

Numerical Methods Course # PHY 203

Chapter #: 6: Least-Square Approximation

Lecture # 9.1: Orthonormality

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- In this lecture, we will discuss about the orthogonality of the vectors.
- The main idea in this chapter is the following: We know a well-defined linear system
 has equal number of linear variables and numbers of conditions/equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}b_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

In matrix form , the above equation can be expressed as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

• Here the $n \times 1$ column matrix x^T is the vector \vec{x} which is unknown, and the right –hand side is also a constant vector \vec{b} which in $n \times 1$ column matrix form.



• In other words, the transformation matrix, A, is a square matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

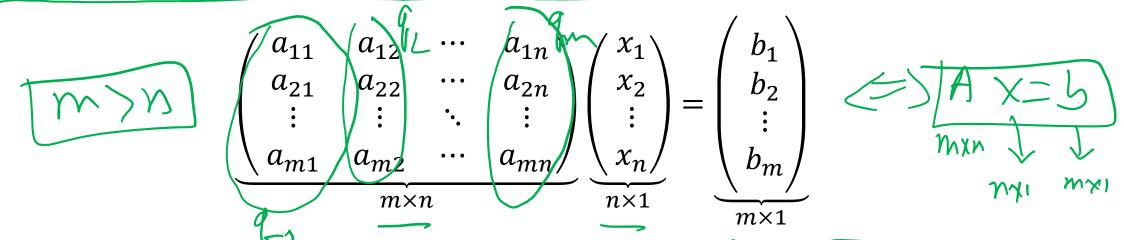
• If $\det A \neq 0$, then there exits only one unique solution:

$$x = A^{-1}b.$$

- However, if we have a linear system where there are more equations to be satisfied than the number of variables, then how do we solve them. We call such a system an over-determined system.
- In such an over-determined system $x = A^{-1}b$ can not give any solution because the matrix A is not invertible (because it is no longer a square matrix).
- For an overdetermined system, the matrix A is now a $m \times n$ matrix. $A = m \times n$
- Here m is the number of rows which is also same as the number of equations to be satisfied.
- And n is the number of columns, which is also the number of unknown variables in the system.



- Least-square approximation method is a way to find an approximate solution of an over-determined system where the transformation matrix A is no longer invertible.
- In an overdetermined system, we have the following matrix equations:



- Let's denote the $m \times n$ matrix by Q instead of A.
- The first column of the matrix Q is q_1 and it is given by

$$q_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$
, which is a $m \times 1$ column matrix.







$$q_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, q_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

• So, we can express the matrix Q as

$$Q = (q_1 \quad q_2 \quad \cdots \quad q_n)$$

- Now we have a very important Theorem:

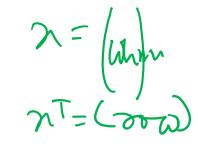
Orthogonality: Let \vec{x} and \vec{y} are two vectors in n-dimensional Euclidean vector space \mathbb{R}^n . That is: $\vec{x}, \vec{y} \in \mathbb{R}^n$.



- In matrix form, we identify an n-dimensional vector an $n \times 1$ column matrix.
- So, the inner product between any two vectors is expresses as

$$(|y|)(|x|) - (|x|) = \vec{x} \cdot \vec{y} \equiv x^T y = \sum_{i=1}^{n} x_i y_i = \text{pumber}$$

$$\chi^T = (|y|)$$



which is known as the scalar or dot product in three dimensions.

The inner product with itself is called the l_2 -norm, or simply norm of the vector is given by:

$$|\vec{x}| = \sqrt{x^T x} = \sqrt{\hat{x}^T \hat{x}}$$

which is known as the absolute value or magnitude in three dimensions.

If θ is the angle between any two vectors, then we write:

$$x^{T}y = |\vec{x}| |\vec{y}| \cos \theta \equiv xy \cos \theta$$

$$(|y|) (|y|)$$



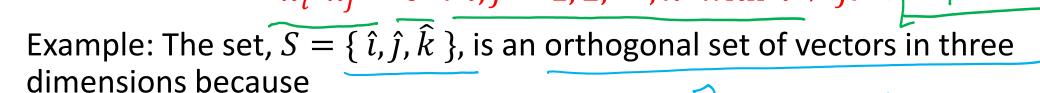




$$x^T y = 0 \Longrightarrow \theta = 90^0$$
.

- That is, they lie at right angles in \mathbb{R}^n . In three dimensions, we say that these two vector are perpendicular to each other.
- Let consider a set of n vectors in \mathbb{R}^n : $S = \{x_1, x_2, \dots, x_n\}$.

The set
$$S$$
 is called the orthogonal set if $x_i^T x_j = 0 \ \forall \ i, j = 1, 2, \dots, n \ \text{with} \ i \neq j.$



$$\hat{\imath} \cdot \hat{\jmath} = 0. = 5. \hat{k} = 0.$$

In matrix form, we can write the following:

$$\hat{i} = (1 \ 0 \ 0)^T$$
, and $\hat{j} = (0 \ 1 \ 0)^T$.

$$2 \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \top$$





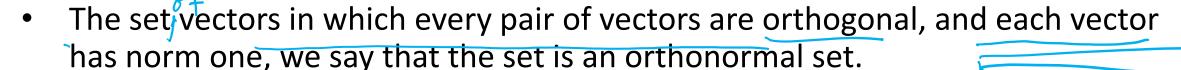
On the other hand, in addition to orthogonality, we also have

$$x_i^T x_i = 1,$$

$$x_i^T x_i = 1$$
, for each $i = 1, 2, \dots, n$

We say that the vectors x_i has norm unity.

In other words, these are all unit vectors.



Mathematically, we write the orthonormal set as

$$S = \{ x_i \mid x_i \in \mathbb{R}^n, x_i^T x_j = \delta_{ij}, i, j = 1, 2, \dots, n \}.$$

Here the Kronecker delta δ_{ij} is defined as

$$\frac{i}{s} \text{ is defined as} \\
\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \text{ and } very \text{ useful when in the next lecture}$$

The above concept will be necessary and very useful when in the next lecture, we discuss the QR-decomposition of an $m \times n$ matrix to solve a linear system.







$$S = \left\{ \frac{1}{\sqrt{5}} (2,1)^T, \frac{1}{\sqrt{5}} (1,-2)^T \right\} \implies \text{in order www.}$$

- Let $u = \frac{1}{\sqrt{5}} (2, 1)^T$ and $v = \frac{1}{\sqrt{5}} (1, -2)^T$.
- We easily see that:

$$u = \frac{1}{\sqrt{5}} (2,1)^{T} \text{ and } v = \frac{1}{\sqrt{5}} (1,-2)^{T}.$$
easily see that:
$$u^{T}u = \left(\frac{1}{5}\right) (2^{2} + 1^{2}) = 1.$$

$$v^{T}v = \left(\frac{1}{5}\right) (1^{2} + (-2)^{2}) = 1.$$

$$v^{T}v = \frac{1}{5} (2 \times 1 - 1 \times 2) = 0.$$

Therefore, S is an orthonormal set in \mathbb{R}^2 , and hence form a basis in twodimensional vector space.

