Fixed point Representation

$$X = \pm \left(d_1 d_2 \cdots d_{k-1} \cdot d_k \cdots d_n\right)_\beta$$

where $d_1 \cdots d_n \in \{0, 1 \cdots \beta - 1\}$

Fixed points or floating points are how numbers are stored / represented in a computer

Example

$X = +(10.1)_2$

$X = -(123.12)_{10}$

Evaluating fixed point numbers in base 10:

$$(10.1)_2 \rightarrow 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1}$$
$$= 2 + 0 + \frac{1}{2}$$
$$= (2.5)_{10}$$

Floating point Representation:

$$F = \{\pm \underbrace{(0.d_1 d_2 \cdots d_n)}_{\text{fraction / mantissa}}{}_\beta \times \beta^e$$

$e \rightarrow$ exponent

$\beta \rightarrow$ base

where $\beta, d_i, e \in \mathbb{Z} \rightarrow$ integers

$0 \leq d_i \leq \beta - 1$

$e_{min} \leq e \leq e_{max}$

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Evaluating ~~floating~~ ~~point~~ numbers ~~in base 10~~

Examples

123.45

$= 12.345 \times 10^1$

$= 1.2345 \times 10^2$

$= 0.12345 \times 10^3$

1001.11

$= \underbrace{0.100111}_{\text{Fraction}} \times \underbrace{2}_{\text{Base}}^{4}$ — exponent

Conventions :

① $\pm (0.\boxed{d_1} d_2 \,----\, d_m)_\beta \, \beta^e$

$\Downarrow$

$d_1 = 1$ always.

Example:

$\beta = 2 \quad e_{min} = -1$

$m = 3 \quad e_{max} = 2$

$\Rightarrow$ highest possible FP number $= (0.111)_2 \times 2^2$.

② Normalized Form:

$\pm (1.d_1 d_2 \,----\, d_m)_\beta \, \beta^e$

$\downarrow$

$\left.\begin{array}{l} d_1 = 1 \\ d_1 \neq 1 \end{array}\right\}$ both correct

Example:

$\beta = 2 \quad e_{min} = -1$

$m = 3 \quad e_{max} = 2$

$\Rightarrow$ highest possible FP number $= (1.111)_2 \times 2^2$

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Denormalized Form

$$\pm (0.1\, d_1 d_2 \ldots d_m)_\beta \, \beta^e$$

Example

$\beta = 2 \quad e_{min} = -1$

$m = 3 \quad e_{max} = 2$

$$(0.1\,111)_2 \times 2^2$$

$\beta = 2 \quad e_{min} = -1 \quad$ convention 1

$m = 3 \quad e_{max} = 2$

Ignoring the $\pm$ sign

Find the smallest and largest non-negative number

$$0.\underbrace{1}_{\text{Fixed}}\ \square\ \square$$

d_2	d_3
0	0
0	1
1	0
1	1

} 4 possible #

e: $-1, 0, 1, 2$ } 4 possible #

$\therefore$ Total possible # that can be represented $= 4 \times 4 = 16$

smallest #

$$\boxed{(0.1\,00)_2} \times 2^{-1} \quad (e_{min})$$

$$= \boxed{(1\times 2^{-1})} \times 2^{-1}$$

$$= \frac{1}{4}$$

largest #

$$(0.111)_2 \times 2^2 \quad (e_{max})$$

$$= (1\times 2^{-1} + 1\times 2^{-2} + 1\times 2^{-3}) \times 2^2$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \times 2^2$$

$$= \frac{7}{2}$$

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Considering sign bit

Smallest possible # $= -\frac{7}{2}$

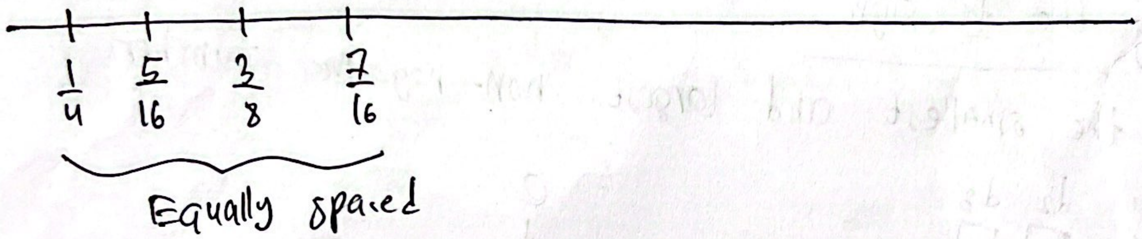
$\boxed{e=-1}$ e = -1

1st smallest non-negative number $= (0.100) \times 2^{-1} = \frac{1}{4}$

2nd " " " $= (0.101) \times 2^{-1} = \left(\frac{1}{4} + \frac{1}{8}\right) \times 2^{-1} = \frac{5}{16}$

3rd " " " $= (0.110) \times 2^{-1} = \frac{3}{8}$

4th " " " $= (0.111) \times 2^{-1} = \frac{7}{16}$



$\frac{5}{16} - \frac{1}{4} = \frac{3}{8} - \frac{5}{16} = \frac{7}{16} - \frac{3}{8} = \frac{1}{16}$ [exponent constant = equally spaced]

$\boxed{e=0}$

$(0.100) \times 2^{0} = \frac{1}{2}$

$(0.101) \times 2^{0} = \frac{5}{8}$

$\vdots$

$\boxed{e=1}$

$(0.100) \times 2^{1} = 1$

$\vdots$

$\boxed{e=2}$

$(0.100) \times 2^{2} = 2$

$\vdots$

$(0.111) \times 2^{2} = \frac{7}{2}$

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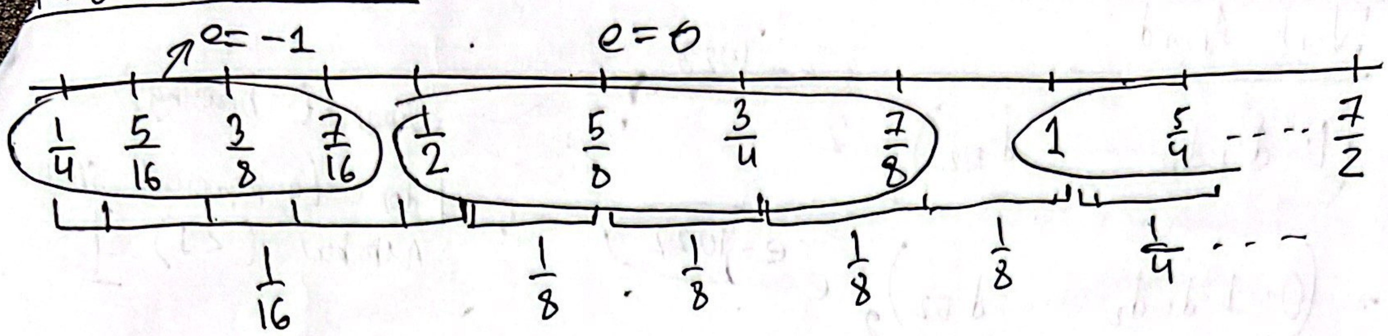
Non-negative number line:



Important

If convention 1/ denormalized form is used, we dont have 0 in our number system. Bcz $(0.1\,0\,0\ldots)$

we have 1 fixed here.

But 0 can be approximated using, for example, $(0.1000\ldots)\,2^{-50} \approx 0$

IEEE standard for double precision

$\beta = 2$

52 - bits for fraction/mantissa

11 - bits for exponent

1 - bit for sign

Starting with Normalized Form:

$$(1.d_1d_2 \cdots d_{52}) \times 2^{e}$$

min value of $e = 0$ [all 11 bits 0]

max value of $e = 2^{11} - 1 = 2047$ [all 11 bits 1]

$e_{min} = 0$

$e_{max} = 2047$

largest possible non-neg num $= (1.11\cdots1)_2 \times 2^{2047}$

smallest " " " $= (1.0\cdots0)_2 \times 2^{0} = 1$ (not that small)

cannot express 0.001 for example.

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Work Around

$$(1.d_1 d_2 \cdots d_{52})_2 \, 2^{e-1023}$$

$$= (0.1\, d_1 d_2 \cdots d_{52})_2 \, e^{e-1022}$$

exponent biasing. [done to represent small numbers (<1)]

previously $e \in [0, 2047]$

Now, with exponent biasing $[-1022, 1025]$

Now, highest possible num $= (0.1\,\underbrace{11\cdots1}_{52\ 1s}) \times 2^{1025} \approx \infty$

smallest possible number $= (0.10\cdots0) \times 2^{-1022} \approx 0$

In IEEE standard, 2 bits from exponent is reserved for ∞ and 0.

Highest possible exponent is used to store infinity

$$2^{1025} \longrightarrow \infty$$

smallest possible exponent is used to store 0

$$2^{-1022} \longrightarrow 0$$

Now, highest possible num, except inf $= (0.1\,11\cdots1)_2 \times 2^{1024}$

$\approx 1.798 \times 10^{308}$

lowest " ", except 0 $= (0.1)_2 \times 2^{-1021}$

$\approx 2.225 \times 10^{-308}$.

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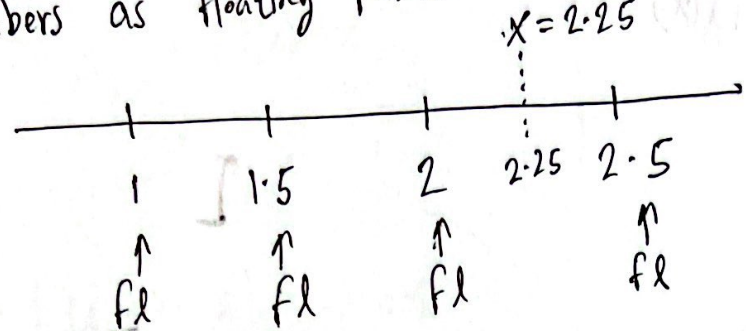
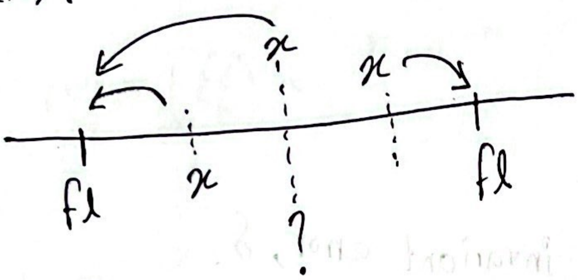
Rounding

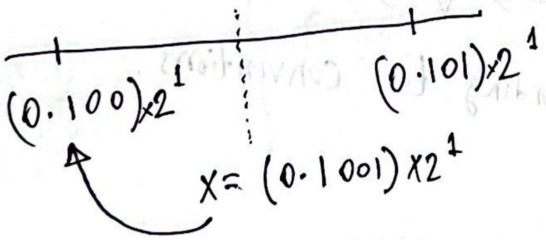
In the number line we can only represent DISCRETE set of numbers as floating points.



~~x is alwa~~ if x is given, x should always be converted to $fl(x)$.



rule → if perfectly in the middle, round it to the nearest even fl.

