Machine Learning

CSE427

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Superhumungous Thanks

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NUL

- So far we let *m* depend on ϵ, δ
- But m was independent of *D* and *h*
- Learnable classes limited to finite VC dimension classes
- Now consider a more general (but weaker) framework
- Let *m* depend on the hypothesis (Nonuniform Learnability)
- Then later let *m* depend on the distribution *D* (Consistency)

NUL

- NUL is a strict relaxation of agnostic PAC
- Sufficient condition for NUL is
$$X = \text{countable union of hypothesis classes } X_n$$
Each X_n is uniformly convergent
- Algorithm implementing NUL
$\Rightarrow$ Structural Risk Minimization (SRM)
- This is just like when ERM implements PAC learning

NUL

- *m* depends on *h* : $m_{\mathcal{H}}(\epsilon, \delta, h)$
- **Why does this make sense?**
- Some hypothesis might need more sample data to validate
- EG :
  $\rightarrow$ 2 pts determine a line
  $\rightarrow$ 3 pts determine a quadratic
  $\rightarrow$ *nH* points determine a *n*-degree polynomial

- Thus in choosing a K-degree polynomial predictor you would expect to need more points as K goes up
- We say that a hypothesis h is (ϵ, δ) competitive with respect to h', y with probability more than $1 - \delta$ that

$$L_D(h) \leq L_D(h') + \epsilon$$

- In PAC, APAC, competitiveness not very useful. This is because looking for hypothesis with absolute low/minimum risk

NUL

Definition : NUL

$\mathcal{H}$ is uniformly learnable if there exists a learning algorithm A and $m_{\mathcal{H}}^{NUL}(\epsilon, \delta, h)$ such that for every ϵ, δ, h if

$$m \geq m_{\mathcal{H}}^{NUL}(\epsilon, \delta, h)$$

then for every distribution D with probability at least $1 - \delta$ over the choice of $S \sim D^m$ that

$$L_D(A(S)) \leq L_D(h) + \epsilon$$

Note :

If $\mathcal{H}$ is APAC then it is also NUL

Characterizing NUL

- For APAC $\rightarrow$ finite VC dimension implied APAC
- For NUL, have the following

Theorem :

A hypothesis class of binary classifiers is NUL if and only if it is a countable union of APAC hypothesis classes

- Countable union means we can label the individual hypothesis classes using a "*Counting*" index n

Characterizing NUL

- This means

$$\mathcal{H} = \bigcup_n \mathcal{H}_n$$

- Proof of the theorem above relies on:

Theorem : **

Let $\mathcal{H}$ be a hypothesis class that can be written as a countable union of hypothesis classes $\mathcal{H} = \bigcup_n \mathcal{H}_n$, where each $\mathcal{H}_n$ is uniformly convergent. Then $\mathcal{H}$ is NUL.

Characterizing NUL

- This theorem generalizes the previous result of $UC \rightarrow APAC$ to nul
- Now prove the first theorem

Proof:

- Assume $\mathcal{H} = \bigcup\limits_{n} \mathcal{H}_n$
- Each $\mathcal{H}_n$ is UC by *Theorem *** , $\mathcal{H}$ is NUL
  Now prove the other way
- Assume $\mathcal{H}$ is NUL using some algorithm A

Characterizing NUL

Proof:

- Let,

$$\mathcal{H}_n = \left\{ h \in \mathcal{H} \mid m_{\mathcal{H}}^{NUL}\left(\frac{1}{8}, \frac{1}{7}, h\right) \le n \right\}$$

- Clearly $\mathcal{H} = \bigcup_n \mathcal{H}_n$
- Using the definition of $m_{\mathcal{H}}^{NUL}$ we know that for any distribution D, with the probability of at least $1 - \delta = \frac{6}{7}$ over $S \sim D^m \Rightarrow L_D(A(S)) \le \frac{1}{8} + L_D(h)$
- Since this is true for each class $\mathcal{H}_n$, each $\mathcal{H}_n$ is APAC

