- Equilibrium: x_{eq} if $f(x_{eq})=0$
- Linearization

$$\dot{x} = f(x_{eq}, u_{eq}) + \overbrace{\frac{\partial f}{\partial x}\Big|_{\substack{x_{eq}\\u_{eq}}}}^{A}(x-x_{eq}) + \overbrace{\frac{\partial f}{\partial u}\Big|_{\substack{x_{eq}\\u_{eq}}}}^{B}(u-u_{eq})$$

- Eigenvalues $\det(A-\lambda I)=0$

$Av=\lambda v$. If $A=A^T$ (symmetric), λs are all real.

Decomposition: $A = Q\Lambda Q^{-1}$ — n lin indep. ev; Λ Diagonal

$Q>0$ $\Updownarrow$ All e-values $Q>0$ real parts $\Updownarrow$ $x^TQx>0$

- Supremum: ~~greatest~~ least upperbound
- Infnimum: greatest lower bound
- Norms

$\|\cdot\|_1 = \sum x_i$ $\quad \|\cdot\|_\infty = \max_i x_i$

$\|\cdot\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ Cauchy-Schwartz: $x^Ty \le |x^Ty| \le \|x\|_2\|y\|_2$

Induced norms

$$\underset{\text{induced}}{\|A\|_i} = \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1}\|Ax\|$$

$\|Ax\| \le \|A\|_i\|x\|$

$\|A\|_i = \sqrt{\lambda_{max}(A^TA)}$

- Openball: $B_a(x_0) = \{x\in V \mid \|x-x_0\|<a\}$
- Cauchy sequence: if $\forall \varepsilon>0, \exists N<\infty$ s.t. $\forall m,n \ge N$ $\|x_n-x_m\|<\varepsilon$ "converges", but can also be bounded
- Contraction mapping

If $\exists$ a constant $0\le c<1$ s.t. $\forall x,y\in X, \|P(x)-P(y)\| \le c\|x-y\|$

$\swarrow$

x^* fixed point $P(x^*)=x^*$ — unique

- Continuous – No "jumps" locally

Function $h: V\to W$ is continuous if @ $x_0\in V$

$\forall \varepsilon>0, \exists \delta>0$ s.t.

$\|x-x_0\|<\delta \longrightarrow \|h(x)-h(x_0)\|<\varepsilon$

$x\in B_\delta(x_0)$

- Lipschitz – derivative is bounded

If at $x_0\in V$ if $\exists r>0, L<\infty$ s.t.

$\forall x,y\in B_r(x_0) \longrightarrow L\|x-y\| \leftarrow \left\|\frac{\partial h(x)}{\partial x}\right\| \le L$

ex. x^2 locally Lipschitz, $|x|$ global, $\sqrt{x}$ not Lipschitz

- Solutions of ODEs
  - Local Existence & Uniqueness — Also if $f(t,x)$ locally Lipschitz, every solution of $\dot{x}=f(t,x), x(t_0)=x_0$ lies entirely in W.

Given:

(a) $\dot{x}=f(t,x(t)), t\ge t_0, x(t_0)=x_0\in\mathbb{R}^n$ $x_0\in W$

(b) f is p.w. continuous w.r.t. t

(c) $\exists T>t_0, r>0, L>0$ s.t. $\|f(t,x)-f(t,y)\| \le L\|x-y\|$ $\forall x,y\in B_r(x_0), \forall t\in[t_0,T)$ — Locally Lipschitz

Then: $\exists \delta>0$ s.t. soln exists & is unique on $[t_0, t_0+\delta]$

- Global Existence & uniqueness $\longrightarrow$ Implies continuous dependence on initial conditions
  - $f(t,x)$ p.w cont in t
  - $\forall T\in[t_0,\infty)$ If $\exists L_T<\infty$ s.t. $\|f(t,x)-f(t,y)\| \le L_T\|x-y\|$ $\underbrace{\forall x,y\in\mathbb{R}^n}_{\text{Global Lipschitz continuity}}, \forall t\in[t_0,T]$

Then: For every $T\in[t_0,\infty)$, $\dot{x}=f(t,x_0), x(t_0)=x_0$ has exactly one soln on $[t_0,T]$

- Bellman-Gronwall Inequality

Suppose $\lambda\in\mathbb{R}$ constant and $\mu:[a,b]\to\mathbb{R}$ is cont. & non-negative function.

If $y(t)\le\lambda+\int_a^t \mu(\tau)y(\tau)d\tau, \forall a\le t\le b$ — Implicit

then $y(t)\le\lambda e^{\int_a^t \mu(\tau)d\tau}, \forall a\le t\le b$ — Explicit

Inequality

- Compact space: closed + bounded
- $\Omega_c = \{x \mid V(x)\le c\}$ Sublevel set
- Stability
  - SISL — unstable otherwise.

If $\forall\varepsilon>0, \exists\delta>0$ s.t. $\|x_0-x_e\|<\delta \longrightarrow \|x(t,x_0)-x_e\|<\varepsilon$ $\forall t\ge 0$ Note: $\delta\le\varepsilon$

- Asymptotically stable if SISL and $\exists\eta>0$ s.t. $\|x_0-x_e\|<\eta \longrightarrow \lim_{t\to\infty}\|x(t,x_0)-x_e\|=0$ "Attractivity"
- Exponentially stable $\|x(t,t_0,x_0)\| \le N\|x_0\|e^{-\gamma(t-t_0)}$

AS is "fragile"; ES is more robust to small perturbations

- Unstable

$\exists\varepsilon>0$ s.t. $\forall\delta>0, \exists x_0$ and $0<\|x_0\|<\delta$, $\sup_{t\ge 0}\|x(t,x_0)\|\ge\varepsilon$

- Antistability > Instability

• Lyapunov's Direct Method

Assume $x_e = 0$ is an eq. point of $\dot{x} = f(x)$, and there exists an open set D about origin s.t.

