CSE330

Theory ASSIGNMENT- 1

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Section: 10

Ans. to the ques. No: 1

(a)

Given that,

$\beta = 2, \ m = 4, \ e = [-3, 6]$

For maximum number, $e_{max} = 6$

Now,

Convention: 1

$$F = \pm (0.d_1 d_2 d_3 \cdots d_m)_\beta \, \beta^e$$

where $d_1 = 1$

Here,

Highest possible number: $(0.1111)_2 \times 2^6$

$$= [(1\times 2^{-1}) + (1\times 2^{-2}) + (1\times 2^{-3}) + (1\times 2^{-4})] \times 2^6$$

$$= \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right] \cancel{\times 2^6} \times 2^6$$

$$= 60$$

Convention: 2

$$F = \pm (1.d_1 d_2 d_3 \cdots d_m)_\beta \, \beta^e$$

Highest possible number: $(1.1111)_2 \times 2^6$

$$\cancel{= [1\times 2^0}$$

$$= [(1\times 2^0) + (1\times 2^{-1}) + (1\times 2^{-2}) + (1\times 2^{-3}) + (1\times 2^{-4})] \times 2^6$$

$$= \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right] \times 2^6$$

$$= 124$$

Convention: 3

$$F = \pm (0 \cdot 1 d_1 d_2 d_3 \cdots d_m)_\beta \beta^e$$

Highest possible ~~max~~ number:

$$(0.11111)_2 \times 2^6$$

$$= [(1\times 2^{-1}) + (1\times 2^{-2}) + (1\cdot\times 2^{-3}) + (1\times 2^{-4}) + (1\times 2^{-5})] \times 2^6$$

$$= \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right] \times 2^6$$

$$= 62$$

So,

Maximum numbers that can be stored in the system are as follows;

Lecture note form : 60

Normalized " : 124

Denormalized " : 62

(b)

Non-negative minimum numbers that can be stored in the system by the three forms are as follows:

For minimum number, $e_{min} = -3$

Lecture Note Form: $F = \pm (0.d_1 d_2 d_3 \ldots d_m)_\beta \beta^e$,

where, $d_1 = 1$

Minimum number:

$$(0.1000)_2 \times 2^{-3}$$

$$= (1 \times 2^{-1}) \times 2^{-3}$$

$$= \frac{1}{2} \times \frac{1}{8}$$

$$= \frac{1}{16}$$

Normalized Form: $F = \pm (1.d_1 d_2 d_3 \ldots d_m)_\beta \beta^e$

Minimum number:

$$(1.0000)_2 \times 2^{-3}$$

$$= 1 \times 2^0 \times 2^{-3}$$

$$= 1 \times \frac{1}{8}$$

$$= \frac{1}{8}$$

Denormalized Form: $F = \pm (0.1 d_1 d_2 d_3 \cdots d_m)_\beta \cdot \beta^e$

Minimum number:

$$(0.10000)_2 \times 2^{-3}$$

$$= (1 \times 2^{-1}) \times 2^{-3}$$

$$= \frac{1}{2} \times \frac{1}{2^3}$$

$$= \frac{1}{16}$$

(c)

Equation: 1

$$F = \pm (0.d_1 d_2 d_3 \cdots d_m)_\beta \, \beta^e$$

$$d_1 = 1, \; e = -1$$

Now,

$$(0.1000)_2 \times 2^{-1} = (1 \times 2^{-1}) \times 2^{-1} = \frac{1}{4}$$

$$(0.1001)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-4})] \times 2^{-1} = \frac{9}{32}$$

$$(0.1010)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-3})] \times 2^{-1} = \frac{5}{16}$$

$$(0.1011)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-3}) + (1 \times 2^{-4})] \times 2^{-1} = \frac{11}{32}$$

