

She is fairer than the sun and surpasses every constellation of the stars.

Compared to light, Wisdom is found more radiant.

From the greatness and the beauty of created things their original author, by analogy, is seen.



**Directed Attributed Multi-Hypergraph Definitions and Isomorphism Criteria**

$$H := (V, E, A, B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v \geq 1} \ v \in V \implies v - 1 \in V, \quad E \subseteq (2^V - \{\emptyset\}) \times V \times \mathbb{N}_0, \quad \forall_{e \geq 1} \ (U, v, e) \in E \implies (U, v, e - 1) \in E, \quad m_e := \max_{(U, v, e) \in E_1 \cup E_2} e, \quad A := \{\alpha \mid \alpha : V \rightarrow \Gamma_\alpha\}, \quad B := \{\beta \mid \beta : E \rightarrow \Gamma_\beta\}, \quad \bar{B} := \left\{ \bar{\beta}_i : (2^{\mathbb{N}_0} - \{\emptyset\}) \times (\mathbb{N}_0)^2 \rightarrow \Gamma_\beta \cup \{\varepsilon_\beta\} \mid \begin{array}{l} \bar{\beta}_i(v) := \begin{cases} \beta_1(U, v, e) & (U, v, e) \in E_1, i = 1 \\ \beta_2(U, v, e) & (U, v, e) \in E_2, i = 2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \end{array} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f: V_1 \leftrightarrow V_2, \{g_{U,v}: [m_e]_0 \leftrightarrow [m_e]_0 \mid U \in (2^{V_1} - \{\emptyset\}), v \in V_1\}} \left( \left( \forall_{U \in (2^{V_1} - \{\emptyset\}), v \in V_1, e \in [m_e]_0} (U, v, e) \in E_1 \iff (\bar{f}(U), f(v), e) \in E_2 \right) \wedge \left( \forall_{v \in V_1} \forall_{\bar{\alpha} \in \bar{A}} \bar{\alpha}_1(v) = \bar{\alpha}_2(f(v)) \right) \wedge \left( \forall_{U \in (2^{V_1} - \{\emptyset\}), v \in V_1, e \in [m_e]_0} \forall_{\bar{\beta} \in \bar{B}} \bar{\beta}_1(U, v, e) = \bar{\beta}_2(\bar{f}(U), f(v), g_{U,v}(e)) \right) \right).$$

**Undirected Attributed Multi-Hypergraph Definitions and Isomorphism Criteria**

$$H := (V, E, A, B), \quad V \subseteq \mathbb{N}_0, \quad \forall_{v \geq 1} \ v \in V \implies v - 1 \in V, \quad E \subseteq (2^V - \{\emptyset\}) \times \mathbb{N}_0, \quad \forall_{e \geq 1} \ (U, e) \in E \implies (U, e - 1) \in E, \quad m_e := \max_{(U, e) \in E_1 \cup E_2} e, \quad A := \{\alpha \mid \alpha : V \rightarrow \Gamma_\alpha\}, \quad B := \{\beta \mid \beta : E \rightarrow \Gamma_\beta\}, \quad \bar{B} := \left\{ \bar{\beta}_i : (2^{\mathbb{N}_0} - \{\emptyset\}) \times \mathbb{N}_0 \rightarrow \Gamma_\beta \cup \{\varepsilon_\beta\} \mid \bar{\beta}_i(v) := \begin{cases} \beta_1(U, e) & (U, e) \in E_1, i = 1 \\ \beta_2(U, e) & (U, e) \in E_2, i = 2 \\ \varepsilon_\beta & \text{otherwise} \end{cases} \right\},$$

$$H_1 \equiv H_2 \iff \exists_{f: V_1 \leftrightarrow V_2, \{g_{U,v}: [m_e]_0 \leftrightarrow [m_e]_0 \mid U \in (2^{V_1} - \{\emptyset\})\}} \left( \left( \forall_{U \in (2^{V_1} - \{\emptyset\}), e \in [m_e]_0} (U, e) \in E_1 \iff (\bar{f}(U), e) \in E_2 \right) \wedge \left( \forall_{v \in V_1} \forall_{\bar{\alpha} \in \bar{A}} \bar{\alpha}_1(v) = \bar{\alpha}_2(f(v)) \right) \wedge \left( \forall_{U \in (2^{V_1} - \{\emptyset\}), e \in [m_e]_0} \forall_{\bar{\beta} \in \bar{B}} \bar{\beta}_1(U, e) = \bar{\beta}_2(\bar{f}(U), g_{U,v}(e)) \right) \right).$$

