Machine Learning

CSE427

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Superhumungous Thanks

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Introduction

- **Goal:** Understand which $\mathcal{H}$ are PAC Learnable
- So far, we saw that,
  - *Finite $\mathcal{H}$ are Learnable.*
  - *Class of all functions over an infinite size domains are not learnable.*
- Examples where, infinite size classes are learnable:
  - *aligned axis predictors*
  - *circle predictors,...*
- Thus, *finiteness* doesn't seem to be a necessary condition for learnability
- We want a better measure of learnability

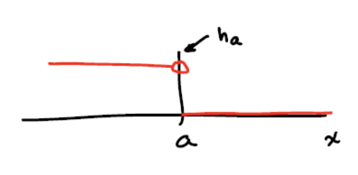
Introduction

- in 1971, *Vapnik and Chervonenkis* in the context of statistics invented the *"VC"* dimension idea
- This was applied to PAC Learning theory by Blomer, etc., in 1989.

Infinite-Size Classes Can Be Learnable

- Let's first show that infinite size classes can be learnable
- Simplest example: *Threshold function*
- Example: $\mathcal{H} = \{h_a \mid a \in \mathbb{R}\}$ where, $h_a : \mathbb{R} \to \{0, 1\}$ such that

$$h_a(\chi) = \underbrace{\mathbb{I}(\chi < a)}_{\text{Indicator Function}}$$



- Since ∞ possible choices for a, $\mathcal{H}$ is infinite

Infinite-Size Classes Can Be Learnable

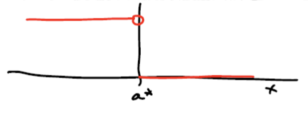
- Let's show that $\mathcal{H}$ is *PAC learnable* with sample complexity

$$m_{\mathcal{H}} \leq \frac{\log \frac{2}{\epsilon}}{\epsilon}$$

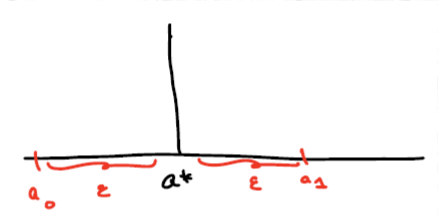
- Let $a^* =$ threshold such that

$$h^*(\chi) = \mathbb{I}(\chi < a^*)$$

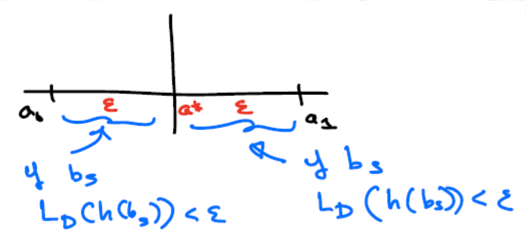
This is the true labeling function and $L_D(h^*) = 0$



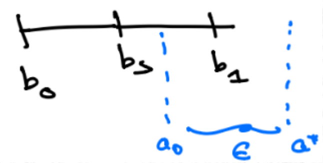
- Now, how can you get an error $< \epsilon$?



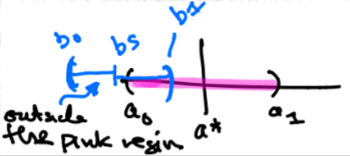
- If the predicted value b_s falls in the region (a_o, a_*) or (a_*, a_1) then the error $< \epsilon$.



- Let $b_0 < b_s < b_1$ be a similar ϵ sized region around b_s just as $a_0 < a^* < a_1$ is an ϵ sized region



- The idea is a^* is associated to the sandwich region (a_0, a_1)
- b_s is associated to the sandwich region (b_0, b_1)
- If $b_s < a_0$ then error $> \epsilon$



Infinite-Size Classes Can Be Learnable

- Similarly if $b_s > a_1$, then error $> \epsilon$

$$\underset{S \sim D^m}{\mathbb{P}}\Big[L_D(h_S) > \epsilon\Big] \leq \underset{S \sim D^m}{\mathbb{P}}\Big[b_s < a_0 \text{ or } b_s > a_1\Big]$$

- Now, use the *Union Bound*,

$$\begin{aligned}\underset{S \sim D^m}{\mathbb{P}}\Big[b_s < a_0 \text{ or } b_s > a_1\Big] &\leq \underset{S \sim D^m}{\mathbb{P}}\Big[b_s < a_0\Big] + \underset{S \sim D^m}{\mathbb{P}}\Big[b_s > a_1\Big] \\ &= (1-\epsilon)^m + (1-\epsilon)^m \\ &= 2(1-\epsilon)^m \\ &\leq 2e^{-\epsilon m}\end{aligned}$$

Infinite-Size Classes Can Be Learnable

- Setting this equal to δ,

$$2e^{-\epsilon m} = \delta$$
$$\Rightarrow m = \frac{1}{\epsilon} \ln \frac{2}{\delta}$$

Thus, if $m \geq \frac{1}{\epsilon} \ln \frac{2}{\delta}$ then $\mathcal{H}$ is *PAC Learnable*.

VC dimension of $\mathcal{H}$ correctly measures it's learnability

Motivation: No Free Lunch Theorem

$$\mathcal{H} = \{\text{all possible functions from } C \subseteq X \text{ to } \{0,1\}\}$$

- If choose no more than $\frac{|C|}{2}$ sample points $\rightarrow$ *average error* $\geq \frac{1}{4}$
- There exists an h with zero error, since $\mathcal{H}$ includes all possible functions from $C \rightarrow \{0,1\}$ in $\mathcal{H}$.
- In the special case $|C| = \infty$, we need ∞ samples for *PAC Learnability*
- Infinite sized $\mathcal{H}$ like this, where all possible hypotheses are included, are not PAC Learnable

VC Dimension

- If we consider distributions like this, that are concentrated on $C \subseteq X$, we should study how X behaves on C.

