She is fairer than the sun and surpasses every constellation of the stars.

Compared to light, Wisdom is found more radiant.

From the greatness and the beauty of created things their original author, by analogy, is seen.

Admissible Heuristic for Polynomial Approximations of a Parametrized Metric on Attributed Multi-Hypergraphs

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Directed Attributed Multi-Hypergraph Definitions and Isomorphism Criteria

$$H \coloneqq (V,E,A,B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v\geq 1}\ v \in V \implies v-1 \in V, \quad E \subseteq \left(2^V - \{\varnothing\}\right) \times V \times \mathbb{N}_0, \quad \forall_{e \geq 1}\ (U,v,e) \in E \implies (U,v,e-1) \in E, \quad m_{\mathrm{e}} \coloneqq \max_{(U,v,e)\in E_1 \cup E_2} e, \quad A \coloneqq \{\alpha \mid \alpha : V \to \Gamma_\alpha\}, \quad B \coloneqq \{\beta \mid \beta : E \to \Gamma_\beta\}, \quad \overline{B} \coloneqq \left\{ \overline{\beta}_i : \left(2^{\mathbb{N}_0} - \{\varnothing\}\right) \times (\mathbb{N}_0)^2 \to \Gamma_\beta \cup \{\varepsilon_\beta\} \;\middle|\; \overline{\beta}_i(v) \coloneqq \begin{cases} \beta_1(U,v,e) & (U,v,e) \in E_1,\ i=1 \\ \beta_2(U,v,e) & (U,v,e)\in E_2,\ i=2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f:V_1 \leftrightarrow V_2,\ \left\{g_{U,v}:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0 \mid U \in \left(2^{V_1}-\{\varnothing\}\right),\ v \in V_1\right\}} \left(\left(\forall_{U\in\left(2^{V_1}-\{\varnothing\}\right),\ v\in V_1,\ e\in[m_{\mathrm{e}}]_0}\ (U,v,e)\in E_1 \iff \left(\overline{f}(U), f(v), e\right) \in E_2\right) \wedge \left(\forall_{v \in V_1}\ \forall_{\overline{\alpha}\in\overline{A}}\ \overline{\alpha}_1(v) = \overline{\alpha}_2(f(v))\right) \wedge \left(\forall_{U\in\left(2^{V_1}-\{\varnothing\}\right),\ v\in V_1,\ e\in[m_{\mathrm{e}}]_0}\ \forall_{\overline{\beta}\in\overline{B}}\ \overline{\beta}_1(U,v,e) = \overline{\beta}_2\left(\overline{f}(U), f(v), g_{U,v}(e)\right)\right)\right).$$

Undirected Attributed Multi-Hypergraph Definitions and Isomorphism Criteria

$$H \coloneqq (V,E,A,B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v\geq 1}\ v \in V \implies v-1 \in V, \quad E \subseteq \left(2^V - \{\varnothing\}\right) \times \mathbb{N}_0, \quad \forall_{e \geq 1}\ (U,e) \in E \implies (U,e-1) \in E, \quad m_{\mathrm{e}} \coloneqq \max_{(U,e)\in E_1 \cup E_2} e, \quad A \coloneqq \{\alpha \mid \alpha : V \to \Gamma_\alpha\}, \quad B \coloneqq \{\beta \mid \beta : E \to \Gamma_\beta\}, \quad \overline{B} \coloneqq \left\{ \overline{\beta}_i : \left(2^{\mathbb{N}_0} - \{\varnothing\}\right) \times \mathbb{N}_0 \to \Gamma_\beta \cup \{\varepsilon_\beta\} \;\middle|\; \overline{\beta}_i(v) \coloneqq \begin{cases} \beta_1(U,e) & (U,e) \in E_1,\ i=1 \\ \beta_2(U,e) & (U,e)\in E_2,\ i=2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f:V_1 \leftrightarrow V_2,\ \left\{g_U:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0 \mid U \in \left(2^{V_1}-\{\varnothing\}\right)\right\}} \left(\left(\forall_{U\in\left(2^{V_1}-\{\varnothing\}\right),\ e\in[m_{\mathrm{e}}]_0}\ (U,e)\in E_1 \iff \left(\overline{f}(U), e\right) \in E_2\right) \wedge \left(\forall_{v \in V_1}\ \forall_{\overline{\alpha}\in\overline{A}}\ \overline{\alpha}_1(v) = \overline{\alpha}_2(f(v))\right) \wedge \left(\forall_{U\in\left(2^{V_1}-\{\varnothing\}\right),\ e\in[m_{\mathrm{e}}]_0}\ \forall_{\overline{\beta}\in\overline{B}}\ \overline{\beta}_1(U,e) = \overline{\beta}_2\left(\overline{f}(U), g_{U,v}(e)\right)\right)\right).$$

