UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS

Mathematics – 2nd cycle

Izak Jenko

K3 SURFACES FROM A DERIVED CATEGORICAL VIEWPOINT

Master's thesis

Supervisor: .

Ljubljana, 2025

UNIVERZA V LJUBLJANI
FAKULTETA ZA MATEMATIKO IN FIZIKO

Matematika – 2. stopnja

Izak Jenko

K3 PLOSKVE Z VIDIKA IZPELJANIH KATEGORIJ

Magistrsko delo

Mentor: .

Ljubljana, 2025

Zahvala

.

Contents

K3 surfaces from a derived categorical viewpoint

Abstract

.

K3 ploskve z vidika izpeljanih kategorij

Povzetek

.

Math. Subj. Class. (2020): .

Keywords: .

Ključne besede: .

1 redistribute: Triangulated categories

In this section we try to make a brief account of the relevant prerequisites needed to discuss derived categories.

1.1 Additive, k-linear and abelian categories

We mainly follow [3].

A categorical *biproduct* of objects X and Y in a category $\mathcal{C}$ is an object $X \oplus Y$ together with morphisms

$$\begin{array}{ll} p_X : X \oplus Y \to X & p_Y : X \oplus Y \to Y \\ i_X : X \to X \oplus Y & i_Y : Y \to X \oplus Y \end{array}$$

for which the pair p_X, p_Y is the categorical product of X and Y and the pair i_X, i_Y is the categorical coproduct. For a pair of morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$ the unique induced morphism into the product $Z \to X \oplus Y$ is denoted by (f_X, f_Y) and for a pair of morphisms $g_X : X \to Z$ and $g_Y : Y \to Z$ the unique induced morphism from the coproduct $X \oplus Y \to Z$ is denoted by $\langle g_X, g_Y \rangle$. For morphisms $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ we introduce notation $f_0 \oplus f_1 : X_0 \oplus X_1 \to Y_0 \oplus Y_1$ to mean either of the two (equal) morphisms

$$(\langle f_0, 0 \rangle, \langle 0, f_1 \rangle) \qquad \text{or} \qquad \langle (f_0, 0), (0, f_1) \rangle$$

also depicted in matrix notation as

$$\begin{pmatrix} f_0 & 0 \\ 0 & f_1 \end{pmatrix}.$$

Let k denote either a field or the ring of integers $\mathbf{Z}$.

Definition 1.1. A category $\mathcal{A}$ is additive (resp. k-linear) if all the hom-sets carry the structure of abelian groups (resp. k-modules) and the following axioms are satisfied

Decide on which terminology to use (everything is covered by k-linear...)

A1 For all all objects X, Y and Z of $\mathcal{A}$ the composition

$$\circ : \operatorname{Hom}_{\mathcal{A}}(Y, Z) \times \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear.

A2 There exists a *zero object* 0, for which $\operatorname{Hom}_{\mathcal{A}}(0, 0) = 0$.

A3 For any two objects X and Y there exists a categorical biproduct of X and Y.

Remark 1.2. (i) The zero object 0 of a k-linear category $\mathcal{A}$ is both the initial and terminal object of $\mathcal{A}$.

(ii) One can recognise k-linear categories as the categories enriched over the category Mod_k of k-modules and k-linear maps.

Definition 1.3. A functor $F : \mathcal{A} \to \mathcal{A}'$ between two additive (resp. k-linear) categories $\mathcal{A}$ and $\mathcal{A}'$ is *additive* (resp. *k-linear*), if its action on morphisms

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}'}(F(X), F(Y))$$

is a group homomorphism (resp. k-linear map).

Traditionally the term *additive* is reserved for **Z**-linear categories and **Z**-linear functors between such categories.

For a morphism $f\colon X \to Y$ in an additive category $\mathcal{A}$ recall, that the *kernel* of f is the equalizer of f and 0 in $\mathcal{A}$, if it exists, and, dually, the *cokernel* of f is the coequalizer of f and 0. It is well-known and easy to verify, that the structure maps $\ker f \hookrightarrow X$ and $Y \twoheadrightarrow \operatorname{coker} f$ are monomophim and epimorphism respectively. We also define the *image* and the *coimage* of f to be

$$\begin{aligned}(\operatorname{im} f \to Y) &:= \ker(Y \to \operatorname{coker} f)\\ (X \to \operatorname{coim} f) &:= \operatorname{coker}(\ker f \to X).\end{aligned}$$

Notice that the image and the coimage, just like the kernel and the cokernel, are defined to be morphisms, not only objects. Sometimes these are called *structure morphisms.*

For a monomorphism $Y \hookrightarrow X$ we will sometimes by abuse of terminology call the cokernel $\operatorname{coker}(Y \to X)$ a *quotient* and denote it by X/Y.

Definition 1.4. A k-linear category $\mathcal{A}$ is *abelian*, if it is closed under kernels and cokernels and satisfies axiom A4.

A4 For any morphism $f\colon X \to Y$ in $\mathcal{A}$ the canonical morphism $\operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f$ is an isomorphism.

$$\begin{tikzcd} \ker f \arrow[r, hook] & X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[r, two heads] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[u, hook] & \end{tikzcd}$$

Axiom A4 essentailly states that abelian categories are those additive categories possesing all kernels and cokernels in which the first isomorphism theorem holds.

Remark 1.5. We obtain the morphism mentioned in axiom A4 in the following way. Due to $(\operatorname{im} f \to Y) = \ker(Y \to \operatorname{coker} f)$ and the composition $X \to Y \to \operatorname{coker} f$ being 0, there is a unique morphism $X \to \operatorname{im} f$ by the universal property of kernels. The composition $\ker f \to X \to \operatorname{im} f$ then equals 0, by the fact that $\operatorname{im} f \hookrightarrow Y$ is mono and $\ker f \to X \to Y$ equals 0. From the universal property of cokernels we obtain a unique morphism $\operatorname{coim} f \to \operatorname{im} f$, since $(X \to \operatorname{coim} f) = \operatorname{coker}(\ker f \to X)$.

Example 1.6. The default examples of abelian categories are the category of abelian groups Ab or more generaly the category of A-modules Mod_A for a commutative ring A and the categories of coherent and quasi-coherent sheaves $\mathsf{coh}(X)$ and $\mathsf{qcoh}(X)$ on a scheme X. On the other hand the category of (real or complex) vector bundles over a manifold of dimension at least 1 is additive, but never abelian.

Definition 1.7. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a sequence of composable morphisms in an abelian category $\mathcal{A}$.

(i) We say this sequence is *exact*, if $g \circ f = 0$ and the induced morphism $\operatorname{im} g \to \ker f$ is an isomorphism.

(ii) Extending (i), a sequence $\cdots \to X^0 \to X^1 \to X^2 \to \cdots$ is *exact*, if any subsequence $X^{i-1} \to X^i \to X^{i+1}$ for $i \in \mathbf{Z}$ is exact.

(iii) Exact sequences of the form $0 \to X \to Y \to Z \to 0$ are called *short exact sequences.*

To relate the definition of exactness with more primitive objects of an abelian category, namely kernels and cokernels, it is not difficult to show that

$$0 \to X \to Y \to Z \text{ is exact, if and only if } (X \to Y) = \ker(Y \to Z),$$

and dually

$$X \to Y \to Z \to 0 \text{ is exact, if and only if } (Y \to Z) = \operatorname{coker}(X \to Y).$$

In the context of abelian categories functors which preserve a bit more than just the k-linear structure are of interest. This brings us to

Definition 1.8. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(i) F is said to be *left exact*, if $0 \to FX \to FY \to FZ$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

(ii) F is said to be *right exact*, if $FX \to FY \to FZ \to 0$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$..

(iii) F is said to be *exact*, if it is both left and right exact.

Remark 1.9. Equivalently, one can also define left exact functors to be exactly those additive functors, which commute with kernels and dually define right exact functors to be additive functors commuting with cokernels.

1.2 Triangulated categories

In order to formulate the definition of a triangulated category more concisely, we introduce some preliminary notions. A *category with translation* is a pair $(\mathcal{D}, T)$, where $\mathcal{D}$ is a category and T is an auto-equivalence $T\colon \mathcal{D} \to \mathcal{D}$ called the *translation functor*. If $\mathcal{D}$ is additive or k-linear, T is moreover assumed to be additive or k-linear. We usually denote its action on objects X with $X[1]$ and likewise its action on morphisms f with $f[1]$.

A *triangle* in a category with translation $(\mathcal{D}, T)$ is a triplet of composable morphisms (f, g, h) of category $\mathcal{D}$ having the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

A *morphism* of triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ is given by a triple of morphisms (u, v, w) for which the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & X[1] \arrow[d, "u[1]"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

One can compose morphisms of triangles in the obvious way and the notion of an isomorphism of triangles is defined as usual. The following definition as stated is originally due to Verdier, who first introduced it in his thesis [1].

Definition 1.10. A *triangulated category* is a k-linear category with translation $(\mathcal{D}, T)$ equipped with a class of *distinguished triangles*, which is subject to the following four axioms.

TR1 (i) Any triangle isomorphic to a distinguished triangle is also itself destinguished.

(ii) For any X the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished.

(iii) For any morphism $f\colon X \to Y$ there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

The object Z is sometimes called the *cone of f*.

TR2 The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

TR3 Given two distinguished triangles and morphisms $u\colon X \to X'$ and $v\colon Y \to Y'$, depicted in the solid diagram below

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, dashed, "w"'] & X[1] \arrow[d, "u[1]"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

satisfying $v \circ f = f' \circ u$, there exists a (in general non-unique) morphism $w\colon Z \to Z'$, for which (u, v, w) is a morphism of triangles i.e. the diagram above commutes.

TR4 ...

Omitting the so-called *octahedral* axiom TR4 we arrive at the definition of a *pre-triangulated category*. These are essentailly the categories we will be working with, since we will never use nor verify the axiom TR4. We will nevertheless use the terminology *"triangulated category"* in part to remain consistent with the existent literature and more importantly because our categories will be honest triangulated categories anyways.