Definition: Restriction of $\mathcal{H}$ to C

Let $\mathcal{H}$ be the class of functions from X to $\{0, 1\}$. Let

$$C = \{C_1, C_2, \ldots, C_n\} \subset X$$

The restriction of $\mathcal{H}$ to C is the set of functions from C to $\{0, 1\}$ that can be derived from $\mathcal{H}$. That is,

$$\mathcal{H}_c = \{h(C_1), h(C_2), \ldots, h(C_m) \mid h \in \mathcal{H}\}$$

where we can represent each function from $C \to \{0, 1\}$ as a vector in $\{0, 1\}^{|C|}$

Shattering

- **Example: Two Points**

  Suppose, $C = \{0, 1\}$ is just two points,

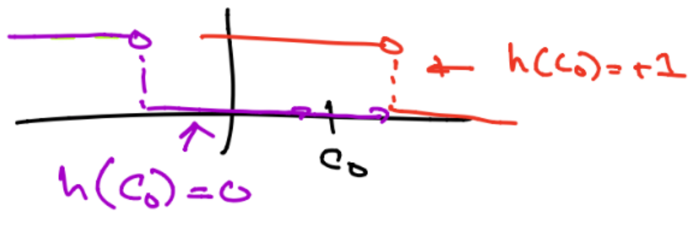
  $$\mathcal{H}_C = \{(0,0), (0,1), (1,0), (1,1)\}$$

  where each set of 0's and 1's is a *hypothesis*.

- If the restriction of $\mathcal{H}$ to C is the set of all functions from C to $\{0, 1\}$, then we say, $\mathcal{H}$ shatters C

- **Example: One Point**

Suppose $C = \{C_0\}$. Take $\mathcal{H}$ = *Class of all hypothesis functions.*



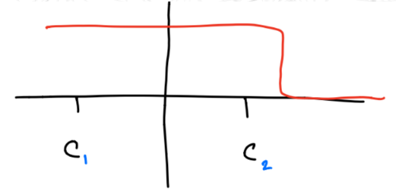
Thus, C_0 gets assigned all possible values. We say that the class of threshold functions shatters $C = \{C_0\}$

Definition: Set Shattering

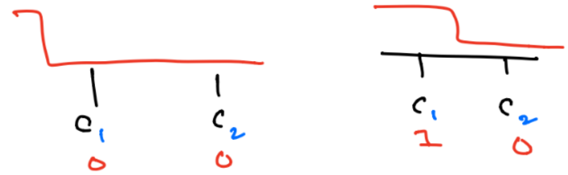
A hypothesis class $\mathcal{H}$ shatters a finite set $C \subset X$, if the restriction of $\mathcal{H}$ to C is the set of all functions from $C \to \{0,1\}$, i.e. $|\mathcal{H}_C| = 2^{|C|}$.

Example: Two points

Let $C = \{C_1, C_2 \mid C_1 \leq C_2\}$. Let's look at the possible labelings with threshold functions in $\mathcal{H}$.



Shattering Examples



- Question: Can we get the labeling $\{0, 1\}$?
- Is $C = \{C_1, C_2\}$ shattered by $\mathcal{H}$?
- Answer: Any labeling assigning 1 to C_2 will assign 1 to C_1 also

$\Rightarrow$ C is not shattered by $\mathcal{H}$

Shattering

- Going back to the *NFL Theorem*:

$\rightarrow$ Whenever some c is shattered by $\mathcal{H}$

$\rightarrow$ the adversary is not restricted by $\mathcal{H}$

$\rightarrow$ the adversary can choose a perfect labeling function and

$\rightarrow$ the adversary can achieve zero loss

Shattered Sets and NFL Theorem

Corollary:

Let $\mathcal{H}$ be a hypothesis class of functions from X to $\{0,1\}$. Let m be the training set size. Assume there exists a set $C \subset X$ of size $2m$ that is shattered by $\mathcal{H}$. Then for any learning algorithm A, there exists a distribution D over $X \times \{0,1\}$ and a predictor $h \in \mathcal{H}$ such that $L_D(h) = 0$, but with probability of at least $\frac{1}{7}$ over the choice of $S \sim D^m$, we have

$$L_D(A(S)) \geq \frac{1}{8}$$

- Thus if $\mathcal{H}$ shatters C, with $C = |2m|$,

$\rightarrow$ then we can't learn $\mathcal{H}$ using m samples.

Shattered Sets and NFL Theorem

- Intuitively, if C shatters $\mathcal{H}$,

$\rightarrow$ a sample of less than half the instances of C, gives no information about the other half of C.

- Any labeling of the rest of the instances in C can be explained by an $h \in \mathcal{H}$

VC Dimension

Definition: VC Dimension

The VC Dimension of $\mathcal{H}$ is the maximal size of a set $C \subset X$ that can be shattered by $\mathcal{H}$. If $\mathcal{H}$ shatters sets of arbitrarily large size, then VC-dim$(\mathcal{H}) \equiv \infty$.

Theorem: Infinite VC Dimension

If VC-dim$(\mathcal{H}) = \infty$, then $\mathcal{H}$ is not PAC Learnable

Proof: Every set of C of size $2m$ is shattered by $\mathcal{H}$, where m is the size of the training set. Now apply NFL to C.

Comments

- It turns out that if VC dim($\mathcal{H} < \infty$), then $\mathcal{H}$ is learnable.

- In order to find the VC-Dim of $\mathcal{H}$, we need to show that
  a) There exists a C of size d, that is shattered by $\mathcal{H}$
  b) Every set C of size $d + 1$ is not shattered by $\mathcal{H}$

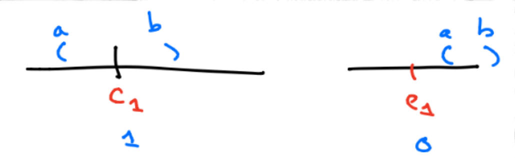
Example: Threshold Functions

- Let's calculate the VC-dim for Threshold Functions.
- $C = \{C_1\}$ is shattered VC-dim$(\mathcal{H}) \geq 1$
- $C = \{C_1, C_2\}$ is *not* shattered. Therefore, VC-dim$(\mathcal{H}) < 2$
- Thus VC-dim$(\mathcal{H}_{threshold}) = 1$
- Thus, it makes sense that the infinite class $\mathcal{H}_{threshold}$ is PAC Learnable

