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K3 SURFACES FROM A DERIVED CATEGORICAL VIEWPOINT

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K3 PLOSKVE Z VIDIKA IZPELJANIH KATEGORIJ

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Introduction

In algebraic geometry we study spaces like schemes and varieties in terms of different sheaves on them. Of special importance are the quasi-coherent and coherent sheaves, which bring us closer to algebra. In this sense we find important geometric data encoded in cohomology of various sheaves. For comparing or manipulating sheaves we often employ different functors, which are unfortunately not always perfectly in tune with cohomology. By this we mean that they are not necessarily exact as one already sees in the case of push-forward and pull-back functors, which are only left and right exact, respectively. For instance let us take a look at the push forward functor along a morphism of schemes $f\colon X \to Y$. It is well-know that this functor is left exact, but in general not always right exact. In more concrete terms this means that after applying the functor f_* to a short exact sequence of shaves $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$ on X we obtain on Y only an exact sequence of the form

$$0 \to f_*\mathcal{F}_0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2.$$

One is then led to extend this sequence by additional terms to the right, which measure the extent to which f_* fails to be exact, to obtain a long exact sequence

$$\begin{aligned} 0 \to f_*\mathcal{F}_0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to \mathbf{R}^1 f_*\mathcal{F}_0 \to \mathbf{R}^1 f_*\mathcal{F}_1 \to \cdots& \\ \cdots \to \mathbf{R}^i f_*\mathcal{F}_0 \to \mathbf{R}^i f_*\mathcal{F}_1 \to \mathbf{R}^i f_*\mathcal{F}_2 \to \mathbf{R}^{i+1} f_*\mathcal{F}_0 \to \cdots& \end{aligned}$$

These additional terms show up as the images of the *higher derived functors* of f_*. It turns out that the collection of all such higher derived functors actually emerges from a single derived functor whose natural domain is the *derived category of coherent sheaves* $\mathsf{D}^b(X)$. The derived category then itself becomes an interesting object of study in part because of its complexity and also since it proved itself to be an important invariant of the space. In particular one is interested in the question of how much data is lost in the process of passing to the derived level? To this end we introduce the term of smooth projective varieties X and Y being *derived equivalent*, if they share equivalent bounded derives categories of coherent sheaves i.e.

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

In certain cases like that of curves the derived category turns out to capture essentially all the information about the curve. In that case if a curve C is derived equivalent to a smooth projective variety X, then necessarily X must also be a curve and is isomorphic to C [Huy06, §5]. A similar phenomenon has been proven to occur by Bondal and Orlov in [BO01], where they showed that a pair of derived equivalent smooth projective varieties X and Y with X admitting either an ample canonical or ample anti-canonical bundle implies that X and Y must necessarily be isomorphic. However contrary to the above there do exist pairs of derived equivalent varieties, which are not isomorphic. A first example of this happening was documented by Mukai in [Muk81], where he showed that an abelian variety and its dual are always derived equivalent however not necessarily isomorphic. This kind of behaviour is observed also for K3 surfaces which are of special importance to us. These are smooth projective surfaces X with a trivial canonical bundles and satisfying $H^1(X, \mathcal{O}_X) = 0$. In this case the Derived Torelli theorem, whose proof following [Orl03] is the principal goal of this thesis, will in particular provide a practical criterion for determining when two K3 surfaces are derived equivalent.

K3 surfaces also show themselves to be intriguing from the point of view of *Homological mirror symmetry*. Since they are exhibited as a special instance of Calabi–Yau manifolds in dimension two, right after elliptic curves in dimension one, they turn out to be the next testing ground in which to tackle different conjectures. In this light the Derived Torelli becomes particularly prominent, because on the algebro-geometric side of homological mirror symmetry, where derived categories make an appearance, it allows one to replace them with a simpler and more convenient Mukai lattice.

The following outline and structure of the thesis tries to highlight some of the most notable themes of each chapter.

The first chapter is devoted to establishing the necessary prerequisites to understand the definition of a derived category. We start with exploring the basics of abelian categories and touch upon exact, left exact and right exact functors between them. Next we introduce *triangulated categories*, which in certain sense formalize the notion of short exact sequences from abelian categories. The chapter also contains some technical results, which we will apply in Chapter 6. Building upon additive and abelian categories we introduce the categories of chain complexes and define the homotopy category of complexes. We will equip the latter with a triangulated structure, exhibiting the first example of a triangulated category we encounter in this thesis. We used [KS06] as the main literature for abelian categories and categories of complexes and [Huy06] for triangulated categories.

The entirety of the second chapter is dedicated to fist constructing the *derived category of an abelian category* and second, to the construction of *derived functors*. What is to be presented is a special case of a process called localization of (triangulated) categories. This method is quite similar to localization of rings and modules, with the central idea being the inversion of certain elements. In our case of derived categories we are led to invert a class of morphisms, which are called *quasi-isomorphisms*, resulting in a category which is well-behaved with respect to cohomology. In a way generalizing the idea of computing cohomology objects of a complex we introduce derived functors. Especially important will be the case of left (resp. right) exact functors between abelian categories, for which we will provide two methods of deriving them. The primary sources in writing this chapter were [GM02; Mil].

In the third chapter we turn our attention to more concrete applications and study in detail the *bounded derived category* $\mathsf{D}^b(X)$ *of coherent sheaves* of a variety X, which is most often assumed to also be smooth and projective. Most notably we show that a connected variety has an indecomposable bounded derived category and that the derived category comes equipped with a spanning class. Both of these results will show up to be useful later on in Chapter 6. In the second part we deal with functors of geometric origin like the global sections functor, the pull-back and push-forward functors along a morphism and others and we will derive them following the methods outlined in Chapter 2. We follow mainly [Huy06, §3] for this part.

Chapter 4 introduces *Fourier–Mukai transforms*. For smooth projective varieties X and Y these functors map between their bounded derived categories – are functors of the form $\mathsf{D}^b(X) \to \mathsf{D}^b(Y)$. They from an important class of examples of triangulated functors between bounded derived categories and are essentially determined by a choice of a bounded complex of sheaves belonging to $\mathsf{D}^b(X \times Y)$ called the *kernel* of a Fourier–Mukai transform. We will see that many well-known triangulated functors between derived categories are actually of Fourier–Mukai type, most surprisingly all fully faithful triangulated functors admitting adjoints are of this sort. This is due to an important theorem of Orlov [Orl03], which shed a new light on how one handles equivalences of derived categories, since it enables one to obtain a more practical description of these equivalences by utilizing their kernels. We will exploit precisely this feature in the proof of the derived Torelli theorem in Chapter 6. With the goal of eventually relating bounded derived categories of K3 surfaces with their cohomology, we will explain the process of first passing Fourier–Mukai transforms to the level of K-theory and afterwards to rational cohomology. This passage relies on the famous Grothendieck–Riemann–Roch theorem and leads to the inception of Mukai vectors. We also loosely skim over the topic of Hodge decompositions of complex projective manifolds and close with examining a nice interaction between Hodge decompositions and Fourier–Mukai transforms on cohomology. All the results on Fourier–Mukai transforms are sourced from [Huy06, §5], which are built on the pioneering work of Mukai [Muk81] and later Orlov [Orl03].

In chapter 5 we meet *K3 surfaces*. Their peculiar name was coined by André Weil in honor of Kodaira, Kummer and Kähler and the K2 mountain in the Karakoram mountain range. We present two points of approach to K3 surfaces, first from the side of algebraic geometry and second through complex geometry and topology. Both viewpoints are fruitful to consider as their interplay lets us calculate the main invariants of these surfaces. We start by adapting Serre duality and computing the Hodge diamond, followed by integral cohomology and the intersection form, which is known to be ubiquitous to the study of 4-manifolds. Lastly, we present the classical Torelli theorem, which underscores how Hodge structures in combination with intersection forms completely determine the geometry of K3 surfaces. The theorem is attributed to the work of Pjateckiĭ–Šapiro and Šafarevič [PŠŠ71]. All of this and so much more on K3 surfaces is packaged concisely in [Huy16].

Through the course of the last chapter, we will prove the derived Torelli theorem, which is due to many authors, most significantly Orlov [Orl03] and Mukai [Muk87]. It states the following.

Derived Torelli theorem. *Let X and Y be K3 surfaces over the field of complex numbers $\mathbf{C}$. Then the following statements are equivalent.*

(i) X and Y share equivalent bounded derived categories of coherent sheaves,

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

(ii) There exists a Hodge isometry $f\colon T_X \to T_Y$ between the transcendental lattices of X and Y.

(iii) Mukai lattices $\tilde{H}(X, \mathbf{Z})$ and $\tilde{H}(Y, \mathbf{Z})$ of X and Y, respectively, are Hodge isometric.

(iv) There exists an isotropic Mukai vector $v \in \tilde{H}(X, \mathbf{Z})$ of divisibility 1, such that Y is isomorphic to a moduli space M_v of stable sheaves on X with Mukai vector v, representing some *moduli functor $\mathcal{M}_v$. Moreover M_v is up to isomorphism the unique K3 surface, for which there exists a Hodge isometry*

$$H^2(M_v, \mathbf{Z}) \simeq v^{\perp}/\mathbf{Z}v.$$

The first highlight of this theorem is the lattice of transcendental cycles. It is exhibited as a Hodge sublattice of $H^2(X, \mathbf{Z})$ and contrasting it, the *Mukai lattice* is displayed as an extension of the ordinary Hodge lattice $H^2(X, \mathbf{Z})$, which encompasses the full integral cohomology of X. Interestingly these two Hodge lattices capture the same data of a complex K3 surface X as is witnessed by the equivalence of points (ii) and (iii). In proving the implication (iii) $\Longrightarrow$ (i) we will invoke Orlov's theorem and follow the approach described in [Orl03]. The other pinnacle of this proof is the *moduli space* of stable sheaves on X. We will bypass its construction, which comes as a result of many people, most notably [GH96; OGr97; Huy06], but will explain precisely how one can use such a moduli space together with a so-called *universal bundle* to construct a triangulated equivalence between the bounded derived categories $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Y)$. The other ties between the equivalent statements will be proven along the way.

In Appendix A we briefly present the concept of a (cohomological) spectral sequence, which we will be using sprinkled in throughout the thesis. We mention the Grothendieck spectral sequence associated to a pair of composable left exact functors between abelian categories from which a multitude of other useful spectral sequences are acquired. Using this machinery we also prove some practical results, which are applied when deriving functors in Chapter 3.

Appendix B on lattice theory is there as support to first establish terminology and second to cover an important result of Nikulin [Nik80] regarding extensions of isometries of certain embedded lattices. We apply the latter in the proof of the derived Torelli theorem in Chapter 6.

Technicalities. A *variety over a field* k is a separated integral scheme of finite type over k. We say that a variety X over k is *smooth*, if the stalks $\mathcal{O}_{X,x}$ are regular[1] local rings for every $x \in X$. A variety X over k is *projective*, if the structure morphism $X \to \operatorname{Spec} k$ is projective, meaning there exists a closed immersion $X \hookrightarrow \mathbb{P}^n_k$ for some $n \in \mathbf{Z}_{\geq 0}$. A variety of dimension two will be called a *surface*. For a scheme X a $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if it is locally isomorphic to a cokernel of a morphism of free $\mathcal{O}_X$-modules in other words meaning that any point $x \in X$ has an open neighbourhood U such that

$$\mathcal{O}_{X|U}^{\oplus I} \to \mathcal{O}_{X|U}^{\oplus J} \to \mathcal{F}|_U \to 0$$

is exact for some index sets I and J. The $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent,* if it is locally isomorphic to a quotient of free *finite rank* $\mathcal{O}_X$-modules, corresponding to the index sets I and J above being finite.

[1] A noetherian local ring $(A, \mathfrak{m})$ is *regular*, if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$, where $\dim A$ denotes the *Krull dimension of* A and k denotes the residue field $A/\mathfrak{m}$.

1 Categorical prerequisites

The present chapter serves as a brief account of the majority of the relevant prerequisites needed to discuss derived categories. In the first section we introduce additive, k-linear and abelian categories together with relevant functors preserving (certain parts of) their structure, following mainly [KS06, §8]. Next we build the framework for triangulated categories of which derived categories will be a special kind and also state and prove some specialized results on triangulated categories, which will be applied later on in chapter 6. Lastly, we lay out the foundations on which derived categories will be constructed – the category of complexes and its quotient homotopy category of complexes. The main pieces of literature for this part were [KS06] and [Huy06, §1].

1.1 Additive, k-linear and abelian categories

With k we denote either a field or the ring of integers $\mathbf{Z}$. A categorical *biproduct* of objects X and Y in a category $\mathcal{C}$ is an object $X \oplus Y$ together with morphisms

$$\begin{array}{ll} p_X \colon X \oplus Y \to X & p_Y \colon X \oplus Y \to Y \\ i_X \colon X \to X \oplus Y & i_Y \colon Y \to X \oplus Y \end{array}$$

for which the pair p_X, p_Y is the categorical product of X and Y and the pair i_X, i_Y is the categorical coproduct. For a pair of morphisms $f_X \colon Z \to X$ and $f_Y \colon Z \to Y$ the unique induced morphism into the product $Z \to X \oplus Y$ is denoted by (f_X, f_Y) and for a pair of morphisms $g_X \colon X \to Z$ and $g_Y \colon Y \to Z$ the unique induced morphism from the coproduct $X \oplus Y \to Z$ is denoted by $\langle g_X, g_Y \rangle$. For morphisms

$$\begin{array}{ll} f_{00} \colon X_0 \to Y_0 & f_{01} \colon X_0 \to Y_1 \\ f_{10} \colon X_1 \to Y_1 & f_{11} \colon X_1 \to Y_1 \end{array}$$

of $\mathcal{C}$, there are two equal[2] morphisms $X_0 \oplus X_1 \to Y_0 \oplus Y_1$, namely

$$(\langle f_{00}, f_{01} \rangle, \langle f_{10}, f_{11} \rangle) \quad \text{and} \quad \langle (f_{00}, f_{10}), (f_{01}, f_{11}) \rangle,$$

which we will be denoting with the matrix

$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} : X_0 \oplus X_1 \to Y_0 \oplus Y_1. \tag{1.1}$$

Definition 1.1. A category $\mathcal{A}$ is additive (resp. k-linear) if all the Hom-sets carry the structure of abelian groups (resp. k-modules) and the following axioms are satisfied

Decide on which terminology to use (everything is covered by k-linear...)

A1 For all all objects X, Y and Z of $\mathcal{A}$ the composition

$$\circ \colon \mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear.

A2 There exists a *zero object* 0, for which $\mathrm{Hom}_{\mathcal{A}}(0, 0) = 0$.

A3 For any two objects X and Y there exists a categorical biproduct of X and Y.

Remark 1.2. (i) The zero object 0 of a k-linear category $\mathcal{A}$ is both the initial and terminal object of $\mathcal{A}$.

[2]This stems from the fact that both morphisms fit into a diagram

(ii) For morphisms $f_0 \colon X_0 \to Y_0$, $f_1 \colon X_1 \to Y_1$ we introduce notation

$$f_0 \oplus f_1 \colon X_0 \oplus X_1 \to Y_0 \oplus Y_1$$

to mean either of the two (equal) morphisms $(\langle f_0, 0\rangle, \langle 0, f_1\rangle)$ or $\langle (f_0, 0), (0, f_1)\rangle$.

Definition 1.3. A functor $F \colon \mathcal{A} \to \mathcal{A}'$ between two additive (resp. k-linear) categories $\mathcal{A}$ and $\mathcal{A}'$ is *additive* (resp. *k-linear*), if its action on morphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}'}(F(X), F(Y))$$

is a group homomorphism (resp. k-linear map).

Traditionally the term *additive* is reserved for $\mathbf{Z}$-linear categories and $\mathbf{Z}$-linear functors between such categories.

For a morphism $f \colon X \to Y$ in an additive category $\mathcal{A}$ recall, that the *kernel* of f is the equalizer of f and 0 in $\mathcal{A}$, if it exists, and, dually, the *cokernel* of f is the coequalizer of f and 0, if it exists. It is well-known and easy to verify, that the structure maps $\ker f \hookrightarrow X$ and $Y \twoheadrightarrow \operatorname{coker} f$ are a monomorphism and an epimorphism respectively. We also define the *image* and the *coimage* of f to be

$$(\operatorname{im} f \to Y) := \ker(Y \to \operatorname{coker} f)$$
$$(X \to \operatorname{coim} f) := \operatorname{coker}(\ker f \to X).$$

Notice that the image and the coimage, just like the kernel and the cokernel, are defined to be morphisms, not only objects. Sometimes these are called their *structure morphisms.*

For a monomorphism $Y \hookrightarrow X$ we will sometimes by abuse of terminology call the cokernel $\operatorname{coker}(Y \to X)$ a *quotient* and denote it by X/Y.

Definition 1.4. A k-linear category $\mathcal{A}$ is *abelian*, if it is closed under kernels and cokernels and satisfies axiom A4.

A4 For any morphism $f \colon X \to Y$ in $\mathcal{A}$ the canonical morphism $\operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f$ is an isomorphism.

$$\begin{tikzcd}
\ker f \arrow[r, hook] & X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[r, two heads] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[u, hook]
\end{tikzcd}$$

Axiom A4 essentially states that abelian categories are those additive categories possessing all kernels and cokernels in which the first isomorphism theorem holds.

Remark 1.5. We obtain the morphism mentioned in axiom A4 in the following way. Due to $(\operatorname{im} f \to Y) = \ker(Y \to \operatorname{coker} f)$ and the composition $X \to Y \to \operatorname{coker} f$ being 0, there is a unique morphism $X \to \operatorname{im} f$ by the universal property of kernels. The composition $\ker f \to X \to \operatorname{im} f$ then equals 0, by $\operatorname{im} f \hookrightarrow Y$ being a monomorphism and $\ker f \to X \to Y$ being equal to 0. From the universal property of cokernels we then obtain a unique morphism $\operatorname{coim} f \to \operatorname{im} f$, because $(X \to \operatorname{coim} f) = \operatorname{coker}(\ker f \to X)$.

Example 1.6. The default examples of abelian categories are the category of abelian groups Ab or more generally the category of A-modules Mod_A for a commutative ring A. The categories of coherent and quasi-coherent sheaves, $\mathsf{coh}(X)$ and $\mathsf{qcoh}(X)$, on a scheme X are also abelian ([Sp25, Tag 01BY] and [Sp25, Tag 077P]).

On the other hand the full subcategory of projective $\mathbf{Z}$-modules is additive, but not abelian, for its lack of cokernels. For example $\mathbf{Z}/n$, for $n > 1$, is not projective even though it is realized as a cokernel of a map between two projectives. Another, more geometric example also showcases the same behaviour. The category of (real or complex) vector bundles for example over the real line $\mathbf{R}$ is additive, but not abelian. The last claim is due to the category of vector bundles over $\mathbf{R}$ not being closed under kernels and cokernels. Through the lens of the Serre–Swan theorem, we observe that the above examples are at least in spirit quite closely related. The theorem states that over a connected compact Hausdorff base space X the categories of real (resp. complex) vector bundles over X and the category of projective finitely generated $\mathcal{C}(X, \mathbf{R})$-modules (resp. $\mathcal{C}(X, \mathbf{C})$-modules) are equivalent as additive categories.

Definition 1.7. (i) Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a sequence of composable morphisms in an abelian category $\mathcal{A}$. We say the sequence is *exact*, if $g \circ f = 0$ and the induced morphism $\operatorname{im} f \to \ker g$ is an isomorphism.

(ii) Extending (i), a sequence $\cdots \to X^0 \to X^1 \to X^2 \to \cdots$ is *exact*, if any subsequence $X^{i-1} \to X^i \to X^{i+1}$ for $i \in \mathbf{Z}$ is exact.

(iii) Exact sequences of the form $0 \to X \to Y \to Z \to 0$ are called *short exact sequences.*

To relate the definition of exactness with more primitive objects of an abelian category, namely kernels and cokernels, it is not difficult to show the following two dual statements.

$$\begin{aligned} 0 \to X \to Y \to Z \quad &\text{is exact, if and only if} \quad (X \to Y) = \ker(Y \to Z). \\ X \to Y \to Z \to 0 \quad &\text{is exact, if and only if} \quad (Y \to Z) = \operatorname{coker}(X \to Y). \end{aligned}$$

In the context of abelian categories, functors, which preserve a bit more than just the k-linear structure, are of interest. This brings us to the notion of exactness of additive functors.

Definition 1.8. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(i) F is said to be *left exact*, if $0 \to FX \to FY \to FZ$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

(ii) F is said to be *right exact*, if $FX \to FY \to FZ \to 0$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$..

(iii) F is said to be *exact*, if it is both left and right exact.

Remark 1.9. Equivalently, one can also define left exact functors to be exactly those additive functors, which commute with kernels and dually define right exact functors to be additive functors commuting with cokernels.

Example 1.10. Let $\mathcal{A}$ be an abelian category and $\mathcal{A}^{\mathrm{op}}$ its opposite category, which is readily seen to be an abelian category as well. Then the Hom-functors

$$\operatorname{Hom}_{\mathcal{A}}(W, -)\colon \mathcal{A} \to \mathsf{Mod}_k \quad \text{and} \quad \operatorname{Hom}_{\mathcal{A}}(-, W)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$$

are both *left* exact for any object W of $\mathcal{A}$. Note that a sequence $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}^{\mathrm{op}}$, if $0 \to Z \to Y \to X \to 0$ is exact in $\mathcal{A}$.

1.2 Triangulated categories

In order to formulate the definition of a triangulated category more concisely, we introduce some preliminary notions. A *category with translation* is a pair $(\mathcal{D}, T)$, where $\mathcal{D}$ is a category and T is an auto-equivalence $T\colon \mathcal{D} \to \mathcal{D}$ called the *translation functor*. If $\mathcal{D}$ is additive or k-linear, T is moreover assumed to be additive or k-linear. We usually denote its action on objects X with $T(X) = X[1]$ and likewise its action on morphisms f with $T(f) = f[1]$. A *triangle* in a category with translation $(\mathcal{D}, T)$ is a triplet of composable morphisms (f, g, h) of category $\mathcal{D}$ taking the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]. \tag{1.2}$$

A *morphism* of triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ is given by a triple of morphisms (u, v, w), for which the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

One can compose morphisms of triangles in the obvious way and the notion of an isomorphism (of triangles) is defined as usual. The following definition as stated is originally due to Verdier, who first introduced it in his thesis [Ver96].

Definition 1.11. A *triangulated category* (*over* k) is a k-linear category with translation $(\mathcal{D}, T)$ equipped with a class of *distinguished triangles*, which is subject to the following four axioms.

TR1 (i) Any triangle isomorphic to a distinguished triangle is also itself distinguished.

(ii) For any X the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished.

(iii) For any morphism $f\colon X \to Y$ there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

The object Z is sometimes called the *cone of* f.

TR2 The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

TR3 Given two distinguished triangles and morphisms $u\colon X \to X'$ and $v\colon Y \to Y'$, depicted in the solid diagram below

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, dashed, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

satisfying $v \circ f = f' \circ u$, there exists a (in general non-unique) morphism $w \colon Z \to Z'$, for which (u, v, w) is a morphism of triangles i.e. the diagram above commutes.

TR4 Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow X[1],$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow Y[1],$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{\ell} Y' \longrightarrow X[1],$$

there exists a distinguished triangle $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$ for which the diagram below is commutative.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, dashed, "u"] & X[1] \arrow[d, equal] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \arrow[r, "\ell"] \arrow[d, equal] & Y' \arrow[r] \arrow[d, dashed, "v"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d, "h"'] & Z \arrow[r, "k"] \arrow[d, "\ell"] & X' \arrow[r] \arrow[d, equal] & Y[1] \arrow[d, "h[1]"] \\
Z' \arrow[r, dashed, "u"] & Y' \arrow[r, dashed, "v"] & X' \arrow[r, dashed, "w"] & Z'[1]
\end{tikzcd}$$

Omitting the so-called *octahedral* axiom TR4 we arrive at the definition of a *pre-triangulated category*. These are essentially the categories we will be working with, since we will never use nor verify the axiom TR4. We will nevertheless use the terminology *"triangulated category"* in part to remain consistent with the existent literature and more importantly because our categories will in fact be honest triangulated categories.

Remark 1.12. The object Z, called the *cone of* f, in axiom TR1(iii) is unique up to isomorphism. This is seen through the use of axiom TR3 in combination with example 1.22 and the five lemma [KS06, Lemma 8.3.13] from homological algebra in Mod_k.

Proposition 1.13. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *be a distinguished triangle in a triangulated category* $\mathcal{D}$. *Then* $g \circ f = 0$.

Proof. First, by axiom TR2, triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished. By axiom TR1 (ii), triangle $Z \xrightarrow{\mathrm{id}_Z} Z \to 0 \to Z[1]$ is distinguished and, by axiom TR3, there exits a morphism from the first triangle to the second one depicted in the diagram below.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[r, "h"] \arrow[d, "\mathrm{id}_Z"] & X[1] \arrow[r, "-f[1]"] \arrow[d, dashed] & Y[1] \arrow[d, "g[1]"'] \\
Z \arrow[r, "\mathrm{id}_Z"] & Z \arrow[r] & 0 \arrow[r] & Z[1]
\end{tikzcd}$$

The right most square says $g[1] \circ (-f[1]) = 0$, from which we conclude that $g \circ f = 0$, after applying a quasi-inverse of the translation functor. □

Definition 1.14. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with translation functors T and T' respectively. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is defined to be *triangulated*, if the following two conditions are satisfied.

(i) There exists a natural isomorphism of functors

$$\eta\colon F \circ T \simeq T' \circ F.$$

(ii) For every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}$, the triangle

$$F(X) \xrightarrow{Ff} F(Y) \xrightarrow{Fg} F(Z) \to F(X)[1]$$

is distinguished in $\mathcal{D}'$, where the last morphism is obtained as the composition $F(Z) \xrightarrow{Fh} F(X[1]) \xrightarrow{\eta_X} F(X)[1]$.

Remark 1.15. The condition on F being an *additive* functor in the above definition is actually unnecessary and follows from conditions (i) and (ii) [Sp25, Tag 05QY].

Example 1.16. The entirety of Chapter 2 and first two sections of Chapter 3 will be devoted to constructing a substantial amount of triangulated categories and triangulated functors. The first triangulated category we will encounter in practice is $\mathsf{K}(\mathcal{A})$, the homotopy category of complexes of an additive category $\mathcal{A}$. More on this can be found already in Section 1.3 of this chapter.

Definition 1.17. Let $\mathcal{D}_0$ and $\mathcal{D}$ be triangulated categories such that $\mathcal{D}_0$ is a subcategory of $\mathcal{D}$. Then $\mathcal{D}_0$ is a *triangulated subcategory* of $\mathcal{D}$, if the inclusion functor $i\colon \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is a triangulated functor.

Proposition 1.18. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}_0 \subseteq \mathcal{D}$ a full additive subcategory of $\mathcal{D}$. Assume that the translation functor T of $\mathcal{D}$ restricts to an autoequivalence T_0 of $\mathcal{D}_0$ and that for every distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}$, where f belongs to $\mathcal{D}_0$, the object Z is isomorphic to some object of $\mathcal{D}_0$. Then $\mathcal{D}_0$ is naturally equipped with a triangulated structure, for which it becomes a* triangulated *subcategory of $\mathcal{D}$.*

Proof. We take T_0 to be the translation functor on $\mathcal{D}_0$ and take distinguished triangles in $\mathcal{D}_0$ to be all the triangles of $\mathcal{D}_0$, for which there exists an isomorphism of triangles to some distinguished triangle of $\mathcal{D}$. Then $\mathcal{D}_0$ is clearly triangulated and the inclusion functor $i\colon \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is triangulated. □

Definition 1.19. Triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ are said to be *equivalent* (as triangulated categories), if there are triangulated functors $F\colon \mathcal{D} \to \mathcal{D}'$ and $G\colon \mathcal{D}' \to \mathcal{D}$, such that $G \circ F \simeq \mathrm{id}_{\mathcal{D}}$ and $F \circ G \simeq \mathrm{id}_{\mathcal{D}'}$ and we call F and G *triangulated equivalences.*

Proposition 1.20. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, which is an equivalence of categories with a quasi-inverse $G\colon \mathcal{D}' \to \mathcal{D}$. Then G is also a triangulated functor.*

Proof. This is a consequence of [Huy06, Proposition 1.41], as equivalences of categories are special instances of adjunctions. □

As a consequence, two triangulated categories are equivalent (as triangulated categories) whenever there exists a fully faithful essentially surjective triangulated functor from one to the other.

Definition 1.21. Let $H\colon \mathcal{D} \to \mathcal{A}$ be an additive functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$. We say H is a *cohomological functor*, if for every distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the induced long sequence in $\mathcal{A}$

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to H(Y[1]) \to H(Z[1]) \to \cdots \tag{1.3}$$

is exact.

Construction of the long sequence (1.3) is extremely simple as opposed to other known long exact sequences assigned to certain short exact sequences (e.g. of sheaves or complexes) as all the complexity is actually captured within the distinguished triangle already. All one has to do is unwrap the triangle $X \to Y \to Z \to X[1]$ into the following chain of composable morphisms

$$\cdots \to Y[-1] \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to Z[1] \to X[2] \to \cdots$$

and apply functor H over it.

Example 1.22. For any object W in a triangulated category $\mathcal{D}$ the functors

$$\mathrm{Hom}_{\mathcal{D}}(W,-)\colon \mathcal{D} \to \mathsf{Mod}_k \text{ and } \mathrm{Hom}_{\mathcal{D}}(-,W)\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Mod}_k$$

are cohomological[3]. Let us verify the first claim. Consider the long sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(W,X) \to \mathrm{Hom}_{\mathcal{D}}(W,Y) \to \mathrm{Hom}_{\mathcal{D}}(W,Z) \to \mathrm{Hom}_{\mathcal{D}}(W,X[1]) \to \cdots$$

arising from a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. Since the translation functor on $\mathcal{D}$ and the axiom TR2 allow us to turn this triangle and still end up with a distinguished triangle, it suffices to verify only that

$$\mathrm{Hom}_{\mathcal{D}}(W,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{D}}(W,Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{D}}(W,Z) \quad \text{is exact.}$$

By proposition 1.13 we see that $\mathrm{im}\, f_* \subseteq \ker g_*$, so we are left to prove the other inclusion. Suppose $v\colon W \to Y$ is in $\ker g_*$. Then axiom TR3 (together with TR2 and TR1(ii)) asserts the existence of a morphism $u\colon W \to X$ making the diagram below commutative.

$$\begin{tikzcd} W \arrow[r, "\mathrm{id}_W"] \arrow[d, dashed, "u"'] & W \arrow[r] \arrow[d, "v"'] & 0 \arrow[r] \arrow[d, "w"'] & W[1] \arrow[d, "u[1]"'] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

From the commutative square on the left it is then clear that $v \in \mathrm{im}\, f_*$, as $v = f \circ u$.

Lemma 1.23. *Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then f is an isomorphism if and only if $Z \simeq 0$.*

Proof. We argue with a chain of equivalences. Observe that f is an isomorphism if and only if $f_*\colon \mathrm{Hom}_{\mathcal{D}}(W,X) \to \mathrm{Hom}_{\mathcal{D}}(W,Y)$ and $f^*\colon \mathrm{Hom}_{\mathcal{D}}(Y,W) \to \mathrm{Hom}_{\mathcal{D}}(X,W)$ are isomorphisms for all objects W of $\mathcal{D}$. From the existence of long exact sequences established in the previous example the latter condition is equivalent to $\mathrm{Hom}_{\mathcal{D}}(W,Z) = 0$ and $\mathrm{Hom}_{\mathcal{D}}(Z,W) = 0$ for all W, which in turn is equivalent to $Z \simeq 0$. □

I think Yoneda can be used here to enable us to consider only f_* instead of both f_* and f^*. i.e. f_* iso for all W iff f iso

[3] With claiming that $\mathrm{Hom}_{\mathcal{D}}(-,W)$ is cohomological we are being slightly imprecise, as we have not clarified what the triangulated structure on $\mathcal{D}^{\mathrm{op}}$ is. We take it to be the most obvious one (cf. [Mil, Chapter 1, §1.2]).

Lemma 1.24. *Let triangles* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *and* $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ *be distinguished in a triangulated category* $\mathcal{D}$. *Then their direct sum*

$$X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} (X \oplus X')[1]$$

is also distinguished in $\mathcal{D}$.

Proof. By TR1(iii) we may extend $f_0 \oplus f_1$ to a distinguished triangle

$$X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \to Q \to (X \oplus X')[1]. \tag{1.4}$$

Axiom TR3 applied to the canonical projections

$$X \xleftarrow{p} X \oplus X' \xrightarrow{p'} X' \quad \text{and} \quad Y \xleftarrow{q} Y \oplus Y' \xrightarrow{q'} Y'$$

then induces morphisms $r\colon Q \to Z$ and $r'\colon Q \to Z'$, such that (p, q, r) is a morphism from the triangle (1.4) to (f, g, h) and similarly (p', q', r') is a morphism from triangle (1.4) to (f', g', h'). By the universal property of products $(r, r')\colon Q \to Z \oplus Z'$ is induced. We thus obtain a morphism of triangles, where the top triangle is distinguished, while the bottom one we are aiming to show is distinguished as well.

$$\begin{tikzcd}
X \oplus X' \arrow[r, "f \oplus f'"] \arrow[d, equal] & Y \oplus Y' \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, "(r,r')"'] & (X \oplus X')[1] \arrow[d, equal] \\
X \oplus X' \arrow[r, "f \oplus f'"] & Y \oplus Y' \arrow[r, "g \oplus g'"] & Z \oplus Z' \arrow[r, "h \oplus h'"] & (X \oplus X')[1]
\end{tikzcd} \tag{1.5}$$

Consider now the cohomological functor $H = \mathrm{Hom}_{\mathcal{D}}(W, -)\colon \mathcal{D} \to \mathsf{Mod}_k$ for an object W of $\mathcal{D}$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$ to the diagram (1.5) yields the following commutative diagram in Mod_k.

$$\begin{tikzcd}
H(X \oplus X') \arrow[r] \arrow[d, equal] & H(Y \oplus Y') \arrow[r] \arrow[d, equal] & H(Q) \arrow[r] \arrow[d, "(r,r')_*"] & H((X \oplus X')[1]) \arrow[r] \arrow[d, equal] & H((Y \oplus Y')[1]) \arrow[d, equal] \\
H(X \oplus X') \arrow[r] & H(Y \oplus Y') \arrow[r] & H(Z \oplus Z') \arrow[r] & H((X \oplus X')[1]) \arrow[r] & H((Y \oplus Y')[1])
\end{tikzcd}$$

By the universal property of products, we have isomorphisms of modules

$$\mathrm{Hom}_{\mathcal{D}}(W, X \oplus X') \simeq \mathrm{Hom}_{\mathcal{D}}(W, X) \oplus \mathrm{Hom}_{\mathcal{D}}(W, X'),$$

natural in X and X'. Thus the bottom row of the above diagram may be viewed as a direct sum of two sequences, which come about through the application of the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$ to a pair of distinguished triangles (f, g, h) and (f', g', h'). Since $\mathrm{Hom}_{\mathcal{D}}(W, -)$ is cohomological and a sum of exact sequences is exact the bottom row is an exact sequence. The top row is exact as well because (1.4) is distinguished. Since all but the middle vertical arrow $(r, r')_*$ are isomorphisms, the *five lemma*[4] implies that $(r, r')_*$ is an isomorphism as well. As this holds for any object W of $\mathcal{D}$, the morphism (r, r') is an isomorphism by Yoneda lemma, more precisely [KS06, §1, Corollary 1.4.7]. This means that the bottom row of (1.5) is distinguished by axiom TR1(i), which is what we wanted to show. □

[4] KS06, §8, Lemma 8.3.13.

Lemma 1.25. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$ and assume $h = 0$. Then there exists an isomorphism $Y \simeq X \oplus Z$, for which the next diagram commutes.*

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X \oplus Z \arrow[r] \arrow[d, "\sim"'] & Z \arrow[r, "0"] \arrow[d, equal] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "0"] & X'[1] \end{tikzcd} \tag{1.6}$$

Proof. First, the triangle $X \xrightarrow{i_X} X \oplus Z \xrightarrow{p_Z} Z \xrightarrow{0} X[1]$ is distinguished by Lemma 1.24, because it is easily seen to be isomorphic to the direct sum of distinguished triangles $X \to X \to 0 \to X[1]$ and $0 \to Z \to Z \to 0[1]$. By axiom TR3, there exists a morphism $X \oplus Z \to Y$ for which diagram (1.6) commutes. A similar use of the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(W, -) \colon \mathcal{D} \to \mathsf{Mod}_k$ as in the proof of Lemma 1.24, arguing with the five lemma and Yoneda lemma, shows that $X \oplus Z \to Y$ is in fact an isomorphism. $\square$

The following are all specialized results characterizing, when a triangulated functor is an equivalence. They will be used in a fundamental way in the proof of the Derived Torelli theorem. In particular we highlight Proposition 1.27, Proposition 1.35 and Corollary 1.31, which will be directly used in the proof. We mention that all the results of this part can be found in [Huy06, §1].

Definition 1.26. Let $\mathcal{D}$ be a triangulated category. A collection of objects Ω in $\mathcal{D}$ forms a *spanning class* of $\mathcal{D}$, if for all objects X of $\mathcal{D}$ the following two conditions are satisfied.

(i) If $\mathrm{Hom}_{\mathcal{D}}(U, X[i]) = 0$ for all $U \in \Omega$ and all $i \in \mathbf{Z}$, then $X \simeq 0$.

(ii) If $\mathrm{Hom}_{\mathcal{D}}(X, U[i]) = 0$ for all $U \in \Omega$ and all $i \in \mathbf{Z}$, then $X \simeq 0$.

Proposition 1.27. *Let $F \colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories. Assume F has left and right adjoints $G \dashv F \dashv H$. Suppose $\mathcal{D}$ contains a spanning class Ω and that F induces isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}}(U, V[i]) \to \mathrm{Hom}_{\mathcal{D}'}(F(U), F(V[i])) \tag{1.7}$$

for all $U, V \in \Omega$. Then F is fully faithful.

Proof. For any X, Y of $\mathcal{D}$ we will show that the morphism action of F

$$\mathrm{Hom}_{\mathcal{D}}(X, Y[i]) \to \mathrm{Hom}_{\mathcal{D}'}(F(X), F(Y[i]))$$

is an isomorphism. The two adjunctions $G \dashv F \dashv H$ give rise to the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(X, Y) \arrow[r, "(\eta_Y)_*"] \arrow[d, "(\varepsilon_X)^*"'] \arrow[dr, "F"] & \mathrm{Hom}_{\mathcal{D}}(X, HF(Y)) \arrow[d, "\sim"] \\ \mathrm{Hom}_{\mathcal{D}}(GF(X), Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{D}'}(F(X), F(Y)), \end{tikzcd} \tag{1.8}$$

where $\eta_Y \colon Y \to HF(Y)$ denotes the unit of the adjunction $F \dashv H$ and $\varepsilon_X \colon GF(X) \to X$ denotes the counit of the adjunction $G \dashv F$.

First we show, that ε_U is an isomorphism for every $U \in \Omega$. Indeed, as ε_U fits into a distinguished triangle

$$GF(U) \xrightarrow{\varepsilon_U} U \longrightarrow Z \longrightarrow (GF(U))[1],$$

the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(-, V)$, for $V \in \Omega$, induces a long exact sequence

$$\begin{aligned} \cdots \leftarrow \mathrm{Hom}_{\mathcal{D}}(GF(U), V) \xleftarrow{(\varepsilon_U)^*} \mathrm{Hom}_{\mathcal{D}}(U, V) \leftarrow \\ \leftarrow \mathrm{Hom}_{\mathcal{D}}(Z, V) \leftarrow \mathrm{Hom}_{\mathcal{D}}(GF(U)[1], V) \leftarrow \cdots . \end{aligned} \tag{1.9}$$

By assumption (1.7) is a bijection for all $U, V \in \Omega$ and $i \in \mathbf{Z}$, thus, considering the diagram (1.8), where we set $X = U$ and $Y = V$, we see that $(\varepsilon_U)^*$, along with $(\varepsilon_U[i])^*$ for all $i \in \mathbf{Z}$, is an isomorphism. From the long exact sequence (1.9) we then conclude, that $\mathrm{Hom}_{\mathcal{D}}(Z[i], V) = 0$ for all $i \in \mathbf{Z}$ and as $V \in \Omega$ belonging to a spanning class of $\mathcal{D}$ was arbitrary, $Z \simeq 0$, showing ε_U is an isomorphism by Lemma 1.23.

Secondly, we show that η_Y is an isomorphism for all objects Y of $\mathcal{D}$. This is done in a very similar manner. Considering now the distinguished triangle

$$Y \xrightarrow{\eta_Y} HF(Y) \longrightarrow Z \longrightarrow Y[1],$$

we apply the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(U, -)$, for $U \in \Omega$, to obtain the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_{\mathcal{D}}(U, Y) \xrightarrow{(\eta_Y)_*} \mathrm{Hom}_{\mathcal{D}}(U, HF(Y)) \to \\ \to \mathrm{Hom}_{\mathcal{D}}(U, Z) \to \mathrm{Hom}_{\mathcal{D}}(U, Y[1]) \to \cdots . \end{aligned} \tag{1.10}$$

Setting $X = U$ in diagram (1.8), we see that ε_U being an isomorphism implies $(\eta_Y)_*$ is an isomorphism. The long exact sequence (1.10) then establishes $\mathrm{Hom}_{\mathcal{D}}(U, Z[i]) = 0$ for all $i \in \mathbf{Z}$, thus showing $Z \simeq 0$. Consequently η_Y is an isomorphism by Lemma 1.23.

Lastly, looking back at diagram (1.8), η_Y being an isomorphism, for all objects Y of $\mathcal{D}$, proves, that the morphism action of F is an isomorphism i.e. F is fully faithful. □

Definition 1.28. Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}_0$, $\mathcal{D}_1$ its triangulated subcategories. We say $\mathcal{D}$ *decomposes* into $\mathcal{D}_0$ and $\mathcal{D}_1$ if the following three conditions are met.

(i) Categories $\mathcal{D}_0$ and $\mathcal{D}_1$ contain objects not isomorphic to 0.

(ii) Every object X of $\mathcal{D}$ fits into a distinguished triangle (in $\mathcal{D}$) of the form

$$Y_0 \to X \to Y_1 \to Y_0[1],$$

where Y_0 and Y_1 belong to $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively.

(iii) For all objects Y_0 of $\mathcal{D}_0$ and Y_1 of $\mathcal{D}_1$ it holds that

$$\mathrm{Hom}_{\mathcal{D}}(Y_0, Y_1) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_0) = 0.$$

In this case we also say that $\mathcal{D}_0$ and $\mathcal{D}_1$ are *orthogonal*.

Additionally, $\mathcal{D}$ is called *indecomposable*, if it can not be decomposed in this way.

Proposition 1.29. *Suppose a triangulated category $\mathcal{D}$ decomposes into $\mathcal{D}_0$ and $\mathcal{D}_1$. Then every object X of $\mathcal{D}$ is isomorphic to $Y_0 \oplus Y_1$ for some objects Y_0 and Y_1 belonging to $\mathcal{D}_0$ and $\mathcal{D}_1$, respectively.*

Proof. This is a direct consequence of the definition and Lemma 1.25. $\square$

Lemma 1.30. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a fully faithful triangulated functor between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$. Suppose F has a right adjoint $F \dashv H\colon \mathcal{D}' \to \mathcal{D}$. Then F is an equivalence if and only if for any object Z in $\mathcal{D}'$ the condition $H(Z) \simeq 0$ implies $Z \simeq 0$.*

Proof. ($\Rightarrow$) Let Y belong to $\mathcal{D}'$. Since F is part of an equivalence, the counit $\varepsilon_Y : FH(Y) \to Y$ is an isomorphism, thus, as $H(Y) \simeq 0$ implies $FH(Y) \simeq 0$, we see that $Y \simeq 0$.

($\Leftarrow$) Recall, that whenever F is part of an adjunction $F \dashv H$, it is fully faithful if and only if the unit $\eta\colon \mathrm{id}_{\mathcal{D}} \Longrightarrow HF$ is an isomorphism (this is easily seen from diagram (1.8) or [Sp25, Tag 07RB]). Thus it suffices to show that the counit $\varepsilon\colon FH \Longrightarrow \mathrm{id}_{\mathcal{D}'}$ is an isomorphism. Pick an object Y of $\mathcal{D}'$ and extend $\varepsilon_Y : FH(Y) \to Y$ to a distinguished triangle $FH(Y) \to Y \to Z \to FH(Y)[1]$ in $\mathcal{D}'$. After applying the triangulated functor H (cf. [Huy06, §1, Proposition 1.41]) to the latter triangle, we obtain

$$HFH(Y) \xrightarrow{H(\varepsilon_Y)} H(Y) \longrightarrow H(Z) \longrightarrow HFH(Y)[1].$$

By the triangle identity relating units and counits of an adjunction [Sp25, Tag 0GLL], we have

$$H(\varepsilon_Y) \circ \eta_{H(Y)} = \mathrm{id}_{H(Y)}, \tag{1.11}$$

and as observed earlier, since $\eta_{H(Y)}$ is an isomorphism, $H(\varepsilon_Y)$ is as well. It follows by 1.23 that $H(Z) \simeq 0$, from which, by the assumption, $Z \simeq 0$ follows. Finally, by 1.23 again, ε_Y is an isomorphism, proving our claim. $\square$

Proposition 1.31. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a fully faithful triangulated functor between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ having both left and right adjoints $G \dashv F \dashv H$. Further assume $\mathcal{D}$ has objects not isomorphic to 0 and that $\mathcal{D}'$ is indecomposable. Then F is an equivalence if and only if for all objects Y in $\mathcal{D}'$ the condition $H(Y) \simeq 0$ implies $G(Y) \simeq 0$.*

Proof. If F is an equivalence, the condition is satisfied because its quasi-inverse is both a left and right adjoint to F and adjoints are unique up to natural isomorphisms. Suppose now that the condition is satisfied and we show, that F is essentially surjective.

We will first construct two triangulated subcategories of $\mathcal{D}'$, where one of which will turn out to contain only trivial objects by indecomposability of $\mathcal{D}'$. Let $\mathcal{D}'_0$ the full subcategory of $\mathcal{D}'$ consisting of all objects Y, for which $H(Y) \simeq 0$. As H is triangulated $\mathcal{D}'_0$ is a full *triangulated* subcategory of $\mathcal{D}'$. We will show this category contains only trivial objects. Let $\mathcal{D}'_1$ be the full subcategory of $\mathcal{D}'$ spanned on objects of the form $F(X)$ for some object X of $\mathcal{D}'$. Similarly, as F is triangulated, $\mathcal{D}'_1$ is a triangulated subcategory of $\mathcal{D}'$, sometimes called the *essential image of F*. We now show, that $\mathcal{D}'_0$ and $\mathcal{D}'_1$ satisfy conditions (ii) and (iii) of Definition 1.28.

To verify (ii) pick any object Y of $\mathcal{D}'$ and consider the counit ε_Y within a distinguished triangle

$$FH(Y) \xrightarrow{\varepsilon_Y} Y \to Z \to FH(Y)[1] \tag{1.12}$$

of $\mathcal{D}'$. By definition $FH(Y)$ belongs to $\mathcal{D}'_1$. On the other hand, to see that Z belongs to $\mathcal{D}'_0$, we map the triangle to $\mathcal{D}$ via the functor H, to conclude that $H(\varepsilon_Y)$ is an isomorphism, by the triangle identity (1.11) and the fact that F is fully faithful (thus $\eta_{H(Y)}$ is an isomorphism). Then $H(Z) \simeq 0$ by Lemma 1.23, showing that Z belongs to $\mathcal{D}'_0$.

To see why (iii) holds true pick objects Y_0 and Y_1 of $\mathcal{D}'_0$ and $\mathcal{D}'_1$ respectively. Then by definition there exists an object X of $\mathcal{D}$, such that $F(X) \simeq Y_1$, so we compute

$$\mathrm{Hom}_{\mathcal{D}'}(Y_1, Y_0) \simeq \mathrm{Hom}_{\mathcal{D}'}(F(X), Y_0) \simeq \mathrm{Hom}_{\mathcal{D}}(X, H(Y_0)) = 0$$

because $H(Y_0) \simeq 0$. As per the assumption, $H(Y_0) \simeq 0$ implies $G(Y_0) \simeq 0$, so we see that

$$\mathrm{Hom}_{\mathcal{D}'}(Y_0, Y_1) \simeq \mathrm{Hom}_{\mathcal{D}'}(Y_0, F(X)) \simeq \mathrm{Hom}_{\mathcal{D}}(G(Y_0), X) = 0$$

Since $\mathcal{D}$ contains non-trivial objects and F is fully faithful, the category $\mathcal{D}'_1$ must contain a non-trivial object, namely an F-image of any non-trivial object of $\mathcal{D}$. Since by assumption $\mathcal{D}'$ is indecomposable, the category $\mathcal{D}'_0$ only contains trivial objects. In other words, if $H(Y) \simeq 0$ for some object Y of $\mathcal{D}'$, then $Y \simeq 0$.

Finally, to show that F is essentially surjective, we prove that the counit $\varepsilon_Y : FH(Y) \to Y$ is an isomorphism for every object Y of $\mathcal{D}'$. Once again considering the image of the distinguished triangle (1.12) via the functor H, we have already established that $H(Z) \simeq 0$. The latter implies $Z \simeq 0$ and finally, by Lemma 1.23, ε_Y is an isomorphism. $\square$

For the last part of this section let k denote a field.

Definition 1.32. Let $\mathcal{D}$ be a triangulated category over k with $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ being a finite dimensional k-vector space for all objects X and Y of $\mathcal{D}$. A *Serre functor* is a triangulated autoequivalence $S \colon \mathcal{D} \to \mathcal{D}$ of $\mathcal{D}$, such that for all objects X and Y there exists an isomorphism of k-vector spaces

$$\sigma_{X,Y} \colon \mathrm{Hom}_{\mathcal{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Y, S(X))^*,$$

which is natural in both arguments X and Y, thus forming a natural isomorphism of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Mod}_k$.

Example 1.33. In Section 3.1 we will see that $\mathsf{D}^b(X)$ – the bounded derived category of coherent sheaves on a smooth and projective variety X over k – comes equipped with a Serre functor owing to the fact that such varieties enjoy Serre duality.

Whenever a triangulated category $\mathcal{D}$ over a filed k is equipped with a Serre functor, we will tacitly assume all the Homs $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ are finite dimensional k-vector spaces for any pair of objects X and Y of $\mathcal{D}$.

Proposition 1.34. *Let $F \colon \mathcal{D} \to \mathcal{D}'$ be an equivalence of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ endowed with Serre functors $S_{\mathcal{D}}$ and $S_{\mathcal{D}'}$, respectively. Then there exists a natural isomorphism of functors*

$$F \circ S_{\mathcal{D}} \simeq S_{\mathcal{D}'} \circ F.$$

Proof. We will show that $S_{\mathcal{D}}^{-1} \circ F^{-1} \circ S_{\mathcal{D}'} \circ F$ is left adjoint to the identity functor $\mathrm{id}_{\mathcal{D}}$. As $\mathrm{id}_{\mathcal{D}}$ is already left adjoint to $\mathrm{id}_{\mathcal{D}}$, this will show that $S_{\mathcal{D}}^{-1} \circ F^{-1} \circ S_{\mathcal{D}'} \circ F$ and $\mathrm{id}_{\mathcal{D}}$

are naturally isomorphic by uniqueness of adjoints. This is indeed so, since for any two objects X, Y of $\mathcal{D}$ we have a chain of isomorphisms, natural in both X and Y

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(S_{\mathcal{D}}^{-1}(F^{-1}(S_{\mathcal{D}'}(F(X)))), Y) &\simeq \mathrm{Hom}_{\mathcal{D}}(F^{-1}(S_{\mathcal{D}'}(F(X))), S_{\mathcal{D}}(Y)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(Y, F^{-1}(S_{\mathcal{D}'}(F(X))))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(F(Y), S_{\mathcal{D}'}(F(X)))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(F(X), (F(Y))) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X, Y). \qquad \square\end{aligned}$$

Proposition 1.35. *Suppose $F\colon \mathcal{D} \to \mathcal{D}'$ is a triangulated functor of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ endowed with Serre functors $S_{\mathcal{D}}$ and $S_{\mathcal{D}'}$ respectively. Assume F has a left adjoint $G \dashv F$. Then $S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}$ is right adjoint to F.*

Proof. For any two objects X of $\mathcal{D}$ and Y of $\mathcal{D}'$ we compute

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}'}(F(X),\ Y) &\simeq \mathrm{Hom}_{\mathcal{D}'}(Y,\ S_{\mathcal{D}'}(F(X)))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(S_{\mathcal{D}'}^{-1}(Y),\ F(X))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y)),\ X)^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X,\ S_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y))))^{**} \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(X,\ S_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y)))).\end{aligned}$$

For this is a chain of natural equivalences of functors in X and Y, it follows that

$$F \dashv S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}. \qquad \square$$

1.3 Categories of complexes

In order to define derived categories of an additive category $\mathcal{A}$ we first introduce the category of complexes and the homotopy category of complexes of $\mathcal{A}$. We will equip the latter with a triangulated structure. We also recall the definition of cohomology of chain complexes and introduce quasi-isomorphisms. Throughout this section $\mathcal{A}$ will be a fixed additive or k-linear category and we also mention that we will be using the cohomological indexing convention.

Category of chain complexes

By a *chain complex* in $\mathcal{A}$ we mean a collection of objects and morphisms

$$A^\bullet = \left((A^i)_{i\in\mathbf{Z}}, (d_A^i\colon A^i \to A^{i+1})_{i\in\mathbf{Z}}\right),$$

where A^i are objects and d^i are morphisms of $\mathcal{A}$, called *differentials*, subject to equations $d^{i+1} \circ d^i = 0$, for all $i \in \mathbf{Z}$. A complex is *bounded from below* (resp. *bounded from above*), if there exists $i_0 \in \mathbf{Z}$ for which $A^i \simeq 0$ for all $i \leq i_0$ (resp. $i \geq i_0$) and is *bounded*, if it is both bounded from below and bounded from above. A *chain map* between two chain complexes $A^\bullet$ and $B^\bullet$ in $\mathcal{A}$ is a collection of morphisms in $\mathcal{A}$

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}},$$

for which $f^{i+1} \circ d_A^i = d_B^i \circ f^i$ holds for all $i \in \mathbf{Z}$. This may diagrammatically be described by the following commutative ladder.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Next we define the *category of chain complexes* in $\mathcal{A}$, denoted by $\mathsf{Ch}(\mathcal{A})$, as the following additive category.

Objects:	chain complexes in $\mathcal{A}$.
Morphisms:	$\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$ is the set of chain maps $A^\bullet \to B^\bullet$, equipped with a group structure inherited from $\mathcal{A}$ by applying operations componentwise.

The composition law is defined componentwise and is clearly associative and bilinear. The identity morphisms $\mathrm{id}_{A^\bullet}$ are defined to be $(\mathrm{id}_{A^i})_{i \in \mathbf{Z}}$. The complex $\cdots \to 0 \to 0 \to \cdots$ plays the role of the zero object in $\mathsf{Ch}(\mathcal{A})$ and the biproduct of complexes $A^\bullet$ and $B^\bullet$ exists and is witnessed by the chain complex

$$A^\bullet \oplus B^\bullet = \left((A^i \oplus B^i)_{i \in \mathbf{Z}}, (d_A^i \oplus d_B^i)_{i \in \mathbf{Z}}\right),$$

together with the canonical projection and injection morphisms arising from direct sums componentwise.

Additionally, we also define the following full additive subcategories of $\mathsf{Ch}(\mathcal{A})$.

$\mathsf{Ch}^+(\mathcal{A})$	*Category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$\mathsf{Ch}^-(\mathcal{A})$	*Category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$\mathsf{Ch}^b(\mathcal{A})$	*Category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

Remark 1.36. Whenever $\mathcal{A}$ is k-linear, all the categories of complexes $\mathsf{Ch}^*(\mathcal{A})$ become k-linear as well in the obvious way.