Remark 1.11. For a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

morphisms f and g are said to be of degree 0 and morphism h is said to be of degree $+1$. They are also sometimes diagramatically depicted as triangles, with the markings on morphisms describing their respective degrees.

$$\begin{tikzcd}
& Z \arrow[dl, "+1"'] & \\
X \arrow[rr, "0"] & & Y \arrow[ul, "0"']
\end{tikzcd}$$

Definition 1.12. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with translation functors T and T' respectively. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is defined to be *triangulated* or *exact*, if the following two conditions are satisfied.

Maybe I will pick only triangulated, to not overload the term exact...

(i) There exists a natural isomorphism of functors

$$\eta\colon F \circ T \simeq T' \circ F.$$

(ii) For every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}$, the triangle

$$F(X) \xrightarrow{Ff} F(Y) \xrightarrow{Fg} F(Z) \to F(X)[1]$$

is distinguished in $\mathcal{D}'$, where the last morphism (of degree 1) is obtained as the composition $F(Z) \xrightarrow{Fh} F(X[1]) \xrightarrow{\eta_X} F(X)[1]$.

Remark 1.13. The condition on F being an *additive* functor in the above definition is actually unnecessary and follows from conditions (i) and (ii) [5, Tag 05QY].

Definition 1.14. Triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ are said to be *equivalent* (as triangulated categories), if there are triangulated functors $F\colon \mathcal{D} \to \mathcal{D}'$ and $G\colon \mathcal{D}' \to \mathcal{D}$, such that $G \circ F \simeq \mathrm{id}_{\mathcal{D}}$ and $F \circ G \simeq \mathrm{id}_{\mathcal{D}'}$ and we call F and G *triangulated equivalences*.

Proposition 1.15. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangualted functor, which is an equivalence of categories with a quasi-inverse $G\colon \mathcal{D}' \to \mathcal{D}$. Then G is also a triangulated functor.*

Proof. This is a consequence of [2, Proposition 1.41], as equivalences of categories are special instances of adjunctions. $\square$

As a consequence, two triangualted categories are equivalent (as triangulated categories) whenever there exists a fully faithful essentailly surjective triangulated functor from one to the other.

Definition 1.16. Let $H\colon \mathcal{D} \to \mathcal{A}$ be an additive functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$. We say H is a *cohomological functor*, if for every distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the induced long sequence in $\mathcal{A}$

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to H(Y[1]) \to H(Z[1]) \to \cdots \tag{1.1}$$

is exact.

Construction of the long sequence (1.1) is extremely simple as opposed to other known long exact sequences assigned to certain short exact sequences (e.g. of sheaves or complexes) as all the complexity is actually captured within the distinguished triangle already. All one has to do is unwrap the triangle $X \to Y \to Z \to X[1]$ into the following chain of composable morphisms

$$\cdots \to Y[-1] \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to Z[1] \to X[2] \to \cdots$$

and apply functor H over it.

Example 1.17. For any object W in a triangulated category $\mathcal{D}$ the functors

$$\mathrm{Hom}_{\mathcal{D}}(W, -)\colon \mathcal{D} \to \mathsf{Mod}_k \text{ and } \mathrm{Hom}_{\mathcal{D}}(-, W)\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Mod}_k$$

are cohomological.

2 **redistribute:** Derived categories

2.1 Categories of complexes

In order to define derived categories of an additive category $\mathcal{A}$ we first introduce the category of complexes and the homotopic category of complexes of $\mathcal{A}$. We will equip these with a triangulated structure and . Throughout this section $\mathcal{A}$ will be a fixed additeve or k-linear category and we also mention that we will be using the cohomological indexing convention.

As a preliminary we introduce graded objects in ??

Category of chain complexes

By a *chain complex* in $\mathcal{A}$ we mean a collection of objects and morphisms

$$A^\bullet = \left((A^i)_{i\in\mathbf{Z}}, (d_A^i\colon A^i \to A^{i+1})_{i\in\mathbf{Z}}\right),$$

where A^i are objects and d^i are morphisms of $\mathcal{A}$, called *differenitials*, subject to equations $d^{i+1}\circ d^i = 0$, for all $i \in \mathbf{Z}$. A complex is *bounded from below* (resp. *bounded from above*), if there exists $i_0 \in \mathbf{Z}$ for which $A^i \simeq 0$ for all $i \leq i_0$ (resp. $i \geq i_0$) and is *bounded*, if it is both bounded from below and bounded from above. A *chain map* between two chain complexes $A^\bullet$ and $B^\bullet$ in $\mathcal{A}$ is a collection of morphisms in $\mathcal{A}$

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}},$$

for which $f^{i+1}\circ d_A^i = d_B^i \circ f^i$ holds for all $i \in \mathbf{Z}$. This may diagramatically be described by the following commutative ladder.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Sometimes we will denote a complex just by A, B,... insted of $A^\bullet$, $B^\bullet$,... to simplify notation.

Next we define the *category of chain complexes* in $\mathcal{A}$, denoted by $\mathrm{Ch}(\mathcal{A})$, as the following additive category.

Objects:	chain complexes in $\mathcal{A}$.
Morphims:	$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$ is the set of chain maps $A \to B$, equipped with a group structure inherited from $\mathcal{A}$ by applying operations componentwise.

The composition law is defined componentwise and is clearly associative and bilinear. The identity morphisms $1_{A^\bullet}$ are defined to be $(1_{A^i})_{i\in\mathbf{Z}}$. The complex $\cdots \to 0 \to 0 \to \cdots$ plays the role of the zero object in $\mathrm{Ch}(\mathcal{A})$ and the biproduct of complexes A and B exists and is witnessed by the chain complex

$$A \oplus B = \left((A^i \oplus B^i)_{i\in\mathbf{Z}}, (d_A^i \oplus d_B^i)_{i\in\mathbf{Z}}\right),$$

together with the cannonical projection and injection morphisms arising from biproducts componentwise.

Additionally, we also define the following full additive subcategories of $\mathrm{Ch}(\mathcal{A})$.

$\mathrm{Ch}^+(\mathcal{A})$	*Category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$\mathrm{Ch}^-(\mathcal{A})$	*Category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$\mathrm{Ch}^b(\mathcal{A})$	*Category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

Remark 2.1. Whenever $\mathcal{A}$ is k-linear, all the categories of complexes $\mathrm{Ch}^*(\mathcal{A})$ become k-linear as well in the obvious way.

On all the categories of complexes mentioned above, we can now define the translation functor

$$T\colon\ \mathrm{Ch}^*(\mathcal{A}) \to \mathrm{Ch}^*(\mathcal{A})$$

given by its action on objects and morphisms as follows.

Objects:	$T(A^\bullet) = A^\bullet[1]$ is the chain complex with $(A^\bullet[1])^i := A^{i+1}$ and differentials $d^i_{A[1]} = -d^{i+1}_A$.
Morphims:	For a chain map $f^\bullet\colon A^\bullet \to B^\bullet$ we define $f^\bullet[1]$ to have component maps $(f^\bullet[1])^i = f^{i+1}$.

The translation functor T thus acts on a complex $A^\bullet$ by twisting its differential by a sign and shifting it one step to the *left*, which is graphically pictured below.

$$\begin{array}{lccccccccccc}
 & \cdots & & -1 & & 0 & & 1 & & 2 & & \cdots \\
A^\bullet & \cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
A^\bullet[1] & \cdots & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow & \cdots
\end{array}$$

Remark 2.2. We remark that the translation functor T is clearly also additive or k-linear, whenever $\mathcal{A}$ is additive or k-linear.

Since T is an auto-equivalence there exists a quasi-inverse T^{-1} to T, which is defined and unique up to a natural isomorphism. We may then speak of T^k for any $k \in \mathbf{Z}$, whose action on a complex $A^\bullet$ is described by $(A^\bullet[k])^i = A^{i+k}$ with differential $d^i_{A[k]} = (-1)^k d^{i+k}_A$.

Homotopy category of complexes

In this subsection we construct the homotopy category of chain complexes associated to a given additive category $\mathcal{A}$ and equip it with a triangulated structure. The main motivation for its introduction in this work is the fact that we will later on use it to construct the derived category of $\mathcal{A}$. In particular the homotopy category of $\mathcal{A}$, as opposed to the category of complexes[1] $\mathrm{Ch}(\mathcal{A})$, can be enhanced with a triangulated structure which will afterwards descend to the level of derived categories.

Definition 2.3. Let $f^\bullet$ and $g^\bullet$ be two chain maps in $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$. We define $f^\bullet$ and $g^\bullet$ to be *homotopic*, if there exists a collection of morphisms $(h^i\colon A^i \to B^{i-1})_{i\in\mathbf{Z}}$, satisfying

$$f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$$

[1] It is still possible to construct the derived category of $\mathcal{A}$ without passing through the homotopy category of complexes, however equipping it with a triangulated structure in that case becomes less elegant.

for all $i \in \mathbf{Z}$. If $f^\bullet$ and $g^\bullet$ are homotopic, we denote it by $f^\bullet \simeq g^\bullet$.

$$
\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] \arrow[dl, "h^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] \arrow[dl, "h^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}
$$

We say $f^\bullet$ is *nullhomotopic*, if $f^\bullet \simeq 0$.

Lemma 2.4. *Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be complexes in* $\mathrm{Ch}(\mathcal{A})$*, and let $f, f' \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$ and $g, g' \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(B^\bullet, C^\bullet)$ be chain maps.*

(i) The subset of all nullhomotopic chain maps in $A^\bullet \to B^\bullet$ forms a submodule of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$*.*

(ii) If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

Pick one notation convention, sometimes its f, f', sometimes its f_0, f_1, ...