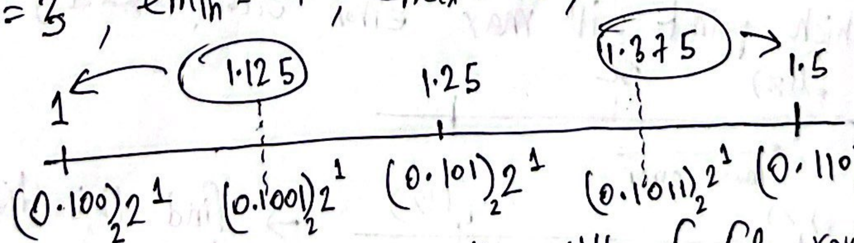


For binary, if number ends in 0 → even
" " 1 → odd

Example

$\beta = 2$, $m = 3$, $e_{min} = -1$, $e_{max} = 2$, convention 1.

$\boxed{\text{when } e = 1}$



If numbers are in the middle of fl, round it off to the ~~nai~~ nearest even.

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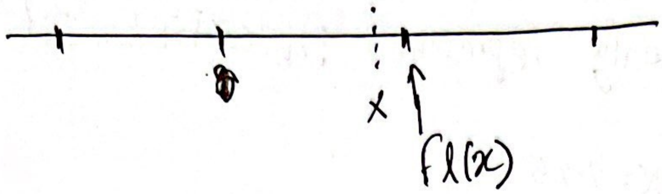
Rounding Error



$$\delta = \frac{|fl(x) - x|}{|x|}$$

$\uparrow$
scale invariant error

$$\delta \cdot x = fl(x) - x$$

$$fl(x) = \delta x + x$$

$$\boxed{fl(x) = x(1+\delta)}$$

Machine Epsilon

→ Maximum possible scale invariant error, δ.

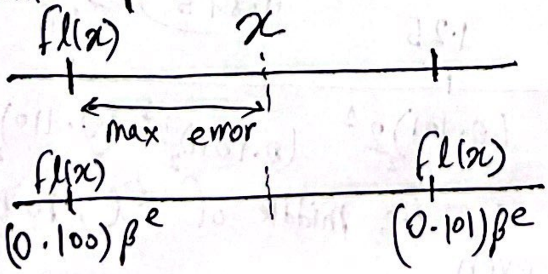
$$\delta = \frac{|fl(x) - x|}{|x|} \quad \begin{matrix} \to \uparrow \\ \to \downarrow \end{matrix} \Big\} \text{ results in maximum } \delta.$$

→ δ would change according to conventions.

① **Convention 1 :**

$(0.d_1 d_2 \cdots d_m)_\beta \times \beta^e$

At which point will max^m error occur? exactly in the middle.



→ find this distance then divide by 2.

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x $\quad$ $fl(x)$

$(0.100)\beta^e$ $\quad$ $(0.101)\beta^e$

max error

$$(0.101)\beta^e - (0.100)\beta^e$$
$$= (0.001)\beta^e$$
$$= \beta^{-3} \cdot \beta^e$$
$$= \beta^{-m}\beta^e$$

$$\therefore \text{max error} = fl(x) - x$$
$$= \frac{1}{2}\beta^{-m}\beta^e$$

$$\delta = \frac{|fl(x) - x|}{|x|} \quad \begin{matrix} \rightarrow \text{max} \\ \rightarrow \text{min} \rightarrow (0.100)\beta^e \rightarrow \beta^{-1}\beta^e \end{matrix}$$

$$\therefore \text{machine epsilon } (\varepsilon_M) = \frac{\frac{1}{2}\beta^{-m}\beta^e}{\beta^{-1}\beta^e} = \boxed{\frac{1}{2}\beta^{1-m}}$$

② **Normalized Form**

$$\varepsilon_M = \frac{1}{2}\beta^{-m}$$

③ **Denormalized form**

$$\varepsilon_M = \frac{1}{2}\beta^{-m}$$

same.

point to notice $\boxed{\delta \le \varepsilon_M}$

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FP Arithmetic with Rounding Error:

conv = 1

$\beta = 2 \quad m = 3 \quad e_{min} = -1 \quad e_{max} = 2$

$x = \frac{5}{8} \qquad y = \frac{7}{8}$

$= (0.101)_2\, 2^0 \qquad = (0.111)_2\, 2^0$

both are already FPs, bcz it matches with above specification.

$\therefore fl(x) = (0.101)_2 \times 2^0 \qquad fl(y) = (0.111)_2\, 2^0$

$\boxed{\text{Find } x * y}$

$x * y = fl(x) \cdot fl(y)$

$= \frac{5}{8} \times \frac{7}{8}$

$= \frac{35}{64}$

$= (\underbrace{0.100}\,\vdots\, 011)_2 \times 2^0$

→ need to take upto d_3 according to specification.

→ should we take 0.100 or 0.101 ?

If $(m+1)$ digit = 1, round it to next number.
" " " = 0, " " " prev ".

2 possible FP

(0.100)   (0.101)

$x * y = 0.100 \vdots \underline{0}11$

If $x * y = 0.100 \vdots \underline{1}11$

if $x * y$ perfectly in middle, round it to nearest even.
0.1001

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$x * y \xrightarrow{\text{mapped to}} fl(xy) = (0.100)_2 \, 2^0$

$= \frac{1}{2}$

$= \frac{32}{64}$

originally $x * y = \frac{35}{64}$. But for toy computer $fl(xy) = \frac{32}{64}$

Bcz of Rounding error

Note:

If initially $fl(x) \neq$, $fl(y) \neq y$

approx x to $fl(x)$

" y to $fl(y)$

Then do arithmetic like $fl(x) + fl(y)$

$= \ldots$

$= \ldots$

$= (\quad)$

then ~~approach~~ approx again $\boxed{fl(x+y)}$

Loss of Significance

previously $x = fl(x)$, $y = fl(y)$

what if $x \neq fl(x)$, $y \neq fl(y)$?

Then $fl(x) = x(1+\delta_1)$ $fl(y) = y(1+\delta_2)$

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Now, we want to calculate $x \pm y$

$$x \pm y \to fl(x) \pm fl(y)$$
$$= x(1+\delta_1) \pm y(1+\delta_2)$$
$$= (x \pm y) \pm x\delta_1 \pm y\delta_2$$
$$= (x \pm y)\left(1 + \underbrace{\frac{x\delta_1 \pm y\delta_2}{x \pm y}}_{\text{Scale invariant error}}\right)$$

If we want to calculate $\boxed{x-y}$:

For scale invarient error, we have

$$\frac{x\delta_1 - y\delta_2}{x-y} \to \text{if } x \approx y, \text{ value } \approx 0, \text{ error would } \text{~~increa~~ increase.}$$

<u>This is called Loss of significance.</u>

<u>How to avoid Los:</u>

$x^2 - 56x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$x_1 = 28 + \sqrt{783} = 55.98$

$x_2 = 28 - \sqrt{783} = 0.01786$

equal

Lets say, my toy computer can only calc upto 4 sf.

$\sqrt{783} = 27.98$

$\therefore x_1 = 28 + 27.98 = 55.98$

$x_2 = \boxed{28} - \boxed{27.98} = 0.02000$

not equal

close numbers.

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~~Example 2~~

$\cancel{f(x) = e^{x} - \cos(x) - x}$

$\cancel{x \in [-5 \times 10}$

Work Around:

$x^2 - 56x + 1$

$x^2 - \underbrace{(\alpha + \beta)}_{\alpha, \beta \text{ are roots.}} x + \alpha\beta$

$x^2 - 56x + 1$

$\alpha\beta = 1$

Find α using $28 + 27.98$

$\therefore \quad \alpha = 55.98$

$\alpha\beta = 1$

$55.98\,\beta = 1$

$\beta = \frac{1}{55.98}$

$= 0.01786$ (same as actual x_2)

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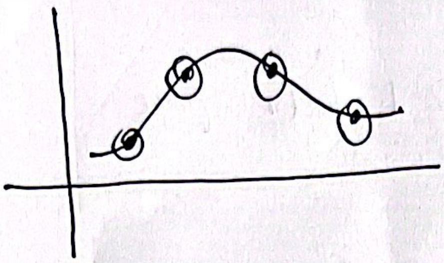
Polynomial Interpolation:



$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

$\uparrow$
degree

$(a_0, a_1, \cdots a_n) \rightarrow$ constants / coefficients

→ $P_n(x)$ has $(n+1)$ coefficients

→ Polynomials can be thought as vectors in a vector space

→ vector space is a set of vectors, that we can

add → $1 + x + 2x^2$

$+3x^3$

$(- - - -) \rightarrow$ new polynomial

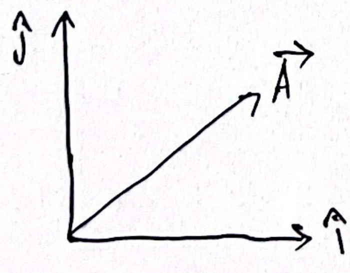
multiply by scalers, c → $5(1+x+2x^2)$ = new polynomial

satisfy 10 axioms

Basis of a vector space:

→ Basis is a set of vectors that spans the vector space

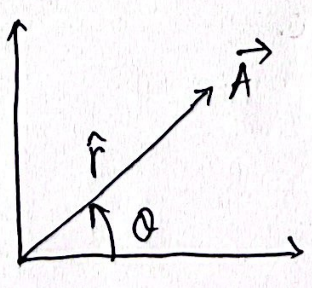
Example 1



$\vec{A} = A_x \hat{i} + A_y \hat{j} \rightarrow$ basis

constants / coefficients

Example-2



$\vec{A} = \hat{r} \angle \theta$

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Any vector in a 2-d vectorspace can be written as a linear combination of:

$$\left.\begin{array}{l} ① \ \{\hat{i}, \hat{j}\} \\ ② \ \{\hat{r}, \hat{\theta}\} \end{array}\right\}$$ both are basis of a 2-d vector space

---

$$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$= a_0 \cdot 1 + a_1 x + a_2 x^2$$

basis $\leftrightarrow \{1, x, x^2\} \rightarrow$ 3 dimensional space

→ By choosing different values of a, we can reproduce any polynomial we want of degree 2.

$$P_n(x) = a_0 \cdot 1 + a_1 x^1 + a_2 x^2 + \cdots a_n x^n$$

basis $\rightarrow \underbrace{\{1, x, x^2 \cdots x^n\}}_{\text{natural basis}} \rightarrow (n+1)$ dimensional space

---

Function Space → is a vector space

Function space can be considered as a vector space bcz

$\left\{\begin{array}{l} \text{add} \\ \text{multiply by scalers, } c \\ \vdots \\ \text{10 axioms} \end{array}\right.$

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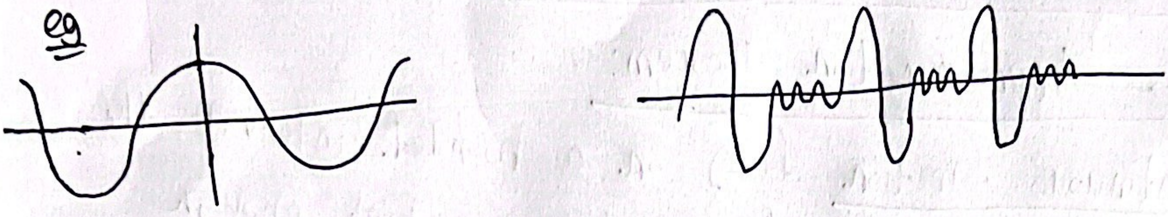
Intuition of fourier Series:

Any function can be constructed using sin and cos

$$f(x) = \Sigma \left(f' \sin(\ldots) + f' (\cos(\ldots)) \right)$$

eg



This implies there are ~~infinity~~ infinite amount of basis elements.

$$\underbrace{\{ \sin(\ldots), \cos(\ldots) \ldots\ldots \}}_{\infty \text{ dimension basis.}} \rightarrow \text{fourier basis}$$

Writing function using natural basis instead of fourier basis:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

example

$$\sin(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots}_{\text{still } \infty \text{ dim vector space, bcz it is a function.}}$$

But for Polynomial, it should stop.

example

$$P_3(x) = x - \frac{x^3}{3!}$$

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$P_n(x) \in V^{(n+1)}$ ← dimension

↑ Vector space

$f(x) \in V^{\infty}$

Weierstrass Approximation Theorem:

For a continuous function $f(x)$, on a bounded interval, the following is always possible if you take a high enough degree polynomial:

For any $f \in C([0,1])$ and any $\epsilon > 0$, there exist a polynomial such that

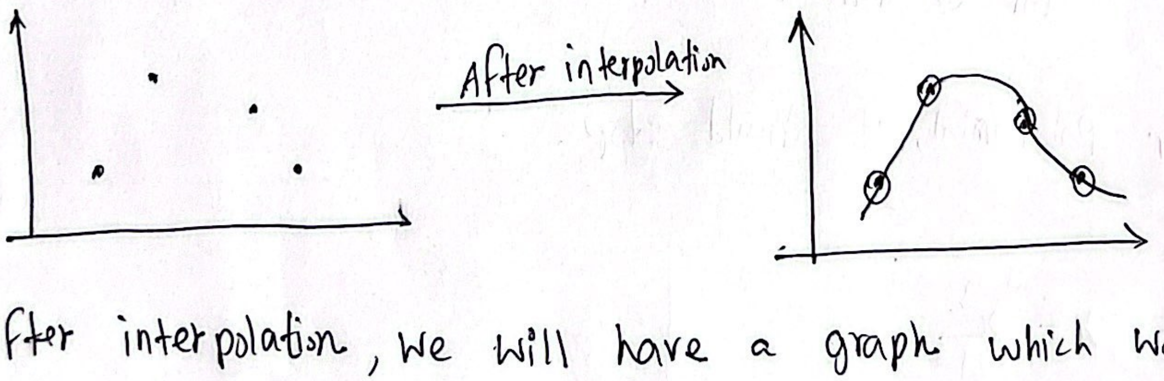
$$\max |f(x) - p(x)| \leq \epsilon$$

> Remember, if $P_n(x) \approx P_\infty(x)$, $P(x)$ will act as $f(x)$

Polynomial Interpolation

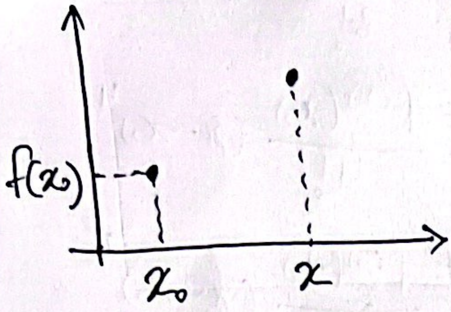
→ There exists a $f(x)$ but we do not know how it looks like

→ But we know ~~some~~ the values of some points (x_1, y_1) (x_2, y_2) ...



After interpolation, we will have a graph which would exactly do through these points.

Taylor Series:



If I know $f^{(1)}(x_0)$

$f^{(2)}(x_0)$

$f^{(3)}(x_0)$

$\vdots$

$f^{(\infty)}(x_0)$

Can I predict the value of the function at other point, x? $\longrightarrow$ Yes.

$$\underbrace{f(x)}_{\text{new value at point } x} = \underbrace{f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots}_{\text{We have all info at point } x_0}$$

Example:

Expand the function $\sin(x)$ using taylor series, centered at $x_0 = 0$

Solution:

$f(x) = \sin(x) \longrightarrow f(0) = \sin(0) = 0$

$f^{(1)}(x) = \cos(x) \longrightarrow f^{(1)}(0) = \cos(0) = 1$

$f^{(2)}(x) = -\sin(x) \longrightarrow f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(x) = -\cos(x) \longrightarrow f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(x) = \sin(x) \longrightarrow f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(x) = \cos(x) \longrightarrow f^{(5)}(0) = \cos(0) = 1$

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$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3$$
$$+ \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$
$$+ \frac{f^{(5)}(x_0)}{5!}(x-x_0)^5$$
$$+ \cdots$$

$\boxed{x_0 = 0}$

$$f(x) = f(0) + f^{(1)}(0)(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3$$
$$+ \frac{f^{(4)}(0)}{4!}(x-0)^4$$
$$+ \frac{f^{(5)}(0)}{5!}(x-0)^5 + \cdots$$

$$= 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4$$
$$+ \frac{1}{5!}(x-0)^5 + \cdots$$

$$\therefore f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

→ we knew all derivatives at $x_0 = 0$

→ Now we can find the value at, lets say for example, $x = 0.1$

If we take the 1st term, $f(x) = x$ ($x \leftarrow 0.1$)

2nd term, $f(x) = x - \frac{1}{3!}x^3$

3rd term, $f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

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$\therefore$ if we want to find the value of the function at $x = 0.1$

Taking 1^{st} term, $f(0.1) = 0.1 = 0.1$

2^{nd} term, $f(0.1) = 0.1 - \frac{1}{3!}(0.1)^3 =$

3^{rd} term, $f(0.1) = 0.1 - \frac{1}{3!}(0.1)^3 + \frac{1}{5!}(0.1)^5 =$

~~Taylors~~ Taylor's Series:

$$f(x) = \underbrace{f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2}_{\text{If we take upto the third term, it becomes a polynomid of degree 2}} + \underbrace{\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots}_{\text{This terms are the part of the error.}}$$

$\Rightarrow$ We can formalize the above thing using Taylor's Theorem

Taylor's Theorem:

Let f be $(n+1)$ times differentiable on $(a+b)$ and let $f^{(n)}$ be continuous on $[a,b]$. If $x, x_0 \in [a,b]$ then there exists $\xi \in (a,b)$, such that:

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor's polynomial of degree, } n} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange form of the remainder}}$$

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Important point:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

→ We do not know the actual value of ξ, if we did, we could find the exact value of the error, and hence, the exact value of the function. We only know that ξ is a value between (a,b), but we dont know the exact value.

→ What we can find, however, is the maximum bound of the error.

→ In other words, if ξ is a value between (a,b), what will be the maximum value of the error?

Example:

using the first 3 terms only.

$$f(x) = \sin(x) = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}} - \frac{x^7}{7!} + \dots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0 \cdot x^6$$

$$P_6(x)$$

Using ~~From~~ Taylor's Theorem:

$$f(x) = \underbrace{P_6(x)}_{\text{Taylor's polynomial of degree 6}} + \underbrace{\frac{f^{(7)}(\xi)}{7!}(x-x_0)^7}_{\text{Lagrange form of the remainder}}$$

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$$\Rightarrow \underbrace{\left| f(x) - P_6(x) \right|}_{\text{difference}} = \underbrace{\left| \frac{f^{(7)}(\xi)}{7!} (x-x_0)^7 \right|}_{\text{error}}$$

The taylor series was found while being centered at $x_0 = 0$

Now, lets say I want to find the value of the function at $x = 0.1$

$$\left| f(x) - P_6(x) \right| = \left| \frac{f^{(7)}(\xi)}{7!} (x-x_0)^7 \right|$$

$$\boxed{\begin{aligned} f(x) &= \sin(x) \\ f^{(1)}(x) &= \cos(x) \\ &\vdots \\ f^{(7)}(x) &= -\sin(x) \end{aligned}}$$

$$\left| f(0.1) - P_6(0.1) \right| = \left| \frac{-\sin(\xi)}{7!} (0.1-0)^7 \right|$$

→ What will be the max^m value of $\sin(\xi)$ between $\overset{a}{0}$ & $\overset{b}{0.1}$?

↳ 1.

$$\left| f(0.1) - P_6(0.1) \right| \leq \frac{1}{5040} (0.1)^7$$

$$\leq 1.984 \times 10^{-11}$$ } estimation correct upto 11 decimal points.

→ the error is coming bcz of the truncation of the taylor series

→ While doing numerical analysis, computer suffers from :

① Truncation error

② Rounding error of fl.

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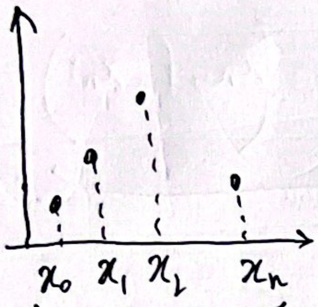
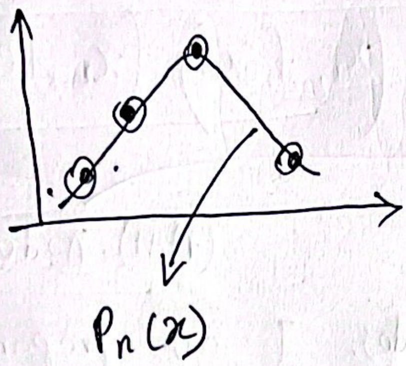
Polynomial Interpolation (Using Vandermonde Matrix)

If I am given $(n+1)$ number of nodes, the polynomial, that I can interpolate will be of degree, n





$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

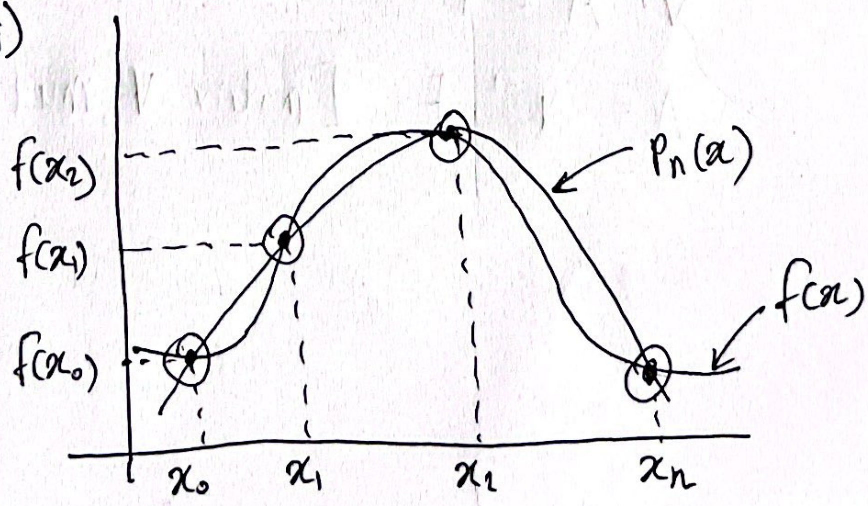
$$= \sum_{k=0}^{n} a_k \, x^k$$

constants/coefficients (a_k) natural basis (x^k)

our task is to find these constants/coefficients.

