Classical Mechanics and Special Relativity

Course # PHY 204

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Example:

- Consider the function : $f(x) = x^2 - 2xe^{-x} + e^{-2x}$. Find the solution of this function $f(x) = 0$ within 10^{-5} by using
  a. Newton's method
  b. By using Aitken acceleration

- Note:
  a. Within 10^{-5} means that the answer must accurate upto five decimal places because $10^{-5} = 0.00001$.
  b. The question is which one is the answer: the $f(x) = 0$ or the root $x_\star$?
  c. If it is $f(x) = 0$, then we reach the answer when $f(x_k) < 0.00001$ where k is the iteration number, we say that $f(x_k) \approx 0$ within the upper bound of the error.
  d. If it is $x_\star$, then if $|x_\star - x_k| < 0.00001$ for some iteration number, then we say that $|x_\star - x_k| \approx 0$, and x_k is the root within the upper bound of the error.
  e. When $x_\star$ is irrational (van not be solved exactly by analytical method, or $f(x) = 0$ can not be solved exactly, then numerical method is the ONLY way to solve.
  f. This means that we look for $f(x_k) = 0$ in general when $x_\star$ is unknown or irrational.

$f(x) \longrightarrow f'(x) = 2x - 2\bar{x}^{2} - 2x(-\bar{e}^{x}) - 2\bar{e}^{2x}$

$\therefore$ Iteration formula (Newton's Method):

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2e^{-x_k} + 2x_k e^{-x_k} - 2e^{-2x_k}}}$$

k = iteration number

Starting point : $x_0 = 1$ ($k=0$)

k	x_k	$f(x_k)$	Is $f(x_k) < 10^{-5}$
0	1	0.399576	NO
1	0.768941	0.093292	NO
2	0.664590	0.022532	NO
3	0.615033	$f(x_3) = 0.005537$	NO
4	0.590884	0.001372	NO

k	x_k	$f(x_k)$	Is $f(x_k) < 10^{-5}$
5	0.378963	$f(x_5) = 0.000342$	NO
6	0.573041	0.000085	NO
7	0.570089	2×10^{-5}	NO
8	0.568615	0.3×10^{-5}	YES

$|x_7 - x_8| \approx 1.4 \times 10^{-3}$

$\boxed{f(x_8) \approx 0}$ ✓

$x_* = \text{LambertW}(1)$
≈ 0.567143