KTU LECTURE NOTES

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LOGIC AND FUZZY SYSTEMS

LECTURE 7

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Linguistic variables

- Fuzzy logic uses linguistic variables.
- The values of a linguistic variables are words or sentences in a natural language.
- It provides approximate characterization of a complex problem.
- The name of the variable, universe of discourse and a fuzzy subset characterize a fuzzy variable.

- A *linguistic variable* is characterized by,
    1. *name* of the variable, x
    2. *term set* of the variable, $t(x)$
    3. *syntactic rule* for generating the values of x
    4. *semantic rule* for associating each value of x with its meaning.
- *Linguistic hedges(Linguistic modifiers)*

Reasoning and Propositions

- *Reasoning* has logic as its basis, whereas *propositions* are text sentences expressed in any languages.
- Proposition is expressed in an canonical form as,

  $$Z \textit{ is } P$$

  where,
  Z is the subject
  P is the predicate designing the characteristics of subject.
- Every proposition has its opposite called *negation*.

Truth tables

- *Truth tables* define logic functions of two propositions.
- Let X and Y be two propositions, either of which can be *True* or *False*.
- The basic logic operations performed over these propositions are:
  1. *Conjunction* ($\wedge$): X AND Y
  2. *Disjunction* ($\vee$): X OR Y
  3. *Implication or Conditional* ($\Rightarrow$): IF X THEN Y
  4. *Bidirectional or Equivalence* ($\Leftrightarrow$): X IF AND ONLY IF Y

X	Y	$X \wedge Y$	$X \vee Y$	$X \Rightarrow Y$	$X \Leftrightarrow Y$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Tautology

- On the basis of these operations on propositions, inference rules can be formed.
- These rules produce certain propositions that are always true irrespective of the truth values of propositions.

Example

Show that,

$$(P \Rightarrow Q) \vee (Q \Rightarrow P)$$

is a tautology.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \vee (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$(P \Rightarrow Q) \vee (Q \Rightarrow P)$ *is a tautology.*

Truth values

- The truth values of propositions are allowed to range over the unit interval $[0, 1]$.
- A truth value in fuzzy logic "*very true*" may be interpreted as a fuzzy set in $[0, 1]$.
- Truth value of proposition " *Z is A*" or truth value of A, $tv(A)$, is defined by a point in $[0, 1]$ called *numerical truth value* or a fuzzy set in $[0, 1]$ called *linguistic truth value*.
- Truth value of a proposition can be obtained from the logic operations of other propositions whose truth values are known.

Operations

- If $tv(X)$ and $tv(Y)$ are numerical truth values of propositions X and Y, then,
  - *Intersection*
    $$tv(X\ AND\ Y) = tv(X) \wedge tv(Y) = min\{tv(X), tv(Y)\}$$
  - *Union*
    $$tv(X\ OR\ Y) = tv(X) \vee tv(Y) = max\{tv(X), tv(Y)\}$$
  - *Complement*
    $$tv(NOT\ X) = 1 - tv(X)$$
  - *Implication*
    $$tv(X \Rightarrow Y) = tv(X) \Rightarrow tv(Y) = max\{1 - tv(X), min\{tv(X), tv(Y)\}\}$$

Fuzzy Propositions

- (1) *Fuzzy predicates*: In fuzzy logic, the predicates can be fuzzy.
- (2) *Fuzzy predicate modifiers*: In fuzzy logic, there exists a wide range of predicate modifiers that acts as hedges.
  They are necessary for generating the values of a linguistic variable.
- (3) *Fuzzy quantifiers*: It can be interpreted as a fuzzy proposition, which provides an imprecise characterization of the cardinality of one or more fuzzy or non-fuzzy sets.
  It can be used to represent the meaning of propositions that containing probabilities.

