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K3 SURFACES FROM A DERIVED CATEGORICAL VIEWPOINT

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K3 PLOSKVE Z VIDIKA IZPELJANIH KATEGORIJ

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1 Triangulated categories

1.1 Additive, k-linear and abelian categories

We mainly follow [1].

A categorical *biproduct* of objects X and Y of a category $\mathcal{C}$ is an object $X \oplus Y$ together with morphisms

$$\begin{aligned} p_X &: X \oplus Y \to X & p_Y &: X \oplus Y \to Y \\ i_X &: X \to X \oplus Y & i_Y &: Y \to X \oplus Y \end{aligned}$$

for which the pair p_X, p_Y is the categorical product of X and Y and the pair i_X, i_Y is the categorical coproduct.

Definition 1.1. A category $\mathcal{A}$ is additive (resp. k-linear) if all the hom-sets carry the structure of abelian groups (resp. k-modules) and the following axioms are satisfied

A1 For all all objects X, Y and Z of $\mathcal{A}$ the composition

$$\circ \colon \operatorname{Hom}_{\mathcal{A}}(Y, Z) \times \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear.

A2 There exists a *zero object* 0, for which $\operatorname{Hom}_{\mathcal{A}}(0, 0) = 0$.

A3 For any two objects X and Y there exists a categorical biproduct of X and Y.

Definition 1.2. A functor $F \colon \mathcal{A} \to \mathcal{A}'$ between two additive (resp. k-linear) categories $\mathcal{A}$ and $\mathcal{A}'$ is *additive* (resp. *k-linear*), if its action on morphisms

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}'}(F(X), F(Y))$$

is a group homomorphism (resp. k-linear map).

For a morphism $f \colon X \to Y$ in an additive category $\mathcal{A}$ recall, that the *kernel* of f is the equalizer of f and 0 in $\mathcal{A}$, if it exists, and, dually, the *cokernel* of f is the coequalizer of f and 0. It is well-known and easy to verify, that the structure maps $\ker f \hookrightarrow X$ and $Y \twoheadrightarrow \operatorname{coker} f$ are mono and epi respectively. We also define the *image* and *coimage* of f to be

$$\begin{aligned} (\operatorname{im} f \to Y) &:= \ker(Y \to \operatorname{coker} f) \\ (X \to \operatorname{coim} f) &:= \operatorname{coker}(\ker f \to X). \end{aligned}$$

Definition 1.3. An additive category $\mathcal{A}$ is *abelian*, if it contains all kernels and cokernels and satisfies axiom A4.

A4 For any morphism $f \colon X \to Y$ in $\mathcal{A}$ the canonical morphism $\operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f$ is an isomorphism.

$$\begin{tikzcd} \ker f \arrow[r, hook] & X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[r, two heads] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[u, hook] & \end{tikzcd}$$

Remark 1.4. We obtain the morphism mentioned in axiom A4 in the following way. Since $(\operatorname{im} f \to Y) = \ker(Y \to \operatorname{coker} f)$ and the composition $X \to Y \to \operatorname{coker} f$ is equal to 0, there is a unique morphism $X \to \operatorname{im} f$ by the universal property of kernels. The composit $\ker f \to X \to \operatorname{im} f$ then equals 0, by the fact that $\operatorname{im} f \hookrightarrow Y$ is mono and $\ker f \to X \to Y$ equals 0. From the universal property of cokernels we obtain a unique morphism $\operatorname{coim} f \to \operatorname{im} f$, since $(X \to \operatorname{coim} f) = \operatorname{coker}(\ker f \to X)$.

Definition 1.5. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a sequence of composable morphisms in an abelian category $\mathcal{A}$ satisfying $g \circ f = 0$. We say it is *exact*, if the canonical morphism $\operatorname{im} g \xrightarrow{\sim} \ker f$ is an isomorphism.

1.2 Triangulated categories

In order to formulate the definition of a triangulated category more concisely, we introduce some preliminary notions. A *category with translation* is a category $\mathcal{D}$ together with an auto-equivalence $T \colon \mathcal{D} \to \mathcal{D}$ called the *translation functor*. If $\mathcal{D}$ is additive or k-linear, T is moreover assumed to be additive or k-linear. We usually denote its action on objects X with $X[1]$ and its action on morphisms f with $f[1]$.

