value of participation against the capital gains tax rate if the investor never defers capital gains taxes in the ordinary account, for both the FD case and the UD case with $\alpha = 0.5$.

of 0.24%. A greater value of η implies a larger deviation from the pecking-order rule, which is associated with a greater loss on the option value. For example, when the investor locates about 30% of her total equity exposure in the pension account, she will lose about 1.82% of the option value. If this proportion further increases to 100% (i.e., similar to the case of "indirect stockholder" in Gomes et al. (2009)), the loss can be as large as 6.72%. These results suggest that suboptimal asset location can indeed cause a significant loss on the value of participation.

Abundant empirical evidence suggests that many individual investors are not tax-sophisticated (see, e.g., Odean (1998) and Dhar and Zhu (2006)). Motivated by this, we examine how naiveté in tax-timing affects the value of participation. To fix the idea, we consider the extreme case where the investor never defers capital gains taxes in her ordinary account. In this case, the value functions are characterized by Proposition B.3 in Appendix B.5.

In Figure 5.3, we plot the proportional change in the value of participation. As can be seen, the option value is not very sensitive to non-deferral of capital gains taxes in the ordinary account.

For an investor with fully-diversified portfolio and a capital gains tax rate of 30%, non-deferral of capital gains tax only results in a decrease of 0.46% in the option value. Interestingly, in the UD case, the subjective value of participation is in fact slightly higher if she does not defer capital gains taxes, as indicated by the negative value of loss in Figure 5.3. The intuition is that participation leads to a lower optimal stock investment in the ordinary account, which reduces the cost of exhibiting tax-timing naiveté and translates into a higher value of participation.

Overall, these results indicate that the investor's subjective valuation of the participation option is not significantly affected by her tax-timing skills.

6 Further Discussions of Lump-sum Distribution

As discussed in Sect. 2.2, the lump-sum distribution of pension wealth, which is assumed in our model, may lead to an increase in the investor's ordinary tax rate. In this section, we consider this factor and replace (2.3) with

$$x_{T_1} = x_{T_1-} + (1-\tau_P) z_{T_1-},$$

where $\tau_P \geq \tau_L$ is the increased tax rate on distribution. In this case, the function $g(t)$ in (3.2) should be modified to

$$g(t) = \frac{(1-\tau_P)e^{r(1-\tau_i)(T_2-T_1)+r(T_1-t)}}{p_1(t)+1}. \tag{6.1}$$

We emphasize that with this change, the propositions and theorems in Sect. 3, except Proposition 3.6, remain valid. Proposition 3.6 may no longer hold true, because the pension wealth is eventually taxed at an increased rate and participation may not be beneficial for the investor. Next, we numerically examine the impact of this change on the investor's optimal policies and subjective valuation of the plan.

6.1 Optimal Investment and Consumption Policies

Notably, the reduced value function h defined in (3.18) is not affected by the higher ordinary tax rate on withdrawal. Consequently, the optimal investment policies in the ordinary account are also unaffected. However, this increased tax rate does have an impact on the optimal investment policies in the pension account and the optimal consumption choice.

The formulas (3.21) and (6.1) imply that the optimal investment in the pension account is

$$\xi_t^* = \left(\frac{(a_2 + \gamma\rho\sigma_1(b_t h_b(y_t, b_t, t) - y_t h_y(y_t, b_t, t) - (1-\tau_g)p(t)y_t))(p_1(t)+1)}{(1-\tau_P)\gamma\sigma_2 e^{r(1-\tau_i)(T_2-T_1)+r(T_1-t)}} \right)^+,$$

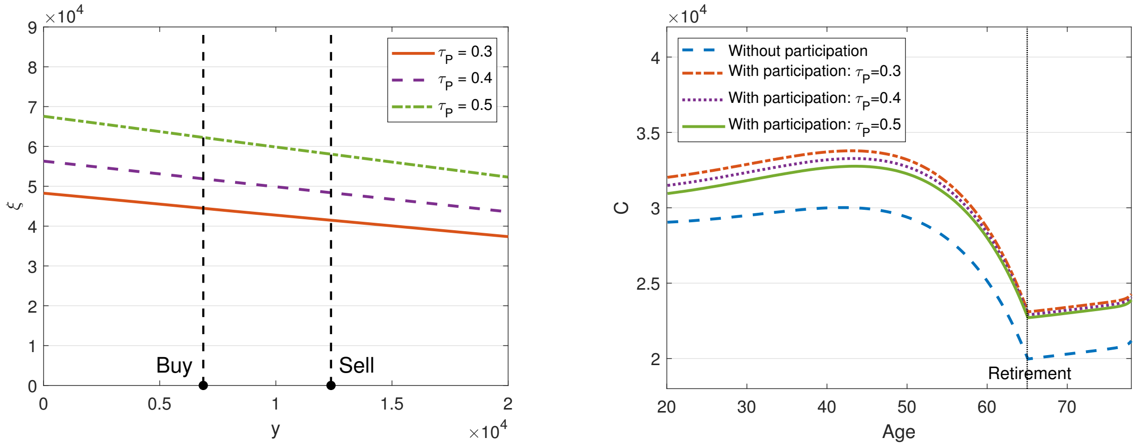
which increases with an increase in τ_P, as shown in Figure 6.1(a). Intuitively, with a higher tax rate on the distribution of pensions, the investor needs to earn more stock risk premia by increasing the risky investment in the pension account, to cover the increased tax costs.

Similarly, by substituting (6.1) into (3.20), we find that the consumption rate decreases with an increase in τ_P, as illustrated in Figure 6.1(b). The intuition is that a higher tax rate on withdrawal implies a lower after-retirement wealth level; hence, the investor needs to reduce her consumption level in response to the consumption-smoothing incentive.

6.2 Value of Participation

With an increased tax rate on distribution, the value of participation can also be decomposed into four parts:

$$\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4.$$



(a) Pension Investment

(b) Consumption Rate

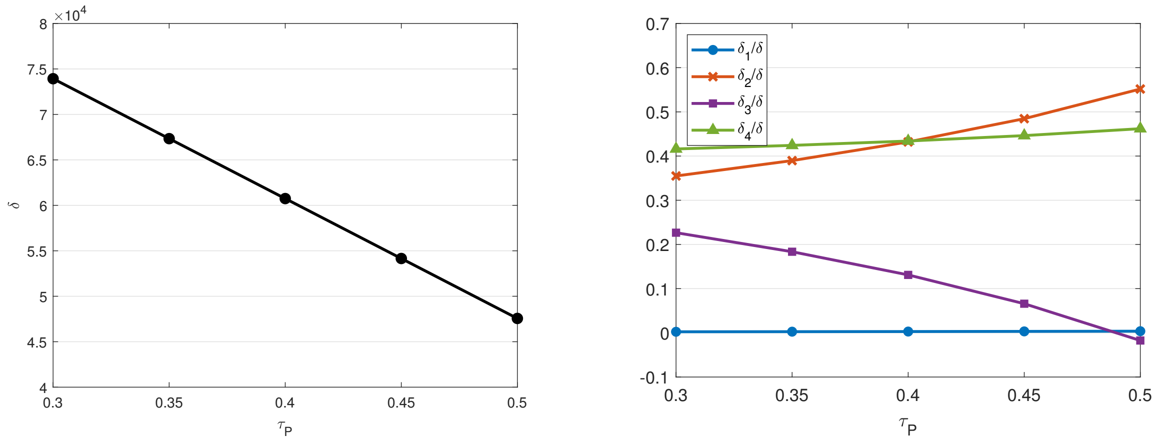
Figure 6.1: Optimal Policies: Impact of Increased Tax Rate on Pension Distribution
The left subfigure shows the investor's optimal stock exposure in the pension account at age 30, when $b = 0.7$. The right subfigure shows the time evolution of the investor's average consumption rate. The results are obtained with the parameter values in the UD case and three tax rates on the distribution of pension wealth.

δ_1 and δ_2 are still given by Equations (5.2) and (5.3), respectively; δ_3 and δ_4 become

$$
\begin{aligned}
\delta_3 =& (1-\tau_P)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds - (1-\tau_L)\int_0^{T_1} L_C(s)e^{-r(1-\tau_i)s}ds, \\
\delta_4 =& \alpha(1-\tau_P)e^{r\tau_i T_1}\int_0^{T_1} L_C(s)e^{-rs}ds,
\end{aligned}
$$

respectively.

Figure 6.2 plots the value of participation against the tax rate on distribution (i.e., τ_P). Intuitively, this value decreases with an increase in τ_P, as shown by Panel (a). Nevertheless, it is worth noting that even with a significant lift in the tax bracket, the value of participation can still be significant. For example, when the tax rate on distribution is as high as $\tau_P = 0.5$, the total value of participation is \$47,561. Panel (b) of Figure 6.2 draws the decomposition of the total value against τ_P. The results suggest that the value of deferring interest taxes on employee contribution can be negative when the tax rate on distribution is sufficiently high (e.g., when $\tau_P = 0.5$) because of the additional tax penalty incurred by the deferred interest. In contrast, the other components remain



(a) Total Value of Participation

(b) Weight of Each Component

Figure 6.2: Value of Participation: Impact of Increased Tax Rate on Pension Distribution
This figure shows the value of participation (left subfigure) and its decomposition (right subfigure) for the UD case with $\alpha = 0.5$ against the tax rate on the distribution of pension wealth.

positive.

7 Conclusion

Defined contribution plans (DCPs) have become many American's primary retirement savings vehicle. Participating in DCPs makes investors eligible for various tax and investment benefits. We propose a continuous-time model for studying optimal life-time portfolio and consumption choice when the investor participates in a typical DCP. Technically, using a novel domain-truncating approach, we overcome the difficulty caused by the unboundedness of the CARA utility function and prove the coincidence between the value function and the minimum viscosity solution to the associated HJB equation. This coincidence enables us to numerically solve for the value function. By performing a comprehensive numerical analysis, we examine the impacts of several factors, including tax-sheltering, employer matching, potential portfolio diversification, and tax consequence of lump-sum withdrawal on the investor's optimal investment and consumption strategies.

Our model also delivers utility-based valuation of DCP participation. It allows us to calculate the participant's utility gains from income tax deferral, employer matching, capital gains tax exemption, and potential diversification benefit separately. Furthermore, we examine the impact of following some suboptimal investment strategies on the value of participation, and find that adopting a suboptimal asset location strategy will cause a significant loss on this value, but exhibiting naive tax-timing skills will not.

Appendix

The content of this Appendix is as follows. In Appendix A, we present the solution to the model in the absence of capital gains tax. In Appendix B, we collect technical details for the model with capital gains tax except Theorem 3.9, which will be left to Appendix C. In Appendix D, we present the details of the numerical methods.

A Proof of Proposition 3.1

In this section, we provide the details of the model without capital gains taxes, including the proof of Proposition 3.1.

Absent capital gains taxes, we choose the wealth levels in the ordinary account, denoted as w_t, and in the pension account, denoted as z_t, as the state variables. Let $\{y_t\}_{t\in[0,T_2]}$ be the dollar amount invested in the stock in the ordinary account, $\{\xi_t\}_{t\in[0,T_2]}$ be the dollar amount invested in the stock in the pension account, and $\{C_t\}_{t\in[0,T_2]}$ be the consumption rate satisfying $\int_0^{T_2} |C_t|dt < +\infty$ almost surely, then we have the following dynamics during the pre-retirement period $t \in [0, T_1)$:

$$
\begin{aligned}
dw_t =& r(1-\tau_i)w_t dt - C_t dt + \big(\mu_1 + q_1(1-\tau_d) - r(1-\tau_i)\big) y_t dt + \sigma_1 y_t dB_{1t} \\
&+ (1-\tau_L)\big(L(t) - L_C(t)\big)dt, \\
dz_t =& \big(rz_t + \xi_t(\mu_2 + q_2 - r)\big)dt + \xi_t \sigma_2(\rho dB_{1t} + \sqrt{1-\rho^2} dB_{2t}) + (1+\alpha)L_C(t)dt,
\end{aligned}
$$

and the following dynamics during the post-retirement period $t \in [T_1, T_2]$:

$$
dw_t = r(1-\tau_i)w_t dt - C_t dt + \big(\mu_1 + q_1(1-\tau_d) - r(1-\tau_i)\big) y_t dt + \sigma_1 y_t dB_{1t}.
$$

At the retirement time T_1, we have

$$
w_{T_1} = w_{T_1-} + (1-\tau_L) z_{T_1-}.
$$

Again, for notational simplicity, we extend z_t and ξ_t to the post-retirement period by setting $z_t \equiv 0$ and $\xi_t \equiv 0$ for $t \in [T_1, T_2]$.

