✱ from previous class

for z-axis based rotation, not with respect to the origin:

(3D)

$$x = r\cos\varphi \qquad x' = x\cos\theta - y\sin\theta \qquad x_c \rightarrow \text{translation in } x\text{-axis}$$
$$y = r\sin\varphi \qquad y' = x\sin\theta + y\cos\theta \qquad y_c \rightarrow \text{translation in } y\text{-axis}$$
$$z' = z \ (\text{remains same})$$

(+Translation)   (Rotation)   (−Translation)

$$
\begin{vmatrix} x' \\ y' \\ z' \\ -1 \end{vmatrix}
=
\begin{vmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{vmatrix}
\cdot
\begin{vmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\cdot
\begin{vmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{vmatrix}
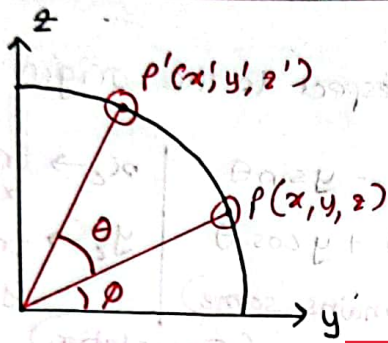\cdot
\begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}
$$

×

$$
=
\begin{bmatrix}
\cos\theta & -\sin\theta & 0 & -x_c\cos\theta + y_c\sin\theta \\
\sin\theta & \cos\theta & 0 & -x_c\sin\theta - y_c\cos\theta \\
0 & 0 & 1 & -z_c \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

×

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
=
\begin{bmatrix}
\cos\theta & -\sin\theta & 0 & (-x_c\cos\theta + y_c\sin\theta) + x_c \\
\sin\theta & \cos\theta & 0 & (-x_c\sin\theta - y_c\cos\theta) + y_c \\
0 & 0 & 1 & -z_c + z_c \rightarrow 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

The composite matrix   original coordinates

(✳) For x-axis based Rotation, not w.r.t origin

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x \quad \text{(same)}$$
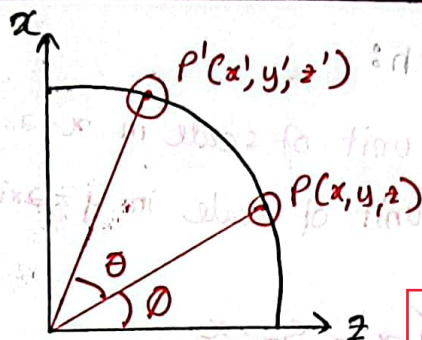
$P'(x', y', z')$

$P(x, y, z)$

Now,

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
=
\underbrace{\begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(+\text{Translation})}
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(\text{Rotation})}
\underbrace{\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(-\text{Translation})}
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos\theta & -\sin\theta & (-y_c\cos\theta + z_c\sin\theta) + y_c \\
0 & \sin\theta & \cos\theta & (-y_c\sin\theta - z_c\cos\theta) + z_c \\
0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$
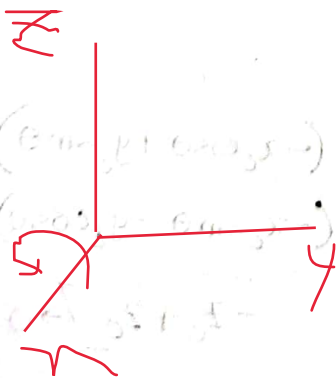
$\circledast$ for y based rotation

$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y \quad (\text{same})$$

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
=
\underbrace{\begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(+T)}
\cdot
\underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(R)}
\cdot
\underbrace{\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(-T)}
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
=
\begin{bmatrix} \cos\theta & 0 & \sin\theta & \left(-x_c\cos\theta - z_c\sin\theta\right)+x_c \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & \left(x_c\sin\theta - z_c\cos\theta\right)+z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$
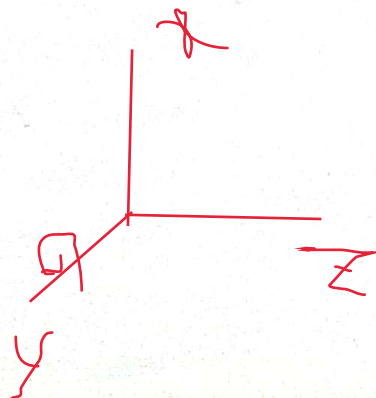
———————————— X ————————————

$\circledast$ for coordinate transformation based on all 3 rotations.

$$|P'| = |T| \cdot |R_{\theta x}| \cdot |R_{\theta y}| \cdot |R_{\theta z}| \cdot |-T| \cdot |P|$$

