A TOPOS THEORETIC APPROACH FORCING

KAY THOMPSON

ABSTRACT. In this talk we develop the categorical notions relevant to topos theory in order to construct a topos that models $ZFC + \neg CH$.

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1. Introduction to Toposes

To begin, we introduce the notion of an elementary topos.

Definition 1.1. An *elementary topos* is a category C such that

- ullet C is finitely complete, i.e., $\mathcal C$ has all finite limits
- ullet C is finitely cocomplete, i.e., C has all finite colimits
- \bullet $\mathcal C$ has exponentiation
- \bullet \mathcal{C} has a subobject classifier

We expand on each of these notions below.

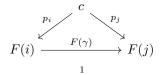
1.1. Finite Limits.

Let \mathcal{C} be a category.

Definition 1.2. A terminal object t is an object of C such that, for every object a of C, there is a unique morphism $\varphi_a: a \to t$.

Let I be a finite category and let $F: I \to \mathcal{C}$ be a functor.

Definition 1.3. A cone on F is an object c of C and a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.



The collection of cones on F form a category $\operatorname{Cone}(F)$ in which the morphisms $(c, \{p_i\}_{i \in I}) \to (d, \{q_i\}_{i \in I})$ are morphisms $\varphi : c \to d$ in \mathcal{C} such that, for every $i \in I$, the following diagram commutes.

$$c \xrightarrow{\varphi} d$$

$$F(i)$$

Definition 1.4. A limit of F is a terminal object in the category Cone(F)

Definition 1.5. A diagram of shape I is a functor $F: I \to \mathcal{C}$. A limit of a diagram is the limit of said functor.

We say that a category \mathcal{C} is finitely complete if every finite diagram of \mathcal{C} has a limit. Equivalently, a category is finitely complete if and only if it has a terminal object and pullbacks.

Definition 1.6. A pullback is a limit of the diagram of shape

$$b \xrightarrow{q} z$$

In **Set**, given a diagram $f: A \to Z \leftarrow B: g$, the pushout is the set $A \times_Z B = \{(a,b) \in A \times B: f(a) = g(b)\}$, together with the restrictions to $A \times_Z B$ of the projection maps π_A, π_B .

1.2. Finite Colimits.

Definition 1.7. An initial object i is an object of C such that, for every object a of C, there is a unique morphism $\varphi_a: i \to a$

Definition 1.8. A cocone of F is an object c of C and a collection of morphisms $\{q_i : F(i) \to c\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.

$$F(i) \xrightarrow{F(\gamma)} F(j)$$

$$\downarrow q_i \qquad \downarrow q_j$$

Cocones on F form a category Cocone(F) in a similar way to cones.

Definition 1.9. A *colimit of* F is an initial object in Cocone(F)

We say that a category \mathcal{C} is finitely cocomplete if every finite diagram of \mathcal{C} has a colimit. Equivalently, a category is finitely cocomplete if and only if it has an initial object and pushouts.

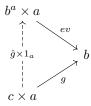
Definition 1.10. A pushout is a limit of the diagram of shape

$$\begin{array}{ccc}
z & \xrightarrow{f} & a \\
\downarrow g & & \\
b & & & \\
\end{array}$$

In **Set**, given a diagram $f: Z \to A$, $g: Z \to B$, the pushout is the set $A \sqcup_Z B = A \sqcup B/\{(a,b) \in A \sqcup B : \exists z \in Z(f(z) = a \land g(z) = b)\}$, together with the inclusions to $A \sqcup B$ composed with the quotient map $A \sqcup B \to A \sqcup_Z B$.

1.3. Exponentiation.

Let \mathcal{C} be a category with binary products. We say that \mathcal{C} has exponentiation if, for all object a, b of \mathcal{C} , there is an object b^a and a morphism $ev: b^a \times a \to b$ such that, for any object c and morphism $g: c \times a \to b$, there is a unique morphism $\hat{g}: c \to b^a$ such that the following diagram commutes



In a category \mathcal{C} with exponentiation, we have a bijection $\operatorname{Hom}_{\mathcal{C}}(c \times b, a) \cong \operatorname{Hom}_{\mathcal{C}}(c, b^a)$

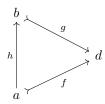
In **Set**, the exponentiation B^A is the set of functions from A to B.

A category with finite limits and exponentiation is called *Cartesian closed*.

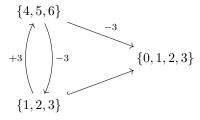
1.4. Subobject Classifier.

