Image Deformation Using Moving Least Squares

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Presented by Teddy Zhang

Credit to Nirup Reddy

Image Deformation

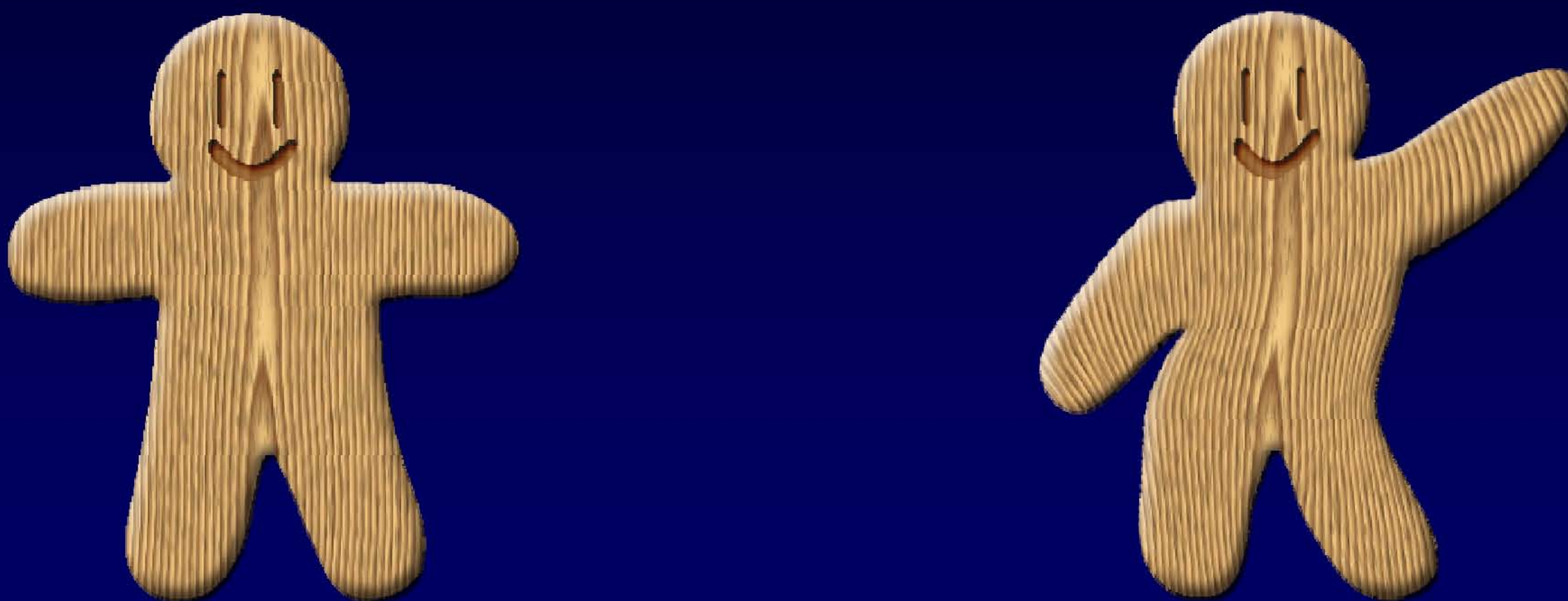
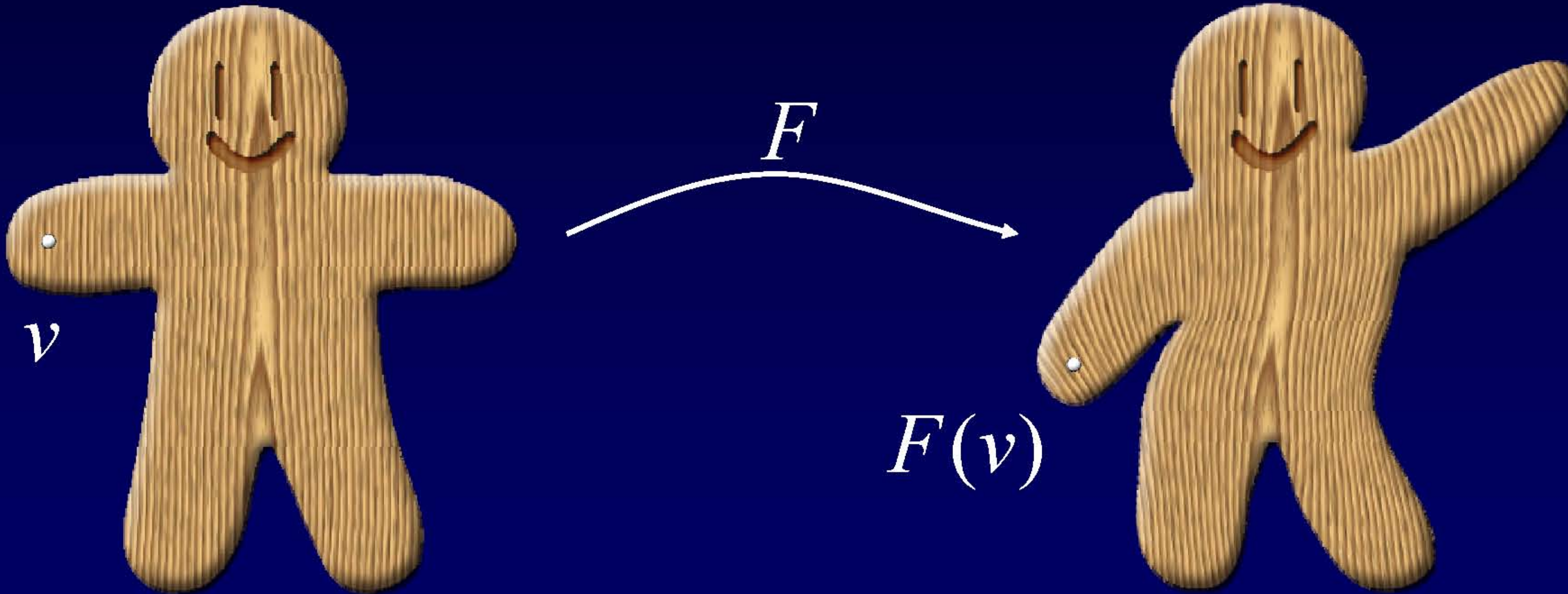
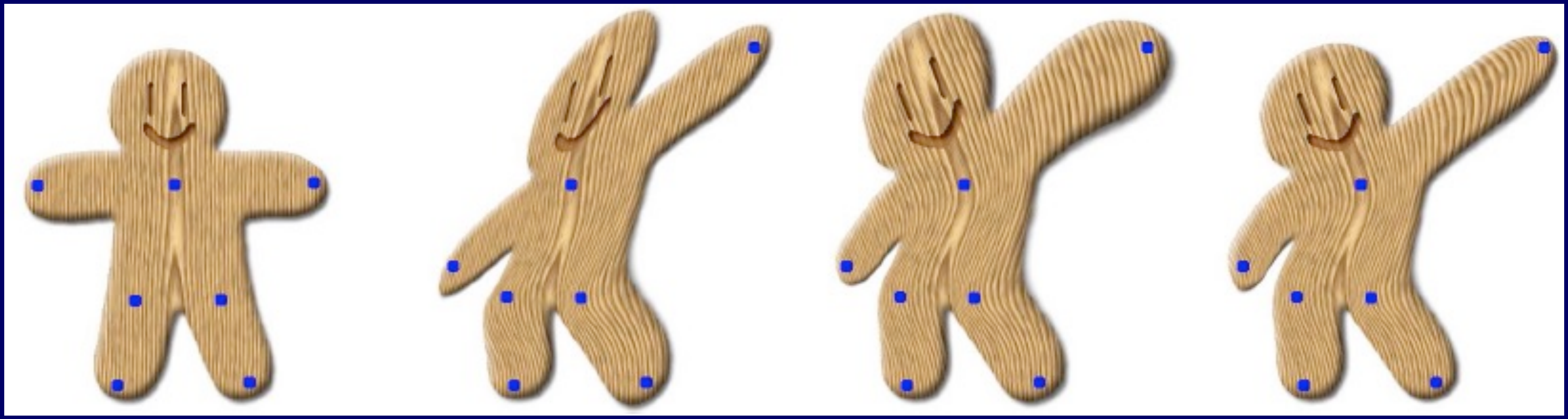