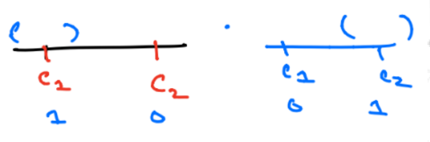
- Let's calculate the VC-dim of Intervals.

$$h_{ab}(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

- $C = \{C_1\}$ is shattered by $\mathcal{H}_{ab}$



- $C = \{C_1, C_2\}$ is shattered

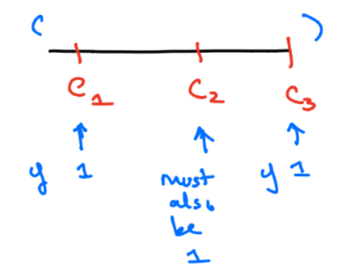


Example: Intervals

- Thus, VC-dim$(\mathcal{H}_{ab}) \geq 2$

Example: Intervals

- $C = \{C_1, C_2, C_3\}$



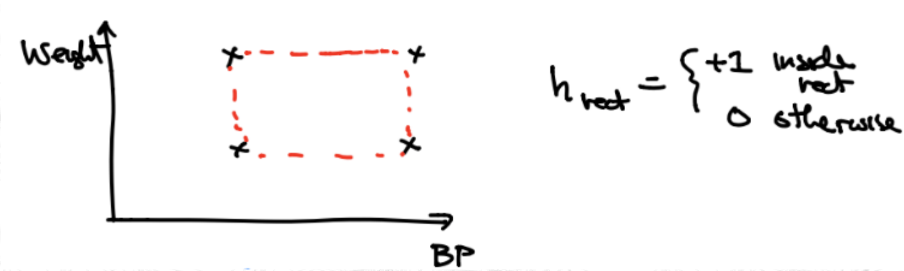
- If $h(C_1) = 1, h(C_3) = 1 \Rightarrow h(C_2) = 1$

- If $h(C_1) = 1, h(C_3) = 1 \Rightarrow h(C_2) = 1$, thus $(1, 0, 1)$ is not possible

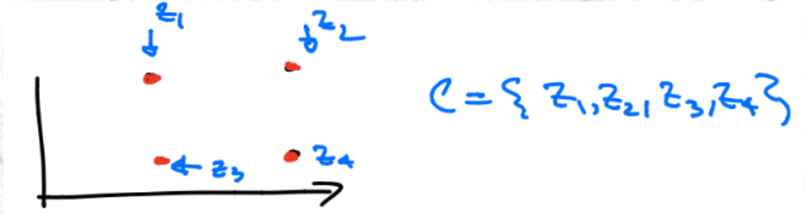
$\Rightarrow$ $VC\ dim(\mathcal{H}_{ab}) < 3$

$\Rightarrow$ $VC\ dim(\mathcal{H}_{ab}) = 2$

Recall, that aligned-axis rectangles look like:

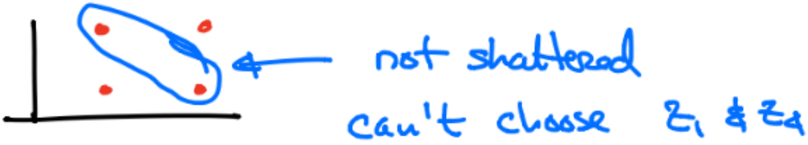


Example: Axis Aligned Rectangles



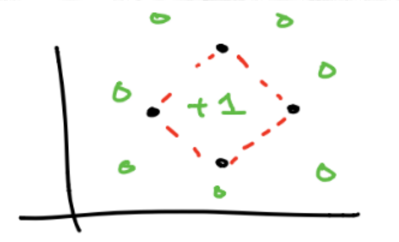
- Label the corner points as $\{z_1\}, \{z_2\}, \{z_3\}, \{z_4\}$
- $\{z_1, z_2\}, \{z_2, z_3\}, \{z_3, z_4\}, \{z_2, z_4\}$ are all shattered

- But, $\{z_1, z_4\}$ and $\{z_2, z_3\}$ are not shattered



Example: Axis Aligned Rectangles

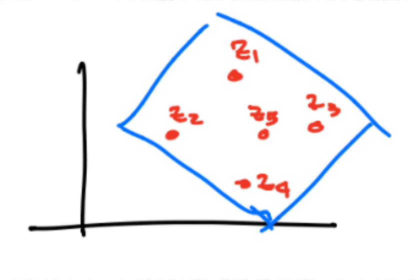
- Solution: rotate rectangle.



- Now, see VC-dim($\mathcal{H}_{rect}$) ≥ 4

Example: Axis Aligned Rectangles

- But if $C = \{z_1, z_2, z_3, z_4, z_5\}$



- The point z_5 is always captured along with the rest of the points.

$\Rightarrow$ VC-dim$(\mathcal{H}_{rect}) < 5$

$\Rightarrow$ VC-dim$(\mathcal{H}_{rect}) = 4$

Remarks: Finite Classes

- Suppose $\mathcal{H}$ is finite.
- Suppose $|C| = k$. This means C has k points.
- Suppose, the labels are binary. Then $|\mathcal{H}| = 2^k$.
- Suppose VC-dim$(\mathcal{H}) = d$.
- The VC-dim of $\mathcal{H}$ restricted to C, can be at most k.
- Thus, $d \leq k$.
- This means
$$|\mathcal{H}| \geq 2^d$$
- Or equivalently, taking logarithms
$$\log_2 |\mathcal{H}| \geq \text{VC-dim}(\mathcal{H}) = d$$

Remarks: Finite Classes

- C can't be shattered if $d < 2^{|C|}$
- Will prove that classes with finite VC dimension are learnable
- Example: Threshold functions: $\mathcal{H}_a = \{h_a \mid a \in \mathbb{R}\}$ is an infinite class. but it has finite VC-dim$(\mathcal{H}_a) = 1$

