THE TOPOS-THEORETIC APPROACH FORCING

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ABSTRACT. We develop the necessary categorical notions to describe an elementary topos and relevant examples, such as categories of sets, bundles, and sheaves. We then examine how taking sheaves over a partial order relates to Cohen's method of forcing and use this to construct a topos which 'models' $ZFC+\neg CH$.

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1. Introduction to Toposes

To begin, we introduce the notion of an elementary topos.

Definition 1.1. An elementary topos is a category C such that

- ullet C is finitely complete, i.e., $\mathcal C$ has all finite limits
- ullet C is finitely cocomplete, i.e., C has all finite colimits
- $\mathcal C$ has exponentiation
- \bullet \mathcal{C} has a subobject classifier

We expand on each of these notions below.

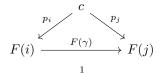
1.1. Finite Limits.

Let \mathcal{C} be a category.

Definition 1.2. A terminal object t is an object of \mathcal{C} such that, for every object a of \mathcal{C} , there is a unique morphism $\varphi_a: a \to t$.

Let I be a finite category and let $F: I \to \mathcal{C}$ be a functor.

Definition 1.3. A cone on F is an object c of C and a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.



The collection of cones on F form a category $\operatorname{Cone}(F)$ in which the morphisms $(c, \{p_i\}_{i \in I}) \to (d, \{q_i\}_{i \in I})$ are morphisms $\varphi : c \to d$ in \mathcal{C} such that, for every $i \in I$, the following diagram commutes.

$$c \xrightarrow{\varphi} d$$

$$F(i)$$

Definition 1.4. A limit of F is a terminal object in the category Cone(F)

Definition 1.5. A diagram of shape I is a functor $F: I \to \mathcal{C}$. A limit of a diagram is the limit of said functor.

We say that a category \mathcal{C} is finitely complete if every finite diagram of \mathcal{C} has a limit. Equivalently, a category is finitely complete if and only if it has a terminal object and pullbacks.

Definition 1.6. A pullback is a limit of the diagram of shape

$$b \xrightarrow{q} z$$

In **Set**, given a diagram $f: A \to Z \leftarrow B: g$, the pullback is the set $A \times_Z B = \{(a, b) \in A \times B: f(a) = g(b)\}$, together with the restrictions to $A \times_Z B$ of the projection maps π_A, π_B .

1.2. Finite Colimits.

Definition 1.7. An initial object i is an object of C such that, for every object a of C, there is a unique morphism $\varphi_a: i \to a$

Definition 1.8. A cocone of F is an object c of C and a collection of morphisms $\{q_i : F(i) \to c\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.

$$F(i) \xrightarrow{F(\gamma)} F(j)$$

$$q_i \xrightarrow{q_j} c$$

Cocones on F form a category Cocone(F) in a similar way to cones.

Definition 1.9. A *colimit of* F is an initial object in Cocone(F)

We say that a category \mathcal{C} is finitely cocomplete if every finite diagram of \mathcal{C} has a colimit. Equivalently, a category is finitely cocomplete if and only if it has an initial object and pushouts.

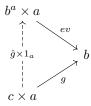
Definition 1.10. A pushout is a limit of the diagram of shape

$$\begin{array}{ccc}
z & \xrightarrow{f} & a \\
\downarrow g & & \\
b & & & \\
\end{array}$$

In **Set**, given a diagram $f: Z \to A$, $g: Z \to B$, the pushout is the set $A \sqcup_Z B = A \sqcup B/\{(a,b) \in A \sqcup B : \exists z \in Z(f(z) = a \land g(z) = b)\}$, together with the inclusions to $A \sqcup B$ composed with the quotient map $A \sqcup B \to A \sqcup_Z B$.

1.3. Exponentiation.

Let \mathcal{C} be a category with binary products. We say that \mathcal{C} has exponentiation if, for all object a, b of \mathcal{C} , there is an object b^a and a morphism $ev: b^a \times a \to b$ such that, for any object c and morphism $g: c \times a \to b$, there is a unique morphism $\hat{g}: c \to b^a$ such that the following diagram commutes



In a category \mathcal{C} with exponentiation, we have a bijection $\operatorname{Hom}_{\mathcal{C}}(c \times b, a) \cong \operatorname{Hom}_{\mathcal{C}}(c, b^a)$

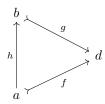
In **Set**, the exponentiation B^A is the set of functions from A to B.

A category with finite limits and exponentiation is called *Cartesian closed*.

