(TRASFORMATIONS CONTINUED)

(*) From previous class

For z-axis based rotation, not with respect to the origin :

(3D)

$$x = r\cos\varphi \qquad y = r\sin\varphi$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z \text{ (remains same)}$$

$x_c \rightarrow$ translation in x-axis

$y_c \rightarrow$ translation in y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{(+Translation)}} \cdot \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{(Rotation)}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{(-Translation)}} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\times$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & -x_c\cos\theta + y_c\sin\theta \\ \sin\theta & \cos\theta & 0 & -x_c\sin\theta - y_c\cos\theta \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\times$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & (-x_c\cos\theta + y_c\sin\theta) + x_c \\ \sin\theta & \cos\theta & 0 & (-x_c\sin\theta - y_c\cos\theta) + y_c \\ 0 & 0 & 1 & -z_c + z_c \rightarrow 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{The composite matrix}} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\text{original coordinates}}$$

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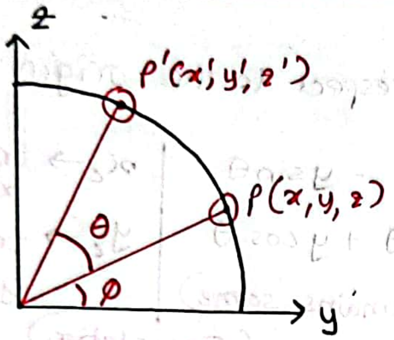
✱ For x-axis based Rotation, not w.r.t origin



$$y' = y\cos\theta - z\sin\theta$$

$$z' = y\sin\theta + z\cos\theta$$

$$x' = x \quad \text{(same)}$$

Now,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \overset{(+\text{Translation})}{\begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{(\text{Rotation})}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{(-\text{Translation})}{\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\times$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & (-y_c\cos\theta + z_c\sin\theta) + y_c \\ 0 & \sin\theta & \cos\theta & (-y_c\sin\theta - z_c\cos\theta) + z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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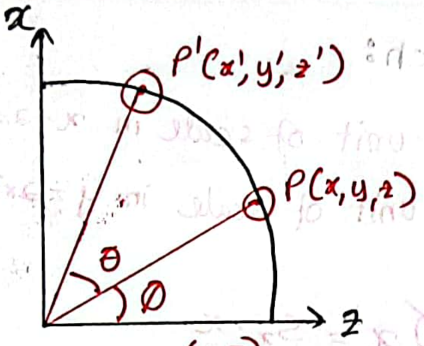
(*) for y based rotation



$z' = z\cos\theta - x\sin\theta$

$x' = z\sin\theta + x\cos\theta$

$y' = y$ (same)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & x_c \\ 0 & 1 & 0 & y_c \\ 0 & 0 & 1 & z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(+T)} \cdot \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(R)} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(-T)} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & (-x_c\cos\theta - z_c\sin\theta) + x_c \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & (x_c\sin\theta - z_c\cos\theta) + z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X

(*) for coordinate transformation based on all 3 rotations.

$$|P'| = |T| \cdot |R_{\theta x}| \cdot |R_{\theta y}| \cdot |R_{\theta z}| \cdot |-T| \cdot |P|$$

same thing, Just the order of rotation is in reverse

The order of the rotations is swapped around if we want to generate P' from P & P from P'

$$|P| = |T| \cdot |R_{\theta z}| \cdot |R_{\theta y}| \cdot |R_{\theta x}| \cdot |-T| \cdot |P'|$$

3) Scaling :

The matrix for scaling is such:

2D, Scale (S_x, S_y), where $S_x \rightarrow$ unit of scale in x-axis

$S_y \rightarrow$ unit of scale in y-axis

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{cases} x' = S_x \cdot x \\ y' = S_y \cdot y \end{cases}$$

Q) Let's assume a point with coordinates (6,3) from a rectangle has to scalled, 3 units in x axis & 5 units in y axis. Calculate the new coordinate?

