Summary of test statistics studied in “Corrected Goodness-of-fit Test in Covariance Structure Analysis”

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Psychological Methods

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In a structural equation modeling, a model-implied covariance matrix $\mathbf{\Sigma}(\boldsymbol{\theta}), (p \times p)$ is fitted to the sample covariance matrix $\mathbf{S}_N$ which is obtained from N observations $(\mathbf{z}_1, ..., \mathbf{z}_N), (p \times 1)$ in order to minimize a discrepancy function. In practice, unknown parameter $\boldsymbol{\theta}$ is typically obtained by minimizing the Wishart likelihood $F_{ML}(\boldsymbol{\theta})$:

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}_{ML} &= \underset{\theta}{\operatorname{argmin}} \, F_{ML}\left(\boldsymbol{\theta}\right), \\
F_{ML}\left(\boldsymbol{\theta}\right) &= \log\left|\mathbf{\Sigma}\left(\boldsymbol{\theta}\right)\right| - \log\left|\mathbf{S}_N\right| + \operatorname{tr}\left(\mathbf{S}_N \mathbf{\Sigma}\left(\boldsymbol{\theta}\right)^{-1}\right) - p.
\end{aligned}$$

The conventional goodness of fit test or a likelihood ratio test is defined as

$$T_{ML} = n \cdot \widehat{F}_{ML} \tag{1}$$

where $\widehat{F}_{ML} = F_{ML}(\widehat{\boldsymbol{\theta}}_{ML})$ and $n = N - 1$.

The reweighted least squares (RLS) goodness-of-fit test due to Browne (1974) is defined as

$$T_{RLS} = \frac{n}{2} \operatorname{tr}\left\{\left(\mathbf{S}_N - \widehat{\mathbf{\Sigma}}\right)\widehat{\mathbf{\Sigma}}^{-1}\right\}^2 = \frac{n}{2}\left\{\mathbf{s}_N - \widehat{\boldsymbol{\sigma}}\right\}' \mathbf{D}_p' \left(\widehat{\mathbf{\Sigma}}^{-1} \otimes \widehat{\mathbf{\Sigma}}^{-1}\right) \mathbf{D}_p \left\{\mathbf{s}_N - \widehat{\boldsymbol{\sigma}}\right\} \tag{2}$$

where $\widehat{\mathbf{\Sigma}} = \mathbf{\Sigma}(\widehat{\boldsymbol{\theta}}_{ML})$, $\widehat{\boldsymbol{\sigma}} = \operatorname{vech}(\widehat{\mathbf{\Sigma}})$, $\mathbf{s}_N = \operatorname{vech}(\mathbf{S}_N)$, and $\mathbf{D}_p$ is a $p^2 \times \frac{p(p+1)}{2}$ duplication matrix such that $\operatorname{vec}(\mathbf{A}) = \mathbf{D}_p \operatorname{vech}(\mathbf{A})$. Browne (1974) has demonstrated that two test statistics T_{ML} and T_{RLS} are related as follows:

$$T_{ML} = T_{RLS} + B \tag{3}$$

where $B = n\sum_{k=3}^{\infty} \frac{1}{k} \operatorname{tr}\left\{\mathbf{I}_p - \mathbf{S}_N \widehat{\mathbf{\Sigma}}^{-1}\right\}^k$.

Under multivariate normality and correct specification for $\mathbf{\Sigma}(\boldsymbol{\theta})$, T_{ML} and T_{RLS} asymptotically follows the chi-square distribution as $N \to \infty$ with p fixed (Browne, 1974, Proposition 7):

$$T_{ML},\ T_{RLS} \xrightarrow{d} \chi^2_{df}$$

where $df = p^* - q$ with $p^* = p(p+1)/2$ and q is the number of free parameters in $\boldsymbol{\theta}$.

