UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS

Mathematics – 2nd cycle

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K3 SURFACES FROM A DERIVED CATEGORICAL VIEWPOINT

Master's thesis

Supervisor: .

Ljubljana, 2025

UNIVERZA V LJUBLJANI
FAKULTETA ZA MATEMATIKO IN FIZIKO

Matematika – 2. stopnja

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K3 PLOSKVE Z VIDIKA IZPELJANIH KATEGORIJ

Magistrsko delo

Mentor: .

Ljubljana, 2025

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Keywords: .
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Introduction

1 Categorical prerequisites

In this section we try to make a brief account of the majority of the relevant prerequisites needed to discuss derived categories. Firstly we introduce additive, k-linear and abelian categories together with relevant functors preserving (certain parts of) their structure. Next we build the framework for triangulated categories of which derived categories will be a special kind and also state and prove some specialized results on triangulated categories, which will be applied later on in chapter 6. Lastly, we lay out the foundations on which derived categories will be constructed on – the (homotopy) category of complexes. The main pieces of literature for this part were [KS06] and [Huy06].

1.1 Additive, k-linear and abelian categories

In this section we mainly follow [KS06, Chapter 8]. With k we denote either a field or the ring of integers $\mathbf{Z}$.

A categorical *biproduct* of objects X and Y in a category $\mathcal{C}$ is an object $X \oplus Y$ together with morphisms

$$\begin{array}{ll} p_X \colon X \oplus Y \to X & p_Y \colon X \oplus Y \to Y \\ i_X \colon X \to X \oplus Y & i_Y \colon Y \to X \oplus Y \end{array}$$

for which the pair p_X, p_Y is the categorical product of X and Y and the pair i_X, i_Y is the categorical coproduct. For a pair of morphisms $f_X \colon Z \to X$ and $f_Y \colon Z \to Y$ the unique induced morphism into the product $Z \to X \oplus Y$ is denoted by (f_X, f_Y) and for a pair of morphisms $g_X \colon X \to Z$ and $g_Y \colon Y \to Z$ the unique induced morphism from the coproduct $X \oplus Y \to Z$ is denoted by $\langle g_X, g_Y \rangle$. For morphisms

$$\begin{array}{ll} f_{00} \colon X_0 \to Y_0 & f_{01} \colon X_0 \to Y_1 \\ f_{10} \colon X_1 \to Y_1 & f_{11} \colon X_1 \to Y_1 \end{array}$$

of $\mathcal{C}$, there are two equal morphisms $X_0 \oplus X_1 \to Y_0 \oplus Y_1$, namely

$$(\langle f_{00}, f_{01} \rangle, \langle f_{10}, f_{11} \rangle) \quad \text{and} \quad \langle (f_{00}, f_{10}), (f_{01}, f_{11}) \rangle ,$$

which we will be denoting with the matrix

$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} : X_0 \oplus X_1 \to Y_0 \oplus Y_1. \tag{1.1}$$

Definition 1.1. A category $\mathcal{A}$ is additive (resp. k-linear) if all the hom-sets carry the structure of abelian groups (resp. k-modules) and the following axioms are satisfied

Decide on which terminology to use (everything is covered by k-linear...)

A1 For all all objects X, Y and Z of $\mathcal{A}$ the composition

$$\circ \colon \operatorname{Hom}_{\mathcal{A}}(Y, Z) \times \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear.

A2 There exists a *zero object* 0, for which $\operatorname{Hom}_{\mathcal{A}}(0, 0) = 0$.

A3 For any two objects X and Y there exists a categorical biproduct of X and Y.

Remark 1.2. (i) The zero object 0 of a k-linear category $\mathcal{A}$ is both the initial and terminal object of $\mathcal{A}$.

(ii) For morphisms $f_0\colon X_0 \to Y_0$, $f_1\colon X_1 \to Y_1$ we introduce notation

$$f_0 \oplus f_1\colon X_0 \oplus X_1 \to Y_0 \oplus Y_1$$

to mean either of the two (equal) morphisms $(\langle f_0, 0\rangle, \langle 0, f_1\rangle)$ or $\langle (f_0, 0), (0, f_1)\rangle$.

(iii) One can recognize k-linear categories as the categories *enriched* over the category Mod_k of k-modules and k-linear maps.

this point may be skipped

Definition 1.3. A functor $F\colon \mathcal{A} \to \mathcal{A}'$ between two additive (resp. k-linear) categories $\mathcal{A}$ and $\mathcal{A}'$ is *additive* (resp. *k-linear*), if its action on morphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}'}(F(X), F(Y))$$

is a group homomorphism (resp. k-linear map).

Traditionally the term *additive* is reserved for $\mathbf{Z}$-linear categories and $\mathbf{Z}$-linear functors between such categories.

For a morphism $f\colon X \to Y$ in an additive category $\mathcal{A}$ recall, that the *kernel* of f is the equalizer of f and 0 in $\mathcal{A}$, if it exists, and, dually, the *cokernel* of f is the coequalizer of f and 0, if it exists. It is well-known and easy to verify, that the structure maps $\ker f \hookrightarrow X$ and $Y \twoheadrightarrow \operatorname{coker} f$ are a monomorphism and an epimorphism respectively. We also define the *image* and the *coimage* of f to be

$$\begin{aligned}(\operatorname{im} f \to Y) &:= \ker(Y \to \operatorname{coker} f)\\ (X \to \operatorname{coim} f) &:= \operatorname{coker}(\ker f \to X).\end{aligned}$$

Notice that the image and the coimage, just like the kernel and the cokernel, are defined to be morphisms, not only objects. Sometimes these are called their *structure morphisms*.

For a monomorphism $Y \hookrightarrow X$ we will sometimes by abuse of terminology call the cokernel $\operatorname{coker}(Y \to X)$ a *quotient* and denote it by X/Y.

Definition 1.4. A k-linear category $\mathcal{A}$ is *abelian*, if it is closed under kernels and cokernels and satisfies axiom A4.

A4 For any morphism $f\colon X \to Y$ in $\mathcal{A}$ the canonical morphism $\operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f$ is an isomorphism.

$$\begin{tikzcd}
\ker f \arrow[r, hook] & X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[r, two heads] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[u, hook] &
\end{tikzcd}$$

Axiom A4 essentially states that abelian categories are those additive categories possessing all kernels and cokernels in which the first isomorphism theorem holds.

Remark 1.5. We obtain the morphism mentioned in axiom A4 in the following way. Due to $(\operatorname{im} f \to Y) = \ker(Y \to \operatorname{coker} f)$ and the composition $X \to Y \to \operatorname{coker} f$ being 0, there is a unique morphism $X \to \operatorname{im} f$ by the universal property of kernels. The composition $\ker f \to X \to \operatorname{im} f$ then equals 0, by $\operatorname{im} f \hookrightarrow Y$ begin a monomorphism and $\ker f \to X \to Y$ equaling 0. From the universal property of cokernels we then obtain a unique morphism $\operatorname{coim} f \to \operatorname{im} f$, because $(X \to \operatorname{coim} f) = \operatorname{coker}(\ker f \to X)$.

Example 1.6. The default examples of abelian categories are the category of abelian groups Ab or more generally the category of A-modules Mod_A for a commutative ring A. The categories of coherent and quasi-coherent sheaves, $\mathsf{coh}(X)$ and $\mathsf{qcoh}(X)$, on a scheme X are also abelian ([The25, Tag 01BY] and [The25, Tag 077P]). On the other hand the category of (real or complex) vector bundles for example over the real line $\mathbf{R}$ is additive, but not abelian. The last claim is due to the category of vector bundles on $\mathbf{R}$ not being closed under kernels and cokernels.

Definition 1.7. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a sequence of composable morphisms in an abelian category $\mathcal{A}$.

(i) We say this sequence is *exact*, if $g \circ f = 0$ and the induced morphism $\operatorname{im} g \to \ker f$ is an isomorphism.

(ii) Extending (i), a sequence $\cdots \to X^0 \to X^1 \to X^2 \to \cdots$ is *exact*, if any subsequence $X^{i-1} \to X^i \to X^{i+1}$ for $i \in \mathbf{Z}$ is exact.

(iii) Exact sequences of the form $0 \to X \to Y \to Z \to 0$ are called *short exact sequences*.

To relate the definition of exactness with more primitive objects of an abelian category, namely kernels and cokernels, it is not difficult to show that

$$0 \to X \to Y \to Z \text{ is exact, if and only if } (X \to Y) = \ker(Y \to Z),$$

and dually

$$X \to Y \to Z \to 0 \text{ is exact, if and only if } (Y \to Z) = \operatorname{coker}(X \to Y).$$

In the context of abelian categories, functors, which preserve a bit more than just the k-linear structure, are of interest. This brings us to the notion of exactness of additive functors.

Definition 1.8. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(i) F is said to be *left exact*, if $0 \to FX \to FY \to FZ$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

(ii) F is said to be *right exact*, if $FX \to FY \to FZ \to 0$ is exact for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$..

(iii) F is said to be *exact*, if it is both left and right exact.

Remark 1.9. Equivalently, one can also define left exact functors to be exactly those additive functors, which commute with kernels and dually define right exact functors to be additive functors commuting with cokernels.

Example 1.10. For an abelian category $\mathcal{A}$ the hom-functors

$$\operatorname{Hom}_{\mathcal{A}}(A, -)\colon \mathcal{A} \to \mathsf{Mod}_k \quad \text{and} \quad \operatorname{Hom}_{\mathcal{A}}(-, A)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$$

are both left exact. Note that a sequence $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}^{\mathrm{op}}$, if $0 \to Z \to Y \to X \to 0$ is exact in $\mathcal{A}$.

1.2 Triangulated categories

In order to formulate the definition of a triangulated category more concisely, we introduce some preliminary notions. A *category with translation* is a pair $(\mathcal{D}, T)$, where $\mathcal{D}$ is a category and T is an auto-equivalence $T\colon \mathcal{D} \to \mathcal{D}$ called the *translation functor*. If $\mathcal{D}$ is additive or k-linear, T is moreover assumed to be additive or k-linear. We usually denote its action on objects X with $T(X) = X[1]$ and likewise its action on morphisms f with $T(f) = f[1]$. A *triangle* in a category with translation $(\mathcal{D}, T)$ is a triplet of composable morphisms (f, g, h) of category $\mathcal{D}$ taking the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]. \tag{1.2}$$

A *morphism* of triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ is given by a triple of morphisms (u, v, w), for which the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

One can compose morphisms of triangles in the obvious way and the notion of an isomorphism (of triangles) is defined as usual. The following definition as stated is originally due to Verdier, who first introduced it in his thesis [Ver96].

Definition 1.11. A *triangulated category* (*over* k) is a k-linear category with translation $(\mathcal{D}, T)$ equipped with a class of *distinguished triangles*, which is subject to the following four axioms.

TR1 (i) Any triangle isomorphic to a distinguished triangle is also itself distinguished.

(ii) For any X the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished.

(iii) For any morphism $f\colon X \to Y$ there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

The object Z is sometimes called the *cone of* f.

TR2 The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

TR3 Given two distinguished triangles and morphisms $u\colon X \to X'$ and $v\colon Y \to Y'$, depicted in the solid diagram below

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, dashed, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

satisfying $v \circ f = f' \circ u$, there exists a (in general non-unique) morphism $w\colon Z \to Z'$, for which (u, v, w) is a morphism of triangles i.e. the diagram above commutes.

TR4

Will be added at some point

Omitting the so-called *octahedral* axiom TR4 we arrive at the definition of a *pre-triangulated category*. These are essentially the categories we will be working with, since we will never use nor verify the axiom TR4. We will nevertheless use the terminology *"triangulated category"* in part to remain consistent with the existent literature and more importantly because our categories will be honest triangulated categories anyway.

Remark 1.12. The object Z, called the *cone of* f, in axiom TR1 (iii) is unique up to isomorphism. This is seen through the use of axiom TR3 in combination with example 1.21 and the five lemma [KS06, Lemma 8.3.13] from homological algebra in Mod_k.

Proposition 1.13. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *be a distinguished triangle in a triangulated category* $\mathcal{D}$*. Then* $g \circ f = 0$*.*

Proof. First, by axiom TR2, triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished. By axiom TR1 (ii), triangle $Z \xrightarrow{\mathrm{id}_Z} Z \to 0 \to Z[1]$ is distinguished and, by axiom TR3, there exits a morphism from the first triangle to the second one depicted in the diagram below.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[r, "h"] \arrow[d, "\mathrm{id}_Z"] & X[1] \arrow[r, "-f[1]"] \arrow[d, dashed] & Y[1] \arrow[d, "g[1]"'] \\
Z \arrow[r, "\mathrm{id}_Z"] & Z \arrow[r] & 0 \arrow[r] & Z[1]
\end{tikzcd}$$

The right most square says $g[1] \circ (-f[1]) = 0$, from which we conclude that $g \circ f = 0$, after applying a quasi-inverse of the translation functor. □

Definition 1.14. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with translation functors T and T' respectively. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is defined to be *triangulated* or *exact*, if the following two conditions are satisfied.

Maybe I will pick only triangulated, to not overload the term exact...

(i) There exists a natural isomorphism of functors

$$\eta\colon F \circ T \simeq T' \circ F.$$

(ii) For every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{D}$, the triangle

$$F(X) \xrightarrow{Ff} F(Y) \xrightarrow{Fg} F(Z) \to F(X)[1]$$

is distinguished in $\mathcal{D}'$, where the last morphism is obtained as the composition $F(Z) \xrightarrow{Fh} F(X[1]) \xrightarrow{\eta_X} F(X)[1]$.

Remark 1.15. The condition on F being an *additive* functor in the above definition is actually unnecessary and follows from conditions (i) and (ii) [The25, Tag 05QY].

this remark may be omitted.

Definition 1.16. Let $\mathcal{D}_0$ and $\mathcal{D}$ be triangulated categories such that $\mathcal{D}_0$ is a subcategory of $\mathcal{D}$. Then $\mathcal{D}_0$ is a *triangulated subcategory* of $\mathcal{D}$, if the inclusion functor $i\colon \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is a triangulated functor.

Proposition 1.17. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}_0 \subseteq \mathcal{D}$ a full additive subcategory of $\mathcal{D}$. Assume that the translation functor T of $\mathcal{D}$ restricts to an autoequivalence T_0 of $\mathcal{D}_0$ and that for every distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}$, where f belongs to $\mathcal{D}_0$, the object Z is isomorphic to some object of $\mathcal{D}_0$. Then $\mathcal{D}_0$ is naturally equipped with a triangulated structure, for which it becomes a* triangulated *subcategory of $\mathcal{D}$.*

Proof. We take T_0 to be the translation functor on $\mathcal{D}_0$ and take distinguished triangles in $\mathcal{D}_0$ to be all the triangles of $\mathcal{D}_0$, for which there exists an isomorphism of triangles to some distinguished triangle of $\mathcal{D}$. Then $\mathcal{D}_0$ is clearly triangulated and the inclusion functor $i\colon \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is triangulated. □

Definition 1.18. Triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ are said to be *equivalent* (as triangulated categories), if there are triangulated functors $F\colon \mathcal{D} \to \mathcal{D}'$ and $G\colon \mathcal{D}' \to \mathcal{D}$, such that $G \circ F \simeq \mathrm{id}_{\mathcal{D}}$ and $F \circ G \simeq \mathrm{id}_{\mathcal{D}'}$ and we call F and G *triangulated equivalences.*

Proposition 1.19. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, which is an equivalence of categories with a quasi-inverse $G\colon \mathcal{D}' \to \mathcal{D}$. Then G is also a triangulated functor.*

Proof. This is a consequence of [Huy06, Proposition 1.41], as equivalences of categories are special instances of adjunctions. □

As a consequence, two triangulated categories are equivalent (as triangulated categories) whenever there exists a fully faithful essentially surjective triangulated functor from one to the other.

Definition 1.20. Let $H\colon \mathcal{D} \to \mathcal{A}$ be an additive functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$. We say H is a *cohomological functor*, if for every distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the induced long sequence in $\mathcal{A}$

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to H(Y[1]) \to H(Z[1]) \to \cdots \tag{1.3}$$

is exact.

Construction of the long sequence (1.3) is extremely simple as opposed to other known long exact sequences assigned to certain short exact sequences (e.g. of sheaves or complexes) as all the complexity is actually captured within the distinguished triangle already. All one has to do is unwrap the triangle $X \to Y \to Z \to X[1]$ into the following chain of composable morphisms

$$\cdots \to Y[-1] \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to Z[1] \to X[2] \to \cdots$$

and apply functor H over it.

Example 1.21. For any object W in a triangulated category $\mathcal{D}$ the functors

$$\mathrm{Hom}_{\mathcal{D}}(W, -)\colon \mathcal{D} \to \mathsf{Mod}_k \text{ and } \mathrm{Hom}_{\mathcal{D}}(-, W)\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Mod}_k$$

are cohomological[1]. Let us verify the first claim. Consider the long sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to \mathrm{Hom}_{\mathcal{D}}(W, X[1]) \to \cdots$$

[1] With claiming that $\mathrm{Hom}_{\mathcal{D}}(-, W)$ is cohomological we are being slightly imprecise, as we have not clarified what the triangulated structure on $\mathcal{D}^{\mathrm{op}}$ is. We take it to be the most obvious one (cf. [Mil, Chapter 1, §1.2]).

arising from a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. Since the translation functor on $\mathcal{D}$ and the axiom TR2 allow us to turn this triangle and still end up with a distinguished triangle, it suffices to verify only that

$$\mathrm{Hom}_{\mathcal{D}}(W, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{D}}(W, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{D}}(W, Z) \quad \text{is exact.}$$

By proposition 1.13 we see that $\operatorname{im} f_* \subseteq \ker g_*$, so we are left to prove the other inclusion. Suppose $v\colon W \to Y$ is in $\ker g_*$. Then axiom TR3 (together with TR2 and TR1 (ii)) asserts the existence of a morphism $u\colon W \to X$ making the diagram below commutative.

As this is the first argument with the axioms of triangulated cats, I could maybe expand it a little bit more.

$$\begin{tikzcd}
W \arrow[r, "\mathrm{id}_W"] \arrow[d, dashed, "u"'] & W \arrow[r] \arrow[d, "v"'] & 0 \arrow[r] \arrow[d, "w"'] & W[1] \arrow[d, "u[1]"'] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1]
\end{tikzcd}$$

From the commutative square on the left it is then clear that $v \in \operatorname{im} f_*$, as $v = f \circ u$.

Lemma 1.22. *Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then f is an isomorphism if and only if $Z \simeq 0$.*

Proof. We argue with a chain of equivalences. Observe that f is an isomorphism if and only if $f_*\colon \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y)$ and $f^*\colon \mathrm{Hom}_{\mathcal{D}}(Y, W) \to \mathrm{Hom}_{\mathcal{D}}(X, W)$ are isomorphisms for all objects W of $\mathcal{D}$. From the existence of long exact sequences established in the previous example the latter condition is equivalent to $\mathrm{Hom}_{\mathcal{D}}(W, Z) = 0$ and $\mathrm{Hom}_{\mathcal{D}}(Z, W) = 0$ for all W, which in turn is equivalent to $Z \simeq 0$. □

I think Yoneda can be used here to enable us to consider only f_* instead of both f_* and f^*. i.e. f_* iso for all W iff f iso

Lemma 1.23. *Let triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ be distinguished in a triangulated category $\mathcal{D}$. Then their direct sum*

$$X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} (X \oplus X')[1]$$

is also distinguished in $\mathcal{D}$.

Proof. □

Lemma 1.24. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$ and assume $h = 0$. Then there exists an isomorphism $Y \simeq X \oplus Z$, for which the next diagram commutes.*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X \oplus Z \arrow[r] \arrow[d, "\sim"'] & Z \arrow[r, "0"] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "0"] & X'[1]
\end{tikzcd}$$

Proof. □

The following are all specialized results characterizing, when a triangulated functor is an equivalence. They will be used in a fundamental way in the proof of the Derived Torelli theorem. In particular we highlight Proposition 1.26, Proposition 1.33 and Corollary 1.31, which will be directly used in the proof. We mention that all the results of this part can be found in [Huy06, Chapter 1].

Definition 1.25. Let $\mathcal{D}$ be a triangulated category. A collection of objects Ω in $\mathcal{D}$ forms a *spanning class* of $\mathcal{D}$, if for all objects X of $\mathcal{D}$ the following two conditions are satisfied.

(i) If $\mathrm{Hom}_{\mathcal{D}}(U, X[i]) = 0$ for all $U \in \Omega$ and all $i \in \mathbf{Z}$, then $X \simeq 0$.

(ii) If $\mathrm{Hom}_{\mathcal{D}}(X, U[i]) = 0$ for all $U \in \Omega$ and all $i \in \mathbf{Z}$, then $X \simeq 0$.

Proposition 1.26. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories. Assume F has left and right adjoints $G \dashv F \dashv H$. Suppose $\mathcal{D}$ contains a spanning class Ω and that F induces isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}}(U, V[i]) \to \mathrm{Hom}_{\mathcal{D}'}(F(U), F(V[i])) \tag{1.4}$$

for all $U, V \in \Omega$. Then F is fully faithful.

Proof. For any X, Y of $\mathcal{D}$ we will show that the morphism action of F

$$\mathrm{Hom}_{\mathcal{D}}(X, Y[i]) \to \mathrm{Hom}_{\mathcal{D}'}(F(X), F(Y[i]))$$

is an isomorphism. The two adjunctions $G \dashv F \dashv H$ give rise to the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(X, Y) \arrow[r, "(\eta_Y)_*"] \arrow[d, "(\varepsilon_X)^*"'] \arrow[dr, "F"] & \mathrm{Hom}_{\mathcal{D}}(X, HF(Y)) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathcal{D}}(GF(X), Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{D}'}(F(X), F(Y)),
\end{tikzcd} \tag{1.5}$$

where $\eta_Y\colon Y \to HF(Y)$ denotes the unit of the adjunction $F \dashv H$ and $\varepsilon_X\colon GF(X) \to X$ denotes the counit of the adjunction $G \dashv F$.