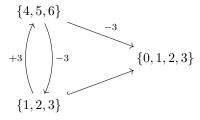
1.4. Subobject Classifier.

Any monomorphism $f: A \rightarrow B$ in the category of sets denotes a subset of B, namely, the image of f, which is isomorphic to B. Similarly, in an arbitary category C, a *subobject* of d is a monomorphism $f: a \rightarrow d$. We can define an 'inclusion' between subobjects

Definition 1.11. Given two subobjects $f: a \rightarrow d$ and $g: b \rightarrow d$ of d, we say that $f \subseteq g$ if there is a (necessarily monic) morphism $h: a \rightarrow b$ such that the following diagram commutes



We note that \subseteq is reflexive and transitive, but not quite antisymmetric. Take for example $a = \{4, 5, 6\}$, $b = \{1, 2, 3\}$, $d = \{0, 1, 2, 3\}$, g inclusion, $f : a \to d$ mapping $x \mapsto x - 3$.

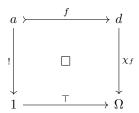


However, when we have such a diagram, i.e., when $f \subseteq g$ and $g \subseteq f$, we have isomorphic subobjects $f \cong g$. Thankfully, \cong is an equivalence relation. We form the collection $\operatorname{Sub}(d) = \{[f] : f \text{ is monic with target } d\}$. As such, we redefine a 'subobject' to be an equivalence class in $\operatorname{Sub}(d)$. In **Set**, we have an isomorphism $\operatorname{Sub}(A) \cong \mathcal{P}(A)$. We now define a general analog to the fact that $2^A \cong \mathcal{P}(A)$.

Definition 1.12. Let \mathcal{C} be a category with a terminal object 1. A *subobject classifier* is an object Ω of \mathcal{C} together with a morphism $\top: 1 \to \Omega$ satisfying the Ω -axiom:

For any subobject $f: a \rightarrowtail d$ there is a unique characteristic map $\chi_f: d \to \Omega$ such that the following diagram is a

pullback square



In any category \mathcal{C} that has a subobject classifier and exponentiation, we have $\mathrm{Sub}(d) \cong \mathrm{Hom}_{\mathcal{C}}(d,\Omega) \cong \Omega^d$. In **Set**, the subobject classifier is any two element set; we let $1 := \{1\}$ be our terminal object and we use $2 := \{0,1\}$ together with the inclusion map $\top : 1 \to 2$ as our classifier.

The Ω -axiom is a topos-theoretic analog to the comprehension axiom of ZFC.

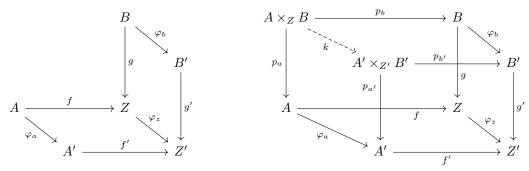
2. Examples

2.1. Set and Set^{\rightarrow} . Clearly Set is a topos (see above).

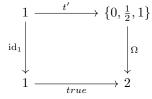
Another example of a topos is the category $\mathbf{Set}^{\rightarrow}$, in which the objects are morphisms in \mathbf{Set} and the morphisms are commuting squares: given $f: A \rightarrow B$ and $g: C \rightarrow D$, a morphism from f to g is a pair of functions $\varphi_a: A \rightarrow C$ and $\varphi_b: B \rightarrow D$ such that

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\varphi_a \downarrow & & \downarrow \varphi_b \\
C & \xrightarrow{g} & D
\end{array}$$

commutes. The terminal object of $\mathbf{Set}^{\rightarrow}$ is the identity morphism $1 \rightarrow 1$ in \mathbf{Set} . A pullback diagram (left) has a limit (right) made by forming the pullbacks of the front and back \mathbf{Set} -diagrams



where k is given by the pullback diagrams in **Set**. The subobject classifier in $\mathbf{Set}^{\rightarrow}$ is the object $\Omega: \{0, \frac{1}{2}, 1\} \rightarrow 2$ together with the morphism $\top: \mathrm{id}_1 \rightarrow \Omega$



2.2. **Bundles.** Let \mathcal{A} be a collection of pairwise disjoint sets. We can index these sets with the set I, such that $\mathcal{A} = \{A_i : i \in I\}$, and let $A = \bigcup_{i \in I} A_i$. We can then visualize our structure with the following image from Goldblatt's *Topoi*.

We have a map $p: A \to I$, where p(a) = i iff $a \in A_i$, which is well defined by the disjointness condition on \mathcal{A} .