Characterizing NUL

Example : NUL is a strict relaxation of APAC

- $\mathcal{X} = \mathbb{R}$
- $\mathcal{H}_n =$ class of nth degree polynomials
- Binary Classifiers
$$h(\chi) = sign(p(\chi))$$
- $\mathcal{H} = \bigcup_n \mathcal{H}_n$
- VC dim $(\mathcal{H}_n) = n + 1 = d_n$ for every set C, with VC dim $= d$, there is a set with higher VC dim
- VC dim $(\mathcal{H}) = \infty$
- Thus $\mathcal{H}$ is not PAC/APAC
- But $\mathcal{H}$ is NUL

Characterizing NUL

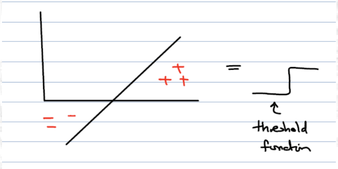


Figure: Linear Classifier

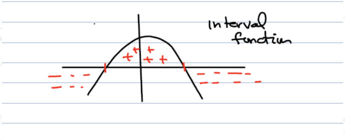


Figure: Quadratic Classifier

Standard Risk Minimization (SRM)

- So far encoded prior knowledge with choice of $\mathcal{H}$
- A more powerful way is to specify preferences over $\mathcal{H}$
- In SRM, assume
  - $\mathcal{H} = \bigcup_n \mathcal{H}_n$
  - Specify a weight factor $w : \mathbb{N} \to [0, 1]$ and $\sum_{n=1}^{\infty} w(n) \leq 1$
  - higher weight $\Rightarrow$ stronger preference

Standard Risk Minimization (SRM)

- one example of a weighting scheme is **Minimum Descriptive length (MDL)**

Define ϵ_n :

$$\epsilon_n(m, \delta) = min\left(\{\epsilon \mid m^{UC}_{\mathcal{H}_n}(\epsilon, \delta) \leq m\}\right)$$

Here,

$$m = \text{Sample size}$$
$$\delta = \text{Confidence}$$
$$UC = \text{Uniform Convergence}$$

  - ϵ_n is the most accuracy you can get by sampling up to m data points

Standard Risk Minimization (SRM)

- From the definition of UC, for every ϵ, δ and $prob \geq 1 - \delta$, we find that
$$\forall h \in \mathcal{H}_n, \mid L_D(h) - L_S(h) \mid \leq \epsilon_n(m, \delta)$$
- If all $\mathcal{H}$ have equal height
$$w(n) = \frac{1}{N} \text{ (total number of classes } = N)$$
- If the highest weight for low n (for example, low degree polynomials) choice is
$$w(n) = \frac{6}{\pi^2 n^2}$$

Note :

$$\sum_{n=1}^{\infty} w(h) = \frac{6}{\pi^2} \sum \frac{1}{n^2} = 1$$

Standard Risk Minimization (SRM)

- The SRM is a bound minimization approach
- It minimizes a certain upper bound on *The true risk*
- The bound in tries to minimize is the following:

Theorem :

Let $w(n)$ be a weight function with $w(n) \leq 1$. Suppose $\mathcal{H} = \underset{n}{\cup} \mathcal{H}_n$, and each $\mathcal{H}$ is UC, with sample complexity $m^{UC}_{\mathcal{H}_n}$. Let ϵ_n be defined as

$$\epsilon_n(m, \delta) = min\left(\{\epsilon \mid m^{UC}_{\mathcal{H}_n}(\epsilon, \delta) \leq m\}\right)$$

Standard Risk Minimization (SRM)

Then for every ϵ, D with probability $\geq 1-\delta$ over the choice of $S \sim D^m$, the following bound holds simultaneously for every n and $\mathcal{H}_n$

$$|L_D(h) - L_S(h)| \leq \epsilon_n(m, w(n) \cdot \delta)$$

Therefore for every δ and D with probability $\geq 1-\delta$, it holds that

$$\forall h \in \mathcal{H}, L_D(h) \leq L_S(h) + \min_{n: h \in \mathcal{H}_n} \epsilon_n(m, w(n) \cdot \delta)$$

[pick the smallest ϵ_n]

Standard Risk Minimization (SRM)

Proof :

