Admissible Heuristic for Polynomial Approximations of a Parametrized Metric on Attributed Multi-Hypergraphs

inż. Kazimierz Wojciechowski, prof. dr hab. inż. Władysław Homenda — September 2022 — Faculty of Mathematics and Information Science — Warsaw University of Technology

Directed Attributed Multi-Hypergraph Definitions and Isomorphism Criteria

$$H \coloneqq (V, E, A, B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v \geq 1} \; v \in V \implies v-1 \in V, \quad E \subseteq \left(2^V - \{\varnothing\}\right) \times V \times \mathbb{N}_0, \quad \forall_{e \geq 1} \; (U, v, e) \in E \implies (U, v, e-1) \in E, \quad m_\mathrm{e} \coloneqq \max_{(U,v,e) \in E_1 \cup E_2} e, \quad A \coloneqq \{\alpha \mid \alpha : V \to \Gamma_\alpha\}, \quad B \coloneqq \{\beta \mid \beta : E \to \Gamma_\beta\}, \quad \overline{B} \coloneqq \left\{ \overline{\beta}_i : \left(2^{\mathbb{N}_0} - \{\varnothing\}\right) \times (\mathbb{N}_0)^2 \to \Gamma_\beta \cup \{\varepsilon_\beta\} \;\middle|\; \overline{\beta}_i(v) \coloneqq \begin{cases} \beta_1(U,v,e) & (U,v,e) \in E_1, \; i=1 \\ \beta_2(U,v,e) & (U,v,e) \in E_2, \; i=2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f : V_1 \leftrightarrow V_2, \; \left\{g_{U,v} : [m_\mathrm{e}]_0 \leftrightarrow [m_\mathrm{e}]_0 \;\mid\; U \in \left(2^{V_1} - \{\varnothing\}\right), \; v \in V_1\right\}} \left(\left(\forall_{U \in \left(2^{V_1} - \{\varnothing\}\right), \; v \in V_1, \; e \in [m_\mathrm{e}]_0} \; (U, v, e) \in E_1 \iff \left(\overline{f}(U), f(v), e\right) \in E_2\right) \wedge \left(\forall_{v \in V_1} \forall_{\overline{\alpha} \in \overline{A}} \; \overline{\alpha}_1(v) = \overline{\alpha}_2(f(v))\right) \wedge \left(\forall_{U \in \left(2^{V_1} - \{\varnothing\}\right), \; v \in V_1, \; e \in [m_e]_0} \forall_{\overline{\beta} \in \overline{B}} \; \overline{\beta}_1(U, v, e) = \overline{\beta}_2\left(\overline{f}(U), f(v), g_{U,v}(e)\right)\right)\right).$$

Undirected Attributed Multi-Hypergraph Definitions and Isomorphism Criteria

$$H \coloneqq (V, E, A, B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v \geq 1} \; v \in V \implies v-1 \in V, \quad E \subseteq \left(2^V - \{\varnothing\}\right) \times \mathbb{N}_0, \quad \forall_{e \geq 1} \; (U, e) \in E \implies (U, e-1) \in E, \quad m_\mathrm{e} \coloneqq \max_{(U,e) \in E_1 \cup E_2} e, \quad A \coloneqq \{\alpha \mid \alpha : V \to \Gamma_\alpha\}, \quad B \coloneqq \{\beta \mid \beta : E \to \Gamma_\beta\}, \quad \overline{B} \coloneqq \left\{ \overline{\beta}_i : \left(2^{\mathbb{N}_0} - \{\varnothing\}\right) \times \mathbb{N}_0 \to \Gamma_\beta \cup \{\varepsilon_\beta\} \;\middle|\; \overline{\beta}_i(v) \coloneqq \begin{cases} \beta_1(U,e) & (U,e) \in E_1, \; i=1 \\ \beta_2(U,e) & (U,e) \in E_2, \; i=2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f : V_1 \leftrightarrow V_2, \; \left\{g_U : [m_\mathrm{e}]_0 \leftrightarrow [m_\mathrm{e}]_0 \;\mid\; U \in \left(2^{V_1} - \{\varnothing\}\right)\right\}} \left(\left(\forall_{U \in \left(2^{V_1} - \{\varnothing\}\right), \; e \in [m_\mathrm{e}]_0} \; (U, e) \in E_1 \iff \left(\overline{f}(U), e\right) \in E_2\right) \wedge \left(\forall_{v \in V_1} \forall_{\overline{\alpha} \in \overline{A}} \; \overline{\alpha}_1(v) = \overline{\alpha}_2(f(v))\right) \wedge \left(\forall_{U \in \left(2^{V_1} - \{\varnothing\}\right), \; e \in [m_e]_0} \forall_{\overline{\beta} \in \overline{B}} \; \overline{\beta}_1(U, e) = \overline{\beta}_2\left(\overline{f}(U), g_{U,v}(e)\right)\right)\right).$$