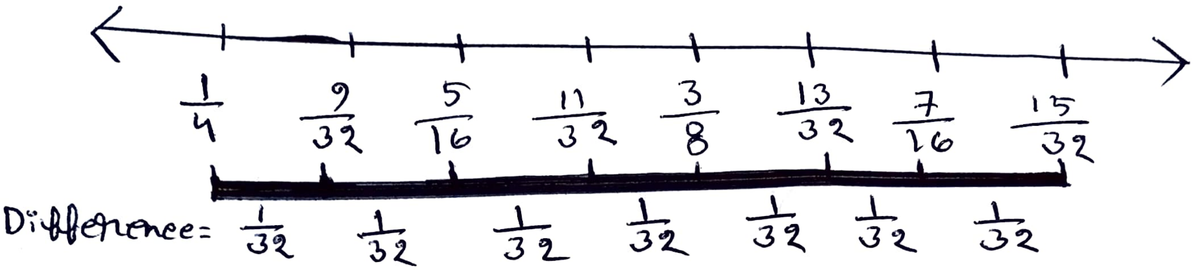
$$(0.1100)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-2})] \times 2^{-1} = \frac{3}{8}$$

$$(0.1101)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-4})] \times 2^{-1} = \frac{13}{32}$$

$$(0.1110)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})] \times 2^{-1} = \frac{7}{16}$$

$$(0.1111)_2 \times 2^{-1} = [(1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})] \times 2^{-1} = \frac{15}{32}$$

Number line:



From the number line we can see that for equation(1), the number line is equally spaced. For one particular exponent there will be equally spaced sets.

Answer to the ques. No: 2

Given that,

$\beta = 2$

$m = 4$

$e = [-1, 2]$

(a)

Normalized Form:

$$F = \pm (1.d_1 d_2 \cdots d_m)_\beta \; \beta^e$$

$$\text{Minimum } |x| = \beta^e$$

$$= 2^{-1} \qquad [e_{min} = -1]$$

$$= \frac{1}{2}$$

(b)

Machine epsilon for normalized form:

$$\epsilon_M = \frac{1}{2} \beta^{-m}$$

$$= \frac{1}{2} \times 2^{-4}$$

$$= \frac{1}{2} \times \frac{1}{2^4}$$

$$= \frac{1}{32}$$

 Date: ..../............/............

(c)

We know, Machine epsilon for,

(1) Convention 1: $\frac{1}{2}\beta^{1-m}$

(2) Normalized form: $\frac{1}{2}\beta^{-m}$

(3) Denormalized form: $\frac{1}{2}\beta^{-m}$

As we can see Machine epsilon does not depend on exponent. There is no relation in between machine epsilon and exponent.

(d)

Denormalized Form:

$$F = \pm (0.1d_1d_2 \cdots d_m)_\beta \beta^e$$

Machine epsilon, ϵ_m for denormalized form:

$$\epsilon_m = \frac{1}{2}\beta^{-m}$$

$$= \frac{1}{2} 2^{-4}$$

$$= \frac{1}{32}$$

(e)

Equation (1)

$$F = \pm (0 \cdot d_1 d_2 d_3 \ldots d_m)_\beta \times \beta^e$$

Here, $d_1 = 1$

$$\text{Delta, } \delta = \frac{|fl(x) - x|}{|x|}$$

We will get maximum delta value when,

$|fl(x) - x|$ is maximum

and $|x|$ is minimum

So, minimum $|x| = \beta^{-1} \beta^e$

Maximum $|fl(x) - x| = \frac{1}{2} \beta^{-m} \beta^e$

At the middle point of two value, the $|fl(x) - x|$ gets maximum

So, $$\delta_{max} = \frac{|fl(x) - x|}{|x|}$$

$$= \frac{\frac{1}{2} \beta^{-m} \beta^e}{\beta^{-1} \beta^e}$$

$$= \frac{1}{2} \beta^{1-m}$$

$$= \frac{1}{2} 2^{1-4} = \frac{1}{16}$$

Ans: $\frac{1}{16}$

Date: ..../........../..........

