LU Decomposition:

- In the previous lecture, we learnt how to transform the $n \times n$ matrix A into a triangular form: either upper or lower triangular form.
- In this lecture, we will learn how the transformation matrix A of a linear system can always be transformed into a product of two triangular matrix: one lower triangular (L) and the other upper triangular (U).
- Hence this process is called LU decomposition: $A = LU$.
- To develop the decomposition method, we just need to use the row multipliers that we defined in the previous lecture to transform the matrix A into triangular form step by step.

- This method depends on the following facts:
- Each row operation changes the values of all matrix elements below 1st row as in the following way:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow{1^{\text{st}} \text{ row operation}} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix}$$

- On the 1st row, a_{11} is the diagonal element. All element below a_{11} (that means, the elements are: $a_{21}, a_{31}, \cdots, a_{n1}$) will be replaced by ZERO after the completion of 1st row operation. Other elements are arbitrary.
- The 1st row operation can be expressed as a left multiplication of the matrix A by a matrix, denoted by $F^{(1)}$.
- The matrix, $F^{(1)}$, is built out of the row multipliers $m_{21}, m_{31}, \cdots, m_{n1}$ only.

- So, the matrix , $F^{(1)}$, for the 1st row operation can be written as:

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

- If we multiply A from left by $F^{(1)}$, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \cdots & a'_{nn} \end{pmatrix}$$

$$\Uparrow$$

- Here we used the following:

$$m_{i1} = \frac{a_{i1}}{a_{11}} \quad \text{and} \quad a'_{i1} = a_{i1} - m_{i1}\, a_{11} \quad \Leftarrow \quad \text{The } 1^{st} \text{ row operation !!!}$$

- From the previous slide, we see the matrix form of the 1st row operation.
- The 1st row operation or equivalently multiplying by the matrix $F^{(1)}$ means that:
  1) The multiple of the first row is subtracted from the row below it such that all new matrix element in the 1st column below the a_{11} become zero.
  2) Other matrix elements may or may not be zero and are denoted by a prime as a superscript (Do not confuse with derivative). Here the prime just indicates that these values are changed by the row operation.
  3) Now, we can apply the 2nd row operation. Certainly, this operation implies that the multiple of the 2nd row will be subtracted from the row below it (that is, from the 3rd ,4th rows etc.)
  4) The 1st row DOES NOT get changed by the 2nd or any other row operations. Only the matrix elements in the 2nd column, but below the a_{22} element will be changed by the row operation and will be again denoted by putting more primes as a superscript. This means that the value of a'_{23} will be changed to a''_{23} after 2nd row operation.

- Similarly, the matrix, $F^{(2)}$, for the 2nd row operation is defined by:

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{pmatrix}$$

- This matrix acts on the result of the 1st row operation, NOT on A:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a''_{n3} & \cdots & a''_{nn} \end{pmatrix}$$

- Note that the values of a'_{ij} do not remain same. For 2nd row operation, the 2nd row values do not change, but the values on the 3rd, 4th, etc. rows get changed, and denoted by double primes.

- From the above, the following can be concluded easily for k^{th} row operation:
  1) For k-th row operation, the matrix element on and above the k-th row DO NOT change (that is the values on the 1st, 2nd, $\cdots$, k^{th} row remain same).
  2) The matrix elements below the diagonal elements AND also on the 1st, 2nd, $\cdots$, k^{th} column changes to ZERO.
  3) All other matrix elements are changed by: $r_{ij}^{(k)} \to r_{ij}^{\prime(k)} = r_{ij}^{(k)} - m_{ik} a_{kj}^{(k-1)}$, here the superscript is the row operation number, and the subscript is the matrix element on a rows on which the row operation is applied.
- After $(n-1)$-th row operation on a $n \times n$ matrix A, we obtain an upper diagonal matrix:

$$F^{(n-1)} \ldots F^{(2)} F^{(1)} A = U.$$

- Denoting $A \equiv A^{(1)}$, we define: $A^{(2)} \equiv F^{(1)} A^{(1)}$, $A^{(3)} \equiv F^{(2)} A^{(2)}$, and so on. This gives us: $A^{(n)} \equiv F^{(n-1)} A^{(n-1)} = U$.
- The above can also be expressed as: $FA = U$, where $F \equiv F^{(n-1)} \ldots F^{(2)} F^{(1)}$. The matrix F encodes all row operations required to convert A into an upper triangular form $U \equiv A^{(n)}$.

- In matrix form, the complete row operation to traingularize is: $FA = U$.
- Here, the matrix for the k-th row operation is:

$$F^{(k)} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & -m_{k+1,k} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -m_{n,k} & \cdots & 0 & 1 \end{pmatrix}$$

- We know that inverse of the product of matrices equals to the reverse order product of the inverse matrices:

$$(ABC\cdots)^{-1} = \cdots C^{-1}\, B^{-1}\, A^{-1}.$$

- Using the above property, we write:

$$A = F^{-1}U = \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1} \cdots \left(F^{(n-1)}\right)^{-1}.$$

- Since $F^{(k)}$ is upper triangular, it's inverse $\left(F^{(k)}\right)^{-1}$ is also upper triangular.
- Remember that the inverse of a non-singular matrix C can be calculated from the following formula:

$$C^{-1} = \frac{1}{|\det C|}\,(\mathrm{Adj}\; C)\,.$$

- Let's take $k = 2, n = 4$. as an exercise you can show by explicit computation that

$$\text{For } F^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 \\ 0 & -m_{42} & 0 & 1 \end{pmatrix} \implies \left(F^{(2)}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m_{32} & 1 & 0 \\ 0 & m_{42} & 0 & 1 \end{pmatrix}$$

  Here m_{32}, m_{42} are the row multipliers (non-zero).
- The general form of inverse matrices is true for any k and any n, even though it is shown here only 4×4 matrix.
- The matrices, $F^{(k)}$, and their inverses are also called unit triangular matrices (triangular matrix with all diagonal elements are one).

