Review of Positivity with Continuous Exposure

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1 INTRODUCTION

1.1 Positivity and continuous exposure

- Here we will review meaning and implication of positivity violation when the exposure of interest is continuous.
- Special thanks to @EpiDancer, @ashley_naimi, and @jfeldman_epi.

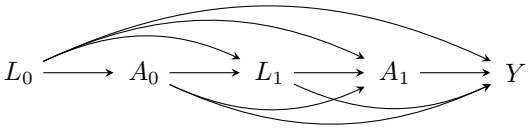
1.2 Notations

Y : Outcome measured at the end of the study
Y^{a_0} : Counterfactual outcome with intervention at time 0 only
Y^{a_0,a_1} : Counterfactual outcome with intervention at time 0 and 1
L_0 : Baseline covariates
A_0 : Baseline treatment assignment
L_1 : Post-baseline covariates
A_1 : Post-baseline treatment assignment

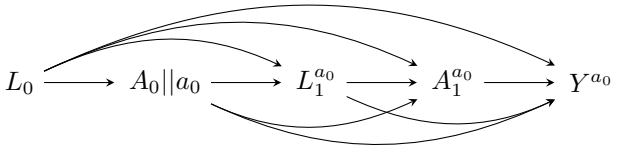
Here we will only consider the causal effect of A_0 and its associated counterfactual Y^{a_0}.

1.3 Causal structure

1. Original DAG



2. Single time point intervention SWIG



The backdoor to A_0 is only through L_0. Thus, conditioning on L_0 is sufficient for exchangeability as explained below.

2 Single time point strategy

2.1 Identifiability Conditions

We will follow the terminologies in [Hernan and Robins, 2019] (Chapter 3).

- **Consistency**: the values of treatment under comparison correspond to well-defined interventions that, in turn, correspond to the versions of treatment in the data

$$Y_i = Y_i^{a_0} \text{ if } A_{0i} = a_0 \text{ for all } a_0$$

- **Exchangeability**: the conditional density of receiving every value of treatment, though not decided by the investigators, depends only on the measured covariates

$$A_0 \perp\!\!\!\perp Y^{a_0} | L_0 \text{ for all } a_0$$

- **Positivity**: the conditional density of receiving every value of treatment is greater than zero, i.e., positive

$$f(A_0 = a_0 | L_0 = l_0) > 0 \text{ for all } a_0, l_0 \text{ where } f(L_0 = l_0) > 0$$

2.2 Identification with all conditions met

Here we will examine how the average causal effect is identified when all three conditions are met. Please note identification REF is under infinite amount of data. This derivation follows the Technical Point 2.3 in [Hernan and Robins, 2019].

$$\begin{aligned}
&\text{By iterative expectation} \\
E[Y^{a_0}] &= E[E[Y^{a_0} | L_0]] \\
&\text{By conditional exchangeability: } Y^{a_0} \perp\!\!\!\perp A_0 | L_0 \\
&= E[E[Y^{a_0} | A_0, L_0]] \\
&\text{By exchangeability, } E[Y^{a_0} | A_0, L_0] = E[Y^{a_0} | A_0 = a_0, L_0] \\
&= E[E[Y^{a_0} | A_0 = a_0, L_0]] \\
&\text{By consistency} \\
&= E[E[Y | A_0 = a_0, L_0]] \\
&\text{Make outer expectation explicit integration} \\
&= \int_{l_0} E[Y | A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0 \\
&= \text{Conditional mean averaged over } L_0
\end{aligned}$$

The role of positivity is to ensure that the last integration is well defined at all l_0 values with $f(L_0 = l_0) > 0$ for any given a_0. As a result, we can identify the mean counterfactual outcome in the population at any given a_0 with observable variables. Thus, any causal contrast of interest $E[Y^{a_0}] - E[Y^{a_0 *}]$ is also identifiable. This also implies the entire counterfactual dose response (functional form of $E[Y^{a_0}]$ as a function of a_0) in the population is identified.

2.3 Non-identification with positivity violation

Here we will examine the implication of positivity violation, which is the complement of the positivity condition stated above.

- **Positivity violation**: the conditional density of receiving some value of treatment is not greater than zero

$$f(A_0 = a_0 | L_0 = l_0) = 0 \text{ for some } a_0, l_0 \text{ where } f(L_0 = l_0) > 0$$

The identification formula is still the same.

$$\begin{aligned}
&\text{By iterative expectation} \\
E[Y^{a_0}] &= E[E[Y^{a_0} | L_0]] \\
&\text{By conditional exchangeability: } Y^{a_0} \perp\!\!\!\perp A_0 | L_0 \\
&= E[E[Y^{a_0} | A_0, L_0]] \\
&\text{By exchangeability, } E[Y^{a_0} | A_0, L_0] = E[Y^{a_0} | A_0 = a_0, L_0] \\
&= E[E[Y^{a_0} | A_0 = a_0, L_0]] \\
&\text{By consistency}
\end{aligned}$$

