THE TOPOS-THEORETIC APPROACH FORCING

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Abstract. We develop the necessary categorical notions to describe an elementary topos and relevant examples, such as categories of sets, bundles, and sheaves. We then examine how taking sheaves over a partial order relates to Cohen's method of forcing and use this to construct a topos which 'models' ZFC+¬CH.

Contents

1. Introduction to Toposes

To begin, we introduce the notion of an elementary topos.

Definition 1.1. An *elementary topos* is a category $\mathcal{C}$ such that

- $\mathcal{C}$ is finitely complete, i.e., $\mathcal{C}$ has all finite limits
- $\mathcal{C}$ is finitely cocomplete, i.e., $\mathcal{C}$ has all finite colimits
- $\mathcal{C}$ has exponentiation
- $\mathcal{C}$ has a subobject classifier

We expand on each of these notions below.

1.1. Finite Limits.

Let $\mathcal{C}$ be a category.

Definition 1.2. A *terminal object* t is an object of $\mathcal{C}$ such that, for every object a of $\mathcal{C}$, there is a unique morphism $\varphi_a : a \to t$.

Let I be a finite category and let $F : I \to \mathcal{C}$ be a functor.

Definition 1.3. A *cone on* F is an object c of $\mathcal{C}$ and a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.

$$\begin{tikzcd} & c \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ F(i) \arrow[rr, "F(\gamma)"] & & F(j) \end{tikzcd}$$

The collection of cones on F form a category $\mathrm{Cone}(F)$ in which the morphisms $(c, \{p_i\}_{i \in I}) \to (d, \{q_i\}_{i \in I})$ are morphisms $\varphi : c \to d$ in $\mathcal{C}$ such that, for every $i \in I$, the following diagram commutes.

$$\begin{tikzcd} c \arrow[rr, "\varphi"] \arrow[dr, "p_i"'] & & d \arrow[dl, "q_i"] \\ & F(i) & \end{tikzcd}$$

Definition 1.4. A *limit of* F is a terminal object in the category $\mathrm{Cone}(F)$

Definition 1.5. A *diagram of shape* I is a functor $F : I \to \mathcal{C}$. A *limit* of a diagram is the limit of said functor.

We say that a category $\mathcal{C}$ is finitely complete if every finite diagram of $\mathcal{C}$ has a limit. Equivalently, a category is finitely complete if and only if it has a terminal object and pullbacks.

Definition 1.6. A *pullback* is a limit of the diagram of shape

$$\begin{tikzcd} & a \arrow[d, "f"] \\ b \arrow[r, "g"'] & z \end{tikzcd}$$

In **Set**, given a diagram $f : A \to Z \leftarrow B : g$, the pullback is the set $A \times_Z B = \{(a, b) \in A \times B : f(a) = g(b)\}$, together with the restrictions to $A \times_Z B$ of the projection maps π_A, π_B.

1.2. **Finite Colimits.**

Definition 1.7. An *initial object* i is an object of $\mathcal{C}$ such that, for every object a of $\mathcal{C}$, there is a unique morphism $\varphi_a : i \to a$

Definition 1.8. A *cocone of* F is an object c of $\mathcal{C}$ and a collection of morphisms $\{q_i : F(i) \to c\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.

$$\begin{tikzcd} F(i) \arrow[rr, "F(\gamma)"] \arrow[dr, "q_i"'] & & F(j) \arrow[dl, "q_j"] \\ & c & \end{tikzcd}$$

Cocones on F form a category $\mathrm{Cocone}(F)$ in a similar way to cones.

Definition 1.9. A *colimit of* F is an initial object in $\mathrm{Cocone}(F)$

We say that a category $\mathcal{C}$ is finitely cocomplete if every finite diagram of $\mathcal{C}$ has a colimit. Equivalently, a category is finitely cocomplete if and only if it has an initial object and pushouts.

Definition 1.10. A *pushout* is a limit of the diagram of shape

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, "g"'] & a \\ b & \end{tikzcd}$$

In **Set**, given a diagram $f : Z \to A$, $g : Z \to B$, the pushout is the set $A \sqcup_Z B = A \sqcup B / \{(a, b) \in A \sqcup B : \exists z \in Z(f(z) = a \wedge g(z) = b)\}$, together with the inclusions to $A \sqcup B$ composed with the quotient map $A \sqcup B \to A \sqcup_Z B$.