$\Rightarrow$ Threshold function class is *learnable*

VC Dimension and the Number of Parameters

- It looks like VC-dim $=$ number of parameters, example: threshold functions parameter by $a \in \mathbb{R}$
- But not always true that number of parameters $=$ VC dimension

The Fundamental Theorem of Statistical Learning

- We saw that infinite VC dimension class is not learnable.
- Opposite statement is also true. This gives rise to:

Theorem 6.7 (The Fundamental Theorem of Statistical Learning). *Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0,1\}$ and let the loss function be the* $0-1$ *loss. Then, the following are equivalent:*

1. *$\mathcal{H}$ has the uniform convergence property.*
2. *Any ERM rule is a successful agnostic PAC learner for $\mathcal{H}$.*
3. *$\mathcal{H}$ is agnostic PAC learnable.*
4. *$\mathcal{H}$ is PAC learnable.*
5. *Any ERM rule is a successful PAC learner for $\mathcal{H}$.*
6. *$\mathcal{H}$ has a finite VC-dimension.*

- VC Dimension determines *Learnability and Sample Complexity*

The Fundamental Theorem of Statistical Learning - Quantitative Version

Theorem 6.8 (The Fundamental Theorem of Statistical Learning – Quantitative Version). *Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0,1\}$ and let the loss function be the* 0−1 *loss. Assume that* $\mathrm{VCdim}(\mathcal{H}) = d < \infty$. *Then, there are absolute constants C_1, C_2 such that*

1. *$\mathcal{H}$ has the uniform convergence property with sample complexity*

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \le m_{\mathcal{H}}^{UC}(\epsilon,\delta) \le C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

2. *$\mathcal{H}$ is agnostic PAC learnable with sample complexity*

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \le m_{\mathcal{H}}(\epsilon,\delta) \le C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

3. *$\mathcal{H}$ is PAC learnable with sample complexity*

$$C_1 \frac{d + \log(1/\delta)}{\epsilon} \le m_{\mathcal{H}}(\epsilon,\delta) \le C_2 \frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon}$$

The proof of this theorem is given in Chapter 28.

Note: *Lower bound and upper bound are determined by the VC Dimension=d*

Proof of Theorem 6.7 **Proof:**

a) $1 \rightarrow 2$, We proved this in *Chapter 4*

b) $2 \rightarrow 3$, $3 \rightarrow 4$, $2 \rightarrow 5$ are clearly true

c) $4 \rightarrow 6$, $5 \rightarrow 6$ follow from *NFL Theorem*

- Difficult part is $1 \rightarrow 6$

Comments:

- Suppose $VC\ dim(\mathcal{H}) = d$, $C \subseteq \chi$
- Then effective size of $\mathcal{H}_C$ is $|\mathcal{H}_C|$

  $|\mathcal{H}_C| = \mathcal{O}(|C|^d)$
- For binary predictors, $|\mathcal{H}_C| = |C|^d$
- As $d = VC\ dim(\mathcal{H})$ grows, m increases

Proof of Theorem 6.7

Lower Bound Analysis For Agnostic PAC Learning

- As $d = VC\ dim(\mathcal{H})$ grows, m increases
- As $\epsilon \to 0$, $m \to 0$
- As $\delta \to 0$, $m \to 0$, but *logarithmically* only
- Look at *(1)* in Theorem 6.8, since
  ERM + Uniform Convergence = PAC Corollary 4.4, page 32, (1)
  says all you need is an ERM Learning Paradigm
- However, as we will see to apply ERM $\to$ need to minimize error over all $h \in \mathcal{H}$
- But then we have to worry about *computational complexity*

$\Rightarrow$ How long will it take to calculate the error for every $h \in \mathcal{H}$

$\Rightarrow$ ∞ time if $|\mathcal{H}| = \infty$

Lower Bound Analysis For Agnostic PAC Learning

- For *agnostic PAC Learning*,

$$\Rightarrow C_1 \frac{d + log\frac{1}{\delta}}{\epsilon^2} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{d + log\frac{1}{\delta}}{\epsilon^2}$$

While for *PAC*,

$$\Rightarrow C_1 \frac{d + log\frac{1}{\delta}}{\epsilon} \leq m_{\mathcal{H}}(\epsilon, \delta) \leq C_2 \frac{dlog\frac{1}{\epsilon} + log\frac{1}{\delta}}{\epsilon}$$

  - This is because for agnostic PAC, there is no guarantee that you will succeed
  - Therefore, we need more samples to get an accurate, reliable predictor

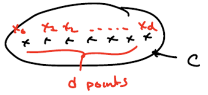
Proof of Theorem 6.7

Lower Bound Analysis For Agnostic PAC Learning

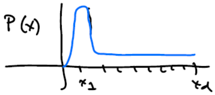
- first consider lower bound
- NFL tells us that
$$\frac{d}{2} = \frac{|c|}{2} < m_{\mathcal{H}}(\epsilon, \delta)$$
- How does the ϵ - dependence of the lower bound come in ?
- $\mathcal{H}$ should work for any distribution on C. Choose the most difficult distribution to predict from.

Lower Bound Analysis For Agnostic PAC Learning

- choose the largest set C that is shattered by $\mathcal{H}$. It has $dim = d$

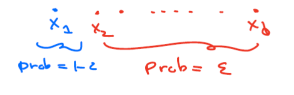


- This set will be very hard to learn from if you keep sampling the same point



Lower Bound Analysis For Agnostic PAC Learning

- Let the probability of sampling x_1 be ϵ, $P(x_1) = 1 - \epsilon, P(x_i = \frac{\epsilon}{d-1})$ $P(x\ outside\ C) = 0$
-



- Then to learn from x_2.........x_d need to sample at least half the points, which is $\frac{d-1}{2}$
- Let m be number of points sample from $x_2, \ldots, x_d$
- the expected number of points in the $\epsilon-$ region is :

$$m\epsilon$$

- Thus need

$$m\epsilon > \frac{d-1}{2}$$

Proof of Theorem 6.7

Lower Bound Analysis For Agnostic PAC Learning

$$\Rightarrow m > \frac{d-1}{2\epsilon}$$

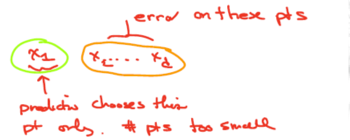
$$m > \mathcal{O}(\frac{d}{\epsilon})$$

- Thus we expect for the PAC case, the lower bound to be $\mathcal{O}(\frac{d}{\epsilon})$
- Can check this with a more complete derivation (Chapter 28)