Definitions Common to Both

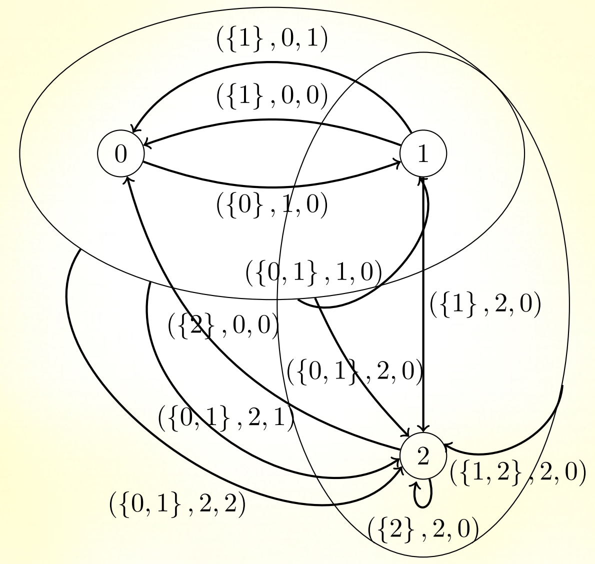
$$P \coloneqq \left\{\rho_{\Gamma_\gamma^\varepsilon} : \left(\Gamma_\gamma^\varepsilon\right)^2 \to \mathbb{R}_0^+ \;\middle|\; \gamma \in A_1 \cup A_2 \cup B_1 \cup B_2,\ \rho \text{ is a metric on the set } \Gamma_\gamma^\varepsilon \equiv \Gamma_{\overline{\gamma}} \coloneqq \Gamma_\gamma \cup \{\varepsilon_\gamma\}\right\}, \quad \overline{A} \coloneqq \left\{\overline{\alpha}_i : \mathbb{N}_0 \to \Gamma_\alpha \cup \{\varepsilon_\alpha\} \;\middle|\; \overline{\alpha}_i(v) \coloneqq \begin{cases} \alpha_1(v) & v \in V_1,\ i=1 \\ \alpha_2(v) & v\in V_2,\ i=2 \\ \varepsilon_\alpha & \text{otherwise}\end{cases}\right\}, \quad m_{\mathrm{v}} \coloneqq \max\{|V_1|, |V_2|\} - 1, \quad \forall_{n,k\in\mathbb{N}_0}\ [n]_k \coloneqq \bigcup_{i=k}^{n} \{i\}, \quad f : A \to B \implies \overline{f} : 2^A \to 2^B,\ \ \overline{f}(C) \coloneqq \{f(c) \mid c \in C\}.$$

Parametrized Metric for Directed and Undirected Attributed Multi-Hypergraphs: Complete and Permute Attributed Multi-Hypergraph Distance

$$\rho(H_1,H_2,P) \coloneqq \min_{f:[m_{\mathrm{v}}]_0 \leftrightarrow [m_{\mathrm{v}}]_0} \left(\sum_{v=0}^{m_{\mathrm{v}}} \sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f(v))\right)\right) + \sum_{U\in\left(2^{V_1\cup V_2}-\{\varnothing\}\right)} \sum_{v=0}^{m_{\mathrm{v}}} \left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U,v,e), \overline{\beta}_2\left(\overline{f}(U), f(v), g(e)\right)\right)\right), \quad \rho(H_1,H_2,P) \coloneqq \min_{f:[m_{\mathrm{v}}]_0 \leftrightarrow [m_{\mathrm{v}}]_0} \left(\sum_{v=0}^{m_{\mathrm{v}}} \sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f(v))\right)\right) + \sum_{U\in\left(2^{V_1\cup V_2}-\{\varnothing\}\right)} \left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U,e), \overline{\beta}_2\left(\overline{f}(U), g(e)\right)\right)\right).$$

Example

$$H_{\text{example}} \coloneqq \left(\{0,1,2\},\ \{(\{0\},1,0), (\{1\},0,0), (\{1\},0,1), (\{1\},2,0), (\{2\},0,0), (\{2\},2,0), (\{0,1\},1,0), (\{0,1\},2,0), (\{0,1\},2,1), (\{0,1\},2,2), (\{1,2\},2,0)\},\ \left\{\text{value}(v) \coloneqq \sqrt{v+\sqrt{v}}\right\} \cup \{\text{exists}(v) \equiv 1\},\ \left\{\text{vector}(U,v,e) \coloneqq \left(\cosh\left(\frac{v^7}{\ln(1+|U|)}\right),\ \exp\left(-\frac{\pi}{2} + e\cdot|U\cap\{v\}|\right),\ \textstyle\sum_{u\in U\cup\{v\}} \sqrt[77]{\text{value}(u)+e}\right)\right\}\right).$$



