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Classical Mechanics and Special Relativity

Course # PHY 204

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Aitken Acceleration

- In this method, we will learn how to accelerate the convergence rate for a linear convergent function.
- That means, this method is for a function for which $\lambda = \text{constant} \neq 0$.
- Note that λ is defined as

$$\lambda = \lim_{k \to \infty} \frac{|x_\star - x_{k+1}|}{|x_\star - x_k|}$$

- For linear convergence, λ is a non-zero constant. So if λ is known, then $x_\star$ can be found for a given k from the following relation

$$x_\star = \frac{x_{k+1} - \lambda x_k}{1 - \lambda} \quad .$$

- But if λ is unknown, then it can be eliminated from the above equation by equating the nearest pairs of iteration points.

- We take three nearest iteration points: x_k, x_{k+1} and x_{k+2}.
- We compute λ using (x_k, x_{k+1}) and (x_{k+1}, x_{k+2}) to eliminate λ:

$$\lambda = \frac{x_\star - x_{k+2}}{x_\star - x_{k+1}} = \frac{x_\star - x_{k+1}}{x_\star - x_k} \quad \text{(since } \lambda \text{ is constant)}$$

- Solving for $x_\star$ yields:

$$(x_\star - x_{k+2})(x_\star - x_k) = (x_\star - x_{k+1})^2$$

$$\Longrightarrow x_\star = \frac{x_{k+2}\, x_k - x_{k+1}^2}{x_{k+2} - 2x_{k+1} + x_k}$$

$$\therefore \quad x_\star = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \; .$$

- Therefore, the iteration formula is:

$$\hat{x}_{k+2} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \ .$$

- Using the forward difference notation, we can also write:

$$\Delta x_k \equiv x_{k+1} - x_k \implies (x_{k+1} - x_k)^2 = (\Delta x_k)^2 \ .$$

$$\begin{aligned} x_{k+2} - 2x_{k+1} + x_k &= (x_{k+2} - x_{k+1}) - (x_{k+1} - x_k) \\ &= \Delta x_{k+1} - \Delta x_k \\ &= \Delta(\Delta x_k) \\ &= \Delta^2 x_k \ . \end{aligned}$$

- Hence, we can also write the iteration formula as:

$$\hat{x}_{k+2} = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k} \ .$$

- Starting from x_0, every two iteration after (that is, every third iteration iteration point has been accelerated.
- Accelerated means that the iterated point become more close to $x_\star$.
- $x_0 \to x_1 \to x_2 \xrightarrow{\text{accelerated}} \hat{x}_2$

$$\Downarrow$$

Same

$$x_2 \to x_3 \to x_4 \xrightarrow{\text{accelerated}} \hat{x}_4$$

$$\Downarrow$$

$$x_4 \to x_5 \to \hat{x}_6$$

- Consider the sequence: $x_2 \to \hat{x}_2 \to x_2$. Here the first x_2 has been accelerated, the value is denoted by $\hat{x}_2$. This new value now has been treated as x_2 and used to compute the 3rd iteration point x_3.
- Now starting from x_2, the third iteration point is x_4, and will be iterated.

- Example: Consider the function, $f(x) = \frac{1}{x} - 0.5$.
- Clearly, $f(2) = 0$. Therefore, the root is $x_\star = 2.0$.
- Let's now construct $g(x)$ to find the fixed point of $g(x)$, such that $f(x) = 0$.
- We consider

$$g(x) = x + \frac{1}{16} f(x) = x + \frac{1}{16}\left(\frac{1}{x} - 0.5\right).$$

- Clearly, $g(2) = 2 \Longrightarrow$ it is a fixed point of $g(x)$.
- We check that:

$$\lambda = g'(x_\star) = \frac{dg}{dx}\bigg|_{x=x_\star} = 1 + \frac{1}{16}\left(-\frac{1}{x^2}\right)\bigg|_{x=2} = 0.984375 < 1.$$

- Since λ is close to one, it is linear convergence, but very slowly converging. And it will take a large number of iteration to converge to the fixed point.