Define : $\delta_n = w(n) \cdot \delta$

- using *UC* for each n with *probability* $\geq 1 - \delta$ and a fixed n,

$$D^m(S \mid \forall h \in \mathcal{H}_n, |L_D(h) - L_S(h)| \leq \epsilon_n) \geq 1 - \delta_n$$

$$\Rightarrow D^m(S \mid \exists h \in \mathcal{H}_n, |L_D(h) - L_S(h)| > \epsilon_n) < \delta_n$$

$$\cup \; D^m(S \mid \exists h \in \mathcal{H}_n, |L_D(h) - L_S(h)| > \epsilon_n)$$

$$\leq \delta_1 + \delta_2 ..... = \sum \delta_n = \delta \sum w(n) \leq \delta$$

Standard Risk Minimization (SRM)

- Now apply the Union bound over all n = 1,2..... so that this holds for every n

$$\forall h \in \mathcal{H}_n, \mid L_D(h) - L_S(h) \mid \leq \epsilon_n(m, \delta_n)$$

- Since

$$1 - \sum_n \delta_n = 1 - \delta \sum w(n)$$

$$\geq 1 - \delta$$

have that with probability $\geq 1 - \delta$

$$|L_D(h) - L_S(h)| \leq \epsilon_n(m, w(n) \cdot \delta)$$

Standard Risk Minimization (SRM)

- Now define

$$n(h) = \min(\{n \mid h \in \mathcal{H}_n\})$$

- $n(h)$ is the smallest n for which h is in a Subclass $\mathcal{H}_n$
  Thus $\min_{n(h)} \epsilon_n(m, \delta_n)$ is the smallest n that gives error ϵ_n
- Then we can rewrite

$$\forall h \in \mathcal{H}, L_D(h) \leq L_S(h) + \min_{n:h \in \mathcal{H}_n} \epsilon_n(m, w(n) \cdot \delta)$$

  as

$$L_D(h) \leq L_S(h) + \epsilon_n(h)(m, w(n(h)) \cdot \delta)$$

- This gives rise to the *SRM* paradigm

Standard Risk Minimization (SRM)

Structural Risk Minimization (SRM)

prior knowledge:
$\mathcal{H} = \bigcup_n \mathcal{H}_n$ where $\mathcal{H}_n$ has uniform convergence with $m^{\mathrm{UC}}_{\mathcal{H}_n}$
$w : \mathbb{N} \rightarrow [0,1]$ where $\sum_n w(n) \leq 1$
define: ϵ_n as in Equation (7.1); $n(h)$ as in Equation (7.4)
input: training set $S \sim \mathcal{D}^m$, confidence δ
output: $h \in \mathrm{argmin}_{h \in \mathcal{H}} \left[L_S(h) + \epsilon_{n(h)}(m, w(n(h)) \cdot \delta) \right]$

Figure: SRM

- In SRM we care about
  1. Sample loss $L_S(h)$
  2. Size of $\epsilon_n(h)$

Standard Risk Minimization (SRM)

- Since ϵ_n increases with n
  $\Rightarrow$ tradeoff between estimation error $\downarrow$ and $\epsilon_n(h)(m, w(n(h)) \cdot \delta)$
- Now we can show that SRM can be used for NUL problems ....

Theorem :

Let $\mathcal{H}$ be a hypothesis class such that $\mathcal{H} = \cup\, \mathcal{H}_n$. Here $\mathcal{H}_n$ is *UC* with saple complexity $m^{UC}_{\mathcal{H}_n}$.
Let $0 \leq w(n) \leq 1$ be such that,

$$w(n) = \frac{6}{n^2 \pi^2}$$

Standard Risk Minimization (SRM)

Then $\mathcal{H}$ is non-uniformly learnable using the SRM role with

$$m_{\mathcal{H}}^{NUL}(\epsilon, \delta, h) \leq m_{\mathcal{H}_{n(h)}}^{UC}\left(\frac{\epsilon}{2}, \frac{6}{(\pi n(h))^2} \cdot \delta\right)$$

Proof :

- $A(s)$ is SRM algorithm with respect to weight function w
- For every $h \in \mathcal{H}$ and $\epsilon, \delta, m_{\mathcal{H}_{n(h)}}^{UC}(\epsilon, w(n(h)) \cdot \delta) \leq m$
- Now use Theorem

$$\forall h \in \mathcal{H}, L_D(h) \leq L_S(h) + \min_{n:h \in \mathcal{H}_n} \epsilon_n(m, w(n) \cdot \delta)$$

with the fact that $\sum_n w(n) = 1$

Standard Risk Minimization (SRM)