1.3. **Exponentiation.**

Let $\mathcal{C}$ be a category with binary products. We say that $\mathcal{C}$ has *exponentiation* if, for all object a, b of $\mathcal{C}$, there is an object b^a and a morphism $ev : b^a \times a \to b$ such that, for any object c and morphism $g : c \times a \to b$, there is a unique morphism $\hat{g} : c \to b^a$ such that the following diagram commutes

$$\begin{tikzcd}
b^a \times a \arrow[dr, "ev"] & \\
& b \\
c \times a \arrow[uu, dashed, "\hat{g} \times 1_a"] \arrow[ur, "g"'] &
\end{tikzcd}$$

In a category $\mathcal{C}$ with exponentiation, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(c \times b, a) \cong \mathrm{Hom}_{\mathcal{C}}(c, b^a)$

In **Set**, the exponentiation B^A is the set of functions from A to B.

A category with finite limits and exponentiation is called *Cartesian closed.*

1.4. **Subobject Classifier.**

Any monomorphism $f : A \rightarrowtail B$ in the category of sets denotes a subset of B, namely, the image of f, which is isomorphic to B. Similarly, in an arbitary category $\mathcal{C}$, a *subobject* of d is a monomorphism $f : a \rightarrowtail d$. We can define an 'inclusion' between subobjects

Definition 1.11. Given two subobjects $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ of d, we say that $f \subseteq g$ if there is a (necessarily monic) morphism $h : a \to b$ such that the following diagram commutes

$$\begin{tikzcd}
b \arrow[dr, tail, "g"] & \\
& d \\
a \arrow[uu, "h"] \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

We note that $\subseteq$ is reflexive and transitive, but not quite antisymmetric. Take for example $a = \{4, 5, 6\}$, $b = \{1, 2, 3\}$, $d = \{0, 1, 2, 3\}$, g inclusion, $f : a \to d$ mapping $x \mapsto x - 3$.

$$\begin{tikzcd}
\{4,5,6\} \arrow[dr, "-3"] \arrow[dd, bend left, "-3"] & \\
& \{0,1,2,3\} \\
\{1,2,3\} \arrow[uu, bend left, "+3"] \arrow[ur, hook] &
\end{tikzcd}$$

However, when we have such a diagram, i.e., when $f \subseteq g$ and $g \subseteq f$, we have isomorphic subobjects $f \cong g$. Thankfully, $\cong$ is an equivalence relation. We form the collection $\mathrm{Sub}(d) = \{[f] : f \text{ is monic with target } d\}$. As such, we redefine a 'subobject' to be an equivalence class in $\mathrm{Sub}(d)$. In **Set**, we have an isomorphism $\mathrm{Sub}(A) \cong \mathcal{P}(A)$. We now define a general analog to the fact that $2^A \cong \mathcal{P}(A)$.

Definition 1.12. Let $\mathcal{C}$ be a category with a terminal object 1. A *subobject classifier* is an object Ω of $\mathcal{C}$ together with a morphism $\top : 1 \to \Omega$ satisfying the Ω-axiom:
For any subobject $f : a \rightarrowtail d$ there is a unique *characteristic map* $\chi_f : d \to \Omega$ such that the following diagram is a

pullback square

$$\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\square"] & d \arrow[d, "\chi_f"] \\
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

In any category $\mathcal{C}$ that has a subobject classifier and exponentiation, we have $\mathrm{Sub}(d) \cong \mathrm{Hom}_{\mathcal{C}}(d, \Omega) \cong \Omega^d$. In **Set**, the subobject classifer is any two element set; we let $1 := \{1\}$ be our terminal object and we use $2 := \{0, 1\}$ together with the inclusion map $\top : 1 \to 2$ as our classifier.
The Ω-axiom is a topos-theoretic analog to the comprehension axiom of ZFC.