Definitions Common to Both

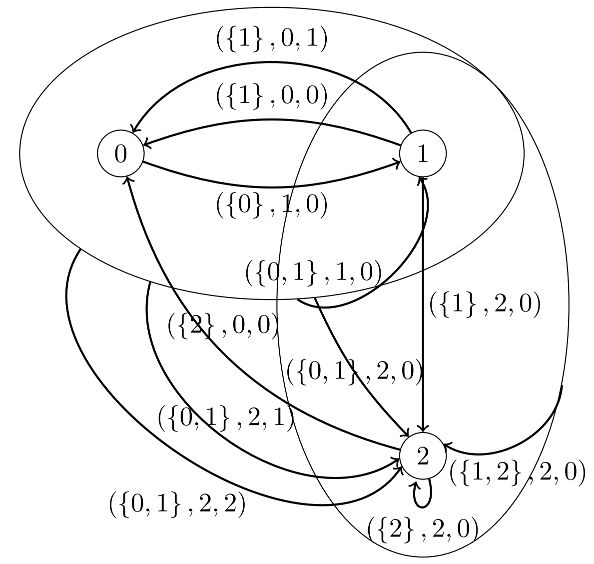
$$P \coloneqq \left\{ \rho_{\Gamma^\varepsilon_\gamma} : \left(\Gamma^\varepsilon_\gamma\right)^2 \to \mathbb{R}^+_0 \;\middle|\; \gamma \in A_1 \cup A_2 \cup B_1 \cup B_2, \; \rho \text{ is a metric on the set } \Gamma^\varepsilon_\gamma \equiv \Gamma_{\overline{\gamma}} \coloneqq \Gamma_\gamma \cup \{\varepsilon_\gamma\} \right\}, \quad \overline{A} \coloneqq \left\{ \overline{\alpha}_i : \mathbb{N}_0 \to \Gamma_\alpha \cup \{\varepsilon_\alpha\} \;\middle|\; \overline{\alpha}_i(v) \coloneqq \begin{cases} \alpha_1(v) & v \in V_1, \; i=1 \\ \alpha_2(v) & v \in V_2, \; i=2 \\ \varepsilon_\alpha & \text{otherwise} \end{cases} \right\}, \quad m_\mathrm{v} \coloneqq \max\{|V_1|, |V_2|\} - 1, \quad \forall_{n,k \in \mathbb{N}_0} \; [n]_k \coloneqq \bigcup_{i=k}^{n} \{i\}, \quad f : A \to B \implies \overline{f} : 2^A \to 2^B, \; \overline{f}(C) \coloneqq \{f(c) \mid c \in C\}.$$

Parametrized Metric for Directed and Undirected Attributed Multi-Hypergraphs: Complete and Permute Attributed Multi-Hypergraph Distance

$$\rho(H_1, H_2, P) \coloneqq \min_{f : [m_\mathrm{v}]_0 \leftrightarrow [m_\mathrm{v}]_0} \left(\sum_{v=0}^{m_\mathrm{v}} \sum_{\overline{\alpha} \in \overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f(v))\right) \right) + \sum_{U \in \left(2^{V_1 \cup V_2} - \{\varnothing\}\right)} \sum_{v=0}^{m_\mathrm{v}} \left(\min_{g : [m_\mathrm{e}]_0 \leftrightarrow [m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U, v, e), \overline{\beta}_2\left(\overline{f}(U), f(v), g(e)\right)\right) \right), \quad \rho(H_1, H_2, P) \coloneqq \min_{f : [m_\mathrm{v}]_0 \leftrightarrow [m_\mathrm{v}]_0} \left(\sum_{v=0}^{m_\mathrm{v}} \sum_{\overline{\alpha} \in \overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f(v))\right) \right) + \sum_{U \in \left(2^{V_1 \cup V_2} - \{\varnothing\}\right)} \left(\min_{g : [m_\mathrm{e}]_0 \leftrightarrow [m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}} \sum_{\overline{\beta} \in \overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U, e), \overline{\beta}_2\left(\overline{f}(U), g(e)\right)\right) \right).$$

Example

$$H_\text{example} \coloneqq \left(\{0,1,2\}, \; \{(\{0\},1,0), (\{1\},0,0), (\{1\},0,1), (\{1\},2,0), (\{2\},0,0), (\{2\},2,0), (\{0,1\},1,0), (\{0,1\},2,0), (\{0,1\},2,1), (\{0,1\},2,2), (\{1,2\},2,0)\}, \; \left\{ \text{value}(v) \coloneqq \sqrt{v + \sqrt{v}} \right\} \cup \{\text{exists}(v) \equiv 1\}, \; \left\{ \text{vector}(U, v, e) \coloneqq \left(\cosh\left(\frac{v^7}{\ln(1 + |U|)}\right), \; \exp\left(-\frac{\pi}{2} + e \cdot |U \cap \{v\}|\right), \; \textstyle\sum_{u \in U \cup \{v\}} \sqrt[77]{\text{value}(u) + e} \right) \right\} \right).$$