Image Deformation





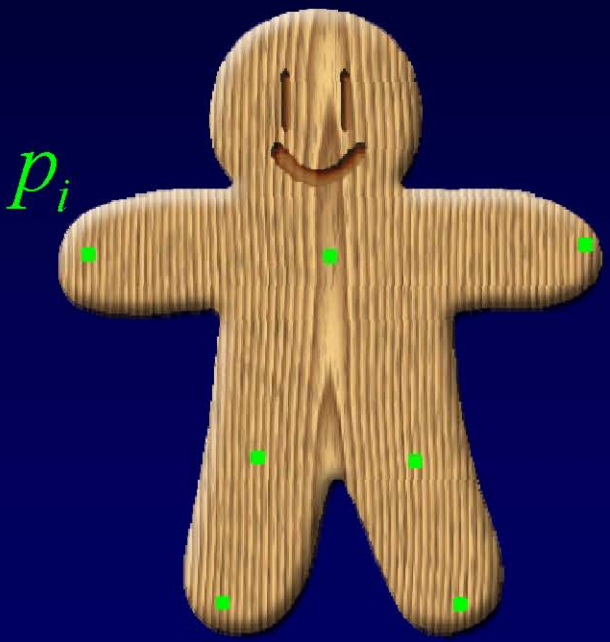
Input Affine Similarity Rigid

Introduction

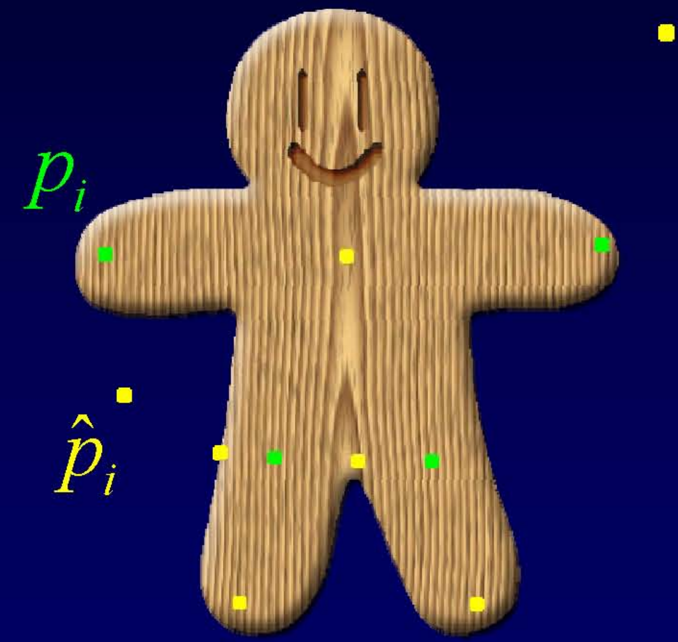
- Image deformation
  - Animation, morphing, medical imaging…
  - Controlled by handles: points, lines…
  - Function f
    - Interpolation: $f(p_i)=q_i$ (handles)
    - Smoothness ~ smooth deformations
    - Identity: $q_i=p_i \rightarrow f(x)=x$
  - Similar to scattered data interpolation