Definition 2.1. We introduce the following terminology:

- (i) The set A_i is called the *stalk* or *fiber* over i
- (ii) The members of A_i are called the *germs* at i
- (iii) The set I is called the base space
- (iv) The set A is called the stalk space

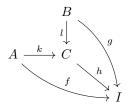
(v) The whole structure is called a bundle over I

Conversely, given any map $p: A \to I$ we can define $A_i := p^{-1}(\{i\})$ for each $i \in I$, and define $\mathcal{A} := \{A_i : i \in I\}$. Then \mathcal{A} is a bundle over I.

We now consider the category $\mathbf{Bn}(I)$ of bundles over I. It is easy to see that this is the same as the category $\mathbf{Set} \downarrow I$. A morphism in $\mathbf{Bn}(I)$, say $k: \mathcal{A} \to \mathcal{B}$, is a morphism in $\mathbf{Set} \ \hat{k}: A \to B$ such that, if $f: A \to I$ and $g: B \to I$ are the functions associated to \mathcal{A} and \mathcal{B} respectively, we have $g \circ \hat{k} = f$.

Proposition 2.1. The category Bn(I) is an elementary topos.

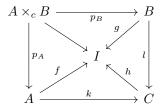
Proof. The terminal object 1 is the pair (I, id_I) . The stalk of this bundle over i is the set $\{i\}$, which is terminal in **Set**. Thus, given any bundle (A, f) over I, the morphism $f : A \to I$ gives rise the unique morphism $A \to 1$. Given a diagram



in $\mathbf{Bn}(I)$, the pullback is given by the pullback in \mathbf{Set} of the square

$$\begin{array}{ccc}
A \times_{c} B & \xrightarrow{p_{B}} B \\
\downarrow^{p_{A}} & \downarrow \downarrow \\
A & \xrightarrow{k} C
\end{array}$$

resulting in the following diagram in $\mathbf{Bn}(I)$



Since $\mathbf{Bn}(I)$ has a terminal object and pullbacks, $\mathbf{Bn}(I)$ is finitely complete. The proof of finite cocompleteness is similar, with the initial object being the 'empty bundle'.

The subobject classifier is the pair $\Omega = (2 \times I, p_I)$, where p_I is the projection onto I. The stalk over i is the set $\Omega_i = \{(0,i),(1,i)\} = 2 \times \{i\}$. We can think of the morphism $T: 1 \to \Omega$ as a bundle of morphisms $true_i: \{i\} \to 2 \times \{i\}$ mapping i to (1,i).

If we consider sets A, B such that $A \subseteq B$, and bundles $\mathcal{A} = (A, f), \mathcal{B} = (B, g)$ over I, we want to know how the characteristic map $\chi_A : \mathcal{B} \to \Omega$ acts. If we think about the characteristic map $\chi_A : B \to 2$ in **Set**, we answer our own question. For any element $x \in B$, we simply map x to $(\chi_A(x), g(x))$.

Note that \top is a section of the bundle Ω , i.e., it picks one germ out of each stalk. This property is true of any map from the terminal object 1 in a category of bundles. So, a map $1 \to \mathcal{A}$ in $\mathbf{Bn}(I)$, is a section of \mathcal{A} . Thus, when we consider the truth values of $\mathbf{Bn}(I)$, i.e., the elements of Ω , we are considering the sections of Ω . But we have an isomorphism $\mathrm{Hom}(1,\Omega) \cong \mathrm{Sub}(1)$, so the truth values in a category of bundles over I are exactly the subsets of I.

2.3. Sheaves. Sheaves are a sort of topological analog to bundles in which the base space I is a topological space. We consider sheaves over a topological space (I, Θ) .

Definition 2.2. A sheaf over I is a pair (A, p) where A is a topological space and $p : A \to I$ is a continuous map that is also a local homeomorphism.

The category $\mathbf{Sh}(I,\Theta)$ of sheaves over (I,Θ) is a topos. We construct the subobject classifier as follows:

For each $i \in I$ we define the equivalence relation \sim_i on Θ by

$$U \sim_i V \iff \exists W \in I : i \in W \text{ and } U \cap W = V \cap W$$

The idea is that $U \sim_i V$ iff they are 'the same' local to i. The equivalence class $[U]_i$ is the germ of i at U, it represents the points of U that are 'close' to i. We take as the stalk over i the set $\Omega_i = \{(i, [U]_i) : U \in \Theta\}$. Letting $\hat{I} = \bigcup \Omega_i$, we have as the subobject classifier the sheaf $\Omega = (\hat{I}, p)$ where $p : \hat{I} \to I$ is the natural map. The topology on \hat{I} has as a basis the sets $[U, V] = \{(i, [U]_i) : i \in V\}$ where $U \subseteq V \in \Theta$.

