10 Line integrals

1. Let $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$. Then $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$ and $\|\mathbf{r}'(t)\| = \sqrt{a^2+c^2}$. Hence
$$s = \int_0^t \sqrt{a^2+c^2}du = (\sqrt{a^2+c^2})t.$$
Thus the arc length parametrization is
$$\mathbf{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2+c^2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2+c^2}}\right)\mathbf{j} + \frac{cs}{\sqrt{a^2+c^2}}\mathbf{k}.$$

2.
$$\int_{C_1} = -\int_{-1}^{+1} \frac{x^2dx}{(1+x^2)^2} = -\int_{-\pi/4}^{+\pi/4} \sin^2\theta = -\pi/4 + \frac{1}{2},$$
$$\int_{C_2} = -\int_{-1}^{+1} \frac{dy}{(1+y^2)^2} = -\int_{-\pi/4}^{+\pi/4} \cos^2\theta d\theta = -\pi/4 - 1/2,$$
$$\int_{C_3} = \int_{-1}^{+1} \frac{-x^2dx}{(1+x^2)}^2 = -\pi/4 + \frac{1}{2},$$
$$\int_{C_4} = \int_{-1}^{+1} \frac{-dy}{(1+y^2)^2} = -\pi/4 - 1/2.$$
Hence
$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -\pi.$$

3. A parametrization of C is $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$. Note that the outward unit normal to the circle at $\mathbf{r}(t)$ is the radial vector $\mathbf{n} = \mathbf{r}(t)$. Also,
$$\nabla(x^2-y^2) = 2x\,\mathbf{i} - 2y\,\mathbf{j}.$$
Thus
$$\begin{aligned}\oint_C \nabla(x^2-y^2)\cdot d\mathbf{n} &= \int_0^{2\pi} (2\cos t\,\mathbf{i} - 2\sin t\mathbf{j})\cdot(-\sin t\mathbf{i} + \cos t\mathbf{j})dt \\ &= \int_0^{2\pi} (-2\sin 2t)dt = 0.\end{aligned}$$

4. Parameterize C as
$$\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j},\ 0 \le t \le 2.$$
Then $\mathbf{r}'(t) = \mathbf{i} + 3t^2\,\mathbf{j}$. Since $\nabla(x^2-y^2) = 2t\mathbf{i} - 2t^3\mathbf{j}$, we have
$$\int_C \nabla(x^2-y^2)\cdot d\mathbf{r} = \int_0^2 (2t - 6t^5)dt = 4 - 64 = -60.$$

5. The required integral is
$$= \int_{C_1} \frac{dx+dy}{|x|+|y|} + \int_{C_2} \frac{dx+dy}{|x|+|y|} + \int_{C_3} \frac{dx+dy}{|x|+|y|} + \int_{C_4} \frac{dx+dy}{|x|+|y|}.$$

Along C_1: $x+y=1$ and $|x|+|y|=x+y=1$. Thus

$$\int_{C_1}\frac{dx+dy}{|x|+|y|}=\int_1^0 dx-\int_1^0 dx=0.$$

Along C_2: $-x+y=1$ and $|x|+|y|=-x+y=1$. Thus

$$\int_{C_2}\frac{dx+dy}{|x|+|y|}=\int_0^{-1} dx+\int_0^{-1} dx=-2.$$

Along C_3: $x+y=-1$ and $|x|+|y|=-x-y=1$. Thus

$$\int_{C_3}\frac{dx+dy}{|x|+|y|}=\int_{-1}^0 dx-\int_{-1}^0 dx=0.$$

Along C_4: $x-y=1$ and $|x|+|y|=x-y=1$. Thus

$$\int_{C_4}\frac{dx+dy}{|x|+|y|}=\int_0^1 dx+\int_0^1 dx=2.$$

Hence

$$\int_C\frac{dx+dy}{|x|+|y|}=2-2=0.$$

6.

$$\begin{aligned}
\text{Work W} &= \int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (xy\mathbf{i}+x^6y^2\mathbf{j})\cdot(\mathbf{i}dx+\mathbf{j}dy)\\
&= \int_0^1 ax^{b+1}dx+\int_0^1 (a^2x^{2b+6})(abx^{b-1})dx\\
&= \frac{a}{b+2}+\frac{a^3b}{3b+6}\\
&= \frac{a}{b+2}\left(1+\frac{a^2b}{3}\right)=a\left(\frac{3+a^2b}{3(b+2)}\right).
\end{aligned}$$

This will be independent of b iff $\frac{dW}{db}=0$ iff $0=\frac{(b+2)a^2-(3+a^2b)}{(b+2)^2}$ iff $a=\sqrt{\frac{3}{2}}$ (as $a>0$).

