# A TOPOS THEORETIC APPROACH FORCING

KAY THOMPSON

Abstract. In this talk we develop the categorical notions relevant to topos theory in order to construct a topos that models $ZFC + \neg CH$.

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## 1. Introduction to Toposes

To begin, we introduce the notion of an elementary topos.

**Definition 1.1.** An *elementary topos* is a category $\mathcal{C}$ such that

- $\mathcal{C}$ is finitely complete, i.e., $\mathcal{C}$ has all finite limits
- $\mathcal{C}$ is finitely cocomplete, i.e., $\mathcal{C}$ has all finite colimits
- $\mathcal{C}$ has exponentiation
- $\mathcal{C}$ has a subobject classifier

We expand on each of these notions below.

### 1.1. Finite Limits.

Let $\mathcal{C}$ be a category.

**Definition 1.2.** A *terminal object* $t$ is an object of $\mathcal{C}$ such that, for every object $a$ of $\mathcal{C}$, there is a unique morphism $\varphi_a : a \to t$.

Let $I$ be a finite category and let $F : I \to \mathcal{C}$ be a functor.

**Definition 1.3.** A *cone on $F$* is an object $c$ of $\mathcal{C}$ and a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in $I$, the following diagram commutes.

$$\begin{tikzcd} & c \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ F(i) \arrow[rr, "F(\gamma)"] & & F(j) \end{tikzcd}$$

The collection of cones on $F$ form a category Cone($F$) in which the morphisms $(c, \{p_i\}_{i\in I}) \to (d, \{q_i\}_{i\in I})$ are morphisms $\varphi : c \to d$ in $\mathcal{C}$ such that, for every $i \in I$, the following diagram commutes.

$$\begin{tikzcd} c \arrow[rr, "\varphi"] \arrow[dr, "p_i"'] & & d \arrow[dl, "q_i"] \\ & F(i) & \end{tikzcd}$$

**Definition 1.4.** A *limit of* $F$ is a terminal object in the category Cone($F$)

**Definition 1.5.** A *diagram of shape* $I$ is a functor $F : I \to \mathcal{C}$. A *limit* of a diagram is the limit of said functor.

We say that a category $\mathcal{C}$ is finitely complete if every finite diagram of $\mathcal{C}$ has a limit. Equivalently, a category is finitely complete if and only if it has a terminal object and pullbacks.

**Definition 1.6.** A *pullback* is a limit of the diagram of shape

$$\begin{tikzcd} & a \arrow[d, "f"] \\ b \arrow[r, "g"'] & z \end{tikzcd}$$

In **Set**, given a diagram $f : A \to Z \leftarrow B : g$, the pushout is the set $A \times_Z B = \{(a, b) \in A \times B : f(a) = g(b)\}$, together with the restrictions to $A \times_Z B$ of the projection maps $\pi_A, \pi_B$.

1.2. **Finite Colimits.**

**Definition 1.7.** An *initial object* $i$ is an object of $\mathcal{C}$ such that, for every object $a$ of $\mathcal{C}$, there is a unique morphism $\varphi_a : i \to a$

**Definition 1.8.** A *cocone of* $F$ is an object $c$ of $\mathcal{C}$ and a collection of morphisms $\{q_i : F(i) \to c\}_{i\in I}$ such that, for every morphism $\gamma : i \to j$ in $I$, the following diagram commutes.

$$\begin{tikzcd} F(i) \arrow[rr, "F(\gamma)"] \arrow[dr, "q_i"'] & & F(j) \arrow[dl, "q_j"] \\ & c & \end{tikzcd}$$

Cocones on $F$ form a category Cocone($F$) in a similar way to cones.

**Definition 1.9.** A *colimit of* $F$ is an initial object in Cocone($F$)

We say that a category $\mathcal{C}$ is finitely cocomplete if every finite diagram of $\mathcal{C}$ has a colimit. Equivalently, a category is finitely cocomplete if and only if it has an initial object and pushouts.

