Ch. 5 : Linear Equations

- In this chapter, we will discuss the following two topics:
  1. Gaussian Elimination method
  2. LU decomposition
- At first, we define a linear system defined by the linear equations.
- A linear system is described by a set of linear equations, and each linear equation is expressed by a set of linear variables.
- The linear variable means that the exponent of all variables must be either zero (constant) or one.
- A simplest solvable linear system has the same numbers of equations and linearly independent variables.
- The variables are denoted by x, y, z or by $x_1, x_2, x_3, x_4, \cdots$ etc.

- Algebraically, a linear system is expressed ~~algebraically~~ as

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\
\vdots \qquad\qquad &= \ \vdots \\
a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}x_n &= b_n\,.
\end{aligned}
$$

- Here a_{ij} are the constant coefficients with $i, j = 1, 2, \cdots, n$. The first subscript runs horizontally as in a `row', and the second subscript runs vertically as in a `column' in a matrix.
- The right–hand side is a constant (variables with zero exponent). If all b_i are zero, it is homogeneous linear system, otherwise ~~nonlinear.~~ non homogeneous
- A linear system is very common in everyday life now-a-days, and generally n is very large: data science, AI applications, weather forecasting, etc.
- The above set of equations can be very nicely expressed in matrix forms or notations:

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- $$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{n\times n \text{ matrix}=A} \times \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{n\times 1 \text{ matrix }=x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{n\times 1 \text{ matrix}=b} \implies \boxed{Ax = b.}$$

- The solution of the linear system is: $\boxed{\boldsymbol{x = A^{-1}\, b}.}$
- The basi~~s~~c properties of the matrix A are the following:
    - A is a square matrix of order $n \times n$.
    - A^T is the transpose of A, hence $(a^T)_{ij} = a_{ji}$.
    - A is symmetric if $A = A^T$.
    - A is non-singular iff $\exists$ a solution $x \in R^n$ for every $b \in R^n$.
    - A is non-singular iff $\det(A) \neq 0$.
    - A is non –singular if and only if there exists a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$.

- The Gaussian elimination method is a technique that transform the matrix A into triangular form and solve $Ax = b$ for x.
- To do so we only use elementary row or column operations.
- The lower and the upper triangular matrices, L and U, are defined as

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}; \quad U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}.$$

- Clearly the determinants of the lower and upper triangular matrices are:

$$\det(L) = \prod_{k=1}^{k=n} l_{kk} = l_{11} l_{22} \cdots l_{nn}; \quad \det(U) = \prod_{k=1}^{k=n} u_{kk} = u_{11} u_{22} \cdots u_{nn}.$$

- If the matrix A is in triangular form, either L or U, then the solution is easy.

- Example: Take $n = 4$ and $A = L$. So the linear equations becomes:

$$\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \Longrightarrow \begin{array}{rcl} l_{11}x_1 & = & b_1 \\ l_{21}x_1 + l_{22}x_2 & = & b_2 \\ l_{31}x_1 + l_{32}x_2 + l_{33}x_3 & = & b_3 \\ l_{41}x_1 + l_{42}x_2 + l_{43}x_3 + l_{44}x_4 & = & b_4 \end{array}$$

- The first equation on the right-side gives: $x_1 = \frac{b_1}{l_{11}}$.
- Putting x_1 into the second equation gives: $x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}$.
- Similarly, the third and fourth equations give:

$$x_3 = \frac{b_3 - l_{31}x_1 - l_{32}x_2}{l_{33}} \quad \text{and} \quad x_4 = \frac{b_4 - l_{41}x_1 - l_{42}x_2 - l_{43}x_3}{l_{44}} \ .$$

- All these solution are well defined because $\det(L) \neq 0$.
- Similarly for any $n \times n$ lower triangular system, $Lx = b$, the solution is:

$$x_j = \frac{b_j - \sum_{k=1}^{j-1} l_{jk} x_k}{l_{jj}}, \qquad j = 1,2,\cdots,n.$$

- This is also known as the forward substitution method.
- Similarly for any $n \times n$ upper triangular system, $Ux = b$, the solution is:

$$x_j = \frac{b_j - \sum_{k=j+1}^{n} u_{jk} x_k}{u_{jj}}, \qquad j = n, n-1, \cdots, 1.$$

- This is also known as the backward substitution method.
- Here we assumed that $\det(U) \neq 0$.

- It is also possible to find out how many operations is needed to complete a calculation to find x_j.
- Recall the following equations from the previous example:

$$\begin{aligned} l_{11}x_1 &= b_1 \\ l_{21}x_1 + l_{22}x_2 &= b_2 \\ l_{31}x_1 + l_{32}x_2 + l_{33}x_3 &= b_3 \\ l_{41}x_1 + l_{42}x_2 + l_{43}x_3 + l_{44}x_4 &= b_4 \end{aligned}$$

- Let's count the number of operation to calculate the x_j's:

i. $x_1 = \frac{b_1}{l_{11}} \Longrightarrow$ 1 division. This for j = 1.

ii. $x_2 = \frac{b_2 - l_{21}x_1}{l_{22}} \Longrightarrow$ 1 division, 1 multiplication and 1 subtraction. This is for j = 2.

iii. $x_3 = \frac{b_3 - l_{31}x_1 - l_{32}x_2}{l_{33}} \Longrightarrow$
1 division, 2 multiplication and 2 subtraction. This is for j = 3.

- Clearly, for x_n, the number of operation needed is

$$1 \text{ division}, (\mathrm{n}-1) \text{ multiplication and } (\mathrm{n}-1) \text{ subtraction.}$$

- Therefore, the total number of operations is the sum of all these:

$$
\begin{aligned}
\#\text{ of Operations} \quad &= \quad \sum_{j=1}^{n} [\,1 + 2\,(j-1)] \\
&= \quad \sum_{j-1}^{n} (2j \;-1) \\
&= \quad 2\sum_{j=1}^{n} j \;- \sum_{j=1}^{n} 1 \\
&= \quad 2 \times \left(\frac{1}{2}\right) n(n+1) - n \\
&= \quad n^2
\end{aligned}
$$

- The number n^2 is called the computational complexity.
- It gives a rough estimate of the computational cost.
- Note that, in reality, time is also needed to write into a memory cell. and read from a memory cell.
- It is also needed to keep in mind that in any triangular matrix, all diagonal elements must be non-zero.
- This ensures that the matrices are non-singular ($\det A \neq 0$).
- Now the important question is: what should we do if the matrix is NOT triangular? Neither upper nor lower?
- In that case, we need to make the matrix ~~diagonal~~ triangular. This process is known as the Gaussian elimination method.
- This is the topic of the second part of the current lecture.