Proof. See []. □

The homotopy category of complexes in $\mathcal{A}$, denoted by $K(\mathcal{A})$, is defined to be an additive category consisting of

$$
\begin{aligned}
\textit{Objects:} &\quad \text{chain complexes in } \mathcal{A}. \\
\textit{Morphims:} &\quad \mathrm{Hom}_{K(\mathcal{A})}(X, Y) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, Y)/_{\simeq}
\end{aligned}
$$

The composition law descends to the quotient by lemma 2.4 (ii), i.e. $[g] \circ [f] := [g \circ f]$, for composable $[f]$ and $[g]$, and for any $A^\bullet$ the identity morphism is defined to be $[1_A^\bullet]$. All the hom-sets $\mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ are k-modules by lemma 2.4 (i) and compositions are k-bilinear maps. The biproduct of two complexes is

define the biproduct as well

As in the case of categories of complexes in $\mathcal{A}$, we can also define the following full additive subcategories of $K(\mathcal{A})$.

$K^+(\mathcal{A})$ — *Homotopy category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.

$K^-(\mathcal{A})$ — *Homotopy category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.

$K^b(\mathcal{A})$ — *Homotopy category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

in practise everywhere unless stated otherwise our complexes in $K^+(\mathcal{A})$ will be supported in $\mathbf{Z}_{\geq 0}$

Cohomology

A very important invariant of a chain complex in the homotopy category, which measures the extent to which it fails to be exact, is its cohomology. Here we are no longer assuming $\mathcal{A}$ is just k-linear, but abelian, since we will need kernels and cokernels to exists. For a chain complex $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$ and $i \in \mathbf{Z}$ we define its *i-th cohomology* to be

$$H^i(A^\bullet) := \mathrm{coker}(\mathrm{im}\, d^{i-1} \to \ker d^i).$$

Remark 2.5. A computation with universal properties inside an abelian category shows, that the following are all equivalent ways of defining the cohomology of a complex as well

$$
\begin{aligned}
H^i(A^\bullet) &:= \mathrm{coker}(\mathrm{im}\, d^{i-1} \to \ker d^i) \simeq \ker(\mathrm{coker}\, d^{i-1} \to \mathrm{im}\, d^i) \\
&\simeq \mathrm{coker}(A^{i-1} \to \ker d^i) \simeq \ker(\mathrm{coker}\, d^{i-1} \to A^i).
\end{aligned}
$$

See [3, Def. 8.3.8. (i)].

For a morphism of complexes $f^\bullet\colon A^\bullet \to B^\bullet$ one can also define a morphism

$$H^i(f^\bullet)\colon H^i(A^\bullet) \to H^i(B^\bullet)$$

in $\mathcal{A}$, because $f^\bullet$ induces maps $\operatorname{im} d_A^{i-1} \to \operatorname{im} d_B^{i-1}$ and $\ker d_A^i \to \ker d_B^i$, which fit into the commutative diagram bellow.

$$\begin{tikzcd}
B^{i-1} \arrow[rrr, "d_B^{i-1}"] \arrow[ddr] & & & B^i \arrow[rrr, "d_B^i"] & & & B^{i+1} \\
A^{i-1} \arrow[u] \arrow[rrr, "d_A^{i-1}"] \arrow[ddr] & & & A^i \arrow[u] \arrow[rrr, "d_A^i"] & & & A^{i+1} \arrow[u] \\
& \operatorname{im} d_B^{i-1} \arrow[uurr] \arrow[rrrr] \arrow[ddrr, "0"] & & & & \ker d_B^i \arrow[uull] \arrow[uur, "0"] \arrow[ddll] & \\
& \operatorname{im} d_A^{i-1} \arrow[u] \arrow[uurr] \arrow[rrrr] \arrow[ddrr, "0"] & & & & \ker d_A^i \arrow[u] \arrow[uull] \arrow[uur, "0"] \arrow[ddll] & \\
& & & H^i(B^\bullet) & & & \\
& & & H^i(A^\bullet) \arrow[u, dashed] & & &
\end{tikzcd}$$

All the induced morphisms come from universal properties and are as such unique for which the diagram commutes. Thus the assignment

$$f^\bullet \mapsto H^i(f^\bullet)\colon \operatorname{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B) \to \operatorname{Hom}_{\mathcal{A}}(H^i(A), H^i(B))$$

is functorial i.e. respects composition and maps identity morphisms to identity morphisms. For the same reasons it is also a k-linear homomorphism, showing that

$$H^i\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$$

a k-linear functor.

It is very fruitful to consider all the cohomology functors $(H^i)_{i\in\mathbf{Z}}$ at once, as is witnessed by the next proposition.

Proposition 2.6. *[3, Theorem 12.3.3.] Let* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *be a short exact sequence in* $\mathrm{Ch}(\mathcal{A})$*. Then there exists a long exact sequence in* $\mathcal{A}$

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots.$$

this is not done...

Functors H^i actually descend to the homotopy category $K(\mathcal{A})$ of $\mathcal{A}$

We now shift our focus to the construction of a triangulated structure on $K^*(\mathcal{A})$. The translation functor $T\colon K^*(\mathcal{A}) \to K^*(\mathcal{A})$ is defined on objects and morphisms in the following way.

$$\begin{aligned} \textit{Objects:} &\quad A^\bullet \longmapsto A^\bullet[1]. \\ \textit{Morphims:} &\quad [f^\bullet] \longmapsto [f^\bullet[1]]. \end{aligned}$$

This assignment is clearly well defined on morphisms, as $f \simeq f'$ implies $f[1] \simeq f'[1]$.

Mapping cones

The other piece of data required to obtain a triangulated category is a collection of distinguished triangles. To describe what distingushed triangles are in our case, we must first introduce the mapping cone of a morphism of complexes and to this end we will for a moment step outside the scope of the homotopy categroy of complexes back into the category of complexes of $\mathcal{A}$.

Definition 2.7. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathrm{Ch}(\mathcal{A})$. The complex $C(f)^\bullet$ is specified by the collection of objects

$$C(f)^i := A^{i+1} \oplus B^i$$

and differentials

$$d^i_{C(f)} := \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1]^i & d^i_B \end{pmatrix} \tag{2.1}$$
$$d^i_{C(f)}\colon A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}$$

for all $i \in \mathbf{Z}$.

Remark 2.8. Using matrix notiaion here might be a bit misleading at first. Formally, matrix (2.1) represents the morphism

$$\Big(\langle -d^{i+1}_A, 0\rangle, \langle f^{i+1}, d^i_B\rangle\Big) = \Big\langle (-d^{i+1}_A, f^{i+1}), (0, d^i_B)\Big\rangle$$

according to our convention for defining morphisms from and into biproducts. A simple computation, using the fact that f is a chain map, shows that $C(f)^\bullet$ is a chain complex.

Along with the cone of a chain map f we also define two chain maps

$$\tau_f\colon B^\bullet \to C(f)^\bullet,$$

given by the collection $\left(\tau^i_f\colon B^i \to A^{i+1} \oplus B^i\right)_{i\in\mathbf{Z}}$, where $\tau^i_f = (0, \mathrm{id}_{B^i})$, and

$$\pi_f\colon C(f)^\bullet \to A^\bullet[1],$$

given by the collection $\left(\pi^i_f\colon A^{i+1} \oplus B^i \to A^{i+1}\right)_{i\in\mathbf{Z}}$, where $\pi^i_f = \langle \mathrm{id}_{A^{i+1}}, 0\rangle$.

The naming convention of course comes from topology, where one can show that the singular chain complex associated to the topological mapping cone $M(f)$ of a continuous map $f\colon X \to Y$ is chain homotopically equivalent to the cone of the chain map induced by f between singular chain complexes of X and Y.

We define any triangle in $K(\mathcal{A})$ isomorphic to a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{\tau_f} C(f) \xrightarrow{\pi_f} A[1]$$

to be distinguished.

Proposition 2.9. *The homotopy category $K(\mathcal{A})$ together with the translation functor $T\colon K(\mathcal{A}) \to K(\mathcal{A})$ and distinguished triangles defined above is a triangulated category.*

Proof. □

Remark 2.10. The proposition still holds true, if we replace $K(\mathcal{A})$ with any of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$.

Lemma 2.11. *Let $f^\bullet\colon A^\bullet \to B^\bullet$ be a quasi-isomorphism and let $g^\bullet\colon C^\bullet \to B^\bullet$ be a morphism. Then there exist a quasi-isomorphism $u^\bullet\colon C_0^\bullet \to C^\bullet$ and a morphism $v^\bullet\colon C_0^\bullet \to A^\bullet$, such that $f \circ u = g \circ v$ i.e. the diagram below commutes.*

$$\begin{tikzcd}
C_0^\bullet \arrow[r, dashed, "u^\bullet", "\sim"'] \arrow[d, dashed, "v^\bullet"'] & C^\bullet \arrow[d, "g^\bullet"] \\
A^\bullet \arrow[r, "f^\bullet", "\sim"'] & B^\bullet
\end{tikzcd}$$

The above also holds in categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$.

Proposition 2.12. *Suppose $\mathcal{A}$ contains enough injectives. Then every $A^\bullet$ in $K^+(\mathcal{A})$ has an injective resolution.*

Proof. □

2.2 Derived functors

The main goal of this section is to assign an appropriate triangulated functor on the level of derived categories, called a *derived functor* of F to a sensible additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between two abelian categories. We will first deal with the easiest case, when F is exact and then move on to the case, where F will only be assumed to be left or right exact and instead require the domain category $\mathcal{A}$ to posses some nice properties (e.g. contain enough injectives or contain an F-adapted class of objects). In practise the additional conditions $\mathcal{A}$ is required to satify are not very restricitve.

We start with some establishing lemmas.