On all the categories of complexes mentioned above, we can now define the translation functor

$$T\colon \mathsf{Ch}^*(\mathcal{A}) \to \mathsf{Ch}^*(\mathcal{A})$$

given by its action on objects and morphisms as follows.

Objects:	$T(A^\bullet) = A[1]^\bullet$ is the chain complex with $(A[1]^\bullet)^i := A^{i+1}$ and differentials $d^i_{A[1]} = -d_A^{i+1}$.
Morphisms:	For a chain map $f^\bullet\colon A^\bullet \to B^\bullet$ we define $f[1]^\bullet$ to have component maps $(f[1]^\bullet)^i = f^{i+1}$.

The translation functor T thus acts on a complex $A^\bullet$ by twisting its differential by a sign and shifting it one step to the *left*, which is pictured below.

$$\begin{array}{ccccccccccc}
 & \cdots & & -1 & & 0 & & 1 & & 2 & & \cdots \\
A^\bullet & \cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
A[1]^\bullet & \cdots & \longrightarrow & A^{0} & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow & \cdots
\end{array}$$

Remark 1.37. We remark that the translation functor T is clearly also additive or k-linear, whenever $\mathcal{A}$ is additive or k-linear.

Since T is an auto-equivalence there exists a quasi-inverse T^{-1} to T, which is defined and unique up to a natural isomorphism. We may then speak of T^k for any $k \in \mathbf{Z}$, whose action on a complex $A^\bullet$ is described by $(A[k]^\bullet)^i = A^{i+k}$ with differential $d^i_{A[k]} = (-1)^k d_A^{i+k}$.

Homotopy category of chain complexes

In this subsection we construct the homotopy category of chain complexes associated to a given additive category $\mathcal{A}$ and equip it with a triangulated structure. The main motivation for its introduction in this work is the fact that we will later on use it to construct the derived category of $\mathcal{A}$. In particular the homotopy category of $\mathcal{A}$, as opposed to the category of complexes[5] $\mathsf{Ch}(\mathcal{A})$, can be enhanced with a triangulated structure which will afterwards descend to the level of derived categories.

Definition 1.38. Let f and g be two chain maps in $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$. We define f and g to be *homotopic*, if there exists a collection of morphisms $\left(h^i \colon A^i \to B^{i-1}\right)_{i\in\mathbf{Z}}$, satisfying

$$f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$$

for all $i \in \mathbf{Z}$. The collection of morphisms $(h^i)_{i\in\mathbf{Z}}$ is called a *homotopy* and we write $f \simeq g$, if f and g are homotopic.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] \arrow[dl, "h^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, "h^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We say f is *null-homotopic*, if $f \simeq 0$.

Lemma 1.39. *Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be complexes in $\mathsf{Ch}(\mathcal{A})$ and let $f, f' \in \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$ and $g, g' \in \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(B^\bullet, C^\bullet)$ be chain maps.*

(i) The subset of all null-homotopic chain maps in $A^\bullet \to B^\bullet$ forms a submodule of $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$.

(ii) If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

Proof. For (i) see [Mil, §3, 1.3.1]. Claim (ii) is a direct consequence of [Mil, §3, 1.3.2]. □

The homotopy category of complexes in $\mathcal{A}$, denoted by $\mathsf{K}(\mathcal{A})$, is defined to be an additive category consisting of

[5] It is still possible to construct the derived category of $\mathcal{A}$ without passing through the homotopy category of complexes, however equipping it with a triangulated structure in that case becomes less elegant.

$$\begin{array}{rl} \textit{Objects:} & \text{chain complexes in } \mathcal{A}. \\ \textit{Morphisms:} & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)/_{\simeq} \end{array}$$

The composition law descends to the quotient by lemma 1.39 (ii), i.e. $[g] \circ [f] := [g \circ f]$, for composable $[f]$ and $[g]$, and for any $A^\bullet$ the identity morphism is defined to be $[\mathrm{id}_{A^\bullet}]$. All the Hom-sets $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y)$ are k-modules by lemma 1.39 (i) and compositions are k-bilinear maps. The direct sum of two complexes $A^\bullet$ and $B^\bullet$ consists of an object $A^\bullet \oplus B^\bullet$ corresponding to the usual biproduct in $\mathsf{Ch}(\mathcal{A})$ together with the homotopy classes of structure maps of its $\mathsf{Ch}(\mathcal{A})$-counterpart.

Triangulated structure on $\mathsf{K}(\mathcal{A})$. We now shift[6] our focus to the construction of a triangulated structure on $\mathsf{K}(\mathcal{A})$. The translation functor $T \colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ is defined on objects and morphisms in the following way.

$$\begin{array}{rl} \textit{Objects:} & A^\bullet \longmapsto A^\bullet[1]. \\ \textit{Morphisms:} & [f^\bullet] \longmapsto [f[1]^\bullet]. \end{array}$$

This assignment is clearly well defined on morphisms, as $f \simeq f'$ implies $f[1] \simeq f'[1]$.

The other piece of data required to obtain a triangulated category is a collection of *distinguished triangles*. To describe what distinguished triangles are in the case of $\mathsf{K}(\mathcal{A})$, we must first introduce the mapping cone of a morphism of complexes and to this end we will for a moment step outside the scope of the homotopy category of complexes back into the category of chain complexes.

Definition 1.40. Let $f \colon A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathsf{Ch}(\mathcal{A})$. The complex $C(f)^\bullet$ is specified by the collection of objects

$$C(f)^i := A^{i+1} \oplus B^i$$

and differentials

$$d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1]^i & d_B^i \end{pmatrix}, \tag{1.13}$$

for all $i \in \mathbf{Z}$, is called the *cone of* f.

Remark 1.41. A simple matrix calculation, using the fact that f is a chain map and d_A, d_B differentials, shows that $C(f)^\bullet$ is indeed a chain complex.

Remark 1.42. The naming convention comes from topology, where one can show that the singular chain complex associated to the topological mapping cone $M(f)$ of a continuous map $f \colon X \to Y$ is chain homotopically equivalent to the cone of the chain map induced by f between singular chain complexes of X and Y.

Along with the cone of a chain map f, we also introduce two chain maps

$$\tau_f \colon B^\bullet \to C(f)^\bullet,$$

given by the collection $\left(\tau_f^i \colon B^i \to A^{i+1} \oplus B^i\right)_{i \in \mathbf{Z}}$, where τ_f^i is the canonical injection into the biproduct for all $i \in \mathbf{Z}$, and

$$\pi_f \colon C(f)^\bullet \to A^\bullet[1],$$

given by the collection $\left(\pi_f^i \colon A^{i+1} \oplus B^i \to A^{i+1}\right)_{i \in \mathbf{Z}}$, where π_f^i is the canonical projection from the biproduct for all $i \in \mathbf{Z}$.

[6] Pun intended.

Definition 1.43. We define any triangle in $\mathsf{K}(\mathcal{A})$ isomorphic to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} C(f)^\bullet \xrightarrow{\pi_f} A[1]^\bullet \tag{1.14}$$

to be distinguished.

Proposition 1.44. *The homotopy category* $\mathsf{K}(\mathcal{A})$ *together with the translation functor* $T\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ *and distinguished triangles defined above is a triangulated category.*

The proof of this proposition, more precisely the verification of axiom TR2, relies heavily on the following very technical lemma. This lemma will also be useful later on in the construction of the derived category $\mathsf{D}(\mathcal{A})$.

Lemma 1.45. *Let* $f\colon A^\bullet \to B^\bullet$ *be a chain map. In notation of Definition 1.43 there exists a homotopy equivalence* $g\colon A[1]^\bullet \to C(\tau)$*, for which the diagram below commutes in* $\mathsf{K}(\mathcal{A})$*.*

$$\begin{tikzcd} B^\bullet \arrow[r, "\tau"] \arrow[d, equal] & C(f)^\bullet \arrow[r, "\pi"] \arrow[d, equal] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, dashed, "g"] & B[1]^\bullet \arrow[d, equal] \\ B^\bullet \arrow[r, "\tau"] & C(f)^\bullet \arrow[r, "\tau_\tau"] & C(\tau)^\bullet \arrow[r, "\pi_\tau"] & B[1]^\bullet \end{tikzcd} \tag{1.15}$$

Proof. The morphism $g\colon A[1]^\bullet \to C(\tau)^\bullet$ is defined componentwise by

$$g^i = \left(-f^{i+1}, \mathrm{id}_A, 0\right) \qquad g^i\colon A^{i+1} \longrightarrow B^{i+1} \oplus A^{i+1} \oplus B^i.$$

The computation below then shows that g is a chain map. Indeed, omitting the indexes for clarity we have

$$d^i_{C(\tau)} g^i = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ -\mathrm{id}_B & -f & d_B \end{pmatrix} \begin{pmatrix} -f \\ \mathrm{id}_A \\ 0 \end{pmatrix} = \begin{pmatrix} d_B f \\ -d_A \\ 0 \end{pmatrix} = \begin{pmatrix} f d_A \\ -d_A \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ \mathrm{id}_A \\ 0 \end{pmatrix} d_{A[1]} = g^{i+1} d^i_{A[1]}.$$

Next we will show that the projection $p\colon C(\tau)^\bullet \to A[1]^\bullet$, given by

$$p^i = (0, \mathrm{id}_A, 0) \qquad p^i\colon B^{i+1} \oplus A^{i+1} \oplus B^i \longrightarrow A^{i+1}$$

is a homotopy inverse to g. This is a left inverse to g already in $\mathsf{Ch}(\mathcal{A})$, so we only need to check $g \circ p \simeq \mathrm{id}_{C(\tau)}$. One computes that $g^i \circ p^i - \mathrm{id}_{C(\tau)}$ is represented by a matrix

$$\begin{pmatrix} -\mathrm{id}_B & -f^{i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_B \end{pmatrix}$$

and that the collection $(h^i\colon C(\tau)^i \to C(\tau)^{i-1})_{i\in\mathbf{Z}}$, given by

$$h^i = \begin{pmatrix} 0 & 0 & \mathrm{id}_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the homotopy witnessing $g \circ p - \mathrm{id}_{C(\tau)} \simeq 0$. It is left to show that the diagram (1.15) commutes in $\mathsf{K}(\mathcal{A})$. Clearly the leftmost square commutes and we have $-f[1] = \pi_\tau \circ g$ already in $\mathsf{Ch}(\mathcal{A})$, so the rightmost square commutes as well. Lastly, we have $p \circ \tau_\tau = \pi$ in $\mathsf{Ch}(\mathcal{A})$ thus post-composing both sides with g gives $\tau_\tau \simeq g \circ p \circ \tau_\tau = g \circ \pi$, making the middle square also commutative, only now in $\mathsf{K}(\mathcal{A})$. ☐

Remark 1.46. The above proposition is particularly important because it tells us that the rotated triangle (1.14), namely

$$B^\bullet \xrightarrow{\tau_f} C(f)^\bullet \xrightarrow{\pi_f} A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet \tag{1.16}$$

is also distinguished in $\mathsf{K}(\mathcal{A})$.

Proof of Proposition 1.44. Omitting TR4, we verify the axioms TR1–TR3 of Definition 1.11 one by one.

TR1(i) is true by definition. For TR1(ii) we see that for any complex $A^\bullet$ of $\mathsf{K}(\mathcal{A})$, we have $C(\mathrm{id}_A)^\bullet \simeq 0$. This is seen to be true by constructing a homotopy witnessing $\mathrm{id}_{C(\mathrm{id}_A)} \simeq 0$. It is not difficult to verify that the homotopy $h^i \colon A^{i+1} \oplus A^i \to A^i \oplus A^{i-1}$ given by $\left(\begin{smallmatrix} 0 & \mathrm{id}_{A^i} \\ 0 & 0 \end{smallmatrix}\right)$ for all $i \in \mathbf{Z}$ achieves this. Axiom TR1(iii) holds also by definition, since any $f \colon A^\bullet \to B^\bullet$ may be extended to a distinguished triangle via its cone.

TR2. Assuming

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1]^\bullet$$

is distinguished, it is isomorphic to a triangle of the form (1.14). Lemma 1.45 asserts that (1.16) is distinguished, therefore triangle $B^\bullet \to C^\bullet \to A[1]^\bullet \to B[1]^\bullet$ is as well. For the opposite implication, if the former triangle were to be distinguished, rotating twice more shows that

$$A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet \xrightarrow{-g[1]} C[1]^\bullet \xrightarrow{-h[1]} A[2]^\bullet$$

is distinguished. It is therefore isomorphic to

$$A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet \to C(-f[1])^\bullet \to A[2]^\bullet$$

and since $C(f)[1]^\bullet = C(-f[1])^\bullet$, we see that $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ is isomorphic to a triangle of the form (1.14) i.e. distinguished.

TR3. It suffices to verify this axiom only on distinguished triangles of the form (1.14). We then have a commutative diagram in $\mathsf{K}(\mathcal{A})$

$$\begin{tikzcd}
A_0^\bullet \arrow[r, "f_0"] \arrow[d, "u"'] & B_0^\bullet \arrow[r] \arrow[d, "v"'] & C(f_0)^\bullet \arrow[r] \arrow[d, dashed, "w"'] & A_0[1]^\bullet \arrow[d, "{u[1]}"'] \\
A_1^\bullet \arrow[r, "f_1"] & B_1^\bullet \arrow[r] & C(f_1)^\bullet \arrow[r] & A_1[1]^\bullet,
\end{tikzcd}$$

where the dashed morphism $w \colon C(f_0)^\bullet \to C(f_1)^\bullet$ is the homotopy class of the chain map $u[1] \oplus v$, given by components

$$u^{i+1} \oplus v^i \colon A_0^{i+1} \oplus B_0^i \to A_1^{i+1} \oplus B_1^i. \qquad \square$$

As in the case of categories of complexes in $\mathcal{A}$, we can also define the following full additive subcategories of $\mathsf{K}(\mathcal{A})$.

- $\mathsf{K}^+(\mathcal{A})$ *Homotopy category of complexes bounded below*, consisting of complexes in $\mathcal{A}$ bounded below.
- $\mathsf{K}^-(\mathcal{A})$ *Homotopy category of complexes bounded above*, consisting of complexes in $\mathcal{A}$ bounded above.
- $\mathsf{K}^b(\mathcal{A})$ *Homotopy category of bounded complexes*, consisting of bounded complexes in $\mathcal{A}$.

For convenience we will in practice assume that unless otherwise stated our complexes $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ will be supported in $\mathbf{Z}_{\geq 0}$ i.e. we will assume $A^i \simeq 0$ for $i < 0$.

Remark 1.47. By Proposition 1.18 all the subcategories $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$, $\mathsf{K}^b(\mathcal{A})$ of $\mathsf{K}(\mathcal{A})$ are triangulated.

Lastly we mention that $\mathcal{A}$ can naturally be seen as a full subcategory of $\mathsf{K}(\mathcal{A})$. This is due to the functor $\mathcal{A} \to \mathsf{K}(\mathcal{A})$, defined by the following action on objects and morphisms, being fully faithful.

$$\begin{array}{rccl} \textit{Objects:} & A & \longmapsto & (\cdots \to 0 \to A \to 0 \to \cdots), \\ \textit{Morphisms:} & f & \longmapsto & [(f^i)_{i\in\mathbf{Z}}], \text{ where } f^0 = f \text{ and } f^i = 0 \text{ for } i \neq 0. \end{array}$$

The proof of this fact is not difficult to verify, yet we skip it for brevity, but leave the interested reader with the reference [Mil, §3, Lemma 1.3.5] for completeness.

Cohomology

A very important invariant of a chain complex in the homotopy category, which measures the extent to which it fails to be exact, is its cohomology. Here we are no longer assuming $\mathcal{A}$ is just k-linear, but abelian, since we will need kernels and cokernels to exists. For a chain complex $A^\bullet$ in $\mathsf{Ch}(\mathcal{A})$ and $i \in \mathbf{Z}$ we define its *i-th cohomology* to be

$$H^i(A^\bullet) := \operatorname{coker}(\operatorname{im} d^{i-1} \to \ker d^i).$$

Remark 1.48. A computation with universal properties inside an abelian category shows, that the following are all equivalent ways of defining the cohomology of a complex as well

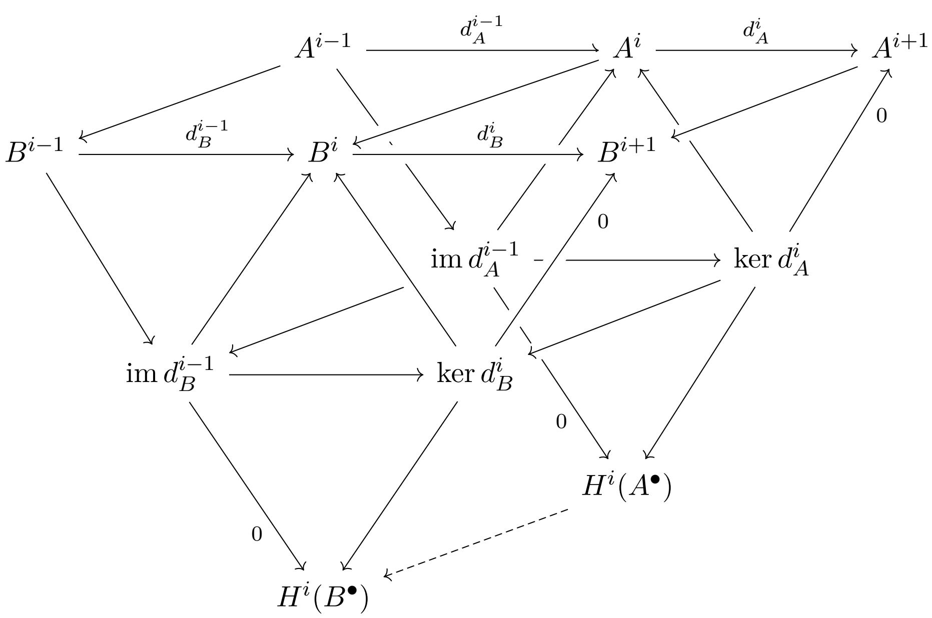
$$\begin{aligned} H^i(A^\bullet) :=& \operatorname{coker}(\operatorname{im} d^{i-1} \to \ker d^i) \simeq \ker(\operatorname{coker} d^{i-1} \to \operatorname{im} d^i) \\ \simeq& \operatorname{coker}(A^{i-1} \to \ker d^i) \simeq \ker(\operatorname{coker} d^{i-1} \to A^i). \end{aligned}$$

See [KS06, §8, Definition 8.3.8.].

For a morphism of complexes $f^\bullet \colon A^\bullet \to B^\bullet$, one can also define a morphism

$$H^i(f^\bullet) \colon H^i(A^\bullet) \to H^i(B^\bullet)$$

in $\mathcal{A}$, because $f^\bullet$ induces maps $\operatorname{im} d_A^{i-1} \to \operatorname{im} d_B^{i-1}$ and $\ker d_A^i \to \ker d_B^i$, which fit into the commutative diagram (1.17).



(1.17)

All the induced morphisms come from universal properties and are as such unique for which the diagram commutes. Thus the assignment

$$f^\bullet \mapsto H^i(f^\bullet)\colon \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A,B) \to \mathrm{Hom}_{\mathcal{A}}(H^i(A), H^i(B))$$

is functorial i.e. respects composition and maps identity morphisms to identity morphisms. For the same reasons it is also a k-linear homomorphism, showing that

$$H^i\colon \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$$

a k-linear functor. Due to the following proposition 1.49, the i-th cohomology functor H^i descends to a well defined additive functor

$$H^i\colon \mathsf{K}(\mathcal{A}) \to \mathcal{A}$$

on the homotopy category $\mathsf{K}(\mathcal{A})$.

Proposition 1.49. *Let $f\colon A^\bullet \to B^\bullet$ be a null-homotopic chain map in $\mathsf{Ch}(\mathcal{A})$. Then f induces the zero map on cohomology, that is $H^i(f) = 0$ for all $i \in \mathbf{Z}$.*

Proof. As f is null-homotopic, there exists a homotopy $(h^i\colon A^i \to B^{i-1})_{i\in Z}$ such that

$$f^i = h^{i+1}d_A^i + d_B^{i-1}h^i \qquad \text{for all } i \in \mathbf{Z}.$$

Let us name the following morphisms from diagram (1.17).

$$\begin{array}{rr} i_A\colon \ker d_A^i \hookrightarrow A^i & i_B\colon \ker d_B^i \hookrightarrow B^i \\ \psi_B\colon B^{i-1} \twoheadrightarrow \mathrm{im}\, d_B^{i-1} & \xi_B\colon \mathrm{im}\, d_B^{i-1} \to \ker d_B^i \\ \pi_B\colon \ker d_B^i \twoheadrightarrow H^i(B^\bullet) & \phi\colon \ker d_A^i \to \ker d_B^i \end{array}$$

We compute $i_B\phi = f^i i_A = (h^{i+1}d_A^i + d_B^{i-1}h^i)i_A = d_B^{i-1}h^i i_A = i_B\xi_B\psi_B h^i i_A$. Since i_B is a monomorphism, we may cancel it on the left to express ϕ as $\xi_B\psi_B h^i i_A$. Then it is clear that $\pi_B\phi = 0$ (as $\pi_B\xi_B = 0$), which means that 0 is the unique map $H^i(A^\bullet) \to H^i(B^\bullet)$ fitting into the commutative diagram (1.17). $\square$

Remark 1.50. The proof of Proposition 1.49 although technical and not very aesthetically pleasing is worth including, since it showcases a technique confined fully in the context of abelian categories and universal properties. Usually these kinds of results are proven first in the category Mod_A of A-modules and then generalized to arbitrary abelian categories as a consequence of the Freyd–Mitchell embedding theorem.

It is very fruitful to consider all the cohomology functors $(H^i)_{i\in\mathbf{Z}}$ at once, as is witnessed by the next proposition.

Proposition 1.51. *[KS06, §12, Theorem 12.3.3] Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence in $\mathsf{Ch}(\mathcal{A})$. Then there exists a long exact sequence in $\mathcal{A}$*

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots,$$

which is functorial in the short exact sequence.

Remark 1.52. Morphisms $H^i(A^\bullet) \to H^i(B^\bullet)$ and $H^i(B^\bullet) \to H^i(C^\bullet)$ are induced by the corresponding morphisms between complexes. But the existence of a *connecting* morphism $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ is part of the assertion of the proposition.

An elementary observation shows that the translation functor T on $\mathsf{Ch}(\mathcal{A})$ gives rise to natural equivalences

$$H^0 \circ T^i \simeq H^i, \tag{1.18}$$

for all $i \in \mathbf{Z}$. Hence the long exact sequence of Proposition 1.51 can be rephrased as

$$\cdots \to H^0(A^\bullet[i]) \to H^0(B^\bullet[i]) \to H^0(C^\bullet[i]) \to H^0(A^\bullet[i+1]) \to \cdots .$$

Proposition 1.53. *With respect to the triangulated structure on $\mathsf{K}(\mathcal{A})$ the cohomology functors H^i are cohomological.*

Proof. First, it is enough to show only that

$$H^0(B^\bullet) \to H^0(C(f)^\bullet) \to H^0(A[1]^\bullet)$$

is exact for any distinguished triangle of the form (1.14). This statement becomes apparent, once we realize that the distinguished triangle yields a short exact sequence

$$0 \longrightarrow B^\bullet \xrightarrow{\tau_f} C(f)^\bullet \xrightarrow{\pi_f} A[1]^\bullet \longrightarrow 0$$

in $\mathsf{Ch}(\mathcal{A})$ to which one then applies proposition 1.51[7].

Now we argue why this is enough to show that all H^i are cohomological. Since every distinguished triangle in $\mathsf{K}(\mathcal{A})$ is isomorphic to a triangle of the form (1.14) we see that the sequence $H^0(B^\bullet) \to H^0(C^\bullet) \to H^0(A[1]^\bullet)$ is exact for any distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ by functoriality of H^i. Rotating the preceding triangle i.e. periodically applying axiom TR2, shows that H^0 is cohomological. Finally, by (1.18) clearly all H^i are cohomological. □

To end this section, we introduce a class of morphisms and a class of objects in $\mathsf{K}(\mathcal{A})$, playing a principal role in the sequel.

Definition 1.54. Let $A^\bullet$ and $B^\bullet$ be objects and $f\colon A^\bullet \to B^\bullet$ a morphism of $\mathsf{K}(\mathcal{A})$.

(i) Chain complex $A^\bullet$ is said to be *acyclic*, if $H^i(A^\bullet) \simeq 0$ for all $i \in \mathbf{Z}$.

(ii) Morphism f is a *quasi-isomorphism*, if $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.

[7] Note that we have actually used much less than what the proposition has to offer, in particular not even the existence of the connecting morphism.

2 Derived categories

The first goal of this section is to construct the derived category of an abelian category $\mathcal{A}$ and equip it with a triangulated structure. Our arguments, although specialized to the homotopy category $\mathsf{K}(\mathcal{A})$ and the class of quasi-isomorphisms, do not differ tremendously from the general theory of localization of categories. A more comprehensive and formal treatment of this topic is laid out in Chapters 7, 10 and 13 of [KS06] or the meticulous [Mil]. The second part covers the construction of derived functors. We see how the established framework of derived categories nicely lends itself for the definition of derived functors and we also relate them back to the classical higher derived functors. To close the chapter, we prove a correspondence between the Hom-sets of $\mathsf{D}^+(\mathcal{A})$ and certain Ext-modules.

2.1 Derived categories of abelian categories

In algebraic geometry cohomology of a geometric object, like a scheme or a variety, X with respect to some coherent sheaf $\mathcal{F}$ plays a very important role. One way of computing $H^i(X, \mathcal{F})$, which we shall also feature in Section 3.2, involves the following. Instead of directly studying the sheaf $\mathcal{F}$, we represent it with a so called *resolution*, which consists of a complex of sheaves $F^\bullet$, built up from sheaves F^i, for $i \in \mathbf{Z}$, belonging to some class of sheaves, which is well behaved under cohomology, and a quasi-isomorphism of the form $F^\bullet \to \mathcal{F}$ or $\mathcal{F} \to F^\bullet$. After noting that the sheaf $\mathcal{F}$ can be seen as a complex concentrated in degree 0 and observing that replacing a resolution of $\mathcal{F}$ with another one results in computing isomorphic cohomology groups, we are motivated not to distinguish the sheaf $\mathcal{F}$ from its resolutions in the ambient (homotopy) category of complexes any longer. Taking a step back, we would like to modify the homotopy category $\mathsf{K}(\mathsf{coh}(X))$ in such a way that $\mathcal{F}$ is identified with all its resolutions, or in other words, we want all the quasi-isomorphisms of $\mathsf{K}(\mathsf{coh}(X))$ to turn into isomorphisms in a "universal way". We will take the latter to be our inspiration for the definition of the derived category $\mathsf{D}(\mathcal{A})$ of a general abelian category $\mathcal{A}$. This is stated more formally in the form of the ensuing universal property.

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $\mathsf{K}(\mathcal{A})$ its homotopy category. A category $\mathsf{D}(\mathcal{A})$ together with a functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is called the *derived category of* $\mathcal{A}$, if it satisfies:

(i) For every quasi-isomorphism s in $\mathsf{K}(\mathcal{A})$, $Q(s)$ is an isomorphism in $\mathsf{D}(\mathcal{A})$.

(ii) For any category $\mathcal{D}$ and any functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$, sending quasi-isomorphisms s in $\mathsf{K}(\mathcal{A})$ to isomorphisms $F(s)$ in $\mathcal{D}$, there exists a functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathcal{D}$, which is unique up to a unique natural isomorphism, such that $F \simeq F_0 \circ Q$. In other words, the diagram below commutes up to natural isomorphism.

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\ \mathsf{D}(\mathcal{A}) \arrow[ur, dashed, "F_0"'] & \end{tikzcd}$$

Remark 2.2. We recognize this definition as a special case of localization of categories [KS06, §7, Definition 7.1.1.] or [GM02, §III.2, Definition 1]. In particular it defines $\mathsf{D}(\mathcal{A})$ to be the *localization of a triangulated category* $\mathsf{K}(\mathcal{A})$ by the family of all quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$.

A naive way of constructing $\mathsf{D}(\mathcal{A})$ out of $\mathsf{K}(\mathcal{A})$ would be to artificially add the inverses to all the quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ and then impose the correct collection of relations on the newly constructed class of morphisms. As this can quickly lead us to some set theoretic problems, we will construct a specific model, which achieves this, instead. Our construction is a priori not going to result in a locally small[8] category, but as we shall soon see in practice all the categories we will encounter are going to be locally small.

2.1.1 Construction

To start, we first need a technical lemma resembling the Ore condition from non-commutative algebra.

Lemma 2.3. *Let $f\colon A^\bullet \to B^\bullet$ and $s\colon C^\bullet \to B^\bullet$ belong to the homotopy category $\mathsf{K}(\mathcal{A})$, with s being a quasi-isomorphism. Then there exists a quasi-isomorphism $u\colon C_0^\bullet \to A^\bullet$ and a morphism $g\colon C_0^\bullet \to C^\bullet$, such that the diagram below commutes in $\mathsf{K}(\mathcal{A})$.*

$$\begin{tikzcd} C_0^\bullet \arrow[r, dashed, "g"] \arrow[d, dashed, "\sim"', "u"] & C^\bullet \arrow[d, "\sim"', "s"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \end{tikzcd}$$

Proof. We are no in a *triangulated* category $\mathsf{K}(\mathcal{A})$ and first extend $s\colon C^\bullet \to B^\bullet$ into a distinguished triangle

$$C^\bullet \xrightarrow{s} B^\bullet \xrightarrow{\tau} C(s)^\bullet \xrightarrow{\pi} C[1]^\bullet.$$

The composition $\tau f\colon A^\bullet \to C(s)^\bullet$ then also extends to a distinguished triangle, for which we have the following commutative diagram with solid arrows

$$\begin{tikzcd} C(\tau f)[-1]^\bullet \arrow[r, "u"] \arrow[d, dashed] & A^\bullet \arrow[r, "\tau f"] \arrow[d, "f"] & C(s)^\bullet \arrow[r] \arrow[d, equal] & C(\tau f)^\bullet \arrow[d, dashed] \\ C^\bullet \arrow[r, "s"] & B^\bullet \arrow[r, "\tau"] & C(s)^\bullet \arrow[r, "\pi"] & C[1]^\bullet \end{tikzcd}$$

By Axiom TR2, there exists a morphism $C(\tau f)^\bullet \to C[1]^\bullet$ making the whole diagram commutative. As the bottom triangle is distinguished an application of the long exact sequence in cohomology, guaranteed by Proposition 1.53, shows that $H^i(C(s)^\bullet) \simeq 0$ for all $i \in \mathbf{Z}$, because s is a quasi-isomorphism. Applying the long exact sequence now to the top triangle shows that $u\colon C(\tau f)[-1]^\bullet \to A^\bullet$ is a quasi-isomorphism. Taking $C_0^\bullet$ to be $C(\tau f)[-1]^\bullet$ proves the lemma. □

Equipped with the preceding lemma, we are now in a position to construct the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. The derived category $\mathsf{D}(\mathcal{A})$ will consist of

Objects of $\mathsf{D}(\mathcal{A})$*:* chain complexes in $\mathcal{A}$,

i.e. the class of objects of $\mathsf{K}(\mathcal{A})$ or $\mathsf{Ch}(\mathcal{A})$, and a class of morphisms, which is quite intricate to define. For fixed complexes $A^\bullet$ and $B^\bullet$ we define the Hom-set[9] $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ in the following way.

[8] A category $\mathcal{C}$ is called *locally small* if for all objects X and Y of $\mathcal{C}$ the Hom-sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are actual sets.

[9] What will be defined here is a priori not necessarily a set, but a class, so $\mathsf{D}(\mathcal{A})$, defined in this section, is not necessarily a locally small category.

Hom-sets. A *left roof spanned on $A^\bullet$ and $B^\bullet$* is a pair of morphisms $s\colon C^\bullet \to A^\bullet$ and $f\colon C^\bullet \to B^\bullet$ in the homotopy category $\mathsf{K}(\mathcal{A})$, where s is a quasi-isomorphism. This roof is depicted in the following diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \tag{2.1}$$

and denoted by (s, f). Dually, one also obtains the notion of a *right roof spanned on $A^\bullet$ and $B^\bullet$*, which is a pair of morphisms $g\colon A^\bullet \to C^\bullet$ and $u\colon B^\bullet \to C^\bullet$, where u is a quasi-isomorphism, and is depicted below.

$$\begin{tikzcd} & C^\bullet & \\ A^\bullet \arrow[ur, "g"'] & & B^\bullet \arrow[ul, "\sim"', "u"] \end{tikzcd}$$

Our construction of $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ will be based on left roofs, for nothing is gained or lost by picking either one of the two. Both work just as well and are in fact equivalent (see [KS06, Remark 7.1.18]). Despite our arbitrary choice, it is still beneficial to consider both, as we will sometimes switch between the two whenever convenient.

Definition 2.4. Two left roofs $A^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} B^\bullet$ and $A^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} B^\bullet$ are defined to be *equivalent*, if there exists a quasi-isomorphism $u\colon C^\bullet \to C_0^\bullet$ and a morphism $g\colon C^\bullet \to C_1^\bullet$ in $\mathsf{K}(\mathcal{A})$, for which the diagram below commutes (in $\mathsf{K}(\mathcal{A})$).

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "\sim"', "u"] \arrow[dr, "g"] & & \\ & C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd} \tag{2.2}$$

We denote this relation by $\equiv$.

Note that since $C^\bullet \to C_0^\bullet \to A^\bullet$ is a quasi-isomorphism, the same is true for the composition $C^\bullet \to C_1^\bullet \to A^\bullet$, concluding that $g\colon C^\bullet \to C_1^\bullet$ is a quasi-isomorphism. Also observe that in the diagram (2.2) we may find a new left roof, namely

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s_0 \circ u"', "\sim"] \arrow[dr, "f_1 \circ g"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

which is also equivalent to the two roofs we started with, (s_0, f_0) and (s_1, f_1).

Lemma 2.5. *The equivalence of left roofs on $A^\bullet$ and $B^\bullet$ is an equivalence relation.*

Proof. The relation is clearly reflexive – we take both u and g to be $\mathrm{id}_{C^\bullet}$. By the note above it is also symmetric, since g is a quasi-isomorphism. It remains to show transitivity. Suppose left roofs $A^\bullet \longleftarrow C_0^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ are equivalent and left roofs $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_2^\bullet \longrightarrow B^\bullet$ are equivalent. This is witnessed by the diagrams

$$\begin{tikzcd} & & D_0^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & \\ & C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & D_1^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet. \end{tikzcd}$$

By Lemma 2.3 the right roof $D_0^\bullet \to C_1^\bullet \leftarrow D_1^\bullet$ may be completed to form a commutative square

$$\begin{tikzcd}
C^\bullet \arrow[r, dashed] \arrow[d, dashed, "\sim"'] & D_1^\bullet \arrow[d, "\sim"] \\
D_0^\bullet \arrow[r] & C_1^\bullet,
\end{tikzcd}$$

proving that $A^\bullet \leftarrow C_0^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are equivalent. □

For a left roof (2.1) we let $[s \setminus f]$ or $\left[A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet\right]$ denote its equivalence class under $\equiv$.

We then define $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ to be the class of left roofs spanned by $A^\bullet$ and $B^\bullet$, quotiented by the relation $\equiv$. That is

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet) := \left\{ \left[\begin{tikzcd}[column sep=small] & C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \right]_\equiv \;\middle|\; \begin{array}{l} C^\bullet \in \mathrm{Ob}\,\mathsf{K}(\mathcal{A}), \\ f \in \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(C^\bullet, B^\bullet), \\ s \in \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(C^\bullet, A^\bullet) \text{ quasi-iso.} \end{array} \right\}.$$

Composition. Next we define the composition operations

$$\circ\colon \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_0^\bullet, A_1^\bullet) \times \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_1^\bullet, A_2^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_0^\bullet, A_2^\bullet),$$

for all objects $A_0^\bullet$, $A_1^\bullet$ and $A_2^\bullet$ of $\mathsf{D}(\mathcal{A})$. Let $\phi_0\colon A_0^\bullet \to A_1^\bullet$ and $\phi_1\colon A_1^\bullet \to A_2^\bullet$ be a pair of composable morphisms in $\mathsf{D}(\mathcal{A})$. Next, pick their respective left roof representatives, $A_0^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} A_1^\bullet$ and $A_1^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} A_2^\bullet$, and concatenate them according to the solid zig-zag diagram below.

$$\begin{tikzcd}[column sep=small]
 & & & C^\bullet \arrow[dl, dashed, "\sim"', "u"] \arrow[dr, dashed, "g"] & & & \\
 & C_0^\bullet \arrow[dl, "\sim"', "s_0"] \arrow[dr, "f_0"] & & & & C_1^\bullet \arrow[dl, "\sim"', "s_1"] \arrow[dr, "f_1"] & \\
A_0^\bullet & & & A_1^\bullet & & & A_2^\bullet
\end{tikzcd}$$

By Lemma 2.3 there are morphisms $u\colon C^\bullet \to C_0^\bullet$ and $g\colon C^\bullet \to C_1^\bullet$, depicted with dashed arrows, completing the diagram in $\mathsf{K}(\mathcal{A})$. In this way we obtain a left roof, formed by a quasi-isomorphism $s_0 \circ u$ and a morphism $f_1 \circ g$, the equivalence class of which we define to be the composition

$$\phi_1 \circ \phi_0 := \left[A_0^\bullet \xleftarrow{s_0 \circ u} C^\bullet \xrightarrow{f_1 \circ g} A_2^\bullet\right].$$

It can be shown that this is a well defined composition, independent of the choice of left roof representatives of ϕ_0 and ϕ_1 and independent of the choice of the peak $C^\bullet$ and morphisms $u\colon C^\bullet \to C_0^\bullet$ and $g\colon C^\bullet \to C_1^\bullet$, for which the square at the top of the diagram commutes. It is also true that the operation is associative. We leave out the proof, because it is routine, but refer the reader to a very detailed account by Miličić [Mil, Ch. 1.3].

Identities. The identity morphism on an object $A^\bullet$ of $\mathsf{D}(\mathcal{A})$ is defined to be the equivalence class of $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet$. It is not difficult to check that these classes play the role of identity morphisms in $\mathsf{D}(\mathcal{A})$.

Functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. The *localization* functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an identity on objects functor with the action on morphisms defined by the assignment

$$\begin{aligned} \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet) &\xrightarrow{Q} \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet) \\ (f\colon A^\bullet \to B^\bullet) &\longmapsto \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{f} B^\bullet\right]. \end{aligned}$$

k-linear structure. To equip the hom-sets of $\mathsf{D}(\mathcal{A})$ with a k-module structure, we consult the following lemma resembling finding a common denominator in the context of left roofs.

Lemma 2.6. *Let $\phi_0\colon A^\bullet \to B^\bullet$ and $\phi_1\colon A^\bullet \to B^\bullet$ be two morphisms in $\mathsf{D}(\mathcal{A})$ represented by left roofs*

$$\begin{tikzcd} & C_0^\bullet \arrow[dl, "s_0"', "\sim"] \arrow[dr, "f_0"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & C_1^\bullet \arrow[dl, "s_1"', "\sim"] \arrow[dr, "f_1"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

Then there is a quasi-isomorphism $s\colon C^\bullet \to A^\bullet$ and morphisms g_0, $g_1\colon C^\bullet \to B^\bullet$, such that

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s"', "\sim"] \arrow[dr, "g_0"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & C^\bullet \arrow[dl, "s"', "\sim"] \arrow[dr, "g_1"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

represent ϕ_0 and ϕ_1, respectively.

Proof. The proof of this lemma relies on Lemma 2.3 and is otherwise not complicated, therefore we omit it. A proof may be found in [Mil, §1, Lemma 1.3.5]. ☐

Using notation of Lemma 2.6, we define the addition operation on $\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ by the rule

$$\phi_0 + \phi_1 := \left[A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{g_0+g_1} B^\bullet\right].$$

It is well-defined, i.e. independent of the choice of $C^\bullet$, s, g_0 or g_1 and depends only on ϕ_0 and ϕ_1. It is also associative, commutative and has a neutral element $0 = Q(0)$. This is all very thoroughly expanded on in [Mil, §1.2]. The action of scalars of the ring k is defined as

$$\lambda\phi_0 := \left[A^\bullet \xleftarrow{s_0} C^\bullet \xrightarrow{\lambda f_0} B^\bullet\right].$$

Moreover, the composition law $\circ$ is k-bilinear and the localization functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an additive functor.

The zero complex 0 plays the role of the zero object of $\mathsf{D}(\mathcal{A})$ and the direct sum of two complexes is seen to exist and is induced from the direct sum in $\mathsf{K}(\mathcal{A})$ by applying the localization functor Q.

Triangulated structure. The translation functor $T\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is defined to be the usual translation functor on objects and a morphism represented by some roof is sent to the equivalence class of that roof on which we have acted with the translation functor of $\mathsf{K}(\mathcal{A})$. This action respects the equivalence relations, which define the hom-sets of $\mathsf{D}(\mathcal{A})$, and thus induces a well defined functor, which is also an additive auto-equivalence, the quasi-inverse being given by translation in the other direction.

The class of distinguished triangles in $\mathsf{D}(\mathcal{A})$ is defined to consist of all triangles $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$, for which there exists a distinguished triangle $A_0^\bullet \to B_0^\bullet \to C_0^\bullet \to A_0[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$, such that $Q(A_0^\bullet) \to Q(B_0^\bullet) \to Q(C_0^\bullet) \to Q(A_0[1]^\bullet)$ is isomorphic to $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $\mathsf{D}(\mathcal{A})$. Spelling this out in the case of left roofs, we see that this condition implies the existence of another distinguished triangle $A_1^\bullet \to B_1^\bullet \to C_1^\bullet \to A_1[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$, such that the following commutative diagram, where vertical arrows are all quasi-isomorphisms, exists in $\mathsf{K}(\mathcal{A})$.

$$\begin{tikzcd}
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A[1]^\bullet \\
A_1^\bullet \arrow[r] \arrow[u] \arrow[d] & B_1^\bullet \arrow[r] \arrow[u] \arrow[d] & C_1^\bullet \arrow[r] \arrow[u] \arrow[d] & A_1[1]^\bullet \arrow[u] \arrow[d] \\
A_0^\bullet \arrow[r] & B_0^\bullet \arrow[r] & C_0^\bullet \arrow[r] & A_0[1]^\bullet
\end{tikzcd}$$

Verification, that the above defines the structure of a triangulated category on $\mathsf{D}(\mathcal{A})$, is left out, but we remark, that it follows naturally from the triangulated structure on $\mathsf{K}(\mathcal{A})$ and refer the reader to [Mil, Chapter 2, Theorem 1.6.1] or [KS06, Chapter 10, Theorem 10.2.3] for more details.

Remark 2.7. All that has been defined and established in this section with left roofs can analogously also be done with right roofs.

Proposition 2.8. *The constructed derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ together with the localization functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ satisfies the universal property of Definition 2.1. Additionally, in notation of Definition 2.1, if category $\mathcal{D}$ and the functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$ are k-linear or triangulated, the induced functor F_0 is as well.*

Proof. For (i) suppose $s\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism. Then the inverse of $Q(s)$ is a morphism $\phi\colon B^\bullet \to A^\bullet$ represented by the left roof $B^\bullet \xleftarrow{s} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet$. Clearly $\phi \circ Q(s) = \mathrm{id}_{A^\bullet}$, whereas $Q(s) \circ \phi$, represented by $B^\bullet \xleftarrow{s} A^\bullet \xrightarrow{s} B^\bullet$, can be seen to equal $\mathrm{id}_{B^\bullet}$ by the following diagram.

$$\begin{tikzcd}
 & & A^\bullet \arrow[dl, equal] \arrow[dr, "s"] & & \\
 & A^\bullet \arrow[dl, "s"'] \arrow[drrr, "s"] & & B^\bullet \arrow[dr, equal] & \\
B^\bullet \arrow[urrr, equal] & & & & B^\bullet
\end{tikzcd}$$

For (ii) the functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ is defined by the assignment on

$$\begin{array}{lccc}
\textit{Objects:} & A^\bullet & \longmapsto & F(A^\bullet). \\
\textit{Morphisms:} & [A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet] & \longmapsto & F(f) \circ F(s)^{-1}.
\end{array}$$

This is well defined and functorial since the functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$ sends quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathcal{D}$. Moreover, F_0 is a k-linear functor, as can quickly be computed by an application of Lemma 2.6. Using the same notation as in the lemma, we have

$$F_0(\phi_0 + \phi_1) = F(g_0 + g_1) \circ F(s)^{-1} = F(g_0) \circ F(s)^{-1} + F(g_1) \circ F(s)^{-1} = F_0(\phi_0) + F_0(\phi_1)$$
$$\text{and} \quad F_0(\lambda\phi_0) = F(\lambda f_0) \circ F(s_0)^{-1} = \lambda(F(f_0) \circ F(s_0)^{-1}) = \lambda F_0(\phi_0).$$

It is clear from the definition, that $F_0 \circ Q = F$ and that F_0 is unique up to a natural isomorphism for which the diagram (2.1) commutes.

Lastly, when $\mathcal{D}$ and F are triangulated, the functor F_0 also commutes with the translation functors of $\mathsf{D}(\mathcal{A})$ and $\mathcal{D}$, because Q as the identity on objects and F commutes with translation functors of $\mathsf{K}(\mathcal{A})$ and $\mathcal{D}$. From the fact that distinguished triangles in $\mathsf{D}(\mathcal{A})$ are by definition exactly those triangles, which are isomorphic to Q-images of distinguished triangles of $\mathsf{K}(\mathcal{A})$, and the fact that F maps distinguished triangles of $\mathsf{D}(\mathcal{A})$ to distinguished triangles of $\mathcal{D}$, we can conclude that F_0 is triangulated as well. □

2.1.2 Cohomology

Since cohomology already appeared as an indispensable tool on the level of the homotopy category $\mathsf{K}(\mathcal{A})$, it would be nice to also have it be accessible at the level of derived categories. We can in fact define cohomology functors on $\mathsf{D}(\mathcal{A})$ by inducing them from the functors $H^i\colon \mathsf{K}(\mathcal{A}) \to \mathcal{A}$. By definition they send quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathcal{A}$, thus using the universal property of Definition 2.1 we obtain k-linear functors

$$H^i\colon \mathsf{D}(\mathcal{A}) \to \mathcal{A}, \qquad i \in \mathbf{Z},$$

which as expected return the i-th cohomology of a complex $A^\bullet$ and return the morphism $H^i(f) \circ H^i(s)^{-1}$ when applied to a morphism of $\mathsf{D}(\mathcal{A})$, represented by a left roof (2.1).

2.1.3 Subcategories of derived categories

As with the homotopy category of complexes $\mathsf{K}(\mathcal{A})$, we also have the bounded versions of the derived category $\mathsf{D}(\mathcal{A})$. They are constructed in the exact same way as $\mathsf{D}(\mathcal{A})$ above, with the adjustment, that we replace every instance of the category $\mathsf{K}(\mathcal{A})$ with either $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ or $\mathsf{K}^b(\mathcal{A})$, to obtain $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ or $\mathsf{D}^b(\mathcal{A})$, respectively. They are clearly all triangulated as well. All three bounded variants naturally come with the inclusion functors

$$\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A}), \qquad \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A}), \qquad \mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A}),$$

defined to be the identity on objects and send a morphism, i.e. equivalence class of some roof representative with respect to the "bounded variant" of the equivalence relation 2.4, where, again, every instance of $\mathsf{K}(\mathcal{A})$ is replaced by either $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ or $\mathsf{K}^b(\mathcal{A})$, to the equivalence class of that same roof representative, but now under the original equivalence relation. All three functors are clearly also triangulated.

There is also another (equivalent) way of obtaining the categories mentioned above, which will sometimes be more convenient to work with. We introduce them in the form of the following proposition.

Proposition 2.9. *Let $\mathcal{A}$ be an abelian category.*

(i) The functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and induces an equivalence between $\mathsf{D}^+(\mathcal{A})$ and the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ spanned on (possibly unbounded) complexes $A^\bullet$, for which there is an integer $n \in \mathbf{Z}$, such that $H^i(A^\bullet) \simeq 0$ for all $i < n$.

(ii) The functor $\mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and induces an equivalence between $\mathsf{D}^-(\mathcal{A})$ and the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ spanned on (possibly unbounded) complexes $A^\bullet$, for which there is an integer $n \in \mathbf{Z}$, such that $H^i(A^\bullet) \simeq 0$ for all $i > n$.

(iii) The functor $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ *is fully faithful and induces an equivalence between* $\mathsf{D}^b(\mathcal{A})$ *and the full triangulated subcategory of* $\mathsf{D}(\mathcal{A})$ *spanned on (possibly unbounded) complexes* $A^\bullet$*, for which there is an integer* $n \in \mathbf{Z}$*, such that* $H^i(A^\bullet) \simeq 0$ *for all* $|i| > n$*.*

To make the proof of this proposition conceptually clearer, we introduce two complexes associated to a complex $A^\bullet$, called its *right and left truncations*

$$\tau_{\leq n}(A^\bullet) = (\cdots \to A^{n-2} \to A^{n-1} \to \ker d^n \to 0 \to \cdots) \quad \text{and}$$
$$\tau_{\geq n}(A^\bullet) = (\cdots \to 0 \to \operatorname{coker} d^{n-1} \to A^{n+1} \to A^{n+2} \to \cdots).$$

They come equipped with the natural inclusion map $j\colon \tau_{\leq n}(A^\bullet) \to A^\bullet$ and the natural quotient map $q\colon A^\bullet \to \tau_{\geq n}(A^\bullet)$, satisfying the following claims

$$H^i(j)\colon H^i(\tau_{\leq n}(A^\bullet)) \to H^i(A^\bullet) \text{ is an isomorphism for all } i \leq n. \tag{2.3}$$
$$H^i(q)\colon H^i(A^\bullet) \to H^i(\tau_{\geq n}(A^\bullet)) \text{ is an isomorphism for all } i \geq n. \tag{2.4}$$

Indeed, claim (2.3) is clearly true for $i < n$, with $i = n$ being the only interesting case. Here the right commutative square of the diagram below

$$\begin{tikzcd} A^{n-1} \arrow[r, "\delta"] \arrow[d, "j^{n-1}"'] & \ker d^n \arrow[r, "0"] \arrow[d, "j^n"'] & 0 \arrow[d, "j^{n+1}"] \\ A^{n-1} \arrow[r, "d^{n-1}"] & A^n \arrow[r, "d^n"] & A^{n+1} \end{tikzcd}$$

induces the identity morphism between the kernels of the horizontal differentials and the left commutative square induces the identity morphism between the images of the horizontal differential. Consequently j induces the identity morphism on the n-th cohomology $H^n(j)\colon H^n(\tau_{\leq n}(A^\bullet)) \to H^n(A^\bullet)$. Dually, the claim (2.4) also holds.

Proof of Proposition 2.9. For (i) the functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is readily seen to be fully faithful by considering morphisms to be represented by *right* roofs. For example in the right roof $A^\bullet \to C^\bullet \leftarrow B^\bullet$ spanned on complexes $A^\bullet$ and $B^\bullet$ of $\mathsf{D}^+(\mathcal{A})$, we first observe that $C^\bullet$ has bounded cohomology from below, thus $q\colon C^\bullet \to \tau_{\geq n}(C^\bullet)$ is a quasi-isomorphism for some $n \in \mathbf{Z}$. Extending the aforementioned right roof with quasi-isomorphism q, to obtain $A^\bullet \to \tau_{\geq n}(C^\bullet) \leftarrow B^\bullet$, then yields an equivalent right roof, whose peak now belongs to $\mathsf{D}^+(\mathcal{A})$ and the equivalence class of which is therefore sent to the morphism represented by $A^\bullet \to C^\bullet \leftarrow B^\bullet$.

To verify this functor is essentially surjective onto the subcategory of complexes with cohomology bounded from below, suppose $A^\bullet$ satisfies $H^i(A^\bullet) \simeq 0$, for all $i < n$. Then by (2.4) the quotient map $q\colon A^\bullet \to \tau_{\geq n}(A^\bullet)$ is a quasi-isomorphism, inducing an isomorphism $A^\bullet \simeq \tau_{\geq n}(A^\bullet)$ in $\mathsf{D}(\mathcal{A})$.

The case (ii) is proven analogously, now considering left roofs and right truncations. For (iii), the functor $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is viewed as the composition $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. Here $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A})$ is seen to be fully faithful via a minor modification of the argument for (ii) and $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful by (i). Essential surjectivity onto the required subcategory then follows as the roof $A^\bullet \to \tau_{\geq n}(A^\bullet) \leftarrow \tau_{\leq n}(\tau_{\geq n}(A^\bullet))$, for some $n \in \mathbf{Z}$, witnesses an isomorphism in $\mathsf{D}(\mathcal{A})$ between a complex $A^\bullet$ with bounded cohomology and a bounded complex $\tau_{\leq n}(\tau_{\geq n}(A^\bullet))$. □

Recall that in Section 1.3 we introduced a functor $\mathcal{A} \to \mathsf{K}(\mathcal{A})$, including an object A of $\mathcal{A}$ to a complex concentrated in degree 0 and having a similar action on morphisms. Postcomposing this functor with the localozation functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ results in a functor $\mathcal{A} \to \mathsf{D}(\mathcal{A})$. Due to the next proposition, we may view $\mathcal{A}$ as a full subcategory of the derived category $\mathsf{D}(\mathcal{A})$.

Proposition 2.10. *The natural functor $\mathcal{A} \to \mathsf{D}(\mathcal{A})$ is fully faithful and identifies $\mathcal{A}$ with the full subcategory of $\mathsf{D}(\mathcal{A})$, consisting of complexes $A^\bullet$, satisfying $H^i(A^\bullet) \simeq 0$, for all $i \neq 0$.*

Proof. For fully faithfulness see [KS06, §13, Proposition 13.1.10]. Essential surjectivity onto the required subcategory is [KS06, §13, Proposition 13.1.12] and follows from a similar argument as in the proof of Proposition 2.9 (iii) by taking n to be 0, thus showing that in $\mathsf{D}(\mathcal{A})$ the complex $A^\bullet$ with trivial cohomology in non-zero degrees is isomorphic to $H^0(A^\bullet)$ concentrated in degree 0. □

2.1.4 Derived categories of abelian categories with enough injectives

After possibly working with proper classes, when constructing the derived category, this section will once again place us back on familiar grounds of set theory. Aside from these concerns the establishing result of this section will have a very important practical application – construction of derived functors. Under an assumption on the abelian category $\mathcal{A}$, we will establish an equivalence of $\mathsf{D}^+(\mathcal{A})$ with the homotopy category of a full additive subcategory of $\mathcal{A}$, spanned on *injective objects*, which we define presently.

Definition 2.11. An object I of an abelian category $\mathcal{A}$ is *injective*[10], if the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$$

is exact. Equivalently, I is injective, whenever for any monomorphism $A \hookrightarrow B$ and morphism $A \to I$ there is a morphism $B \to I$, for which the following diagram commutes.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

A category $\mathcal{A}$ is said to *have enough injectives*, if every object A of $\mathcal{A}$ embeds into an injective object i.e. there is a monomorphism $A \hookrightarrow I$ for some injective object I.

Remark 2.12. A simple verification shows, that the full subcategory of an abelian or k-linear category $\mathcal{A}$, spanned on all injective objects of $\mathcal{A}$ is a k-linear category, as the zero object 0 is injective and the biproduct of two injective objects is again injective. We denote this category by $\mathcal{J}$.

Example 2.13. The category of abelian groups Ab contains enough injectives. As the latter is equivalent to $\mathsf{qcoh}(\operatorname{Spec}\mathbf{Z})$, it also contains enough injectives. However, since injective abelian groups are divisible, and non-trivial divisible groups are never finitely generated, the full abelian subcategory of finitely generated abelian groups does not contain enough injectives. The latter category is equivalent to $\mathsf{coh}(\operatorname{Spec}\mathbf{Z})$, so it also does not contain enough injectives.