An admissible consumption-investment strategy for state (w, z), starting from $t \in [0, T_2)$, is a triple (C, y, ξ) satisfying the no-short-selling constraint:

$$
y_s \geq 0, \xi_s \geq 0, \text{ for } s \in [t, T_2].
$$

We denote by $\bar{\mathscr{A}_t}(w,z)$ ($\bar{\mathscr{A}_t}(w)$ resp.) if $t \in [0, T_1)$ ($t \in [T_1, T_2]$ resp.) the set of all admissible strategies.

The investor aims to maximize expected utility derived from life-time consumption plus bequest by choosing her optimal consumption-investment strategy over all the admissible ones, that is,

$$\max_{(C,y,\xi)\in\bar{\mathscr{A}}_0(w,0)} E_0^{w,0}\left[\omega\int_0^{T_2} e^{-\beta s}U(C_s)ds+(1-\omega)e^{-\beta T_2}U(w_{T_2})\right],$$

where $E_t^{w,z}$ is the expectation under the condition $(w_t,z_t)=(w,z)$. For $(w,z)\in\mathbb{R}^2$, denote by $\bar{V}$ the value function during the pre-retirement period $[0,T_1)$:

$$\bar{V}(w,z,t)=\max_{(C,y,\xi)\in\bar{\mathscr{A}}_t(w,z)} E_t^{w,z}\left[\omega\int_t^{T_2} e^{-\beta(s-t)}U(C_s)ds+(1-\omega)e^{-\beta(T_2-t)}U(w_{T_2})\right] \tag{A.1}$$

and denote by $\bar{V}^R$ the value function during the post-retirement period $[T_1,T_2]$:

$$\bar{V}^R(w,t)=\max_{(C,y,\xi)\in\bar{\mathscr{A}}_t(w)} E_t^{w}\left[\omega\int_t^{T_2} e^{-\beta(s-t)}U(C_s)ds+(1-\omega)e^{-\beta(T_2-t)}U(w_{T_2})\right]. \tag{A.2}$$

Formally, $\bar{V}^R(w,t)$ satisfies the following HJB equation for all $w\in\mathbb{R}$,

$$\begin{cases} \bar{V}_t^R+\max\limits_{y,C}\left\{\frac{1}{2}\sigma_1^2y^2\bar{V}_{ww}^R+\big(r(1-\tau_i)w-C+\sigma_1a_1y\big)\bar{V}_w^R-\beta\bar{V}^R-\omega e^{-\gamma C}\right\}=0, \\ \hfill \forall t\in[T_1,T_2], \\ \bar{V}^R(w,T_2)=-(1-\omega)e^{-\gamma w}. \end{cases}$$

As a standard Merton's problem with a CARA utility function, its solution is

$$\bar{V}^R(w,t)=-\,e^{-\gamma(p(t)w+\bar{h}(t))},t\in[T_1,T_2],$$

where

$$\begin{aligned}\bar{h}(t) =& \frac{p(t)(T_2-t)-\log(p_1(t)+1)}{\gamma} - \frac{p_1(t)(\log\omega+1)+\log(1-\omega)}{\gamma(p_1(t)+1)} \\ &+ \frac{p_2(t)+T_2-t}{p_1(t)+1}A_0.\end{aligned}$$

with $A_0 = \frac{\beta}{\gamma} + \frac{a_1^2}{2\gamma}$ and $p_2(t) := \int_t^{T_2} p_1(s)ds$. Therefore, the first-order conditions for C_t^* and y_t^* are

$$\sigma_1^2 y \bar{V}_{ww}^R + \sigma_1 a_1 \bar{V}_w^R = 0, -\bar{V}_w^R + \omega\gamma e^{-\gamma C} = 0,$$

solving which yields (3.8).

Analogously, the function $\bar{V}(w,z,t)$ satisfies

$$\begin{cases} \bar{V}_t + \max\limits_{y,C,\xi}\Big\{\frac{1}{2}\sigma_1^2 y^2 \bar{V}_{ww} + \rho\sigma_1\sigma_2\xi y \bar{V}_{wz} + \frac{1}{2}\sigma_2^2\xi^2\bar{V}_{zz} \\ \qquad + \Big(r(1-\tau_i)w - C + \sigma_1 a_1 y + (1-\tau_L)\big(L(t)-L_C(t)\big)\Big)\bar{V}_w \\ \qquad + \big(rz + \xi\sigma_2 a_2 + (1+\alpha)L_C(t)\big)\bar{V}_z - \beta\bar{V} - \omega e^{-\gamma C}\Big\} = 0, \forall t \in [0,T_1], \\ \bar{V}(w,z,T_1) = -e^{-\gamma(p(T_1)(w+(1-\tau_L)z)) + \bar{h}(T_1)}. \end{cases}$$

for $(w,z) \in \mathbb{R}^2$. The solution to the above equation is

$$\bar{V}(w,z,t) = -e^{-\gamma(p(t)(w+f_1(t))+g(t)(z+f_2(t))+\bar{h}_2(t))}, t \in [0,T_1),$$

where

$$\begin{aligned}\bar{h}_2(t) =& \frac{p(t)(T_2-t)-\log(p_1(t)+1)}{\gamma} - \frac{p_1(t)(\log\omega+1)+\log(1-\omega)}{\gamma(p_1(t)+1)} \\ &+ \frac{p_2(t)+T_2-t}{p_1(t)+1}A - \frac{p_2(T_1)+T_2-T_1}{p_1(t)+1}(A-A_0),\end{aligned}$$

with $A = \frac{\beta}{\gamma} + \frac{a_1^2+a_2^2-2\rho a_1 a_2}{2\gamma(1-\rho^2)}\mathbb{I}_{\{a_1-\rho a_2>0, a_2-\rho a_1>0\}} + \frac{a_2^2}{2\gamma}\mathbb{I}_{\{a_1-\rho a_2\leq 0\}} + \frac{a_1^2}{2\gamma}\mathbb{I}_{\{a_2-\rho a_1\leq 0\}}$. It follows that the first-order conditions for C_t^*, y_t^*, and ξ_t^* are

$$\begin{cases} \sigma_1^2 y \bar{V}_{ww} + \sigma_1\sigma_2\rho\xi\bar{V}_{wz} + \sigma_1 a_1 \bar{V}_w = 0, \\ \sigma_2^2 \xi \bar{V}_{zz} + \sigma_1\sigma_2\rho y \bar{V}_{wz} + \sigma_2 a_2 \bar{V}_z = 0, \\ -\bar{V}_w + \omega\gamma e^{-\gamma C} = 0, \end{cases}$$

solving which yields (3.5)-(3.7).

For the notational simplicity, we define

$$\begin{aligned} \bar{h}(t) = & \frac{p(t)(T_2-t) - \log(p_1(t)+1)}{\gamma} - \frac{p_1(t)(\log\omega+1) + \log(1-\omega)}{\gamma(p_1(t)+1)} \\ & + \begin{cases} \frac{p_2(t)+T_2-t}{p_1(t)+1}A - \frac{p_2(T_1)+T_2-T_1}{p_1(t)+1}(A-A_0), & t\in[0,T_1), \\ \frac{p_2(t)+T_2-t}{p_1(t)+1}A_0, & t\in[T_1,T_2]. \end{cases} \end{aligned} \tag{A.3}$$

Then, the value functions can be expressed in the following explicit forms

$$\begin{aligned} \bar{V}(w,z,t) &= -e^{-\gamma(p(t)(w+f_1(t))+g(t)(z+f_2(t))+\bar{h}(t))}, && t\in[0,T_1), \\ \bar{V}^R(w,t) &= -e^{-\gamma(p(t)w+\bar{h}(t))}, && t\in[T_1,T_2]. \end{aligned}$$

B Technical Details for the Model with Capital Gains Tax

In this Appendix, we present technical details in the model with capital gains tax.

For later use, we define the consumption-investment criterion for $(x,y,k,z)\in\mathbb{R}^4$ with $y\geq 0$

and $k \geq 0$, $t \in [0,T_1)$, and $\Theta = (C,I,D,\xi) \in \mathscr{A}_t(x,y,k,z)$,

$$J(x,y,k,z,t;\Theta) := E_t^{x,y,k,z}\left[\omega \int_t^{T_2} e^{-\beta s} U(C_s) ds + (1-\omega) e^{-\beta T_2} U\big(G(x_{T_2}, y_{T_2}, k_{T_2}; \iota)\big)\right],$$

where $\{(x_s,y_s,k_s,z_s)\}_{s\in[t,T_2]}$ represents the subsequent portfolio by following strategy Θ with $(x_t,y_t,k_t,z_t) = (x,y,k,z)$. Additionally, in the capital gains tax-free market, for $(w,z) \in \mathbb{R}^2$, $t \in [0,T_1)$, and $\bar{\Theta} = (C,y,\xi) \in \bar{\mathscr{A}_t}(w,z)$, consider the consumption-investment criterion as follows

$$\bar{J}(w,z,t;\bar{\Theta}) := E_t^{w,z}\left[\omega \int_t^{T_2} e^{-\beta s} U(C_s) ds + (1-\omega) e^{-\beta T_2} U(w_{T_2})\right],$$

where $\{(w_s,z_s)\}_{s\in[t,T_2]}$ represents the subsequent portfolio in the tax-free market by following strategy $\bar{\Theta}$ with $(w_t,z_t) = (w,z)$. Then

$$V(x,y,k,z,t) = \max_{\Theta\in\mathscr{A}_t(x,y,k,z)} J(x,y,k,z,t;\Theta),$$

$$\bar{V}(w,z,t) = \max_{\bar{\Theta}\in\bar{\mathscr{A}_t}(w,z)} \bar{J}(w,z,t;\bar{\Theta}).$$

B.1 Proof of Proposition 3.4

We only prove part (2) as the proof of part (1) is similar to that of Proposition 4.1 in Ben Tahar et al. (2010). In addition, we only provide the proof for post-retirement period as the proof for pre-retirement period is almost the same. Fix $(x,y,k) \in \mathbb{R}^3$ with $y \geq 0$ and $k \geq 0$, $t \in [T_1,T_2)$, and let $\Theta = (C,I,D,\xi) \in \mathscr{A}_t(x,y,k)$ be an arbitrarily chosen admissible strategy with subsequent portfolio $\{(x_s,y_s,k_s)\}_{s\in[t,T_2]}$ such that $y_s \geq k_s$ for all $s > t$ (note that we can select such a strategy due to the optimality of wash-sale; see Proposition B.2 in Appendix B.4). Denote by $w_s := f(x_s,y_s,k_s)$ the net wealth in the ordinary account before death.

At the beginning time t, if $y < k$, we immediately implement a wash-sale, which changes the dollar amount invested in the bond to $x+\tau_g(k-y)^+$. We next construct the following admissible strategy $\bar{\Theta} := (\bar{C}, \bar{y})$ for wealth position $x+y+\tau_g(k-y)^+$ starting from time t in the capital gains tax-free market:

$$\bar{C}_s = C_s, \bar{y}_s = y_s.$$

We denote as $\{\bar{w}_s\}_{s\in[t,T_2]}$ the resulting wealth process, and as $\{\bar{x}_s\}_{s\in[t,T_2]}$ the resulting dollar amount invested in the bond. We will show that

$$\bar{w}_{T_2} \geq G(x_{T_2}, y_{T_2}, k_{T_2}; 1). \tag{B.1}$$

In fact, noticing that the right hand side of (B.1) equals $x_{T_2}+y_{T_2}$ from the construction requirement that $y_s \geq k_s$ for $s>t$, it suffices to show that $\bar{w}_{T_2} \geq x_{T_2}+y_{T_2}$. Given $y_s = \bar{y}_s$, we only need to show $\bar{x}_{T_2} \geq x_{T_2}$. To see this, note that we have

$$d\bar{x}_s - dx_s = r(1-\tau_i)(\bar{x}_{s-} - x_{s-})ds + \big(y_{s-} - f(0, y_{s-}, k_{s-})\big)ds,$$

$\bar{x}_t = x+\tau_g(k-y)^+ = x_t$, and $k_s \leq y_s$ for $s>t$, which establish the inequality (B.1). The proposition then follows from the identity of the two consumption processes $\bar{C}$ and C, the arbitrariness of the strategy Θ, and the optimality of the wash-sale.