Deterministic Iterative Polynomial Approximations for Complete and Permute Directed and Undirected Attributed Multi-Hypergraph Distance Metric: an Orb-Weaver Elimination Heuristic

$$X_0 \coloneqq \varnothing, \quad X_{n+1} \coloneqq X_n \cup \left\{ \operatorname*{argmax}_{\left\{(u,f_u) \in \left([m_\mathrm{v}]_0\right)^2 - X_n \;\middle|\; \left(\exists_{x \in [m_\mathrm{v}]_0 - \{u\}} \; (x, f_u) \notin X_n\right) \wedge \left(\exists_{x \in [m_\mathrm{v}]_0 - \{f_u\}} \; (u, x) \notin X_n\right)\right\}} \rho\left(H_1, H_2, P, X_n \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{u\}} (x, f_u)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_u\}} (u, x)\right)\right) \right\},$$

$$RCD_0 \coloneqq \varnothing, \quad RCD_{n+1} \coloneqq RCD_n \cup \left\{ \lim_{\epsilon \to 0^+} \operatorname*{argmax}_{\left\{(u,f_u) \in \left([m_\mathrm{v}]_0\right)^2 - RCD_n \;\middle|\; \left(\exists_{x \in [m_\mathrm{v}]_0 - \{u\}} \; (x, f_u) \notin RCD_n\right) \wedge \left(\exists_{x \in [m_\mathrm{v}]_0 - \{f_u\}} \; (u, x) \notin RCD_n\right)\right\}} \max\{R_n(u, f_u), C_n(u, f_u), D_n(u, f_u)\} + \epsilon \cdot \rho\left(H_1, H_2, P, RCD_n \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{u\}} (x, f_u)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_u\}} (u, x)\right)\right) + \epsilon^2 \cdot D_n(u, f_u) + \epsilon^3 \cdot R_n(u, f_u) + \epsilon^4 \cdot C_n(u, f_u) \right\},$$

$$R_n(u, f_u) \coloneqq \max_{v \in [m_\mathrm{v}]_0 - \{u\}} \min_{\left\{f_v \in [m_\mathrm{v}]_0 - \{f_u\} \;\middle|\; (v, f_v) \notin RCD_n\right\}} \rho\left(H_1, H_2, P, RCD_n \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{u\}} (x, f_u)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_u\}} (u, x)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f_v)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_v\}} (v, x)\right)\right),$$

$$C_n(u, f_u) \coloneqq \max_{f_v \in [m_\mathrm{v}]_0 - \{f_u\}} \min_{\left\{v \in [m_\mathrm{v}]_0 - \{u\} \;\middle|\; (v, f_v) \notin RCD_n\right\}} \rho\left(H_1, H_2, P, RCD_n \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{u\}} (x, f_u)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_u\}} (u, x)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f_v)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_v\}} (v, x)\right)\right),$$

$$D_n(u, f_u) \coloneqq \min_{f : \left([m_\mathrm{v}]_0 - \{u\}\right) \leftrightarrow \left([m_\mathrm{v}]_0 - \{f_u\}\right)} \sum_{v \in \left([m_\mathrm{v}]_0 - \{u\}\right)} \frac{1}{m_\mathrm{v}} \cdot \rho\left(H_1, H_2, P, RCD_n \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{u\}} (x, f_u)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_u\}} (u, x)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f(v))\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f(v)\}} (v, x)\right)\right),$$

$$M_\mathrm{X} \coloneqq \left([m_\mathrm{v}]_0\right)^2 - X_{(m_\mathrm{v}+1)^2 - (m_\mathrm{v}+1)}, \quad M_\mathrm{RCD} \coloneqq \left([m_\mathrm{v}]_0\right)^2 - RCD_{(m_\mathrm{v}+1)^2 - (m_\mathrm{v}+1)}, \quad \exists_{X \subseteq \left([m_\mathrm{v}]_0\right)^2, \; |X| = (m_\mathrm{v}+1)^2 - (m_\mathrm{v}+1)} \; \rho(H_1, H_2, P) = \rho(H_1, H_2, P, X),$$

$$\rho(H_1, H_2, P) \geq RCD \coloneqq \max\left\{ \max_{v \in [m_\mathrm{v}]_0} \min_{f_v \in [m_\mathrm{v}]_0} \rho\left(H_1, H_2, P, \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f_v)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_v\}} (v, x)\right)\right), \; \max_{f_v \in [m_\mathrm{v}]_0} \min_{v \in [m_\mathrm{v}]_0} \rho\left(H_1, H_2, P, \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f_v)\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f_v\}} (v, x)\right)\right), \; \min_{f : [m_\mathrm{v}]_0 \leftrightarrow [m_\mathrm{v}]_0} \sum_{v \in [m_\mathrm{v}]_0} \frac{1}{m_\mathrm{v} + 1} \cdot \rho\left(H_1, H_2, P, \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{v\}} (x, f(v))\right) \cup \left(\bigcup_{x \in [m_\mathrm{v}]_0 - \{f(v)\}} (v, x)\right)\right) \right\} \geq \rho(H_1, H_2, P, \varnothing).$$

Admissible Heuristic for a Parametrized Metric on Directed Attributed Multi-Hypergraphs

