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K3 SURFACES FROM A DERIVED CATEGORICAL VIEWPOINT

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K3 PLOSKVE Z IZPELJANO KATEGORIČNEGA POGLEDA

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1 Spectral sequences and how to use them

In this chapter we will first define cohomological spectral sequences and discuss their meaning through applications connected with derived categories.

At first, when encountering spectral sequences, one might think of them as just book-keeping devices encoding a tremendous amount of data, but we will soon see how elegantly one can infer certain properties related to derived categorical claims exploiting the fact that they can naturally be encoded with spectral sequences. One of the most important spectral sequences, which is also very general within our scope of inspection, will be the Grothendieck spectral sequence relating the higher derived functors of two composable functors with the higher derived functors of their composition. Later on we will see that many useful and well-known spectral sequences occur as special cases of the Grothendieck spectral sequence. What follows was gathered mostly from []

For the time being we fix an abelian category $\mathcal{A}$ and start off with a definition.

Definition 1.1. A *cohomological spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data on which we further impose two convergence conditions.

- *Sequence of pages.* A sequence of bi-graded objects $(E_r^{\bullet,\bullet})_{r\in\mathbf{Z}_{\geq 0}}$ equipped with differentials of bi-degree $(r, 1-r)$. The r-th term of this sequence is called the r-th *page* and it consists of a lattice of objects $E_r^{p,q}$ of $\mathcal{A}$, for $p, q \in \mathbf{Z}$, and differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

  satisfying

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0,$$

  for each $p, q \in \mathbf{Z}$.

- *Isomorphisms.* A collection of isomorphisms

$$\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q},$$

  for all $p, q \in \mathbf{Z}$ and $r \in \mathbf{Z}_{\geq 0}$, where

$$H^{p,q}(E_r) := \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}),$$

  which allow us to turn the pages.

- *Transfinite page.* A bi-graded object $E_\infty^{\bullet,\bullet}$.

- *Goal of computation.* A sequence of objects $(E^n)_{n\in\mathbf{Z}}$ of the category $\mathcal{A}$.

The above collection of data also has to satisfy the following two convergence conditions.

(a) For each pair (p, q), there exists $r_0 \geq 0$, such that for all $r \geq r_0$ we have

$$d_r^{p,q} = 0 \quad \text{and} \quad d_r^{p+r,q-r+1} = 0$$

  and the isomorphism $\alpha_r^{p,q}$ becomes the identity. We then say that the (p, q)-term stabilizes after page r_0 and we denote $E_{r_0}^{p,q}$ (along with all the subsequent $E_r^{p,q}$ for $r \geq r_0$) by $E_\infty^{p,q}$.

(b) For each $n \in \mathbf{Z}$ there is a decreasing regular[1] filtration of E^n

$$E^n \supseteq \cdots \supseteq F^p E^n \supseteq F^{p+1} E^n \supseteq \cdots \supseteq 0$$

and isomorphisms

$$\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^p E^{p+q} / F^{p+1} E^{p+q}$$

for all $p, q \in \mathbf{Z}$.

In this case we also denote the existence of such a spectral sequence by

$$E_r^{p,q} \implies E^n.$$

Remark 1.2. A few words are in order to justify us naming the sequence $(E^n)_{n\in\mathbf{Z}}$ our *goal of computation*. Usually one is given a starting page or a small number of them and the first goal is to identify the transfinite page – we are refering to convergence condition (a). Often one is able to infer the differentials degenerate after a number of turns of the pages from context or by observing the shape of the spectral sequence. For example *first quadrant spectral sequences*, i.e. the ones with non-trivial $E_r^{p,q}$ only for (p, q) lying in the first quadrant, always satisfy condition (a).

The second part of the computation is concerned with relating objects from the transfinte page $E_\infty^{\bullet,\bullet}$ with objects E^n. This is captured in the convergence condition (b), from which we can clearly observe that the intermediate quotients of the filtration $(F^p E^n)_{p\in\mathbf{Z}}$ for a fixed term E^n lie on the anti-diagonal of the transfinite page passing through e.g. $E_\infty^{n,0}$.

In condition (b) the existence of isomorphisms $\beta^{p,q}$ can also be restated by saying that $E_\infty^{p,q}$ fits into a short exact sequence

$$0 \to F^{p+1} E^n \to F^p E^n \to E_\infty^{p,q} \to 0.$$

This observation becomes very fruitful when considering properties of objects of the category $\mathcal{A}$ which are closed under extensions, especially when the filtration of E^n is finite.

Thoku paper

[1]In our case the filtration $(F^p E^n)_{p\in\mathbf{Z}}$ is *regular*, whenever $\bigcap_p F^p E^n = \lim_p F^p E^n = 0$ and $\bigcup_p F^p E^n = \operatorname{colim}_p F^p E^n = E^n$.

Razširjeni povzetek