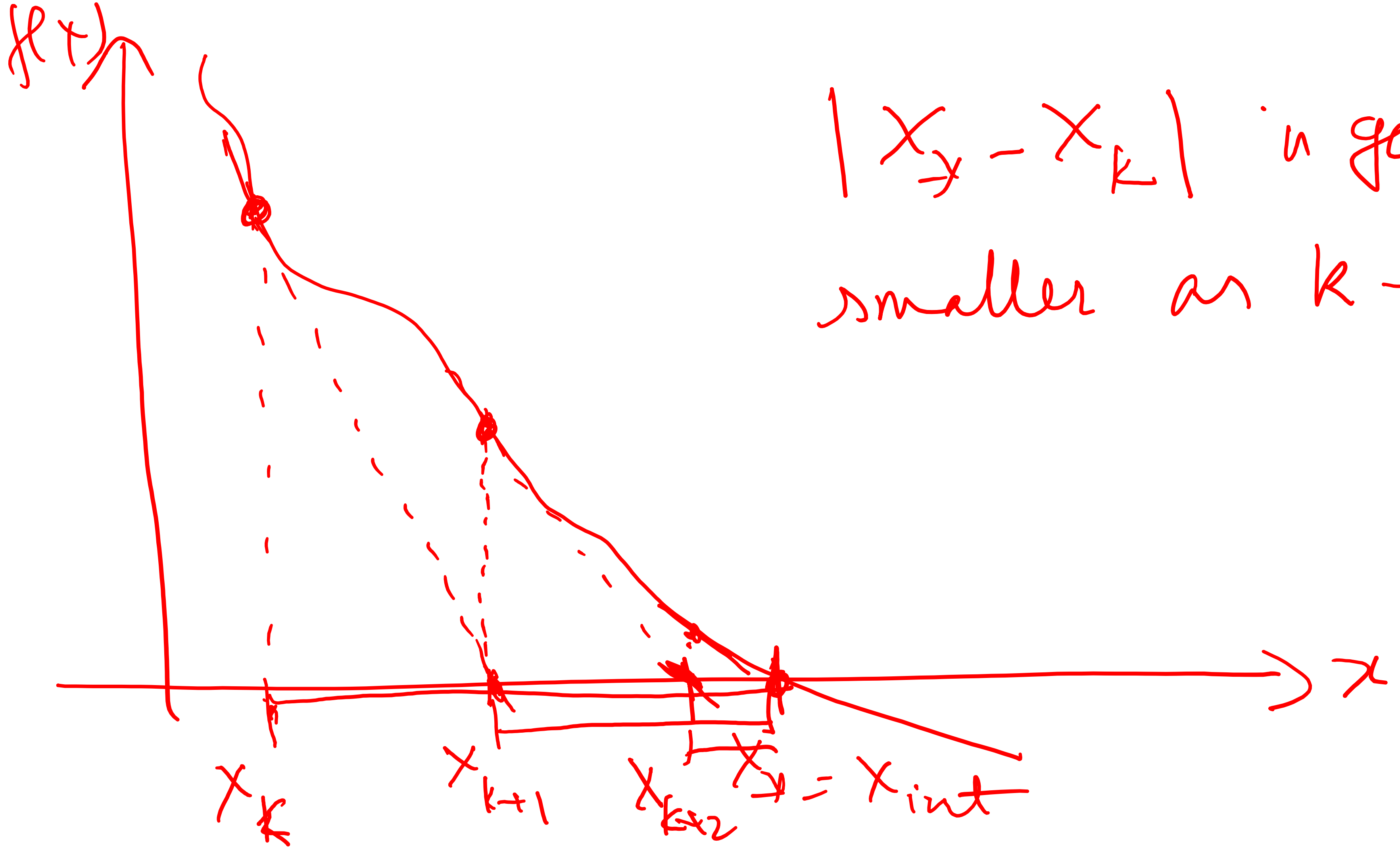
Newton’s Method:

- This is another method to find the root of the function $f(x)$.
- Conceptually it is a Fixed Point Method with superlinear convergence.
- The difference is it’s method to find the converging mapping or the function $g(x)$.
- The basic idea is the same as in the previous lecture: choose two points, and after each iteration compute the error or distance between the points before and after the iteration.
- But the points we choose is NOT the same as in the fixed point method.
- This method is a very widely used method for nonlinear equations.

- Geometrically the Newton’s method is pretty simple.
- The idea is explained below:



$|x_* - x_k|$ is getting smaller as $k \to \infty$.

- According to the discussion on the previous slide, it is clear that both x_k and x_{k+1} are on the same tangent line.
- Therefore, the slope of the tangent line is:

$$\text{Slope} = \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k} = -\frac{f(x_k)}{x_{k+1}-x_k}$$

Here, $f(x_{k+1}) = 0$ because x_{k+1} is the x_{int}, hence $f(x_{k+1}) = 0$.

- But, by definition, the slope of the tangent line to the function $f(x)$ at point x_k is the first derivative of $f(x)$ at $x = x_k$.
- Therefore, we can write:

$$f'(x_k) = -\frac{f(x_k)}{x_{k+1}-x_k} \implies \boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \boxed{\equiv g(x_k).}$$

- Clearly, at $x_k = x_\star$, then $f(x_k)|_{x_k=x_\star} = f(x_\star) = 0$. So, $x_{k+1} = x_k$.