Deterministic Iterative Polynomial Approximations for Complete and Permute Directed and Undirected Attributed Multi-Hypergraph Distance Metric: an Orb-Weaver Elimination Heuristic

$$X_0 \coloneqq \varnothing, \quad X_{n+1} \coloneqq X_n \cup \left\{ \operatorname*{argmax}_{\left\{(u,f_u)\in\left([m_{\mathrm{v}}]_0\right)^2 - X_n \;\middle|\; \left(\exists_{x\in[m_{\mathrm{v}}]_0-\{u\}}\ (x,f_u)\notin X_n\right) \wedge \left(\exists_{x\in[m_{\mathrm{v}}]_0-\{f_u\}}\ (u,x)\notin X_n\right)\right\}} \rho\left(H_1,H_2,P,X_n \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{u\}} (x,f_u)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_u\}} (u,x)\right)\right)\right\},$$

$$RCD_0 \coloneqq \varnothing, \quad RCD_{n+1} \coloneqq RCD_n \cup \left\{ \lim_{\epsilon\to 0^+} \operatorname*{argmax}_{\left\{(u,f_u)\in\left([m_{\mathrm{v}}]_0\right)^2 - RCD_n \;\middle|\; \left(\exists_{x\in[m_{\mathrm{v}}]_0-\{u\}}\ (x,f_u)\notin RCD_n\right) \wedge \left(\exists_{x\in[m_{\mathrm{v}}]_0-\{f_u\}}\ (u,x)\notin RCD_n\right)\right\}} \max\{R_n(u,f_u), C_n(u,f_u), D_n(u,f_u)\} + \epsilon\cdot\rho\left(H_1,H_2,P,RCD_n \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{u\}} (x,f_u)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_u\}} (u,x)\right)\right) + \epsilon^2\cdot D_n(u,f_u) + \epsilon^3 \cdot R_n(u,f_u) + \epsilon^4 \cdot C_n(u,f_u)\right\},$$

$$R_n(u,f_u) \coloneqq \max_{v\in[m_{\mathrm{v}}]_0-\{u\}} \min_{\left\{f_v\in[m_{\mathrm{v}}]_0-\{f_u\} \;\middle|\; (v,f_v)\notin RCD_n\right\}} \rho\left(H_1,H_2,P,RCD_n \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{u\}} (x,f_u)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_u\}} (u,x)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f_v)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_v\}} (v,x)\right)\right),$$

$$C_n(u,f_u) \coloneqq \max_{f_v\in[m_{\mathrm{v}}]_0-\{f_u\}} \min_{\left\{v\in[m_{\mathrm{v}}]_0-\{u\} \;\middle|\; (v,f_v)\notin RCD_n\right\}} \rho\left(H_1,H_2,P,RCD_n \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{u\}} (x,f_u)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_u\}} (u,x)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f_v)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_v\}} (v,x)\right)\right),$$

$$D_n(u,f_u) \coloneqq \min_{f:\left([m_{\mathrm{v}}]_0-\{u\}\right)\leftrightarrow\left([m_{\mathrm{v}}]_0-\{f_u\}\right)} \sum_{v\in\left([m_{\mathrm{v}}]_0-\{u\}\right)} \frac{1}{m_{\mathrm{v}}}\cdot\rho\left(H_1,H_2,P,RCD_n \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{u\}} (x,f_u)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_u\}} (u,x)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f(v))\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f(v)\}} (v,x)\right)\right),$$

$$M_{\mathrm{X}} \coloneqq \left([m_{\mathrm{v}}]_0\right)^2 - X_{(m_{\mathrm{v}}+1)^2-(m_{\mathrm{v}}+1)}, \quad M_{\mathrm{RCD}} \coloneqq \left([m_{\mathrm{v}}]_0\right)^2 - RCD_{(m_{\mathrm{v}}+1)^2-(m_{\mathrm{v}}+1)}, \quad \exists_{X\subseteq\left([m_{\mathrm{v}}]_0\right)^2,\ |X|=(m_{\mathrm{v}}+1)^2-(m_{\mathrm{v}}+1)}\ \rho(H_1,H_2,P) = \rho(H_1,H_2,P,X),$$

$$\rho(H_1,H_2,P) \geq RCD \coloneqq \max\left\{ \max_{v\in[m_{\mathrm{v}}]_0} \min_{f_v\in[m_{\mathrm{v}}]_0} \rho\left(H_1,H_2,P,\left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f_v)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_v\}} (v,x)\right)\right),\ \max_{f_v\in[m_{\mathrm{v}}]_0} \min_{v\in[m_{\mathrm{v}}]_0} \rho\left(H_1,H_2,P,\left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f_v)\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f_v\}} (v,x)\right)\right),\ \min_{f:[m_{\mathrm{v}}]_0\leftrightarrow[m_{\mathrm{v}}]_0} \sum_{v\in[m_{\mathrm{v}}]_0} \frac{1}{m_{\mathrm{v}}+1}\cdot\rho\left(H_1,H_2,P,\left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{v\}} (x,f(v))\right) \cup \left(\bigcup_{x\in[m_{\mathrm{v}}]_0-\{f(v)\}} (v,x)\right)\right)\right\} \geq \rho(H_1,H_2,P,\varnothing).$$