Note: error for less than $\frac{d-1}{2}$ point is $\frac{1}{4} \times \epsilon = \frac{\epsilon}{4}$
Note: $\underset{S\sim}{\mathbb{E}}(L_D(A(S))) \geq \frac{\epsilon}{4}$ of $m \leq \frac{d-1}{2\epsilon}$ for any ϵ

- think about the points x_2.........x_d as the points where the classifier makes a mistake

Lower Bound Analysis For Agnostic PAC Learning

- 

  Now give argument for Agnostic PAC that lower bound $\sim \theta(\frac{1}{\epsilon^2})$
- **Why are things harder in Agnostic case ?**
- Recall, PAC case one h has zero error. Now no such assumption
- Consider a Binomial process
  $\rightarrow$ like trying to figure out the probability of candidate A getting a vote instead of candidate B

Proof of Theorem 6.7

Lower Bound Analysis For Agnostic PAC Learning

- 2 candidates $A$and B
- want to predict probability of winning p
- want accuracy ϵ
  $\Rightarrow$ thus, $\hat{p} - \epsilon \leq p \leq \hat{p} + \epsilon$
  $\Rightarrow$ here $\hat{p}$ is the estimate of p
- For binomial process

$$\mathrm{Var}(x) = p(1-p)$$

$$\text{Average Var } = \frac{p(1-p)}{n}$$

$$\text{std-dev } = \frac{\sqrt{p(1-p)}}{n}$$

Lower Bound Analysis For Agnostic PAC Learning

- graph of $p(1-p) =$

- max of $p(1-p)$ is at $p = \frac{1}{2}$

$$\sqrt{p(1-p)} = \sqrt{\frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{2}$$

- consider the 2 sigma range

Proof of Theorem 6.7

Lower Bound Analysis For Agnostic PAC Learning 2σ range is

$\hat{p} \pm \frac{2\sqrt{p(1-p)}}{\sqrt{n}}$

$\Rightarrow \hat{p} \pm \frac{1}{\sqrt{n}}$

- accuracy $= \frac{1}{\sqrt{n}}$

$$\epsilon = \frac{1}{\sqrt{n}} \Rightarrow n = \frac{1}{\epsilon^2}$$

- Thus for more than $\frac{1}{\epsilon^2}$ points accuracy $\leq \epsilon$
- That's why for Agnostic PAC lower bound $\sim \frac{1}{\epsilon^2}$

Proof of Theorem 6.7

Upper Bound Analysis For Agnostic PAC Learning:
Sauer-Shelah-Perles Lemma

- Lower bound analysis is less interesting because it doesn't tell you how many samples you need for machine learning
- It just says the number of samples is at least ..... this many
- Lower bound analysis uses probability theory. Doesn't use the VC dimension. Only way we used VC dimension $\Rightarrow$ number of points in a set that has all possible behaviours.
- Upper bounds depend on a very special property of VC dimension $\Rightarrow$ Sauer Lemma

Upper Bound Analysis For Agnostic PAC Learning:
Sauer-Shelah-Perles Lemma

- Define the Growth function

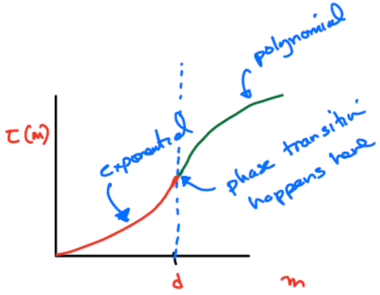
$$\tau_{\mathcal{H}}(m) = \max_{C \subset X, |C| = m} \left(|\mathcal{H}_C| \right)$$

- $\tau_{\mathcal{H}}$ is the maximum number of different functions from C which has size m to $\{0, 1\}$
- $\mathcal{H}_C$ = class restricted to C
- If $\text{VCdim}(\mathcal{H}) = d$ then for any $m \leq d$

$$\tau_{\mathcal{H}}(m) = 2^m$$

Upper Bound Analysis For Agnostic PAC Learning:
Sauer-Shelah-Perles Lemma

- Then $\mathcal{H}_C$ includes all possible function from $C \to \{0, 1\}$
- Sauer lemma tells us what happens when $m > d$



Sauer-Shelah-Perles Lemma

Sauer-Shelah-Perles Lemma

Let $\mathcal{H}$ be a hypothesis class with $VCdim(\mathcal{H}) \leq d < \infty$. Then for all m

$$\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{d} \binom{m}{i}$$

In particular if $m > d$ then

$$\tau_{\mathcal{H}}(m) \leq \left(\frac{em}{d}\right)^{d} \sim m^{d}$$

Proof of Sauer-Shelah-Perles Lemma

Proof:

- Let $C = \{c_1, ...c_m\}$
- It's enough to prove for all $\mathcal{H}$

$$|\mathcal{H}_C| \leq \left| \left\{ \begin{array}{l} \textit{all the subsets } B \textit{ of } C, \\ \textit{that are shattered by } \mathcal{H} \end{array} \right\} \right|$$

  more concisely,

$$|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$$

- This is sufficient to prove Sauer because if
  a $VCdim(\mathcal{H}) \leq d$
  then no set with size $> d$ is shattered by $\mathcal{H}$

Proof of Sauer-Shelah-Perles Lemma

Therefore

$$|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$$

$\leq$ number of subsets of size up to d of C, where

$$|C| = m$$
$$= \sum_{i=0}^{d} \binom{m}{i}$$

- when $m > d$ then

$$\sum_{i=0}^{d} \binom{m}{i} \leq \left(\frac{em}{d}\right)^d$$

(thus this is a technical algebraic fact $\rightarrow$ Appendix: A-5)