Important point:

$$P_n(x_i) = f(x_i)$$



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If I have $(n+1)$ nodes, I will have to satisfy $(n+1)$ conditions, I need to find $(n+1)$ nodes.

Lets say, I am given the following $(n+1)$ nodes:

$$\underbrace{\big(x_0, f(x_0)\big)\ \big(x_1, f(x_1)\big)\ \big(x_2, f(x_2)\big) \dots \big(x_n, f(x_n)\big)}_{(n+1)\text{ nodes}}$$

From the above $(n+1)$ nodes, I can prepare the following vandermonde matrix:

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{\text{Vandermonde Matrix, } V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}}_{F}$$

$$V \cdot A = F$$

$A = V^{-1} \cdot F$ [matrix V must be invertible

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Example:

$x_0 = 0$ $\quad$ $x_1 = \frac{\pi}{2}$ $\quad$ $x_2 = \pi$

$f(x_0) = 1$ $\quad$ $f(x_1) = 0$ $\quad$ $f(x_2) = -1$

3 nodes $\longrightarrow P_2(x)$

$\downarrow$

(3X3) Vandermonde Matrix

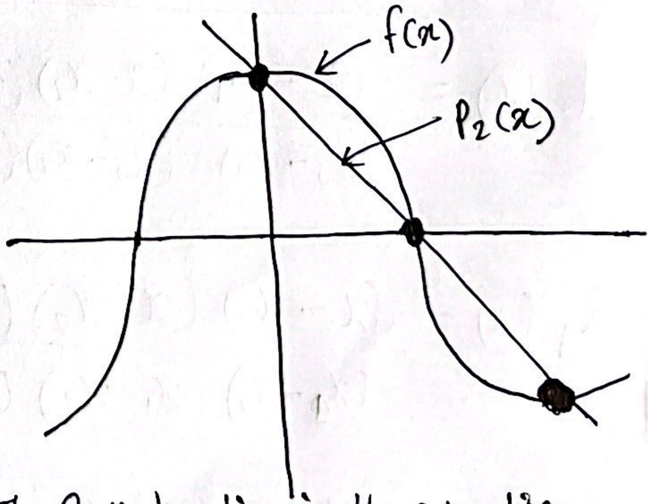
$$\begin{bmatrix} 1 & 0 & 0^2 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & \pi^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{\pi}{2} & \frac{\pi^2}{4} \\ 1 & \pi & \pi^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{\pi} \\ 0 \end{bmatrix}$$

$$\begin{aligned} P_2(x) &= a_0 \cdot 1 + a_1 x^1 + a_2 x^2 \\ &= 1 + \left(-\frac{2}{\pi}\right)x + 0\,(x)^2 \\ &= 1 - \frac{2}{\pi}x \end{aligned}$$

The above 3 nodes came from the function $\cos(x)$.



The 3 nodes lie in the same line, hence we got a straight line ($P_1(x)$) instead of a degree 2 polynomial, $P_2(x)$

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Lagrange Basis:

Natural basis $\rightarrow \{1, x, x^2 \dots\}$

~~lagrange basis~~ $\cancel{\rightarrow \{ l_0(x), l_1, l_2 \dots \}}$

lagrange basis $\rightarrow \{ l_0(x), l_1(x), l_2(x) \dots \}$

Previously, using natural basis:

$$P_n(x) = \sum_{k=0}^{n} \underbrace{a_k}_{\text{coefficients}} \underbrace{x^k}_{\text{basis}}$$

coefficients [We need to calculate the coefficients]

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots a_n x^n$$

Now, using lagrange basis:

basis [We need to calculate the basis]

$$P_n(x) = \sum_{k=0}^{n} \underbrace{f(x_k)}_{\text{coefficients}} \underbrace{l_k(x)}_{\text{basis}}$$

$$= f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x) + \dots f(x_n)\, l_n(x)$$

How to calculate Lagrange basis:

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3) \dots (x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3) \dots (x_0-x_n)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3) \dots (x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3) \dots (x_1-x_n)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3) \dots (x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3) \dots (x_2-x_n)}$$

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will be provided in the question

$$P_n(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \cdots + f(x_n)\, l_n(x)$$

need to calculate

Example

$x_0 = -\frac{\pi}{4}$ $\quad x_1 = 0$ $\quad x_2 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad f(x_1) = 1$ $\quad f(x_2) = \frac{1}{\sqrt{2}}$

3 nodes $\rightarrow P_2(x)$

$\downarrow$

3 coefficients

$\downarrow$

3 $l_k(x)$ s.

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-\frac{\pi}{4})}{(-\frac{\pi}{4})(-\frac{\pi}{2})} = \frac{8}{\pi^2}\, x\left(x-\frac{\pi}{4}\right)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+\frac{\pi}{4})(x-\frac{\pi}{4})}{(-\frac{\pi}{4})(\frac{\pi}{4})} = -\frac{16}{\pi^2}\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+\frac{\pi}{4})(x-0)}{(\frac{\pi}{2})(\frac{\pi}{4})} = \frac{8}{\pi^2}\, x\left(x+\frac{\pi}{4}\right)$$

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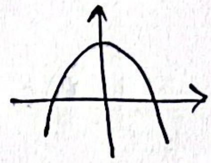
$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x)$$

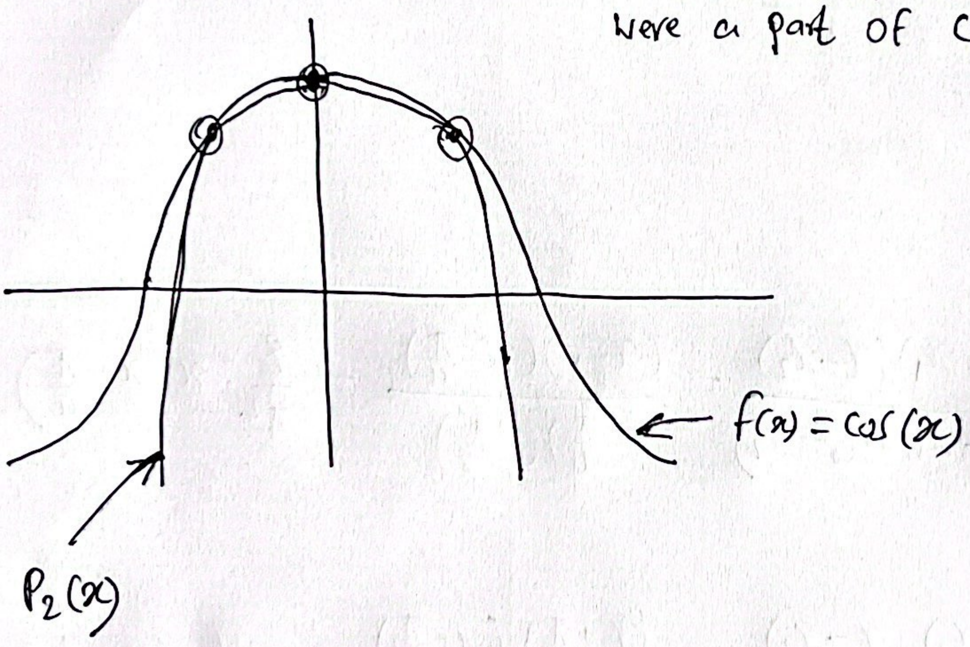
$$= \frac{1}{\sqrt{2}} \left(\frac{8}{\pi^2}\right)(x)\left(x-\frac{\pi}{4}\right) + 1\cdot\left(\frac{-16}{\pi^2}\right)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{8}{\pi^2}\right)(x)\left(x+\frac{\pi}{4}\right)$$

$$= \underbrace{\frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}}-1\right)}_{-\text{ve number}} x^2 + 1 \quad \rightsquigarrow$$



The 3 nodes which was initially given were a part of cos (x) graph



<u>Problem of Lagrange:</u>

→ New nodes cannot be added

→ If added, need to do calculation of $l_k(x)$ newly again

<u>Advantage of Lagrange:</u>

→ No need to inverse a matrix (Inversing a matrix is computionally very expensive)

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Newton's Polynomial:

$n_0(x) = 1$

$n_1(x) = (x - x_0)$

$n_2(x) = (x - x_0)(x - x_1)$

$n_3(x) = (x - x_0)(x - x_1)(x - x_2)$

$\vdots$

→ x is variable

→ $x_0, x_1, x_2 \ldots$ are values/constants which can be found from the given nodes

Creating $P_n(x)$ using Newton's polynomial as basis:

$$P_n(x) = \sum_{k=0}^{n} a_k \, n_k(x)$$

$$= a_0 \cancel{n_0(x)}^{1} + a_1 \cancel{n_1(x)}^{(x-x_0)} + a_2 \cancel{n_2(x)}^{(x-x_0)(x-x_1)} + \ldots + a_n n_n(x)$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

How to find a?

Lets introduce a new notation

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$\vdots$

$a_n = f[x_0, x_1 \ldots x_n]$

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$$P_n(x) = \underset{f[x_0]}{\cancel{a_0}} + \underset{f[x_0,x_1]}{\cancel{a_1}}(x-x_0) + \underset{f[x_0,x_1,x_2]}{\cancel{a_2}}(x-x_0)(x-x_1) + \cdots$$

$$= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots$$

Example:

$x_0 = -1$ $\quad x_1 = 0$ $\quad x_2 = 1$ $\quad x_3 = 2$

$f(x_0) = 5$ $\quad f(x_1) = 1$ $\quad f(x_2) = 1$ $\quad f(x_3) = 11$

4 nodes $\longrightarrow P_3(x)$

$\downarrow$

find till a_3

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= f[x_0] + f[x_0,x_1](x-x_0)$$
$$+ f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+ f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

Need to find $f[x_0]$

$f[x_0,x_1]$

$f[x_0,x_1,x_2]$

$f[x_0,x_1,x_2,x_3]$

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$x_0 = -1 \quad f[x_0] = 5$

$f[x_0, x_1] = \frac{1-5}{0-(-1)} = -4$

$x_1 = 0 \quad f[x_1] = 1$

$f[x_0, x_1, x_2] = \frac{0-(-4)}{1-(-1)} = 2$

$f[x_1, x_2] = \frac{1-1}{1-0} = 0$

$f[x_0, x_1, x_2, x_3] = \frac{5-2}{2-(-1)} = 1$

$x_2 = 1 \quad f[x_2] = 1$

$f[x_1, x_2, x_3] = \frac{10-0}{2-0} = 5$

$f[x_2, x_3] = \frac{11-1}{2-1} = 10$

$x_3 = 2 \quad f[x_3] = 11$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= 5 + (-4)(x+1) + 2(x+1)(x-0) + 1(x+1)(x-0)(x-1)$$

degree 3 polynomial.

<u>Advantage of Newton's divided difference</u>

→ can keep on adding nodes, but no need to do calculations from very beginning.