B.2 Separation Property of the Value Functions

Lemma B.1. *The value function has the following properties for* $(x,y,k,z) \in \mathbb{R}^4$ *with* $y \geq 0$ *and* $k \geq 0$*:*

$$
\begin{aligned}
V(x,y,k,z,t) =& e^{-\gamma(p(t)x+g(t)z)}V(0,y,k,0,t), && t \in [0,T_1), \\
V^R(x,y,k,t) =& e^{-\gamma p(t)x}V^R(0,y,k,t), && t \in [T_1,T_2].
\end{aligned}
$$

Proof. We only prove for the pre-retirement period. Fix $(x,y,k,z) \in \mathbb{R}^4$ with $y \geq 0$ and $k \geq 0$, $t \in [0,T_1)$, and let $\Theta = (C,I,D,\xi) \in \mathscr{A}_t(0,y,k,0)$, which results in the subsequent portfolio $\{(x_{0s},y_{0s},k_{0s},z_{0s})\}_{s\in[t,T_2]}$. We construct a consumption-investment strategy $\tilde{\Theta} := \big(C + p(t)x + g(t)z, I, D, \xi\big)$ for starting position (x,y,k,z) from t, which results in the following portfolio $\{(x_s,y_s,k_s,z_s)\}_{s\in[t,T_2]}$. Then, clearly, $y_s = y_{0s}$ and $k_s = k_{0s}$ for $s \in [0,T_2]$, and thus $\tilde{\Theta} \in \mathscr{A}_t(x,y,k,z)$. Let

$$
\hat{x}_s = x_s - x_{0s}, \hat{z}_s = z_s - z_{0s}, \text{ for } \forall s \in [t,T_2],
$$

with $(\hat{x}_t,\hat{z}_t) = (x,z)$. We obtain from the dynamics of the state variables and the relationship between strategies Θ and $\tilde{\Theta}$ that

$$
de^{r(T_1-s)}\hat{z}_s = 0,
$$

$$
de^{r(1-\tau_i)(T_2-s)}\hat{x}_s = -e^{r(1-\tau_i)(T_2-s)}\big(p(t)x + g(t)z\big)ds, \forall s \in [t,T_2]\backslash\{T_1\},
$$

and thus,

$$
\hat{z}_{T_1-} = e^{r(T_1-t)}z,
$$

$$
\begin{aligned}
\hat{x}_{T_2} &= e^{r(1-\tau_i)(T_2-t)}x - \big(p(t)x + g(t)z\big)p_1(t) + e^{r(1-\tau_i)(T_2-T_1)}(\hat{x}_{T_1} - \hat{x}_{T_1-}) \\
&= p(t)x - g(t)p_1(t)z + (1-\tau_L)e^{r(1-\tau_i)(T_2-T_1)+r(T_1-t)}z \\
&= p(t)x + g(t)z.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
J(x,y,k,z,t;\tilde{\Theta}) = & e^{-\gamma(p(t)x+g(t)z)}\bigg[\omega\int_t^{T_2} e^{-\beta(s-t)}U(C_s)ds \\
& \qquad + (1-\omega)e^{-\beta(T_2-t)}U\big(G(x_{0T_2}, y_{0T_2}, k_{0T_2}; \iota)\big)\bigg].
\end{aligned}
$$

Since the map from Θ to $\tilde{\Theta}$ is a bijiection, we get the desired result by taking maximum over the above equation. $\square$

B.3 Proof of Proposition 3.6

Consider an admissible strategy $\Theta_0 = (C, I, D)$ for initial position $(x,y,k) \in \mathbb{R}^3$ with $y \geq 0$ and $k \geq 0$ starting at time 0 when the investor only invests in the ordinary account and denote by $\{(x_t^0, y_t^0, k_t^0)\}_{t\in[0,T_2]}$ the following portfolio processes. Note that strategy Θ_0 is admissible for a non-participating investor if the constraint (2.5) is satisfied with $z_t = \xi_t \equiv 0$ in $[0, T_2]$. Setting $\xi_t \equiv 0$ for $t \in [0, T_1)$, we see that $\Theta := (C, I, D, \xi) \in \mathscr{A}_0(x,y,k,0)$ when the investor invests in both the ordinary account and the tax-deferred pension accounts. Let $\{(x_t, y_t, k_t, z_t)\}_{t\in[0,T_2]}$ be the subsequent portfolio sequence by following strategy Θ. It then follows that $y_t = y_t^0$ and $k_t = k_t^0$ for $\forall t \in [0, T_2]$. We obtain from the dynamics of the state variables and the relationship between strategies Θ_0 and Θ that

$$
d(e^{-rt}z_t) = (1+\alpha)e^{-rt}L_C(t)dt, \qquad \forall t \in [0, T_1),
$$

and

$$
\begin{aligned}
d\big(e^{-r(1-\tau_i)t}(x_t - x_t^0)\big) =& -(1-\tau_L)e^{-r(1-\tau_i)t}L_C(t)dt, && \forall t \in [0, T_1),\\
d\big(e^{-r(1-\tau_i)t}(x_t - x_t^0)\big) =& 0, && \forall t \in [T_1, T_2].
\end{aligned}
$$

During pre-retirement period $[0, T_1)$, we have

$$
p(t)(x_t - x_t^0) + g(t)z_t = \frac{1-\tau_L}{p_1(t)+1}\int_0^t \big((1+\alpha)e^{r\tau_i(T_1-s)} - 1\big)e^{r(1-\tau_i)(T_2-s)}L_C(s)ds \geq 0.
$$

During post-retirement period $[T_1, T_2]$, it follows

$$
\begin{aligned}
x_t - x_t^0 =& e^{r(1-\tau_i)(t-T_1)}(x_{T_1} - x_{T_1}^0)\\
=& e^{r(1-\tau_i)(t-T_1)}\big(x_{T_1-} - x_{T_1-}^0 + (1-\tau_L)z_{T_1-}\big)\\
=& \big(p(T_1)(x_{T_1-} - x_{T_1-}^0) + g(T_1)z_{T_1-}\big)(p_1(T_1)+1)e^{-r(1-\tau_i)(T_2-t)} \geq 0. \qquad \text{(B.2)}
\end{aligned}
$$

Inequality (B.2) indicates $x_{T_2} \geq x_{T_2}^0$ and thus $G(x_{T_2}, y_{T_2}, k_{T_2}; \iota) \geq G(x_{T_2}^0, y_{T_2}^0, k_{T_2}^0; \iota)$. Put differently, the investor can raise her terminal wealth level by participating in the DCP while maintaining her consumption level over the life-cycle. Therefore, $V(x, y, k, 0, 0) \geq V_0(x, y, k, 0)$ follows due to the arbitrariness of Θ_0.

B.4 Optimality of Wash-Sale in the Ordinary Investment Account

The following proposition is an extension of Proposition 3.4 in Ben Tahar et al. (2010) on the optimality of wash-sale. Its proof is omitted to save space.

Proposition B.2. *For $(x, y, k, z) \in \mathbb{R}^4$ with $y \geq 0$ and $k \geq 0$ and $\Theta \in \mathscr{A}_0(x, y, k, z)$, denote by $\{(x_t, y_t, k_t, z_t)\}_{t\in[0,T_2]}$ the following state processes. If $k_\tau > y_\tau$ a.s. for some finite stopping time τ,* *then there exists another admissible strategy* $\tilde{\Theta}$ *with the resulting process* $\{(X_t, Y_t, K_t, Z_t)\}_{t\in[0,T_2]}$, *such that for* $\forall t \in [0, T_2]$,

$$Y_t = y_t, d\tilde{D}_t = dD_t(1 - \mathbb{I}_{\{t=\tau\}}) + \mathbb{I}_{\{t=\tau\}}, \tilde{\xi}_t = \xi_t,$$

and the investor has larger expected life-time consumption and bequest in the sense that

$$J(x, y, k, z, 0; \tilde{\Theta}) > J(x, y, k, z, 0; \Theta).$$

B.5 Value Functions without Tax Deferral

The next proposition characterizes the investor's value function without deferring capital gains tax in the ordinary account (assuming taxes are not forgiven at the time of death). Its proof is similar to that of Proposition 4.2 in Ben Tahar et al. (2010), and we omit the details.

Proposition B.3. *Assume* $\iota = 0$ *and let* $\tilde{\mu}_1 = (1 - \tau_g)\mu_1$ *and* $\tilde{\sigma}_1 = (1 - \tau_g)\sigma_1$. *Then, without deferring capital gains realizations in the ordinary account, the investor's highest achievable utility level is* $\bar{V}\left(x + (1 - \tau_g)y + \tau_g k, z, t; \tilde{\mu}_1, \tilde{\sigma}_1\right)$ *for* $t \in [0, T_1)$, *and* $\bar{V}^R\left(x + (1 - \tau_g)y + \tau_g k, t; \tilde{\mu}_1, \tilde{\sigma}_1\right)$ *for* $t \in [T_1, T_2]$. *Here, the notations* $\bar{V}(w, z, t; \mu_1, \sigma_1)$ *and* $\bar{V}^R(w, t; \mu_1, \sigma_1)$ *indicate the dependency of* $\bar{V}$ *and* $\bar{V}^R$ *on the parameters* μ_1 *and* σ_1.

Remark B.4. It is well known that the investor can benefit from optimally deferring capital gains tax through interest saving. Proposition B.3 in fact presents the lower bounds of the value functions with optimal tax deferral.

C Proof of Theorem 3.9

We will prove that the value function $\tilde{h}(y,k,t;\Lambda)$ is the minimum viscosity solution, which is locally uniformly Lipschitz in (y,k), to the corresponding partial differential equation (PDE); and that the value function we concerned, $\tilde{h}(y,k,t)$, is the limit $\lim_{\Lambda\to\infty}\tilde{h}(y,k,t;\Lambda)$.

To achieve this, we first consider the corresponding regular control problem with an upper bound λ on the transaction speed and a compulsory sell condition beyond the boundary $y^2+k^2=\Lambda^2$ and show the associated value function, denoted as $\tilde{h}(y,k,t;\Lambda,\lambda)$, is continuous. After that, we show $\lim_{\lambda\to\infty}\tilde{h}(y,k,t;\Lambda,\lambda)=\tilde{h}(y,k,t;\Lambda)$ and $\lim_{\Lambda\to\infty}\tilde{h}(y,k,t;\Lambda)=\tilde{h}(y,k,t)$. Third, we prove $\tilde{h}(y,k,t;\Lambda)$ is continuous and locally uniformly Lipschitz in (y,k), which guarantees that it is one of the viscosity solutions to the HJB PDE associated with the singular control problem. Fourth, we prove a comparison principle, which guarantees that any viscosity solution, which is locally uniformly Lipschitz in (y,k), to the corresponding PDE satisfied by $\tilde{h}(y,k,t;\Lambda)$ must be no less than $\tilde{h}(y,k,t;\Lambda,\lambda)$. Finally, we prove $\tilde{h}(y,k,t;\Lambda)$ is the minimum one among all such solutions.

For expositional simplicity, we only consider the case with $t\in[T_1,T_2]$, $r>0$, and $\tau_i=q_1=\iota=L(t)=L_c(t)=0$. The proof for the general case is similar. With a slight abuse of notation, we let $T_1=0$ and $T_2=T$ for notational simplicity.