Proof of Sauer-Shelah-Perles Lemma

- Now to prove $|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$, use induction
- $m = 0 \rightarrow$ one point

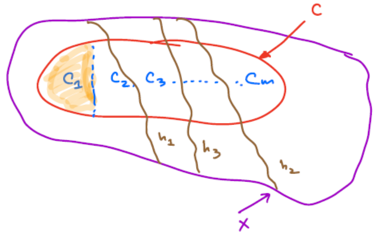
$$|\mathcal{H}_C| = 1$$

$$|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}| = |\phi| = 1$$

- If start induction at $m = 1$ then $|\mathcal{H}_C = 2|$,
  $\{B \subseteq C \mid B \; \textit{shattered}\} = 2$
- **Induction Step :**
  Assume $|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$ holds for sets of size $k < m$
- **Need to prove :**
  $|\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$ holds for sets of size m

Proof of Sauer-Shelah-Perles Lemma

- fix $\mathcal{H}$
- $C = \{c_1 .....c_m\}$
  $C' = \{c_2 .....c_m\}$



Let Y_0 be the set of the functions $\{c_2, ...c_m\}$

Proof of Sauer-Shelah-Perles Lemma

$$Y_0 = \{\text{the functions in } \mathcal{H} \text{ restricted to } \mathcal{H}_{C'}\}$$
$$= \text{ these are the functions which map}$$

$$\{f : (c_2....c_m) \to \{0,1\}\}$$

- these functions are $m-1$ dimensional vectors

$$\begin{pmatrix} h(c_2) \\ h_(c_3) \\ . \\ . \\ . \\ h(c_m) \end{pmatrix}$$

Proof of Sauer-Shelah-Perles Lemma

- Now define

$$Y_1 = \left\{ \begin{array}{l} \text{the set of all functions on } C' \\ \text{that can be extended to } c_1 \end{array} \right\}$$

$$= \left\{ (y_2, y_3 \ldots y_m) \mid \begin{array}{l} (0, y_2 \ldots y_m) \epsilon \mathcal{H}_C \text{ and} \\ (1, y_2 \ldots y_m) \epsilon \mathcal{H}_C \end{array} \right\}$$

- Then

$$|\mathcal{H}_C| = |Y_0| + |Y_1|$$

Here,
$Y_0 =$ the number of functions living only in C'
$Y_1 =$ the number of functions in C'that can be extended to c_1

Proof of Sauer-Shelah-Perles Lemma

- Now using the induction step

$$|Y_0| = |\mathcal{H}_C| \leq |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|$$

$$= |\{B \subseteq C' \mid c_1 \notin B \text{ and } \mathcal{H} \text{ shatters } B\}|$$

- Now define $\mathcal{H}' \subseteq \mathcal{H}$ to be the pairs of functions that agree on C' but differ on c_1

$$\begin{aligned} \mathcal{H}' = \{h, h' \in \mathcal{H} &\text{ such that} \\ &(1 - h'(c_1), h'(c_2) ... h'(c_m)) \\ = &\ (h(c_1), h(c_2) .... h(c_m)) \\ &(h(c_1), h(c_2), ... h(c_m))\} \end{aligned}$$

Proof of Sauer-Shelah-Perles Lemma

- Therefore,
  $\Rightarrow$ if $\mathcal{H}'$ shatters $B \subseteq C$, then it also shatters $\{c_1\} \cup B$ and *vice versa*
- Now

$$Y_1 = \mathcal{H}'_{C'}$$

- Therefore :

$$|Y_1| = |\mathcal{H}'_{C'}| \leq \left| \left\{ B \subseteq C' \mid \mathcal{H} \text{ shatters } B \right\} \right|$$

$$= \left| \left\{ B \subseteq C' \mid \mathcal{H} \text{ shatters } B \cup \{c_1\} \right\} \right|$$

$$= \left| \left\{ B \subseteq C \mid c_1 \in B \text{ and } \mathcal{H}' \text{ shatters } B \right\} \right|$$

$$\leq \left| \{ B \subseteq C \mid c_1 \in B \text{ and } \mathcal{H} \text{ shatters } B \} \right|$$

Proof of Theorem 6.7

Proof of Sauer-Shelah-Perles Lemma

- Putting everything together

$$
\begin{aligned}
|\mathcal{H}_C| &= |Y_0| + |Y_1| \\
&\le |\{B \subseteq C \mid c_1 \notin B \text{ and } \mathcal{H} \text{ shatters } B\}| \\
&\le |\{B \subseteq C \mid c_1 \in B \text{ and } \mathcal{H} \text{ shatters } B\}| \\
&= |\{B \subseteq C \mid \mathcal{H} \text{ shatters } B\}|
\end{aligned}
$$

This is what we wanted to prove !

Proof of Sauer-Shelah-Perles Lemma

- **What are the consequences of this?**
  $\tau_{\mathcal{H}}(m) = \max\limits_{|C|=m} |\mathcal{H}_C|$ The max is taken over all sets $C \subseteq \chi$ such that $|C| = m$
- Note different C with $|C| = m$ may have different $|\mathcal{H}_C|$
  Example: $\mathcal{H} =$ *aligned axis rectangle*

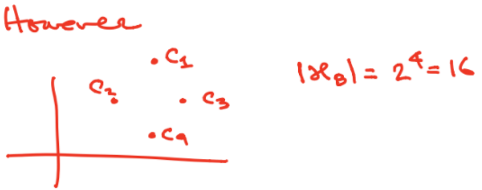
$C = \{c_1, c_2, c_3, c_4\}$

$|\mathcal{H}_B| = 2^4 - 2 = 14$

because and $\emptyset$ not possible

Proof of Theorem 6.7

Proof of Sauer-Shelah-Perles Lemma



- If $\tau_{\mathcal{H}}(m) \sim m^d$ for $m > d$, then
- $$D^m(S|L_{D,f}(h) > \epsilon) leq |\mathcal{H}_B| e^{-\epsilon m} \tag{1}$$

  where $|\mathcal{H}_B| e^{-\epsilon m} = \textit{number of bad hypothesis}$

Proof of Sauer-Shelah-Perles Lemma

- Recall that,

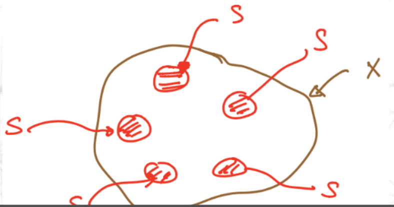
  $|\mathcal{H}_B| =$

$$\left\{ \begin{array}{c} \text{number of bad hypothesis} \\ \text{generated by sampling from} \\ \text{misleading training sets} \end{array} \right\}$$

- How do you choose a training set?