- Then with probability of at least $1 - \delta$

$$L_D(h') \leq L_S(h') + \epsilon_{n(h')}(m, w(n(h')) \cdot \delta)$$

- This also holds for the $A(S)$ returned by the SRM rule
- By definition of SRM

$$L_D(A(S)) \leq \min_{h' \in \mathcal{H}} L_S(h') + \epsilon_{n(h')}(m, w(n(h')) \cdot \delta)$$

$$\leq L_S(h) + \epsilon_{n(h)}(m, w(n(h)) \cdot \delta)$$

- Here h is the minimum of $L_S(h') + \epsilon_{n(h')}(m, w(n(h')) \cdot \delta)$

Standard Risk Minimization (SRM)

- Consider

$$m^{UC}_{\mathcal{H}_{n(h)}}\left(\frac{\epsilon}{2}, \frac{6}{(\pi n(h))^2}\cdot\delta\right)$$

if

$$m \geq m^{UC}_{\mathcal{H}_{n(h)}}\left(\frac{\epsilon}{2}, \frac{6}{(\pi n(h))^2}\cdot\delta\right)$$

then plugging this higher m into $\epsilon_{n(h)}$

$$\epsilon_{n(h)}(m, w(n(h))\cdot\delta) \leq \frac{\epsilon}{2}$$

Standard Risk Minimization (SRM)

- Since each $\mathcal{H}$ is UC

$$L_D(h) \leq L_S(h) + \frac{\epsilon}{2}$$

- The

$$L_D A(S) \leq L_S(A(S)) + \frac{\epsilon}{2} \text{ (by UC)}$$

$$\leq L_S(h') + \frac{\epsilon}{2} \text{ (by SRM)}$$

$$\leq L_D(h') + \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (by UC)}$$

Standard Risk Minimization (SRM)

- $\Rightarrow L_D(A(S)) \leq L_D(h) + \epsilon$
  Thus $\mathcal{H}$ is NUL

Remarks :

- One can show that for any infinite domain set the hypothesis class of a binary predictor is not equal to a countable union

$$\mathcal{H} \neq \bigcup_n \text{(classes of finite VC dimension)}$$

- Thus NFL holds for NUL also
  $\Rightarrow \infty$ domain $\Rightarrow$ no universal learner $+ \exists$ perfect adversary

Standard Risk Minimization (SRM)

- More samples are needed in NUL than APAC
  $\Rightarrow$ NUL must search over all $\mathcal{H}_n$
  $\Rightarrow$ APAC, search over one $\mathcal{H}_n$
  (consequence of having more prior knowledge)
- Assume for all n

$$n = VCdim(\mathcal{H}_n)$$

$$m_{\mathcal{H}_n}(\epsilon, \delta) = C\ \frac{n + \log \frac{1}{\delta}}{\epsilon^2}$$

Standard Risk Minimization (SRM)

- one can then show

$$m_{\mathcal{H}}^{NUL}(\epsilon, \delta, h) - m_{\mathcal{H}_n}^{UC}\left(\frac{\epsilon}{2}, \delta\right) \leq 4C \cdot \frac{2\log(2n)}{\epsilon^2}$$

$\Rightarrow$ Thus NUL needs many more samples
$\Rightarrow$ grows logarithmically with V dim of $\mathcal{H}_n$
$\Rightarrow$ grows as $\frac{1}{\epsilon^2}$

Minimum Description Length and Occam's Razor

- NUL is actually a practical and widely used learning paradigm
- But what is $w(n)$?
  How to express preferences with respect to different hypotheses?
- Consider the case of singleton classes

$$\mathcal{H}_1 = \left\{ h_z \mid z \in \mathcal{X}, h_z(\chi) = \left\{ \begin{array}{l} 1 \text{ if } \chi = z \\ 0 \text{ otherwise} \end{array} \right\} \right\}$$

$$\mathcal{H}_2 = \left\{ h_{z_1, z_2} \mid z_1, z_2 \in \mathcal{X}, h_z = \begin{array}{l} \text{zero unless} \\ \chi = z_1 \text{ or } \chi = z_2 \end{array} \right\}$$