First we show, that ε_U is an isomorphism for every $U \in \Omega$. Indeed, as ε_U fits into a distinguished triangle

$$GF(U) \xrightarrow{\varepsilon_U} U \longrightarrow Z \longrightarrow (GF(U))[1],$$

the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(-, V)$, for $V \in \Omega$, induces a long exact sequence

$$\begin{aligned}\cdots \leftarrow \mathrm{Hom}_{\mathcal{D}}(GF(U), V) \xleftarrow{(\varepsilon_U)^*} \mathrm{Hom}_{\mathcal{D}}(U, V) \leftarrow \\ \leftarrow \mathrm{Hom}_{\mathcal{D}}(Z, V) \leftarrow \mathrm{Hom}_{\mathcal{D}}(GF(U)[1], V) \leftarrow \cdots .\end{aligned} \tag{1.6}$$

By assumption (1.4) is a bijection for all $U, V \in \Omega$ and $i \in \mathbf{Z}$, thus, considering the diagram (1.5), where we set $X = U$ and $Y = V$, we see that $(\varepsilon_U)^*$, along with $(\varepsilon_U[i])^*$ for all $i \in \mathbf{Z}$, is an isomorphism. From the long exact sequence (1.6) we then conclude, that $\mathrm{Hom}_{\mathcal{D}}(Z[i], V) = 0$ for all $i \in \mathbf{Z}$ and as $V \in \Omega$ belonging to a spanning class of $\mathcal{D}$ was arbitrary, $Z \simeq 0$, showing ε_U is an isomorphism by Lemma 1.22.

Secondly, we show that η_Y is an isomorphism for all objects Y of $\mathcal{D}$. This is done in a very similar manner. Considering now the distinguished triangle

$$Y \xrightarrow{\eta_Y} HF(Y) \longrightarrow Z \longrightarrow Y[1],$$

we apply the cohomological functor $\mathrm{Hom}_{\mathcal{D}}(U, -)$, for $U \in \Omega$, to obtain the long exact sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathcal{D}}(U, Y) \xrightarrow{(\eta_Y)_*} \mathrm{Hom}_{\mathcal{D}}(U, HF(Y)) \to \\ \to \mathrm{Hom}_{\mathcal{D}}(U, Z) \to \mathrm{Hom}_{\mathcal{D}}(U, Y[1]) \to \cdots .\end{aligned} \tag{1.7}$$

Setting $X = U$ in diagram (1.5), we see that ε_U being an isomorphism implies $(\eta_Y)_*$ is an isomorphism. The long exact sequence (1.7) then establishes $\mathrm{Hom}_{\mathcal{D}}(U, Z[i]) = 0$ for all $i \in \mathbf{Z}$, thus showing $Z \simeq 0$. Consequently η_Y is an isomorphism by Lemma 1.22.

Lastly, looking back at diagram (1.5), η_Y being an isomorphism, for all objects Y of $\mathcal{D}$, proves, that the morphism action of F is an isomorphism i.e. F is fully faithful. $\square$

Definition 1.27. Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}_0$, $\mathcal{D}_1$ its triangulated subcategories. We say $\mathcal{D}$ *decomposes* into $\mathcal{D}_0$ and $\mathcal{D}_1$ if the following three conditions are met.

(i) Categories $\mathcal{D}_0$ and $\mathcal{D}_1$ contain objects not isomorphic to 0.

(ii) Every object X of $\mathcal{D}$ fits into a distinguished triangle (in $\mathcal{D}$) of the form

$$Y_0 \to X \to Y_1 \to Y_0[1],$$

where Y_0 and Y_1 belong to $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively.

(iii) For all objects Y_0 of $\mathcal{D}_0$ and Y_1 of $\mathcal{D}_1$ it holds that

$$\mathrm{Hom}_{\mathcal{D}}(Y_0, Y_1) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_0) = 0.$$

Additionally, $\mathcal{D}$ is called *indecomposable*, if it can not be decomposed in this way.

Proposition 1.28. *Suppose a triangulated category $\mathcal{D}$ decomposes into $\mathcal{D}_0$ and $\mathcal{D}_1$. Then every object X of $\mathcal{D}$ is isomorphic to $Y_0 \oplus Y_1$ for some objects Y_0 and Y_1 belonging to $\mathcal{D}_0$ and $\mathcal{D}_1$, respectively.*

Proof. This is a direct consequence of the definition and Lemma 1.24. $\square$

Lemma 1.29. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a fully faithful triangulated functor between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$. Suppose F has a right adjoint $F \dashv H\colon \mathcal{D}' \to \mathcal{D}$. Then F is an equivalence if and only if for any object Z in $\mathcal{D}'$ the condition $H(Z) \simeq 0$ implies $Z \simeq 0$.*

Proof. ($\Rightarrow$) Let Y belong to $\mathcal{D}'$. Since F is part of an equivalence, the counit $\varepsilon_Y : FH(Y) \to Y$ is an isomorphism, thus, as $H(Y) \simeq 0$ implies $FH(Y) \simeq 0$, we see that $Y \simeq 0$.
($\Leftarrow$) Recall, that whenever F is part of an adjunction $F \dashv H$, it is fully faithful if and only if the unit $\eta\colon \mathrm{id}_{\mathcal{D}} \Longrightarrow HF$ is an isomorphism (this is easily seen from digram (1.5) or [The25, Tag 07RB]). Thus it suffices to show that the counit $\varepsilon\colon FH \Longrightarrow \mathrm{id}_{\mathcal{D}'}$ is an isomorphism. Pick an object Y of $\mathcal{D}'$ and extend $\varepsilon_Y : FH(Y) \to Y$ to a distinguished triangle $FH(Y) \to Y \to Z \to FH(Y)[1]$ in $\mathcal{D}'$. After applying the triangulated functor H (cf. [Huy06, Prop. 1.41]) to the latter triangle, we obtain

$$HFH(Y) \xrightarrow{H(\varepsilon_Y)} H(Y) \longrightarrow H(Z) \longrightarrow HFH(Y)[1].$$

By the triangle identity relating units and counits of an adjunction [The25, Tag 0GLL], we have

$$H(\varepsilon_Y) \circ \eta_{H(Y)} = \mathrm{id}_{H(Y)}, \tag{1.8}$$

and as observed earlier, since $\eta_{H(Y)}$ is an isomorphism, $H(\varepsilon_Y)$ is as well. It follows by 1.22 that $H(Z) \simeq 0$, from which, by the assumption, $Z \simeq 0$ follows. Finally, by 1.22 again, ε_Y is an isomorphism, proving our claim. $\square$

Proposition 1.30. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a fully faithful triangulated functor between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ having both left and right adjoints $G \dashv F \dashv H$. Further assume $\mathcal{D}$ has objects not isomorphic to* 0 *and that $\mathcal{D}'$ is indecomposable. Then F is an equivalence if and only if for all objects Y in $\mathcal{D}'$ the condition $H(Y) \simeq 0$ implies $G(Y) \simeq 0$.*

Proof. If F is an equivalence, the condition is satisfied because its quasi-inverse is both a left and right adjoint to F and adjoints are unique up to natural isomorphisms. Suppose now that the condition is satisfied and we show, that F is essentially surjective.

We will first construct two triangulated subcategories of $\mathcal{D}'$, where one of which will turn out to contain only trivial objects by indecomposability of $\mathcal{D}'$. Let $\mathcal{D}'_0$ the full subcategory of $\mathcal{D}'$ consisting of all objects Y, for which $H(Y) \simeq 0$. As H is triangulated $\mathcal{D}'_0$ is a full *triangulated* subcategory of $\mathcal{D}'$. We will show this category contains only trivial objects. Let $\mathcal{D}'_1$ be the full subcategory of $\mathcal{D}'$ spanned on objects of the form $F(X)$ for some object X of $\mathcal{D}'$. Similarly, as F is triangulated, $\mathcal{D}'_1$ is a triangulated subcategory of $\mathcal{D}'$, sometimes called the *essential image of F*. We now show, that $\mathcal{D}'_0$ and $\mathcal{D}'_1$ satisfy conditions (ii) and (iii) of Definition 1.27.

To verify (ii) pick any object Y of $\mathcal{D}'$ and consider the counit ε_Y within a distinguished triangle

$$FH(Y) \xrightarrow{\varepsilon_Y} Y \to Z \to FH(Y)[1] \tag{1.9}$$

of $\mathcal{D}'$. By definition $FH(Y)$ belongs to $\mathcal{D}'_1$. On the other hand, to see that Z belongs to $\mathcal{D}'_0$, we map the triangle to $\mathcal{D}$ via the functor H, to conclude that $H(\varepsilon_Y)$ is an isomorphism, by the triangle identity (1.8) and the fact that F is fully faithful (thus $\eta_{H(Y)}$ is an isomorphism). Then $H(Z) \simeq 0$ by Lemma 1.22, showing that Z belongs to $\mathcal{D}'_0$.

To see why (iii) holds true pick objects Y_0 and Y_1 of $\mathcal{D}'_0$ and $\mathcal{D}'_1$ respectively. Then by definition there exists an object X of $\mathcal{D}$, such that $F(X) \simeq Y_1$, so we compute

$$\operatorname{Hom}_{\mathcal{D}'}(Y_1, Y_0) \simeq \operatorname{Hom}_{\mathcal{D}'}(F(X), Y_0) \simeq \operatorname{Hom}_{\mathcal{D}}(X, H(Y_0)) = 0$$

because $H(Y_0) \simeq 0$. As per the assumption, $H(Y_0) \simeq 0$ implies $G(Y_0) \simeq 0$, so we see that

$$\operatorname{Hom}_{\mathcal{D}'}(Y_0, Y_1) \simeq \operatorname{Hom}_{\mathcal{D}'}(Y_0, F(X)) \simeq \operatorname{Hom}_{\mathcal{D}}(G(Y_0), X) = 0$$

Since $\mathcal{D}$ contains non-trivial objects and F is fully faithful, the category $\mathcal{D}'_1$ must contain a non-trivial object, namely an F-image of any non-trivial object of $\mathcal{D}$. Since by assumption $\mathcal{D}'$ is indecomposable, the category $\mathcal{D}'_0$ only contains trivial objects. In other words, if $H(Y) \simeq 0$ for some object Y of $\mathcal{D}'$, then $Y \simeq 0$.

Finally, to show that F is essentially surjective, we prove that the counit $\varepsilon_Y\colon FH(Y) \to Y$ is an isomorphism for every object Y of $\mathcal{D}'$. Once again considering the image of the distinguished triangle (1.9) via the functor H, we have already established that $H(Z) \simeq 0$. The latter implies $Z \simeq 0$ and finally, by Lemma 1.22, ε_Y is an isomorphism. □

Corollary 1.31. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a fully faithful triangulated functor between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ with $G\colon \mathcal{D}' \to \mathcal{D}$ simultaneously being its left and right adjoint $G \dashv F \dashv G$. Further assume $\mathcal{D}$ has objects not isomorphic to* 0 *and that $\mathcal{D}'$ is indecomposable. Then F is an equivalence.*

For the last part of this section let k denote a field.

Definition 1.32. Let $\mathcal{D}$ be a triangulated category over k with $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ being a finite dimensional k-vector space for all objects X and Y of $\mathcal{D}$. A *Serre functor* is a triangulated autoequivalence $S\colon \mathcal{D} \to \mathcal{D}$ of $\mathcal{D}$, such that for all objects X and Y there exists an isomorphism of k-vector spaces

$$\sigma_{X,Y}\colon \operatorname{Hom}_{\mathcal{D}}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(Y, S(X))^*,$$

which is natural in both arguments X and Y, thus forming a natural isomorphism of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Mod}_k$.

Proposition 1.33. *Suppose $F\colon \mathcal{D} \to \mathcal{D}'$ is a triangulated functor of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ endowed with Serre functors $S_{\mathcal{D}}$ and $S_{\mathcal{D}'}$ respectively. Assume F has a left adjoint $G \dashv F$. Then $S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}$ is right adjoint to F.*

Proof. For any two objects X of $\mathcal{D}$ and Y of $\mathcal{D}'$ we compute

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}'}(F(X),\ Y) &\simeq \mathrm{Hom}_{\mathcal{D}'}(Y,\ S_{\mathcal{D}'}(F(X)))^* \\
&\simeq \mathrm{Hom}_{\mathcal{D}'}(S_{\mathcal{D}'}^{-1}(Y),\ F(X))^* \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y)),\ X)^* \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(X,\ S_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y))))^{**} \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(X,\ S_{\mathcal{D}}(G(S_{\mathcal{D}'}^{-1}(Y)))).
\end{aligned}$$

For this is a chain of natural equivalences of functors in X and Y, it follows that

$$F \dashv S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}. \qquad \square$$

1.3 Categories of complexes

In order to define derived categories of an additive category $\mathcal{A}$ we first introduce the category of complexes and the homotopic category of complexes of $\mathcal{A}$. We will equip the latter with a triangulated structure. We also recall cohomology of chain complexes and introduce quasi-isomorphisms. Throughout this section $\mathcal{A}$ will be a fixed additive or k-linear category and we also mention that we will be using the cohomological indexing convention.

As a preliminary we introduce the *graded objects* of $\mathcal{A}$ to be $\mathbf{Z}$-indexed sequences $A^\bullet = (A^i)_{i\in\mathbf{Z}}$ of objects A^i of $\mathcal{A}$. A *morphism* $f\colon A^\bullet \to B^\bullet$ *of degree* $k \in \mathbf{Z}$ is a sequence $(f^i\colon A^i \to B^{i+k})_{i\in\mathbf{Z}}$ of morphisms of $\mathcal{A}$. The set of all degree k morphisms will be denoted with $\mathrm{Hom}^k_{\mathcal{A}}(A^\bullet, B^\bullet)$. Bundling up all this data, we obtain the **Z**-*graded category of* $\mathcal{A}$ denoted by $\mathcal{A}^{\mathbf{Z}}$, consisting of graded objects of $\mathcal{A}$ and morphisms of all integer degrees, thus

$$\mathrm{Hom}_{\mathcal{A}^{\mathbf{Z}}}(A^\bullet, B^\bullet) = \prod_{k\in\mathbf{Z}} \mathrm{Hom}^k_{\mathcal{A}}(A^\bullet, B^\bullet).$$

The category $\mathcal{A}^{\mathbf{Z}}$ can also be thought of as the functor category from the free category of $\mathbf{Z}$ to $\mathcal{A}$, hence the notation $\mathcal{A}^{\mathbf{Z}}$.

This is introduced so I can say the differential is a morphism of degree 1, homotopy is a morphism of degree -1, and makes defining $\mathrm{Hom}^\bullet$ just slightly easier.

Category of chain complexes

By a *chain complex* in $\mathcal{A}$ we mean a collection of objects and morphisms

$$A^\bullet = \left((A^i)_{i\in\mathbf{Z}}, (d^i_A\colon A^i \to A^{i+1})_{i\in\mathbf{Z}}\right),$$

where A^i are objects and d^i are morphisms of $\mathcal{A}$, called *differentials*, subject to equations $d^{i+1} \circ d^i = 0$, for all $i \in \mathbf{Z}$. A complex is *bounded from below* (resp. *bounded from above*), if there exists $i_0 \in \mathbf{Z}$ for which $A^i \simeq 0$ for all $i \leq i_0$ (resp. $i \geq i_0$) and is *bounded*, if it is both bounded from below and bounded from above. A *chain map* between two chain complexes $A^\bullet$ and $B^\bullet$ in $\mathcal{A}$ is a collection of morphisms in $\mathcal{A}$

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}},$$

for which $f^{i+1} \circ d_A^i = d_B^i \circ f^i$ holds for all $i \in \mathbf{Z}$. This may diagrammatically be described by the following commutative ladder.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Sometimes we will denote a complex just by A, B,... insted of $A^\bullet$, $B^\bullet$,... to simplify notation.

Next we define the *category of chain complexes* in $\mathcal{A}$, denoted by $\mathsf{Ch}(\mathcal{A})$, as the following additive category.

Objects:	chain complexes in $\mathcal{A}$.
Morphisms:	$\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A, B)$ is the set of chain maps $A \to B$, equipped with a group structure inherited from $\mathcal{A}$ by applying operations componentwise.

The composition law is defined componentwise and is clearly associative and bilinear. The identity morphisms $\mathrm{id}_{A^\bullet}$ are defined to be $(\mathrm{id}_{A^i})_{i\in\mathbf{Z}}$. The complex $\cdots \to 0 \to 0 \to \cdots$ plays the role of the zero object in $\mathsf{Ch}(\mathcal{A})$ and the biproduct of complexes A and B exists and is witnessed by the chain complex

$$A \oplus B = \Big((A^i \oplus B^i)_{i\in\mathbf{Z}}, (d_A^i \oplus d_B^i)_{i\in\mathbf{Z}} \Big),$$

together with the canonical projection and injection morphisms arising from biproducts componentwise.

Additionally, we also define the following full additive subcategories of $\mathsf{Ch}(\mathcal{A})$.

$\mathsf{Ch}^+(\mathcal{A})$	*Category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$\mathsf{Ch}^-(\mathcal{A})$	*Category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$\mathsf{Ch}^b(\mathcal{A})$	*Category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

Remark 1.34. Whenever $\mathcal{A}$ is k-linear, all the categories of complexes $\mathsf{Ch}^*(\mathcal{A})$ become k-linear as well in the obvious way.

On all the categories of complexes mentioned above, we can now define the translation functor

$$T\colon \mathsf{Ch}^*(\mathcal{A}) \to \mathsf{Ch}^*(\mathcal{A})$$

given by its action on objects and morphisms as follows.

Objects:	$T(A^\bullet) = A^\bullet[1]$ is the chain complex with $(A^\bullet[1])^i := A^{i+1}$ and differentials $d_{A[1]}^i = -d_A^{i+1}$.
Morphisms:	For a chain map $f^\bullet\colon A^\bullet \to B^\bullet$ we define $f^\bullet[1]$ to have component maps $(f^\bullet[1])^i = f^{i+1}$.

The translation functor T thus acts on a complex $A^\bullet$ by twisting its differential by a sign and shifting it one step to the *left*, which is pictured below.

$$
\begin{array}{cccccccccccc}
 & \cdots & & -1 & & 0 & & 1 & & 2 & & \cdots \\
A^\bullet & \cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
A^\bullet[1] & \cdots & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow & \cdots
\end{array}
$$

Remark 1.35. We remark that the translation functor T is clearly also additive or k-linear, whenever $\mathcal{A}$ is additive or k-linear.

Since T is an auto-equivalence there exists a quasi-inverse T^{-1} to T, which is defined and unique up to a natural isomorphism. We may then speak of T^k for any $k \in \mathbf{Z}$, whose action on a complex $A^\bullet$ is described by $(A^\bullet[k])^i = A^{i+k}$ with differential $d^i_{A[k]} = (-1)^k d^{i+k}_A$.

Homotopy category of chain complexes

In this subsection we construct the homotopy category of chain complexes associated to a given additive category $\mathcal{A}$ and equip it with a triangulated structure. The main motivation for its introduction in this work is the fact that we will later on use it to construct the derived category of $\mathcal{A}$. In particular the homotopy category of $\mathcal{A}$, as opposed to the category of complexes[2] $\mathsf{Ch}(\mathcal{A})$, can be enhanced with a triangulated structure which will afterwards descend to the level of derived categories.

Definition 1.36. Let $f^\bullet$ and $g^\bullet$ be two chain maps in $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$. We define $f^\bullet$ and $g^\bullet$ to be *homotopic*, if there exists a collection of morphisms $(h^i \colon A^i \to B^{i-1})_{i\in\mathbf{Z}}$, satisfying

$$f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$$

for all $i \in \mathbf{Z}$. The collection of morphisms $(h^i)_{i\in\mathbf{Z}}$ is called a *homotopy* and we denote $f^\bullet$ and $g^\bullet$ being homotopic by $f^\bullet \simeq g^\bullet$.

$$
\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}_A"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d^i_A"] \arrow[d, "f^i"'] \arrow[dl, "h^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, "h^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}_B"] & B^i \arrow[r, "d^i_B"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}
$$

We say $f^\bullet$ is *nullhomotopic*, if $f^\bullet \simeq 0$.

Lemma 1.37. *Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be complexes in $\mathsf{Ch}(\mathcal{A})$ and let $f, f' \in \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$ and $g, g' \in \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(B^\bullet, C^\bullet)$ be chain maps.*

(i) The subset of all nullhomotopic chain maps in $A^\bullet \to B^\bullet$ forms a submodule of $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$.

(ii) If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

Pick one notation convention, sometimes its f, f', sometimes its f_0, f_1, ...

Proof. For (i) see [Mil, Chapter 3, 1.3.1]. Claim (ii) is a direct consequence of [Mil, Chapter 3, 1.3.2]. □

[2] It is still possible to construct the derived category of $\mathcal{A}$ without passing through the homotopy category of complexes, however equipping it with a triangulated structure in that case becomes less elegant.

The homotopy category of complexes in $\mathcal{A}$, denoted by $\mathsf{K}(\mathcal{A})$, is defined to be an additive category consisting of

$$\begin{aligned} \textit{Objects:} \quad & \text{chain complexes in } \mathcal{A}. \\ \textit{Morphisms:} \quad & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y) := \mathrm{Hom}_{\mathsf{Ch}(\mathcal{A})}(X,Y)/_{\simeq} \end{aligned}$$

The composition law descends to the quotient by lemma 1.37 (ii), i.e. $[g]\circ[f] := [g\circ f]$, for composable $[f]$ and $[g]$, and for any $A^\bullet$ the identity morphism is defined to be $[\mathrm{id}_{A^\bullet}]$. All the hom-sets $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y)$ are k-modules by lemma 1.37 (i) and compositions are k-bilinear maps. The biproduct of two complexes $A^\bullet$ and $B^\bullet$ consists of an object $A^\bullet \oplus B^\bullet$ corresponding to the usual biproduct in $\mathsf{Ch}(\mathcal{A})$ together with the homotopy classes of structure maps of its $\mathsf{Ch}(\mathcal{A})$-counterpart.

Triangulated structure on $\mathsf{K}(\mathcal{A})$. We now shift[3] our focus to the construction of a triangulated structure on $\mathsf{K}(\mathcal{A})$. The translation functor $T\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ is defined on objects and morphisms in the following way.

$$\begin{aligned} \textit{Objects:} \quad & A^\bullet \longmapsto A^\bullet[1]. \\ \textit{Morphisms:} \quad & [f^\bullet] \longmapsto [f^\bullet[1]]. \end{aligned}$$

This assignment is clearly well defined on morphisms, as $f \simeq f'$ implies $f[1] \simeq f'[1]$.

The other piece of data required to obtain a triangulated category is a collection of *distinguished triangles*. To describe what distinguished triangles are in the case of $\mathsf{K}(\mathcal{A})$, we must first introduce the mapping cone of a morphism of complexes and to this end we will for a moment step outside the scope of the homotopy category of complexes back into the category of chain complexes.

Definition 1.38. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathsf{Ch}(\mathcal{A})$. The complex $C(f)^\bullet$ is specified by the collection of objects

$$C(f)^i := A^{i+1} \oplus B^i$$

and differentials

$$d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1]^i & d_B^i \end{pmatrix}, \tag{1.10}$$

for all $i \in \mathbf{Z}$, is called the *cone of* f.