- Let's start the iteration with $x_0 = 1.5$ which is close to $x_\star = 2$.
- We choose to consider upto 7 sig.fig.
- This means that the 8th digit will be rounded off and the error comparing the first 7 digits will be $0.000000\cdots \approx 10^{-6}$.

$$x_0 = 1.5$$

$$\begin{array}{ccccc}
x_1 & = & g(x_0) = 1.510417 & \Longrightarrow & |x_1 - x_\star| = 0.489583 \\
x_2 & = & g(x_1) = 1.520546 & \Longrightarrow & |x_2 - x_\star| = 0.479454 \\
x_3 & = & g(x_2) = 1.530400 & \Longrightarrow & |x_3 - x_\star| = 0.469600 \\
x_4 & = & g(x_3) = 1.539989 & \Longrightarrow & |x_4 - x_\star| = 0.460011 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{818} & = & g(x_{817}) = 1.999999 & \Longrightarrow & |x_{818} - x_\star| = 0.000000
\end{array}$$

- We have to perform 818 iteration to get the error within 10^{-6} or matching upto 7 sig.fig.

- We now apply Aitken acceleration:

$$
\begin{array}{rcll}
 & & x_0 = 1.5 & \\
x_1 & = & g(x_0) = 1.510417 & \Longrightarrow \quad |x_1 - x_\star| = 0.489583 \\
x_2 & = & g(x_1) = 1.520546 & \Longrightarrow \quad |x_2 - x_\star| = 0.479454 \\
\hat{x}_2 & = & x_0 - \dfrac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 1.877604 & \Longrightarrow \quad |\hat{x}_2 - x_\star| = {\color{red}0.122396} \\
x_3 & = & g(\hat{x}_2) = 1.879641 & \Longrightarrow \quad |x_3 - x_\star| = 0.120359 \\
x_4 & = & g(x_3) = 1.881642 & \Longrightarrow \quad |x_4 - x_\star| = 0.118358 \\
\hat{x}_4 & = & \hat{x}_2 - \dfrac{(x_3 - \hat{x}_2)^2}{x_4 - 2x_3 + \hat{x}_2} = 1.992634 & \Longrightarrow \quad |\hat{x}_4 - x_\star| = {\color{red}0.007366} \\
\vdots & \vdots & \vdots & \vdots \qquad\qquad \vdots \\
\hat{x}_8 & = & g(x_7) = 2.000000 & \Longrightarrow \quad |\hat{x}_8 - x_\star| = 0.000000
\end{array}
$$

- ONLY 8 iterations needed to get the same answer within 10^{-6} error bound.

Secant Method:

- Recall the Newton's method of finding the root. The iteration is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \; .$$

- The above requires that at each iteration point, we need to calculate the first derivative and check if it is zero. This is a drawback. Moreover if $f'(x_k)$ is very small, then loss of significance may occur.
- To avoid this drawback, we replace $f'(x_k)$ by an easily computable function g_k which is approximately equal to $f'(x_k)$. This technique is known as the Quasi-Newton Method.
- In particular if we choose g_k to be the backward difference between the nearest iteration points, then it is called the Secant Method.
- There are other Quasi-Newton method, as for example, Steffensen's method, etc. which we will not discuss here.

- The backward difference for the nearest iteration points is

$$g_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

- Therefore the iteration formula for the Secant method become

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

- Clearly, the iteration starts with two initial values: x_0 and x_1. So, the first iterated point is x_2, and the iteration continues.
- Secondly, only one function to be evaluated per iteration (not two functions as in Newton's method).

- Example: $f(x) = \frac{1}{x} - 0.5$. Let's choose: $x_0 = 0.25$ and $x_1 = 0.5$.

k	x_k	$\frac{\lvert x_\star - x_k \rvert}{\lvert x_\star - x_{k-1} \rvert}$
2	0.6875	0.75
3	1.01562	0.75
4	1.3540	0.65625
5	1.68205	0.492188
6	1.8973	0.322998
7	1.98367	0.158976
8	1.99916	0.0513488

- Here the convergence is very fast, and in 12 iteration ϵ_M is achieved.
- Since, the error ratio is decreasing, it is super linear convergence.

