CSE330

Theory ASSIGNMENT- 2

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Section: 10

Answer to the ques. No: 1

(a)

x	$f(x)$
0	1
0.5	1.6487
1	2.7183

Nodes = 3
Degree = 2

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

Now,

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x - 0.5}{0 - 0.5} \times \frac{x - 1}{0 - 1}$$

$$= \frac{(x-0.5)(x-1)}{0.5}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2} = \frac{x - 0}{0.5 - 0} \times \frac{x - 1}{0.5 - 1}$$

$$= \frac{x(x-1)}{-0.25}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} = \frac{x - 0}{1 - 0} \times \frac{x - 0.5}{1 - 0.5} = \frac{x(x-0.5)}{0.5}$$

(b)

Now,

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= \left[\frac{(x-0.5)(x-1)}{0.5} \times 1\right] + \left[\frac{x(x-1)}{-0.25} \times 1.6487\right] + \left[\frac{x(x-0.5)}{0.5} \times 2.7183\right]$$

$$\Rightarrow P_2(0.2) = \left[\frac{(0.2-0.5)(0.2-1)}{0.5}\right] - \left[\frac{0.2(0.2-1)}{0.25} \times 1.6487\right] + \left[\frac{0.2(0.2-0.5)}{0.5} \times 2.7183\right]$$

$$\Rightarrow P_2(0.2) = 0.48 + 1.0552 - 0.3262$$

$$\Rightarrow P_2(0.2) = 1.209$$

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(c)

Given,

$f(x) = e^x$

So, $f(0.2) = e^{0.2}$

$= 1.2214$

From 1(b), we got that,

$P_2(0.2) = 1.209$

So, Using weierstrass approximation,

Maximum error $= \left| f(0.2) - P_2(0.2) \right|$

$= 0.0124$

(d)

Lagrange method is better than Vandermonde matrix method, because, if we have large amount of data it will be tough and very complex if ~~to use~~ we use Vandermonde matrix. This is not the case for lagrange method.

Also, Lagrange method can be used to find the value of the function even when the arguments are not equally spaced. This method can work with large data set too.

Therefore, it is prefered to use lagrange method over vandermonde matrix method to find interpolating polynomial.

Ans. to the ques. No:- 2

(a)

x	$f(x)$
-1	8
0	4
1	16

Nodes: 3
Degree: 2

$$P_2(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$

$a_0 = f[x_0]$, $a_1 = f[x_0, x_1]$, $a_2 = f[x_0, x_1, x_2]$

$x_0 = -1$ $\quad f(x_0) = 8$

$f[x_0, x_1] = \frac{4-8}{0+1} = \frac{-4}{1} = -4$

$f[x_0, x_1, x_2] = \frac{12+4}{1+1} = 8$

$x_1 = 0$ $\quad f(x_1) = 4$

$f[x_1, x_2] = \frac{16-4}{1-0} = 12$

$x_2 = 1$ $\quad f(x_2) = 16$

So, $a_0 = f[x_0] = 8$, $a_1 = f[x_0, x_1] = -4$,
$a_2 = f[x_0, x_1, x_2] = 8$

So, $P_2(x) = 8 - 4(x+1) + 8(x+1)(x-0)$

$= 8 - 4(x+1) + 8(x+1)x$

(b)

Using the polynomial from 2(a):

$P_2(0.5) = 8 - 4(0.5+1) + 8(0.5+1)(0.5)$

$= 2 + 6 = 8$

$P_2(-0.9) = 8 - 4(-0.9+1) + 8(-0.9+1)(-0.9)$

$= 8 - 0.4 - 0.72$

$= 6.88$

Ans to the ques. No: 3

(a)

$f(x) = \sin^2\left(\frac{x}{2}\right)$

$x_0 = -\frac{\pi}{3}$ | nodes = 3

$x_1 = 0$ | Degree, $n = 2$

$x_2 = \frac{\pi}{3}$

interval = $[-1.2, 1.2]$

Using cauchy's theorem to compute upper bound error:

$$|f(x) - P_n(x)| = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$$

$$\Rightarrow |f(x) - P_2(x)| = \frac{f^3(\xi)}{3!}\left(x+\frac{\pi}{3}\right)(x-0)\left(x-\frac{\pi}{3}\right) \quad \cdots\cdots (1)$$

Now,

$$f(x) = \sin^2\left(\frac{x}{2}\right)$$

$$f'(x) = \frac{\sin(x)}{2}$$

$$f''(x) = \frac{\cos(x)}{2}$$

$$f'''(x) = -\frac{\sin(x)}{2}$$

Now,

$$\left|f(x) - P_2(x)\right| = \left|\frac{f^3(\xi)}{3!}\right|\left|\left(x+\frac{\pi}{3}\right)(x)\left(x-\frac{\pi}{3}\right)\right|$$

$$= \left|\frac{-\frac{\sin(\xi)}{2}}{6}\right|\left|\left(x+\frac{\pi}{3}\right)(x)\left(x-\frac{\pi}{3}\right)\right|$$

$$= \left|-\frac{\sin(\xi)}{12}\right|\left|\left(x+\frac{\pi}{3}\right)(x)\left(x-\frac{\pi}{3}\right)\right|$$

$$= \left|-\frac{\sin(1.2)}{12}\right| \left|x\left(x^2-\frac{\pi^2}{9}\right)\right|$$

$$= \left|-\frac{\sin(1.2)}{12}\right| \left|\left(x^3-\frac{\pi^2 x}{9}\right)\right|$$