Remark 1.39. Using matrix notation here might be a bit misleading at first. Formally, matrix (1.10) represents the morphism

$$\big(\langle -d_A^{i+1}, 0\rangle, \langle f^{i+1}, d_B^i\rangle\big) = \Big\langle (-d_A^{i+1}, f^{i+1}), (0, d_B^i)\Big\rangle$$

according to our convention for defining morphisms from and into biproducts. A simple matrix calculation, using the fact that f is a chain map and d_A, d_B differentials, shows that $C(f)^\bullet$ is indeed a chain complex.

Remark 1.40. The naming convention of course comes from topology, where one can show that the singular chain complex associated to the topological mapping cone $M(f)$ of a continuous map $f\colon X \to Y$ is chain homotopically equivalent to the cone of the chain map induced by f between singular chain complexes of X and Y.

this remark can be skipped

[3] Pun intended.

Along with the cone of a chain map f we also introduce two chain maps

$$\tau_f \colon B^\bullet \to C(f)^\bullet,$$

given by the collection $\left(\tau_f^i \colon B^i \to A^{i+1} \oplus B^i\right)_{i \in \mathbf{Z}}$, where τ_f^i is the canonical injection into the biproduct for all $i \in \mathbf{Z}$, and

$$\pi_f \colon C(f)^\bullet \to A^\bullet[1],$$

given by the collection $\left(\pi_f^i \colon A^{i+1} \oplus B^i \to A^{i+1}\right)_{i \in \mathbf{Z}}$, where π_f^i is the canonical projection from the biproduct for all $i \in \mathbf{Z}$.

Definition 1.41. We define any triangle in $\mathsf{K}(\mathcal{A})$ isomorphic to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau_f} C(f)^\bullet \xrightarrow{\pi_f} A[1]^\bullet \tag{1.11}$$

to be distinguished.

Proposition 1.42. *The homotopy category* $\mathsf{K}(\mathcal{A})$ *together with the translation functor* $T \colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ *and distinguished triangles defined above is a triangulated category.*

Proof. □

Very detailed proof is given in [Mil, Ch. 3, 2.1.]

As in the case of categories of complexes in $\mathcal{A}$, we can also define the following full additive subcategories of $\mathsf{K}(\mathcal{A})$.

$\mathsf{K}^+(\mathcal{A})$	*Homotopy category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$\mathsf{K}^-(\mathcal{A})$	*Homotopy category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$\mathsf{K}^b(\mathcal{A})$	*Homotopy category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

For convenience we will in practice assume that unless otherwise stated our complexes $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ will be supported in $\mathbf{Z}_{\geq 0}$ i.e. we will assume $A^i \simeq 0$ for $i < 0$.

Remark 1.43. By proposition 1.17 all the subcategories $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$, $\mathsf{K}^b(\mathcal{A})$ of $\mathsf{K}(\mathcal{A})$ are triangulated.

Cohomology

A very important invariant of a chain complex in the homotopy category, which measures the extent to which it fails to be exact, is its cohomology. Here we are no longer assuming $\mathcal{A}$ is just k-linear, but abelian, since we will need kernels and cokernels to exists. For a chain complex $A^\bullet$ in $\mathsf{Ch}(\mathcal{A})$ and $i \in \mathbf{Z}$ we define its *i-th cohomology* to be

$$H^i(A^\bullet) := \operatorname{coker}(\operatorname{im} d^{i-1} \to \ker d^i).$$

Remark 1.44. A computation with universal properties inside an abelian category shows, that the following are all equivalent ways of defining the cohomology of a complex as well

$$\begin{aligned} H^i(A^\bullet) &:= \operatorname{coker}(\operatorname{im} d^{i-1} \to \ker d^i) \simeq \ker(\operatorname{coker} d^{i-1} \to \operatorname{im} d^i) \\ &\simeq \operatorname{coker}(A^{i-1} \to \ker d^i) \simeq \ker(\operatorname{coker} d^{i-1} \to A^i). \end{aligned}$$

See [KS06, Def. 8.3.8. (i)].

For a morphism of complexes $f^\bullet \colon A^\bullet \to B^\bullet$ one can also define a morphism

$$H^i(f^\bullet) \colon H^i(A^\bullet) \to H^i(B^\bullet)$$

in $\mathcal{A}$, because $f^\bullet$ induces maps $\operatorname{im} d_A^{i-1} \to \operatorname{im} d_B^{i-1}$ and $\ker d_A^i \to \ker d_B^i$, which fit into the commutative diagram (1.12).

$$\begin{tikzcd}
 & & A^{i-1} \arrow[rr, "d_A^{i-1}"] \arrow[dll] \arrow[ddr] & & A^i \arrow[rr, "d_A^i"] \arrow[dll] & & A^{i+1} \arrow[dll] \\
B^{i-1} \arrow[rr, "d_B^{i-1}"] \arrow[ddr] & & B^i \arrow[rr, "d_B^i"] & & B^{i+1} & & \\
 & & & \operatorname{im} d_A^{i-1} \arrow[rr] \arrow[uur] \arrow[dll] \arrow[ddr, "0"] & & \ker d_A^i \arrow[uul] \arrow[uur, "0"'] \arrow[dll] \arrow[ddl] & \\
 & \operatorname{im} d_B^{i-1} \arrow[rr] \arrow[uur] \arrow[ddr, "0"'] & & \ker d_B^i \arrow[uul] \arrow[uur, "0"] \arrow[ddl] & & & \\
 & & & & H^i(A^\bullet) \arrow[dll, dashed] & & \\
 & & H^i(B^\bullet) & & & &
\end{tikzcd} \tag{1.12}$$

All the induced morphisms come from universal properties and are as such unique for which the diagram commutes. Thus the assignment

$$f^\bullet \mapsto H^i(f^\bullet) \colon \operatorname{Hom}_{\mathsf{Ch}(\mathcal{A})}(A, B) \to \operatorname{Hom}_{\mathcal{A}}(H^i(A), H^i(B))$$

is functorial i.e. respects composition and maps identity morphisms to identity morphisms. For the same reasons it is also a k-linear homomorphism, showing that

$$H^i \colon \mathsf{Ch}(\mathcal{A}) \to \mathcal{A}$$

a k-linear functor. Due to the following proposition 1.45, the i-th cohomology functor H^i descends to a well defined additive functor

$$H^i \colon \mathsf{K}(\mathcal{A}) \to \mathcal{A}$$

on the homotopy category $\mathsf{K}(\mathcal{A})$.

Proposition 1.45. *Let $f \colon A^\bullet \to B^\bullet$ be a nullhomotopic chain map in $\mathsf{Ch}(\mathcal{A})$. Then f induces the zero map on cohomology, that is $H^i(f) = 0$ for all $i \in \mathbf{Z}$.*

Proof. As f is nullhomotopic, there exists a homotopy $(h^i \colon A^i \to B^{i-1})_{i \in Z}$ such that

$$f^i = h^{i+1} d_A^i + d_B^{i-1} h^i \quad \text{for all } i \in \mathbf{Z}.$$

Let us name the following morphisms from diagram (1.12).

$$\begin{aligned}
i_A &\colon \ker d_A^i \hookrightarrow A^i & \qquad i_B &\colon \ker d_B^i \hookrightarrow B^i \\
\psi_B &\colon B^{i-1} \twoheadrightarrow \operatorname{im} d_B^{i-1} & \qquad \xi_B &\colon \operatorname{im} d_B^{i-1} \to \ker d_B^i \\
\pi_B &\colon \ker d_B^i \twoheadrightarrow H^i(B^\bullet) & \qquad \phi &\colon \ker d_A^i \to \ker d_B^i
\end{aligned}$$

We compute $i_B\phi = f^i i_A = (h^{i+1}d^i_A + d^{i-1}_B h^i)i_A = d^{i-1}_B h^i i_A = i_B\xi_B\psi_B h^i i_A$. Since i_B is a monomorphism, we may cancel it on the left, to express ϕ as $\xi_B\psi_B h^i i_A$. Then it is clear, that $\pi_B\phi = 0$ (as $\pi_B\xi_B = 0$), which means that 0 is the unique map $H^i(A^\bullet) \to H^i(B^\bullet)$ fitting into the commutative diagram (1.12). $\square$

ugly but uses only universal properties, no elements

It is very fruitful to consider all the cohomology functors $(H^i)_{i\in\mathbf{Z}}$ at once, as is witnessed by the next proposition.

Proposition 1.46. *[KS06, Theorem 12.3.3.] Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence in $\mathsf{Ch}(\mathcal{A})$. Then there exists a long exact sequence in $\mathcal{A}$*

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots,$$

which is functorial in the short exact sequence.

Remark 1.47. Morphisms $H^i(A^\bullet) \to H^i(B^\bullet)$ and $H^i(B^\bullet) \to H^i(C^\bullet)$ are induced by the corresponding morphisms between complexes. But the existence of a *connecting* morphism $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ is part of the assertion of the proposition.

An elementary observation shows that for all $i \in \mathbf{Z}$ the following natural equivalence exists

$$H^0 \circ T^i \simeq H^i, \tag{1.13}$$

so the long exact sequence of proposition 1.46 can be rephrased as

$$\cdots \to H^0(A^\bullet[i]) \to H^0(B^\bullet[i]) \to H^0(C^\bullet[i]) \to H^0(A^\bullet[i+1]) \to \cdots.$$

Proposition 1.48. *With respect to the triangulated structure on $\mathsf{K}(\mathcal{A})$ the cohomology functors H^i are cohomological.*

Proof. First, it is enough to show only that

$$H^0(B^\bullet) \to H^0(C(f)^\bullet) \to H^0(A[1]^\bullet)$$

is exact for any distinguished triangle of the form (1.11). This statement becomes apparent, once we realize that the distinguished triangle yields a short exact sequence

$$0 \longrightarrow B^\bullet \xrightarrow{\tau_f} C(f)^\bullet \xrightarrow{\pi_f} A[1]^\bullet \longrightarrow 0$$

in $\mathsf{Ch}(\mathcal{A})$ to which one then applies proposition 1.46[4].

Now we argue why this is enough to show that all H^i are cohomological. Since every distinguished triangle in $\mathsf{K}(\mathcal{A})$ is isomorphic to a triangle of the form (1.11) we see that the sequence $H^0(B^\bullet) \to H^0(C^\bullet) \to H^0(A[1]^\bullet)$ is exact for any distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ by functoriality of H^i. Rotating the preceding triangle i.e. periodically applying axiom TR2, shows that H^0 is cohomological. Finally, by (1.13) clearly all H^i are cohomological. $\square$

To end this section, we introduce a class of morphisms and a class of objects in $\mathsf{K}(\mathcal{A})$, playing a principal role in the sequel.

Definition 1.49. Let $A^\bullet$ and $B^\bullet$ be objects and $f\colon A^\bullet \to B^\bullet$ a morphism of $\mathsf{K}(\mathcal{A})$.

(i) Chain complex $A^\bullet$ is said to be *acyclic*, if $H^i(A^\bullet) \simeq 0$ for all $i \in \mathbf{Z}$.

(ii) Morphism f is a *quasi-isomorphism*, if $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.

[4] Note that we have actually used much less than what the proposition has to offer, in particular not even the existence of the connecting morphism.

2 Derived categories

The first goal of this section is to construct the derived category of an abelian category $\mathcal{A}$ and equip it with a triangulated structure. Our arguments, although specialized to the homotopy category $\mathsf{K}(\mathcal{A})$ and the class of quasi-isomorphisms, do not differ tremendously from the general theory of localization of categories. A more comprehensive and formal treatment of this topic is laid out in [KS06, Chapters 7, 10, 13] or [Mil]. The second part covers the construction of derived functors. We see how the established framework of derived categories nicely lends itself for the definition of derived functors and we also relate them back to the classical higher derived functors. To close the chapter, we prove a correspondence between the hom-sets of $\mathsf{D}^+(\mathcal{A})$ and certain Ext-modules.

2.1 Derived categories of abelian categories

In algebraic geometry cohomology of a geometric object, like a scheme or a variety, X with respect to some coherent sheaf $\mathcal{F}$ plays a very important role. One way of computing $H^i(X, \mathcal{F})$, which we shall also feature in section 3, involves the following. Instead of intrinsically studying the sheaf $\mathcal{F}$, we represent it with a so called *resolution*, which consists of a complex of sheaves $F^\bullet$, built up from sheaves F^i, for $i \in \mathbf{Z}$, belonging to some class of sheaves, which is well behaved under cohomology, and a quasi-isomorphism of the form $F^\bullet \to \mathcal{F}$ or $\mathcal{F} \to F^\bullet$. After noting that the sheaf $\mathcal{F}$ can be seen as a complex concentrated in degree 0 and observing that replacing a resolution of $\mathcal{F}$ with another one results in computing isomorphic cohomology groups, we are motivated not to distinguish the sheaf $\mathcal{F}$ from its resolutions in the ambient (homotopy) category of complexes any longer. Taking a step back, we would like to modify the homotopy category $\mathsf{K}(\mathsf{coh}(X))$ in such a way that $\mathcal{F}$ is identified with all its resolutions, or in other words, we want all the quasi-isomorphisms of $\mathsf{K}(\mathsf{coh}(X))$ to turn into isomorphisms in a "universal way". We will take the latter to be our inspiration for the definition of the derived category $\mathsf{D}(\mathcal{A})$ of a general abelian category $\mathcal{A}$. This is stated more formally in the form of the ensuing universal property.

intrinsically might not be the best word as resolutions in certain cases do represent a sheaf intrinsically in the sense of generators and relations... (Hilbert syzygy thm)

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $\mathsf{K}(\mathcal{A})$ its homotopy category. A category $\mathsf{D}(\mathcal{A})$ together with a functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is the *derived category of* $\mathcal{A}$, if it satisfies:

(i) For every quasi-isomorphism s in $\mathsf{K}(\mathcal{A})$, $Q(s)$ is an isomorphism in $\mathsf{D}(\mathcal{A})$.

(ii) For any category $\mathcal{D}$ and any functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$, sending quasi-isomorphisms s in $\mathsf{K}(\mathcal{A})$ to isomorphisms $F(s)$ in $\mathcal{D}$, there exists a functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathcal{D}$, which is unique up to a unique natural isomorphism, such that $F \simeq F_0 \circ Q$. In other words, the diagram below commutes up to natural isomorphism.

$$\begin{tikzcd}
\mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\
\mathsf{D}(\mathcal{A}) \arrow[ur, dashed, "F_0"'] &
\end{tikzcd}$$

Remark 2.2. We recognize this definition as a special case of localization of categories [KS06, §7, Definition 7.1.1.] or [GM02, §III.2, Definition 1]. In particular it defines $\mathsf{D}(\mathcal{A})$ to be the *localization of a triangulated category* $\mathsf{K}(\mathcal{A})$ by the family of all quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$.

A naive way of constructing $\mathsf{D}(\mathcal{A})$ out of $\mathsf{K}(\mathcal{A})$ would be to artificially add the inverses to all the quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ and then impose the correct collection of relations on the newly constructed class of morphisms. As this can quickly lead us to some set theoretic problems, we will construct a specific model, which achieves this, instead. Our construction is a priori not going to result in a locally small[5] category, but as we shall soon see in practice all the categories we will be concerned with will be locally small.

2.1.1 Construction

To start, we first need a technical lemma resembling the Ore condition from non-commutative algebra.

Lemma 2.3. *Let $f\colon A^\bullet \to B^\bullet$ and $s\colon C^\bullet \to B^\bullet$ belong to the homotopy category $\mathsf{K}(\mathcal{A})$, with s being a quasi-isomorphism. Then there exists a quasi-isomorphism $u\colon C_0^\bullet \to A^\bullet$ and a morphism $g\colon C_0^\bullet \to C^\bullet$, such that the diagram below commutes in $\mathsf{K}(\mathcal{A})$.*

$$\begin{tikzcd}
C_0^\bullet \arrow[r, dashed, "g"] \arrow[d, dashed, "\sim"', "u"] & C^\bullet \arrow[d, "\sim"', "s"] \\
A^\bullet \arrow[r, "f"] & B^\bullet
\end{tikzcd}$$

Proof. $\square$ add proof

Equipped with the preceding lemma, we are now in a position to construct the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. The derived category $\mathsf{D}(\mathcal{A})$ will consist of

$$\textit{Objects of } \mathsf{D}(\mathcal{A})\textit{:}\quad \text{chain complexes in } \mathcal{A},$$

i.e. the class of objects of $\mathsf{K}(\mathcal{A})$ or $\mathsf{Ch}(\mathcal{A})$, and a class of morphisms, which is quite intricate to define. For fixed complexes $A^\bullet$ and $B^\bullet$ we define the hom-set[6] $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ in the following way.

Hom-sets. A *left roof spanned on $A^\bullet$ and $B^\bullet$* is a pair of morphisms $s\colon A^\bullet \to C^\bullet$ and $f\colon C^\bullet \to B^\bullet$ in the homotopy category $\mathsf{K}(\mathcal{A})$, where s is a quasi-isomorphism. This roof is depicted in the following diagram

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\
A^\bullet & & B^\bullet
\end{tikzcd} \tag{2.1}$$

and denoted by (s, f). Dually, one also obtains the notion of a *right roof spanned on $A^\bullet$ and $B^\bullet$*, which is a pair of morphisms $g\colon A^\bullet \to C^\bullet$ and $u\colon B^\bullet \to C^\bullet$, where u is a quasi-isomorphism, and is depicted below.

$$\begin{tikzcd}
& C^\bullet & \\
A^\bullet \arrow[ur, "g"'] & & B^\bullet \arrow[ul, "\sim"', "u"]
\end{tikzcd}$$

[5] A category $\mathcal{C}$ is called *locally small* if for all objects X and Y of $\mathcal{C}$ the hom-sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are actual sets.

[6] What will be defined here is a priori not necessarily a set, but a class, so $\mathsf{D}(\mathcal{A})$, defined in this section, is not a category in the usual sense. We will however prove that in specific cases some variants of the derived category, especially concrete ones used later on, will from categories in the usual sense.

Our construction of $\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ will be based on left roofs, for nothing is gained or lost by picking either one of the two. Both work just as well and are in fact equivalent (see [KS06, Remark 7.1.18]). Despite our arbitrary choice, it is still beneficial to consider both, as we will sometimes switch between the two whenever convenient.

Definition 2.4. Two left roofs $A^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} B^\bullet$ and $A^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} B^\bullet$ are defined to be *equivalent*, if there exists a quasi-isomorphism $u\colon C^\bullet \to C_0^\bullet$ and a morphism $g\colon C^\bullet \to C_1^\bullet$ in $\mathsf{K}(\mathcal{A})$, for which the diagram below commutes (in $\mathsf{K}(\mathcal{A})$).

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "u"', "\sim"] \arrow[dr, "g"] & & \\
& C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd} \tag{2.2}$$

We denote this relation by $\equiv$.

Note that since $C^\bullet \to C_0^\bullet \to A^\bullet$ is a quasi-isomorphism, the same is true for the composition $C^\bullet \to C_1^\bullet \to A^\bullet$, concluding that $g\colon C^\bullet \to C_1^\bullet$ is a quasi-isomorphism. Also observe that in the diagram (2.2) we may find a new left roof, namely

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "s_0 \circ u"', "\sim"] \arrow[dr, "f_1 \circ g"] & \\
A^\bullet & & B^\bullet,
\end{tikzcd}$$

which is also equivalent to the two roofs we started with (s_0, f_0) and (s_1, f_1).

Lemma 2.5. *The equivalence of left roofs on $A^\bullet$ and $B^\bullet$ is an equivalence relation.*

Proof. The relation is clearly reflexive. We take both u and g to be $\mathrm{id}_{C^\bullet}$. By the note above it is also symmetric, for g is a quasi-isomorphism. It remains to show transitivity. Suppose left roofs $A^\bullet \longleftarrow C_0^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ are equivalent and left roofs $A^\bullet \longleftarrow C_1^\bullet \longrightarrow B^\bullet$ and $A^\bullet \longleftarrow C_2^\bullet \longrightarrow B^\bullet$ are equivalent. This is witnessed by the diagrams

$$\begin{tikzcd}
& & D_0^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & \\
& C_0^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_1^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & D_1^\bullet \arrow[dl, "\sim"'] \arrow[dr] & & \\
& C_1^\bullet \arrow[dl, "\sim"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\sim"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet.
\end{tikzcd}$$

By lemma 2.3 the right roof $D_0^\bullet \to C_1^\bullet \leftarrow D_1^\bullet$ may be completed to form a commutative square

$$\begin{tikzcd}
C^\bullet \arrow[r, dashed] \arrow[d, dashed, "\sim"'] & D_1^\bullet \arrow[d, "\sim"] \\
D_0^\bullet \arrow[r] & C_1^\bullet,
\end{tikzcd}$$

proving that $A^\bullet \leftarrow C_0^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are equivalent. $\square$

For a left roof (2.1) we let $[s \setminus f]$ or $\left[A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet\right]$ denote its equivalence class under $\equiv$.

We then define $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ to be the class of left roofs spanned by $A^\bullet$ and $B^\bullet$, quotiented by the relation $\equiv$. That is

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet) := \left\{ \left[\begin{tikzcd}[column sep=small] & C^\bullet \arrow[dl, "\sim"', "s"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \right]_{\equiv} \;\middle|\; \begin{array}{l} C^\bullet \in \mathrm{Ob}\,\mathsf{K}(\mathcal{A}), \\ f \in \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(C^\bullet, B^\bullet), \\ s \in \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(C^\bullet, A^\bullet) \text{ quasi-iso.} \end{array} \right\}.$$

Composition. Next we define the composition operations

$$\circ\colon \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_0^\bullet, A_1^\bullet) \times \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_1^\bullet, A_2^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A_0^\bullet, A_2^\bullet),$$

for all objects $A_0^\bullet$, $A_1^\bullet$ and $A_2^\bullet$ of $\mathsf{D}(\mathcal{A})$. Let $\phi_0\colon A_0^\bullet \to A_1^\bullet$ and $\phi_1\colon A_1^\bullet \to A_2^\bullet$ be a pair of composable morphisms in $\mathsf{D}(\mathcal{A})$. Next, pick their respective left roof representatives, $A_0^\bullet \xleftarrow{s_0} C_0^\bullet \xrightarrow{f_0} A_1^\bullet$ and $A_1^\bullet \xleftarrow{s_1} C_1^\bullet \xrightarrow{f_1} A_2^\bullet$, and concatenate them according to the solid zig-zag diagram below.

$$\begin{tikzcd}[column sep=small] & & C^\bullet \arrow[dl, dashed, "\sim"', "u"] \arrow[dr, dashed, "g"] & & \\ & C_0^\bullet \arrow[dl, "\sim"', "s_0"] \arrow[dr, "f_0"] & & C_1^\bullet \arrow[dl, "\sim"', "s_1"] \arrow[dr, "f_1"] & \\ A_0^\bullet & & A_1^\bullet & & A_2^\bullet \end{tikzcd}$$

By Lemma 2.3 there are morphisms $u\colon C^\bullet \to C_0^\bullet$ and $g\colon C^\bullet \to C_1^\bullet$, depicted with dashed arrows, completing the diagram in $\mathsf{K}(\mathcal{A})$. In this way we obtain a left roof, formed by a quasi-isomorphism $s_0 \circ u$ and a morphism $f_1 \circ g$, the equivalence class of which we define to be the composition

$$\phi_1 \circ \phi_0 := \left[A_0^\bullet \xleftarrow{s_0 \circ u} C^\bullet \xrightarrow{f_1 \circ g} A_2^\bullet \right].$$

It can be shown that this is a well defined composition, independent of the choice of left roof representatives of ϕ_0 and ϕ_1 and independent of the choice of the peak $C^\bullet$ and morphisms $u\colon C^\bullet \to C_0^\bullet$ and $g\colon C^\bullet \to C_1^\bullet$, for which the square at the top of the diagram commutes. It is also true that the operation is associative. We leave out the proof, because it is routine, but refer the reader to a very detailed account by Miličić [Mil, Ch. 1.3].