(1) $f: D \to \mathbb{R}^n$ locally Lipschitz

(2) $\exists V \in C^1$, $V: D \to \mathbb{R}$ s.t.

(a) $V(0) = 0$ Then $x_e = 0$ is SISL

(b) $V(x) > 0$, $x \neq 0$, $x \in D$

(c) $\dot{V}(x) \leq 0$, $\forall x \in D$ $\longrightarrow$ $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x)$

$= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$

(3) AS if $\dot{V}(x) < 0$

• Global Asymptotic Stability (GAS)

$x_e = 0$ if GAS if it is SISL and $\forall x_0 \in \mathbb{R}^n$,

$\lim_{t \to \infty} \| x(t, x_0) \| = 0$ Replace D (open set) with all of $\mathbb{R}^n$

(3) + $V(x) \to \infty$ as $\|x\| \to \infty$) "radially unbounded" — observability property

• Exponentially stable if $\dot{V}(x) \leq -\gamma V(x)$

Quadratic Lyapunov functions

P = $n \times n$ real matrix, $V(x) = x^T P x$

$$P = \underbrace{\frac{P + P^T}{2}}_{\substack{\text{symmetric} \\ A^T = A}} + \underbrace{\frac{P - P^T}{2}}_{\substack{\text{anti-symmetric} \\ A^T = -A}} \quad \text{For any matrix}$$

$x^T P x = x^T \left(\frac{P + P^T}{2} \right) x$ Make any matrix symmetric

• Lyapunov's Indirect Method (Don't use Lyapunov functions) Justification of linear control theory

Consider $\dot{x} = f(x)$, $f \in C^1$, $f(x_e) = 0$

(a) If linearization of system about x_e has only e-values w/ negative real parts, then x_e is locally AS eq point of $\dot{x} = f(x)$

(b) If linearization has at least one e-value w/ positive real part, then x_e is unstable for $\dot{x} = f(x)$

(c) If all e-values of linearization A have real parts ≤ 0, and at least one e-value has real part $= 0$, then no conclusion can be made on stability of x_e.

• Converse Lyapunov theorem

If ~~these~~ system is GAS, $\exists V: \mathbb{R}^n \to \mathbb{R}$, $V \in C^1$, radially unbounded

$V(0) = 0$, $V(x) > 0$, $x \neq 0$. $\dot{V}(x) < 0$, $x \neq 0$ for $\dot{x} = f(x)$)

• Lasalle's invariance principle

Given $f(x)$, $f(0) = 0$. If $\exists C^1$ function $V(x)$

s.t. $\dot{V}(x) \leq 0$, $V(x) > 0$ $\forall x \neq 0$, $V(0) = 0$

If the only function $\phi: [0, T] \to \mathbb{R}^n$ that satisfies: $S = \{ x \in D \mid \dot{V}(x) = 0 \}$

· $\dot{\phi}(t) = f(\phi(t))$

· $\phi(t) \in S$, $\forall t$

is $\phi(t) \equiv 0$, then eq. point $x_e = 0$ is AS

• Chitaev's Instability Theorem

$V(0) = 0$, $\forall \delta > 0$, $\exists x_0 \in B_\delta(0)$ s.t. $V(x_0) > 0$

$\exists r > 0$ st. $\dot{V}(x) > 0$ on $U = \{ x \in B_r(0) \mid V(x) > 0 \} \subset D$

Then $x_e = 0$ is unstable

• Region of attraction

$R_A(x_e) = \{ x_0 \in \mathbb{R}^n \mid \lim_{t \to \infty} x(t, x_0) = x_e \}$

Estimate by fixing $V(x) = x^T P x$, taking sublevel se (conservative guarantee)

$\Omega_c = \{ x \in D \mid V(x) \leq c \}$ "curvature"

• Large axis of ellipse $\to$ e-vec of λ_{min}

Largest ball in ellipse: $c^* = (r^*)^2 \lambda_{max}$ Inscribed

Smallest ball around ellipse: $c^* = (r^*)^2 \lambda_{min}$ Circumscribed

• Finite escape time: If solution $\to \infty$ in finite time

ex. $x(t) = \tan(t)$, $x(t) = \frac{1}{1-t}$

$$\lambda_{min}(P) x^T x \leq x^T P x \leq \lambda_{max}(P) x^T x$$

• Estimating region of attraction:

· Fix $V(x)$ as quadratic $\to V(x) = x^T P x$

· Pick $Q > 0$

· Solve $A^T P + PA = -Q$ for P — lyap(A^T, P)

Linearize system to $f(x) = Ax + R(x)$

• See what needs to be true for $P > 0$. $\to$ Some conditions on x

· Find sublevel set of V (Use inscribed/circumscribed circle equations) Also $\dot{V}(x) < 0$, $V(x) > 0$

• SOS — sum of squares

Show that

$V(x) - \varepsilon x^T x \in \Sigma[x] \geq 0$

$-\nabla V(x) \cdot f(x) \in \Sigma[x] \geq 0$