A *triangle* in a category $\mathcal{D}$ with translation T is a triplet of composable morphisms (u, v, w) of category $\mathcal{D}$ having the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A *morphism* of triangles $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$ and $X' \overset{u'}{\to} Y' \overset{v'}{\to} Z' \overset{w'}{\to} X'[1]$ is given by a triple of morphisms (α, β, γ) for which the diagram below commutes.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & X[1] \arrow[d, "\alpha[1]"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

One can compose morphisms of triangles in the obvious way and the notion of an isomorphism of triangles is defined as usual.

Definition 1.6. A *triangulated category* is an additive or k-linear category $\mathcal{D}$ with translation T equipped with a class of *distinguished triangles*, which is subject to the following axioms.

TR1 (i) Any triangle isomorphic to a distinguished triangle is also itself destinguished.

(ii) For any X the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished.

(iii) For any morphism $f \colon X \to Y$ there is a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

TR2

TR3

TR4

Remark 1.7. For a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

morphisms u and v are said to be of degree 0 and morphism w is said to be of degree $+1$. They are also sometimes diagramatically depicted as triangles, with the markings on morphisms describing their respective degrees.

$$\begin{tikzcd} & Z \arrow[dl, "+1"'] & \\ X \arrow[rr, "0"] & & Y \arrow[ul, "0"'] \end{tikzcd}$$

Definition 1.8. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with translation functors T and T' respectively. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is defined to be *triangulated* or *exact*, if the following two conditions are satisfied.

(i) There exists a natural isomorphism of functors

$$\eta\colon F \circ T \simeq T' \circ F.$$

(ii) For every distinguished triangle

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$$

in $\mathcal{D}$, the triangle

$$F(X) \to F(Y) \to F(Z) \to F(X)[1]$$

is distinguished in $\mathcal{D}'$, where the last morphism (of degree 1) is obtained as the composition $F(Z) \xrightarrow{Fw} F(X[1]) \xrightarrow{\eta_X} F(X)[1]$.

Definition 1.9. Let $H\colon \mathcal{D} \to \mathcal{A}$ be an additive functor between a triangulated category $\mathcal{D}$ and an abelian category $\mathcal{A}$. We say H is a *cohomological functor* if for every distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the induced long sequence in $\mathcal{A}$

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to H(Y[1]) \to H(Z[1]) \to \cdots$$

is exact.

Example 1.10. For any object W in a triangulated category $\mathcal{D}$ the functors

$$\mathrm{Hom}_{\mathcal{D}}(W, -)\colon \mathcal{D} \to \mathsf{Ab} \text{ and } \mathrm{Hom}_{\mathcal{D}}(-, W)\colon \mathcal{D}^{\mathrm{op}} \to \mathsf{Ab}$$

are cohomological.

2 Derived categories

2.1 Categories of complexes

In order to define derived categories of an additive category $\mathcal{A}$ we first introduce the category of complexes and the homotopic category of complexes of $\mathcal{A}$. We will equip these with a triangulated structure and . Throughout this section $\mathcal{A}$ will be a fixed additeve or k-linear category and we also mention that we will be using the cohomological indexing convention.

As a preliminary we introduce graded objects in ??

Category of complexes

By a *chain complex* in $\mathcal{A}$ we mean a collection of objects and morphisms

$$A^\bullet = \left((A^i)_{i\in\mathbf{Z}}, (d_A^i\colon A^i \to A^{i+1})_{i\in\mathbf{Z}}\right),$$

where A^i are objects and d^i are morphisms of $\mathcal{A}$, called *differenitials*, subject to equations $d^{i+1}\circ d^i = 0$, for all $i \in \mathbf{Z}$. A complex is *bounded from below* (resp. *bounded from above*), if there exists $i_0 \in \mathbf{Z}$ for which $A^i = 0$ for all $i \leq i_0$ (resp. $i \geq i_0$) and is *bounded*, if it is both bounded from below and bounded from above. A *chain map* between two chain complexes $A^\bullet$ and $B^\bullet$ in $\mathcal{A}$ is a collection of morphisms in $\mathcal{A}$

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}},$$

for which $f^{i+1}\circ d_A^i = d_B^i \circ f^i$ holds for all $i \in \mathbf{Z}$. This may diagramatically be described by the following commutative ladder.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

We then define the *category of chain complexes* in $\mathcal{A}$, denoted by $\mathrm{Ch}(\mathcal{A})$, to be the following additive category.