The order of the rotations is swapped around if we want to generate P' from P & P from P'

same thing, just the order of rotation is in reverse

$$|P| = |T| \cdot |R_{\theta z}| \cdot |R_{\theta y}| \cdot |R_{\theta x}| \cdot |-T| \cdot |P'|$$

3) Scaling :

The matrix for scaling is such:

2D, Scale (S_x, S_y), where $S_x \rightarrow$ unit of scale in x-axis

$\qquad\qquad\qquad\qquad S_y \rightarrow$ unit of scale in y-axis

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{cases} x' = S_x \cdot x \\ y' = S_y \cdot y \end{cases}$$

Q) Let's assume a point with coordinates $(6,3)$ from a rectangle has to scaled, 3 units in x axis & 5 units in y axis. Calculate the new coordinate?

$$P' = S \cdot P \qquad\qquad \text{where } s \text{ is the scaling matrix}$$

$$\Rightarrow P' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 18 \\ 15 \end{bmatrix} \quad \text{Ans:}$$

Scale only with x-axis

$$\begin{vmatrix} sx & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Scale only with y-axis

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

4) Reflection

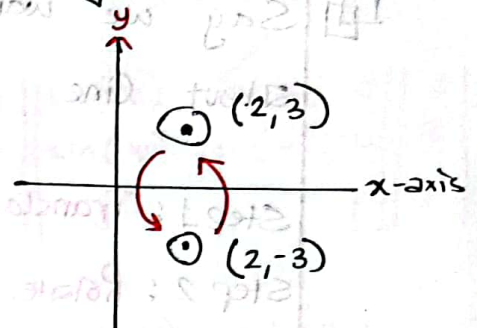(*) For reflection about x-axis, the value for x remains same, but the y-axis value goes to negative

Hence,

$$\boxed{\begin{aligned} x' &= x \\ y' &= -y \end{aligned}}$$

So the matrix representation stands as:

$$\begin{bmatrix} x' \\ y' \\ 01 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Refl x

(*) For reflection about y-axis, the value for y remains same, but the x-axis value goes to negative.
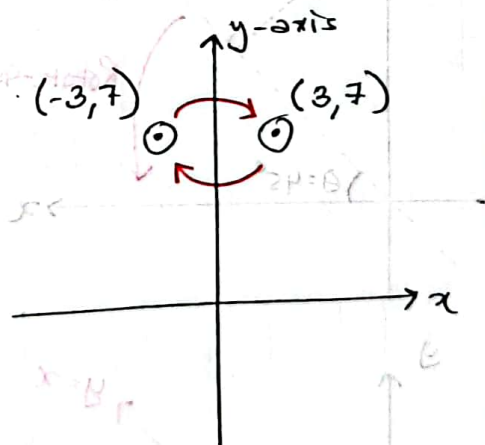
Hence,

$$\boxed{\begin{aligned} x' &= -x \\ y' &= y \end{aligned}}$$

Matrix Representation.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Refl y

Ø Reflection about a line $(y = mx + c)$

⬚ Say we want to reflect the point $(2, 8)$

about line $\boxed{y = x}$. How to do it?

