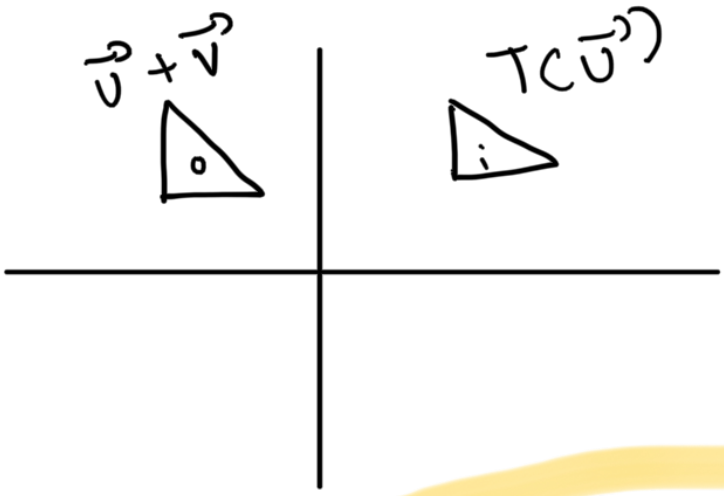
Week 7 Linear Transformation 3

$\boxed{\text{Translation}}$ moving object from one place to another.



In L.T we cannot move the origin

$$T(\vec{u}) = \vec{u} + \vec{v}$$

$$T(\vec{0}) = \vec{0} + \vec{v}$$

$$\boxed{T(\vec{u}) = A\vec{u}}$$

A ↑ matrix, $\vec{u}$ ↑ vector

→ # $T(\vec{u}) = A\vec{u} + \vec{v}$ | $A\vec{u} + 0$, $T(\vec{u}) = A\vec{u}$

Affine Transformation

$\boxed{T(\vec{u}) = A\vec{u}}$ composite transformation / multiple " at a time

$$T(\vec{u}) = A\vec{u} + \vec{v}$$

Solution: Introducing homogeneous coordinate system.

N dimension ⟶ N+1 dimension.

2D or 3D

$(10, 5) \rightarrow (\underline{10, 5, 1})$

In general,

$(X, Y) \rightarrow (x, y, w)$ → can be of any value. for now, $w = \underline{1}$

$w = 5$

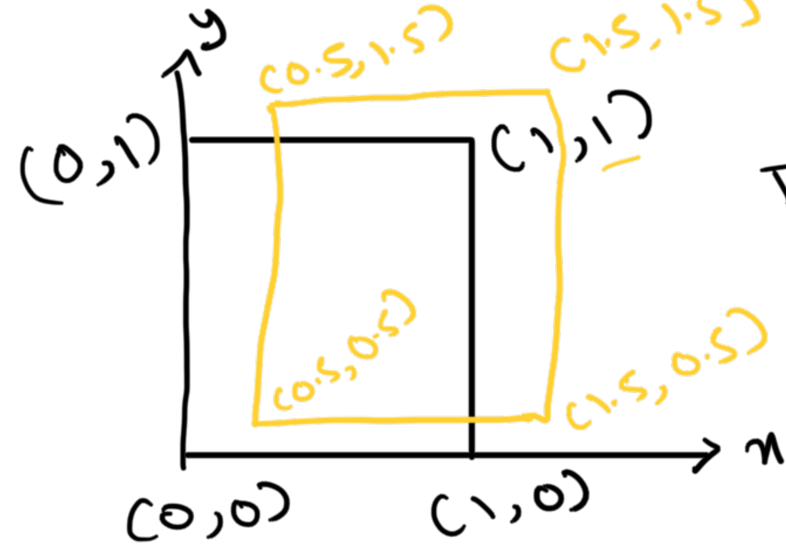
$$\underset{X,\ Y,\ w}{(10, 5, 1)} \rightleftarrows \underset{x,\ \ y,\ \ w}{\left(\frac{50}{5}, \frac{25}{5}, 5\right)}$$

