Fixed Point Iteration

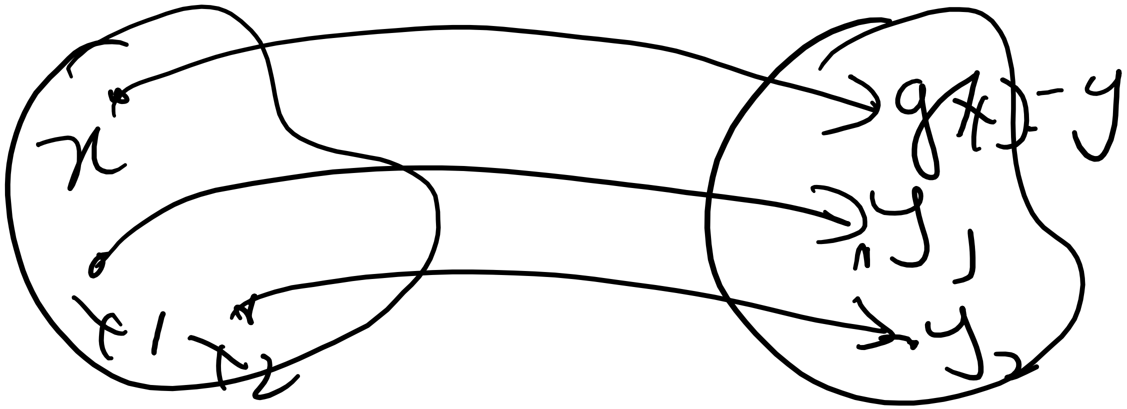
The motivation is the same:

- Want to solve: $f(x) = 0$.
- Need to transform, $f(x) = 0$, into a new form, $g(x) - x = 0$, by using algebraic operations only.
- Conditions: Both $f(x)$ and $g(x)$ must be continuous on some interval $I = [a, b]$, where a and b are real numbers.
- Advantage: In this method, presence of multiple roots can be taken care of, which was not allowed in the Interval Bisection method.

- The existence of the solution for $f(x) = 0$ implies that

$$g(x) = x.$$

- This is known as the fixed point equation or mapping.
- Any point x that remains same under a nontrivial mapping g is called the fixed point under that mapping.
- Note that the mapping g is NOT a trivial mapping (meaning that it is not an identity transformation. In other words, it is NOT a multiplication by one).



$y = g(x) \neq x.$

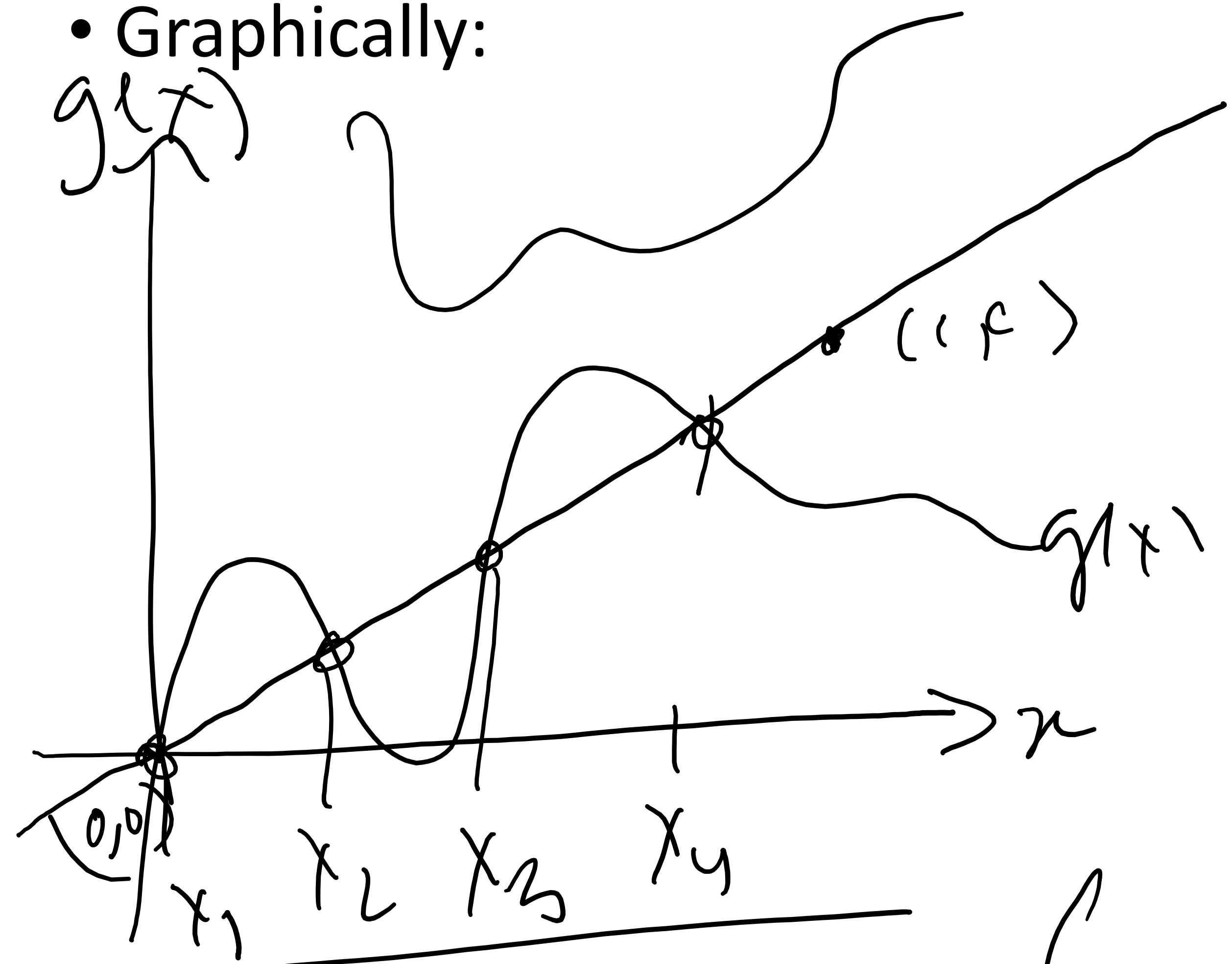
If $\exists\, x_0 \in I \Rightarrow \boxed{g(x_0) = x_0}$