[10] Dually, an object P of $\mathcal{A}$ is *projective*, if P is injective in $\mathcal{A}^{\mathrm{op}}$, or more explicitly, the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathsf{Mod}_k$ is exact. Category $\mathcal{A}$ is said to *have enough projectives*, if every object A of $\mathcal{A}$ is a quotient of some projective object i.e. there is an epimorphism $P \twoheadrightarrow A$ for some projective object P.

Injective objects will be utilized as building blocks for representing complexes. They are the preferred class of objects to work with in this context because they enable a favourable interplay between cohomology and homotopy. For example, as a consequence of lemma 2.18, every acyclic bounded below complex of injectives $I^\bullet$ is actually null-homotopic i.e. isomorphic to the object 0 in $\mathsf{K}^+(\mathcal{A})$. In light of this, consider a complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$. A complex of injectives $I^\bullet$ of $\mathsf{K}^+(\mathcal{J})$ together with a quasi-isomorphism $f\colon A^\bullet \to I^\bullet$ is called an *injective resolution* of $A^\bullet$. Later we will also see that this injective resolution is unique up to homotopy equivalence.

Proposition 2.14. *Suppose $\mathcal{A}$ contains enough injectives. Then for every $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ there is a quasi-isomorphism $f\colon A^\bullet \to I^\bullet$, where $I^\bullet \in \operatorname{Ob}\mathsf{K}^+(\mathcal{A})$ is a complex of injectives.*

Proof. We will inductively construct a complex $I^\bullet$ in $\mathsf{K}^+(\mathcal{A})$, built up from injectives, and a quasi-isomorphism $f\colon A^\bullet \to I^\bullet$. For simplicity assume $A^i = 0$ for $i < 0$.

The base case is trivial. Define $I^i = 0$ and $f^i = 0$ for $i < 0$ and take $f^0\colon A^0 \to I^0$ to be the morphism obtained from the assumption about $\mathcal{A}$ having enough injectives.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 & &
\end{tikzcd}$$

For the induction step suppose we have constructed injective objects I^i, with differentials $\delta^i\colon I^i \to I^{i+1}$, and morphisms f^i for all i up to n, such that the diagram below commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r, "\delta^{n-1}"] & I^n & &
\end{tikzcd}$$

Consider the cokernel of the differential[11] $d^{n-1}_{C(f)}$

$$A^n \oplus I^{n-1} \xrightarrow{d^{n-1}_{C(f)}} A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d^{n-1}_{C(f)}$$

and denote with $j\colon \operatorname{coker} d^{n-1}_{C(f)} \hookrightarrow I^{n+1}$ an embedding to an injective object I^{n+1}. After precomposing the morphism

$$A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d^{n-1}_{C(f)} \xrightarrow{j} I^{n+1}$$

with canonical embeddings of A^{n+1} and I^n into their biproduct $A^{n+1} \oplus I^n$, we obtain morphisms $\delta^n\colon I^n \to I^{n+1}$ and $f^{n+1}\colon A^{n+1} \to I^{n+1}$. As $j \circ e$ is a morphism fitting into the following coproduct diagram,

$$\begin{tikzcd}
 & I^n \arrow[d] \arrow[dr, bend left, "\delta^n"] & \\
A^{n+1} \arrow[r] \arrow[rr, bend right, "f^{n+1}"', start anchor=south, end anchor=west] & A^{n+1} \oplus I^n \arrow[r, dashed, "j \circ e"'] & I^{n+1}
\end{tikzcd}$$

[11]Strictly speaking this is not a differential of a complex yet, because we have not specified the chain map f entirely. But this is easily fixed by setting $I^i = 0$ and f^i to be the zero morphism $A^i \to 0$, for $i > n$.

we see that $j \circ e = \langle f^{n+1}, \delta^n \rangle$. Next, we see that $\delta^n \delta^{n-1} = 0$ and $\delta^n f^n = f^{n+1} d^n$ from the computation

$$0 = j \circ e \circ d^{n-1}_{C(f)} = \langle f^{n+1},\ \delta^n \rangle \begin{pmatrix} -d^n & 0 \\ f^n & \delta^{n-1} \end{pmatrix} = \langle \delta^n f^n - f^{n+1} d^n,\ \delta^n \delta^{n-1} \rangle$$

Thus far we have constructed a chain complex $I^\bullet$ of injective objects and a chain map $f \colon A^\bullet \to I^\bullet$. It remains to be shown that f is a quasi-isomorphism. Recall, that f is a quasi-isomorphism if and only if its cone $C(f)^\bullet$ is acyclic. By Remark 1.48, we known

$$H^n(C(f)^\bullet) = \ker(\operatorname{coker} d^{n-1}_{C(f)} \xrightarrow{d} A^{n+2} \oplus I^{n+1}),$$

where d is obtained from the universal property of $\operatorname{coker} d^{n-1}_{C(f)}$, induced by $d^n_{C(f)}$. Thus it suffices to show that d is a monomorphism. Consider the following diagram, which we will show to commute.

$$\begin{tikzcd}
A^n \oplus I^{n-1} \arrow[r, "d^{n-1}_{C(f)}"] & A^{n+1} \oplus I^n \arrow[rr, "d^n_{C(f)}"] \arrow[dr, two heads, "e"'] & & A^{n+2} \oplus I^{n+1} \arrow[dr, "p"] & \\
& & \operatorname{coker} d^{n-1}_{C(f)} \arrow[ur, dashed, "d"] \arrow[rr, hook, "j"] & & I^{n+1}
\end{tikzcd}$$

Once we show, that its right most triangle is commutative, we see that d is monomorphic, because j is monomorphic. To see that the triangle in question commutes, we compute

$$p \circ d \circ e = p \circ d^n_{C(f)} = \langle f^{n+1}, \delta^n \rangle = j \circ e.$$

As e is an epimorphism, we may cancel it on the right, to arrive at $p \circ d = j$, which ends the proof. □

Remark 2.15. As is evident from the proof by close inspection we have only used two facts about the class of all injective objects of $\mathcal{A}$ – that every object of $\mathcal{A}$ embeds into some injective object and that the class of injectives is closed under finite direct sums. This will later on be used in subsection 2.2.2, when constructing a right derived functor of a given functor F in the presence of an F-adapted class and also in the proof of Proposition 3.3.

Theorem 2.16. *Assume $\mathcal{A}$ contains enough injectives and let $\mathcal{J} \subseteq \mathcal{A}$ denote the full subcategory on injecitve objects of $\mathcal{A}$. Then the inclusion $\mathsf{K}^+(\mathcal{J}) \hookrightarrow \mathsf{K}^+(\mathcal{A})$ induces an equivalence of triangulated categories*

$$\mathsf{K}^+(\mathcal{J}) \simeq \mathsf{D}^+(\mathcal{A}).$$

Remark 2.17. When $\mathcal{A}$ contains enough injectives, Theorem 2.16 in particular shows that $\mathsf{D}^+(\mathcal{A})$ is a locally small category, i.e. all its hom-sets are in fact sets.

Lemma 2.18. *Let $A^\bullet$ be any acyclic complex in $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet$ a complex of injectives from $\mathsf{K}^+(\mathcal{J})$. Then*

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) = 0.$$

In other words every morphism from an acyclic complex to an injective one is null-homotopic.

Proof. We will show that $f \simeq 0$ for any chain map $f\colon A^\bullet \to I^\bullet$ by inductively constructing an appropriate homotopy h. For simplicity assume $A^i = 0$ and $I^i = 0$ for $i < 0$ and define $h^i\colon A^i \to I^{i-1}$ to be 0 for $i \leq 0$.

As $A^\bullet$ is acyclic, the first non-trivial differential $d^0\colon A^0 \to A^1$ is a monomorphism. From I^0 being injective, we obtain a morphism $h^1\colon A^1 \to I^0$, satisfying

$$f^0 = h^1 \circ d^0.$$

Note that the above can actually be rewritten to $f^0 = h^1 d^0 + \delta^{-1} h^0$, as $h^0 = 0$.

For the induction step assume that we have already constructed $h^i\colon A^i \to I^{i-1}$, with $f^{i-1} = h^i d^{i-1} + \delta^{i-2} h^{i-1}$ for all $i \leq n$. We are aiming to construct $h^{n+1}\colon A^{n+1} \to I^n$, for which $f^n = h^{n+1} d^n + \delta^{n-1} h^n$ holds. First, expand the differential d^n into a composition of the canonical epimorphism $e\colon A^n \to \operatorname{coker} d^{n-1}$, followed by $j\colon \operatorname{coker} d^{n-1} \to A^{n+1}$ induced by the universal property of $\operatorname{coker} d^{n-1}$ by the map $d^n\colon A^n \to A^{n+1}$. Morphism j is actually a monomorphism by acyclicity of $A^\bullet$, since we know that

$$0 = H^n(A^\bullet) \simeq \ker(j\colon \operatorname{coker} d^{n-1} \to A^{n+1}).$$

Next, we see that by the universal property of $\operatorname{coker} d^{n-1}$, morphism $f^n - \delta^{n-1} h^n\colon A^n \to I^n$ induces a morphism $g\colon \operatorname{coker} d^{n-1} \to I^n$, because

$$\begin{aligned}
(f^n - \delta^{n-1} h^n) d^{n-1} &= f^n d^{n-1} - \delta^{n-1} h^n d^{n-1} \\
&= f^n d^{n-1} + \delta^{n-1} \delta^{n-2} h^{n-1} - \delta^{n-1} f^{n-1} \\
&= 0.
\end{aligned}$$

The second equality follows from the inductive hypothesis and the third one from f being a chain map and δ^{n-1}, δ^{n-2} being a differentials.

Lastly, for I^n is injecive and j is a monomorphism, there is a morphism $h^{n+1}\colon A^{n+1} \to I^n$, satisfying $h^{n+1} \circ j = g$, thus after precomposing both sides with e, we arrive at $h^{n+1} d^n = f^n - \delta^{n-1} h^n$, which can be rewritten as

$$f^n = h^{n+1} d^n + \delta^{n-1} h^n. \qquad \square$$

Lemma 2.19. *Let $A^\bullet$ and $B^\bullet$ belong to $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet \in \mathsf{K}^+(\mathcal{J})$. Let $f\colon B^\bullet \to A^\bullet$ be a quasi-isomorphism, then*

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet)$$

is an isomorphism of k-modules.

Proof. In $\mathsf{K}^+(\mathcal{A})$ we have a distinguished triangle $B^\bullet \to A^\bullet \to C(f)^\bullet \to B[1]^\bullet$, which induces a long exact sequence (of k-modules)

$$\begin{aligned}
\cdots \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)^\bullet, I^\bullet) \longrightarrow \\
\longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet) \longrightarrow \\
\longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)[-1]^\bullet, I^\bullet) \longrightarrow \cdots,
\end{aligned}$$

since $\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(-, I^\bullet)$ is a cohomological functor by Example 1.22. As f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic along with all its shifts, thus showing that

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)[i]^\bullet, I^\bullet) = 0,$$

for all $i \in \mathbf{Z}$ by Lemma 2.18, since $I^\bullet$ is a complex of injectives. From the long exact sequence it then clearly follows that f^* is an isomorphism. $\square$

Lemma 2.20. *Let $A^\bullet$ belong to $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet$ to $\mathsf{K}^+(\mathcal{J})$. Then the morphism action*

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \tag{2.5}$$

of the localization functor Q is an isomorphism of k-modules.

Proof. We already know this is a k-module homomorphism, thus it is enough to show that it is bijective. This is accomplished by constructing its inverse. Let $\phi\colon A^\bullet \to I^\bullet$ be a morphism in $\mathsf{D}^+(\mathcal{A})$ and let $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ be its left roof representative. The morphism s being a quasi-isomorphism implies that $s^*\colon \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet)$ is bijective by lemma 2.19, so there is a unique morphism $g\colon A^\bullet \to I^\bullet$, such that $f = g \circ s$ in $\mathsf{K}^+(\mathcal{A})$. The inverse to (2.5) is then defined by sending ϕ to g.

First let's argue why this map is well-defined, i.e. independent of the choice of a left roof representative for ϕ. We pick two left roof representatives $A^\bullet \xleftarrow{s_0} B_0^\bullet \xrightarrow{f_0} I^\bullet$ and $A^\bullet \xleftarrow{s_1} B_1^\bullet \xrightarrow{f_1} I^\bullet$ for ϕ and let g_0 and g_1 be such that $f_i = g_i \circ s_i$ for both i. As the roofs are equivalent there are quasi-isomorphisms $t_0\colon C^\bullet \to B_0^\bullet$ and $t_1\colon C^\bullet \to B_1^\bullet$, which fit into the following commutative diagram.

$$\begin{tikzcd}
 & & C^\bullet \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & & \\
 & B_0^\bullet \arrow[dl, "s_0"'] \arrow[drrr, "f_0"' near end] & & B_1^\bullet \arrow[dlll, "s_1" near end] \arrow[dr, "f_1"] & \\
A^\bullet & & & & I^\bullet
\end{tikzcd}$$

Therefore $g_0 s_0 t_0 = f_0 t_0 = f_1 t_1 = g_1 s_1 t_1 = g_1 s_0 t_0$. As $s_0 t_0$ is a quasi-isomorphism, we see that $g_0 = g_1$ by applying Lemma 2.19.

Following the diagram below, it is clear why the constructed map is a right inverse to the morphism action of Q (2.5)

$$\begin{gathered}
\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{Q} \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \\
(g\colon A^\bullet \to I^\bullet) \longmapsto \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right] \longmapsto (g\colon A^\bullet \to I^\bullet)\,.
\end{gathered}$$

Lastly we see that it is also a left inverse by the following diagram

$$\begin{gathered}
\operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{Q} \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \\
\left[A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet\right] \longmapsto (g\colon A^\bullet \to I^\bullet) \longmapsto \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right]
\end{gathered}$$

along with observing that $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet$ and $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ are equivalent roofs. □

Proof of Theorem 2.16. As $\mathsf{K}^+(\mathcal{J}) \to \mathsf{D}^+(\mathcal{A})$ is a triangulated functor, we only need to show that it is fully faithful and essentially surjective. Let $I^\bullet$ and $J^\bullet$ be objects of $\mathsf{K}^+(\mathcal{J})$. Then

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{J})}(I^\bullet, J^\bullet) \xrightarrow{=} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(I^\bullet, J^\bullet) \xrightarrow{(2.20)} \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$$

is a bijection, showing fully faithfulness (the last map is a bijection by Lemma 2.20). Essential surjectivity is clear from the existence of injective resolutions (cf. Proposition 2.14), because quasi-isomorphisms now play the role of isomorphisms in $\mathsf{D}^+(\mathcal{A})$. □

Within the scope of an abelian category $\mathcal{A}$ with enough injectives, Theorem 2.16 yields several well-known classical results of homological algebra.

Corollary 2.21. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(i) Any two injective resolutions of a complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$ are homotopically equivalent i.e. isomorphic in $\mathsf{K}^+(\mathcal{A})$

(ii) For any morphism of complexes $f\colon A^\bullet \to B^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ and any injective resolutions $A^\bullet \to I_A^\bullet$ and $B^\bullet \to I_B^\bullet$ of $A^\bullet$ and $B^\bullet$, respectively, there is a chain map $I_A^\bullet \to I_B^\bullet$, unique up to homotopy, for which the square below commutes in $\mathsf{K}^+(\mathcal{A})$.

$$\begin{tikzcd} I_A^\bullet \arrow[d, dashed] & A^\bullet \arrow[l] \arrow[d, "f"] \\ I_B^\bullet & B^\bullet \arrow[l] \end{tikzcd}$$

2.2 Derived functors

The main goal of this section is to assign to a sensible additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between two abelian categories an appropriate *triangulated* functor on the level of derived categories, called a *derived functor*. We will first deal with the easiest case, when F is exact and then move on to the case, where F will only be assumed to be left or right exact and instead require the domain category $\mathcal{A}$ to posses some nice properties (e.g. contain enough injectives or contain an F-adapted class of objects). In practice the additional conditions $\mathcal{A}$ is required to satisfy are not very restrictive.

2.2.1 Derived functors of exact functors

We start with some establishing lemmas.

Lemma 2.22. *Let $f\colon A^\bullet \to B^\bullet$ be a chain map. Then f is a quasi-isomorphism if and only if its associated cone $C(f)^\bullet$ is acyclic.*

Proof. One only needs to consider the distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$ and its associated long exact sequence in cohomology. □

Lemma 2.23. *Let $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ be a triangulated functor, then the following two conditions are equivalent.*

(i) For every quasi-isomorphism s in $\mathsf{K}(\mathcal{A})$, $F(s)$ is a quasi-isomorphism in $\mathsf{K}(\mathcal{B})$.

(ii) For every acyclic complex $A^\bullet$ of $\mathsf{K}(\mathcal{A})$, $F(A^\bullet)$ is acyclic.

Proof. ($\Rightarrow$) Let $A^\bullet$ be an acyclic complex. Then the zero morphism $A^\bullet \to 0$ is a quasi-isomorphism and is by assumption mapped to a quasi-isomorphism $F(A^\bullet) \to 0$ by F. This means $F(A^\bullet)$ is acyclic.

($\Leftarrow$) Let $f\colon A^\bullet \to B^\bullet$ be a quasi-isomorphism. The image of a distinguished triangle $A^\bullet \to B^\bullet \to C(f)^\bullet \to A^\bullet[1]$ in $\mathsf{K}(\mathcal{A})$ by the triangulated functor F is a distinguished triangle $FA^\bullet \to FB^\bullet \to F(C(f)^\bullet) \to F(A^\bullet)[1]$ in $\mathsf{K}(\mathcal{B})$. By Lemma 2.22 $C(f)^\bullet$ is acyclic, implying $F(C(f)^\bullet)$ is acyclic by the assumption (ii). Hence again by Lemma 2.22 $F(f)$ is a quasi-isomorphism. □

Proposition 2.24. *Let $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ be a functor and assume it satisfies one of the equivalent conditions of Lemma 2.23. Then there exists a unique functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ on the level of derived categories, for which the following diagram of functors commutes.*

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathsf{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathsf{D}(\mathcal{A}) \arrow[r, "F_0"] & \mathsf{D}(\mathcal{B}) \end{tikzcd} \tag{2.6}$$

Uniqueness of the functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ and commutativity of the square are both meant up to natural isomorphism. If $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ is assumed to be k-linear or triangulated, the induced functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ is as well.

Proof. The claim follows from the universal property of $\mathsf{D}(\mathcal{A})$ (cf. proposition 2.8), as $Q_{\mathcal{B}} \circ F$ sends quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathsf{D}(\mathcal{B})$. □

Example 2.25. This proposition gives us another way of defining the translation functor on the derived category $\mathsf{D}(\mathcal{A})$. The translation functor $T\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ clearly satisfies the equivalent conditions of Lemma 2.23, thus it induces a functor on the level of derived categories $T\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. The fact that the induced translation functor on $\mathsf{D}(\mathcal{A})$ is an autoequivalence follows from the assertion about uniqueness.

Every k-linear functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a triangulated functor on the level of homotopy categories $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$, defined by the assignment.

$$\begin{aligned} \textit{Objects:} &\quad A^\bullet \longmapsto (\cdots \to F(A^i) \xrightarrow{F(d^i)} F(A^{i+1}) \to \cdots). \\ \textit{Morphisms:} &\quad [f\colon A^\bullet \to B^\bullet] \longmapsto \Big[\big(F(f^i)\colon F(A^i) \to F(B^i)\big)_{i\in\mathbf{Z}}\Big]. \end{aligned}$$

If moreover F is assumed to be exact, then $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ satisfies the assumptions of lemma 2.24 and thus induces a triangulated functor $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ on the level of derived categories. We call this the *derived functor of* F and often denote it with F as well. We emphasise again that this convention will be used only in the case when $F\colon \mathcal{A} \to \mathcal{B}$ is an *exact* functor between abelian categories.

2.2.2 Derived functors of left (resp. right) exact functors

Assume $F\colon \mathcal{A} \to \mathcal{B}$ is a l*eft* exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. The theory for right exact functors is obtained in parallel by dualizing everything. A crucial downside of functors which are no longer exact, is that their induced functors on the category of complexes no longer satisfy the two equivalent conditions of Lemma 2.23. The same idea with the universal property of Definition 2.1, which worked in the case of exact functors, therefore cannot be applied in this case. Instead, we will present two other more restrictive methods of constructing right derived functors, which are the following:

- ▷ Assuming $\mathcal{A}$ has enough injectives, construct a *right derived functor*

  $$\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$$

  for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

- ▷ Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ and assume the category $\mathcal{A}$ contains a so-called *F-adapted class*, then construct a *right derived functor* $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$.

These methods are however no less restrictive than the case of exact functors, when viewed through the perspective of our applications later on. Lastly we note, that the first method is in some sense a special instance of the second one, which will also be explained in this subsection.

METHOD 1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories with $\mathcal{A}$ having enough injectives. By Theorem 2.16 we know that the composition of $\mathsf{K}^+(\mathcal{J}) \to \mathsf{K}^+(\mathcal{A})$ followed by the localization functor $Q_{\mathcal{A}}\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A})$ is an equivalence of triangulated categories. The *right derived functor of F* is then defined to be the composition

$$\mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{K}^+(\mathcal{J}) \hookrightarrow \mathsf{K}^+(\mathcal{A}) \xrightarrow{F} \mathsf{K}^+(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}^+(\mathcal{B}) \tag{2.7}$$

The functor $\mathbf{R}F$ is clearly triangulated for it is a composition of triangulated functors. Note that in this case the functor $\mathbf{R}F$ does not necessarily make the square (2.6) commutative, but rather satisfies another kind of universal property, which we will mention towards the end of this subsection.

We will use this construction many times through the course of the thesis, so we reiterate the above definition by providing a simple recipe for computing the image of $\mathbf{R}F$ on complexes $A^\bullet$ of $\mathsf{D}^+(\mathcal{A})$.

1. Pick an injective resolution $A^\bullet \to I_A^\bullet$ of $A^\bullet$.
2. Apply the functor F to the complex $I_A^\bullet$ term-wise to obtain $F(I_A^\bullet)$. As complexes we have
$$\mathbf{R}F(A^\bullet) \simeq F(I_A^\bullet).$$

Remark 2.26. In the slightly more general situation, if one is already given a triangulated functor $F\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathcal{B})$ at the level of homotopy categories and assumes that $\mathcal{A}$ contains enough injectives, procedure (2.7) again yields a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$. This remark will be especially useful when considering the *differentialy graded inner* $\mathrm{Hom}^\bullet$-functor of subsection 2.2.4, which does not appear as a functor induced by some left exact functor $\mathcal{A} \to \mathcal{B}$ on the level of abelian categories.

Remark 2.27. When considering a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$, we instead assume the category $\mathcal{A}$ contains enough projectives. In this case we use the dualized version of Theorem 2.16, stating that $\mathsf{K}^-(\mathcal{P}) \simeq \mathsf{D}^-(\mathcal{A})$, where $\mathcal{P}$ denotes the full subcategory of $\mathcal{A}$, spanned on projective objects of $\mathcal{A}$. The left derived functor is then defined as the composition

$$\mathbf{L}F\colon \quad \mathsf{D}^-(\mathcal{A}) \xrightarrow{\simeq} \mathsf{K}^-(\mathcal{P}) \hookrightarrow \mathsf{K}^-(\mathcal{A}) \xrightarrow{F} \mathsf{K}^-(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}^-(\mathcal{B}).$$

Again, as a composition of triangulated functors, we see that $\mathbf{L}F$ is triangulated as well.

METHOD 2. Unfortunately some of our categories will *not* contain enough injectives, as can already be seen with the category of coherent sheaves $\mathsf{coh}(X)$ of a scheme X (see Example 2.13). In this case METHOD 1 of constructing right derived functors will not work. Luckily however, there exists another way of obtaining a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$, assigned to a left exact functor $F : \mathcal{A} \to \mathcal{B}$, when $\mathcal{A}$ does not contain enough injectives, but instead possesses a broader and less restrictive class of objects. To this end we first introduce *F-adapted classes.*

Definition 2.28. A class of objects $\mathcal{I}_F \subseteq \mathcal{A}$ is *adapted* to a *left exact*[12] functor $F\colon \mathcal{A} \to \mathcal{B}$ if the following three conditions are satisfied.

(i) $\mathcal{I}_F$ is stable under finite direct sums.

I started calling biproducts direct sums now...

(ii) Every object A of $\mathcal{A}$ *embeds* into some object of $\mathcal{I}_F$, i.e. there exists an object $I \in \mathcal{I}_F$ and a monomorphism $A \hookrightarrow I$.

(iii) For every acyclic complex $I^\bullet$ in $\mathsf{K}^+(\mathcal{A})$, with $I^i \in \mathcal{I}_F$ for all $i \in \mathbf{Z}$, its image $F(I^\bullet)$ under F is also acyclic.

Example 2.29. We observe, that whenever $\mathcal{A}$ contains enough injectives, the class of all injective objects forms an F-adapted class for *every* left exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Points (i) and (ii) are clearly true for the class of injective objects and point (iii) follows from Lemma 2.18. Indeed, it tells us that an acyclic complex of injectives $I^\bullet$ is null-homotopic, which implies $F(I^\bullet)$ is null-homotopic as well, therefore, in particular, also acyclic.

In the presence of an F-adapted class $\mathcal{I}$ we will now construct the right derived functor of F. Firstly we upgrade the class of objects $\mathcal{I}$ to a full additive subcategory of $\mathcal{A}$, also denoted by $\mathcal{I}$, having its class of objects be precisely the F-adapted class $\mathcal{I}$. Then extending F term-wise to a functor on the level of homotopy categories, also denoted by $F\colon \mathsf{K}^+(\mathcal{I}) \to \mathsf{K}^+(\mathcal{A})$, utilizing the procedure outlined at the end of subsection 2.2.1, we obtain a functor, which maps acyclic complexes to acyclic complexes because by definition the F-adapted class $\mathcal{I}$ satisfies condition (iii) of Definition 2.28. By Proposition 2.24 the functor F then descends to a well defined functor on the level of derived categories

$$\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B}). \tag{2.8}$$

Since we want to define the derived functor $\mathbf{R}F$, whose domain is $\mathsf{D}^+(\mathcal{A})$, it remains to construct a functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{I})$, which we can then post-compose with $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B})$ to obtain $\mathbf{R}F$. What we will show instead is the following.

Proposition 2.30. *The inclusion of categories $\mathcal{I} \hookrightarrow \mathcal{A}$ induces an equivalence of triangulated categories*

$$\mathsf{D}^+(\mathcal{I}) \simeq \mathsf{D}^+(\mathcal{A}).$$

The proof of Proposition 2.30 hinges on the following lemma, closely resembling Lemma 2.14. In the same spirit, we call the complex $I^\bullet$ belonging to $\mathsf{K}^+(\mathcal{I})$ together with a quasi-isomorphism $A^\bullet \to I^\bullet$ an *$\mathcal{I}$-resolution* of the complex $A^\bullet$.

Lemma 2.31. *Every complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$ has an $\mathcal{I}$-resolution.*

Proof. A close inspection of the proof of Proposition 2.14 reveals that formally only properties (i) and (ii) of Definition 2.28, which the class of all injective objects clearly satisfies, were used. Thus the same proof can be copied for this lemma. $\square$

Proof of Proposition 2.30. First, there exists a triangulated functor $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{A})$ by Proposition 2.24, since the inclusion $\mathcal{I} \hookrightarrow \mathcal{A}$ induces an inclusion $\mathsf{K}^+(\mathcal{I}) \hookrightarrow \mathsf{K}^+(\mathcal{A})$ sending acyclic complexes to acyclic ones. We claim that functor $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{A})$ is an equivalence, so we will show only that it is full and faithful, since essential surjectivity is already taken care of by Lemma 2.31

[12] Dually, a class of objects $\mathcal{P}_F$ of $\mathcal{A}$ is *adapted* to a *right exact* functor $F\colon \mathcal{A} \to \mathcal{B}$ if it is closed under finite direct sums, every object of $\mathcal{A}$ is a quotient of some object of $\mathcal{P}_F$ and for every acyclic complex $P^\bullet$ in $\mathsf{K}^-(\mathcal{A})$, with $P^i \in \mathcal{P}_F$, its image $F(P^\bullet)$ under F is also acyclic.

First, we consider the morphism action

$$\operatorname{Hom}_{\mathsf{D}^+(\mathcal{I})}(I^\bullet, J^\bullet) \to \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet) \tag{2.9}$$

and show it to be injective for all objects $I^\bullet$, $J^\bullet$ of $\mathsf{D}^+(\mathcal{I})$. Suppose two morphisms ϕ_0, $\phi_1 \colon I^\bullet \to J^\bullet$, represented by *right* roofs $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ and $I^\bullet \to I_1^\bullet \xleftarrow{\sim} J^\bullet$, are sent to the same morphism in $\operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$, meaning that there is the following commutative diagram in $\mathsf{K}^+(\mathcal{A})$.

$$\begin{tikzcd}
& & A^\bullet & & \\
& I_0^\bullet \arrow[ur] & & I_1^\bullet \arrow[ul, "\sim"'] & \\
I^\bullet \arrow[ur] \arrow[urrr] & & & & J^\bullet \arrow[ul, "\sim"'] \arrow[ulll, "\sim"]
\end{tikzcd}$$

Then simply extending the diagram by a quasi-isomorphism $A^\bullet \to I_2^\bullet$, whose existence is ensured by Lemma 2.31, proves that ϕ_0 and ϕ_1 were in fact equal and that the action (2.9) is faithful.

Next, we show that the action on morphisms is also surjective, i.e. the homomorphism (2.9) is surjective. Pick any $\psi \colon I^\bullet \to J^\bullet$ in $\mathsf{D}^+(\mathcal{A})$, which is represented by a right roof $I^\bullet \to A^\bullet \xleftarrow{\sim} J^\bullet$. As before, extending the roof by a quasi-isomorphism $A^\bullet \to I_0^\bullet$ from Lemma 2.31, leaves us with a representative $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ of some morphism $\phi \colon I^\bullet \to J^\bullet$ in $\mathsf{D}^+(\mathcal{I})$.

$$\begin{tikzcd}
& & I_0^\bullet & & \\
& I_0^\bullet \arrow[ur, equal] & & A^\bullet \arrow[ul, "\sim"'] & \\
I^\bullet \arrow[ur] \arrow[urrr] & & & & J^\bullet \arrow[ul, "\sim"'] \arrow[ulll, "\sim"]
\end{tikzcd}$$

The above diagram then shows, that ϕ is sent to ψ and thus the action on morphisms is surjective. $\square$

Remark 2.32. Upon close inspection, we recognise Theorem 2.16 as a special case of Proposition 2.30, namely take the F-adapted class $\mathcal{I}$ to be the class of all injectives of $\mathcal{A}$, provided there is enough of them. Indeed, it can be shown that any quasi-isomorphism $f \colon I^\bullet \to J^\bullet$ between two bounded below complexes of injectives is an isomorphism in $\mathsf{K}^+(\mathcal{I})$. This is done by showing both $f^* \colon \operatorname{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{K}^+(\mathcal{I})}(I^\bullet, I^\bullet)$ and $f_* \colon \operatorname{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, J^\bullet)$ are isomorphisms. Lemma 2.19 already shows us that f^* is an isomorphism and by a very similar argument as in the proof of the same lemma f_* is as well.

We can now define the *right derived functor* $\mathbf{R}F$ as the composition

$$\mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{D}^+(\mathcal{I}) \longrightarrow D^+(B),$$

where $\mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{D}^+(\mathcal{I})$ denotes a quasi-inverse to the equivalence $\mathsf{D}^+(\mathcal{I}) \simeq \mathsf{D}^+(\mathcal{A})$ established in Proposition 2.30 and $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B})$ is the functor (2.8). The procedure for computing the right derived functor $\mathbf{R}F(A^\bullet)$ of a complex $A^\bullet$ in $\mathsf{D}^+(\mathcal{A})$ is then very similar to Method 1. One takes an $\mathcal{I}$-resolution $A^\bullet \to I_A^\bullet$ of $A^\bullet$ and then applies the functor F to $I_A^\bullet$, i.e.

$$\mathbf{R}F(A^\bullet) = F(I_A^\bullet).$$

We mention that both methods give rise to right derived functors satisfying the following defining universal property. We only state this here, but refer the reader to [GM02, §III.6] for the claim and its proof or [KS06, §13.3, §10.3, §7.3] for a more comprehensive and high level treatment using the formalism of localization.

Definition 2.33. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. The *right derived functor of* F is a triangulated functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ together with a natural transformation $\varepsilon\colon Q_{\mathcal{B}} \circ F \Longrightarrow \mathbf{R}F \circ Q_{\mathcal{A}}$, such that for any triangulated functor $G\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ and any natural transformation $\theta\colon Q_{\mathcal{B}} \circ F \Longrightarrow G \circ Q_{\mathcal{A}}$, there exists a unique natural transformation $\eta\colon \mathbf{R}F \Longrightarrow G$, for which

$$\begin{tikzcd} & Q_{\mathcal{B}} \circ F \arrow[dl, Rightarrow, "\varepsilon"'] \arrow[dr, Rightarrow, "\theta"] & \\ \mathbf{R}F \circ Q_{\mathcal{A}} \arrow[rr, Rightarrow, "\eta Q_{\mathcal{A}}"] & & G \circ Q_{\mathcal{A}} \end{tikzcd}$$

is commutative in the functor category $\mathrm{Fct}(\mathsf{K}^+(\mathcal{A}), \mathsf{D}^+(\mathcal{B}))$.

For two composable right exact functors the following proposition tells us that first deriving and then composing, or first composing and then deriving gives essentially the same result.

Proposition 2.34. *[GM02, §III.7, Theorem 1] Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Suppose category $\mathcal{A}$ admits an F-adapted class $\mathcal{I}_F$, category $\mathcal{B}$ admits a G-adapted class $\mathcal{I}_G$ and assume that every object of $\mathcal{I}_F$ is sent to $\mathcal{I}_G$ by F. Then $\mathbf{R}F$, $\mathbf{R}G$ and $\mathbf{R}(G \circ F)$ exist and there is a natural isomorphism of functors*

$$\mathbf{R}(G \circ F) \simeq \mathbf{R}G \circ \mathbf{R}F\colon \qquad \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{C}).$$

Proof. Right derived functors $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ and $\mathbf{R}G\colon \mathsf{D}^+(\mathcal{B}) \to \mathsf{D}^+(\mathcal{C})$ exist because $\mathcal{A}$ (resp. $\mathcal{B}$) admits an F-adapted (resp. G-adapted) class. To see that $\mathbf{R}(G \circ F)$ exists we show that $\mathcal{I}_F$ is also an $(G \circ F)$-adapted class. Indeed, $\mathcal{I}_F$ is stable under finite sums and for every object A of $\mathcal{A}$ there is an embedding $A \hookrightarrow I$ into some object I belonging to $\mathcal{I}_F$. Further, assuming $I^\bullet$ is an acyclic complex with I^i from $\mathcal{I}_F$ for all $i \in \mathbf{Z}$, we see that $F(I^\bullet)$ is acyclic, because $\mathcal{I}_F$ is F-adapted. For all $i \in \mathbf{Z}$ we know $F(I^i)$ belong to $\mathcal{I}_G$ by assumption, therefore we see that $G(F(I^\bullet))$ is acyclic, because $\mathcal{I}_G$ is G-adapted.

Showing that $\mathbf{R}(G \circ F)$ and $\mathbf{R}G \circ \mathbf{R}F$ are naturally isomorphic will be an exercise in using the universal property of Definition 2.33. First denote the following natural transformations

$$\alpha\colon Q_{\mathcal{B}} \circ F \implies \mathbf{R}F \circ Q_{\mathcal{A}}, \qquad \beta\colon Q_{\mathcal{C}} \circ G \implies \mathbf{R}G \circ Q_{\mathcal{B}},$$
$$\varepsilon\colon Q_{\mathcal{C}} \circ (G \circ F) \implies \mathbf{R}(G \circ F) \circ Q_{\mathcal{A}}.$$

Define a natural transformation $\theta := \mathbf{R}G\alpha \circ \beta F$, this means $\theta_{A^\bullet} = \mathbf{R}G(\alpha_{A^\bullet}) \circ \beta_{F(A^\bullet)}$ for any object $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$. By the universal property of $\mathbf{R}(G \circ F)$ there exists a natural transformation $\eta\colon \mathbf{R}(G \circ F) \implies \mathbf{R}G \circ \mathbf{R}F$ such that the following diagram of natural transformations commutes.

$$\begin{tikzcd} & Q_{\mathcal{C}} \circ (G \circ F) \arrow[dl, Rightarrow, "\varepsilon"'] \arrow[dr, Rightarrow, "\theta"] & \\ \mathbf{R}(G \circ F) \circ Q_{\mathcal{A}} \arrow[rr, Rightarrow, "\eta Q_{\mathcal{A}}"] & & (\mathbf{R}G \circ \mathbf{R}F) \circ Q_{\mathcal{A}} \end{tikzcd}$$

We will show that η is an isomorphism and we will do so by showing that $\eta_{A^\bullet}$ is an isomorphism for every $A^\bullet$ of $\mathsf{D}^+(\mathcal{A})$. Since $A^\bullet$ has an $\mathcal{I}_F$-resolution $A^\bullet \to I^\bullet$ and η is a natural transformation, we may show that for any $I^\bullet$ belonging to $\mathsf{D}^+(\mathcal{I}_F)$, $\eta_{I^\bullet}$ is an isomorphism instead. In that case we have the following commutative diagram in $\mathsf{D}^+(\mathcal{C})$.

$$\begin{tikzcd}
(Q_{\mathcal{C}} \circ G \circ F)(I^\bullet) \arrow[r, "\beta_{F(I^\bullet)}"] \arrow[d, "\varepsilon_{I^\bullet}"'] & (\mathbf{R}G \circ Q_{\mathcal{B}})(F(I^\bullet)) \arrow[d, "\mathbf{R}G(\alpha_{I^\bullet})"] \\
\mathbf{R}(G \circ F)(Q_{\mathcal{A}}(I^\bullet)) \arrow[r, "\eta_{I^\bullet}"] & (\mathbf{R}G \circ \mathbf{R}F)(Q_{\mathcal{A}}(I^\bullet))
\end{tikzcd}$$

From the construction of the "structural" natural transformations α, β and ε [GM02, §III, 6.4] one sees that whenever these are applied to appropriately addapted resolutions, they become isomorphisms. In our case, since $\mathcal{I}_F$ is both F-adapted and $(G \circ F)$-adapted and $\mathcal{I}_G$ is G-adapted, this means that $\varepsilon_{I^\bullet}$, $\beta_{F(I^\bullet)}$ and $\alpha_{I^\bullet}$ are all isomorphisms, allowing us to conclude that $\eta_{I^\bullet}$ is an isomorphism as well. □

2.2.3 Higher derived functors

Classically derived functors of a left exact functor $F: \mathcal{A} \to \mathcal{B}$ first appeared as a sequence of additive functors $\mathcal{A} \to \mathcal{B}$. We introduce them in the following definition, using our established viewpoint.

Definition 2.35. For a right derived functor $\mathbf{R}F: \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ of a left exact functor $F: \mathcal{A} \to \mathcal{B}$, we define the *higher derived functors of* F as compositions

$$\mathbf{R}^i F := H^i \circ \mathbf{R}F: \quad \mathsf{D}^+(\mathcal{A}) \to \mathcal{B}$$

for all $i \in \mathbf{Z}$.

Frequently we will be interested in the image of a higher derived functor $\mathbf{R}^i F$ of just a single object A belonging to category $\mathcal{A}$. In that case we will always interpret it as a bounded complex concentrated in degree 0 according to the fully faithful functor $\mathcal{A} \to \mathsf{D}(\mathcal{A})$ of Proposition 2.10.

Example 2.36. For an object A of $\mathcal{A}$, equipped with an F-adapted class $\mathcal{I}$, we can compute that $\mathbf{R}^i F(A) \simeq 0$, for $i < 0$, and $\mathbf{R}^0 F(A) \simeq F(A)$. Indeed, consider an $\mathcal{I}$-resolution $I^\bullet$ of A. Then, as $I^i = 0$ for $i < 0$, we have that $\mathbf{R}^i F(A) = H^i(F(I^\bullet)) \simeq 0$ for $i < 0$. For the second claim, because $I^\bullet$ is quasi-isomorphic to A, we have $\ker(I^0 \to I^1) \simeq A$, and since F is left exact, we conclude that $F(A) \simeq \ker(F(I^0) \to F(I^1)) \simeq H^0(F(I^\bullet)) = \mathbf{R}^0 F(A)$.

Definition 2.37. Suppose the right derived functor $\mathbf{R}F$ exists. We say an object A of $\mathcal{A}$ is F*-acyclic*, if $\mathbf{R}^i F(A) \simeq 0$ for all $i \neq 0$.

A useful tool for gathering information about the action of the higher derived functors of a left exact functor F on objects will be the long exact sequence associated to F introduced in the following proposition. In view of this proposition we also interpret higher derived functors of F as measuring the extent to which F fails to be exact.

Proposition 2.38. *Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact functor and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then there exists a long exact sequence*

$$\begin{aligned} 0 \to F(A) \to F(B) \to F(C) \to \mathbf{R}^1 F(A) \to \mathbf{R}^1 F(B) \to \mathbf{R}^1 F(C) \to \cdots \\ \cdots \to \mathbf{R}^i F(A) \to \mathbf{R}^i F(B) \to \mathbf{R}^i F(C) \to \mathbf{R}^{i+1} F(A) \to \cdots . \end{aligned} \tag{2.10}$$

Proof. First, we describe how the short exact sequence gives rise to a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $\mathsf{D}^+(\mathcal{A})$, where $A^\bullet$, $B^\bullet$ and $C^\bullet$ are objects A, B and C, respectively, considered as complexes concentrated in degree 0. First denote the morphisms $f\colon A \to B$ and $g\colon B \to C$. Then the cone $C(f)^\bullet$ can be used as a peak for a roof spanned on $C^\bullet$ and $A[1]^\bullet$ as follows.

$$\begin{tikzcd}
C^\bullet & \cdots \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots \\
C(f)^\bullet \arrow[u] \arrow[d] & \cdots \arrow[r] & A \arrow[r, "f"] \arrow[u] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "d^0"] \arrow[u, "g"'] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
A[1]^\bullet & \cdots \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Then the morphism $C(f)^\bullet \to C^\bullet$ is a quasi-isomorphism. Indeed, since f is a monomorphism, we have $H^{-1}(C(f)^\bullet) \simeq 0$, thus the induced map on cohomology in degree -1 is the trivial morphism, which in this case an isomorphism. Since $B \simeq \ker d^0$, we have $H^0(C(f)^\bullet) \simeq \operatorname{coker} f$. As $0 \to A \to B \to C \to 0$ is exact, we know that g induces an isomorphism $\operatorname{coker} f \to C$, but this is precisely the morphism $H^0(C(f)^\bullet \to C^\bullet)$. Therefore $C(f)^\bullet \to C^\bullet$ is a quasi-isomorphism and the left roof

$$C^\bullet \longleftarrow C(f)^\bullet \longrightarrow A[1]^\bullet$$

represents a morphism $\phi\colon C^\bullet \to A[1]^\bullet$ in $\mathsf{D}^+(\mathcal{A})$. In $\mathsf{D}^+(\mathcal{A})$ we thus obtain a commutative diagram, where the vertical arrows are isomorphisms.

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, equal] & B^\bullet \arrow[r] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, "\sim"'] & A[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r, "\phi"] & A[1]^\bullet
\end{tikzcd}$$

This shows that the bottom triangle is distinguished.

We can now succesively apply the triangulated functor $\mathbf{R}F$ and the cohomological functor H^0 over the distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ to obtain the following long exact sequence in cohomology by Proposition 1.51.

$$\begin{aligned}
&0 \to H^0(\mathbf{R}F(A)) \to H^0(\mathbf{R}F(B)) \to H^0(\mathbf{R}F(C)) \to H^0(\mathbf{R}F(A)[1]) \to \cdots \\
&\cdots \to H^0(\mathbf{R}F(A)[i]) \to H^0(\mathbf{R}F(B)[i]) \to H^0(\mathbf{R}F(C)[i]) \to H^0(\mathbf{R}F(A)[i+1]) \to \cdots
\end{aligned}$$

The terms of this sequence are then readily seen to be isomorphic to the terms of (2.10). □

The rest of this section is devoted to proving some useful results relating F-acyclic and F-adapted objects.

Lemma 2.39. *Every object of an F-adapted class is also F-acyclic.*

Proof. We only have to recall Definition 2.37. An object I belonging to an F-adapted class $\mathcal{I}$ trivially forms its own $\mathcal{I}$-resolution $I^\bullet = (\cdots \to 0 \to I \to 0 \to \cdots)$, which allows us to compute $\mathbf{R}^iF(I) = H^i(\mathbf{R}F(I^\bullet)) = H^i(F(I^\bullet)) \simeq 0$, whenever $i \neq 0$ (and also $F(I)$ for $i = 0$). □

Proposition 2.40. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. Let $\mathcal{I} \subset \mathcal{A}$ be an F-adapted class of objects in $\mathcal{A}$. Then the class of all F-acyclic objects of $\mathcal{A}$, denoted by $\mathcal{I}_F$, is also F-adapted.*

Proof. We verify conditions (i)–(iii) of Definition 2.28. (i) Since the higher derived functors $\mathbf{R}^i F$ are additive, $\mathcal{I}_F$ is stable under finite sums. (ii) Holds by Lemma 2.39. (iii) Suppose $A^\bullet$ is an acyclic complex in $K^+(\mathcal{A})$ with $A^i \in \mathcal{I}_F$ for all i. Then using acyclicity of $A^\bullet$, we may break this complex up into a series of short exact sequences

$$0 \to \ker d^i \to A^i \to \ker d^{i+1} \to 0 \quad \text{ for } i \in \mathbf{Z}. \tag{2.11}$$

As our complex $A^\bullet$ is supported on $\mathbf{Z}_{\geq 0}$, the first interesting short exact sequence happens at index $i = 1$, where $\ker d^1 = \operatorname{im} d^0 \simeq A^0$ It yields the following long exact sequence associated to F

$$\begin{aligned} 0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to \mathbf{R}^1 F(A^0) \to \cdots \\ \cdots \to \mathbf{R}^i F(A^0) \to \mathbf{R}^i F(A^1) \to \mathbf{R}^i F(\ker d^2) \to \mathbf{R}^{i+1} F(A^0) \to \cdots . \end{aligned}$$

Exactness of this sequence together with F-acyclicity of A^0 and A^1 shows that $\ker d^2$ is also F-acyclic and by only considering the beginning few terms of the sequence, we obtain the short exact sequence

$$0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to 0.$$

After inductively applying the same reasoning for each of the original short exact sequences of (2.11) for $i > 1$, we are left with short exact sequences[13]

$$0 \to \ker F d^i \to F(A^i) \to \ker F d^{i+1} \to 0 \quad \text{ for } i \geq 1.$$

Collecting them all back together we conclude that the complex $F(A^\bullet)$ is acyclic. □

2.2.4 Ext functors

As a first application of the established theory of derived functors we consider the Hom-functor

$$\operatorname{Hom}_{\mathcal{A}}(A, -) \colon \mathcal{A} \to \mathsf{Mod}_k,$$

where A is an object of the abelian category $\mathcal{A}$. Assume that category $\mathcal{A}$ contains enough injectives. By Example 1.10, we know that $\operatorname{Hom}_{\mathcal{A}}(A, -)$ is a left exact functor, so we can right derive it, to obtain the functor

$$\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(A, -) \colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Mod}_k).$$

Taking the i-th cohomology, we obtain higher derived functors of $\operatorname{Hom}_{\mathcal{A}}(A, -)$, called the *Ext-functors*

$$\operatorname{Ext}^i_{\mathcal{A}}(A, -) := \mathbf{R}^i\operatorname{Hom}_{\mathcal{A}}(A, -) = H^i(\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(A, -)).$$

There exists a very beautiful connection relating the Ext-functors of category $\mathcal{A}$ and the Hom-functors of the bounded below derived category $\mathsf{D}^+(\mathcal{A})$ captured in the next proposion, given without proof, for in a moment we will present its generalized version.

[13] To be more precise, we have used that F is left exact in order to swap the order of F and ker to reach

$$\ker F d^i \simeq F(\ker d^i).$$

Proposition 2.41. *Let $\mathcal{A}$ be an abelian category with enough injectives and let A and B belong to $\mathcal{A}$. Then for all $i \in \mathbf{Z}$ there exist isomorphisms*

$$\operatorname{Ext}^i_{\mathcal{A}}(A, B) \simeq \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A, B[i]).$$

As a preliminary we introduce the *graded objects* of $\mathcal{A}$ to be $\mathbf{Z}$-indexed sequences $A^\bullet = (A^j)_{j\in\mathbf{Z}}$ of objects A^j of $\mathcal{A}$. A *morphism* $f\colon A^\bullet \to B^\bullet$ *of degree* $i \in \mathbf{Z}$ is a sequence $(f^j\colon A^j \to B^{i+j})_{j\in\mathbf{Z}}$ of morphisms of $\mathcal{A}$. The set of all degree i morphisms will be denoted with $\operatorname{Hom}^i_{\mathcal{A}}(A^\bullet, B^\bullet)$. Bundling up all this data, we obtain the **Z**-*graded category of* $\mathcal{A}$ denoted by $\mathcal{A}^{\mathbf{Z}}$, consisting of graded objects of $\mathcal{A}$ and morphisms of all integer degrees, thus

$$\operatorname{Hom}_{\mathcal{A}^{\mathbf{Z}}}(A^\bullet, B^\bullet) = \prod_{i\in\mathbf{Z}} \operatorname{Hom}^i_{\mathcal{A}}(A^\bullet, B^\bullet).$$

The category $\mathcal{A}^{\mathbf{Z}}$ can also be thought of as the functor category from the free category of $\mathbf{Z}$ to $\mathcal{A}$, hence the notation $\mathcal{A}^{\mathbf{Z}}$.

Definition 2.42. Let $A^\bullet$ and $B^\bullet$ be complexes of $\mathcal{A}$. We define the *Hom complex* $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ of k-modules given by terms

$$\operatorname{Hom}^i_{\mathcal{A}}(A^\bullet, B^\bullet) = \prod_{j\in\mathbf{Z}} \operatorname{Hom}_{\mathcal{A}}(A^j, B^{i+j}),$$

i.e. the set of all degree i maps from $A^\bullet$ to $B^\bullet$, considered as graded objects of $\mathcal{A}^{\mathbf{Z}}$, and differentials

$$\delta^i\colon \operatorname{Hom}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \to \operatorname{Hom}^{i+1}_{\mathcal{A}}(A^\bullet, B^\bullet), \quad \delta^i(f) = \left(d_B^{i+j} \circ f^j + (-1)^{i+1} f^{j+1} \circ d_A^j\right)_{j\in\mathbf{Z}}.$$

In a moment, when specifying the dependence of the differential δ^i on the complex $B^\bullet$ will become important, we will denote it by δ^i_B instead.

Set $\operatorname{Hom}^\bullet(A^\bullet, -)(B^\bullet) = \operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ and for a chain map $u\colon B^\bullet \to C^\bullet$ in $\mathsf{Ch}(\mathcal{A})$ define $u_* = \operatorname{Hom}^\bullet(A^\bullet, -)(u)\colon \operatorname{Hom}^\bullet(A^\bullet, B^\bullet) \to \operatorname{Hom}^\bullet(A^\bullet, C^\bullet)$ to be the chain map given by the collection $(u^i_*)_{i\in\mathbf{Z}}$, where u^i_* sends $f = (f^j)_{j\in\mathbf{Z}} \in \operatorname{Hom}^i(A^\bullet, B^\bullet)$ to $(u^{i+j} \circ f^j)_{j\in\mathbf{Z}}$. Then we obtain a functor

$$\operatorname{Hom}^\bullet(A^\bullet, -)\colon \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Mod}_k).$$

Indeed, clearly u_* is a graded morphism of degree 0, which moreover commutes with the differentials, according to the following computation

$$\begin{aligned}
u^{i+1}_*\left(\delta^i_B(f)\right) &= u^{i+1}_*\left(\left(d_B^{i+j} f^j + (-1)^{i+1} f^{j+1} d_A^j\right)_j\right) \\
&= \left(u^{i+j+1} d_B^{i+j} f^j + (-1)^{i+1} u^{i+j+1} f^{j+1} d_A^j\right)_j \\
&= \left(d_C^{i+j} u^{i+j} f^j + (-1)^{i+1} u^{i+j+1} f^{j+1} d_A^j\right)_j \\
&= \delta^i_C\left(\left(u^{i+j} f^j\right)_j\right) \\
&= \delta^i_C\left(u^i_*(f)\right),
\end{aligned}$$

for all $f \in \operatorname{Hom}^i(A^\bullet, B^\bullet)$. While not difficult to check, the verification of functoriality is omitted.