**Definitions Common to Both**

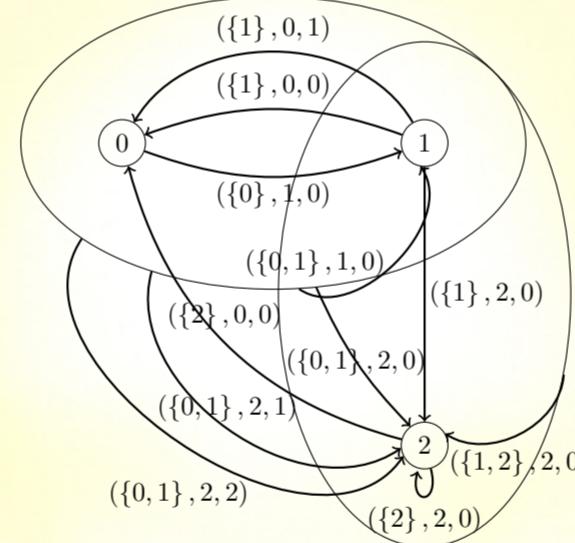
$$P := \left\{ \rho_{\Gamma_\gamma^\varepsilon} : (\Gamma_\gamma^\varepsilon)^2 \rightarrow \mathbb{R}_0^+ \mid \gamma \in A_1 \cup A_2 \cup B_1 \cup B_2, \rho \text{ is a metric on the set } \Gamma_\gamma^\varepsilon \equiv \Gamma_\gamma := \Gamma_\gamma \cup \{\varepsilon_\gamma\} \right\}, \quad \bar{A} := \left\{ \bar{\alpha}_i : \mathbb{N}_0 \rightarrow \Gamma_\alpha \cup \{\varepsilon_\alpha\} \mid \bar{\alpha}_i(v) := \begin{cases} \alpha_1(v) & v \in V_1, i = 1 \\ \alpha_2(v) & v \in V_2, i = 2 \\ \varepsilon_\alpha & \text{otherwise} \end{cases} \right\}, \quad m_v := \max \{|V_1|, |V_2|\} - 1, \quad \forall_{n,k \in \mathbb{N}_0} [n]_k := \bigcup_{i=k}^n \{i\}, \quad f: A \rightarrow B \implies \bar{f}: 2^A \rightarrow 2^B, \quad \bar{f}(C) := \{f(c) \mid c \in C\}.$$

**Parametrized Metric for Directed and Undirected Attributed Multi-Hypergraphs: Complete and Permute Attributed Multi-Hypergraph Distance**

$$\rho(H_1, H_2, P) := \min_{f: [m_v]_0 \leftrightarrow [m_v]_0} \left( \sum_{v=0}^{m_v} \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f(v))) \right) + \sum_{U \in (2^{V_1 \cup V_2} - \{\emptyset\})} \sum_{v=0}^{m_v} \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(U, v, e), \bar{\beta}_2(\bar{f}(U), f(v), g(e))) \right), \quad \rho(H_1, H_2, P) := \min_{f: [m_v]_0 \leftrightarrow [m_v]_0} \left( \sum_{v=0}^{m_v} \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f(v))) \right) + \sum_{U \in (2^{V_1 \cup V_2} - \{\emptyset\})} \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(U, e), \bar{\beta}_2(\bar{f}(U), g(e))) \right).$$

**Example**

$$H_{\text{example}} := \left( \{0, 1, 2\}, \{(\{0\}, 1, 0), (\{1\}, 0, 0), (\{1\}, 0, 1), (\{1\}, 2, 0), (\{2\}, 0, 0), (\{2\}, 2, 0), (\{0, 1\}, 1, 0), (\{0, 1\}, 2, 0), (\{0, 1\}, 2, 2), (\{1, 2\}, 2, 0)\}, \left\{ \text{value}(v) := \sqrt{v + \sqrt{v}} \cup \{\text{exists}(v) \equiv 1\}, \text{vector}(U, v, e) := \left( \cosh\left(\frac{v^7}{\ln(1 + |U|)}\right), \exp\left(-\frac{\pi}{2} + e \cdot |U \cap \{v\}|\right), \sum_{u \in U \cup \{v\}} \sqrt[7]{\text{value}(u) + e} \right) \right\} \right).$$