$$
\begin{aligned}
\rho(H_1, H_2, P, X) \coloneqq \sup_{b_\varepsilon \in \mathbb{R}} & \sup_{\substack{\mathrm{a,b,c,d}:(V_1\cup V_2)^4 \to \mathbb{R} \\ \left(u \notin N^+_{E_1}(v) \vee f_u \notin N_{E_2}(f_v)\right) \implies \left(\mathrm{a}(v,u,f_v,f_u)=\mathrm{a}(u,v,f_u,f_v)=\mathrm{c}(v,u,f_v,f_u)=\mathrm{c}(u,v,f_u,f_v)=\mathrm{d}(v,u,f_v,f_u)=\mathrm{d}(u,v,f_u,f_v)=1 \,\wedge\, \mathrm{b}(v,u,f_v,f_u)=\mathrm{b}(u,v,f_u,f_v)=b_\varepsilon\right)}} \\
& \min_{\substack{f_{\mathrm{v}} : [m_{\mathrm{v}}]_0 \leftrightarrow [m_{\mathrm{v}}]_0 \\ \forall_{(x,f_x)\in X}\ f_{\mathrm{v}}(x) \neq f_x}} \sum_{v=0}^{m_{\mathrm{v}}} \left(\left(\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}} \left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_{\mathrm{v}}(v))\right) \right) + \left(\min_{g:[m_{\mathrm{e}}]_0 \leftrightarrow [m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}} \left(\overline{\beta}_1(\{v\}, v, e), \overline{\beta}_2(\{f_{\mathrm{v}}(v)\}, f_{\mathrm{v}}(v), g(e))\right) \right) + rs^{\mathrm{e}}_{\min}(v, f_{\mathrm{v}}(v), v, f_{\mathrm{v}}(v)) \right) \cdot \left(1 - \frac{\min\left\{m_{\mathrm{v}}, \left|N^+_{E_1}(v) - \{v\}\right| + \left|N_{E_2}(f_{\mathrm{v}}(v)) - \{f_{\mathrm{v}}(v)\}\right|\right\}}{|V_1\cup V_2|}\right) + \right. \\
& \min_{\substack{f_{\mathrm{u}}:\Psi\leftrightarrow\Omega \\ [m_{\mathrm{v}}]_0 - \{v\} \supseteq \Psi \supseteq N^+_{E_1}(v) - \{v\} \\ [m_{\mathrm{v}}]_0 - \{f_{\mathrm{v}}(v)\} \supseteq \Omega \supseteq N_{E_2}(f_{\mathrm{v}}(v)) - \{f_{\mathrm{v}}(v)\} \\ |\Psi| = |\Omega| = \min\left\{m_{\mathrm{v}}, \left|N^+_{E_1}(v)-\{v\}\right| + \left|N_{E_2}(f_{\mathrm{v}}(v)) - \{f_{\mathrm{v}}(v)\}\right|\right\}}} \sum_{u\in\Psi}
\begin{cases}
0 & u \notin N^+_{E_1}(v) \ \wedge\ f_{\mathrm{u}}(u) \notin N_{E_2}(f_{\mathrm{v}}(v)) \\[2ex]
+\infty & u \in N^+_{E_1}(v) \ \wedge\ f_{\mathrm{u}}(u) \in N_{E_2}(f_{\mathrm{v}}(v)) \ \wedge\ (u, f_{\mathrm{u}}(u)) \in X \\[2ex]
\left(\begin{gathered}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_{\mathrm{v}}(v))\right)\right) \cdot \frac{1 + \mathrm{a}(u, v, f_{\mathrm{u}}(u), f_{\mathrm{v}}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(u), \overline{\alpha}_2(f_{\mathrm{u}}(u))\right)\right) \cdot \frac{1 - \mathrm{a}(v, u, f_{\mathrm{v}}(v), f_{\mathrm{u}}(u))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\}, v, e), \overline{\beta}_2(\{f_{\mathrm{u}}(u)\}, f_{\mathrm{v}}(v), g(e))\right)\right) \cdot \frac{1 + \mathrm{b}(u, v, f_{\mathrm{u}}(u), f_{\mathrm{v}}(v))}{2} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\}, u, e), \overline{\beta}_2(\{f_{\mathrm{v}}(v)\}, f_{\mathrm{u}}(u), g(e))\right)\right) \cdot \frac{1 - \mathrm{b}(v, u, f_{\mathrm{v}}(v), f_{\mathrm{u}}(u))}{2} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\}, v, e), \overline{\beta}_2(\{f_{\mathrm{v}}(v)\}, f_{\mathrm{v}}(v), g(e))\right)\right) \cdot \frac{1 + \mathrm{c}(u, v, f_{\mathrm{u}}(u), f_{\mathrm{v}}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\}, u, e), \overline{\beta}_2(\{f_{\mathrm{u}}(u)\}, f_{\mathrm{u}}(u), g(e))\right)\right) \cdot \frac{1 - \mathrm{c}(v, u, f_{\mathrm{v}}(v), f_{\mathrm{u}}(u))}{2\cdot|V_1\cup V_2|} + \\
rs^{\mathrm{e}}_{\min}(v, f_{\mathrm{v}}(v), v, f_{\mathrm{v}}(v)) \cdot \frac{1 + \mathrm{d}(u, v, f_{\mathrm{u}}(u), f_{\mathrm{v}}(v))}{2\cdot|V_1\cup V_2|} + \\
rs^{\mathrm{e}}_{\min}(u, f_{\mathrm{u}}(u), u, f_{\mathrm{u}}(u)) \cdot \frac{1 - \mathrm{d}(v, u, f_{\mathrm{v}}(v), f_{\mathrm{u}}(u))}{2\cdot|V_1\cup V_2|} + \\
rs^{\mathrm{e}}_{\min}(u, f_{\mathrm{u}}(u), v, f_{\mathrm{v}}(v))
\end{gathered}\right) & u \in N^+_{E_1}(v) \ \wedge\ f_{\mathrm{u}}(u) \in N_{E_2}(f_{\mathrm{v}}(v)) \ \wedge\ (u, f_{\mathrm{u}}(u)) \notin X \\[2ex]
\left(\begin{gathered}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v), \overline{\alpha}_2(f_{\mathrm{v}}(v))\right)\right) \cdot \frac{1}{|V_1\cup V_2|} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\}, v, e), \overline{\beta}_2(\{f_{\mathrm{u}}(u)\}, f_{\mathrm{v}}(v), g(e))\right)\right) \cdot \frac{1 + b_\varepsilon}{2} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\}, u, e), \overline{\beta}_2(\{f_{\mathrm{v}}(v)\}, f_{\mathrm{u}}(u), g(e))\right)\right) \cdot \frac{1 - b_\varepsilon}{2} + \\
\left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\}, v, e), \overline{\beta}_2(\{f_{\mathrm{v}}(v)\}, f_{\mathrm{v}}(v), g(e))\right)\right) \cdot \frac{1}{|V_1\cup V_2|} + \\
rs^{\mathrm{e}}_{\min}(v, f_{\mathrm{v}}(v), v, f_{\mathrm{v}}(v)) \cdot \frac{1}{|V_1\cup V_2|} + \\
rs^{\mathrm{e}}_{\min}(u, f_{\mathrm{u}}(u), v, f_{\mathrm{v}}(v))
\end{gathered}\right) & \text{otherwise}
\end{cases}
\end{aligned}
$$