Then x_0 is the fixed point.

- Graphically how do these look like?
- Recall the following for the graph of $f(x) = 0$:

- Let’s consider the graph of $g(x) = x$:
- It is clear that a line through the origin (0,0) and (c, c) must have slope 1 (one).
- Therefore, any graph of $g(x)$ vs. x that interest the line with slope one must have a fixed point.
- For straight line: Slope = $\frac{c-0}{c-0}$ = 1.
- If the graph of $g(x)$ intersect the line at $x = d$, then we must have $g(d) = d$.
- Hence, $x = d$ is a fixed point of $g(x)$.
- It also means that $x_\star = x = d$ is a root of the function $f(x)$. Hence: we have a solution: $f(d) = 0$.

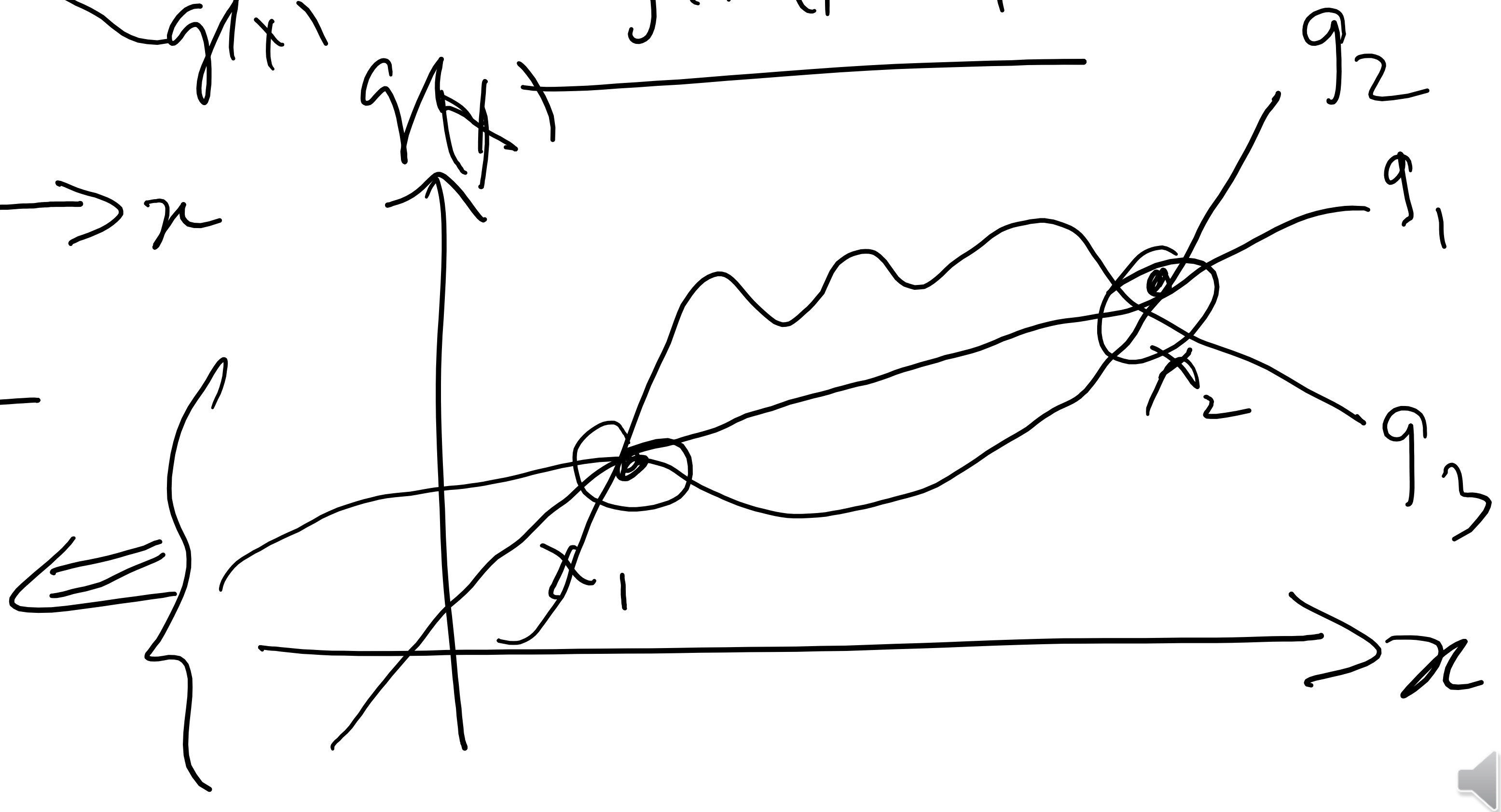
- Graphically:



