1 Spline path

$$P(t) = \sum_{i=0}^{n} P_i \times b_i(\frac{t - t_0}{T})$$

where $t \in [t_0, t_0 + T]$, (b_i) is a basis of the polynoms of degree less or equal to n and (P_i) is a set of points.

Derivative

$$P^{(k)}(t) = \sum_{i=0}^{n} \frac{P_i}{T^k} \times b_i^{(k)}(\frac{t - t_0}{T})$$

Integral of derivatives

$$\begin{aligned}
\int_{t_{min}}^{t_{min}+T} P^{(k)}(t)^T . P^{(k)}(t) dt &= \sum_{i,j} \frac{P_i^T P_j}{T^{2k}} \int_{t_{min}}^{t_{min}+T} b_i^{(k)}(\frac{t - t_0}{T}) b_j^{(k)}(\frac{t - t_0}{T}) dt \\
&= \sum_{i,j} \frac{P_i^T P_j}{T^{2k}} T \int_0^1 b_i^{(k)}(u) b_j^{(k)}(u) du \\
&= T^{1-2k} \sum_{i,j} P_i^T P_j \int_0^1 b_i^{(k)}(u) b_j^{(k)}(u) du
\end{aligned}$$

2 Definiton of some basis of polynoms

We consider $u \in [0, 1]$.

2.1 Canonical polynom basis

$$b_i(u) = u^i$$

Its k-th derivative is, for $k \leq i$:

$$b_i^{(k)}(u) = \frac{i!}{(i-k)!} u^{i-k}$$

Finally,

$$\begin{aligned}
\int_0^1 b_i^{(k)}(u) b_j^{(k)}(u) du &= \frac{i!}{(i-k)!} \frac{j!}{(j-k)!} \int_0^1 u^{i+j-2k} du \\
&= \frac{i!}{(i-k)!} \frac{j!}{(j-k)!} \frac{1}{i+j-2k+1}
\end{aligned}$$

2.2 Bernstein polynoms

The i-th Bernstein polynom of degree n is:

$$b_{i,n}(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

Its k-th derivative is, for $k \leq n$:

$$\begin{aligned}
b_{i,n}^{(k)}(u) &= n! \sum_{p=\max(0,k+i-n)}^{\min(k,i)} \binom{k}{p} (-1)^{k-p} \frac{u^{i-p}(1-u)^{n-k-(i-p)}}{(i-p)!(n-k-(i-p))!} \\
&= \frac{n!}{(n-k)!} \sum_{p=\max(0,k+i-n)}^{\min(k,i)} \binom{k}{p} (-1)^{k-p} b_{i-p,n-k}(u)
\end{aligned}$$

Integral of product of derivatives We seek to compute

$$\begin{aligned}
\alpha_{i,j,n}^{(k)} &:= \int_0^1 b_{i,n}^{(k)}(u) b_{j,n}^{(k)}(u) dt \\
&= \left(\frac{n!}{(n-k)!}\right)^2 \sum_{p \in I(i,k,n), q \in I(j,k,n)} \binom{k}{p} \binom{k}{q} (-1)^{2k-(p+q)} \alpha_{i-p,j-q,n-k}^{(0)}
\end{aligned}$$

with $I(i,k,n) = [\max(0, k+i-n), \min(k,i)]$

Calculation of $\alpha_{i,j,n}^{(0)}$

$$\begin{aligned}
\alpha_{i,j,n}^{(0)} &= \binom{n}{i} \binom{n}{j} \binom{2n}{i+j}^{-1} \int_0^1 b_{i+j,2n}(u) dt \\
&= \binom{n}{i} \binom{n}{j} \binom{2n}{i+j}^{-1} \frac{1}{2n+1}
\end{aligned}$$

Maximum on an interval Let $I = [t_0, t_1] \subset [0,1]$.

We seek $M_{i,n}(I) = \max_{u \in I} |b_{i,n}(u)|$. Clearly, we have

$$M_{0,n}(I) = b_{0,n}(t_0)$$

and

$$M_{n,n}(I) = b_{n,n}(t_1)$$

As $b'_{i,n}(u) = \binom{n}{i}(i-un)u^{i-1}(1-u)^{n-i-1}$, we have, $\forall i \in [1, n-1]$, $M_{i,n}(I) = b_{i,n}(t_M(I))$ where

$$t_M(I) = \begin{cases} t_0 & \textbf{if } \frac{i}{n} > t_0 \\ t_1 & \textbf{if } t_1 < \frac{i}{n} \\ \frac{i}{n} & \textbf{otherwise} \end{cases} \tag{1}$$

The above formula is also true for $i = 0$ and $i = n$.

We seek $M'_{i,n}(I) = \max_{u \in I} |b'_{i,n}(u)|$.

Case $i = 0$ **or** $i = n$: $b'_{i,n}(u)$ is monotone on I.

Case $i = 1$: $b'_{1,n}(u) = n(1 - un)(1 - u)^{n-2}$ and

$$b''_{1,n}(u) = n(n-1)\left[nu - 2\right](1-u)^{n-3}$$

The extremun, in $u = \frac{2}{n}$, is $\frac{(n-2)^{n-2}}{n^{n-3}}$.

Case $i = n - 1$: $b'_{n-1,n}(u) = n(n - 1 - un)u^{n-2}$ and

$$b''_{n-1,n}(u) = n(n-1)\left[n - 2 - nu\right]u^{n-3}$$

The extremun, in $u = \frac{n-2}{n}$, is $\frac{(n-2)^{n-2}}{n^{n-3}}$, as for $i = 1$.

General case :

$$b''_{i,n}(u) = \binom{n}{i}\left[(n-1)^2u^2 - (2i-1)(n-1)u + i(i-1)\right]u^{i-2}(1-u)^{n-i-2}$$

Let $\alpha(u) = (n-1)^2u^2 - (n-1)(2i-1)u + i(i-1)$. As $\Delta = (n-1)^2((2i-1)^2 - 4i(i-1)) = (n-1)^2$, the roots of α are $u^-_{i,n} = \frac{i-1}{n-1}$ and $u^+_{i,n} = \frac{i}{n-1}$. As $(n-1)^2 > 0$, $b'_{i,n}$ increases on $\left[0, u^-_{i,n}\right]$ and on $\left[u^+_{i,n}, 1\right]$ and decreases on $\left[u^-_{i,n}, u^+_{i,n}\right]$.

$$b'_{i,n}(u^-_{i,n}) = \binom{n}{i}\frac{(n-i)^{n-i}(i-1)^{i-1}}{(n-1)^{n-1}}$$

$$b'_{i,n}(u^+_{i,n}) = -\binom{n}{i}\frac{(i)^{i}(n-i-1)^{n-i-1}}{(n-1)^{n-1}}$$

Then, the following procedure computes M:

1. $M'_{i,n}(I) = \max(|b'_{i,n}(t_0)|, |b'_{i,n}(t_1)|)$
2. If $u^+_{i,n} \in I$, then $M'_{i,n}(I) = \max(M'_{i,n}(I), |b'_{i,n}(u^+_{i,n})|)$
3. If $u^-_{i,n} \in I$, then $M'_{i,n}(I) = \max(M'_{i,n}(I), |b'_{i,n}(u^-_{i,n})|)$