$$
rs^{\mathrm{e}}_{\min}(u_0, f_{u_0}, v, f_v) \coloneqq
\begin{cases}
r^{\mathrm{e}+\infty}_{\min}\left(u_0 + 1, f_{u_0} + 1, \{u_0\}, \{f_{u_0}\}, v, f_v, 1\right) & \forall_{i\in[m_{\mathrm{v}}]_0} \ \left(\{v\}, i, 0\right) \notin E_1 \wedge \left(\{i\}, v, 0\right) \notin E_1 \wedge \left(\{f_v\}, f_{\mathrm{u}}(i), 0\right) \notin E_2 \wedge \left(\{f_{\mathrm{u}}(i)\}, f_v, 0\right) \notin E_2 \\[2ex]
r^{\mathrm{e}+\infty}_{\min}\left(u_0 + 1, 0, \{u_0\}, \{f_{u_0}\}, v, f_v, 1\right) & \text{otherwise}
\end{cases}
$$

$$
r^{\mathrm{e}+\infty}_{\min}(s, f_s, U, f_U, v, f_v, j) \coloneqq \min_{\substack{f_j:[m_{\mathrm{v}}]_s \xrightarrow{1:1} [m_{\mathrm{v}}]_{f_s} - f_U \\ v \geq s \implies f_j(v) \coloneqq f_v}}
\begin{cases}
\displaystyle\sum_{i=s}^{m_{\mathrm{v}}} r^{\mathrm{e}+\infty}_{\min}\left(i+1, f_j(i)+1, U \cup \{i\}, f_U \cup \{f_j(i)\}, v, f_v, j+1\right) & \forall_{s \leq i \leq m_{\mathrm{v}}} \ \left(U\cup\{i\}, v, 0\right) \notin E_1 \wedge \left(f_U \cup \{f_j(i)\}, f_v, 0\right) \notin E_2 \\[2ex]
\displaystyle\sum_{i=s}^{m_{\mathrm{v}}} \left(\min_{g:[m_{\mathrm{e}}]_0\leftrightarrow[m_{\mathrm{e}}]_0} \sum_{e=0}^{m_{\mathrm{e}}} \sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U\cup\{i\}, v, e), \overline{\beta}_2(f_U \cup \{f_j(i)\}, f_v, g(e))\right)\right) + r^{\mathrm{e}+\infty}_{\min}\left(i+1, 0, U\cup\{i\}, f_U\cup\{f_j(i)\}, v, f_v, j+1\right) & \text{otherwise}
\end{cases}
$$

$$
\begin{aligned}
N^+_E(v) &\coloneqq \left\{u \in V_1 \cup V_2 \;\middle|\; \left(\{v\}, u, 0\right) \in E \vee \left(\exists_{(V_1\cup V_2)\supseteq U_0 \supseteq \{u\}} \ \left(\forall_{w\in U_0} \ w \geq u\right) \wedge \left(U_0, v, 0\right) \in E\right)\right\} \\
N_E(v) &\coloneqq \left\{u \in V_1 \cup V_2 \;\middle|\; \left(\{v\}, u, 0\right) \in E \vee \left(\exists_{(V_1\cup V_2)\supseteq U_0 \supseteq \{u\}} \ \left(U_0, v, 0\right) \in E\right)\right\}
\end{aligned}
$$

$$
c(H_1, H_2) = \mathcal{O}\left(\left\lceil \frac{|V_1\cup V_2|^2}{p} \right\rceil \cdot \left(|V_1\cup V_2|^3 + |V_1\cup V_2|^{2+2\cdot\left(\max_{(U,v,0)\in(E_1\cup E_2)} |U|\right)} \cdot \left(\max_{(U,v,e)\in E_1\cup E_2} e\right)^3\right)\right), \qquad \min_{f:A\xrightarrow{1:1}B} \sum_{a\in A} \gamma(a, f(a)) \coloneqq \min_{\substack{f:A\cup C\leftrightarrow B \\ |A\cup C| = |B| \\ A\cap C = \varnothing}} \sum_{a\in A\cup C} \begin{cases} \gamma(a, f(a)) & a \in A \\ 0 & a \in C \end{cases}, \qquad \min_{\substack{f:[n]_0\leftrightarrow[n]_0 \\ \forall_{(x,f_x)\in X} \ f(x)\neq f_x}} \sum_{k=0}^{n} \gamma(k, f(k)) \coloneqq \min_{f:[n]_0\leftrightarrow[n]_0} \sum_{k=0}^{n} \begin{cases} \gamma(k, f(k)) & (k, f(k)) \notin X \\ +\infty & (k, f(k)) \in X \end{cases}.
$$