- (4) *Fuzzy qualifiers*:
  - (a) *Fuzzy truth qualification*

    $$x \text{ is } \tau$$

    in which τ is a fuzzy truth value.
  - (b) *Fuzzy probability qualification*

    $$x \text{ is } \lambda$$

    where λ is fuzzy probability.
  - (c) *Fuzzy possibility qualification*

    $$x \text{ is } \pi$$

    where π is a fuzzy possibility.
  - (d) *Fuzzy usuality qualification*

    $$usually(X) = usually(X \text{ is } F)$$

    in which the subject X is a variable taking values in a U and the predicate F is a fuzzy subset of U.

    $$U(X) = F$$

Formation of fuzzy rules

- The general way of representing human knowledge is by forming natural language expressions given by,

  IF antecedent THEN consequent

- There are three general forms that exist for any linguistic variable:
  1. assignment statements
  2. conditional statements
  3. unconditional statements

Assignment statements

- These statement utilizes " = " for assignment.
- The *assignment statements* limit the value of a variable to a specific quantity.
- They are of the form:

  $y = brown$

  $Orange\ color = orange$

  $a = s$

  Paul is not tall and not very short

  $Climate = autumn$

  $Outside\ temperature = normal$

Conditional statements

- The *conditional statements* use the "*IF–THEN*" rule based form.
- Example:
  IF y is very cool THEN stop.
  IF A is high THEN B is low ELSE B is not low.
  IF temperature is high THEN climate is hot.

Unconditional statements

- They can be of the form:

  Goto sum.

  Stop.

  Divide by a.

  Turn the pressure low.

Canonical form of fuzzy rule—based system

Rule 1:	IF condition C_1, THEN restriction R_1
Rule 2:	IF condition C_2, THEN restriction R_2
.	
.	
.	
Rule n:	IF condition C_n, THEN restriction R_n

Decomposition of Rules(Compound Rules)

- A *compound rule* is a collection of many simple rules combined together.
- Any compound rule structure may be decomposed and reduced to a number of simple canonical rule forms.
- Rules are generally based on natural language representations.
- *Methods*
    1. Multiple conjunctive antecedents
    2. Multiple disjunctive antecedents
    3. Conditional statements
    4. Nested–IF–THEN rules

Multiple conjunctive antecedents

$$IF\ x\ is\ A_1, A_2, \cdots, A_n\ THEN\ y\ is\ B_m$$

Assume a new fuzzy subset A_m defined as,

$$A_m = A_1 \cap A_2 \cap \cdots \cap A_n$$

Expressed by means of membership function,

$$\mu_{A_m}(x) = min[\mu_{A_1}(x), \mu_{A_2}(x), \cdots, \mu_{A_n}(x)]$$

The compound rule may be rewritten as,

$$IF\ x\ is\ A_m\ THEN\ y\ is\ B_m$$

Multiple disjunctive antecedents

$$\textit{IF } x \textit{ is } A_1, A_2, \cdots, A_n \textit{ THEN } y \textit{ is } B_m$$

Assume a new fuzzy subset A_m defined as,

$$A_m = A_1 \cup A_2 \cup \cdots \cup A_n$$

Expressed by means of membership function,

$$\mu_{A_m}(x) = max[\mu_{A_1}(x), \mu_{A_2}(x), \cdots, \mu_{A_n}(x)]$$

The compound rule may be rewritten as,

$$\textit{IF } x \textit{ is } A_m \textit{ THEN } y \textit{ is } B_m$$

Conditional statements(with ELSE and UNLESS)

Statements of the kind

$$IF\ A_1\ THEN\ (B_1\ ELSE\ B_2)$$

Can be decomposed into two simple canonical rule forms, connected by "OR":

$$IF\ A_1\ THEN\ B_1$$
$$OR$$
$$IF\ NOT\ A_1\ THEN\ B_2$$
$$IF\ A_1\ (THEN\ B_1)\ UNLESS\ A_2$$

Can be decomposed as,

$$IF\ A_1\ THEN\ B_1$$
$$OR$$
$$IF\ A_2\ THEN\ NOT\ B_1$$
$$IF\ A_1\ THEN\ (B_1)\ ELSE\ IF\ A_2\ THEN\ (B_2)$$

Conditional statements(with ELSE and UNLESS)

Statements of the kind

$$IF\ A_1\ THEN\ (B_1\ ELSE\ B_2)$$

Can be decomposed into two simple canonical rule forms, connected by "OR":

$$\begin{gathered} IF\ A_1\ THEN\ B_1 \\ OR \\ IF\ NOT\ A_1\ THEN\ B_2 \end{gathered}$$

$$IF\ A_1\ (THEN\ B_1)\ UNLESS\ A_2$$

Can be decomposed as,

$$\begin{gathered} IF\ A_1\ THEN\ B_1 \\ OR \\ IF\ A_2\ THEN\ NOT\ B_1 \end{gathered}$$

$$IF\ A_1\ THEN\ (B_1)\ ELSE\ IF\ A_2\ THEN\ (B_2)$$

Can be decomposed into the form,

$$\begin{gathered} IF\ A_1\ THEN\ B_1 \\ OR \\ IF\ NOT\ A_1\ AND\ IF\ A_2 \\ THEN\ B_2 \end{gathered}$$