C.1 Regular Control Problem and Continuity of the Value Function

In what follows, we prove the continuity of the value function arising from a regular control problem. For two arbitrary positive constants λ and Λ, the regular control state processes corresponding to (2.4) are

$$\begin{cases} d\hat{x}_s = r\hat{x}_s ds - c_s ds - i_s ds + f(0,\hat{y}_s,\hat{k}_s)(j_s ds + d\hat{M}_s), \\ d\hat{y}_s = \mu_1 \hat{y}_s ds + \sigma_1 \hat{y}_s dB_{1s} + i_s ds - \hat{y}_s(j_s ds + d\hat{M}_s), \\ d\hat{k}_s = i_s ds - \hat{k}_s(j_s ds + d\hat{M}_s) \end{cases} \tag{C.1}$$

with $(\hat{x}_t,\hat{y}_t,\hat{k}_t) = (x,y,k)$, where $0 \le i_s, j_s \le \lambda$, $|c_s| \le \lambda$ are controls and $\hat{M}_s$ is the local time on the boundary $y^2 + k^2 = \Lambda^2$ such that $\hat{y}_s^2 + \hat{k}_s^2 \le \Lambda^2$, $\forall t \le s \le T$. Denote the set of admissible regular controls $\{(i_s, j_s, c_s)\}_{s\in[t,T]}$ as $\mathscr{A}_t(x,y,k;\lambda)$. Then, we have the following proposition.

Proposition C.1. *For any admissible strategy in $\mathscr{A}_t(x,y,k;\lambda)$ and parameters $\hat{\gamma}$, λ, Λ, $T > 0$, there exist positive constants $C(\hat{\gamma},\lambda,\Lambda,T)$ and $C(\hat{\gamma},sgn(x),T)$ such that*

$$\begin{aligned} &E_t^{x,y,k}\left[\exp\left(-\hat{\gamma} f(\hat{x}_T,\hat{y}_T,\hat{k}_T)\right)\right] \\ \le& e^{C(\hat{\gamma},\lambda,\Lambda,T)+C(\hat{\gamma},sgn(x),T)x} \exp\Big(-\hat{\gamma}\big(x + (1-\tau_g)y + \tau_g k\big)\Big). \end{aligned}$$

Proof. Without loss of generality, we assume the starting time is $t = 0$. From (C.1), we have $d\hat{x}_t \ge r\hat{x}_t dt - 2\lambda dt$, which implies

$$\hat{x}_t \ge e^{rt}\left(x - 2\lambda\frac{1-e^{-rt}}{r}\right) \ge e^{rt}x - 2\lambda\frac{e^{rT}}{r} \ge \begin{cases} e^{rT}x - 2\lambda\frac{e^{rT}}{r}, & \text{when } x \le 0, \\ x - 2\lambda\frac{e^{rT}}{r}, & \text{when } x > 0. \end{cases}$$

Define $F_t = \exp\Big(-\hat{\gamma}\big(\hat{x}_t + (1-\tau_g)\hat{y}_t + \tau_g\hat{k}_t\big)\Big)$, then since $\hat{y}_t \le \Lambda$, we have

$$\begin{aligned} dF_t/F_t =& -\hat{\gamma}\big(r\hat{x}_t - c_t + \mu_1(1-\tau_g)\hat{y}_t\big)dt + \frac{\hat{\gamma}^2(1-\tau_g)^2}{2}\sigma_1^2\hat{y}_t^2 dt - \hat{\gamma}\sigma_1(1-\tau_g)\hat{y}_t dB_{1t} \\ \le& \Big(D(\hat{\gamma},\lambda,\Lambda,T) + D\big(\hat{\gamma},sgn(x),T\big)x\Big)dt - \hat{\gamma}\sigma_1(1-\tau_g)\hat{y}_t dB_{1t}, \end{aligned}$$

where $D(\hat{\gamma},\lambda,\Lambda,T)=\hat{\gamma}\lambda+2\hat{\gamma}\lambda e^{rT}+\frac{\hat{\gamma}^2(1-\tau_g)^2}{2}\sigma_1^2\Lambda^2$ and

$$D(\hat{\gamma},sgn(x),T)=\begin{cases}-r\hat{\gamma}e^{rT} & \text{when } x\leq 0,\\ -r\hat{\gamma}, & \text{when } x>0.\end{cases}$$

Then, we have from the Novikov condition (Corollary 5.13 in Karatzas and Shreve (2012)),

$$\frac{d}{dt}E[F_t]\leq\Big(D(\hat{\gamma},\lambda,\Lambda,T)+D\big(\hat{\gamma},sgn(x),T\big)x\Big)E[F_t],$$

and thus

$$\begin{aligned}&E_0^{x,y,k}\Big[\exp\Big(-\hat{\gamma}\big(\hat{x}_T+(1-\tau_g)\hat{y}_T+\tau_g\hat{k}_T\big)\Big)\Big]\\ \leq&e^{(D(\hat{\gamma},\lambda,\Lambda,T)+D(\hat{\gamma},sgn(x),T)x)T}\exp\Big(-\hat{\gamma}\big(x+(1-\tau_g)y+\tau_g k\big)\Big).\end{aligned}$$

Therefore, $C(\hat{\gamma},\lambda,\Lambda,T)=D(\hat{\gamma},\lambda,\Lambda,T)T$ and $C\big(\hat{\gamma},sgn(x),T\big)=D\big(\hat{\gamma},sgn(x),T\big)T$ is what we need.

□

Define the value function in the regular control problem as

$$\begin{aligned}V^R(x,y,k,t;\Lambda,\lambda):=\max_{\{(c_s,i_s,j_s)\}_{s\in[t,T]}\in\mathscr{A}_t(x,y,k;\lambda)}E_t^{x,y,k}\Big[&\omega\int_t^T e^{-\beta(s-t)}U(c_s)ds\\ &+(1-\omega)e^{-\beta(T-t)}U\big(f(\hat{x}_T,\hat{y}_T,\hat{k}_T)\big)\Big].\end{aligned}$$

We have the following theorem regarding the continuity and boundedness of this value function.

Lemma C.2. *For fixed $\lambda,\Lambda>0$, the function $V^R(x,y,k,t;\Lambda,\lambda)$ is continuous in the solvency region*

$\mathscr{S}_\Lambda := \{(x,y,k,t) \in \mathbb{R}^3 \times [0,T] : y,k \geq 0, y^2+k^2 \leq \Lambda^2\}$. *Moreover, for any $\lambda > 1$, we have the bound*

$$|V^R(x,y,k,t;\Lambda,\lambda)| \leq e^{\gamma}T + e^{C(\gamma,\lambda,\Lambda,T)+C(\gamma,sgn(x),T)x}\exp\big(-\gamma f(x,y,k)\big). \tag{C.2}$$

Proof. We first show the upper-semicontinuity. Given $\varepsilon > 0$, consider a sequence $(x_n,y_n,k_n,t_n) \to (x,y,k,t)$ and the corresponding admissible regular control policies $\pi_n = (c^n,i^n,j^n) \in \mathscr{A}_{t_n}(x_n,y_n,k_n;\lambda)$ such that under these policies, the resultant expected utility can achieve the level of $V^R(x_n,y_n,k_n,t_n;\Lambda,\lambda) - \varepsilon$. We employ the same strategy π_n on the starting point (x,y,k) in $[\max\{t,t_n\},T]$, and if $t < t_n$, we set $c_s = i_s = j_s = 0$ for $s \in [t,t_n)$. We denote as $\{(x_{n,s},y_{n,s},k_{n,s})\}_{s\in[t_n,T]}$, $\{(x_s,y_s,k_s)\}_{s\in[t,T]}$ the corresponding state processes and

$$\{(\Delta x_s,\Delta y_s,\Delta k_s)\}_{s\in[\max\{t,t_n\},T]} := \{(x_{n,s}-x_s,y_{n,s}-y_s,k_{n,s}-k_s)\}_{s\in[\max\{t,t_n\},T]}$$

the difference in the state processes. Then, we have for $s \in [\max\{t,t_n\},T]$,

$$\begin{aligned}
d\Delta x_s =& r\Delta x_s ds + f(0,\Delta y_s,\Delta k_s)j_{n,s}ds + f(0,y_{n,s},k_{n,s})dM_{n,s} - f(0,y_s,k_s)d\hat{M}_s,\\
d\Delta y_s =& \mu_1\Delta y_s ds + \sigma_1\Delta y_s dB_{1s} - \Delta y_s j_{n,s}ds + y_{n,s}dM_{n,s} - y_s dM_s,\\
d\Delta k_s =& -\Delta k_s j_{n,s}ds + k_{n,s}dM_{n,s} - k_s dM_s.
\end{aligned}$$

It can be seen that

$$\lim_{n\to\infty} \Delta x_T, \Delta y_T, \Delta k_T = 0, a.s. \tag{C.3}$$

According to the selection of the strategy π_n, we have

$$
\begin{aligned}
& e^{-\beta t_n}\big(V^R(x_n,y_n,k_n,t_n;\Lambda,\lambda)-\varepsilon\big)-e^{-\beta t}V^R(x,y,k,t;\Lambda,\lambda) \\
\leq & \,\omega\left(\int_t^{t_n} e^{-\beta s}|U(c_s)|ds\right)\mathbb{I}_{\{t<t_n\}} \\
& +(1-\omega)e^{-\beta T}\left(E_{t_n}^{x_n,y_n,k_n}\left[e^{-\gamma f(x_{n,T},y_{n,T},k_{n,T})}\right]-E_t^{x,y,k}\left[e^{-\gamma f(x_T,y_T,k_T)}\right]\right).
\end{aligned}
$$

Given the L_p boundedness from Proposition C.1 (by selecting $p=\hat{\gamma}/\gamma\in(1,\infty)$) and the almost surely convergence result (C.3), we have

$$
\limsup_{n\to+\infty} V^R(x_n,y_n,k_n,t_n;\Lambda,\lambda)-\varepsilon-V^R(x,y,k,t;\Lambda,\lambda)\leq 0.
$$

By the arbitrariness of $\varepsilon>0$, we obtain the upper-semicontinuity. We can analogously establish the lower-semicontinuity.

Lastly, we show (C.2). Note that $V^R(x,y,k,t;\Lambda,\lambda)$ is negative and monotonically increasing in λ, according to Proposition C.1, we have for $\lambda>1$ that

$$
\begin{aligned}
& |V^R(x,y,k,t;\Lambda,\lambda)| \\
\leq & |V^R(x,y,k,t;\Lambda,1)| \\
\leq & E_t^{x,y,k}\left[\omega\int_t^T e^{-\beta(s-t)}|U(c_s)|ds+(1-\omega)e^{-\beta(T-t)}\big|U\big(f(x_T,y_T,k_T)\big)\big|\right] \\
\leq & \,\omega e^{\gamma}T+(1-\omega)E_t^{x,y,k}\left[\exp\big(-\gamma f(x_T,y_T,k_T)\big)\right] \\
\leq & e^{\gamma}T+e^{C(\gamma,\lambda,\Lambda,T)+C(\gamma,sgn(x),T)x}\exp\Big(-\gamma\big(x+(1-\tau_g)y+\tau_g k\big)\Big).
\end{aligned}
$$

□

C.2 Convergence to the Singular Control Value Function

In this subsection, we prove for any (x,y,k,t),

$$\lim_{\lambda\to\infty} V^R(x,y,k,t;\Lambda,\lambda) = V^R(x,y,k,t;\Lambda), \tag{C.4}$$

$$\lim_{\Lambda\to\infty} V^R(x,y,k,t;\Lambda) = V^R(x,y,k,t), \tag{C.5}$$

where $V^R(x,y,k,t;\Lambda)$ represents the singular control value function with sell condition beyond the boundary $\Gamma_\Lambda := \{(x,y,k,t) \in \mathscr{S}_\Lambda : y,k \geq 0, y^2+k^2 = \Lambda^2\}$.

