Newton's Method

December 1, 2022

1 Newton's Method Notes (for optional programming problem)

Problem: solve the equation $f(x) = 0$ for x. Newton's method attacks this problem using tangent lines.

To explain the idea, here's a warmup:

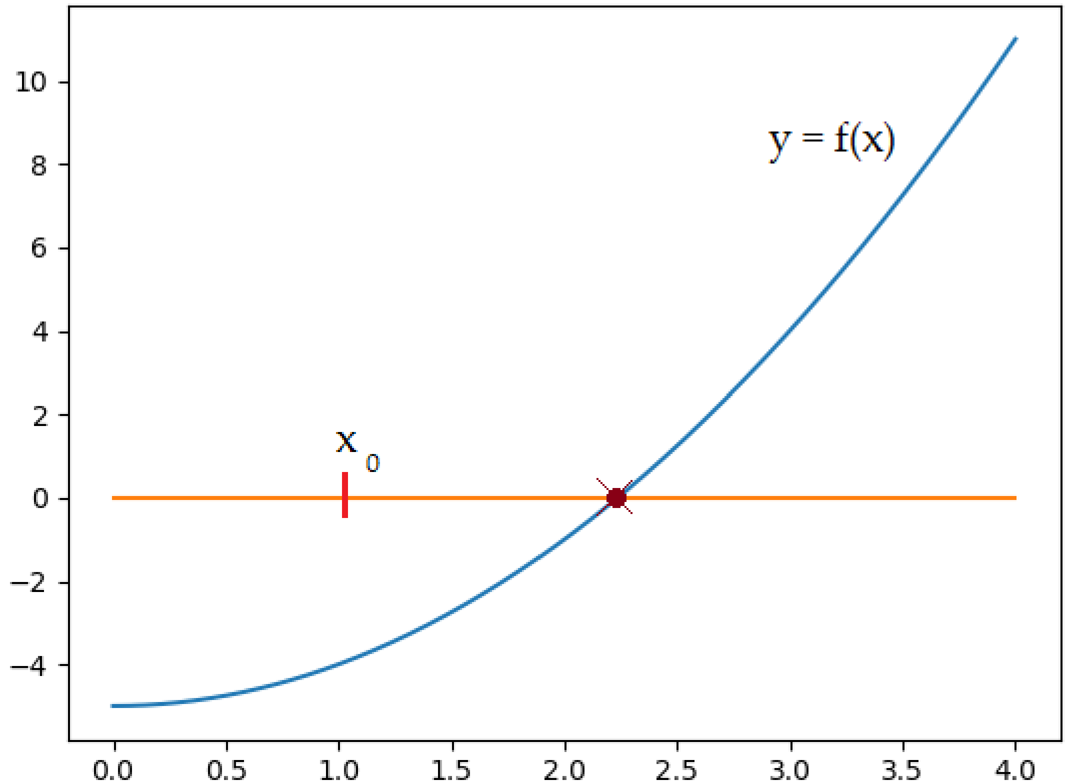
$g(x) = x^2$

$h(x) = 6x - 9$

When $x = 3$, what is $g(x)$? $h(x)$? How about when $x = 3.1$? Or when $x = 3.01$?

That each pair is close isn't an accident. $y = 6x - 9$ is the equation of the tangent line to $y = x^2$ at the point $(3, 9)$; or, otherwise put, $h(x)$ is the *linearization* of $g(x)$ at $x = 3$. This means that, at least when $x \approx 3$, $h(x)$ is a good replacement for $g(x)$. Furthermore, the linear function $h(x)$ is easier to work with, and solve equations with. This is the idea behind Newton's method.

Let's work with a concrete example: $f(x) = x^2 - 5$. Here's a graph of this function:



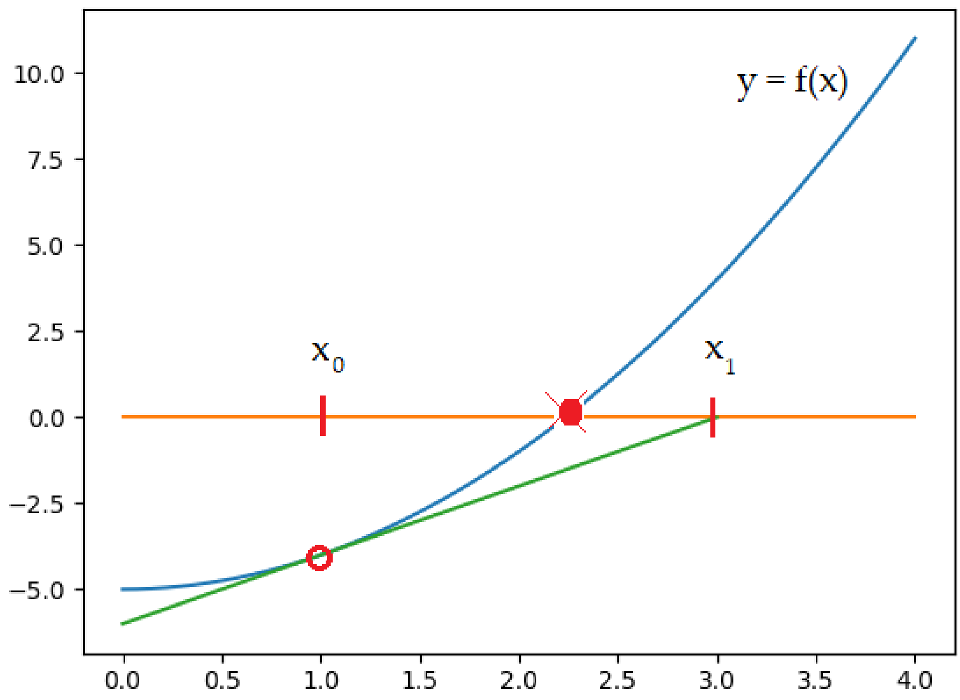
We want to solve $x^2 - 5 = 0$, which is the same as finding the x-intercept of the graph. We of course have simple ways to do this example, but it is one of the easiest examples to illustrate Newton's method with. The method works almost identically with harder examples.