Nested–IF–THEN rules

The rule

$$IF\ A_1\ \ THEN\ [IF\ A_2\ \ THEN\ (B_1)]$$

Can be of the form

$$IF\ A_1\ \ AND\ A_2\ \ THEN\ (B_1)$$

Aggregation of Fuzzy Rules

- *Aggregation of rule* is the process of obtaining the overall consequences from the individual consequences provided by each rule.
- *Methods*
  1. Conjunctive system of rules
  2. Disjunctive system of rules

Conjunctive system of rules

- For a system of rules to be jointly satisfied.
- Rules are connected by " *and* " connectives.
- Aggregated output, y is determined by,

$$y = y_1 \ and \ y_2 \ and \cdots and \ y_n$$

$$or$$

$$y = y_1 \cap y_2 \cap \cdots \cap y_n$$

- This aggregated output can be defined by the membership function,

$$\mu_y(y) = min[\mu_{y_1}(y), \mu_{y_2}(y), \cdots, \mu_{y_n}(y)] \ for \ y \in Y$$

Disjunctive system of rules

- The satisfaction of at least one rule is required.
- Rules are connected by "*or*" connectives.
- Aggregated output, y is determined by,

$$y = y_1 \;\; or \; y_2 \;\; or \cdots \; or \; y_n$$

$$or$$

$$y = y_1 \cup y_2 \cup \cdots \cup y_n$$

- This aggregated output can be defined by the membership function,

$$\mu_y(y) = max[\mu_{y_1}(y), \mu_{y_2}(y), \cdots, \mu_{y_n}(y)] \; for \; y \in Y$$

Linguistic Hedges

In linguistics, fundamental terms are often modified with adjectives or adverbs like very, low, slightly, more or less, fairly, almost, mostly, approximately.....

- The singular meaning of an atomic term is modified or hedged, from its original interpretation.
- Using fuzzy sets as the calculus of interpretation, these linguistic hedges have the effect of modifying the membership function for a basic atomic term.

Let the basic linguistic atom be α, and subject it to some hedges.
Define,

$$\alpha = \int_y \frac{\mu_\alpha(y)}{y}$$

Then,

$$\text{" very "}\ \alpha = \alpha^2 = \int_y \frac{\mu_\alpha(y)^2}{y}$$
$$\text{" very, very "}\ \alpha = \alpha^4$$
$$\text{" plus "}\ \alpha = \alpha^{1.25}$$

$$\text{" very " } \alpha = \alpha^2 = \int_y \frac{\mu_\alpha(y)^2}{y}$$
$$\text{" very, very " } \alpha = \alpha^4$$
$$\text{" plus " } \alpha = \alpha^{1.25}$$

These linguistic hedges known as Concentrations. Concentrations tend to concentrate elements of a fuzzy set by reducing the degree of membership of all elements that are only " partly " in the set.

$$\text{" slightly "}\ \alpha = \sqrt{\alpha} = \int_y \frac{\mu_\alpha(y)^{0.5}}{y}$$
$$\text{" minus "}\ \alpha = \alpha^{0.75}$$
$$\text{" fairly "}\ \alpha = \alpha^{2/3}$$

These linguistic hedges known as Dilations.
Dilations dilate a fuzzy set by increasing the degree of membership of all elements that are only " partly " in the set.

Intensification This operation acts in a combination of concentration and dilation.
It increases the degree of membership of those elements in the set with original membership value greater than 0.5.
It decreases the degree of membership of those elements in the set with original membership value less than 0.5.

$$\text{" intensify "}\alpha = \begin{cases} 2\mu_\alpha^2(y) \ \textit{for}\ 0 \leq \mu_\alpha(y) \leq 0.5 \\ 1 - 2[1-\mu_\alpha(y)]^2 \ \textit{for}\ 0.5 \leq \mu_\alpha(y) \leq 1 \end{cases}$$