Admissible Heuristic for a Parametrized Metric on Undirected Attributed Multi-Hypergraphs

$$
\begin{aligned}
\rho(H_1,H_2,P,X) \coloneqq & \sup_{\substack{\mathrm{a},\mathrm{b}:(V_1\cup V_2)^4\to\mathbb{R}\\ \left(u\notin N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\,\vee\, f_u\notin N_{E_2}(f_v)\right)\implies(\mathrm{a}(v,u,f_v,f_u)=\mathrm{a}(u,v,f_u,f_v)=\mathrm{b}(v,u,f_v,f_u)=\mathrm{b}(u,v,f_u,f_v)=1)}} \; \min_{\substack{f_\mathrm{v}:[m_\mathrm{v}]_0\leftrightarrow[m_\mathrm{v}]_0\\ \forall_{(x,f_x)\in X}\; f_\mathrm{v}(x)\neq f_x}} \sum_{v=0}^{m_\mathrm{v}} \left(\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right) + \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\right)\cdot\left(1-\frac{\min\left\{m_\mathrm{v}-v,\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\}}{|V_1\cup V_2|}\right) + \\
& \min_{\substack{f_\mathrm{u}:\Psi\xrightarrow{1:1}\Omega\\ [m_\mathrm{v}]_{v+1}\supseteq\Psi\supseteq\left(N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right)\\ \left([m_\mathrm{v}]_0-\{f_\mathrm{v}(v)\}\right)\supseteq\Omega\supseteq N_{E_2}(f_\mathrm{v}(v))\\ |\Psi|=\min\left\{m_\mathrm{v}-v,\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\}\\ |\Omega|=\min\left\{m_\mathrm{v},\left|N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\right|+\left|N_{E_2}(f_\mathrm{v}(v))\right|\right\}}} \sum_{u\in\Psi}
\begin{cases}
0 & u\notin N^+_{E_1}(v)\;\wedge\;f_\mathrm{u}(u)\notin N_{E_2}(f_\mathrm{v}(v))\\
+\infty & u\in N^+_{E_1}(v)\;\wedge\;f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v))\;\wedge\;(u,f_\mathrm{u}(u))\in X\\
\left(\begin{array}{c}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right)\cdot\frac{1+\mathrm{a}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(u),\overline{\alpha}_2(f_\mathrm{u}(u))\right)\right)\cdot\frac{1-\mathrm{a}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\cdot\frac{1+\mathrm{b}(u,v,f_\mathrm{u}(u),f_\mathrm{v}(v))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},e),\overline{\beta}_2(\{f_\mathrm{u}(u)\},g(e))\right)\right)\cdot\frac{1-\mathrm{b}(v,u,f_\mathrm{v}(v),f_\mathrm{u}(u))}{2\cdot|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u,v\},e),\overline{\beta}_2(\{f_\mathrm{u}(u),f_\mathrm{v}(v)\},g(e))\right)\right) + \\
rs^\mathrm{e}_{\min}(u,f_\mathrm{u}(u),v,f_\mathrm{v}(v))
\end{array}\right) & u\in N^+_{E_1}(v)\;\wedge\;f_\mathrm{u}(u)\in N_{E_2}(f_\mathrm{v}(v))\;\wedge\;(u,f_\mathrm{u}(u))\notin X\\
\left(\begin{array}{c}
\left(\sum_{\overline{\alpha}\in\overline{A}} \rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_\mathrm{v}(v))\right)\right)\cdot\frac{1}{|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},e),\overline{\beta}_2(\{f_\mathrm{v}(v)\},g(e))\right)\right)\cdot\frac{1}{|V_1\cup V_2|} + \\
\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u,v\},e),\overline{\beta}_2(\{f_\mathrm{u}(u),f_\mathrm{v}(v)\},g(e))\right)\right) + \\
rs^\mathrm{e}_{\min}(u,f_\mathrm{u}(u),v,f_\mathrm{v}(v))
\end{array}\right) & \text{otherwise}
\end{cases}
\end{aligned}
$$

$$
rs^\mathrm{e}_{\min}(u_0,f_{u_0},v,f_v) \coloneqq \begin{cases}
r^{\mathrm{e}+\infty}_{\min}(u_0+1,f_{u_0}+1,\{u_0\},\{f_{u_0}\},v,f_v,1) & \forall_{i\in[m_\mathrm{v}]_0}\;\left(\{v,i\},0\right)\notin E_1\;\wedge\;\left(\{f_v,f_\mathrm{u}(i)\},0\right)\notin E_2\\
r^{\mathrm{e}+\infty}_{\min}(u_0+1,0,\{u_0\},\{f_{u_0}\},v,f_v,1) & \text{otherwise}
\end{cases}
$$