2. Examples

2.1. Set and Set$^{\to}$.

Clearly **Set** is a topos (see above).
Another example of a topos is the category **Set**$^{\to}$, in which the objects are morphisms in **Set** and the morphisms are commuting squares: given $f : A \to B$ and $g : C \to D$, a morphism from f to g is a pair of functions $\varphi_a : A \to C$ and $\varphi_b : B \to D$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi_a"'] & B \arrow[d, "\varphi_b"] \\
C \arrow[r, "g"] & D
\end{tikzcd}$$

commutes. The terminal object of **Set**$^{\to}$ is the identity morphism $1 \to 1$ in **Set**. A pullback diagram (left) has a limit (right) made by forming the pullbacks of the front and back **Set**-diagrams

$$\begin{tikzcd}
 & & B \arrow[dr, "\varphi_b"] \arrow[dd, "g"] & \\
 & & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\varphi_a"'] & & Z \arrow[dr, "\varphi_z"] & \\
 & A' \arrow[rr, "f'"] & & Z'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times_Z B \arrow[rr, "p_b"] \arrow[dr, dashed, "k"'] \arrow[dd, "p_a"'] & & B \arrow[dr, "\varphi_b"] \arrow[dd, "g"] & \\
 & A' \times_{Z'} B' \arrow[rr, "p_{b'}"] \arrow[dd, "p_{a'}"'] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\varphi_a"'] & & Z \arrow[dr, "\varphi_z"] & \\
 & A' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

where k is given by the pullback diagrams in **Set**. The subobject classifier in **Set**$^{\to}$ is the object $\Omega : \{0, \frac{1}{2}, 1\} \to 2$ together with the morphism $\top : \mathrm{id}_1 \to \Omega$

$$\begin{tikzcd}
1 \arrow[r, "t'"] \arrow[d, "\mathrm{id}_1"'] & \{0, \frac{1}{2}, 1\} \arrow[d, "\Omega"] \\
1 \arrow[r, "true"'] & 2
\end{tikzcd}$$

2.2. Bundles.

Let $\mathcal{A}$ be a collection of pairwise disjoint sets. We can index these sets with the set I, such that $\mathcal{A} = \{A_i : i \in I\}$, and let $A = \bigcup \mathcal{A} = \bigcup_{i \in I} A_i$. We can then visualize our structure with the following image from Goldblatt's *Topoi*.

We have a map $p : A \to I$, where $p(a) = i$ iff $a \in A_i$, which is well defined by the disjointness condition on $\mathcal{A}$.

Definition 2.1. We introduce the following terminology:

(i) The set A_i is called the *stalk* or *fiber* over i
(ii) The members of A_i are called the *germs* at i
(iii) The set I is called the base space
(iv) The set A is called the stalk space

(v) The whole structure is called a *bundle* over I

Conversely, given any map $p : A \to I$ we can define $A_i := p^{-1}(\{i\})$ for each $i \in I$, and define $\mathcal{A} := \{A_i : i \in I\}$. Then $\mathcal{A}$ is a bundle over I.
We now consider the category $\mathbf{Bn}(I)$ of bundles over I. It is easy to see that this is the same as the category $\mathbf{Set} \downarrow I$. A morphism in $\mathbf{Bn}(I)$, say $k : \mathcal{A} \to \mathcal{B}$, is a morphism in $\mathbf{Set}$ $\hat{k} : A \to B$ such that, if $f : A \to I$ and $g : B \to I$ are the functions associated to $\mathcal{A}$ and $\mathcal{B}$ respectively, we have $g \circ \hat{k} = f$.

Proposition 2.1. *The category* $\mathbf{Bn}(I)$ *is an elementary topos.*

Proof. The terminal object 1 is the pair (I, id_I). The stalk of this bundle over i is the set $\{i\}$, which is terminal in $\mathbf{Set}$. Thus, given any bundle (A, f) over I, the morphism $f : A \to I$ gives rise the the unique morphism $\mathcal{A} \to 1$.
Given a diagram

$$\begin{tikzcd} & B \arrow[d, "l"'] \arrow[dr, bend left, "g"] & \\ A \arrow[r, "k"] \arrow[rr, bend right, "f"'] & C \arrow[r, "h"] & I \end{tikzcd}$$

in $\mathbf{Bn}(I)$, the pullback is given by the pullback in $\mathbf{Set}$ of the square

$$\begin{tikzcd} A \times_c B \arrow[r, "p_B"'] \arrow[d, "p_A"] & B \arrow[d, "l"'] \\ A \arrow[r, "k"] & C \end{tikzcd}$$

resulting in the following diagram in $\mathbf{Bn}(I)$

$$\begin{tikzcd} A \times_c B \arrow[rr, "p_B"'] \arrow[dr] \arrow[dd, "p_A"] & & B \arrow[dl, "g"'] \arrow[dd, "l"] \\ & I & \\ A \arrow[ur, "f"] \arrow[rr, "k"] & & C \arrow[ul, "h"'] \end{tikzcd}$$