Problem

(1) *The membership functions for the linguistic variables " tall " and " short " are :*

$$\text{" } tall \text{ "} = \{\frac{0.2}{5} + \frac{0.3}{7} + \frac{0.7}{9} + \frac{0.9}{11} + \frac{1.0}{12}\}$$

$$\text{" } short \text{ "} = \{\frac{0.3}{0} + \frac{0}{30} + \frac{1}{60} + \frac{0.5}{90} + \frac{0}{120}\}$$

Develop membership functions for the following linguistic phrases:

(*a*) *very tall*

(*b*) *not very short*

(*c*) *fairly tall*

(*d*) *intensely short*

(a)

$$\text{" very tall "} = tall^2 = \{\frac{0.04}{5} + \frac{0.09}{7} + \frac{0.49}{9} + \frac{0.81}{11} + \frac{1.0}{12}\}$$

(b)

$$\text{" very short "} = short^2 = \{\frac{0.09}{0} + \frac{0}{30} + \frac{1}{60} + \frac{0.25}{90} + \frac{0}{120}\}$$

$$\text{" not very short "} = 1 - \text{" very short "}$$

$$= \{\frac{0.91}{0} + \frac{1}{30} + \frac{0}{60} + \frac{0.75}{90} + \frac{1}{120}\}$$

(c)

$$\text{"} \textit{ fairly tall } \text{"} = tall^{2/3} = \{\frac{0.34}{5} + \frac{0.45}{7} + \frac{0.79}{9} + \frac{0.93}{11} + \frac{1.0}{12}\}$$

(d)

$$\text{"} \textit{ intensely short } \text{"}$$
$$= \{\frac{2 \times (0.3)^2}{0} + \frac{0}{30} + \frac{1 - 2[1-1]^2}{60} + \frac{2 \times (0.5)^2}{90} + \frac{0}{120}\}$$
$$= \{\frac{0.18}{0} + \frac{0}{30} + \frac{1}{60} + \frac{0.5}{90} + \frac{0}{120}\}$$

Fuzzy Inference Systems (FIS)

- The key unit of a fuzzy logic system is FIS.
- *Fuzzy rule–based systems*, *fuzzy models*, and *fuzzy expert systems* are generally known as *fuzzy inference systems*.
- The primary work of this system is decision making.
- FIS uses "$IF \cdots THEN$" rules along with connectors "OR" or "AND" for making necessary decision rules.
- The input may be fuzzy or crisp, but the output is always a fuzzy set.