Objects:	chain complexes in $\mathcal{A}$.
Morphims:	$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$ is the set of chain maps $A \to B$, equipped with a group structure inherited from $\mathcal{A}$ by applying operations componentwise.

The composition law is defined componentwise and is clearly associative and bilinear, and the identity morphisms $1_{A^\bullet}$ are defined to be $(1_{A^i})_{i\in\mathbf{Z}}$. The complex $\cdots \to 0 \to 0 \to \cdots$ plays the role of the zero object in $\mathrm{Ch}(\mathcal{A})$ and the biproduct of complexes A and B exists and is witnessed by the chain complex

$$A \oplus B = \left((A^i \oplus B^i)_{i\in\mathbf{Z}}, (d_A^i \oplus d_B^i)_{i\in\mathbf{Z}}\right),$$

together with the cannonical projection and injection morphisms arising from biproducts componentwise.

Additionally, we also define the following full additive subcategories of $\mathrm{Ch}(\mathcal{A})$.

$\mathrm{Ch}^+(\mathcal{A})$	*Category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$\mathrm{Ch}^-(\mathcal{A})$	*Category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$\mathrm{Ch}^b(\mathcal{A})$	*Category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

Remark 2.1. Whenever $\mathcal{A}$ is k-linear, all the categories of complexes $\mathrm{Ch}^*(\mathcal{A})$ become k-linear as well in the obvious way.

On all the categories of complexes mentioned above, we can now define the translation functor

$$T\colon\ \mathrm{Ch}^*(\mathcal{A}) \to \mathrm{Ch}^*(\mathcal{A})$$

given by its action on objects and morphisms as follows.

Objects:	$T(A^\bullet) = A^\bullet[1]$ is the chain complex with $(A^\bullet[1])^i := A^{i+1}$ and differentials $d^i_{A[1]} = -d^{i+1}_A$.
Morphims:	For a chain map $f^\bullet\colon A^\bullet \to B^\bullet$ we define $f^\bullet[1]$ to have component maps $(f^\bullet[1])^i = f^{i+1}$.

The translation functor T thus acts on a complex $A^\bullet$ by twisting its differential by a sign and shifting it one step to the *left*, which is graphically pictured below.

$$\begin{array}{lccccccccccc}
 & \cdots & & -1 & & 0 & & 1 & & 2 & & \cdots \\
A^\bullet & \cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \cdots \\
A^\bullet[1] & \cdots & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow & \cdots
\end{array}$$

Remark 2.2. We remark that the translation functor T is clearly also additive or k-linear, whenever $\mathcal{A}$ is additive or k-linear.

Since T is an auto-equivalence there exists a quasi-inverse T^{-1} to T, which is defined and unique up to a natural isomorphism. We may then speak of T^k for any $k \in \mathbf{Z}$, whose action on a complex $A^\bullet$ is described by $(A^\bullet[k])^i = A^{i+k}$ with differential $d^i_{A[k]} = (-1)^k d^{i+k}_A$.

Homotopy category of complexes

In this subsection we construct the homotopy category of chain complexes associated to a given additive category $\mathcal{A}$ and equip it with a triangulated structure. The main motivation for its introduction in this work is the fact that we will later on use it to construct the derived category of $\mathcal{A}$. In particular the homotopy category of $\mathcal{A}$, as opposed to the category of complexes[1] $\mathrm{Ch}(\mathcal{A})$, can be enhanced with a triangulated structure which will afterwards descend to the level of derived categories.