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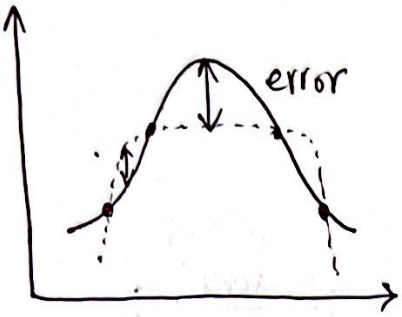
Interpolation Error:



error $\rightarrow$ $|f(x) - P_n(x)|$

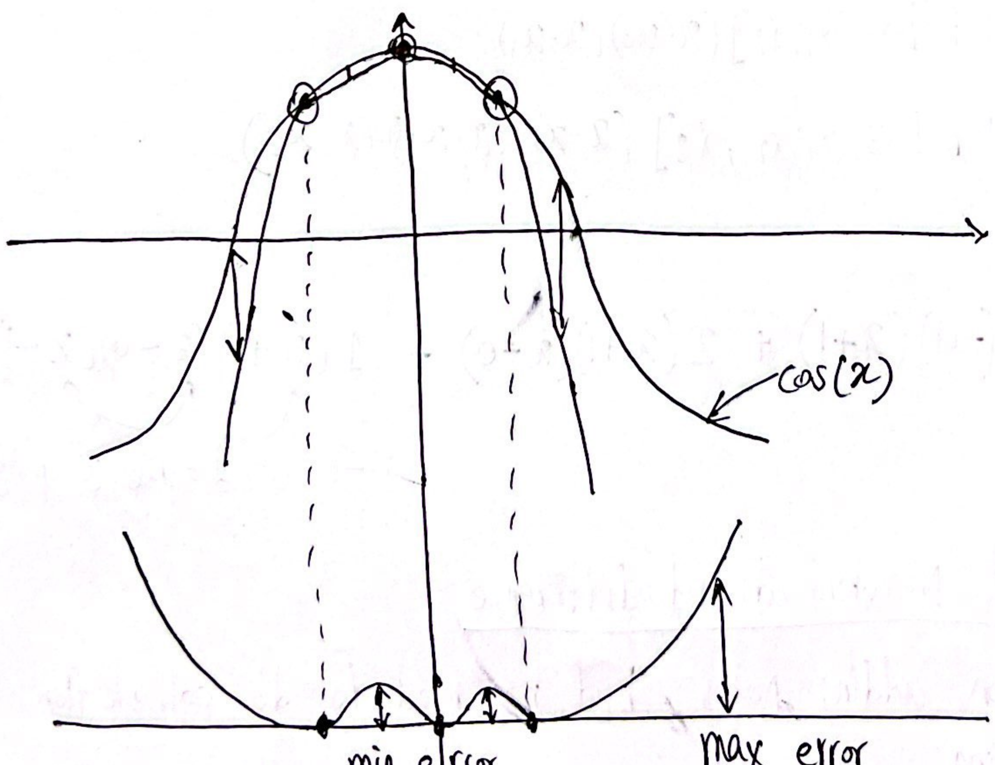
error at nodes $= 0$

Example

$f(x) = \cos x$

$P_2(x) = \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 + 1$

$$|f(x) - P_2(x)| = \left|\cos(x) - \frac{16}{\pi^2}\left(\frac{1}{\sqrt{2}} - 1\right)x^2 - 1\right|$$



error goes to zero at the nodes

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Cauchy's Theorem

Let $P_n \in \mathcal{P}_n$ be the unique polynomial interpolating $f(x)$ at the $(n+1)$ distinct nodes $x_0, x_1 \cdots x_n \in [a,b]$ and let f be continuous on $[a,b]$ with $(n+1)$ continuous derivatives on (a,b). Then for each $x \in [a,b]$, there exists a $\xi \in (a,b)$, such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1) \cdots (x-x_n)}$$

→ error goes to zero at the nodes $(x_0, x_1 \ldots x_n)$

→ $(x-x_0)(x-x_1) \cdots (x-x_n)$
so when $x = x_0$ or x_1 or x_2, error goes to zero

Similar to taylors theorem, we could've found the exact error at any point if we knew the value of ξ. But all we know is that ξ is a value between a & b. So we ~~have~~ can find the maximum value of the function ~~at~~ betwen a & b instead.

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$$\left|f(x)-P_n(x)\right| = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Example

$f(x) = \cos(x)$

$x_0 = -\frac{\pi}{4}$ $\quad$ $x_1 = 0$ $\quad$ $x_1 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ $\quad$ $f(x_1) = 1$ $\quad$ $f(x_2) = \frac{1}{\sqrt{2}}$

3 nodes $\rightarrow$ $P_2(x)$ [find $P_2(x)$ using either vandermonde matrix, lagrange, or newtons divided difference form.

$$\left|f(x)-P_2(x)\right| = \frac{f^{(3)}(\xi)}{3!}\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)(x)$$

$$= \boxed{\frac{\sin(\xi)}{6}(x)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)}$$

We need to find the maximum of this whole thing inside of an interval.

example $\quad \xi \in [-1, 1] \leftarrow$ interval will be given in the question.

$\rightarrow$ if interval is not given, use the lowest value of the nodes as 'a', highest value of the nodes as 'b'

Important! nodes x_0, x_1, x_2 must lie within interval, example $[-1, 1]$

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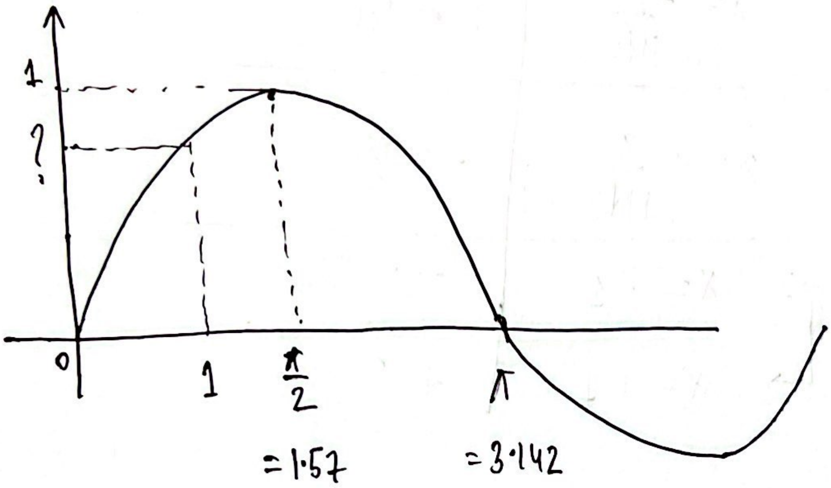
We will use the interval $\xi \in [-1, 1]$ for the ~~problem~~ rest of the problem.

$$\underbrace{\frac{\sin(\xi)}{6}}_{\text{Need to find max}^m \text{ of this in } [-1,1] \text{ interval}} \qquad \underbrace{(x)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)}_{\text{Need to find max}^m \text{ of this in } [-1,1] \text{ interval}}$$



Max^m value of $\sin(\xi)$ in interval $[-1, 1]$ is $\sin(1)$

$$\frac{\sin(1)}{6} =$$

$$W(x) = x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)$$

$$= x\left(x^2 - \frac{\pi^2}{16}\right)$$

$$= x^3 - \frac{\pi^2}{16}x$$

$$W'(x) = 3x^2 - \frac{\pi^2}{16} = 0$$

$$x = \pm \frac{\pi}{4\sqrt{3}}$$

General rule.

Find maximum of $W(x)$ in interval.

1. set $W'(x) = 0$
2. solve for x
3. if $W''(x) < 0$, maxima
   $W''(x) > 0$, minima

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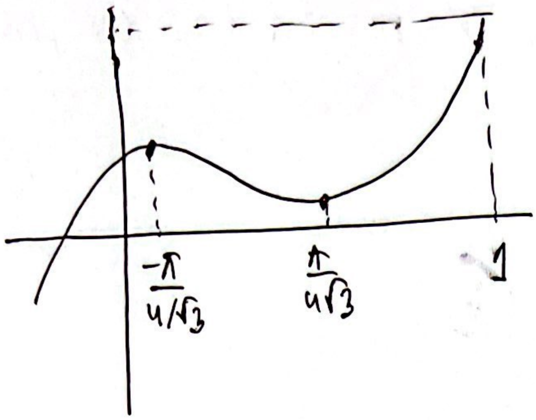

$\therefore$ find $w(x)$ at

x	$w(x)$
$x = -\frac{\pi}{4\sqrt{3}}$	$+0.186$
$x = +\frac{\pi}{4\sqrt{3}}$	-0.186
$x = -1$	-0.383
$x = +1$	$+0.383$

include the intervals too

$$\therefore \left| f(x) - P_2(x) \right| = \frac{\sin(\xi)}{6} (x) \left(x + \frac{\pi}{4}\right) \left(x - \frac{\pi}{4}\right)$$

Max error $= \frac{\sin(1)}{6} * 0.383$

If we did not have a sin(x) function in the first part, but we got another polynomial function instead, we would have to do the same calculation of $w(x)$ twice for both part.

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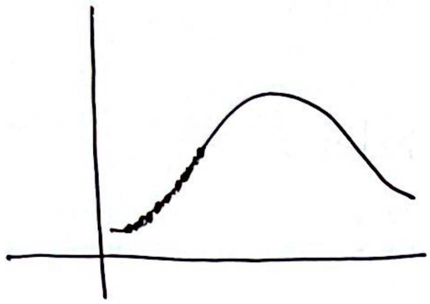
Chebyshev Nodes:

Expected:

if $n \to \infty$, error $\to 0$



But there are some functions which do not show the above properties. Those functions are called "<u>Runge Functions</u>"

Example:

$$f(x) = \frac{1}{1+25x^2} \quad \text{on } [-1, 1]$$

Runge function

It gives big errors on the corners.

$$\boxed{n \to \infty, \quad \text{error} \to \infty}$$

when nodes are equally spaced.

Work around:

→ Do not take equally spaced nodes

→ Take more nodes at the corner.

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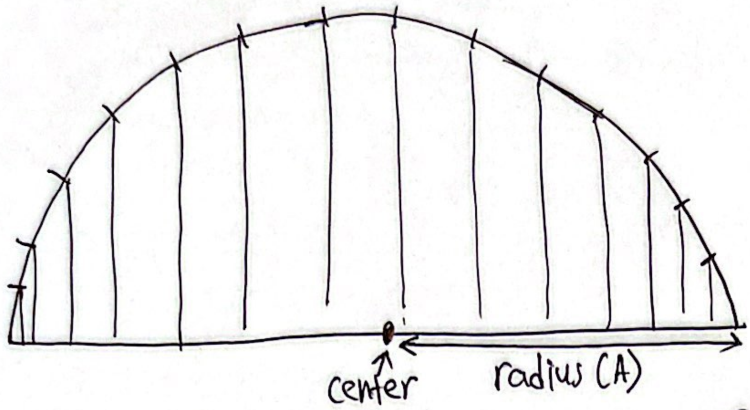
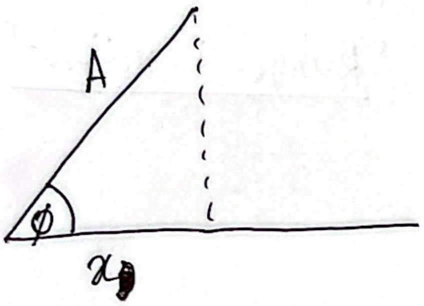


→ Take equally spaced angles instead of nodes.

