Interval Bisection Method

- This method depends on the following Theorem:

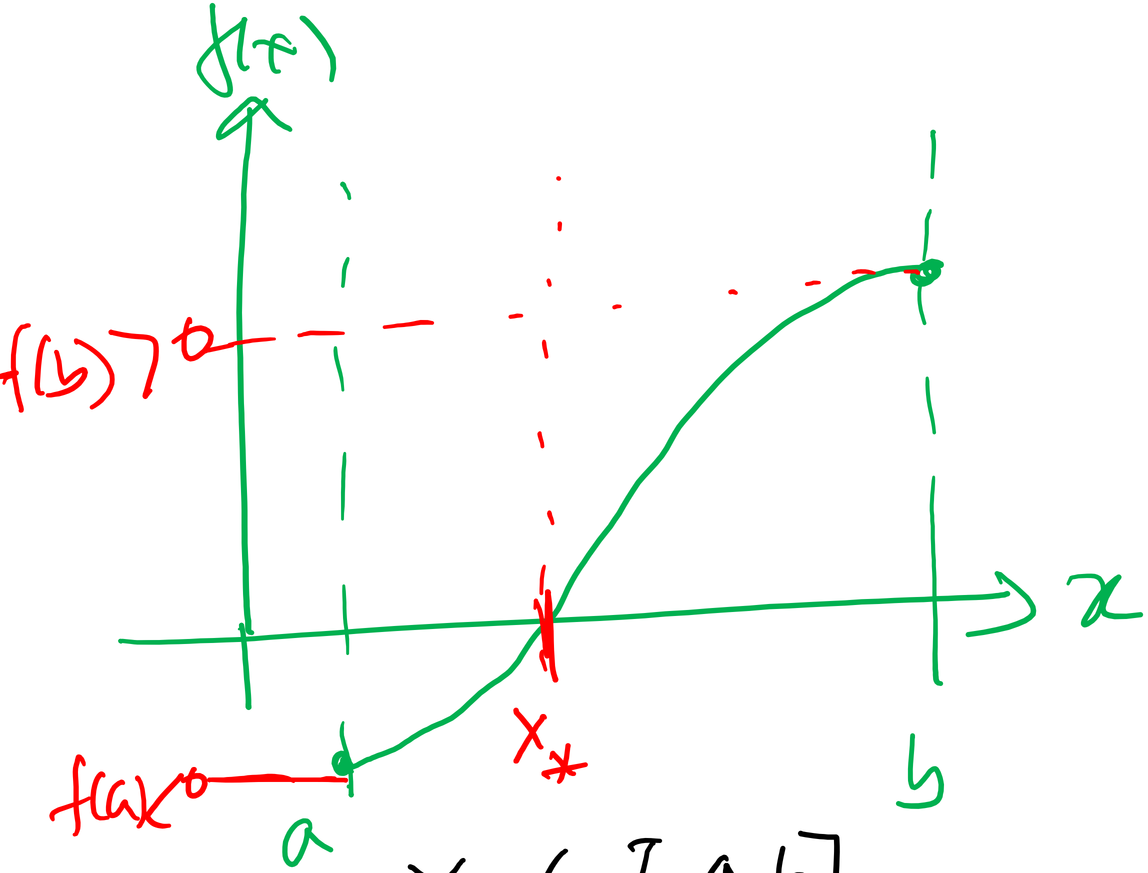
  If a function $f(x)$ is continuous on an interval $I = [a, b]$, then for each $x \in I$, there exists a real number c, such that $\underline{f(x) = c}$.

- This theorem is known as the `Intermediate Value Theorem'.
- How does this theorem help us?
- If $c = 0$, then by this theorem, then there exists a point $x \in I$, such that $\underline{f(x) = 0}$. This point is the root $\underline{x_\star}$ of $f(x)$.
- Hence, we must have (because the function crosses the x-axis):

$$\left.\begin{array}{ll} \text{either} & f(a) > 0 \text{ and } f(b) < 0. \\ \text{or} & f(a) < 0 \text{ and } f(b) > 0. \end{array}\right\} \boxed{f(a)f(b) < 0}$$

- The converse is also true. This means that if the values of a function at the two end points of an interval have opposite signs, then there is zero of the function on that interval.