Identities. The identity morphism on an object $A^\bullet$ of $\mathsf{D}(\mathcal{A})$ is defined to be the equivalence class of $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet$. It is not difficult to check that these classes play the role of identity morphisms in $\mathsf{D}(\mathcal{A})$.

Functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. The *localization* functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an identity on objects functor with the action on morphisms defined by the assignment

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet) \xrightarrow{Q} \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$$
$$(f\colon A^\bullet \to B^\bullet) \longmapsto \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{f} B^\bullet \right].$$

k-linear structure. To equip the hom-sets of $\mathsf{D}(\mathcal{A})$ with a k-module structure, we consult the following lemma resembling finding a common denominator in the context of left roofs.

Lemma 2.6. *Let $\phi_0\colon A^\bullet \to B^\bullet$ and $\phi_1\colon A^\bullet \to B^\bullet$ be two morphisms in $\mathsf{D}(\mathcal{A})$ represented by left roofs*

$$
\begin{tikzcd}
& C_0^\bullet \arrow[dl, "s_0"', "\sim"] \arrow[dr, "f_0"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& C_1^\bullet \arrow[dl, "s_1"', "\sim"] \arrow[dr, "f_1"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
$$

Then there is a quasi-isomorphism $s\colon C^\bullet \to A^\bullet$ and morphisms g_0, $g_1\colon C^\bullet \to B^\bullet$, such that

$$
\begin{tikzcd}
& C^\bullet \arrow[dl, "s"', "\sim"] \arrow[dr, "g_0"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& C^\bullet \arrow[dl, "s"', "\sim"] \arrow[dr, "g_1"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
$$

represent ϕ_0 and ϕ_1 respectively.

Proof. $\square$ add proof

Using notation from lemma 2.6, we define the addition operation on $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ by the rule

$$\phi_0 + \phi_1 := \left[A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{g_0+g_1} B^\bullet\right].$$

It is well defined, associative, commutative and has a neutral element $0 = Q(0)$. The action of scalars of the ring k is defined as

$$\lambda\phi_0 := \left[A^\bullet \xleftarrow{s_0} C^\bullet \xrightarrow{\lambda f_0} B^\bullet\right].$$

Moreover, the composition law $\circ$ is k-bilinear and the localization functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an additive functor.

The zero complex $0^\bullet$ plays the role of the zero object of $\mathsf{D}(\mathcal{A})$ and the direct sum of two complexes is seen to exist and is induced from the direct sum in $\mathsf{K}(\mathcal{A})$ by applying the localization functor Q.

Triangulated structure. The translation functor $T\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is defined to be the usual translation functor on objects and a morphism represented by some roof is sent to the equivalence class of that roof on which we have acted with the translation functor of $\mathsf{K}(\mathcal{A})$. This action respects the equivalence relations, which define the hom-sets of $\mathsf{D}(\mathcal{A})$, and thus induces a well defined functor, which is also an additive auto-equivalence, the quasi-inverse being given by translation in the other direction.

The class of distinguished triangles in $\mathsf{D}(\mathcal{A})$ is defined to consist of all triangles $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$, for which there exists a distinguished triangle $A_0^\bullet \to B_0^\bullet \to C_0^\bullet \to A_0[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$, such that $Q(A_0^\bullet) \to Q(B_0^\bullet) \to Q(C_0^\bullet) \to Q(A_0[1]^\bullet)$ is isomorphic to $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $\mathsf{D}(\mathcal{A})$. Spelling this out in the case of left roofs, we see that this condition implies the existence of another distinguished triangle $A_1^\bullet \to B_1^\bullet \to C_1^\bullet \to A_1[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$, such that the following commutative diagram, where vertical arrows are all quasi-isomorphisms exists in $\mathsf{K}(\mathcal{A})$.

$$
\begin{tikzcd}
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A[1]^\bullet \\
A_1^\bullet \arrow[r] \arrow[u] \arrow[d] & B_1^\bullet \arrow[r] \arrow[u] \arrow[d] & C_1^\bullet \arrow[r] \arrow[u] \arrow[d] & A_1[1]^\bullet \arrow[u] \arrow[d] \\
A_0^\bullet \arrow[r] & B_0^\bullet \arrow[r] & C_0^\bullet \arrow[r] & A_0[1]^\bullet
\end{tikzcd}
$$

Verification, that the above defines the structure of a triangulated category on $\mathsf{D}(\mathcal{A})$, is left out, but we remark, that it follows naturally from the triangulated structure on $\mathsf{K}(\mathcal{A})$

and refer the reader to [Mil, Chapter 2, Theorem 1.6.1] or [KS06, Chapter 10, Theorem 10.2.3].

Remark 2.7. All that has been defined and established in this section with left roofs can analogously also be done with right roofs.

Proposition 2.8. *The constructed derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ together with the localization functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ satisfies the universal property of Definition 2.1. Additionally, in notation of Definition 2.1, if category $\mathcal{D}$ and the functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$ are k-linear or triangulated, the induced functor F_0 is as well.*

Proof. For (i) suppose $s\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism. Then the inverse of $Q(s)$ is a morphism $\phi\colon B^\bullet \to A^\bullet$ represented by the left roof $B^\bullet \xleftarrow{s} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet$. Clearly $\phi \circ Q(s) = \mathrm{id}_{A^\bullet}$, whereas $Q(s) \circ \phi$, represented by $B^\bullet \xleftarrow{s} A^\bullet \xrightarrow{s} B^\bullet$, can be seen to equal $\mathrm{id}_{B^\bullet}$ by the following diagram.

$$\begin{tikzcd}
 & & A^\bullet \arrow[dl, equal] \arrow[dr, "s"] & & \\
 & A^\bullet \arrow[dl, "s"'] \arrow[drrr, "s"] & & B^\bullet \arrow[dlll, equal] \arrow[dr, equal] & \\
B^\bullet & & & & B^\bullet
\end{tikzcd}$$

For (ii) the functor $F_0\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ is defined by the assignment on

$$\begin{array}{rrcl}
\textit{Objects:} & A^\bullet & \longmapsto & F(A^\bullet). \\
\textit{Morphisms:} & [A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet] & \longmapsto & F(f) \circ F(s)^{-1}.
\end{array}$$

This is well defined and functorial for the functor $F\colon \mathsf{K}(\mathcal{A}) \to \mathcal{D}$ sends quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathcal{D}$. Moreover, F_0 is a k-linear functor, as can quickly be computed by an application of Lemma 2.6. Using the same notation as in the lemma, we have

$$F_0(\phi_0 + \phi_1) = F(g_0 + g_1) \circ F(s)^{-1} = F(g_0) \circ F(s)^{-1} + F(g_1) \circ F(s)^{-1} = F_0(\phi_0) + F_1(\phi_1)$$
$$\text{and} \quad F_0(\lambda\phi_0) = F(\lambda f_0) \circ F(s_0)^{-1} = \lambda(F(f_0) \circ F(s_0)^{-1}) = \lambda F_0(\phi_0).$$

It is clear from the definition, that $F_0 \circ Q = F$ and that F_0 is unique up to a natural isomorphism for which the diagram (2.1) commutes.

Lastly, when $\mathcal{D}$ and F are triangulated, the functor F_0 also commutes with the translation functors of $\mathsf{D}(\mathcal{A})$ and $\mathcal{D}$, because Q as the identity on objects and F commutes with translation functors of $\mathsf{K}(\mathcal{A})$ and $\mathcal{D}$. From the fact that distinguished triangles in $\mathsf{D}(\mathcal{A})$ are by definition exactly those triangles, which are isomorphic to Q-images of distinguished triangles of $\mathsf{K}(\mathcal{A})$, and the fact that F maps distinguished triangles of $\mathsf{D}(\mathcal{A})$ to distinguished triangles of $\mathcal{D}$, we can conclude that F_0 is triangulated as well. □

2.1.2 Cohomology

Since cohomology already appeared as an indispensable tool on the level of the homotopy category $\mathsf{K}(\mathcal{A})$, it would be nice to also have it be accessible at the level of derived categories. We can in fact define cohomology functors on $\mathsf{D}(\mathcal{A})$ by inducing them from the functors $H^i\colon \mathsf{K}(\mathcal{A}) \to \mathcal{A}$, which by definition send quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathcal{A}$, using the universal property of Definition 2.1. In this way we obtain k-linear functors

$$H^i\colon \mathsf{D}(\mathcal{A}) \to \mathcal{A}, \qquad i \in \mathbf{Z}$$

for $i \in \mathbf{Z}$, which as expected return the i-th cohomology of a complex $A^\bullet$ and return the morphism $H^i(f) \circ H^i(s)^{-1}$ when applied to a morphism in $\mathsf{D}(\mathcal{A})$, represented by a left roof (2.1).

2.1.3 Subcategories of derived categories

As with the homotopy category of complexes $\mathsf{K}(\mathcal{A})$, we also have the bounded versions of the derived category $\mathsf{D}(\mathcal{A})$. They are constructed in the exact same way as $\mathsf{D}(\mathcal{A})$ above, with the adjustment, that we replace every instance of the category $\mathsf{K}(\mathcal{A})$ with either $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ or $\mathsf{K}^b(\mathcal{A})$, to obtain $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ or $\mathsf{D}^b(\mathcal{A})$, respectively. They are clearly all triangulated as well. All three bounded variants naturally come with the inclusion functors

$$\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A}), \qquad \mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A}), \qquad \mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A}),$$

defined to be the identity on objects and send a morphism, i.e. equivalence class of some roof representative with respect to the "bounded variant" of the equivalence relation 2.4, where, again, every instance of $\mathsf{K}(\mathcal{A})$ is replaced by either $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ or $\mathsf{K}^b(\mathcal{A})$, to the equivalence class of that same roof representative, but now under the original equivalence relation. All three functors are clearly also triangulated.

There is also another (equivalent) way of obtaining the categories mentioned above, which will sometimes be more convenient to work with. We introduce them in the form of the following proposition.

Proposition 2.9. *Let $\mathcal{A}$ be an abelian category.*

(i) The functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and induces an equivalence between $\mathsf{D}^+(\mathcal{A})$ and the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ spanned on (possibly unbounded) complexes $A^\bullet$, for which there is an integer $n \in \mathbf{Z}$, such that $H^i(A^\bullet) \simeq 0$ for all $i < n$.

(ii) The functor $\mathsf{D}^-(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and induces an equivalence between $\mathsf{D}^-(\mathcal{A})$ and the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ spanned on (possibly unbounded) complexes $A^\bullet$, for which there is an integer $n \in \mathbf{Z}$, such that $H^i(A^\bullet) \simeq 0$ for all $i > n$.

(iii) The functor $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful and induces an equivalence between $\mathsf{D}^b(\mathcal{A})$ and the full triangulated subcategory of $\mathsf{D}(\mathcal{A})$ spanned on (possibly unbounded) complexes $A^\bullet$, for which there is an integer $n \in \mathbf{Z}$, such that $H^i(A^\bullet) \simeq 0$ for all $|i| > n$.

To make the proof of this proposition conceptually clearer, we introduce two complexes associated to a complex $A^\bullet$, called its *right and left truncations*

$$\tau_{\leq n}(A^\bullet) = (\cdots \to A^{n-2} \to A^{n-1} \to \ker d^n \to 0 \to \cdots) \quad \text{and}$$
$$\tau_{\geq n}(A^\bullet) = (\cdots \to 0 \to \operatorname{coker} d^{n-1} \to A^{n+1} \to A^{n+2} \to \cdots).$$

They come equipped with the natural inclusion map $j\colon \tau_{\leq n}(A^\bullet) \to A^\bullet$ and the natural quotient map $q\colon A^\bullet \to \tau_{\geq n}(A^\bullet)$, satisfying the following claims

$$H^i(j)\colon H^i(\tau_{\leq n}(A^\bullet)) \to H^i(A^\bullet) \text{ is an isomorphism for all } i \leq n. \tag{2.3}$$
$$H^i(q)\colon H^i(A^\bullet) \to H^i(\tau_{\geq n}(A^\bullet)) \text{ is an isomorphism for all } i \geq n. \tag{2.4}$$

Indeed, claim (2.3) is clearly true for $i < n$, with $i = n$ being the only interesting case. Here the right commutative square of the diagram below

$$\begin{tikzcd}
A^{n-1} \arrow[r, "\delta"] \arrow[d, "j^{n-1}"'] & \ker d^n \arrow[r, "0"] \arrow[d, "j^n"'] & 0 \arrow[d, "j^{n+1}"] \\
A^{n-1} \arrow[r, "d^{n-1}"] & A^n \arrow[r, "d^n"] & A^{n+1}
\end{tikzcd}$$

induces the identity morphism between the kernels of the horizontal differentials and the left commutative square induces the identity morphism between the images of the horizontal differential. Consequently j induces the identity morphism on the n-th cohomology $H^n(j)\colon H^n(\tau_{\leq n}(A^\bullet)) \to H^n(A^\bullet)$. Dually, the claim (2.4) also holds.

Proof of Proposition 2.9. For (i) the functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is readily seen to be fully faithful by considering morphisms to be represented by *right* roofs. For example in the right roof $A^\bullet \to C^\bullet \leftarrow B^\bullet$ spanned on complexes $A^\bullet$ and $B^\bullet$ of $\mathsf{D}^+(\mathcal{A})$, we first observe that $C^\bullet$ has bounded cohomology from below, thus $q\colon C^\bullet \to \tau_{\geq n}(C^\bullet)$ is a quasi-isomorphism for some $n \in \mathbf{Z}$. Extending the aforementioned right roof with quasi-isomorphism q, to obtain $A^\bullet \to \tau_{\geq n}(C^\bullet) \leftarrow B^\bullet$, then yields an equivalent right roof, whose peak now belongs to $\mathsf{D}^+(\mathcal{A})$ and the equivalence class of which is therefore sent to the morphism represented by $A^\bullet \to C^\bullet \leftarrow B^\bullet$.

To verify this functor is essentially surjective onto the subcategory of complexes with cohomology bounded from below, suppose $A^\bullet$ satisfies $H^i(A^\bullet) \simeq 0$, for all $i < n$. Then by (2.4) the quotient map $q\colon A^\bullet \to \tau_{\geq n}(A^\bullet)$ is a quasi-isomorphism, inducing an isomorphism $A^\bullet \simeq \tau_{\geq n}(A^\bullet)$ in $\mathsf{D}(\mathcal{A})$.

The case (ii) is proven analogously, now considering left roofs and right truncations. For (iii), the functor $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is viewed as the composition $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. Here $\mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A})$ is seen to be fully faithful via a minor modification of the argument for (ii) and $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is fully faithful by (i). Essential surjectivity onto the required subcategory then follows as the roof $A^\bullet \to \tau_{\geq n}(A^\bullet) \leftarrow \tau_{\leq n}(\tau_{\geq n}(A^\bullet))$, for some $n \in \mathbf{Z}$, witnesses an isomorphism in $\mathsf{D}(\mathcal{A})$ between a complex $A^\bullet$ with bounded cohomology and a bounded complex $\tau_{\leq n}(\tau_{\geq n}(A^\bullet))$. $\square$

Move footnote about $\mathcal{A} \to \mathsf{K}(\mathcal{A})$ being fully faithful over here.

Proposition 2.10. *The natural functor $\mathcal{A} \to \mathsf{D}(\mathcal{A})$ is fully faithful and identifies $\mathcal{A}$ with the full subcategory of $\mathsf{D}(\mathcal{A})$, spanned by complexes $A^\bullet$, satisfying $H^i(A^\bullet) \simeq 0$, for all $i \neq 0$.*

Proof. For fully faithfulness see [KS06, Chapter 13, Proposition 13.1.10]. Essential surjectivity onto the required subcategory is [KS06, Chapter 13, Proposition 13.1.12] and follows from a similar argument as in the proof of Proposition 2.9 (iii) by taking n to be 0, thus showing that in $\mathsf{D}(\mathcal{A})$ the complex $A^\bullet$ with trivial cohomology in non-zero degrees is isomorphic to $H^0(A^\bullet)$ concentrated in degree 0. $\square$

2.1.4 Derived categories of abelian categories with enough injectives

After possibly working with proper classes, when constructing the derived category, this section will once again place us back on familiar grounds of set theory. Aside from these concerns the establishing result of this section will have a very important practical application – construction of derived functors. Under an assumption on the abelian category $\mathcal{A}$, we will establish an equivalence of $\mathsf{D}^+(\mathcal{A})$ with the homotopy category of a full additive subcategory of $\mathcal{A}$, spanned on *injective objects*, which we define presently.

Definition 2.11. An object I of an abelian category $\mathcal{A}$ is *injective*[7], if the functor

$$\operatorname{Hom}_{\mathcal{A}}(-, I) \colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Mod}_k$$

is exact. Equivalently, I is injective, whenever for any monomorphism $A \hookrightarrow B$ and morphism $A \to I$ there is a morphism $B \to I$, for which the following diagram commutes.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

A category $\mathcal{A}$ is said to *have enough injectives*, if every object A of $\mathcal{A}$ embeds into an injective object i.e. there is a monomorphism $A \hookrightarrow I$ for some injective object I.

Remark 2.12. A simple verification shows, that the full subcategory of an abelian or k-linear category $\mathcal{A}$, spanned on all injective objects of $\mathcal{A}$ is a k-linear category, as the zero object 0 is injective and the biproduct of two injective objects is again injective. We denote this category by $\mathcal{J}$.

Injective objects will be utilized as building blocks for representing complexes. They are the preferred class of objects to work with in this context because they enable a favourable interplay between cohomology and homotopy. For example, as a consequence of lemma 2.17, every acyclic bounded below complex of injectives $I^\bullet$ is actually nullhomotopic i.e. isomorphic to the object 0 in $\mathsf{K}^+(\mathcal{A})$. In light of this, consider a complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$. A complex of injectives $I^\bullet$ of $\mathsf{K}^+(\mathcal{J})$ together with a quasi-isomorphism $f \colon A^\bullet \to I^\bullet$ is called an *injective resolution* of $A^\bullet$. Later we will also see that this injective resolution is unique up to homotopy equivalence.

Proposition 2.13. *Suppose $\mathcal{A}$ contains enough injectives. Then for every $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ there is a quasi-isomorphism $f \colon A^\bullet \to I^\bullet$, where $I^\bullet \in \operatorname{Ob} \mathsf{K}^+(\mathcal{A})$ is a complex of injectives.*

Proof. We will inductively construct a complex $I^\bullet$ in $\mathsf{K}^+(\mathcal{A})$, built up from injectives, and a quasi-isomorphism $f \colon A^\bullet \to I^\bullet$. For simplicity assume $A^i = 0$ for $i < 0$.

The base case is trivial. Define $I^i = 0$ and $f^i = 0$ for $i < 0$ and take $f^0 \colon A^0 \to I^0$ to be the morphism obtained from the assumption about $\mathcal{A}$ having enough injectives.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & I^0 & & \end{tikzcd}$$

For the induction step suppose we have constructed injective objects I^i, with differentials $\delta^i \colon I^i \to I^{i+1}$, and morphisms f^i for all i up to n, such that the diagram below commutes.

$$\begin{tikzcd} \cdots \arrow[r] & A^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r] & \cdots \\ \cdots \arrow[r] & I^{n-1} \arrow[r, "\delta^{n-1}"] & I^n & & \end{tikzcd}$$

[7]Dually, an object P of $\mathcal{A}$ is *projective*, if P is injective in $\mathcal{A}^{\mathrm{op}}$, or more explicitly, the functor $\operatorname{Hom}_{\mathcal{A}}(P, -) \colon \mathcal{A} \to \mathsf{Mod}_k$ is exact. Category $\mathcal{A}$ is said to *have enough projectives*, if every object A of $\mathcal{A}$ is a quotient of some projective object i.e. there is an epimorphism $P \twoheadrightarrow A$ for some projective object P.