2.4. **Presheaves.** Let \mathcal{C} be a small category. Then the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} = \hat{\mathcal{C}}$ ("presheaves over \mathcal{C} ") is a topos. The objects are functors $F: \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and the morphisms are natural transformations between functors

 \mathcal{C}^{op} $\underbrace{\alpha}_{F'}$ **Set** . Recall a natural transformation assigns to each object a of \mathcal{C} a morphism $\alpha_a : F(a) \to F'(a)$

such that the following diagram commutes for every $f \in \text{Hom}_{\mathbf{Set}}(a, b)$

$$F(a) \xrightarrow{F(f)} F(b)$$

$$\downarrow^{\alpha_a} \qquad \downarrow^{\alpha_b}$$

$$F'(a) \xrightarrow{F'(f)} F'(b)$$

Definition 2.3. For any small category C we have the Yoneda embedding

$$y: \mathcal{C} \to \hat{\mathcal{C}}$$

 $X \mapsto \operatorname{Hom}_{\mathcal{C}}(-, X)$

For each X, the functor $y(X): \mathcal{C}^{\text{op}\to\mathbf{Set}}$ is the representable presheaf

For the purposes of forcing, we are interested in presheaves over a poset \mathbb{P}

3. HEYTING AND BOOLEAN ALGEBRAS

Write up more-for the talk, just mention:

- In a boolean algebra, such as the lattice of subsets $\mathcal{P}(A)$ every element x has a complement $\neg x$ such that $x \land \neg x = 0, \ x \lor \neg x = 1$
- in a heyting algebra, such as the algebra of truth values in the category of sheaves over a space, this is not the case. an open set U has a pseudocomplement $Int(U^c)$, which is not the complement of U unless U is clopen. So taking the join does not necessarily give us the whole space (which is 1 in the algebra). This is why topoi are great for modeling intuitionistic logic.

4. Logic in Toposes

Write up more-for the talk, just mention

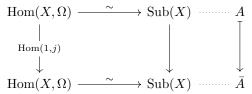
• The internal logic of a topos is given by the elements of the subobject classifier Ω , called the truth values. Since this forms a closed lattice, we can do logic using meets, joins, complements/psuedocomplements etc. So some topoi have an intuitionistic internal logic, whereas some, like **Set** have a classical internal logic

5. Lawvere-Tierney Topology

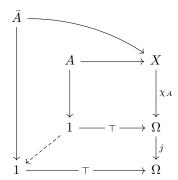
Given an elementary topos \mathcal{C} , we can define a categorical analog to a topology

Definition 5.1. A Lawvere-Tierney topology is a map $j: \Omega \to \Omega$ such that $j \circ \top = \top$, $j \circ j = j$, and $j \circ \wedge = \wedge \circ (j \times j)$, i.e., the following diagrams commute.

A Lawvere-Tierney topology j determines a unary operator, the closure, $A \mapsto \bar{A}$ on the subobjects $A \mapsto X$ for every object X in \mathcal{C} via the following diagram



The closure \bar{A} is the subobject of X with characteristic morphism $j \circ \chi_A$



An equivalent statement to the definition of a Lawvere-Tierney topology is as follows: for every object $A, A \subseteq \bar{A}$, $\bar{A} = \bar{A}$, and $\bar{A} \cap \bar{B} = A \cap \bar{B}$.

We can take sheaves over Lawvere-Tierney topologies. We first introduce the notion of a dense subobject: a subobject $A \rightarrow X$ is dense in X if $\bar{A} = X$. In this case, we call the morphism $A \rightarrow X$ a dense monomorphism.

Definition 5.2. An object F of C is a *sheaf* for j, or a j-sheaf if, for every dense monomorphism $m:A \rightarrow X$, composition with m induces an isomorphism $m^*: \operatorname{Hom}_{\mathcal{C}}(X,F) \rightarrow \operatorname{Hom}_{\mathcal{C}}(A,F)$, i.e., we have the following commutative diagram:

$$A \longrightarrow F$$

$$\downarrow^{\text{dense}} Y$$

The particular Lawvere-Tierney topology that we are interested in is the double negation topology $\neg\neg:\Omega\to\Omega$, also called the dense topology. This is of interest to us because taking sheaves over $\neg\neg$ lets us pass from a topos with arbitrary internal logic to a topos with a classical internal logic. This is easy to see: for a sheaf for the double negation topology, the identity $\neg\neg S = S$ necessarily holds!