Least Squares Deformation

Least Squares Deformation



Least Squares Deformation



Least Squares Deformation

- Find the best affine transformation that maps p to $\hat{p}$
- Unique, closed-form solution
- Apply transformation to each point in the image

$$\sum_i | F(p_i) - \hat{p}_i |^2$$

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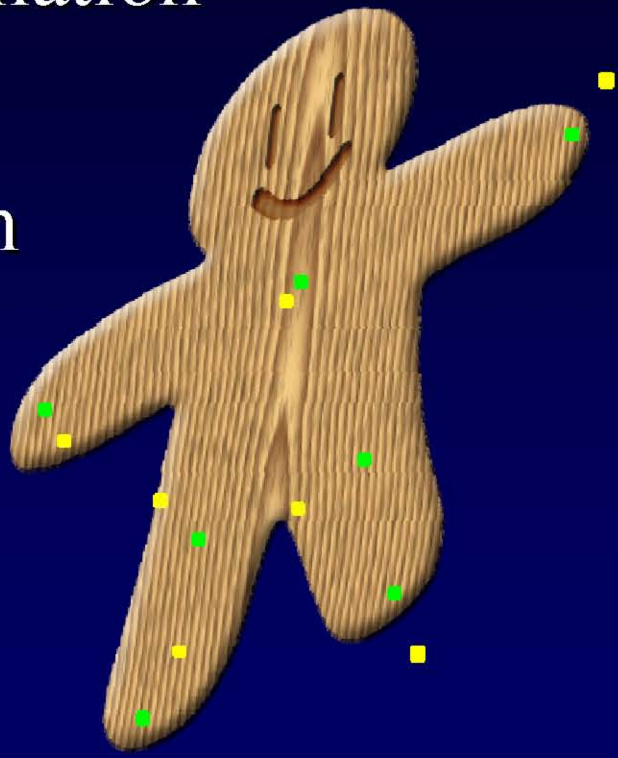
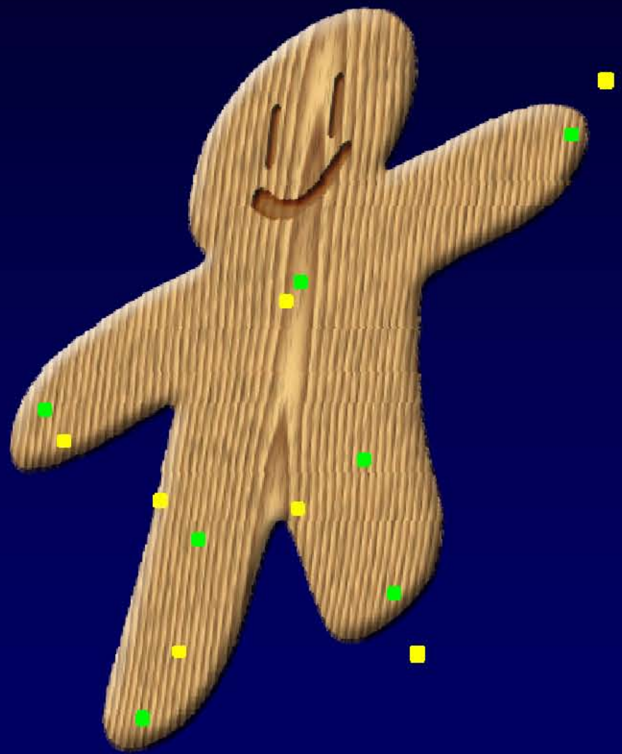


Moving Least Squares Deformation

- Find the best transformation that maps p to $\hat{p}$
- Weight each term in least squares problem by $\frac{1}{|p_i - v|}$
- Different transformation for each point in image

$$\sum_i w_i \,|\, F(p_i) - \hat{p}_i \,|^2$$



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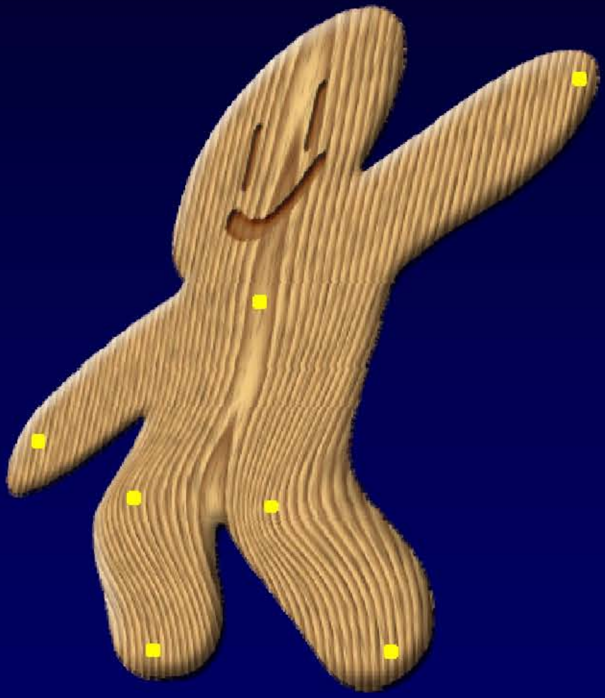