We first prove (C.4). For any $\varepsilon > 0$, we can choose an admissible singular control policy $\{(C_s,I_s,D_s)\}_{t\leq s\leq T} \in \mathscr{A}_t(x,y,k)$, such that adopting it ensures y_s, $k_s \leq \Lambda$, the indirect utility level is higher than $V^R(x,y,k,t;\Lambda)-\varepsilon$, and C_s is uniformly bounded for all time and Brownian motion paths. Then we can construct regular control approximation $\{(C_s,i_s,j_s)\}_{s\in[t,T]}$ as in Appendix E of Bian et al. (2021). Define the wealth process

$$w_s := x_s + (1-\tau_g)y_s + \tau_g k_s.$$

Then, we have

$$dw_s = \big(rx_s - C_s + \mu_1(1-\tau_g)y_s\big)ds + \sigma_1(1-\tau_g)y_s dB_{1s},$$

and

$$\begin{aligned} d(e^{-rt}w_s) =& e^{-rs}\Big(-r\big((1-\tau_g)y_s+\tau_g k_s\big) - C_s + \mu_1(1-\tau_g)y_s\Big)ds \\ &+ e^{-rt}\sigma_1(1-\tau_g)y_s dB_{1s}. \end{aligned} \tag{C.6}$$

Analogously, for the regular control problem, we have

$$
\begin{aligned}
d(e^{-rt}\hat{w}_s) =& e^{-rs}\Big(-r\big((1-\tau_g)\hat{y}_s+\tau_g\hat{k}_s\big)-C_s+\mu_1(1-\tau_g)\hat{y}_s\Big)ds \\
&+e^{-rs}\sigma_1(1-\tau_g)\hat{y}_s dB_{1s}.
\end{aligned} \tag{C.7}
$$

Since $0\leq y_s,\hat{y}_s,k_s,\hat{k}_s\leq\Lambda$ and $|C_s|$ are uniformly bounded for all Brownian motion paths, we have the following proposition for both the singular control and regular control wealth processes.

Lemma C.3. *For a given initial position $(x,y,k,t)\in\mathscr{S}_\Lambda$,*

$$
P\Big[\min_{t\leq s\leq T} w_s\leq-\zeta\Big], P\Big[\min_{t\leq s\leq T}\hat{w}_s\leq-\zeta\Big]\lesssim O(e^{-\delta_1\zeta^2}),
$$

as $\zeta\to\infty$ for some sufficiently small $\delta_1>0$. This asymptotic order is uniform for all $\lambda>0$.

Proof. Define $Z_s=e^{-rs}w_s$, then from (C.6) and (C.7) we have

$$
dZ_s\geq -C_1ds+f_sdB_{1s},
$$

where C_1 is a positive constant, and f_s is a bounded process. Without loss of generality, we also set $0\leq f_s\leq C_1$. We only need to show

$$
P\Big[\min_{t\leq s\leq T} Z_s\leq-\zeta\Big]\lesssim O(e^{-\delta_1\zeta^2}).
$$

Without loss of generality, we assume

$$
dZ_s=f_sdB_{1s}
$$

since the term involving C_1 does not influence the asymptotic order $e^{-\delta_1\zeta^2}$. By setting $u=$

$\int_t^s \sqrt{f_v}dv$, we have $\{Z_v\}_{v\in[t,s]}$ is identical in distribution to $\{Z_0+B_v\}_{v\in[0,u]}$, where B_v is a standard Brownian motion. Therefore,

$$
\begin{aligned}
P\Big[\min_{t\leq s\leq T} Z_s \leq -\zeta\Big] =& P\Big[\min_{0\leq u\leq \int_t^T \sqrt{f_v}dv} B_u \leq -\zeta - Z_0\Big] \\
\leq& P\Big[\min_{0\leq u\leq \sqrt{C_1}(T-t)} B_s \leq -\zeta - Z_0\Big].
\end{aligned}
$$

Applying the reflection principle, the right hand side equals

$$
2P[B_{\sqrt{C_1}(T-t)} \geq \zeta + Z_0] = 2P\Big[B_1 \geq \frac{\zeta+Z_0}{C_1^{1/4}\sqrt{T-t}}\Big],
$$

while for any $x>0$, $P[B_1>x] = \int_x^\infty e^{-\frac{s^2}{2}}ds \leq \int_x^\infty \frac{s}{x}e^{-\frac{s^2}{2}}ds = \frac{1}{x}e^{-\frac{x^2}{2}}$. The proof for $\hat{w}_t$ is similar. $\square$

Next, we prove (C.4). On the one hand, we have

$$
V^R(x,y,k,t;\Lambda,\lambda) \leq V^R(x,y,k,t;\Lambda).
$$

On the other hand, given constant $\zeta>0$, we can derive from the dynamic programming principle that

$$
\begin{aligned}
& V^R(x,y,k,t;\Lambda) - V^R(x,y,k,t;\Lambda,\lambda) \\
\leq& \varepsilon + E_t^{x,y,k}\Big[\Big(\omega\int_t^T e^{-\beta(s-t)}U(C_s)ds + (1-\omega)e^{-\beta(T-t)}U\big(f(x_T,y_T,k_T)\big)\Big)\mathbb{I}_{\{\tau_\zeta>T\}}\Big] \\
& + \big|E_t^{x,y,k}\big[e^{-\beta(\tau_\zeta-t)}V^R(x_{\tau_\zeta},y_{\tau_\zeta},k_{\tau_\zeta},\tau_\zeta;\Lambda)\mathbb{I}_{\{\tau_\zeta\leq T\}}\big]\big| \\
& - E_t^{x,y,k}\Big[\Big(\omega\int_t^T e^{-\beta(s-t)}U(C_s)ds + (1-\omega)e^{-\beta(T-t)}U\big(f(\hat{x}_T,\hat{y}_T,\hat{k}_T)\big)\Big)\mathbb{I}_{\{\hat{\tau}_\zeta>T\}}\Big] \\
& + \big|E_t^{x,y,k}\big[e^{-\beta(\hat{\tau}_\zeta-t)}V^R(\hat{x}_{\hat{\tau}_\zeta},\hat{y}_{\hat{\tau}_\zeta},\hat{k}_{\hat{\tau}_\zeta},\hat{\tau}_\zeta;\Lambda,\lambda)\mathbb{I}_{\{\hat{\tau}_\zeta\leq T\}}\big]\big|
\end{aligned}
$$

where τ_ζ and $\hat{\tau}_\zeta$ are respectively the hitting time of $w_s=-\zeta$ and $\hat{w}_s=-\zeta$.

As $\lambda \to \infty$, we have $\hat{\tau}_\zeta \to \tau_\zeta$, $\mathbb{I}_{\{\hat{\tau}_\zeta \le T\}} \to \mathbb{I}_{\{\tau_\zeta \le T\}}$ almost surely from the definition of the regular control. Then, due to Proposition C.1 and the boundedness of C_s, we can apply the dominated convergence theorem to establish

$$\begin{aligned}
&E_t^{x,y,k}\left[\left(\omega\int_t^T e^{-\beta(s-t)}U(C_s)ds+(1-\omega)e^{-\beta(T-t)}U\big(f(\hat{x}_T,\hat{y}_T,\hat{k}_T)\big)\right)\mathbb{I}_{\{\hat{\tau}_\zeta\le T\}}\right]\\
\to &E_t^{x,y,k}\left[\left(\omega\int_t^T e^{-\beta(s-t)}U(C_s)ds+(1-\omega)e^{-\beta(T-t)}U\big(f(x_T,y_T,k_T)\big)\right)\mathbb{I}_{\{\tau_\zeta\le T\}}\right].
\end{aligned}$$

Recalling Lemma C.2, Lemma C.3 and Proposition C.1, since $V^R(x,y,k,t;\Lambda,\lambda)$ is monotonically increasing in λ, for any $\lambda > 1$ we have

$$\begin{aligned}
&E_t^{x,y,k}\left[e^{-\beta(\tau_\zeta-t)}|V(x_{\tau_\zeta},y_{\tau_\zeta},k_{\tau_\zeta},\tau_\zeta;\Lambda)|\right]\\
\le &E_t^{x,y,k}\left[e^{-\beta(\tau_\zeta-t)}|V(x_{\tau_\zeta},y_{\tau_\zeta},k_{\tau_\zeta},\tau_\zeta;\Lambda,\lambda)|\right]\\
\le &Ce^{-\delta_1\zeta^2}\max_{f(x,y,k)=-\zeta,0\le t\le T}|V^R(x,y,k,t;\Lambda,\lambda)|\\
\le &Ce^{-\delta_1\zeta^2}\max_{f(x,y,k)=-\zeta,0\le t\le T}|V(x,y,k,t;\Lambda,1)|\\
\le &Ce^{-\delta_1\zeta^2}\left(e^{\gamma}T+e^{C(\gamma,1,\Lambda,T)+C(\gamma,sng(x),T)x}\exp(\gamma\zeta)\right).
\end{aligned}$$

Noticing $|x| = |w-(1-\tau_g)y-\tau_g k| \le 2\Lambda+\zeta$ when $w = -\zeta$, we have (C.4) by noticing that ζ can be arbitrarily large.

We proceed to considering (C.5). Given any strategy $(C,I,D) \in \mathscr{A}_t(x,y,k)$ of the original optimization problem, we employ it in the case with the compulsory sell boundary Γ_Λ until this boundary is hit; afterwards we follow the optimal strategy. The difference of the generated utility levels is no more than

$$E_t^{x,y,k}\left[e^{-\beta(\tau_\Lambda-t)}\big(V(x_{\tau_\Lambda},y_{\tau_\Lambda},k_{\tau_\Lambda},\tau_\Lambda)-V(x_{\tau_\Lambda},y_{\tau_\Lambda},k_{\tau_\Lambda},\tau_\Lambda;\Lambda)\big)\right], \tag{C.8}$$

where τ_Λ is the hitting time of Γ_Λ.

Lemma C.4. $\max\limits_{(x,y,k,t)\in\Gamma_\Lambda, 0\le t\le T} \frac{V(x,y,k,t;\Lambda)}{V(x,y,k,t)}$ *is uniformly bounded for all* $\Lambda > 1$.

Proof. Similar to Lemma B.1, we have

$$
\begin{aligned}
&|V(x,y,k,t;\Lambda)| \le |V(w,0,0,t;\Lambda)| = e^{-\gamma p(t)w}|V(0,0,0,t;\Lambda)| \\
\le& e^{-\gamma p(t)w}\big((1-\omega)|V(0,0,0,T;\Lambda)| + \omega(T-t)\big) \le e^{-\gamma p(t)w}(T+1),
\end{aligned}
$$

where the first inequality is obtained from considering the strategy of no transaction or consumption during $[t,T]$. It is straightforward to derive $\tilde{h}(y,k,t) \le \bar{h}(t)$ from Proposition 3.4, which implies a uniform upper bound C_U. Then, we have

$$
|V(x,y,k,t)| \ge e^{-\gamma p(t)w} e^{-\gamma C_U}
$$

and the uniform bound can be obtained accordingly. □

Applying this lemma yields that the term defined by (C.8) is no more than

$$
CE_t^{x,y,k}\left[e^{-\beta(\tau_\Lambda - t)} V(x_{\tau_\Lambda}, y_{\tau_\Lambda}, k_{\tau_\Lambda}, \tau_\Lambda)\mathbb{I}_{\{\tau_\Lambda < T\}}\right]
$$

for some constant C. Since $\lim\limits_{\Lambda\to\infty} \tau_\Lambda = \infty$, a.s. and the expectation is dominated by

$$
E_t^{x,y,k}\left[\left(\omega\int_t^T e^{-\beta(s-t)}U(C_s)ds + (1-\omega)e^{-\beta(T-t)}U\big(f(x_T,y_T,k_T)\big)\right)\mathbb{I}_{\{\tau_\Lambda<T\}}\right],
$$

we obtain (C.5).

C.3 Continuity of $V^R(x,y,k,t)$ and $V^R(x,y,k,t;\Lambda)$

We first show that $V(x,y,k,t)$ is locally uniformly Lipschitz continuous in (x,y,k). On the one hand, it is obvious that V is increasing in each argument x, y, k. On the other, given $x' < x$, $y' < y$, and $k' < k$, from Lemma B.1 we have

$$\begin{aligned} &V(x',y',k',t) \geq V(x'-\Delta,y'+\Delta,k'+\Delta,t) \\ =&e^{\gamma p(t)(x+\Delta-x')}V(x,y'+\Delta,k'+\Delta,t) \\ \geq&e^{\gamma p(t)(x+\Delta-x')}V(x,y,k,t) \end{aligned}$$

if $\Delta = \max\{y-y',k-k'\}$. That implies locally uniformly Lipschitz continuity in (x,y,k).