Lemma 2.13. *Let $f\colon A^\bullet \to B^\bullet$ be a chain map. Then f is a quasi-isomorphism if and only if its associated cone $C(f)^\bullet$ is acyclic.*

Proof. One only needs to consider the distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$ in $K(\mathcal{A})$ and its associated long exact sequence in cohomology. □

Lemma 2.14. *Let $F\colon K(\mathcal{A}) \to K(\mathcal{B})$ be a triangulated functor, then the following are equivalent.*

(i) *F preserves quasi-isomorphisms.*

(ii) *F preserves acyclic complexes.*

Proof. ($\Rightarrow$) Let $A^\bullet$ be an acyclic complex. Then the zero morphism $A^\bullet \to 0$ is a quasi-isomorphism and is by assumption mapped to a quasi-isomorphism $F(A^\bullet) \to 0$ by F. This means $F(A^\bullet)$ is acyclic.

($\Leftarrow$) Let $f\colon A^\bullet \to B^\bullet$ be a quasi-isomorphism. The image of a distinguished triangle $A^\bullet \to B^\bullet \to C(f)^\bullet \to A^\bullet[1]$ in $K(\mathcal{A})$ by the triangulated functor F is a distinguished triangle $FA^\bullet \to FB^\bullet \to F(C(f)^\bullet) \to F(A^\bullet)[1]$ in $K(\mathcal{B})$. By lemma 2.13 $C(f)^\bullet$ is acyclic, implying $F(C(f)^\bullet)$ is acyclic by the assumption (ii). Hence again by lemma 2.13 $F(f)$ is a quasi-isomorphism. □

Proposition 2.15. *Let $F\colon K(\mathcal{A}) \to K(\mathcal{B})$ be a triangulated functor and assume it satisfies one of the equivalent conditions of lemma 2.14. Then there exists a triangulated* *functor $D(\mathcal{A}) \to D(\mathcal{B})$ on the level of derived categories, for which the following diagram of triangulated functors commutes.*

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] & K(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D(\mathcal{A}) \arrow[r] & D(\mathcal{B}) \end{tikzcd} \tag{2.2}$$

Proof. The functor $D(\mathcal{A}) \to D(\mathcal{B})$ is defined on objects by the assignment $A^\bullet \mapsto F(A^\bullet)$ and on morphisms by the action

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{D(\mathcal{B})}(F(A^\bullet), F(B^\bullet))$$

sending a morphism $\phi\colon A^\bullet \to B^\bullet$, represented by a roof $A^\bullet \xleftarrow{\sim} C^\bullet \to B^\bullet$, to the equivalence class of the roof $F(A^\bullet) \xleftarrow{\sim} F(C^\bullet) \to F(B^\bullet)$. At this point it was crutial that F preserves quasi-isomorphisms. By functoriality of F it is evident, that this map is well defined. Clearly $D(\mathcal{A}) \to D(\mathcal{B})$ fits into the commutative diagram (2.2), is also k-linear by the k-linearity assumption on F, commutes with the respective translation functors and maps distinguished triangles to distingushed triangles, because of commutativity of diagram (2.2), making it a triangulated functor. □

Every k-linear functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a triangulated functor on the level of homotopy categories $K(F)\colon K(\mathcal{A}) \to K(\mathcal{B})$ and if moreover F is assumed to be exact, then $K(F)$ satisfies the assumptions of lemma 2.15 and thus induces a triangulated functor on the level of derived categories. In the next subsection we will deal with the case, when F is only left or right exact.

2.2.1 F-adapted classes

Unfortunately our categories at hand will sometimes not contain enough injectives, as can already be seen with the category of coherent sheaves $\mathsf{coh}(X)$ on a scheme X. In this case our previously defined method of constructing right derived functors will not work. Luckly however, there exists a method of obtaining a right derived functor $\mathbf{R}F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$, given a left exact functor $F : \mathcal{A} \to \mathcal{B}$, even when $\mathcal{A}$ does not contain enough injectives, but posseses a milder class of objects. To this end we first introduce F-adapted classes.

Definition 2.16. A class of objects $\mathcal{I} \subseteq \mathcal{A}$ is *adapted* to a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, if the following three conditions are satisfied.

(i) $\mathcal{I}$ is stable under finite sums.

(ii) Every object A of $\mathcal{A}$ *embeds* into some object of $\mathcal{I}$, i.e. there exists an object $I \in \mathcal{I}$ and a monomorphism $A \hookrightarrow I$.

(iii) For every acyclic complex $I^\bullet$ in $K^+(\mathcal{A})$, with $I^i \in \mathcal{I}$ for all $i \in \mathbf{Z}$, its image $F(I^\bullet)$ under F is also acyclic.

Remark 2.17. We observe, that whenever $\mathcal{A}$ contains enough injectives, the class of all injective objects $\mathcal{I}_{\mathcal{A}}$ forms an F-adapted class for *every* left exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Points (i) and (ii) are clearly true for $\mathcal{I}_{\mathcal{A}}$ and point (iii) follows from lemma 3.13. Indeed it tells that an acyclic complex of injectives $I^\bullet$ is nulhomotopic, which implies $F(I^\bullet)$ is nulhomotopic and in particular acyclic.

In the presence of an F-adapted class $\mathcal{I}$ we will now construct the right derived functor of F. Firstly one can upgrade the class of objects $\mathcal{I}$ to a full additive subcategory $\mathcal{J}$ of $\mathcal{A}$ having its class of objects be preciesly $\mathcal{I}$. This is done by declaring $\mathrm{Hom}_{\mathcal{J}}(X, Y) := \mathrm{Hom}_{\mathcal{A}}(X, Y)$ for all objects X, Y of the class $\mathcal{I}$. Then we can define a functor $K^+(F)\colon K^+(\mathcal{J}) \to K^+(\mathcal{B})$ between triangulated categories, which acts on objects of $K^+(\mathcal{J})$ as

$$\Big(\cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots\Big) \longmapsto \Big(\cdots \to F(A^i) \xrightarrow{F(d^i)} F(A^{i+1}) \to \cdots\Big)$$

and on morphisms as

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}} \longmapsto (F(f^i))_{i\in\mathbf{Z}}.$$

It is then clear that $K^+(F)$ is triangulated and maps acyclic complexes to acyclic one, thus it descends to a well defined functor on the level of derived categories

$$D^+(\mathcal{J}) \to D^+(\mathcal{B})$$

by proposition 2.15. Since we want to define the derived functor $\mathbf{R}F$, whose domain is $D^+(\mathcal{A})$, it remains to construct a functor $D^+(\mathcal{A}) \to D^+(\mathcal{J})$, which we can then post-compose with $K^+(F)$ to obtain $\mathbf{R}F$. What we will show instead is the following.

Proposition 2.18. *The inclusion of categories $\mathcal{J} \hookrightarrow \mathcal{A}$ induces an equivalence of triangulated categories $D^+(\mathcal{J}) \simeq D^+(\mathcal{A})$.*

This hinges on the following lemma.

Lemma 2.19. *For every complex $A^\bullet$ in $K^+(\mathcal{A})$ there is a quasi-isomorphism $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex in $K^+(\mathcal{J})$.*

Proof. By inspecting the proof of proposition 3.9 we see that only conditions (i) and (ii) of definition 2.16 were actually used, so the proof of this lemma can be copied from the proof of said proposition. $\square$

Proof of proposition 2.18. First, there exists a triangulated functor $D^+(\mathcal{J}) \to D^+(\mathcal{A})$ by proposition 2.15, since the inclusion $\mathcal{J} \hookrightarrow \mathcal{A}$ sends acyclic complexes to acyclic ones. We claim, that this functor is an equivalence, thus we will show that it is full, faithful and essentially surjective. By lemma 2.19 we see that it is essentially surjective. Next we show that it is faithful i.e. that the morphism action

$$\mathrm{Hom}_{D^+(\mathcal{J})}(I^\bullet, J^\bullet) \to \mathrm{Hom}_{D^+(\mathcal{A})}(I^\bullet, J^\bullet) \tag{2.3}$$

is injective for all objects $I^\bullet$, $J^\bullet$ of $D^+(\mathcal{J})$. Suppose two morphisms ϕ_0, $\phi_1\colon I^\bullet \to J^\bullet$, repersented by right roofs $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ and $I^\bullet \to I_1^\bullet \xleftarrow{\sim} J^\bullet$, are sent to the same morphism in $\mathrm{Hom}_{D^+(\mathcal{A})}(I^\bullet, J^\bullet)$, meaning that there is the following commutative diagram in $K^+(\mathcal{A})$.

$$\begin{tikzcd}
 & A^\bullet & \\
I_0^\bullet \arrow[ur] & & I_1^\bullet \arrow[ul, "\sim"'] \\
I^\bullet \arrow[u] \arrow[urr] & & J^\bullet \arrow[u, "\sim"'] \arrow[ull, "\sim"]
\end{tikzcd}$$

Then simply extending the diagram by a quasi-isomorphism $A^\bullet \to I_2^\bullet$, ensured by lemma 2.19, proves that ϕ_0 and ϕ_1 were in fact equal and that the action is faithful.