$\boxed{X = x/w}$

$$\boxed{Y = y/w}$$

$(10, 5) = (10, 5, 1)$

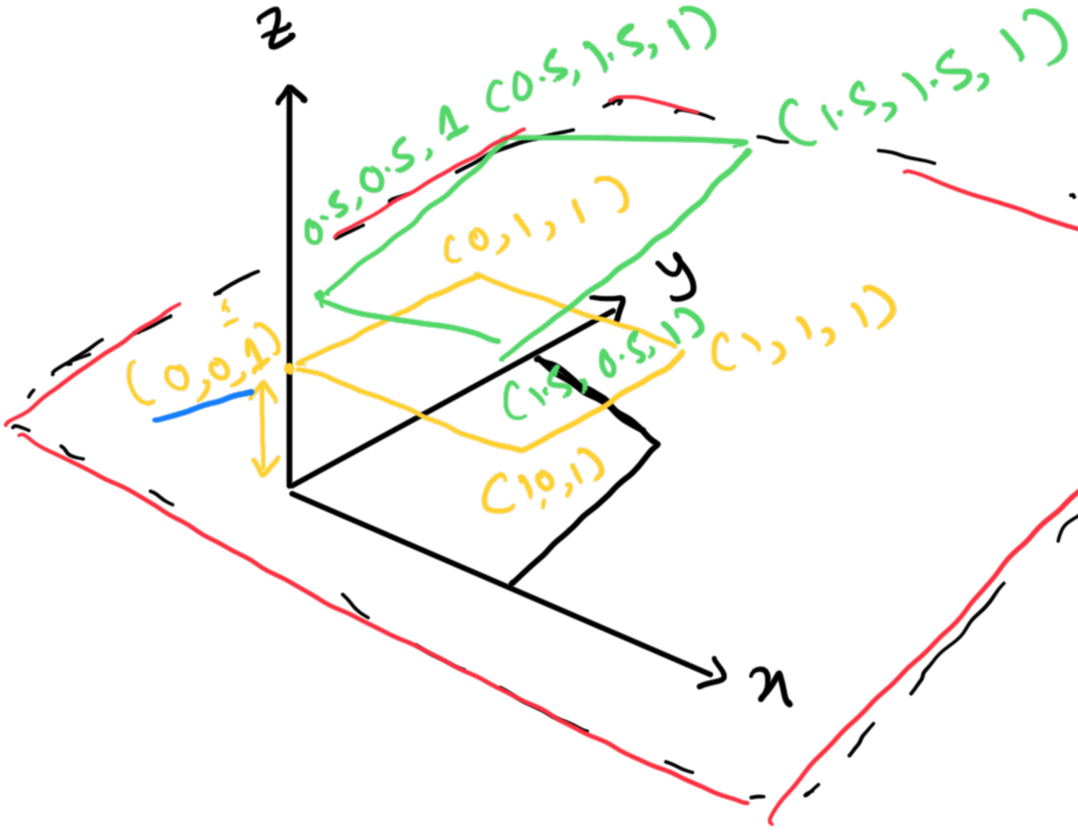
$(10, 5, 3) = (10, 5, 3, 1)$



Translate by vector $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.5 & 1.5 & 1.5 \\ 0.5 & 1.5 & 1.5 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

2D Transformations matrices with homogenues. transformation

T : translate (dx, dy) $\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \end{bmatrix}$

$$0 \quad 0 \quad 1$$

T^{-1}: translate $(-dx, -dy)$ $\begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$

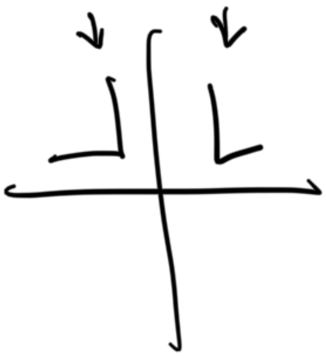
T: rotate $(\theta)_z$ through z-axis $\begin{array}{cc} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$

T: rotate $(\theta)_x$ = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

T: rotate $(\theta)_y$ = $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

T^{-1}: rotate $(\theta)_z$ = $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T: reflect (x-axis) = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



T^{-1}: reflect (x-axis) = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

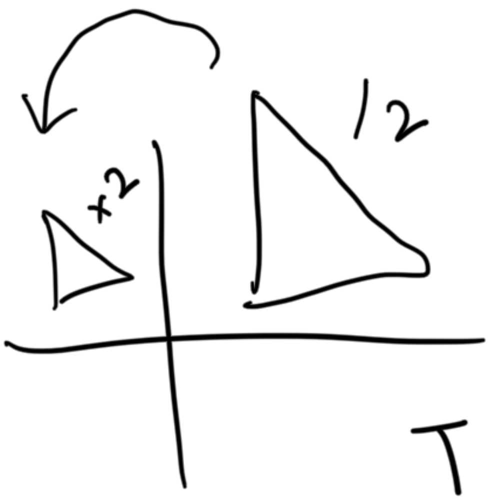
T: reflect (Y-axis) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T^{-1}