Consider the cokernel of the differential[8] $d_{C(f)}^{n-1}$

$$A^n \oplus I^{n-1} \xrightarrow{d_{C(f)}^{n-1}} A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d_{C(f)}^{n-1}$$

and denote with $j\colon \operatorname{coker} d_{C(f)}^{n-1} \hookrightarrow I^{n+1}$ an embedding to an injective object I^{n+1}. After precomposing the morphism

$$A^{n+1} \oplus I^n \xrightarrow{e} \operatorname{coker} d_{C(f)}^{n-1} \xrightarrow{j} I^{n+1}$$

with canonical embeddings of A^{n+1} and I^n into their biproduct $A^{n+1} \oplus I^n$, we obtain morphisms $\delta^n\colon I^n \to I^{n+1}$ and $f^{n+1}\colon A^{n+1} \to I^{n+1}$. As $j \circ e$ is a morphism fitting into the following coproduct diagram,

$$\begin{tikzcd}
 & I^n \arrow[d] \arrow[ddr, bend left, "\delta^n"] & \\
A^{n+1} \arrow[r] \arrow[drr, bend right, "f^{n+1}"'] & A^{n+1} \oplus I^n \arrow[dr, dashed, "j\circ e"'] & \\
 & & I^{n+1}
\end{tikzcd}$$

we see that $j \circ e = \langle f^{n+1}, \delta^n \rangle$. Next, we see that $\delta^n\delta^{n-1} = 0$ and $\delta^n f^n = f^{n+1}d^n$ from the computation

$$0 = j \circ e \circ d_{C(f)}^{n-1} = \langle f^{n+1},\ \delta^n \rangle \begin{pmatrix} -d^n & 0 \\ f^n & \delta^{n-1} \end{pmatrix} = \langle \delta^n f^n - f^{n+1}d^n,\ \delta^n\delta^{n-1} \rangle$$

Thus far we have constructed a chain complex $I^\bullet$ of injective objects and a chain map $f\colon A^\bullet \to I^\bullet$. It remanis to be shown that f is a quasi-isomorphism. Recall, that f is a quasi-isomorphism if and only if its cone $C(f)^\bullet$ is acyclic. By Remark 1.44, we known

$$H^n(C(f)^\bullet) = \ker(\operatorname{coker} d_{C(f)}^{n-1} \xrightarrow{d} A^{n+2} \oplus I^{n+1}),$$

where d is obtained from the universal property of $\operatorname{coker} d_{C(f)}^{n-1}$, induced by $d_{C(f)}^n$. Thus it suffices to show, that d is a monomorphism. Consider the following diagram, which we will show to commute.

$$\begin{tikzcd}
A^n \oplus I^{n-1} \arrow[r, "d_{C(f)}^{n-1}"] & A^{n+1} \oplus I^n \arrow[rr, "d_{C(f)}^n"] \arrow[dr, two heads, "e"] & & A^{n+2} \oplus I^{n+1} \arrow[dr, "p"] & \\
 & & \operatorname{coker} d_{C(f)}^{n-1} \arrow[ur, dashed, "d"] \arrow[rr, hook, "j"] & & I^{n+1}
\end{tikzcd}$$

Once we show, that its right most triangle is commutative, we see that d is monomorphic, because j is monomorphic. To see that the triangle in question commutes, we compute

$$p \circ d \circ e = p \circ d_{C(f)}^n = \langle f^{n+1}, \delta^n \rangle = j \circ e.$$

As e is epic, we may cancel it on the right, to arrive at $p \circ d = j$, which ends the proof. $\square$

[8] Strictly speaking this is not a differential of a complex yet, because we have not specified the chain map f entirely. But this is easily fixed by setting $I^i = 0$ and f^i to be the zero morphism $A^i \to 0$, for $i > n$.

Remark 2.14. As is evident from the proof by close inspection we have only used two facts about the class of all injective objects of $\mathcal{A}$ – that every object of $\mathcal{A}$ embeds into some injective object and that the class of injectives is closed under finite direct sums. This will later on be used in subsection **??** when constructing a right derived functor of a given functor F in the presence of an F-adapted class.

Theorem 2.15. *Assume $\mathcal{A}$ contains enough injectives and let $\mathcal{J} \subseteq \mathcal{A}$ denote the full subcategory on injecitve objects of $\mathcal{A}$. Then the inclusion $\mathsf{K}^+(\mathcal{J}) \hookrightarrow \mathsf{K}^+(\mathcal{A})$ induces an equivalence of triangulated categories*

$$\mathsf{K}^+(\mathcal{J}) \simeq \mathsf{D}^+(\mathcal{A}).$$

Remark 2.16. Theorem 2.15 in particular shows that $\mathsf{D}^+(\mathcal{A})$ is a category in the usual sense, i.e. all its hom-sets are in fact sets.

maybe I replace "usual" with locally small everywhere this comes up.

Lemma 2.17. *Let $A^\bullet$ be any acyclic complex in $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet$ a complex of injectives from $\mathsf{K}^+(\mathcal{J})$. Then*

$$\mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) = 0.$$

In other words every morphism from an acyclic complex to an injective one is nulhomotopic.

Proof. We will show that $f \simeq 0$ for any chain map $f\colon A^\bullet \to I^\bullet$ by inductively constructing an appropriate homotopy h. For simplicity assume $A^i = 0$ and $I^i = 0$ for $i < 0$ and define $h^i\colon A^i \to I^{i-1}$ to be 0 for $i \leq 0$.

As $A^\bullet$ is acyclic, the first non-trivial differential $d^0\colon A^0 \to A^1$ is mono. From I^0 being injective, we obtain a morphism $h^1\colon A^1 \to I^0$, satisfying

$$f^0 = h^1 \circ d^0.$$

Note that the above can actually be rewritten to $f^0 = h^1 d^0 + \delta^{-1} h^0$, as $h^0 = 0$.

For the induction step assume that we have already constructed $h^i\colon A^i \to I^{i-1}$, with $f^{i-1} = h^i d^{i-1} + \delta^{i-2} h^{i-1}$ for all $i \leq n$. We are aiming to construct $h^{n+1}\colon A^{n+1} \to I^n$, for which $f^n = h^{n+1} d^n + \delta^{n-1} h^n$ holds. First, expand the differential d^n into a composition of the canonical epimorphism $e\colon A^n \to \operatorname{coker} d^{n-1}$, followed by $j\colon \operatorname{coker} d^{n-1} \to A^{n+1}$ induced by the universal property of $\operatorname{coker} d^{n-1}$ by the map $d^n\colon A^n \to A^{n+1}$. Morphism j is actually a monomorphism by acyclicity of $A^\bullet$, since we know that

$$0 = H^n(A^\bullet) \simeq \ker(j\colon \operatorname{coker} d^{n-1} \to A^{n+1}).$$

Next, we see that by the universal property of $\operatorname{coker} d^{n-1}$, morphism $f^n - \delta^{n-1} h^n\colon A^n \to I^n$ induces a morphism $g\colon \operatorname{coker} d^{n-1} \to I^n$, because

$$\begin{aligned}(f^n - \delta^{n-1} h^n) d^{n-1} &= f^n d^{n-1} - \delta^{n-1} h^n d^{n-1} \\ &= f^n d^{n-1} + \delta^{n-1} \delta^{n-2} h^{n-1} - \delta^{n-1} f^{n-1} \\ &= 0.\end{aligned}$$

The second equality follows from the inductive hypothesis and the third one from f being a chain map and δ^{n-1}, δ^{n-2} being a differentials.

Lastly, for I^n is injecive and j mono, there is a morphism $h^{n+1}\colon A^{n+1} \to I^n$, satisfying $h^{n+1} \circ j = g$, thus after precomposing both sides with e, we arrive at $h^{n+1} d^n = f^n - \delta^{n-1} h^n$, which can be rewritten as

$$f^n = h^{n+1} d^n + \delta^{n-1} h^n.$$

□

Lemma 2.18. *Let $A^\bullet$ and $B^\bullet$ belong to $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet \in \mathsf{K}^+(\mathcal{J})$. Let $f\colon B^\bullet \to A^\bullet$ be a quasi-isomorphism, then*

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet)$$

is an isomorphism of k-modules.

Proof. In $\mathsf{K}^+(\mathcal{A})$ we have a distinguished triangle $B^\bullet \to A^\bullet \to C(f)^\bullet \to B[1]^\bullet$, which induces a long exact sequence (of k-modules)

$$\begin{aligned} \cdots \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)^\bullet, I^\bullet) \longrightarrow& \\ \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet) \longrightarrow& \\ \longrightarrow \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)[-1]^\bullet, I^\bullet) \longrightarrow \cdots ,& \end{aligned}$$

since $\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(-, I^\bullet)$ is a cohomological functor by example 1.21. As f is a quasi-isomorphism, the cone $C(f)^\bullet$ is acyclic along with all its shifts, thus showing that

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(C(f)[i]^\bullet, I^\bullet) = 0,$$

for all $i \in \mathbf{Z}$ by Lemma 2.17, since $I^\bullet$ is a complex of injectives. From the long exact sequence it then clearly follows that f^* is an isomorphism. □

Lemma 2.19. *Let $A^\bullet$ belong to $\mathsf{K}^+(\mathcal{A})$ and $I^\bullet$ to $\mathsf{K}^+(\mathcal{J})$. Then the morphism action*

$$\operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \tag{2.5}$$

of the localization functor Q is an isomorphism of k-modules.

Proof. We already know this is a k-module homomorphism, thus it is enough to show that it is bijective. This is accomplished by constructing its inverse. Let $\phi\colon A^\bullet \to I^\bullet$ be a morphism in $\mathsf{D}^+(\mathcal{A})$ and let $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ be its left roof representative. The morphism s being a quasi-isomorphism implies that $s^*\colon \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(B^\bullet, I^\bullet)$ is bijective by lemma 2.18, so there is a unique morphism $g\colon A^\bullet \to I^\bullet$, such that $f = g \circ s$ in $\mathsf{K}^+(\mathcal{A})$. The inverse to (2.5) is then defined by sending ϕ to g.

First let's argue why this map is well defined i.e. independent of the choice of left roof representative for ϕ. We pick two left roof representatives $A^\bullet \xleftarrow{s_0} B_0^\bullet \xrightarrow{f_0} I^\bullet$ and $A^\bullet \xleftarrow{s_1} B_1^\bullet \xrightarrow{f_1} I^\bullet$ for ϕ and let g_0 and g_1 be such that $f_i = g_i \circ s_i$ for both i. As the roofs are equivalent there are quasi-isomorphisms $t_0\colon C^\bullet \to B_0^\bullet$ and $t_1\colon C^\bullet \to B_1^\bullet$, which fit into the following commutative diagram.

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "t_0"'] \arrow[dr, "t_1"] & & \\ & B_0^\bullet \arrow[dl, "s_0"'] \arrow[drrr, "f_0" near end] & & B_1^\bullet \arrow[dlll, "s_1" near end] \arrow[dr, "f_1"] & \\ A^\bullet & & & & I^\bullet \end{tikzcd}$$

Therefore $g_0 s_0 t_0 = f_0 t_0 = f_1 t_1 = g_1 s_1 t_1 = g_1 s_0 t_0$. As $s_0 t_0$ is a quasi-isomorphism, we see that $g_0 = g_1$ by applying lemma 2.18.

Following the diagram below, it is clear why the constructed map is a right inverse to (2.5)

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) & \xrightarrow{Q} & \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) & \longrightarrow & \operatorname{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) \\ (g\colon A^\bullet \to I^\bullet) & \longmapsto & \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right] & \longmapsto & (g\colon A^\bullet \to I^\bullet)\,. \end{array}$$

Lastly we see that it is also a left inverse by the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(A^\bullet, I^\bullet) & \xrightarrow{Q} & \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I^\bullet) \\ \left[A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet\right] & \longmapsto & (g \colon A^\bullet \to I^\bullet) & \longmapsto & \left[A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet\right] \end{array}$$

along with observing that $A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{g} I^\bullet$ and $A^\bullet \xleftarrow{s} B^\bullet \xrightarrow{f} I^\bullet$ are equivalent roofs. □

Proof of theorem 2.15. As $\mathsf{K}^+(\mathcal{J}) \to \mathsf{D}^+(\mathcal{A})$ is a triangulated functor, we only need to show that it is fully faithful and essentially surjective. Let $I^\bullet$ and $J^\bullet$ be objects of $\mathsf{K}^+(\mathcal{J})$. Then

$$\mathrm{Hom}_{\mathsf{K}^+(\mathcal{J})}(I^\bullet, J^\bullet) \xrightarrow{=} \mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(I^\bullet, J^\bullet) \xrightarrow{(2.19)} \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$$

is a bijection, showing fully faithfulness (the last map is a bijection by Lemma 2.19). Essential surjectivity is clear from the existence of injective resolutions (cf. Proposition 2.13) because quasi-isomorphisms now play the role of isomorphisms in $\mathsf{D}^+(\mathcal{A})$. □

Within the scope of an abelian category $\mathcal{A}$ with enough injectives, theorem 2.15 now offers some well known and classical results of homological algebra.

Corollary 2.20. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(i) Any two injective resolutions of a complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$ are homotopically equivalent i.e. isomorphic in $\mathsf{K}^+(\mathcal{A})$

(ii) For any morphism of complexes $f \colon A^\bullet \to B^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ and any injective resolutions $A^\bullet \to I_A^\bullet$ and $B^\bullet \to I_B^\bullet$ of $A^\bullet$ and $B^\bullet$ respectively, there is a unique up to homotopy chain map $I_A^\bullet \to I_B^\bullet$ for which the square below commutes up to homotopy.

$$\begin{tikzcd} I_A^\bullet \arrow[d, dashed] & A^\bullet \arrow[l] \arrow[d, "f"] \\ I_B^\bullet & B^\bullet \arrow[l] \end{tikzcd}$$

2.2 Derived functors

The main goal of this section is to assign to a sensible additive functor $F \colon \mathcal{A} \to \mathcal{B}$ between two abelian categories an appropriate *triangulated* functor on the level of derived categories, called a *derived functor*. We will first deal with the easiest case, when F is exact and then move on to the case, where F will only be assumed to be left or right exact and instead require the domain category $\mathcal{A}$ to posses some nice properties (e.g. contain enough injectives or contain an F-adapted class of objects). In practise the additional conditions $\mathcal{A}$ is required to satisfy are not very restrictive.

2.2.1 Derived functors of exact functors

We start with some establishing lemmas.

Lemma 2.21. *Let $f \colon A^\bullet \to B^\bullet$ be a chain map. Then f is a quasi-isomorphism if and only if its associated cone $C(f)^\bullet$ is acyclic.*

proof can reference a proposition from section on triangulated cats...

Proof. One only needs to consider the distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$ in $\mathsf{K}(\mathcal{A})$ and its associated long exact sequence in cohomology. $\square$

Lemma 2.22. *Let $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ be a triangulated functor, then the following two conditions are equivalent.*

(i) For every quasi-isomorphism s in $\mathsf{K}(\mathcal{A})$, $F(s)$ is a quasi-isomorphism in $\mathsf{K}(\mathcal{B})$.

(ii) For every acyclic complex $A^\bullet$ of $\mathsf{K}(\mathcal{A})$, $F(A^\bullet)$ is acyclic.

Proof. ($\Rightarrow$) Let $A^\bullet$ be an acyclic complex. Then the zero morphism $A^\bullet \to 0$ is a quasi-isomorphism and is by assumption mapped to a quasi-isomorphism $F(A^\bullet) \to 0$ by F. This means $F(A^\bullet)$ is acyclic.

($\Leftarrow$) Let $f\colon A^\bullet \to B^\bullet$ be a quasi-isomorphism. The image of a distinguished triangle $A^\bullet \to B^\bullet \to C(f)^\bullet \to A^\bullet[1]$ in $\mathsf{K}(\mathcal{A})$ by the triangulated functor F is a distinguished triangle $FA^\bullet \to FB^\bullet \to F(C(f)^\bullet) \to F(A^\bullet)[1]$ in $\mathsf{K}(\mathcal{B})$. By Lemma 2.21 $C(f)^\bullet$ is acyclic, implying $F(C(f)^\bullet)$ is acyclic by the assumption (ii). Hence again by Lemma 2.21 $F(f)$ is a quasi-isomorphism. $\square$

Proposition 2.23. *Let $F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ be a triangulated functor and assume it satisfies one of the equivalent conditions of Lemma 2.22. Then there exists a triangulated functor $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ on the level of derived categories, for which the following diagram of triangulated functors commutes.*

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathsf{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathsf{D}(\mathcal{A}) \arrow[r] & \mathsf{D}(\mathcal{B}) \end{tikzcd} \tag{2.6}$$

Proof. The claim follows from the universal property of $\mathsf{D}(\mathcal{A})$ (cf. proposition 2.8), as $Q_{\mathcal{B}} \circ F$ sends quasi-isomorphisms of $\mathsf{K}(\mathcal{A})$ to isomorphisms of $\mathsf{D}(\mathcal{B})$. $\square$

Every k-linear functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a triangulated functor on the level of homotopy categories $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$, defined by the assignment.

$$\begin{aligned} \textit{Objects:} \quad & A^\bullet \longmapsto \Big(\cdots \to F(A^i) \xrightarrow{F(d^i)} F(A^{i+1}) \to \cdots\Big). \\ \textit{Morphisms:} \quad & (f\colon A^\bullet \to B^\bullet) \longmapsto \big(F(f^i)\colon F(A^i) \to F(B^i)\big)_{i\in\mathbf{Z}}. \end{aligned}$$

If moreover F is assumed to be exact, then $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ satisfies the assumptions of lemma 2.23 and thus induces a triangulated functor $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ on the level of derived categories. We call this the *derived functor of F* and often denote it with F as well. We emphasise again that this convention will be used only in the case when $F\colon \mathcal{A} \to \mathcal{B}$ is an *exact* functor between abelian categories.

2.2.2 Derived functors of left (right) exact functors

Assume $F\colon \mathcal{A} \to \mathcal{B}$ is a l*eft* exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. The theory for right exact functors is obtained in parallel by dualizing everything. A crucial downside of functors, which are no longer exact, is that their induced functors on the category of complexes no longer satisfy the two equivalent conditions of Lemma 2.22, so the same idea with the universal property of Definition 2.1, which worked in the case of exact functors, cannot be applied in this case. Instead we will present two other more restrictive methods of constructing right derived functors, which are the following:

▷ Assuming $\mathcal{A}$ has enough injectives, construct a *right derived functor*

$$\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$$

for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$.

▷ Fix a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ and assume the category $\mathcal{A}$ contains a so-called *F-adapted class* then construct a *right derived functor* $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$.

These methods are however no less restrictive than the case of exact functors with our applications in mind. These methods are however no less restrictive than the case of exact functors, when viewed through the perspective of our applications later on. Lastly we note, that the first method is in some sense a special instance of the second one, which will also be explained in this subsection.

Method 1. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories with $\mathcal{A}$ having enough injectives. By Theorem 2.15 we know that the composition of $\mathsf{K}^+(\mathcal{J}) \to \mathsf{K}^+(\mathcal{A})$ followed by the localization functor $Q_{\mathcal{A}}\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A})$ is an equivalence of triangulated categories. The *right derived functor of F* is then defined to be the composition

$$\mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{K}^+(\mathcal{J}) \hookrightarrow \mathsf{K}^+(\mathcal{A}) \xrightarrow{F} \mathsf{K}^+(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}^+(\mathcal{B}) \tag{2.7}$$

The functor $\mathbf{R}F$ is clearly triangulated for it is a composition of triangulated functors. Note that in this case the functor $\mathbf{R}F$ does not necessarily make the square (2.6) commutative.

We will use this construction many times through the course of the thesis, so we reiterate the above definition by providing a simple recipe for computing the image of $\mathbf{R}F$ on complexes $A^\bullet$ of $\mathsf{D}^+(\mathcal{A})$.

1. Pick an injective resolution $A^\bullet \to I_A^\bullet$ of $A^\bullet$.

2. Apply the functor F to the complex $I_A^\bullet$ term-wise to obtain $F(I_A^\bullet)$. As a complex

$$\mathbf{R}F(A^\bullet) = F(I_A^\bullet).$$

Remark 2.24. In the slightly more general situation, if one is already given a triangulated functor $F\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathcal{B})$ at the level of homotopy categories and assumes that $\mathcal{A}$ contains enough injectives, procedure (2.7) again yields a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$. This remark will be esspecially useful when considering the *differentialy graded inner* $\mathrm{Hom}^\bullet$-functor of subsection 2.2.3, which does not appear as a functor induced by some left exact functor $\mathcal{A} \to \mathcal{B}$ on the level of abelian categories.

Remark 2.25. When considering a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$, we instead assume the category $\mathcal{A}$ contains enough projectives. In this case we use the dualized version of Theorem 2.15, stating that $\mathsf{K}^-(\mathcal{P}) \simeq \mathsf{D}^-(\mathcal{A})$, where $\mathcal{P}$ denotes the full subcategory of $\mathcal{A}$, spanned on projective objects of $\mathcal{A}$. The left derived functor is then defined as the composition

$$\mathbf{L}F\colon \quad \mathsf{D}^-(\mathcal{A}) \xrightarrow{\simeq} \mathsf{K}^-(\mathcal{P}) \hookrightarrow \mathsf{K}^-(\mathcal{A}) \xrightarrow{F} \mathsf{K}^-(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}^-(\mathcal{B}).$$

Again, as a composition of triangulated functors, we see that $\mathbf{L}F$ is triangulated as well.

Method 2. Unfortunately some of our categories will *not* contain enough injectives, as can already be seen with the category of coherent sheaves $\mathsf{coh}(X)$ of a scheme X , which is not a point. In this case Method 1 of constructing right derived functors will not work. Luckily however, there exists another way of obtaining a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$, assigned to a left exact functor $F : \mathcal{A} \to \mathcal{B}$, when $\mathcal{A}$ does not contain enough injectives, but instead possesses a broader and less restrictive class of objects. To this end we first introduce *F-adapted classes.*

Definition 2.26. A class of objects $\mathcal{I}_F \subseteq \mathcal{A}$ is *adapted* to a *left exact*[9] functor $F\colon \mathcal{A} \to \mathcal{B}$, if the following three conditions are satisfied.

(i) $\mathcal{I}_F$ is stable under finite direct sums.

I started calling biproducts direct sums now...

(ii) Every object A of $\mathcal{A}$ *embeds* into some object of $\mathcal{I}_F$, i.e. there exists an object $I \in \mathcal{I}_F$ and a monomorphism $A \hookrightarrow I$.

(iii) For every acyclic complex $I^\bullet$ in $\mathsf{K}^+(\mathcal{A})$, with $I^i \in \mathcal{I}_F$ for all $i \in \mathbf{Z}$, its image $F(I^\bullet)$ under F is also acyclic.

Example 2.27. We observe, that whenever $\mathcal{A}$ contains enough injectives, the class of all injective objects forms an F-adapted class for *every* left exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Points (i) and (ii) are clearly true for the class of injective objects and point (iii) follows from Lemma 2.17. Indeed, it tells us that an acyclic complex of injectives $I^\bullet$ is nullhomotopic, which implies $F(I^\bullet)$ is nullhomotopic as well, therefore, in particular, also acyclic.

In the presence of an F-adapted class $\mathcal{I}$ we will now construct the right derived functor of F. Firstly we upgrade the class of objects $\mathcal{I}$ to a full additive subcategory of $\mathcal{A}$, also denoted by $\mathcal{I}$, having its class of objects be precisely the F-adapted class $\mathcal{I}$. Then extending F term-wise to a functor on the level of homotopy categories, also denoted by $F\colon \mathsf{K}^+(\mathcal{I}) \to \mathsf{K}^+(\mathcal{A})$, utilizing the procedure outlined at the end of subsection 2.2.1, we obtain a functor, which maps acyclic complexes to acyclic complexes because by definition the F-adapted class $\mathcal{I}$ satisfies condition (iii) of Definition 2.26. By Proposition 2.23 the functor F then descends to a well defined functor on the level of derived categories

$$\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B}). \tag{2.8}$$

Since we want to define the derived functor $\mathbf{R}F$, whose domain is $\mathsf{D}^+(\mathcal{A})$, it remains to construct a functor $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{I})$, which we can then post-compose with $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B})$ to obtain $\mathbf{R}F$. What we will show instead is the following.