Lastly we show that the action on morphisms is also surjective, i.e. the homomorphism of (2.3) is surjective. Pick any $\psi \colon I^\bullet \to J^\bullet$ in $D^+(\mathcal{A})$, which is represented by a right roof $I^\bullet \to A^\bullet \xleftarrow{\sim} J^\bullet$. As before, extending the roof by a quasi-isomorphism $A^\bullet \to I_0^\bullet$ by lemma 2.19, leaves us with a representative $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ of some morphism $\phi \colon I^\bullet \to J^\bullet$ in $D^+(\mathcal{J})$.

$$\begin{tikzcd}
 & & I_0^\bullet & & \\
 & I_0^\bullet \arrow[ur, equal] & & A^\bullet \arrow[ul, "\sim"'] & \\
I^\bullet \arrow[ur] \arrow[urrr] & & & & J^\bullet \arrow[ul, "\sim"'] \arrow[ulll, "\sim"]
\end{tikzcd}$$

The above diagram then shows, that ϕ is sent to ψ and that the action on morphisms is surjective. $\square$

We can now define the right dervied functor $\mathbf{R}F$ as the compositon

$$\mathbf{R}F\colon \quad D^+(\mathcal{A}) \xrightarrow{\simeq} D^+(\mathcal{J}) \longrightarrow D^+(B),$$

where $D^+(\mathcal{A}) \xrightarrow{\simeq} D^+(\mathcal{J})$ denotes a quasi-inverse to the equivalence $D^+(\mathcal{J}) \simeq D^+(\mathcal{A})$ established in proposition 2.18. Informally to compute the right derived functor $\mathbf{R}F(A^\bullet)$ of a complex $A^\bullet$ in $D^+(\mathcal{A})$, one takes its F-adapted resolution $A^\bullet \to I_A^\bullet$ and then applies the functor $K^+(F)$ on $I_A^\bullet$, i.e.

$$\mathbf{R}F(A^\bullet) = K^+(F)(I_A^\bullet).$$

Proposition 2.20. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelin categories. Let $\mathcal{I} \subset \mathcal{A}$ be an F-adapted class of objects in $\mathcal{A}$. Then the class of all F-acyclic objects of $\mathcal{A}$, denoted by $\mathcal{I}_F$, is also F-adapted.*

Lemma 2.21. *Every object of an F-adapted class is also F-acyclic.*

Proof. We only have to recall definition **??**. An F-adapted object I of $\mathcal{J}$ trivially forms its own resolution $I^\bullet = (\cdots \to 0 \to I \to 0 \to \cdots)$, which allows us to compute $\mathbf{R}^i F(I) = H^i(\mathbf{R}F(I^\bullet))$ to be 0, whenever $i \neq 0$ and I for $i = 0$. $\square$

Proof of proposition 2.20. We verify conditions (i)–(iii) of definition 2.16. (i) As the higher derived functors R^iF are clearly additive, $\mathcal{I}_F$ is stable under finite sums. (ii) Holds by lemma 2.21. (iii) Suppose $A^\bullet$ is an acyclic complex in $K^+(\mathcal{A})$ with $A^i \in \mathcal{I}_F$ for all i. Then using acyclicity of $A^\bullet$, we may break this complex up into a series of short exact sequences

$$0 \to \ker d^i \to A^i \to \ker d^{i+1} \to 0 \qquad \text{for } i \in \mathbf{Z}. \tag{2.4}$$

As our complex $A^\bullet$ is supported on $\mathbf{Z}_{\geq 0}$, the first true short exact sequence happens at index $i = 1$, where $\ker d^1 = \operatorname{im} d^0 \simeq A^0$ It yields the following long exact sequence associated to F

$$\begin{aligned} 0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to \mathbf{R}^1 F(A^0) \to \cdots \\ \cdots \to \mathbf{R}^p F(A^0) \to \mathbf{R}^p F(A^1) \to \mathbf{R}^p F(\ker d^2) \to \mathbf{R}^{p+1} F(A^0) \to \cdots . \end{aligned}$$

Exacness of this sequence together with F-acyclicity of A^0 and A^1 shows that $\ker d^2$ is also F-acyclic and by only considering the begining few terms of the sequence, we obtain the short exact sequence

$$0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to 0.$$

After inductively applying the same reasoning for each of the original short exact sequences of (2.4) for $i > 1$, we are left with short exact sequences[2]

$$0 \to \ker Fd^i \to F(A^i) \to \ker Fd^{i+1} \to 0 \quad \text{for } i \geq 1.$$

Collecting them all back together we conclude that the complex $F(A^\bullet)$ is acyclic. $\square$

There are inconsistencies with $F(A^\bullet) = K^+(F)(A^\bullet)$ and so on. $\to$ I am going to abuse notation and use $F(A^\bullet)$, bc this is also done in practise e.g. $f_*\mathcal{F}^\bullet$, not $K^+(f_*)(\mathcal{F}^\bullet)$

[2]To be more precise we have used that F is left exact in order to swap the order of F and ker to reach

$$\ker Fd^i \simeq F(\ker d^i)$$

.

3 Derived categories

The first goal of this section is to construct the derived category of an abelian category $\mathcal{A}$ and equip it with a triangulated structure. Our arguments will try to be minimal and will not relay on the already established theory of localization of categories. Our approach might at times seem less elegant, but we include the exposition for completness nonetheless. A more comprehensive and formal treatment of this topic can be found in [3, §7] or [4].

The second part covers the construction of derived functors. We also see how this holostic approach

3.1 Derived category of an abelian category

So our goal is fairly simple, modify the homotopy category $\mathrm{K}(\mathcal{A})$ in such a way that all the quasi-isomorphisms become isomorphisms in a universal way. This can be stated more formally in the form of a universal property.

Definition 3.1. Let $\mathcal{A}$ be an abelian category and $\mathrm{K}(\mathcal{A})$ its homotopy category. A triangulated category $\mathrm{D}(\mathcal{A})$ together with a triangulated functor $Q\colon \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is the *derived category of* $\mathcal{A}$, if it satisfies:

(i) For every quasi-isomorphism s in $\mathrm{K}(\mathcal{A})$, $Q(s)$ is an isomorphism in $\mathrm{D}(\mathcal{A})$.

(ii) For any triangulated category $\mathcal{D}$ and any functor $F\colon \mathrm{K}(\mathcal{A}) \to \mathcal{D}$, sending quasi-isomorphisms s in $\mathrm{K}(\mathcal{A})$ to isomorphisms $F(s)$ in $\mathcal{D}$, there exists a triangulated functor $F_0\colon D(\mathcal{A}) \to \mathcal{D}$, which is unique up to a unique natural isomorphism, such that $F \simeq F_0 \circ Q$. In other words, the diagram below commutes up to natural isomorphism.

$$\begin{tikzcd} \mathrm{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\ \mathrm{D}(\mathcal{A}) \arrow[ur, dashed, "F_0"'] & \end{tikzcd}$$

Remark 3.2. We recognise this definition as a special case of localization [3, Def. 7.1.1.]. In particular it says that $\mathrm{D}(\mathcal{A})$ is the localization of triangulated category $\mathrm{K}(\mathcal{A})$ by the family of all quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$.

Instead we will construct a specific model, which achieves this. We will first need a technical lemma resembling the Ore condition from non-commutative algebra.

categorical motivation and universal property of $\mathrm{D}(\mathcal{A})$

Lemma 3.3. *Let $f\colon A^\bullet \to B^\bullet$ and $s\colon C^\bullet \to B^\bullet$ belong to the homotopy category* $\mathrm{K}(\mathcal{A})$*, with s being a quasi-isomorphism. Then there exists a quasi-isomorphism $u\colon C_0^\bullet \to A^\bullet$ and a morphism $g\colon C_0^\bullet \to C^\bullet$, such that the diagram below commutes in $K(\mathcal{A})$.*

$$\begin{tikzcd} C_0^\bullet \arrow[r, dashed, "g"] \arrow[d, dashed, "u", "\sim"'] & C^\bullet \arrow[d, "s", "\sim"'] \\ A^\bullet \arrow[r, "f"] & B^\bullet \end{tikzcd}$$

Proof. □

add proof

We are now in a position to construct the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. It consists of

$$\textit{Objects of } D(\mathcal{A})\text{:}\quad \text{chain complexes in } \mathcal{A},$$

i.e. the class of objects of $\mathrm{K}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{A})$, and a class of morphisms, which is a bit more intricate to define. For fixed complexes $A^\bullet$ and $B^\bullet$ we define the hom-set[3] $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ in the following way.

Hom-sets. A *left roof spanned on* $A^\bullet$ *and* $B^\bullet$ is a pair of morphisms $s\colon A^\bullet \to C^\bullet$ and $f\colon C^\bullet \to B^\bullet$ in the homotopy category $K(\mathcal{A})$, where s is a quasi-isomorphism. This roof is depicted in the following diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \tag{3.1}$$

or denoted by (s, f). Dually, one also obtains the notion of a *right roof spanned on* $A^\bullet$ *and* $B^\bullet$, which is a pair of morphisms $g\colon A^\bullet \to C^\bullet$ and $u\colon B^\bullet \to C^\bullet$, where u is a quasi-isomorphism, and is depicted below.

$$\begin{tikzcd} & C^\bullet & \\ A^\bullet \arrow[ur, "g"'] & & B^\bullet \arrow[ul, "\sim"', "u"] \end{tikzcd}$$

Our construction of $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ will be based on left roofs, for nothing is gained or lost by picking either one of the two. Both work just as well and are in fact equvalent. Despite our arbitrary choice, it is still benefitial to consider both, as we will sometimes switch between the two whenever convenient.

Definition 3.4. Two left roofs $A^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} B^\bullet$ and $A^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} B^\bullet$ are defined to be *equivalent*, if there exists a quasi-isomorphism $u\colon C^\bullet \to C_0^\bullet$ and a morphism $g\colon C^\bullet \to C_1^\bullet$ in $K(\mathcal{A})$, for which the diagram below commutes (in $K(\mathcal{A})$).

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "\sim"', "u"] \arrow[dr, "g"] & & \\ & C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd} \tag{3.2}$$

We denote this relation by $\equiv$.

Note that since $C^\bullet \to C_0^\bullet \to A^\bullet$ is a quasi-isomorphism, the same is true for the composition $C^\bullet \to C_1^\bullet \to A^\bullet$, concluding that $g\colon C^\bullet \to C_1^\bullet$ is a quasi-isomorphism. Also observe that in the diagram (3.2) we may find a new left roof, namely

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s_0 \circ u"', "\sim"] \arrow[dr, "f_1 \circ g"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

which is also equivalent to the two roofs we started with (s_0, f_0) and (s_1, f_1).

[3] What will be defined here is a priori not necessarily a set, but a class, so $D(\mathcal{A})$, defined in this section, is not a category in the usual sense. We will however prove that in specific cases some variants of the derived category, especially concrete ones used later on, will from categories in the usual sense.