Since $\mathbf{Bn}(I)$ has a terminal object and pullbacks, $\mathbf{Bn}(I)$ is finitely complete. The proof of finite cocompleteness is similar, with the initial object being the 'empty bundle'.
The subobject classifier is the pair $\Omega = (2 \times I, p_I)$, where p_I is the projection onto I. The stalk over i is the set $\Omega_i = \{(0, i), (1, i)\} = 2 \times \{i\}$. We can think of the morphism $\top : 1 \to \Omega$ as a bundle of morphisms $true_i : \{i\} \to 2 \times \{i\}$ mapping i to $(1, i)$. $\square$

If we consider sets A, B such that $A \subseteq B$, and bundles $\mathcal{A} = (A, f), \mathcal{B} = (B, g)$ over I, we want to know how the characteristic map $\chi_{\mathcal{A}} : \mathcal{B} \to \Omega$ acts. If we think about the characteristic map $\chi_A : B \to 2$ in $\mathbf{Set}$, we answer our own question. For any element $x \in B$, we simply map x to $(\chi_A(x), g(x))$.

Note that $\top$ is a section of the bundle Ω, i.e., it picks one germ out of each stalk. This property is true of any map from the terminal object 1 in a category of bundles. So, a map $1 \to \mathcal{A}$ in $\mathbf{Bn}(I)$, is a section of $\mathcal{A}$. Thus, when we consider the truth values of $\mathbf{Bn}(I)$, i.e., the elements of Ω, we are considering the sections of Ω. But we have an isomorphism $\mathrm{Hom}(1, \Omega) \cong \mathrm{Sub}(1)$, so the truth values in a category of bundles over I are exactly the subsets of I.

2.3. **Sheaves.** Sheaves are a sort of topological analog to bundles in which the base space I is a topological space. We consider sheaves over a topological space (I, Θ).

Definition 2.2. A *sheaf* over I is a pair (A, p) where A is a topological space and $p : A \to I$ is a continuous map that is also a local homeomorphism.

The category $\mathbf{Sh}(I, \Theta)$ of sheaves over (I, Θ) is a topos. We construct the subobject classifier as follows:

For each $i \in I$ we define the equivalence relation $\sim_i$ on Θ by

$$U \sim_i V \iff \exists W \in I : i \in W \text{ and } U \cap W = V \cap W$$

The idea is that $U \sim_i V$ iff they are 'the same' local to i. The equivalence class $[U]_i$ is the germ of i at U, it represents the points of U that are 'close' to i. We take as the stalk over i the set $\Omega_i = \{(i, [U]_i) : U \in \Theta\}$. Letting $\hat{I} = \bigcup \Omega_i$, we have as the subobject classifier the sheaf $\Omega = (\hat{I}, p)$ where $p : \hat{I} \to I$ is the natural map. The topology on $\hat{I}$ has as a basis the sets $[U, V] = \{(i, [U]_i) : i \in V\}$ where $U \subseteq V \in \Theta$.

2.4. **Presheaves.** Let $\mathcal{C}$ be a small category. Then the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} = \hat{\mathcal{C}}$ ("presheaves over $\mathcal{C}$") is a topos. The objects are functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and the morphisms are natural transformations between functors

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbf{Set} \end{tikzcd}$$

. Recall a natural transformation assigns to each object a of $\mathcal{C}$ a morphism $\alpha_a : F(a) \to F'(a)$ such that the following diagram commutes for every $f \in \mathrm{Hom}_{\mathbf{Set}}(a, b)$

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"'] \arrow[d, "\alpha_a"'] & F(b) \arrow[d, "\alpha_b"] \\ F'(a) \arrow[r, "F'(f)"'] & F'(b) \end{tikzcd}$$

Definition 2.3. For any small category $\mathcal{C}$ we have the *Yoneda embedding*

$$\begin{aligned} y : \mathcal{C} &\to \hat{\mathcal{C}} \\ X &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, X) \end{aligned}$$