Under the violation of multivariate normality, T_{ML} and T_{RLS} do not follow the standard chi-square distribution. Corrected tests that approximately follow the standard chi-square distribution are defined as

$$\dot{T}_{ML} = \frac{df}{\widehat{a}_1}T_{ML}, \qquad \ddot{T}_{ML} = \frac{\widehat{a}_1}{\widehat{a}_2}T_{ML}, \qquad \ddot{T}^c_{ML} = \frac{\widehat{a}_1}{\widehat{a}^c_2}T_{ML}, \tag{4}$$

$$\dot{T}_{RLS} = \frac{df}{\widehat{a}_1}T_{RLS}, \qquad \ddot{T}_{RLS} = \frac{\widehat{a}_1}{\widehat{a}_2}T_{RLS}, \qquad \ddot{T}^c_{RLS} = \frac{\widehat{a}_1}{\widehat{a}^c_2}T_{RLS}, \tag{5}$$

where

$$\begin{aligned}
\widehat{a}_1 &= \mathrm{tr}(\widehat{\mathbf{V}}), \qquad \widehat{a}_2 = \mathrm{tr}(\widehat{\mathbf{V}}^2), \\
\widehat{a}^c_2 &= \frac{(N-2)(N-1)\,\mathrm{tr}\left(\mathbf{H}^2\right) - N(N-1)\,\mathrm{tr}\left(\mathbf{D}^2\right) + [\mathrm{tr}\,(\mathbf{H})]^2}{N(N-1)(N-2)(N-3)}
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathbf{V}} &= \widehat{\mathbf{U}}^{1/2}\widehat{\boldsymbol{\Omega}}\widehat{\mathbf{U}}^{1/2}, \qquad \widehat{\mathbf{U}} = \widehat{\mathbf{W}} - \widehat{\mathbf{W}}\widehat{\mathbf{G}}\left(\widehat{\mathbf{G}}'\widehat{\mathbf{W}}\widehat{\mathbf{G}}\right)^{-1}\widehat{\mathbf{G}}'\widehat{\mathbf{W}}, \qquad \widehat{\mathbf{W}} = \frac{1}{2}\mathbf{D}'_p\left(\widehat{\boldsymbol{\Sigma}}^{-1}\otimes\widehat{\boldsymbol{\Sigma}}^{-1}\right)\mathbf{D}_p, \\
\widehat{\mathbf{G}} &= \frac{\partial\boldsymbol{\sigma}(\widehat{\boldsymbol{\theta}}_{ML})}{\partial\boldsymbol{\theta}'}, \quad \widehat{\boldsymbol{\Omega}} = \frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{s}_i-\bar{\mathbf{s}})(\mathbf{s}_i-\bar{\mathbf{s}})', \quad \mathbf{s}_i = \mathrm{vech}\{(\mathbf{z}_i-\bar{\mathbf{z}})(\mathbf{z}_i-\bar{\mathbf{z}})'\}, \quad \bar{\mathbf{s}} = N^{-1}\sum_{i=1}^{N}\mathbf{s}_i, \\
\mathbf{H} &= \sum_{i=1}^{N}\mathbf{y}_i\mathbf{y}'_i, \quad \mathbf{D} = \mathrm{diag}\left(\mathbf{y}'_1\mathbf{y}_1, ..., \mathbf{y}'_N\mathbf{y}_N\right), \quad \mathbf{y}_i = \widehat{\mathbf{w}}_i - \bar{\widehat{\mathbf{w}}}, \quad \bar{\widehat{\mathbf{w}}} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\mathbf{w}}_i, \quad \widehat{\mathbf{w}}_i = \widehat{\mathbf{U}}^{1/2}\mathbf{s}_i.
\end{aligned}$$

The notations used in R code and in the paper are summarized in the following table.

Table 1: Notations in R code and paper

R code	Paper
`T_ML`	T_{ML}
`T_RLS`	T_{RLS}
`T_SB1_ML`	$\dot{T}_{ML}$
`T_SB1_RLS`	$\dot{T}_{RLS}$
`T_SB2_ML`	$\ddot{T}_{ML}$
`T_SB2_RLS`	$\ddot{T}_{RLS}$
`T_SB2_ML_c`	$\ddot{T}^c_{ML}$
`T_SB2_RLS_c`	$\ddot{T}^c_{RLS}$

References

Browne, M. W. (1974) "Generalized Least Squares Estimators in the Analysis of Covariances," *South African Statistical Journal*, 8, p. 24.