$$
r^{\mathrm{e}+\infty}_{\min}(s,f_s,U,f_U,v,f_v,j) \coloneqq \min_{f_j:\left([m_\mathrm{v}]_s-\{v\}\right)\xrightarrow{1:1}\left([m_\mathrm{v}]_{f_s}-f_U-\{f_v\}\right)} \begin{cases}
\sum_{i=s}^{m_\mathrm{v}} r^{\mathrm{e}+\infty}_{\min}(i+1,f_j(i)+1,U\cup\{i\},f_U\cup\{f_j(i)\},v,f_v,j+1) & \forall_{s\leq i\leq m_\mathrm{v}}\;\left(U\cup\{i,v\},0\right)\notin E_1\;\wedge\;\left(f_U\cup\{f_j(i),f_v\},0\right)\notin E_2\\
\sum_{i=s}^{m_\mathrm{v}} \left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0} \sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}} \rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(U\cup\{i,v\},e),\overline{\beta}_2(f_U\cup\{f_j(i),f_v\},g(e))\right)\right) + r^{\mathrm{e}+\infty}_{\min}(i+1,0,U\cup\{i\},f_U\cup\{f_j(i)\},v,f_v,j+1) & \text{otherwise}
\end{cases}
$$

$$
\begin{aligned}
N^+_E(v) &\coloneqq \left\{u\in(V_1\cup V_2)-\{v\}\;\middle|\;\exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u,v\}}\;\left(\forall_{w\in U_0-\{v\}}\;w\geq u\right)\;\wedge\;(U_0,0)\in E\right\},\\
N_E(v) &\coloneqq \left\{u\in(V_1\cup V_2)-\{v\}\;\middle|\;\exists_{(V_1\cup V_2)\supseteq U_0\supseteq\{u,v\}}\;(U_0,0)\in E\right\},
\end{aligned}
$$

$$
c(H_1,H_2)=\mathcal{O}\left(\left\lceil\frac{|V_1\cup V_2|^2}{p}\right\rceil\cdot\left(|V_1\cup V_2|^3+|V_1\cup V_2|^{2+2\cdot\left(\max_{(U,0)\in(E_1\cup E_2)}\;|U|\right)}\cdot\left(\max_{(U,e)\in E_1\cup E_2} e\right)^3\right)\right),\qquad
\min_{f:A\xrightarrow{1:1}B}\sum_{a\in A}\gamma(a,f(a)) \coloneqq \min_{\substack{f:A\cup C\leftrightarrow B\\ |A\cup C|=|B|\\ A\cap C=\varnothing}} \sum_{a\in A\cup C}\begin{cases}\gamma(a,f(a)) & a\in A\\ 0 & a\in C\end{cases},\qquad
\min_{\substack{f:[n]_0\leftrightarrow[n]_0\\ \forall_{(x,f_x)\in X}\;f(x)\neq f_x}}\sum_{k=0}^{n}\gamma(k,f(k)) \coloneqq \min_{f:[n]_0\leftrightarrow[n]_0}\sum_{k=0}^{n}\begin{cases}\gamma(k,f(k)) & (k,f(k))\notin X\\ +\infty & (k,f(k))\in X\end{cases}.
$$

Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Directed Attributed Multi-Hypergraphs

$$\mathrm{a}_0(u,v,f_u,f_v)=\mathrm{c}_0(u,v,f_u,f_v)=\mathrm{d}_0(u,v,f_u,f_v)\coloneqq\begin{cases}1 & u\notin N^+_{E_1}(v)\vee v\notin N^+_{E_1}(u)\vee f_u\notin N_{E_2}(f_v)\\ 0 & \text{otherwise}\end{cases},\quad\forall_{u,v,f_u,f_v\in[m_\mathrm{v}]_0}\quad \mathrm{b}_0(u,v,f_u,f_v)=b_{\varepsilon,0}\coloneqq 0,$$

$$\mathrm{a}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{a}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2\cdot|V_1\cup V_2|}\cdot\left(\displaystyle\sum_{\overline{\alpha}\in\overline{A}}\rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_v)\right)\right)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\wedge v\in N^+_{E_1}(u)\wedge u\neq v\wedge f_u\in N_{E_2}(f_v)\\ 1 & \text{otherwise}\end{cases},$$

$$\mathrm{b}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{b}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2}\cdot\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\displaystyle\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},v,e),\overline{\beta}_2(\{f_u\},f_v,g(e))\right)\right)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\wedge v\in N^+_{E_1}(u)\wedge u\neq v\wedge f_u\in N_{E_2}(f_v)\\ b_{\varepsilon,n} & \text{otherwise}\end{cases},$$

$$\mathrm{c}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{c}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2\cdot|V_1\cup V_2|}\cdot\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\displaystyle\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},v,e),\overline{\beta}_2(\{f_v\},f_v,g(e))\right)\right)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\wedge v\in N^+_{E_1}(u)\wedge u\neq v\wedge f_u\in N_{E_2}(f_v)\\ 1 & \text{otherwise}\end{cases},$$

$$\mathrm{d}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{d}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2\cdot|V_1\cup V_2|}\cdot rs^{\mathrm{e}}_{\min}(v,f_v,v,f_v)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\wedge v\in N^+_{E_1}(u)\wedge u\neq v\wedge f_u\in N_{E_2}(f_v)\\ 1 & \text{otherwise}\end{cases},$$

