- In the previous lecture, we learned the fixed point method.
- There still remains two unresolved issues:
  a) How to choose x_0 to start the iteration?
  b) Which form of $g(x)$ is convergent?
- Both are answered by the ``Contraction Mapping Theorem".
- For simplicity, we are NOT going to prove the theorem and then apply.
- Instead, we will go back to the examples in the previous lecture, and try to understand the behavior of the function $g(x)$.
- From these analysis, we will be able to apply the Theorem, and finally find the fixed point under $g(x)$.

- For the function, $f(x) = x^2 - 2x - 3$, we found that $g(x)$ can be written in many ways.
- For this function, we already have:

$$g(x) = \begin{cases} \sqrt{2x+3} \\ x^2 - x - 3 \\ \dfrac{(x^2+3)}{2x-2} \end{cases}$$

- We computed the iteration formula, $x_{k+1} = g(x_k)$, starting with $x_0 = 0$. Now, we will recalculate the same starting with $x_0 = 42$.

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- We copy from the previous lecture for $g(x) = \sqrt{2x+3}$ and upto 3 significant figures:

$g(0) = 1.73$		
$g(1.73) = 2.54$		1.21
$g(2.54) = 2.84$		0.30
$g(2.84) = 2.95$		0.11
$g(2.95) = 2.98$		0.03
$g(2.98) = 2.99$		0.01
$g(2.99) = 3.00$		0.01
		0.000

$g(42) = 9.33$		
$g(9.33) = 4.65$		4.68
$g(4.65) = 3.51$		1.14
$g(3.51) = 3.17$		0.340
$g(3.17) = 3.06$		0.110
$g(3.06) = 3.02$		0.040
$g(3.02) = 3.01$		0.010
$g(3.01) = 3.00$		0.010
		0.000

- For both values of x_0, the function $g(x)$ converges to a root 3.00.
- Note that even though $x_0 = 0$ is nearer to the root -1, the iteration converge to 3, NOT -1. The reason is that the ratio λ is less than 1 requires that $x > -1$.

- For the second case, $g(x) = x^2 - x - 3$, we get:

$$
\begin{aligned}
g(0) &= -3.00 \\
g(-3) &= 9.00 \\
g(9) &= 69.0 \\
g(69) &= 4.69 \times 10^3
\end{aligned}
\qquad
\begin{aligned}
g(42) &= 1.72 \times 10^3 \\
g(1.72 \times 10^3) &= 2.95 \times 10^6 \\
g(2.95 \times 10^6) &= 8.72 \times 10^{12}
\end{aligned}
$$

12

66

10^6

10^{12}

- For both choices for x_0, $g(x)$ diverges very rapidly.
- Note that even though $g(-1) = -1$ and $g(3) = 3$, the ratio is getting bigger and bigger.
- Therefore, it is not possible to obtain a fixed point in this case.

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- Let's take the 3rd expression: $g(x) = \frac{x^2+3}{2x-2}$.
- Starting from $x_0 = 0$ and 42 and also upto 3 sig. fig., we get:

$$\begin{aligned} g(0) &= -1.50 \\ g(-1.50) &= -1.05 \\ g(-1.05) &= -1.00 \end{aligned}$$

0.450

0.850

0.000

$$\begin{aligned} g(42) &= 21.6 \\ g(21.6) &= 11.4 \\ g(11.4) &= 6.39 \\ g(6.39) &= 4.07 \\ g(4.07) &= 3.19 \\ g(3.19) &= 3.01 \\ g(3.01) &= 3.00 \end{aligned}$$

10.2

5.01

2.32

0.180

0.010

0.010

0.000

- Now $g(x)$ converges to the other fixed point.
- Note that $x_0 = 0$ converges to the nearest root which is -1, and $x_0 = 42$ converges to 3 because it is closer to 3 than from -1.