Now,

Let,

$$|w(x)| = \left|x^3 - \frac{\pi^2 x}{9}\right|$$

$$\Rightarrow w'(x) = 3x^2 - \frac{\pi^2}{9} = 0$$

$$\Rightarrow 3x^2 = \frac{\pi^2}{9}$$

$$\Rightarrow x^2 = \frac{\pi^2}{27}$$

$$\Rightarrow x = \pm\sqrt{\frac{\pi^2}{27}}$$

$$\Rightarrow x = \pm\frac{\pi}{3\sqrt{3}}$$

Now,

x	$\lvert w(x) \rvert$
$\frac{\pi}{3\sqrt{3}}$	0.442
$-\frac{\pi}{3\sqrt{3}}$	0.442
-1.2	0.4121
1.2	0.4121

$$\left[w(x) = x^3 - \frac{\pi^2 x}{9} \right]$$

Here, 0.442 is the highest value.

So, upper bound:

$$|f(x) - P_2(x)| = \left| -\sin(1.2) \frac{1}{12} \right| \times \left| \left(x^3 - \frac{\pi^2 x}{9} \right) \right|$$

$$= \frac{\sin(1.2)}{12} \times 0.442$$

$$= 0.03433$$

(Ans)

(b)

In case of runge functions, corner points are needed to be highlighted. Then, tradiontal approaches does not work well. So, we have take equal angles nodes. In other names, chebyshev nodes.

Chebyshev nodes are more of an optial choice in iterpolation because it takes care of corner points. In traditional way corner points are ignored. So, chebyshev nodes takes values in such a way that the amount of error in corner parts are minimam.

Therefore, chebysher nodes are an optimal ~~ed~~ choice in interpolation.

Ans. to the ques. No:- 4

x	$f(x)$	$f'(x)$
0.1	−0.62050	3.58502
0.2	−0.28340	3.14033

Nodes = 2

So, n = nodes − 1
$= 1$

Degree $= 2n+1$
$= (2\times1)+1$
$= 3$

(a)

$$P_3(x) = h_0(x) f(x_0) + h_1(x) f(x_1) + \hat{h}_0(x) f'(x_0) + \hat{h}_1(x) f'(x_1)$$

Now,

$$h_0(x) = \{1 - 2(x - x_0)\, l_0'(x_0)\}\{l_0(x)\}^2$$

Now,

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.2}{0.1 - 0.2} = -\frac{x - 0.2}{0.1}$$

$$l_0'(x) = \frac{d}{dx}\left(-\frac{x - 0.2}{0.1}\right)$$

$$= \frac{1}{-0.1}\frac{d}{dx}(x - 0.2)$$

$$= \frac{1}{-0.1}\, 1$$

$$\Rightarrow l_0'(x_0) = -10$$

So,

$$h_0(x) = \{1 - 2(x - 0.1) \times (-10)\} \left(-\frac{x-0.2}{0.1}\right)^2$$

$$= \{1 - (2x - 0.2)(-10)\} \frac{(x-0.2)^2}{0.01}$$

$$= \{1 - (-20x + 2)\} \frac{(x-0.2)^2}{0.01}$$

$$= (1 + 20x - 2) \frac{(x-0.2)^2}{0.01}$$

$$= (20x - 1) \times \frac{(x-0.2)^2}{0.01}$$

$$h_1(x) = \{1 - 2(x - x_1)\, l_1'(x_1)\} \{l_1(x)\}^2$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0.1}{0.2 - 0.1} = \frac{x - 0.1}{0.1}$$

$$l_1'(x) = \frac{1}{0.1} \times 1$$

$$\Rightarrow l_1'(x_1) = 10$$

Now,

$$h_1(x) = \{1 - 2(x - 0.2)(10)\} \frac{(x-0.1)^2}{0.01}$$

$$= (1 - 20x + 4) \frac{(x-0.1)^2}{0.01}$$

$$= (5 - 20x) \frac{(x-0.1)^2}{0.01}$$

$$\hat{h}_0(x) = (x - x_0)\{l_0(x)\}^2$$

$$= (x - 0.1)\left(-\frac{x - 0.2}{0.1}\right)^2$$

$$= (x - 0.1)\frac{(x - 0.2)^2}{0.01}$$

$$\hat{h}_1(x) = (x - x_1)\{l_1(x)\}^2$$

$$= (x - 0.2)\frac{(x - 0.1)^2}{0.01}$$

(b)

$x = 0.15$

So, $P_3(0.15) = h_0(x) f(x_0) + h_1(x) f(x_1) +$
$\hat{h}_0(x) f'(x_0) + \hat{h}_1(x) f'(x_1)$

$$= \{(20 \times 0.15) - 1\} \times \frac{(0.15 - 0.2)^2}{0.01} \times (-0.62050) +$$

$$(5 - (20 \times 0.15)) \times \frac{(0.15 - 0.1)^2}{0.01} \times (-0.28340) +$$

$$(0.15 - 0.1) \times \frac{(0.15 - 0.2)^2}{0.01} \times 3.58502 +$$

$$(0.15 - 0.2) \times \frac{(0.15 - 0.1)^2}{0.01} \times 3.14033$$

$= -0.31025 - 0.1417 + 0.044813 - 0.03925$

$= -0.446387$

(Ans)