[9]Dually, a class of objects $\mathcal{P}_F$ of $\mathcal{A}$ is *adapted* to a *right exact* functor $F\colon \mathcal{A} \to \mathcal{B}$ if it is closed under finite direct sums, every object of $\mathcal{A}$ is a quotient of some object of $\mathcal{P}_F$ and for every acyclic complex $P^\bullet$ in $\mathsf{K}^-(\mathcal{A})$, with $P^i \in \mathcal{P}_F$, its image $F(P^\bullet)$ under F is also acyclic.

Proposition 2.28. *The inclusion of categories $\mathcal{I} \hookrightarrow \mathcal{A}$ induces an equivalence of triangulated categories*

$$\mathsf{D}^+(\mathcal{I}) \simeq \mathsf{D}^+(\mathcal{A}).$$

The proof of Proposition 2.28 hinges on the following lemma, closely resembling Lemma 2.13. In the same sprit, we call the complex $I^\bullet$ belonging to $\mathsf{K}^+(\mathcal{I})$ together with a quasi-isomorphism $A^\bullet \to I^\bullet$ an $\mathcal{I}$-resolution of the complex $A^\bullet$.

Lemma 2.29. *Every complex $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$ has an $\mathcal{I}$-resolution.*

Proof. A close inspection of the proof of Proposition 2.13 reveals, that formally only properties (i) and (ii) of Definition 2.26, which the class of all injective objects clearly satisfies, were used. Thus the same proof can be copied for this lemma to be true. □

Proof of Proposition 2.28. First, there exists a triangulated functor $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{A})$ by Proposition 2.23, since the inclusion $\mathcal{I} \hookrightarrow \mathcal{A}$ induces an inclusion $\mathsf{K}^+(\mathcal{I}) \hookrightarrow \mathsf{K}^+(\mathcal{A})$ sending acyclic complexes to acyclic ones. We claim, that functor $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{A})$ is an equivalence, so we will show that it is full, faithful and essentially surjective.

First, we consider the morphism action

$$\mathrm{Hom}_{\mathsf{D}^+(\mathcal{I})}(I^\bullet, J^\bullet) \to \mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet) \tag{2.9}$$

and show it to be injective for all objects $I^\bullet$, $J^\bullet$ of $\mathsf{D}^+(\mathcal{I})$. Suppose two morphisms ϕ_0, $\phi_1 \colon I^\bullet \to J^\bullet$, represented by *right* roofs $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ and $I^\bullet \to I_1^\bullet \xleftarrow{\sim} J^\bullet$, are sent to the same morphism in $\mathrm{Hom}_{\mathsf{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$, meaning that there is the following commutative diagram in $\mathsf{K}^+(\mathcal{A})$.

$$\begin{tikzcd}
 & & A^\bullet & & \\
 & I_0^\bullet \arrow[ur] & & I_1^\bullet \arrow[ul, "\sim"'] & \\
I^\bullet \arrow[ur] \arrow[urrr] & & & & J^\bullet \arrow[ul, "\sim"'] \arrow[ulll, "\sim"]
\end{tikzcd}$$

Then simply extending the diagram by a quasi-isomorphism $A^\bullet \to I_2^\bullet$, whose existence is ensured by Lemma 2.29, proves that ϕ_0 and ϕ_1 were in fact equal and that the action (2.9) is faithful.

Next, we show that the action on morphisms is also surjective, i.e. the homomorphism (2.9) is surjective. Pick any $\psi \colon I^\bullet \to J^\bullet$ in $\mathsf{D}^+(\mathcal{A})$, which is represented by a right roof $I^\bullet \to A^\bullet \xleftarrow{\sim} J^\bullet$. As before, extending the roof by a quasi-isomorphism $A^\bullet \to I_0^\bullet$ from Lemma 2.29, leaves us with a representative $I^\bullet \to I_0^\bullet \xleftarrow{\sim} J^\bullet$ of some morphism $\phi \colon I^\bullet \to J^\bullet$ in $\mathsf{D}^+(\mathcal{I})$.

$$\begin{tikzcd}
 & & I_0^\bullet & & \\
 & I_0^\bullet \arrow[ur, equal] & & A^\bullet \arrow[ul, "\sim"'] & \\
I^\bullet \arrow[ur] \arrow[urrr] & & & & J^\bullet \arrow[ul, "\sim"'] \arrow[ulll, "\sim"]
\end{tikzcd}$$

The above diagram then shows, that ϕ is sent to ψ and thus the action on morphisms is surjective.

Lastly, Lemma 2.29 asserts that our functor is essentially surjective. □

Remark 2.30. Upon close inspection, we recognise Theorem 2.15 as a special case of Proposition 2.28, namely take the F-adapted class $\mathcal{I}$ to be the class of all injectives of $\mathcal{A}$, provided there is enough of them. Indeed, it can be shown that any quasi-isomorphism

$f\colon I^\bullet \to J^\bullet$ between two bounded below complexes of injectives is an isomorphism in $\mathsf{K}^+(\mathcal{I})$. This is done by showing both $f^*\colon \mathrm{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathsf{K}^+(\mathcal{I})}(I^\bullet, I^\bullet)$ and $f_*\colon \mathrm{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathsf{K}^+(\mathcal{I})}(J^\bullet, J^\bullet)$ are isomorphisms. Lemma 2.18 already shows us that f^* is an isomorphism and by a very similar argument as in the proof of the same lemma f_* is as well.

We can now define the *right derived functor* $\mathbf{R}F$ as the composition

$$\mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{D}^+(\mathcal{I}) \longrightarrow D^+(B),$$

where $\mathsf{D}^+(\mathcal{A}) \xrightarrow{\simeq} \mathsf{D}^+(\mathcal{I})$ denotes a quasi-inverse to the equivalence $\mathsf{D}^+(\mathcal{I}) \simeq \mathsf{D}^+(\mathcal{A})$ established in Proposition 2.28 and $\mathsf{D}^+(\mathcal{I}) \to \mathsf{D}^+(\mathcal{B})$ is the functor (2.8). The procedure for computing the right derived functor $\mathbf{R}F(A^\bullet)$ of a complex $A^\bullet$ in $\mathsf{D}^+(\mathcal{A})$ is then very similar to Method 1. One takes an $\mathcal{I}$-resolution $A^\bullet \to I_A^\bullet$ of $A^\bullet$ and then applies the functor F to $I_A^\bullet$, i.e.

$$\mathbf{R}F(A^\bullet) = F(I_A^\bullet).$$

We mention that both methods give rise to right derived functors satisfying the following defining universal property. We only state this here, but refer the reader to [GM02, §III.6] for the claim and its proof or [KS06, §13.3, §10.3, §7.3] for a more comprehensive and high level treatment using the formalism of localization.

Definition 2.31. [GM02, §III.6, Definition 6] Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. The *right derived functor of* F is a triangulated functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ together with a natural transformation $\varepsilon_F\colon Q_{\mathcal{B}} \circ F \Longrightarrow \mathbf{R}F \circ Q_{\mathcal{A}}$, such that for any triangulated functor $G\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ and any natural transformation $\varepsilon\colon Q_{\mathcal{B}} \circ F \Longrightarrow G \circ Q_{\mathcal{A}}$, there exists a unique natural transformation $\eta\colon \mathbf{R}F \Longrightarrow G$, for which

$$\begin{tikzcd} & Q_{\mathcal{B}} \circ F \arrow[dl, Rightarrow, "\varepsilon_F"'] \arrow[dr, Rightarrow, "\varepsilon"] & \\ \mathbf{R}F \circ Q_{\mathcal{A}} \arrow[rr, Rightarrow, "\eta Q_{\mathcal{A}}"] & & G \circ Q_{\mathcal{A}} \end{tikzcd}$$

is commutative in the functor category $\mathrm{Fct}(\mathsf{D}^+(\mathcal{A}), \mathsf{D}^+(\mathcal{B}))$.

For two composable the following proposition tells us that first deriving and then composing or first composing and then deriving results in the same functor.

Proposition 2.32.

Proof. □

Higher derived functors

Classically derived functors of a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ first appeared as a sequence of additive functors $\mathcal{A} \to \mathcal{B}$. We introduce them in the following definition, using our established (modern) viewpoint.

Definition 2.33. For a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ of a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, we define the *higher derived functors of* F as compositions

$$\mathbf{R}^i F := H^i \circ \mathbf{R}F\colon \quad \mathsf{D}^+(\mathcal{A}) \to \mathcal{B}$$

for all $i \in \mathbf{Z}$.

Frequently we will be interested in the image of a higher derived functor $\mathbf{R}^i F$ of just a single object A belonging to category $\mathcal{A}$. In that case we will always interpret[10] it as a bounded complex concentrated in degree 0.

Example 2.34. For an object A of $\mathcal{A}$, equipped with an F-adapted class $\mathcal{I}$, we can compute that $\mathbf{R}^i F(A) \simeq 0$, for $i < 0$, and $\mathbf{R}^0 F(A) \simeq F(A)$. Indeed, consider an $\mathcal{I}$-resolution $I^\bullet$ of A. Then, as $I^i = 0$ for $i < 0$, we have that $\mathbf{R}^i F(A) = H^i(F(I^\bullet)) \simeq 0$ for $i < 0$. For the second claim, because $I^\bullet$ is quasi-isomorphic to A, we have $\ker(I^0 \to I^1) \simeq A$, and since F is left exact, we conclude that $F(A) \simeq \ker(F(I^0) \to F(I^1)) \simeq H^0(F(I^\bullet)) = \mathbf{R}^0 F(A)$.

Definition 2.35. Suppose the right derived functor $\mathbf{R}F$ exists. We say an object A of $\mathcal{A}$ is *F-acyclic*, if $\mathbf{R}^i F(A) \simeq 0$ for all $i \neq 0$.

A useful tool for gathering information about the action of the higher derived functors of a left exact functor F on objects will be the long exact sequence associated to F introduced in the following proposition. In view of this proposition we also interpret higher derived functors of F as measuring the extent to which F fails to be exact.

Proposition 2.36. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then there exists a long exact sequence*

$$\begin{aligned} 0 \to F(A) \to F(B) \to F(C) \to \mathbf{R}^1 F(A) \to \mathbf{R}^1 F(B) \to \mathbf{R}^1 F(C) \to \cdots \\ \cdots \to \mathbf{R}^i F(A) \to \mathbf{R}^i F(B) \to \mathbf{R}^i F(C) \to \mathbf{R}^{i+1} F(A) \to \cdots . \end{aligned} \tag{2.10}$$

Proof. First, we describe how the short exact sequence gives rise to a distinguished triangle $A \to B \to C \to A[1]$ in $\mathsf{D}^+(\mathcal{A})$, where A, B and C are considered as complexes concentrated in degree 0. finish this

After first applying the triangulated functor $\mathbf{R}F$ and then the cohomological functor H^0 to the triangle $A \to B \to C \to A[1]$, we obtain a long exact sequence

$$\begin{aligned} 0 \to H^0(\mathbf{R}F(A)) \to H^0(\mathbf{R}F(B)) \to H^0(\mathbf{R}F(C)) \to H^0(\mathbf{R}F(A)[1]) \to \cdots \\ \cdots H^0(\mathbf{R}F(A)[i]) \to H^0(\mathbf{R}F(B)[i]) \to H^0(\mathbf{R}F(C)[i]) \to H^0(\mathbf{R}F(A)[i+1]) \to \cdots \end{aligned}$$

by Proposition 1.46, whose terms are readily seen to be isomorphic to the terms of (2.10). ∎

The rest of this section is devoted to proving some useful results relating F-acyclic and F-adapted objects.

Lemma 2.37. *Every object of an F-adapted class is also F-acyclic.*

Proof. We only have to recall definition 2.35. An object I belonging to an F-adapted class $\mathcal{I}$ trivially forms its own $\mathcal{I}$-resolution $I^\bullet = (\cdots \to 0 \to I \to 0 \to \cdots)$, which allows us to compute $\mathbf{R}^i F(I) = H^i(\mathbf{R}F(I^\bullet)) = H^i(F(I^\bullet)) \simeq 0$, whenever $i \neq 0$ (and also $F(I)$ for $i = 0$). ∎

Proposition 2.38. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. Let $\mathcal{I} \subset \mathcal{A}$ be an F-adapted class of objects in $\mathcal{A}$. Then the class of all F-acyclic objects of $\mathcal{A}$, denoted by $\mathcal{I}_F$, is also F-adapted.*

[10] More precisely, there is always a fully faithful functor $\mathcal{A} \to \mathsf{K}(\mathcal{A})$ sending an object A to a complex $\cdots \to 0 \to A \to 0 \to \cdots$ concentrated in degree 0 and a morphism $f\colon A \to B$ to the homotopy class of the chain map defined by $f^0 = f$ and $f^i = 0$, for $i \neq 0$ [Mil, Chapter 3, Lemma 1.3.5].

Proof. We verify conditions (i)–(iii) of definition 2.26. (i) Since the higher derived functors $\mathbf{R}^i F$ are additive, $\mathcal{I}_F$ is stable under finite sums. (ii) Holds by lemma 2.37. (iii) Suppose $A^\bullet$ is an acyclic complex in $K^+(\mathcal{A})$ with $A^i \in \mathcal{I}_F$ for all i. Then using acyclicity of $A^\bullet$, we may break this complex up into a series of short exact sequences

$$0 \to \ker d^i \to A^i \to \ker d^{i+1} \to 0 \quad \text{for } i \in \mathbf{Z}. \tag{2.11}$$

As our complex $A^\bullet$ is supported on $\mathbf{Z}_{\geq 0}$, the first interesting short exact sequence happens at index $i = 1$, where $\ker d^1 = \operatorname{im} d^0 \simeq A^0$ It yields the following long exact sequence associated to F

$$\begin{aligned} 0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to \mathbf{R}^1 F(A^0) \to \cdots \\ \cdots \to \mathbf{R}^i F(A^0) \to \mathbf{R}^i F(A^1) \to \mathbf{R}^i F(\ker d^2) \to \mathbf{R}^{i+1} F(A^0) \to \cdots . \end{aligned}$$

Exactness of this sequence together with F-acyclicity of A^0 and A^1 shows that $\ker d^2$ is also F-acyclic and by only considering the beginning few terms of the sequence, we obtain the short exact sequence

$$0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to 0.$$

After inductively applying the same reasoning for each of the original short exact sequences of (2.11) for $i > 1$, we are left with short exact sequences[11]

$$0 \to \ker Fd^i \to F(A^i) \to \ker Fd^{i+1} \to 0 \quad \text{for } i \geq 1.$$

Collecting them all back together we conclude that the complex $F(A^\bullet)$ is acyclic. □

2.2.3 Ext functors

As a first application of the established theory of derived functors we consider the Hom-functor

$$\operatorname{Hom}_{\mathcal{A}}(A, -) \colon \mathcal{A} \to \mathsf{Mod}_k,$$

where A is an object of an abelian category $\mathcal{A}$. Assume that category $\mathcal{A}$ contains enough injectives. By Example 1.10, we know that $\operatorname{Hom}_{\mathcal{A}}(A, -)$ is a left exact functor, so we can right derive it, to obtain the functor

$$\mathbf{R} \operatorname{Hom}_{\mathcal{A}}(A, -) \colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Mod}_k).$$

Taking the i-th cohomology, we obtain higher derived functors of $\operatorname{Hom}_{\mathcal{A}}(A, -)$, called the *Ext-functors*

$$\operatorname{Ext}^i_{\mathcal{A}}(A, -) := \mathbf{R}^i \operatorname{Hom}_{\mathcal{A}}(A, -) = H^i(\mathbf{R} \operatorname{Hom}_{\mathcal{A}}(A, -)).$$

There exists a very beautiful connection relating the Ext-functors of category $\mathcal{A}$ and the hom-functors of the bounded below derived category $\mathsf{D}^+(\mathcal{A})$ captured in the next proposion, given without proof, for in a moment we will present its generalized verison.

Proposition 2.39. *Let $\mathcal{A}$ be an abelian category with enough injectives and let A and B belong to $\mathcal{A}$. Then for all $i \in \mathbf{Z}$ there exist isomorphisms*

$$\operatorname{Ext}^i_{\mathcal{A}}(A, B) \simeq \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A, B[i]).$$

[11] To be more precise, we have used that F is left exact in order to swap the order of F and ker to reach

$$\ker Fd^i \simeq F(\ker d^i)$$

.

For any two complexes $A^\bullet$ and $B^\bullet$ we can define a complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ of k-modules given by terms

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) = \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, B^{j+i}),$$

i.e. the set of all degree i maps from $A^\bullet$ to $B^\bullet$, considered as graded objects of $\mathcal{A}^{\mathbf{Z}}$, and differentials

$$d^i\colon \mathrm{Hom}^i(A^\bullet, B^\bullet) \to \mathrm{Hom}^{i+1}(A^\bullet, B^\bullet), \qquad d^i(f) = \left(d_B^{i+j} \circ f^j + (-1)^{i+1} f^{j+1} \circ d_A^j\right)_{j\in\mathbf{Z}}.$$

This differential will also be denoted by δ_B^i.

By setting $\mathrm{Hom}^\bullet(A^\bullet, -)(B^\bullet) = \mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ and for a chain map $u\colon B^\bullet \to C^\bullet$ in $\mathsf{Ch}(\mathcal{A})$ defining $u_* = \mathrm{Hom}^\bullet(A^\bullet, -)(u)\colon \mathrm{Hom}^\bullet(A^\bullet, B^\bullet) \to \mathrm{Hom}^\bullet(A^\bullet, C^\bullet)$ to be the chain map given by the collection $(u_*^i)_{i\in\mathbf{Z}}$, where u_*^i sends $f = (f^j)_{j\in\mathbf{Z}} \in \mathrm{Hom}^i(A^\bullet, B^\bullet)$ to $u \circ f = (u^{i+j} \circ f^j)_{j\in\mathbf{Z}}$, we obtain a functor

$$\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Mod}_k).$$

Indeed, clearly u_* is a graded morphism of degree 0, which moreover commutes with the differentials, according to the following computation

$$\begin{aligned}
u_*^{i+1}\left(d^i_{\mathrm{Hom}^\bullet(A,B)}(f)\right) &= u_*^{i+1}\left(\left(d_B^{i+j} f^j + (-1)^{i+1} f^{j+1} d_A^j\right)_j\right)\\
&= \left(u^{i+j+1} d_B^{i+j} f^j + (-1)^{i+1} u^{i+j+1} f^{j+1} d_A^j\right)_j\\
&= \left(d_C^{i+j} u^{i+j} f^j + (-1)^{i+1} u^{i+j+1} f^{j+1} d_A^j\right)_j\\
&= d^i_{\mathrm{Hom}^\bullet(A,C)}\left(\left(u^{i+j} f^j\right)_j\right)\\
&= d^i_{\mathrm{Hom}^\bullet(A,C)}\left(u_*^i(f)\right),
\end{aligned}$$

for all $f \in \mathrm{Hom}^i(A^\bullet, B^\bullet)$. While not difficult to check, the verification of functoriality is omitted.

Next, we would like $\mathrm{Hom}^\bullet(A^\bullet, -)$ to descend to a homotopy functor $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{Mod}_k)$. This is in fact so, as a null-homotopic chain map $u\colon B^\bullet \to C^\bullet$ is sent to a null-homotopic chain map $u_* \simeq 0$. Indeed, suppose homotopy $h \in \mathrm{Hom}^{-1}(B^\bullet, C^\bullet)$ witnesses $u \simeq 0$, then for all $i \in \mathbf{Z}$, we have

$$u^i = d_C^{i-1} \circ h^i + h^{i+1} \circ d_B^i.$$

We let h_*^i denote the morphism $\mathrm{Hom}^i(A^\bullet, B^\bullet) \to \mathrm{Hom}^{i-1}(A^\bullet, C^\bullet)$, given by sending f to $(h^{i+j} \circ f^j)_{j\in\mathbf{Z}}$, and then compute

$$\begin{aligned}
(\delta_C^{i-1} \circ h_*^i + h_*^{i+1} \circ \delta_B^i)(f) &= \delta_C^{i-1}\left((h^{i+j} f^j)_j\right) + h_*^{i+1}\left((d_B^{i+j} f^j + (-1)^{i+1} f^{j+1} d_A^j)_j\right)\\
&= \left(d_C^{i+j-1} h^{i+j} f^j + (-1)^i h^{i+j+1} f^{j+1} d_A^j\right)_j +\\
&\qquad + \left(h^{i+j+1} d_B^{i+j} f^j + (-1)^{i+1} h^{i+j+1} f^{j+1} d_A^j\right)_j\\
&= \left(d_C^{i+j-1} h^{i+j} f^j + h^{i+j+1} d_B^{i+j} f^j\right)_j\\
&= \left((d_C^{i+j-1} h^{i+j} + h^{i+j+1} d_B^{i+j}) \circ f^j\right)_j\\
&= \left(u^{i+j} \circ f^j\right)_j\\
&= u_*^i(f).
\end{aligned}$$

Thus the homotopy $h_* = (h^i_*)_{i\in\mathbf{Z}}$ whitnesses $u_* \simeq 0$, allowing us to conclude that

$$\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{Mod}_k)$$

is well defined. The preceding functor is also triangulated, because it commutes with the translation functors and it sends distinguished triangles of $\mathsf{K}(\mathcal{A})$ to distinguished triangles of $\mathsf{K}(\mathsf{Mod}_k)$ This is because of the following isomorphism of chain complexes of k-modules,

$$C(u_*)^\bullet = C(\mathrm{Hom}^\bullet(A^\bullet, -)(u))^\bullet \cong \mathrm{Hom}^\bullet(A^\bullet, -)(C(u)^\bullet),$$

which holds for any chain map $u\colon B^\bullet \to C^\bullet$. Indeed, we have

$$\begin{aligned}
C(u_*)^i &= \mathrm{Hom}^{i+1}(A^\bullet, B^\bullet) \oplus \mathrm{Hom}^i(A^\bullet, C^\bullet)\\
&= \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1}) \oplus \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, C^{i+j})\\
&\cong \prod_{j\in\mathbf{Z}} \Big(\mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1}) \oplus \mathrm{Hom}_{\mathcal{A}}(A^j, C^{i+j})\Big)\\
&\cong \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, B^{i+j+1} \oplus C^{i+j})\\
&= \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(A^j, C(u)^{i+j})\\
&= \mathrm{Hom}^i(A^\bullet, C(u)^\bullet),
\end{aligned}$$

where these isomorphisms also commute with the differentials. Pick $f \in \mathrm{Hom}^i(A^\bullet, C(u)^\bullet)$ which is given by a pair f

At last, for $A^\bullet$ belonging to $\mathsf{K}^-(\mathcal{A})$ i.e. bounded from *above*[12], we have a triangulated functor $\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathsf{Mod}_k)$, which we may derive, in accordance with Remark 2.24, of course assuming $\mathcal{A}$ contains enough injectives, to obtain

$$\mathbf{R}\,\mathrm{Hom}^\bullet(A^\bullet, -)\colon \mathsf{D}^+(\mathcal{A}) \longrightarrow \mathsf{D}^+(\mathsf{Mod}_k).$$

Its higher derived counterpart will be denoted by $\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, -) := \mathbf{R}^i\,\mathrm{Hom}^\bullet(A^\bullet, -)$.