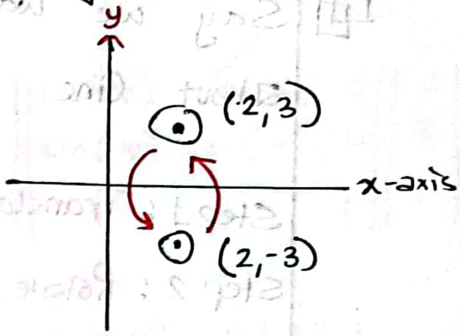
$$P' = S \cdot P$$ where s is the scaling matrix

$$\Rightarrow P' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 15 \end{bmatrix}$$ Ans:

4) Reflection

⊛ For reflection about x-axis, the value for x remains same, but the y-axis value goes to negative



Hence,

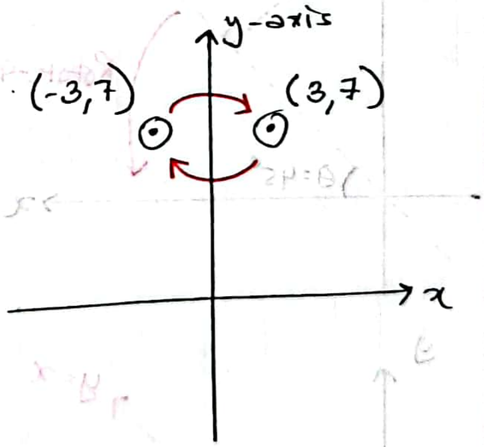
$$x' = x$$
$$y' = -y$$

So the matrix representation stands as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$Refl_x$

⊛ For reflection about y-axis, the value for y remains same, but the x-axis value goes to negative.



Hence,

$$x' = -x$$
$$y' = y$$

Matrix Representation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$Refl_y$

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Ø Reflection about a line $(y = mx + c)$

Say we want to reflect the point $(2,8)$ about line $y = x$. How to do it?

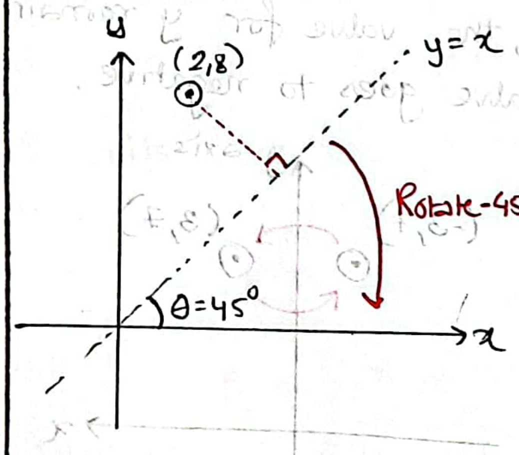
Step 1: Translate if nessesary

Step 2: Rotate $(-\theta)$

Step 3: Reflect about x-axis

Step 4: Rotate $(+\theta)$

Step 5: Translate back

1


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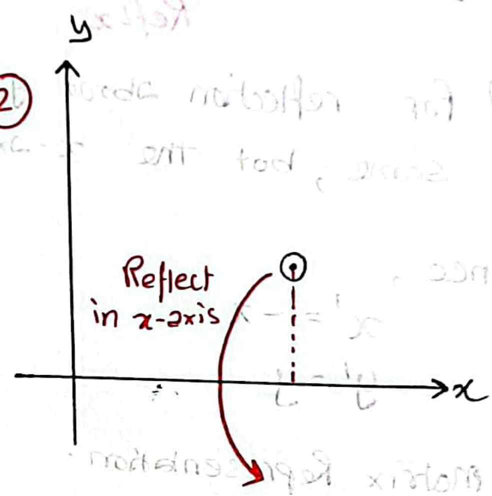

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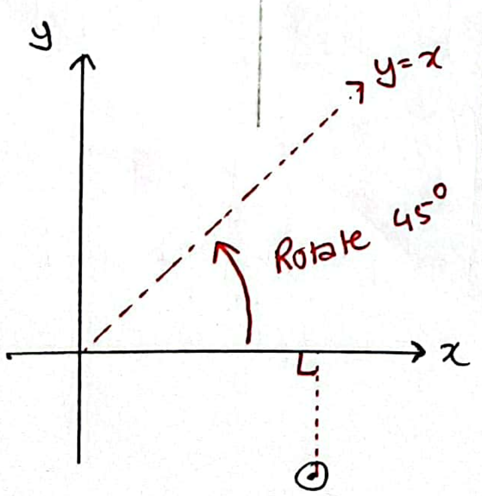

→ and we get the answer.