→ Then take their projection on the x-axis.



$$\cos(\phi) = \frac{x_j}{A}$$

$$x_j = A\cos(\phi)$$

$$\boxed{\phi_j = \frac{(2j+1)\pi}{2(n+1)}, \quad j = 0, 1, 2, \ldots, n}$$

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Formula:

$$x_j = A\cos(\phi_j) + \text{center}$$

$$= A\cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] + \text{center} \qquad j = 0, 1, 2 \dots, n$$

Example:

Runge function

$$f(x) = \frac{1}{1+25x^2} \quad \text{on the Interval } [-1, 1]$$

The above function is to be interpolated with a polynomial of degree 3.

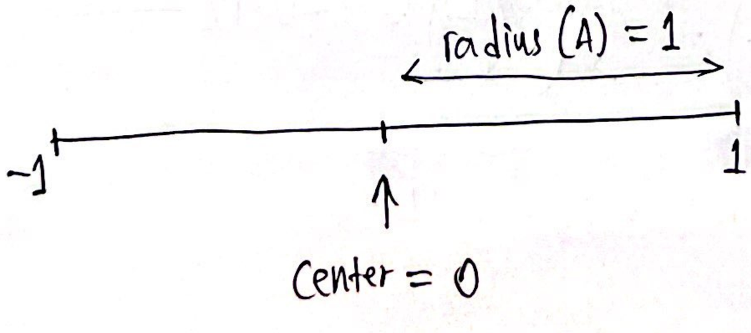
Find suitable nodes with which you would want to perform the polynomial interpolation.

Solution:

degree $(n) = 3$

$\therefore$ number of nodes required $= n+1 = 4$.

Interval: $[-1, 1]$



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$$x_j = A\cos(\phi_j) + \text{center}$$

$$= 1 \cdot \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] + 0$$

$$= \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] \qquad j = 0, 1 \ldots, n$$

$= 0, 1, 2, 3$ ← since $n=3$ in our question.

<u>When $j=0$</u>

$$x_0 = \cos\left(\frac{[2(0)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{\pi}{8}\right) = 0.92$$

<u>When $j=1$</u>

$$x_1 = \cos\left(\frac{[2(1)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{3\pi}{8}\right) = 0.38$$

<u>When $j=2$</u>

$$x_2 = \cos\left(\frac{[2(2)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{5\pi}{8}\right) = -0.38$$

<u>When $j=3$</u>

$$x_3 = \cos\left(\frac{[2(3)+1]\pi}{2(3+1)}\right) = \cos\left(\frac{7\pi}{8}\right) = -0.92$$

$\therefore$ The chebyshev nodes are: $x_0 = 0.92$
$x_1 = 0.38$
$x_2 = -0.38$
$x_3 = -0.92$

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Hermitte Interpolation:

Previously only one condition used to be fulfilled:

$$p(x_i) = f(x_i)$$

Now, along with the previous condition, one more condition is to be fulfilled:

$$p'(x_i) = f'(x_i)$$

Previously,

If I am given $(n+1)$ nodes, degree of polynomial was $P_n(x)$

Now, using Hermitte Interpolation:

If I am given $(n+1)$ nodes, degree of polynomial will be $P_{2n+1}(x)$

Using Natural Basis:

$$P_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 x^0 + a_1 x^1 + \cdots a_n x^n$$

Using Lagrange Basis

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \cdots$$

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Using Hermitte Basis:

$$P_{2n+1}(x) = f(x_k)\,h_k(x) + f'(x_k)\,\hat{h}_k(x)$$

$$\boxed{\begin{aligned} h_k(x) &= \left[1 - 2(x - x_k)\,\ell_k'(x_k)\right] \ell_k^2(x) \\ \hat{h}_k(x) &= (x - x_k)\,\ell_k^2(x) \end{aligned}}$$

Example:

$f(x) = \sin(x)$

$x_0 = 0 \qquad x_1 = \frac{\pi}{2}$

$f(x_0) = 0 \qquad f(x_1) = 1$

$f'(x_0) = 1 \qquad f'(x_1) = 0$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f'(x_0) = f'(0) = \cos(0) = 1$

$f'(x_1) = f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$$\rightarrow \left.\begin{aligned} \text{Number of nodes} &= 2 \\ n+1 &= 2 \\ n &= 1 \end{aligned}\right\} \quad P_{2n+1}(x) = P_{2(1)+1}(x) = P_3(x)$$

$$P_3(x) = \cancelto{0}{f(x_0)}\,h_0(x) + \cancelto{1}{f'(x_0)}\,\hat{h}_0(x) + \cancelto{1}{f(x_1)}\,h_1(x) + \cancelto{0}{f'(x_1)}\,\hat{h}_1(x)$$

$$P_3(x) = \hat{h}_0(x) + h_1(x)$$

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$$h_k(x) = \left[1 - 2(x - x_k)\, l_k'(x_k)\right] l_k^2(x)$$

$$h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)$$

$x_0 = 0 \quad , x_1 = \frac{\pi}{2}$

$\uparrow$ (from question)

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi} x$$

$$l_1'(x) = \frac{2}{\pi}$$

$$\therefore h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)$$

$$= \left[1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left[\frac{2}{\pi} x\right]^2$$

$$= \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

$$\hat{h}_k(x) = (x - x_k)\, l_k^2(x)$$

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{\pi}{2}}{0 - \frac{\pi}{2}} = 1 - \frac{2}{\pi} x$$

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$= (x - 0)\left(1 - \frac{2}{\pi} x\right)^2$$

$$= x\left(1 - \frac{2}{\pi} x\right)^2$$

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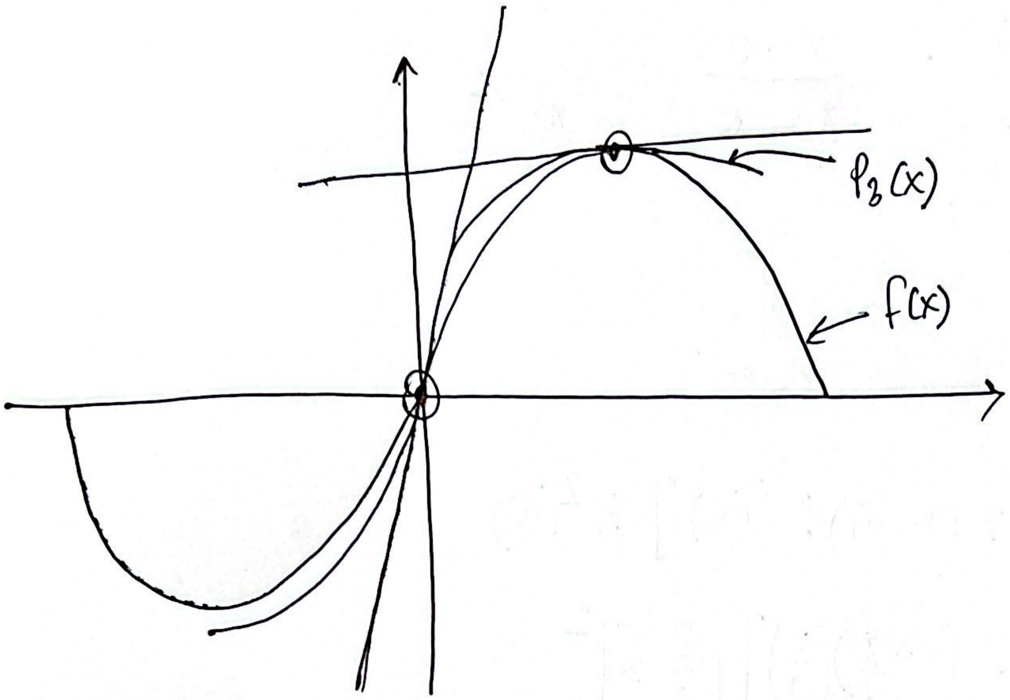
$$P_3(x) = \hat{h}_0(x) + h_1(x)$$

$$= x\left(1-\frac{2}{\pi}x\right)^2 + \frac{4}{\pi^2}x^2\left(3-\frac{4}{\pi}x\right)$$



$$P_n(x_i) = f(x_i)$$

$$P_n'(x_i) = f'(x_i)$$

Very good interpolation with 2 nodes only.

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☐ Differentiation:

$$\frac{d}{dx}(c \cdot x^n) = n \cdot c \cdot x^{n-1}$$

$$f(x) = x^3 - 4x + 1$$

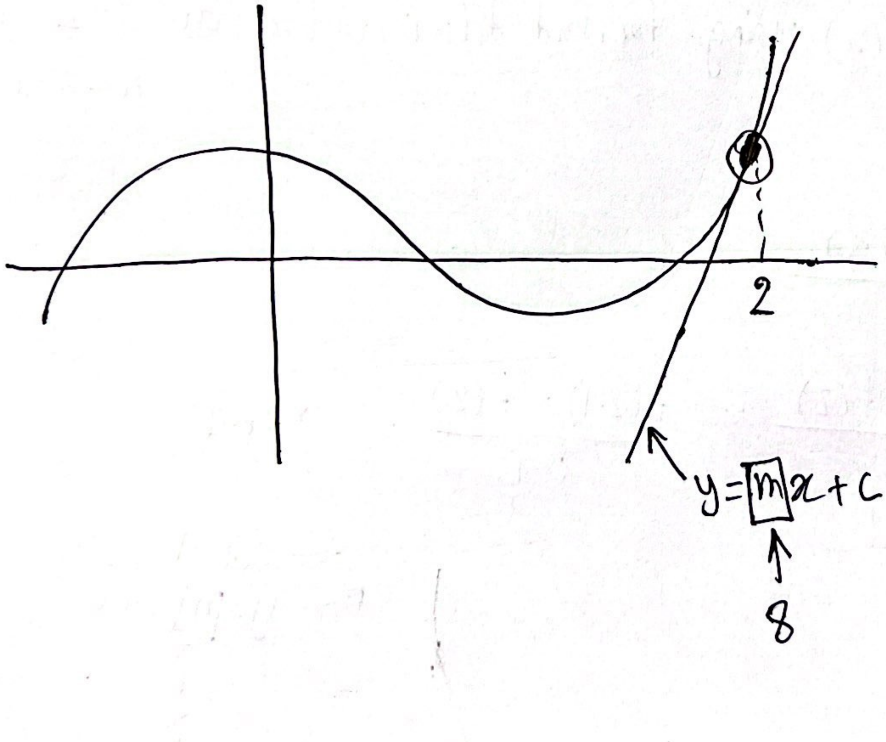
$$f'(x) = 3x^2 - 4$$

$$f'(2) = 3(2)^2 - 4$$

$$= 8$$

↳ what is the meaning of this number, 8?

Understanding its significance through a graph.



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Numerical Approach:

Forward Differentiation:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$h \rightarrow$ step size

$\hookrightarrow$ Assign some small value to h.

