# Nonlinear Equations:

- The function, $f(x)$, is nonlinear. This means that the function is NOT restricted to be degree one Polynomial (which is a straight line). $\Rightarrow f(x) = P_1(x) = a_0 + a_1 x$.
- The function could be a series, polynomial, rational function, etc. There is no restriction on the type of a function.
- The function MUST be continuous on some interval $I = [\,a\,, b\,]$.
- We want to solve a equation: $f(x) = 0$.

1. The solution of $f(x) = 0$ is called the ROOT of the function $f(x)$, and is denoted by $x_\star$. This is also called the `Zero' of the function.
2. This means that $f(x_\star) = 0$. $\Longrightarrow$ $\boxed{\text{Exact Solution.}}$
3. This also means that $(x - x_\star)$ is a factor of the function $f(x)$.
4. Since $f(x_\star) = 0$, by definition, $x_\star$ is also the x-intercept of the function $f(x)$.

- How does the graph of $f(x)$ look like?

There are only two possibilities:

  - Since $f(x_\star) = 0$, the function may cross the $x$-axis at $x = x_\star$. So the function changes sign as it crosses the $x$-axis at $x = x_\star$.
  - The other possibility is that the point $x = x_\star$ is the touching point (This implies that the first derivative changes sign at $x = x_\star$).
  - Here we will assume that the graph of $f(x)$ crosses the horizontal axis only once for $x \in I = [a, b]$.
  - In other words, the factor $(x - x_\star)$ has multiplicity 1 (one).
  - Note that we are looking for approximate solutions: $f(x) \approx 0$. This also implies that: $x - x_\star \approx 0$.

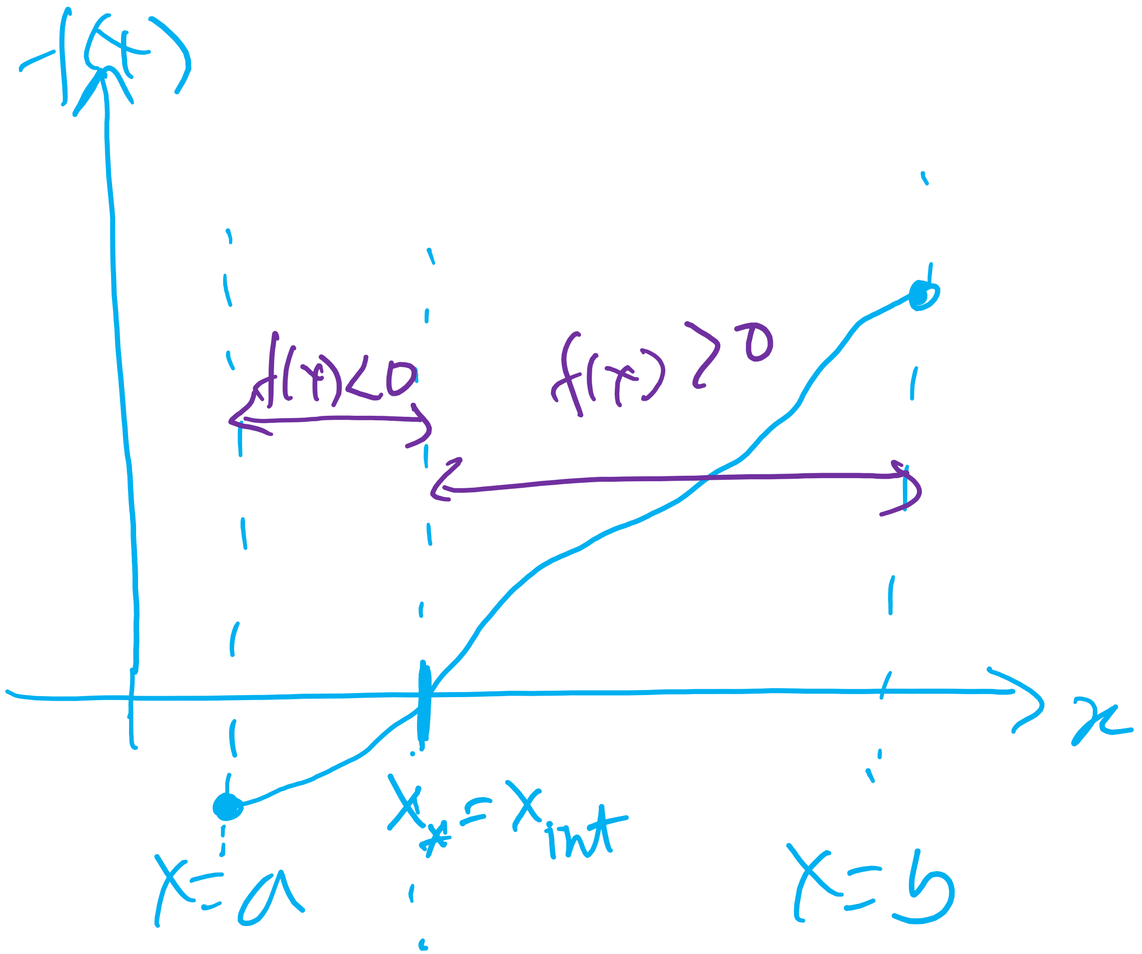
- Since we are considering function that crosses the $x$-axis only once, then the following questions arise:
    1. How to deal with function with higher multiplicities?
    2. How to deal with functions that do not cross the $x$-axis, but touches it?