- The above numerical analysis suggest that if $g(x)$ is convergent, then it converges to the nearest fixed point from x_0.
- For the convergent $g(x)$, the difference between the successive iterated values are decreasing.
- This suggest that $\left|\frac{\Delta g(x_k)}{\Delta x}\right| \equiv \left|\frac{g(x_{k+1})-g(x_k)}{x_{k+1}-x_k}\right| < 1.$
- This also suggest that, x_0 may not be chosen arbitrarily. It has a certain region (or subinterval) to be chosen from.
- These brings about the `Contraction Mapping Theorem'.

- The Contraction Mapping Theorem states that:
- If g is a contraction mapping on $L = [a, b]$, then
    1. There exists a unique fixed point $x_\star \in L$ with $g(x_\star) = x_\star$.
    2. For any $x_0 \in L$, the iteration $x_{k+1} = g(x_k)$ will converge to $x_\star$ as $k \to \infty$.
- To understand the theorem, let's take two points $x, y \in L$, and consider their mapping from L to L.
- Since to obtain a fixed point, the mapping g has to be converging, the distance between the points on L must be decreasing.

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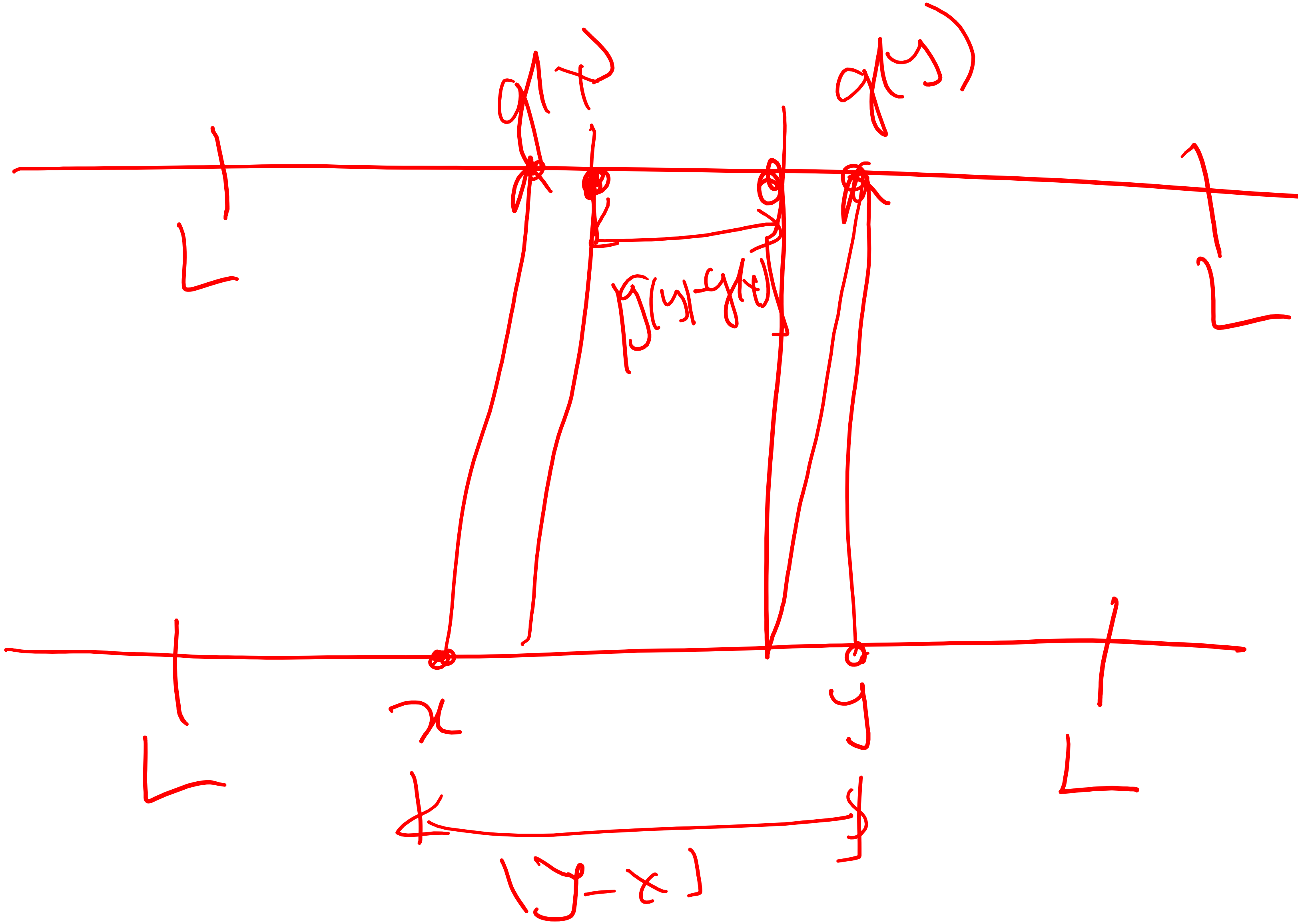
- Diagrammatically, it looks like:



$$\frac{|g(y_k)-g(x_k)|}{|y-x|}$$

- The distance before the mapping = $|y - x|$.
- The distance after the mapping = $|g(y) - g(x)|$.
- Since g is converging, we must have:

$$|g(y) - g(x)| < |y - x|.$$
$$|g(y) - g(x)| = \lambda|y - x|.$$

- Here λ is real number and less than one (fraction).
- After k-th iteration, we have:

$$|g(y_k) - g(x_k)| = \lambda^{k+1}|y - x| \to 0 \text{ as } k \to \infty.$$

- It should be noted that if both x, y are fixed point, then $\lambda = 1$. The length will not decrease.
- If any one or both point are not fixed points, $\lambda < 1$.

- Now rewriting, we have:

$$\frac{|g(y) - g(x)|}{|y - x|} < 1$$

- Now, Let x is a fixed point and y a point near to x. Since as $k \to \infty$, the distance $|y - x| \to 0$, we can write,

$$\lambda = \lim_{y \to x} \frac{|g(y) - g(x)|}{|y - x|} = |g'(x)| < 1\ .$$

- The above result shows that if the derivative of g is less than one at the fixed point, then the iteration starting at any point near to the fixed point will converge to the fixed point.
- The parameter, λ, is called the rate or ratio.
- In summary, if g is a contraction mapping and the iteration start at x_0, the iteration will converge to the nearest fixed point from x_0.

- Let’s apply this to different forms of $g(x)$:
  1. For $g(x) = \sqrt{2x+3}$, we find:
     $$\lambda = \left|\frac{dg}{dx}\right| = \left|\left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{2x+3}}\right)\right| = \left|\frac{1}{\sqrt{2x+3}}\right|.$$
     Clearly, $\lambda|_{x=3} = \frac{1}{3} < 1$. Also, for $x > -1$, $\lambda < 1$. That’s why this converging to 3 for both values of x_0.
  2. For $g(x) = x^2 - x - 3$, we find:
     $$\lambda = \left|\frac{dg}{dx}\right| = |2x - 1|.$$
     Clearly, $\lambda|_{x=3} = 5 > 1$, and$\lambda|_{x=-1} = 3 > 1$. So this diverges.

3. The last choice was $g(x) = \frac{x^2+3}{2x-2}$. For this, we have:

$$\lambda = \left|\frac{d}{dx}\left(\frac{x^2+3}{2x-2}\right)\right| = \left|\frac{\mathrm{x}^2-2\mathrm{x}-3}{2(\mathrm{x}-1)^2}\right|.$$

$$\therefore \quad \lambda\Big|_{x=-1} = 0 < 1 \quad \text{and} \quad \lambda\Big|_{x=3} = 0 < 1\,.$$

Note that since $x_0 = 0$ is closer to -1, the iteration converged to -1. and similarly, since $x_0 = 42$ is closer to 3, the iteration converges to 3.