Moving Least Squares Deformation

- Build image deformations based on handles (p_i, q_i)
  - Interpolation
  - Smoothness
  - Identity
- Given a point v, solve for the best $l_v(x)$:

$$\sum_i w_i \,| l_v(p_i) - q_i |^2 \qquad w_i = \frac{1}{| p_i - v |^{2\alpha}}$$

- Deformation function $f(v)=l_v(v)$
  - Locally defined --- MOVING Least Squares
  - Satisfy the three properties

Affine Transform

- Affine transform: $l_v(x)=xM+T$
- Translation can be removed: $T=q_*-p_*M$

$$p_* = \frac{\sum_i w_i p_i}{\sum_i w_i} \qquad q_* = \frac{\sum_i w_i q_i}{\sum_i w_i}$$

- So, $l_v(x)=(x-p_*)M+q_*$
- The new cost function:

$$\sum_i w_i \,|\, \hat{p}_i M - \hat{q}_i \,|^2$$

$$\hat{p}_i = p_i - p_* \qquad \hat{q}_i = q_i - q_*$$

- M could be different class of transformations

Affine Deformations

- Solution

$$M = \left(\sum_i \hat{p}_i^T w_i \hat{p}_i\right)^{-1} \sum_j w_j \hat{p}_j^T \hat{q}_j$$

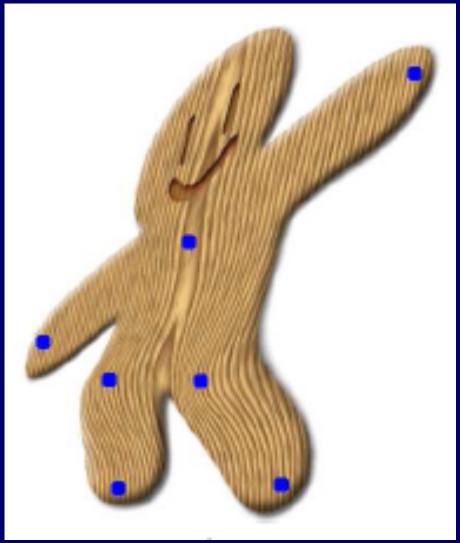
- The deformation function

$$f_a(v) = \sum_j A_j \hat{q}_j + q_*$$

$$A_j = (v - p_*)\left(\sum_i \hat{p}_i^T w_i \hat{p}_i\right)^{-1} \hat{p}_i^T$$

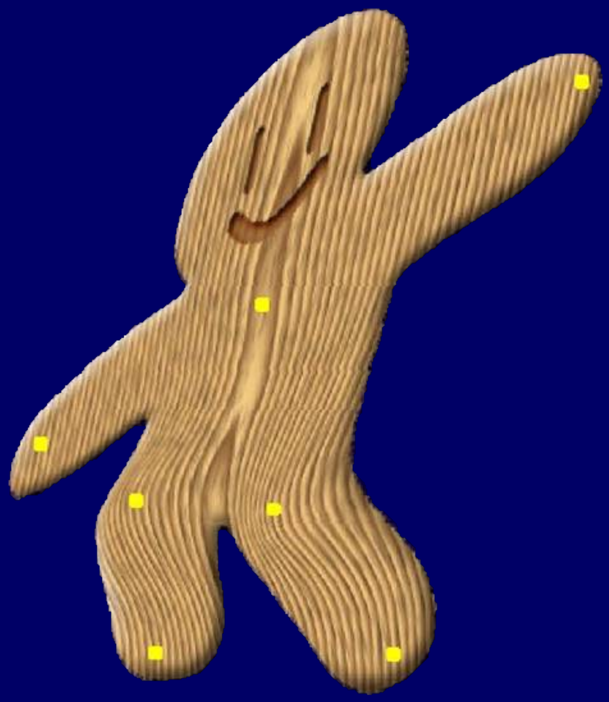
  - A_j can be precomputed

- Contains non-uniform scaling and shear



Similarity Deformations

- Only include translation, rotation, and uniform scaling
- Remove shear from deformation



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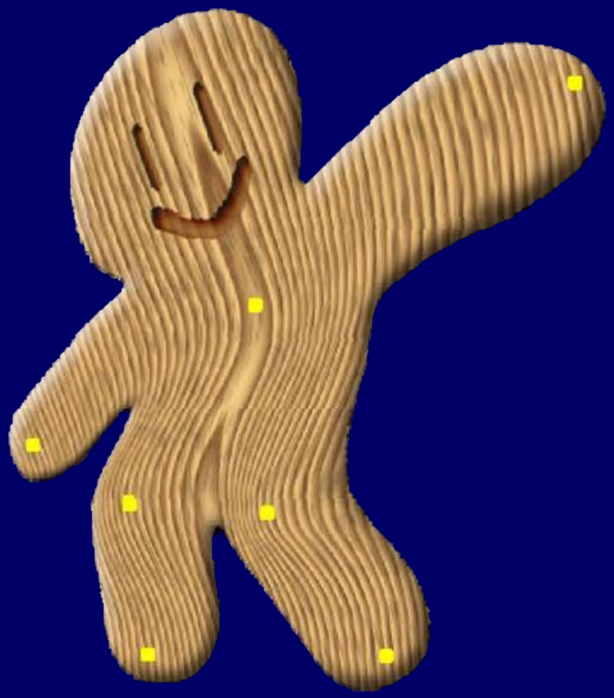


Similarity Deformations (Cont.)