- Since the product of triangular matrix is also triangular, we get:

$$F^{-1} \equiv \left(F^{(1)}\right)^{-1}\left(F^{(2)}\right)^{-1}\cdots\left(F^{(n-1)}\right)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ m_{21} & 1 & \ddots & \cdots & \cdots & \vdots \\ m_{31} & m_{32} & 1 & & \cdots & \vdots \\ m_{41} & m_{42} & m_{43} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$$

- This matrix clearly is a unit lower triangular matrix. Henceforth, instead of F^{-1}, it is denoted by L, the unit lower triangular matrix :

$$L \;:= F^{-1} \equiv \left(F^{(1)}\right)^{-1}\left(F^{(2)}\right)^{-1}\cdots\left(F^{(n-1)}\right)^{-1}.$$

- Hence, we can write, $A = LU$. This means that any $n \times n$ matrix A of a linear system can always be expressed as a product of a unit lower triangular matrix L and an upper triangular matrix U. This is the reason, we call it `LU Decomposition'.

- This is stated as Theorem 5.2 in the required text for this course.

- How does this LU decomposition help us? Ley's find it out !!!
- The linear system in matrix form is: $Ax = b$.
- Using the decomposition, we get: $L\underbrace{Ux} = b$.
- Now, let $\boxed{Ux = y} \implies \boxed{Ly = b}$.

    1. Since L is lower ~~diagonal,~~ triangular we solve $Ly = b$ to obtain y. Here we use the formula developed for lower triangular matrix in the Gaussian Elimination lecture.
    2. Since U and y are known, we solve $Ux = y$ to obtain x. Here we use the formula developed for upper triangular matrix in the Gaussian Elimination lecture.

- The main advantage in this method is that it can be used to solve several linear system that differ by the values in b only. We need to compute L and U only once.
- But in Gaussian Elimination method, if b changes we needed to restart the row operation, and this is a big disadvantage.
- But in LU method, we do not need to repeat the row operations. This is a big advantage and saves a lot of time.

- Example: Let's reconsider the example in the previous lecture:
- The linear ~~Lear~~ system is

$$x_1 + 2x_2 + x_3 = 0, \qquad x_1 - 2x_2 + 2x_3 = 4 \quad \text{and} \quad 2x_1 + 12x_2 - 2x_3 = 4.$$

- In matrix form, we write:

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}}_{b}.$$

- Here, the 1st multipliers are: $m_{21} = 1$, and $m_{31} = 2$. Therefore, the 1st row operation matrix is

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

- Therefore, we obtain:

$$A^{(2)} = F^{(1)}A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{pmatrix}.$$

- The 2nd row multiplier is : $m_{32} = -\frac{4}{8} = -2$. The matrix, $F^{(2)}$, is

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

- Therefore, the upper triangular matrix is:

$$U := A^{(3)} = F^{(2)} A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

- The lower triangular matrix is:

$$L := \left(F^{(1)}\right)^{-1} \left(F^{(2)}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

- It can be easily checked that

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix} = A.$$

- We now have both $\cancel{U}$ L and U. We now solve $Ly = b$ first:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \Longrightarrow$$

$$\boldsymbol{y_1 = 0}.$$

$$\boldsymbol{y_1 + y_2 = 4. \therefore y_2 = 4 - y_1 = 4.}$$

$$\boldsymbol{2y_1 - 2y_2 + y_3 = 4 \ \therefore y_3 = 4 - 2y_1 - 2y_2 = 12}.$$

- Now we solve: $Ux = y$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}.$$

- Therefore, the solutions are:
  - Last row: $-2x_3 = 12 \Longrightarrow x_3 = -6.$
  - Middle row: $-4x_2 + x_3 = 4 \Longrightarrow \ x_2 = -\frac{4 - x_3}{4} = -\frac{5}{2}$.
  - First row: $x_1 + 2x_2 + x_3 = 0 \ \Longrightarrow \ x_1 = -2x_2 - x_3 = 11.$

Pivoting:

- The `Pivoting’ is a technique to avoid the appearance of ZERO along the diagonal elements in Gaussian Elimination and LU-decomposition method.
- If ZERO appears along the diagonal, then the multiplier for that row operation will be undefined (dividing by ZERO). That’s why both method fails.
- By Pivoting, this problem can be avoided.
- The diagonal elements should be of same order of magnitudes. If the elements along the diagonal differ by large order of magnitude (like 10^3), then in the solution loss of significance may occur. This problem can also be avoided by Pivoting.
- In simple terms, Pivoting means to swap two rows or two columns so that the diagonal elements do not have any ZEROs , and also the order of magnitudes of the diagonal elements are similar or close (see examples on page.55 of the required lecture note).