1 Hypergeometric

http://planetmath.org/proofofvarianceofthehypergeometricdistribution

$$\text{pmf}$$
$$f(x; N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$$

$$\text{Expected value}$$
$$\begin{aligned}
E[X] &= \sum_{x=0}^{K} x f(x; N, M, K) \\
&= \sum_{x=1}^{K} x f(x; N, M, K) \\
&= \sum_{x=1}^{K} x \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} \\
&= \sum_{x=1}^{K} \frac{\left(x\binom{M}{x}\right)\binom{N-M}{K-x}}{\binom{N}{K}} \\
&\text{Using } x\frac{M!}{x!(M-x)!} = M\frac{(M-1)!}{(x-1)!(M-x)!} \\
&\text{and } \frac{N!}{K!(N-K)!} = \frac{N}{K}\frac{(N-1)!}{(K-1)!(N-K)!} \\
&= \sum_{x=1}^{K} \frac{\left(M\binom{M-1}{x-1}\right)\binom{N-M}{K-x}}{\frac{N}{K}\binom{N-1}{K-1}} \\
&= \frac{MK}{N} \sum_{x=1}^{K} \frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}} \\
&\text{Using } y = x-1 \\
&= \frac{MK}{N} \sum_{y=0}^{K-1} \frac{\binom{M-1}{y}\binom{N-M}{K-(y+1)}}{\binom{N-1}{K-1}} \\
&= \frac{MK}{N} \sum_{y=0}^{K-1} f(y; N-1, M-1, K-1) \\
&= \frac{MK}{N} F(Y = K-1; N-1, M-1, K-1) \\
&= \frac{MK}{N}(1) \\
&= \frac{MK}{N}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{x=0}^{K} x^2 f(x; N, M, K) \\
&= \sum_{x=1}^{K} x^2 f(x; N, M, K) \\
&= \sum_{x=1}^{K} x^2 \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x=1}^{K} x \frac{\left(M\binom{M-1}{x-1}\right)\binom{N-M}{K-x}}{\frac{N}{K}\binom{N-1}{K-1}} \\
&= \frac{MK}{N} \sum_{x=1}^{K} x \frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}} \\
&= \frac{MK}{N} \sum_{x=1}^{K} \left[(x-1)\frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}} + \frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}} \right] \\
&= \frac{MK}{N} \sum_{x=1}^{K} (x-1)\frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}} + \frac{MK}{N} \sum_{x=1}^{K} \frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}}
\end{aligned}
$$

Last term is 1 as previously shown

$$
= \frac{MK}{N} + \frac{MK}{N} \sum_{x=1}^{K} (x-1)\frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}}
$$

Index can start at $x = 2$ because at $x = 1$ it is 0

$$
= \frac{MK}{N} + \frac{MK}{N} \sum_{x=2}^{K} (x-1)\frac{\binom{M-1}{x-1}\binom{N-M}{K-x}}{\binom{N-1}{K-1}}
$$

Using $(x-1)\dfrac{(M-1)!}{(x-1)!(M-x)!} = (M-1)\dfrac{(M-2)!}{(x-2)!(M-x)!}$

and $\dfrac{(N-1)!}{(K-1)!(N-K)!} = \dfrac{N-1}{K-1}\dfrac{(N-2)!}{(K-2)!(N-K)!}$

$$
\begin{aligned}
&= \frac{MK}{N} + \frac{MK}{N} \sum_{x=2}^{K} \frac{(M-1)\binom{M-2}{x-2}\binom{N-M}{K-x}}{\frac{N-1}{K-1}\binom{N-2}{K-2}} \\
&= \frac{MK}{N} + \frac{MK}{N}\frac{(M-1)(K-1)}{N-1} \sum_{x=2}^{K} \frac{\binom{M-2}{x-2}\binom{N-M}{K-x}}{\binom{N-2}{K-2}}
\end{aligned}
$$

Using $z = x - 2$

$$
\begin{aligned}
&= \frac{MK}{N} + \frac{MK}{N}\frac{(M-1)(K-1)}{N-1} \sum_{z=0}^{K-2} \frac{\binom{M-2}{z}\binom{N-M}{K-(x+2)}}{\binom{N-2}{K-2}} \\
&= \frac{MK}{N} + \frac{MK}{N}\frac{(M-1)(K-1)}{N-1} \sum_{z=0}^{K-2} f(x; N-2, M-2, K-2) \\
&= \frac{MK}{N} + \frac{MK}{N}\frac{(M-1)(K-1)}{N-1}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \frac{MK}{N} + \frac{MK}{N}\frac{(M-1)(K-1)}{N-1} - \left(\frac{MK}{N}\right)^2 \\
&= \frac{MK}{N}\left[1 + \frac{(M-1)(K-1)}{N-1} - \frac{MK}{N}\right] \\
&= \frac{MK}{N}\frac{1}{N(N-1)}\left[N(N-1) + N(M-1)(K-1) - MK(N-1)\right] \\
&= \frac{MK}{N}\frac{1}{N(N-1)}[N^2 - N + NMK - NK - NM + N - NMK + MK] \\
&= \frac{MK}{N}\frac{1}{N(N-1)}[N^2 - NK - NM + MK] \\
&= \frac{MK}{N}\frac{1}{N(N-1)}(N-K)(N-M)
\end{aligned}
$$

$$= \frac{MK}{N} \frac{(N-K)(N-M)}{N(N-1)}$$