- Alternate derivation:
- The same iteration formula can also be understood by Taylor's Theorem.
- Recall that a function, $f(x)$, can be expanded in a Taylor series (infinite series) around point y (which is near to the point x).
- But if the function is $(n+1)$-th times differentiable, then the Taylor series can also be expressed as

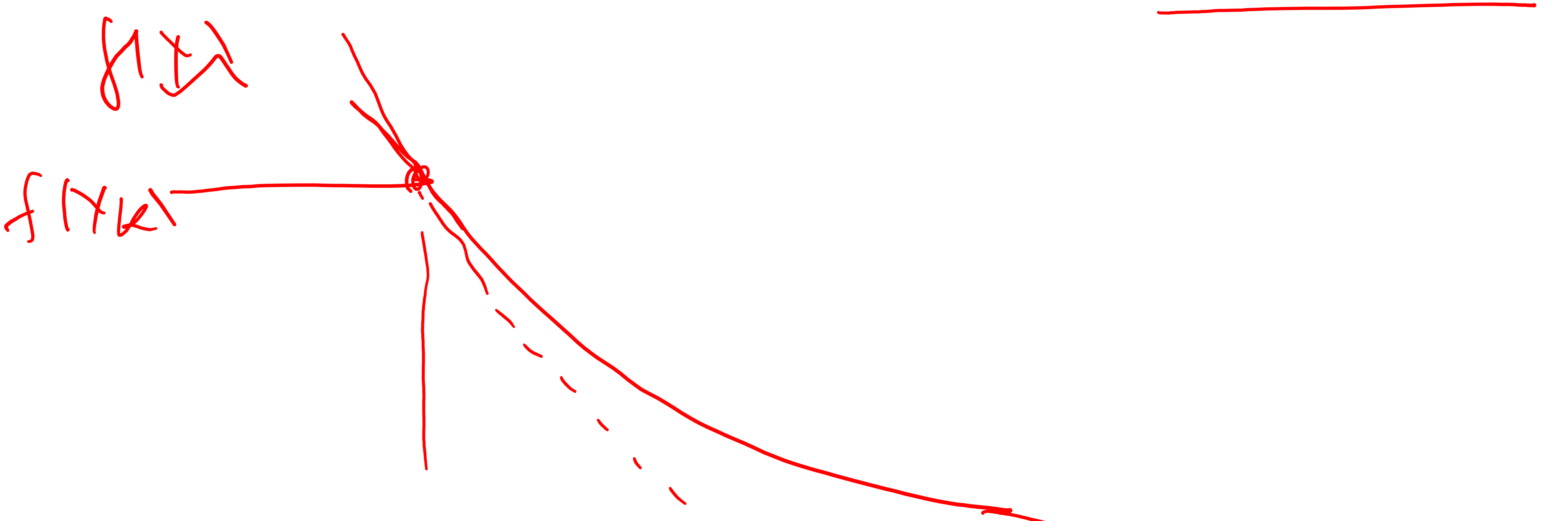
$$f(x) = f(y) + (x-y)f'(x) + \cdots + \frac{1}{n!}(x-y)^n f^{(n)}(x)$$
$$+ \frac{1}{(n+1)!} \; (x-y)^{n+1} f^{(n+1)}(\xi)\,,$$

$$= \sum_{k=0}^{k=n} \frac{f^{(k)}(x)}{k!}\,(x-y)^k + \frac{f^{(n+1)}(\overset{\xi}{x})}{(n+1)!}(x-y)^{n+1}.$$

- This is known as the Taylor's Theorem.
- The term (the summation term) is a polynomial of degree n, and the last term is called the remainder.
- More appropriately, the last term is the error term whenever we approximate a given function by a polynomial, and $\xi \in [x, y]$.
- We apply the same technique/idea here, but for $n = 1$.
- We also take: $x = x_{k+1}$ and $y = x_k$, such that $(x_{k+1} - x_k)$ is very small.
- Therefore, we can write:

$$f(x_{k+1}) = f(x_k) + (x_{k+1} - x_k)f'(x_k) + \frac{1}{2}(x_{k+1} - x_k)^2 f''(\xi(x)).$$

- We apply this equation to find the Newton's method.
- We take the point x_k on the graph of $f(x)$, and the x_{k+1} is the x_{int} of the tangent line to the graph of $f(x)$ at $x = x_k \implies \underline{f(x_{k+1}) = 0}$.



- Since, the point x_{k+1} is NOT on the graph of $f(x)$, the last term $\frac{1}{2}\underline{(x_{k+1} - x_k)}^2 f''(\xi(x))$ represent the error term.

- The points, x_{k+1} and x_k, are very close to each other.
- So, $|x_{k+1} - x_k| \ll 1$. Therefore, the error term $\frac{1}{2}(x_{k+1} - x_k)^2 f''(\xi)$ is also negligibly small. Neglecting the error term, we can write:

$$0 = f(x_{k+1}) \approx f(x_k) + (x_{k+1} - x_k) f'(x_k).$$

- Rearranging, we find:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

- Note that, for Newton's method to work, we must have: $f'(x_k) \neq 0 \; \forall \, k$.
- Compare to the fixed point method, we easily identify:

$$g(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$$

- Clearly, we see that as $x_k \to x_\star$, $f(x_k)|_{x_k \to x_\star} \to 0.$
- Therefore, we find:

$$x_{k+1} = g(x_k) = x_k = x_\star$$

$\Longrightarrow$ Fixed point of $g(x)$, and hence a root of $f(x)$.

- The rate of convergence is

$$g'(x) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right)$$

$$= 1 - \frac{f'(x)\,f'(x) - f(x)f''(x)}{\left(f'(x)\right)^2} = \frac{f(x)f''(x)}{\left(f'(x)\right)^2}$$

$$\lambda \equiv g'(x_\star) = \left.\frac{f(x)f''(x)}{\left(f'(x)\right)^2}\right|_{x=x_\star} = \frac{f(x_\star)f''(x_\star)}{\left(f'(x_\star)\right)^2} = 0.$$

- Since $\lambda = 0$, the convergence is superlinear.