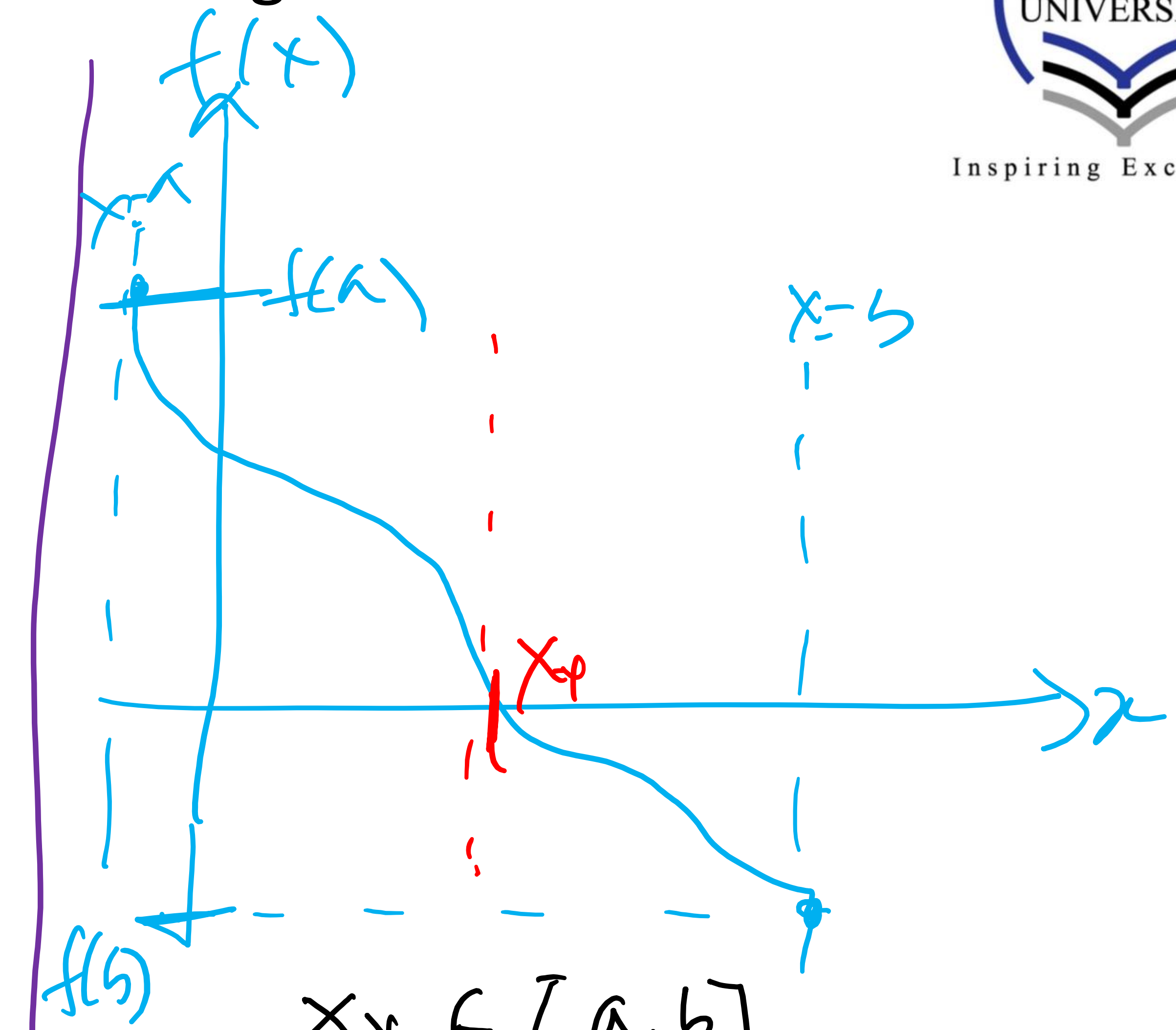
- Graphically, it looks like as in the following:



$x_* \in [a,b]$

And $f(a)<0$ & $f(b)>0$

$\therefore \boxed{f(a)f(b)<0}$

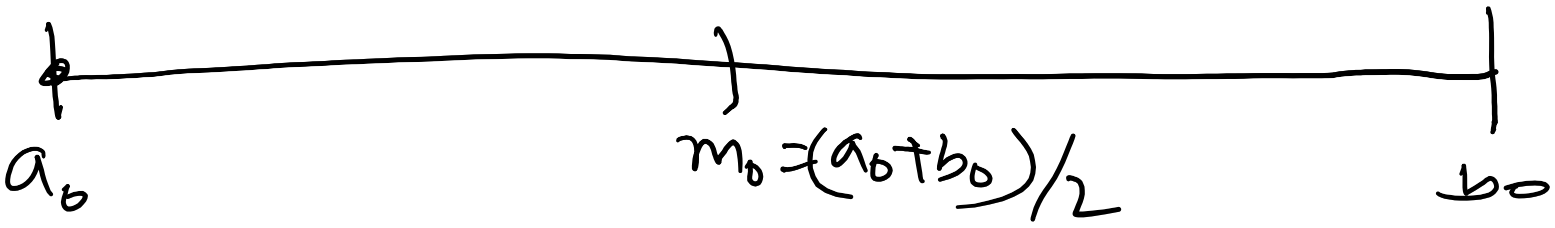


$x_* \in [a,b]$

And $f(a)>0$ & $f(b)<0$

$\Rightarrow \boxed{f(a)f(b)<0}$

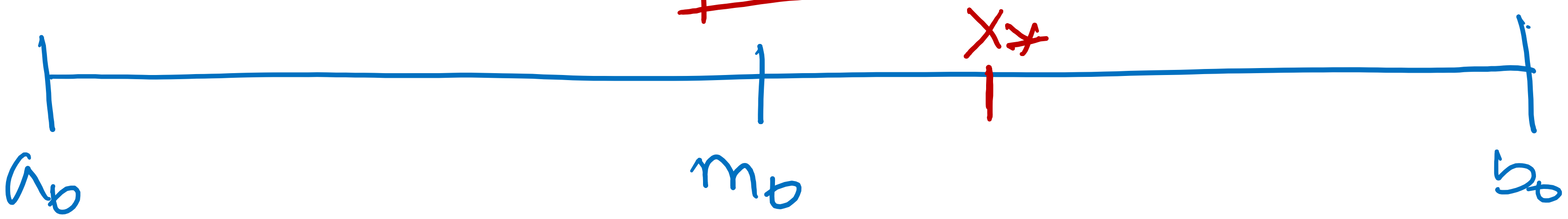
- Let’s find how does the `Interval Bisection Method’ works:
- Let $I = [a_0 , b_0]$ is the interval. We also assume that $f(a_0) > 0$ and $f(b_0) < 0$. Now find the middle point, $m_0 = (a_0 + b_0)/2$.
- The interval I breaks down as : $[a_0, b_0] \to [a_0, m_0] \cup [m_0, b_0]$.



- Since there is only one root in the interval I, the roots must be in the subinterval $[a_0, m_0]$ or $[m_0, b_0]$.
- To find which subinterval, we compute $f(m_0)$ and check if $f(m_0)$ is greater than, less than or equal to zero.
- If $f(m_0) = 0$, then it is the solution and $x_\star = m_0$ is the root.

- Let $f(m_0) > 0$, then the solution or root must be in the interval $[m_0, b_0]$. And the error must be half the length of the interval I or less:

$$\boxed{|x_\star - m_0| \le \frac{1}{2}|b_0 - a_0|.}$$



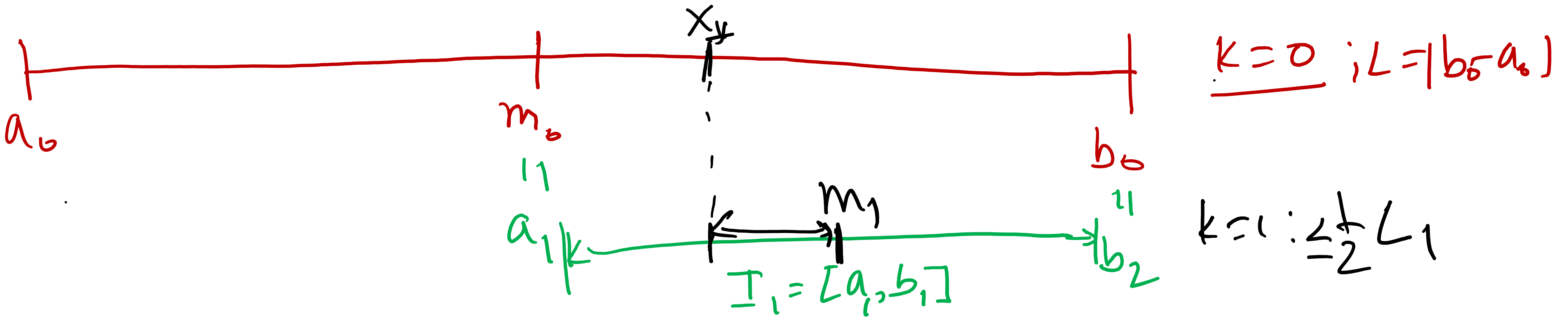
$L = |b_0 - a_0|$

$|a_0 - m_0| = |m_0 - b_0| = \frac{1}{2}L$

$\boxed{k = 0}$

- If the error is within the bound, then m_0 is the root. So, NO iteration.
- If the error is NOT within the error bound, we repeat the above process. This is the first iteration, and we label it by putting $k = 1$. Here k is the iteration number.

- For the first iteration, the new interval is $I_1 = [a_1, b_1]$, where $a_1 = m_0$ and $b_1 = b_0$.
- The previous process is repeated with $m_1 = \frac{1}{2}(b_1 + a_1)$.
- Hence, $I_1 = [a_1, m_1] \cup [m_1, b_1]$.
- Let $f(m_1) < 0$. So the root must be in the interval $[a_1, m_1]$.



- The error now: $|x_\star - m_1| \leq \frac{1}{2}|b_1 - a_1| = \frac{1}{2}|b_0 - m_0| = \frac{|b_0 - a_0|}{4}$.
- If the error is within the bound, m_1 is the root, and if not, iteration continues.

- After n-th iteration, the error is

$$\boxed{|x_{\star} - m_n| \le \frac{|b_0 - a_0|}{2^{n+1}}\,.}$$

- Let δ is the desired error bound, then we must have:

$$\frac{|b_0 - a_0|}{2^{n+1}} \le \delta \quad \Longrightarrow \quad \boxed{n \ge \frac{\log(|b_0 - a_0|) - \log(\delta)}{\log(2)} - 1\,.}$$

- If $a_0 = 1.5$, $b_0 = 3$ and $\delta = \epsilon_M = 1.1 \times 10^{-16}$ (Machine epsilon), then we easily find that the number of iterations required to find the root is

$$n \ge \frac{\log(3-1.5) - \log\left(1.1\times10^{-16}\right)}{\log 2} - 1 \quad \Longrightarrow \quad n \ge 53 \text{ iterations.}$$

Example:

- Let $f(x) = \frac{1}{x} - 0.5$ and $I = [1.5, 3]$.
- Here: $a_0 = 1.5$, and $f(a_0) = 0.1666$. Also $b_0 = 3$, and $f(b_0) = -0.1666$.
- So, there exists a solution in [1.5,3]. Now: $m_0 = \frac{a_0+b_0}{2} = 2.25$, and $f(m_0) = -0.0555 < 0$.
- Clearly, the roots lies in $[a_0, m_0] \equiv [a_1, b_1] = [1.5, 2.25]$.
- Now: $m_1 = \frac{a_1+b_1}{2} = 1.875$, and $f(m_1) = 0.0333 > 0$.
- Now, the solution is in the interval $[m_1, b_1] = [1.875, 2.25]$.
- If the error is $\delta = 1.0 \times 10^{-5}$, then we can easily show that the minimum number of iteration required is

$$n \geq \frac{\log(3 - 1.5) - \log(1.0 \times 10^{-5})}{\log(2)} - 1 \implies \boxed{n \geq 16.}$$