Definition 2.3. Let $f^\bullet$ and $g^\bullet$ be two chain maps in $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)$. We define $f^\bullet$ and $g^\bullet$ to be *homotopic*, if there exists a collection of morphisms $(h^i\colon A^i \to B^{i-1})_{i\in\mathbf{Z}}$, satisfying

$$f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$$

[1] It is still possible to construct the derived category of $\mathcal{A}$ without passing through the homotopy category of complexes, however equipping it with a triangulated structure in this case becomes less elegant.

for all $i \in \mathbf{Z}$ and denote it by $f^\bullet \simeq g^\bullet$. We say $f^\bullet$ is *nullhomotopic*, if $f^\bullet \simeq 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] \arrow[dl, "h^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, "h^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 2.4. *Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be complexes in $\mathrm{Ch}(\mathcal{A})$, and let $f, f' \in \mathrm{Hom}(A^\bullet, B^\bullet)$ and $g, g' \in \mathrm{Hom}(B^\bullet, C^\bullet)$ be chain maps.*

(i) The set of all nullhomotopic chain maps $A^\bullet \to B^\bullet$ forms a subgroup of $\mathrm{Hom}(A^\bullet, B^\bullet)$.

(ii) If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.

The homotopy category of complexes in $\mathcal{A}$, denoted by $K(\mathcal{A})$, is defined to be the additive category consisting of

$$\begin{aligned} &\textit{Objects:} && \text{chain complexes in } \mathcal{A}. \\ &\textit{Morphims:} && \mathrm{Hom}_{K(\mathcal{A})}(X,Y) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X,Y)/_{\simeq} \end{aligned}$$

The composition law descends to the quotient by lemma 2.4 (ii), i.e. $[g] \circ [f] := [g \circ f]$, for composable $[f]$ and $[g]$, and for any $A^\bullet$ the identity morphism is defined to be $[1_A^\bullet]$. All the hom-sets $\mathrm{Hom}_{K(\mathcal{A})}(X,Y)$ are abelian groups by lemma 2.4 (i) and compositions are bilinear maps.

As in the case of categories of complexes in $\mathcal{A}$, we can also define the following full additive subcategories of $K(\mathcal{A})$.

$K^+(\mathcal{A})$	*Homotopy category of complexes bounded below*, spanned on complexes in $\mathcal{A}$ bounded below.
$K^-(\mathcal{A})$	*Homotopy category of complexes bounded above*, spanned on complexes in $\mathcal{A}$ bounded above.
$K^b(\mathcal{A})$	*Homotopy category of bounded complexes*, spanned on bounded complexes in $\mathcal{A}$.

We now shift our focus to the construction of a triangulated structure on $K^*(\mathcal{A})$. The translation functor $T\colon K^*(\mathcal{A}) \to K^*(\mathcal{A})$ is defined on objects and morphisms in the following way.

$$\begin{aligned} &\textit{Objects:} && A^\bullet \longmapsto A^\bullet[1]. \\ &\textit{Morphims:} && [f^\bullet] \longmapsto [f^\bullet[1]]. \end{aligned}$$

We define any triangle in $K(\mathcal{A})$ isomorphic to a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{\pi} C(f) \xrightarrow{\tau} A[1]$$

to be distinguished.

Proposition 2.5. *The homotopy category $K(\mathcal{A})$ together with the translation functor $T\colon K(\mathcal{A}) \to K(\mathcal{A})$ and distinguished triangles defined above is a triangulated category.*

Remark 2.6. The proposition still holds true, if we replace $K(\mathcal{A})$ with any of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$.

Lemma 2.7. *Let $f^\bullet\colon A^\bullet \to B^\bullet$ be a quasi-isomorphism and let $g^\bullet\colon C^\bullet \to B^\bullet$ be a morphism. Then there exist a quasi-isomorphism $u^\bullet\colon C_0^\bullet \to C^\bullet$ and a morphism $v^\bullet\colon C_0^\bullet \to A^\bullet$, such that $f \circ u = g \circ v$ i.e. the diagram below commutes.*

$$\begin{tikzcd} C_0^\bullet \arrow[r, dashed, "u^\bullet", "\sim"'] \arrow[d, dashed, "v^\bullet"'] & C^\bullet \arrow[d, "g^\bullet"] \\ A^\bullet \arrow[r, "f^\bullet", "\sim"'] & B^\bullet \end{tikzcd}$$

The above also holds in categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$.