Remark 2.40. Behind the scenes of this whole process one actually consideres an additive bifunctor $F\colon \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$, with $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ abelian, where the latter is assumed to contain countable products. From this bifunctor we induce another additive bifunctor on the level of chain complexes $F_\pi\colon \mathsf{Ch}(\mathcal{A}) \times \mathsf{Ch}(\mathcal{A}') \to \mathsf{Ch}(\mathcal{A}'')$ via the total complex of a double complex as described in [KS06, §11.6]. It then turns out that F_π induces a well defined triangulated bifunctor on the level of the homotopy category

$$\mathsf{K}(\mathcal{A}) \times \mathsf{K}(\mathcal{A}') \to \mathsf{K}(\mathcal{A}'').$$

Proposition 2.41. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for every $A^\bullet$ of $\mathsf{K}^-(\mathcal{A})$ and $B^\bullet$ belonging to $\mathsf{K}^+(\mathcal{A})$ there natural isomorphisms*

$$\alpha_{A^\bullet, B^\bullet} : \mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A^\bullet, B[i]^\bullet).$$

Proposition 2.42. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for every $A^\bullet$ and $B^\bullet$ belonging to $\mathsf{D}^b(\mathcal{A})$ there natural isomorphisms*

I think this is true, but I won't be able to show it as I need to use $\mathsf{D}^+(\mathcal{A}) \simeq \mathsf{K}^+(\mathcal{J})$, since $A^\bullet$ is not bounded from below.

$$\alpha_{A^\bullet,B^\bullet} : \operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \simeq \operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B[i]^\bullet).$$

Lemma 2.43. *Let $A^\bullet$ and $B^\bullet$ be chain complexes in $\mathsf{K}(\mathcal{A})$ and let $i \in \mathbf{Z}$. Then*

$$H^i(\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)) = \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B[i]^\bullet).$$

Proof. This is practically the definition of $\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B[i]^\bullet)$. The cohomology k-module $H^i(\operatorname{Hom}^\bullet(A^\bullet, B^\bullet))$ is defined to be the quotient $\ker \delta^i / \operatorname{im} \delta^{i-1}$, thus let us identify $\ker \delta^i$ and $\operatorname{im} \delta^{i-1}$. Notice, that for $f \in \operatorname{Hom}^i(A^\bullet, B^\bullet)$ we have

$$\delta^i(f) = \left(d_B^{i+j} \circ f^j + (-1)^{i+1} f^{j+1} \circ d_A^j\right)_{j\in\mathbf{Z}} = (-1)^i \left(d^j_{B[i]} \circ f^j - f^{j+1} \circ d^j_A\right)_{j\in\mathbf{Z}},$$

thus $\ker \delta^i$ is the set of all chain maps $A^\bullet \to B[i]^\bullet$, and $\operatorname{im} \delta^{i-1}$ the set of all null-homotopic chain maps. □

Proof of Proposition 2.42. Let $B^\bullet \to I^\bullet$ be an injective resolution for $B^\bullet$. We follow the chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{D}^b(\mathcal{A})}(A^\bullet, B[i]^\bullet) &= \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, B[i]^\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{D}^+(\mathcal{A})}(A^\bullet, I[i]^\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{K}^+(\mathcal{J})}(A^\bullet, I[i]^\bullet) && \text{(Theorem 2.15)} \\
&= H^i(\operatorname{Hom}^\bullet(A^\bullet, I^\bullet)) && \text{(Lemma 2.43)} \\
&= \operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet).
\end{aligned}$$

□

[12] This assumption cannot be skipped, for otherwise we cannot claim, that the functor $\operatorname{Hom}^\bullet(A^\bullet, -)$ maps into the bounded *below* homotopy category $\mathsf{K}^+(\mathcal{A})$.

3 Derived categories in geometry

3.1 Derived category of coherent sheaves

One of the main invariants of a scheme X over k is its category of coherent sheaves $\mathsf{coh}(X)$. We can think of this category as a slight extension of the category of locally free $\mathcal{O}_X$-modules of finite rank also known as k-vector bundles on X in the sense that $\mathsf{coh}(X)$ is abelian, where as the former very often is not. In this chapter we restrict ourself to noetherian schemes and our primary object of study will be the bounded derived category of coherent sheaves

$$\mathsf{D}^b(X) := \mathsf{D}^b(\mathsf{coh}(X)).$$

Proposition 3.1. *The category of quasi-coherent sheaves $\mathsf{qcoh}(X)$ of a noetherian scheme X contains enough injectives.*

Proposition 3.2. *For a noetherian scheme X the inclusion $\mathsf{D}^b(X) \hookrightarrow \mathsf{D}^b(\mathsf{qcoh}(X))$ induces an equivalence of triangulated categories*

$$\mathsf{D}^b(X) \simeq \mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(X)),$$

where $\mathsf{D}^b_{\mathsf{coh}}(\mathsf{qcoh}(X))$ is the full triangulated subcategory of $\mathsf{D}^b(\mathsf{qcoh}(X))$, spanned on bounded complexes of quasi-coherent shaves on X with coherent cohomology.

Proposition 3.3 ([Huy06, §3, Proposition 3.17] or [Bri98, Example 2.2])**.** *Let X be a smooth projective variety over k. The set of all skyscraper sheaves $k(x)$, for closed points $x \in X$, forms a spanning class of the bounded derived category $\mathsf{D}^b(X)$.*

Proof. □

Theorem 3.4 ([Huy06, §3, Proposition 3.10])**.** *The bounded derived category $\mathsf{D}^b(X)$ of a connected noetherian scheme X is indecomposable.*

Proof. □

Serre's vanishing results

3.2 Derived functors in algebraic geometry

In this subsection we will derive some functors occuring in algebraic geometry and later state some important facts relating them with each other. To derive these functors we will either deal with injective sheaves and access the realm of quasi-coherent sheaves, following the first part of section **??**, or introduce certain special classes of coherent sheaves depending on the functor we wish to derive and taking up the role of adapted classes, mirroring what was done in subsection **??**.

3.2.1 Global sections functor

Arguably the most common functor in algebraic geometry is the global sections functor. Associated to a scheme X, the functor of *global sections*

$$\Gamma \colon \mathsf{Mod}_{\mathcal{O}_X} \to \mathsf{Mod}_k$$

assigns to each $\mathcal{O}_X$-module $\mathcal{F}$ its module of global sections $\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ and to a morphism of $\mathcal{O}_X$-modules $\alpha \colon \mathcal{F} \to \mathcal{G}$ a homomorphism $\alpha_X \colon \mathcal{F}(X) \to \mathcal{G}(X)$. By abuse of notation we use Γ to also denote the restrictions of the global sections functor to subcategories $\mathsf{qcoh}(X)$ and $\mathsf{coh}(X)$ of $\mathsf{Mod}_{\mathcal{O}_X}$.

As is commonly known, Γ is a left exact functor, so our aim is to construct its right derived counterpart. Due to $\mathsf{coh}(X)$ not having enough injectives, we resort to $\mathsf{qcoh}(X)$, on which we may define

$$\mathbf{R}\Gamma \colon \mathsf{D}^+(\mathsf{qcoh}(X)) \to \mathsf{D}^+(\mathsf{Mod}_k),$$

according to 2.2.2. Precomposing the latter functor with the inclusion $\mathsf{D}^+(\mathsf{coh}(X)) \to \mathsf{D}^+(\mathsf{qcoh}(X))$ leaves us with the derived functor

$$\mathbf{R}\Gamma \colon \mathsf{D}^+(\mathsf{coh}(X)) \to \mathsf{D}^+(\mathsf{Mod}_k).$$

3.2.2 Push-forward f_*

3.2.3 Inner hom

3.2.4 Tensor product

3.3 Interactions between derived functors

projection formula, base change, GV-duality

4 Fourier-Mukai transforms

Example 4.1. (i) $\mathrm{id}_{\mathsf{D}^b(X)} \simeq \Phi_{\mathcal{O}_{\Delta_X}}$. Recall that $\mathcal{O}_{\Delta_X}$ is defined to be push-forward $\Delta_*\mathcal{O}_X$, of the structure sheaf of X along the diagonal embedding $\Delta\colon X \to X \times X$. Then for any object E of $\mathsf{D}^b(X)$ we compute

$$\begin{aligned}\Phi_{\mathcal{O}_{\Delta_X}} &= p_*(\mathcal{O}_{\Delta_X} \otimes_{\mathcal{O}_{X\times X}} p^*(E))\\ &= p_*(\Delta_*\mathcal{O}_X \otimes_{\mathcal{O}_{X\times X}} p^*(E))\\ &\simeq p_*(\Delta_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} \Delta^*p^*(E)))\\ &\simeq \mathrm{id}_{\mathsf{D}^b(X)}(E).\end{aligned}$$

All isomorphisms are natural in E, thus proving $\mathrm{id}_{\mathsf{D}^b(X)} \simeq \Phi_{\mathcal{O}_{\Delta_X}}$.

(ii) $(-)[1] \simeq \Phi_{\mathcal{O}_{\Delta_X}[1]}$. A similar computation as above also identifies the translation functor as a Fourier-Mukai transform

Theorem 4.2 ([Orl97, Theorem 2.2])**.** *Let X and Y be smooth projective varieties over a field k. Suppose $F\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ is a fully faithful functor admitting a right (and, consequently, left) adjoint functor. Then there exists up to an isomorphism a unique object E of $\mathsf{D}^b(X \times Y)$ such that F and Φ_E are naturally isomorphic.*

Proposition 4.3. *Let $\Phi_{\mathcal{E}^\bullet}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ and $\Phi_{\mathcal{F}^\bullet}\colon \mathsf{D}^b(Y) \to \mathsf{D}^b(Z)$ be Fourier-Mukai transforms with kernels $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ belonging to $\mathsf{D}^b(X \times Y)$ and $\mathsf{D}^b(Y \times Z)$ respectively. Then the composition $\Phi_{\mathcal{F}^\bullet} \circ \Phi_{\mathcal{E}^\bullet}\colon \mathsf{D}^b(X) \to \mathsf{D}^b(Z)$ is also a Fourier-Mukai transform with kernel $\mathcal{R}^\bullet = ...$, which is sometimes also called the* convolution *of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$.*

4.1 on K-groups

4.2 on rational cohomology

Proposition 4.4. *Let $f^{\mathcal{E}^\bullet}\colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q})$ and $f^{\mathcal{F}^\bullet}\colon H^\bullet(Y, \mathbf{Q}) \to H^\bullet(Z, \mathbf{Q})$ be cohomological Fourier-Mukai transforms for $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ belonging to $\mathsf{D}^b(X \times Y)$ and $\mathsf{D}^b(Y \times Z)$ respectively. Then the composition $f^{\mathcal{F}^\bullet} \circ f^{\mathcal{E}^\bullet}$ is also a cohomological Fourier-Mukai transform with kernel $\mathcal{R}^\bullet$ of Proposition* **??**

Theorem 4.5 (Grothendieck-Riemann-Roch)**.** *Let*

Remark 4.6. What's the deal with Chow groups $A_{\mathbf{Q}}(X)$

Corollary 4.7 (Hirzebruch-Riemann-Roch)**.**

$$\chi(X, \mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\,\mathrm{td}_X\,.$$

Remark 4.8. Meaning of integral $\int_X \mathrm{ch}(\mathcal{E})\,\mathrm{td}_X = \langle \mathrm{ch}(\mathcal{E})\,\mathrm{td}_X, [X]\rangle$

Proof. from GRR. □

Proposition 4.9. *Let $f\colon X \to Y$ be a continuous map between closed orientable manifolds X and Y. Then for all $\alpha \in H^*(X, \mathbf{Z})$ and $\beta \in H^*(Y, \mathbf{Z})$*

$$f_*\alpha \smile \beta = f_*(\alpha \smile f^*\beta).$$

Proof. □

Proposition 4.10. *Then for every pair of integers (p, q)*

$$f^E_{\mathbf{C}}\left(H^{p,q}(X)\right) \subseteq \bigoplus_{s-t=p-q} H^{s,t}(Y). \tag{4.1}$$

5 K3 surfaces

5.1 Algebraic and complex

5.2 Main invariants

5.2.1 Cohomology

5.2.2 Intersection pairing

Proposition 5.1. *Let X be a K3 surface. Then the intersection pairing on $H^2(X, \mathbf{Z})$ given by the $\smile$-product is an even lattice.*

5.2.3 Hodge structure

5.3 Two important theorems

Remark 5.2. For a topological space X we will distinguish between the collection of all cohomology groups, considered concisely as a graded object $H^\bullet(X, \mathbf{Z}) = \bigoplus_n H^n(X, \mathbf{Z})$, and its (graded) cohomology ring, which we will denote by $H^*(X, \mathbf{Z})$.

6 Derived Torelli theorem

In this last chapter we will again be dealing with K3 surfaces over the field of complex numbers $\mathbf{C}$. In particular we will characterize when two K3 surfaces X and Y have equivalent bounded derived categories, through their internal structure, namely their so called *transcendental lattices*. For a K3 surface X, its *transcendental lattice* is defined to be the orthogonal complement $\mathrm{NS}(X)^{\perp} \subseteq H^2(X, \mathbf{Z})$ with respect to the intersection pairing $\langle -, - \rangle$ on $H^2(X, \mathbf{Z})$, introduced in the previous chapter. The transcendental lattice of X will be denoted by T_X. Naturally, T_X also comes equipped with a weight 2 Hodge structure induced from that of $H^2(X, \mathbf{Z})$.

todd class of K3

Theorem 6.1. *Let X and Y be K3 surfaces over the field of complex numbers $\mathbf{C}$. Then $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$ if and only if there exists a Hodge isometry $f \colon T_X \to T_Y$ between their transcendental lattices.*

We will be using Fourier-Mukai transforms in an essential way. When applied to the case of K3 surfaces adjoints of Fourier-Mukai transforms take up an interesting form.

Thus we see that in the case of K3 surfaces every left and right adjoints of Fourier-Mukai transforms agree. Here we have of course used one of the defining properties of K3 surfaces – triviality of their canonical bundle.

6.1 Mukai lattice

In order to consider all the cohomology groups of a K3 surface X at once Mukai introduced the following pairing on $H^\bullet(X, \mathbf{Z})$. For

For this lattice to nicely fit into our context of Hodge lattices, we also equip it with a compatible weight 2 Hodge structure. This is achieved through the introduction of the *Mukai lattice*.

Definition 6.2. The *Mukai lattice* of a K3 surface X over $\mathbf{C}$, denoted $\widetilde{H}(X, \mathbf{Z})$, consists of the free abelian group $H^\bullet(X, \mathbf{Z})$ together with the bilinear pairing introduced above in (**??**) equipped with a weight 2 Hodge structure described by

$$\begin{array}{rcc} \widetilde{H}^{2,0}(X) & = & H^{2,0}(X) \\ \widetilde{H}^{1,1}(X) & = & H^0(X, \mathbf{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbf{C}) \\ \widetilde{H}^{0,2}(X) & = & H^{0,2}(X). \end{array}$$

6.2 Moduli space of sheaves on a K3 surface

6.3 Proof

recall what the projections p and q are.

Implication $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y) \implies \widetilde{H}(X, \mathbf{Z}) \simeq \widetilde{H}(Y, \mathbf{Z})$. After first establishing a few preliminary results, we will prove Proposition 6.3 by borrowing the kernel of a Fourier-Mukai transform appearing as the equivalence $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$, which Orlov's result 4.2 enables us to do, and then show that its corresponding cohomological version gives a Hodge isometry $\widetilde{H}(X, \mathbf{Z}) \simeq \widetilde{H}(Y, \mathbf{Z})$.

Proposition 6.3. *Suppose there exists an equivalence of triangulated categories $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$, then there exists a Hodge isometry $\widetilde{H}(X, \mathbf{Z}) \simeq \widetilde{H}(Y, \mathbf{Z})$.*

Lemma 6.4. *For any object $\mathcal{E}^\bullet$ of $\mathsf{D}^b(X \times Y)$, its Chern character $\mathrm{ch}(\mathcal{E}^\bullet)$ is integral, i.e. belongs to $H^\bullet(X \times Y, \mathbf{Z}) \subseteq H^\bullet(X \times Y, \mathbf{Q})$. Consequently the Mukai vector $v(\mathcal{E}^\bullet)$ is integral as well.*

Proof. We will use the Künneth formula in an essential way so we recall it here. Since X and Y, as K3 surfaces, have torsion-free cohomology the n-th cohomology of the product $X \times Y$ decomposes as

$$H^n(X \times Y, \mathbf{Z}) \simeq \bigoplus_{p+q=n} H^p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} H^q(Y, \mathbf{Z}),$$

for all $n \in \mathbf{Z}_{\geq 0}$.

First, rewrite the Chern character with respect to the ordinary grading on cohomology $H^\bullet(X \times Y, \mathbf{Q}) = \bigoplus_{i=0}^4 H^{2i}(X \times Y, \mathbf{Q})$ as the vector

$$\mathrm{ch}(\mathcal{E}^\bullet) = \left(\mathrm{rk}(\mathcal{E}^\bullet), \mathrm{c}_1(\mathcal{E}^\bullet), \tfrac{1}{2}(\mathrm{c}_1(\mathcal{E}^\bullet)^2 - 2\mathrm{c}_2(\mathcal{E}^\bullet)), \mathrm{ch}^2(\mathcal{E}^\bullet), \mathrm{ch}^3(\mathcal{E}^\bullet)\right).$$

Components $\mathrm{rk}(\mathcal{E}^\bullet)$ and $\mathrm{c}_1(\mathcal{E}^\bullet)$ are integral by definition. The first Chern class $\mathrm{c}_1(\mathcal{E}^\bullet) \in H^2(X \times Y, \mathbf{Z})$ may be rewritten, in accordance with the Künneth decomposition

$$H^2(X \times Y, \mathbf{Z}) \simeq H^2(X, \mathbf{Z}) \oplus H^2(Y, \mathbf{Z}),$$

in the form $\mathrm{c}_1(\mathcal{E}^\bullet) = p^*\alpha + q^*\beta$, for some $\alpha \in H^2(X, \mathbf{Z})$ and $\beta \in H^2(Y, \mathbf{Z})$. Therefore, as K3 surfaces X and Y have *even* intersection pairings by Proposition 5.1, we see that $\mathrm{c}_1(\mathcal{E}^\bullet)^2 = p^*\alpha^2 + 2p^*\alpha.q^*\beta + q^*\beta^2$ is an even multiple of some class from $H^4(X \times Y, \mathbf{Z})$. Along with $\mathrm{c}_2(\mathcal{E}^\bullet)$ being integral this shows the $H^4(X \times Y, \mathbf{Q})$-component of $\mathrm{ch}(\mathcal{E}^\bullet)$ is integral.

□

Lemma 6.5. *Let X be a K3 surface over $\mathbf{C}$ and $\nu\colon X \to \operatorname{Spec} \mathbf{C}$ its structure map. Then for all α, $\alpha' \in \widetilde{H}(X, \mathbf{Z})$*

$$\langle \alpha, \alpha' \rangle = \nu_*(\alpha \smile \alpha')$$

and

$$\nu_*(\alpha^\vee) = \nu_*(\alpha).$$

Proof. □

Proof of Proposition 6.3. By Orlov's Theorem 4.2, the functor witnessing the equivalence $\mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$ is naturally isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{E}^\bullet}$ for some kernel $\mathcal{E}^\bullet$ of the category $\mathsf{D}^b(X \times Y)$. We will show that the cohomological Fourier-Mukai transform $f^{\mathcal{E}^\bullet}\colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q})$ of section 4.2 induces a Hodge isometry between the Mukai lattices

$$f^{\mathcal{E}^\bullet}\colon \widetilde{H}(X, \mathbf{Z}) \to \widetilde{H}(Y, \mathbf{Z}).$$

First recall that by Definition **??**, the cohomologiacl Fourier-Mukai transform $f^{\mathcal{E}^\bullet}$ is defined to be

$$f^{\mathcal{E}^\bullet}\colon H^\bullet(X, \mathbf{Q}) \to H^\bullet(Y, \mathbf{Q}) \qquad \alpha \mapsto p_*(v(\mathcal{E}^\bullet) \smile q^*(\alpha)).$$

By lemma 6.4 the Mukai vector $v(\mathcal{E}^\bullet)$ lies in $H^\bullet(X \times Y, \mathbf{Z}) \subseteq H^\bullet(X \times Y, \mathbf{Q})$, thus $f^{\mathcal{E}^\bullet}$ may be restricted to the integral parts to form an additive map

$$f^{\mathcal{E}^\bullet}\colon H^\bullet(X, \mathbf{Z}) \to H^\bullet(Y, \mathbf{Z}).$$

Next we verify that $f^{\mathcal{E}^\bullet}$ is bijective. We do so by constructing its right inverse. For $\Phi_{\mathcal{E}^\bullet}$ is an equivalence, the Fourier-Mukai transform associated to the the kernel $\mathcal{E}^\bullet_L$, of its left adjoint, is actually its quasi-inverse. Thus from $\Phi_{\mathcal{E}^\bullet_L} \circ \Phi_{\mathcal{E}^\bullet} \simeq \mathrm{id}_{\mathsf{D}^b(X)}$ along with uniqueness of kernels and Example 4.1, we see that $\mathcal{O}_{\Delta_X} \simeq \mathcal{E}^\bullet_L * \mathcal{E}^\bullet$. As cohomological Fourier-Mukai transforms compose just like the categorical ones do, according to Proposition 4.4, we see that $f^{\mathcal{E}_L^\bullet} \circ f^{\mathcal{E}^\bullet} = f^{\mathcal{O}_{\Delta_X}}$, thus it suffices to show that $f^{\mathcal{O}_{\Delta_X}} = \mathrm{id}_{H^\bullet(X,\mathbf{Z})}$.