The principle is: pick a random point x_0, linearize $f(x)$ at $x = x_0$, solve the easier linearized equation to get x_1, which is hopefully closer to a real solution. Then, repeat!

So, to start, let $x_0 = 1$. This is our more-or-less random first guess; I chose it mostly because I knew it was an easy number to start with, and because it's slightly close to the true solution.

Now, let's find the tangent line to $y = x^2 - 5$ at the point $x = 1$: that would be $y = 2x - 6$.

To get the next step: instead of solving $x^2 - 5 = 0$ (harder), we'll solve $2x - 6 = 0$ (easier), and get $x = 3$. This x value will be our x_1: $x_1 = 3$. Obviously, that's not the answer – we changed the problem! *However*, observe that this x-value is closer to the true intercept.

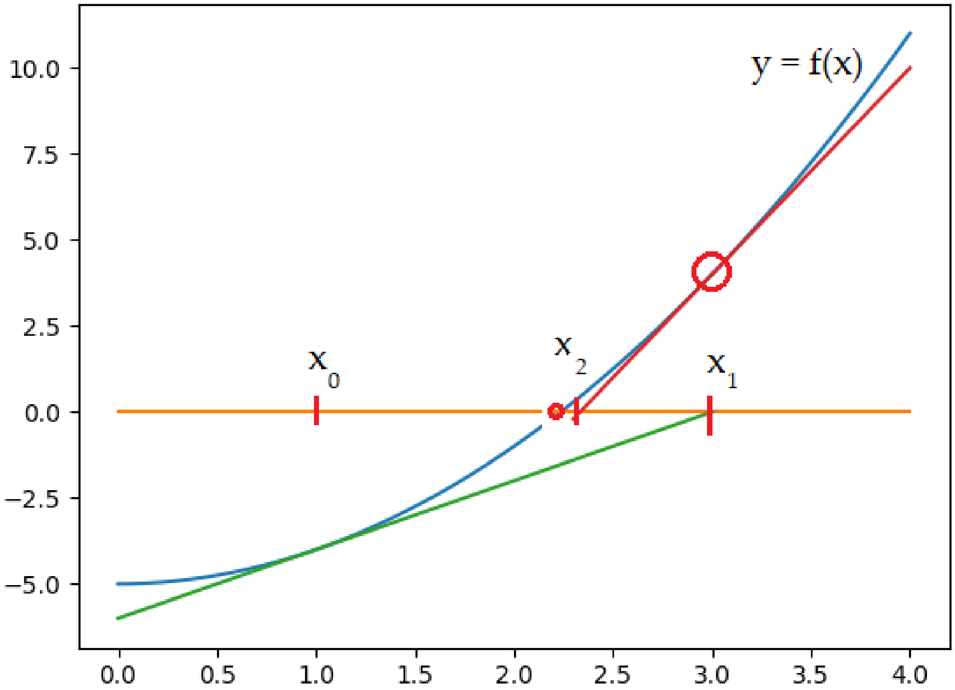


So now: do this all again, except this time, start with the better guess we just obtained, $x_1 = 3$.

Tangent line: $y = 6x - 14$

Intercept of tangent line: $x = \frac{14}{6} \approx 2.3333 = x_2$

Graph:



And then repeat, until you are sufficiently close.

In general, to estimate a solution to $f(x) = 0$:

Start with a guess x_0.

Find the equation of the tangent line at the point where $x = x_0$: $y - f(x_0) = f'(x_0)(x - x_0)$

Find the intercept of that tangent line, and call that x_1: $x = x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$

Then to get x_2, you do the same things starting with x_1 instead of x_0. You end up with $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. And so on.

This is **Newton's Method**.

To estimate a solution to $f(x) = 0$:

- Take a guess $x = x_0$.
- Then, calculate improved estimates x_i using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Keep repeating until you deem the value of $f(x_n)$ sufficiently close to 0.

Note that Newton's method is not a completely robust algorithm. If your guess happens to be a critical number of $f(x)$, Newton's method will fail; sometimes the method ends up going back and forth between two points; and of course, if there are multiple solutions, Newton's method will only converge to one of them. However, in certain cases (for example, when you are dealing with a *convex* function), it works rather nicely, and rather quickly.