Construction and Working Principle of FIS

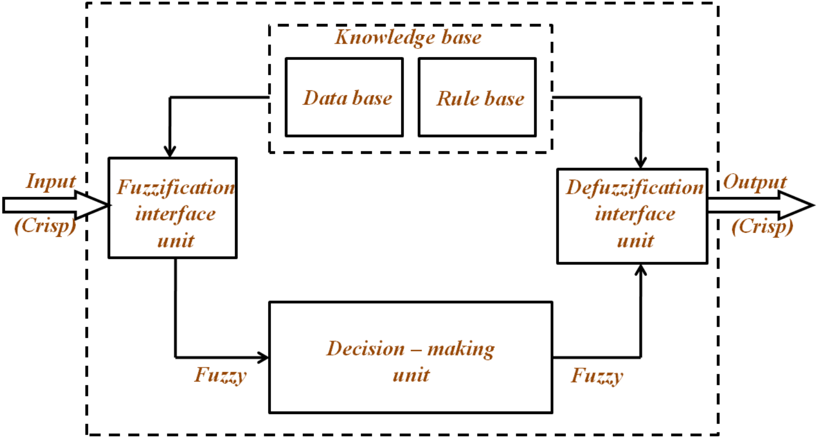


Figure 7.1: Block diagram of FIS

Methods of FIS

1. *Mamdani FIS*
2. *Sugeno FIS*

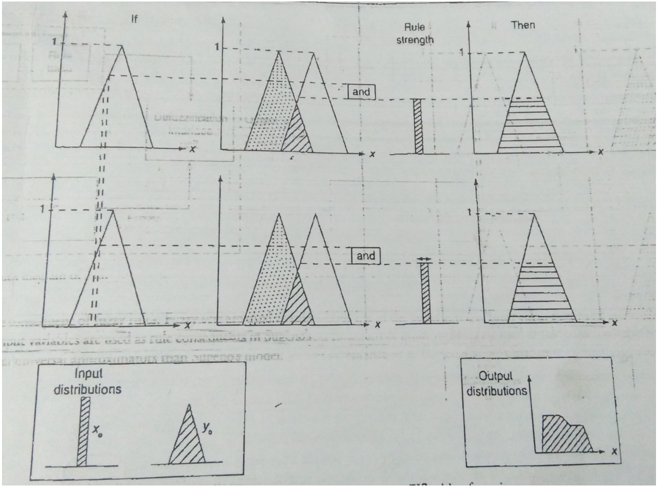
Mamdani FIS

- The output membership functions are expected to be fuzzy sets.
- After aggregation process, each output variable contain a fuzzy set, hence defuzzification is important at the output stage.

- The following steps have to be followed to compute the output:
  1. *Step 1* : Determine a set of fuzzy rules.
  2. *Step 2* : Make the inputs fuzzy using input membership functions.
  3. *Step 3* : Combine the fuzzified inputs according to the fuzzy rules for establishing a rule strength.
  4. *Step 4* : Determine the consequent of the rule by combining the rule strength and the output membership function.
  5. *Step 5* : Combine all the consequents to get an output distribution.
  6. *Step 6* : Finally, a defuzzified output distribution is obtained.

- The fuzzy rules are formed using " IF – THEN " statements and " AND/OR " connectives.
- The consequence of the rule can be obtained in two steps:
  1. by computing the rule strength completely using the fuzzified inputs from the fuzzy combination.
  2. by clipping the output membership function at the rule strength.
- The output of all the fuzzy rules are combined to obtain one fuzzy output distribution

A two–input, two–rule Mamdani FIS with a fuzzy input



Advantages of Mamdani FIS

- It has widespread acceptance.
- it is well suited for human input.
- It is intuitive.

Takagi–Sugeno Fuzzy Model(TS Method)

- The format of the fuzzy rue of a Sugeno fuzzy model is given by,

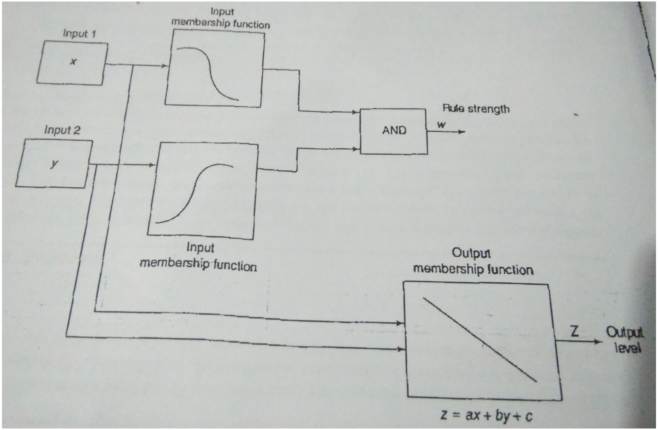
$$IF\ x\ is\ A\ and\ y\ is\ B\ THEN\ z = f(x, y)$$

where
A and B are fuzzy sets in the antecedents
$z = f(x, y)$ is a crisp function in the consequent.

- Generally, $f(x, y)$ is a polynomial in the input variables x and y.
- If $z = f(x, y)$ is a first order polynomial, we get *first– order Sugeno fuzzy model.*
- If f is a constant, we get *zero– order Sugeno fuzzy model.*

- The main steps of the fuzzy inference process namely,
  1. fuzzifying the inputs
  2. applying the fuzzy operator
- Sugeno output membership functions are either linear or constant.

Sugeno rule: IF $x = 3$ and $y = 5$ then output is $z = ax + by + c$



Advantages of TS Method

- It is computationally efficient.
- It is compact and works well with linear technique, optimization technique and adaptive technique.
- It is best suited for mathematical analysis.
- It has a guaranteed continuity on the output surface.

Comparison between Mamdani and Sugeno Method

- Output membership functions.
- Consequences of their fuzzy rules.
- A large number of fuzzy rules must be employed in Sugeno method.
- The configuration of Sugeno fuzzy systems can be reduced and it becomes smaller than that of Mamdani fuzzy systems.
- Sugeno controllers have more adjustable parameters in the rule consequent and number of parameters grow exponentially with the increase in the number of input variables.
- Formation of Mamdani FIS is more easier than Sugeno FIS.

END