練習問題 4.4 解答

(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to+0}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to+0}\left[2\sqrt{x}\right]_a^1 = \lim_{a\to+0}\left(2-2\sqrt{a}\right) = 2$

(2) $\displaystyle\int_0^1 \frac{dx}{x} = \lim_{a\to+0}\int_a^1 \frac{dx}{x} = \lim_{a\to+0}\left[\log|x|\right]_a^1 = \lim_{a\to+0}(-\log a) = \infty$

(3) $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{a\to+0}\int_a^1 \frac{dx}{x^2} = \lim_{a\to+0}\left[-\frac{1}{x}\right]_a^1 = \lim_{a\to+0}\left(\frac{1}{a}-1\right) = \infty$

(4) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty}\left(2\sqrt{b}-2\right) = \infty$

(5) $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\left[\log|x|\right]_1^b = \lim_{b\to\infty}\log b = \infty$

(6) $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$

(7) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 = \lim_{a\to-\infty}\left(-\tan^{-1}a\right) = \frac{\pi}{2}$$
$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}$$

よって，$\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$

(8) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2} = \int_{-\infty}^0 \frac{dx}{x^2} + \int_0^\infty \frac{dx}{x^2} = \int_{-\infty}^{-1} \frac{dx}{x^2} + \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2}$

(3) より，$\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2}$ は発散する．

(9) $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}\,dx = \int_{-1}^0 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\int_{-1}^0 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{a\to-1+0}\left[-\sqrt{1-x^2}\right]_a^0 = \lim_{a\to-1+0}\left(-1+\sqrt{1-a^2}\right) = -1$$
$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{b\to1-0}\left[-\sqrt{1-x^2}\right]_0^b = \lim_{b\to1-0}\left(-\sqrt{1-b^2}+1\right) = 1$$

よって，$\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}\,dx = -1+1 = 0.$

(10) $\displaystyle\int_{-\infty}^{\infty} x\ dx = \int_{-\infty}^{0} x\ dx + \int_{0}^{\infty} x\ dx$

$$\int_0^{\infty} x\ dx = \lim_{b\to\infty}\int_0^b x\ dx = \lim_{b\to\infty}\left[\frac{x^2}{2}\right]_0^b = \lim_{b\to\infty}\frac{b^2}{2} = \infty$$

よって，$\displaystyle\int_{-\infty}^{\infty} x\ dx$ は発散する．

(11) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan x\ dx = \int_{-\frac{\pi}{2}}^{0} \tan x\ dx + \int_{0}^{\frac{\pi}{2}} \tan x\ dx$

$$\int_0^{\frac{\pi}{2}} \tan x\ dx = \lim_{b\to\frac{\pi}{2}-0}\left[-\log|\cos x|\right]_0^b = \lim_{b\to\frac{\pi}{2}-0}(-\log|\cos b|) = \infty$$

よって，$\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan x\ dx$ は発散する．

(12)

$$\begin{aligned}\int_0^1 x\log x\ dx &= \lim_{a\to+0}\int_a^1 x\log x\ dx = \lim_{a\to+0}\left[\frac{x^2}{2}\log x - \frac{x}{4}\right]_a^1 \\ &= -\frac{1}{4}+\frac{1}{2}\lim_{a\to+0}a^2\log a = -\frac{1}{4}+\frac{1}{2}\lim_{a\to+0}\frac{\log a}{\frac{1}{a^2}} \\ &= -\frac{1}{4}+\frac{1}{2}\lim_{a\to+0}\frac{\frac{1}{a}}{-\frac{2}{a^3}} = -\frac{1}{4}-\frac{1}{2}\lim_{a\to+0}a^2 = -\frac{1}{4}\end{aligned}$$

(13) $x\in[0,1]$ で $\displaystyle\sqrt{1+\frac{1}{x^4}} \geq \sqrt{\frac{1}{x^4}} = \frac{1}{x^2}$ なので，(3) から発散する．
（これは曲線 $y=\frac{1}{x}$ $(0\leq x\leq 1)$ の長さが発散することを表している．）

(14) $\displaystyle\int_0^{\infty} xe^{-x^2}\ dx = \lim_{b\to\infty}\left[-\frac{e^{-x^2}}{2}\right]_0^b = \lim_{b\to\infty}\left(\frac{1}{2}-\frac{e^{-b^2}}{2}\right) = \frac{1}{2}$

(15) $\displaystyle\int_0^{\infty} e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^{\infty} e^{-x^2}\ dx$
e^{-x^2} は $[0,1]$ で連続なので，$\displaystyle\int_0^1 e^{-x^2}\ dx$ は通常の定積分である．一方，$x\geq 1$ で $e^{-x^2}\leq xe^{-x^2}$ であり，

$$\int_1^{\infty} xe^{-x^2}dx = \lim_{b\to\infty}\left[-\frac{e^{-x^2}}{2}\right]_1^b = \lim_{b\to\infty}\left(\frac{e^{-1}}{2}-\frac{e^{-b^2}}{2}\right) = \frac{e^{-1}}{2}$$

より，$\displaystyle\int_1^{\infty} e^{-x^2}\ dx$ は収束する．よって，$\displaystyle\int_0^{\infty} e^{-x^2}\ dx$ は収束する．