- In the previous lecture, the general algebraic formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

has been derived. Here k is the iteration number.

- The above formula is valid if and only if $f'(x_k) \neq 0$ for any value of k.
- It was also shown that this method is super ~~b~~rlinear, and hence the iteration will converge to a fixed point if the starting point x_0 is close enough to $x_\star$.

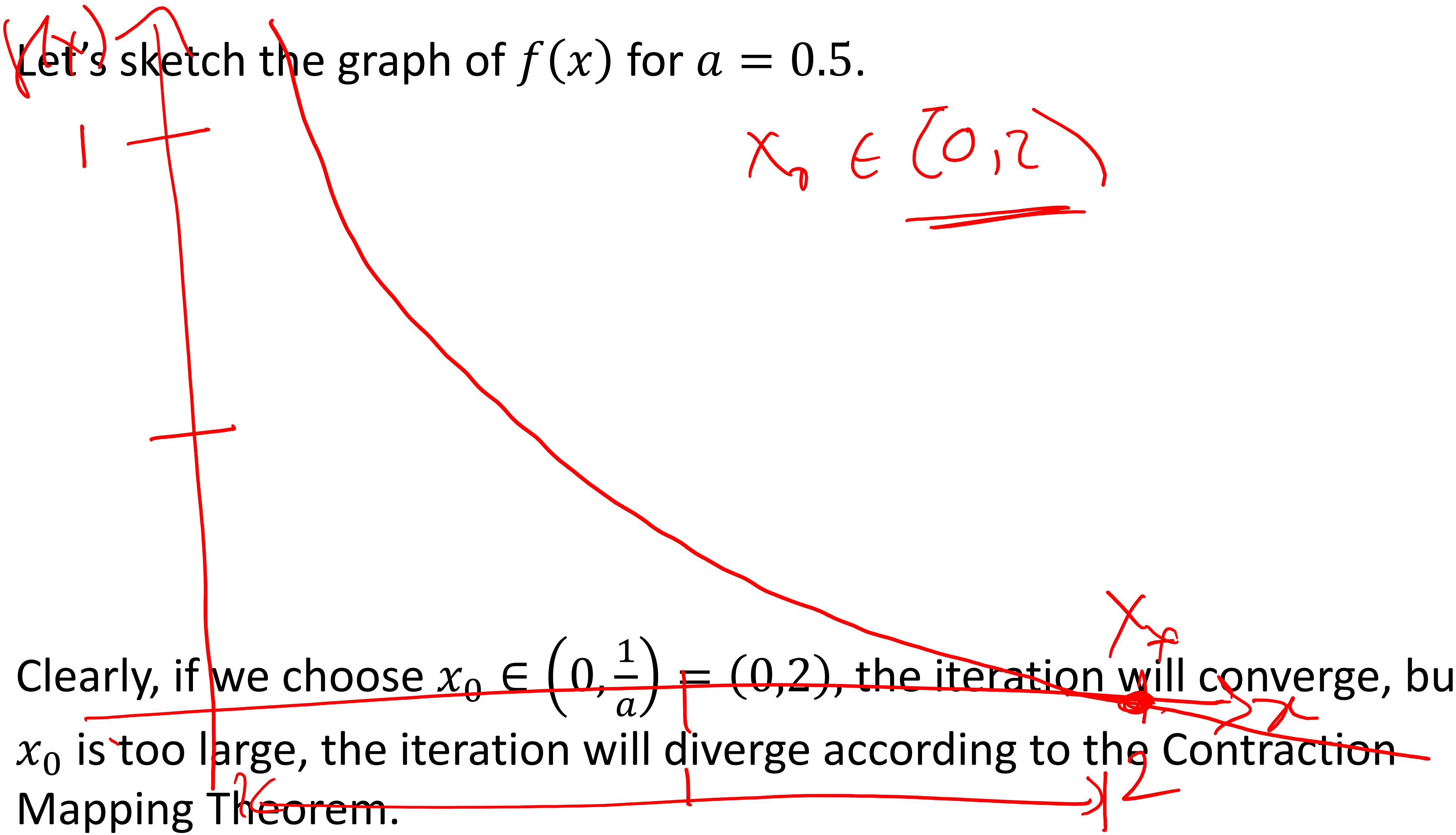
- Example: Let's find the solution of the function,

$$f(x) = \frac{1}{x} - a \quad \text{for} \quad a > 0.$$

- Clearly, the graph of $f(x)$ intersect the x-axis at $x = \frac{1}{a}$. That is, $x_\star = \frac{1}{a}$ is the root of the function $f(x)$.

- Let's sketch the graph of $f(x)$ for $a = 0.5$.