Next, we would like $\operatorname{Hom}^\bullet(A^\bullet, -)$ to descend to a homotopy functor $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{Mod}_k)$. This is in fact so, as a null-homotopic chain map $u\colon B^\bullet \to C^\bullet$ is sent to a null-homotopic chain map $u_* \simeq 0$. Indeed, suppose homotopy $h \in \mathrm{Hom}^{-1}(B^\bullet, C^\bullet)$ witnesses $u \simeq 0$, then for all $i \in \mathbf{Z}$, we have

$$u^i = d_C^{i-1} \circ h^i + h^{i+1} \circ d_B^i.$$

We let h_*^i denote the morphism $\mathrm{Hom}^i(A^\bullet, B^\bullet) \to \mathrm{Hom}^{i-1}(A^\bullet, C^\bullet)$, given by sending f to $(h^{i+j} \circ f^j)_{j\in\mathbf{Z}}$, and then compute

$$\begin{aligned}
(\delta_C^{i-1} \circ h_*^i + h_*^{i+1} \circ \delta_B^i)(f) &= \delta_C^{i-1}\left((h^{i+j} f^j)_j\right) + h_*^{i+1}\left((d_B^{i+j} f^j + (-1)^{i+1} f^{j+1} d_A^j)_j\right) \\
&= \left(d_C^{i+j-1} h^{i+j} f^j + (-1)^i h^{i+j+1} f^{j+1} d_A^j\right)_j + \\
&\qquad + \left(h^{i+j+1} d_B^{i+j} f^j + (-1)^{i+1} h^{i+j+1} f^{j+1} d_A^j\right)_j \\
&= \left(d_C^{i+j-1} h^{i+j} f^j + h^{i+j+1} d_B^{i+j} f^j\right)_j \\
&= \left((d_C^{i+j-1} h^{i+j} + h^{i+j+1} d_B^{i+j}) \circ f^j\right)_j \\
&= \left(u^{i+j} \circ f^j\right)_j \\
&= u_*^i(f).
\end{aligned}$$

Thus the homotopy $h_* = (h_*^i)_{i\in\mathbf{Z}}$ whitnesses $u_* \simeq 0$, allowing us to conclude that

$$\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{Mod}_k)$$

is well defined. The preceding functor is also triangulated, because it commutes with the translation functors and it sends distinguished triangles of $\mathsf{K}(\mathcal{A})$ to distinguished triangles of $\mathsf{K}(\mathsf{Mod}_k)$. This is because of the following isomorphism of chain complexes of k-modules,

$$C(u_*)^\bullet = C(\mathrm{Hom}^\bullet(A^\bullet, -)(u))^\bullet \simeq \mathrm{Hom}^\bullet(A^\bullet, -)(C(u)^\bullet) = \mathrm{Hom}^\bullet(A^\bullet, C(u)^\bullet),$$

which holds for any chain map $u\colon B^\bullet \to C^\bullet$. Indeed, we first have for all $i \in \mathbf{Z}$ the following isomorphisms going between the terms of each complex

$$\begin{aligned}
\mathrm{Hom}^i(A^\bullet, C(u)^\bullet) &= \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, C(u)^{i+j}) \\
&\simeq \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1} \oplus C^{i+j}) \\
&\simeq \prod_{j\in\mathbf{Z}} \left(\mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1}) \oplus \mathrm{Hom}_{\mathcal{A}}(A^j, C^{i+j})\right) \\
&= \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1}) \oplus \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, C^{i+j}) \\
&= \mathrm{Hom}^{i+1}(A^\bullet, B^\bullet) \oplus \mathrm{Hom}^i(A^\bullet, C^\bullet) \\
&= C(u_*)^i.
\end{aligned}$$

To show that this collection of isomorphisms witnesses two complexes being isomorphic, we have to show that they are also chain maps, i.e. they commute with the differentials. Let

$$\delta_{C(u)}^i\colon \mathrm{Hom}^i(A^\bullet, C(u)^\bullet) \to \mathrm{Hom}^{i+1}(A^\bullet, C(u)^\bullet)$$

denote the differential of $\mathrm{Hom}^\bullet(A^\bullet, C(u)^\bullet)$ and

$$d^i\colon C(u_*)^i \to C(u_*)^{i+1} \qquad d^i = \begin{pmatrix} -\delta_B^{i+1} & 0 \\ u_*^{i+1} & \delta_C^i \end{pmatrix}$$

the differential of $C(u_*)^\bullet$. Pick an element $f = ((f_B^j, f_C^j))_{j\in\mathbf{Z}} \in \mathrm{Hom}^i(A^\bullet, C(u)^\bullet)$. Under the isomorphism above f is sent to the pair $(f_B, f_C) \in C(u_*)^i$, where

$$f_B = (f_B^j\colon A^j \to B^{i+j+1})_{j\in\mathbf{Z}} \quad \text{and} \quad f_C = (f_C^j\colon A^j \to C^{i+j})_{j\in\mathbf{Z}}.$$

Then the computation below proves that the isomorphisms commute with the differentials, because $\delta^i_{C(u)}(f)$ and $d^i((f_B, f_C))$ coincide at every coordinate $j \in \mathbf{Z}$.

$$\begin{aligned}
[\delta^i_{C(u)}(f)]_j &= d^{i+j}_{C(u)} \circ f^j + (-1)^{i+1} f^{j+1} \circ d^j_A \\
&= \begin{pmatrix} d^{i+j}_{B[1]} & 0 \\ u^{i+j+1} & d^{i+j}_C \end{pmatrix} \circ \begin{pmatrix} f_B^j \\ f_C^j \end{pmatrix} + (-1)^{i+1} \begin{pmatrix} f_B^{j+1} \\ f_C^{j+1} \end{pmatrix} \circ d^j_A \\
&= \begin{pmatrix} d^{i+j}_{B[1]} f_B^j + (-1)^{i+1} f_B^{j+1} d^j_A \\ u^{i+j+1} f_B^j + d^{i+j}_C f_C^j + (-1)^{i+1} f_C^{j+1} d^j_A \end{pmatrix} \\
&= \begin{pmatrix} [\delta^i_{B[1]}(f_B)]_j \\ [u_*^{i+1}(f_B)]_j + [\delta^i_C(f_C)]_j \end{pmatrix} \\
&= \left[\begin{pmatrix} -\delta_B^{i+1} & 0 \\ u_*^{i+1} & \delta_C^i \end{pmatrix} \begin{pmatrix} f_B \\ f_C \end{pmatrix} \right]_j
\end{aligned}$$

Here we have used the notation $[-]_j$ to mean the j-th component of a tuple.

At last, for $A^\bullet$ belonging to $\mathsf{K}^-(\mathcal{A})$ i.e. bounded from *above*[14], we have a triangulated functor $\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathsf{Mod}_k)$, which we may derive, in accordance with Remark 2.26, of course assuming $\mathcal{A}$ contains enough injectives, to obtain

$$\mathbf{R}\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{D}^+(\mathcal{A}) \longrightarrow \mathsf{D}^+(\mathsf{Mod}_k).$$

Its higher derived counterpart will be denoted by $\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, -) := \mathbf{R}^i\mathrm{Hom}^\bullet(A^\bullet, -)$.

Remark 2.43. Behind the scenes of this whole procedure one actually consideres an additive bifunctor $F\colon \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$, with $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ abelian, where the latter is assumed to contain countable products. From this bifunctor we induce another additive bifunctor on the level of chain complexes $F_\pi\colon \mathsf{Ch}(\mathcal{A}) \times \mathsf{Ch}(\mathcal{A}') \to \mathsf{Ch}(\mathcal{A}'')$ via the total complex of a double complex as described in [KS06, §11.6]. It then turns out that F_π induces a well defined triangulated bifunctor on the level of the homotopy category

$$\mathsf{K}(\mathcal{A}) \times \mathsf{K}(\mathcal{A}') \to \mathsf{K}(\mathcal{A}'').$$

Theorem 2.44. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for any two bounded complexes $A^\bullet$ and $B^\bullet$ belonging to $\mathsf{D}^b(\mathcal{A})$ there are natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B[i]^\bullet).$$

Remark 2.45. Notice that in the theorem we require the complex $A^\bullet$ to be bounded even though we have so far only assumed it to be bounded from above. This is necessary as we will use Lemma 2.20, which requires $A^\bullet$ to be bounded from below.

Lemma 2.46. *Let $A^\bullet$ and $B^\bullet$ be chain complexes in $\mathsf{K}(\mathcal{A})$ and let $i \in \mathbf{Z}$. Then*

$$H^i(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) = \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B[i]^\bullet).$$

[14] This assumption cannot be omitted, for otherwise we cannot claim, that the functor $\mathrm{Hom}^\bullet(A^\bullet, -)$ maps into the bounded *below* homotopy category $\mathsf{K}^+(\mathsf{Mod}_k)$.

Proof. This is practically the definition of $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B[i]^\bullet)$. The cohomology k-module $H^i(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$ is defined to be the quotient $\ker \delta_B^i / \operatorname{im} \delta_B^{i-1}$. Thus let us identify $\ker \delta_B^i$ and $\operatorname{im} \delta_B^{i-1}$. Notice, that for $f \in \mathrm{Hom}^i(A^\bullet, B^\bullet)$ we have

$$\delta^i(f) = \left(d_B^{i+j} \circ f^j + (-1)^{i+1} f^{j+1} \circ d_A^j\right)_{j\in\mathbf{Z}} = (-1)^i \left(d_{B[i]}^j \circ f^j - f^{j+1} \circ d_A^j\right)_{j\in\mathbf{Z}}.$$

Hence $\ker \delta_B^i$ is the set of all chain maps $A^\bullet \to B[i]^\bullet$, and $\operatorname{im} \delta_B^{i-1}$ the set of all null-homotopic chain maps $A^\bullet \to B[i]^\bullet$. □

Proof of Proposition 2.44. Let $B^\bullet \to I^\bullet$ be an injective resolution for $B^\bullet$. We follow the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B[i]^\bullet) &= \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, B[i]^\bullet) \\
&\simeq \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I[i]^\bullet) \\
&\simeq \mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I[i]^\bullet) && \text{(Lemma 2.20)} \\
&= H^i(\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)) && \text{(Lemma 2.46)} \\
&= \mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet).
\end{aligned}$$

□

3 Derived categories in geometry

This chapter will be devoted to applying results of Chapter 2 to the abelian category $\mathsf{coh}(X)$ of coherent sheaves on X. We will first define the bounded derived category of coherent sheaves $\mathsf{D}^b(X)$ of a scheme X over k and discuss how this category fits into the broader context of other derived categories. We prove that under some assumptions $\mathsf{D}^b(X)$ is indecomposable and admits a spanning class, which will play an important role later on. Generalizing Serre duality we also equip $\mathsf{D}^b(X)$ with a Serre functor

The second section will cover the construction of various different derived functors having their origin in geometry. These will include the global sections functor, the push-forward along a morphism, different Hom-functors and Ext-functors, the tensor product and Tor-functors, and the pull-back along a morphism. We will conclude this chapter by providing a collection of results showcasing relationships that hold between the derived functors of the previous section. These will serve as convenient computational tools especially in Chapter 4 on Fourier-Mukai transforms.

3.1 Derived category of coherent sheaves

One of the main invariants of a scheme X over k is its category of coherent sheaves $\mathsf{coh}(X)$. We can think of this category as a slight extension of the category of locally free $\mathcal{O}_X$-modules of finite rank also known as k-vector bundles on X in the sense that $\mathsf{coh}(X)$ is abelian, where as the former generally is not. In this chapter we restrict ourself to noetherian schemes and later to smooth projective varieties over an algebraically closed field k. Our primary object of study will be the bounded derived category of coherent sheaves

$$\mathsf{D}^b(X) := \mathsf{D}^b(\mathsf{coh}(X)).$$

In this section some of the results will have to be given without proof but we will provide precise references when required. Certain results will serve as the groundwork on which the theory is then further developed, for example statements like certain functors map coherent sheaves into coherent shaves, while others, like Grothendieck-Verdier duality, lie well outside the scope of this thesis, but are relevant for establishing certain relations among the derived functors. We also mainly focus on showcasing the methods and interactions between the derived functors and not that much on being as general as one can be.

Notation. Copying Huybrechts [Huy06], we will denote by $\mathcal{H}^i(\mathcal{F}^\bullet)$ the i-th cohomology of a complex $\mathcal{F}^\bullet$, not to mistake it for sheaf cohomology $H^i(X, \mathcal{F})$ of X with coefficients in a sheaf $\mathcal{F}$.

As observed in Chapter 2 the construction of derived functors relied heavily on the existence of some special class of objects in the domain category. The most special of them all being the class of injective objects. As the category of coherent sheaves on X often does not contain enough injectives[15], we are forced to expand our scope. Luckily, the contrary is true, if we restrict ourself to noetherian schemes and replace coherent sheaves with quasi-coherent ones.

Proposition 3.1. *The category of quasi-coherent sheaves $\mathsf{qcoh}(X)$ of a noetherian scheme X contains enough injectives.*

[15]Intuitively this statement may be mirrored with the analogous statement that injective A-modules almost never appear to be finitely generated.

Proof. This is a consequence of [Har66, §II, Theorem 7.18], which states that every quasi-coherent sheaf on a noetherian scheme X can be embedded into a quasi-coherent sheaf, which is an injective $\mathcal{O}_X$-module. □

Proposition 3.2. *For a noetherian scheme X the inclusion $\mathsf{D}^b(X) \hookrightarrow \mathsf{D}^b(\mathsf{qcoh}(X))$ induces an equivalence of triangulated categories*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(X)),$$

where $\mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(X))$ is the full triangulated subcategory of $\mathsf{D}^b(\mathsf{qcoh}(X))$, spanned on bounded complexes of quasi-coherent shaves on X with coherent cohomology.

Proof. For a proof see [Huy06, §3, Proposition 3.5]. □

Proposition 3.3. *Let X be a smooth projective variety over k. Then any complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ is isomorphic in $\mathsf{D}^b(X)$ to a bounded complex $\mathcal{E}^\bullet$ of locally free $\mathcal{O}_X$-modules of finite rank also known as* vector bundles*.*

Proof. To prove this proposition we will need the following two facts concerning locally free sheaves on smooth projective varieties.

- ▷ Any coherent sheaf $\mathcal{F}$ on X is a quotient of a locally free $\mathcal{O}_X$-module of finite rank $\mathcal{E}$, i.e. there is an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$ [Har77, §II, Corollary 5.18].
- ▷ Let n denote the dimension of X. For any coherent sheaf $\mathcal{F}$ on X admitting an exact sequence of coherent sheaves

  $$0 \to \mathcal{E}^{-n} \to \mathcal{E}^{-n+1} \to \cdots \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{F} \to 0.$$

  If $\mathcal{E}^0$, $\mathcal{E}^{-1}$,..., $\mathcal{E}^{-n+1}$ are locally free, the sheaf $\mathcal{E}^{-n}$ is locally free as well. See [Ful98, §B.8.3] and note that this is essentially a local statement since a coherent sheaf $\mathcal{E}$ is a vector bundle if and only if all its stalks $\mathcal{E}_x$ are free $\mathcal{O}_{X,x}$-modules [Wei13, §1, Lemma 5.1.3].

Pick a bounded complex of coherent sheaves $\mathcal{F}^\bullet$. Without loss of generality assume $\mathcal{F}^i \simeq 0$ for $i \leq 1$, otherwise replace $\mathcal{F}^\bullet$ by an appropriate shift of itself. By the first fact and a dual argument to the proof of Proposition 2.14 we see that there exists a complex $\mathcal{E}^\bullet$ of vector bundles bounded from above and a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{E}^{-n} \arrow[r] \arrow[d] & \mathcal{E}^{-n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E}^0 \arrow[r] \arrow[d] & \mathcal{E}^1 \arrow[r] \arrow[d] & \mathcal{E}^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E}^m \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathcal{F}^2 \arrow[r] & \cdots \arrow[r] & \mathcal{F}^m \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Hence $\mathcal{H}^i(\mathcal{E}^\bullet) \simeq 0$ for $i \leq 1$ and the right truncation $\tau_{\leq 1}(\mathcal{E}^\bullet)$ is acyclic. Truncating once more on the *left* at degree $-n$, where n is the dimension of X, we obtain an exact sequence $\tau_{\geq -n}(\tau_{\leq 1}(\mathcal{E}^\bullet))$, depicted below

$$0 \to \operatorname{coker} d_{\mathcal{E}}^{-n-1} \to \mathcal{E}^{-n+1} \to \cdots \to \mathcal{E}^0 \to \ker d_{\mathcal{E}}^1 \to 0.$$

This is now a resolution of a coherent sheaf $\ker d_{\mathcal{E}}^1$ by coherent sheaves, where $\mathcal{E}^0$, $\mathcal{E}^{-1}$,..., $\mathcal{E}^{-n+1}$ are vector bundles. By the second fact we see that $\operatorname{coker} d_{\mathcal{E}}^{-n-1}$ is then a vector bundle, therefore $\tau_{\geq -n}(\mathcal{E}^\bullet)$ is a complex of vector bundles quasi-isomorphic to $\mathcal{E}^\bullet$ and consequently also quasi-isomorphic to $\mathcal{F}^\bullet$. □

Definition 3.4. Let X be a scheme. The *support* of an $\mathcal{O}_X$-module $\mathcal{F}$ is defined to be the set

$$\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \not\simeq 0\}.$$

The *support* of a complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ is defined to be the union

$$\operatorname{supp}(\mathcal{F}^\bullet) = \bigcup_{i\in\mathbf{Z}} \operatorname{supp}(\mathcal{H}^i(\mathcal{F}^\bullet)).$$

Remark 3.5. The support $\operatorname{supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F} \in \mathsf{coh}(X)$ is a closed subset of X, therefore the support of any complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$, being a finite union of closed sets, is closed as well.

Theorem 3.6 ([Huy06, §3, Proposition 3.10])**.** *The bounded derived category $\mathsf{D}^b(X)$ of a connected scheme X over k is indecomposable.*

Proof. For the sake of contradiction, assume $\mathsf{D}^b(X)$ decomposes into triangulated subcategories $\mathcal{D}_0$ and $\mathcal{D}_1$. We will first prove that the structure sheaf $\mathcal{O}_X$ belongs to either $\mathcal{D}_0$ or $\mathcal{D}_1$. By Proposition 1.29, the structure sheaf $\mathcal{O}_X$ is isomorphic to a direct sum $\mathcal{F}_0^\bullet \oplus \mathcal{F}_1^\bullet$, with complexes $\mathcal{F}_i^\bullet$ belonging to $\mathcal{D}_i$. Since $\mathcal{O}_X$ clearly has cohomology supported only in degree 0, the same is true for $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$, thus, by Proposition 2.10, we may replace them with coherent sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. By the same proposition this means that there is an isomorphism $\mathcal{O}_X \simeq \mathcal{F}_0 \oplus \mathcal{F}_1$ in $\mathsf{coh}(X)$. As the functor computing stalks of a sheaf at any given point $x \in X$ is exact, we obtain a direct sum decomposition $\mathcal{O}_{X,x} \simeq \mathcal{F}_{0,x} \oplus \mathcal{F}_{1,x}$, enabling us to write

$$X = \operatorname{supp}(\mathcal{O}_X) = \operatorname{supp}(\mathcal{F}_0) \cup \operatorname{supp}(\mathcal{F}_1).$$

As $\mathcal{F}_i$ are coherent sheaves, their supports are closed in X, according to Remark 3.5. Since $\mathcal{O}_{X,x}$ is a local ring, and therefore *indecomposable*[16], for every $x \in X$, either $\mathcal{F}_{0,x} = 0$ or $\mathcal{F}_{1,x} = 0$, meaning $\operatorname{supp}(\mathcal{F}_0) \cap \operatorname{supp}(\mathcal{F}_1) = \varnothing$. By connectedness of X, it follows that either $\mathcal{F}_0 = 0$ or $\mathcal{F}_1 = 0$, thus, without loss of generality, we may assume that $\mathcal{O}_X$ belongs to $\mathcal{D}_0$.

Secondly picking any closed point $x \in X$, we see that $k(x)$ belongs to $\mathcal{D}_0$ as well. Indeed, as with $\mathcal{O}_X$, we may decompose $k(x)$ as $k(x) \simeq \mathcal{F}_0 \oplus \mathcal{F}_1$ in $\mathsf{coh}(X)$ for some coherent sheaves $\mathcal{F}_i$ belonging to $\mathcal{D}_i$. Then firstly $\operatorname{supp}(\mathcal{F}_i) \subset \{x\}$ and computing the stalk at x shows either $\mathcal{F}_0$ or $\mathcal{F}_1$ is trivial. If $\mathcal{F}_1 = 0$, $k(x)$ belongs to $\mathcal{D}_0$, but if $\mathcal{F}_0 = 0$, then the isomorphism $k(x) \simeq \mathcal{F}_1$ allows us to construct a non-zero homomorphism of sheaves $\mathcal{O}_X \to k(x) \to \mathcal{F}_1$, contradicting orthogonality of $\mathcal{D}_0$ and $\mathcal{D}_1$. In the latter composition the morphism $\mathcal{O}_X \to k(x)$ comes from the morphism $\operatorname{Spec} k \to X$, which includes the closed point x into X.

To finish off the proof, we now show that $\mathcal{D}_1$ contains only trivial objects. Suppose $\mathcal{F}^\bullet$ is an object of $\mathcal{D}_1$. If $\mathcal{F}^\bullet$ is not trivial, let $m \in \mathbf{Z}$ be maximal with the property $\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. Then $\operatorname{supp}(\mathcal{H}^m(\mathcal{F}^\bullet)) \neq \varnothing$ and there is a *closed* point $x \in X$, which is also contained in $\operatorname{supp}(\mathcal{H}^m(\mathcal{F}^\bullet))$[17]. As before this means there is a non-zero map $\mathcal{H}^m(\mathcal{F}^\bullet) \to k(x)$. Using this we will construct a non-zero morphism $\mathcal{F}^\bullet \to k(x)[m]$ in $\mathsf{D}^b(X)$, once again

[16] A ring R is *indecomposable*, if it can not be written in the form $R \simeq I \oplus J$ for two non-zero R-modules I and J. If such a decomposition exists, I and J can be identified with *proper* ideals of R. Now assuming R is local with a maximal ideal $\mathfrak{m}$, we end up with a contradiction $R \subseteq I + J \subseteq \mathfrak{m} \subsetneq R$.

[17] If $x_0 \in \operatorname{supp}(\mathcal{H})$ is not a closed point, consider it inside an affine open chart $U \simeq \operatorname{Spec} A$. Then $\operatorname{supp}(\mathcal{H}) \cap U$ corresponds in $\operatorname{Spec} A$ to a non-empty closed subset of the form $V(I)$ for some ideal $I \trianglelefteq A$. Since every ideal I is contained in some maximal ideal, we obtain a closed point of $V(I)$ in this way.

contradicting orthogonality of $\mathcal{D}_0$ and $\mathcal{D}_1$. By (2.3), we know the inclusion $\tau_{\leq m}(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet$ is a quasi-isomorphism. If we view $\mathcal{H}^m(\mathcal{F}^\bullet)$ as $\operatorname{coker}(\mathcal{F}^{m-1} \to \ker d^m)$, we obtain a commutative diagram

$$\begin{array}{lccccccccc}
\tau_{\leq m}(\mathcal{F}^\bullet) & \cdots \longrightarrow & \mathcal{F}^{m-1} & \longrightarrow & \ker d^m & \longrightarrow & 0 & \longrightarrow \cdots \\
& & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{H}^m(\mathcal{F}^\bullet)[m] & \cdots \longrightarrow & 0 & \longrightarrow & \mathcal{H}^m(\mathcal{F}^\bullet) & \longrightarrow & 0 & \longrightarrow \cdots ,
\end{array}$$

which gives rise to a chain map $\tau_{\leq m}(\mathcal{F}^\bullet) \to \mathcal{H}^m(\mathcal{F}^\bullet)[m]$. The left roof (3.1) below then represents a non-zero morphism $\mathcal{F}^\bullet \to k(x)[m]$ in $\mathsf{D}^b(X)$, as it is clearly non-zero once the m-th cohomology functor is applied.

$$\mathcal{F}^\bullet \xleftarrow{\sim} \tau_{\leq m}(\mathcal{F}^\bullet) \to \mathcal{H}^m(\mathcal{F}^\bullet) \to k(x)[m] \tag{3.1}$$

As this cannot be so, $\mathcal{F}^\bullet$ has trivial cohomology making it and the category $\mathcal{D}_1$ trivial. This contradicts our initial assumption of $\mathcal{D}_0$ and $\mathcal{D}_1$ forming a decomposition of $\mathsf{D}^b(X)$ and proves that $\mathsf{D}^b(X)$ is indecomposable. □

Serre functors

Let X be a smooth projective scheme over k. In order to define a Serre functor on $\mathsf{D}^b(X)$ we first establish finite-dimensionality of its Hom-sets.

Proposition 3.7. *Let X be a projective variety over k. For any two bounded complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ the Hom-set $\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is a finite dimensional k-vector space.*

The above proposition relies on a finiteness result of Serre found in [Har77, §III, Theorem 5.2]. As we will actually only need a very special case of this result, we also include a separate reference to the special case [Har77, §II, Theorem 5.19].

Theorem 3.8. *Let X be a projective variety over k and $\mathcal{F}$ a coherent sheaf on X. Then $H^i(X, \mathcal{F})$ are finite dimensional k-vector spaces for all $i \in \mathbf{Z}$.*

Proof of Proposition 3.7. Since $\mathsf{D}^b(X) \to \mathsf{D}^b(\mathsf{qcoh}(X))$ is fully faithful and $\mathsf{qcoh}(X)$ contains enough injectives, Proposition 2.44, tells us that

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \simeq \operatorname{Ext}^0_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = H^0(\operatorname{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)).$$

Recall that $\operatorname{Hom}^i_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_{j \in \mathbf{Z}} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}^j, \mathcal{F}^{i+j})$, where the direct sum is actually finite due to $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ being bounded complexes. Momentarily borrowing the internal Hom of two sheaves from Section 3.2.3, [EGA, II, §9, Proposition 9.1.1] lets us state that the internal Hom of two coherent sheaves is again coherent. Thus for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ the module of global sections $\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is finite dimensional by Theorem 3.8. In turn this implies $\operatorname{Hom}^i_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ are finitely dimensional k-vector spaces for all $i \in \mathbf{Z}$, meaning $H^0(\operatorname{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet))$ is finite dimensional as well. □

Recall, that for a smooth and projective scheme X over k of dimension n, Serre duality [Har77, §III, Theorem 7.6] states that for any coherent sheaf $\mathcal{F}$ on X there is an isomorphism

$$\operatorname{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \simeq H^i(X, \mathcal{F})^*, \tag{3.2}$$

where ω_X denotes the canonical bundle of X i.e. the n-th exterior power of the sheaf of Kähler differentials, $\omega_X = \bigwedge^n \Omega_X$. Generalizing Serre's duality theorem for smooth projective schemes over k to the context of complexes leads us to define Serre functors. This also clarifies why we called such functors *Serre functors* already in Section 1.2, where no geometry was present.

Definition 3.9. For a smooth and projective variety X over k define the *Serre functor* $S_X \colon \mathsf{D}^b(X) \to \mathsf{D}^b(X)$ to be the composition

$$S_X = (-) \otimes_{\mathcal{O}_X} \omega_X[\dim X].$$

Remark 3.10. This remark serves as a quick justification, why S_X may be defined in this way. The canonical bundle ω_X is a line bundle, thus in particular flat, so the additive functor $(-) \otimes_{\mathcal{O}_X} \omega_X \colon \mathsf{coh}(X) \to \mathsf{coh}(X)$ is exact. The induced homotopy functor $\mathsf{K}^b(\mathsf{coh}(X)) \to \mathsf{K}^b(\mathsf{coh}(X))$ then satisfies the conditions of Lemma 2.23 and therefore induces a triangulated functor

$$(-) \otimes_{\mathcal{O}_X} \omega_X \colon \mathsf{D}^b(X) \to \mathsf{D}^b(X)$$

by Proposition 2.24. The Serre functor S_X is then obtained by composing the latter functor with the translation functor $\dim X$ times.

Theorem 3.11 (Serre duality for derived categories)**.** *Let X be a smooth projective variety over a field k and let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ belong to $\mathsf{D}^b(X)$. The functor S_X is a Serre functor in the sense of Definition 1.32 i.e. there is an isomorphism*

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}^\bullet, S_X(\mathcal{F}^\bullet))^*,$$

which is natural in $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$.

Proof. See [Huy06, §3.4, Remark 3.37] for an elegant proof using Grothendieck–Verdier duality. One is able to derive this version of Serre duality also from the classical one (3.2). $\square$

A spanning class

Suppose X is a variety over an algebraically closed field k and let $x \in X$ be a closed point. Then we know it is a k-point and there is a corresponding morphism of schemes over k, namely $x\colon \operatorname{Spec} k \to X$. We define the skyscraper sheaf $k(x)$ to be the push-forward of the structure sheaf $\mathcal{O}_{\operatorname{Spec} k}$ along the map x,

$$k(x) := x_* \mathcal{O}_{\operatorname{Spec} k}.$$

The set of all such skyscrapers will turn out to form a spanning class for the triangulated category $\mathsf{D}^b(X)$, when X is smooth and projective. The following proposition can be found in [Huy06, §3, Proposition 3.17] or in Bridgeland's paper [Bri98, Example 2.2]. As we shall see, it is proven by a very neat application of the spectral sequence (A.10).

Proposition 3.12. *Suppose X is a smooth projective variety over k. The set of all skyscraper sheaves $k(x)$, for closed points $x \in X$, forms a spanning class of the bounded derived category $\mathsf{D}^b(X)$.*

Proof. By virtue of contradiction we will show that for any non-trivial complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ there are closed points x_0, $x_1 \in X$ and integers i_0, $i_1 \in \mathbf{Z}$, for which

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet, k(x_0)[i_0]) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{\mathsf{D}^b(X)}(k(x_1), \mathcal{F}^\bullet[i_1]) \neq 0.$$

By Serre duality 3.11, we see it suffices to find only x_0 and i_0, since

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}^b(X)}(k(x_1), \mathcal{F}^\bullet[i_1]) &\simeq \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet[i_1], k(x) \otimes_{\mathcal{O}_X} \omega_X[\dim X])^* \\ &\simeq \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet, k(x)[\dim X - i_1])^*. \end{aligned}$$

As $\mathcal{F}^\bullet$ is non-trivial, let $m \in \mathbf{Z}$ be maximal with the property that $\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. As in the proof of Theorem .16, pick a closed point in the non-empty support of $\mathcal{H}^m(\mathcal{F}^\bullet)$, for which we then know there is a non-zero morphism $\mathcal{H}^m(\mathcal{F}^\bullet) \to k(x)$ in $\mathsf{D}^b(X)$. Consider now the spectral sequence (A.10)

$$E_2^{p,q} = \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{H}^{-q}(\mathcal{F}^\bullet), k(x)[p]) \Rightarrow \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet, k(x)[p+q]) = E^{p+q}.$$

This is a spectral sequence lying in the right half plane, since $E_2^{p,q} = 0$ for $p < 0$ as negative Ext-modules are always trivial. A portion of the second page of the spectral sequence is depicted below.

$$\begin{tikzcd}
0 \arrow[ddr] & * & * & * \\
0 & * & * & * \\
0 & E_2^{0,-m} \arrow[ddr] & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{tikzcd}$$

The highlighted arrows are the two trivial differentials, mapping into and out of $E_2^{0,-m}$, meaning that $E_3^{0,-m}$ is isomorphic to $E_2^{0,-m}$, thus also non-trivial. The same may be deduced for the $(0,-m)$-th therm of each page, allowing us to conclude that $E_\infty^{0,-m}$ is non-trivial. The filtration for $E^{-m} = \mathrm{Hom}_{\mathsf{D}^b(X)}(\mathcal{F}^\bullet, k(x)[-m])$ therefore contains a non-trivial intermediary quotient, making it itself non-trivial and proving our claim. $\square$

3.2 Derived functors in algebraic geometry

In this section we will derive some functors occurring in algebraic geometry To derive these functors we will pursue one of the two following options. Either deal with injective sheaves and access the realm of quasi-coherent sheaves, following the first part of Subsection 2.2.2, or introduce certain special classes of coherent sheaves depending on the functor we wish to derive, which are going to take the role of adapted classes, mirroring what was done in the second part of that subsection.

3.2.1 Global sections functor

Arguably the most common functor in algebraic geometry is the global sections functor. Associated to a noetherian scheme X, the functor of *global sections*

$$\Gamma \colon \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_k$$

assigns to each $\mathcal{O}_X$-module $\mathcal{F}$ its module of global sections $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ and to a morphism of $\mathcal{O}_X$-modules $\alpha\colon \mathcal{F} \to \mathcal{G}$ a homomorphism $\alpha_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$. By abuse of notation we use Γ to also denote the restrictions of the global sections functor to subcategories $\mathsf{qcoh}(X)$ and $\mathsf{coh}(X)$ of $\mathsf{Mod}_{\mathcal{O}_X}$. When we want to emphasise that Γ is associated to a scheme X, we write $\Gamma(X, -)$ to denote the global sections functor on X.

As is commonly known, Γ is a left exact functor, so our aim is to construct its right derived counterpart. Due to $\mathsf{coh}(X)$ not having enough injectives, we resort to $\mathsf{qcoh}(X)$[18], on which we may define

$$\mathbf{R}\Gamma\colon \mathsf{D}^+(\mathsf{qcoh}(X)) \to \mathsf{D}^+(\mathsf{Mod}_k),$$

according to method 1 of Subsection 2.2.2. Precomposing the latter functor with the inclusion $\mathsf{D}^+(\mathsf{coh}(X)) \to \mathsf{D}^+(\mathsf{qcoh}(X))$ leaves us with the derived functor

$$\mathbf{R}\Gamma\colon \mathsf{D}^+(\mathsf{coh}(X)) \to \mathsf{D}^+(\mathsf{Mod}_k).$$

Utilizing the cohomology functors, we obtain higher derived functors of Γ, denoted by

$$\mathbf{R}^i\Gamma = H^i(X, -),$$

for $i \in \mathbf{Z}$. Applied to a complex of sheaves $\mathcal{F}^\bullet$ the resulting modules $\mathbf{R}^i\Gamma(\mathcal{F}^\bullet) = H^i(X, \mathcal{F}^\bullet)$ are sometimes called the *hypercohomology modules* of $\mathcal{F}^\bullet$. Plugging in just a single coherent sheaf $\mathcal{F}$, we get the familiar *sheaf cohomology* of X with respect to a sheaf $\mathcal{F}$, denoted with $H^i(X, \mathcal{F})$.

Example 3.13. As a nice application of the abstract machinery of Chapter 2 we obtain the famous long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to \cdots \\ \cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{E}) \to \cdots \end{aligned} \tag{3.3}$$

associated to a short exact sequence of coherent sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$. This is as an intermediate corollary of Proposition 2.38.

The above definition of $\mathbf{R}\Gamma$ will suffice for most cases, but we can go slightly further, utilizing a vanishing result of Grothendieck, to be found in [Har77, §III, Theorem 2.7].

Theorem 3.14. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a noetherian scheme X of dimension n. Then $H^i(X, \mathcal{F}) = 0$ for all $i > n$.*

If X is a noetherian scheme of dimension n, the modules $\mathbf{R}^i\Gamma(\mathcal{F})$ are non-trivial only for finitely many $i \in \mathbf{Z}$, for any coherent sheaf $\mathcal{F}$. By Proposition A.8 (i) we thus obtain a functor

$$\mathbf{R}\Gamma\colon \mathsf{D}^b(X) \to \mathsf{D}^b(\mathsf{Mod}_k).$$

3.2.2 Push-forward f_*

A generalization of the global sections functor is the *push-forward* or the *direct image* functor along a morphism of schemes $f\colon X \to Y$. The *push-forward* functor

$$f_*\colon \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\mathcal{O}_Y}$$

[18] The scheme X is noetherian, so $\mathsf{qcoh}(X)$ contains enough injectives by Proposition 3.1

is defined on objects by assigning an $\mathcal{O}_X$-module $\mathcal{F}$ the $\mathcal{O}_Y$-module $f_*\mathcal{F}$ defined on open subsets $V \subseteq Y$ to be $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ with the obvious restriction homomorphisms of $\mathcal{F}$. On a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, f_* is defined to act as $(f_*\alpha)_V = \alpha_{f^{-1}(V)}$ for all open $V \subseteq Y$.

We are going to strengthen our initial assumption that our schemes are noetherian to them being of finite type over k and consider only morphisms $f\colon X \to Y$ between schemes of this kind. To induce a functor between the quasi-coherent and coherent categories of X and Y, we need to recall the following proposition first.

Proposition 3.15. *Let $f\colon X \to Y$ be a morphism of schemes of finite type over k. Then for every quasi-coherent sheaf $\mathcal{F}$ on X, $f_*\mathcal{F}$ is quasi-coherent on Y. If f is assumed to be projective or more generally proper and $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.*

Proof. The quasi-coherent part is [Har77, §II, Proposition 5.8], because schemes of finite type over k are noetherian. For the coherent part see [Har77, §II, Proposition 5.20] in the case that f is projective and [EGA, Part III, §3.2, Theorem 3.2.1], when f is proper. □

It is a well known fact that f_* is left exact, because it is a right adjoint and right adjoints preserve limits. As $\mathsf{qcoh}(X)$ contains enough injectives, we may right derive f_* to obtain

$$\mathbf{R}f_*\colon \mathsf{D}^+(\mathsf{qcoh}(X)) \to \mathsf{D}^+(\mathsf{qcoh}(Y)).$$

This allows us to introduce the higher derived direct image functors

$$\mathbf{R}^i f_* = \mathcal{H}^i \circ \mathbf{R}f_*$$

for all $i \in \mathbf{Z}$.

Next, we would like to introduce the derived push-forward also between the *bounded* derived categories of quasi-coherent sheaves. The next proposition will enable us to do so.

Proposition 3.16. *Let $f\colon X \to Y$ be a morphism of schemes of finite type over k and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathbf{R}^i f_*\mathcal{F} \simeq 0$ for all $i > n$, where n is the dimension of X.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ and let δ denote the differential of $\mathcal{I}^\bullet$. Then

$$\mathbf{R}^i f_*\mathcal{F} = \mathcal{H}^i(\mathbf{R}f_*\mathcal{F}) = \mathcal{H}^i(f_*\mathcal{I}^\bullet) = \operatorname{coker}(\operatorname{im}(f_*\delta^{i-1}) \to \ker(f_*\delta^i)).$$

We recognise the cokernel at the end as the sheafification of the presheaf described on open subsets of Y by the assignment

$$V \mapsto \operatorname{coker}(\operatorname{im}(\delta^{i-1}_{f^{-1}(V)}) \to \ker(\delta^i_{f^{-1}(V)})). \tag{3.4}$$

For every open $V \subseteq Y$, the open subset $f^{-1}(V)$ determines a noetherian subscheme of Y whose dimension is at most n. Therefore, by Grothendieck's vanishing Theorem 3.14, the presheaf given by (3.4) is trivial for $i > n$, because

$$\begin{aligned}\operatorname{coker}(\operatorname{im}(\delta^{i-1}_{f^{-1}(V)}) \to \ker(\delta^i_{f^{-1}(V)})) &= H^i(\Gamma(f^{-1}(V), \mathcal{I}^\bullet)) = \\ &= \mathbf{R}^i\Gamma(f^{-1}(V), \mathcal{I}^\bullet) \simeq H^i(f^{-1}(V), \mathcal{F}).\end{aligned}$$

□

Applying Proposition A.8 (i), we may induce the derived direct image functor

$$\mathbf{R}f_*\colon \mathsf{D}^b(\mathsf{qcoh}(X)) \to \mathsf{D}^b(\mathsf{qcoh}(Y)). \tag{3.5}$$

Finally, we obtain the derived direct image functor between the bounded derived categories of coherent sheaves formulated in the following proposition.

Proposition 3.17. *Let $f\colon X \to Y$ be a proper morphism between schemes X and Y of finite type over k. Then the derived direct image functor (3.5) induces*

$$\mathbf{R}f_*\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y).$$

Proof. By [EGA, Part III, §3.2, Theorem 3.2.1] all the higher direct images $\mathbf{R}^i f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ are again coherent. As the category of coherent sheaves $\mathsf{coh}(X)$ is thick in $\mathsf{qcoh}(X)$, according to [Har77, §II, Proposition 5.7], Proposition A.8 (ii) tells us that the derived direct image $\mathbf{R}f_*\mathcal{F}^\bullet$ has coherent cohomology for any complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$. Functor $\mathbf{R}f_*$ thus maps complexes of $\mathsf{D}^b(X)$ to complexes of $\mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(Y))$, inducing

$$\mathbf{R}f_*\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y),$$

by Proposition 3.2. □

Remark 3.18. Whenever $f\colon X \to Y$ is an isomorphism of schemes of finite type over k, the push-forward functor $f_*\colon \mathsf{coh}(X) \to \mathsf{coh}(Y)$ is an equivalence of categories. Therefore it also induces an equivalence on the level of derived categories $\mathsf{D}^b(X) \to \mathsf{D}^b(Y)$. This nearly obvious but nonetheless important observation shows that $\mathsf{D}^b(-)$ is an invariant of the isomorphism class of a scheme of finite type over k.

Example 3.19. An example of a direct image functor we will encounter very often is pushing-forward along a projection $p\colon X \times Y \to X$ for smooth and projective varieties X and Y over a field k. In this case the projection p is proper because the properness condition is stable under base change [Sp25, Tag 01W4] and the structure map $Y \to \operatorname{Spec} k$ of a projective variety Y is by definition proper.

3.2.3 Hom functors

Let X be a noetherian scheme over a field k. For any two quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we denote with $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ the k-vector space of morphisms in the abelian category $\mathsf{qcoh}(X)$. As with any category it gives rise to a functor

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},-)\colon \mathsf{qcoh}(X) \to \mathsf{Mod}_k$$

for any quasi-coherent sheaf $\mathcal{F}$ on X. Since $\mathsf{qcoh}(X)$ contains enough injectives by Proposition 3.1, we can derive this Hom functor according to Section 2.2.4 to obtain

$$\mathbf{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},-)\colon \mathsf{D}^+(\mathsf{qcoh}(X)) \to \mathsf{D}^+(\mathsf{Mod}_k).$$

In this way we also obtain all its higher derived functors, namely the *Ext-functors*

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},-) = \mathbf{R}^i\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$$

for all $i \in \mathbf{Z}$.

Internal Hom

The *internal Hom* or *sheaf Hom* of two sheaves $\mathcal{F}$ and $\mathcal{G}$ on X is defined to be a sheaf on X given on open sets $U \subseteq X$ by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U),$$

with the restriction homomorphisms being the usual restrictions of homomorphisms of sheaves. When $\mathcal{F}$ and $\mathcal{G}$ are taken to be $\mathcal{O}_X$-modules, the internal Hom may also be equipped with an $\mathcal{O}_X$-action making it an $\mathcal{O}_X$-module. We denote it by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ when we want to clarify that we are working over a base space X and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ otherwise. Applying the global sections functor $\Gamma(X, -)$ to $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ it is clear that

$$\Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

The internal Hom also defines a bifunctor

$$\mathcal{H}om(-,-)\colon \mathsf{Mod}_{\mathcal{O}_X}^{\mathrm{op}} \times \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\mathcal{O}_X}. \tag{3.6}$$

For a fixed $\mathcal{O}_X$-module $\mathcal{G}$ the $\mathcal{H}om(-, \mathcal{G})$-action on a $\mathcal{O}_X$-module homomorphism $\alpha\colon \mathcal{F}_1 \to \mathcal{F}_0$ is defined as an $\mathcal{O}_X$-module homomorphism

$$\alpha^*\colon \mathcal{H}om(\mathcal{F}_0, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}_1, \mathcal{G}),$$

given on open subsets $U \subseteq X$ by $\Gamma(U, \mathcal{O}_X)$-linear maps

$$(\alpha^*)_U = \left(\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_0|_U, \mathcal{G}|_U) \xrightarrow{(\alpha_U)^*} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_1|_U, \mathcal{G}|_U)\right).$$

This action is clearly functorial and the same may be said for the similarly defined $\mathcal{H}om(\mathcal{F}, -)$-action, for a fixed $\mathcal{O}_X$-module $\mathcal{F}$. By [EGA, II, §9, Proposition 9.1.1] the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. Therefore (3.6) may be restricted to a bifunctor

$$\mathcal{H}om(-,-)\colon \mathsf{coh}(X)^{\mathrm{op}} \times \mathsf{coh}(X) \to \mathsf{coh}(X).$$

Analogous to the Hom complex of Section 2.2.4 (Definition 2.42) we can also define the following complex. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be bounded complexes of coherent sheaves on X. The *internal Hom complex* $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is then defined by its terms

$$\mathcal{H}om^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \bigoplus_{j \in \mathbf{Z}} \mathcal{H}om(\mathcal{F}^j, \mathcal{G}^{i+j})$$

and differentials

$$\delta^i\colon \mathcal{H}om^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \mathcal{H}om^{i+1}(\mathcal{F}^\bullet, \mathcal{G}^\bullet),$$

which one can formally described as follows. Let $\pi_j\colon \mathcal{H}om^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \mathcal{H}om(\mathcal{F}^j, \mathcal{G}^{i+j})$ denote the canonical projection onto the j-th component. Then δ^i is the unique map obtained by the universal property of the product $\mathcal{H}om^{i+1}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$, induced by the family of morphisms

$$(d_{\mathcal{G}}^{i+j})_* \circ \pi_j + (-1)^i (d_{\mathcal{F}}^j)^* \circ \pi_{j+1}\colon \quad \mathcal{H}om^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \mathcal{H}om(\mathcal{F}^j, \mathcal{G}^{i+j+1}).$$

Not very different from how we showed in Section 2.2.4 that the Hom complex induces a functor or even a bifunctor according to Remark 2.43, we obtain a bifunctor

$$\mathcal{H}om^\bullet\colon \mathsf{K}^-(\mathsf{coh}(X))^{\mathrm{op}} \times \mathsf{K}^+(\mathsf{coh}(X)) \to \mathsf{K}^+(\mathsf{coh}(X)).$$

When X is smooth and projective and Proposition 3.3 is at ones disposal, it is possible to derive the internal Hom complex functor above to obtain a bifunctor

$$\mathbf{R}\mathcal{H}om^\bullet(-,-)\colon \mathsf{D}^b(X)^{\mathrm{op}} \times \mathsf{D}^b(X) \to \mathsf{D}^b(X).$$

The main ingredients for how one goes about proving the above are described in [Huy06, pp. 75–77]. On this note we also mention that Proposition 3.3 greatly simplifies how we compute $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{F}^\bullet,-)$ for some complex $\mathcal{F}^\bullet$ from $\mathsf{D}^b(X)$. This is because $\mathcal{F}^\bullet$ can be replaced by an isomorphic (in $\mathsf{D}^b(X)$) complex $\mathcal{E}^\bullet$ of vector bundles, for which $\mathbf{R}\mathcal{H}om^\bullet(\mathcal{E}^\bullet,-)$ and $\mathcal{H}om^\bullet(\mathcal{E}^\bullet,-)$ coincide. The *dual* of a complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ will be taken to be

$$\mathcal{F}^{\bullet\vee} := \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_X).$$

In general, even if $\mathcal{F}^\bullet$ is just a coherent sheaf concentrated in degree 0, its dual may be an honest complex.

3.2.4 Tensor product

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ their *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined as the sheafification of the presheaf given on open subsets $U \subseteq X$ by

$$U \mapsto \Gamma(U,\mathcal{F}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,\mathcal{G}).$$

Using this description $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is equipped with the $\mathcal{O}_X$-action in the obvious way making it into an $\mathcal{O}_X$-module. Moreover this definition is functorial in both arguments $\mathcal{F}$ and $\mathcal{G}$ in the sense that a morphism of $\mathcal{O}_X$-modules $\alpha\colon \mathcal{F}_0 \to \mathcal{F}_1$ induces a morphism $\alpha \otimes \mathrm{id}_{\mathcal{G}}\colon \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}$ in a functorial way. The morphism $\alpha \otimes \mathrm{id}_{\mathcal{G}}$ comes from the universal property of sheafification, such that for every open subset $U \subseteq X$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma(U,\mathcal{F}_0) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,\mathcal{G}) \arrow[r] \arrow[d, "\alpha_U \otimes (\mathrm{id}_{\mathcal{G}})_U"'] & \Gamma(U,\mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G}) \arrow[d, dashed, "(\alpha \otimes \mathrm{id}_{\mathcal{G}})_U"] \\
\Gamma(U,\mathcal{F}_1) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,\mathcal{G}) \arrow[r] & \Gamma(U,\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G})
\end{tikzcd}$$

In exactly the same way one also sees that the tensor product is functorial in the other variable. Hence the tensor product assembles a bifunctor

$$(-) \otimes_{\mathcal{O}_X} (-)\colon \mathsf{Mod}_{\mathcal{O}_X} \times \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_{\mathcal{O}_X}.$$

According to [Sp25, Tag 01CE] tensor product of two coherent sheaves is again coherent, so we also have a bifunctor

$$(-) \otimes_{\mathcal{O}_X} (-)\colon \mathsf{coh}(X) \times \mathsf{coh}(X) \to \mathsf{coh}(X).$$

Partially evaluating this functor at a coherent sheaf $\mathcal{F}$ on X gives a functor $\mathcal{F} \otimes_{\mathcal{O}_X} (-)\colon \mathsf{coh}(X) \to \mathsf{coh}(X)$, which we know is left exact (for example due to the tensor-Hom adjunction [Sp25, Tag 01CN]). We also mention that tensor product commutes with the stalks functor [Sp25, Tag 01CB], meaning that for any $x \in X$ there are natural isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

In order to derive the functor $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$, we need an adapted class, which is provided by the following proposition.

Proposition 3.20. *Let X be a smooth projective variety and $\mathcal{F}$ a coherent sheaf on X. Then the class of vector bundles on X forms a $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$-adapted class.*

Proof. The class of vector bundles is obviously closed under finite direct sums and we already know that every coherent sheaf is a quotient of some vector bundle. Thus it remains to show that given an acyclic complex of vector bundles $\mathcal{E}^\bullet$, which is bounded from above, the tensor product $\mathcal{F} \otimes \mathcal{E}^\bullet$ is again acyclic. This is verified on stalks, where localizing at a point $x \in X$ gives

$$\cdots \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x^{i-1} \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x^{i} \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x^{i+1} \to \cdots .$$

This complex is acyclic, because the complex $\mathcal{E}_x^\bullet$ is acyclic and consists of free $\mathcal{O}_{X,x}$-modules. $\square$

On a smooth projective variety X Proposition 3.20 allows us to derive $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$ and obtain the left derived tensor product functor

$$\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \colon \mathsf{D}^-(X) \to \mathsf{D}^-(X).$$

Along with it also come its higher derived companions, called the *internal Tor sheaves*

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{H}^{-i}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}).$$

Generalizing the tensor product of sheaves, we can define also a tensor product of complexes of sheaves in an analogous way to what was done with the internal Hom complex. For bounded complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ of coherent sheaves on X we introduce the *internal tensor complex* $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet$ to consist of terms

$$(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^i = \bigoplus_{p+q=i} \mathcal{F}^p \otimes_{\mathcal{O}_X} \mathcal{G}^q$$

and differentials $\delta^i \colon (\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^i \to (\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^{i+1}$ formally described as the unique map induced by the family of morphisms, indexed by p, q, satisfying $p + q = i + 1$

$$(d_{\mathcal{F}}^{p-1} \otimes \mathrm{id}_{\mathcal{G}^q}) \circ \pi_{p-1,q} + (-1)^i (\mathrm{id}_{\mathcal{F}^p} \otimes d_{\mathcal{G}}^{q-1}) \circ \pi_{p,q-1} \colon \quad (\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^i \to \mathcal{F}^p \otimes_{\mathcal{O}_X} \mathcal{G}^q$$

by universal property of the product $(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^{i+1}$. Here $\pi_{p,q}$ denote the canonical projections $(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)^i \to \mathcal{F}^p \otimes_{\mathcal{O}_X} \mathcal{G}^q$ for $p + q = i$.

The contents of [Huy16, pp. 79–80] allow us to first form a bifunctor out of the internal tensor complex and provided we are working over a smooth projective variety X derive it into a bifunctor

$$(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \colon \mathsf{D}^b(X) \times \mathsf{D}^b(X) \to \mathsf{D}^b(X).$$

We remark that due to Proposition 3.3 one can always replace bounded complexes of coherent sheaves with bounded complexes of vector bundles and compute the derived tensor product as the ordinary complex tensor product of the corresponding complexes of vector bundles.

3.2.5 Pull-back f^*

Let $f\colon X\to Y$ be a morphism of smooth projective varieties. The *pull-back* or the *inverse image* functor along f is defined to be

$$f^*:\mathsf{Mod}_{\mathcal{O}_Y}\to\mathsf{Mod}_{\mathcal{O}_X}\qquad f^*=\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}(-).$$

As with the tensor product, the pull-back f^* is also *left exact* because it is a composition of an exact functor $f^{-1}\colon\mathsf{Mod}_{\mathcal{O}_Y}\to\mathsf{Mod}_{f^{-1}\mathcal{O}_Y}$ and a left exact functor $\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}(-)\colon\mathsf{Mod}_{f^{-1}\mathcal{O}_Y}\to\mathsf{Mod}_{\mathcal{O}_X}$.

Proposition 3.21. *Let $f\colon X\to Y$ be a morphism of schemes.*

(i) If $\mathcal{G}$ is a quasi-coherent sheaf on Y, then $f^*\mathcal{G}$ is a quasi-coherent sheaf on X

(ii) If X and Y are noetherain and $\mathcal{G}$ is a coherent sheaf on Y, then $f^*\mathcal{G}$ is a coherent sheaf on X.

Proof. See [Har77, §II, Proposition 5.8]. □

Clearly we may thus restrict the pull-back functor to the categories of coherent sheaves

$$f^*\colon\mathsf{coh}(Y)\to\mathsf{coh}(X).$$

To derive it we will utilize the class of vector bundles on X

Proposition 3.22. *Let $f\colon X\to Y$ be a morphism of smooth projective varieties. Then the class of vector bundles on Y forms an f^*-adapted class.*

Proof. Similarly as in the proof of Proposition 3.20, it remains to show that given a bounded above acyclic complex of vector bundles $\mathcal{E}^\bullet$ the pull-back $f^*\mathcal{E}^\bullet$ is still acyclic. This is verified on stalks, where one first sees that upon localizing at any $y\in Y$, the complex $\mathcal{E}^\bullet$ becomes an acyclic complex of free $\mathcal{O}_{Y,y}$-modules

$$\cdots\to\mathcal{E}^{i-1}_y\to\mathcal{E}^i_y\to\mathcal{E}^{i+1}_y\to\cdots.$$

Since localization and f^* naturally commute, we see that the complex $f^*\mathcal{E}^\bullet$ localized at a point $x\in X$ becomes

$$\cdots\to\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{Y,f(x)}}\mathcal{E}^{i-1}_{f(x)}\to\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{Y,f(x)}}\mathcal{E}^{i}_{f(x)}\to\mathcal{O}_{X,x}\otimes_{\mathcal{O}_{Y,f(x)}}\mathcal{E}^{i+1}_{f(x)}\to\cdots.$$

As $\mathcal{E}^i_{f(x)}$ are free $\mathcal{O}_{Y,f(x)}$-modules the sequence stays acyclic. This holds true for any $x\in X$, therefore $f^*\mathcal{E}^\bullet$ is acyclic. □

Corollary 3.23. *Let $f\colon X\to Y$ be a morphism of smooth projective varieties. Then any vector bundle on Y is f^*-acyclic.*

Proof. This is a consequence of Proposition 3.22 and Lemma 2.39. □

Under the assumption that $f\colon X\to Y$ is a morphism of smooth projective varieties the presence of an f^*-adapted class ensured by Proposition 3.22 allows us to derive the pull-back functor to obtain

$$f^*\colon\mathsf{D}^-(Y)\to\mathsf{D}^-(X).$$

Since any bounded complex in $\mathsf{D}^b(Y)$ is isomorphic to a *bounded* complex of vector bundles by Proposition 3.3 it is also clear that the functor above may be restricted to the bounded derived categories

$$f^*\colon\mathsf{D}^b(Y)\to\mathsf{D}^b(X).$$

Remark 3.24. In many cases the morphism $f: X \to Y$ will be *flat*. By definition this means that for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is flat as a $\mathcal{O}_{Y,f(x)}$-module, which in turn translates to the functor $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} (-)$ being exact. For any such flat morphism f the pull-back functor $f^*: \mathsf{coh}(Y) \to \mathsf{coh}(X)$ is exact and allows us to induce the derived functor $f^*: \mathsf{D}^b(Y) \to \mathsf{D}^b(X)$ simply by Proposition 2.24.

3.3 Interactions between derived functors

In this section we will present two interactions that appear between the derived functors of geometric origin. We will need them in the next chapter on Fourier–Mukai transforms, in particular when dealing with compositions of two such transforms.

Proposition 3.25 (Flat base change formula)**.** *Let the following be a pull-back square of smooth projective varieties*

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Assuming $f: X \to Y$ is flat and $g: Y' \to Y$ is proper, then $u: X' \to Y'$ is flat and there is a natural isomorphism of functors $\mathsf{D}^b(Y') \to \mathsf{D}^b(X)$

$$f^* \circ \mathbf{R}g_* \simeq \mathbf{R}v_* \circ u^*. \tag{3.7}$$

Proof. See [Sp25, Tag 02KH]. □

The following projection formula we will see is just a derived incarnation of the ordinary projection formula, which is known to hold for sheaves.