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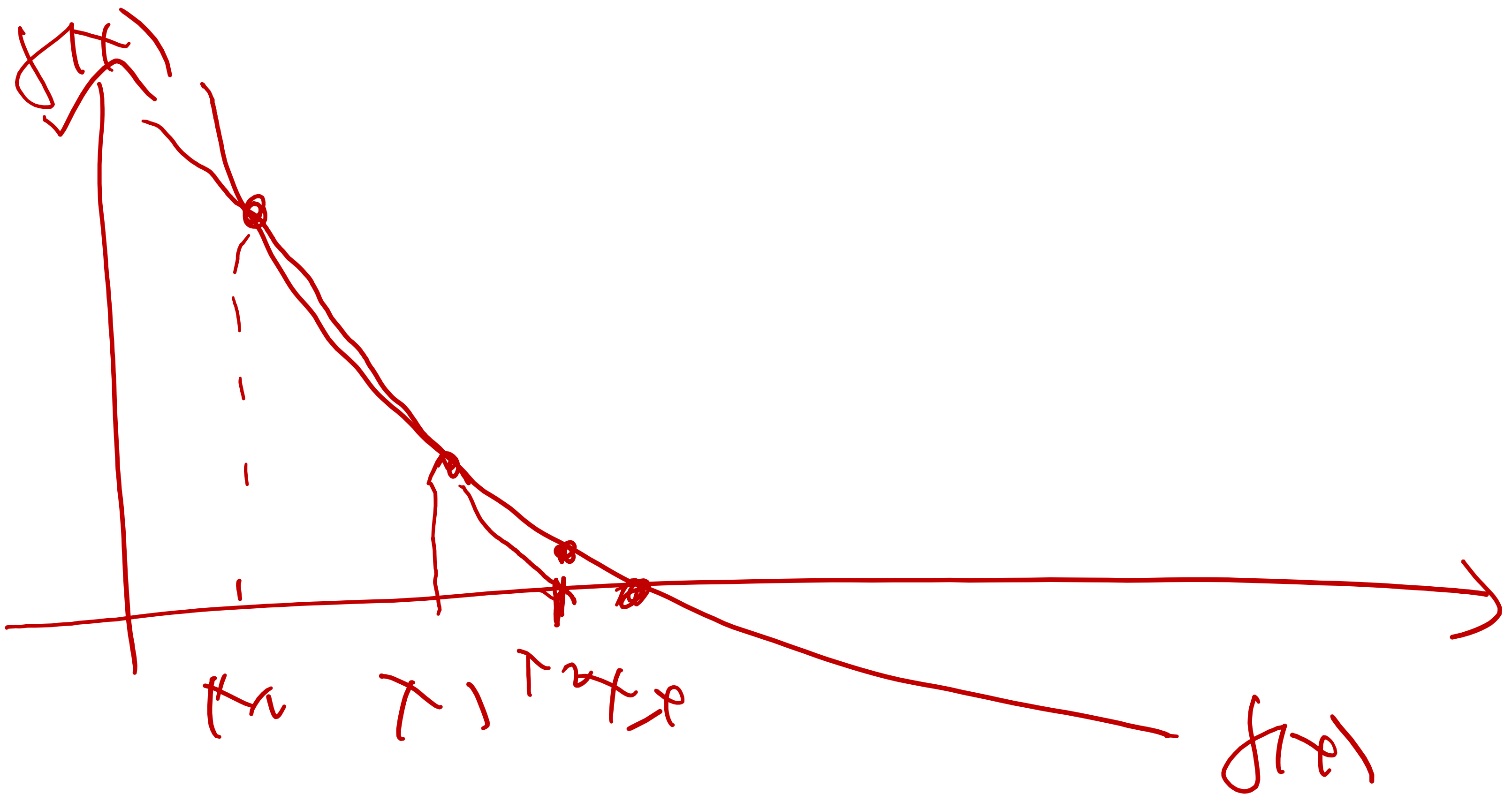
- Graphically, the Secant method looks as in the following:



- Finally, we state a theorem and end this chapter. Theorem 4.6 states that the order of convergence is $\frac{(1+\sqrt{5})}{2} = 1.618\cdots$, if $f'(x_\star) \neq 0$ and for x_0 and x_1 sufficiently close to $x_\star$.