- A special subset of affine transforms
  - Translation
  - Rotation
  - Constraints: Uniform-scaling
- Requirement: $M^T M=\lambda^2 I$
- Define $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$, where $M_2=M_1^{\perp}$
  - Cost function (Least squares problem) still quadratic in M_1

$$\sum_i w_i \left| \begin{pmatrix} \hat{p}_i \\ -\hat{p}_i^{\perp} \end{pmatrix} M_1 - \hat{q}_i^T \right|^2$$

where $(x,y)^{\perp}=(-y,x)$

Similarity Deformations (Cont.)

- Solution for matrix *M*

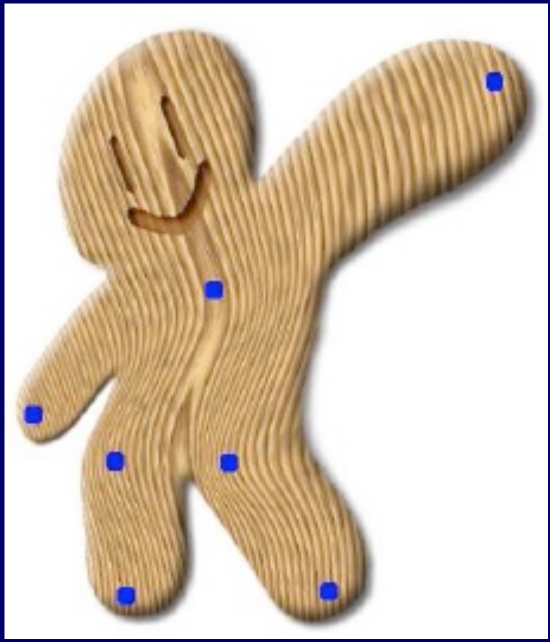
$$M = \frac{1}{\mu_s} \sum_i w_i \begin{pmatrix} \hat{p}_i \\ -\hat{p}_i^{\perp} \end{pmatrix} \begin{pmatrix} \hat{q}_i^T & -\hat{q}_i^{\perp T} \end{pmatrix}$$

where $\mu_s = \sum_i w_i \hat{p}_i \hat{p}_i^T$

- Solution for deformation function

$$f_s(v) = \sum_i \hat{q}_i \left(\frac{1}{\mu_s} A_i\right) + q_*$$

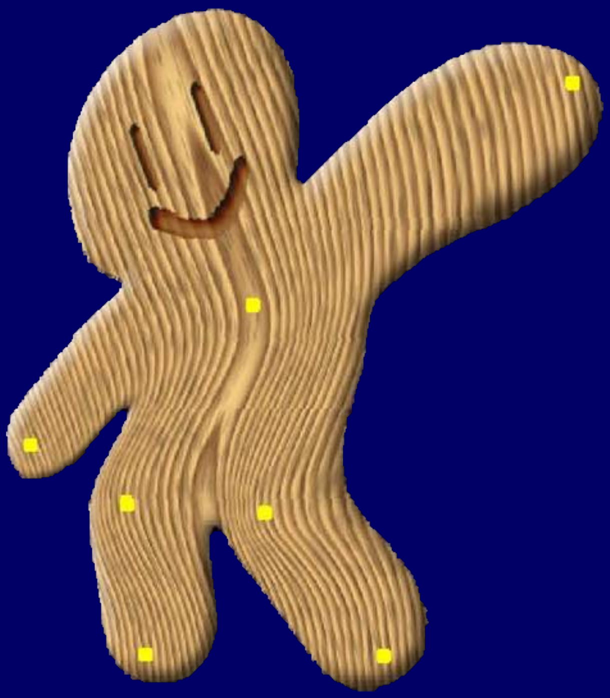
where $A_i = w_i \begin{pmatrix} \hat{p}_i \\ -\hat{p}_i^{\perp} \end{pmatrix} \begin{pmatrix} v - p_* \\ -(v - p_*)^{\perp} \end{pmatrix}^T$



- Property
  - Preserves angles better than affine deformations
  - Local scaling can hurt realism

Rigid Deformations

- Deformations should not include scaling and shear
  - [Alexa 2000; Igarashi et al. 2005]
- Best rigid transformation can be found from best similarity transformation
  - Remove local uniform scaling $M^T M = I$



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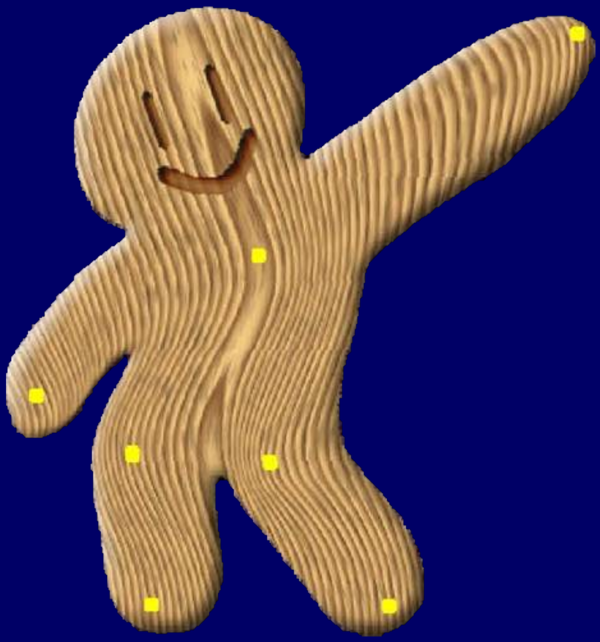


Rigid Deformations (Cont.)

