Orders of Convergence

- From the previous lecture, we learnt that not all forms of $g(x)$ are convergent.
- By using the `Contraction Mapping Theorem', we can find/choose the convergent function.
- This theorem tells us that if $x_0 \in L$, then the mapping $g(x)$ starting with $x = x_0$ will converge to a point after enough iterations.
- If x_0 is closer to the fixed point, then iteration number is also the smaller.
- The next question is: how fast the iteration is?
- The condition, $\lambda < 1$, just tells that there is convergence, but it does not tell how fast the convergence is.

- In this lecture, we will classify the types of convergences.
- In the previous lecture, we already learned that for convergence, we must have the following conditions satisfied:

$$\lambda = \left|\frac{dg}{dx}\right|_{x=x_\star} < 1.$$

- Let us now explore a bit more to find out the rate of convergence.
- Note from previous lecture that for a pair of points $(x, y) \in L$, we have,

$$\lambda = \frac{|g(y) - g(x)|}{|y - x|} = 1$$

if and only if both x and y are fixed of g.

- If $x = x_\star$ and $y = x_k$, then as $k \to \infty$, we wrote:

$$\lambda = \left| \lim_{k \to \infty} \frac{g(x_\star) - g(x_k)}{x_\star \; - x_k} \right| = \left| \lim_{x_k \to x_\star} \frac{g(x_\star) - g(x_k)}{x_\star \; - x_k} \right| = \text{Constant} < 1.$$

- Note that this is constant only in the limiting sense. But for any finite value of k, λ must be less than 1, but may not constant.
- Before, exploring when and how λ might constant or not, let's recall facts or notions from the computations:

$|x_\star \; - x_k| =$ Error after k-th iteration = Distance or Length between these two points

$|g(x_\star) - g(x_k)| \equiv |x_\star - x_{k+1}|$= Error after $(k+1)$-th iteration

$\left|\frac{g(x_\star) - g(x_k)}{x_\star - x_k}\right| < 1$ for each k, and decreases as k increases.

x_k	$\lvert 3 - x_k \rvert$	$\dfrac{\lvert 3 - x_k \rvert}{\lvert 3 - x_{k-1} \rvert}$
0.000000000	3.0000000000	$---$
1.7320508076	1.2679491924	0.4226497308
2.5424597568	0.4575402432	0.3608506129
2.8433992885	0.1566007115	0.3422665304
2.9473375404	0.0526624596	0.3362849319
2.9823941860	0.0176058140	0.3343143126
2.9941256440	0.0058743560	0.3336600063
$\downarrow$	$\downarrow$	$\downarrow$
3.0000000000	0.0000000000	$\dfrac{1}{3}$

- Note that in the last row above, we only considered 11 sig. fig (or error bound of $1.0000000000 \times 10^{-11}$. Clearly, the last column the ratio approached λ= 1/3 only in limiting sense only.

x_k	$\lvert(-1) - x_k\rvert$	$\dfrac{\lvert(-1) - x_k\rvert}{\lvert(-1) - x_{k-1}\rvert}$
0.0000000000	1.0000000000	$---$
-1.5000000000	0.5000000000	0.5000000000
-1.0500000000	0.0500000000	0.1000000000
-1.0006297561	0.0006097561	0.0121951220
-1.0000000929	0.0000000929	0.0001523926
$\downarrow$	$\downarrow$	$\downarrow$
$\downarrow$	$\downarrow$	$\downarrow$
-1.0000000000	0.00000000000	0.00000000000

- Clearly, the first column approaches the fixed point, the second column approached the zero error and the last column approaches λ= 0 within the same error bound as in the previous slide.

- From the above analysis, we have three possible scenario for the rate or ratio, λ :
  1. If $\lambda = 0$, the convergence is very fast, and this is called the Superlinear Convergence. This is the fastest convergence.
  2. If $0 < \lambda < 1$, the convergence is certain, but not as fast as in previous case. This is known as Linear Convergence.
  3. If $\lambda = 1$, the points are the fixed points. For this case, the numerical approximation method is redundant, and hence is completely ignored here.
  4. Finally, if $\lambda > 1$, the mapping $g(x)$ is diverging, and hence no fixed point is can be obtained or fixed point does not exist.
- The first two are nicely summarized by the Theorem 4.5.

- Theorem 4.5: Let g' be continuous in the neighborhood of a fixed point $x_\star = g(x_\star)$, and suppose that $x_{k+1} = g(x_k)$ converges to $x_\star$ as $k \to \infty$. Then, the following must hold:
  1. If $|g'(x_\star)| \neq 0$ then the convergence will be linear with rate $\lambda = |g'(x_\star)|$.
  2. If $|g'(x_\star) = 0$ then the convergence will be superlinear.
- Examples: We recall the $g(x)$ that we used before:
  1. For $g(x) = \sqrt{2x+3}$, we find:
  $$g'(3) = \frac{d}{dx}\left(\sqrt{2x+3}\right)\Big|_{x=3} = \frac{1}{\sqrt{2x+3}}\Big|_{x=3} = \frac{1}{3} < 1 \implies \text{Linear}$$
  2. For $g(x) = \frac{x^2+3}{2x-2}$, we similarly find:
  $$g'(-1) = \frac{d}{dx}\left(\frac{x^2+3}{2x-2}\right)\Big|_{x=-1} = \frac{x^2-2x-3}{2(x-1)^2}\Big|_{x=-1} = 0 \implies \text{Superlinear}$$
  3. Clearly, $\lambda = \frac{1}{2}$, for the Interval Bisection Method.