**Definition 1.10.** A *pushout* is a limit of the diagram of shape

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "g"'] & a \\ b & \end{tikzcd}$$

In **Set**, given a diagram $f : Z \to A$, $g : Z \to B$, the pushout is the set $A \sqcup_Z B = A \sqcup B/\{(a, b) \in A \sqcup B : \exists z \in Z(f(z) = a \wedge g(z) = b)\}$, together with the inclusions to $A \sqcup B$ composed with the quotient map $A \sqcup B \to A \sqcup_Z B$.

1.3. **Exponentiation.**

Let $\mathcal{C}$ be a category with binary products. We say that $\mathcal{C}$ has *exponentiation* if, for all object $a, b$ of $\mathcal{C}$, there is an object $b^a$ and a morphism $ev : b^a \times a \to b$ such that, for any object $c$ and morphism $g : c \times a \to b$, there is a unique morphism $\hat{g} : c \to b^a$ such that the following diagram commutes

$$\begin{tikzcd}
b^a \times a \arrow[dr, "ev"] & \\
 & b \\
c \times a \arrow[uu, dashed, "\hat{g}\times 1_a"] \arrow[ur, "g"'] &
\end{tikzcd}$$

In a category $\mathcal{C}$ with exponentiation, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(c \times b, a) \cong \mathrm{Hom}_{\mathcal{C}}(c, b^a)$

In **Set**, the exponentiation $B^A$ is the set of functions from $A$ to $B$.

A category with finite limits and exponentiation is called *Cartesian closed.*

1.4. **Subobject Classifier.**

Any monomorphism $f : A \rightarrowtail B$ in the category of sets denotes a subset of $B$, namely, the image of $f$, which is isomorphic to $B$. Similarly, in an arbitary category $\mathcal{C}$, a *subobject* of $d$ is a monomorphism $f : a \rightarrowtail d$. We can define an 'inclusion' between subobjects

**Definition 1.11.** Given two subobjects $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ of $d$, we say that $f \subseteq g$ if there is a (necessarily monic) morphism $h : a \to b$ such that the following diagram commutes

$$\begin{tikzcd}
b \arrow[dr, tail, "g"] & \\
 & d \\
a \arrow[uu, "h"] \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

We note that $\subseteq$ is reflexive and transitive, but not quite antisymmetric. Take for example $a = \{4, 5, 6\}$, $b = \{1, 2, 3\}$, $d = \{0, 1, 2, 3\}$, $g$ inclusion, $f : a \to d$ mapping $x \mapsto x - 3$.

$$\begin{tikzcd}
\{4,5,6\} \arrow[dr, "-3"] \arrow[dd, bend left, "-3"] & \\
 & \{0,1,2,3\} \\
\{1,2,3\} \arrow[uu, bend left, "+3"] \arrow[ur, hook] &
\end{tikzcd}$$

However, when we have such a diagram, i.e., when $f \subseteq g$ and $g \subseteq f$, we have isomorphic subobjects $f \cong g$. Thankfully, $\cong$ is an equivalence relation. We form the collection $\mathrm{Sub}(d) = \{[f] : f \text{ is monic with target } d\}$. As such, we redefine a 'subobject' to be an equivalence class in $\mathrm{Sub}(d)$. In **Set**, we have an isomorphism $\mathrm{Sub}(A) \cong \mathcal{P}(A)$. We now define a general analog to the fact that $2^A \cong \mathcal{P}(A)$.