We next show the continuity in the temporal variable t. Given an admissible strategy (C_t,I_t,D_t) for the initial state (x,y,k,t), we implement it on $(x,y,k,t-\delta)$ by setting $C_s^\delta = C_{s+\delta}$, $I_s^\delta = I_{s+\delta}$, $D_s^\delta = D_{s+\delta}$, $\forall t-\delta \leq s \leq T-\delta$. After time $T-\delta$, $C_s^\delta = I_s^\delta = D_s^\delta = 0$. Then, applying the Fatou's lemma yields

$$\limsup_{\delta\to 0} V(x,y,k,t-\delta) \leq V(x,y,k,t).$$

Analogously, we have

$$\limsup_{\delta\to 0} V(x,y,k,t+\delta) \leq V(x,y,k,t).$$

These two inequalities imply the upper semicontinuity. Given equations (C.4), (C.5) and Lemma C.2, $V^R(x,y,k,t;\Lambda)$ is the limit of a monotone increasing sequence of continuous functions, which must be lower-semicontinuous; $V(x,y,k,t)$ is the limit of a monotone increasing sequence of lower-semicontinuous functions ($V^R(x,y,k,t;\Lambda)$), which must also be lower-semicontinuous.

Analogously, $V^R(x,y,k,t;\Lambda)$ is also continuous in all its arguments and locally uniformly Lipchitz continuous in (x,y,k).

C.4 Comparison Principle

Define $\tilde{h}(y,k,t;\Lambda,\lambda)$ as in Proposition 3.5, i.e.,

$$\tilde{h}(y,k,t;\Lambda,\lambda) := -\frac{\log(-V^R(x,y,k,t;\Lambda,\lambda))}{\gamma} - p(t)f(x,y,k).$$

Then, the main differential operator in the interior of the solution domain degenerates to

$$\tilde{h}_t + \mathscr{L}_\lambda \tilde{h} = \tilde{h}_t + \mathscr{L}_0 \tilde{h} + \lambda(\mathscr{B}_0 \tilde{h})^+ + \lambda(\mathscr{S}_0 \tilde{h})^+.$$

Formally, $\tilde{h}(y,k,t;\Lambda,\lambda)$ should satisfy the PDE

$$\tilde{h}_t + \mathscr{L}_\lambda \tilde{h} = 0, \forall y,k > 0, y^2+k^2 < \Lambda^2, 0 \leq t < T \tag{C.9}$$

with the boundary conditions

$$\mathscr{S}_0 \tilde{h} = 0, \qquad \text{when } (y,k,t) \in \Gamma'_\Lambda, \tag{C.10}$$

$$\tilde{h}_t + \mathscr{L}_\lambda \tilde{h} \geq 0, \qquad \text{when } yk = 0, 0 \leq y,k < \Lambda, \tag{C.11}$$

and terminal condition

$$\tilde{h}(y,k,T;\Lambda,\lambda) = -\frac{1}{\gamma}\log(1-\omega), \qquad \forall y,k \geq 0, y^2+k^2 \leq \Lambda^2. \tag{C.12}$$

As usual, we will show that $\tilde{h}(y,k,t;\Lambda,\lambda)$ is a viscosity solution. In order to define the viscosity solution, we first define subsolution and supersolution as follows:

Definition C.5. A continuous function $p(y,k,t)$ defined on the region $\mathscr{D}_\Lambda$ is a subsolution to the PDE problem (C.9)-(C.12) if it satisfies:

(1) For any $g(y,k,t) \in C^{2,1}(\mathscr{D}_\Lambda)$ such that $g-p$ achieves a local minimum of 0 at some point $(y_0,k_0,t_0) \in \mathscr{D}_\Lambda \setminus \Gamma'_\Lambda$, we have

$$g_t + \mathscr{L}_\lambda g \geq 0$$

at (y_0,k_0,t_0).

(2) $\limsup\limits_{\delta \to 0} \frac{p(y,k,t) - p((1-\delta)y,(1-\delta)k,t)}{\delta} \leq 0$ on Γ'_Λ.

(3) $p(y,k,T;\Lambda,\lambda) \leq -\frac{1}{\gamma}\log(1-\omega)$ when $y,k \geq 0$, $y^2+k^2 \leq \Lambda^2$.

Definition C.6. A continuous function $q(y,k,t)$ defined in the region $\mathscr{D}_\Lambda$ to is a supersolution to the PDE problem (C.9)-(C.12) if it satisfies:

(1) For any $g(y,k,t) \in C^{2,1}(\mathscr{D}_\Lambda)$ such that $g-q$ achieves a local maximum of 0 at some point $(y_0,k_0,t_0) \in \mathscr{D}_\Lambda \setminus \Gamma'_\Lambda$, we have

$$g_t + \mathscr{L}_\lambda g \leq 0$$

at (y_0,k_0,t_0).

(2) $\liminf\limits_{\delta \to 0} \frac{q(y,k,t) - q((1-\delta)y,(1-\delta)k,t)}{\delta} \geq 0$ on Γ'_Λ.

(3) $q(y,k,T;\Lambda,\lambda) \geq -\frac{1}{\gamma}\log(1-\omega)$ when $y,k \geq 0$, $y^2+k^2 \leq \Lambda^2$

Then, we have the following definition for viscosity solution:

Definition C.7. A function is a viscosity solution if it is both a subsolution and a supersolution.

Similar to Bian et al. (2021) and Fleming and Soner (2006), we can prove the following theorem:

Theorem C.8. *$\tilde{h}(y,k,t;\Lambda,\lambda)$ is a viscosity solution to the PDE problem* (C.9)-(C.12).

We prove the uniqueness of viscosity solution by the following comparison principle.

Theorem C.9. *Suppose P, Q are respectively a subsolution and supersolution in $\mathscr{D}_\Lambda$ to the PDE problem* (C.9) *with boundary conditions* (C.10), (C.11) *and the terminal condition* (C.12). *If Q is locally uniformly Lipschitz continuous in y, then $P \le Q$.*

Proof. We prove by contradiction. Assume otherwise, then there is $(\hat{y},\hat{k},\hat{t})$ such that $P(\hat{y},\hat{k},\hat{t}) - Q(\hat{y},\hat{k},\hat{t}) > 0$. Given the continuity of P and Q, we can assume $\hat{t} > 0$. We prove the result by the following steps.

1. Set $\mathbf{n} = (1,1,0)$. Denote $\xi = (y,k,t)$, $\eta = (u,v,s)$. For any $i \in \mathbb{N}^+$ and $\delta,\varepsilon > 0$, let

$$
\begin{aligned}
&\Phi(\xi,\eta) = |i(\xi-\eta)+\delta\mathbf{n}|^2, \phi_i(\xi,\eta) = \Phi(\xi,\eta) + \varepsilon(k+y) + \varepsilon(u+v),\\
&\varphi_i(\xi,\eta) = P(\xi) - Q(\eta) - \phi_i(\xi,\eta) - \frac{\delta}{t}, (\xi_i,\eta_i) = \arg\max_{\xi,\eta\in\mathscr{D}_\Lambda} \varphi_i(\xi,\eta).
\end{aligned}
$$

We have $\{(\xi_i,\eta_i,\Phi(\xi_i,\eta_i))\}_{i\ge 1}$ is bounded in $\mathbb{R}^7$. Therefore, there is a subsequence of (ξ_i,η_i) which converges to some $(\bar{\xi},\bar{\eta})$. Without loss of generality, we assume the whole sequence converges. Moreover, for sufficiently small ε and δ we have

$$
\begin{aligned}
P(\xi_i) - Q(\eta_i) &\ge \varphi_i(\xi_i,\eta_i) \ge \varphi_i\big((\hat{y},\hat{k},\hat{t}),(\hat{y},\hat{k},\hat{t})\big)\\
&= P(\hat{y},\hat{k},\hat{t}) - Q(\hat{y},\hat{k},\hat{t}) - |\delta\mathbf{n}|^2 - 2\varepsilon(\hat{y}+\hat{k}) - \frac{\delta}{\hat{t}} > 0.
\end{aligned}
\tag{C.13}
$$

Due to the optimality of (ξ_i,η_i), we have

$$
\varphi_i(\xi_i,\eta_i) \ge \varphi_i(\xi_i,\xi_i),
$$

which implies

$$
\begin{aligned}
|i(\xi_i-\eta_i)+\delta\mathbf{n}|^2 \leq & Q(\xi_i)-P(\xi_i)+\varepsilon(y_i+k_i)+\varepsilon(y_i+k_i) \\
& +P(\xi_i)-Q(\eta_i)-\varepsilon(y_i+k_i)-\varepsilon(u_i+v_i)+2\delta^2 \\
= & Q(\xi_i)-Q(\eta_i)+\varepsilon\big(y_i+k_i-(u_i+v_i)\big)+2\delta^2.
\end{aligned} \tag{C.14}
$$

Since the right hand side is bounded, $|i(\xi_i-\eta_i)+\delta\mathbf{n}|^2$ is uniformly bounded for $i \geq 1$. Therefore, we must have $\bar{\eta}=\bar{\xi}$. Then, according to the continuity of Q, we have from (C.14) that

$$
\lim_{i\to\infty} |i(\xi_i-\eta_i)+\delta\mathbf{n}| \leq \sqrt{2}\delta. \tag{C.15}
$$

2. Denote $g(\xi)=\varepsilon(y+k)$, from Ishii's lemma, there are two matrices $(m_{ij})_{3\times3}$ and $(M_{ij})_{3\times3}$ such that

$$
\big(\Phi_\xi(\xi_i,\eta_i),m\big) \in J^{2+}\left(P-g-\frac{\delta}{t}\right)(\xi_i), \tag{C.16}
$$

$$
\big(-\Phi_\eta(\xi_i,\eta_i),M\big) \in J^{2-}(Q+g)(\eta_i), \tag{C.17}
$$

$$
\begin{pmatrix} m & 0 \\ 0 & -M \end{pmatrix} \leqslant D^2\Phi(\xi_i,\eta_i)+\frac{1}{4i^2}\big(D^2\Phi(\xi_i,\eta_i)\big)^2, \tag{C.18}
$$

where by direct calculation we have

$$
\begin{aligned}
&\Phi_\xi = -\Phi_\eta = 2i^2(\xi-\eta)+2i\delta\mathbf{n}, \\
&\Phi_{\xi\xi} = 2i^2 I, \Phi_{\xi\eta} = -2i^2 I, \Phi_{\eta\eta} = 2i^2 I
\end{aligned} \tag{C.19}
$$

with I being the 3×3 identity matrix. We multiply both sides of (C.18) by the vector $(y,0,0,u,0,0)$, and have from (C.15)

$$y^2 m_{11} - u^2 M_{11} \le 4i^2(y-u)^2 \le 4|i(\xi_i - \eta_i)|^2 \le 4(\sqrt{2}\delta + \delta|\mathbf{n}|)^2 = 32\delta^2. \tag{C.20}$$

3. There are two possible cases.

(i) Given other parameters, $\bar{u}^2 + \bar{v}^2 = \Lambda^2$ for any sufficiently small $\delta > 0$.