$x_0 \in (0,2)$



- Clearly, if we choose $x_0 \in \left(0, \frac{1}{a}\right) = (0,2)$, the iteration will converge, but if x_0 is too large, the iteration will diverge according to the Contraction Mapping Theorem.

- Applying Newton's method, the iteration formula is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{\frac{1}{x_k} - 0.5}{\frac{d}{dx}\left(\frac{1}{x_k} - 0.5\right)}$$

$$\therefore \boxed{x_{k+1} = 2x_k - 0.5x_k^2}.$$

- Let start the iteration at $x_0 = 1$ (which is between 0 and 2) and show the results in the following table. Note that the last column is added to further clarify the order of convergence.

- Using $x_\star = 2$, we get the following:

x_k	$\lvert 2 - x_k \rvert$	$\dfrac{\lvert 2 - x_k \rvert}{\lvert 2 - x_{k-1} \rvert}$	$\dfrac{\lvert 2 - x_k \rvert}{\lvert 2 - x_{k-1} \rvert^2}$
1.0	1.0	———	0.5
1.5	0.5	0.5	0.5
1.875	0.125	0.25	0.5
1.9921875	0.0078125	0.0625	0.5
1.999969482	$3.0517578 \times 10^{-05}$	0.00390625	0.5
2.0	$4.656612873 \times 10^{-10}$	$1.5258789 \times 10^{-05}$	0.5
2.0	$1.084202172 \times 10^{-19}$	$2.3283964379 \times 10^{-05}$	0.5

- The second column gives the error for each k. The third column is also very familiar. It gives the ratio or rate for each k, which goes to zero very fast as k increases, implying that this is superlinear convergence.

- The fourth column is new. This is related to the order of convergence.
- To understand its significance, we need to go back to the behavior of a differentiable functions, and the convergence rate λ.
- Note that for any function $h(x)$, the first derivative $h'(x)$ is zero means that $\Delta h \to 0$ faster than $\Delta x \to 0$. This is the case for superlinear convergence.
- But for linear convergence $(0 < \lambda < 1)$, both Δg and Δx changes at the same rate asymptotically, and hence their ratio becomes a nonzero constant.
- The question is now: is it somehow possible to get a finite nonzero value, but still less than one for the superlinear case (as in the linear case)?
- The answer is YES!! This is done by defining the `Order of convergence'.

- Recall the definition of convergence rate or ratio~~n~~:

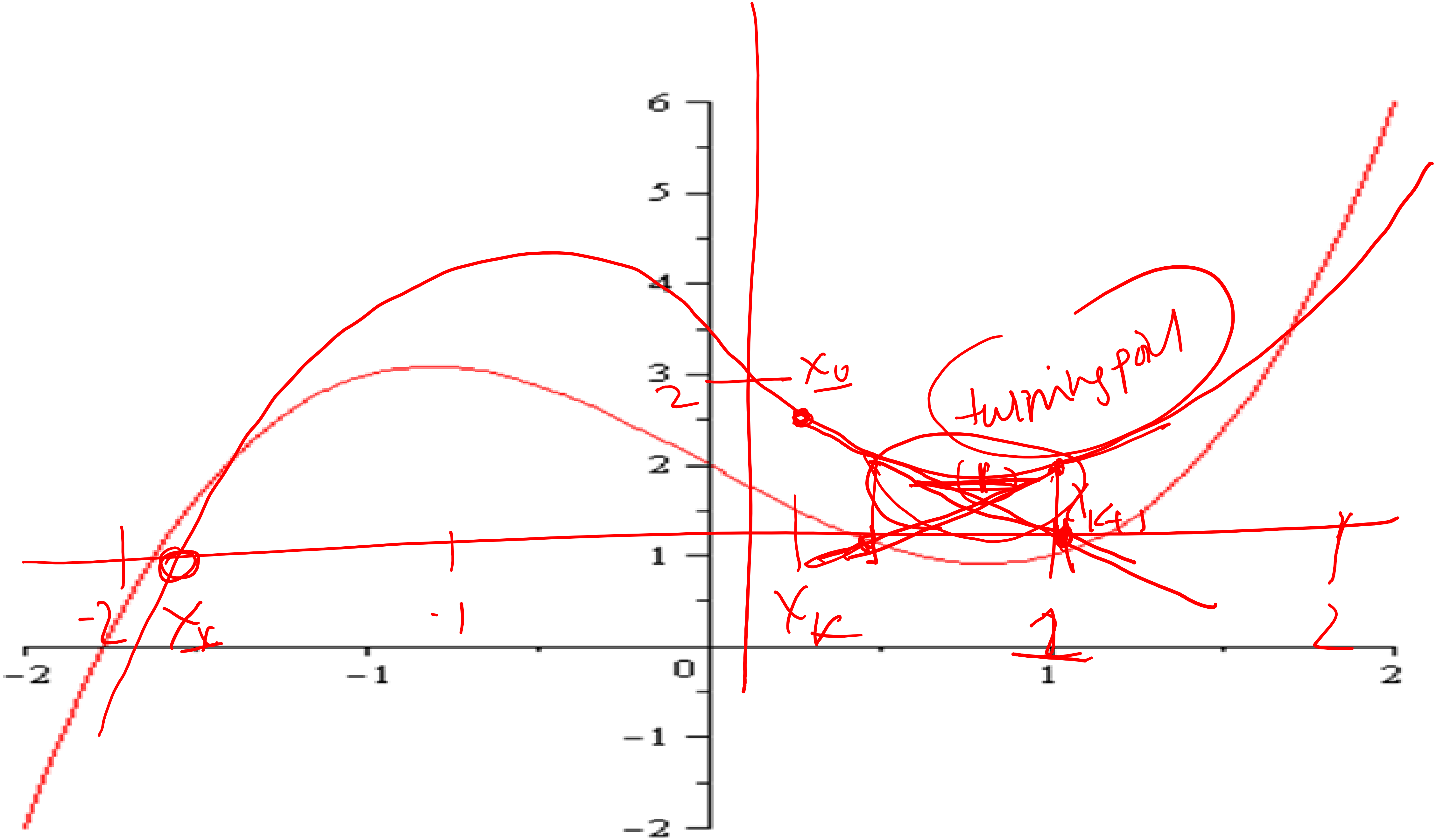
$$\lambda \equiv |g'(x_\star)| = \left| \lim_{x_k \to x_\star} \frac{g(x_k) - g(x_\star)}{x_k - x_\star} \right|.$$