As there is no constant

Step 1: Translate if nessesary

Step 2: Rotate $(-\theta)$ → -45

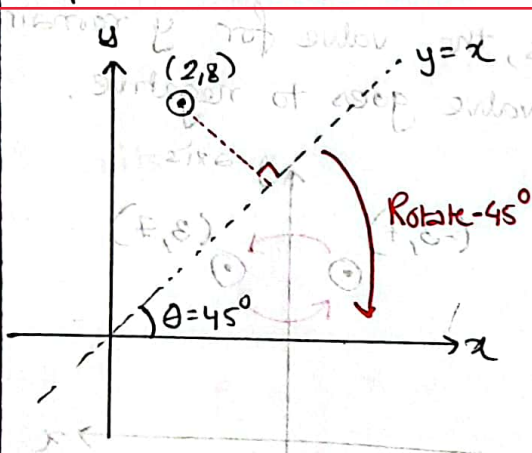Step 3: Reflect about x-axis

Step 4: Rotate $(+\theta)$ → 45

Step 5: Translate back

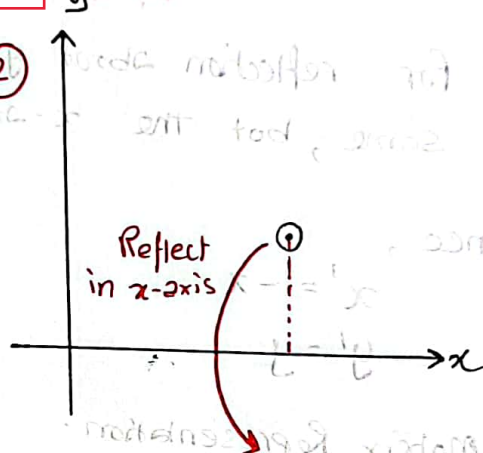if needed, not mandatory

Mandatory steps

①

$(2, 8)$
$y = x$
Rotate $-45°$
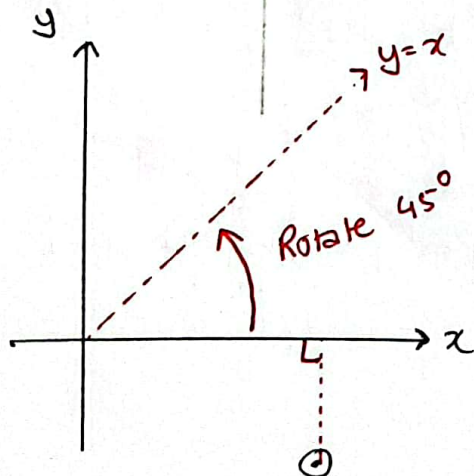$\theta = 45°$

②

Reflect in x-axis

③

$y = x$
Rotate $45°$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & & \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

→ and we get the answer.

Ø Here we don't need any translation since the line goes through the origin and there is no x/y intercept

3rd Rotate 45°
(anticlockwise)

2nd Reflect on x-axis

1st rotate -45° (clockwise)

Solution

$$P' = |R_{45}| \cdot |Refl_x| \cdot |R_{-45}| \cdot |P|$$

no translation nessesary

$$= \begin{vmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos 45 & \sin 45 & 0 \\ \sin 45 & -\cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

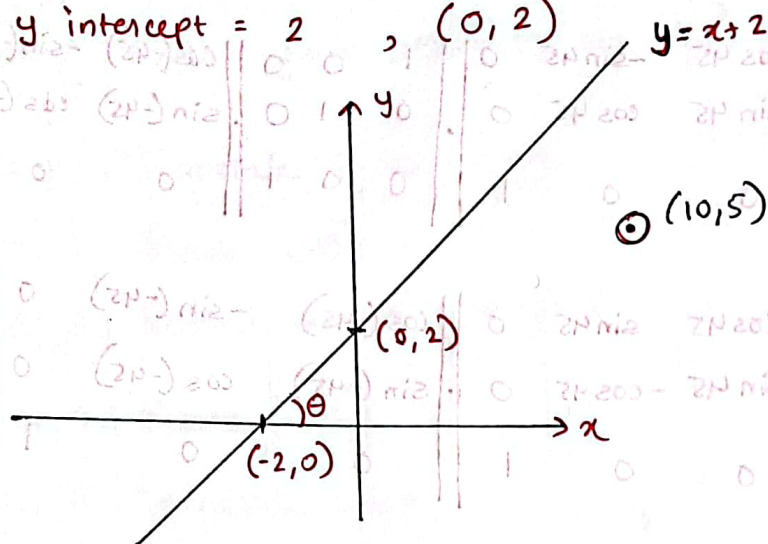$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \quad \text{(Ans :)}$$

DIY

Problem that requires translation

Q) Reflect the point $(10,5)$ to the line $y = x + 2$?

Ans

x intercept $= -2$, $(-2, 0)$

y intercept $= 2$, $(0, 2)$

Slope, $m = 1$

$\theta = \tan^{-1} 1$

$\theta = 45$

Step 1 : Translate to origin $T \longrightarrow (0, -2)$

Step 2 : rotate $(-45°)$, clockwise

step 3 : Reflect in x-axis

Step 4 : rotate $(45°)$, anticlockwise

Step 5 : Translate to local $T \longrightarrow (0, 2)$

$$\overbrace{P' = |T_{0,2}| \cdot |R_{45}| \cdot |Refl_x| \cdot |R_{-45}| \cdot |T_{0,-2}|}^{M_H} \cdot |P|$$

⑤ ← ④ ← ③ ← ② ← ①

P.T.O

Ø first calculate the composite matrix M_H.

$$M_H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$M_H = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