Lemma 3.5. *The equivalence of left roofs on $A^\bullet$ and $B^\bullet$ is an equivalence relation.*

Proof. The relation is clearly reflexive. We take both u and g to be $\mathrm{id}_{C^\bullet}$. By the note above it is also symmetric, for g is a quasi-isomorphism. It remains to show transitivity. Suppose left roofs $A^\bullet \longleftarrow C_0^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ are equivalent and left roofs $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_2^\bullet \longrightarrow B^\bullet$ are equivalent. This is witnessed by the solid diagram below.

$$\begin{tikzcd}
 & & & C^\bullet \arrow[dl, dashed, "\sim"'] \arrow[dr, dashed] & & & \\
 & & D_0^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & D_1^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & \\
 & C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrrrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[drrr] & & C_2^\bullet \arrow[dlllll, "\sim"] \arrow[dr] & \\
A^\bullet & & & & & & B^\bullet
\end{tikzcd}$$

By lemma 3.3 this diagram may be completed with the dashed arrows at the top, proving that $A^\bullet \leftarrow C_0^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are equivalent. □

I don't like how this diagram looks...

For a left roof (3.1) we let $[s \setminus f]$ or $\left[A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet\right]$ denote its equivalence class under $\equiv$.

We then define $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ to be the class of left roofs spanned by $A^\bullet$ and $B^\bullet$, quotiented by the relation $\equiv$. That is

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet) := \left\{ \left[\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \right]_{\equiv} \;\middle|\; \begin{array}{l} C^\bullet \in \mathrm{Ob}\, K(A), \\ f \in \mathrm{Hom}_{K(A)}(C^\bullet, B^\bullet), \\ s \in \mathrm{Hom}_{K(A)}(C^\bullet, A^\bullet) \text{ quasi-iso.} \end{array} \right\}.$$

Composition. Next we define the compositon operations

$$\circ\colon \mathrm{Hom}_{D(\mathcal{A})}(A_0^\bullet, A_1^\bullet) \times \mathrm{Hom}_{D(\mathcal{A})}(A_1^\bullet, A_2^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(A_0^\bullet, A_2^\bullet)$$

for all objects $A_0^\bullet$, $A_1^\bullet$ and $A_2^\bullet$ of $D(\mathcal{A})$. Let $\phi_0\colon A_0^\bullet \to A_1^\bullet$ and $\phi_1\colon A_1^\bullet \to A_2^\bullet$ be a pair of composable morphisms in $D(\mathcal{A})$. Next, pick their respective left roof representatives, $A_0^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} A_1^\bullet$ and $A_1^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} A_2^\bullet$, and concatenate them according to the solid zig-zag diagram below.

$$\begin{tikzcd}
 & & C^\bullet \arrow[dl, dashed, "\sim"', "u"] \arrow[dr, dashed, "g"'] & & \\
 & C_0^\bullet \arrow[dl, "\sim"', "s_0"] \arrow[dr, "f_0"'] & & C_1^\bullet \arrow[dl, "\sim"', "s_1"] \arrow[dr, "f_1"'] & \\
A_0^\bullet & & A_1^\bullet & & A_2^\bullet
\end{tikzcd}$$

By lemma 3.3 there are morphisms $u\colon C^\bullet \to C_0^\bullet$ and $g\colon C^\bullet \to C_1^\bullet$, depicted with dashed arrows, completing the diagram in $K(\mathcal{A})$. In this way we obtain a left roof, formed by a quasi-isomorphism $s_0 \circ u$ and a morphism $f_1 \circ g$, the equivalence class of which we define to be the composition

$$\phi_1 \circ \phi_0 := \left[A_0^\bullet \xleftarrow{s_0 \circ u} C^\bullet \xrightarrow{f_1 \circ g} A_2^\bullet\right].$$

It can be shown that this is a well defined composition that is also associative. We leave out the proof, because it is routine, but refer the reader to a very detailed account by Miličić [4, §1.3].

Identities. The identity morphism on an object $A^\bullet$ of $D(\mathcal{A})$ is defined to be the equivalence class of $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet$. It is not difficult to check that these play the role of identity morphisms in $\mathcal{D}(\mathcal{A})$.

need to pick a consistent choice of notation for derived cats.

k-linear structure.

finish subsection

Triangulated structure. The shift functor $T\colon \mathrm{D}(()\mathcal{A}) \to \mathrm{D}(()\mathcal{A})$ is defined

Functor $Q\colon \mathrm{K}(\mathcal{A}) \to \mathrm{D}(()\mathcal{A})$.

introduce also the localization functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$.

Remark 3.6. All that has been defined and established in this section with left roofs can be analogously done also with right roofs.

As with the homotopy category of complexes $K(\mathcal{A})$, we also have the following bounded versions of the derived category $D(\mathcal{A})$, appearing as full triangulated subcategories of the unbounded variant $D(\mathcal{A})$.

$D^+(\mathcal{A})$	spanned on complexes in $\mathcal{A}$ bounded below.
$D^-(\mathcal{A})$	spanned on complexes in $\mathcal{A}$ bounded above.
$D^b(\mathcal{A})$	spanned on bounded complexes in $\mathcal{A}$.

3.1.1 Derived category of an abelian category with enough injectives

Definition 3.7. An object I of an abelian category $\mathcal{A}$ is called *injective*, if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$ is exact. Equivalently whenever for any monomorphism $A \hookrightarrow B$ and $A \to I$ there is a morphism $B \to I$, for which the following diagram commutes.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

A category $\mathcal{A}$ is said to *have enough injectives* if every object A of $\mathcal{A}$ embeds into an injective object i.e. there is a monomorphism $A \hookrightarrow I$ for some injective object I.

Remark 3.8. A simple verification shows, that the full subcategory of an abelian or k-linear category $\mathcal{A}$, spanend on all injective objects of $\mathcal{A}$ is a k-linear category, as the zero object 0 is injective and the biproduct of two injective objects is again injective. We denote this category by $\mathcal{J}$.

We will use injectives to

We call a quasi-isomorphism $f\colon A^\bullet \to I^\bullet$ an *injective resolution* of the complex $A^\bullet$. Later we will also see that this injective resolution is unique up to homotopy i.e. any two injective resolutions of $A^\bullet$ are homotopically equivalent. Throughout this section differenitials of $I^\bullet$ will be denoted with $\delta^i\colon I^i \to I^{i+1}$.

Proposition 3.9. *Suppose $\mathcal{A}$ contains enough injectives. Then for every $A^\bullet$ in $K^+(\mathcal{A})$ there is a quasi-isomorphism $f^\bullet\colon A^\bullet \to I^\bullet$, where $I^\bullet \in \mathrm{Ob}\, K^+(\mathcal{A})$ is a complex of injectives.*

Proof. We will inductively construct a complex $I^\bullet$ in $K^+(\mathcal{A})$, built up from injectives, and a quasi-isomorphism $f^\bullet \colon A^\bullet \to I^\bullet$. For simplicity assume $A^i = 0$ for $i < 0$.

The base case is trivial. Define $I^i = 0$ and $f^i = 0$ for $i < 0$ and take $f^0 \colon A^0 \to I^0$ to be the morphism obtained from the assumption about $\mathcal{A}$ having enough injectives.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 & &
\end{tikzcd}$$

For the induction step suppose we have constructed the complex $I^\bullet$ and the morphism $f^\bullet$ up to index n.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r, "\delta^{n-1}"] & I^n & &
\end{tikzcd}$$

Consider the cokernel of $d^{n-1}_{C(f)}$

$$A^n \oplus I^{n-1} \xrightarrow{d^{n-1}_{C(f)}} A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d^{n-1}_{C(f)}$$

and denote with $j \colon \operatorname{coker} d^{n-1}_{C(f)} \hookrightarrow I^{n+1}$ an embedding to an injective object I^{n+1}. After precomposing the morphism

$$A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d^{n-1}_{C(f)} \xrightarrow{j} I^{n+1}$$

with canonical embeddings of A^{n+1} and I^n into their biproduct $A^{n+1} \oplus I^n$, we obtain morphisms $\delta^n \colon I^n \to I^{n+1}$ and $f^{n+1} \colon A^{n+1} \to I^{n+1}$. As $j \circ e$ is a morphism fitting into the following coproduct diagram,

$$\begin{tikzcd}
A^{n+1} \arrow[dr] \arrow[ddr, bend right, "f^{n+1}"'] & & I^n \arrow[dl] \arrow[ddl, bend left, "\delta^n"] \\
& A^{n+1} \oplus I^n \arrow[d, "j \circ e"] & \\
& I^{n+1} &
\end{tikzcd}$$

we see that $j \circ e = \langle f^{n+1}, \delta^n \rangle$. Next, we see that $\delta^n \delta^{n-1} = 0$ and $\delta^n f^n = f^{n+1} d^n$ from the computation

$$0 = j \circ e \circ d^{n-1}_{C(f)} = \langle f^{n+1},\ \delta^n \rangle \begin{pmatrix} -d^n & 0 \\ f^n & \delta^{n-1} \end{pmatrix} = \langle \delta^n f^n - f^{n+1} d^n,\ \delta^n \delta^{n-1} \rangle$$

We have now constructed a chain complex $I^\bullet$ of injective objects and a chain map $f \colon A^\bullet \to I^\bullet$. $\blacksquare$

It remanis to be shown that $f \colon A^\bullet \to I^\bullet$ is a quasi-isomorphism. Recall, that f is a quasi-isomorphism if and only if its cone $C(f)^\bullet$ is acyclic. By remark 2.5, we known

$$H^n(C(f)^\bullet) = \ker(\operatorname{coker} d^{n-1}_{C(f)} \xrightarrow{d} A^{n+2} \oplus I^{n+1}),$$

where d is obtained from the universal property of $\operatorname{coker} d^{n-1}_{C(f)}$, induced by $d^n_{C(f)}$. Thus it suffices to show, that d is a monomorphism. Consider the following diagram, which we will show to commute.

$$\begin{tikzcd}
A^n \oplus I^{n-1} \arrow[r, "d^{n-1}_{C(f)}"] & A^{n+1} \oplus I^n \arrow[rr, "d^n_{C(f)}"] \arrow[dr, two heads, "e"] & & A^{n+2} \oplus I^{n+1} \arrow[dr, "p"] & \\
& & \operatorname{coker} d^{n-1}_{C(f)} \arrow[ur, dashed, "d"] \arrow[rr, hook, "j"] & & I^{n+1}
\end{tikzcd}$$

Once we show, that its right most triangle is commutative, we see that d is mono, because j is mono. We compute

$$p \circ d \circ e = p \circ d^n_{C(f)} = \langle f^{n+1}, \delta^n \rangle = j \circ e.$$

As e is epic, we may cancel it on the right, to arrive at $p \circ d = j$, which ends the proof. $\square$

Remark 3.10. As is evident from the proof by close inspection we have only used two facts about the class of all injective objects of $\mathcal{A}$ – that every object of $\mathcal{A}$ embeds into some injective object and that the class of injectives is closed under finite direct sums. This will later on be used in subsection 2.2.1 when constructing a right derived functor of a given functor F in the presence of an F-adapted class.

