Numerical Methods
Course # CSE330
Lecture #: 9.4: QR-Decomposition Application

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In the previous lecture, we learned the QR-decomposition method. This method allows us to do the following:

- For a linear system, $Ax = b$, with A being an $m \times n$ matrix with $m \geq n$, x an unknown $n \times 1$ column vector and b a~~n~~ constant $m \times 1$ vector, we can always rewrite the matrix A as a product of two matrices
$$A = QR$$
known as the QR-decomposition of A.
- The columns of A must be linearly independent.
- Q is an $m \times n$ matrix with its columns being mutually orthonormal, and the R is an $n \times n$ upper triangular matrix with non-zero diagonal elements.
- Now we use the above decomposition to the following normal equations
$$A^T A\, x = A^T b$$
to solve the (over-determined) linear system.
- Putting $A = QR$ in the above normal equation, we find:

$$(QR)^T(QR)x = (QR)^T b$$

$$\implies R^T \underbrace{(Q^T Q)}_{=I} Rx = R^T Q^T b$$

(lecture 9.1)

$$\therefore \boxed{Rx = Q^T b}$$

$(R^{-1}Q^Tb) \rightarrow$ Ans.

- Since R is upper triangular with non-zero diagonal elements, it is invertible, and hence the above equation has unique solutions. The above equation can also be solved by back-substitutions that we learned previously during the lectures of LU-decomposition method.
- Example: Lets solve the same problem that we solve in the 2nd lecture of this module, the least-square straight-line fitting to the data: $f(-3) = f(0) = 0$, $f(6) = 2$.
- Here: $n = 1$ (because straight line), $m = 2$ (three data point), and two unknown variables a_0 and a_1 (because of wo parameters for degree one polynomial).

- The Vandermonde matrix form of the linear system $Ax = b$ becomes:

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} \iff \boxed{Ax = b}$$

$A = 3\times 2$ $\quad x = 2\times 1$ $\quad b = 3\times 1$

u_1

$$\underbrace{\begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{b}$$

Need: $\boxed{A = QR}$

- Here the linearly independent columns of A are:

$$\boxed{u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix}}$$

- We now use Gram-Schmidt process to construct the orthonormal set of vectors.

- First, we identify the first vector:

$$p_1 \equiv u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \boxed{q_1 = \frac{p_1}{|p_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$|p_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

This is the first member of the orthonormal set.

- For the 2nd (and the last as well) member of the orthonormal set, we obtain:

$$\boxed{p_2 = u_2 - \left(u_2^T\, q_1\right) q_1}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \frac{1}{\sqrt{3}}\left[\overbrace{\begin{pmatrix} -3 & 0 & 6 \end{pmatrix}}^{u_2^T} \overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{q_1}\right] \overbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{q_1} = \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

- Therefore, the 2nd member of the orthonormal set is:

$$q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{42}}\begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

$|p_2| = \sqrt{(-4)^2+1^2+(5)^2}$
$= \sqrt{42}$

- Therefore the 3×2 matrix Q with orthonormal columns is

$$Q = \begin{pmatrix} \overbrace{\frac{1}{\sqrt{3}}}^{q_1} & \overbrace{-\frac{4}{\sqrt{42}}}^{q_2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \end{pmatrix} = \{ q_1 \mid q_2 \}$$

$$u_1^T q_1 = (1 \;\; 1 \;\; 1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1+1+1}{\sqrt{3}} = \sqrt{3}$$

- And the 2×2 upper triangular matrix R becomes

$$R = \begin{pmatrix} u_1^T q_1 & u_2^T q_1 \\ 0 & u_2^T q_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{pmatrix}$$

$$u_2^T q_2 = (-3 \;\; 0 \;\; 6) \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix} \frac{1}{\sqrt{42}} = \sqrt{42}.$$

- Using these, the matrix equation, $Rx = Q^T b$ becomes:

$$\underbrace{\begin{pmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}}_{x} = \overbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{4}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \end{pmatrix}}^{Q^T} \overbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}^{b}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \sqrt{3}\, a_0 + \sqrt{3}\, a_1 \\ \sqrt{42}\, a_1 \end{pmatrix}}_{Rx} = \underbrace{\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{10}{\sqrt{42}} \end{pmatrix}}_{Q^Tb}$$

- Therefore, comparing the 2nd row, we obtain: $\boxed{a_1 = \frac{10}{42} = \frac{5}{21}.}$
- Ans the 1st row gives: $\sqrt{3}a_0 + \sqrt{3}a_1 = \frac{2}{\sqrt{3}} \Rightarrow \boxed{a_0 = \frac{1}{\sqrt{3}}\left(\frac{2}{\sqrt{3}} - \sqrt{3} \times \frac{5}{21}\right) = \frac{3}{7}.}$
- This is exactly the same result as we found earlier.

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