Admissible Heuristic for a Parametrized Metric on Directed Attributed Multi-Hypergraphs

$$
\begin{aligned}
\rho(H_1, H_2, P, X) \coloneqq \sup_{b_\varepsilon\in\mathbb{R}} \; & \sup_{\substack{\mathrm{a,b,c,d}:(V_1\cup V_2)^4\to\mathbb{R} \\ \left(u\notin N^+_{E_1}(v) \vee f_u\notin N_{E_2}(f_v)\right) \implies (\mathrm{a}(v,u,f_v,f_u)=\mathrm{a}(u,v,f_u,f_v)=\mathrm{c}(v,u,f_v,f_u)=\mathrm{c}(u,v,f_u,f_v)=\mathrm{d}(v,u,f_v,f_u)=\mathrm{d}(u,v,f_u,f_v)=1 \,\wedge\, \mathrm{b}(v,u,f_v,f_u)=\mathrm{b}(u,v,f_u,f_v)=b_\varepsilon)}} \\
& \min_{\substack{f_\mathrm{v}:[m_\mathrm{v}]_0\leftrightarrow[m_\mathrm{v}]_0 \\ \forall_{(x,f_x)\in X}\; f_\mathrm{v}(x)\neq f_x}} \sum_{v=0}^{m_\mathrm{v}} \left(\left(\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_\mathrm{v}(v))\right)\right) + \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},v,e), \overline{\beta}_2(\{f_\mathrm{v}(v)\}, f_\mathrm{v}(v), g(e))\right)\right) + rs^\mathrm{e}_{\min}(v, f_\mathrm{v}(v), v, f_\mathrm{v}(v))\right) \cdot \left(1 - \frac{\min\left\{m_\mathrm{v}, \left|N^+_{E_1}(v)-\{v\}\right| + \left|N_{E_2}(f_\mathrm{v}(v))-\{f_\mathrm{v}(v)\}\right|\right\}}{|V_1\cup V_2|}\right) + \right. \\
& \min_{\substack{f_\mathrm{u}:\Psi\leftrightarrow\Omega \\ [m_\mathrm{v}]_0-\{v\}\supseteq\Psi\supseteq N^+_{E_1}(v)-\{v\} \\ [m_\mathrm{v}]_0-\{f_\mathrm{v}(v)\}\supseteq\Omega\supseteq N_{E_2}(f_\mathrm{v}(v))-\{f_\mathrm{v}(v)\} \\ |\Psi|=|\Omega|=\min\left\{m_\mathrm{v}, \left|N^+_{E_1}(v)-\{v\}\right| + \left|N_{E_2}(f_\mathrm{v}(v))-\{f_\mathrm{v}(v)\}\right|\right\}}} \sum_{u\in\Psi}
\begin{cases}
0 & u\notin N^+_{E_1}(v) \;\wedge\; f_\mathrm{u}(u)\notin N_{E_2}(f_\mathrm{v}(v)) \\[6pt]
+\infty & u\in N^+_{E_1}(v) \;\wedge\; f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v)) \;\wedge\; (u, f_\mathrm{u}(u))\in X \\[6pt]
\left(\begin{gathered}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_\mathrm{v}(v))\right)\right) \cdot \frac{1+\mathrm{a}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(u), \overline{\alpha}_2(f_\mathrm{u}(u))\right)\right) \cdot \frac{1-\mathrm{a}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},v,e), \overline{\beta}_2(\{f_\mathrm{u}(u)\}, f_\mathrm{v}(v), g(e))\right)\right) \cdot \frac{1+\mathrm{b}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},u,e), \overline{\beta}_2(\{f_\mathrm{v}(v)\}, f_\mathrm{u}(u), g(e))\right)\right) \cdot \frac{1-\mathrm{b}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},v,e), \overline{\beta}_2(\{f_\mathrm{v}(v)\}, f_\mathrm{v}(v), g(e))\right)\right) \cdot \frac{1+\mathrm{c}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},u,e), \overline{\beta}_2(\{f_\mathrm{u}(u)\}, f_\mathrm{u}(u), g(e))\right)\right) \cdot \frac{1-\mathrm{c}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\
rs^\mathrm{e}_{\min}(v, f_\mathrm{v}(v), v, f_\mathrm{v}(v)) \cdot \frac{1+\mathrm{d}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\
rs^\mathrm{e}_{\min}(u, f_\mathrm{u}(u), u, f_\mathrm{u}(u)) \cdot \frac{1-\mathrm{d}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\
rs^\mathrm{e}_{\min}(u, f_\mathrm{u}(u), v, f_\mathrm{v}(v))
\end{gathered}\right) & u\in N^+_{E_1}(v) \;\wedge\; f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v)) \;\wedge\; (u, f_\mathrm{u}(u))\notin X \\[6pt]
\left(\begin{gathered}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_\mathrm{v}(v))\right)\right) \cdot \frac{1}{|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},v,e), \overline{\beta}_2(\{f_\mathrm{u}(u)\}, f_\mathrm{v}(v), g(e))\right)\right) \cdot \frac{1+b_\varepsilon}{2} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},u,e), \overline{\beta}_2(\{f_\mathrm{v}(v)\}, f_\mathrm{u}(u), g(e))\right)\right) \cdot \frac{1-b_\varepsilon}{2} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},v,e), \overline{\beta}_2(\{f_\mathrm{v}(v)\}, f_\mathrm{v}(v), g(e))\right)\right) \cdot \frac{1}{|V_1\cup V_2|} + \\
rs^\mathrm{e}_{\min}(v, f_\mathrm{v}(v), v, f_\mathrm{v}(v)) \cdot \frac{1}{|V_1\cup V_2|} + \\
rs^\mathrm{e}_{\min}(u, f_\mathrm{u}(u), v, f_\mathrm{v}(v))
\end{gathered}\right) & \text{otherwise}
\end{cases}
\end{aligned}
$$