Proposition 3.26 (Projection formula)**.** *Let $f: X \to Y$ be a proper map between projective schemes over k. Let $\mathcal{F}^\bullet$ and $\mathcal{E}^\bullet$ denote complexes belonging to $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Y)$ respectively. Then there is an isomorphism of complexes in $\mathsf{D}^b(Y)$*

$$\mathbf{R}f_*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{L}f^*\mathcal{E}^\bullet) \simeq \mathbf{R}f_*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{E}^\bullet. \tag{3.8}$$

Proof. We first recall the underived version of the projection formula, from which we will derive (3.8). It states that for any morphism of $f: X \to Y$, coherent sheaf $\mathcal{F}$ on Y and locally free sheaf $\mathcal{E}$ on X

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}. \tag{3.9}$$

By Proposition 3.3 we may assume $\mathcal{E}^\bullet$ is a complex of locally free sheaves on Y, otherwise we just replace it by an isomorphic complex of this sort. Then $f^*\mathcal{E}^\bullet$ is a complex of locally free sheaves on X (we can check local freeness on stalks because of the useful criterion found in [Wei13, §1, Lemma 5.1.3]). This already implies that it is enough to only show

$$\mathbf{R}f_*(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet) \simeq \mathbf{R}f_*\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet,$$

where the pull-back and the tensor product are underived. Let $\mathcal{I}^\bullet \to \mathcal{F}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$, momentarily accessing the category of quasi-coherent sheaves on X. Since $f^*\mathcal{E}^\bullet$ is locally free, we see that

$$\mathcal{I}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet \to \mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet$$

is an injective resolution of $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet$, because $(-) \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet$ is exact and tensoring injectives with locally free sheaves again results in an injective sheaf [Sp25, Tag 01E7]. This means that we can compute $\mathbf{R}f_*$ using this resolution. Utilizing (3.9) on each term of the complex now lets us see that

$$\mathbf{R}f_*(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet) \simeq f_*(\mathcal{I}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{E}^\bullet) \simeq f_*\mathcal{I}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet \simeq \mathbf{R}f_*\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{E}^\bullet. \qquad \square$$

4 Fourier-Mukai transforms

Fourier–Mukai transforms are named after two prominent mathematicians – Fourier and Mukai. The second one naming them after the first and in the process of studying them also getting his name attached to the term. In his paper [Muk81] Mukai introduced these transforms to study bounded derived categories of coherent sheaves on abelian varieties and their duals, in the process discovering non-isomorphic varieties which share equivalent bounded derived categories. He named his transforms after Fourier because of their resemblance to classical Fourier transforms of analysis and also provided a dictionary to pass between the two sides of his analogy.

In this chapter we will first define Fourier–Mukai transforms on the level of derived categories, explore some of their properties and state an important theorem of Orlov, which will be indispensable later on. The remaining part of the chapter will be devoted to passing Fourier–Mukai transforms first to the level of K-groups and lastly to rational cohomology. We will examine the interactions of these three views with the use of Mukai vectors.

Notation. Let X and Y be smooth projective varieties over k and denote the canonical projections

$$p\colon X \times Y \to X \quad \text{and} \quad q\colon X \times Y \to Y.$$

We will also transition from using $\mathcal{F}^\bullet$ to denote complexes of sheaves to the lighter notation $\mathcal{F}$, which may refer either to a single sheaf or to a complex of sheaves.

Definition 4.1. The *Fourier–Mukai transform* associated to a complex $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$ is defined to be the functor

$$\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y) \qquad \Phi_{\mathcal{E}} := \mathbf{R}q_*(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} \mathbf{L}p^*(-)).$$

The complex $\mathcal{E}$ is called the *kernel* of $\Phi_{\mathcal{E}}$.

Remark 4.2. We see that $\Phi_{\mathcal{E}}$ is a composition of three triangulated functors,

$$\Phi_{\mathcal{E}}\colon \quad \mathsf{D}^b(X) \xrightarrow{\mathbf{L}p^*} \mathsf{D}^b(X \times Y) \xrightarrow{\mathcal{E}\otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}}(-)} \mathsf{D}^b(X \times Y) \xrightarrow{\mathbf{R}q_*} \mathsf{D}^b(Y)$$

meaning it is also triangulated itself.

Note that as the projection $p\colon X \times Y \to X$ is flat, the pull-back functor p^* is exact and therefore needs not be derived. Many examples of known functors are seen to be of Fourier–Mukai type.

Example 4.3. In the following examples we will write derived functors only wherever truly necessary. For instance tensoring with a vector bundle, pushing-forward along a closed embedding, pulling-back along a flat morphism are all exact functors, which need not be derived.

(i) $\mathrm{id}_{\mathsf{D}^b(X)} \simeq \Phi_{\mathcal{O}_\Delta}$. Recall that $\mathcal{O}_\Delta$ is defined to be the push-forward $\Delta_*\mathcal{O}_X$, of the structure sheaf of X along the diagonal embedding $\Delta\colon X \to X \times X$. Then for any object $\mathcal{F}$ of $\mathsf{D}^b(X)$ we compute

$$\begin{aligned}
\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) &= \mathbf{R}q_*(\mathcal{O}_\Delta \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} p^*(\mathcal{F})) \\
&\simeq \mathbf{R}q_*(\Delta_*\mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} p^*(\mathcal{F})) \\
&\simeq \mathbf{R}q_*(\Delta_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathbf{L}\Delta^*(p^*(\mathcal{F})))) \\
&\simeq \mathbf{R}(q \circ \Delta)_*(\mathbf{L}(p \circ \Delta)^*(\mathcal{F})) \\
&\simeq \mathcal{F}.
\end{aligned}$$

All isomorphisms are natural in $\mathcal{F}$, thus proving $\mathrm{id}_{\mathsf{D}^b(X)} \simeq \Phi_{\mathcal{O}_\Delta}$.

(ii) $(-)[1] \simeq \Phi_{\mathcal{O}_\Delta[1]}$. While noting that all the functors involved are triangulated, a similar computation as above identifies the translation functor as a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$.

(iii) $\mathcal{L} \otimes (-) \simeq \Phi_{\Delta_*\mathcal{L}}$. Let $\mathcal{L}$ denote a line bundle on X. Then utilizing the diagonal embedding $\Delta\colon X \to X \times X$ again, we can compute for any complex $\mathcal{F}$ of $\mathsf{D}^b(X)$

$$\begin{aligned}\Phi_{\Delta_*\mathcal{L}}(\mathcal{F}) &= \mathbf{R}q_*(\Delta_*\mathcal{L} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times X}} p^*(\mathcal{F}))\\ &\simeq \mathbf{R}q_*(\Delta_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathbf{L}\Delta^*(q^*(\mathcal{F}))))\\ &\simeq \mathbf{R}(q\circ\Delta)_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathbf{L}(p\circ\Delta)^*(\mathcal{F}))\\ &\simeq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}.\end{aligned}$$

Since all the isomorphisms are natural, we have $\mathcal{L} \otimes_{\mathcal{O}_X} (-) \simeq \Phi_{\Delta_*\mathcal{L}}$.

(iv) $S_X \simeq \Phi_{\Delta_*\omega_X[n]}$. As a consequence of (ii) and (iii), we see that the Serre functor $S_X = (-)\otimes\omega_X[n]$, where n denotes the dimension of X, is a Fourier–Mukai transform with kernel $\Delta_*\omega_X[n]$.

(v) $\mathbf{R}f_* \simeq \Phi_{\mathcal{O}_{\Gamma_f}}$. For a morphism $f\colon X \to Y$, let $\mathcal{O}_{\Gamma_f}$ denote the push-forward of the sheaf $\mathcal{O}_X$ along the morphism $i\colon X \to X \times Y$ induced by the identity $\mathrm{id}_X\colon X \to X$ and $f\colon X \to Y$. Then for any object $\mathcal{F}$ of $\mathsf{D}^b(X)$ we have

$$\begin{aligned}\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{F}) &= \mathbf{R}q_*(\mathcal{O}_{\Gamma_f} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} p^*(\mathcal{F}))\\ &\simeq \mathbf{R}q_*(i_*\mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} p^*(\mathcal{F}))\\ &\simeq \mathbf{R}q_*(i_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathbf{L}i^*(p^*(\mathcal{F}))))\\ &\simeq \mathbf{R}(q\circ i)_*(\mathbf{L}(p\circ i)^*(\mathcal{F}))\\ &\simeq \mathbf{R}f_*\mathcal{F}.\end{aligned}$$

Note that we have used $p\circ i = \mathrm{id}_X$ and $q \circ i = f$ passing to the last line. As all the isomorphisms are natural, we have $\mathbf{R}f_* \simeq \Phi_{\mathcal{O}_{\Gamma_f}}$.

Remark 4.4. From now on we will consider all our functors on derived categories to be derived and follow the convention of not explicitly denoting them as being derived. For example we will write just f_* to mean $\mathbf{R}f_*$ etc.

In his paper [Muk81] Mukai discovered that every Fourier-Mukai transform admits left and right adjoints.

Proposition 4.5 (Mukai)**.** *Let $\Phi_\mathcal{E}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ be a Fourier-Mukai transform associated to an object $\mathcal{E}$ of $\mathsf{D}^b(X\times Y)$. Then $\Phi_\mathcal{E}$ admits left and right adjoints*

$$\Phi_{\mathcal{E}_\mathrm{L}}\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(X) \quad \textit{and} \quad \Phi_{\mathcal{E}_\mathrm{R}}\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(X),$$

which are evidently also Fourier-Mukai with their respective kernels

$$\mathcal{E}_\mathrm{L} = \mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} q^*\omega_Y[\dim Y] \quad \textit{and} \quad \mathcal{E}_\mathrm{R} = \mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} p^*\omega_X[\dim X].$$

Proof. See [Huy06, §5, Proposition 5.9]. This follows from a highly non-trivial fact that the derived push-forward $\mathbf{R}f_*\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ along a proper map $f\colon X \to Y$ admits a *right* adjoint $f^! = \mathbf{L}f^*(-)\otimes_{\mathcal{O}_X}\omega_f[\dim f]$, where $\omega_f = \omega_X \otimes_{\mathcal{O}_X} f^*\omega_Y^*$ and $\dim f = \dim Y - \dim X$. This is a consequence of Grothendieck–Verdier duality [Har66, §VII, Corollary 4.3]. Since X and Y are projective the projection $p\colon X\times Y \to X$ is proper. ∎

Remark 4.6. Using Serre functors we are able to write down an alternative description of left and right adjoints of $\Phi_{\mathcal{E}}$, namely

$$\Phi_{\mathcal{E}_{\mathrm{L}}} = \Phi_{\mathcal{E}^\vee} \circ S_Y \quad \text{and} \quad \Phi_{\mathcal{E}_{\mathrm{R}}} = S_X \circ \Phi_{\mathcal{E}^\vee}. \tag{4.1}$$

Indeed, a straight forward computation, using the fact that pull-backs commute with tensor products, shows

$$\begin{aligned}
\Phi_{E_{\mathrm{L}}}(\mathcal{F}) &= p_*(\mathcal{E}_{\mathrm{L}} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{F}) \\
&= p_*(\mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} q^*\omega_Y[\dim Y] \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{F}) \\
&\simeq p_*(\mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} q^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \omega_Y[\dim Y])) \\
&= \Phi_{\mathcal{E}^\vee}(S_Y(\mathcal{F}))
\end{aligned}$$

for any $\mathcal{F}$ of $\mathsf{D}^b(Y)$. For the other, using the projection formula (3.8), we have

$$\begin{aligned}
\Phi_{E_{\mathrm{R}}}(\mathcal{F}) &= p_*(\mathcal{E}_{\mathrm{R}} \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{F}) \\
&= p_*(\mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} p^*\omega_X[\dim X] \otimes_{\mathcal{O}_Y} q^*\mathcal{F}) \\
&\simeq p_*(\mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} q^*\mathcal{F}) \otimes_{\mathcal{O}_X} \omega_X[\dim X] \\
&= S_X(\Phi_{\mathcal{E}^\vee}(\mathcal{F}))
\end{aligned}$$

for any $\mathcal{F}$ of $\mathsf{D}^b(X)$. The equations (4.1) also agree with the more general method of Proposition 1.35 for obtaining a right adjoint of a functor provided that a left adjoint is known, when Serre functors are present.

Proposition 4.7. *Let $\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ and $\Phi_{\mathcal{F}}\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(Z)$ be Fourier-Mukai transforms with kernels $\mathcal{E}$ and $\mathcal{F}$ belonging to $\mathsf{D}^b(X \times Y)$ and $\mathsf{D}^b(Y \times Z)$ respectively. Then the composition $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Z)$ is also a Fourier-Mukai transform with kernel*

$$\mathcal{G} = (\pi_{XZ})_*(\pi_{XY}^*\mathcal{E} \otimes \pi_{YZ}^*\mathcal{F}),$$

which is sometimes also called the convolution *of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$. Morphisms π_{XY}, π_{YZ} and π_{XZ} denote the canonical projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$ and $X \times Z$, respectively.*

Proof. See [Huy06, §5, 5.10]. The proof utilizes the projection formula (3.8) and the flat base change formula (3.7). Essentially this means that it is far less sophisticated than the proof of the existence of adjoints. □

The following is a celebrated theorem by Orlov, which shed a new light on the topic of triangulated functors between bounded derived categories of coherent sheaves on smooth projective varieties. Its main advantage lies in the fact that instead of studying fully faithful triangulated functors on their own, one can study them in simpler terms using their Fourier–Mukai kernels.

Theorem 4.8 ([Orl03, Theorem 3.2.2])**.** *Let X and Y be smooth projective varieties over a field k. Suppose $F\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ is a triangulated fully faithful functor admitting a left (or right) adjoint functor. Then there exists up to an isomorphism a unique object $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$ such that F and $\Phi_{\mathcal{E}}$ are naturally isomorphic.*

Remark 4.9. (i) Theorem 4.8 may be applied to any triangulated equivalence F between bounded derived categories $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Y)$. This is what we will do most often.

(ii) Due to bounded derived categories being equipped with Serre functors (Proposition 3.9), the assumptions that F admits a left or a right adjoint are equivalent by Proposition 1.35.

(iii) Even though this theorem was generally believed to be false without the fully faithfulness assumption, the first example of a non-Fourer–Mukai triangulated functor between bounded derived categories of coherent sheaves of two smooth projective varieties was given only in 2019 [RBN19].

We say that varieties X and Y are *derived equivalent*, if there exists a triangulated equivalence $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$. Due to the above theorem by Orlov, varieties X and Y are in that case called *Fourier–Mukai partners.* This leads us to the following beautiful application of Orlov's theorem. It is also an important statement by itself, providing evidence for why the bounded derived category of coherent sheaves is a nicely behaved invariant for smooth projective varieties. In particular it says that $\mathsf{D}^b(-)$ detects the dimension and triviality of the canonical bundle.

Theorem 4.10 ([Huy06, §4, Proposition 4.1]). *Suppose X and Y are smooth projective varieties over k. Assume*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

Then $\dim X = \dim Y$ *and if the canonical bundle ω_X of X is trivial, the canonical bundle ω_Y of Y is trivial as well.*

Proof. By Orlov's Theorem 4.8 the equivalence $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$ is witnessed by a Fourier-Mukai transform $\Phi_{\mathcal{E}}$ with a kernel $\mathcal{E}$ belonging to $\mathsf{D}^b(X \times Y)$. Since $\Phi_{\mathcal{E}}$ is an equivalence its left and right adjoints agree. As they are both given by Fourier-Mukai transforms their respective kernels $\mathcal{E}_{\mathrm{L}}$ and $\mathcal{E}_{\mathrm{R}}$, introduced in Proposition 4.5, must be isomorphic in $\mathsf{D}^b(X \times Y)$. Writing this out we see that

$$\mathcal{E}^\vee \simeq \mathcal{E}^\vee \otimes (q^*\omega_X \otimes p^*\omega_Y)[\dim Y - \dim X].$$

As tensoring by a line bundle, namely $q^*\omega_X \otimes p^*\omega_Y$ is exact, both tensor products are underived, but more importantly, the cohomology of complexes $\mathcal{E}^\vee$ and $\mathcal{E}^\vee \otimes (q^*\omega_X \otimes p^*\omega_Y)$ agree in all degrees. Since $\mathcal{E}$ is a bounded complex of coherent sheaves, its derived dual $\mathcal{E}^\vee = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{E}, \mathcal{O}_{X\times Y})$ is bounded as well. Then comparing cohomology intermediately[19] gives

$$\dim X = \dim Y.$$

For the second part, the assumption $\omega_X \simeq \mathcal{O}_X$, simplifies the Serre functor S_X of $\mathsf{D}^b(X)$ to

$$S_X \colon \mathsf{D}^b(X) \to \mathsf{D}^b(X) \qquad S_X \simeq (-)[n],$$

where n denotes the dimensions of both X and Y. Since equivalences of triangulated categories are known to commute with Serre functors by Proposition 1.34, we deduce from $\Phi_{\mathcal{E}} \circ S_X \simeq S_Y \circ \Phi_{\mathcal{E}}$ that the Serre functor S_Y of $\mathsf{D}^b(Y)$ also simplifies to $(-)[n]$. By Definition 3.9 this may be expressed as

$$(-)[n] \simeq S_Y = (-) \otimes_{\mathcal{O}_Y} \omega_Y[n].$$

Plugging in the structure sheaf $\mathcal{O}_Y$, we see that $\omega_Y \simeq \mathcal{O}_Y$ holds in $\mathsf{coh}(Y)$. The last assertion follows from fully faithfulness of the inclusion $\mathsf{coh}(Y) \to \mathsf{D}^b(Y)$ according to Proposition 2.10. □

[19] As $\Phi_{\mathcal{E}}$ is an equivalence, the kernel $\mathcal{E}$ cannot be isomorphic to the zero object in $\mathsf{D}^b(X \times Y)$, implying that $\mathcal{E}^\vee$ has non-trivial cohomology at least in some degrees.

4.1 ...on K-groups

We introduce the K-theoretic Fourier-Mukai transforms as a passageway between categorical and cohomological Fourier-Mukai transforms. Throughout this section we assume our schemes are smooth projective complex varieties and often that our morphisms are proper or flat. We do this so that we have for example access to propositions like 3.3, 3.17, etc.

Given a smooth projective complex variety X we define its K*-group* as follows. Consider the free abelian group generated by isomorphism classes of coherent sheaves[20] on X. The quotient of this group with its subgroup generated by expressions of the form $[\mathcal{E}]-[\mathcal{F}]+[\mathcal{G}]$, when there is a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, is defined to be the K-group of X. We denote it by $K_\circ(X)$. Extending this notion of additivity on short exact sequences, we define for any bounded complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$

$$[\mathcal{F}^\bullet] := \sum_{i\in\mathbf{Z}} (-1)^i [\mathcal{F}^i] \in K_\circ(X).$$

Remark 4.11. Observe that any complex $\mathcal{F}^\bullet$ may be split up into a series of short exact sequences of the form

$$0 \to \ker d^i_{\mathcal{F}} \to \mathcal{F}^i \to \operatorname{im} d^i_{\mathcal{F}} \to 0.$$

In $K_\circ(X)$ this means that $[\mathcal{F}^i] = [\ker d^i_{\mathcal{F}}] + [\operatorname{im} d^i_{\mathcal{F}}]$. From the short exact sequence defining the cohomology sheaf $\mathcal{H}^i(\mathcal{F}^\bullet)$ we also see that $[\mathcal{H}^i(\mathcal{F}^\bullet)] = [\ker d^i_{\mathcal{F}}] - [\operatorname{im} d^{i-1}_{\mathcal{F}}]$. Using these two sets of formulas one reparenthesizes the definition of $[\mathcal{F}^\bullet]$ to obtain

$$[\mathcal{F}^\bullet] = \sum_{i\in\mathbf{Z}} (-1)^i [\mathcal{H}^i(\mathcal{F}^\bullet)]. \tag{4.2}$$

In particular this shows that the operation $[-]$ of forming a class in $K_\circ(X)$ out of a complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ is an invariant of its isomorphism class in $\mathsf{D}^b(X)$.

The group $K_\circ(X)$ also comes equipped with a multiplication operation. On isomorphism classes of locally free sheaves $\mathcal{E}$ and $\mathcal{E}'$ this operation is given by

$$[\mathcal{E}] \cdot [\mathcal{E}'] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'].$$

For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ it turns out that one has to take the *derived* tensor product $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}]$ to accommodate the relations in $K_\circ(X)$ coming from short exact sequences. It turns out that extending this operation to chain complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ of $\mathsf{D}^b(X)$ also amounts to taking the derived tensor product but now of complexes

$$[\mathcal{F}^\bullet] \cdot [\mathcal{G}^\bullet] = [\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet]. \tag{4.3}$$

Remark 4.12. The above is seen to be true by replacing both complexes with their respective isomorphic complexes of locally free coherent sheaves, whose existence is due to Proposition 3.3.

[20] An attentive reader might express a set-theoretic concern at this point – does the collection of all isomorphism classes of coherent sheaves on X even form a set? On every noetherian scheme X this is indeed tha case. As was pointed out to me by Domen Zevnik this vaguely follows from the fact that X can be covered by a (finite) set of affine schemes of some noetherian rings and over noetherian rings there are only set-many isomorphism classes of finitely generated modules, because all such modules are finitely presented.

Definition 4.13. Let $f\colon X \to Y$ be a morphism between smooth projective varieties over $\mathbf{C}$. The *pull-back map* is defined to be

$$f^*\colon K_\circ(Y) \to K_\circ(X), \qquad f^*([\mathcal{G}]) = \sum_{i\in\mathbf{Z}} (-1)^i [\mathbf{L}^i f^* \mathcal{G}].$$

Remark 4.14. First, all the higher inverse image sheaves $\mathbf{L}^i f^*\mathcal{G}$ are coherent. This is because $\mathbf{L} f^*$ may be computed using locally free resolutions and by Proposition 3.21, $f^*\mathcal{E}$ is coherent for any coherent sheaf $\mathcal{E}$. Second, the existence of a long exact sequence for derived functors associated to a short exact sequence of coherent sheaves (Proposition 2.38), makes f^* well-defined.

In many occasions the morphism $f\colon X \to Y$ will be flat, which means that it induces an *exact* pull-back functor $f^*\colon \mathsf{coh}(Y) \to \mathsf{coh}(X)$ on the level of categories of coherent sheaves. In this case the above definition greatly simplifies to $f^*([\mathcal{G}]) = [f^*\mathcal{G}]$. This will for instance be the case for the canonical projection $X \times Y \to X$, which is known to be flat.

Definition 4.15. Let $f\colon X \to Y$ be a proper morphism between smooth projective varieties over $\mathbf{C}$. The *push-forward map* defined on equivalence classes of coherent sheaves is

$$f_!\colon K_\circ(X) \to K_\circ(Y), \qquad f_!([\mathcal{F}]) = \sum_{i\in\mathbf{Z}} (-1)^i [\mathbf{R}^i f_* \mathcal{F}].$$

By the following remark this definition is sound.

Remark 4.16. First, by [EGA, §III.3.2, Theorem 3.2.1] all the higher direct images $\mathbf{R}^i f_*\mathcal{F}$ are coherent. Second, existence of a long exact sequence for right derived functors associated to a short exact sequence of coherent sheaves (Proposition 2.38) implies that $f_!$ is well-defined. See also [Wei13, §II, Lemma 6.2.6].

The following proposition captures compatibilities of categorical derived push-forward and derived pull-back functors with their respective K-theoretical counterparts.

Proposition 4.17. *Let $f\colon X \to Y$ be a proper morphism between smooth projective complex varieties. For every bounded complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$ and $\mathcal{G}^\bullet$ of $\mathsf{D}^b(Y)$, we have*

$$f_!([\mathcal{F}^\bullet]) = [\mathbf{R} f_* \mathcal{F}^\bullet] \qquad \textit{and} \qquad f^*([\mathcal{G}^\bullet]) = [\mathbf{L} f^* \mathcal{G}^\bullet].$$

Proof. We prove the first equality by applying the *Leray spectral sequence*, which is a special instance of the spectral sequence (A.5)

$$E_2^{p,q} = \mathbf{R}^p f_* \mathcal{H}^q(\mathcal{F}^\bullet) \implies \mathbf{R}^{p+q} f_* \mathcal{F}^\bullet = E^{p+q}.$$

Observe that

$$f_!([\mathcal{F}^\bullet]) = \sum_{j\in\mathbf{Z}} (-1)^j f_!([\mathcal{H}^j(\mathcal{F}^\bullet)]) = \sum_{j\in\mathbf{Z}} (-1)^j \sum_{i\in\mathbf{Z}} (-1)^i [\mathbf{R}^i f_* \mathcal{H}^j(\mathcal{F}^\bullet)] = \sum_{i,j\in\mathbf{Z}} (-1)^{i+j} [E_2^{i,j}]$$

and

$$[\mathbf{R} f_* \mathcal{F}^\bullet] = \sum_{n\in\mathbf{Z}} (-1)^n [\mathbf{R}^n f_* \mathcal{F}^\bullet] = \sum_{n\in\mathbf{Z}} (-1)^n [E^n].$$

Thus we only need to show that

$$\sum_{i,j\in\mathbf{Z}} (-1)^{i+j} [E_2^{i,j}] = \sum_{n\in\mathbf{Z}} (-1)^n [E^n].$$

After recalling that $H^{p,q}(E_r) = \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-q})$, and realizing that each page E_r of the spectral sequence consists of a collection of chain complexes, one utilizes (4.2) to inductively compute

$$\sum_{i,j\in\mathbf{Z}} (-1)^{i+j}[E_2^{i,j}] = \sum_{i,j\in\mathbf{Z}} (-1)^{i+j}[E_3^{i,j}] = \cdots = \sum_{i,j\in\mathbf{Z}} (-1)^{i+j}[E_\infty^{i,j}].$$

Since $E_\infty^{i,j} \simeq F^i E^{i+j}/F^{i+1}E^{i+j}$, we see that

$$\begin{aligned}\sum_{i,j\in\mathbf{Z}} (-1)^{i+j}[E_\infty^{i,j}] &= \sum_{i,j\in\mathbf{Z}} (-1)^{i+j}([F^i E^{i+j}] - [F^{i+1}E^{i+j}]) \\ &= \sum_{p,n\in\mathbf{Z}} (-1)^n([F^p E^n] - [F^{p+1}E^n]) \\ &= \sum_{n\in\mathbf{Z}} (-1)^n[E^n],\end{aligned}$$

where the last equation is seen to follow by telescoping along the filtration $(F^pE^n)_p$ of E^n.

We see the second equality holds true, because $\mathcal{G}^\bullet$ may be replaced by an isomorphic (in $\mathsf{D}^b(X)$) complex $\mathcal{E}^\bullet$ of vector bundles, guaranteed to exist by Proposition 3.3. Since vector bundles are f^*-acyclic by Corollary 3.23, we compute

$$f^*([\mathcal{G}^\bullet]) = f^*([\mathcal{E}^\bullet]) = \sum_{i\in\mathbf{Z}}(-1)^i[\mathbf{L}^i f^*\mathcal{E}^i] = \sum_{i\in\mathbf{Z}}(-1)^i[f^*\mathcal{E}^i] = [f^*\mathcal{E}^\bullet] = [\mathbf{L}f^*\mathcal{G}^\bullet]. \qquad \square$$

Definition 4.18. Let X and Y be smooth projective complex varieties. The *K-theoretic Fourier–Mukai transform* with kernel $\xi \in K_\circ(X\times Y)$ is defined to be

$$\Psi_\xi\colon K_\circ(X) \to K_\circ(Y), \qquad \Psi_\xi := q_!(\xi \cdot p^*(-)).$$

Corollary 4.19. *Let X and Y be smooth projective complex varieties and $\mathcal{E}^\bullet$ a bounded complex of $\mathsf{D}^b(X\times Y)$. Then*

$$\Psi_{[\mathcal{E}^\bullet]}([-]) = [\Phi_{\mathcal{E}^\bullet}(-)].$$

Proof. This is a direct consequence of the compatibility between the product on $K_\circ(X)$ and the derived tensor product on $\mathsf{D}^b(X)$, according to (4.3), and Proposition 4.17. $\square$

4.2 ...on rational cohomology

In order to understand how a Fourier–Mukai transform descends to rational cohomology, we must first explain what is meant by *rational cohomology*. We will take it to be the singular cohomology with rational coefficients, but applying it directly to the underlying topological spaces of a smooth projective variety over $\mathbf{C}$ is *not* the appropriate notion one has to consider. Instead we have to resort to the famous *GAGA principle* of Serre [Ser56], which allows one to identify a smooth projective variety X over $\mathbf{C}$ with its complex analytic space X^{an}, whose underlying set of points consists of precisely the $\mathbf{C}$-points of X. In the case that X is a smooth projective variety over $\mathbf{C}$, the associated complex analytic space X^{an} is a complex projective[21] manifold. Moreover this identification is functorial, meaning that it provides an equivalence between the category of smooth projective varieties over $\mathbf{C}$,

[21] A complex manifold X is *projective* if it admits an embedding into complex projective space $\mathbf{C}\mathrm{P}^n$ for some $n \in \mathbf{Z}_{\geq 0}$.

considered as a full subcategory of $\mathsf{Sch}_{/\mathbf{C}}$, and the category of complex projective manifolds together with holomorphic maps. Moreover it also states that the categories of coherent sheaves on X and the category of *analytic coherent sheaves* on the corresponding complex analytic space X^{an} are equivalent. A good reference for this topic is [GA84]. Henceforth whenever we will be dealing with smooth projective varieties X over $\mathbf{C}$ and considering the rational cohomology groups $H^n(X, \mathbf{Q})$, we will implicitly pass to their complex analytic spaces X^{an} and interpret the cohomology groups as $H^n(X^{\mathrm{an}}, \mathbf{Q})$.

To introduce a Fourier–Mukai type map between the rational cohomology modules of X and Y associated to any cohomology class $\alpha \in H^*(X \times Y, \mathbf{Q})$, we must first define the correct analog of the "push-forward map" on cohomology. Again we are assuming that X and Y are smooth projective varieties over $\mathbf{C}$ and we identify them with their respective complex analytic spaces, which are compact connected complex manifolds. As such they are orientable as real manifolds, allowing Poincaré duality to relate their homology and cohomology. Let n denote the *real* dimension of X and let $[X] \in H_n(X, \mathbf{Q})$ denote the (rational) fundamental class of X. By Poincaré duality [Bre93, §VI, Theorem 8.3] the homomorphism

$$[X] \frown - \colon H^k(X, \mathbf{Q}) \to H_{n-k}(X, \mathbf{Q})$$

is an isomorphism and we denote its inverse with $D_X \colon H_{n-k}(X, \mathbf{Q}) \to H^k(X, \mathbf{Q})$.

Definition 4.20. Let $f \colon X \to Y$ be a continuous map between two compact connected complex manifolds. Let n and m denote the real dimensions of X and Y, respectively. The *Gysin map*[22] $f_! \colon H^k(X, \mathbf{Q}) \to H^{m-n+k}(Y, \mathbf{Q})$ is defined to be the composition

$$f_! \colon \quad H^k(X, \mathbf{Q}) \xrightarrow{[X]\frown -} H_{n-k}(X, \mathbf{Q}) \xrightarrow{f_*} H_{n-k}(Y, \mathbf{Q}) \xrightarrow{D_Y} H^{m-n+k}(Y, \mathbf{Q}).$$

Remark 4.21. In other words $f_!$ is precisely the map, which fits into the commutative diagram below

$$\begin{tikzcd}
H^k(X, \mathbf{Q}) \arrow[r, "{[X]\frown -}"] \arrow[d, "f_!"'] & H_{n-k}(X, \mathbf{Q}) \arrow[d, "f_*"] \\
H^{m-n+k}(X, \mathbf{Q}) \arrow[r, "{[Y]\frown -}"] & H_{n-k}(X, \mathbf{Q}).
\end{tikzcd}$$

Remark 4.22. For a topological space X we will distinguish between the collection of all cohomology groups, considered concisely as a graded object $H^\bullet(X, \mathbf{Z}) = \bigoplus_{i \in \mathbf{Z}} H^i(X, \mathbf{Z})$, and its (graded) cohomology ring, which we will denote by $H^*(X, \mathbf{Z})$. In that regard $f^* \colon H^\bullet(Y, \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$ is a graded map of degree 0 and $f_! \colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q})$ is a graded map of degree $m - n$.

Proposition 4.23 (Cohomological projection formula)**.** *Let $f \colon X \to Y$ be a continuous map between closed orientable manifolds X and Y. Then for all $\alpha \in H^*(X, \mathbf{Z})$ and $\beta \in H^*(Y, \mathbf{Z})$*

$$f_!(\alpha) \smile \beta = f_!(\alpha \smile f^*\beta). \tag{4.4}$$

Remark 4.24. In the proof of the above proposition we use the following formulas, which are easily seen to follow by unwrapping the definitions of operations $\smile$ and $\frown$ on singular cohomology and homology. They may also be found in [Bre93, §VI, Theorem 5.2].

[22] This map is sometimes also called the *umkehr map*. The German word "umkehr" pointing out the fact that the map goes in the opposite direction of the standard pull-back.

(i) Let $\eta \in H^k(X, \mathbf{Z})$, $\theta \in H^\ell(X, \mathbf{Z})$ and $\sigma \in H_{k+\ell+m}(X, \mathbf{Z})$, then

$$(\sigma \frown \eta) \frown \theta = \sigma \frown (\eta \smile \theta)$$

holds in $H_m(X, \mathbf{Z})$.

(ii) Let $\sigma \in H_{k+\ell}(X, \mathbf{Z})$ and $\eta \in H^k(Y, \mathbf{Z})$ then

$$f_*(\sigma \frown f^*\eta) = f_*\sigma \frown \eta$$

holds in $H_\ell(Y, \mathbf{Z})$.

Proof of Proposition 4.23. We compute that

$$\begin{aligned} f_!(\alpha \smile f^*\beta) &= D_Y(f_*([X] \frown (\alpha \smile f^*\beta))) \\ &= D_Y(f_*(([X] \frown \alpha) \frown f^*\beta)) \\ &= D_Y(f_*([X] \frown \alpha) \frown \beta) \\ &= D_Y(([Y] \frown f_!(\alpha)) \frown \beta) \\ &= D_Y([Y] \frown (f_!(\alpha) \smile \beta)) \\ &= f_!(\alpha) \smile \beta. \end{aligned}$$

□

Proposition 4.25 (Cohomological base change formula)**.** *Consider the following pull-back square of closed orientable manifolds and assume g and v are fibrations.*

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then

$$f^* \circ g_! = v_! \circ u^* \tag{4.5}$$

as homomorphisms $H^\bullet(Y', \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$.

Proof. □

needs reference

As before let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ denote the canonical projections, now in the category of complex manifolds.

Definition 4.26. The *cohomological Fourier-Mukai transform* associated to a cohomology class $\alpha \in H^\bullet(X \times Y, \mathbf{Q})$ is defined to be the homomorphism

$$f^\alpha\colon H^\bullet(X, \mathbf{Q}) \longrightarrow H^\bullet(Y, \mathbf{Q}) \qquad \beta \longmapsto q_!(\alpha \smile p^*\beta).$$

We would now like to relate this cohomological Fourier-Mukai transform back to the K-theoretic and the categorical one. This will be achieved by applying the famous Grothendieck–Riemann–Roch theorem, for which we first need to introduce the Chern character and the Todd class.

Grothendieck-Riemann-Roch theorem

In order to relate Fourier–Mukai functors on bounded derived categories of smooth projective varieties over $\mathbf{C}$ with their rational cohomology counterparts we will employ the famous Grothendieck–Riemann–Roch theorem. For us to be able to state it, we first introduce the Chern character and the Todd class of a smooth complex variety.

Definition 4.27. Let X be a smooth projective variety over $\mathbf{C}$. The *Chern character* is a ring homomorphism

$$\mathrm{ch}\colon K_\circ(X) \to H^*(X, \mathbf{Q})$$

enjoying many desirable properties, such as

(i) For a line bundle $\mathcal{L}$ on X: $\mathrm{ch}(\mathcal{L}) = e^{\mathrm{c}_1(\mathcal{L})} = 1 + \mathrm{c}_1(\mathcal{L}) + \frac{1}{2!}\mathrm{c}_1(\mathcal{L})^2 + \frac{1}{3!}\mathrm{c}_1(\mathcal{L})^3 + \cdots$.

(ii) [23]For any coherent sheaf $\mathcal{E}$ on X: $\mathrm{ch}(\mathcal{E}^\vee) = \mathrm{ch}(\mathcal{E})^\vee$.

(iii) Naturality for pull-back: $f^*\mathrm{ch}(e) = \mathrm{ch}(f^*e)$ for any proper morphism $f\colon X \to Y$.

(iv) The Chern character takes values in $H^{2\bullet}(X, \mathbf{Q})$, i.e. cohomology groups of *even* degree. We let $\mathrm{ch}_i(e)$ denote the component of $\mathrm{ch}(e)$ living in $H^{2i}(X, \mathbf{Q})$.

Remark 4.28. Such a homomorphism in fact exists and a construction may be found in [Ful98, §3.2]. The Chern character is usually constructed by first specifying it on line bundles as in (i). Then utilizing the so-called *Splitting principle* [Ful98, §3.2, Remark 3.2.3] it is possible to extend the Chern character to arbitrary vector bundles on X. Further, one defines the Chern character of a coherent sheaf $\mathcal{E}$ on X using a bounded locally free resolution, mentioned already in Proposition 3.3. Lastly, one extends the Chern character to arbitrary bounded complexes of coherent sheaves by additivity i.e.

$$\mathrm{ch}(\mathcal{F}^\bullet) = \sum_{i \in \mathbf{Z}} (-1)^i \mathrm{ch}(\mathcal{F}^i).$$

Along with the Chern character also come Chern classes of vector bundles. For a vector bundle $\mathcal{E}$ on X the i-th Chern class will be an element $\mathrm{c}_i(\mathcal{E})$ of $H^{2i}(X, \mathbf{Z})$. The 0-th Chern class $\mathrm{c}_0(\mathcal{E})$ equals the rank of the vector bundle $\mathcal{E}$ and is also denoted by $\mathrm{rk}(\mathcal{E})$. For a construction and overview on Chern classes we again refer the reader to [Ful98, §3.2]. We introduced Chern classes to get a more practical description of the Chern character. Indeed, it turns out that the Chern character evaluated on a vector bundle $\mathcal{E}$ is expressible in terms of the Chern classes $\mathrm{c}_i = \mathrm{c}_i(\mathcal{E})$ of $\mathcal{E}$ as the following series to be found in [Ful98, §3, Example 3.2.3]

$$\begin{aligned}\mathrm{ch}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + &\frac{1}{2}\Big(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})\Big) + \\ &+ \frac{1}{6}\Big(\mathrm{c}_1(\mathcal{E})^3 - 3\mathrm{c}_1(\mathcal{E})\,\mathrm{c}_2(\mathcal{E}) + 3\mathrm{c}_3(\mathcal{E})\Big) + \cdots. \qquad (4.6)\end{aligned}$$

The Chern classes of X are defined to be the Chern classes $\mathrm{c}_i(\mathcal{T}_X)$ of the tangent sheaf $\mathcal{T}_X = \Omega_X^\vee$ of X and denoted simply by $\mathrm{c}_i(X)$.

The *Todd class* of a smooth complex projective variety X is a cohomology class, living in even degrees, denoted by

$$\mathrm{td}_X \in H^{2\bullet}(X, \mathbf{Q}).$$

[23]See 4.39 for the definiton of $(-)^\vee$ and [vBr20, Lemma C.7] for a proof.

As with the Chern character and Chern classes we refrain from properly defining or constructing td_X, but give the first few terms of its series expansion in terms of Chern classes of X. It may be found in [Ful98, §3, Example 3.2.4] and goes as follows

$$\mathrm{td}_X = 1 + \frac{1}{2}\mathrm{c}_1(X) + \frac{1}{12}\Big(\mathrm{c}_1(X)^2 + \mathrm{c}_2(X)\Big) + \frac{1}{24}\mathrm{c}_1(X)\mathrm{c}_2(X) + \cdots . \tag{4.7}$$

Theorem 4.29 (Grothendieck–Riemann–Roch)**.** *Let $f\colon X \to Y$ be a proper morphism of smooth complex varieties. Then for all $e \in K_\circ(X)$, equation*

$$\mathrm{ch}(f_!(e)) \smile \mathrm{td}_Y = f_!(\mathrm{ch}(e) \smile \mathrm{td}_X) \tag{4.8}$$

holds in $H^*(Y, \mathbf{Q})$.

Proof. A proof may be found in [Ful98, §15]. □

Remark 4.30. In the formulation of the Grothendieck–Riemann–Roch theorem in [Ful98, §15, Theorem 15.2] the equation (4.8) holds inside the so-called (rational) *Chow ring* $A(Y)_{\mathbf{Q}} = A(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ instead of $H^*(Y, \mathbf{Q})$. We will not define this ring, but mention that there exists a push-forward map also on these Chow rings $f_*\colon A(X) \to A(Y)$ and homomorphisms $\mathrm{cl}\colon A(X) \to H^*(X, \mathbf{Z})$, for each smooth complex variety X, called *cycle maps*, which are covariant in X and compatible with Chern classes. This enables us to state the theorem in the way that we did. More on this topic can be found in [Ful98, §19].

For a coherent sheaf $\mathcal{E}$ on X set $h^i(X, \mathcal{E}) = \dim_{\mathbf{C}} H^i(X, \mathcal{E})$, which we know to be finite by Theorem 3.8. We also know that $H^i(X, \mathcal{E}) = 0$ for $i > \dim X$ by a vanishing result of Grothendieck 3.14. The concept of an *Euler characteristic* of $\mathcal{E}$, set out to be

$$\chi(X, \mathcal{E}) = \sum_{i \in \mathbf{Z}} (-1)^i h^i(X, \mathcal{E}),$$

is then well-defined. From the existence of a long exact sequence in cohomology (3.3), we see that $\chi(X, -)$ is additive on short exact sequences i.e. for a short exact sequence of coherent sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, we have

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{E}) + \chi(X, \mathcal{G}).$$

Precisely this property allows us to extend $\chi(X, -)$ to a group homomorphism on $K_\circ(X)$.

Theorem 4.31 (Hirzebruch–Riemann–Roch)**.** *Let X be a smooth complex projective variety and $\mathcal{E}$ a coherent sheaf on X. Then*

$$\chi(X, \mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\, \mathrm{td}_X . \tag{4.9}$$

Remark 4.32. The integral on the right hand side we will take to be $\int_X \mathrm{ch}(\mathcal{E})\,\mathrm{td}_X = \langle (\mathrm{ch}(\mathcal{E})\,\mathrm{td}_X)_n, [X] \rangle$, where n denotes the real dimension of X. Computing the integral then amounts to calculating the n-th degree contributions of the product $\mathrm{ch}(\mathcal{E})\,\mathrm{td}_X$ and evaluating them into a rational number through the isomorphism $H^n(X, \mathbf{Q}) \simeq \mathbf{Q}$, given by the choice of a fundamental class $[X]$.

Proof. We will use the Grothendieck–Riemann–Roch formula for the structure morphism $\varepsilon\colon X \to \operatorname{Spec} \mathbf{C}$. The Chern character on a point is completely determined by its image of the trivial line bundle $\mathcal{L} = \mathcal{O}_{\operatorname{Spec}\mathbf{C}}$. This is because $\mathsf{coh}(\operatorname{Spec}\mathbf{C})$ is equivalent to the category of finite dimensional $\mathbf{C}$-vector spaces and ch is additive. In this case we have

$$\operatorname{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = e^0 = 1.$$

Therefore after identifying the ring $H^*(\operatorname{Spec}\mathbf{C}, \mathbf{Q})$ with $\mathbf{Q}$, we see that $\operatorname{ch}(-) = \dim_{\mathbf{C}}(-)$. We also note that $\operatorname{td}_{\operatorname{Spec}\mathbf{C}} = 1$ in this cohomology ring. The left-hand side of (4.8) is then

$$\operatorname{ch}(\varepsilon_!(\mathcal{E})) \smile 1 = \operatorname{ch}\left(\sum_{i\in\mathbf{Z}}(-1)^i[\mathbf{R}^i\varepsilon_*\mathcal{E}]\right) = \sum_{i\in\mathbf{Z}}(-1)^i h^i(X,\mathcal{E}) = \chi(X,\mathcal{E}).$$

Evaluating the right-hand side only amounts to interpreting the meaning of $\varepsilon_!$ according to remark before this proof. □

Remark 4.33. Notice that since the left-hand side of equation (4.9) is an integer, the integral on the right must be as well. We also remark that since both sides of (4.9) are additive in $\mathcal{E}$, the formula actually describes an equality of two group homomorphisms $K_\circ(X) \to \mathbf{Z}$. This is no surprise, as this same behaviour was already present in the Grothendieck–Riemann–Roch formula, from which we have seen the Hirzebruch–Riemann–Roch formula follows.

Mukai vector

Of special importance for relating the K-theoretic Fourier–Mukai transforms with their cohomological counterparts is the Mukai vector.

Definition 4.34. The *Mukai vector* of a class $e \in K_\circ(X)$ is defined to be

$$v(e) := \operatorname{ch}(e)\sqrt{\operatorname{td}_X}.$$

The Mukai vector of a complex $\mathcal{E}^\bullet$ of $\mathsf{D}^b(X)$ is defined to be the Mukai vector of the corresponding class $[\mathcal{E}^\bullet] \in K_\circ(X)$, so $v(\mathcal{E}^\bullet) := v([\mathcal{E}^\bullet])$.

Remark 4.35. Recall that for compact manifolds X the cohomology ring $H^*(X,\mathbf{Q})$ is finite dimensional. Thus we can get away with defining the square root of a cohomology class of the form $1 + \alpha \in H^*(X,\mathbf{Q})$, where α contains no degree 0 homogeneous term, by a power series

$$\sqrt{1+\alpha} = 1 + \tfrac{1}{2}\alpha - \tfrac{1}{8}\alpha^2 + \tfrac{1}{16}\alpha^3 + \cdots \in H^*(X,\mathbf{Q}). \tag{4.10}$$

In particular this lets us define $\sqrt{\operatorname{td}_X}$.

Example 4.36. Let us compute the Mukai vector of a skyscraper sheaf $k(x)$ associeted to a closed point x of a smooth n-dimensional projective variety X over $\mathbf{C}$. Let $x\colon \operatorname{Spec}\mathbf{C} \to X$ also denote the corresponding morphism and let $\mu_X \in H^n(X,\mathbf{Q}) \simeq \mathbf{Q}$ denote the generator for which $\langle \mu_X, [X]\rangle = 1$. By the Grothendieck–Riemann–Roch formula we compute

$$\operatorname{ch}(k(x))\operatorname{td}_X = \operatorname{ch}(x_*\mathcal{O}_{\operatorname{Spec}\mathbf{C}})\operatorname{td}_X = x_!(\operatorname{ch}(\mathcal{O}_{\operatorname{Spec}\mathbf{C}})\operatorname{td}_{\operatorname{Spec}\mathbf{C}}) = x_!(1) = \mu_X.$$

Thus we see that

$$v(k(x)) = \mu_X \smile \sqrt{\operatorname{td}_X}^{-1}.$$

Proposition 4.37. *For each kernel $\xi \in K_\circ(X\times Y)$ and any $e \in K_\circ(X)$ we have*

$$f^{v(\xi)}(v(e)) = v(\Psi_\xi(e))$$

Proof. This equation follows from the following computation

$$\begin{aligned}
f^{v(\xi)}(v(e)) &= q_!(v(\xi) \smile p^* v(e)) \\
&= q_!(\operatorname{ch}(\xi)\sqrt{\operatorname{td}_{X\times Y}} \smile p^*(\operatorname{ch}(e)\sqrt{\operatorname{td}_X})) \\
&= q_!(\operatorname{ch}(\xi) p^*\sqrt{\operatorname{td}_X} \smile q^*\sqrt{\operatorname{td}_Y} \smile \operatorname{ch}(p^*(e))\, p^*\sqrt{\operatorname{td}_X}) && \text{(Def. 4.27)} \\
&= q_!(\operatorname{ch}(\xi\cdot p^*(e))\operatorname{td}_{X\times Y} \smile q^*\sqrt{\operatorname{td}_Y}^{-1}) && \text{(Formula (3.8))} \\
&= q_!(\operatorname{ch}(\xi\cdot p^*(e))\operatorname{td}_{X\times Y}) \smile \sqrt{\operatorname{td}_Y}^{-1} && \text{(Formula (4.8))} \\
&= \operatorname{ch}(q_!(\xi\cdot p^*(e)))\sqrt{\operatorname{td}_Y} \\
&= v(\Psi_\xi(e)).
\end{aligned}$$

□

From now on a cohomological Fourier–Mukai transform, whose kernel is a Mukai vector $v(\mathcal{E})$ of some complex $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$, will be denoted by $f^{\mathcal{E}}\colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q})$ instead of $f^{v(\mathcal{E})}$.

Proposition 4.38. *Let $f^{\mathcal{E}}\colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q})$ and $f^{\mathcal{F}}\colon H^\bullet(Y, \mathbf{Q}) \to H^\bullet(Z, \mathbf{Q})$ be cohomological Fourier–Mukai transforms for kernels $\mathcal{E}$ and $\mathcal{F}$ belonging to $\mathsf{D}^b(X \times Y)$ and $\mathsf{D}^b(Y \times Z)$ respectively. Then the composition $f^{\mathcal{F}} \circ f^{\mathcal{E}}$ is also a cohomological Fourier-Mukai transform given by the kernel $\mathcal{G}$ of Proposition 4.7.*

Proof. The proof of this result is analogous to the proof of Proposition 4.7, where instead we use the cohomological projection and base change formulas, (4.4) and (4.5), respectively. We also mention that the Mukai vector $v(\mathcal{G})$ turns out being equal to the class $(\pi_{XZ})_!(\pi_{XY}^* v(\mathcal{E}) \smile \pi_{YZ}^* v(\mathcal{F}))$ in $H^*(X \times Z, \mathbf{Q})$, which is also an ingredient we need for the computation above, but we omit the tedious verification. □

Definition 4.39. For a vector $v \in H^{2\bullet}(X, \mathbf{Q}) = \bigoplus_{i\in\mathbf{Z}} H^{2i}(X, \mathbf{Q})$ confined in *even* degrees, with components $v = (v_0, v_1, v_2, v_3, \dots)$, we define $v^\vee = (v_0, -v_1, v_2, -v_3, \dots)$.

This operation is clearly natural for pull-backs and fixes all cohomology classes living in degrees, which are multiples of 4.

Definition 4.40. For any v, $w \in H^{2\bullet}(X, \mathbf{Q})$ coming from the even part, we define the *Mukai pairing* on $H^{2\bullet}(X, \mathbf{Q})$ to be

$$\langle v, w\rangle = \int_X v^\vee \smile w.$$

Since the Mukai vector of any coherent sheaf $\mathcal{E}$ is spread out only in the even cohomology groups, we can pair their Mukai vectors with each other in this way. Another way of pairing up coherent sheaves is given by the *Euler pairing*, defined for coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on X to be the alternating sum

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_{i\in\mathbf{Z}} (-1)^i \dim_{\mathbf{C}} \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}). \tag{4.11}$$

The following proposition says that these two pairings of coherent sheaves are the same.

Proposition 4.41. *Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X. Then*

$$\langle v(\mathcal{E}), v(\mathcal{F})\rangle = \chi(\mathcal{E}, \mathcal{F}). \tag{4.12}$$

Proof. We first prove this equality when $\mathcal{E}$ is a vector bundle and then argue why this is sufficient. First, we note that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$ and this isomorphism is functorial in $\mathcal{F}$. Moreover, since $\mathcal{E}^\vee$ is a vector bandle, the functor $\mathcal{E}^\vee \otimes -$ is exact. Using Proposition 2.34 we compute

$$\begin{aligned} \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) &= H^i(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},-)(\mathcal{F})) \\ &\simeq H^i((\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},-))(\mathcal{F})) \\ &\simeq H^i((\mathbf{R}\Gamma \circ (\mathcal{E}^\vee \otimes -))(\mathcal{F})) \\ &\simeq \mathbf{R}^i\Gamma(\mathcal{E}^\vee \otimes \mathcal{F}) \\ &\simeq H^i(X, \mathcal{E}^\vee \otimes \mathcal{F}). \end{aligned}$$

Applying the Hirzebruch–Riemann–Roch formula (4.9), we see that

$$\langle v(\mathcal{E}), v(\mathcal{F})\rangle = \int_X \mathrm{ch}(\mathcal{E})^\vee \,\mathrm{ch}(\mathcal{F})\,\mathrm{td}_X = \int_X \mathrm{ch}(\mathcal{E}^\vee \otimes \mathcal{F})\,\mathrm{td}_X = \chi(X, \mathcal{E}^\vee \otimes \mathcal{F}) = \chi(\mathcal{E},\mathcal{F}).$$

Since both sides of the equation (4.12) are additive on short exact sequences in the varible $\mathcal{E}$, the general formula follows from this special case of vector bundles. This is because every coherent sheaf on X has a bounded locally free resolution. ☐

Hodge structures

This short section serves as a quick recollection on the very basics of Hodge theory and ends with a proposition describing a relationship between a cohomological Fourier–Mukai transform and some Hodge decompositions.

Definition 4.42. A *Hodge structure of weight* $n \in \mathbf{Z}$ on a finitely generated free abelian group or a finite dimensional $\mathbf{Q}$-vector space V is a direct sum decomposition of $V_\mathbf{C} = V \otimes_\mathbf{Z} \mathbf{C}$, consisting of subspaces $V^{p,q} \subseteq V_\mathbf{C}$ for p, $q \in \mathbf{Z}$ of the form

$$V_\mathbf{C} = \bigoplus_{p+q=n} V^{p,q},$$

satisfying $\overline{V^{p,q}} = V^{q,p}$ for all pairs p, $q \in \mathbf{Z}$. If V is a $\mathbf{Z}$-module, we say it is equipped with an *integral* Hodge structure and if V is a $\mathbf{Q}$-vector space, we say the Hodge structure is *rational*.

A $\mathbf{Z}$-linear (resp. $\mathbf{Q}$-linear) homomorphism $f\colon V \to W$ between two integral (resp. rational) Hodge structures of weight n is said to *preserve the Hodge structures*, if

$$f_\mathbf{C}(V^{p,q}) \subseteq W^{p,q}$$

for all p, $q \in \mathbf{Z}$, with $p + q = n$, where $f_\mathbf{C}$ denotes the complexification $f \otimes \mathrm{id}_\mathbf{C}$.

Remark 4.43. Complex conjugation on $V_\mathbf{C} = V \otimes_\mathbf{Z} \mathbf{C}$ is defined on elementary tensors as $\overline{v \otimes \lambda} = v \otimes \bar{\lambda}$.

A prototypical example of a Hodge structure of weight n comes from complex geometry and is given by the following decomposition of the n-th cohomology group $H^n(X,\mathbf{C})$ of a complex projective manifold X

$$H^n(X,\mathbf{Q})_\mathbf{C} = \bigoplus_{p+q=n} H^{p,q}(X). \tag{4.13}$$

The groups $H^{p,q}(X)$ are defined to be the sheaf cohomology groups $H^q(X, \Omega_X^p)$. Here Ω_X^p denotes the sheaf of holomorphic p-forms or more precisely the p-th exterior power $\bigwedge^p \Omega_X$ of the cotangent sheaf Ω_X on X. The groups $H^{p,q}(X)$ turn out to be isomorphic to the so-called *Dolbeault cohomology groups* $H_{\bar{\partial}}^{p,q}(X)$, which are in turn easier to identify with certain subspaces of globally defined (p,q)-forms, which are called *harmonic* forms. This shift of perspective, from cohomology classes to globally defined differntial forms, then makes the origin of a Hodge decomposition of $H^n(X, \mathbf{C})$ at least somewhat reasonable and we mention that it arises as a result of the *Hodge decomposition theorem* [GH94, §0.6]. We collect a few facts, which one is able to deduce from this viewpoint. They can be found in standard texts like [GH94] or [Voi02].