For each X, the functor $y(X) : \mathcal{C}^{\mathrm{op} \to \mathbf{Set}}$ is the *representable presheaf*

For the purposes of forcing, we are interested in presheaves over a poset $\mathbb{P}$

3. Heyting and Boolean Algebras

Write up more–for the talk, just mention:

- In a boolean algebra, such as the lattice of subsets $\mathcal{P}(A)$ every element x has a complement $\neg x$ such that $x \wedge \neg x = 0$, $x \vee \neg x = 1$
- in a heyting algebra, such as the algebra of truth values in the category of sheaves over a space, this is not the case. an open set U has a pseudocomplement $Int(U^c)$, which is not the complement of U unless U is clopen. So taking the join does not necessarily give us the whole space (which is 1 in the algebra). This is why topoi are great for modeling intuitionistic logic.

4. Logic in Toposes

Write up more–for the talk, just mention

- The internal logic of a topos is given by the elements of the subobject classifier Ω, called the truth values. Since this forms a closed lattice, we can do logic using meets, joins, complements/psuedocomplements etc. So some topoi have an intuitionistic internal logic, whereas some, like **Set** have a classical internal logic

5. Lawvere-Tierney Topology

Given an elementary topos $\mathcal{C}$, we can define a categorical analog to a topology

Definition 5.1. A *Lawvere-Tierney topology* is a map $j : \Omega \to \Omega$ such that $j \circ \top = \top$, $j \circ j = j$, and $j \circ \wedge = \wedge \circ (j \times j)$, i.e., the following diagrams commute.

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

A Lawvere-Tierney topology j determines a unary operator, the closure, $A \mapsto \bar{A}$ on the subobjects $A \rightarrowtail X$ for every object X in $\mathcal{C}$ via the following diagram

$$
\begin{tikzcd}
\mathrm{Hom}(X,\Omega) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}(1,j)"] & \mathrm{Sub}(X) \arrow[d] \arrow[r, dotted, no head] & A \arrow[d, mapsto] \\
\mathrm{Hom}(X,\Omega) \arrow[r, "\sim"] & \mathrm{Sub}(X) \arrow[r, dotted, no head] & \bar{A}
\end{tikzcd}
$$

The closure $\bar{A}$ is the subobject of X with characteristic morphism $j \circ \chi_A$

$$
\begin{tikzcd}
\bar{A} \arrow[rrd, bend left] \arrow[ddd] & & \\
& A \arrow[r] \arrow[d] & X \arrow[d, "\chi_A"] \\
& 1 \arrow[r, "\top"] \arrow[dl, dashed] & \Omega \arrow[d, "j"] \\
1 \arrow[rr, "\top"] & & \Omega
\end{tikzcd}
$$

An equivalent statement to the definition of a Lawvere-Tierney topology is as follows: for every object A, $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, and $\bar{A} \cap \bar{B} = A \bar{\cap} B$.

We can take sheaves over Lawvere-Tierney topologies. We first introduce the notion of a dense subobject: a subobject $A \rightarrowtail X$ is dense in X if $\bar{A} = X$. In this case, we call the morphism $A \rightarrowtail X$ a dense monomorphism.

Definition 5.2. An object F of $\mathcal{C}$ is a *sheaf* for j, or a *j-sheaf* if, for every dense monomorphism $m : A \rightarrowtail X$, composition with m induces an isomorphism $m^* : \mathrm{Hom}_{\mathcal{C}}(X, F) \to \mathrm{Hom}_{\mathcal{C}}(A, F)$, i.e., we have the following commutative diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "\text{dense}"'] & F \\
X \arrow[ur, dashed, "!"'] &
\end{tikzcd}
$$

The particular Lawvere-Tierney topology that we are interested in is the double negation topology $\neg\neg : \Omega \to \Omega$, also called the dense topology. This is of interest to us because taking sheaves over $\neg\neg$ lets us pass from a topos with arbitrary internal logic to a topos with a classical internal logic. This is easy to see: for a sheaf for the double negation topology, the identity $\neg\neg S = S$ necessarily holds!