$g(x_1) = x_1$

$\vdots$

$g(x_4) = x_4$

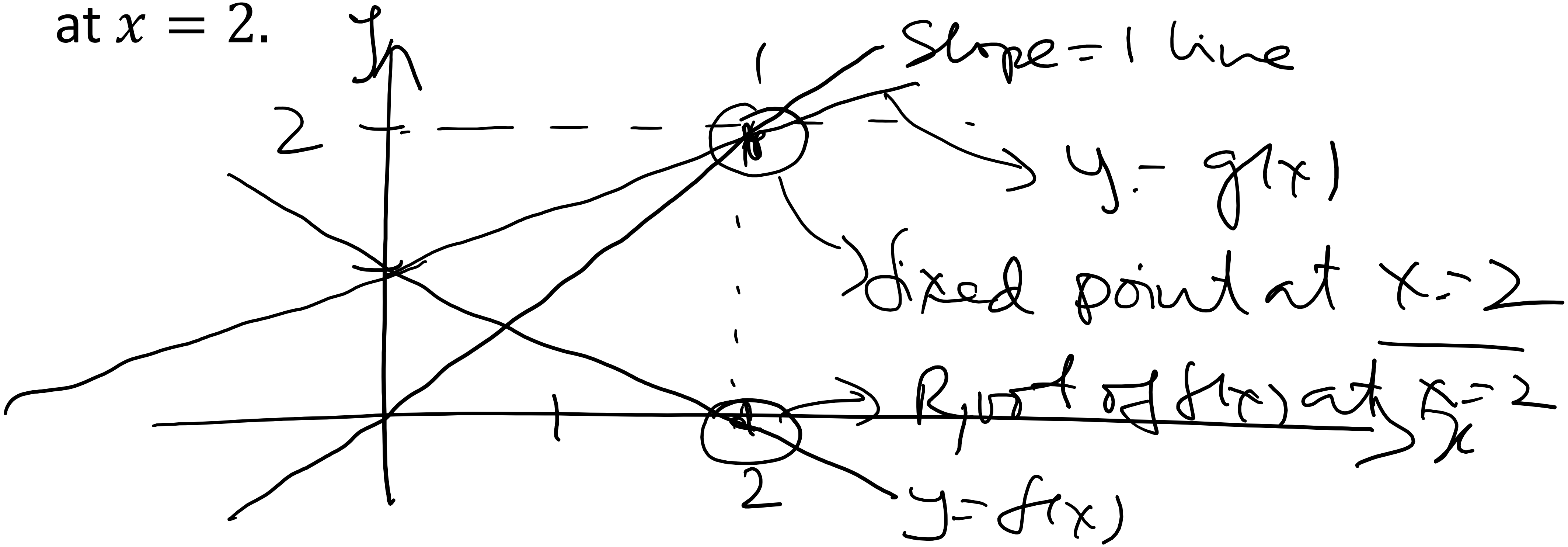


$g_1(x_1) = x_1$

$g_2(x_1) = x_1$

$g_3(x_1) = x_2$

- Lets take an example: $g(x) = \frac{x+2}{2}$.
  - $g(0) = 1 \neq 0 \Longrightarrow x = 0$ is not a fixed point.
  - $g(1) = 1.5 \neq 1 \Longrightarrow x = 1$ is not a fixed point either.
  - $g(2) = 2 \quad \Longrightarrow \quad x = 2$ is a fixed point.
- Therefore, $f(x) = g(x) - x = \frac{x+2}{2} - x = -\frac{1}{2}x + 1$ has a root at $x = 2$.