- Solution for matrix *M*

$$M = \frac{\sum_i w_i \begin{pmatrix} \hat{p}_i \\ \hat{p}_i^{\perp} \end{pmatrix} \begin{pmatrix} \hat{q}_i^T & \hat{q}_i^{\perp T} \end{pmatrix}}{\sqrt{\left(\sum_i w_i \hat{q}_i \hat{p}_i^T\right)^2 + \left(\sum_i w_i \hat{q}_i \hat{p}_i^{\perp T}\right)^2}}$$

- Solution for deformation function

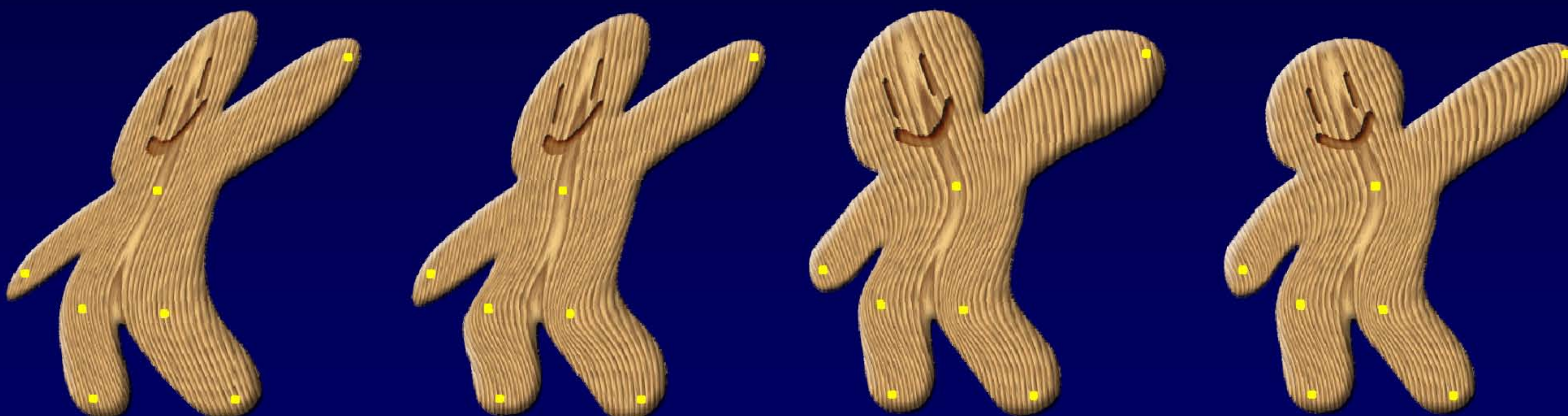
$$f_r(v) = |v - p_*| \frac{\vec{f}_r(v)}{|\vec{f}_r(v)|} + q_*.$$

where $\vec{f}_r(v) = \sum_i \hat{q}_i A_i$ and A_i is as in similarity deformations.

- Limited precomputation

Comparison

- Rigid transformations create the most realistic looking deformations



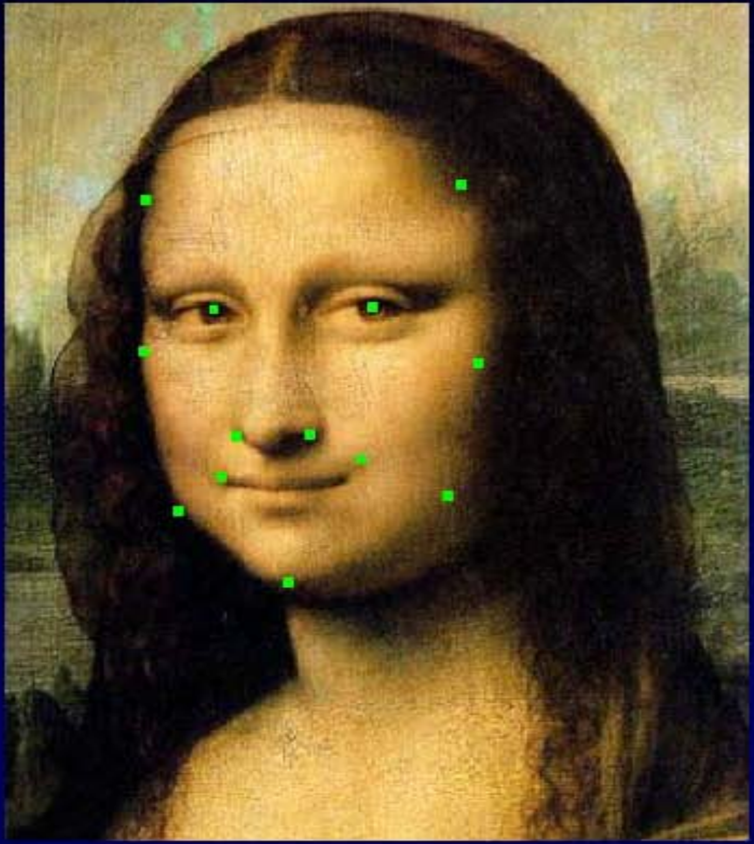
Thin-Plate [Bookstein 1989] | Affine MLS | Similarity MLS | Rigid MLS