7. First we observe that the cylinder is given by

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}.$$

From the equations of the sphere and the cylinder we have that, on the intersection C,

$$z^2=a^2-ax.$$

Noting the requirement $z\geq 0$, a parametrization of C is given by

$$x=\frac{a}{2}+\frac{a}{2}\cos\theta,\qquad y=\frac{a}{2}\sin\theta,\ z=a\sin\frac{\theta}{2};\ 0\leq\theta\leq 2\pi.$$

Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} [\left(\frac{a^2}{4}\sin^2\theta\right)(-\frac{a}{2}\sin\theta) + (a^2\sin^2\frac{\theta}{2})(\frac{a}{2}\cos\theta) \\
&\qquad + \left(\frac{a^2}{4} + \frac{a^2}{4}\cos^2\theta + \frac{a^2}{2}\cos\theta\right)(\frac{a}{2}\cos\frac{\theta}{2})]d\theta \\
&= \int_0^{2\pi} [-\frac{a^3}{8}\sin^3\theta + \frac{a^3}{2}\sin^2\frac{\theta}{2}\cos\theta + \frac{a^3}{8}\cos\frac{\theta}{2} + \frac{a^3}{8}\cos^2\theta\cos\frac{\theta}{2} \\
&\qquad + \frac{a^3}{4}\cos\theta\cos\frac{\theta}{2}]d\theta \\
&= -\frac{\pi a^3}{4}.
\end{aligned}$$

8. $\dfrac{\partial f_1}{\partial y} = 3x$, $\dfrac{\partial f_2}{\partial x} = 3x^2 y$ where $(f_1, f_2) = f$. Now

$$\begin{aligned}
\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \quad &\text{iff} \quad 3x = 3x^2 y \\
&\text{iff} \quad \text{either } x = 0 \text{ or } xy = 1.
\end{aligned}$$

Since the sets $\{(x,y)|x=0\}, \{(x,y)|xy=1\}$ are not open, $\mathbf{F}(x,y)$ is not the gradient of a scalar field on any open subset of $\mathbb{R}^2$.

9.

$$\frac{\partial f_1}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial f_2}{\partial x} \text{ on } \mathbb{R}^2 \setminus \{(0,0)\}.$$

However, $\mathbf{F} \neq \nabla f$ for any f. Indeed, let C to be the unit circle $x^2 + y^2 = 1$, oriented anticlockwise. Then one has

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin t\mathbf{i} + \cos t\mathbf{j})\cdot(-\sin t\mathbf{i} + \cos t\mathbf{j})dt \\
&= 2\pi \neq 0
\end{aligned}$$

10. Suppose $\mathbf{v} = \nabla\phi$ for some ϕ.
Then

$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \quad \Rightarrow \quad \phi(x,y,z) = x^2y + z^3x + f(y,z)$$

for some $f(y,z)$. Assuming f has partial derivatives, we get

$$\begin{aligned}
\frac{\partial\phi}{\partial y} &= x^2 + \frac{\partial f}{\partial y} = x^2 \\
\text{so that } \frac{\partial f}{\partial y} &= 0
\end{aligned}$$

and $f(y,z)$ depends only on z

Let $f(y,z) = g(z)$. Then $\phi(x,y,z) = x^2y + z^3x + g(z) \Rightarrow \dfrac{\partial\phi}{\partial z} = 3z^2x + g'(z) = 3z^2x \Rightarrow g'(z) = 0$. Let us select $g(z) = 0$. It can be checked that $\phi(x,y,z) = x^2y + z^3x$ satisfies $\nabla\phi = \mathbf{v}$. Hence

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = 0$$

for every smooth closed curve C.

11. $\mathbf{F}(x,y,z) = f(r)\mathbf{r} = f(r)x\ \mathbf{i} + f(r)y\ \mathbf{j} + f(r)z\ \mathbf{k}$.
Since

$$r = (x^2+y^2+z^2)^{1/2},$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

If $\mathbf{F}$ is to be $\nabla\phi$ for some ϕ, then we must have $\phi_x = f(r)x, \phi_y = f(r)y, \phi_z = f(r)z$; that is,

$$\begin{aligned}
\phi_x = xf(r) &= \frac{x}{r}rf(r) = \frac{\partial r}{\partial x}rf(r),\\
\phi_y = yf(r) &= \frac{y}{r}rf(r) = \frac{\partial r}{\partial y}rf(r),\\
\phi_z = zf(r) &= \frac{z}{r}rf(r) = \frac{\partial r}{\partial z}rf(r).
\end{aligned}$$

Now it can be seen that $\phi(x,y,z) = \displaystyle\int_{t_0}^{r} tf(t)dt$, with some t_0 fixed, satisfies all the desired equations.