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CS_59000_RL

Probability theory

- Probability space
- Conditional Probability
- Integration and Expectation

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consider a set of element Ω $\{5, 1, 2, 3, 6, 4\} \to \{1,2,3,4,5,6\}$

Now, ~~and~~ consider G as a collection of subsets of Ω : $G: \{\{1,2\}, \{2,3\} \dots\}$

Def. A Collection G of subset of a set Ω is said to be a topology in Ω if G has the following three properties:

(i) $\phi \in G$ and $\Omega \in G$

(ii) If $A_i \in G$ for $i=1, \dots n$, then $\bigcap_{i=1}^{n} A_i \in G$

(iii) If $\{A_\alpha\}$ is an arbitrary collection of members of G (finite, countable, or uncountable) then $\bigcup_\alpha A_\alpha \in G$

If G is a topology in Ω, then Ω is called a topological space, and the members of G are called ~~the~~ open set in Ω.

- (a,b)

$[a,b]$

If Ω and Ω' are two topological spaces and if f is a mapping of Ω into Ω', the f is said to be continuous if $f^{-1}(A)$ is an open set in Ω for every open set A in Ω'.

Again consider a set Ω. e.g. $\{1, 2, 3, \ldots 6\}$ Call it out come space.

Now, consider a collection of subsets of Ω.

Def: A collection $\mathcal{F}$ of sub sets of a set Ω is said to be a σ-algebra is Ω if $\mathcal{F}$ has the following properties:

(i) $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) For $A_n \in \mathcal{F}$, $n = 1 \ldots$, if $A = \bigcup_{n=1}^{\infty} A_n$ then $A \in \mathcal{F}$

If $\mathcal{F}$ is a σ-algebra in Ω, the Ω is called a measurable space. and the members of $\mathcal{F}$ are called the measurable set in Ω.

Example: $\{\phi, \Omega\}$

If Ω is a measurable space, Ω' is a topological space, and if f is a mapping of Ω into Ω', the f is said to be measurable, if $f^{-1}(A)$ is a measurable set in Ω, for every open set A in Ω'.

Borel sets: ($\mathcal{B}$): For a topological space Ω, the smallest σ-algebra in Ω that also contains all open sets in Ω is called Borel ~~set~~ space. The members of $\mathcal{B}$ are called the Borel sets.

$\mathcal{B}(R)$: A collection of all open intervals.

$(\Omega, \mathcal{F})$ is called measurable space.

Def: A measure (positive) is a function μ, defined on a σ-algebra $\mathcal{F}$, ~~where~~ whose range is in $[0, \infty)$ and which is countablly additive. This means that if $\{A_i\}$ is a disjoint countable collection of members of $\mathcal{F}$, the $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

(~~There~~ Ther exist at least one $A \in \mathcal{F}$ s.t. $\mu(A) < \infty$)

A measurable space is a ~~mea~~ measurable space which has a measure defined on the σ-algebra of its measurable sets.

4)

$(\Omega, \mathcal{F}, \mu)$ is called a measure space.

If $\mu(\Omega) = 1 \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ is called probability space.

Bayes rule. For $A, B \in \mathcal{F}$, If $\mathbb{P}(B) > 0$

$$\rightarrow \mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

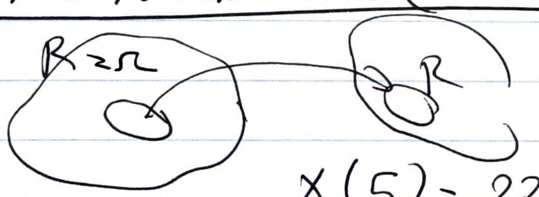
Independence: we say A, and B are independant if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

HW: $E[XY] = E[X]\,E[Y]$

Def: A random varible on a measurable space $(\Omega, \mathcal{F})$ is a function $X: \Omega \rightarrow \mathbb{R}$ such that $X^{-1}(A) \in \mathcal{F}$ for all A in $\mathcal{B}(\mathbb{R})$

For a probability ~~spa~~ space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X, P_X is the law of X if, $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R})$

Pushforwar measure



$X(5) = 22$

$X((9,25)) = (\ )$

5)

If $\mathcal{F}$ is a σ-algebra, and $\mathcal{G} \subset \mathcal{F}$ is also a σ-algebra, then, we say $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$.

$\mathcal{F} = \{\{1\}, \{2\}, \{1,2\} \cdots \{1,2,3\}\}$

$\mathcal{G} = \{\phi, \Omega, \{1,2,3\}, \{4,5,6\}\}$

If $\mathbb{P}$ is a measure on $\mathcal{F}$, the restriction of $\mathbb{P}$ to $\mathcal{G}$ is a ~~mean~~ measure $\mathbb{P}_{|\mathcal{G}}$ on $\mathcal{G}$, such that

$$\mathbb{P}_{|\mathcal{G}}(A) = \mathbb{P}(A) \quad \text{for all } A \in \mathcal{G}$$

Consider throwing a coin n times:

what are the possible outcomes. $\{0,1\}$

$x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_n$

$\Omega \{ (0, 0, 0 \ldots 0, 0)$

$(0, 0, 0, \ldots 0, 1)$

$(0, 0, 0, \ldots 1, 0)$

$\vdots$

$0, 1, \ldots$

$0, 1$

$(1, 0 \ldots 0, 0)$

$(1, 0, \ldots, 0, 1)$

$(1, 0, \ldots, 1, 0)$

$\vdots$

$1, 0$

$1, 1$

$(1, 1 \ldots 1)\}$

$\mathcal{F} = 2^{\Omega}$

$\mathcal{F}_0 = \{\phi, \Omega\}$

$\mathcal{F}_1 = \{\phi, \Omega,$ first half, second half$\}$

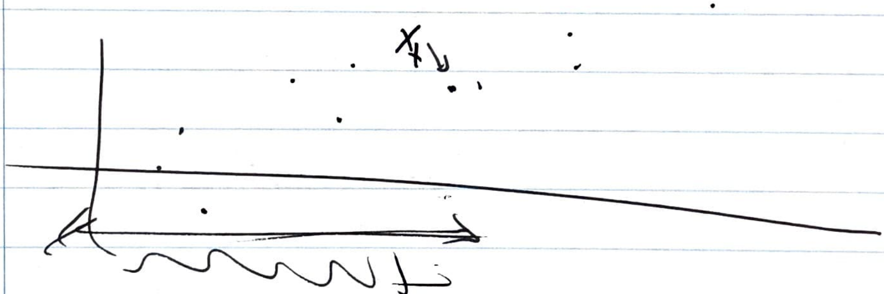
$\mathcal{F}_2 = \{\phi, \Omega,$ firsts half, second half, $1/4, 2/4, 3/4, 4/4\}$

6)

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 - \subseteq \mathcal{F}_n \subseteq \mathcal{F}$$

Given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of sub-σ-algebras of $\mathcal{F}$, wher $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t < n$ (n can be ∞)

A sequence of random variables $(X_t)_{t=1}^{n}$ is adapted to filtration $\mathbb{F} = (\mathcal{F}_t)_{t=1}^{n}$, if X_t is $\mathcal{F}_t$-measurable at each $1 \le t \le n$.



$\mathcal{B}(\mathbb{R})^n$