Minimum Description Length and Occam's Razor

- Suppose we don't have a true predictor
  $\Rightarrow$ APAC predictor needed
- Use Hoeffding's inequality or
  Find them of Statistical Learnining

$$m^{UC}(\epsilon, \delta) = \frac{\log\left(\frac{2}{\delta}\right)}{\epsilon^2}$$

- Now invert this to

$$m = \frac{\log \frac{2}{\delta}}{\epsilon_n^2}$$

$$\Rightarrow \epsilon_n(m, \delta) = \sqrt{\frac{\frac{2}{\delta}}{2m}}$$

$$\Rightarrow \delta \Rightarrow \delta \cdot w(n)$$

- Thus

  $\epsilon_{n(h)}(m, w(n(h)) \cdot \delta)$

  $= \sqrt{\frac{\log \frac{2}{w \cdot \delta}}{2m}}$

  $= \sqrt{\frac{-\log w(n) + \log \frac{2}{\delta}}{2m}}$

- The SRM rule then becomes

$$h \in \underset{h \in \mathcal{H}}{\text{argmin}} \left(L_S(h) + \sqrt{\frac{-\log w(h) + \log \frac{2}{\delta}}{2m}} \right)$$

- A convenient way to assign weights is to use a complexity measure for the hypotheses

Minimum Description Length and Occam's Razor

- Consider the bit-length required to describe a hypothesis
  $\Rightarrow$ the more complicated the *h* , the longer the description
- Thus fix a description language
  $\Rightarrow$ can be English/Bangla/Python/Math formulas/etc...
- A description consists of a finite string of symbols from a fixed alphabet
- Fix a finite set $\sum$, of symbols.
  Call them *"characters"*
- For example, $\sum = \{0, 1\}$
- $\sigma =$ string of symbols from $\sum$
  Suppose
  $\sigma = (0, 1, 1, 1, 1)$
  $|\sigma| = 5$ (length of string)

Minimum Description Length and Occam's Razor

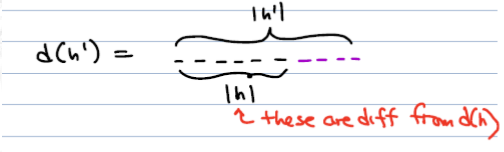
- The set of all *finite* length string is denoted $\sum^*$
- A descriptive language for $\mathcal{H}$ is
$$d : \mathcal{H} \to \sum^*$$
maps every $h \in \mathcal{H}$ to a string $d(h)$
- $d(h) \equiv$ description of h
$|h| =$ length of h
- To make sure that each $d(h)$ uniquely describes a string h
$\Rightarrow$ require the description language to be **Prefix-free**

Minimum Description Length and Occam's Razor

- Prefix-free means if

$$h' \neq h$$

then the first $|h|$ symbols of $d(h')$ cannot be $d(h)$ or vice -versa



- Prefix-free strings can be used to weight different hypotheses

Minimum Description Length and Occam's Razor

Kraft Inequality :

If $S \subseteq \{0,1\}^*$ is a prefix-free set of strings then

$$\sum_{\sigma \in S} \frac{1}{2^{|\sigma|}} \leq 1$$

Proof :

- Suppose S is a binary prefix-free set of string S
- given a string σ, what is the probability of randomly choosing σ?
  $\Rightarrow$ Suppose $P(0) = P(1) = \frac{1}{2}$
  $\Rightarrow$ because S is prefix-free once the first $|\sigma|$ flips match σ,
  we know that this sequence of coin flips uniquely corresponds to σ

- Thus,

$$P(\sigma) = \frac{1}{2^{|\sigma|}}$$

  for every $\sigma \in S$
- Since the $P(\sigma)$ add up to 1, proof is finished

- There we can weight h as

$$w(h) = \frac{1}{2^{|h|}}$$

- Inserting this weight into the

$$\sqrt{-\log w(h) + \log \frac{2}{\delta}}$$

Minimum Description Length and Occam's Razor

Theorem :

Let $\mathcal{H}$ be a hypothesis class, and let d be a descriptive language

$$d : \mathcal{H} \to \{0,1\}^{* \leftarrow \text{finite}}$$

that is prefix-free.