Ø Here we don't need any translation since the line goes through the origin and there is no x/y intercept

3rd Rotate 45° (anticlockwise) 2nd Reflect on x-axis 1st rotate -45° (clockwise)

Solution

$$P' = |R_{45}| \cdot |Ref_x| \cdot |R_{-45}| \cdot |P|$$

no translation nessesary

$$= \begin{vmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos 45 & \sin 45 & 0 \\ \sin 45 & -\cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 8 \\ 1 \end{vmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix} \text{ (Ans:)}$$

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Problem that requires translation

Q) Reflect the point $(10,5)$ to the line $y = x+2$?

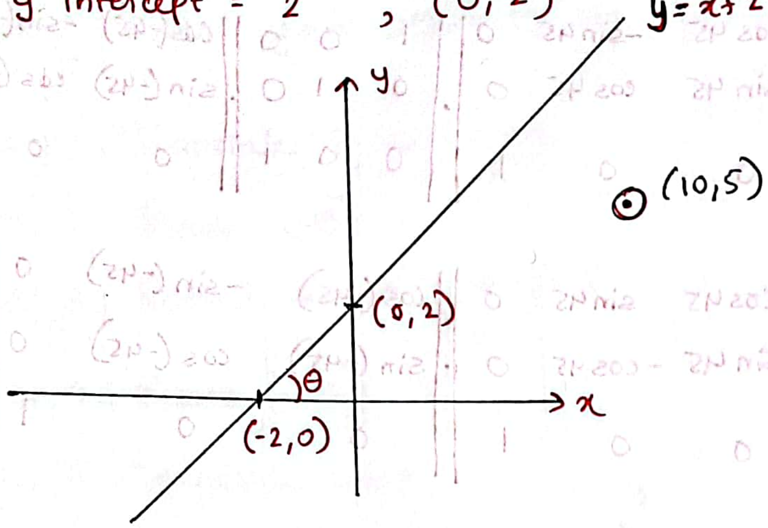
Ans x intercept $= -2$, $(-2,0)$

y intercept $= 2$, $(0,2)$

Slope, $m = 1$

$\theta = \tan^{-1} 1$

$\theta = 45$



Step 1 : Translate to origin $T \rightarrow (0,-2)$

Step 2 : rotate (-45°), clockwise

Step 3 : Reflect in x-axis

Step 4 : rotate (45°), anticlockwise

Step 5 : Translate to local $T \rightarrow (0,2)$

$$P' = \underbrace{|T_{0,2}| \cdot |R_{45}| \cdot |Refl_x| \cdot |R_{-45}| \cdot |T_{0,-2}|}_{M_H} \cdot |P|$$

$$⑤ \leftarrow ④ \leftarrow ③ \leftarrow ② \leftarrow ①$$

P.T.O

Ø First calculate the composite matrix M_H.

$$M_H = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos 45 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$M_H = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

Ø Now, multiplying the composite matrix, with the original point, to derive the new point.

$\therefore P' = M_H . P$

$$P' = \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 10 \\ 5 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 \\ 12 \\ 1 \end{vmatrix}$$