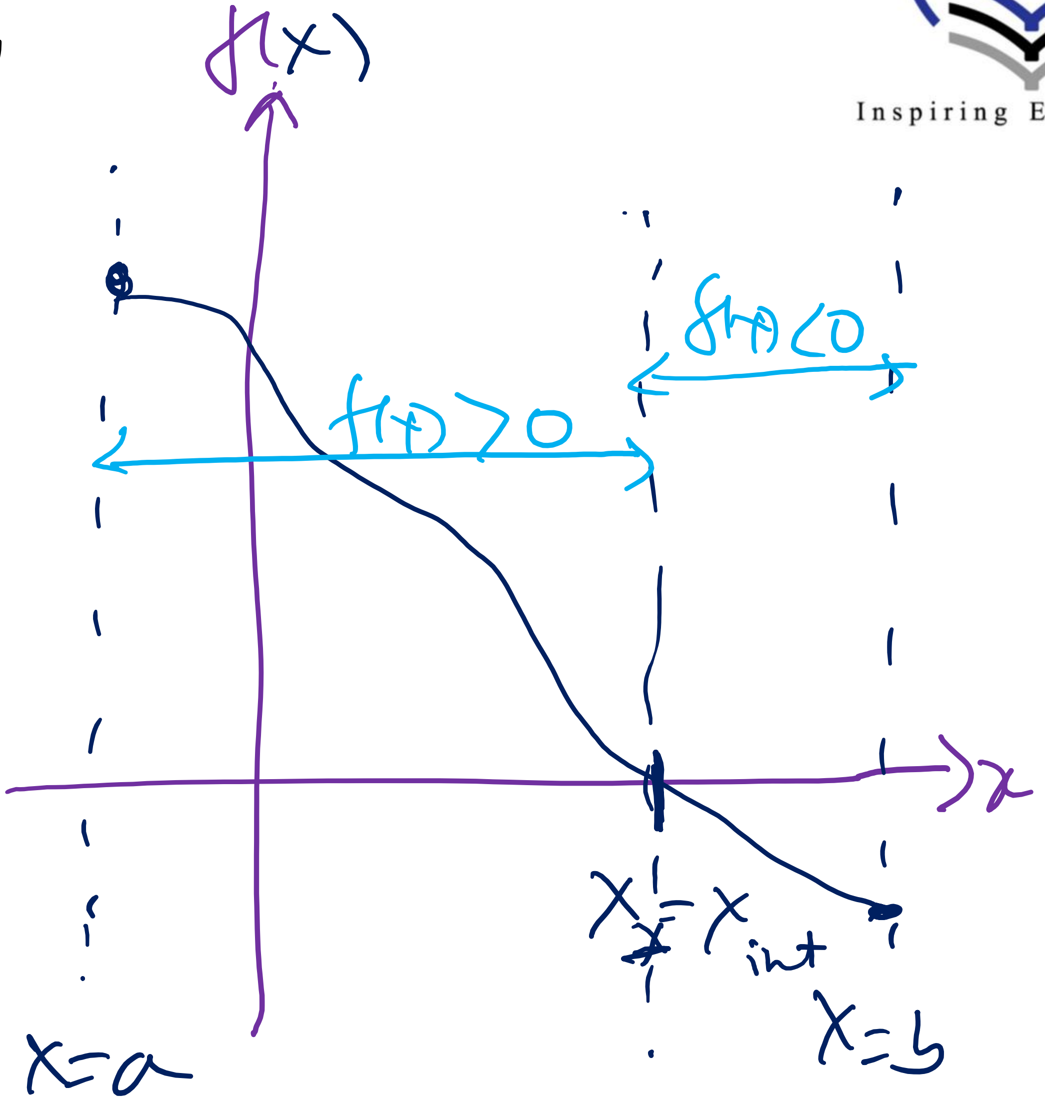
- The answer to the above two questions are very simple:
    1. For the functions with higher multiplicity, we divide the domain of the function by their turning points:

$$[a, b] = [a, c_1] \cup \cdots \cup [c_n, b] \equiv I_1 \cup \cdots \cup I_n,$$

       where $c_1, \cdots, c_n$ are turning points.
    2. For the function with touching points only, we do the same as in the previous case.

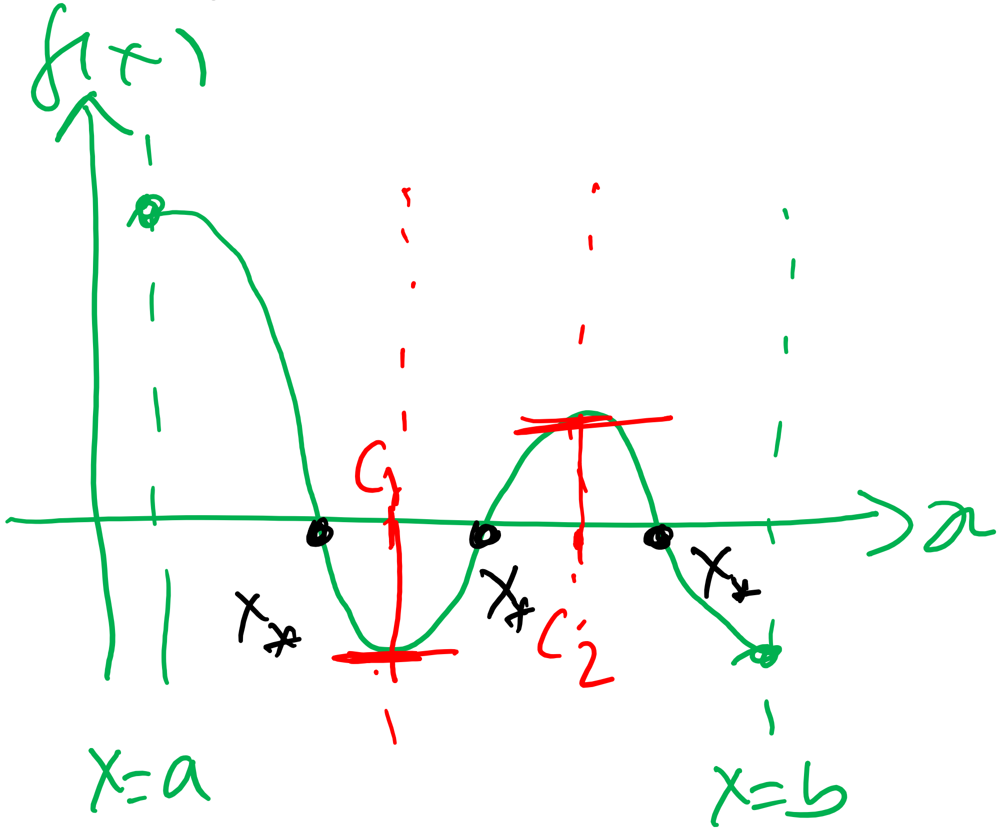
- Graphs of functions with only one $x$-intercept:

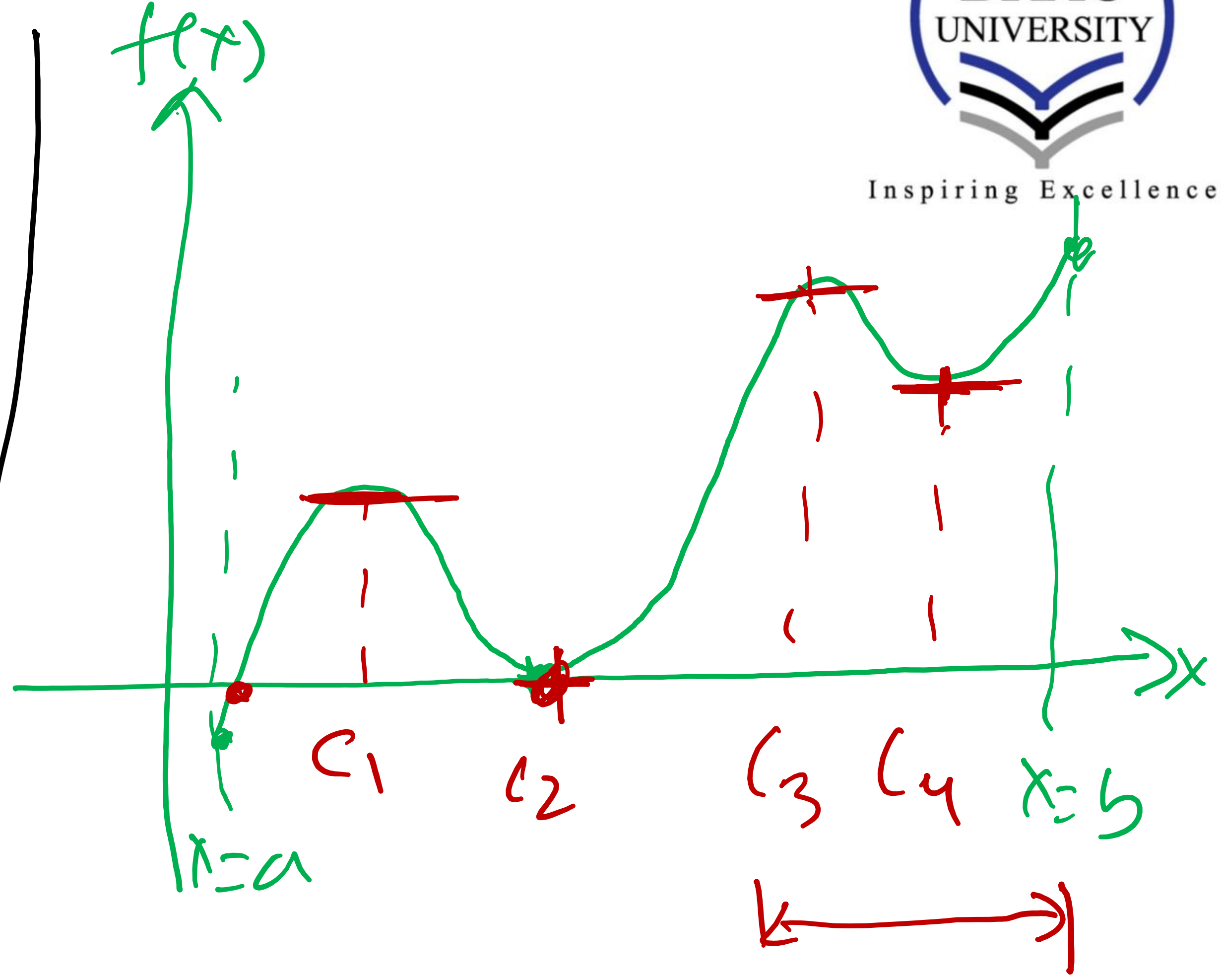


- Graphs of functions with multiple $x$-intercepts:






$$I = I_1 \cup I_2 \cup I_3$$

$$\Rightarrow [a,b] = [a,c_1] \cup [c_1,c_2] \cup [c_2,b]$$

$$[a,b] = [a,c_1] \cup [c_1,c_2] \cup [c_2,c_3]$$