Any monomorphism $f: A \rightarrow B$ in the category of sets denotes a subset of B, namely, the image of f, which is isomorphic to B. Similarly, in an arbitary category C, a *subobject* of d is a monomorphism $f: a \rightarrow d$. We can define an 'inclusion' between subobjects

Definition 1.11. Given two subobjects $f: a \rightarrow d$ and $g: b \rightarrow d$ of d, we say that $f \subseteq g$ if there is a (necessarily monic) morphism $h: a \rightarrow b$ such that the following diagram commutes



We note that \subseteq is reflexive and transitive, but not quite antisymmetric. Take for example $a = \{4, 5, 6\}$, $b = \{1, 2, 3\}$, $d = \{0, 1, 2, 3\}$, g inclusion, $f : a \to d$ mapping $x \mapsto x - 3$.

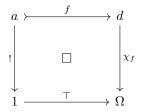


However, when we have such a diagram, i.e., when $f \subseteq g$ and $g \subseteq f$, we have isomorphic subobjects $f \cong g$. Thankfully, \cong is an equivalence relation. We form the collection $\operatorname{Sub}(d) = \{[f] : f \text{ is monic with target } d\}$. As such, we redefine a 'subobject' to be an equivalence class in $\operatorname{Sub}(d)$. In **Set**, we have an isomorphism $\operatorname{Sub}(A) \cong \mathcal{P}(A)$. We now define a general analog to the fact that $2^A \cong \mathcal{P}(A)$.

Definition 1.12. Let \mathcal{C} be a category with a terminal object 1. A *subobject classifier* is an object Ω of \mathcal{C} together with a morphism $\top : 1 \to \Omega$ satisfying the Ω -axiom:

For any subobject $f: a \rightarrow d$ there is a unique characteristic map $\chi_f: d \rightarrow \Omega$ such that the following diagram is a

pullback square



In any category \mathcal{C} that has a subobject classifier, we have $\mathrm{Sub}(d) \cong \mathrm{Hom}_{\mathcal{C}}(d,\Omega) \cong \Omega^d$. In **Set**, the subobject classifer is any two element set; we let $1 := \{1\}$ be our terminal object and we use $2 := \{0,1\}$ together with the inclusion map $\top : 1 \to 2$ as our classifier.

The Ω -axiom is a topos-theoretic analog to the comprehension axiom of ZFC.

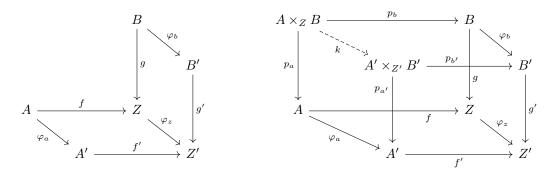
1.5. Examples.

Clearly **Set** is a topos (see above).

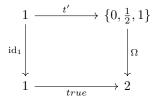
Another example of a topos is the category $\mathbf{Set}^{\rightarrow}$, in which the objects are morphisms in \mathbf{Set} and the morphisms are commuting squares: given $f: A \rightarrow B$ and $g: C \rightarrow D$, a morphism from f to g is a pair of functions $\varphi_a: A \rightarrow C$ and $\varphi_b: B \rightarrow D$ such that

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\varphi_a \downarrow & & \downarrow \varphi_b \\
C & \xrightarrow{g} & D
\end{array}$$

commutes. The terminal object of $\mathbf{Set}^{\rightarrow}$ is the identity morphism $1 \rightarrow 1$ in \mathbf{Set} . A pullback diagram (left) has a limit (right) made by forming the pullbacks of the front and back \mathbf{Set} -diagrams



where k is given by the pullback diagrams in **Set**. The subobject classifier in $\mathbf{Set}^{\rightarrow}$ is the object $\Omega: \{0, \frac{1}{2}, 1\} \rightarrow 2$ together with the morphism $\top: \mathrm{id}_1 \rightarrow \Omega$



Let \mathcal{C} be a small category. Then the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} = \hat{\mathcal{C}}$ ("presheaves over \mathcal{C} ") is a topos. The objects are functors $F: \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and the morphisms are natural transformations between functors $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathbf{Set}$.

Recall a natural transformation assigns to each object a of \mathcal{C} a morphism $\alpha_a: F(a) \to F'(a)$ such that the following

diagram commutes for every $f \in \text{Hom}_{\mathbf{Set}}(a, b)$

$$F(a) \xrightarrow{F(f)} F(b)$$

$$\downarrow^{\alpha_a} \qquad \qquad \downarrow^{\alpha_b}$$

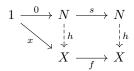
$$F'(a) \xrightarrow{F'(f)} F'(b)$$

2. Logic in Toposes

3. Lawvere-Tierney Topology

4. Forcing

We introduce the notion of a natural numbers object $1 \xrightarrow{0} N \xrightarrow{s} N$, for which the following diagram commutes for any $1 \xrightarrow{x} X \xrightarrow{f} X$



5. References

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