- ▷ For any p, $q \in \mathbf{Z}$: $\overline{H^{p,q}(X)} = H^{q,p}(X)$ [Voi02, §II, Corollary 6.12].
- ▷ The exterior product of a (p,q)-form α with a (r,s)-from β is a $(p+r, q+s)$ form $\alpha \wedge \beta$. Consequently, if n denotes the complex dimension of X and if either $p+r > n$ or $q+s > n$, then $\alpha \wedge \beta = 0$ [Voi02, §II, Corollary 6.15].
- ▷ For any holomorphic map $f\colon X \to Y$, the pull-back $f^*\colon H^\bullet(Y, \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$ preserves the Hodge structures, meaning that if β is a (p,q)-from on Y, $f^*\beta$ is a (p,q)-from on X. See [Voi02, §7.3.2].
- ▷ Let n and m denote the *complex* dimensions of compact complex manifolds X and Y, respectively and set $k = m - n$. Then for any holomorphic map $f\colon X \to Y$, the Gysin map $f_!\colon H^\bullet(X, \mathbf{C}) \to H^\bullet(Y, \mathbf{C})[2k]$ is of bi-degree (k,k), meaning that a (p,q)-from α on X is sent to a $(p+k, q+k)$-form $f_*\alpha$ on Y. In that regard $f_!$ does not preserve the Hodge structure unless X and Y are equidmensional. See [Voi02, §7.3.2].
- ▷ Characteristic classes like the Chern character and the Todd class are *algebraic*, meaning they live in
$$\bigoplus_{p \in \mathbf{Z}} H^{p,p}(X) \cap H^{2p}(X, \mathbf{Q}).$$

A useful invariant related to the Hodge decomposition (4.13) are the *Hodge numbers*

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^{p,q}(X).$$

Remark 4.44. In this section the meaning of cohomology groups $H^n(X, \mathbf{C})$ is a bit blurred. At times we take it to be the singular cohomology with complex coefficients, this makes it easy to understand how $H^n(X, \mathbf{Q})$ might be a subgroup of $H^n(X, \mathbf{C})$. At other times we take it to be the complex de Rham cohomology, which is better suited for Hodge decompositions. In any case our worries may be remedied by the complex analogue of *de Rham's theorem*, which states that for compact complex manifolds X there is a natural isomorphism of graded rings

$$H^*_{\mathrm{dR}}(X, \mathbf{C}) \to H^*_{\mathrm{sing}}(X, \mathbf{C}).$$

For a proof using a sheaf-theoretic argument see [GH94, p. 43–45].

Cohomological Fourier–Mukai transforms unfortunately do not preserve the Hodge structures in general[24], but they interact with Hodge structures in a particular way. At least

[24] In the same breath we have to mention that for K3 surfaces Fourier–Mukai transforms actually *do* preserve the Hodge structures, as on can deduce from Lemma 6.7. However note that one uses also a certain lattice structure on the cohomology ring to prove this fact.

when they arise from equivalences of the corresponding bounded derived categories, they preserve another kind of grading.

Proposition 4.45. *Let X and Y be complex projective varieties over $\mathbf{C}$. Consider the cohomological Fourier–Mukai transform $f^{\mathcal{E}}\colon H^\bullet(X,\mathbf{Q}) \to H^\bullet(Y,\mathbf{Q})$ associated to a kernel $\mathcal{E}$ belonging to $\mathsf{D}^b(X\times Y)$, for which $\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ is an equivalence. Then for every $i \in \mathbf{Z}$, transform $f^{\mathcal{E}}$ induces isomorphisms*

$$\bigoplus_{p-q=i} H^{p,q}(X) \simeq \bigoplus_{p-q=i} H^{p,q}(Y).$$

Proof. First we show that $f^{\mathcal{E}}$ is an isomorphism. As $\Phi_{\mathcal{E}}$ is an equivalence, its left adjoint $\Phi_{\mathcal{E}_{\mathrm{L}}}$ is its quasi-inverse, so there is a natural isomorphism of functors $\Phi_{\mathcal{E}_{\mathrm{L}}} \circ \Phi_{\mathcal{E}} \simeq \mathrm{id}_{\mathsf{D}^b(X)}$. On the other hand, By Proposition 4.7, we know that the composition $\Phi_{\mathcal{E}_{\mathrm{L}}} \circ \Phi_{\mathcal{E}}$ is also a Fourier–Mukai transform, given by some kernel $\mathcal{G}$ of $\mathsf{D}^b(X\times X)$. Hence by Orlov's theorem 4.8, stipulating uniqueness of kernels of Fourier–Mukai equivalences, we obtain $\mathcal{G} \simeq \mathcal{O}_\Delta$, because $\mathrm{id}_{\mathsf{D}^b(X)} \simeq \Phi_{\mathcal{O}_\Delta}$. On the level of cohomology we therefore see that $f^{\mathcal{E}_{\mathrm{L}}} \circ f^{\mathcal{E}} = f^{\mathcal{O}_\Delta}$. Borrowing Lemma 6.13 from a future chapter and noting that the exact same proof works in our slightly more general case, we get $f^{\mathcal{O}_\Delta} = \mathrm{id}_{H^\bullet(X,\mathbf{Q})}$. This means that $f^{\mathcal{E}}$ is an isomorphism. Now it is enough to prove that for each p, $q \in \mathbf{Z}$ the following inclusion holds

$$f^{\mathcal{E}}_{\mathbf{C}}\left(H^{p,q}(X)\right) \subseteq \bigoplus_{r-s=p-q} H^{r,s}(Y). \tag{4.14}$$

The Mukai vector $v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_{X\times Y}}$ being a product of two algebraic classes is also algebraic, meaning that it is a sum of classes of type (p,p), i.e.

$$v(\mathcal{E}) \in \bigoplus_{p\in\mathbf{Z}} H^{p,p}(X\times Y) \cap H^{2p}(X\times Y,\mathbf{Q}).$$

Consider the complexified[25] cohomological Fourier–Mukai transform $f^{\mathcal{E}}$ as the composition

$$f^{\mathcal{E}}_{\mathbf{C}}\colon \quad H^\bullet(X,\mathbf{C}) \xrightarrow{p^*} H^\bullet(X\times Y,\mathbf{C}) \xrightarrow{v(\mathcal{E})\smile -} H^\bullet(X\times Y,\mathbf{C}) \xrightarrow{q_!} H^\bullet(Y,\mathbf{C}).$$

Pick any class $\theta \in H^{k,\ell}(X)$. Since p^* is of bi-degree $(0,0)$, $p^*\theta \in H^{k,\ell}(X\times Y)$. Multiplying by $v(\mathcal{E})$ results in a sum of classes of type $(k+t,\ell+t)$ for $t\in\mathbf{Z}$, i.e.

$$v(\mathcal{E}) \smile p^*\theta \in \bigoplus_{t\in\mathbf{Z}} H^{k+t,\ell+t}(X\times Y)$$

Lastly, since the push-forward $q_!$ is of bi-degree (n,n), where n denotes the dimension of X, we end up with a sum of classes of type $(k+t+n,\ell+t+n)$ for $t\in\mathbf{Z}$. Re-indexing shows that $f^{\mathcal{E}}(\theta)$ is a sum of classes of type (r,s), where r, s run over the integers, satisfying $r-s=k-\ell$, i.e.

$$f^{\mathcal{E}}(\theta) = q_!(v(\mathcal{E})\smile p^*\theta) \in \bigoplus_{\substack{r,s\in\mathbf{Z}\\ r-s=k-\ell}} H^{r,s}(Y). \qquad \square$$

[25] The universal coefficients theorem allows us to just swap $\mathbf{Q}$ for $\mathbf{C}$ everywhere, because the isomorphisms $H^n(X,\mathbf{Q})\otimes\mathbf{C} \simeq H^n(X,\mathbf{C})$ and $H_n(X,\mathbf{Q})\otimes\mathbf{C} \simeq H_n(X,\mathbf{C})$ are natural.

5 K3 surfaces

This chapter serves as a gentle introduction and overview of the main properties of K3 surfaces. We will start off by defining algebraic K3 surfaces and their complex counterparts. Next, using a combination of both viewpoints we will derive the most important invariants of K3 surfaces, beginning with the computation of the Chern and Todd classes, which will enable us to compute the so-called Hodge diamond of a K3 surface. Using some topological methods we will identify all the integral cohomology groups. Along the way we will introduce the Picard and Néron–Severi group and see that they are isomorphic. We will conclude the chapter with examining the intersection pairing and finally stating the global Torelli theorem at the end characterizing K3 surfaces through their Hodge lattices.

5.1 Algebraic and complex

K3 surfaces come in two flavours, algebraic and complex, essentially equivalent, but still formally different to allow us to exploit both the methods of algebraic geometry as well as topology in combination with complex geometry. We start with the algebraic ones.

Definition 5.1. A smooth projective surface X over a field k is a *K3 surface*, if

$$\omega_X \simeq \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

Example 5.2. Let X be a smooth quartic in $\mathbb{P}^3$, meaning a smooth projective variety given by some homogeneous polynomial equation of degree four. We will show that any such variety is a K3 surface. Let $f \in \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$ be the defining homogeneous polynomial for X of degree four. Since X is a hypersurface in $\mathbb{P}^3$ it is of dimension two i.e. a surface. Let $i\colon X \hookrightarrow \mathbb{P}^3$ denote the closed embedding of X into $\mathbb{P}^3$ and consider the following short exact sequence of coherent sheaves on $\mathbb{P}^3$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0.$$

The first non-trivial morphism is given by multiplication by f. The second non-trivial morphism arises as the unit of the adjunction $i^{-1} \dashv i_*$, since by definition we set $\mathcal{O}_X$ to be $i^{-1}\mathcal{O}_{\mathbb{P}^3}$. First consider a portion of the long exact sequence in cohomology (3.3)

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(\mathbb{P}^3, i_*\mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots.$$

Since the groups on the edge are known to vanish by [Har77, §III, Theorem 5.1], the middle one does as well. Noting that the push-forward i_* is exact, since i is an embedding, we see that Proposition 2.34 yields

$$\begin{aligned} H^1(\mathbb{P}^3, i_*\mathcal{O}_X) &= H^1(\mathbf{R}\Gamma_{\mathbb{P}^3}(i_*\mathcal{O}_X)) \\ &\simeq H^1((\mathbf{R}\Gamma_{\mathbb{P}^3} \circ i_*)(\mathcal{O}_X)) \\ &\simeq H^1(\mathbf{R}\Gamma_X(\mathcal{O}_X)) \\ &= H^1(X, \mathcal{O}_X). \end{aligned}$$

Therefore $H^1(X, \mathcal{O}_X) = 0$. Triviality of the canonical bundle ω_X is seen from the *conormal sequence*

$$0 \to \mathcal{O}_X(-4) \to i^*\Omega_{\mathbb{P}^3} \to \Omega_X \to 0.$$

After taking determinants we obtain

$$i^*\omega_{\mathbb{P}^3} \simeq \mathcal{O}_X(-4) \otimes \omega_X.$$

Upon tensoring with the inverse of the pull-back of $\omega_{\mathbb{P}^3} \simeq \mathcal{O}_{\mathbb{P}^3}(-4)$, we conclude with $\omega_X \simeq \mathcal{O}_X$.

Example 5.3. A very concrete example of a smooth quartic in $\mathbb{P}^3$ is the *Fermat quartic*[26], which is a surface cut out by the homogeneous polynomial equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

Complex analytic K3 surfaces come with essentially the same definition as the algebraic ones. Of course we are now interpreting the structure sheaf $\mathcal{O}_X$ as the sheaf of *holomorphic* functions on X and ω_X as the holomorphic canonical bundle.

Definition 5.4. A connected complex projective manifold X of dimension two is a *K3 surface*, if

$$\omega_X \simeq \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

Remark 5.5. To give a little intuition about the condition $H^1(X, \mathcal{O}_X) = 0$ in the definition, we will shortly see that also $H^1(X, \mathbf{Z}) = 0$. The latter is a necessary condition for X to be simply connected, however not sufficient. But in the case of complex analytic K3 surfaces all of them turn out to be simply connected. This is a deep fact we will not go into, but mention that it is proven using another profound result, namely that all analytic K3 surfaces are diffeomorphic and then showing that the Fermat quartic is simply connected. More on this can be found in [Huy16, §7.1].

As already mentioned in the introduction to Section 4.2 one can pass between the algebraic and the analytic categories using Serre's GAGA principle [Ser56]. We also mentioned the existence of an equivalence between categories of coherent sheaves on the algebraic side and analytic coherent sheaves on the complex analytic side preserving cohomology. This passage is also well behaved in the sense that the associated analytic coherent sheaf to the structure sheaf $\mathcal{O}_X$ is the sheaf of holomorphic functions $\mathcal{O}_{X^{\mathrm{an}}}$ on X^{an} and the sheaf associated to the sheaf of Kähler differentials Ω_X on X is the sheaf of holomorphic 1-forms $\Omega^1_{X^{\mathrm{an}}}$ on X^{an}. From this we see that the complex analytic space associated to an algebraic K3 surface is a complex K3 surface in the sense of Definition 5.4.

Remark 5.6. The *projective* assumption is usually not present in the definition. In fact it is replaced by a more general assumption of compactness. We made this choice, because we will only be dealing only with algebraic K3 surfaces and thus every complex analytic K3 surface will have its algebraic counterpart. It is also worth noting that there do exist non-projective complex K3 surfaces and that they show up in important places. One of which is in a proof of the global Torelli theorem 5.24, which will make an appearance at the end of this chapter, but whose proof we omit, instead referring to [Huy16, §7].

Notation. K3 surfaces are complex manifolds therefore orientable, implying $H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$. Until the rest of the chapter we fix a generator $\mu_X \in H^4(X, \mathbf{Z})$ for which $\langle \mu_X, [X] \rangle = 1$, where $[X] \in H_4(X, \mathbf{Z})$ is a chosen fundamental class of X. Throughout the remainder of this chapter we will be accessing the language of lattice theory. For its main themes we refer the reader to Appendix B.

5.2 Main invariants

The first vital tool for our examination is Serre duality (3.2). Given a vector bundle $\mathcal{E}$ on X it gives us isomorphisms $\mathrm{Ext}^{2-i}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \simeq H^i(X, \mathcal{E})^*$ for all $i \in \mathbf{Z}$. Here we have

[26] It is easily seen to be smooth by the Jacobian criterion for smoothness.

already used a defining property of a K3 surface, namely that $\omega_X \simeq \mathcal{O}_X$. The left side of the duality can be slightly modified to obtain

$$H^{2-i}(X, \mathcal{E}^\vee) \simeq H^i(X, \mathcal{E})^*. \tag{5.1}$$

Next we recall the general fact that the pairing

$$\Omega_X \otimes_{\mathcal{O}_X} \Omega_X \to \bigwedge^2 \Omega_X$$

is *perfect.* By definition this means that the morphism $\Omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \bigwedge^2 \Omega_X)$, associated to the pairing above, is an isomorphism. Since the canonical bundle $\omega_X = \bigwedge^2 \Omega_X$ of a K3 surface X is trivial, we see that

$$\Omega_X \simeq \Omega_X^\vee = \mathcal{T}_X.$$

This useful observation will be key in the computation of Chern and Todd classes of K3 surfaces.

Proposition 5.7. *Let X be a complex K3 surface, then*

$$\mathrm{c}_0(X) = 2 \qquad \mathrm{c}_1(X) = 0 \qquad \mathrm{c}_2(X) = 24\mu_X \qquad \mathrm{td}_X = 1 + 2\mu_X.$$

Proof. Tangent sheaf $\mathcal{T}_X$ is a vector bundle of rank 2, so $\mathrm{c}_0(X) = 2$. Since the sheaf of differentials Ω_X is self-dual, the tangent sheaf $\mathcal{T}_X$ is as well. First, we see that $\mathrm{ch}_1(\mathcal{T}_X) = \mathrm{c}_1(X)$. Then property (ii) of Definition 4.27 applied to $\mathcal{T}_X \simeq \mathcal{T}_X^\vee$ shows that $\mathrm{c}_1(X) = -\mathrm{c}_1(X)$. In Proposition 5.15 we will see that $H^2(X, \mathbf{Z})$ is torsion-free, thus we reach $\mathrm{c}_1(X) = 0$. The second Chern class is computed using the Hirzebruch–Riemann–Roch formula applied to the trivial line bundle $\mathcal{O}_X$. Utilizing (5.1) we compute its Euler characteristic to be

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 - 0 + 1 = 2.$$

Noting that $\mathrm{ch}(\mathcal{O}_X) = 1$ and simplifying (4.7) to $\mathrm{td}_X = 1 + \frac{1}{12}\mathrm{c}_2(X)$, Theorem 4.31 implies

$$2 = \chi(X, \mathcal{O}_X) = \int_X \mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}_X = \langle \tfrac{1}{12}\mathrm{c}_2(X), [X] \rangle.$$

Thus $\mathrm{c}_2(X) = 24\mu_X$ and $\mathrm{td}_X = 1 + 2\mu_X$. $\square$

For later we record the following K3 version of the Hirzebruch–Riemann–Roch formula. Let $\mathcal{E}$ be a coherent sheaf on X then

$$\chi(X, \mathcal{E}) = \langle \mathrm{ch}_2(\mathcal{E}), [X] \rangle + 2\,\mathrm{rk}(\mathcal{E}) = \langle \tfrac{1}{2}(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})), [X] \rangle + 2\,\mathrm{rk}(\mathcal{E}). \tag{5.2}$$

This already allows us to compute the Hodge numbers of a K3 surface X.

Proposition 5.8. *Let X be a K3 surface. Arranged in a so-called* Hodge diamond *the Hodge numbers of X are the following:*

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \qquad\qquad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Proof. Due to the symmetry $h^{p,q} = h^{q,p}$ coming from $\overline{H^{p,q}(X)} = H^{q,p}(X)$, we will only compute $h^{p,q}$ for $p \geq q$. Starting from the bottom $h^{0,0}(X) = h^0(X, \mathcal{O}_X) = 1$. By definition we have $h^{1,0}(X) = h^1(X, \mathcal{O}_X) = 0$. By Serre duality (5.1) we get $h^{2,0}(X) = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$ and $h^{2,2}(X) = h^2(X, \Omega_X^2) = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$. Next $h^{2,1}(X) = h^{1,2}(X) = h^1(X, \Omega_X^2) = h^1(X, \mathcal{O}_X) = 0$. Finally, for the center of the diamond, we use the Hirzebruch–Riemann–Roch formula (5.2) applied to the tangent sheaf $\mathcal{T}_X \simeq \Omega_X^\vee \simeq \Omega_X$, to obtain

$$h^{0,1}(X) - h^{1,1}(X) + h^{2,1}(X) = \langle \tfrac{1}{2}(\mathrm{c}_1(\Omega_X)^2 - 2\mathrm{c}_2(\Omega_X)), [X] \rangle + 2\,\mathrm{rk}(\Omega_X) = -20.$$

Thus, since $h^{0,1}(X) = h^{2,1}(X) = 0$, we get $h^{1,1}(X) = 20$. □

Notice that Proposition 4.45, says in particular that a Fourier–Mukai transform, which arises from an equivalence of derived categories, preserves sums of all the columns in the Hodge diamond. As a consequence of the Hodge numbers just computed above, we prove that the bounded derived category $\mathsf{D}^b(-)$ "detects K3 surfaces".

Theorem 5.9. *Suppose X and Y are smooth projective varieties over $\mathbf{C}$. Assume X is a K3 surface and*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

Then Y is a K3 surface as well.

Proof. By Theorem 4.10, we already know Y is a surface with a trivial canonical bundle. It is left to show that $H^1(Y, \mathcal{O}_Y) = 0$. By Proposition 4.45, we see that

$$h^{0,1}(X) + h^{1,2}(X) = h^{0,1}(Y) + h^{1,2}(Y).$$

However $h^{0,1}(X) = 0$ and $h^{1,2}(X) = h^{2,1}(X) = 0$. Thus $h^{0,1}(Y) = 0$ proves that Y is a K3 surface. □

Cohomology and intersection pairing

Now we turn our attention to cohomology of K3 surfaces. The passage to complex K3 surfaces will unlock techniques of singular and Čech cohomology, which will turn out to be really useful here. Fluidly transitioning to the complex analytic side of things, let X be a complex K3 surface, let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X and define $\mathcal{O}_X^*$ to be the sheaf of nowhere vanishing holomorphic functions on X. Consider the *exponential sequence*

$$0 \to \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

The first term $\mathbf{Z}$ is seen as the sheaf of locally constant functions taking values in the ring of integers. The first non-trivial morphism is just the inclusion and the second one is given locally by $f \mapsto e^{2\pi i f}$. The associated long exact sequence in cohomology then reads

$$\cdots \to H^i(X, \mathbf{Z}) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X^*) \to H^{i+1}(X, \mathbf{Z}) \to \cdots. \tag{5.3}$$

Definition 5.10. The *Picard group* of a complex K3 surface X is defined to be the group of isomorphism classes of line bundles on X with multiplication given by the tensor product of sheaves. We denote it with $\mathrm{Pic}(X)$.

Remark 5.11. Based on the cocycle representation of line bundles, one is able to construct a natural isomorphism from $\mathrm{Pic}(X)$ to the first *Čech cohomology* group $H^1(X, \mathcal{O}_X^*)$ [Voi02, §4, Theorem 4.49]. The Čech cohomology groups are also known to be isomorphic to the ususal sheaf cohomology groups we have been using all along [Voi02, §4, Theorem 4.44].

Based on the previous remark we are able to bring new meaning to the second connecting homomorphism $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$ of (5.3). The value of a line bundle $\mathcal{L}$ on X under the composition

$$\mathrm{Pic}(X) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbf{Z})$$

can actually be taken to be the definition of its *first Chern class* $\mathrm{c}_1(\mathcal{L})$.

Definition 5.12. The image of the connecting homomorphism $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$ is defined to be the *Néron–Severi group* of X, denoted by $\mathrm{NS}(X)$.

Remark 5.13. By an already listed property of characteristic classes, when discussing Hodge decompositions in Section 4.2, we note that the Néron–Severi group $\mathrm{NS}(X)$ can be seen as a subgroup of $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$. The converse inclusion is also true and follows from the Lefschetz Theorem on $(1,1)$-classes [GH94, §1.2]. Thus we have an equality

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbf{Z}). \tag{5.4}$$

Proposition 5.14. *Let X be a complex K3 surafce. The Picard group $\mathrm{Pic}(X)$ and the Néron–Severi group $\mathrm{NS}(X)$ are isomorphic. They are torsion-free and carry a symmetric non-degenerate integral bilinear form called the intersection form.*

Proof. Observe that $H^1(X, \mathcal{O}_X) = 0$ in (5.3) shows that the connecting homomorphism is injective, therefore $\mathrm{Pic}(X) \simeq \mathrm{NS}(X)$. For the definition of the bilinear form on $\mathrm{Pic}(X)$ and the rest see [Huy16, §1, Proposition 2.4]. □

Proposition 5.15. *Let X be a complex K3 surface. The integral cohomology of X is*

$$H^\bullet(X, \mathbf{Z}): \qquad \mathbf{Z} \quad 0 \quad \mathbf{Z}^{22} \quad 0 \quad \mathbf{Z}. \tag{5.5}$$

Proof. We already know $H^0(X, \mathbf{Z}) \simeq \mathbf{Z}$ and $H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$ for X is connected and orientable. Looking at the Hodge numbers we can already conclude that the cohomology groups are as in (5.5), but only up to torsion. We now show why they are in fact all torsion-free. For $H^1(X, \mathbf{Z})$ consider the following part of (5.3)

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbf{Z}) \to H^1(X, \mathcal{O}_X).$$

The two groups on the left are $\mathbf{C}$ and $\mathbf{C}^*$, respectively, since X is compact. The morphism relating them is the surjective exponential function, thus the connecting homomorphism going into $H^1(X, \mathbf{Z})$ is trivial, This makes $H^1(X, \mathbf{Z})$ into a subgroup of the trivial group $H^1(X, \mathcal{O}_X) = 0$. Identifying $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$ we have an exact sequence

$$0 \to \mathrm{Pic}(X) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X).$$

By Serre duality (5.1) we see that $H^2(X, \mathcal{O}_X) \simeq \mathbf{C}$. Since $\mathrm{Pic}(X)$ is torsion free by Proposition 5.14, two torsion free groups surround $H^2(X, \mathbf{Z})$ in an exact sequence, preventing it from having torsion either. By the universal coefficients theorem the torsion subgroup of $H^3(X, \mathbf{Z})$ is isomorphic to the torsion subgroup of the homology $H_2(X, \mathbf{Z})$. As the latter is torsion free, being isomorphic to $H^2(X, \mathbf{Z})$ by Poincaré duality, the former must be torsion free as well. □

The middle cohomology group $H^2(X, \mathbf{Z})$ now being free, enables us to introduce a lattice structure on it. The pairing it comes equipped with is called the *intersection pairing* and is naturality induced by the cohomological $\smile$-product.

Definition 5.16. The *intersection pairing* on $H^2(X, \mathbf{Z})$ is defined to be

$$(\alpha.\beta) = \langle \alpha \smile \beta, [X] \rangle.$$

Proposition 5.17. *Let X be a K3 surface. Then the intersection pairing $(\cdot.\cdot)$ makes $H^2(X, \mathbf{Z})$ into an even unimodular lattice of rank 22 and signature $(3, 19)$, making it isomorphic to the following orthogonal direct sum*

$$H^2(X, \mathbf{Z}) \simeq E_8(-1)^{\oplus 2} \oplus \mathcal{U}^{\oplus 3}. \tag{5.6}$$

Sketch of proof. Since $H^2(X, \mathbf{Z})$ is free, the Kronecker pairing from the universal coefficients theorem together with Poincaré duality show that the intersection pairing is unimodular. To see that the intersection pairing is even one considers the mod 2 reduction in the coefficients $H^2(X, \mathbf{Z}) \to H^2(X, \mathbf{Z}/2)$ and shows that the pairing on $H^2(X, \mathbf{Z}/2)$ is always 0. By Proposition 5.7 we know that the first Chern class $\mathrm{c}_1(X)$ is trivial and we also know that it reduces to the second Stiefel–Whitney class $w_2(X)$ (mod 2), which is therefore also trivial. Using the Wu formula, one is able to show that

$$x \smile x = w_2(X) \smile x$$

holds for all classes $x \in H^2(X, \mathbf{Z}/2)$ from which the desired result follows. Proposition 5.15 shows that $H^2(X, \mathbf{Z})$ is free of rank 22. The claim about the signature of this lattice is proven in [Huy16, §1, Proposition 3.5]. Lastly employing Milnor's classification result of even indefinite unimodular lattices (Theorem B.14), we obtain (5.6). ☐

Remark 5.18. When $\mathrm{NS}(X)$ is equipped with the lattice structure of Proposition 5.14, the inclusion $\mathrm{NS}(X) \hookrightarrow H^2(X, \mathbf{Z})$ is an isometry. This makes it possible to interpret the intersection pairing of two (isomorphism classes of) line bundles $\mathcal{L}$ and $\mathcal{L}'$ on X as

$$(\mathcal{L}.\mathcal{L}') = (\mathrm{c}_1(\mathcal{L}).\mathrm{c}_1(\mathcal{L}')).$$

Corollary 5.19. *Let X be a K3 surface. The Chern character $\mathrm{ch}(\mathcal{E})$ of any coherent sheaf $\mathcal{E}$ on X is integral, meaning that it lies in $H^\bullet(X, \mathbf{Z}) \subseteq H^\bullet(X, \mathbf{Q})$.*

Proof. We see that the Chern character on a K3 surface reduces to the following

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + \frac{1}{2}\Big(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})\Big).$$

Since the intersection pairing is even, $\mathrm{c}_1^2(\mathcal{E})$ is twice a multiple of some cohomology class from $H^4(X, \mathbf{Z})$. ☐

Remark 5.20. The Chern character of a coherent sheaf $\mathcal{E}$ being integral implies also that its Mukai vector $v(\mathcal{E})$ is integral. Indeed, using the power series expansion (4.10) (or just guessing), we see that

$$\sqrt{\mathrm{td}_X} = 1 + \mu_X.$$

Thus, one easily sees that the Mukai vector on a K3 surface is integral because it has the following from

$$\begin{aligned} v(\mathcal{E}) &= \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \big(\mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_0(\mathcal{E})\,\mu_X\big) \\ &= \mathrm{rk}(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}) + \frac{1}{2}\Big(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})\Big) + \mathrm{rk}(\mathcal{E})\mu_X. \end{aligned} \tag{5.7}$$

We also remark that on a K3 surface the inverse of the square root of the Todd class $(\sqrt{\mathrm{td}_X})^{-1}$ equals $1 - \mu_X$ and is therefore integral as well.

With the Néron–Severi group seen as a sublattice of $H^2(X, \mathbf{Z})$ one can consider also its orthogonal coplement. The resulting group will play a very important role in the last chapter in characterizing derived equivalent K3 surfaces.

Definition 5.21. The *transcendental lattice* of a complex K3 surface X is defined to be the orthogonal complement

$$T_X = \mathrm{NS}(X)^\perp = \{x \in H^2(X, \mathbf{Z}) \mid (x.y) = 0 \text{ for all } y \in \mathrm{NS}(X)\}.$$

Remark 5.22. By Lemma B.7 the transcendental lattice T_X is indeed a lattice with the bilinear form inherited from $H^2(X, \mathbf{Z})$. It is also a primitive sublattice of $H^2(X, \mathbf{Z})$ by Example B.9.

Global Torelli theorem

Lastly we mention the global Torelli theorem. It will be utilized in the last chapter to prove that a K3 surface Y is isomorphic to a moduli space of certain sheaves on its Fourier–Mukai partner. We have already seen that $H^2(X, \mathbf{Z})$ comes equipped with a lattice structure given by the intersection pairing. Moreover due to X being a complex projective manifold it also posseses a Hodge structure. We will call such lattices *Hodge lattices* and two Hodge lattices will be considered isomorphic, if there is a *Hodge isometry* between them.

Definition 5.23. A *Hodge isometry* between two Hodge lattices Λ and Λ' is a Hodge structure preserving isomorphism of lattices $f\colon \Lambda \xrightarrow{\sim} \Lambda'$.

Theorem 5.24 (Global Torelli theorem)**.** *Let X and Y be K3 surfaces over $\mathbf{C}$. Then X and Y are isomorphic if and only if there exists a Hodge isometry*

$$f\colon H^2(X, \mathbf{Z}) \xrightarrow{\sim} H^2(Y, \mathbf{Z}).$$

Proof. For a proof see [Huy16, §7, Theorem 5.3]. □

Remark 5.25. The global Torelli theorem emphasises that the Hodge lattice $H^2(X, \mathbf{Z})$ of a K3 surface X is a complete invariant of its isomorphism type.

6 Derived Torelli theorem

In this last chapter we will again be dealing with K3 surfaces over the field of complex numbers $\mathbf{C}$. We will characterize when two K3 surfaces X and Y have equivalent bounded derived categories, through their internal structure, namely their *transcendental lattices* and yet to be defined, *Mukai lattices*. In the process of proving these results we will encounter other realizations of smooth projective varieties over $\mathbf{C}$, disguised as *moduli spaces* of certain sheaves on K3 surfaces, which will themself turn out to be K3 surfaces as well. Pretty much in all aspects Fourier–Mukai transforms whether categorical or cohomological will be present. They will serve as crucial constituents of the proof facilitating a way to connect either various different Hodge lattices or the bounded derived categories. It is worth mentioning here that Orlov's paper [Orl03], especially Chapter 3, was the primary inspiration to govern the course of this chapter followed closely by the pioneering paper of Mukai [Muk87].

We will let the following theorem be our initial guide for this chapter.

Theorem 6.1. *Let X and Y be complex K3 surfaces. Then the following two statements are equivalent*

i. There exists a Hodge isometry $f\colon T_X \to T_Y$ between the lattices of transcendental cycles of X and Y.

ii. There exists a triangulated equivalence $\mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ between bounded derived categories of coherent sheaves on X and Y.

Over the span of this chapter we will encounter two more equivalent statements to add to the theorem above culminating in Theorem 6.27 at the end.

Throughout X and Y are reserved to denote K3 surfaces over the field of complex numbers $\mathbf{C}$ and all sheaves in this chapter are assumed to be coherent.

6.1 The Mukai lattice and one side of the proof

Recall that the *transcendental lattice* T_X of a complex K3 surface X is defined to be the orthogonal complement of the Néron-Severi group $\mathrm{NS}(X) \subseteq H^2(X, \mathbf{Z})$ with respect to the intersection pairing $(\cdot.\cdot)$ on $H^2(X, \mathbf{Z})$. Naturally, T_X also comes equipped with a weight 2 Hodge structure induced from that of $H^2(X, \mathbf{Z})$, by declaring $T_X^{p,q} = H^{p,q}(X) \cap (T_X \otimes_{\mathbf{Z}} \mathbf{C})$. By inspecting the Hodge structure of T_X a bit closer we see that

$$\begin{array}{rcl} T^{2,0}(X) & = & H^{2,0}(X) \\ T^{1,1}(X) & = & H^{1,1}(X) \cap (T_X \otimes_{\mathbf{Z}} \mathbf{C}) \\ T^{0,2}(X) & = & H^{0,2}(X). \end{array}$$

This is because on a K3 surface X we have $\mathrm{NS}(X) \simeq \mathrm{Pic}(X) \simeq H^{1,1}(X) \cap H^2(X, \mathbf{Z})$, which is orthogonal to both $H^{2,0}(X)$ and $H^{0,2}(X)$. The Hodge structure together with the restriction of the pairing $(\cdot.\cdot)$ to T_X, make T_X into a Hodge lattice.

With the purpose of examining not only the smaller transcendental lattice, but taking into account the whole cohomology group $H^\bullet(X, \mathbf{Z})$ of X, Mukai in [Muk87] devised the following. First, he equipped $H^\bullet(X, \mathbf{Z})$ with the pairing introduced in Definition 4.40. To be more explicit, letting $v = v_0 + v_1 + v_2$ and $w = w_0 + w_1 + w_2$ belong to $H^\bullet(X, \mathbf{Z})$, with the ordinary grading $H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z})$, the Mukai pairing is expressed as

$$\langle v, w \rangle = \langle v_0 w_2 + v_2 w_0 - v_1 \smile w_1, [X] \rangle.$$

Put more simply, if we identify $H^0(X, \mathbf{Z})$ and $H^4(X, \mathbf{Z})$ with $\mathbf{Z}$ and under this identification interpret components v_0, w_0 and v_2, w_2 as ordinary integers, the Mukai pairing is given by

$$\langle v, w\rangle = v_0 w_2 + v_2 w_0 - (v_1.w_1).$$

Remark 6.2. This kind of identification will be done quite often during the span of this chapter. On this note $[X] \in H_4(X, \mathbf{Z})$ will denote a fixed chosen fundamental class of a K3 surface X and $\mu_X \in H^4(X, \mathbf{Z})$ its dual with respect to the Kronecker pairing $\langle \mu_X, [X]\rangle = 1$. Then we have the following bijective correspondence between vectors $v \in \mathbf{Z} \oplus H^2(X, \mathbf{Z}) \oplus \mathbf{Z}$ and cohomology classes of $H^\bullet(X, \mathbf{Z})$

$$(v_0, v_1, v_2) \quad \leftrightarrow \quad v_0 \smile 1 + v_1 + v_2 \smile \mu_X,$$

where $1 \in H^0(X, \mathbf{Z})$ denotes the neutral element for multiplication in the cohomology ring of X.

Remark 6.3. Notice that upon restricting the Mukai pairing $\langle \cdot, \cdot\rangle$ onto $H^2(X, \mathbf{Z})$, the usual intersection pairing $(\cdot . \cdot)$ of $H^2(X, \mathbf{Z})$ differs from it by a sign. This makes the lattice $H^\bullet(X, \mathbf{Z})$ abstractly isomorphic to the orthogonal direct sum

$$H^\bullet(X, \mathbf{Z}) \simeq H^2(X, \mathbf{Z})(-1) \oplus \mathcal{U} \simeq E_8^{\oplus 2} \oplus \mathcal{U}^{\oplus 4}.$$

Note that $\mathcal{U}(-1) \simeq \mathcal{U}$. The above also shows that the lattice structure on $H^\bullet(X, \mathbf{Z})$ is an even unimodular lattice.

Endowing $H^\bullet(X, \mathbf{Z})$ with the Mukai pairing and the ensuing Hodge structure makes the group $H^\bullet(X, \mathbf{Z})$ into a *Mukai lattice.*

Definition 6.4. The *Mukai lattice* of a K3 surface X over $\mathbf{C}$, denoted by $\widetilde{H}(X, \mathbf{Z})$, is an even unimodular lattice $H^\bullet(X, \mathbf{Z})$ given by the Mukai pairing $\langle \cdot, \cdot\rangle$ equipped with a Hodge structure of weight 2 described by

$$\begin{array}{rcc} \widetilde{H}^{2,0}(X) & = & H^{2,0}(X) \\ \widetilde{H}^{1,1}(X) & = & H^0(X, \mathbf{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbf{C}) \\ \widetilde{H}^{0,2}(X) & = & H^{0,2}(X). \end{array}$$

Remark 6.5. This Hodge decomposition is *not* orthogonal with respect to the Mukai pairing on $\widetilde{H}(X, \mathbf{Z}) \otimes \mathbf{C}$, but we have some orthogonality relations among the components nonetheless.

$$\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{2,0}(X) \qquad \widetilde{H}^{0,2}(X) \perp \widetilde{H}^{0,2}(X) \tag{6.1}$$

$$\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{1,1}(X) \qquad \widetilde{H}^{0,2}(X) \perp \widetilde{H}^{1,1}(X) \tag{6.2}$$

These relations essentially all stem from observing the following facts regarding the Hodge decomposition of a compact complex manifold.

- Each cohomology class $x \in H^{p,q}(X) = H^q(X, \Omega_X^p)$ corresponds bijectively to a certain *harmonic* differential form, which is a smooth complex valued (p, q)-form.
- The wedge product of a (p, q)-form with a (r, s)-form results in a $(p + r, q + s)$-from.
- Singular cohomology ring with complex coefficients and complex de Rham cohomology ring of a complex manifold X are isomorphic as rings.

The reason for considering Mukai lattices ends up being beneficial is showcased in the following reconstruction technique allowing us to relate the Mukai lattice $\tilde{H}(X, \mathbf{Z})$ back to the original Hodge lattice $H^2(X, \mathbf{Z})$. Let $e \in \tilde{H}(X, \mathbf{Z})$ denote the Mukai vector $(0, 0, 1)$. The orthogonal complement of e is then easily seen to be

$$\begin{aligned} e^\perp &= \{v \in \tilde{H}(X, \mathbf{Z}) \mid \langle e, v \rangle = 0\} \\ &= \{(v_0, v_1, v_2) \in \tilde{H}(X, \mathbf{Z}) \mid v_0 = 0\} \\ &= H^2(X, \mathbf{Z}) \oplus \mathbf{Z}e. \end{aligned}$$

Next consider the quotient $e^\perp/\mathbf{Z}e$. It is clearly isomorphic to $H^2(X, \mathbf{Z})$ as a free finite rank $\mathbf{Z}$-module. Actually this isomorphism preserves much more than just the additive structure once we equip $e^\perp/\mathbf{Z}e$ with lattice and Hodge structures. Since e is isotropic, the quotient naturally inherits a bilinear from from the Mukai lattice $\tilde{H}(X, \mathbf{Z})$. It is simply defined by

$$\langle v + \mathbf{Z}e, w + \mathbf{Z}e \rangle = \langle v, w \rangle.$$

The quotient $e^\perp/\mathbf{Z}e$ is also naturally equipped with a weight 2 Hodge structure, since it is a quotient of two free finite rank $\mathbf{Z}$-modules endowed with weight 2 Hodge structures[27]. This makes $e^\perp/\mathbf{Z}e$ into a Hodge lattice and the isomorphism

$$e^\perp/\mathbf{Z}e \simeq H^2(X, \mathbf{Z})$$

an isomorphism of Hodge lattices. This notion now enables us to abstractly identify a Hodge lattice of a K3 surface Y in terms of certain vectors in the Mukai lattice of another K3 surface X, provided there exists a Hodge isometry $\tilde{H}(X, \mathbf{Z}) \simeq \tilde{H}(Y, \mathbf{Z})$.

Proposition 6.6. *Let X and Y be complex K3 surfaces. Assume $f\colon \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$ is a Hodge isometry. Let $e \in \tilde{H}(Y, \mathbf{Z})$ denote the vector $(0, 0, 1)$ and let $v \in \tilde{H}(X, \mathbf{Z})$ be such that $f(v) = e$. Then we have an isomorphism of Hodge lattices*

$$v^\perp/\mathbf{Z}v \simeq H^2(Y, \mathbf{Z}).$$

Proof. Since f is an isometry it restricts to a well defined map on the orthogonal complements $v^\perp \to e^\perp$ and is moreover an isomorphism of additive structures. It then clearly descends to a well-defined additive isomorphism of the quotients $v^\perp/\mathbf{Z}v \to e^\perp/\mathbf{Z}e$. This makes $v^\perp/\mathbf{Z}v$ into a free finite rank $\mathbf{Z}$-module. Since v is isotropic, $v^\perp/\mathbf{Z}v$ inherits a bilinear form from $\tilde{H}(X, \mathbf{Z})$. The Hodge structure on $v^\perp/\mathbf{Z}v$ is also induced from $\tilde{H}(X, \mathbf{Z})$, because both $v^\perp$ and $\mathbf{Z}v$ come equipped with weight 2 Hodge structures. The reason for this being that f preserves the Hodge structures of $\tilde{H}(X, \mathbf{Z})$ and $\tilde{H}(Y, \mathbf{Z})$. The induced map between the quotients then preserves the bilinear form and the Hodge structure, in turn proving our claim. □

Lemma 6.7. *Let X and Y be complex K3 surfaces and $\phi\colon \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$ an isomorphism of lattices satisfying only*

$$\phi_{\mathbf{C}}(\widetilde{H}^{2,0}(X)) \subseteq \widetilde{H}^{2,0}(Y). \tag{6.3}$$

Then ϕ is a Hodge isometry, i.e. preserves Hodge structures.

[27] $(e^\perp)_{\mathbf{C}}$ decomposes into $H^{2,0}(X)$, $H^{1,1}(X) \oplus \mathbf{C}e$ and $H^{0,2}(X)$ of bi-degrees $(2, 0)$, $(1, 1)$ and $(0, 2)$, respectively. $\mathbf{Z}e$ has a weight 2 Hodge structure concentrated only in bi-degree $(1, 1)$.

Proof. First, if $\phi_{\mathbf{C}}(\widetilde{H}^{2,0}(X)) \subseteq \widetilde{H}^{2,0}(Y)$, then $\phi_{\mathbf{C}}(\widetilde{H}^{0,2}(X)) \subseteq \widetilde{H}^{0,2}(Y)$ is obtained by conjugating

$$\phi_{\mathbf{C}}\left(\widetilde{H}^{0,2}(X)\right) = \phi_{\mathbf{C}}\left(\overline{\widetilde{H}^{2,0}(X)}\right) = \overline{\phi_{\mathbf{C}}\left(\widetilde{H}^{2,0}(X)\right)} \subseteq \overline{\widetilde{H}^{2,0}(Y)} = \widetilde{H}^{0,2}(Y).$$

To see $\phi_{\mathbf{C}}(\widetilde{H}^{1,1}(X)) \subseteq \widetilde{H}^{1,1}(Y)$, we pick a non-zero class $x \in \widetilde{H}^{1,1}(X)$ and map it over to $y = \phi_{\mathbf{C}}(x) \in \widetilde{H}(X,\mathbf{Z}) \otimes \mathbf{C}$. Then for any $y^{2,0} \in \widetilde{H}^{2,0}(Y)$, we have $\langle y, y^{2,0}\rangle = 0$. Indeed, first as $\phi_{\mathbf{C}}$ is an isomorphism of $\mathbf{C}$-vector spaces and $\widetilde{H}^{2,0}(X)$ and $\widetilde{H}^{2,0}(Y)$ are one dimensional, the inclusion (6.3) is actually an equality[28], thus we may represent $y^{2,0}$ as $\phi_{\mathbf{C}}(x^{2,0})$ for some $x^{2,0} \in \widetilde{H}^{2,0}(X)$. As $\widetilde{H}^{2,0}(X) \perp \widetilde{H}^{1,1}(X)$ and $\phi_{\mathbf{C}}$ is an isometry, we obtain $0 = \langle x, x^{2,0}\rangle = \langle y, y^{2,0}\rangle$ to conclude $y \in \widetilde{H}^{2,0}(Y)^{\perp}$. Symmetrically one obtains $y \in \widetilde{H}^{0,2}(Y)^{\perp}$.

Finally, once we verify that $\widetilde{H}^{2,0}(Y)^{\perp} \cap \widetilde{H}^{0,2}(Y)^{\perp} \subseteq \widetilde{H}^{1,1}(Y)$, we may conclude that $y \in \widetilde{H}^{1,1}(Y)$, which will prove our last claim. This is indeed the case for once we write a class $z \in \widetilde{H}^{2,0}(Y)^{\perp} \cap \widetilde{H}^{0,2}(Y)^{\perp}$ with respect to the Hodge decomposition as $z = z^{2,0} + z^{1,1} + z^{0,2}$, we see that $z^{2,0} = z^{0,2} = 0$, by non-degeneracy of the pairing $\langle\cdot,\cdot\rangle$. For example, for any $w = w^{2,0} + w^{1,1} + w^{0,2} \in \widetilde{H}(Y,\mathbf{Z})$, written out with respect to the Hodge decomposition, we have

- $\langle z^{2,0}, w^{2,0}\rangle = 0$, by the first orthogonality relation of (6.1).
- $\langle z^{2,0}, w^{1,1}\rangle = 0$, by the first orthogonality relation of (6.2).
- $\langle z^{2,0}, w^{0,2}\rangle = 0$, because $z^{2,0} = z - z^{1,1} - z^{0,2}$ and all the terms on the right-hand side of the equation lie in $\widetilde{H}^{0,2}(Y)^{\perp}$.

This implies that

$$\langle z^{2,0}, w\rangle = \langle z^{2,0}, w^{2,0}\rangle + \langle z^{2,0}, w^{1,1}\rangle + \langle z^{2,0}, w^{0,2}\rangle = 0,$$

which shows that $z^{2,0} = 0$. Symmetrically one also shows $z^{0,2} = 0$ to conclude that $z = z^{1,1} \in \widetilde{H}^{1,1}(Y)$. □

It is beneficial to introduce the Néron-Severi group analog for the Mukai lattice. We call it the *extended Néron-Severi group* and it is defined to be

$$\mathrm{N}(X) = \mathrm{NS}(X) \oplus H^0(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z}) = \widetilde{H}^{1,1}(X) \cap \widetilde{H}(X,\mathbf{Z}).$$

We will utilize it in the following proposition, which explains why our sudden jump away from transcendental lattices to Mukai lattices was reasonable.

Proposition 6.8. *Let X and Y be complex K3 surfaces. Then their transcendental lattices are Hodge isometric if and only if their Mukai lattices are Hodge isometric. Schematically,*

$$T_X \simeq T_Y \quad\Longleftrightarrow\quad \widetilde{H}(X,\mathbf{Z}) \simeq \widetilde{H}(Y,\mathbf{Z}).$$

Proof. ($\Rightarrow$) First note that T_X is primitively embedded into $H^2(X,\mathbf{Z})$ by Example B.9 and the latter is clearly also primitively embedded into $\widetilde{H}(X,\mathbf{Z})$. By Proposition B.10, $T_X \hookrightarrow \widetilde{H}(X,\mathbf{Z})$ is therefore also primitive and the same holds true for $T_Y \hookrightarrow \widetilde{H}(Y,\mathbf{Z})$. Since T_X is embedded into $H^2(X,\mathbf{Z})$, it is orthogonal to $H^0(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z}) \simeq \mathcal{U}$. This means that one can find a copy of the hyperbolic lattice in the orthogonal complement $T_X^{\perp}$ inside of $\widetilde{H}(X,\mathbf{Z})$. By Corollary B.12 one can extend any isomorphism of lattices

[28] This is one of the specialities of working with K3 surfaces, they have a one-dimensional $H^{2,0}(X)$.

$f\colon T_X \to T_Y$ to an isomorphism $\phi\colon \tilde{H}(X,\mathbf{Z}) \to \tilde{H}(Y,\mathbf{Z})$. It is left to show that ϕ also preserves the Hodge structures. By Lemma 6.7 it turns out that for this to be the case, it is sufficient only that the inclusion

$$\phi_{\mathbf{C}}(\widetilde{H}^{2,0}(X)) \subseteq \widetilde{H}^{2,0}(Y),$$

holds. But in our case this is satisfied as $\phi_{\mathbf{C}}$ extends $f_{\mathbf{C}}$, $\widetilde{H}^{2,0}(X) = H^{2,0}(X) = T^{2,0}(X)$ and $\widetilde{H}^{0,2}(Y) = T^{0,2}(Y)$.

($\Leftarrow$) Assuming we have a Hodge isometry $\phi\colon \tilde{H}(X,\mathbf{Z}) \to \tilde{H}(Y,\mathbf{Z})$ we need to show that T_X is sent to T_Y. Observe that T_X is actually also the orthogonal complement of $\mathrm{N}(X)$ in the Mukai lattice $\tilde{H}(X,\mathbf{Z})$. Indeed, since $\mathrm{NS}(X) \subseteq \mathrm{N}(X)$, we have $\mathrm{N}(X)^{\perp} \subseteq T_X$. For the converse observe that since $T_X \subseteq H^2(X,\mathbf{Z})$, T_X is automatically orthogonal to $H^0(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z})$ with respect to the Mukai pairing $\langle\cdot,\cdot\rangle$. Therefore $T_X \subseteq \mathrm{N}(X)^{\perp}$.

Lastly, since $\phi(\mathrm{N}(X)) = \phi_{\mathbf{C}}(\widetilde{H}^{1,1}(X) \cap \tilde{H}(X,\mathbf{Z})) = \mathrm{N}(Y)$, it is clear that ϕ maps transcendental cycles of T_X to cycles T_Y. This allows us to restrict the Hodge isometry ϕ to a Hodge isometry

$$\phi|_{T_X} : T_X \to T_Y. \qquad \square$$

Proposition 6.8 thus lets us reformulate our guiding Theorem 6.1 into the following.

Theorem 6.9. *Let X and Y be K3 surfaces over the field of complex numbers $\mathbf{C}$. Then the bounded derived categories $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Y)$ are equivalent as triangulated categories if and only if there exists a Hodge isometry $f\colon \tilde{H}(X,\mathbf{Z}) \to \tilde{H}(Y,\mathbf{Z})$ between their Mukai lattices. Schematically,*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y) \quad\Longleftrightarrow\quad \tilde{H}(X,\mathbf{Z}) \simeq \tilde{H}(Y,\mathbf{Z}).$$

The remainder of this section will be devoted to proving the left-to-right implication of Theorem 6.9. For this purpose we first develop some technical preliminary results.

Technical preliminaries

Lemma 6.10. *For any object $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$, its Chern character $\mathrm{ch}(\mathcal{E})$ is integral, i.e. belongs to $H^\bullet(X \times Y,\mathbf{Z}) \subseteq H^\bullet(X \times Y,\mathbf{Q})$.*

Proof. We will use the Künneth formula in an essential way, so we recall it again. Since X and Y, as K3 surfaces, have torsion-free integral cohomology the n-th cohomology of the product $X \times Y$ decomposes as

$$H^n(X \times Y,\mathbf{Z}) \simeq \bigoplus_{k+\ell=n} H^k(X,\mathbf{Z}) \otimes_{\mathbf{Z}} H^\ell(Y,\mathbf{Z}),$$

for all $n \in \mathbf{Z}_{\geq 0}$. Let p and q, as always, denote the canonical projections from $X \times Y$ onto X and Y, respectively.

First, rewrite the Chern character with respect to the ordinary grading on cohomology $H^\bullet(X \times Y,\mathbf{Q}) = \bigoplus_{i=0}^4 H^{2i}(X \times Y,\mathbf{Q})$ as the vector

$$\mathrm{ch}(\mathcal{E}) = \Big(\mathrm{rk}(\mathcal{E}), \mathrm{c}_1(\mathcal{E}), \tfrac{1}{2}(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})), \mathrm{ch}_3(\mathcal{E}), \mathrm{ch}_4(\mathcal{E})\Big).$$

Components $\mathrm{rk}(\mathcal{E})$ and $\mathrm{c}_1(\mathcal{E})$ are integral by definition. The first Chern class $\mathrm{c}_1(\mathcal{E}) \in H^2(X \times Y,\mathbf{Z})$ may be rewritten, in accordance with the Künneth decomposition

$$H^2(X \times Y,\mathbf{Z}) \simeq H^2(X,\mathbf{Z}) \oplus H^2(Y,\mathbf{Z}),$$

in the form $\mathrm{c}_1(\mathcal{E}) = p^*\alpha + q^*\beta$, for some $\alpha \in H^2(X,\mathbf{Z})$ and $\beta \in H^2(Y,\mathbf{Z})$. Therefore, as K3 surfaces X and Y have *even* intersection pairings by Proposition 5.17, we see that $\mathrm{c}_1(\mathcal{E})^2 = p^*\alpha^2 + 2p^*\alpha \smile q^*\beta + q^*\beta^2$ is an even multiple of some class from $H^4(X \times Y, \mathbf{Z})$. Along with $\mathrm{c}_2(\mathcal{E})$ being integral this shows the $H^4(X \times Y, \mathbf{Q})$-component of $\mathrm{ch}(\mathcal{E})$ is integral.

To show $\mathrm{ch}_3(\mathcal{E})$ and $\mathrm{ch}_4(\mathcal{E})$ are integral, we first modify the Grothendieck–Riemann–Roch formula a bit. Recalling that $\mathrm{td}_{X\times Y} = p^*\,\mathrm{td}_X \smile q^*\,\mathrm{td}_Y$, we compute

$$\begin{aligned}\mathrm{ch}(q_!\mathcal{E})\,\mathrm{td}_Y &= q_!(\mathrm{ch}(\mathcal{E})\,\mathrm{td}_{X\times Y})\\ &= q_!(\mathrm{ch}(\mathcal{E}) \smile p^*\,\mathrm{td}_X \smile q^*\,\mathrm{td}_Y)\\ &= q_!(\mathrm{ch}(\mathcal{E})\,p^*\,\mathrm{td}_X)\,\mathrm{td}_Y\,.\end{aligned}$$

Thus $\mathrm{ch}(q_!\mathcal{E}) = q_!(\mathrm{ch}(\mathcal{E})\,p^*\,\mathrm{td}_X)$, because $\mathrm{td}_Y = 1 + 2\mu_Y$ is invertible in $H^*(Y,\mathbf{Q})$. This will be very useful as the left side of the equation is the Chern character of some sheaf on Y, which we already know to be integral by Remark **??**. According to the Künneth decomposition write the Chern character as

$$\mathrm{ch}(\mathcal{E}) = \sum_{k,\ell\leq 4} e^{k,\ell},$$

where $e^{k,\ell} = p^*\alpha^{k,\ell} \smile q^*\beta^{k,\ell} \in H^{k+\ell}(X \times Y, \mathbf{Q})$ for some $\alpha^{k,\ell} \in H^k(X,\mathbf{Q})$ and $\beta^{k,\ell} \in H^\ell(Y,\mathbf{Q})$. As $H^6(X \times Y,\mathbf{Q})$ decomposes into a direct sum of $H^4(X,\mathbf{Q}) \otimes H^2(Y,\mathbf{Q})$ and $H^2(X,\mathbf{Q}) \otimes H^4(Y,\mathbf{Q})$, we have

$$\mathrm{ch}_3(\mathcal{E}) = e^{4,2} + e^{2,4}.$$

We show only that $e^{4,2}$ is integral, the case of $e^{2,4}$ follows by a symmetric argument. First note that $q_!$ lowers degrees by 4, then using the fact that $\mathrm{td}_X = 1 + 2\mu_X$ and our modified Grothendieck–Riemann–Roch formula, we calculate its $H^2(Y,\mathbf{Q})$-contributions to be

$$\begin{aligned}\mathrm{ch}_1(q_!\mathcal{E}) &= q_!(\mathrm{ch}_3(\mathcal{E}) + 2\mathrm{ch}_1(\mathcal{E})\,p^*\mu_X)\\ &= q_!(e^{4,2} + e^{2,4} + 2(e^{2,0} + e^{0,2}) \smile p^*\mu_X)\\ &= q_!(e^{4,2}) + 2q_!(e^{0,2} \smile p^*\mu_X).\end{aligned}$$

Terms $q_!(e^{2,4})$ and $q_!(2e^{2,0} \smile p^*\mu_X)$ are seen to vanish for dimension reasons after applying the projection formula. We already know $e^{0,2}$ and $\mathrm{ch}_1(q_!\mathcal{E})$ are integral so $q_!(e^{4,2}) \in H^2(Y,\mathbf{Z})$ is integral as well. To show that $e^{4,2}$ is integral write $\alpha^{4,2} = a\mu_X$ and $\beta^{4,2} = \sum_i b_i y_i$ for some $a,\ b_i \in \mathbf{Q}$, where y_i form an integral basis of $H^2(Y,\mathbf{Z})$. First note that $q_!(p^*\mu_X) = 1$, then compute

$$q_!(e^{4,2}) = q_!(p^*\alpha^{4,2} \smile q^*\beta^{4,2}) = q_!p^*(\alpha^{4,2}) \smile \beta^{4,2} = a \cdot q_!p^*(\mu_X) \smile \sum_i b_i y_i = \sum_i a b_i y_i.$$

Since we have deduced $q_!(e^{4,2})$ is integral all the products ab_i are integers. Thus $e^{4,2}$ is integral as

$$e^{4,2} = p^*\alpha^{4,2} \smile q^*\beta^{4,2} = \sum_i a b_i p^*\mu_X \smile q^* y_i$$

and $p^*\mu_X \smile q^* y_i \in H^6(X \times Y, \mathbf{Z})$. This shows that $\mathrm{ch}_3(\mathcal{E})$ is integral.