$$
rs^\mathrm{e}_{\min}(u_0, f_{u_0}, v, f_v) \coloneqq
\begin{cases}
r^{\mathrm{e}+\infty}_{\min}(u_0+1, f_{u_0}+1, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \forall_{i\in[m_\mathrm{v}]_0}\; (\{v\}, i, 0)\notin E_1 \wedge (\{i\}, v, 0)\notin E_1 \wedge (\{f_v\}, f_\mathrm{u}(i), 0)\notin E_2 \wedge (\{f_\mathrm{u}(i)\}, f_v, 0)\notin E_2 \\[6pt]
r^{\mathrm{e}+\infty}_{\min}(u_0+1, 0, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \text{otherwise}
\end{cases}
$$

$$
r^{\mathrm{e}+\infty}_{\min}(s, f_s, U, f_U, v, f_v, j) \coloneqq \min_{\substack{f_j:[m_\mathrm{v}]_s \xrightarrow{1:1} [m_\mathrm{v}]_{f_s} - f_U \\ v\geq s \implies f_j(v) := f_v}}
\begin{cases}
\sum_{i=s}^{m_\mathrm{v}} r^{\mathrm{e}+\infty}_{\min}(i+1, f_j(i)+1, U\cup\{i\}, f_U\cup\{f_j(i)\}, v, f_v, j+1) & \forall_{s\leq i\leq m_\mathrm{v}}\; (U\cup\{i\}, v, 0)\notin E_1 \wedge (f_U\cup\{f_j(i)\}, f_v, 0)\notin E_2 \\[6pt]
\sum_{i=s}^{m_\mathrm{v}} \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U\cup\{i\}, v, e), \overline{\beta}_2(f_U\cup\{f_j(i)\}, f_v, g(e))\right)\right) + r^{\mathrm{e}+\infty}_{\min}(i+1, 0, U\cup\{i\}, f_U\cup\{f_j(i)\}, v, f_v, j+1) & \text{otherwise}
\end{cases}
$$

$$
N^+_E(v) \coloneqq \left\{u\in V_1\cup V_2 \;\middle|\; (\{v\}, u, 0)\in E \vee \left(\exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u\}}\; (\forall_{w\in U_0}\; w\geq u) \wedge (U_0, v, 0)\in E\right)\right\}
$$
$$
N_E(v) \coloneqq \left\{u\in V_1\cup V_2 \;\middle|\; (\{v\}, u, 0)\in E \vee \left(\exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u\}}\; (U_0, v, 0)\in E\right)\right\}
$$

$$
c(H_1, H_2) = \mathcal{O}\left(\left\lceil\frac{|V_1\cup V_2|^2}{p}\right\rceil \cdot \left(|V_1\cup V_2|^3 + |V_1\cup V_2|^{2+2\cdot\left(\max_{(U,v,0)\in(E_1\cup E_2)}|U|\right)} \cdot \left(\max_{(U,v,e)\in E_1\cup E_2} e\right)^3\right)\right), \quad \min_{f:A\xrightarrow{1:1}B} \sum_{a\in A}\gamma(a, f(a)) \coloneqq \min_{\substack{f:A\cup C\leftrightarrow B \\ |A\cup C|=|B| \\ A\cap C=\varnothing}} \sum_{a\in A\cup C} \begin{cases}\gamma(a, f(a)) & a\in A \\ 0 & a\in C\end{cases}, \quad \min_{\substack{f:[n]_0\leftrightarrow[n]_0 \\ \forall_{(x,f_x)\in X}\; f(x)\neq f_x}} \sum_{k=0}^{n}\gamma(k, f(k)) \coloneqq \min_{f:[n]_0\leftrightarrow[n]_0} \sum_{k=0}^{n} \begin{cases}\gamma(k, f(k)) & (k, f(k))\notin X \\ +\infty & (k, f(k))\in X\end{cases}.
$$

