Gaussian Elimination Method

- This method is basically a technique to obtain triangular matrix.
- We want the following:

$$\underbrace{A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{n\times n} \implies U = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a'_{nn} \end{pmatrix}}_{n\times n}$$

- To achieve the above we apply the `row operation' by column wise.
- The 1st row operation means that every element in the first column below a_{11} will be made zero.
- The 2nd row operation means every element in the second column below a_{22} will be made zero, and so on.

- Each row operation involves multiple arithmetic operations.
  1. To make a_{21} element zero, we nee to multiply the first row with a multiplier m_{21} such that $m_{21}a_{11}$ element equals a_{21}.
  2. Then the new first row is subtracted from the second, and the result replaces the entire second row:
$$r_2 \to r_2' = r_2 - m_{21}r_1.$$
  3. Here is how it looks like:
$$m_{21} = \frac{a_{21}}{a_{11}}. \quad \therefore m_{21}r_1 = m_{21}(a_{11}, a_{12}, \cdots, a_{nn})$$
$$= (a_{21}, m_{21}a_{12}, \cdots, m_{21}a_{nn}) \equiv (a_{21}, a_{12}', \cdots, a_{nn}')$$
$$\therefore \quad r_2' = (a_{21}, a_{22}, \cdots, a_{2n}) - (a_{21}, a_{12}', \cdots, a_{nn}') = (0, a_{22}', \cdots, a_{2n}')$$
- To make a_{31} zero, we repeat the above operation between first and third rows. Here the multiplier is m_{31}, and the new row is $\mathrm{r}_3 \to r_3' = r_3 - m_{31}r_1$.
- The above is continued until the a_{n1} in the first column become zero. This completes the first row operation.

- After the first row operation, the matrix looks like the following:

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{n\times n} \rightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \cdots \end{pmatrix}}_{n\times n}$$

$$\downarrow$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \cdots & a'_{nn} \end{pmatrix}}_{n\times n} \leftarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ a_{41} & a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}}_{n\times n}$$

- Below is the order how the matrix looks like after each row operation:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \xrightarrow{1^{\text{st}} \text{ row operation}} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ 0 & a'_{32} & \cdots & a'_{3n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$$\downarrow 2^{\text{nd}} \text{ row operation}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a'_{nn} \end{pmatrix} \xleftarrow{\text{After } (\text{n}-1)^{\text{th}} \text{ row operation}} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & a'_{42} & \cdots & a'_{4n} \\ 0 & 0 & \vdots & \cdots & a'_{nn} \end{pmatrix}$$

- Once the matrix A has been transformed into triangular form (upper or lower), the finding of the solution is straight forward.

- Note that when the row operation is performed, the b-matrix must also be included. That is $b_1 \to b'_{j} = b_1 m_{21}$ etc. In general: $b_j \to b'_j = b_j - m_{pj} b_j, \quad p = j+1, \cdots, n.$

- Let’s summarize formally the discussions above:
  - Define the row multipliers

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}\ ; \qquad i = k+1, k+2, \cdots, n.$$

where the super ~~sub~~script (k) is the k-th row operation and the subscripts ik and kk are the matrix element indices.

  - Use these multipliers to eliminate the elements in entire k-th column below a_{kk} element by performing the following for $i, j = k+1, \cdots, n$:

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik}\, a_{kj}^{(k)}, \qquad b_i^{(k+1)} = b_i^{(k)} - m_{ik}\, b_k^{(k)}.$$

  - After performing all row operations, the final matrix $A^{(n)} = U$ will be an upper diagonal matrix

- Example: A linear system is described by the following:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 0 \\
x_1 - 2x_2 + 2x_3 &= 4 \\
2x_1 + 12x_2 - 2x_3 &= 4
\end{aligned}
$$

- The matrices are:

$$
A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}.
$$

- Sometimes it is helpful to combine these matrices to form an Augmented matrix (matrix b is the fourth column) as:

$$
\begin{pmatrix} 1 & 2 & 1 & \vdots & 0 \\ 1 & -2 & 2 & \vdots & 4 \\ 2 & 12 & -2 & \vdots & 4 \end{pmatrix}
$$

- We perform the row operations on the Augmented matrix. It takes care of both A and b matri~~x~~ces.

- The row multiplier is $m_{21} = \frac{a_{21}}{a_{11}} = 1$. So, the 1st row operation is:

  $r_2 - 1 \times r_1 = (1 \quad -2 \quad 2 \quad 4) - 1 \times (1 \quad 2 \quad 1 \quad 0) = (0 \quad -4 \quad 1 \quad 4)$

  This is the new second row.

- The Augmented matrix now looks like:

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 0 \\ 0 & -4 & 1 & \vdots & 4 \\ 2 & 12 & -2 & \vdots & 4 \end{pmatrix}$$

- The row multiplier is $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$. So the 3rd row will change as

  $r_3 - 2 \times r_1 = (2 \quad 12 \quad -2 \quad 4) - 2 \times (1 \quad 2 \quad 1 \quad 0) = (0 \quad 8 \quad -4 \quad 4)$

  This is the new 3rd row. The Augmented matrix now takes the form

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 0 \\ 0 & -4 & 1 & \vdots & 4 \\ 0 & 8 & -4 & \vdots & 4 \end{pmatrix}$$

- The 1st row operation is complete.

- For the 2nd row operation, the multiplier is

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{8}{-4} = -2.$$

- So, the 3rd row now further changes as

$$r_3 - (-2) \times r_2 = (0 \quad 8 \quad -4 \quad 4) - (-2)(0 \quad -4 \quad 1 \quad 4)$$
$$= (0 \quad 0 \quad -2 \quad 12)$$

- The final form of the Augmented matrix is (the last column is the b-matrix)

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 0 \\ 0 & -4 & 1 & \vdots & 4 \\ 0 & 0 & -2 & \vdots & 12 \end{pmatrix}$$

- Hence the upper triangular form of the linear system is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}.$$

- Comparing both sides, the 3rd row gives:
$$-2\,x_3 = 12 \quad \Longrightarrow \quad x_3 = -6\,.$$
- The 2nd row gives us: $-4x_2 + x_3 = 4 \implies x_2 = \frac{4-x_3}{-4} = \frac{4-(-6)}{-4} = -\frac{5}{2}$.
- Finally the 1st row yields the following: $x_1 + 2x_2 + x_3 = 0 \implies x_1 = -2x_2 - x_3$
$$\therefore \;\; x_1 = \; -2\left(-\frac{5}{2}\right) - (-6) = 11.$$
- We now find the number of operations required:
  - For i-th multiplier, we need $(n-(k+1)+1) = (n-k)$ number of divisions.
  - For the (ij)-th element, each index i has $(n-(k+1)+1) = (n-k)$ number of subtractions and also multiplications. For the index j also need $(n-k)$ number of subtractions and multiplications.
  - So for the (ij)-th element, we need $(n-k)^2$ number of subtractions, and $(n-k)^2$ number of multiplications.

- So, the total number of operations required is:

$$N = \sum_{k=1}^{n-1} [\, 2\,(n-k)^2 + (n-k)]$$

$$= n(2n+1)\sum_{k=1}^{n-1} 1 - (4n+1)\sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1} k^2$$

$$= n(2n+1)(n-1) - (4n+1)\left(\frac{1}{2}\right)(n-1)n$$

$$+2\left(\frac{1}{6}\right)n(n+1(2n+1)$$

$$= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n\,.$$