The smaller the value, the more closer to actual value (more accurate)

Example:

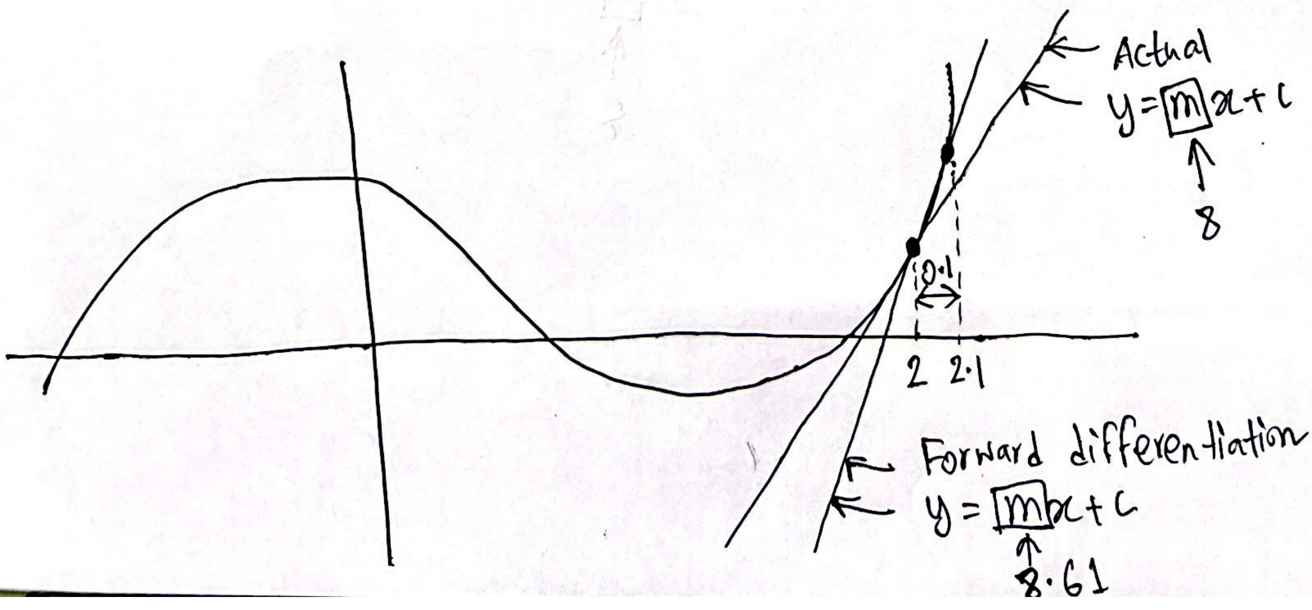
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using forward differentiation at $x=2$, and $h=0.1$.

Solution:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(2) = \frac{f(2+0.1) - f(2)}{0.1} = \frac{f(2.1) - f(2)}{0.1} = 8.61$$



Backward Differentiation:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

Example:

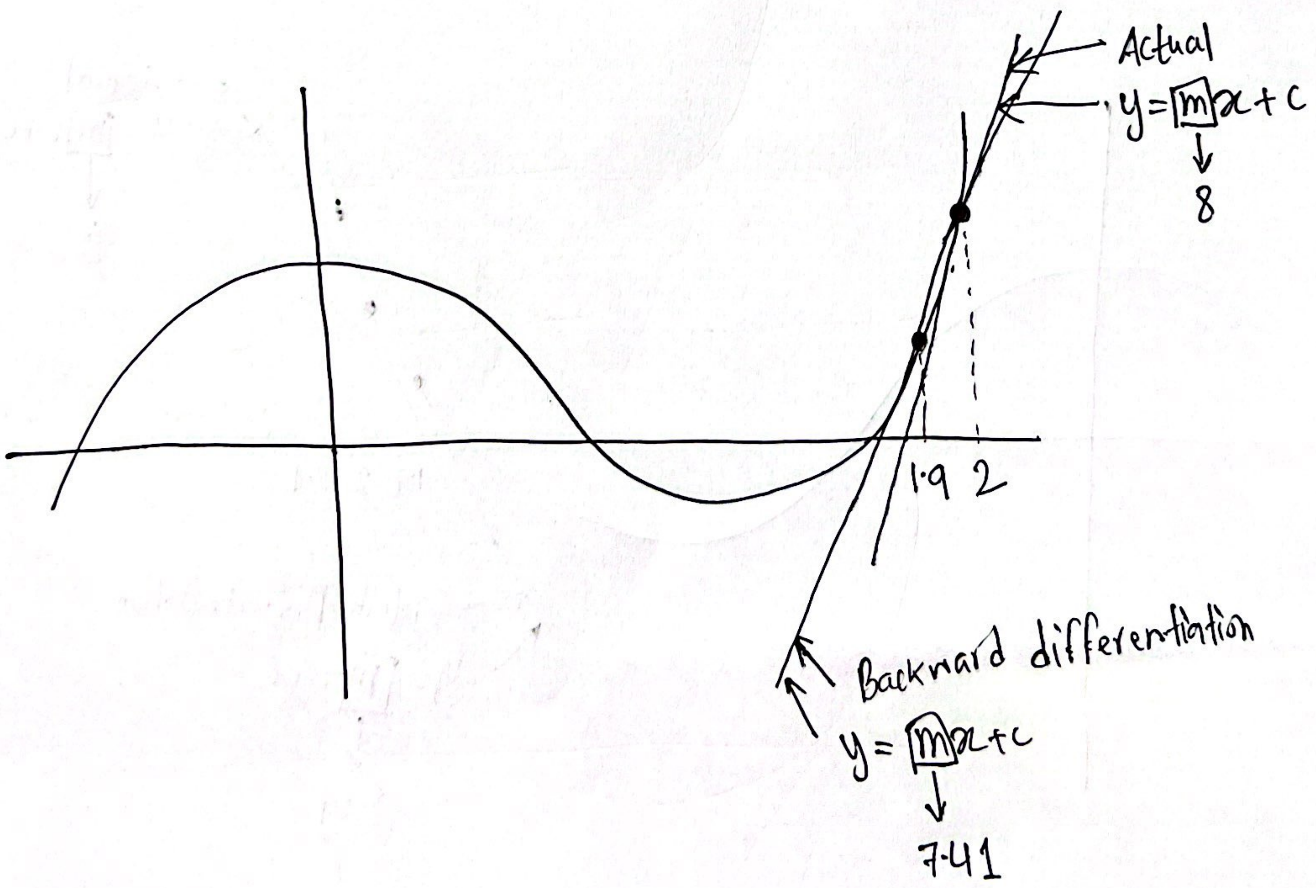
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using Backward Differentiation at $x = 2$, $h = 0.1$

Solution:

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$f'(2) = \frac{f(2) - f(2-0.1)}{0.1} = \frac{f(2) - f(1.9)}{0.1} = 7.41$$



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Central Differentiation:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Central difference gives less error than forward and backward differentiation.

Example:

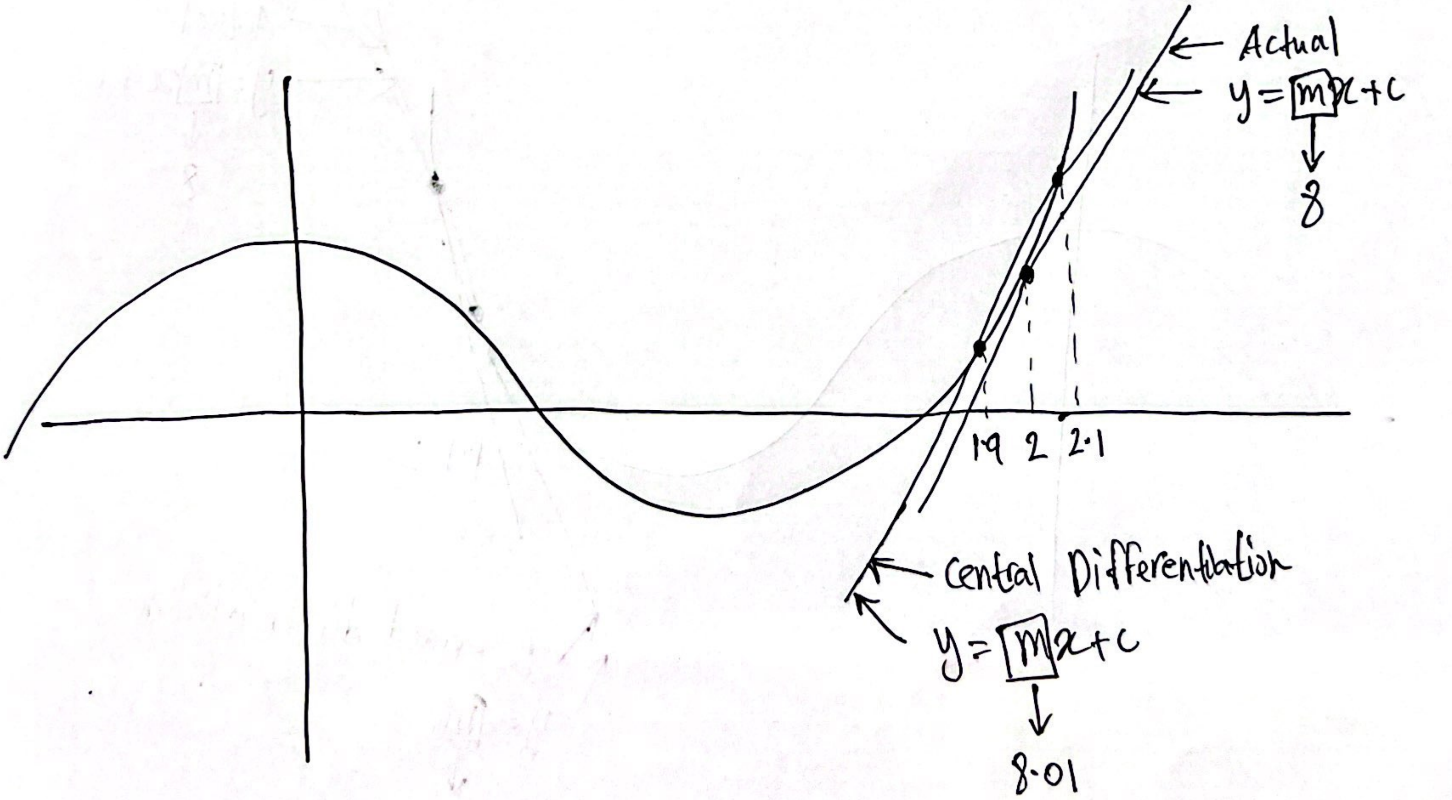
$f(x) = x^3 - 4x + 1$

Find the value of $f'(x)$ using centrall differentiation at $x = 2$
$h = 0.1$

Solution:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(2) = \frac{f(2+0.1) - f(2-0.1)}{2(0.1)} = \frac{f(2.1) - f(1.9)}{0.2} = 8.01$$



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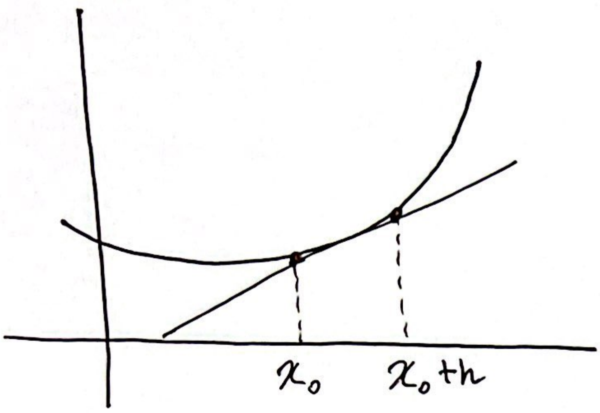
Forward Difference



$$\begin{array}{cc} x_0 & x_1 \\ \downarrow & \downarrow \end{array}$$

Interpolating a polynomial using $x_0,\ x_0+h$

$$P_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$= \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$f(x) = \underbrace{\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)}_{P_1(x)} + \underbrace{\overbrace{\frac{f''(\xi)}{2}}^{u}\,\overbrace{(x-x_0)(x-x_1)}^{v}}_{\text{error}}$$

$$f'(x) = \frac{1}{x_0-x_1} f(x_0) + \frac{1}{x_1-x_0} f(x_1) + \frac{f'''(\xi)}{2}\,\frac{d\xi}{dx}\,\cancelto{0}{(x-x_0)}(x-x_1)$$

$$+ \frac{f''(\xi)}{2}(2x - x_0 - x_1)$$

Plugging $x = x_0$

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{f''(\xi)}{2}(x_0 - x_1)$$

$x_1 = x_0+h$

$$= \frac{f(x_0+h) - f(x_0)}{h} + \boxed{\frac{f''(\xi)}{2}(-h)}$$

truncation error.