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Admissible Heuristic for a Parametrized Metric on Undirected Attributed Multi-Hypergraphs

$$\rho(H_1,H_2,P,X) \coloneqq \sup_{\substack{\mathrm{a},\mathrm{b}:(V_1\cup V_2)^4\to\mathbb{R} \\ \left(u\notin N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\vee f_u\notin N_{E_2}(f_v)\right)\implies(\mathrm{a}(v,u,f_v,f_u)=\mathrm{a}(u,v,f_u,f_v)=\mathrm{b}(v,u,f_v,f_u)=\mathrm{b}(u,v,f_u,f_v)=1)}} \min_{\substack{f_\mathrm{v}:[m_\mathrm{v}]_0\leftrightarrow[m_\mathrm{v}]_0 \\ \forall_{(x,f_x)\in X}\ f_\mathrm{v}(x)\neq f_x}} \sum_{v=0}^{m_\mathrm{v}} \left(\left(\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right) + \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\right)\cdot\left(1-\frac{\min\left\{m_\mathrm{v}-v,\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\}}{|V_1\cup V_2|}\right) + \right.$$

$$\left.\min_{\substack{f_\mathrm{u}:\Psi\xrightarrow{1:1}\Omega \\ [m_\mathrm{v}]_{v+1}\supseteq\Psi\supseteq\left(N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right) \\ \left([m_\mathrm{v}]_0-\{f_\mathrm{v}(v)\}\right)\supseteq\Omega\supseteq N_{E_2}(f_\mathrm{v}(v)) \\ |\Psi|=\min\left\{m_\mathrm{v}-v,\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\} \\ |\Omega|=\min\left\{m_\mathrm{v},\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\}}} \sum_{u\in\Psi} \begin{cases} 0 & u\notin N^+_{E_1}(v)\ \wedge\ f_\mathrm{u}(u)\notin N_{E_2}(f_\mathrm{v}(v)) \\ +\infty & u\in N^+_{E_1}(v)\ \wedge\ f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v))\ \wedge\ (u,f_\mathrm{u}(u))\in X \\ \left(\begin{array}{c} \left(\sum_{\overline{\alpha}\in\overline{A}}\rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right)\cdot\frac{1+\mathrm{a}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\ \left(\sum_{\overline{\alpha}\in\overline{A}}\rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(u),\overline{\alpha}_2(f_\mathrm{u}(u))\right)\right)\cdot\frac{1-\mathrm{a}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\ \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\cdot\frac{1+\mathrm{b}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\ \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},e),\overline{\beta}_2(\{f_\mathrm{u}(u)\},g(e))\right)\right)\cdot\frac{1-\mathrm{b}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\ \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u,v\},e),\overline{\beta}_2(\{f_\mathrm{u}(u),f_\mathrm{v}(v)\},g(e))\right)\right) + \\ rs^{\mathrm{e}}_{\min}(u,f_\mathrm{u}(u),v,f_\mathrm{v}(v)) \end{array}\right) & u\in N^+_{E_1}(v)\ \wedge\ f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v))\ \wedge\ (u,f_\mathrm{u}(u))\notin X \\ \left(\begin{array}{c} \left(\sum_{\overline{\alpha}\in\overline{A}}\rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right)\cdot\frac{1}{|V_1\cup V_2|} + \\ \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\cdot\frac{1}{|V_1\cup V_2|} + \\ \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u,v\},e),\overline{\beta}_2(\{f_\mathrm{u}(u),f_\mathrm{v}(v)\},g(e))\right)\right) + \\ rs^{\mathrm{e}}_{\min}(u,f_\mathrm{u}(u),v,f_\mathrm{v}(v)) \end{array}\right) & \text{otherwise} \end{cases}\right)$$

$$rs^{\mathrm{e}}_{\min}(u_0,f_{u_0},v,f_v) \coloneqq \begin{cases} r^{\mathrm{e}+\infty}_{\min}\left(u_0+1,f_{u_0}+1,\{u_0\},\{f_{u_0}\},v,f_v,1\right) & \forall_{i\in[m_\mathrm{v}]_0}\ \left(\{v,i\},0\right)\notin E_1\ \wedge\ \left(\{f_v,f_\mathrm{u}(i)\},0\right)\notin E_2 \\ r^{\mathrm{e}+\infty}_{\min}\left(u_0+1,0,\{u_0\},\{f_{u_0}\},v,f_v,1\right) & \text{otherwise} \end{cases}$$

$$r^{\mathrm{e}+\infty}_{\min}(s,f_s,U,f_U,v,f_v,j) \coloneqq \min_{f_j:\left([m_\mathrm{v}]_s-\{v\}\right)\xrightarrow{1:1}\left([m_\mathrm{v}]_{f_s}-f_U-\{f_v\}\right)} \begin{cases} \sum_{i=s}^{m_\mathrm{v}} r^{\mathrm{e}+\infty}_{\min}\left(i+1,f_j(i)+1,U\cup\{i\},f_U\cup\{f_j(i)\},v,f_v,j+1\right) & \forall_{s\leq i\leq m_\mathrm{v}}\ \left(U\cup\{i,v\},0\right)\notin E_1\ \wedge\ \left(f_U\cup\{f_j(i),f_v\},0\right)\notin E_2 \\ \sum_{i=s}^{m_\mathrm{v}} \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U\cup\{i,v\},e),\overline{\beta}_2(f_U\cup\{f_j(i),f_v\},g(e))\right)\right) + r^{\mathrm{e}+\infty}_{\min}\left(i+1,0,U\cup\{i\},f_U\cup\{f_j(i)\},v,f_v,j+1\right) & \text{otherwise} \end{cases}$$