For clarity, given $\delta > 0$, we denote by $\bar{\eta}^\delta = (\bar{u}^\delta, \bar{v}^\delta, \bar{s}^\delta)$, ξ_i^δ and η_i^δ the corresponding points. First, we have

$$\begin{aligned}
&\limsup_{i\to\infty} \varphi_i(\xi_i^\delta, \eta_i^\delta) \\
=&\limsup_{i\to\infty} P(\xi_i^\delta) - Q(\eta_i^\delta) - |i(\xi_i^\delta - \eta_i^\delta) + \delta\mathbf{n}|^2 - \varepsilon(k_i^\delta + y_i^\delta) - \varepsilon(u_i^\delta + v_i^\delta) - \frac{\delta}{t_i^\delta} \\
\le&\limsup_{i\to\infty} P(\xi_i^\delta) - Q(\eta_i^\delta) - 2\varepsilon(\bar{u}^\delta + \bar{v}^\delta) - \frac{\delta}{\bar{s}^\delta} \\
=&P(\bar{\eta}^\delta) - Q(\bar{\eta}^\delta) - 2\varepsilon(\bar{u}^\delta + \bar{v}^\delta) - \frac{\delta}{\bar{s}^\delta}.
\end{aligned} \tag{C.21}$$

Second, as $\delta \to 0$, there is a convergent subsequence of $\bar{\eta}^\delta$ which converges to some $\bar{\eta}^0$. Without loss of generality, we assume the whole sequence converges. Denoting unit vector $\mathbf{n}_1 := \frac{(\bar{u}^0, \bar{v}^0)}{\Lambda}$, we have

$$\begin{aligned}
&\varphi_i(\bar{\eta}^0 - \Delta\mathbf{n}_1, \bar{\eta}^0 - \Delta\mathbf{n}_1) \\
=&P(\bar{\eta}^0 - \Delta\mathbf{n}_1) - Q(\bar{\eta}^0 - \Delta\mathbf{n}_1) - |\delta\mathbf{n}|^2 - 2\varepsilon(\bar{u}^0 + \bar{v}^0) + 2\varepsilon\frac{\Delta}{\Lambda}(\bar{u}^0 + \bar{v}^0) - \frac{\delta}{\bar{s}^0}.
\end{aligned} \tag{C.22}$$

Using the boundary condition (3.15), we can infer that for sufficiently small Δ, which is independent of δ, we have

$$P(\bar{\eta}^0) - Q(\bar{\eta}^0) \le P(\bar{\eta}^0 - \Delta\mathbf{n}_1) - Q(\bar{\eta}^0 - \Delta\mathbf{n}_1) + \frac{\varepsilon}{2}\Delta. \tag{C.23}$$

Therefore, by noticing $\bar{u}^0 + \bar{v}^0 \geq \sqrt{(\bar{u}^0)^2 + (\bar{v}^0)^2} = \Lambda$, we can infer from (C.21), (C.22), and (C.23) that

$$\begin{aligned}
&\lim_{\delta \to 0} \liminf_{i \to \infty} \varphi_i(\bar{\eta} - \Delta \mathbf{n}_1, \bar{\eta} - \Delta \mathbf{n}_1) - \varphi_i(\xi_i, \eta_i) \\
\geq &\lim_{\delta \to 0} -\frac{\varepsilon}{2}\Delta - |\delta \mathbf{n}|^2 - 2\varepsilon(\bar{u}^0 + \bar{v}^0) + 2\varepsilon \frac{\Delta}{\Lambda}(\bar{u}^0 + \bar{v}^0) - \frac{\delta}{\bar{s}^0} + 2\varepsilon(\bar{u}^\delta + \bar{v}^\delta) + \frac{\delta}{\bar{s}^\delta} \\
= &-\frac{\varepsilon}{2}\Delta + 2\varepsilon \frac{\Delta}{\Lambda}(\bar{u}^0 + \bar{v}^0) \\
> &0,
\end{aligned}$$

which contradicts with the optimality of ξ_i, η_i.

(ii). There is a sequence $\delta \to 0$, along which $\bar{u}^2 + \bar{v}^2 < \Lambda^2$.

Due to the optimality of (ξ_i, η_i), we have

$$\varphi_i(\xi_i, \eta_i) \geq \varphi_i\left(\bar{\xi}, \bar{\xi} + \frac{\delta \mathbf{n}}{i}\right),$$

which implies

$$\begin{aligned}
|i(\xi_i - \eta_i) + \delta \mathbf{n}|^2 \leq & Q\left(\bar{\xi} + \frac{\delta \mathbf{n}}{i}\right) - P(\bar{\xi}) + \varepsilon(\bar{y} + \bar{k}) + \varepsilon\left(\bar{y} + \bar{k} + \frac{2\delta}{i}\right) + \frac{\delta}{\bar{t}} \\
& + P(\xi_i) - Q(\eta_i) - \varepsilon(y_i + k_i) - \varepsilon(u_i + v_i) - \frac{\delta}{t_i}.
\end{aligned}$$

According to the continuity of P and Q, we have

$$\lim_{i \to \infty} |i(\xi_i - \eta_i) + \delta \mathbf{n}| = 0.$$

Therefore, for $i \gg 1$,

$$\eta_i = \xi_i + \frac{\delta \mathbf{n} + o(1)}{i}$$

and $u_i, v_i > 0$.

For simplicity of notation, we omit i in the following. According to (C.9), (C.11), (C.16), and (C.17), we have

$$\begin{aligned}
&\mathscr{L}_\lambda\left[\xi, P, \Phi_\xi + g_\xi + \left(\frac{\delta}{t}\right)_\xi, m + D^2\left(\frac{\delta}{t}\right)\right] \geq 0,\\
&\mathscr{L}_\lambda[\eta, Q, -\Phi_\eta - g_\eta, M] \leq 0.
\end{aligned}$$

By taking difference of these two terms, we have

$$\begin{aligned}
0 \leq &\Big(-\frac{\delta}{t^2} - p(t)(P-Q) + 2\big(\mu_1 - \gamma\sigma_1^2(1-\tau_g)p(\bar{t})\bar{y}\big)\bar{y}\varepsilon\Big)\\
&+ \frac{1}{2}\sigma_1^2(y^2 m_{11} - u^2 M_{11})\\
&- \frac{1}{2}\sigma_1^2\gamma\Big(y^2\big(2i^2(y-u)+2i\delta+\varepsilon\big)^2 - u^2\big(2i^2(y-u)+2i\delta-\varepsilon\big)^2\Big)\\
&+ \lambda\Big(\big(2i^2(y+k-u-v)+4i\delta+2\varepsilon\big)^+ - \big(2i^2(y+k-u-v)+4i\delta-2\varepsilon\big)^+\Big)\\
&+ \lambda\bigg(\Big(-y\big(2i^2(y-u)+2i\delta+\varepsilon\big) - k\big(2i^2(k-v)+2i\delta+\varepsilon\big)\Big)^+\\
&\qquad - \Big(-y\big(2i^2(y-u)+2i\delta-\varepsilon\big) - k\big(2i^2(k-v)+2i\delta-\varepsilon\big)\Big)^+\bigg)\\
&+ o(1)\\
= &I_1 + I_2 + I_3 + I_4 + I_5 + o(1).
\end{aligned} \tag{C.24}$$

We can infer from $\bar{y} \leq \Lambda$ that

$$I_1 = -\frac{\delta}{t^2} - p(t)(P-Q) + 2\big(\mu_1 - \gamma\sigma_1^2(1-\tau_g)p(\bar{t})\bar{y}\big)\bar{y}\varepsilon \leq -p(t)(P-Q) + 2\mu_1\Lambda\varepsilon.$$

From (C.20), we can infer that

$$I_2 = \frac{1}{2}\sigma_1^2(y^2 m_{11} - u^2 M_{11}) \leq 16\sigma_1^2\delta^2.$$

Denote the uniform Lipschitz constant of Q around $\bar{\eta}$ by L, then from (C.17) we have

$$|\Phi_\eta + g_\eta| \leq L.$$

Given (C.19) and $g_\eta = \varepsilon$, we denote $\zeta = i(y-u) + \delta$ and have

$$|2i\zeta - \varepsilon| \leq L.$$

Therefore, we have

$$\begin{aligned}
& y^2\big(2i^2(y-u) + 2i\delta + \varepsilon\big)^2 - u^2\big(2i^2(y-u) + 2i\delta - \varepsilon\big)^2 \\
=& y^2(2i\zeta + \varepsilon)^2 - u^2(2i\zeta - \varepsilon)^2 \\
=& (y^2 - u^2)(2i\zeta + \varepsilon)^2 + u^2\big((2i\zeta + \varepsilon)^2 - (2i\zeta - \varepsilon)^2\big) \\
\geq& o(1) + u^2 8i\zeta\varepsilon \\
\geq& -4\Lambda^2(L + \varepsilon)\varepsilon + o(1).
\end{aligned}$$

Then $I_3 \leq 2\sigma_1^2\gamma\Lambda^2(L+\varepsilon)\varepsilon$. Since $c_1^+ - c_2^+ \leq \max\{0, c_1 - c_2\}$ for any real numbers c_1, c_2, we have

$$I_4 = \lambda\Big(\big(2i^2(y+k-u-v)+4i\delta+2\varepsilon\big)^+ - \big(2i^2(y+k-u-v)+4i\delta-2\varepsilon\big)^+\Big) \leq 4\varepsilon\lambda$$

and

$$\begin{aligned} I_5 = \lambda\bigg(&\Big(-y\big(2i^2(y-u)+2i\delta+\varepsilon\big)-k\big(2i^2(k-v)+2i\delta+\varepsilon\big)\Big)^+ \\ &-\Big(-y\big(2i^2(y-u)+2i\delta-\varepsilon\big)-k\big(2i^2(k-v)+2i\delta-\varepsilon\big)\Big)^+\bigg) \leq 0. \end{aligned}$$

To sum up, from (C.24) we have

$$0 \leq -p(t)(P-Q)+2\mu_1\Lambda\varepsilon+16\sigma_1^2\delta^2+2\sigma_1^2\gamma\Lambda^2(L+\varepsilon)\varepsilon+4\lambda\varepsilon+o(1).$$

From (C.13), since $-p(t)$ is uniformly negative, we can set δ and ε small enough such that the right hand side is negative, which causes a contradiction.

□

C.5 Minimum Viscosity Solution

In this subsection, we prove that $\tilde{h}(y,k,t;\Lambda)$ is the minimum viscosity solution as described in Theorem 3.9.

First, from Sect. C.3, we know that $\tilde{h}(y,k,t;\Lambda)$ is continuous and locally uniformly Lipschitz in (y,k). Thus, $\tilde{h}(y,k,t;\Lambda)$ is a viscosity solution to this PDE problem from the dynamic programming principle, see, e.g., Fleming and Soner (2006). Next, according to the comparison principle established in Theorem C.9, any viscosity solution $v(y,k,t)$ to the PDE problem (3.14) with boundary conditions (3.15)–(3.17) will be no less than $\tilde{h}(y,k,t;\Lambda,\lambda)$, as long as it is locally uniformly Lipschitz continuous in (y,k). Since (3.14) and (3.16) imply (C.9) and (C.11), and (3.15) and

(3.17) imply (C.10) and (C.12), we can infer that $v(y,k,t)$ must be a supersolution in $\mathscr{D}_\Lambda$ to the PDE problem (C.9) with boundary conditions (C.10)–(C.12).

Lastly, from (C.4) we can infer that

$$v(y,k,t) \geq \lim_{\lambda\to\infty} \tilde{h}(y,k,t;\Lambda,\lambda) = \tilde{h}(y,k,t;\Lambda).$$

Therefore, $\tilde{h}(y,k,t;\Lambda)$ is indeed the minimum viscosity solution.

D Numerical Method

In this section, we present the details of the numerical method used for obtaining the results in Sect. 4.

