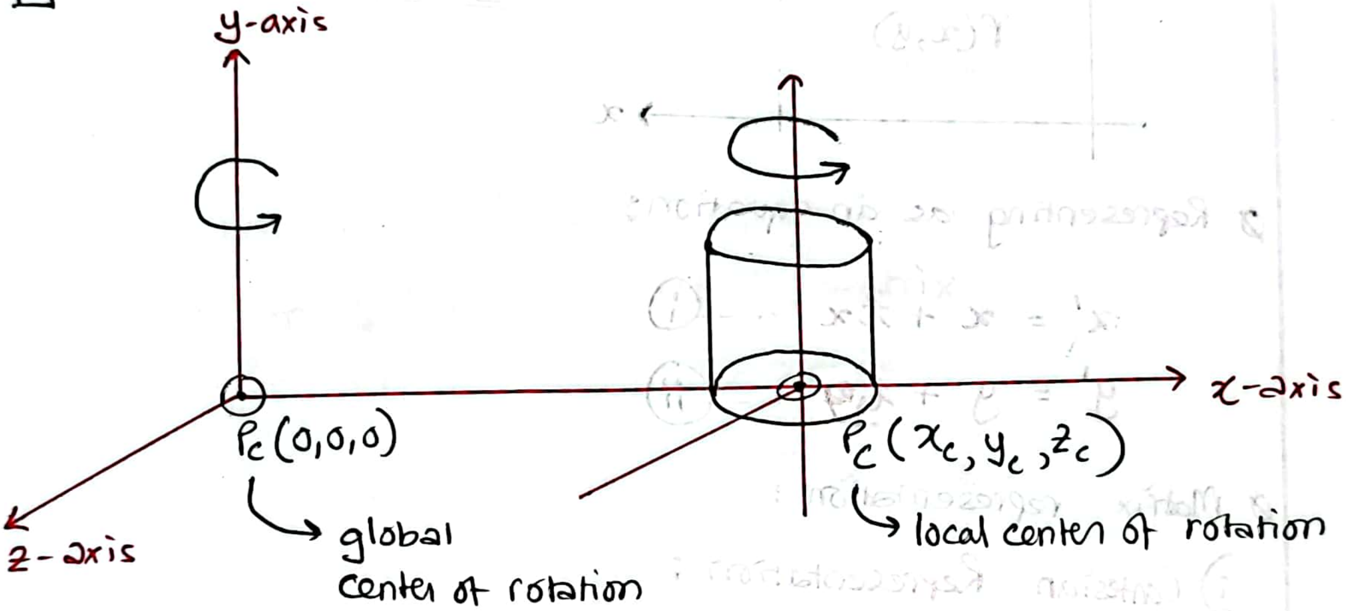
GEOMETRIC (TRANSFORMATIONS)

LECTURE 10

→ Repositioning & resizing of an object.

Ø Classification of transformation →
- i) Geometric Transformation (like revolution) (center of rotation is global)
- ii) Coordinate Transformation (like rotation) (center of rotation is local)

Based on Center of Rotation

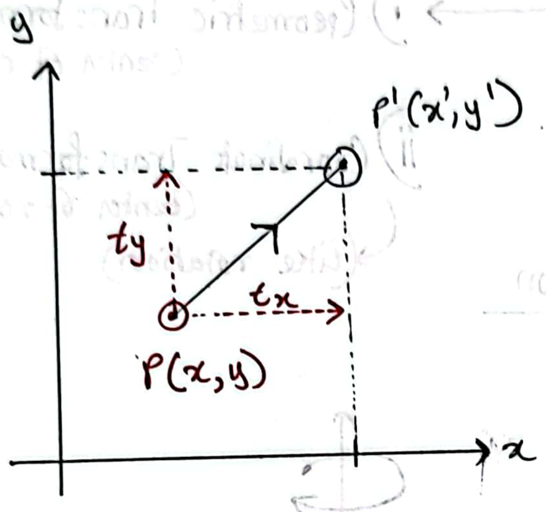


Ø Categories of Transformation:

1) Translation
2) Rotation
3) Scaling
4) Reflection
5) Shearing

1) Translation :

- It is the repositionging of an object / pixel in a straight line



- t_x, t_y are translation across x axis & y axis

$$\therefore \underline{\underline{P' = P + t}}$$

- Representing as an equation:

$$x' = x + t_x \quad \text{--- (i)}$$

$$y' = y + t_y \quad \text{--- (ii)}$$

- Matrix representation :

i) Cartesian Representation :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

ii) Homogeneous Representation :

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Q) We mainly prefer to use Homogeneous coordinate system in Computer Graphics. But why?

→ Using Homogeneous coordinate system adds an extra dimension to the 2D/3D Cartesian system. This allows us to do addition/subtraction, but also allows us to perform multiplication & division. It can be used to generate composite Matrices combining multiple transformations that can be represented into a single matrix to reduce computations.

Ø. So in Homogeneous Coordinate system:

→ for 2D we use (3x3) matrix
→ for 3D we use (4x4) matrix

3D Translation matrix representation (x, y, z)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Q) Translate the point $(3, 2, 7)$, 3 units in x-axis 4 units in y-axis & -3 units in the z-axis?

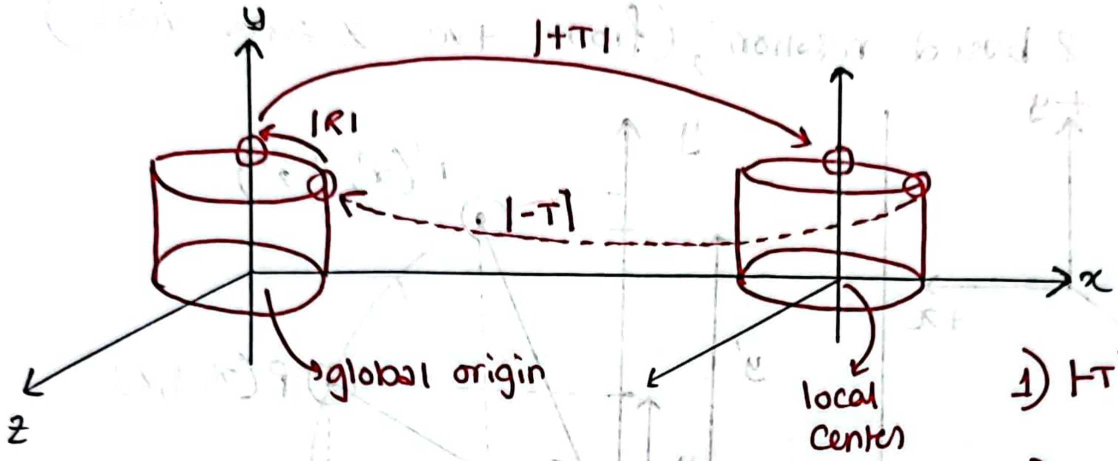
Ans $t_x = 3$, $t_y = 4$, $t_z = -3$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 \\ 2+4 \\ 7-3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 4 \\ 1 \end{bmatrix} = (6, 6, 4) \text{ Ans:}$$

2) Rotation :

How do we work on Rotation ?



1) |-T| translate to origin
2) |R| Rotate
3) |+T| translate to local origin/center.

$$|P'| = \overbrace{|+T|\cdot|R|\cdot|-T|}^{\text{generates a composite matrix}}\cdot|P|$$