  $\Rightarrow$ Select a $|C| = m$ subset of χ

  $\Rightarrow$ Therefore, the number of ways to choose a bad hypothesis is $|\mathcal{H}_B| \leq |\mathcal{H}|_C$, where $C = S$



Proof of Sauer-Shelah-Perles Lemma

- $$\max_{\substack{S\subseteq\chi \\ |S|=m}} |\mathcal{H}_B| \leq \max_{\substack{S\subseteq\chi \\ |S|=m}} |\mathcal{H}|_S = \tau(m)$$

- Therefore equation (1) becomes,

$$D^m(S|L_{D,f}(h) > \epsilon) leq \tau(m) e^{-m\epsilon}$$

- Thus we see that, if

$$\tau(m) \sim polynomial$$

Then

$$\tau(m)e^{-m\epsilon} \Rightarrow 0$$

as $m \Rightarrow \infty$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size Now lets go back to proving $1 \to 6$ in **Theorem 6.7**

- to prove this we need to show that finite VCdim leads to uniform convergence
- Recall that if a class $\mathcal{H}$ is uniformly convergent if for every distribution D , there is a $m_{\mathcal{H}}(\epsilon, \delta)$ such that S is ϵ-representative
- And recall that S is ϵ-representative if for every $h \epsilon \mathcal{H}$

$$|L_S(h) - L_D(h)| \leq \epsilon$$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- Since $|L_S(h) - L_D(h)| \leq \epsilon$ should hold for every $h \in \mathcal{H}$, choose the h fro which

$$|L_S(h) - L_D(h)|$$

  is largest
- Formally, this is

$$\underset{h \in \mathcal{H}}{\text{SUP}} |L_S(h) - L_D(h)|$$

  SUP $\equiv$ least upper hand

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size The expected max of $|L_S(h) - L_D(h)|$ is

$$\mathop{\mathbb{E}}_{S\sim D^m}\left[\sup_{h\in\mathcal{H}}|L_D(h) - L_S(h)|\right]$$

- One thing we have been doing up to this point is
- We constructed our prediction after seeing the data
- This is generally an illegal move in statistics

Uniform Convergence for Classes of Small Effective Size

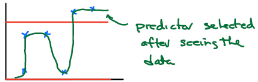
- For example
  a. Suppose I construct predictors
     $f_0 =$ all zero
     $f_1 =$ all one
     Then I evaluate the error by evaluating it over the data

Then we evaluate the true error

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- b Suppose, instead we look at the data first and the construct select our predictors. Then we might select



- The predictor that is selected will then look overfitted and the true error will not really be zero, even if it appears so.
- One way to select the predictor after seeing the data is to evaluate the predictor's true error $L_D(h)$ on a data set disjoint/independent from the data set S
  This is call the **double-sampling trick**

Uniform Convergence for Classes of Small Effective Size

- Evaluate h on a set S'

$$L_D(h) = \mathop{\mathbb{E}}_{S' \sim D^m} (L_{S'}(h))$$

then

$$\mathop{\mathbb{E}}_{S \sim D^m} \left[\sup_{h \in \mathcal{H}} |L_D(h) - L_S(h)| \right]$$

$$\mathop{\mathbb{E}}_{S \sim D^m} \left[\sup_{h \in \mathcal{H}} \left| \mathop{\mathbb{E}}_{S' \sim D^m} (L_{S'}(h)) - L_S(h) \right| \right]$$

$$\leq \mathop{\mathbb{E}}_{S \sim D^m} \sup_{h \in \mathcal{H}} \mathop{\mathbb{E}}_{S' \sim D^m} |(L_{S'}(h)) - L_S(h)|$$

(using triangle inequality $|x + y + z| \leq |x| + |y| + |z|$)

Uniform Convergence for Classes of Small Effective Size

$$\leq \underset{S\sim D^m}{\mathbb{E}} \underset{S'\sim D^m}{\mathbb{E}} \sup_{h\in\mathcal{H}} |(L_{S'}(h)) - L_S(h)|$$

$$\text{(using } \sup \mathbb{E} \leq \mathbb{E} \ \sup\text{)}$$

- we have reduced this to a double sample over S and S'
- Since $L_{S'}(h)$ and $L_S(h)$ are empirical errors

$$\underset{S\sim D^m}{\mathbb{E}} \left(\sup_{h\in\mathcal{H}} |L_D(h) - L_S(h)| \right)$$

$$\leq \underset{S',S\sim D^m}{\mathbb{E}} \left(\sup_{h\in\mathcal{H}} |L_{S'}(h) - L_S(h)| \right)$$

$$= \underset{S',S\sim D^m}{\mathbb{E}} \left(\sup \frac{1}{m} \,|\sum_{i=1}^{m} \ell(h, z_i') - \ell(h, z_i) \right) \ldots\ldots (1)$$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- the sample S

$$\{z'_1 = (x'_1, y'_1),\ z'_2 = (x'_2, y'_2)......, z'_m\}$$

and

$$\{z_1 = (x_1, y_1), ... \ z_m = (x_m, y_m)\}$$

are $2m$ vectors

- Our answer is symmetric in z_i and z_i' if we replace

$$z_i \leftrightarrow z_i'$$

then,

$$l(h, z_i') - l(h, z_i) \leftrightarrow -(l(h, z_i') - l(h, z_i))$$

We can write equation (1) as

$$\mathbb{E}_{S \in D^m} \sup_{h \in \mathcal{H}} |L_D(h) - L_S(h)|$$

$$\leq \mathbb{E}_{S, S' \sim D^m} \sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \delta_i (l(h, z_i') - l(h, z_i))$$