**Deterministic Iterative Polynomial Approximations for Complete and Permute Directed and Undirected Attributed Multi-Hypergraph Distance Metric: an Orb-Weaver Elimination Heuristic**

$$X_0 := \emptyset, \quad X_{n+1} := X_n \cup \left\{ \underset{\{(u, f_u) \in ([m_v]_0)^2 - X_n \mid (\exists_{x \in [m_v]_0 - \{u\}} (x, f_u) \notin X_n) \wedge (\exists_{x \in [m_v]_0 - \{f_u\}} (u, x) \notin X_n)} \operatorname{argmax} \rho \left( H_1, H_2, P, X_n \cup \left( \bigcup_{x \in [m_v]_0 - \{u\}} (x, f_u) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_u\}} (u, x) \right) \right) \right\},$$

$$RCD_0 := \emptyset, \quad RCD_{n+1} := RCD_n \cup \left\{ \lim_{\epsilon \rightarrow 0^+} \underset{\{(u, f_u) \in ([m_v]_0)^2 - RCD_n \mid (\exists_{x \in [m_v]_0 - \{u\}} (x, f_u) \notin RCD_n) \wedge (\exists_{x \in [m_v]_0 - \{f_u\}} (u, x) \notin RCD_n)} \operatorname{argmax} \max \{R_n(u, f_u), C_n(u, f_u), D_n(u, f_u)\} + \epsilon \cdot \rho \left( H_1, H_2, P, RCD_n \cup \left( \bigcup_{x \in [m_v]_0 - \{u\}} (x, f_u) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_u\}} (u, x) \right) \right) + \epsilon^2 \cdot D_n(u, f_u) + \epsilon^3 \cdot R_n(u, f_u) + \epsilon^4 \cdot C_n(u, f_u) \right\},$$

$$R_n(u, f_u) := \max_{v \in [m_v]_0 - \{u\}} \min_{\{f_v \in [m_v]_0 - \{u\} \mid (v, f_v) \notin RCD_n\}} \rho \left( H_1, H_2, P, RCD_n \cup \left( \bigcup_{x \in [m_v]_0 - \{u\}} (x, f_u) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_u\}} (u, x) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f_v) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_v\}} (v, x) \right) \right),$$

$$C_n(u, f_u) := \max_{f_v \in [m_v]_0 - \{f_u\}} \min_{\{v \in [m_v]_0 - \{u\} \mid (v, f_v) \notin RCD_n\}} \rho \left( H_1, H_2, P, RCD_n \cup \left( \bigcup_{x \in [m_v]_0 - \{u\}} (x, f_u) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_u\}} (u, x) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f_v) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_v\}} (v, x) \right) \right),$$

$$D_n(u, f_u) := \min_{f: ([m_v]_0 - \{u\}) \leftrightarrow ([m_v]_0 - \{f_u\})} \sum_{v \in ([m_v]_0 - \{u\})} \frac{1}{m_v} \cdot \rho \left( H_1, H_2, P, RCD_n \cup \left( \bigcup_{x \in [m_v]_0 - \{u\}} (x, f_u) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_u\}} (u, x) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f(v)) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f(v)\}} (v, x) \right) \right),$$

$$M_X := ([m_v]_0)^2 - X_{(m_v+1)^2 - (m_v+1)}, \quad M_{RCD} := ([m_v]_0)^2 - RCD_{(m_v+1)^2 - (m_v+1)}, \quad \exists_{X \subseteq ([m_v]_0)^2, |X| = (m_v+1)^2 - (m_v+1)} \rho(H_1, H_2, P) = \rho(H_1, H_2, P, X),$$

$$\rho(H_1, H_2, P) \geq RCD := \max \left\{ \max_{v \in [m_v]_0} \min_{f_v \in [m_v]_0} \rho \left( H_1, H_2, P, \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f_v) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_v\}} (v, x) \right) \right), \min_{f_v \in [m_v]_0} \max_{v \in [m_v]_0} \rho \left( H_1, H_2, P, \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f_v) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f_v\}} (v, x) \right) \right), \min_{f: [m_v]_0 \leftrightarrow [m_v]_0} \sum_{v \in [m_v]_0} \frac{1}{m_v + 1} \cdot \rho \left( H_1, H_2, P, \left( \bigcup_{x \in [m_v]_0 - \{v\}} (x, f(v)) \right) \cup \left( \bigcup_{x \in [m_v]_0 - \{f(v)\}} (v, x) \right) \right) \right\} \geq \rho(H_1, H_2, P, \emptyset).$$