Ans. to the ques. No:-3

(a)

Given that,

$f(x) = xe^{x}$

$x_0 = 0$

$f(x) = xe^{x}$

$f'(x) = e^{x} + xe^{x}$

$f''(x) = e^{x} + e^{x} + xe^{x} = 2e^{x} + xe^{x}$

$f'''(x) = 2e^{x} + e^{x} + xe^{x} = 3e^{x} + xe^{x}$

$f^{iv}(x) = 4e^{x} + xe^{x}$, $f^{v}(x) = 5e^{x} + xe^{x}$

Now,

Taylor series,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{n}(x_0)}{n!}(x-x_0)^n$$

Now,

$f(x_0) = 0$

$f'(x_0) = 1 + 0 = 1$

$f''(x_0) = 2 + 0 = 2$

$f'''(x_0) = 3 + 0 = 3$

$f^{iv}(x_0) = 4 + 0 = 4$

$f^{v}(x_0) = 5 + 0 = 5$

So, Taylor expansion of $f(x) = e^x x$ is:

$$f(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \cdots$$

(b)

From 3(a) we got,

$$\phi(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots$$

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

So,

we get,

$a_0 = 0$

$a_1 = 1$

$a_2 = 1$

$a_3 = \frac{1}{2}$

(c)

Now,

$$f(x) = xe^{x}$$

$$\Rightarrow f(0.1) = (0.1) \times e^{0.1}$$

$$= 0.1105171$$ [upto seven significant digit]

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= x + x^2 + \frac{1}{2} x^3 \quad [a_0 = 0,\ a_1 = a_2 = 1,\ a_3 = \tfrac{1}{2}]$$

$$P_3(0.1) = 0.1 + (0.1)^2 + \frac{(0.1)^3}{2}$$

$$= 0.1105000$$ [upto seven significant digit]

(Ans)

(d)

Percent error:

$$\frac{|f(0.1) - P_3(0.1)|}{|f(0.1)|} \times 100$$

$$= \frac{|0.1105171 - 0.1105000|}{|0.1105171|} \times 100$$

$$= 0.0155\%$$

(Ans)

Ans. to the ques. No:- 4

(a)

Given,

$f(x) = xe^x$

Nodes Given : $-1, 0, 1$

Total nodes $= 3$

So, Total coefficients $= 3$

And, Degree of polynomial $=$ nodes $- 1$

Text

$= 3 - 1$

$= 2$

$P_2(x) = a_0 + a_1 x + a_2 x^2$

Coefficients: a_0, a_1, a_2

Now,

$$P_2(-1) = a_0 - a_1 + a_2 = f(-1)$$

$$P_2(0) = a_0 = f(0)$$

$$P_2(1) = a_0 + a_1 + a_2 = f(1)$$

Vandarmonde matrix:

$$V = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(b)

$$\det(V) = \det\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= [1 \times 0] - [(-1) \times 1] + [1 \times 1]$$

$$= 1 + 1$$

$$= 2$$

c

$$V = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{\det(V)} \operatorname{adj.}(V)$$

Now,

$$V^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\operatorname{adj.}(V) = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

So, $V^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

(Ans)

(d)

We know,

$$A = V^{-1} F$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix}$$ [from b(a)]

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} -e^{-1} \\ 0 \\ e^{1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \times \begin{bmatrix} -0.36788 \\ 0.00000 \\ 2.718282 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.54308 \\ 1.175201 \end{bmatrix}$$

So, The values of A, [upto 5 decimal places]

$a_0 = 0$

$a_1 = 1.54308$

$a_2 = 1.17520$

(Ans)

(e)

Expression for $P_2(x) = a_0 + a_1 x + a_2 x^2$

$P_2(0.25) = 0 + (1.54308 \times 0.25) + [1.17520 \times (0.25)^2]$

$= 0.38577 + 0.07345$

$= 0.45922$

$f(x) = x e^x$

$f(0.25) = 0.25 \times e^{0.25}$

$= 0.32101$

(Ans)

(f)

Percent error of interpolating

$$\frac{|f(0.25) - P_2(0.25)|}{|f(0.25)|} \times 100$$

$$\Rightarrow \frac{|0.32101 - 0.45922|}{|0.32101|} \times 100$$

$$= 43.05\%$$

(Ans)