6. Forcing

We introduce the notion of a *natural numbers object* $1 \xrightarrow{0} N \xrightarrow{s} N$, for which the following diagram commutes for any $1 \xrightarrow{x} X \xrightarrow{f} X$

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\
& X \arrow[r, "f"'] & X
\end{tikzcd}
$$

We start with a countable transitive model $M \models ZFC$, which by the axiom of infinity will have a natural numbers object N.

Lemma 6.1. *If $\mathcal{C}$ is a topos with a natural numbers object, and $\mathcal{D}$ is a topos with functors*

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, bend left]
\end{tikzcd}
$$

Then $\mathcal{D}$ has a natural numbers object as well.

Corollaries of this lemma will help us in the construction of new toposes. The first is that the category of presheaves over a topos that models ZFC has a natural numbers object:

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "\Delta"]
\end{tikzcd}
$$

Where Γ takes the global section of each presheaf and Δ assigns each set to the constant functor from $\mathcal{C}^{\mathrm{op}}$ to that set.

Further, if a topos $\mathcal{C}$ has a natural numbers object, then the category of sheaves over a Lawvere-Tierney topology j has a natural numbers object:

$$\begin{tikzcd} \mathbf{Sh}_j\mathcal{C} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, bend left, "sh"] \end{tikzcd}$$

where sh is the 'sheafification' functor, which is left adjoint to the inclusion functor.
For the purposes of forcing, we will using the Cohen Poset $\mathbb{P}$, defined below.
In our model M of ZFC we want to force a set B in between the sets N and $\mathrm{Sub}(N) \cong 2^N$. We do so by approximating a monomorphism $g : B \to \mathcal{P}(N)$. For a subset $F_p \subseteq B \times N$ and a function $p : F_p \to 2$, the pair (F_p, p) is called a condition. We define an order on these conditions

$$q \leq p \iff F_q \supseteq F_p \text{ and } q|_{F_p} = p$$

i.e., q extends p.

We now take presheaves $M^{\mathbb{P}^{\mathrm{op}}}$. By the above, this category has a natural numbers object. We build a subobject A of the constant functor $\Delta(B \times N)$, where $A(p) = \{(b, n) : p(b, n) = 0\}$. A is in fact a closed subobject of $\Delta(B \times N)$ with respect to the double negation topology, i.e. $\neg\neg A = A$ in $\mathrm{Sub}(\Delta(B \times N)$.

Letting Ω be the subobject classifier for $M^{\mathbb{P}^{\mathrm{op}}}$ and $\Omega_{\neg\neg}$ the subobject classifier for $\mathbf{Sh}(\mathbb{P}, \neg\neg)$. Because A is closed, the characteristic map of A $\chi_A : \Delta(B \times N) \to \Omega$ factors through $\Omega_{\neg\neg}$

$$\begin{tikzcd} \neg\neg A \arrow[r, equal] & A \arrow[r] \arrow[dd, "!"'] & \Delta(B \times N) \arrow[d, "f"] \\ & & \Omega_{\neg\neg} \arrow[d] \\ & 1 \arrow[r, "\top"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega_{\neg\neg} \arrow[r] & \Omega \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\neg\neg"'] & \Omega \end{tikzcd}$$

So we have a map $f : \Delta B \times \Delta N \to \Omega_{\neg\neg}$, from which we can obtain a map $g : \Delta(B) \to \Omega_{\neg\neg}^{\Delta N}$, which is a monomorphism.

The sheafification functor $sh_{\neg\neg}$ is left exact and thus preserves monomorphisms. As such, our morphism g in the category of presheaves is sent to a monomorphism $\hat{g} : \hat{B} \to \Omega_{\neg\neg}^{\hat{N}}$

We thus have $\hat{N} \rightarrowtail \hat{B} \rightarrowtail P(\hat{N}) = \hat{2}^{\hat{N}}$. Although our monomorphism $B \rightarrowtail \hat{2}^{\hat{N}}$ may not be 'strict', through some additional very involved methods (expand in these notes later) we can find that, given strict inequalities $N < 2^N < B$ in our original model M, taking sheaves results in strict inequalities $\hat{N} < 2^{\hat{N}} < \hat{B}$ in the new topos. Then from the two results above, we will have $\hat{N} < 2^{\hat{N}} < \hat{B} \leq \hat{2}^{\hat{N}}$

7. References

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