**Admissible Heuristic for a Parametrized Metric on Directed Attributed Multi-Hypergraphs**

$$\rho(H_1, H_2, P, X) := \sup_{b \in \mathbb{R}} \sup_{\substack{\alpha, b, c, d: (V_1 \cup V_2)^4 \rightarrow \mathbb{R} \\ (u \notin N_{E_1}^+(v) \vee f_u \notin N_{E_2}(f_v)) \implies (\text{a}(v, u, f_v, f_u) = \text{a}(u, v, f_u, f_v) = \text{c}(v, u, f_v, f_u) = \text{c}(u, v, f_u, f_v) = \text{d}(v, u, f_v, f_u) = \text{d}(u, v, f_u, f_v) = 1 \wedge \text{b}(v, u, f_v, f_u) = \text{b}(u, v, f_u, f_v) = b)}}$$

$$\min_{\substack{f_v: [m_v]_0 \leftrightarrow [m_v]_0 \\ \forall (x, f_x) \in X \\ f_v(x) \neq f_x}} \sum_{v=0}^{m_v} \left( \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \right) + \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, v, e), \bar{\beta}_2(\{f_v(v)\}, f_v(v), g(e))) \right) + rs_{\min}^e(v, f_v(v), v, f_v(v)) \right) + \left( 1 - \frac{\min\{m_v, |N_{E_1}^+(v) - \{v\}| + |N_{E_2}(f_v(v)) - \{f_v(v)\}|\}}{|V_1 \cup V_2|} \right) +$$

$$\begin{cases} 0 & u \notin N_{E_1}^+(v) \wedge f_u(u) \notin N_{E_2}(f_v(v)) \\ +\infty & u \in N_{E_1}^+(v) \wedge f_u(u) \in N_{E_2}(f_v(v)) \wedge (u, f_u(u)) \in X \\ & u \in N_{E_1}^+(v) \wedge f_u(u) \in N_{E_2}(f_v(v)) \wedge (u, f_u(u)) \notin X \\ & u \in N_{E_1}^+(v) \wedge f_u(u) \in N_{E_2}(f_v(v)) \wedge (u, f_u(u)) \in X \\ & u \in N_{E_1}^+(v) \wedge f_u(u) \in N_{E_2}(f_v(v)) \wedge (u, f_u(u)) \notin X \end{cases}$$

$$\min_{\substack{f_u: \Psi \rightarrow \Omega \\ [m_v]_0 - \{v\} \supseteq \Psi \supseteq N_{E_1}^+(v) - \{v\} \\ [\Psi] = \Omega = \min\{m_v, |N_{E_1}^+(v) - \{v\}| + |N_{E_2}(f_v(v)) - \{f_v(v)\}|\}}} \sum_{u \in \Psi} \left( \sum_{\substack{g: [m_e]_0 \leftrightarrow [m_e]_0 \\ |\Psi| = |\Omega| = \min\{m_v, |N_{E_1}^+(v) - \{v\}| + |N_{E_2}(f_v(v)) - \{f_v(v)\}|\}}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \cdot \frac{1 + \text{a}(u, v, f_u(u), f_v(v))}{2 \cdot |V_1 \cup V_2|} + \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(u), \bar{\alpha}_2(f_u(u))) \cdot \frac{1 - \text{a}(v, u, f_v(v), f_u(u))}{2 \cdot |V_1 \cup V_2|} + \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, v, e), \bar{\beta}_2(\{f_u(u)\}, f_v(v), g(e))) \cdot \frac{1 + \text{b}(u, v, f_u(u), f_v(v))}{2} + \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, u, e), \bar{\beta}_2(\{f_v(v)\}, f_u(u), g(e))) \cdot \frac{1 - \text{b}(v, u, f_v(v), f_u(u))}{2} + \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, u, e), \bar{\beta}_2(\{f_v(v)\}, f_v(v), g(e))) \cdot \frac{1 + \text{c}(u, v, f_u(u), f_v(v))}{2 \cdot |V_1 \cup V_2|} + \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, u, e), \bar{\beta}_2(\{f_u(u)\}, f_u(u), g(e))) \cdot \frac{1 - \text{c}(v, u, f_v(v), f_u(u))}{2 \cdot |V_1 \cup V_2|} + rs_{\min}^e(v, f_v(v), v, f_v(v)) \cdot \frac{1 + \text{d}(u, v, f_u(u), f_v(v))}{2 \cdot |V_1 \cup V_2|} + rs_{\min}^e(u, f_u(u), u, f_u(u)) \cdot \frac{1 - \text{d}(v, u, f_v(v), f_u(u))}{2 \cdot |V_1 \cup V_2|} + rs_{\min}^e(u, f_u(u), v, f_v(v)) \end{cases}$$