$$b_{\varepsilon,n+1}\coloneqq b_{\varepsilon,n}+\frac{\eta_{n+1}}{2}\cdot\left(\sum_{v=0}^{m_\mathrm{v}}\sum_{\left\{u\in\Psi\;\middle|\;u\notin N^+_{E_1}(v)\vee v\notin N^+_{E_1}(u)\vee f^{\min}_{\mathrm{u},n}(u)\notin N_{E_2}\left(f^{\min}_{\mathrm{v},n}(v)\right)\right\}}\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{u\},v,e),\overline{\beta}_2\left(\left\{f^{\min}_{\mathrm{u},n}(u)\right\},f^{\min}_{\mathrm{v},n}(v),g(e)\right)\right)\right)-\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(\{v\},u,e),\overline{\beta}_2\left(\left\{f^{\min}_{\mathrm{v},n}(v)\right\},f^{\min}_{\mathrm{u},n}(u),g(e)\right)\right)\right)\right).$$

Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Undirected Attributed Multi-Hypergraphs

$$\mathrm{a}_0(u,v,f_u,f_v)=\mathrm{b}_0(u,v,f_u,f_v)\coloneqq\begin{cases}1 & u\notin N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\vee v\notin N^+_{E_1}(u)\cap[m_\mathrm{v}]_{u+1}\vee f_u\notin N_{E_2}(f_v)\\ 0 & \text{otherwise}\end{cases},$$

$$\mathrm{a}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{a}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2\cdot|V_1\cup V_2|}\cdot\left(\displaystyle\sum_{\overline{\alpha}\in\overline{A}}\rho_{\Gamma_{\overline{\alpha}}}\left(\overline{\alpha}_1(v),\overline{\alpha}_2(f_v)\right)\right)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\wedge v\in N^+_{E_1}(u)\cap[m_\mathrm{v}]_{u+1}\wedge f_u\in N_{E_2}(f_v)\\ 1 & \text{otherwise}\end{cases},$$

$$\mathrm{b}_{n+1}(u,v,f_u,f_v)\coloneqq\begin{cases}\left(\mathrm{b}_n(u,v,f_u,f_v)+\dfrac{\eta_{n+1}}{2\cdot|V_1\cup V_2|}\cdot\left(\min_{g:[m_\mathrm{e}]_0\leftrightarrow[m_\mathrm{e}]_0}\displaystyle\sum_{e=0}^{m_\mathrm{e}}\sum_{\overline{\beta}\in\overline{B}}\rho_{\Gamma_{\overline{\beta}}}\left(\overline{\beta}_1(v,e),\overline{\beta}_2(f_v,g(e))\right)\right)\cdot\begin{cases}1 & f_u=f^{\min}_{\mathrm{u},n}(u)\wedge f_v=f^{\min}_{\mathrm{v},n}(v)\wedge\left(f_v\neq f^{\min}_{\mathrm{u},n}(v)\vee f_u\neq f^{\min}_{\mathrm{v},n}(u)\right)\\ -1 & f_v=f^{\min}_{\mathrm{u},n}(v)\wedge f_u=f^{\min}_{\mathrm{v},n}(u)\wedge\left(f_u\neq f^{\min}_{\mathrm{u},n}(u)\vee f_v\neq f^{\min}_{\mathrm{v},n}(v)\right)\\ 0 & \text{otherwise}\end{cases}\right) & u\in N^+_{E_1}(v)\cap[m_\mathrm{v}]_{v+1}\wedge v\in N^+_{E_1}(u)\cap[m_\mathrm{v}]_{u+1}\wedge f_u\in N_{E_2}(f_v)\\ 1 & \text{otherwise}\end{cases}.$$

For independent comparison, the following is a notation-adjusted discovery by prof. Kaspar Riesen (h-index: 32) and prof. Horst Bunke (h-index: 96) published in 2009 having 737 citations as of September 2022

$$\rho_{\text{(Riesen, Bunke)}}\left(G_1, G_2\right) \coloneqq \min_{f_{\text{v}}:[|V_1|+|V_2|-1]_0 \leftrightarrow [|V_1|+|V_2|-1]_0} \sum_{v=0}^{|V_1|+|V_2|-1} c_{\text{vertex}}\left(\overline{\alpha}_1\left(v\right), \overline{\alpha}_2\left(f_{\text{v}}(v)\right)\right) + \left(\min_{\substack{f_{\text{u}}:[|V_1|+|V_2|-1]_0 \leftrightarrow [|V_1|+|V_2|-1]_0 \\ f_{\text{u}}(v) \coloneqq f_{\text{v}}(v) \text{ (optimization not specified in paper)}}} \sum_{u=0}^{|V_1|+|V_2|-1} c_{\text{edge}}\left(\overline{\beta}_1\left(v, u\right), \overline{\beta}_2\left(f_{\text{v}}(v), f_{\text{u}}(u)\right)\right) \right).$$

If c_{vertex} and c_{edge} are metric functions, $\rho_{\text{(Riesen, Bunke)}}$ becomes a special case of ρ:

$$\rho\left(G_1, G_2, P\right) \geq \rho\left(G_1, G_2, P, \varnothing\right) \geq \rho_{\text{(Riesen, Bunke)}}\left(G_1, G_2\right)$$

She is fairer than the sun and surpasses every constellation of the stars.

Compared to light, Wisdom is found more radiant.

From the greatness and the beauty of created things their original author, by analogy, is seen.