Ø Now, multiplying the composite matrix, with the original point, to derive the new point.
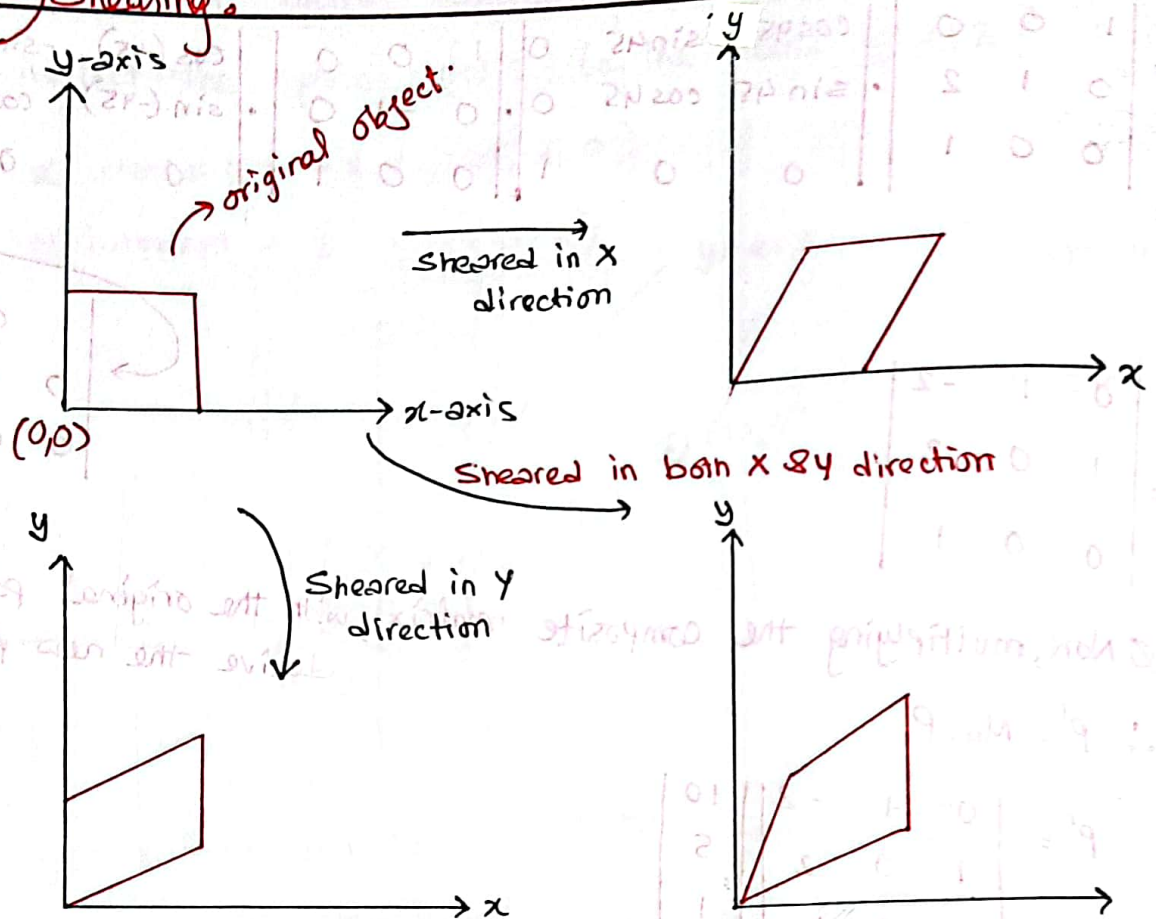
$\therefore P' = M_H \cdot P$

$$P' = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 10 \\ 5 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 \\ 12 \\ 1 \end{vmatrix}$$

Ans: New point = $\left(3, 12\right)$

$$SH_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

5) Shearing:

y-axis

→ original object.

```
Sheared in x
direction
```

(0,0) → x-axis

Sheared in both x & y direction

Sheared in y direction

let   a = shear in x direction
       b = shear in y direction

∅ for only shear in x direction

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∅ For only shear in y direction :

$$SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∅ For shear in both x & y direction :

The x and y direction shear

$$SH_{xy} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If there is any local center, we have to perform translation first and the rotation/ shear or anything

⊛ ∅ For shear with respect to points (X_c, Y_c) in both direction :

$$|P'| = |T_{(x_c, y_c)}| \cdot |SH_{xy}| \cdot |T_{(-x_c, -y_c)}| \cdot |P|$$

1st translate to origin through negative translation

3rd , translate back to local origin through positive translation.

2nd shear in both x & y direction

Q) Shear the point $(4,7)$, 2 units in x direction & 5 units in y direction, with respect to the point $(2,-1)$?

Ans:

$$P' = T_{(2,-1)} \times SH_{(2,5)} \times T_{(-2,+1)} \times P$$

$$P' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 20 \\ 17 \\ 1 \end{bmatrix}$$

$$P' = (20, 17) \quad \underline{Ans}$$