**Definition 1.12.** Let $\mathcal{C}$ be a category with a terminal object 1. A *subobject classifier* is an object $\Omega$ of $\mathcal{C}$ together with a morphism $\top : 1 \to \Omega$ satisfying the $\Omega$-axiom:
For any subobject $f : a \rightarrowtail d$ there is a unique *characteristic map* $\chi_f : d \to \Omega$ such that the following diagram is a

pullback square

$$\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\square"] & d \arrow[d, "\chi_f"] \\
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

In any category $\mathcal{C}$ that has a subobject classifier, we have $\mathrm{Sub}(d) \cong \mathrm{Hom}_{\mathcal{C}}(d, \Omega) \cong \Omega^d$. In **Set**, the subobject classifer is any two element set; we let $1 := \{1\}$ be our terminal object and we use $2 := \{0, 1\}$ together with the inclusion map $\top : 1 \to 2$ as our classifier.
The $\Omega$-axiom is a topos-theoretic analog to the comprehension axiom of $ZFC$.

1.5. **Examples.**

Clearly **Set** is a topos (see above).
Another example of a topos is the category $\mathbf{Set}^{\to}$, in which the objects are morphisms in **Set** and the morphisms are commuting squares: given $f : A \to B$ and $g : C \to D$, a morphism from $f$ to $g$ is a pair of functions $\varphi_a : A \to C$ and $\varphi_b : B \to D$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi_a"'] & B \arrow[d, "\varphi_b"] \\
C \arrow[r, "g"] & D
\end{tikzcd}$$

commutes. The terminal object of $\mathbf{Set}^{\to}$ is the identity morphism $1 \to 1$ in **Set**. A pullback diagram (left) has a limit (right) made by forming the pullbacks of the front and back **Set**-diagrams

$$\begin{tikzcd}
 & B \arrow[d, "g"] \arrow[dr, "\varphi_b"] & \\
 & {} & B' \arrow[dd, "g'"] \\
A \arrow[r, "f"] \arrow[dr, "\varphi_a"'] & Z \arrow[dr, "\varphi_z"] & \\
 & A' \arrow[r, "f'"] & Z'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times_Z B \arrow[rr, "p_b"] \arrow[dd, "p_a"'] \arrow[dr, dashed, "k"'] & & B \arrow[dd, "g"] \arrow[dr, "\varphi_b"] & \\
 & A' \times_{Z'} B' \arrow[rr, "p_{b'}"] \arrow[dd, "p_{a'}"'] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\varphi_a"'] & & Z \arrow[dr, "\varphi_z"] & \\
 & A' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

where $k$ is given by the pullback diagrams in **Set**. The subobject classifier in $\mathbf{Set}^{\to}$ is the object $\Omega : \{0, \frac{1}{2}, 1\} \to 2$ together with the morphism $\top : \mathrm{id}_1 \to \Omega$

$$\begin{tikzcd}
1 \arrow[r, "t'"] \arrow[d, "\mathrm{id}_1"'] & \{0, \frac{1}{2}, 1\} \arrow[d, "\Omega"] \\
1 \arrow[r, "true"'] & 2
\end{tikzcd}$$

Let $\mathcal{C}$ be a small category. Then the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} = \hat{\mathcal{C}}$ ("presheaves over $\mathcal{C}$") is a topos. The objects are functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and the morphisms are natural transformations between functors $\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbf{Set} \end{tikzcd}$.
Recall a natural transformation assigns to each object $a$ of $\mathcal{C}$ a morphism $\alpha_a : F(a) \to F'(a)$ such that the following

diagram commutes for every $f \in \mathrm{Hom}_{\mathbf{Set}}(a, b)$

$$\begin{tikzcd}
F(a) \arrow[r, "F(f)"'] \arrow[d, "\alpha_a"'] & F(b) \arrow[d, "\alpha_b"] \\
F'(a) \arrow[r, "F'(f)"'] & F'(b)
\end{tikzcd}$$

## 2. Logic in Toposes

## 3. Lawvere-Tierney Topology

## 4. Forcing

We introduce the notion of a *natural numbers object* $1 \xrightarrow{0} N \xrightarrow{s} N$ , for which the following diagram commutes for any $1 \xrightarrow{x} X \xrightarrow{f} X$

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\
& X \arrow[r, "f"'] & X
\end{tikzcd}$$

## 5. References

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