- This is known as the convergence of order one (hence called linear or superlinear convergence depending ion whether λ is a fraction or zero).
- We define the order α of convergence as:

$$\lambda \equiv |g'(x_\star)| = \left| \lim_{x_k \to x_\star} \frac{g(x_k) - g(x_\star)}{(x_k - x_\star)^\alpha} \right| < \infty$$

  for some value of α.
- Clearly for linear convergence, the order of convergence is $\alpha = 1$.
- In the fourth column of the example $\alpha = 2$. Therefore, the convergence is quadratic or second order.
- Note that the order parameter α dos not need to be an integer.

- There is one more fact we need to very careful about.
- Since the Newton's method requires that $f'(x_k) \neq 0$, we need to take extra precaution to choose the initial point x_0. Otherwise we might run into an infinite loop.
- While computing the iterations, $x_0 \to x_1 \to \cdots \to x_k \to x_{k+1} \to \cdots$, we need to make sure that the turning point of the function $f(x)$ does not fall between the successive iterated points, as for example, between x_k and x_{k+1}, for any $k \geq 0$.
- The reason is that if the turning point is between the points x_k and x_{k+1}, then the terms, $f'(x_k)$ and $f'(x_{k+1})$, will have opposite signs.
- As a result, instead of converging to a fixed point or root, the iteration will fluctuate around the turning point indefinitely.
- For an example, consider the function: $f(x) = x^3 - 2x + 2$. The graph is shown in the next slide:



6
5
4
3
2
1
0
−1
−2
−2
−1
1
2
x_0
turning point
x_k
x_{k+1}

- Clearly, $x_0 = 0$ or -0.5 lead to an infinite loop.
- To find the correct value of x_0, we need to find the turning points. These are:

$$f'(x) = \frac{d}{dx}(x^3 - 2x + 2) = 3x^2 - 2 = 0 \implies x = \pm\sqrt{\frac{2}{3}}.$$

- Clearly, $x_0 < -\sqrt{\frac{2}{3}} \approx -0.81649\cdots$, otherwise the iteration will lead to infinite loop.
- As for example, if we choose $x_0 = 0$, the iteration gives, $x_1 = 0 - \frac{f(0)}{f'(0)} = 1$, and then, $x_2 = 1 - \frac{f(1)}{f'(1)} = 0$, leading to the infinite loop. This because the turning point $+0.81649\cdots$ lie between 0 and 1.
- Similarly, $x_0 = -0.5$ also lead to an infinite loop.

- Hence, $x_0 = -1$ would be a correct choice to start the iteration.
- The point $x_0 < -0.816\cdots$ is also more close to the fixed point which is approximately $x_\star \approx -1.76929\cdots$.
- This is also consistent with the statement that if x_0 is close to $x_\star$, then the iteration will converge to the nearest fixed point.
- The above example clearly indicates that the choice of x_0 still requires more attention.
- Not all value of x_0 lead to converging iteration even though it may be relatively close to the fixed point.
- The turning points must not fall in between the iterated points.
- In the next lecture, we will learn how to make the Newton's Method even faster even though it is already superlinear.