$$\begin{cases} \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \right) \cdot \frac{1}{|V_1 \cup V_2|} + & \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, v, e), \bar{\beta}_2(\{f_u(u)\}, f_v(v), g(e))) \right) \cdot \frac{1 + b_{\varepsilon}}{2} + & \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, u, e), \bar{\beta}_2(\{f_v(v)\}, f_u(u), g(e))) \right) \cdot \frac{1 - b_{\varepsilon}}{2} + & \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, u, e), \bar{\beta}_2(\{f_v(v)\}, f_v(v), g(e))) \right) \cdot \frac{1}{|V_1 \cup V_2|} + & \\ rs_{\min}^e(v, f_v(v), v, f_v(v)) \cdot \frac{1}{|V_1 \cup V_2|} + & \\ rs_{\min}^e(u, f_u(u), v, f_v(v)) \end{cases} \quad \text{otherwise}$$

$$rs_{\min}^e(u_0, f_{u_0}, v, f_v) := \begin{cases} r_{\min}^{e+\infty}(u_0 + 1, f_{u_0} + 1, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \forall i \in [m_v]_0 \quad (\{v\}, i, 0) \notin E_1 \wedge (\{i\}, v, 0) \notin E_1 \wedge (\{f_v\}, f_u(i), 0) \notin E_2 \wedge (\{f_u(i)\}, f_v, 0) \notin E_2 \\ r_{\min}^{e+\infty}(u_0 + 1, 0, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \text{otherwise} \end{cases}$$

$$r_{\min}^{e+\infty}(s, f_s, U, f_U, v, f_v, j) := \begin{cases} \min_{\substack{f_j: [m_v]_s \xrightarrow{1:1} [m_v]_{f_s} - f_U \\ v \geq s \implies f_j(v) := f_v}} \sum_{i=s}^{m_v} r_{\min}^{e+\infty}(i+1, f_j(i)+1, U \cup \{i\}, f_U \cup \{f_j(i)\}, v, f_v, j+1) & \forall s \leq i \leq m_v \quad (U \cup \{i\}, v, 0) \notin E_1 \wedge (f_U \cup \{f_j(i)\}, f_v, 0) \notin E_2 \\ \sum_{i=s}^{m_v} \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(U \cup \{i\}, v, e), \bar{\beta}_2(f_U \cup \{f_j(i)\}, f_v, g(e))) \right) + r_{\min}^{e+\infty}(i+1, 0, U \cup \{i\}, f_U \cup \{f_j(i)\}, v, f_v, j+1) & \text{otherwise} \end{cases}$$

$$N_E^+(v) := \{u \in V_1 \cup V_2 \mid (\{v\}, u, 0) \in E \vee (\exists_{(V_1 \cup V_2) \supseteq U_0 \supseteq \{u\}} (\forall_{w \in U_0} w \geq u) \wedge (U_0, v, 0) \in E)\}$$

$$N_E(v) := \{u \in V_1 \cup V_2 \mid (\{v\}, u, 0) \in E \vee (\exists_{(V_1 \cup V_2) \supseteq U_0 \supseteq \{u\}} (U_0, v, 0) \in E)\}$$

$$c(H_1, H_2) = \mathcal{O} \left( \left[ \frac{|V_1 \cup V_2|^2}{p} \right] \cdot \left( |V_1 \cup V_2|^3 + |V_1 \cup V_2|^{2+2 \cdot (\max_{(U, v, e) \in (E_1 \cup E_2)} |U|)} \cdot \left( \max_{(U, v, e) \in E_1 \cup E_2} e \right)^3 \right) \right), \quad \min_{f: A \xrightarrow{1:1} B} \sum_{a \in A} \gamma(a, f(a)) := \min_{\substack{f: A \cup C \rightarrow B \\ A \cap C = \emptyset}} \sum_{a \in A \cup C} \begin{cases} \gamma(a, f(a)) & a \in A \\ 0 & a \in C \end{cases}, \quad \min_{\substack{f: [n]_0 \leftrightarrow [n]_0 \\ \forall (x, f_x) \in X \\ f(x) \neq f_x}} \sum_{k=0}^n \gamma(k, f(k)) := \min_{f: [n]_0 \leftrightarrow [n]_0} \sum_{k=0}^n \begin{cases} \gamma(k, f(k)) & (k, f(k)) \notin X \\ +\infty & (k, f(k)) \in X \end{cases}$$