Proposition 2.8. *Suppose $\mathcal{A}$ contains enough injectives. Then every $A^\bullet$ in $K^+(\mathcal{A})$ has an injective resolution.*

Proof. □

Remark 2.9. As is evident from the proof by close inspection we have only used two facts about the class of all injective objects of $\mathcal{A}$ – that every object of $\mathcal{A}$ embeds into some injective object and that the class of injectives is closed under finite direct sums. This will later on be used in subsection 2.2.1 when constructing a right derived functor of a given functor F in the presence of an F-adapted class.

Lemma 2.10. *Let $A^\bullet$ be any acyclic complex in $K^+(\mathcal{A})$ and $I^\bullet$ a complex of injectives from $K^+(J)$. Then*

$$\mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) = 0.$$

In other words every morphism from an acyclic complex to an injective one is nulhomotopic.

Proof. □

Theorem 2.11. *Assume $\mathcal{A}$ contains enough injectives and let $\mathcal{J} \subseteq \mathcal{A}$ denote the full subcategory on injecitve objects of $\mathcal{A}$. Then the inclusion $K^+(\mathcal{J}) \hookrightarrow K^+(\mathcal{A})$ induces an equivalence of categories*

$$K^+(\mathcal{J}) \simeq D^+(\mathcal{A}).$$

2.2 Derived functors

2.2.1 F-adapted classes

Unfortunately our categories at hand will sometimes not contain enough injectives, as can already be seen with the category of coherent sheaves $\mathsf{Coh}(X)$ on a scheme X, which is not a point. In this case our previously defined method of constructing right derived functors will not work. Luckly however, there exists a method of obtaining a right derived functor $\mathrm{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, given a left exact functor $F : \mathcal{A} \to \mathcal{B}$, even when $\mathcal{A}$ does not contain enough injectives. To this end we first introduce F-adapted classes.

Definition 2.12. A class of objects $\mathcal{I} \subset \mathcal{A}$ is *adapted* to a left exact functor $F : \mathcal{A} \to \mathcal{B}$, if the following three conditions are satisfied.

(i) $\mathcal{I}$ is stable under finite sums.

(ii) Every object A of $\mathcal{A}$ embeds into some object of $\mathcal{I}$, i.e. there exists an object $I \in \mathcal{I}$ and a monomorphism $A \hookrightarrow I$.

(iii) For every acyclic complex $I^\bullet$ in $K^+(\mathcal{A})$, with $I^i \in \mathcal{I}$ for all $i \in \mathbf{Z}$, its image $F(I^\bullet)$ under F is also acyclic.

Remark 2.13. We imediately observe that whenever $\mathcal{A}$ contains enough injectives, the class of all injective objects forms an F-adapted class for *every* functor $F\colon \mathcal{A} \to \mathcal{B}$.

In the presence of an F-adapted class $\mathcal{I}$ we will now construct the right derived functor of F. Firstly one can upgrade the class of objects $\mathcal{I}$ to a full additive subcategory $\mathcal{J}$ of $\mathcal{A}$ having its class of objects be preciesly $\mathcal{I}$. This is done by declaring $\operatorname{Hom}_{\mathcal{J}}(X,Y) := \operatorname{Hom}_{\mathcal{A}}(X,Y)$ for all objects X, Y of the class $\mathcal{I}$. Then we can define a functor $K^+(F)\colon K^+(\mathcal{J}) \to K^+(\mathcal{B})$ between triangulated categories, which acts on objects of $K^+(\mathcal{J})$ as

$$\Big(\cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots\Big) \longmapsto \Big(\cdots \to F(A^i) \xrightarrow{F(d^i)} F(A^{i+1}) \to \cdots\Big)$$

and on morphisms as

$$f^\bullet = (f^i\colon A^i \to B^i)_{i\in\mathbf{Z}} \longmapsto (F(f^i))_{i\in\mathbf{Z}}.$$

It is then clear that $K^+(F)$ is exact because of condition (ii) of definition 2.12 and it descends to a well defined functor on the level of derived categories

$$D^+(\mathcal{J}) \to D^+(\mathcal{B}).$$

Since we want to define the derived functor $\mathrm{R}F$, whose domain is $D^+(\mathcal{A})$, it remains to construct a functor $D^+(\mathcal{A}) \to D^+(\mathcal{J})$, which we can then post-compose with $K^+(F)$ to obtain $\mathrm{R}F$. What we will show instead is that the inclusion of categories $\mathcal{J} \hookrightarrow \mathcal{A}$ induces an exact equivalence of triangulated categories $D^+(\mathcal{J})$ and $D^+(\mathcal{A})$.