To show $\mathrm{ch}_4(\mathcal{E})$ is integral observe that $H^8(X \times Y,\mathbf{Q}) \simeq H^4(X,\mathbf{Q}) \otimes H^4(Y,\mathbf{Q})$, thus $\mathrm{ch}_4(\mathcal{E}) = e^{4,4}$. Using the modified Grothendieck–Riemann–Roch formula, its $H^4(Y,\mathbf{Q})$-contributions are

$$\begin{aligned}\mathrm{ch}_4(q_!\mathcal{E}) &= q_!(\mathrm{ch}_4(\mathcal{E}) + 2\mathrm{ch}_2(\mathcal{E})\,p^*\mu_X)\\ &= q_!(e^{4,4} + 2(e^{4,0} + e^{2,2} + e^{0,4}) \smile p^*\mu_X)\\ &= q_!(e^{4,4}) + 2q_!(e^{0,4} \smile p^*\mu_X).\end{aligned}$$

Similarly as before the missing terms vanish for dimension reasons. The class $e^{0,4}$ is integral either by observing $H^0(Y, \mathbf{Q})$-contributions of the modified Grothendieck–Riemann–Roch or noting that $\mathrm{ch}_2(\mathcal{E})$ is integral. Then an analogous argument as with $e^{4,2}$ shows that $e^{4,4}$ is integral, because $\mathrm{ch}_4(q_!\mathcal{E}) = 0$ and $e^{0,4}$ is integral. □

Remark 6.11. Any bounded complex of sheaves $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$ has a Mukai vector

$$v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_{X\times Y}}.$$

We know that $\sqrt{\mathrm{td}_X} = 1 + \mu_X$ and likewise $\sqrt{\mathrm{td}_Y} = 1 + \mu_X$, as X and Y are both K3 surfaces. Thus $\sqrt{\mathrm{td}_{X\times Y}}$ is integral since it may be expressed as

$$\sqrt{\mathrm{td}_{X\times Y}} = p^*\sqrt{\mathrm{td}_X} \smile q^*\sqrt{\mathrm{td}_Y} = p^*(1+\mu_X) \smile q^*(1+\mu_Y).$$

The Mukai vector $v(\mathcal{E})$ is therefore an integral vector, which allows us to restrict our cohomological Fourier–Mukai transforms from rational to integral cohomology. A Fourier–Mukai transform $f^{\mathcal{E}}$ associated to a kernel $\mathcal{E}$ of $\mathsf{D}^b(X \times Y)$ will thus from now on always mean a homomorphism between *integral* cohomology groups

$$f^{\mathcal{E}}\colon H^\bullet(X, \mathbf{Z}) \to H^\bullet(Y, \mathbf{Z}).$$

Lemma 6.12. *Let X be a K3 surface over $\mathbf{C}$ and $\varepsilon\colon X \to \operatorname{Spec}\mathbf{C}$ its structure map. Then for all v, $w \in \tilde{H}(X, \mathbf{Z})$*

$$\langle v, w\rangle = \varepsilon_!(v^\vee \smile w) \qquad \textit{and} \qquad \varepsilon_!(v^\vee) = \varepsilon_!(v).$$

Proof. We only have to recall Definition 4.20 of the Gysin map. Since $\operatorname{Spec}\mathbf{C}$ is topologically just a point, we see that $\varepsilon_!$ is non-trivial only in degree 4 for dimension reasons. The first formula is then evident from the definition of the Mukai pairing and the second one by the $(-)^\vee$-operation leaving the degree 4 component of a Mukai vector invariant. □

Lemma 6.13. *Let $\mathcal{O}_\Delta$ denote the push-forward $\Delta_*\mathcal{O}_X$ of the structure sheaf $\mathcal{O}_X$ along the diagonal embedding $\Delta\colon X \to X \times X$. Then*

$$f^{\mathcal{O}_\Delta} = \mathrm{id}_{H^\bullet(X,\mathbf{Z})}.$$

Proof. We start by computing the Mukai vector of $\mathcal{O}_\Delta$. By the Grothendieck–Riemann–Roch formula (4.8), we see that

$$\begin{aligned}\mathrm{ch}(\mathcal{O}_\Delta)\,\mathrm{td}_{X\times X} &= \mathrm{ch}(\Delta_*\mathcal{O}_X)\,\mathrm{td}_{X\times X} = \\ &= \mathrm{ch}(\Delta_!\mathcal{O}_X)\,\mathrm{td}_{X\times X} = \Delta_*(\mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}_X) = \Delta_*\,\mathrm{td}_X\,.\end{aligned}$$

In the second equality, we have used that $[\Delta_*\mathcal{O}_X] = [\Delta_!\mathcal{O}_X]$ in $K_\circ(X \times X)$, because for any closed embedding, such as $\Delta\colon X \to X \times X$, its push-forward Δ_* is exact. For the last equality we have used $\mathrm{ch}(\mathcal{O}_X) = \exp(\mathrm{c}_1(\mathcal{O}_X)) = 1$. Next, we observe, that from $\mathrm{td}_{X\times X} = p^*\,\mathrm{td}_X \smile p^*\,\mathrm{td}_X$, equation $\Delta^*\sqrt{\mathrm{td}_{X\times X}} = \mathrm{td}_X$ follows. By the cohomological projection formula (cf. Proposition 4.23), we further compute

$$\Delta_!\,\mathrm{td}_X = \Delta_!\Delta^*\sqrt{\mathrm{td}_{X\times X}} = \Delta_!(1)\sqrt{\mathrm{td}_{X\times X}}.$$

This implies that $v(\mathcal{O}_\Delta) = \Delta_!(1)$. Finally, for any $\alpha \in H^\bullet(X, \mathbf{Z})$, again utilizing the projection formula, we arrive at

$$f^{\mathcal{O}_\Delta}(\alpha) = q_!(v(\mathcal{O}_{\Delta_X}) \smile p^*\alpha) = q_!(\Delta_!(1) \smile p^*\alpha) = q_!(\Delta_!(1 \smile \Delta^*p^*(\alpha))) = \alpha. \qquad □$$

Proof for ($\Rightarrow$) *of Theorem 6.9.* By Orlov's Theorem 4.8, the functor witnessing the equivalence $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$ is naturally isomorphic to a Fourier–Mukai transform $\Phi_{\mathcal{E}}$ for some kernel $\mathcal{E}$ of the category $\mathsf{D}^b(X \times Y)$. By Remark 6.11 the cohomological Fourier–Mukai transform associated to $\mathcal{E}$, defined as

$$f^{\mathcal{E}} \colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q}) \qquad \alpha \mapsto p_*(v(\mathcal{E}) \smile q^*(\alpha)),$$

restricts to integral cohomology, forming a homomorphism

$$f^{\mathcal{E}^\bullet} \colon H^\bullet(X, \mathbf{Z}) \to H^\bullet(Y, \mathbf{Z}).$$

In the following three parts we show $f^{\mathcal{E}}$ is a bijective Hodge structure preserving isometry.

Bijection. We first tackle bijectivity by proving $f^{\mathcal{E}}$ has a right inverse. Since $\Phi_{\mathcal{E}}$ is an equivalence, its left adjoint $\Phi_{\mathcal{E}_{\mathrm{L}}}$ is also its quasi-inverse. Thus $\Phi_{\mathcal{E}_{\mathrm{L}}} \circ \Phi_{\mathcal{E}} \simeq \mathrm{id}_{\mathsf{D}^b(X)}$ and by Proposition 4.7 there is a kernel $\mathcal{G}$ from $\mathsf{D}^b(X \times X)$, for which $\Phi_{\mathcal{E}_{\mathrm{L}}} \circ \Phi_{\mathcal{E}} \simeq \Phi_{\mathcal{G}}$. By uniqueness of kernels, ensured by Theorem 4.8, we see that $\mathcal{G} \simeq \mathcal{O}_\Delta$, since $\mathcal{O}_\Delta$ was computed to be the kernel of $\mathrm{id}_{\mathsf{D}^b(X)}$ in Example 4.3. As cohomological Fourier–Mukai transforms compose just like the categorical ones do, according to Proposition 4.38, we see that $f^{\mathcal{E}_{\mathrm{L}}} \circ f^{\mathcal{E}} = f^{\mathcal{O}_\Delta}$. According to Lemma 6.13, $f^{\mathcal{O}_\Delta} = \mathrm{id}_{H^\bullet(X, \mathbf{Z})}$, thus $f^{\mathcal{E}}$ has a right inverse. Since $f^{\mathcal{E}}$ is a homomorphism of free abelian groups of equal rank admitting a right inverse it must be an isomorphism[29].

Isometry. To see that $f^{\mathcal{E}}$ is an isometry, we do two preliminary computations. Let $\alpha \in \tilde{H}(X, \mathbf{Z})$ and $\beta \in \tilde{H}(Y, \mathbf{Z})$ and let $\gamma \colon X \to \operatorname{Spec} \mathbf{C}$, $\varepsilon \colon Y \to \operatorname{Spec} \mathbf{C}$ and $\eta \colon X \times Y \to \operatorname{Spec} \mathbf{C}$ denote the structure maps. Then, by the first formula of Lemma 6.12, we see that

$$\begin{aligned}
\langle f^{\mathcal{E}}(\alpha), \beta \rangle_Y &= \varepsilon_!(\beta^\vee \smile f^{\mathcal{E}}(\alpha)) \\
&= \varepsilon_! \left(\beta^\vee \smile p_* \left(v(\mathcal{E}) \smile q^*\alpha\right)\right) \\
&= \varepsilon_! \left(\beta^\vee \smile p_* \left(\mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{td}_{X \times Y}} \smile q^*\alpha\right)\right) \\
&= \varepsilon_! \left(p_* \left(p^*\beta^\vee \smile \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{td}_{X \times Y}} \smile q^*\alpha\right)\right) \\
&= \eta_! \left(p^*\beta^\vee \smile \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{td}_{X \times Y}} \smile q^*\alpha\right).
\end{aligned}$$

Similarly, using the property $\mathrm{ch}(\mathcal{F}^\vee) = \mathrm{ch}(\mathcal{F})^\vee$ of the Chern character, one computes

$$\langle \alpha, f^{\mathcal{E}_{\mathrm{L}}}(\beta) \rangle_X = \eta_! \left(q^*\alpha^\vee \smile \mathrm{ch}(\mathcal{E})^\vee \sqrt{\mathrm{td}_{X \times Y}} \smile p^*\beta\right).$$

Using the second formula of Lemma 6.12 shows $\langle f^{\mathcal{E}}(\alpha), \beta \rangle_Y = \langle \alpha, f^{\mathcal{E}_{\mathrm{L}}}(\beta) \rangle_X$. Thus for any α, $\alpha' \in \tilde{H}(X, \mathbf{Z})$ we obtain

$$\langle f^{\mathcal{E}}(\alpha), f^{\mathcal{E}}(\alpha') \rangle_Y = \langle \alpha, f^{\mathcal{E}_{\mathrm{L}}}(f^{\mathcal{E}}(\alpha')) \rangle_X = \langle \alpha, \alpha' \rangle_X,$$

proving that $f^{\mathcal{E}}$ is an isometry.

Hodge structures. Lastly, we show that $f^{\mathcal{E}}$ preserves the Hodge structures i.e.

$$f^{\mathcal{E}}_{\mathbf{C}}(\widetilde{H}^{p,q}(X)) \subseteq \widetilde{H}^{p,q}(Y)$$

[29] If $f \colon A \to B$ is a homomorphism of free abelian groups of equal (finite) rank, admitting a right inverse $r \colon B \to A$, f is in particular injective and the short exact sequence $0 \to A \to B \to \operatorname{coker} f \to 0$ splits. The cokernel $\operatorname{coker} f$ can therefore only be finite, hence it is trivial, because $B \simeq A \oplus \operatorname{coker} f$.

for all p, q, with $p + q = 2$. Since $f^{\mathcal{E}}$ is a cohomological Fourier–Mukai transform, the condition (4.14) of Proposition 4.45, conveniently specializes to

$$f^{\mathcal{E}}_{\mathbf{C}}(\widetilde{H}^{2,0}(X)) \subseteq \widetilde{H}^{2,0}(Y). \tag{6.4}$$

Then we have already shown in Lemma 6.7 that the bijective isometry $f^{\mathcal{E}}$ preserves the Hodge structures. □

For convenience and later referencing we record the following corollary, which states essentially what we have just proven.

Corollary 6.14. *Let $\mathcal{E}$ denote a complex belonging to $\mathsf{D}^b(X \times Y)$ such that the corresponding Fourier–Mukai transform $\Phi_{\mathcal{E}} \colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ is an equivalence. Then the the cohomological Fourier–Mukai transform*

$$f^{\mathcal{E}} \colon \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$$

is an isomorphism of Hodge lattices.

6.2 Moduli spaces of sheaves and the other side of the proof

The last subsection is devoted to proving the right-to-left implication of Theorem 6.9, captured in the ensuing proposition.

Proposition 6.15. *Let X and Y be K3 surfaces over $\mathbf{C}$ and suppose there exists a Hodge isometry*

$$f \colon \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$$

of their Mukai lattices. Then there exists an equivalence of their bounded derived categories of coherent sheaves $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$.

The method of our proof will lead us to introduce another K3 surface M, which will turn out to be isomorphic to Y through an application of the classical Torelli theorem 5.24, and whose derived category we will show to be equivalent to $\mathsf{D}^b(X)$. Not only will the K3 surface M itself be important, but also how it is constructed or what it *represents*. We will expand on this viewpoint to some extent, but will have to refer the reader to [HL10] or [vBr20] for details and proofs, as this part is unfortunately outside the scope of our thesis. We also mention that this line of thinking was guided by the influential paper of Mukai [Muk87]. This approach will serve as an invaluable tool for obtaining a kernel for a Fourier–Mukai transform of the from $\mathsf{D}^b(M) \to \mathsf{D}^b(X)$, which we will later prove to be an equivalence. In proving the latter claim we will exploit a convenient characterization of fully faithfulness for triangulated functors between derived categories of smooth projective varieties attributed to Bondal and Orlov [BO95]. The characterization utilizes the fact that skyscraper sheaves $k(x)$, for closed points $x \in X$ form a spanning class for $\mathsf{D}^b(X)$, when X is smooth and projective. Essential surjectivity will be proven via the use of Corollary 1.31, which we have established towards the latter parts of Section 1.2.

A categorical digression. For a moment let $\mathcal{C}$ be any category. Recall that each object X of $\mathcal{C}$ gives rise to a functor

$$h_X = \mathrm{Hom}_{\mathcal{C}}(-, X) \colon \quad \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}.$$

A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is said to be *representable*, if there is some object X of $\mathcal{C}$, for which F and h_X are naturally isomorphic. Moreover the Yoneda lemma then asserts that the map

$$\mathrm{Hom}_{\mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathsf{Set})}(h_X, F) \to F(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X), \tag{6.5}$$

is an isomorphism for every object X of $\mathcal{C}$ and any functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, and it is natural both in X and F.

From now on we focus on a fixed K3 surface X over the field of complex numbers $\mathbf{C}$ and consider the category $\mathsf{Sch}_{/\mathbf{C}}$ of schemes over $\mathbf{C}$. The main idea for conceiving a new space out of X, we will employ, is an attempt to try and perceive the collection of all coherent sheaves on X as a space itself. What was said above is not precise nor fully correct, as the collection of all coherent sheaves on X is way too large to consider all at once, so we will instead restrict ourself to a smaller class of sheaves by imposing some additional constraints, which we present presently.

Huybrechts restricts himself to schemes of finite type. But Huybrechts, Lehn do not.

Firstly, in the spirit of deformation theory, we might consider our sheaves arising as members of certain families of sheaves on X "parametrized" by points of some other scheme S over $\mathbf{C}$. The notion of such a family is encoded in a single coherent sheaf $\mathcal{F} \in \mathsf{coh}(X \times S)$, defined on the product $X \times S$. Indeed, consider a closed point $s \in S$, to which there is a corresponding morphism of schemes over $\mathbf{C}$, also denoted $s\colon \operatorname{Spec}\mathbf{C} \to S$. Then we define the morphism $i_s\colon X \to X \times S$ to be the composition

$$i_s\colon \quad X \longrightarrow X \times \operatorname{Spec}\mathbf{C} \xrightarrow{\mathrm{id}_X \times s} X \times S, \tag{6.6}$$

where the first morphism into the product is induced by id_X and the structure map $X \to \operatorname{Spec}\mathbf{C}$, and introduce $\mathcal{F}|_s$ to mean the pull-back sheaf $i_s^*\mathcal{F}$ on X. In particular, this will allow us to focus only on those sheaves on X, parametrized by S, which have a fixed Mukai vector. Since we would like the geometry of our scheme S to have a greater impact on the parametrization of sheaves on X, we would like our sheaves $\mathcal{F}|_s$ from the family $\mathcal{F}$ to vary in a somewhat continuous manner with respect to the parameter $s \in S$. To this end we impose the following *flatness* condition on $\mathcal{F}$.

Definition 6.16. For any scheme S over $\mathbf{C}$, a sheaf $\mathcal{F} \in \mathsf{coh}(X \times S)$ is said to be *flat over* S, if $\mathcal{F}$ is a flat $\pi_S^{-1}\mathcal{O}_S$-module, where $\pi_S\colon X \times S \to S$ is the canonical projection.

A condition, which restricts the scope of available sheaves even further is *stability*. We quickly glance over the main components, which enable us to define what a stable sheaf is in the first place and give a property analogous to Schur's lemma, but for stable sheaves, which will become relevant in the proof of Proposition 6.15.

- ▷ Let $\mathcal{O}_X(1)$ be an *ample*[30] invertible sheaf on X. As X is projective an example of such a sheaf would be the the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$ along an inclusion $X \hookrightarrow \mathbb{P}^n$. With respect to $\mathcal{O}_X(1)$, define $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m}$ for $\mathcal{F} \in \mathsf{coh}(X)$ and $m \in \mathbf{Z}_{\geq 0}$.
- ▷ The *Hilbert polynomial* of a coherent sheaf $\mathcal{F} \in \mathsf{coh}(X)$ is defined as

$$P(\mathcal{F}, -)\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z} \qquad m \mapsto \chi(X, \mathcal{F}(m)),$$

[30] According to [Har77, §II.7], an invertible sheaf $\mathcal{L}$ on a noetherian scheme X is *ample* if for every coherent sheaf $\mathcal{F}$ on X there is an integer $n_{\mathcal{F}} \in \mathbf{Z}_{\geq 0}$, for which the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for every $n \geq n_{\mathcal{F}}$. By [Har77, §II, Example 7.6.1], the pull-back $i_*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_X(1)$ along an inclusion $i\colon X \hookrightarrow \mathbb{P}^n$ is ample.

which is know to be a polynomial function uniquely expressible in the form

$$P(\mathcal{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \frac{m^i}{i!},$$

for some rational coefficients $\alpha_i(\mathcal{F})$ and $d \in \mathbf{Z}_{\geq 0}$ (cf. [HL10, §I.1.2, Lemma 1.2.1]). Let $P(\mathcal{F})$ denote the corresponding polynomial. The *reduced Hilbert polynomial* of $\mathcal{F}$ is defined to be $p(\mathcal{F}) = P(\mathcal{F})/\alpha_d(\mathcal{F})$.

- ▷ Equip the set of all polynomials $\mathbf{Q}[t]$ with the lexicographical ordering of their coefficients $<$, meaning that for polynomials f and g we set $f < g$, if the leading coefficient of $g - f$ is positive.

- ▷ We declare the *dimension* of a sheaf to be the dimension of its support. A sheaf is said to be *pure* if all its non-trivial subsheaves have the same dimension.

- ▷ A pure sheaf $\mathcal{F} \in \mathsf{coh}(X)$ is called *stable* (resp. *semi-stable*) if for each proper non-trivial subsheaf $\mathcal{G} \subseteq \mathcal{F}$, $p(\mathcal{G}) < p(\mathcal{F})$ (resp. $p(\mathcal{G}) \leq p(\mathcal{F})$) holds. Notice that just as the Hilbert polynomial subtly depends on the choice of an ample invertible sheaf $\mathcal{O}_X(1)$, so does the stability condition above. In this context a choice of an ample invertible sheaf on X is called a *polarization*.

- ▷ Any homomorphism $\mathcal{F} \to \mathcal{G}$ between two stable sheaves with $P(\mathcal{F}) = P(\mathcal{G})$ is either 0 or an isomorphism. Consequently, for two stable sheaves $\mathcal{F}$ and $\mathcal{G}$, we have

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \begin{cases} \mathbf{C}, & \text{if } \mathcal{F} \simeq \mathcal{G}, \\ 0, & \text{if } \mathcal{F} \not\simeq \mathcal{G}. \end{cases} \tag{6.7}$$

  For the proof one may consult Proposition 1.2.7 and Corollary 1.2.8 of [HL10, §I.1.2].

- ▷ For a coherent sheaf $\mathcal{F}$, it is possible to compute its Hilbert polynomial using the Hirzebruch–Riemann–Roch Theorem 4.31. In the case that X is a K3 surface the result of the computation is

$$P(\mathcal{F}, m) = (2\mathrm{ch}_0(\mathcal{F}) + \mathrm{ch}_2(\mathcal{F})) + \langle h, \mathrm{ch}_1(\mathcal{F})\rangle m + \tfrac{1}{2}\langle h, h\rangle \mathrm{ch}_0(\mathcal{F})\, m^2,$$

  where $h = \mathrm{c}_1(\mathcal{O}_X(1))$. Recalling the formula (5.7) we therefore see that the Mukai vector $v(\mathcal{F})$ determines the Hilbert polynomial $P(\mathcal{F})$.

With all the necessary terminology now established, we can introduce the following moduli functor.

Definition 6.17. Let X be a K3 surface over $\mathbf{C}$ with a prescribed Mukai vector $v \in \tilde{H}(X, \mathbf{Z})$. The *moduli functor*

$$\mathcal{M}_v \colon \mathsf{Sch}_{/\mathbf{C}}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

is defined on objects as

$$S \longmapsto \left\{ \mathcal{F} \in \mathsf{coh}(X \times S) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is flat over } S, \text{ and for each closed point } s \in S, \\ \mathcal{F}|_s \text{ is a semi-stable sheaf on } X, \text{ with } v(\mathcal{F}|_s) = v. \end{array} \right\}_{/\sim}.$$

Here $\mathcal{F} \sim \mathcal{F}'$ is defined to hold precisely when there exist a line bundle $\mathcal{L}$ on S, for which

$\mathcal{F} \simeq \mathcal{F}' \otimes_{\mathcal{O}_{X\times S}} \pi_S^*\mathcal{L}$. On morphisms[31] $\mathcal{M}_v$ is defined as

$$(f\colon S' \to S) \longmapsto \left((\mathrm{id}_X \times f)^* \colon \quad \begin{array}{l} \mathcal{M}_v(S) \to \mathcal{M}_v(S') \\ [\mathcal{F}]_\sim \mapsto [(\mathrm{id}_X \times f)^*\mathcal{F}]_\sim \end{array} \right).$$

There is also a *stable* variant of the moduli functor, denoted by $\mathcal{M}_v^s \colon \mathsf{Sch}_{/\mathbf{C}}^{\mathrm{op}} \to \mathsf{Set}$, where every instance of the word "semi-stable" in the definition of $\mathcal{M}_v$ is replaced by "stable".

Remark 6.18. Our moduli functor is constrained by a prescribed Mukai vector as opposed to [vBr20, §4.2, Proposition 4.15], where they prescribe the Chern character. As Mukai vectors and Chern characters determine each other on a K3 surface, the definitions are in fact equivalent.

Definition 6.19. Let X be a complex K3 surface. A vector $v = (v_0, v_1, v_2) \in \tilde{H}(X, \mathbf{Z})$ is *effective* if either $v_0 > 0$ or $v_0 = 0$ and v_1 is an effective divisor[32]. The *divisibility* of a vector $v \in \tilde{H}(X, \mathbf{Z})$ is defined to be the greatest common divisor of the set of integers $\{\langle v, v'\rangle \in \mathbf{Z} \mid v' \in \mathrm{N}(X)\}$ and denoted by $\mathrm{div}(v)$.

The following theorem, based on the pioneering work of Mukai [Muk87] is due to many authors [GH96; Huy06; OGr97]. A modern exposition can be found in [BM14].

Theorem 6.20. *[GH96; Huy06; OGr97; HL10; BM14] Let $v \in \tilde{H}(X, \mathbf{Z})$ be an effective isotropic vector for which there exists a class $v' \in \mathrm{N}(X)$, satisfying $\langle v, v'\rangle = 1$. Then there exists a polarization on X for which all semi-stable sheaves are stable, so the functors $\mathcal{M}_v$ and $\mathcal{M}_v^s$ coincide, and the moduli functor $\mathcal{M}_v \colon \mathsf{Sch}_{/\mathbf{C}}^{op} \to \mathsf{Set}$ is representable. The moduli functor $\mathcal{M}_v$ is then represented by a smooth projective complex surface M_v, called a* fine moduli space*.*

Proof. This is a very sophisticated and deep result lying fully outside the scope of this thesis. Not only is the construction of a moduli space M_v in a certain weaker sense, called *universal corepresentability*, very complicated also showing that it has the desired properties is very difficult. □

Circling back to the original idea of constructing a space, which parametrizes isomorphism classes of stable coherent shaves on X with a prescribed Mukai vector, we now argue as to why M_v achieves this. This is done in its entirety through unwrapping the definition of what it means for a moduli functor $\mathcal{M}_v$ to be representable. Since the set of all closed points of M_v is in bijection with the set of morphisms $\mathrm{Spec}\,\mathbf{C} \to M_v$ in $\mathsf{Sch}_{/\mathbf{C}}$, we obtain the following chain of natural bijections

$$\mathrm{Hom}_{\mathsf{Sch}_{/\mathbf{C}}}(\mathrm{Spec}\,\mathbf{C}, M_v) \xrightarrow{=} h_{M_v}(\mathrm{Spec}\,\mathbf{C}) \xrightarrow{\simeq} \mathcal{M}_v(\mathrm{Spec}\,\mathbf{C}). \tag{6.8}$$

Thus every closed point of M_v represents and is represented by a unique class $[\mathcal{F}]_\sim \in \mathcal{M}_v(\mathrm{Spec}\,\mathbf{C})$ of coherent sheaves on $X \times \mathrm{Spec}\,\mathbf{C}$. As $\mathrm{Spec}\,\mathbf{C}$ carries only the trivial line bundle $\mathcal{O}_{\mathrm{Spec}\,\mathbf{C}}$, the equivalence relation $\sim$ of Definition .19 is enhanced to distinguishing

[31] The assignment is well defined because

$$(\mathrm{id}_X \times f)^*(\mathcal{F} \otimes \pi_S^*\mathcal{L}) \simeq (\mathrm{id}_X \times f)^*\mathcal{F} \otimes (\mathrm{id}_X \times f)^*\pi_S^*\mathcal{L} \simeq (\mathrm{id}_X \times f)^*\mathcal{F} \otimes \pi_{S'}^* f^*\mathcal{L}.$$

[32] A divisor $D = \sum_i n_i[X_i]$ is *effective*, if $n_i \geq 0$, for each i. Here X_i are closed integral subschemes of codimension one in X.

only *non-isomorphic* sheaves on $X \times \operatorname{Spec} \mathbf{C}$. Since clearly $X \simeq X \times \operatorname{Spec} \mathbf{C}$, we arrive at the bijective correspondence

$$\{\text{closed points of } M_v\} \leftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes } [\mathcal{F}]_\simeq \\ \text{of stable sheaves on } X \text{ with } v(\mathcal{F}) = v \end{array} \right\}. \tag{6.9}$$

This correspondence ties in perfectly into our initial idea of M_v parametrizing certain sheaves on X.

Another key ingredient, playing a principal part in the proof of Proposition 6.15, is the *universal bundle* defined by the subsequent category theoretic magic.

Definition 6.21. The *universal bundle (on X)* is any representative $\mathcal{E} \in \mathsf{coh}(X \times M_v)$ of the unique equivalence class of $\mathcal{M}_v(M_v)$, to which the isomorphism

$$\operatorname{Hom}_{\mathrm{Fct}(\mathsf{Sch}^{\mathrm{op}}_{/\mathbf{C}}, \mathsf{Set})}(h_{M_v}, \mathcal{M}_v) \to \mathcal{M}_v(M_v)$$

of Yoneda lemma sends the natural isomorphism of functors $\eta\colon h_M \Longrightarrow \mathcal{M}_v$, which detects representability of the moduli functor $\mathcal{M}_v$. In other words

$$[\mathcal{E}]_\sim = \eta_{M_v}(\mathrm{id}_{M_v}).$$

Remark 6.22. As a consequence of a more general fact [HL10, §6.1, Remark 6.1.9], we see that the universal bundle $\mathcal{E}$ is a *vector bundle* i.e. a locally free $\mathcal{O}_{X \times M_v}$-module of finite rank. This is just a justification for why we call $\mathcal{E}$ a bundle as we will never actually use this fact.

Remark 6.23. Utilizing the universal bundle $\mathcal{E}$ we can provide another way of interpreting the isomorphism classes of stable sheaves on X with Mukai vector v i.e. elements of $\mathcal{M}_v(\operatorname{Spec} \mathbf{C})$. Under the natural bijection $\eta_{\operatorname{Spec} \mathbf{C}}$ of (6.8) every isomorphism class $[\mathcal{F}]_\simeq \in \mathcal{M}_v(\operatorname{Spec} \mathbf{C})$ corresponds to a unique morphism $s\colon \operatorname{Spec} \mathbf{C} \to M_v$. By naturality of η, we obtain the following commutative square

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{Sch}_{/\mathbf{C}}}(M_v, M_v) \arrow[r, "\eta_{M_v}"] \arrow[d, "- \circ s"'] & \mathcal{M}_v(M_v) \arrow[d, "\mathcal{M}_v(s)"] \\
\operatorname{Hom}_{\mathsf{Sch}_{/\mathbf{C}}}(\operatorname{Spec} \mathbf{C}, M_v) \arrow[r, "\eta_{\operatorname{Spec} \mathbf{C}}"] & \mathcal{M}_v(\operatorname{Spec} \mathbf{C}),
\end{tikzcd}$$

which spells out that

$$\begin{aligned}
[\mathcal{F}]_\simeq = \eta_{\operatorname{Spec} \mathbf{C}}(s) = \eta_{\operatorname{Spec} \mathbf{C}}(((-) \circ s)(\mathrm{id}_M)) = \\
= \mathcal{M}_v(s)(\eta_M(\mathrm{id}_M)) = \mathcal{M}_v(s)([\mathcal{E}]_\sim) = [(\mathrm{id}_X \times s)^* \mathcal{E}]_\simeq.
\end{aligned}$$

Taking the liberty of identifying $s\colon \operatorname{Spec} \mathbf{C} \to M_v$ with its image – a closed point corresponding to a stable sheaf $\mathcal{F} \in \mathsf{coh}(X)$, with $v(\mathcal{F}) = v$, as in (6.9), and denoting it with $[\mathcal{F}]$, we see that the above calculation reads

$$\mathcal{E}|_{[\mathcal{F}]} \simeq \mathcal{F}.$$

Lastly we need the following theorem, originally due to Bondal and Orlov [BO95], and slightly amplified by Bridgeland [Bri98], which characterizes fully faithfulness of a Fourier–Mukai transform.

Theorem 6.24 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties over $\mathbf{C}$ and let $\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ be a Fourier-Mukai transform associated to a kernel $\mathcal{E}$ of category $\mathsf{D}^b(X \times Y)$. Then $\Phi_{\mathcal{E}}$ is fully faithful if and only if for any two closed points x, $y \in X$ the following condition is satisfied*

$$\mathrm{Hom}_{\mathsf{D}^b(Y)}(\Phi_{\mathcal{E}}(k(x)), \Phi_{\mathcal{E}}(k(y))[i]) \simeq \begin{cases} \mathbf{C}, & \text{if } x = y \text{ and } i = 0, \\ 0, & \text{if } x \neq y \text{ or } i \notin [0, \dim X]. \end{cases} \tag{6.10}$$

Proof. A proof may be found in [Huy06, §7, Proposition 7.1] or [BO95, Theorem 1.1] or [Bri98, Theorem 5.1]. □

Remark 6.25. The assumption of picking $\mathbf{C}$ to be our ground field in Theorem 6.24 is not very far from the full generality, where the underlying field is required to be algebraically closed and of characteristic zero.

Let us give some evidence for why one might expect such a result to hold and why it is important in our case. The theorem is essentially an improvement of Proposition 1.27 applied to a functor $\mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ and the spanning class of $\mathsf{D}^b(X)$ consisting of skyscraper sheaves $k(x)$ associated to closed points $x \in X$. Its main advantage and importance lies in the fact that it is actually not necessary to check that $\Phi_{\mathcal{E}}$ acts as an isomorphism on *all* the Hom-sets, but only on the ones, where this kind of verification is easy. For example, in our case, when X is a K3 surface, one can compute that for any pair of closed points x, $y \in X$ and any integer $i \in \mathbf{Z}$

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(k(x), k(y)[i]) \simeq \begin{cases} \mathbf{C}, & \text{if } x = y \text{ and } i \in \{0, 2\}, \\ \mathbf{C}^2, & \text{if } x = y \text{ and } i = 1, \\ 0, & \text{else.} \end{cases} \tag{6.11}$$

Then assuming conditions (0.6) hold it is not very difficult to see that $\Phi_{\mathcal{E}}$ acts on Hom-sets as an isomorphism

$$\mathrm{Hom}_{\mathsf{D}^b(X)}(k(x), k(y)[i]) \to \mathrm{Hom}_{\mathsf{D}^b(Y)}(\Phi_{\mathcal{E}}(k(x)), \Phi_{\mathcal{E}}(k(y))[i])$$

for all x, $y \in X$ and $i \in \mathbf{Z}$, except for the case when $x = y$ and $i \in \{1, 2\}$. In the specific circumstance of our proof the case $i = 2$ can be shown to be an isomorphism through an application of Serre duality, but case $i = 1$ turns out to be more tricky. The main difficulty being that $\mathrm{Hom}_{\mathsf{D}^b(X)}(k(x), k(y)[1])$ is now two dimensional and there are no obvious choices for what its basis should be and how $\Phi_{\mathcal{E}}$ acts on that basis. We touch upon this approach in some more detail after the proof Proposition 6.15.

With all the preparations now out of the way, we may begin with the proof.

Proof of Proposition 6.15. Assume $f\colon \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$ is a Hodge isometry. Let $e \in \tilde{H}(Y, \mathbf{Z})$ denote the Mukai vector $(0, 0, 1)$. Denote with $v = (v_0, v_1, v_2) \in \tilde{H}(X, \mathbf{Z})$ the Mukai vector for which $f(v) = e$. We may always assume that $v_0 > 0$ by the following casework.

- If $v_1 = 0$, then either $v_0 \neq 0$ or $v_2 \neq 0$. Then using a combination of the following two obvious Hodge isometries of $\tilde{H}(X, \mathbf{Z})$ one is able to achieve $v_0 > 0$.

$$(w_0, w_1, w_2) \mapsto (w_2, w_1, w_0) \qquad \text{and} \qquad (w_0, w_1, w_2) \mapsto (-w_0, w_1, -w_2).$$

- If $v_1 \neq 0$ and either $v_0 \neq 0$ or $v_2 \neq 0$, proceed as in the case above, otherwise for $v_0 = 0$ and $v_2 = 0$ consider the Fourier–Mukai transform, whose kernel is the sheaf $\Delta_*\mathcal{L}$ on $X \times X$ for some line bundle $\mathcal{L}$ on X. We have shown in Example 4.3 (iii), that $\Phi_{\Delta_*\mathcal{L}}$ is naturally isomorphic to $\mathcal{L} \otimes (-)$. This is clearly an equivalence, thus by Corollary 6.14 the associated cohomological Fourier–Mukai $f^{\Delta_*\mathcal{L}}$ is a Hodge isometry. In a very similar way as in the proof of Lemma 6.13 one computes that $f^{\Delta_*\mathcal{L}}$ is given by multiplication with the Chern character $\mathrm{ch}(\mathcal{L})$, i.e.

$$f^{\Delta_*\mathcal{L}}(w) = \mathrm{ch}(\mathcal{L}) \smile w = \left(w_0, w_1 + w_0\mathrm{c}_1(\mathcal{L}), w_2 + (w_1.\mathrm{c}_1(\mathcal{L})) + \tfrac{1}{2}w_0\mathrm{c}_1^2(\mathcal{L})\right).$$

  Since $e \in \widetilde{H}^{1,1}(Y)$ and $f\colon \tilde{H}(X,\mathbf{Z}) \to \tilde{H}(Y,\mathbf{Z})$ is a Hodge isometry, we may conclude that $v_1 \in H^{1,1}(X) \cap H^2(X,\mathbf{Z})$. The latter group we know to be $\mathrm{NS}(X) \simeq \mathrm{Pic}(X)$ and the intersection pairing $(\cdot.\cdot)$ restricted to $\mathrm{Pic}(X)$ is non-degenerate by Proposition 5.14. Hence there always exists a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, such that $(v_1.\mathrm{c}_1(\mathcal{L})) \neq 0$. Together with the Hodge isometries of the previous case, we can achieve $v_0 > 0$.

The Mukai vector v also satisfies the following two properties, both consequences of f being a Hodge isometry.

- ▷ v is isotropic, since $\langle v, v\rangle = \langle f(v), f(v)\rangle = \langle e, e\rangle = 0$.
- ▷ v is of divisibility 1. Indeed let $e' = (1,0,0)$ and $v' \in \tilde{H}(X,\mathbf{Z})$, for which $f(v') = e'$. Then, since f is a Hodge isometry, $v' \in \mathrm{N}(X)$ and $\langle v, v'\rangle = \langle f(v), f(v')\rangle = \langle e, e'\rangle = 1$.

By Theorem .20 there exists a complex smooth projective surface M_v and a universal bundle $\mathcal{E}$ on $X \times M_v$, such that M_v represents the moduli functor $\mathcal{M}_v$. For brevity we set $M = M_v$. Denote with p and q the canonical projections from $X \times M$ to X and M, respectively

$$X \xleftarrow{p} X \times M \xrightarrow{q} M.$$

We will now show that the Fourier–Mukai transform

$$\Phi_{\mathcal{E}}\colon \mathsf{D}^b(M) \to \mathsf{D}^b(X) \qquad \Phi_{\mathcal{E}} := \mathbf{R}p_*(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times M}} \mathbf{L}q^*(-)),$$

whose kernel is the universal bundle $\mathcal{E}$, is an equivalence of triangulated categories.

Fully faithfulness. Utilizing Bondal and Orlov's characterization, Theorem 6.24 will give us fully faithfulness of $\Phi_{\mathcal{E}}$, provided the spaces $\mathrm{Hom}_{\mathsf{D}^b(X)}(\Phi_{\mathcal{E}}(k(s)), \Phi_{\mathcal{E}}(k(t))[i])$, for closed points s, $t \in M$ and $i \in \mathbf{Z}$, satisfy conditions (0.6). To compute what the Hom-sets are, we first need a more practical description of $\Phi_{\mathcal{E}}(k(s))$. As usual, let $s\colon \mathrm{Spec}\,\mathbf{C} \to M$ denote the morphism corresponding to a closed point s and let $i_s\colon X \to X \times M$ be defined like (6.6). Then the following is a pull-back square, as X may be seen as the fibre of the projection $q\colon X \times M \to M$ above the closed point s.

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times M \arrow[d, "q"] \\ \mathrm{Spec}\,\mathbf{C} \arrow[r, "s"] & M \end{tikzcd}$$

As the projections are always flat, by the underived flat base change formula 3.25 and the underived projection formula 3.26, we compute

$$\begin{aligned}
\Phi_{\mathcal{E}}(k(s)) &= \mathbf{R}p_*(\mathcal{E} \otimes_{\mathcal{O}_{X\times M}} q^*(k(s))) \\
&= \mathbf{R}p_*(\mathcal{E} \otimes_{\mathcal{O}_{X\times M}} q^*(s_*\mathcal{O}_{\operatorname{Spec}\mathbf{C}})) \\
&\simeq \mathbf{R}p_*(\mathcal{E} \otimes_{\mathcal{O}_{X\times M}} i_{s*}\varepsilon^*(\mathcal{O}_{\operatorname{Spec}\mathbf{C}})) && \text{(Base change 3.25)} \\
&\simeq \mathbf{R}p_*(i_{s*}(i_s^*\mathcal{E} \otimes_{\mathcal{O}_X} \varepsilon^*\mathcal{O}_{\operatorname{Spec}\mathbf{C}})) && \text{(Projection formula 3.26)} \\
&\simeq i_s^*\mathcal{E} \otimes_{\mathcal{O}_X} \varepsilon^*\mathcal{O}_{\operatorname{Spec}\mathbf{C}} && (p \circ i_s = \mathrm{id}_X) \\
&\simeq i_s^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X \\
&\simeq i_s^*\mathcal{E} \\
&= \mathcal{E}|_s.
\end{aligned}$$

Here only the push-forward along the projection p is derived. More precisely the tensor product is underived because $\mathcal{E}$ is locally free and by exactness of push-forward along the closed immersion i_s, the former is also underived. To compute the Hom-sets we consider the following cases.

Case $s \neq t$. For $i < 0$ or $i > 2$, we know by Proposition 2.44 that $\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t[i]) \simeq \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t[i])$, which are trivial for $i < 0$. They are also trivial for $i > 2$, by Serre duality 3.11 as

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t[i]) \simeq \operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_t[i], \mathcal{E}|_s[2])^* \simeq \operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_t, \mathcal{E}|_s[2-i])^*.$$

When $i = 0$, we claim that $\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t)$ is trivial. First, fully faithfulness of $\mathsf{coh}(X) \to \mathsf{D}^b(X)$ (cf. Proposition 2.10) allows us to transport the Hom-set back to $\mathsf{coh}(X)$ and show triviality of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t)$ instead. Next, following the discussion of Remark 6.23, points s and t of M naturally correspond to isomorphism classes of stable sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, respectively. In this case we have

$$\mathcal{F} \simeq \mathcal{E}|_{[\mathcal{F}]} \simeq \mathcal{E}|_s \quad \text{and} \quad \mathcal{G} \simeq \mathcal{E}|_{[\mathcal{G}]} \simeq \mathcal{E}|_t.$$

Because the points s and t are distinct, sheaves $\mathcal{F}$ and $\mathcal{G}$ are non-isomorphic, however they share the same Mukai vector. Since the Mukai vector of a sheaf determines its Hilbert polynomial, $P(\mathcal{F}) = P(\mathcal{G})$ follows. Applying (6.7) yields

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0.$$

By Serre duality 3.11 and the freshly established case of $i = 0$, we see

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t[2]) = 0.$$

Lastly, for $i = 1$, we have $\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t[1]) \simeq \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t)$. This is because v is isotropic and by Proposition 4.41 we have

$$\langle v, v\rangle = \chi(\mathcal{E}|_s, \mathcal{E}|_t) = \dim \operatorname{Ext}^0_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t) - \dim \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t) + \dim \operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t).$$

And since both $\operatorname{Ext}^0_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t)$ and $\operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t)$ are trivial, $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{E}|_s, \mathcal{E}|_t)$ is trivial as well.

Case $s = t$. If $i < 0$ or $i > 2$, $\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_s[i])$ is trivial, following the same argument as in the case of $s \neq t$. In the remaining case of $i = 0$, stability of $\mathcal{E}|_s$ and (6.7) show that

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_s) \simeq \mathbf{C}.$$

In conclusion we have shown

$$\operatorname{Hom}_{\mathsf{D}^b(X)}(\mathcal{E}|_s, \mathcal{E}|_t[i]) \simeq \begin{cases} \mathbf{C}, & \text{if } s = t \text{ and } i = 0, \\ 0, & \text{if } s \neq t \text{ or } i \notin \{0,1,2\}. \end{cases} \tag{6.12}$$

By Theorem 6.24 this is enough to conclude that $\Phi_{\mathcal{E}}$ is fully faithful.

Essential surjectivity. Proposition 1.31 will show us that $\Phi_{\mathcal{E}}$ is essentially surjective. Indeed, the functor $\Phi_{\mathcal{E}}$ is fully faithful, $\mathsf{D}^b(M)$ contains non-zero objects, namely the skyscrapers $k(s)$ for closed points $s \in M$, and $\mathsf{D}^b(X)$ is indecomposable by Theorem .16. By Proposition 4.5 the functor $\Phi_{\mathcal{E}}$ has both a left and a right adjoint, which we recall can be neatly expressed as

$$\Phi_{\mathcal{E}_{\mathrm{L}}} = \Phi_{\mathcal{E}^\vee} \circ S_X \quad \text{and} \quad \Phi_{\mathcal{E}_{\mathrm{R}}} = S_M \circ \Phi_{\mathcal{E}^\vee}.$$

Therefore the condition $\Phi_{\mathcal{E}_{\mathrm{R}}}(\mathcal{F}^\bullet) \simeq 0$ clearly implies $\Phi_{\mathcal{E}_{\mathrm{L}}}(\mathcal{F}^\bullet) \simeq 0$ for any complex $\mathcal{F}^\bullet$ of $\mathsf{D}^b(X)$, allowing us to conclude that $\Phi_{\mathcal{E}}$ is indeed an equivalence.

Identifying M_v and Y. As a consequence of Theorem 5.9, we first conclude that the fine moduli space M_v is also a complex K3 surface! It is left to show that $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Y)$ are equivalent. We will prove this by showing an important result by itself, that

$$M_v \text{ and } Y \text{ are isomorphic K3 surfaces.}$$

Consider the cohomological Fourier–Mukai transform $f^{\mathcal{E}} \colon \tilde{H}(M_v, \mathbf{Z}) \to \tilde{H}(X, \mathbf{Z})$ given by the kernel $v(\mathcal{E})$. We then compute

$$f^{\mathcal{E}}(e) = f^{\mathcal{E}}(v(k(s))) = v(\Phi_{\mathcal{E}}(k(s))) = v(\mathcal{E}|_s) = v.$$

In the first equality we have used that the Mukai vector of any skyscraper $k(s)$, for a closed point $s \in M_v$ is precisely $e = (0,0,1)$ by Example 4.36. The second equality is Proposition 4.37. Transform $\Phi_{\mathcal{E}}$ being an equivalence, shows $f^{\mathcal{E}}$ is a Hodge isometry by Corollary 6.14. Since $f^{\mathcal{E}}(e) = v$ this lets us identify

$$v^\perp/\mathbf{Z}v \simeq H^2(M_v, \mathbf{Z})$$

as Hodge lattices by Proposition 6.6. By the same proposition an analogous identification also happens between the Hodge lattice $v^\perp/\mathbf{Z}v$ and $H^2(Y, \mathbf{Z})$. This allows us to construct a Hodge isometry

$$H^2(Y, \mathbf{Z}) \xrightarrow{\sim} v^\perp/\mathbf{Z}v \xrightarrow{\sim} H^2(M_v, \mathbf{Z}).$$

By the classical Torelli Theorem 5.24 the existence of such a Hodge isometry implies that Y and M_v are isomorphic K3 surfaces. Lastly, if we let $\psi \colon Y \to M_v$ denote such an isomorphism, then

$$\mathsf{D}^b(Y) \xrightarrow{\psi_*} \mathsf{D}^b(M_v) \xrightarrow{\Phi_{\mathcal{E}}} \mathsf{D}^b(X)$$

is an equivalence of bounded derived categories of X and Y. $\square$

Remark 6.26. Notice that during the course of this proof we also showed the following two statements.

(i) Assuming $\tilde{H}(X, \mathbf{Z})$ and $\tilde{H}(Y, \mathbf{Z})$ are Hodge isometric, the K3 surface Y is isomorphic to a moduli space of stable sheaves with a prescribed Mukai vector $v \in \tilde{H}(X, \mathbf{Z})$, which is isotropic and satisfies $\operatorname{div}(v) = 1$. Moreover Y is the unique K3 surface with

$$v^{\perp}/\mathbf{Z}v \simeq H^2(Y, \mathbf{Z}).$$

(ii) If Y is isomorphic to a moduli space of stable sheaves on X with a prescribed isotropic Mukai vector $v \in \tilde{H}(X, \mathbf{Z})$ with $\operatorname{div}(v) = 1$, then $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$.

The contents of these two sections finally allow as to conclude with the following theorem, which we set out to prove in the introduction.

Theorem 6.27 (Derived Torelli theorem)**.** *Let X and Y be K3 surfaces over the field of complex numbers $\mathbf{C}$. Then the following statements are equivalent.*

(i) X and Y share equivalent bounded derived categories of coherent sheaves,

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

(ii) There exists a Hodge isometry $f : T_X \to T_Y$ between the transcendental lattices of X and Y.

(iii) Mukai lattices $\tilde{H}(X, \mathbf{Z})$ and $\tilde{H}(Y, \mathbf{Z})$ of X and Y, respectively, are Hodge isometric.

(iv) There exists an isotropic Mukai vector $v \in \tilde{H}(X, \mathbf{Z})$ of divisibility 1, such that Y is isomorphic to a moduli space M_v of stable sheaves on X with Mukai vector v, representing some *moduli functor $\mathcal{M}_v$. Moreover M_v is up to isomorphism the unique K3 surface, for which there exists a Hodge isometry*

$$H^2(M_v, \mathbf{Z}) \simeq v^{\perp}/\mathbf{Z}v.$$

A comment

Lastly we touch upon an attempt of proving fully faithfulness of $\Phi_{\mathcal{E}} : \mathsf{D}^b(M) \to \mathsf{D}^b(X)$, which did not quite work.

maybe I will add this if time permits.

A Spectral sequences and how to use them

In this chapter we will first define cohomological spectral sequences and then discuss their meaning through applications related with derived categories.

At first, when encountering spectral sequences, one might think of them as just bookkeeping devices encoding a tremendous amount of data, but we will soon see how elegantly one can infer certain properties related to derived categorical claims exploiting the fact that they can naturally be encoded with spectral sequences. One of the most important spectral sequences, which is also very general within our scope of inspection, will be the Grothendieck spectral sequence relating the higher derived functors of two composable functors with the higher derived functors of their composition. Later on we will see that many useful and well-known spectral sequences occur as special cases of the Grothendieck spectral sequence. What follows was gathered mostly from [Huy06, Chapter 2, §2.1] and [GM02, Chapter III.7].

Definition A.1. A *(cohomological) spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data on which we further impose two convergence conditions.

- *Sequence of pages.* A sequence of bi-graded objects $(E_r^{\bullet,\bullet})_{r\in\mathbf{Z}_{\geq 0}}$ equipped with differentials of bi-degree $(r, 1-r)$. The r-th term of this sequence is called the r-th *page* and it consists of a lattice of objects $E_r^{p,q}$ of $\mathcal{A}$, for $p, q \in \mathbf{Z}$, and differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

  satisfying

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0,$$

  for each $p, q \in \mathbf{Z}$.

- *Isomorphisms.* A collection of isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q},$$

  for all $p, q \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, where

$$H^{p,q}(E_r) := \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}),$$

  which allow us to turn the pages.

- *Transfinite page.* A bi-graded object $E_\infty^{\bullet,\bullet}$.

- *Goal of computation.* A sequence of objects $(E^n)_{n\in\mathbf{Z}}$ of the category $\mathcal{A}$.

The above collection of data also has to satisfy the following two convergence conditions.

1. For each pair (p, q), there exists $r_0 \geq 0$, such that for all $r \geq r_0$ we have

$$d_r^{p,q} = 0 \quad \text{and} \quad d_r^{p+r,q-r+1} = 0$$

   and the isomorphism $\alpha_r^{p,q}$ can be taken to be the identity. We then say that the (p, q)-term *stabilizes* after page r_0 and we denote $E_{r_0}^{p,q}$ (along with all the subsequent $E_r^{p,q}$ for $r \geq r_0$) by $E_\infty^{p,q}$.

2. For each $n \in \mathbf{Z}$ there is a *regular*[33] decreasing filtration of E^n

$$E^n \supseteq \cdots \supseteq F^p E^n \supseteq F^{p+1} E^n \supseteq \cdots \supseteq 0 \tag{A.1}$$

[33] In our case the filtration $(F^p E^n)_{p\in\mathbf{Z}}$ is *regular*, whenever $\bigcap_p F^p E^n = \lim_p F^p E^n = 0$ and $\bigcup_p F^p E^n = \operatorname{colim}_p F^p E^n = E^n$.

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^p E^{p+q}/F^{p+1}E^{p+q}$$

for all $p, q \in \mathbf{Z}$.

In this case we also denote the existence of such a spectral sequence by

$$E_r^{p,q} \implies E^{p+q}.$$

Remark A.2. A few words are in order to justify us naming the sequence $(E^n)_{n\in\mathbf{Z}}$ our *goal of computation*. Usually one is given a starting page or a small number of them and the first goal is to identify the transfinite page – we are referring to convergence condition (a). Often one is able to infer the differentials degenerate after a number of turns of the pages from context or by observing the shape of the spectral sequence. For example *first quadrant spectral sequences*, i.e. the ones with non-trivial $E_r^{p,q}$ only for (p, q) lying in the first quadrant, always satisfy condition (a).

The second part of the computation is concerned with relating objects from the transfinite page $E_\infty^{\bullet,\bullet}$ with objects E^n. This is captured in the convergence condition (b), from which we can clearly observe that the intermediate quotients of the filtration $(F^pE^n)_{p\in\mathbf{Z}}$, for a fixed term E^n, lie on the anti-diagonal of the transfinite page passing through e.g. $E_\infty^{n,0}$. Explicitly these are terms $\dots, E_\infty^{n-1,1}, E_\infty^{n,0}, E_\infty^{n+1,-1}, \dots$

In condition (b) the existence of isomorphisms $\beta^{p,q}$ can also be restated by saying that $E_\infty^{p,q}$ fits into a short exact sequence

$$0 \to F^{p+1}E^n \to F^pE^n \to E_\infty^{p,q} \to 0. \tag{A.2}$$

This observation becomes very useful when considering properties of objects of the category $\mathcal{A}$ which are closed under extensions, especially when the filtration of E^n is finite.

Lemma A.3. *Let $a \le b$ be integers and consider a regular decreasing filtration*

$$F \supseteq \dots F^{p-1} \supseteq F^p \supseteq F^{p+1} \supseteq \dots \supseteq 0 \tag{A.3}$$

of an object F. Assume that the quotients F^p/F^{p+1} vanish for $p \notin [a, b]$, then $F^p \simeq F$ for all $p \le a$ and $F^p \simeq 0$ for all $p \ge b$, so (A.3) is a finite filtration of F. In particular, when $a = b$, we have $F \simeq 0$.

Proof. Since the quotients F^p/F^{p+1} vanish for $p \notin [a, b]$, all the inclusions $F^{p+1} \subseteq F^p$ turn into isomorphisms. Then by regularity it follows that $F = \operatorname{colim}_p F^p \simeq \operatorname{colim}_{p\le a} F^p = F^a$ and similarly $0 \simeq \lim_p F^p \simeq \lim_{p\ge b} F^p = F^b$. □

In his famous Tôhoku paper [Gro57] Grothendieck devised the following spectral sequence relating higher derived functors of a composition of two left exact functors with the composition of their higher derived functors. For a proof see either [Huy06, §2, Proposition 2.66] (merely a sketch) or [Gro57, §2.4].