$$N^+_E(v) \coloneqq \left\{u\in(V_1\cup V_2)-\{v\}\ \middle|\ \exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u,v\}}\ \left(\forall_{w\in U_0-\{v\}}\ w\geq u\right)\ \wedge\ (U_0,0)\in E\right\},$$
$$N_E(v) \coloneqq \left\{u\in(V_1\cup V_2)-\{v\}\ \middle|\ \exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u,v\}}\ (U_0,0)\in E\right\},$$

$$c(H_1,H_2)=\mathcal{O}\left(\left\lceil\frac{|V_1\cup V_2|^2}{p}\right\rceil\cdot\left(|V_1\cup V_2|^3+|V_1\cup V_2|^{2+2\cdot\left(\max_{(U,0)\in(E_1\cup E_2)}|U|\right)}\cdot\left(\max_{(U,e)\in E_1\cup E_2}e\right)^3\right)\right),\qquad \min_{f:A\xrightarrow{1:1}B}\sum_{a\in A}\gamma(a,f(a)) \coloneqq \min_{\substack{f:A\cup C\leftrightarrow B \\ |A\cup C|=|B| \\ A\cap C=\varnothing}}\sum_{a\in A\cup C}\begin{cases}\gamma(a,f(a)) & a\in A \\ 0 & a\in C\end{cases},\qquad \min_{\substack{f:[n]_0\leftrightarrow[n]_0 \\ \forall_{(x,f_x)\in X}\ f(x)\neq f_x}}\sum_{k=0}^{n}\gamma(k,f(k)) \coloneqq \min_{f:[n]_0\leftrightarrow[n]_0}\sum_{k=0}^{n}\begin{cases}\gamma(k,f(k)) & (k,f(k))\notin X \\ +\infty & (k,f(k))\in X\end{cases}.$$

Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Directed Attributed Multi-Hypergraphs

$$\mathrm{a}_0\left(u,v,f_u,f_v\right) = \mathrm{c}_0\left(u,v,f_u,f_v\right) = \mathrm{d}_0\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} 1 & u \notin N^+_{E_1}(v) \vee v \notin N^+_{E_1}(u) \vee f_u \notin N_{E_2}\left(f_v\right) \\ 0 & \text{otherwise} \end{cases}, \quad \forall_{u,v,f_u,f_v \in [m_{\mathrm{v}}]_0} \;\; \mathrm{b}_0\left(u,v,f_u,f_v\right) = b_{\varepsilon,0} \coloneqq 0,$$

$$\mathrm{a}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{a}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2 \cdot \left|V_1 \cup V_2\right|} \cdot \left(\sum\limits_{\overline{\alpha} \in \overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1\left(v\right), \overline{\alpha}_2\left(f_v\right)\right)\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \wedge v \in N^+_{E_1}(u) \wedge u \neq v \wedge f_u \in N_{E_2}\left(f_v\right) \\ 1 & \text{otherwise} \end{cases},$$

$$\mathrm{b}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{b}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2} \cdot \left(\min\limits_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum\limits_{e=0}^{m_{\mathrm{e}}} \sum\limits_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1\left(\{u\}, v, e\right), \overline{\beta}_2\left(\{f_u\}, f_v, g(e)\right)\right)\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \wedge v \in N^+_{E_1}(u) \wedge u \neq v \wedge f_u \in N_{E_2}\left(f_v\right) \\ b_{\varepsilon,n} & \text{otherwise} \end{cases},$$

$$\mathrm{c}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{c}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2 \cdot \left|V_1 \cup V_2\right|} \cdot \left(\min\limits_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum\limits_{e=0}^{m_{\mathrm{e}}} \sum\limits_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1\left(\{v\}, v, e\right), \overline{\beta}_2\left(\{f_v\}, f_v, g(e)\right)\right)\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \wedge v \in N^+_{E_1}(u) \wedge u \neq v \wedge f_u \in N_{E_2}\left(f_v\right) \\ 1 & \text{otherwise} \end{cases},$$

$$\mathrm{d}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{d}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2 \cdot \left|V_1 \cup V_2\right|} \cdot rs^{\mathrm{e}}_{\min}\left(v, f_v, v, f_v\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \wedge v \in N^+_{E_1}(u) \wedge u \neq v \wedge f_u \in N_{E_2}\left(f_v\right) \\ 1 & \text{otherwise} \end{cases},$$

$$b_{\varepsilon,n+1} \coloneqq b_{\varepsilon,n} + \frac{\eta_{n+1}}{2} \cdot \left(\sum_{v=0}^{m_{\mathrm{v}}} \sum_{\left\{u \in \Psi \;\middle|\; u \notin N^+_{E_1}(v) \vee v \notin N^+_{E_1}(u) \vee f^{\min}_{\mathrm{u},n}(u) \notin N_{E_2}\left(f^{\min}_{\mathrm{v},n}(v)\right)\right\}} \left(\min_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1\left(\{u\}, v, e\right), \overline{\beta}_2\left(\left\{f^{\min}_{\mathrm{u},n}(u)\right\}, f^{\min}_{\mathrm{v},n}(v), g(e)\right)\right)\right) - \left(\min_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1\left(\{v\}, u, e\right), \overline{\beta}_2\left(\left\{f^{\min}_{\mathrm{v},n}(v)\right\}, f^{\min}_{\mathrm{u},n}(u), g(e)\right)\right)\right)\right).$$

Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Undirected Attributed Multi-Hypergraphs

$$\mathrm{a}_0\left(u,v,f_u,f_v\right) = \mathrm{b}_0\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} 1 & u \notin N^+_{E_1}(v) \cap [m_{\mathrm{v}}]_{v+1} \vee v \notin N^+_{E_1}(u) \cap [m_{\mathrm{v}}]_{u+1} \vee f_u \notin N_{E_2}\left(f_v\right) \\ 0 & \text{otherwise} \end{cases},$$

$$\mathrm{a}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{a}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2 \cdot \left|V_1 \cup V_2\right|} \cdot \left(\sum\limits_{\overline{\alpha} \in \overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1\left(v\right), \overline{\alpha}_2\left(f_v\right)\right)\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \cap [m_{\mathrm{v}}]_{v+1} \wedge v \in N^+_{E_1}(u) \cap [m_{\mathrm{v}}]_{u+1} \wedge f_u \in N_{E_2}\left(f_v\right) \\ 1 & \text{otherwise} \end{cases},$$

$$\mathrm{b}_{n+1}\left(u,v,f_u,f_v\right) \coloneqq \begin{cases} \left(\mathrm{b}_n\left(u,v,f_u,f_v\right) + \frac{\eta_{n+1}}{2 \cdot \left|V_1 \cup V_2\right|} \cdot \left(\min\limits_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum\limits_{e=0}^{m_{\mathrm{e}}} \sum\limits_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1\left(v, e\right), \overline{\beta}_2\left(f_v, g(e)\right)\right)\right) \cdot \begin{cases} 1 & f_u = f^{\min}_{\mathrm{u},n}(u) \wedge f_v = f^{\min}_{\mathrm{v},n}(v) \wedge \left(f_v \neq f^{\min}_{\mathrm{u},n}(v) \vee f_u \neq f^{\min}_{\mathrm{v},n}(u)\right) \\ -1 & f_v = f^{\min}_{\mathrm{u},n}(v) \wedge f_u = f^{\min}_{\mathrm{v},n}(u) \wedge \left(f_u \neq f^{\min}_{\mathrm{u},n}(u) \vee f_v \neq f^{\min}_{\mathrm{v},n}(v)\right) \\ 0 & \text{otherwise} \end{cases}\right) & u \in N^+_{E_1}(v) \cap [m_{\mathrm{v}}]_{v+1} \wedge v \in N^+_{E_1}(u) \cap [m_{\mathrm{v}}]_{u+1} \wedge f_u \in N_{E_2}\left(f_v\right) \\ 1 & \text{otherwise} \end{cases}.$$

For independent comparison, the following is a notation-adjusted discovery by prof. Kaspar Riesen (h-index: 32) and prof. Horst Bunke (h-index: 96) published in 2009 having 737 citations as of September 2022

$$\rho_{\text{(Riesen, Bunke)}}\left(G_1, G_2\right) \coloneqq \min_{f_{\mathrm{v}}:\left[|V_1|+|V_2|-1\right]_0 \leftrightarrow \left[|V_1|+|V_2|-1\right]_0} \sum_{v=0}^{|V_1|+|V_2|-1} c_{\text{vertex}}\left(\overline{\alpha}_1\left(v\right), \overline{\alpha}_2\left(f_{\mathrm{v}}(v)\right)\right) + \left(\min_{\substack{f_{\mathrm{u}}:\left[|V_1|+|V_2|-1\right]_0 \leftrightarrow \left[|V_1|+|V_2|-1\right]_0 \\ f_{\mathrm{u}}(v) \coloneqq f_{\mathrm{v}}(v) \text{ (optimization not specified in paper)}}} \sum_{u=0}^{|V_1|+|V_2|-1} c_{\text{edge}}\left(\overline{\beta}_1\left(v, u\right), \overline{\beta}_2\left(f_{\mathrm{v}}(v), f_{\mathrm{u}}(u)\right)\right)\right).$$

If $c_{\textbf{vertex}}$ and $c_{\textbf{edge}}$ are metric functions, $\rho_{\textbf{(Riesen, Bunke)}}$ becomes a special case of ρ:

$$\rho\left(G_1, G_2, P\right) \geq \rho\left(G_1, G_2, P, \varnothing\right) \geq \rho_{\text{(Riesen, Bunke)}}\left(G_1, G_2\right)$$