$$= E[E[Y|A_0 = a_0, L_0]]$$

Make outer expectation explicit integration

$$= \int_{l_0} E[Y|A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0$$

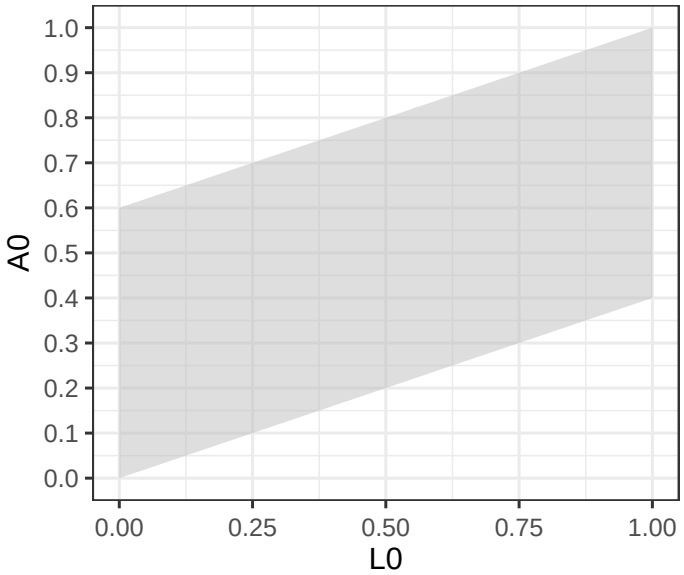
$$= \text{Conditional mean averaged over } L_0$$

The last integration is not always well defined. At some values of a_0, $E[Y|A_0 = a_0, L_0 = l_0]$ is undefined or unobservable at some l_0 where $f(L_0 = l_0) > 0$ because $f(A_0 = a_0|L_0 = l_0) = 0$. Therefore, the counterfactual dose response in the population is not identifiable. However, some specific causal contrasts may be identifiable. For two specific exposure values a_0 and a_0* that both satisfy positivity at all l_0 where $f(L_0 = l_0) > 0$, we can identify $E[Y^{a_0}] - E[Y^{a_0*}]$.

2.4 Illustration with a toy example

Consider a continuous exposure A_0 and a continuous L_0 that both take on values between 0 and 1. We assume $f(L_0 = l_0) > 0$ for all values between 0 and 1. We further assume the exposure density is positive only in the shaded area in the figure.

```
library(tidyverse)
tibble(x = seq(from = 0, to = 1, by = 0.01),
       ymin = 0   + 0.4 * x,
       ymax = 0.6 + 0.4 * x) %>%
ggplot(mapping = aes(x = x)) +
  geom_ribbon(mapping = aes(ymin = ymin, ymax = ymax),
              alpha = 0.5, fill = "gray") +
  labs(y = "A0", x = "L0") +
  scale_y_continuous(breaks = seq(from = 0, to = 1, by = 0.1)) +
  theme_bw() +
  theme(axis.text.x = element_text(angle = 0, vjust = 0.5),
        legend.key = element_blank(),
        plot.title = element_text(hjust = 0.5),
        strip.background = element_blank())
```



At $L_0 = 0$, A_0 can only take on values between 0 and 0.6. At $L_0 = 1$, A_0 can only take on values between 0.4 and 1.0. That is, although A_0 does vary over the full range between 0 and 1 in the population, the range of values that A_0 can take on is restricted at any given $L_0 = l_0$ value. We will consider what counterfactual quantities and causal effects are identifiable.

1. $E[Y^{a_0}]$ for $a_0 < 0.4$

$$E[Y^{a_0}] = \int_{l_0} E[Y|A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0$$

For exposure values below 0.4, the above identification formula is not well defined as the shaded positive area does not span the entire width of L_0 distribution (lower triangular white area violates positivity) and some of the conditional expectations are undefined.

2. $E[Y^{a_0}]$ for $a_0 > 0.6$

$$E[Y^{a_0}] = \int_{l_0} E[Y|A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0$$

For exposure values above 0.6, the above identification formula is not well defined as the shaded positive area does not span the entire width of L_0 distribution (upper triangular white area violates positivity) and some of the conditional expectations are undefined.

3. $E[Y^{a_0}]$ for $0.4 \leq a_0 \leq 0.6$

$$E[Y^{a_0}] = \int_{l_0} E[Y|A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0$$

For exposure values between 0.4 and 0.6, the above identification formula is well defined as the shaded positive area does span the entire width of L_0 distribution. All necessary conditional expectations are defined and observable. Therefore, the population counterfactual dose response in this restricted [0.4, 0.6] range is identifiable. Any causal contrast $E[Y^{a_0}] - E[Y^{a_0 *}]$ for specific values a_0 and $a_0 *$ both of which are within the [0.4, 0.6] range is identified.

4. $E[Y^{a_0}|L_0 \leq 0.5]$

$$E[Y^{a_0}|L_0 \leq 0.5] = \int_{0}^{0.5} E[Y|A_0 = a_0, L_0 = l_0] f(L_0 = l_0) \mathrm{d}l_0$$

If we limit the inference to a subset of the population where $L_0 \leq 0.5$, the range of values for a_0 where the mean counterfactual outcome is identified expands.

For exposure values between 0.2 and 0.6, the above identification formula is well defined as the shaded positive area does span the left half of of L_0 distribution. All necessary conditional expectations are defined and observable. Therefore, the subpopulation counterfactual dose response in this restricted [0.2, 0.6] range is identifiable. Any causal contrast $E[Y^{a_0}|L_0 \leq 0.5] - E[Y^{a_0 *}|L_0 \leq 0.5]$ for specific values a_0 and $a_0 *$ both of which are within the [0.2, 0.6] range is identified.

3 Bibliography Part

3.1 Bibliography

[Hernan and Robins, 2019] Hernan, M. A. and Robins, J. M. (2019). *Causal Inference*. Chapman & Hall/CRC.