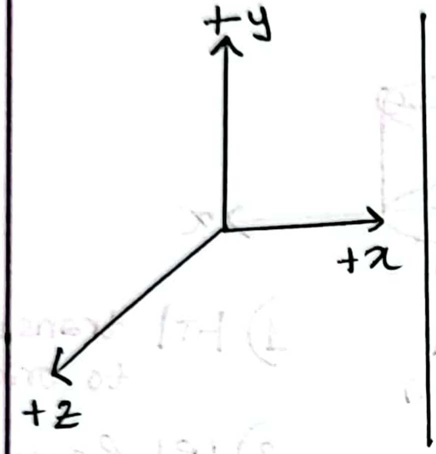
① translate to global origin

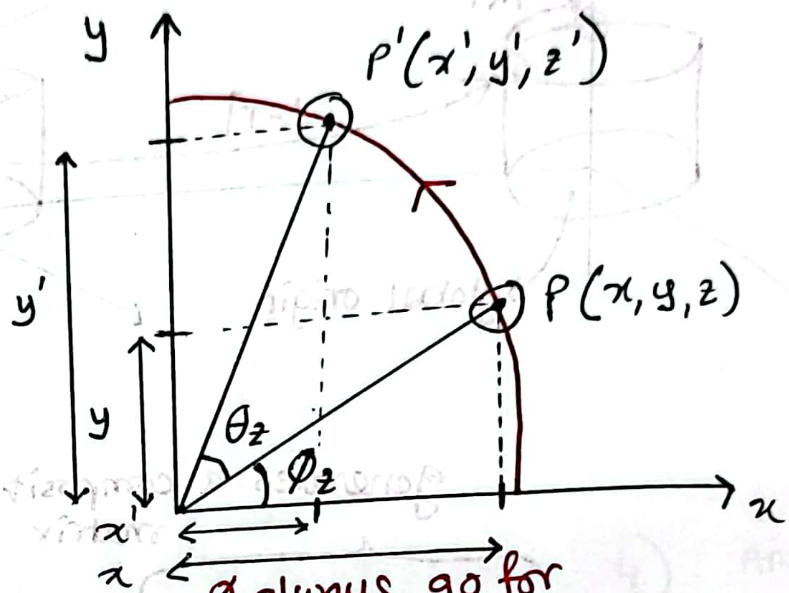
② Rotate

③ Bring to local origin.

(Mathematices Of Rotation)

Ø for z based rotation, (from +ve z axis view)





Ø The value of z-axis is constant, since the rotation is based on z-axis.

Ø always go for counter clockwise motion for rotation

Ø from +z axis, counter clockwise

Ø from −z axis, clockwise

Here,

$$\left.\begin{aligned} x &= r\cos\phi_z \\ y &= r\sin\phi_z \end{aligned}\right] \rightarrow P$$

$$\left.\begin{aligned} x' &= r\cos(\theta_z + \phi_z) \\ y' &= r\sin(\theta_z + \phi_z) \end{aligned}\right] \rightarrow P'$$

$z' = z$ (remains constant)

We know,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \cos A \sin B - \sin A \cos B$$

Since,

$$\left.\begin{aligned} x' &= r\cos(\phi+\theta) \\ y' &= r\sin(\phi+\theta) \end{aligned}\right\} \dashrightarrow \text{(i)}$$

$\downarrow$ expansion

So,

$$\left.\begin{aligned} x' &= r\{(\cos\phi \cdot \cos\theta) - (\sin\phi \cdot \sin\theta)\} \\ y' &= r\{(\cos\phi \cdot \sin\theta) + (\sin\phi \cdot \cos\theta)\} \\ z' &= z \end{aligned}\right) \text{---- ii}$$

We know,

$$x = r\cos\phi$$
$$y = r\sin\phi$$

Then,

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}\right) \text{---- iii}$$

So, the (4×4) representation is : (Homogeneous)