Theorem A.4 (Grothendieck, [Gro57])**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Let $\mathcal{I}_F$ be an F-adapted class in $\mathcal{A}$, $\mathcal{I}_G$ a G-adapted class of objects in $\mathcal{B}$ and suppose every object of $\mathcal{I}_F$ is sent to a G-acyclic object by the functor F. Then for every $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ there exists a spectral sequence*

$$E_2^{p,q} = \mathbf{R}^pG(\mathbf{R}^qF(A^\bullet)) \implies \mathbf{R}^{p+q}(G \circ F)(A^\bullet) = E^{p+q}. \tag{A.4}$$

Remark A.5. Taking the functor F to be the identity in Theorem A.4 we obtain as a consequence that for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, admitting a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ there is a spectral sequence

$$E_2^{p,q} = \mathbf{R}^p F(H^q(A^\bullet)) \implies \mathbf{R}^{p+q} F(A^\bullet) = E^{p+q}, \tag{A.5}$$

for every $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$.

Definition A.6. Let $\mathcal{B}$ be an abelian category and $\mathcal{C}$ its *full abelian subcategory*, meaning that $\mathcal{C}$ is abelian, a full subcategory of $\mathcal{B}$ and the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{B}$ is exact. Then $\mathcal{C}$ is called *thick*, if every extension in $\mathcal{B}$ of objects in $\mathcal{C}$ is again in $\mathcal{C}$. In other words, assuming

$$0 \to B_0 \to B_1 \to B_2 \to 0$$

is a short exact sequence in $\mathcal{B}$, if B_0 and B_2 belong to $\mathcal{C}$, then so does B_1.

Example A.7. The subcategory of finitely generated abelian groups forms a thick subcategory of the category of abelian groups Ab.

Proposition A.8. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor admitting a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$.*

(i) Suppose that for every object A of $\mathcal{A}$ the objects $\mathbf{R}^i F(A)$ are non-trivial only for finitely many $i \in \mathbf{Z}$. Then $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^b(\mathcal{B})$ for any bounded complex $A^\bullet$ of $\mathsf{D}^b(\mathcal{A})$, consequently inducing a right derived functor on the level of bounded derived categories

$$\mathbf{R}F\colon \mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^b(\mathcal{B}).$$

(ii) Let $\mathcal{C} \subset \mathcal{B}$ be a thick subcategory of $\mathcal{B}$ and assume that for every $A^\bullet$ of $\mathsf{D}^+(\mathcal{A})$, $\mathbf{R}^i F(A)$ belong to $\mathcal{C}$ for all A of $\mathcal{A}$ and $i \in \mathbf{Z}$. Moreover assume there exists $m \in \mathbf{Z}_{\geq 0}$, such that $\mathbf{R}^i F(A) = 0$, for all A of $\mathcal{A}$ and $i > m$. Then $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^+_{\mathcal{C}}(\mathcal{B})$ for any $A^\bullet$ in $\mathsf{D}^+(\mathcal{A})$, therefore inducing the functor

$$\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+_{\mathcal{C}}(\mathcal{B}).$$

Proof. For (i) consider $A^\bullet$ in $\mathsf{D}^b(\mathcal{A})$. We will show that $\mathbf{R}^i F(A^\bullet)$ is non-trivial only for finitely many $i \in \mathbf{Z}$. To do so, we will use spectral sequence (A.5). As the q-th cohomology $H^q(A^\bullet)$ is an object of $\mathcal{A}$, the complex $\mathbf{R}F(H^q(A^\bullet))$ is bounded by assumption and thus $\mathbf{R}^p F(H^q(A^\bullet))$ is non-trivial only for finitely many $p \in \mathbf{Z}$. Since $A^\bullet$ is also a bounded complex we see that all the non-trivial objects of page E_2 may be collected inside of a large enough square, say $D = [-a, a]^2$ for a positive integer $a \in \mathbf{Z}$. In other words $E_2^{p,q} \simeq 0$ for $(p, q) \notin D$, implying that also $E_\infty^{p,q} \simeq 0$ for $(p, q) \notin D$. This means that all the intermediate quotients of the filtration of E^n, i.e. the terms $E_\infty^{p,q}$, with $p+q = n$, are trivial, provided $|n| > 2a$. By regularity of the filtrations, we deduce that $\mathbf{R}^n F(A^\bullet) = E^n \simeq 0$ for $|n| > 2a$, implying that $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^b(\mathcal{B})$.

For (ii) we will apply the same idea and use the spectral sequence (A.5) to show that $\mathbf{R}^i F(A^\bullet)$ belong to $\mathcal{C}$ for all $i \in \mathbf{Z}$. This time we see that $\mathbf{R}^p F(H^q(A^\bullet)) = E_2^{p,q} \simeq 0$ for $p \notin [0, m]$, thus the same goes for $E_\infty^{p,q}$. Since every anti-diagonal (line with slope -1) going through a point with integral coordinates passes through only $m + 1$ other points with integral components lying in the infinite strip $[0,m]\times \mathbf{Z}$, all the objects E^n have finite regular filtrations by Lemma A.3. They look like this

$$E^n = F^0E^n \supseteq_{E_\infty^{0,n}} F^1E^n \supseteq_{E_\infty^{1,n-1}} F^2E^n \supseteq \cdots \supseteq F^mE^n \supseteq_{E_\infty^{m,n-1}} F^{m+1}E^n = 0. \tag{A.6}$$

The intermediate quotients, inscribed in the subscript of $\supseteq$[34], of said filtration are precisely those $E_\infty^{p,q}$, with $p+q=n$ and $(p,q)\in[0,m]\times\mathbf{Z}$. Since $\mathcal{C}$ is thick, in particular closed under kernels and quotients, and all the objects of page E_2 belong to $\mathcal{C}$ by assumption, then all the objects from all the subsequent pages, including the transfinite one E_∞, also belong to $\mathcal{C}$. Inductively, using short exact sequences (A.2) and thickness of $\mathcal{C}$, we then see, that all the terms of the finite filtration (A.6) belong to $\mathcal{C}$, including $E^n = \mathbf{R}^nF(A^\bullet)$. □

Proposition A.9. *Let $\mathcal{A}$ be an abelian category with enough injectives. Suppose $A^\bullet$ belongs to $\mathsf{K}^-(\mathcal{A})$ and $B^\bullet$ belongs to $\mathsf{K}^+(\mathcal{A})$. Then there exist spectral sequences*

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{A}}(A^\bullet, H^q(B^\bullet)) \implies \operatorname{Ext}^{p+q}_{\mathcal{A}}(A^\bullet, B^\bullet) = E^{p+q} \tag{A.7}$$

and

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{A}}(H^{-q}(A^\bullet), B^\bullet) \implies \operatorname{Ext}^{p+q}_{\mathcal{A}}(A^\bullet, B^\bullet) = E^{p+q}. \tag{A.8}$$

Proof. See [Huy06, §2, Example 2.70]. □

Remark A.10. Using Proposition 2.44, and taking $A^\bullet$ and $B^\bullet$ to be bounded complexes of $\mathsf{D}^b(\mathcal{A})$ the spectral sequences (A.7) and (A.8) specialize to

$$E_2^{p,q} = \operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, H^q(B^\bullet)[p]) \Rightarrow \operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) = E^{p+q}, \tag{A.9}$$
$$E_2^{p,q} = \operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(H^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) = E^{p+q}. \tag{A.10}$$

B Lattice theory

This short chapter's purpose is mainly to establish the language of lattice theory. It contains two highly non-trivial results as well. One on extending certain isometries due to Nikulin [Nik80], which we are generously utilizing in the proof of Proposition 6.8. And another due to Milnor, one which the first one builds upon, classifying indefinite even unimodular lattices. The second one is not strictly necessary anywhere in this thesis, but serves as a pleasant way to make the intersection pairing on $H^2(X,\mathbf{Z})$ and later the Mukai pairing on $\tilde{H}(X,\mathbf{Z})$ of a K3 surface X a bit more concrete. Our main sources were [Huy16, §14] and [Nik80].

Definition B.1. A free finitely generated $\mathbf{Z}$-module Λ together with a bilinear form $(\cdot.\cdot)\colon \Lambda\times\Lambda\to\mathbf{Z}$, which is

- *symmetric:* $(x.y)=(y.x)$, for all x, $y\in\Lambda$,
- *non-degenerate:* if $(x.y)=0$ for all $y\in\Lambda$, then $x=0$,

is called a *lattice*. Lattice Λ is said to be *even* if $(x.x)$ is an even integer for all $x\in\Lambda$. An element $x\in\Lambda$ is *isotropic*, if $(x.x)=0$.

[34] For example $F^0E^n \supseteq_{E_\infty^{0,n}} F^1E^n$ is supposed to mean that the quotient F^0E^n/F^1E^n is isomorphic to $E_\infty^{0,n}$.

By picking an integral basis of Λ one can represent a lattice by a symmetric integral matrix, called the *intersection matrix*. If $x_1, \ldots, x_n$ form an integral basis of Λ the intersection matrix is

$$\big((x_i.x_j)\big)_{1 \le i,j \le n}.$$

Determinant of the intersection matrix is called the *discriminant* of the lattice Λ, denoted $\operatorname{disc}\Lambda$, and is independent of the choice of the integral basis for Λ. Upon tensoring Λ with $\mathbf{R}$ and $\mathbf{R}$-linearly extending the bilinear form $(\cdot.\cdot)$, we obtain a non-degenerate symmetric bilinear form on $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. These are known to be diagonalizable with only 1 or -1 on the diagonal and we let $n_\pm$ denote the number of ± 1 on the diagonal. The quantities $n_\pm$ are readily seen to be invariants of Λ. We call the pair (n_+, n_-) the *signature* of Λ and the $n_+ + n_- = \operatorname{rk}(\Lambda)$ the *rank* of Λ. The lattice Λ is called *definite*, if either $n_+ = 0$ or $n_- = 0$ and *indefinite* otherwise.

Definition B.2. A lattice Λ is called *unimodular*, if the morphism

$$i_\Lambda \colon \Lambda \to \Lambda^* = \operatorname{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}), \qquad x \mapsto (x.-)$$

is an isomorphism of $\mathbf{Z}$-modules or equivalently[35] $\operatorname{disc}\Lambda = \pm 1$.

Remark B.3. The morphism $i_\Lambda \colon \Lambda \to \Lambda^*$ is always injective, because the bilinear form $(\cdot.\cdot)$ on Λ is non-degenerate.

Example B.4. (i) The *hyperbolic* lattice is defined by the intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote this indefinite unimodular lattice of rank two with $\mathcal{U}$.

(ii) The E_8*-lattice* is given by the intersection matrix

$$\begin{pmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & -1 & -1 & & & \\
 & & -1 & 2 & 0 & & & \\
 & & -1 & 0 & 2 & -1 & & \\
 & & & & -1 & 2 & -1 & \\
 & & & & & -1 & 2 & -1 \\
 & & & & & & -1 & 2
\end{pmatrix}.$$

The E_8-lattice is the unique even unimodular positive definite lattice of rank 8 and we denote it by E_8.

Definition B.5. The orthogonal direct sum of lattices Λ and Λ' is the free $\mathbf{Z}$-module $\Lambda \oplus \Lambda'$ equipped with the following bilinear form

$$((x, x').(y, y')) := (x.y)_\Lambda + (x'.y')_{\Lambda'}.$$

A *twist* of a lattice Λ by an integer $m \in \mathbf{Z}$ is the lattice $\Lambda(m)$ given by the same underlying $\mathbf{Z}$-module Λ, but equipped with the *twisted* bilinear form

$$(\cdot.\cdot)_{\Lambda(m)} = m \cdot (\cdot.\cdot)_\Lambda.$$

[35] Λ^* is also a free $\mathbf{Z}$-module and by non-degeneracy of $(\cdot.\cdot)$, i_Λ is an embedding. Then e.g. using the Smith normal form allows one to see that $|\operatorname{disc}\Lambda|$ equals the index $[\Lambda^* : \Lambda]$.

Definition B.6. Let Λ and Λ' be two lattices. A morphism of lattices or an *isometry* is a $\mathbf{Z}$-linear map $\phi\colon \Lambda \to \Lambda'$ satisfying

$$(\phi(x).\phi(y))_{\Lambda'} = (x.y)_{\Lambda'},$$

for all x, $y \in \Lambda$. A lattice L is called a *sublattice* of Λ, if $L \subseteq \Lambda$ as underlying $\mathbf{Z}$-modules and the embedding $L \hookrightarrow \Lambda$ is an isometry. The *orthogonal complement* of a sublattice L in Λ is defined to be

$$L^{\perp} = \{x \in \Lambda \mid (x.y) = 0, \text{ for all } y \in L\}.$$

Lemma B.7. *Let L be a sublattice of a lattice Λ. Then the orthogonal complement $L^{\perp}$ with the inherited bilinear from on Λ is also a sublattice.*

Proof. First, $L^{\perp}$ is a subgroup of a free finite rank abelian group Λ, thus it is itself also free of finite rank. It is left to show that the inherited form on $L^{\perp}$ is also non-degenerate. For this we consider the rational extensions $L_{\mathbf{Q}} \subseteq \Lambda_{\mathbf{Q}}$ and show instead that the extended form on $L_{\mathbf{Q}}^{\perp}$ is non-degenerate. We denote with W and V the $\mathbf{Q}$-vector spaces $L_{\mathbf{Q}}$ and $\Lambda_{\mathbf{Q}}$, respectively. The adjoint map $V \to V^*$, given by $v \mapsto (v.-)$, is injective and thus an isomorphism, therefore the linear map

$$V \longrightarrow W^*, \qquad v \mapsto (v.-)|_W$$

is surjective. Its kernel is $W^{\perp}$, so $\dim W^{\perp} + \dim W = \dim V$. Thus $W^{\perp} \oplus W = V$ is an orthogonal direct sum decomposition. Assume now that $w \in W^{\perp}$ is such that $(w.w') = 0$ for all $w \in W^{\perp}$. Then $(w.v) = 0$ for all $v \in V$ and by non-degeneracy of the form on V this implies $v = 0$. □

Definition B.8. An embedding of lattices $L \hookrightarrow \Lambda$ is said to be *primitive*, if its cokernel $\operatorname{coker}(L \hookrightarrow \Lambda) = \Lambda/L$ is free. In that case we also call L a *primitive sublattice* of Λ.

Example B.9. Let L be any sublattice of a lattice Λ. Then the orthogonal complement $L^{\perp}$ is a primitive sublattice of Λ. Indeed, suppose $x \in \Lambda$ represents a torsion element in $\Lambda/L^{\perp}$. Then $nx \in L^{\perp}$ for some positive integer $n \in \mathbf{Z}$. But this implies that $x \in L^{\perp}$, so x represents the zero element in $\Lambda/L^{\perp}$. Hence $\Lambda/L^{\perp}$ is free.

Lemma B.10. *Let $i\colon L \hookrightarrow \Lambda$ and $i'\colon \Lambda \hookrightarrow \Lambda'$ be a pair of primitive embeddings. Then their composition $i' \circ i\colon L \hookrightarrow \Lambda'$ is also a primitive embedding.*

Proof. Since Λ/L and Λ'/Λ are free abelian, the short exact sequences $0 \to L \to \Lambda \to \Lambda/L \to 0$ and $0 \to \Lambda \to \Lambda' \to \Lambda'/\Lambda \to 0$ are split. This means that there are retracts $r\colon \Lambda \to L$ and $r'\colon \Lambda' \to \Lambda$, thus the short exact sequence $0 \to L \to \Lambda' \to \Lambda'/L \to 0$ also splits, which makes Λ'/L free as a subgroup of Λ'. □

Proposition B.11 (Nikulin, [Nik80])**.** *Let L and Λ be even lattices and assume Λ is unimodular. Let i_0 and i_1 denote two primitive embeddings $L \hookrightarrow \Lambda$. Suppose the orthogonal complement $i_0(L)^{\perp}$ in Λ contains a copy of the hyperbolic lattice $\mathcal{U}$. Then there exists an isometry $\phi\colon \Lambda \to \Lambda$ such that $i_1 = \phi \circ i_0$.*

$$\begin{tikzcd} & L \arrow[dl, "i_0"'] \arrow[dr, "i_1"] & \\ \Lambda \arrow[rr, "\phi"] & & \Lambda \end{tikzcd}$$

Proof. See [Huy16, §14, Theorem 1.12] and in particular point (iii) in the subsequent remark. For a more general statement, from which this one follows, consult [Nik80, Theorem 1.14.4]. ☐

Corollary B.12. *Let $i_0\colon L_0 \hookrightarrow \Lambda_0$ and $i_1\colon L_1 \hookrightarrow \Lambda_1$ denote two primitive embeddings of even lattices L_0 and L_1 into isomorphic even unimodular lattices Λ_0 and Λ_1, respectively. Suppose that there is an isomorphism of lattices $\psi\colon L_0 \to L_1$ and assume the orthogonal complement $i_0(L_0)^\perp$ in Λ_0 contains a copy of the hyperbolic lattice $\mathcal{U}$. Then there is an isometry $\phi\colon \Lambda_0 \to \Lambda_1$, for which $\phi \circ i_0 = i_1 \circ \psi$.*

$$\begin{tikzcd} L_0 \arrow[r, "\psi"] \arrow[d, hook, "i_0"'] & L_1 \arrow[d, hook, "i_1"] \\ \Lambda_0 \arrow[r, "\phi"] & \Lambda_1 \end{tikzcd}$$

Proof. Let $\theta\colon \Lambda_1 \to \Lambda_0$ be an isomorphism. Then $\theta \circ i_1 \circ \psi\colon L_0 \to \Lambda_0$ is a primitive embedding and we can use Proposition B.11. ☐

Theorem B.13 (Milnor, [Nik80, Theorem 1.1.1])**.** *Let (n_+, n_-) be a pair of non-negative integers. Then there exists an even unimodular lattice of signature (n_+, n_-), if and only if $n_+ - n_- \equiv 0\ (8)$. If n_+, $n_- > 0$ then the (indefinite) lattice is unique up to isomorphism.*

Its easy yet powerful corollary allow us to abstractly identify the lattice structures on $H^2(X, \mathbf{Z})$ and $\tilde{H}(X, \mathbf{Z})$.

Theorem B.14. *Let Λ be an indefinite even unimodular lattice of signature (n_+, n_-). Setting $\tau = n_+ - n_-$ to be the index of Λ, then $\tau \equiv 0\ (8)$ and*

$$\begin{aligned} &\text{if } \tau \geq 0, \text{ then} \quad \Lambda \simeq E_8^{\oplus \frac{\tau}{8}} \oplus \mathcal{U}^{\oplus n_-}, \\ &\text{if } \tau \leq 0, \text{ then} \quad \Lambda \simeq E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus \mathcal{U}^{\oplus n_+}. \end{aligned}$$

Proof. The E_8 lattice has signature $(8, 0)$, and the hyperbolic lattice $\mathcal{U}$ has signature $(1, 1)$, thus the signature of $E_8^{\oplus \frac{\tau}{8}} \oplus \mathcal{U}^{\oplus n_-}$ equals $(8 \cdot \frac{\tau}{8} + n_-, n_-) = (n_+, n_-)$. Now use uniqueness from Theorem B.13. ☐

Remark B.15. We know from Proposition 5.17 that for a K3 surface X the Hodge lattice $H^2(X, \mathbf{Z})$ has signature $(3, 19)$ and from Remark 6.3 that $\tilde{H}(X, \mathbf{Z})$ has signature $(20, 4)$, therefore Theorem B.14 tells us that

$$H^2(X, \mathbf{Z}) \simeq E_8(-1)^{\oplus 2} \oplus \mathcal{U}^{\oplus 3} \quad \text{and} \quad \tilde{H}(X, \mathbf{Z}) \simeq E_8^{\oplus 2} \oplus \mathcal{U}^{\oplus 4}.$$

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Razširjeni povzetek

V algebraični geometriji preučujemo prostore kot so sheme ali raznoterosti preko snopov na njih. Še posebej do izraza pridejo kvazi-koherentni in koherentni snopi, ki nas v tem geometrijskem kotektstu privedejo bližje algebri. V tem smislu lahko najdemo pomembne lastnosti našega prostora zakodirane v kohomologiji glede na določene snope. Za prehajanje in računanje s snopi nam služijo kot orodje določeni funktorji, ki pa niso vedno s kohomologijo najbolje *usklajeni*. Z usklajenostjo mislimo eksaktnost funktorjev in kot vemo že na primeru potiska in povleka vzdolž nekega morfizma ta dva funktorja v splošnem nista eksaktna, temveč sta le levo oz. desno eksaktna. Nekoliko podrobneje si oglejmo primer funktorja potiska f_* vzdolž nekega morfizma $f\colon X \to Y$. Leva eksaktnost f_* se odraža v tem, da za kratko eksaktno zaporedje snopov $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$ na X v splošnem na Y dobimo le eksaktno zaporedje

$$0 \to f_*\mathcal{F}_0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2.$$

To nas povede, da vpeljemo dodatne člene, ki zaporedje eksaktno nadaljujejo v desno in v grobem merijo kako oddaljen od eksaktnosti je funktor f_*

$$\begin{aligned} 0 \to f_*\mathcal{F}_0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to \mathbf{R}^1 f_*\mathcal{F}_0 \to \mathbf{R}^1 f_*\mathcal{F}_1 \to \cdots \\ \cdots \to \mathbf{R}^i f_*\mathcal{F}_0 \to \mathbf{R}^i f_*\mathcal{F}_1 \to \mathbf{R}^i f_*\mathcal{F}_2 \to \mathbf{R}^{i+1} f_*\mathcal{F}_0 \to \cdots . \end{aligned}$$

Dodani členi se pravzaprav pojavijo kot slike višjih izpeljanih funktorjev potiska f_* in izkaže se, da vsi ti izhajajo iz enega izpeljanega funktorja definiranega na *omejeni izplejani kategoriji koherentnih snopov*

$$\mathsf{D}^b(X) = \mathsf{D}^b(\mathsf{coh}(X)).$$

Izpeljana kategorija se potem sama po sebi izkaže za zanimiv objekt preiskave zaradi njene kompleksnosti in ker se pojavi kot obetavna invarianta gladkih projektivnih raznoterosti. V posebnem postane pomenljivo vprašanje v kolikšnji mere se informacija o raznoterosti porazgubi, ko preidemo na njeno izpeljano kategorijo? Izkaže se, da je v določenih primerih mogoče raznoterost popolnoma prepoznati zgolj na podlagi njenega odtisa na izpeljani kategoriji. O tem priča izrek Bondala in Orlova [BO01], ki v primeru gladkih projektivnih raznoterosti X in Y, kjer predpostavljamo, da je bodisi kanoničen bodisi anti-kanoničen sveženj X obilen[36], pravi da sta X in Y izomorfni natanko tedaj, ko obstaja triangulirana ekvivalenca med njunima izpeljanima kategorijama. Obstajajo pa tudi primeri, ko tovrstni zaključki niso resnični. V članku [Muk81] je Mukai na primeru abelovih raznoterosti pokazal, da obstajajo neizomorfne raznoterosti, ki pa imajo vendarle ekvivalentni izpeljani kategoriji. Podobno obnašanje se odraža na primeru K3 ploskev, ki so gladke projektivne ploskve X s trivialnim kanoničnim svežnjem in trivialno grupo $H^1(X, \mathcal{O}_X)$. V tem primeru nam bo *izpeljani Torellijev izrek*, katerega obravnava po vzoru [Orl03] je glavni cilj tega dela, podal natančen in praktičen kriterij za določanje ekvivalentnosti izpeljanih kategorij dveh K3 ploskev.

K3 ploskve se izkažejo za zanimive tudi s stališča *Homološke zrcalne simetrije*, saj so kot poseben primer Calabi–Yau mnogoterosti takoj za enodimenzionalnimi eliptičnimi krivuljami, naslednji oprijemljivi vir tovrstnih mnogoterosti za testiranje domnev. Izpeljani Torellijev izrek pa v tej luči še posebej pride do izraza, ker na algebraično geometrični strani, kjer nastopajo izpeljane kategorije, omogoča slednjo invarianto zamenjati z bolj prikladno in preprosto Mukaijevo mrežo.

[36] Sveženj premic $\mathcal{L}$ na X je *obilen* (angl. *ample*), če za vsak koherentni snop $\mathcal{F}$ na X obstaja število $n_{\mathcal{F}} \in \mathbf{Z}_{\geq 0}$, da je za vse $n \geq n_{\mathcal{F}}$ snop $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generiran z globalnimi prerezi.

Izpeljane kategorije

Zelo pomembno vlogo v algebraični geometriji igra kohomologija geometrijskega objekta X glede na, denimo, koherentni snop $\mathcal{F}$ na X. Eden izmed načinov računanja kohomoloških grup $H^i(X, \mathcal{F})$, ki smo ga opisali v poglavju 2, vključuje sledeče. Namesto, da neposredno obravnavamo snop $\mathcal{F}$, ga predstavimo s t. i. *resolucijo* ali *predstavitvijo*, ki jo sestavljata kompleks snopov $\mathcal{E}^\bullet$, členi katerega pripadajo nekemu razredu snopov, ki ima glede na kohomologijo določene ugodne lastnosti, in kvazi-izomorfizem $\mathcal{E}^\bullet \to \mathcal{F}$ ali $\mathcal{F} \to \mathcal{E}^\bullet$. Ker sprememba resolucije na kohomologijo ne bo imela vpliva in ker lahko vsak snop zase vidimo tudi kot kompleks zgoščen v stopnji 0, želimo snop $\mathcal{F}$ obravnavati enako kot vse njegove resolucije. Pogledano od daleč, želimo homotopsko kategorijo $\mathsf{K}(\mathsf{coh}(X))$ spremeniti tako, da se snop $\mathcal{F}$ identificira z vsemi svojimi resolucijami, oz. z drugimi besedami, želimo vse kvazi-izomorfizme v $\mathsf{K}(\mathsf{coh}(X))$ spremeniti v izomorfizme. Slednje bo naše vodilo, da za splošno abelovo kategorijo $\mathcal{A}$ vpeljemo njej prirejeno izpeljano kategorijo $\mathsf{D}(\mathcal{A})$. Poglejmo si glavne sestavine, ki nam to omogočajo.

Triangulirana kategorija je k-linearna kategorija $\mathcal{D}$ opremljena s k-linearno ekvivalenco $(-)[1]\colon \mathcal{D} \to \mathcal{D}$, imenovano *funktor zamika*, in razredom *odlikovanih trikotnikov*, ki so trojice kompozabilnih morfizmov iz $\mathcal{D}$, oblike

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]. \tag{0.1}$$

Poleg tega mora funktor zamika in razred odlikovanih trikotnikov zadoščati tudi določenim pogojem, to so aksiomi TR1–TR4. Eden od njih na primer pravi, da lahko odlikovani trikotnik (0.1) zasukamo v naslednjem smislu in še zmeraj dobimo odlikovani trikotnik

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1].$$

Pomembno vlogo bodo imeli tudi funktorji med trianguliranimi kategorijami, ki ohranjajo triangulirano strukturo. V grobem so to takšni funktorji, ki komutirajo s funktorjema zamika in preslikajo odlikovane trikotnike v odlikovane trikotnike. Pravimo jim triangulirani funktorji.

Najprej veljemo kategorijo verižnih kompleksov $\mathsf{Ch}(\mathcal{A})$, prirejeno abelovi kategoriji $\mathcal{A}$. *Verižni kompleks* v abelovi kategoriji $\mathcal{A}$ je podan kot zaporedje objektov iz $\mathcal{A}$ in morfizmov

$$A^\bullet: \quad \cdots A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \to \cdots,$$

ki zadoščajo $d_A^i \circ d_A^{i-1} = 0$ za vse $i \in \mathbf{Z}$. Morfizmom d^i pravimo tudi *diferenciali*. *Verižna preslikava* med kompleksoma $A^\bullet$ in $B^\bullet$ je družina morfizmov $f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}}$ iz $\mathcal{A}$, ki zadoščajo $f^{i+1} \circ d_A^i = d_B^i \circ f^i$. Skupek vseh verižnih preslikav tvori razred morfizmov v kategoriji $\mathsf{Ch}(\mathcal{A})$. Poleg same kategorije kompleksov bodo pomembne tudi njene omejene polne podkategorije $\mathsf{Ch}^+(\mathcal{A})$, $\mathsf{Ch}^-(\mathcal{A})$, $\mathsf{Ch}^b(\mathcal{A})$, ki zaporedoma zaobjemajo navzdol omejene, navzgor omejene in omejene komplekse. Pomemben bo tudi funktor zamika $(-)[1]\colon \mathsf{Ch}^*(\mathcal{A}) \to \mathsf{Ch}^*(\mathcal{A})$, ki je v vseh štirih primerih podan s predpisom

na objektih: Za verižni kompleks $A^\bullet$ je $A[1]^\bullet$ podan s členi $(A[1]^\bullet)^i = A^{i+1}$ in diferenciali $d^i_{A[1]} = -d_A^{i+1}$.

na morfizmih: Za verižno preslikavo $f^\bullet\colon A^\bullet \to B^\bullet$ definiramo $f[1]^\bullet$ po komponentah s $(f[1]^\bullet)^i = f^{i+1}$.

Nadalje iz kategorije kompleksov $\mathsf{Ch}(\mathcal{A})$ ustvarimo njej pridruženo homotopsko različico – *homotopsko kategorijo kompleksov* $\mathsf{K}(\mathcal{A})$. Za dve verižni preslikavi $f, g\colon A^\bullet \to B^\bullet$ pravimo, da sta homotopni, kadar obstaja družina morfizmo $(h^i \colon A^i \to B^{i-1})_{i\in\mathbf{Z}}$ v $\mathcal{A}$, da velja

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

za vse $i \in \mathbf{Z}$. Ta relacija na $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$ je ekvivalenčna in označimo jo s $\simeq$. Homotopsko kategorijo kompleksov $\mathsf{K}(\mathcal{A})$ tedaj sestavlja isti razred objektov kot $\mathsf{Ch}(\mathcal{A})$ množica morfizmov $A^\bullet \to B^\bullet$ v $\mathsf{K}(\mathcal{A})$ pa je definirana kot kvocient

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)/_{\simeq}.$$

Funktor zamika se naravno inducira in za razliko od kategorije kopleksov je možno $\mathsf{K}(\mathcal{A})$ opremiti s strukturo triangulirane kategorije. Pomembno vlogo pri tem igra *stožec* verižne preslikave $f \colon A^\bullet \to B^\bullet$, definiran kot kompleks z naslednjimi komponentami in diferenciali

$$C(f)^i := A^{i+1} \oplus B^i \qquad \text{in} \qquad d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1]^i & d_B^i \end{pmatrix}.$$

Ta nam namreč omogoči vpeljati odlikovane trikotnike v $\mathsf{K}(\mathcal{A})$ kot vse tiste trikotnike, ki so izomorfni

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f^\bullet) \longrightarrow A[1]^\bullet.$$

Tukaj sta drugi in tretji morfizem podana po komponentah s kanonično inkluzijo oz. projekcijo. Ta podrobnost, ki razlikuje $\mathsf{K}(\mathcal{A})$ od $\mathsf{Ch}(\mathcal{A})$, se izkaže za zelo pomembno pri definiciji izpeljane kategorije, ki jo sedaj orišemo.

Kohomologijo verižnega kompleksa $A^\bullet$ v stopnji $i \in \mathbf{Z}$ definiramo kot kojedro naravne inkluzije $\operatorname{im} d^{i-1} \to \ker d^i$ in jo označimo s $H^i(A^\bullet)$. Vsaka verižna preslikava $f \colon A^\bullet \to B^\bullet$ inducira morfizem $H^i(f) \colon H^i(A^\bullet) \to H^i(B^\bullet)$ na kohomologiji in kot se izkaže je ta odvisen le od homotopskega razreda verižne preslikave f. Verižna preslikava $f \colon A^\bullet \to B^\bullet$ je *kvazi-izomorfizem*, če f povsod na kohomologiji inducira izomorfizme. Razred vseh kvazi-izomorfizmov je tisti, ki mu v izpeljani kategoriji želimo prirediti inverze. Slednjo konstruiramo na naslednji način. Za razred objektov $\mathsf{D}(\mathcal{A})$ ponovno vzamemo razred vseh verižnih kopleksov, "množico" morfizmov $A^\bullet \to B^\bullet$ v $\mathsf{D}(\mathcal{A})$ pa je nekoliko težje opisati. Vsak morfizem $A^\bullet \to B^\bullet$ v $\mathsf{D}(\mathcal{A})$ je namreč podan s predstavnikom, t. i. *levo streho*, ki je par morfizmov iz $\mathsf{K}(\mathcal{A})$

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

ob tem pa je s kvazi-izomorfizem. Če na ustrezen način (kot je opisano v poglavju 2) definiramo, kdaj sta dve levi strehi ekvivalentni in kako komponirati ekvivalenčne razrede le teh, postane $\mathsf{D}(\mathcal{A})$ kategorija. Še več, triangulirana struktura na $\mathsf{K}(\mathcal{A})$ se preko lokalizacijskega funktorja $Q_\mathcal{A} \colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ naravno spusti na $\mathsf{D}(\mathcal{A})$ tako da tudi ta postane triangulirana kategorija. Kot pri $\mathsf{K}(\mathcal{A})$ se tudi tu pojavijo omejene različice $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ in $\mathsf{D}^b(\mathcal{A})$ kot polne podkategorije $\mathsf{D}(\mathcal{A})$, ki jih zaporedoma sestavljajo navzdol omejeni, navzgor omejeni in omejeni kompleksi. Omenimo še da tako kostruirana izpeljana kategorija zadošča naslednji univerzalni lastnosti.

Definicija 0.1. Naj bo $\mathcal{A}$ abelova kategorija in $\mathsf{K}(\mathcal{A})$ njena homotopska kategorija. Kategorija $\mathsf{D}(\mathcal{A})$ skupaj s funktorjem $Q \colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ je izpeljana kategorija kategorije $\mathcal{A}$, če zadošča:

(i) Za vsak kvazi-izomorfizem s v $\mathsf{K}(\mathcal{A})$, je $Q(s)$ izomorfizem v $\mathsf{D}(\mathcal{A})$

(ii) Za vsako kategorijo $\mathcal{D}$ in funktor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$, ki slika kvazi-izomorfizme s iz $\mathsf{K}(\mathcal{A})$ v izomorfozme $\mathcal{D}$, obstaja do naravnega izomorfizma natančno enoličen funktor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathcal{D}$, za katerega $F \simeq F_0 \circ Q$. Z drugimi besedami spodnji diagram komutira do naravnega izomorfizma natančno

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\ \mathsf{D}(\mathcal{A}) \arrow[ur, dashed, "F_0"'] & \end{tikzcd}$$

Eden izmed razlogov za vpeljavo izpeljanih kategorij je njihova vloga kot naravna domena na katerih definiramo izpeljane funktorje. Naj $F\colon \mathcal{A} \to \mathcal{B}$ označuje aditiven funktor med abelovima kategorijama. Če je F eksakten je razmeroma preprosti definirati izpeljani funktor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ preko zgornje univerzalne lastnosti. Bolj zanimiva je situacija, kadar je F zgolj levo oz. desno eksakten. Tedaj smo primorani privzeti dodatne lastnosti na domenski kategoriji $\mathcal{A}$. Ena možnost je zahtevati obstoj zadostne količine *injektivnih objektov* v $\mathcal{A}$, kar pomeni, da lahko vsak objekt A kategorije $\mathcal{A}$ vložimo v neki injektiven objekt I. Objekt I kategorije $\mathcal{A}$ je *injektiven*, če je funktor $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$ eksakten oz. z drugimi besedami kadar za vsak monomorfizem $A \hookrightarrow B$ in vsak morfizem $A \to I$ obstaja morfizem $B \to I$ za katerega komutira naslednji diagram.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

Razred vseh injektivnih objektov je zaprt za končne direktne vsote, zato naj $\mathcal{J}$ označuje polno aditivno podkategorijo, ki jo sestavljajo vsi injektivni objekti v $\mathcal{A}$. Ob predpostavki, da kategorija $\mathcal{A}$ vsebuje dovolj injektivnih objektov, se izkaže, da sta triangulirani kategoriji $\mathsf{D}^+(\mathcal{A})$ in $\mathsf{K}^+(\mathcal{I})$ ekvivalentni. To nam omogoča definirati izpeljani funktor funktorja F kot sledečo kompozicijo trianguliranih funktorjev

$$\mathsf{D}^+(\mathcal{A}) \xrightarrow{\sim} \mathsf{K}^+(\mathcal{I}) \hookrightarrow \mathsf{K}^+(\mathcal{A}) \xrightarrow{F} \mathsf{K}^+(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}^+(\mathcal{B}).$$

Dobimo funktor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$, ki mu pravimo *desni izpeljani funktor F*. Definiramo lahko tudi njegove višje izpeljane različice, ki smo jih omenjali že v uvodu, kot

$$\mathbf{R}^i F := H^i \circ \mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \to \mathcal{B}$$

za vse $i \in \mathbf{Z}$. To pripelje do nadvse uporabnega dolgega eksaktnega zaporedja pridruženega levo eksaktnemu funktorju F.

Trditev 0.2. *Naj bo $F\colon \mathcal{A} \to \mathcal{B}$ levo eksakten funktor med abelovima kategorijama. Naj bo $0 \to A \to B \to C \to 0$ kratko eksaktno zaporedje v $\mathcal{A}$. Potem obstaja dolgo eksaktno zaporedje*

$$\begin{aligned} 0 \to F(A) \to F(B) \to F(C) \to \mathbf{R}^1F(A) \to \mathbf{R}^1F(B) \to \mathbf{R}^1F(C) \to \cdots \\ \cdots \to \mathbf{R}^iF(A) \to \mathbf{R}^iF(B) \to \mathbf{R}^iF(C) \to \mathbf{R}^{i+1}F(A) \to \cdots. \end{aligned}$$

Trditev pokaže konstrukcija odlikovanega trikotnika $A \to B \to C \to A[1]$ v $\mathsf{D}^+(\mathcal{A})$ pridružena prvotnemu kratkemu eksaktnemu zaporedju na katerem potem uporabimo najprej izpeljani funktor $\mathbf{R}F$ in nato kohomološki funktor H^0.

V posebnem nam ta teorija omogoča izpeljati naslednje Hom-funktorje $\mathrm{Hom}_{\mathcal{A}}(A, -)$ za poljuben objekt A iz abelove kategorije $\mathcal{A}$ in tako definirati klasične *Ext-funktorje* $\mathrm{Ext}^i_{\mathcal{A}}(A, -)$ kot $\mathbf{R}^i\mathrm{Hom}_{\mathcal{A}}(A, -)$. Omenimo še čudovito zvezo, ki poveže slednje Ext-funktorje s Hom-funktorji v izpeljani kategoriji

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \simeq \mathrm{Hom}_{\mathsf{D}^b(\mathcal{A})}(A, B[i]).$$

Izpeljane kategorije v geometriji

Naj bo X shema. Spomnimo se da je snop $\mathcal{O}_X$-modulov $\mathcal{F}$ *qvazi-koherenten*, če ga je lokalno mogoče predstaviti kot kojedro morfizma med dvema prostima $\mathcal{O}_X$-moduloma. Z drugimi besedami to pomeni, da za vsako točko $x \in X$ obstaja odprta okolica $U \subseteq X$, ki vsebuje x, in obstaja eksaktno zaporedje oblike

$$\mathcal{O}_{X|U}^{\oplus I} \to \mathcal{O}_{X|U}^{\oplus J} \to \mathcal{F}|_U \to 0.$$

Snop $\mathcal{O}_X$-modulov je *koherenten*, kadar ga je lokalno mogoče predstaviti kot kojedro morfizma med dvema prostima $\mathcal{O}_X$-moduloma *končnega ranga.* V tem primeru se v zgornji predstavitvi privzema, da sta množici I in J končni.

Skupka vseh kohernetnih oz. kvazi-kohernetnih snopov tvorita polni abelovi podkategoriji v kategoriji $\mathcal{O}_X$-modulov $\mathsf{Mod}_{\mathcal{O}_X}$. Označimo ju z $\mathsf{coh}(X)$ oz. $\mathsf{qcoh}(X)$. *Izpeljana kategorija* sheme X je tedaj definitana kot

$$\mathsf{D}^b(X) = \mathsf{D}^b(\mathsf{coh}(X)).$$

Privzeli bomo da so naše sheme noetherske tj. kvazi-kompaktne in lokalno predstavljive s spektri noetherskih kolobarjev, saj tedaj velja, da kategorija kvazi-kohernetnih snopov $\mathsf{qcoh}(X)$ vsebuje dovolj injektivnih objektov. Slednje se s pridom uporabi za kostrukcijo mnogih izpeljanih funktorjev na $\mathsf{D}^b(X)$, kajti izkaže se, da vložitev $\mathsf{D}^b(\mathsf{coh}(X)) \hookrightarrow \mathsf{D}^b(\mathsf{qcoh}(X))$ porodi ekvivalenco trianguliranih kategorij

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(X)),$$

kjer kategorija na desni označuje polno podkategorijo $\mathsf{D}^b(\mathsf{qcoh}(X))$ na družini vseh omejenih kopleksov kvazi-koherentnih snopov s koherntno kohomologijo.

Če dodatno privzamemo še da je X gladka in projektivna lahko dokažemo, da je vsak objekt $\mathsf{D}^b(X)$ torej omejen kompleks koherentnih snopov $\mathcal{F}^\bullet$ znotraj kategorije $\mathsf{D}^b(X)$ izomorfen omejenemu kompleksu vektorskih svežnjev. Vektorski sveženj bo za nas pomenil lokalno prost snop $\mathcal{O}_X$-modulov končnega ranga. Ta trditev v svojem bistvu temelji na naslednjih dveh dejstvih.

▷ Vsak koherentni snop $\mathcal{F}$ na X je kvocient nekega vektorskega svežnja $\mathcal{E}$ tj. obstaja epimorfizem $\mathcal{E} \to \mathcal{F}$.

▷ Če n označuje dimenzijo raznoterosti X, potem za koherentni snop $\mathcal{F}$, ki dopušča eksaktno zaporedje koherentnih snopov

Theorem .16 ([Huy06, §3, Proposition 3.10])**.** *The bounded derived category* $\mathsf{D}^b(X)$ *of a connected scheme* X *over* k *is indecomposable.*

Proposition .17. *Suppose* X *is a smooth projective variety over* k*. The set of all skyscraper sheaves* $k(x)$*, for closed points* $x \in X$*, forms a spanning class of the bounded derived category* $\mathsf{D}^b(X)$*.*

Fourier-Mukaijeve transformacije

Naj bosta X in Y gladki projektivni raznoterosti nad poljem k. Označimo kanonični projekciji $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$.

Definicija 0.3. *Fourier–Mukaijeva transformacija* prirejena kompleksu $\mathcal{E}$ iz $\mathsf{D}^b(X \times Y)$, ki mu pravimo *jedro* je funktor

$$\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y) \qquad \Phi_{\mathcal{E}} := \mathbf{R}q_*(\mathcal{E} \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} \mathbf{L}p^*(-)).$$

Kot kompozicija samih trianguliranih funktorjev, vidimo, da je tudi $\Phi_{\mathcal{E}}$ triangulirana. Izkaže se, da je mnogo geometrijskih trianguliranih funktorjev med izpeljanima kategorijama X in Y te oblike. To so na primer izpeljani funktor potiska $\mathbf{R}f_*$, identični funktor $\mathrm{id}_{\mathsf{D}^b(X)}$ in funktor zamika $(-)[1]$ na $\mathsf{D}^b(X)$. Pomembni in zanimivi primeri Fourier–Mukaijevih funktorjev so tudi Serrov funktor S_X na $\mathsf{D}^b(X)$ in funktor tenzoriranja s svežnjem premic. Čar teh funktorjev leži v dejstvu, da jih lahko preučujemo preko njihovih jeder, ki so v primerjavi s funktorji samimi bolj enostavni objekti. K plodovitnosti te ideje še dodatno pripomore slavni izrek Orlova [Orl03], ki zagotovi sledeče.

Izrek 0.4. *Naj bo funktor* $F\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ *trianguliran zvest in poln in naj obstaja bodisi levi bodisi desni adjunkt* F*. Tedaj obstaja do izomorfizma natančno določeno jedro* $\mathcal{E}$ *iz* $\mathsf{D}^b(X \times Y)$*, da je* F *naravno izomorfen Fourier–Mukaijevi trnsformaciji* $\Phi_{\mathcal{E}}$*.*

Ta pristop je s pridom uporabil Orlov [Orl03] pri dokazu izpeljanega Torellijevega izreka, ki ga obravnavamo v zadnjem poglavju. Omenimo tudi da je kompozicija Fourier–Mukaijevih transformacij spet Fourier–Mukaijevega tipa. Presenetljivo vse Fourer–Mukaijeve transformacje $\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ dopuščajo tudi leva in desna adjunkta, ki sta pravtako Fourier–Mukaijevi transformaciji z jedri

$$\mathcal{E}_{\mathrm{L}} = \mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} q^*\omega_Y[\dim Y] \quad \text{in} \quad \mathcal{E}_{\mathrm{R}} = \mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times Y}} p^*\omega_X[\dim X].$$

Tukaj $\mathcal{E}^\vee$ označuje izpeljani dual $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{E}, \mathcal{O}_{X\times Y})$. Zaradi prisotnosti Serrovih funktorjev S_X in S_Y se adjunkta izražata kot

$$\Phi_{\mathcal{E}_{\mathrm{L}}} = \Phi_{\mathcal{E}^\vee} \circ S_Y \quad \text{and} \quad \Phi_{\mathcal{E}_{\mathrm{R}}} = S_X \circ \Phi_{\mathcal{E}^\vee}.$$

Pravimo da sta gladki projektivni raznoterosti X in Y *izpeljano ekvivalentni*, če obstaja triangulirana ekvivalenca $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$. V tem primeru X in Y imenujemo tudi *Fourer–Mukaijeva partnerja.* Izrek Orlova nam že omogiči sklepati, da si Fourier–Mukaijeva partnerja delita nekaj skupnih lastnosti. To povzema naslednji izrek.

Izrek 0.5. *Naj bosta* X *in* Y *Fourier–Mukaijeva partnerja. Tedaj je* $\dim X = \dim Y$ *in če ima* X *trivialen kanonični sveženj ga ima tudi* Y*.*

Eventuelno bomo Fourier–Mukaijeve transformacije inducirali tudi na racionalni kohomologiji, pred tem pa se je potrebno spustiti še skozi K-teorijo. Gladki projektivni raznoterosti X priredimo K-grupo, ki je podana kot kvocient proste grupe genrirane na vseh izomorfnostnih razredih koherentnih snopov na X po podgrupi generirani z elementi oblike $[\mathcal{E}] - [\mathcal{F}] + [\mathcal{G}]$, za katere obstaja kratko eksaktno zaporedje $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$. Označimo jo s $K_\circ(X)$. S predpisom $[\mathcal{F}^\bullet] = \sum_{i\in\mathbf{Z}}(-1)^i[\mathcal{F}^i]$ je mogoče razumeti tudi verižne komplekse kot elemente $K_\circ(X)$. Na ta način dobimo preslikavo, ki slika objekte izpeljane kategorije $\mathsf{D}^b(X)$ v razrede abelove grupe $K_\circ(X)$. Izkaže se, da je $K_\circ(X)$ možno opremiti tudi s produktom, ki je na ekvivalenčnih razredih lokalno prostih snopih podan s predpisom

$$[\mathcal{E}_0] \cdot [\mathcal{E}_1] = [\mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{E}_1].$$

Ta se razširi do dobro definiranega produkta kjer pa za razliko od prej uporabimo izpeljani tenzorski produkt dveh kompleksov $[\mathcal{F}^\bullet]\cdot[\mathcal{G}^\bullet] = [\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet]$. Za pravi morfizem $f\colon X \to Y$ je možno definirati tudi potisk in povlek na K-grupah in sicer na sledeč način. Povlek je preslikava

$$f^*\colon K_\circ(Y) \to K_\circ(X), \qquad f^*([\mathcal{G}]) = \sum_{i\in\mathbf{Z}}(-1)^i[\mathbf{L}^i f^*\mathcal{G}].$$

Z uporabo dolgega eksaktnega zaporedja na kohomologiji se lahko prepričamo, da je ta predpis dobro definiran.

Izrek 0.6 (Grothendieck–Riemann–Roch)**.** *Naj bo $f\colon X \to Y$ pravi morfizem med gladkima projektivnima raznoterostima. Tedaj za vse $e \in K_\circ(X)$ velja*

$$\operatorname{ch}(f_!(e)) \smile \operatorname{td}_Y = f_!(\operatorname{ch}(e) \smile \operatorname{td}_X) \tag{0.2}$$

znotraj kolobarja $H^*(Y, \mathbf{Q})$.

K3 ploskve

Definicija 0.7. Gladka projektivna ploskev X nad poljem kopleksnim števil $\mathbf{C}$ je *K3 ploskev*, če velja

$$\omega_X \simeq \mathcal{O}_X \quad \text{in} \quad H^1(X, \mathcal{O}_X) = 0.$$

Na drugi strani imamo še kopleksno geometrično različico zgornje definicije.

Definicija 0.8. Kompleksna povezana projektivna mnogoterost X dimenzije dva je *K3 ploskev*, če velja

$$\omega_X \simeq \mathcal{O}_X \quad \text{in} \quad H^1(X, \mathcal{O}_X) = 0.$$

$$\chi(X, \mathcal{E}) = \langle \operatorname{ch}_2(\mathcal{E}), [X]\rangle + 2\operatorname{rk}(\mathcal{E}) = \langle \tfrac{1}{2}(\mathrm{c}_1(\mathcal{E})^2 - 2\mathrm{c}_2(\mathcal{E})), [X]\rangle + 2\operatorname{rk}(\mathcal{E}). \tag{0.3}$$

Proposition .18. *Naj bo X K3 ploskev. Njena Hodgeva števila zložena v Hodgev diamant so sledeča:*

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Izrek 0.9. *Naj bosta X in Y gladki projektivni raznoterosti nad poljem kompleksnih števil $\mathbf{C}$. Če je X K3 ploskev in velja*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y),$$

potem je tudi Y K3 ploskev.

Trditev 0.10. *Naj bo X kopleksna K3 ploskev. Njene celoštevilske kohomološke grupe so sledeče*

$$H^\bullet(X,\mathbf{Z}): \qquad \mathbf{Z} \quad 0 \quad \mathbf{Z}^{22} \quad 0 \quad \mathbf{Z}. \tag{0.4}$$

Trditev 0.11. *Naj bo X K3 ploskev. Potem je presečna forma na kohomologiji $H^2(X,\mathbf{Z})$ soda in unimodularna. Rang mreže $H^2(X,\mathbf{Z})$ je tedaj 22 njena signatura pa $(3,19)$. Abstraktno je izomorfna*

$$H^2(X,\mathbf{Z}) \simeq E_8(-1)^{\oplus 2} \oplus \mathcal{U}^{\oplus 3}. \tag{0.5}$$

Izrek 0.12 (Global Torelli theorem)**.** *Naj bosta X in Y kompleksni K3 ploskvi. Tedaj sta X in Y izomorfni natanko tedaj, ko obstaja Hodgeva izometrija med njunima presešnima formama*

$$f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z}).$$

Izpeljani Torellijev izrek

Izpeljani Torellijev izrek povezuje mnoge koncepte

Definition .19. Let X be a K3 surface over $\mathbf{C}$ with a prescribed Mukai vector $v \in \tilde{H}(X,\mathbf{Z})$. The *moduli functor*

$$\mathcal{M}_v\colon \mathsf{Sch}^{\mathrm{op}}_{/\mathbf{C}} \longrightarrow \mathsf{Set}$$

is defined on objects as

$$S \longmapsto \left\{ \mathcal{F} \in \mathsf{coh}(X\times S) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is flat over } S, \text{ and for each closed point } s\in S, \\ \mathcal{F}|_s \text{ is a semi-stable sheaf on } X, \text{ with } v(\mathcal{F}|_s) = v. \end{array} \right\}_{/\sim}.$$

Here $\mathcal{F} \sim \mathcal{F}'$ is defined to hold precisely when there exist a line bundle $\mathcal{L}$ on S, for which $\mathcal{F} \simeq \mathcal{F}' \otimes_{\mathcal{O}_{X\times S}} \pi_S^*\mathcal{L}$. On morphisms[37] $\mathcal{M}_v$ is defined as

$$(f\colon S' \to S) \longmapsto \left((\mathrm{id}_X \times f)^*\colon \quad \begin{array}{l} \mathcal{M}_v(S) \to \mathcal{M}_v(S') \\ {[\mathcal{F}]_\sim} \mapsto [(\mathrm{id}_X \times f)^*\mathcal{F}]_\sim \end{array} \right).$$

There is also a *stable* variant of the moduli functor, denoted by $\mathcal{M}_v^s\colon \mathsf{Sch}^{\mathrm{op}}_{/\mathbf{C}} \to \mathsf{Set}$, where every instance of the word "semi-stable" in the definition of $\mathcal{M}_v$ is replaced by "stable".

Theorem .20. *[GH96; Huy06; OGr97; HL10; BM14] Let $v \in \tilde{H}(X,\mathbf{Z})$ be an effective isotropic vector for which there exists a class $v' \in \mathrm{N}(X)$, satisfying $\langle v, v'\rangle = 1$. Then there exists a polarization on X for which all semi-stable sheaves are stable, so the functors $\mathcal{M}_v$ and $\mathcal{M}_v^s$ coincide, and the moduli functor $\mathcal{M}_v\colon \mathsf{Sch}^{op}_{/\mathbf{C}} \to \mathsf{Set}$ is representable. The moduli functor $\mathcal{M}_v$ is then represented by a smooth projective complex surface M_v, called a* fine moduli space*.*

Theorem .21 (Derived Torelli theorem)**.** *Let X and Y be K3 surfaces over the field of complex numbers* $\mathbf{C}$*. Then the following statements are equivalent.*

(i) X and Y share equivalent bounded derived categories of coherent sheaves,

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y).$$

[37] The assignment is well defined because

$$(\mathrm{id}_X \times f)^*(\mathcal{F} \otimes \pi_S^*\mathcal{L}) \simeq (\mathrm{id}_X \times f)^*\mathcal{F} \otimes (\mathrm{id}_X \times f)^*\pi_S^*\mathcal{L} \simeq (\mathrm{id}_X \times f)^*\mathcal{F} \otimes \pi_{S'}^* f^*\mathcal{L}.$$

(ii) There exists a Hodge isometry $f\colon T_X \to T_Y$ between the transcendental lattices of X and Y.

(iii) Mukai lattices $\tilde{H}(X,\mathbf{Z})$ and $\tilde{H}(Y,\mathbf{Z})$ of X and Y, respectively, are Hodge isometric.

(iv) There exists an isotropic Mukai vector $v \in \tilde{H}(X,\mathbf{Z})$ of divisibility 1, such that Y is isomorphic to a moduli space M_v of stable sheaves on X with Mukai vector v, representing some *moduli functor $\mathcal{M}_v$. Moreover M_v is up to isomorphism the unique K3 surface, for which there exists a Hodge isometry*

$$H^2(M_v,\mathbf{Z}) \simeq v^{\perp}/\mathbf{Z}v.$$

Theorem .22 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties over $\mathbf{C}$ and let $\Phi_{\mathcal{E}}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ be a Fourier-Mukai transform associated to a kernel $\mathcal{E}$ of category $\mathsf{D}^b(X \times Y)$. Then $\Phi_{\mathcal{E}}$ is fully faithful if and only if for any two closed points x, $y \in X$ the following condition is satisfied*

$$\operatorname{Hom}_{\mathsf{D}^b(Y)}(\Phi_{\mathcal{E}}(k(x)), \Phi_{\mathcal{E}}(k(y))[i]) \simeq \begin{cases} \mathbf{C}, & \text{if } x = y \text{ and } i = 0, \\ 0, & \text{if } x \neq y \text{ or } i \notin [0, \dim X]. \end{cases} \tag{0.6}$$