Then for every m, S, D with probability $\geq 1-\delta$ over $S \sim D^m$, we have

$$\forall h \in \mathcal{H} \; L_D(h) \leq L_S(h) + \sqrt{\frac{|h| + \ln \frac{2}{\delta}}{2m}}$$

where $|h|$ is the length of $d(h)$

Minimum Description Length and Occam's Razor

Proof :

use $\forall h \in \mathcal{H}, L_D(h) \leq L_S(h) + \min_{n:h\in\mathcal{H}_n} \epsilon_n(m, w(n)\cdot\delta)$ and let $w(h) = 2^{-|h|}$, use

$$\epsilon_n(m,\delta) = \sqrt{\frac{\ln\frac{2}{\delta}}{2m}}$$

and

$$\ln 2^{|h|} = |h| \ln 2 < |h|$$

Minimum Description Length and Occam's Razor

- From this we can construct a new learning paradigm
  1. Find a training set S
  2. Search for a hypothesis h
     Minimizing the bound

$$L_S(h) + \sqrt{\frac{|h| + \ln \frac{2}{\delta}}{2m}}$$

- This trades low empirical risk $L_S(h)$ for low description length

Minimum Description Length (MDL)

prior knowledge:
$\mathcal{H}$ is a countable hypothesis class
$\mathcal{H}$ is described by a prefix-free language over $\{0,1\}$
For every $h \in \mathcal{H}$, $|h|$ is the length of the representation of h
input: A training set $S \sim \mathcal{D}^m$, confidence δ
output: $h \in \operatorname{argmin}_{h \in \mathcal{H}} \left[L_S(h) + \sqrt{\frac{|h| + \ln(2/\delta)}{2m}} \right]$

Figure: MDL

Minimum Description Length and Occam's Razor

- For example $|h|$ could be the length of a program (in binary).

  Assume compiler stops if program 1 *is a header for program* 2

Occam's Razor

- Prefer hypotheses with shorter lengths : less complex
- Two hypotheses with equal sample error
  $\Rightarrow$ prefer the one with shorter *MDL* length
- Choice of language is one way of implementing prior knowledge
  $\Rightarrow$ For example use the more complex language of relativity to describe physics instead of Galilean/Newtonian relativity

- Let m depend on underlying distribution D

Definition : (Consistency)

Let Z be a domain set.
Let P a set of distributions over Z.
Let $\mathcal{H}$ be a hypothesis class.
A learning role is <u>*consistent*</u> with respect to $\mathcal{H}$ and P, if $\exists$ a $m(\epsilon, \delta, h, D)$ such that for every $(\epsilon, \delta) \in (0,1)$, every $h \in \mathcal{H}$, every $D \in P$ that if $m \geq m_{\mathcal{H}}^{NUL}(\epsilon, \delta, h, D)$ then with probability $\geq 1 - \delta$ that

$$L_D(A(S)) \leq L_D(h) + \epsilon$$

If P is the set of all distributions, we say that A is universally consistent with respect to $\mathcal{H}$

Other Notions of Learnability - Consistency

- Consistency is a strict relaxation of NUL
- If $A(S)$ is NUL with respect to $\mathcal{H}$, it is universally consistent with respect to $\mathcal{H}$
- Consistency is desirable, but not powerful
- Suppose $A_1(S)$ and $A_2(S)$ are NUL
  But $A_1(S)$ is consistent, $A_2(S)$ is not.
- East to make $A_2(S)$ consistent
  $\Rightarrow$ Just memorize the sample data and output the most frequent y value of x, y
  x appears in the sample
  $\Rightarrow$ This memorization also can be shown to be consistent

Other Notions of Learnability - Consistency

- Same subtlely regarding the NFL theorem for consistent algorithms

  $\rightarrow$ in APAC/NUL fix training set size m, then find a distribution and labeling the function for this training set size

  $\rightarrow$ in Consistency guarantees, first fix the distribution and the labeling function and then find a training set size that works for learning this part distribution and labeling function