T^{-1}: reflect (y-axis) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T: Scale (a, b) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T^{-1}: Scale $(1/a, 1/b)$ $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1 \end{bmatrix}$



T: Shear (c, d) $\begin{bmatrix} 1 & c & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \overset{A}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad 2D$$

<u>Determinant of a matrix</u>

↳ how an area or volume is affected due to transformation.

$D = 2$

2D – area
3D – volume

area = 10

new area = $2 \times 10 = 20$ unit^2

$D = -2$

new area = $-2 \times 10 = -20$

↓

there is a change in orientation.

To combine multiple transformations together we need $T(\vec{U}) = A\vec{U}$

Apply composite transformation

A_0 = Rotation
A_1 = Reflection
A_2 = Scaling

$$\boxed{T_0(\vec{U}) = A_0\vec{U}}$$

$$T_1(T_0(\vec{U})) = A_1(A_0(\vec{U}))$$

$$T_2(T_1(T_0(\vec{U}))) = A_2(A_1(A_0(\vec{U})))$$

$$= \underbrace{A_2 \cdot A_1 \cdot A_0}_{A_c\vec{U}}(\vec{U})$$

$$\cancel{T(\vec{U}) = (A(\vec{U}) + V)(\quad)}$$

First translate then scale

For ex:

Scale & translate $(10, 15) \rightarrow (10, 15, 1)$
$(2,2)$ $(3,1)$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(\vec{U}) = A_S \cdot A_T \vec{U}$$

$$T(\vec{U}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 15 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+3 \\ 30+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 23 \\ 31 \\ 1 \end{bmatrix}$$

Ans

To remember

Matrix multiplication is NOT Commutative

$$A \cdot B \neq B \cdot A$$

ORDER OF TRANS. IS IMPORTANT

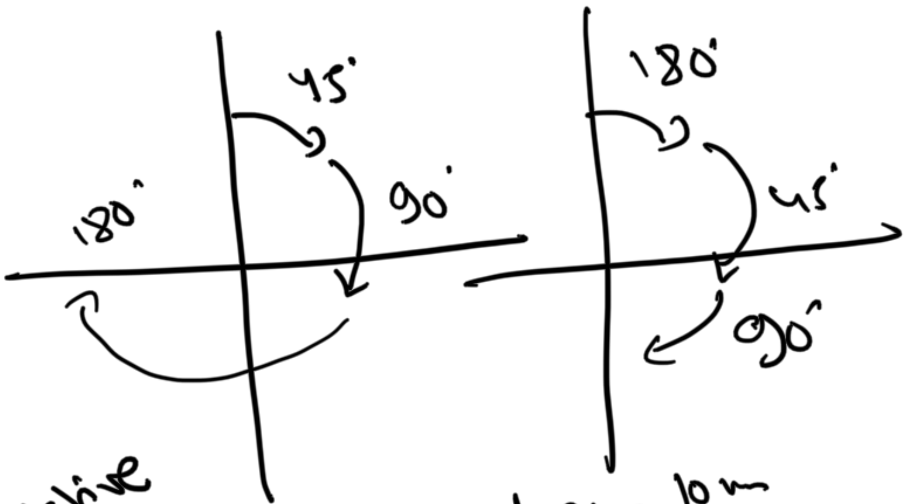
$$T \cdot S = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S \cdot T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 R_2 R_3$ 2D