Comparison:

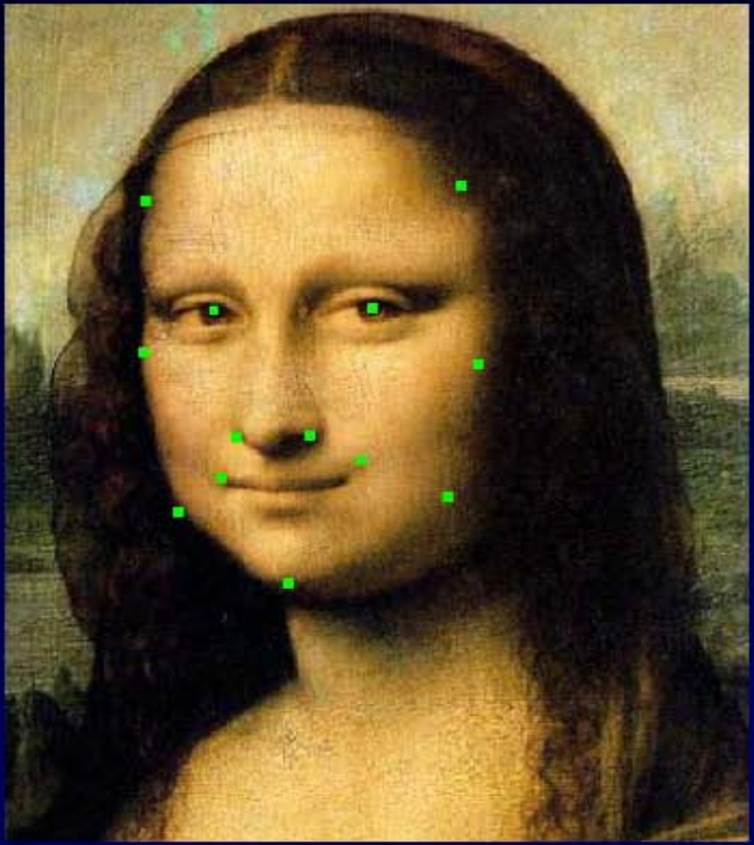
Affine	Similarity	Rigid
Non-uniform scaling and shear	Local scaling can often lead to undesirable Deformations But **Preserves angles**	Deformation is quite realistic
Computation time very low abt 1.5ms	Highest was computed as 3.4ms	Highest was computed 3.8ms