It remains to be shown that the inclusion of categories $\mathcal{J} \hookrightarrow \mathcal{A}$ induces an the equivalence of categories $D^+(\mathcal{J})$ and $D^+(\mathcal{A})$. Firstly we remark that this inclusion clearly descends to a well-defined functor on the level of derived categories

This hinges on the following lemma.

Lemma 2.14. *For every complex $A^\bullet$ in $K^+(\mathcal{A})$ there is a quasi-isomorphism $A^\bullet \to I^\bullet$ where $I^\bullet$ is a complex in $K^+(\mathcal{J})$.*

Proof. By inspecting the proof of proposition 2.8 we see that only conditions (i) and (ii) of definition 2.12 were actually used, so the proof of this lemma follows mutatis mutandis from said proposition. □

Proposition 2.15. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelin categories. Let $\mathcal{I} \subset \mathcal{A}$ be an F-adapted class of objects in $\mathcal{A}$. Then the class of all F-acyclic objects of $\mathcal{A}$, denoted by $\mathcal{I}_F$, is also F-adapted.*

Lemma 2.16. *Every object of an F-adapted class is also F-acyclic.*

Proof. We only have to recall the definition □

Proof of proposition 2.15. We verify conditions (i)–(iii) of definition 2.12. (i) As the higher derived functors R^iF are clearly additive, $\mathcal{I}_F$ is stable under finite sums. (ii) Holds by lemma 2.16. (iii) Suppose $A^\bullet$ is an acyclic complex in $K^+(\mathcal{A})$ with $A^i \in \mathcal{I}_F$ for all i. Then using acyclicity of $A^\bullet$, we may break this complex up into a series of short exact sequences as follows.

$$\begin{aligned} 0 \to A^0 \to &A^1 \to \ker d^2 \to 0 \\ 0 \to \ker d^2 \to &A^2 \to \ker d^3 \to 0 \\ &\vdots \end{aligned}$$

The first short exact sequence yields the following long exact sequence associated to F.

$$\begin{aligned} 0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to \mathrm{R}^1 F(A^0) \to \cdots \\ \cdots \to \mathrm{R}^i F(A^0) \to \mathrm{R}^i F(A^1) \to \mathrm{R}^i F(\ker d^2) \to \mathrm{R}^{i+1} F(A^0) \to \cdots . \end{aligned}$$

Exacness of the sequence together with F-acyclicity of A^0 and A^1 shows that $\ker d^2$ is also F-acyclic and by only considering the begining few terms of the sequence we obtain the short exact sequence

$$0 \to F(A^0) \to F(A^1) \to F(\ker d^2) \to 0.$$

After inductively applying the same reasoning for each of the original short exact sequences of (**??**), we are left with the following set of short exact sequences[2].

$$\begin{aligned} 0 \to F(A^0) \to &F(A^1) \to \ker Fd^2 \to 0 \\ 0 \to \ker Fd^2 \to &F(A^2) \to \ker Fd^3 \to 0 \\ &\vdots \end{aligned}$$

Collecting them all together we can conclude that the complex $F(A^\bullet)$ is acyclic. $\square$

[2] To be more precise we have used that F is left exact in order to swap the order of F and ker to reach

$$\ker Fd^i \simeq F(\ker d^i)$$

.

3 Spectral sequences and how to use them

In this chapter we will first define cohomological spectral sequences and discuss their meaning through applications connected with derived categories.

At first, when encountering spectral sequences, one might think of them as just bookkeeping devices encoding a tremendous amount of data, but we will soon see how elegantly one can infer certain properties related to derived categorical claims exploiting the fact that they can naturally be encoded with spectral sequences. One of the most important spectral sequences, which is also very general within our scope of inspection, will be the Grothendieck spectral sequence relating the higher derived functors of two composable functors with the higher derived functors of their composition. Later on we will see that many useful and well-known spectral sequences occur as special cases of the Grothendieck spectral sequence. What follows was gathered mostly from []

For the time being we fix an abelian category $\mathcal{A}$ and start off with a definition.

