Homework 3
(due Thursday, September 27th, 2018, 02:00)

I: The Groups of StrangeBrew

There are n odd citizens living in StrangeBrew. Their main occupation was forming various groups (with "We're strange doesn't mean we are strangers" being their motto), which at some point started threatening the very survival of the city. In order to limit the number of groups, the city council decreed the following innocent-looking rules:

- Each group has to have an *odd* number of members.
- Every two groups must have an *even* number of members in common.

Prove that under these rules, it is impossible to form more groups than n, the number of citizens. You must use matrix properties to prove this theorem. [Hint: Consider defining an $m \times n$ matrix A (for n citizens and m groups $G_1, G_2, \ldots, G_m$) by $a_{ij} = 1$ if $j \in G_i$ and 0 otherwise.]

II: Matrix Norms:

- Find a vector $\mathbf{x} \in \mathbb{R}^n$ such that the upper bound for the matrix norm objective functions $\max_{\mathbf{x} \in \mathbb{R}^n} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq \|A\|_1 = \max_j \sum_{i=1}^m |a_{ij}|$ is reached for an $m \times n$ matrix A.
- Prove that $\max_{\mathbf{x} \in \mathbf{R}^n} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \leq \max_i \sum_{j=1}^n |a_{ij}|$.
- Find a vector $\mathbf{x} \in \mathbb{R}^n$ such that the upper bound for the matrix norm objective function $\max_{\mathbf{x} \in \mathbf{R}^n} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \leq \|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ is reached for an $m \times n$ matrix A.

III: SVD-based image reconstruction:

- Load the hendrix_final.png image and extract the R, G and B channels. (Convert each channel image to double precision.)
- Execute the SVD separately on the R, G and B channels of the image. (Hint: Call [u,s,v]=svd(im) in MATLAB or equivalent where 'im' is a chosen channel.) Plot (using a log-log plot) the non-zero singular values for the R channel. Comment on the nature of the plot.
- Plot the Frobenius norm (look it up) of the reconstruction error matrix for each channel w.r.t. the dimension (increasing from 1 to the rank).
- Give your own criterion as to how many dimensions you would pick (the same number for all three channels) to get the best trade off between reconstruction error and image fidelity (to the original).
- Display the original and final reconstructed images (combined from R, G and B reconstructions and using your criterion) side by side. You may reduce the size of the original image (at the very outset) in order to ease the computational burden. Comment on your criterion for the choice of reduced size (if any).

Cheat Sheet

Taylor series approximation: $f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^i f^{(i)}(x_0)}{i!} + \frac{(x-x_0)^{n+1} f^{(n+1)}(\theta_{[x_0,x]})}{(n+1)!}$ with $\theta_{[x_0,x]}$ in the interval $[x_0, x]$. Also, $f(x+h) = \sum_{i=0}^{n} \frac{h^i f^{(i)}(x)}{i!} + \frac{h^{n+1} f^{(n+1)}(\theta)}{(n+1)!}$ where $x \le \theta \le (x+h)$. $f^{(0)}(t) \stackrel{\text{def}}{=} f(t)$.

Integral form of the remainder: $R_n(x) = \frac{1}{n!} \int_{x_0}^{x} (x-t)^n f^{(n+1)}(t) dt$.

Chain rule for differentiation: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$. $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

Difference Approximations of Derivatives: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, $f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$.

Nonsingular matrices: An $n \times n$ matrix is said to be *nonsingular* if it satisfies any one of the following equivalent conditions:

1. A has an inverse (i.e., there is a matrix, denoted by A^{-1}, such that $AA^{-1} = A^{-1}A = I$, the identity matrix.
2. $\det(A) \neq 0$ (i.e., the determinant of A is nonzero).
3. $\text{rank}(A) = n$ (the rank of a matrix is the maximum number of linearly independent rows or columns it contains). For an n-vector $\mathbf{b} \notin \text{span}\,(A)$, there does not exist a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.
4. For any vector $\mathbf{z} \neq \mathbf{0}$, $A\mathbf{z} \neq \mathbf{0}$ (i.e., A annihilates no nontrivial vector).

Matrix norm: The matrix norm of an $m \times n$ matrix A is defined as $\|A\| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$ where $\|A\mathbf{x}\|$ and $\|\mathbf{x}\|$ are vector norms. For an $m \times n$ matrix A with matrix elements $\{a_{ij}\}$, $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$, $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

Vector norm: For an integer $p > 0$ and an n-vector $\mathbf{x}$, the vector p -norm is defined by $\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$. $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$. For any vector p-norm, $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ for all vectors $\mathbf{x}, \mathbf{y}$.