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$\boxed{\text{Error} \propto h}$

Example:

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2} = 0.5$

Using Forward differentiation:

h	$f'(x_0)$	Truncation Error
	$\frac{f(x_0+h) - f(x_0)}{h}$	$\lvert$ Actual value - Forward diff value $\rvert$
	$\rightarrow \frac{\ln(2+h) - \ln(2)}{h}$	$\rightarrow \lvert 0.5$ - Forward diff value $\rvert$
1	0.405465	0.0945349
0.1	0.487902	0.0120984
0.01	0.498754	0.00124585
0.001	0.499875	0.00012

Decreasing on a ~~scalee~~ scale of 10, just like h

[if h is divided by 10, error also gets divided by 10]

[$\therefore$ error $\propto h$]

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Backward Differentiation:

Error $\propto h$ [Derivation same as Forward Difference]

Central Differentiation:

Error $\propto h^2$ $\left[\frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(\xi)}{3!} h^2 \right]$

For central differentiation, error becomes small when $h < 1$.

Example

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2} = 0.5$

h	$f'(x_0)$	Truncation Error
	$\boxed{\frac{f(x_0+h) - f(x_0-h)}{2h}}$	|Actual value − central diff value|
1	0.549306	0.0493061
0.1	0.500417	0.000417293
0.01	0.5000004	4.16673×10^{-6}
0.001	0.5000000	4.16666×10^{-8}

[If h is divided by 10, error gets divided by 100
$\therefore$ error $\propto h^2$]

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Rounding Error:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

→ smaller step size, h, better result.

→ However, if h is very small, $f(x+h)$ and $f(x-h)$ will have similar values. Example $f(2+0.001) \approx f(2-0.001)$

→ In floating point chapter, we learned about ~~too~~ loss of significance

→ When we subtract 2 numbers which are close to each other, there are large errors.

→ When h tends to 0 (very small), $f(x+h)$ & $f(x-h)$ are numbers which are closer to each other.

→ Therefore $\dfrac{f(x+h) - f(x-h)}{2h}$ will give large error (large Rounding error)

$$\boxed{fl(x) = (1+\delta_1)\, x}$$ ← From chapter 1.

$$fl[f(x_1+h)] = (1+\delta_1)\, f(x_1+h)$$

$$fl[f(x_1-h)] = (1+\delta_2)\, f(x_1-h)$$

$$\boxed{|\delta_1|, |\delta_2| \le \epsilon_M}$$ ← from chapter 1

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Error:

$$\left| \text{Actual value of differentiation} - \begin{array}{c}\text{Value of differentiation}\\ \text{by numerical Approach}\end{array} \right|$$

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!}h^2 - \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h} \right|$$

cancel

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!}h^2 - \frac{(1+\delta_1) f(x_1+h) - (1+\delta_2) f(x_1-h)}{2h} \right|$$

$$= \left| -\frac{f'''(\xi)}{6} h^2 - \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$$|a+b| \leq |a| + |b| \longrightarrow \begin{array}{l} |5 + (-1)| \leq |5| + |-1| \\ |4| \leq 5 + 1 \\ |4| \leq 6 \end{array}$$

$$\leq \left|\frac{f'''(\xi)}{6}\right| h^2 + \frac{\left|\delta_1 \cdot f(x_1+h) - \delta_2 \cdot f(x_1-h)\right|}{2h}$$

$$|\delta_1|, |\delta_2| \leq \epsilon_M$$

$$\leq \underbrace{\frac{|f'''(\xi)|}{6} h^2}_{} + \underbrace{\epsilon_M \frac{|f(x_1+h)| + |f(x_1-h)|}{2h}}_{}$$

→ This term comes from the truncation of the series.
→ Smaller h, lesser error

→ smaller h, larger error (Rounding error).
→ since h is in the denominator.

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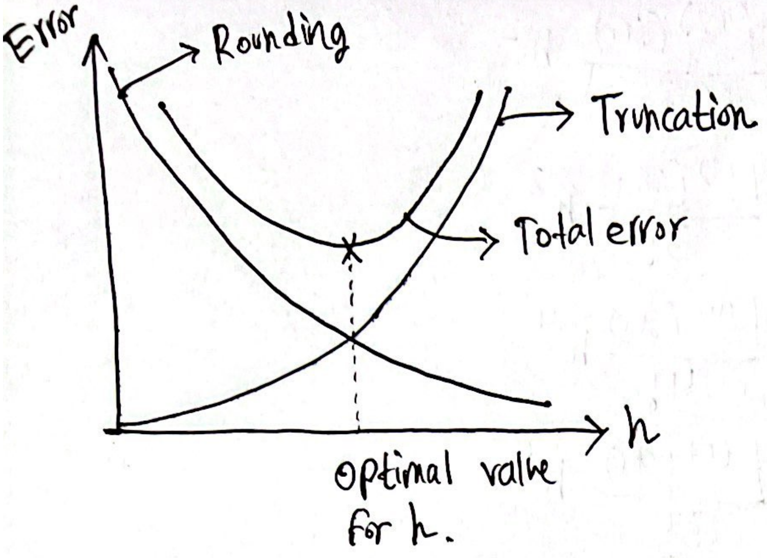

Richardson Extrapolation:

$$D_h := \frac{f(x_1+h) - f(x_1-h)}{2h}$$

central differentiation (D_h)

$$\boxed{D_{\frac{h}{2}} = \frac{f(x_1+\frac{h}{2}) - f(x_1-\frac{h}{2})}{2(\frac{h}{2})}}$$

Taylor Series:

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}}{2!}(x_0)(x-x_0)^2 + \cdots$$

Centering at x_1:

$$f(x) = f(x_1) + f^{(1)}(x_1)(x-x_1) + \frac{f^{(2)}}{2!}(x_1)(x-x_1)^2 + \cdots$$

$$f(x_1+h) = f(x_1) + f^{(1)}(x_1)(x_1+h-x_1) + \frac{f^{(2)}}{2!}(x_1)(x_1+h-x_1)^2 + \cdots$$

$$= f(x_1) + f^{(1)}(x_1)(h) + \frac{f^{(2)}(x_1)}{2!}(h)^2 + \frac{f^{(3)}(x_1)}{3!}(h^3) + \frac{f^{(4)}(x_1)}{4!}(h^4) + \frac{f^{(5)}(x_1)}{5!}(h^5) + O(h^6) \quad \text{--- (1)}$$

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$$f(x_1 - h) = f(x_1) - f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2!}h^2 - \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 - \frac{f^{(5)}(x_1)}{5!}h^5 + O(h^6) \quad \text{—— (II)}$$

$$D_h = \frac{f(x_1+h) - f(x_1-h)}{2h}$$

$$D_h = \frac{1}{2h}\left(\text{I} - \text{II}\right)$$

$$= \frac{1}{\cancel{2}\cancel{h}}\left(\cancel{2} f^{(1)}(x_1)\cancel{h} + \frac{\cancel{2} f^{(3)}(x_1)h^{\cancel{3}}}{3!} + \frac{\cancel{2} f^{(5)}(x_1)}{5!}h^{\cancel{5}} + O(h^7)\right)$$

$$= \underbrace{\boxed{f^{(1)}(x_1)}}_{\text{exact value}} + \underbrace{\boxed{\frac{f^{(3)}(x_1)}{3!}h^2 + \frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)}}_{\text{error}}$$

→ Error is of order h^2, because $h^4, h^6 \ldots$ are less dominant than h^2.

→ Hence proving again that error $\sim h^2$ for central difference

→ Can we make it better?

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$$D_h = f^{(1)}(x_1) + \boxed{\frac{f^{(3)}(x_1)}{3!} h^2} + \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$D_{\frac{h}{2}} = f^{(1)}(x_1) + \boxed{\frac{f^{(3)}(x_1)}{3!} \left(\frac{h}{2}\right)^2} + \frac{f^{(5)}(x_1)}{5!} \left(\frac{h}{2}\right)^4 + O(h^6)$$

→ Take combination in such a way that h^2 term goes away.

→ So that we are left with h^4.

$$2^2 D_{\frac{h}{2}} - D_h = 2^2 f^{(1)}(x_1) - f^{(1)}(x_1) + 2^2 \frac{f^{(5)}(x_1)}{5!} \times \frac{1}{2^{4}} h^4$$

$$- \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$2^2 D_{\frac{h}{2}} - D_h = (2^2 - 1) f^{(1)}(x_1) + \left(\frac{1}{2^2} - 1\right) \frac{f^{(5)}(x_1)}{5!} h^4 + O(h^6)$$

$$\boxed{\frac{2^2 D_{\frac{h}{2}} - D_h}{2^2 - 1}} = f^{(1)}(x_1) + \frac{\left(\frac{1}{2^2} - 1\right)}{(2^2 - 1)\, 5!} f^{(5)}(x_1) h^4 + O(h^6)$$

If we ~~t~~ take this combination, error gets reduced to an order of 4.

→ Lets consider this as $D_h^{(1)}$.

→ calculate $D_{\frac{h}{2}}^{(1)}$

→ Then take combination in such a way that h^4 goes away. So now we can have an error of order h^6. It will be called $D_h^{(2)}$ then.

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$$D_h = f'(x_1) + \boxed{c}h^n + O(h^{n+1})$$

could be anything like $\dfrac{f^{(n)}(x_1)}{n!}$

$$D_{\frac{h}{2}} = f'(x_1) + c\left(\frac{h}{2}\right)^n + O(h^{n+1})$$

multiply $D_{\frac{h}{2}}$ with 2^n

$$D_h^{(\,)} = \frac{2^n D_{\frac{h}{2}} - D_h}{2^n - 1}$$

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