解析学前期 03 いろいろな不定積分 課題

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締め切り 2025/04/29

03-01

(1) $\int e^x \cos x dx$

$$= e^x \cos x + \int e^x \sin x dx$$
$$= e^x \cos x + e^x \sin x - \int e^x \cos x dx$$
$$= \frac{1}{2} e^x (\sin x + \cos x) + C$$

(2) $\int e^{-x} \sin x dx$

$$= e^{-x} \sin x - \int e^{-x} \cos x dx$$
$$= e^{-x} \sin x - (-e^{-x} \cos x - \int e^{-x} \sin x dx)$$
$$= e^{-x} \sin x + e^{-x} \cos x - \int e^{-x} \sin x dx$$
$$= \frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

(3) $\displaystyle\int \sin(\log x)\, dx$

$t = \log x, \quad x = e^t, \quad dx = e^t dt$

$\displaystyle\int \sin(\log x)\, dx = \int e^t \sin t\, dt$

$\displaystyle\int e^t \sin t\, dt = e^t \sin t - \int e^t \cos t\, dt$

さらに: $\displaystyle\int e^t \cos t\, dt = e^t \cos t + \int e^t \sin t\, dt$

$\displaystyle\Rightarrow \int e^t \sin t\, dt - \int e^t \sin t\, dt = e^t \sin t - e^t \cos t$

$\displaystyle 2 \int e^t \sin t\, dt = e^t(\sin t - \cos t)$

$\displaystyle\int e^t \sin t\, dt = \tfrac{1}{2} e^t(\sin t - \cos t) + C$

$\displaystyle\therefore \int \sin(\log x)\, dx = \tfrac{1}{2} x\big(\sin(\log x) - \cos(\log x)\big) + C$

03-02 $x + \sqrt{x^2+1} = t$、$\int \frac{1}{\sqrt{x^2+1}} dx$

両辺を微分すると: $1 + \dfrac{x}{\sqrt{x^2+1}} = \dfrac{dt}{dx}$

したがって: $\dfrac{dt}{dx} = \dfrac{x + \sqrt{x^2+1}}{\sqrt{x^2+1}} = t$

よって: $dx = \dfrac{dt}{t}$ また: $\sqrt{x^2+1} = t - x$

積分は次のように変形される: $\displaystyle\int \frac{1}{\sqrt{x^2+1}} dx = \int \frac{1}{t-x} \cdot \frac{dt}{t}$

$\displaystyle\int \frac{1}{\sqrt{x^2+1}} dx = \int \frac{1}{t} dt = \log|t| + C$

元の変数に戻して: $\log|x + \sqrt{x^2+1}| + C$

03-03 $\frac{x+3}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$ a, b, c の値を定めよ。$\int \frac{x+3}{x(x-1)^2} dx$

$$\begin{aligned} x+3 &= a(x-1)^2 + b\,x(x-1) + c\,x \\ &= (a+b)x^2 + (-2a-b+c)x + a \end{aligned}$$

係数比較より: $a + b = 0, \quad -2a - b + c = 1, \quad a = 3$

$\Rightarrow a = 3,\ b = -3,\ c = 4$

よって: $\displaystyle\int \frac{x+3}{x(x-1)^2}\, dx = \int \Big(\frac{3}{x} - \frac{3}{x-1} + \frac{4}{(x-1)^2}\Big)\, dx$

$\displaystyle = 3\log|x| - 3\log|x-1| - \frac{4}{x-1} + C$

03-04

(1) $\int \frac{3x^2-2x}{(x+2)(x-1)^2} dx$

$$\frac{3x^2-2x}{(x+2)(x-1)^2} = \frac{\frac{16}{9}}{x+2} + \frac{\frac{11}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2}$$

(与式) $\displaystyle = \frac{16}{9}\log|x+2| + \frac{11}{9}\log|x-1|$

$\displaystyle - \frac{1}{3(x-1)} + C$

(2) $\int \frac{1}{x(x^2+1)}dx$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1},$$

$$(\text{与式}) = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \log|x| - \frac{1}{2}\log|x^2+1| + C.$$

03-05

(1) $\int \frac{x^2+x+1}{x^2+1}dx$

$$= \int \left(1 + \frac{x}{x^2+1}\right) dx = \int 1\, dx + \int \frac{x}{x^2+1}\, dx = x + \frac{1}{2}\log|x^2+1| + C.$$

(2) $\int \frac{x^4}{x^2-1}dx$

$$\frac{x^4}{x^2-1} = x^2+1+\frac{1}{x^2-1}$$

$$\int \frac{x^4}{x^2-1}dx = \int (x^2+1)dx + \int \frac{1}{x^2-1}dx = \frac{x^3}{3} + x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

(3) $\int \frac{x^3}{x^2-4}dx$

$$\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}$$

$$\int \frac{x^3}{x^2-4}dx = \int x\, dx + 4\int \frac{x}{x^2-4}dx = \frac{x^2}{2} + 4\cdot\frac{1}{2}\log\left|x^2-4\right| + C = \frac{x^2}{2} + 2\log\left|x^2-4\right| + C.$$

03-06 $\tan\frac{x}{2} = t$ とおき $\int \frac{1}{\sin x - 1}dx$ を求めよ

$$\tan\frac{x}{2} = t, \quad \sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

$$\sin(x-1) = \frac{2t}{1+t^2} - 1 = \frac{2t-1-t^2}{1+t^2} = -\frac{(t-1)^2}{1+t^2}.$$

よって, $\displaystyle\int \frac{1}{\sin x - 1}\, dx = \int \frac{1+t^2}{-(t-1)^2}\cdot\frac{2dt}{1+t^2}$

$$= -2\int \frac{dt}{(t-1)^2}.$$

ここで $u = t-1$ とおくと, $du = dt$。

$\displaystyle\int \frac{du}{u^2} = -\frac{1}{u} + C$、戻して,

$$\int \frac{1}{\sin x - 1}\, dx = \frac{2}{\tan\frac{x}{2} - 1} + C.$$

03-07

(1) $\int \frac{1}{1+\tan x}dx$

$$= \int \frac{\cos x}{\sin x + \cos x}\, dx = \frac{1}{2}\int \frac{(\sin x + \cos x) + (\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int\left(1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right) dx$$

$$= \frac{x}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

ここで $u = \sin x + \cos x, \quad du = (\cos x - \sin x)dx,$

$$\Rightarrow \quad \int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \log|u| + C = \log|\sin x + \cos x| + C,$$

$$\therefore \quad \int \frac{1}{1+\tan x}\, dx = \frac{x}{2} + \frac{1}{2}\log|\sin x + \cos x| + C.$$

(2) $\int \frac{1}{3\sin x + 4\cos x}dx$

$$3\sin x + 4\cos x = 5\sin\left(x + \arcsin\frac{4}{5}\right),$$

$$\int \frac{dx}{3\sin x + 4\cos x} = \frac{1}{5}\int \csc\left(x + \arcsin\frac{4}{5}\right)dx = \frac{1}{5}\ln\left|\tan\left(\frac{x + \arcsin\frac{4}{5}}{2}\right)\right| + C.$$