Theorem 3.11. *Assume $\mathcal{A}$ contains enough injectives and let $\mathcal{J} \subseteq \mathcal{A}$ denote the full subcategory on injecitve objects of $\mathcal{A}$. Then the inclusion $K^+(\mathcal{J}) \hookrightarrow K^+(\mathcal{A})$ induces an equivalence of triangulated categories*

$$K^+(\mathcal{J}) \simeq D^+(\mathcal{A}).$$

Remark 3.12. Theorem 3.11 in particular shows that $D^+(\mathcal{A})$ is a category in the usual sense, i.e. all its hom-sets are in fact sets.

maybe I replace "usual" with locally small everywhere this comes up.

Lemma 3.13. *Let $A^\bullet$ be any acyclic complex in $K^+(\mathcal{A})$ and $I^\bullet$ a complex of injectives from $K^+(J)$. Then*

$$\operatorname{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) = 0.$$

In other words every morphism from an acyclic complex to an injective one is nulhomotopic.

Proof. We will show that $f \simeq 0$ for any chain map $f : A^\bullet \to I^\bullet$ by inductively constructing an appropriate homotopy h. For simplicity assume $A^i = 0$ and $I^i = 0$ for $i < 0$ and define $h^i : A^i \to I^{i-1}$ to be 0 for $i \leq 0$.

As $A^\bullet$ is acyclic, the first non-trivial differential $d^0 : A^0 \to A^1$ is mono. From I^0 being injective, we obtain a morphism $h^1 : A^1 \to I^0$, satisfying

$$f^0 = h^1 \circ d^0.$$

Note that the above can actually be rewritten to $f^0 = h^1 d^0 + \delta^{-1} h^0$, as $h^0 = 0$.

For the induction step assume that we have already constructed $h^i : A^i \to I^{i-1}$, with $f^{i-1} = h^i d^{i-1} + \delta^{i-2} h^{i-1}$ for all $i \leq n$. We are aiming to construct $h^{n+1} : A^{n+1} \to I^n$, for which $f^n = h^{n+1} d^n + \delta^{n-1} h^n$ holds. First, expand the differential d^n into a composition of the canonical epimorphism $e : A^n \to \operatorname{coker} d^{n-1}$, followed by $j : \operatorname{coker} d^{n-1} \to A^{n+1}$ induced by the universal property of $\operatorname{coker} d^{n-1}$ by the map $d^n : A^n \to A^{n+1}$. Morphism j is actually a monomorphism by acyclicity of $A^\bullet$, since we know that

$$0 = H^n(A^\bullet) \simeq \ker(j : \operatorname{coker} d^{n-1} \to A^{n+1}).$$

Next, we see that by the universal property of coker d^{n-1}, morphism $f^n - \delta^{n-1}h^n \colon A^n \to I^n$ induces a morphism $g\colon$ coker $d^{n-1} \to I^n$, because

$$\begin{aligned}(f^n - \delta^{n-1}h^n)d^{n-1} &= f^n d^{n-1} - \delta^{n-1}h^n d^{n-1} \\ &= f^n d^{n-1} + \delta^{n-1}\delta^{n-2}h^{n-1} - \delta^{n-1}f^{n-1} \\ &= 0.\end{aligned}$$

The second equality follows from the inductive hypothesis and the third one from f being a chain map and δ^{n-1}, δ^{n-2} being a differentials.

Lastly, for I^n is injecive and j mono, there is a morphism $h^{n+1}\colon A^{n+1} \to I^n$, satisfying $h^{n+1} \circ j = g$, thus after precomposing both sides with e, we arrive at $h^{n+1}d^n = f^n - \delta^{n-1}h^n$, which can be rewritten as

$$f^n = h^{n+1}d^n + \delta^{n-1}h^n.$$

□

Lemma 3.14. *Let $A^\bullet$ and $B^\bullet$ belong to $K^+(\mathcal{A})$ and $I^\bullet \in K^+(\mathcal{J})$. Let $f\colon B^\bullet \to A^\bullet$ be a quasi-isomorphism, then*

$$\mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \mathrm{Hom}_{K^+(\mathcal{A})}(B^\bullet, I^\bullet)$$

is an isomorphism of k-modules.

Proof. In $K^+(\mathcal{A})$ we have a distinguished triangle $B^\bullet \to A^\bullet \to C(f)^\bullet \to B[1]^\bullet$, which induces a long exact sequence (of k-modules)

$$\begin{aligned}\cdots \longrightarrow \mathrm{Hom}_{K^+(\mathcal{A})}(C(f)[-1]^\bullet, I^\bullet) \longrightarrow \\ \longrightarrow \mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \mathrm{Hom}_{K^+(\mathcal{A})}(B^\bullet, I^\bullet) \longrightarrow \\ \longrightarrow \mathrm{Hom}_{K^+(\mathcal{A})}(C(f)^\bullet, I^\bullet) \longrightarrow \cdots .\end{aligned}$$

The reason being $\mathrm{Hom}_{K^+(\mathcal{A})}(-, I^\bullet)$ is a cohomological functor by example 1.17. As f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic along with all its shifts. Since $I^\bullet$ is a complex of injectives, f^* is clearly an isomorphism by lemma 3.13. □

Lemma 3.15. *Let $A^\bullet$ belong to $K^+(\mathcal{A})$ and $I^\bullet$ to $K^+(\mathcal{J})$. Then the morphism action*

$$\mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, I^\bullet) \tag{3.3}$$

of the localization functor Q is an isomorphism of k-modules.

Proof. We already know this is a k-module homomorphism, thus it is enough to show that it is bijective. This is accomplished by constructing its inverse. Let $\phi\colon A^\bullet \to I^\bullet$ be a morphism in $D^+(\mathcal{A})$ and let $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ be its left roof representative. The morphism s being a quasi-isomorphism implies that $s^*\colon \mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \mathrm{Hom}_{K^+(\mathcal{A})}(B^\bullet, I^\bullet)$ is bijective by lemma 3.14, so there is a unique morphism $g\colon A^\bullet \to I^\bullet$, such that $f = g \circ s$ in $K^+(\mathcal{A})$. The inverse to (3.3) is then defined by sending ϕ to g.

First let's argue why this map is well-defined i.e. independent of the choice of left roof representative for ϕ. We pick two left roof representatives $A^\bullet \xleftarrow{s_0} B_0^\bullet \xrightarrow{f_0} I^\bullet$ and $A^\bullet \xleftarrow{s_1} B_1^\bullet \xrightarrow{f_1} I^\bullet$ for ϕ and let g_0 and g_1 be such that $f_i = g_i \circ s_i$ for both i. As the roofs are equivalent there are quasi-isomorphisms $t_0\colon C^\bullet \to B_0^\bullet$ and $t_1\colon C^\bullet \to B_1^\bullet$, which fit into the following commutative diagram.

$$\begin{tikzcd}
 & & C^\bullet \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & & \\
 & B_0^\bullet \arrow[dl, "s_0"'] \arrow[drrr, "f_0"' near end] & & B_1^\bullet \arrow[dlll, "s_1" near end] \arrow[dr, "f_1"] & \\
A^\bullet & & & & I^\bullet
\end{tikzcd}$$

Therefore $g_0 s_0 t_0 = f_0 t_0 = f_1 t_1 = g_1 s_1 t_1 = g_1 s_0 t_0$. As $s_0 t_0$ is a quasi-isomorphism, we see that $g_0 = g_1$ by applying lemma 3.14.

Following the diagram below, it is clear why the constructed map is a right inverse to (3.3)

$$\begin{array}{ccccc}
\mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) & \xrightarrow{Q} & \mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \\
(g\colon A^\bullet \to I^\bullet) & \longmapsto & \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right] & \longmapsto & (g\colon A^\bullet \to I^\bullet)\,.
\end{array}$$

Lastly we see that it is also a left inverse by the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) & \xrightarrow{Q} & \mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, I^\bullet) \\
\left[A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet\right] & \longmapsto & (g\colon A^\bullet \to I^\bullet) & \longmapsto & \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right]
\end{array}$$

along with observing that $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet$ and $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ are equivalent roofs. □

Proof of theorem 3.11. As $K^+(\mathcal{J}) \to D^+(\mathcal{A})$ is a triangulated functor, we only need to show that it is fully faithful and essentially surjective. Let $I^\bullet$ and $J^\bullet$ be objects of $K^+(\mathcal{J})$. Then

$$\mathrm{Hom}_{K^+(\mathcal{J})}(I^\bullet, J^\bullet) \xrightarrow{=} \mathrm{Hom}_{K^+(\mathcal{A})}(I^\bullet, J^\bullet) \xrightarrow{(3.15)} \mathrm{Hom}_{D^+(\mathcal{A})}(I^\bullet, J^\bullet)$$

is a bijection, showing fully faithfulness (the last map is a bijection by lemma 3.15). Essential surjectivity is clear from the existence of injective resolutions (cf. proposition 3.9) because quasi-isomorphisms now play the role of isomorphisms in $D^+(\mathcal{A})$. □

add a little argument for why injective res. are unique up to htpy.