This is Commutative

3D Rotation Non Commutative

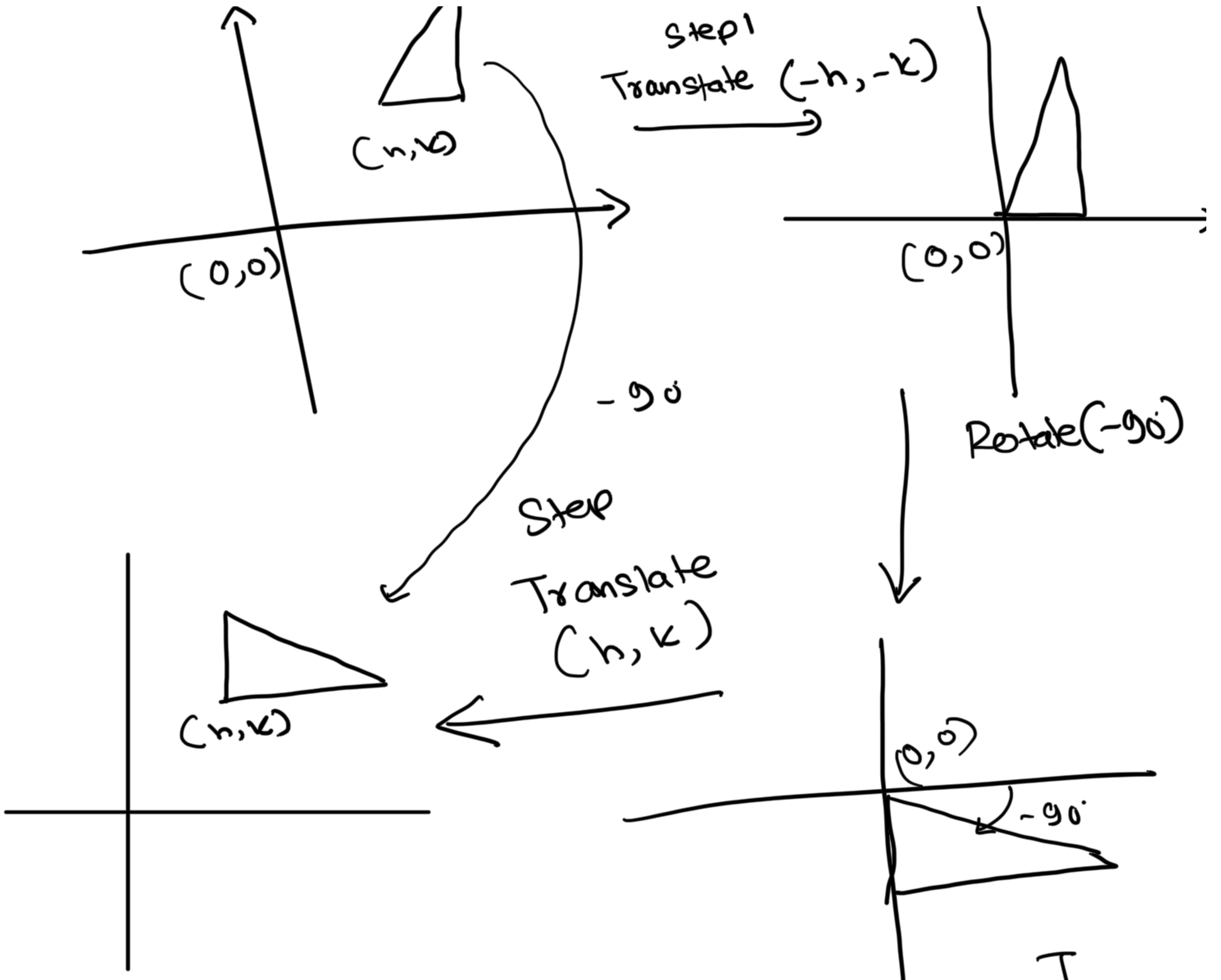


$\dfrac{2km}{10} \quad \dfrac{10km}{2} = 12km$

Two translations are commutative.

Combination of translation & rotation.

non-commutative.



$$\text{Composite matrix, } M = T_{(h,k)} \times R(\theta) \times T_{(-h,-k)}$$

2D

Z-axis

$T(\vec{v}) =$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{}$$

$$\vec{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3D

$$\begin{bmatrix} x' \\ y' \\ \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ \end{bmatrix} \overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \end{bmatrix}$$

$$\begin{bmatrix} z' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

z-axis

y-axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

x-axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Sum 17

$$\begin{array}{ccc} x & y & z \\ (100, & 20, & -79.5) \end{array} \qquad \theta = 45^\circ$$

Centre $(50, 50, 1)$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 & 0 \\ \sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -50 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 20 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 29 \\ -29.5 \\ 1 \end{bmatrix}$$