Ans: New point $= (3, 12)$

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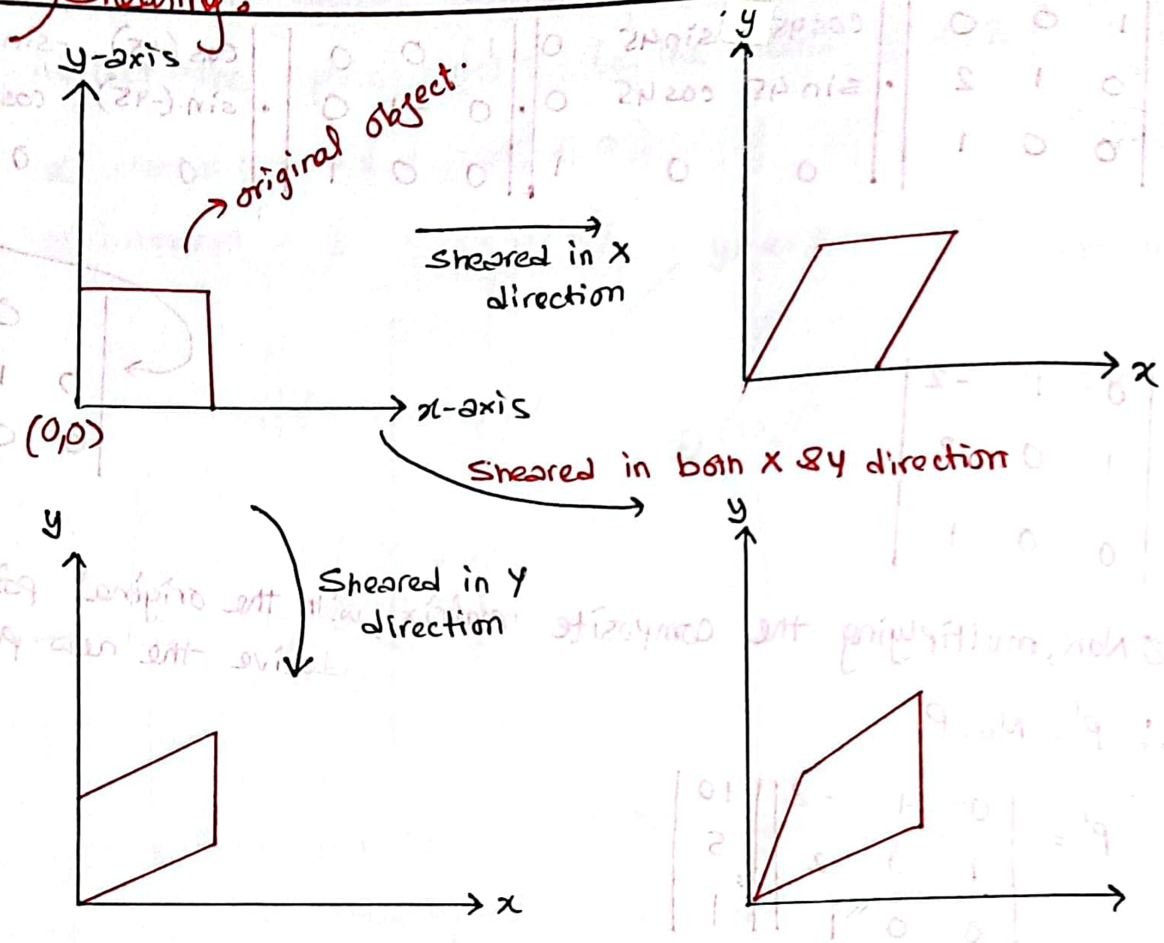
5) Shearing :



let a = shear in x direction

b = shear in y direction

* for only shear in x direction

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ø For only shear in y direction :

$$SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ø For shear in both x & y direction :

$$SH_{xy} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(*) Ø For shear with respect to points (x_c, y_c) in both direction :

$$|P'| = \underbrace{|T_{(x_c,y_c)}|}_{\text{3rd, translate back to local origin through positive translation.}} \cdot \underbrace{|SH_{xy}|}_{\text{2nd shear in both x \& y direction}} \cdot \underbrace{|T_{(-x_c,-y_c)}|}_{\text{1st translate to origin through negative translation}} \cdot |P|$$

Q) Shear the point $(4,7)$, 2 units in x direction & 5 units in y direction, with respect to the point $(2,-1)$?

Ans:

$$\Rightarrow P' = T_{(2,-1)} \times SH_{(2,5)} \times T_{(-2,+1)} \times P$$

$$\Rightarrow P' = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\times} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$\Rightarrow P' = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$\Rightarrow P' = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$\Rightarrow P' = \begin{bmatrix} 20 \\ 17 \\ 1 \end{bmatrix}$$

$P' = (20, 17)$ Ans

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