We start by computing the Mukai vector of $\mathcal{O}_{\Delta_X}$. Consulting the help of Grothendieck-Riemann-Roch 4.5, we see that

$$\begin{aligned}\operatorname{ch}(\mathcal{O}_{\Delta_X})\,\mathrm{td}_{X\times X} &= \operatorname{ch}(\Delta_*\mathcal{O}_X)\,\mathrm{td}_{X\times X} = \\ &= \operatorname{ch}(\Delta_!\mathcal{O}_X)\,\mathrm{td}_{X\times X} = \Delta_*(\operatorname{ch}(\mathcal{O}_X)\,\mathrm{td}_X) = \Delta_*\,\mathrm{td}_X,\end{aligned}$$

holds true in $H^*(X\times X,\mathbf{Q})$. In the second equality, we have used the relation $\Delta_*\mathcal{O}_X = \Delta_!\mathcal{O}_X$ in $K(X\times X)$, because for any closed embedding, such as $\Delta\colon X\to X\times X$, its push-forward Δ_* is exact, and separately in the last equality $\operatorname{ch}(\mathcal{O}_X) = \exp(\mathrm{c}_1(\mathcal{O}_X)) = 1$, because $\mathcal{O}_X$ is a line bundle. Next, we observe, that from $\mathrm{td}_{X\times X} = p^*\,\mathrm{td}_X \smile p^*\,\mathrm{td}_X$, equation $\Delta^*\sqrt{\mathrm{td}_{X\times X}} = \mathrm{td}_X$ follows. By the cohomological projection formula (cf. Proposition 4.9) for the map Δ, we further compute

$$\Delta_*\,\mathrm{td}_X = \Delta_*\Delta^*\sqrt{\mathrm{td}_{X\times X}} = \Delta_*(1)\sqrt{\mathrm{td}_{X\times X}},$$

implying that $v(\mathcal{O}_{\Delta_X}) = \Delta_*(1)$. Lastly, for any $\alpha\in H^\bullet(X,\mathbf{Z})$, using the projection formula again, we arrive at

$$f^{\mathcal{O}_{\Delta_X}}(\alpha) = p_*(v(\mathcal{O}_{\Delta_X})\smile p^*\alpha) = p_*(\Delta_*(1)\smile p^*\alpha) = p_*(\Delta_*(1\smile \Delta^*p^*(\alpha))) = \alpha.$$

The map $f^{\mathcal{E}^\bullet}$ is now a homomorphism of free abelian groups admitting a right inverse, therefore it is an isomorphism of abelian groups.

We are left to show that $f^{\mathcal{E}^\bullet}$ is an isometry and that it preserves the Hodge structure. To see that $f^{\mathcal{E}^\bullet}$ is an isometry, we do two preliminary computations. Let $\alpha\in\widetilde{H}(X,\mathbf{Z})$ and $\beta\in\widetilde{H}(Y,\mathbf{Z})$ and let $\nu_X\colon X\to\operatorname{Spec}\mathbf{C}$, $\nu_Y\colon Y\to\operatorname{Spec}\mathbf{C}$ and $\nu_{X\times Y}\colon X\times Y\to\operatorname{Spec}\mathbf{C}$ denote the structure maps, then by Lemma 6.5

$$\begin{aligned}\langle f^{\mathcal{E}^\bullet}(\alpha),\beta\rangle_Y &= \nu_{Y,*}(\beta^\vee\smile f^{\mathcal{E}^\bullet}(\alpha))\\ &= \nu_{Y,*}\left(\beta^\vee\smile p_*\left(v(\mathcal{E}^\bullet)\smile q^*\alpha\right)\right)\\ &= \nu_{Y,*}\left(\beta^\vee\smile p_*\left(\operatorname{ch}(\mathcal{E}^\bullet)\sqrt{\mathrm{td}_{X\times Y}}\smile q^*\alpha\right)\right)\\ &= \nu_{Y,*}\left(p_*\left(p^*\beta^\vee\smile \operatorname{ch}(\mathcal{E}^\bullet)\sqrt{\mathrm{td}_{X\times Y}}\smile q^*\alpha\right)\right)\\ &= \nu_{X\times Y,*}\left(p^*\beta^\vee\smile \operatorname{ch}(\mathcal{E}^\bullet)\sqrt{\mathrm{td}_{X\times Y}}\smile q^*\alpha\right)\end{aligned}$$

and similarly one computes

ch($E^\vee$) = ch(E)$^\vee$?

$$\langle\alpha, f^{\mathcal{E}_L^\bullet}(\beta)\rangle_X = \nu_{X\times Y,*}\left(q^*\alpha^\vee\smile \operatorname{ch}(\mathcal{E}^\bullet)^\vee\sqrt{\mathrm{td}_{X\times Y}}\smile p^*\beta\right)$$

Then clearly for all α, $\alpha'\in\widetilde{H}(X,\mathbf{Z})$

$$\langle f^{\mathcal{E}^\bullet}(\alpha), f^{\mathcal{E}^\bullet}(\alpha')\rangle_Y = \langle\alpha, f^{\mathcal{E}_L^\bullet}(f^{\mathcal{E}^\bullet}(\alpha'))\rangle_X = \langle\alpha,\alpha'\rangle_X,$$

proving that $f^{\mathcal{E}^\bullet}$ is an isometry.

Lastly, we show that $f^{\mathcal{E}^\bullet}$ preserves the Hodge structure i.e.

$$f^{\mathcal{E}^\bullet}_{\mathbf{C}}(\widetilde{H}^{p,q}(X))\subseteq\widetilde{H}^{p,q}(Y)$$

for all p, q, with $p+q=2$. By Proposition **??**, the condition (**??**) specializes to

$$f_{\mathbf{C}}(\widetilde{H}^{2,0}(X))\subseteq\widetilde{H}^{2,0}(Y)$$

and in fact this turns out to be sufficient for f to preserve the Hodge structures. This is a very special coincidence, which happens as a consequence of the Hodge structure on the Mukai lattice of a K3 surface. Indeed, if $f_{\mathbf{C}}(\widetilde{H}^{2,0}(X)) \subseteq \widetilde{H}^{2,0}(Y)$, then $f_{\mathbf{C}}(\widetilde{H}^{0,2}(X)) \subseteq \widetilde{H}^{0,2}(Y)$ is obtained through conjugation

$$f_{\mathbf{C}}\left(\widetilde{H}^{0,2}(X)\right) = f_{\mathbf{C}}\left(\overline{\widetilde{H}^{2,0}(X)}\right) = \overline{f_{\mathbf{C}}\left(\widetilde{H}^{2,0}(X)\right)} \subseteq \overline{\widetilde{H}^{2,0}(Y)} = \widetilde{H}^{0,2}(Y).$$

And, to see $f_{\mathbf{C}}(\widetilde{H}^{1,1}(X)) \subseteq \widetilde{H}^{1,1}(Y)$, we pick a class $x \in \widetilde{H}^{1,1}(X)$ and map it over to $f_{\mathbf{C}}(x) \in \widetilde{H}(X, \mathbf{Z}) \otimes \mathbf{C}$.

$\square$

A Spectral sequences and how to use them

In this chapter we will first define cohomological spectral sequences and then discuss their meaning through applications related with derived categories.

At first, when encountering spectral sequences, one might think of them as just book-keeping devices encoding a tremendous amount of data, but we will soon see how elegantly one can infer certain properties related to derived categorical claims exploiting the fact that they can naturally be encoded with spectral sequences. One of the most important spectral sequences, which is also very general within our scope of inspection, will be the Grothendieck spectral sequence relating the higher derived functors of two composable functors with the higher derived functors of their composition. Later on we will see that many useful and well-known spectral sequences occur as special cases of the Grothendieck spectral sequence. What follows was gathered mostly from [Huy06, Chapter 2, §2.1] and [GM02, Chapter III.7].

Definition A.1. A *(cohomological) spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data on which we further impose two convergence conditions.

- *Sequence of pages.* A sequence of bi-graded objects $(E_r^{\bullet,\bullet})_{r\in\mathbf{Z}_{\geq 0}}$ equipped with differentials of bi-degree $(r, 1-r)$. The r-th term of this sequence is called the r-th *page* and it consists of a lattice of objects $E_r^{p,q}$ of $\mathcal{A}$, for $p, q \in \mathbf{Z}$, and differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

  satisfying

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0,$$

  for each $p, q \in \mathbf{Z}$.

- *Isomorphisms.* A collection of isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q},$$

  for all $p, q \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, where

$$H^{p,q}(E_r) := \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}),$$

  which allow us to turn the pages.

- *Transfinite page.* A bi-graded object $E_\infty^{\bullet,\bullet}$.

- *Goal of computation.* A sequence of objects $(E^n)_{n\in\mathbf{Z}}$ of the category $\mathcal{A}$.

The above collection of data also has to satisfy the following two convergence conditions.

1. For each pair (p, q), there exists $r_0 \geq 0$, such that for all $r \geq r_0$ we have

$$d_r^{p,q} = 0 \quad \text{and} \quad d_r^{p+r,q-r+1} = 0$$

   and the isomorphism $\alpha_r^{p,q}$ can be taken to be the identity. We then say that the (p, q)-term *stabilizes* after page r_0 and we denote $E_{r_0}^{p,q}$ (along with all the subsequent $E_r^{p,q}$ for $r \geq r_0$) by $E_\infty^{p,q}$.

2. For each $n \in \mathbf{Z}$ there is a *regular*[13] decreasing filtration of E^n

$$E^n \supseteq \cdots \supseteq F^p E^n \supseteq F^{p+1} E^n \supseteq \cdots \supseteq 0 \tag{A.1}$$

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^p E^{p+q} / F^{p+1} E^{p+q}$$

for all $p, q \in \mathbf{Z}$.

In this case we also denote the existence of such a spectral sequence by

$$E_r^{p,q} \implies E^{p+q}.$$

Remark A.2. A few words are in order to justify us naming the sequence $(E^n)_{n\in\mathbf{Z}}$ our *goal of computation.* Usually one is given a starting page or a small number of them and the first goal is to identify the transfinite page – we are referring to convergence condition (a). Often one is able to infer the differentials degenerate after a number of turns of the pages from context or by observing the shape of the spectral sequence. For example *first quadrant spectral sequences*, i.e. the ones with non-trivial $E_r^{p,q}$ only for (p,q) lying in the first quadrant, always satisfy condition (a).

The second part of the computation is concerned with relating objects from the transfinite page $E_\infty^{\bullet,\bullet}$ with objects E^n. This is captured in the convergence condition (b), from which we can clearly observe that the intermediate quotients of the filtration $(F^p E^n)_{p\in\mathbf{Z}}$, for a fixed term E^n, lie on the anti-diagonal of the transfinite page passing through e.g. $E_\infty^{n,0}$. Explicitly these are terms $\ldots, E_\infty^{n-1,1}, E_\infty^{n,0}, E_\infty^{n+1,-1}, \ldots$

In condition (b) the existence of isomorphisms $\beta^{p,q}$ can also be restated by saying that $E_\infty^{p,q}$ fits into a short exact sequence

$$0 \to F^{p+1} E^n \to F^p E^n \to E_\infty^{p,q} \to 0. \tag{A.2}$$

This observation becomes very useful when considering properties of objects of the category $\mathcal{A}$ which are closed under extensions, especially when the filtration of E^n is finite.

Lemma A.3. *Let $a \le b$ be integers and consider the regular decreasing filtration*

$$F \supseteq \ldots F^{p-1} \supseteq F^p \supseteq F^{p+1} \supseteq \cdots \supseteq 0 \tag{A.3}$$

of an object F. Assume that the quotients F^p/F^{p+1} vanish for $p \notin [a,b]$, then $F^p \simeq F$ for $p \le a$ and $F^p \simeq 0$ for $p \ge b$, so (A.3) is a finite filtration of F. In particular, when $a = b$, we have $F \simeq 0$.

Proof. For the quotients F^p/F^{p+1} vanish for $p \notin [a,b]$, all the inclusions $F^{p+1} \subseteq F^p$ turn into isomorphisms. Then by regularity it follows that $F = \operatorname{colim}_p F^p \simeq \operatorname{colim}_{p\le a} F^p = F^a$ and similarly $0 \simeq \lim_p F^p \simeq \lim_{p\ge b} F^p = F^b$. $\square$

In his famous "Tohoku paper" [] Grothendieck devised the following spectral sequence relating higher derived functors of a composition of two functors with the composition of their higher derived functors.

Theorem A.4 (Grothendieck)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Let $\mathcal{I}_F$ be an F-adapted class in $\mathcal{A}$, $\mathcal{I}_G$ a G-adapted class of objects in $\mathcal{B}$ and suppose every object of $\mathcal{I}_F$ is sent to a G-acyclic object by the functor F. Then for every $A^\bullet$ in $\mathsf{K}^+(\mathcal{A})$ there exists a spectral sequence*

$$E_2^{p,q} = \mathbf{R}^p G(\mathbf{R}^q F(A^\bullet)) \implies \mathbf{R}^{p+q}(G \circ F)(A^\bullet) = E^{p+q}. \tag{A.4}$$

[13] In our case the filtration $(F^p E^n)_{p\in\mathbf{Z}}$ is *regular*, whenever $\bigcap_p F^p E^n = \lim_p F^p E^n = 0$ and $\bigcup_p F^p E^n = \operatorname{colim}_p F^p E^n = E^n$.

Remark A.5. Taking the functor F to be the identity in Theorem A.4 we obtain as a consequence that for any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, admitting a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ there is a spectral sequence

$$E_2^{p,q} = \mathbf{R}^p F(H^q(A^\bullet)) \implies \mathbf{R}^{p+q}F(A^\bullet) = E^{p+q}, \tag{A.5}$$

for every $A^\bullet$ of $\mathsf{K}^+(\mathcal{A})$.

Proposition A.6. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor admitting a right derived functor $\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$.*

(i) Suppose that for every object A of $\mathcal{A}$ the complex $\mathbf{R}^iF(A)$ is non-trivial only for finitely many $i \in \mathbf{Z}$. Then $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^b(\mathcal{B})$ for any bounded complex $A^\bullet$ of $\mathsf{D}^b(\mathcal{A})$, consequently inducing a right derived functor on the level of bounded derived categories

$$\mathbf{R}F\colon \mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^b(\mathcal{B}).$$

(ii) Let $\mathcal{C} \subset \mathcal{B}$ be a thick subcategory of $\mathcal{B}$ and assume that for every $A^\bullet$ of $\mathsf{D}^+(\mathcal{A})$, $\mathbf{R}^iF(A)$ belong to $\mathcal{C}$ for all A of $\mathcal{A}$ and $i \in \mathbf{Z}$. Moreover assume there exists $m \in \mathbf{Z}_{\geq 0}$, such that $\mathbf{R}^iF(A) = 0$, for all A of $\mathcal{A}$ and $i > m$. Then $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^+_{\mathcal{C}}(\mathcal{B})$ for any $A^\bullet$ in $\mathsf{D}^+(\mathcal{A})$, therefore inducing the functor

$$\mathbf{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+_{\mathcal{C}}(\mathcal{B}).$$

Proof. For (i) consider $A^\bullet$ in $\mathsf{D}^b(\mathcal{A})$. We will show that $\mathbf{R}^iF(A^\bullet)$ is non-trivial only for finitely many $i \in \mathbf{Z}$. To do so, we will use spectral sequence (A.5). As the q-th cohomology $H^q(A^\bullet)$ is an object of $\mathcal{A}$, the complex $\mathbf{R}F(H^q(A^\bullet))$ is bounded by assumption and thus $\mathbf{R}^pF(H^q(A^\bullet))$ is non-trivial only for finitely many $p \in \mathbf{Z}$. Since $A^\bullet$ is also a bounded complex we see that all the non-trivial objects of page E_2 may be collected inside of a large enough square, say $D = [-a, a]^2$ for a positive integer $a \in \mathbf{Z}$. In other words $E_2^{p,q} \simeq 0$ for $(p, q) \notin D$, implying that also $E_\infty^{p,q} \simeq 0$ for $(p, q) \notin D$. This means that all the intermediate quotients of the filtration of E^n, i.e. the terms $E_\infty^{p,q}$, with $p+q = n$, are trivial, provided $|n| > 2a$. By regularity of the filtrations, we deduce that $\mathbf{R}^nF(A^\bullet) = E^n \simeq 0$ for $|n| > 2a$, implying that $\mathbf{R}F(A^\bullet)$ belongs to $\mathsf{D}^b(\mathcal{B})$.

For (ii) we will apply the same idea and use the spectral sequence (A.5) to show that $\mathbf{R}^iF(A^\bullet)$ belong to $\mathcal{C}$ for all $i \in \mathbf{Z}$. This time we see that $\mathbf{R}^pF(H^q(A^\bullet)) = E_2^{p,q} \simeq 0$ for $p \notin [0, m]$, thus the same goes for $E_\infty^{p,q}$. Since every anti-diagonal (line with slope -1) going through a point with integral coordinates passes through only $m + 1$ other points with integral components lying in the infinite strip $[0, m] \times \mathbf{Z}$, all the objects E^n have finite regular filtrations by Lemma A.3. They look like this

$$E^n = F^0E^n \supseteq_{E_\infty^{0,n}} F^1E^n \supseteq_{E_\infty^{1,n-1}} F^2E^n \supseteq \cdots \supseteq F^mE^n \supseteq_{E_\infty^{m,n-1}} F^{m+1}E^n = 0. \tag{A.6}$$

The intermediate quotients, inscribed in the subscript of $\supseteq$[14], of said filtration are precisely those $E_\infty^{p,q}$, with $p + q = n$ and $(p, q) \in [0, m] \times \mathbf{Z}$. Since $\mathcal{C}$ is thick, in particular closed under kernels and quotients, and all the objects of page E_2 belong to $\mathcal{C}$ by assumption, then all the objects from all the subsequent pages, including the transfinite one E_∞, also belong to $\mathcal{C}$. Inductively, using short exact sequences (A.2) and thickness of $\mathcal{C}$, we then see, that all the terms of the finite filtration (A.6) belong to $\mathcal{C}$, including $E^n = \mathbf{R}^nF(A^\bullet)$. □

[14]For example $F^0E^n \supseteq_{E_\infty^{0,n}} F^1E^n$ is supposed to mean that the quotient F^0E^n/F^1E^n is isomorphic to $E_\infty^{0,n}$.

Proposition A.7. *Let $\mathcal{A}$ be an abelian category with enough injectives. Suppose $A^\bullet$ belongs to $\mathsf{K}^-(\mathcal{A})$ and $B^\bullet$ belongs to $\mathsf{K}^+(\mathcal{A})$. Then there exists a spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p(H^{-q}(A^\bullet), B^\bullet) \implies \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet) = E^{p+q}.$$

Proof. See [Huy06, §2, Example 2.70]. □

B Lattice theory

Theorem B.1 (Milnor)**.** *Let (n_+, n_-) be a pair of non-negative integers. Then there exists an even unimodular lattice of signature (n_+, n_-), if and only if $n_+ - n_- \equiv 0\ (8)$. If n_+, $n_- > 0$ then the (indefinite) lattice is unique up to isomorphism.*

Corollary B.2. *Let Λ_0 and Λ_1 be positive definite even unimodular lattices with $\mathrm{rk}(\Lambda_0) = \mathrm{rk}(\Lambda_1)$, then*

$$\Lambda_0 \oplus \mathcal{U} \simeq \Lambda_1 \oplus \mathcal{U}.$$

Proof. Signature of Λ_0 and Λ_1 is $(\mathrm{rk}(\Lambda_0), 0)$, thus the signature of $\Lambda_0 \oplus \mathcal{U}$ and $\Lambda_1 \oplus \mathcal{U}$ is $(\mathrm{rk}(\Lambda_0) + 1, 1)$, so the result follows by Milnor's Theorem B.1. □

Corollary B.3. *Let Λ be an indefinite even unimodular lattice of signature (n_+, n_-). Setting $\tau = n_+ - n_-$ to be the index of Λ, then $\tau \equiv 0\ (8)$ and*

$$\begin{aligned} &\textit{if } \tau \geq 0, \textit{ then} \quad \Lambda \simeq E_8^{\oplus \frac{\tau}{8}} \oplus \mathcal{U}^{\oplus n_-}, \\ &\textit{if } \tau \leq 0, \textit{ then} \quad \Lambda \simeq E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus \mathcal{U}^{\oplus n_+}. \end{aligned}$$

Proof. The E_8 lattice has signature $(8, 0)$, and the hyperbolic lattice $\mathcal{U}$ has signature $(1, 1)$, thus the signature of $E_8^{\oplus \frac{\tau}{8}} \oplus \mathcal{U}^{\oplus n_-}$ equals $(8 \cdot \frac{\tau}{8} + n_-, n_-) = (n_+, n_-)$. □

These three results are from Huybrects K3

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Razširjeni povzetek

Zelo pomembno vlogo v algebraični geometriji igra kohomologija geometrijskega objekta, kot je shema ali raznoterost, X glede na, denimo, koherentni snop $\mathcal{F}$ na X. Eden izmed načinov računanja kohomoloških grup $H^i(X, \mathcal{F})$, ki ga bomo tudi spoznali v poglavju **??**, vključuje sledeče. Namesto, da intrinzično obravnavamo snop $\mathcal{F}$, ga predstavimo s t. i. *resolucijo* ali *predstavitvijo*, ki jo sestavljata kompleks snopov $F^\bullet$, členi katerega pripadajo nekemu razredu snopov, ki ima glede na kohomologijo določene ugodne lastnosti, in kvazi-izomorfizem $F^\bullet \to \mathcal{F}$ ali $\mathcal{F} \to F^\bullet$. Ker sprememba resolucije na kohomologijo ne bo imela vpliva in ker lahko vsak snop zase vidimo tudi kot kompleks zgoščen v stopnji 0, želimo snop $\mathcal{F}$ obravnavati enako kot vse njegove resolucije. Pogledano od daleč, želimo homotopsko kategorijo $\mathsf{K}(\mathsf{coh}(X))$ spremeniti tako, da se snop $\mathcal{F}$ identificira z vsemi svojimi resolucijami, oz. z drugimi besedami, želimo vse kvazi-izomorfizme v $\mathsf{K}(\mathsf{coh}(X))$ spremeniti v izomorfizme. Slednje bo naše vodilo, da za splošno abelovo kategorijo $\mathcal{A}$ vpeljemo njej prirejeno izpeljano kategorijo $\mathsf{D}(\mathcal{A})$.

morda intrinzično ni prava beseda, ker so klasično predstavitve zajele ravno informacijo o generatorjih, relacijah,...