Homework 1
(due Wednesday, August 29th, 2018)

I: Linear Systems

1. Give a simple example of a 3×3 matrix A of rank 2 and demonstrate that a non-zero vector $\mathbf{z}$ exists for which $A\mathbf{z} = \mathbf{0}$.

2. The condition number of a matrix A is defined as

$$\mathrm{cond}(A) = \|A\| \cdot \|A^{-1}\|. \tag{1}$$

   Beginning with the definition of the matrix norm given in the cheat sheet, show that the condition number of A can be re-expressed as

$$\mathrm{cond}(A) = \left(\max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}\right)\left(\min_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}\right)^{-1}. \tag{2}$$

II. Taylor series approximation

1. What is the second-order Taylor polynomial $p_2(x)$ for $f(x) = x^{\frac{1}{x}}$ about $x_0 = 1$.

2. Plot $f(x)$ and $p_2(x)$ in the interval $(0, 5]$ using MATLAB or equivalent.

3. Where does $f(x)$ attain its maximum value in $(0, \infty)$? Do this analytically and computationally by writing a MATLAB program (or equivalent) to find the maximum.

4. Evaluate $f'(x)$ analytically and approximately using a finite difference approximation. Plot both the analytic and computational derivatives in the interval $(0, 5]$.

III. Computing with finite fields

1. In $GF(p)$ where $p = 43$, show (i) by direct computational checking through all possible field elements and (ii) by using `gfdiv` (or non-MATLAB equivalent) that each element of $GF(43)$ except 0 has a multiplicative inverse in the same field.

Cheat Sheet

Taylor series approximation: $f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^i f^{(i)}(x_0)}{i!} + \frac{(x-x_0)^{n+1} f^{(n+1)}(\theta_{[x_0,x]})}{(n+1)!}$ with $\theta_{[x_0,x]}$ in the interval $[x_0, x]$. Also, $f(x+h) = \sum_{i=0}^{n} \frac{h^i f^{(i)}(x)}{i!} + \frac{h^{n+1} f^{(n+1)}(\theta)}{(n+1)!}$ where $x \leq \theta \leq (x+h)$. $f^{(0)}(t) \stackrel{\text{def}}{=} f(t)$.
Integral form of the remainder: $R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)dt$.
Chain rule for differentiation: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$.
Difference Approximations of Derivatives: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, $f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$.
Nonsingular matrices: An $n \times n$ matrix is said to be *nonsingular* if it satisfies any one of the following equivalent conditions:

1. A has an inverse (i.e., there is a matrix, denoted by A^{-1}, such that $AA^{-1} = A^{-1}A = I$, the identity matrix.

2. $\det(A) \neq 0$ (i.e., the determinant of A is nonzero).

3. $\text{rank}(A) = n$ (the rank of a matrix is the maximum number of linearly independent rows or columns it contains). For an n-vector $\mathbf{b} \notin \text{span}\,(A)$, there does not exist a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

4. For any vector $\mathbf{z} \neq \mathbf{0}$, $A\mathbf{z} \neq \mathbf{0}$ (i.e., A annihilates no nontrivial vector).

Matrix norm: The matrix norm of an $m \times n$ matrix A is defined as $\|A\| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$ where $\|A\mathbf{x}\|$ and $\|\mathbf{x}\|$ are vector norms. For an $m \times n$ matrix A with matrix elements $\{a_{ij}\}$, $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$, $\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$
Vector norm: For an integer $p > 0$ and an n-vector $\mathbf{x}$, the vector p -norm is defined by $\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$. For any vector p-norm, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all vectors $\mathbf{x}, \mathbf{y}$.