Uniform Convergence for Classes of Small Effective Size

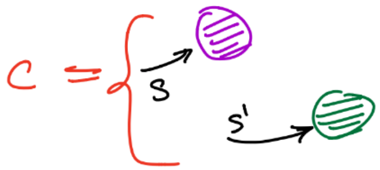
Here, $\delta_i = \pm 1$
$l(h, z_i) - l(h, z_i) = positive$
Here, The sign is isolated in $\delta_i = \{+1, -1\}$
Since we are looking for the *expected* value, we can overage over δ_i
Let the distribution for δ_i be $u_\pm$. Then we have equivalently,

$$\mathop{\mathbb{E}}_{\delta \sim u_\pm^m} \mathop{\mathbb{E}}_{S,S' \in D^m} SUP \frac{1}{m} |\sum_{i=1}^{m} \delta_i (l(h, z_i' - l(h, z_i))|$$

$$\mathop{\mathbb{E}}_{S,S' \in D^m} \mathop{\mathbb{E}}_{\delta \sim u_\pm^m} SUP \frac{1}{m} |\sum_{i=1}^{m} \delta_i (l(h, z_i' - l(h, z_i))|$$

Uniform Convergence for Classes of Small Effective Size

Now fix S and S. Let C be the set of S and S



- $|C| = 2m$
- Take the supremum over $h \epsilon \mathcal{H}_C$

Example of Supremum

-

$$G = \{\frac{1}{n} | n \in N\}$$

$$\text{SUP G} = \text{max value of the set G}$$
$$= \text{max G}$$

Uniform Convergence for Classes of Small Effective Size

- Sometimes the supremum of G isn't in the set G
  For example for,

$$G = \{X \mid X \in \mathbb{R} \text{ and } X < 1\}$$

  then $\text{SUP } G = 1,$ but $1 \notin G$
- when the SUP of G is in G then $\text{SUP } G = \max G$
- therefore

$$\mathop{\mathbb{E}}_{\delta \sim u_{\pm}^{m}} \left(\mathop{\text{SUP}}_{h \in \mathcal{H}} \frac{1}{m} \left| \sum_{i} \right| \right)$$

$$\mathop{\mathbb{E}}_{\delta \sim u_{\pm}^{m}} \left(\mathop{max}_{h \in \mathcal{H}_C} \frac{1}{m} \left| \sum_{i} \right| \right)$$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- now pick h in $\mathcal{H}_C$ and define

$$\theta_n = \frac{1}{m} \sum_{i=1}^{m} \sigma_i(\ell(h, z_i') - \ell(h, z_i))$$

- Since

$$\mathbb{E}(\theta_n) = 0$$

- θ_n is an average of variables
- range of $\theta_n \in [-1, +1]$
- we can then use Hoeffding's inequality for $e > 0$

$$\mathbb{P}[|\theta_n| > e] \leq 2e^{-2m\rho^2}$$

Uniform Convergence for Classes of Small Effective Size

- Apply the union bound

$$\mathbb{P}\left(\max_{h \in \mathcal{H}_C} |\theta_n| \geq \rho \right) \leq 2|\mathcal{H}_C| e^{-2m\rho^2}$$

- use Appendix A.4 if

$$\mathbb{P}(|X - X_0| > t) \leq 2be^{\frac{-t^2}{a^2}}$$

$$(\text{Here, } X_0 \in \mathbb{R}, t \geq 0, a > 0\ )$$

then

$$\mathbb{E}(|X - X_0|) \leq a\left(2 + \sqrt{\log b}\right)$$

$$(\text{Here, } b \geq e)$$

Uniform Convergence for Classes of Small Effective Size

- $$\underset{S\sim D^m}{\mathbb{E}}\left(\underset{h\in\mathcal{H}}{\text{SUP}}|L_D(h)-L_S(h)|\leq\frac{4+\sqrt{\log(\tau_{\mathcal{H}}(2m))}}{\sqrt{2m}}\right)$$
- Implications for m
  $\Rightarrow$ for $m > d$
  $$\tau_{\mathcal{H}}(2m)\leq\left(\frac{2em}{d}\right)^d$$
- $$\mathbb{E}(\text{SUP}|L_S(h)-L_D(h)|)$$
  $\rightarrow$ $=$ (probability of large$|L_S - L_D|$) $\times$ $|L_S - L_D|$

Uniform Convergence for Classes of Small Effective Size

- $$\delta|L_S(h) - L_D(h)| \leq \frac{4 + \sqrt{d \log\left(\frac{2em}{d}\right)}}{\sqrt{2m}}$$
- $$|L_S(h) - L_D(h)| \leq \frac{4 + \sqrt{d \log\left(\frac{2em}{d}\right)}}{\delta\sqrt{2m}}$$
- $$\text{Assume } \sqrt{d \log\left(\frac{2em}{d}\right)} \geq 4$$

$$\Rightarrow \ |L_S(h) - L_D(h)| \leq \frac{1}{\delta}\sqrt{\frac{2d \log\left(\frac{2em}{d}\right)}{m}}$$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- If

$$|L_S(h) - L_D(h)| \leq \epsilon$$

then

$$\frac{1}{\delta}\sqrt{\frac{2d\log\left(\frac{2em}{d}\right)}{m}} \leq \epsilon$$

$$\Rightarrow \quad \frac{1}{\delta^2}\frac{2d\log\frac{2em}{d}}{\epsilon^2} \leq m$$

Proof of Theorem 6.7

Uniform Convergence for Classes of Small Effective Size

- do some Algebra (lemma A.2)

$$m \geq 4\frac{2d}{(\epsilon\delta)^2}\log\frac{2d}{(\epsilon\delta)^2} + \frac{4d\log\frac{2e}{d}}{(\epsilon\delta)^2}$$

- Thus if $\mathrm{VCdim}(\mathcal{H}) = d < \infty$ then there is an m to give PAC learning