**Admissible Heuristic for a Parametrized Metric on Undirected Attributed Multi-Hypergraphs**

$$\begin{aligned}
 \rho(H_1, H_2, P, X) := & \sup_{\substack{\mathbf{a}, \mathbf{b}: (V_1 \cup V_2)^4 \rightarrow \mathbb{R} \\ (u \notin N_{E_1}^+(v) \cap [m_v]_{v+1} \vee f_u \notin N_{E_2}(f_v)) \implies (\mathbf{a}(v, u, f_v, f_u) = \mathbf{b}(v, u, f_v, f_u) = \mathbf{b}(u, v, f_u, f_v) = 1)}} \min_{\substack{f_v: [m_v]_0 \leftrightarrow [m_v]_0 \\ \forall (x, f_x) \in X \\ f_v(x) \neq f_x}} \sum_{v=0}^{m_v} \left( \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \right) + \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, e), \bar{\beta}_2(\{f_v(v)\}, g(e))) \right) \right) \cdot \left( 1 - \frac{\min\{m_v - v, |N_{E_1}^+(v) \cap [m_v]_{v+1}| + |N_{E_2}(f_v(v))|\}}{|V_1 \cup V_2|} \right) + \\
 & \begin{cases} 0 & u \notin N_{E_1}^+(v) \wedge f_u(u) \notin N_{E_2}(f_v(v)) \\ +\infty & u \in N_{E_1}^+(v) \wedge f_u(u) \in N_{E_2}(f_v(v)) \wedge (u, f_u(u)) \in X \end{cases} \\
 & \sum_{\substack{f_u, \Psi \xrightarrow{1:1} \Omega \\ [\Psi]_{v+1} \supseteq \Psi \supseteq (N_{E_1}^+(v) \cap [m_v]_{v+1}) \\ ([m_v]_0 - \{f_v(v)\}) \supseteq \Omega \supseteq N_{E_2}(f_v(v)) \\ |\Psi| = \min\{m_v - v, |N_{E_1}^+(v) \cap [m_v]_{v+1}| + |N_{E_2}(f_v(v))|\} \\ |\Omega| = \min\{m_v, |N_{E_1}^+(v) \cap [m_v]_{v+1}| + |N_{E_2}(f_v(v))|\}}} \min_{\substack{f_u, \Psi \xrightarrow{1:1} \Omega \\ [\Psi]_{v+1} \supseteq \Psi \supseteq (N_{E_1}^+(v) \cap [m_v]_{v+1}) \\ ([m_v]_0 - \{f_v(v)\}) \supseteq \Omega \supseteq N_{E_2}(f_v(v)) \\ |\Psi| = \min\{m_v - v, |N_{E_1}^+(v) \cap [m_v]_{v+1}| + |N_{E_2}(f_v(v))|\} \\ |\Omega| = \min\{m_v, |N_{E_1}^+(v) \cap [m_v]_{v+1}| + |N_{E_2}(f_v(v))|\}}} \sum_{u \in \Psi} \left( \begin{array}{l} \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \right) \cdot \frac{1 + \mathbf{a}(u, v, f_u(u), f_v(v))}{2 \cdot |V_1 \cup V_2|} + \\ \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(u), \bar{\alpha}_2(f_u(u))) \right) \cdot \frac{1 - \mathbf{a}(v, u, f_v(v), f_u(u))}{2 \cdot |V_1 \cup V_2|} + \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, e), \bar{\beta}_2(\{f_v(v)\}, g(e))) \right) \cdot \frac{1 + \mathbf{b}(u, v, f_u(u), f_v(v))}{2 \cdot |V_1 \cup V_2|} + \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, e), \bar{\beta}_2(\{f_u(u)\}, g(e))) \right) \cdot \frac{1 - \mathbf{b}(v, u, f_v(v), f_u(u))}{2 \cdot |V_1 \cup V_2|} + \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u, v\}, e), \bar{\beta}_2(\{f_u(u), f_v(v)\}, g(e))) \right) + \\ rs_{\min}^e(u, f_u(u), v, f_v(v)) \end{array} \right) \\ \left( \begin{array}{l} \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v(v))) \right) \cdot \frac{1}{|V_1 \cup V_2|} + \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, e), \bar{\beta}_2(\{f_v(v)\}, g(e))) \right) \cdot \frac{1}{|V_1 \cup V_2|} + \\ \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u, v\}, e), \bar{\beta}_2(\{f_u(u), f_v(v)\}, g(e))) \right) + \\ rs_{\min}^e(u, f_u(u), v, f_v(v)) \end{array} \right) \quad \text{otherwise} \end{cases} \\
 & rs_{\min}^e(u_0, f_{u_0}, v, f_v) := \begin{cases} r_{\min}^{e+\infty}(u_0 + 1, f_{u_0} + 1, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \forall_{i \in [m_v]_0} (\{v, i\}, 0) \notin E_1 \wedge (\{f_v, f_u(i)\}, 0) \notin E_2 \\ r_{\min}^{e+\infty}(u_0 + 1, 0, \{u_0\}, \{f_{u_0}\}, v, f_v, 1) & \text{otherwise} \end{cases} \\
 & \forall_{s \leq i \leq m_v} (U \cup \{i, v\}, 0) \notin E_1 \wedge (f_U \cup \{f_j(i), f_v\}, 0) \notin E_2 \\
 & r_{\min}^{e+\infty}(s, f_s, U, f_U, v, f_v, j) := \min_{f_j: ([m_v]_s - \{v\}) \xrightarrow{1:1} ([m_v]_{f_s} - f_U - \{f_v\})} \begin{cases} \sum_{i=s}^{m_v} r_{\min}^{e+\infty}(i + 1, f_j(i) + 1, U \cup \{i\}, f_U \cup \{f_j(i)\}, v, f_v, j + 1) & \forall_{s \leq i \leq m_v} (U \cup \{i, v\}, 0) \notin E_1 \wedge (f_U \cup \{f_j(i), f_v\}, 0) \notin E_2 \\ \sum_{i=s}^{m_v} \left( \min_{g: [m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(U \cup \{i, v\}, e), \bar{\beta}_2(f_U \cup \{f_j(i), f_v\}, g(e))) \right) + r_{\min}^{e+\infty}(i + 1, 0, U \cup \{i\}, f_U \cup \{f_j(i)\}, v, f_v, j + 1) & \text{otherwise} \end{cases} \\
 & N_E^+(v) := \{u \in (V_1 \cup V_2) - \{v\} \mid \exists_{(V_1 \cup V_2) \supseteq U_0 \supseteq \{u, v\}} (\forall_{w \in U_0 - \{v\}} w \geq u) \wedge (U_0, 0) \in E\}, \\
 & N_E(v) := \{u \in (V_1 \cup V_2) - \{v\} \mid \exists_{(V_1 \cup V_2) \supseteq U_0 \supseteq \{u, v\}} (U_0, 0) \in E\}, \\
 & c(H_1, H_2) = \mathcal{O} \left( \left[ \frac{|V_1 \cup V_2|^2}{p} \right] \cdot \left( |V_1 \cup V_2|^3 + |V_1 \cup V_2|^{2+2 \cdot (\max_{(U, 0) \in (E_1 \cup E_2)} |U|)} \cdot \left( \max_{(U, e) \in E_1 \cup E_2} e \right)^3 \right) \right), \quad \min_{f: A \xrightarrow{1:1} B} \sum_{a \in A} \gamma(a, f(a)) := \min_{\substack{f: A \cup C \rightarrow B \\ |A \cup C| = |B| \\ A \cap C = \emptyset}} \sum_{a \in A \cup C} \begin{cases} \gamma(a, f(a)) & a \in A \\ 0 & a \in C \end{cases}, \quad \min_{\substack{f: [n]_0 \leftrightarrow [n]_0 \\ \forall (x, f_x) \in X \\ f(x) \neq f_x}} \sum_{k=0}^n \gamma(k, f(k)) := \min_{\substack{f: [n]_0 \leftrightarrow [n]_0 \\ \forall (x, f_x) \in X \\ f(x) \neq f_x}} \sum_{k=0}^n \begin{cases} \gamma(k, f(k)) & (k, f(k)) \notin X \\ +\infty & (k, f(k)) \in X \end{cases}.
 \end{aligned}$$

**Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Directed Attributed Multi-Hypergraphs**

$$\begin{aligned}
 a_0(u, v, f_u, f_v) &= c_0(u, v, f_u, f_v) = d_0(u, v, f_u, f_v) := \begin{cases} 1 & u \notin N_{E_1}^+(v) \vee v \notin N_{E_1}^+(u) \vee f_u \notin N_{E_2}(f_v), \\ 0 & \text{otherwise} \end{cases}, \quad \forall_{u,v,f_u,f_v \in [m_v]_0} b_0(u, v, f_u, f_v) = b_{\varepsilon,0} := 0, \\
 a_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( a_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2 \cdot |V_1 \cup V_2|} \cdot \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v)) \right) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \wedge v \in N_{E_1}^+(u) \wedge u \neq v \wedge f_u \in N_{E_2}(f_v), \\ 1 & \text{otherwise} \end{cases}, \\
 b_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( b_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2} \cdot \left( \min_{g:[m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, v, e), \bar{\beta}_2(\{f_u\}, f_v, g(e))) \right) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \wedge v \in N_{E_1}^+(u) \wedge u \neq v \wedge f_u \in N_{E_2}(f_v), \\ b_{\varepsilon,n} & \text{otherwise} \end{cases}, \\
 c_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( c_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2 \cdot |V_1 \cup V_2|} \cdot \left( \min_{g:[m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, v, e), \bar{\beta}_2(\{f_v\}, f_v, g(e))) \right) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \wedge v \in N_{E_1}^+(u) \wedge u \neq v \wedge f_u \in N_{E_2}(f_v), \\ 1 & \text{otherwise} \end{cases}, \\
 d_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( d_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2 \cdot |V_1 \cup V_2|} \cdot r s_{\min}^e(v, f_v, v, f_v) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \wedge v \in N_{E_1}^+(u) \wedge u \neq v \wedge f_u \in N_{E_2}(f_v), \\ 1 & \text{otherwise} \end{cases}, \\
 b_{\varepsilon,n+1} &:= b_{\varepsilon,n} + \frac{\eta_{n+1}}{2} \cdot \left( \sum_{v=0}^{m_v} \sum_{\{u \in \Psi \mid u \notin N_{E_1}^+(v) \vee v \notin N_{E_1}^+(u) \vee f_{u,n}^{\min}(u) \notin N_{E_2}(f_{v,n}^{\min}(v))\}} \left( \min_{g:[m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{u\}, v, e), \bar{\beta}_2(\{f_{u,n}^{\min}(u)\}, f_{v,n}^{\min}(v), g(e))) \right) - \left( \min_{g:[m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(\{v\}, u, e), \bar{\beta}_2(\{f_{v,n}^{\min}(v)\}, f_{u,n}^{\min}(u), g(e))) \right) \right).
 \end{aligned}$$

**Gradient Ascent-inspired Optimization of Parameters for the Admissible Heuristic for Undirected Attributed Multi-Hypergraphs**

$$\begin{aligned}
 a_0(u, v, f_u, f_v) &= b_0(u, v, f_u, f_v) := \begin{cases} 1 & u \notin N_{E_1}^+(v) \cap [m_v]_{v+1} \vee v \notin N_{E_1}^+(u) \cap [m_v]_{u+1} \vee f_u \notin N_{E_2}(f_v), \\ 0 & \text{otherwise} \end{cases}, \\
 a_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( a_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2 \cdot |V_1 \cup V_2|} \cdot \left( \sum_{\bar{\alpha} \in \bar{A}} \rho_{\Gamma_{\bar{\alpha}}}(\bar{\alpha}_1(v), \bar{\alpha}_2(f_v)) \right) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \cap [m_v]_{v+1} \wedge v \in N_{E_1}^+(u) \cap [m_v]_{u+1} \wedge f_u \in N_{E_2}(f_v), \\ 1 & \text{otherwise} \end{cases}, \\
 b_{n+1}(u, v, f_u, f_v) &:= \begin{cases} \left( b_n(u, v, f_u, f_v) + \frac{\eta_{n+1}}{2 \cdot |V_1 \cup V_2|} \cdot \left( \min_{g:[m_e]_0 \leftrightarrow [m_e]_0} \sum_{e=0}^{m_e} \sum_{\bar{\beta} \in \bar{B}} \rho_{\Gamma_{\bar{\beta}}}(\bar{\beta}_1(v, e), \bar{\beta}_2(f_v, g(e))) \right) \cdot \begin{cases} 1 & f_u = f_{u,n}^{\min}(u) \wedge f_v = f_{v,n}^{\min}(v) \wedge (f_v \neq f_{u,n}^{\min}(v) \vee f_u \neq f_{v,n}^{\min}(u)) \\ -1 & f_v = f_{u,n}^{\min}(v) \wedge f_u = f_{v,n}^{\min}(u) \wedge (f_u \neq f_{u,n}^{\min}(u) \vee f_v \neq f_{v,n}^{\min}(v)) \\ 0 & \text{otherwise} \end{cases} \right) & u \in N_{E_1}^+(v) \cap [m_v]_{v+1} \wedge v \in N_{E_1}^+(u) \cap [m_v]_{u+1} \wedge f_u \in N_{E_2}(f_v), \\ 1 & \text{otherwise} \end{cases}.
 \end{aligned}$$