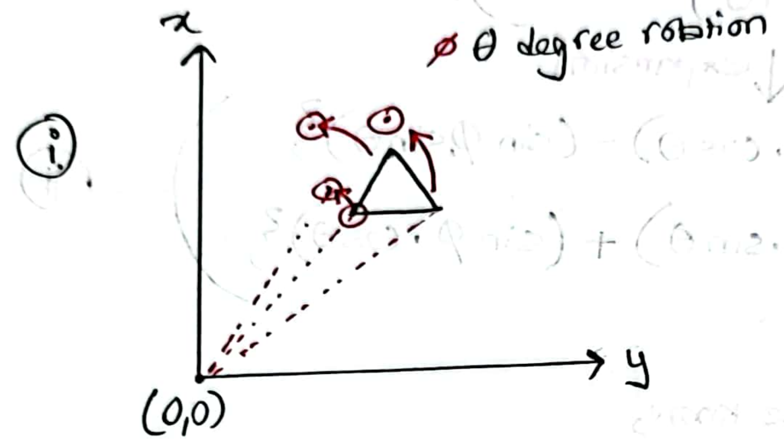
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

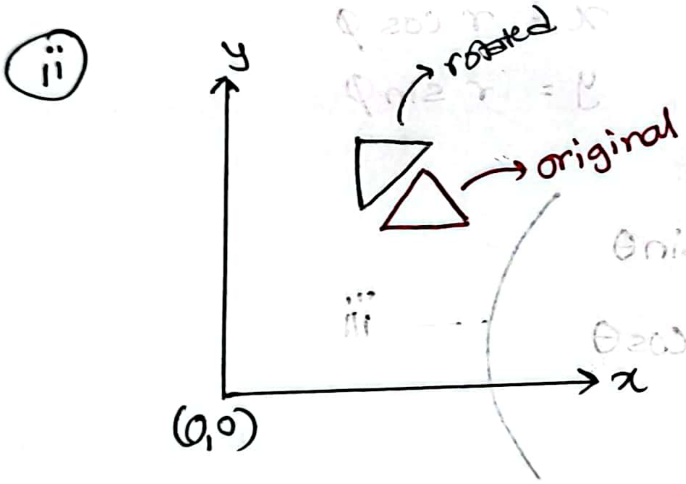
Remember, this rotation is for (rotation with respect to the origin)

hence we don't need the $|+T|$ & $|-T|$ matrices.

(IDEA behind rotation) → 2D

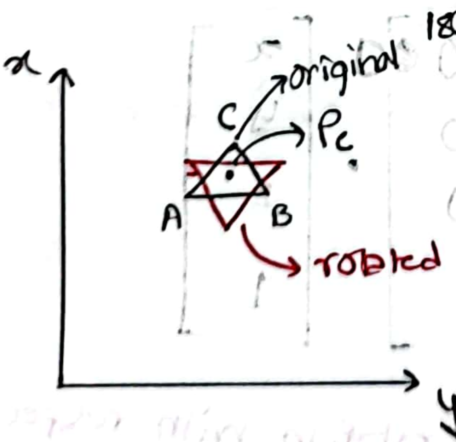
1) rotation with respect to origin:





2) Rotation with respect to a point:

180° rotation about P_c (x_c, y_c)



↳ In this case for the rotation we need to translate to origin first, rotate, then translate back to local origin.

Q) Rotate the point (9, 2) at an angle of 45^0, with respect to origin an find the result?

→ since, we are dealing with a 2d point, we can eliminate the use of z axis for this.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 9\cos 45 - 2\sin 45 \\ 9\sin 45 + 2\cos 45 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 4.95 \\ 7.78 \\ 1 \end{bmatrix}$$ (Ans:)

Q) Rotate the point $(6,2)$ at an angle of 90° with respect to the point $(2,2)$?

So,
$$P' = |+T| \cdot |R| \cdot |-T| \cdot |P|$$

$$\Rightarrow P' = T_{(2,2)} \cdot R_{(90)} \cdot T_{(-2,-2)} \cdot P$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 90 & -\sin 90 & 2 \\ \sin 90 & \cos 90 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 90 & -\sin 90 & -2\cos 90 + 2\sin 90 + 2 \\ \sin 90 & \cos 90 & -2\sin 90 - 2\cos 90 + 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -2+4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$ (Ans.)