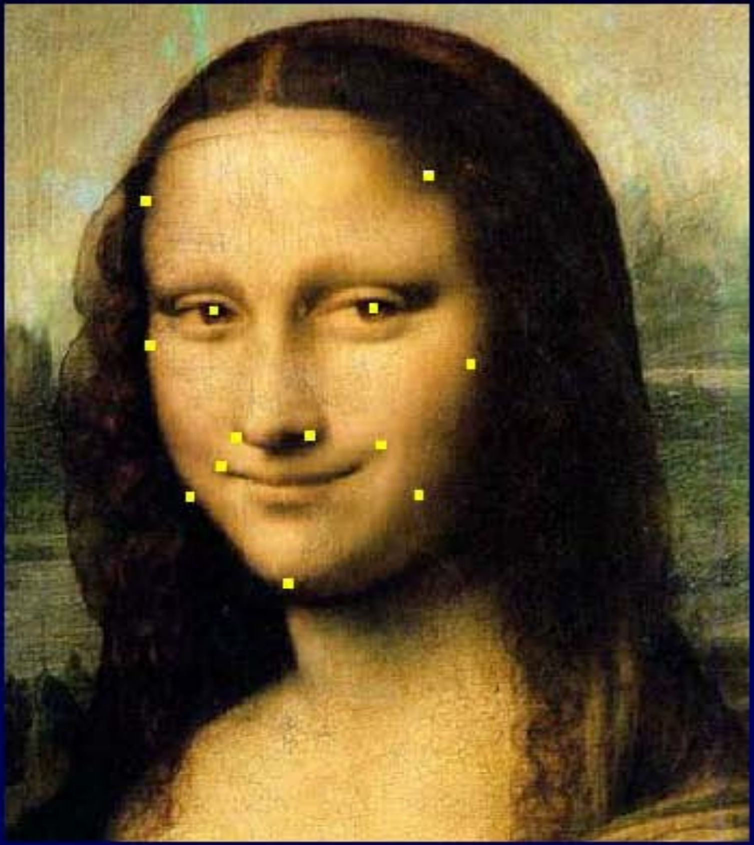
Examples

Examples



Examples



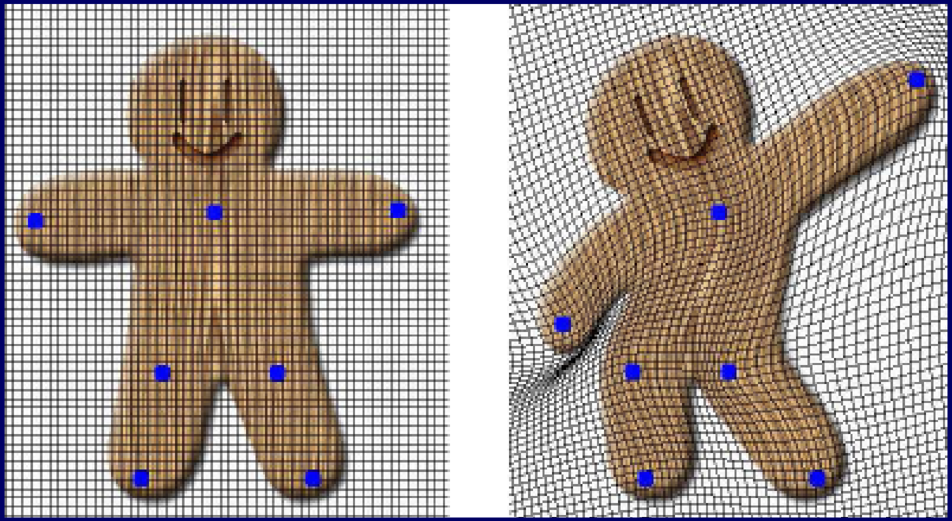


Examples

Some Rigid Deformation Results

Implementation

- Pixel by pixel --- expensive
- Deformation on a downsampled grid
- Fill quads using bilinear interpolation

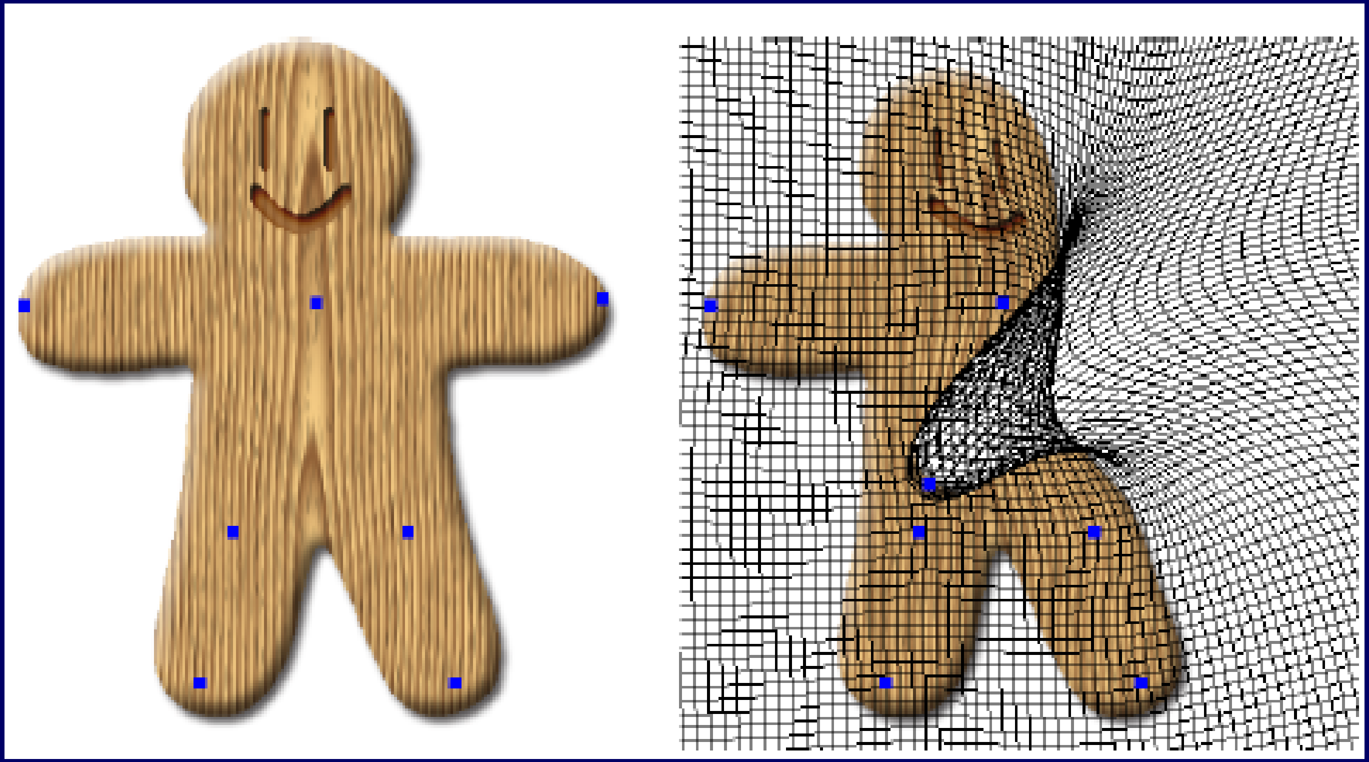


Timing

Method	Figure 1 (7 points)	Figure 4 (11 points)	Figure 5 (7 lines)
Affine MLS	1.5 ms	2.2 ms	1.5 ms
Similarity MLS	2.3 ms	3.4 ms	1.6 ms
Rigid MLS	2.6 ms	3.8 ms	3.3 ms

Grid size: 100x100

Limitation: Foldbacks



These situations occur when the sign of the Jacobian of f changes.

Conclusions

- Moving Least Squares
  - 2x2 linear system at each grid point
    - Real-time
- Similarity and rigid transformations
  - More realistic results
  - Extension: line segments
  - Closed-form expressions

Conclusions (Cont.)

- Future Work
  - Adding topological information
  - Generalizing to 3D to deform surfaces
  - Handles can be any curves