D.1 The Penalty Method

We adopt the penalty method with finite-difference discretization to solve the reduced HJB equations (see Bian et al. (2021) and Dai et al. (2015)). The penalty approximation to the HJB equation associated with function h is

$$\hat{h}_t + \mathscr{L}\hat{h} + \lambda(\mathscr{B}\hat{h})^+ + \lambda(\mathscr{S}\hat{h})^+ = 0, \qquad t \in [0,T_1), \tag{D.1}$$

$$\hat{h}_t + \mathscr{L}_1\hat{h} + \lambda(\mathscr{B}\hat{h})^+ + \lambda(\mathscr{S}\hat{h})^+ = 0, \qquad t \in [T_1,T_2), \tag{D.2}$$

$$\hat{h}(y,b,T_2) = \iota\tau_g y(1-b)^+ - \frac{1}{\gamma}\log(1-\omega), \tag{D.3}$$

where λ is a large, positive penalty parameter, operators $\mathscr{B}$ and $\mathscr{S}$ are given by Equation (3.19), and operators $\mathscr{L}$ and $\mathscr{L}_1$ are given by

$$
\begin{aligned}
\mathscr{L}\hat{h} =& -p(t)\hat{h} + \big(\mu_1 - \gamma\sigma_1^2(1-\tau_g)p(t)y - \gamma\rho\sigma_1\sigma_2\xi_* g(t)\big)y\hat{h}_y \\
&+ \big(\sigma_1^2 - \mu_1 + \gamma\sigma_1^2(1-\tau_g)p(t)y + \gamma\rho\sigma_1\sigma_2\xi_* g(t)\big)b\hat{h}_b \\
&+ \frac{1}{2}\sigma_1^2 y^2\big(\hat{h}_{yy} - \gamma(\hat{h}_y)^2\big) + \frac{1}{2}\sigma_1^2 b^2\big(\hat{h}_{bb} - \gamma(\hat{h}_b)^2\big) - \sigma_1^2 yb\big(\hat{h}_{yb} - \gamma\hat{h}_y\hat{h}_b\big) \\
&+ \big(\mu_2 + q_2 - r - \gamma\rho\sigma_1\sigma_2(1-\tau_g)p(t)y\big)\xi_* g(t) - \frac{1}{2}\gamma\sigma_2^2\xi_*^2 g(t)^2 \\
&+ \Big(\mu_1(1-\tau_g) + q_1(1-\tau_d) - r(1-\tau_i)\big(1-\tau_g(1-b)\big)\Big)p(t)y \\
&- \frac{\gamma}{2}\sigma_1^2 p(t)^2(1-\tau_g)^2 y^2 + \frac{p(t)}{\gamma}\left(\log\frac{p(t)}{\omega} - 1\right) + \frac{\beta}{\gamma},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{L}_1\hat{h} =& -p(t)\hat{h} + \big(\mu_1 - \gamma\sigma_1^2(1-\tau_g)p(t)y\big)y\hat{h}_y + \big(\sigma_1^2 - \mu_1 + \gamma\sigma_1^2(1-\tau_g)p(t)y\big)b\hat{h}_b \\
&+ \frac{1}{2}\sigma_1^2 y^2\big(\hat{h}_{yy} - \gamma(\hat{h}_y)^2\big) + \frac{1}{2}\sigma_1^2 b^2\big(\hat{h}_{bb} - \gamma(\hat{h}_b)^2\big) - \sigma_1^2 yb\big(\hat{h}_{yb} - \gamma\hat{h}_y\hat{h}_b\big) \\
&+ \Big(\mu_1(1-\tau_g) + q_1(1-\tau_d) - r(1-\tau_i)\big(1-\tau_g(1-b)\big)\Big)p(t)y \\
&- \frac{\gamma}{2}\sigma_1^2 p(t)^2(1-\tau_g)^2 y^2 + \frac{p(t)}{\gamma}\left(\log\frac{p(t)}{\omega} - 1\right) + \frac{\beta}{\gamma}.
\end{aligned}
$$

D.1.1 Computation Domain and Boundary Conditions

In the theoretical analysis presented in Sect. 3.2.3, we impose the sell condition on the artificial boundary $y^2 + k^2 = \Lambda^2$. In the numerical scheme, we instead consider the boundaries $y = \Lambda_1$ and $y/k = \Lambda_2$ for sufficiently large positive constants Λ_1 and Λ_2. In other words, we restrict the computation to the truncated spatial domain in the $y-b$ plane:

$$
D = [0, \Lambda_1] \times [0, \Lambda_2],
$$

and impose the following boundary condition

$$\mathscr{S}h = 0, \text{ at } y = \Lambda_1 \text{ or } b = \Lambda_2.$$

As we elaborate in Sect. 4.2, when y or b is sufficiently high, selling the stock is optimal; thus, these boundary conditions hold numerically. We set Λ_1 and Λ_2 to be sufficiently large so that the no-trade region is a subset of the computation domain D.

In theory, the solution to the penalty approximation (D.1)-(D.3), with sell condition on the boundaries $y = \Lambda_1$ and $b = \Lambda_2$, converges to $\hat{h}(y,b,t;\Lambda_1,\Lambda_2)$ as λ goes to infinity. Here, $\hat{h}(y,b,t;\Lambda_1,\Lambda_2)$ is the solution to the HJB equation associated with h, but with sell condition imposed on the boundaries $y = \Lambda_1$ and $b = \Lambda_2$. We select a sufficiently large λ such that the numerical results remain stable (disregarding variations within a specified small tolerance) when we further increase λ. In particular, we set $\lambda = 10^6$.

On the boundary $y = 0$ $(0 \le b < \Lambda_2)$, we impose the buy condition $\mathscr{B}\hat{h} = 0$, which degenerates to $(1-b)\hat{h}_b = 0$. However, this condition further degenerates and provides no information when $b = 1$. At the point $(y,b) = (0,1)$, we let

$$\hat{h}(0,1,t) = \max_{\tilde{y}\in(0,\Lambda_1]} \hat{h}(\tilde{y},1,t), t \in [0,T_2). \tag{D.4}$$

The reason for imposing the condition (D.4) at $(0,1)$ is that it corresponds to

$$V(x,0,0,z,t) = \max_{\tilde{y}\in(0,\Lambda_1]} V(x-\tilde{y},\tilde{y},\tilde{y},z,t) \tag{D.5}$$

for all $x \in R$, $z \in R$, and $t \in [0,T_1)$, and

$$V^R(x,0,0,t) = \max_{\tilde{y}\in(0,\Lambda_1]} V^R(x-\tilde{y},\tilde{y},\tilde{y},t) \tag{D.6}$$

for all $x \in R$ and $t \in [T_1, T_2)$, due to homotheticity. Equations (D.5)-(D.6) reveal the intuition that the optimal investment policy at $y = 0$ is to rebuild the optimal stock position immediately.

Finally, at the boundary $b = 0$ $(0 < y < \Lambda_1)$, we directly use the degenerated versions of Equations (D.1)-(D.2) as the boundary conditions.

D.1.2 Numerical Scheme

The numerical scheme is identical to the one used in Dai and Zhong (2010), except that we need to update ξ_* in each iteration. In particular, we use the fully implicit finite-difference method to discretize the linear terms (e.g., $\hat{h}_y$ and $\hat{h}_{yy}$) and apply the Newton iteration to discretize the nonlinear terms (e.g., $(\hat{h}_y)^2$ and $(\mathscr{B}\hat{h})^+$). We take $\Delta t = \frac{1}{12}$ as the time step. Additionally, we choose $\Delta y = \frac{\Lambda_1}{400}$ and $\Delta b = \frac{1}{100}$ as the mesh sizes in the y and b directions, respectively. Let ξ_*^l and $\hat{h}^l$ denote the l^{th} estimates for ξ_* and $\hat{h}$, respectively. Given the estimate ξ_*^l, we numerically solve for $\hat{h}^{l+1}$. We then calculate ξ_*^{l+1} using (3.21) by replacing h_y and h_b with the central differences of $\hat{h}^{l+1}$ with respect to y and b, respectively, until convergence.

D.2 Monte Carlo Simulations

We apply Monte Carlo simulations to estimate the investor's average consumption rate, average cumulative tax bills, and distribution of wealth at retirement, given her initial state $(x_{0-}, y_{0-}, k_{0-}, z_{0-})$.

Let $t_n = n\Delta t$ for $n = 0, 1, \cdots, \frac{T_2}{\Delta t}$, and assume the investor takes action on these discrete time points. For those n with $t_n < T_1$, given the state $(x_{t_n-}, y_{t_n-}, k_{t_n-}, z_{t_n-})$, the corresponding state variables in the $y - b$ plane are $(y_{t_n-}, \frac{k_{t_n-}}{y_{t_n-}})$. Accordingly, we can numerically solve the optimal consumption-investment policies using the method described in Appendix D.1. In particular, let

(y_{t_n}, b_{t_n}) be the after-trade state in the $y-b$ plane at time t_n, which leads to

$$
\begin{aligned}
dI_{t_n} =& (y_{t_n} - y_{t_n-})^+ \mathbb{I}_{\{b_{t_n-} \leq 1\}} + y_{t_n} \mathbb{I}_{\{b_{t_n-} > 1\}}, && \text{(D.7)} \\
dD_{t_n} =& \left(\frac{y_{t_n-} - y_{t_n}}{y_{t_n-}} \right)^+ \mathbb{I}_{\{b_{t_n-} \leq 1\}} + \mathbb{I}_{\{b_{t_n-} > 1\}}. && \text{(D.8)}
\end{aligned}
$$

Thus, we have

$$
(x_{t_n}, y_{t_n}, k_{t_n}, z_{t_n}) = \big(x_{t_n-} - dI_{t_n} + f(0, y_{t_n-}, k_{t_n-}) dD_{t_n}, y_{t_n}, y_{t_n} b_{t_n}, z_{t_n-} \big)
$$

according to (2.2). Let C_{t_n} and ξ_{t_n} be the optimal consumption and the dollar amount invested in stock S_2 given the state (y_{t_n}, b_{t_n}), respectively. We take a random draw (z_{1n}, z_{2n}) from the two-dimensional standard normal distribution. Then, we update the state at t_{n+1-} using

$$
\begin{aligned}
x_{t_{n+1-}} =& x_{t_n} + \big(r(1-\tau_i) x_{t_n} - C_{t_n} + q_1 (1-\tau_d) y_{t_n} \big) \Delta t \\
& + (1-\tau_L) \int_{t_n}^{t_{n+1}} \big(L(t) - L_C(t) \big) dt, \\
y_{t_{n+1-}} =& y_{t_n} + \mu_1 y_{t_n} \Delta t + \sigma_1 y_{t_n} \sqrt{\Delta t} z_{1n}, \\
k_{t_{n+1-}} =& k_{t_n}, \\
z_{t_{n+1-}} =& z_{t_n} + \big(r z_{t_n} + \xi_{t_n} (\mu_2 + q_2 - r) \big) \Delta t + \xi_{t_n} \sigma_2 \sqrt{\Delta t} (\rho z_{1n} + \sqrt{1-\rho^2} z_{2n}) \\
& + (1+\alpha) \int_{t_n}^{t_{n+1}} L_C(t) dt.
\end{aligned}
$$

The cumulative ordinary income tax and capital gains tax up to time t_n are

$$
\sum_{j=0}^{n-1} \left((r \tau_i x_{t_j} + q_1 \tau_d y_{t_j}) \Delta t + \tau_L \int_{t_j}^{t_{j+1}} \big(L(t) - L_C(t) \big) dt \right)
$$

and

$$\sum_{j=0}^{n} \tau_g (y_{t_j-} - k_{t_j-}) dD_{t_j}$$

respectively.

For the n with $t_n = T_1$, the wealth in the pension account is fully distributed to the investor, so the after-distribution state is

$$\big(x_{t_n-} + (1-\tau_L) z_{t_n-}, y_{t_n-}, k_{t_n-}, 0\big),$$

and the total wealth at retirement is

$$x_{t_n-} + (1-\tau_L) z_{t_n-} + (1-\tau_g) y_{t_n-} + \tau_g k_{t_n-}.$$

Then, the investor optimally adjusts the stock position in the ordinary account by following the post-retirement investment policy. We still denote (y_{t_n}, b_{t_n}) as the after-trade state in the $y-b$ plane at time t_n, then dI_{t_n} and dD_{t_n} are given by (D.7)-(D.8). The position in the ordinary account is thus

$$\big(x_{t_n-} + (1-\tau_L) z_{t_n-} - dI_{t_n} + f(0, y_{t_n-}, k_{t_n-}) dD_{t_n}, y_{t_n}, y_{t_n} b_{t_n}\big).$$

The cumulative ordinary income tax and capital gains tax up to time t_n are

$$\sum_{j=0}^{n-1} \left((r\tau_i x_{t_j} + q_1 \tau_d y_{t_j}) \Delta t + \tau_L \int_{t_j}^{t_{j+1}} \big(L(t) - L_C(t)\big) dt \right) + \tau_L z_{t_n-}$$

and

$$\sum_{j=0}^{n} \tau_g (y_{t_j-} - k_{t_j-}) dD_{t_j}$$

respectively.

For those n with $t_n > T_1$, the consumption rate and cumulative tax bills can be simulated similar to the pre-retirement period by setting $L(t) = L_C(t) = 0$.

We simulate $50,000$ paths for generating the simulation results reported in the paper.

Acknowledgements

We thank the Editor, an anonymous Associate Editor, and two anonymous Reviewers for very constructive comments on the manuscript. Dai acknowledges the supports of Hong Kong Research Grants Council (15217123, 15212324, and T32-615/24-R), the Key Project of National Natural Science Foundation of China (72432005), and The Hong Kong Polytechnic University Research Grant (P0042708). Qian acknowledges the supports of the start-up grant of the Hong Kong University of Science and Technology (B000-0172-R9863).

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