3.1.2 Subcategories of derived categories

We have already seen $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ be defined as full triangulated subcategories of $D(\mathcal{A})$ on the class of certain bounded complexes taking values in $\mathcal{A}$.

3.2 Derived functors

F-adapted classes

4 Derived categories in geometry

4.1 Derived category of coherent sheaves

One of the main invariants of a scheme X over k is its category of coherent sheaves $\mathsf{coh}(X)$. One can think of this category as a slight extension of the category of locally free $\mathcal{O}_X$-modules of finite rank also known as k-vector bundles on X in the sense that $\mathsf{coh}(X)$ is abelian, where as the former very often is not. In this chapter we restrict ourself to noetherian schemes and our primary object of study will be the bounded derived category of coherent sheaves

$$D^b(X) := D^b(\mathsf{coh}(X)).$$

Proposition 4.1. *For a noetherian scheme X, its category of quasi-coherent shaves $\mathsf{qcoh}(X)$ contains enough injectives.*

Proposition 4.2. *For a noetherian scheme X the inclusion $D^b(X) \hookrightarrow D^b(\mathsf{qcoh}(X))$ induces an equivalence of triangulated categories*

$$D^b(X) \simeq D^b_{\mathsf{c}}(\mathsf{qcoh}(X)),$$

where $D^b_{\mathsf{c}}(\mathsf{qcoh}(X))$ is the full triangulated subcategory of $D^b(\mathsf{qcoh}(X))$ spanned on bounded complexes of quasi-coherent shaves on X with coherent cohomology.

Proposition 4.3.

4.2 Derived functors in algebraic geometry

In this subsection we will derive some functors occuring in algebraic geometry and later state some important facts relating them with each other. To derive these functors we will either deal with injective sheaves and access the realm of quasi-coherent sheaves, following the first part of section **??**, or introduce certain special classes of coherent sheaves depending on the functor we wish to derive and taking up the role of adapted classes, mirroring what was done in subsection **??**.

4.2.1 Global sections functor

Arguably the most common functor in algebraic geometry is the global secitons functor. Associated to a scheme X, the functor of *global sections*

$$\Gamma\colon \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_k$$

assigns to each $\mathcal{O}_X$-module $\mathcal{F}$ its module of global sections $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ and to a morphism of $\mathcal{O}_X$-modules $\alpha\colon \mathcal{F} \to \mathcal{G}$ a homomorphism $\alpha_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$. By abuse of notation we use Γ to also denote the restrictions of the global sections functor to subcategories $\mathsf{qcoh}(X)$ and $\mathsf{coh}(X)$ of $\mathsf{Mod}_{\mathcal{O}_X}$.

As is commonly known, Γ is a left exact functor, so our aim is to construct its right derived counterpart. Due to $\mathsf{coh}(X)$ not having enough injectives, we resort to $\mathsf{qcoh}(X)$, on which we may define

$$\mathbf{R}\Gamma\colon D^+(\mathsf{qcoh}(X)) \to D^+(\mathsf{Mod}_k).$$

Precomposing it with the inclusion $D^+(\mathsf{coh}(X)) \to D^+(\mathsf{qcoh}(X))$

4.2.2 Push-forward f_*

4.2.3 Pull-back f^*

4.2.4 Inner hom

4.2.5 Tensor product

5 Fourier-Mukai transforms

5.1 on K-groups

5.2 on rational cohomology

6 K3 surfaces

6.1 Algebraic and complex

6.2 Main invariants

6.2.1 Cohomology

6.2.2 Intersection pairing

6.2.3 Hodge structure

6.3 Two important theorems

7 Derived Torelli theorem

Theorem 7.1. *Let X and Y be K3 surfaces over the field of complex numbers* $\mathbf{C}$*. Then they are D-equivalent if and only if there exists a Hodge isometry $f\colon T(X) \to T(Y)$ between their transcendental lattices.*

7.1 Mukai lattice

7.2 Moduli space of sheaves on a K3 surface

7.3 Proof

A Spectral sequences and how to use them

In this chapter we will first define cohomological spectral sequences and discuss their meaning through applications related with derived categories.

At first, when encountering spectral sequences, one might think of them as just book-keeping devices encoding a tremendous amount of data, but we will soon see how elegantly one can infer certain properties related to derived categorical claims exploiting the fact that they can naturally be encoded with spectral sequences. One of the most important spectral sequences, which is also very general within our scope of inspection, will be the Grothendieck spectral sequence relating the higher derived functors of two composable functors with the higher derived functors of their composition. Later on we will see that many useful and well-known spectral sequences occur as special cases of the Grothendieck spectral sequence. What follows was gathered mostly from []

• Huybrechts FMinAG : chapter on ss
• Gelfand, Manin : III.7
• McCleary

For the time being we fix an abelian category $\mathcal{A}$ and start off with a definition.

Definition A.1. A *(cohomological) spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data on which we further impose two convergence conditions.

- *Sequence of pages.* A sequence of bi-graded objects $(E_r^{\bullet,\bullet})_{r\in\mathbf{Z}_{\geq 0}}$ equipped with differentials of bi-degree $(r, 1-r)$. The r-th term of this sequence is called the r-th *page* and it consists of a lattice of objects $E_r^{p,q}$ of $\mathcal{A}$, for $p, q \in \mathbf{Z}$, and differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

  satisfying

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0,$$

  for each $p, q \in \mathbf{Z}$.

- *Isomorphisms.* A collection of isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q},$$

  for all $p, q \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, where

$$H^{p,q}(E_r) := \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}),$$

  which allow us to turn the pages.

- *Transfinite page.* A bi-graded object $E_\infty^{\bullet,\bullet}$.

- *Goal of computation.* A sequence of objects $(E^n)_{n\in\mathbf{Z}}$ of the category $\mathcal{A}$.

The above collection of data also has to satisfy the following two convergence conditions.

(a) For each pair (p, q), there exists $r_0 \geq 0$, such that for all $r \geq r_0$ we have

$$d_r^{p,q} = 0 \quad \text{and} \quad d_r^{p+r,q-r+1} = 0$$

and the isomorphism $\alpha_r^{p,q}$ can be taken to be the identity. We then say that the (p, q)-term *stabilizes* after page r_0 and we denote $E_{r_0}^{p,q}$ (along with all the subsequent $E_r^{p,q}$ for $r \geq r_0$) by $E_\infty^{p,q}$.

(b) For each $n \in \mathbf{Z}$ there is a decreasing *regular*[4] filtration of E^n

$$E^n \supseteq \cdots \supseteq F^pE^n \supseteq F^{p+1}E^n \supseteq \cdots \supseteq 0$$

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^pE^{p+q}/F^{p+1}E^{p+q}$$

for all $p, q \in \mathbf{Z}$.

In this case we also denote the existence of such a spectral sequence by

$$E_r^{p,q} \implies E^n.$$

Remark A.2. A few words are in order to justify us naming the sequence $(E^n)_{n\in\mathbf{Z}}$ our *goal of computation*. Usually one is given a starting page or a small number of them and the first goal is to identify the transfinite page – we are refering to convergence condition (a). Often one is able to infer the differentials degenerate after a number of turns of the pages from context or by observing the shape of the spectral sequence. For example *first quadrant spectral sequences*, i.e. the ones with non-trivial $E_r^{p,q}$ only for (p, q) lying in the first quadrant, always satisfy condition (a).

The second part of the computation is concerned with relating objects from the transfinte page $E_\infty^{\bullet,\bullet}$ with objects E^n. This is captured in the convergence condition (b), from which we can clearly observe that the intermediate quotients of the filtration $(F^pE^n)_{p\in\mathbf{Z}}$ for a fixed term E^n lie on the anti-diagonal of the transfinite page passing through e.g. $E_\infty^{n,0}$.

In condition (b) the existence of isomorphisms $\beta^{p,q}$ can also be restated by saying that $E_\infty^{p,q}$ fits into a short exact sequence

$$0 \to F^{p+1}E^n \to F^pE^n \to E_\infty^{p,q} \to 0.$$

This observation becomes very useful when considering properties of objects of the category $\mathcal{A}$ which are closed under extensions, especially when the filtration of E^n is finite.

mention Tohoku paper

Theorem A.3 (Grothendieck)**.** *Let $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Let $\mathcal{I}_F$ be an F-adapted class in $\mathcal{A}$, $\mathcal{I}_G$ a G-adapted class of objects in $\mathcal{B}$ and suppose every object of $\mathcal{I}_F$ is sent to a G-acyclic object by the functor F. Then for every $A^\bullet$ in $K^+(\mathcal{A})$ there exists a spectral sequence*

$$E_2^{p,q} = \mathbf{R}^pG(\mathbf{R}^qF(A^\bullet)) \implies \mathbf{R}^{p+q}(G \circ F)(A^\bullet).$$

B Lattice theory

[4] In our case the filtration $(F^pE^n)_{p\in\mathbf{Z}}$ is *regular*, whenever $\bigcap_p F^pE^n = \lim_p F^pE^n = 0$ and $\bigcup_p F^pE^n = \operatorname{colim}_p F^pE^n = E^n$.

References

[1] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, fr (ed. by M. Georges), Astérisque, Société mathématique de France, 1996, available from https://www.numdam.org/item/AST_1996__239__R1_0/.

[2] D. Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Clarendon Press, Oxford University Press, Oxford, UK, 2006, DOI: 10.1093/acprof:oso/9780199296866.001.0001, available from https://doi.org/10.1093/acprof:oso/9780199296866.001.0001.

[3] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften **332**, Springer, 2006, DOI: 10.1007/3-540-27950-4, available from https://doi.org/10.1007/3-540-27950-4.

[4] D. Miličić, *Lectures on Derived Categories*, Lecture notes, University of Utah, Available at https://www.math.utah.edu/~milicic/Eprints/dercat.pdf.

[5] T. S. project authors, *The Stacks project*, https://stacks.math.columbia.edu, 2025.

Razširjeni povzetek