6. Forcing

We introduce the notion of a natural numbers object $1 \xrightarrow{0} N \xrightarrow{s} N$, for which the following diagram commutes for any $1 \xrightarrow{x} X \xrightarrow{f} X$

We start with a countable transitive model $M \models ZFC$, which by the axiom of infinity will have a natural numbers object N.

Lemma 6.1. If C is a topos with a natural numbers object, and D is a topos with functors

$$\mathcal{D} \rightleftarrows \mathcal{C}$$

Then \mathcal{D} has a natural numbers object as well.

Corollaries of this lemma will help us in the construction of new toposes. The first is that the category of presheaves over a topos that models ZFC has a natural numbers object:

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \overset{\Gamma}{\underset{\Delta}{\longleftarrow}} \mathbf{Set}$$

Where Γ takes the global section of each presheaf and Δ assigns each set to the constant functor from \mathcal{C}^{op} to that set.

Further, if a topos C has a natural numbers object, then the category of sheaves over a Lawvere-Tierney topology j has a natural numbers object:

$$\mathbf{Sh}_{j}\mathcal{C} \overset{i}{\underset{sh}{\longleftarrow}} \mathcal{C}$$

where sh is the 'sheafification' functor, which is left adjoint to the inclusion functor.

For the purposes of forcing, we will using the Cohen Poset \mathbb{P} , defined below.

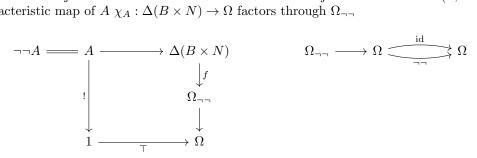
In our model M of ZFC we want to force a set B in between the sets N and $Sub(N) \cong 2^N$. We do so by approximating a monomorphism $g: B \to \mathcal{P}(N)$. For a subset $F_p \subseteq B \times N$ and a function $p: F_p \to 2$, the pair (F_p, p) is called a condition. We define an order on these conditions

$$q \leq p \iff F_q \supseteq F_p \text{ and } q|_{F_p} = p$$

i.e., q extends p.

We now take presheaves $M^{\mathbb{P}^{op}}$. By the above, this category has a natural numbers object. We build a subobject A of the constant functor $\Delta(B \times N)$, where $A(p) = \{(b,n) : p(b,n) = 0\}$. A is in fact a closed subobject of $\Delta(B \times N)$ with respect to the double negation topology, i.e. $\neg \neg A = A$ in $\operatorname{Sub}(\Delta(B \times N))$. Letting Ω be the subobject classifier for $M^{\mathbb{P}^{op}}$ and $\Omega_{\neg \neg}$ the subobject classifier for $\operatorname{Sh}(\mathbb{P}, \neg \neg)$. Because A is

Letting Ω be the subobject classifier for $M^{\mathbb{P}^{op}}$ and $\Omega_{\neg\neg}$ the subobject classifier for $\mathbf{Sh}(\mathbb{P}, \neg\neg)$. Because A is closed, the characteristic map of A $\chi_A : \Delta(B \times N) \to \Omega$ factors through $\Omega_{\neg\neg}$



So we have a map $f: \Delta B \times \Delta N \to \Omega_{\neg\neg}$, from which we can obtain a map $g: \Delta(B) \to \Omega_{\neg\neg}^{\Delta N}$, which is a monomorphism.

The sheafification functor $sh_{\neg\neg}$ is left exact and thus preserves monomorphisms. As such, our morphism g in the category of presheaves is sent to a monomorphism $\hat{g}: \hat{B} \to \Omega_{\neg\neg}^{\hat{N}}$

We thus have $\hat{N} \mapsto \hat{B} \mapsto P(\hat{N}) = \hat{2}^{\hat{N}}$. Although our monomorphism $B \mapsto \hat{2}^{\hat{N}}$ may not be 'strict', through some additional very involved methods (expand in these notes later) we can find that, given strict inequalities $N < 2^N < B$ in our original model M, taking sheaves results in strict inequalities $\hat{N} < 2^{\hat{N}} < \hat{B}$ in the new topos. Then from the two results above, we will have $\hat{N} < 2^{\hat{N}} < \hat{B} < \hat{2}^{\hat{N}}$

7. References

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