- The question is: how to find the fixed point numerically?
- What is the iteration process:
- We start with an arbitrary initial point $x_0 \in I = [a, b]$.
- Compute: $g(x_0)$
  - If $g(x_0) = x_0$, it is a fixed point. Done
  - If not, define: $x_1 = g(x_0)$. $|x_1 - x_0| >$ Error bound.
- Compute: $g(x_1)$
  - If $g(x_1) = x_1$, it is a fixed point. Done
  - If not, define: $x_2 = g(x_1)$. $|x_2 - x_1| >$ Error bound.
- After, k-th iteration:
  - $x_{k+1} = g(x_k)$. If $|x_{k+1} - x_k| \leq \delta$, then we take $|x_{k+1} - x_k| \approx 0$.
  - So, x_k is the fixed point of $g(x)$, and hence $x_\star = x_k$ is the root of the function $f(x)$ within the error bound δ.

- Example: Let’s take $f(x) = x^2 - 2x - 3 = 0$.
- Since $x^2 - 2x - 3 = (x-3)(x+1)$, $f(x)$ has two roots at $x_\star = -1, 3$.
- In the following, we will try to find these roots by rewriting $f(x) = 0$ in terms of a new function $g(x) = x$:
  1) $x^2 - 2x - 3 = 0 \Longrightarrow x = \sqrt{2x+3} \equiv g(x)$.
  2) $x^2 - 2x - 3 = x(x-2) = 3 \Longrightarrow x = \frac{3}{x-2} \equiv g(x)$.
  3) $x^2 - 2x - 3 = 0 \Longrightarrow x = x^2 - x - 3 \equiv g(x)$.
  4) $x^2 - 2x - 3 = 0 \Longrightarrow 2x^2 - 2x = x^2 + 3 \Longrightarrow x = \frac{(x^2+3)}{2x-2} \equiv g(x)$.
- Note that all four form of $g(x)$ above satisfy: $g(-1) = -1$ ~~and~~ or $g(3) = 3$. But when we start the iteration with $x_0 \neq -1, 3$, the fixed point cannot be obtained always.

- Let's consider the first case starting with $x_0 = 0$.
- Using the function: $g(x) = \sqrt{2x+3}$ and upto 3 sig. fig.:

$$
\begin{aligned}
g(0) &= 1.73\\
g(1.73) &= 2.54\\
g(2.54) &= 2.84\\
g(2.84) &= 2.95\\
g(2.95) &= 2.98\\
g(2.98) &= 2.99\\
g(2.99) &= 3.00
\end{aligned}
$$

- After 7th iteration: $x_8 - x_7 \approx 0$ (within 3 sig. fig. or $\delta = 1.00 \times 10^{-3}$).
- So, $x_7 = 3.00$ is the fixed point of $g(x)$ and it is also the root of $f(x)$.

- Let's take the 3rd expression: $g(x) = x^2 - x - 3$.
- Starting from $x_0 = 0$ and upto 3 sig. fig, we get:

$$
\begin{aligned}
g(0) &= -300 \quad -3.00 \\
g(-3) &= 9.00 \\
g(9) &= 69.0 \\
g(69) &= 4.69 \times 10^3
\end{aligned}
$$

- Clearly, as the iteration number increases, $g(x)$ increases indefinitely.
- So, the iteration does not converge to a single value.

- Let's take the 4th expression: $g(x) = \frac{x^2+3}{2x-2}$.
- Starting from $x_0 = 0$ and upto 3 sig. fig., we get:

$$
\begin{aligned}
g(0) &= -1.50 \\
g(-1.50) &= -1.05 \\
g(-1.05) &= -1.00
\end{aligned}
$$

- After 3rd iteration: $x_4 - x_3 \approx 0$ (within 3 sig. fig. or $\delta = 1.00 \times 10^{-3}$).
- So, $x_3 = -1.00$ is the fixed point of $g(x)$ and it is also the root of $f(x)$.

- These examples clearly shows how to find the fixed points.
- But there are two questions that need to be answered:
    1. How to choose x_0? Does all values convergent to a same fixed number (if not divergent)?
    2. Since there are many ways to construct $g(x)$ for the same $f(x)$, how to find out which form of $g(x)$ will lead to a fixed point?
- These questions will be answered in the next video lecture where we will discuss about the Contraction Mapping Theorem.