Definition 3.1. A *cohomological spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data on which we further impose two convergence conditions.

- *Sequence of pages.* A sequence of bi-graded objects $(E_r^{\bullet,\bullet})_{r\in\mathbf{Z}_{\geq 0}}$ equipped with differentials of bi-degree $(r, 1-r)$. The r-th term of this sequence is called the r-th *page* and it consists of a lattice of objects $E_r^{p,q}$ of $\mathcal{A}$, for $p, q \in \mathbf{Z}$, and differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

  satisfying

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0,$$

  for each $p, q \in \mathbf{Z}$.

- *Isomorphisms.* A collection of isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q},$$

  for all $p, q \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, where

$$H^{p,q}(E_r) := \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}),$$

  which allow us to turn the pages.

- *Transfinite page.* A bi-graded object $E_\infty^{\bullet,\bullet}$.

- *Goal of computation.* A sequence of objects $(E^n)_{n\in\mathbf{Z}}$ of the category $\mathcal{A}$.

The above collection of data also has to satisfy the following two convergence conditions.

(a) For each pair (p, q), there exists $r_0 \geq 0$, such that for all $r \geq r_0$ we have

$$d_r^{p,q} = 0 \quad \text{and} \quad d_r^{p+r,q-r+1} = 0$$

and the isomorphism $\alpha_r^{p,q}$ can be taken to be the identity. We then say that the (p, q)-term stabilizes after page r_0 and we denote $E_{r_0}^{p,q}$ (along with all the subsequent $E_r^{p,q}$ for $r \geq r_0$) by $E_\infty^{p,q}$.

(b) For each $n \in \mathbf{Z}$ there is a decreasing regular[3] filtration of E^n

$$E^n \supseteq \cdots \supseteq F^pE^n \supseteq F^{p+1}E^n \supseteq \cdots \supseteq 0$$

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^pE^{p+q}/F^{p+1}E^{p+q}$$

for all $p, q \in \mathbf{Z}$.

In this case we also denote the existence of such a spectral sequence by

$$E_r^{p,q} \implies E^n.$$

Remark 3.2. A few words are in order to justify us naming the sequence $(E^n)_{n\in\mathbf{Z}}$ our *goal of computation*. Usually one is given a starting page or a small number of them and the first goal is to identify the transfinite page – we are refering to convergence condition (a). Often one is able to infer the differentials degenerate after a number of turns of the pages from context or by observing the shape of the spectral sequence. For example *first quadrant spectral sequences*, i.e. the ones with non-trivial $E_r^{p,q}$ only for (p, q) lying in the first quadrant, always satisfy condition (a).

The second part of the computation is concerned with relating objects from the transfinte page $E_\infty^{\bullet,\bullet}$ with objects E^n. This is captured in the convergence condition (b), from which we can clearly observe that the intermediate quotients of the filtration $(F^pE^n)_{p\in\mathbf{Z}}$ for a fixed term E^n lie on the anti-diagonal of the transfinite page passing through e.g. $E_\infty^{n,0}$.

In condition (b) the existence of isomorphisms $\beta^{p,q}$ can also be restated by saying that $E_\infty^{p,q}$ fits into a short exact sequence

$$0 \to F^{p+1}E^n \to F^pE^n \to E_\infty^{p,q} \to 0.$$

This observation becomes very fruitful when considering properties of objects of the category $\mathcal{A}$ which are closed under extensions, especially when the filtration of E^n is finite.

Thoku paper

[3]In our case the filtration $(F^pE^n)_{p\in\mathbf{Z}}$ is *regular*, whenever $\bigcap_p F^pE^n = \lim_p F^pE^n = 0$ and $\bigcup_p F^pE^n = \operatorname{colim}_p F^pE^n = E^n$.

References

[1] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der mathematischen Wissenschaften **332**, Springer, 2006, DOI: 10.1007/3-540-27950-4, available from https://doi.org/10.1007/3-540-27950-4.

Razširjeni povzetek