積分基本問題集壱

次の積分または広義積分を求めよう．

(1) $\displaystyle\int_{-1}^{5} \left(3x^3-5x-1\right)\ dx$

(2) $\displaystyle\int_{0}^{-2} (3x-5)^6\ dx$

(3) $\displaystyle\int_{1}^{4} \frac{dx}{(2x+1)^2}$

(4) $\displaystyle\int_{0}^{\pi} \sin x\ dx$

(5) $\displaystyle\int_{-\pi}^{\pi} \cos x\ dx$

(6) $\displaystyle\int_{-\pi/6}^{\pi/4} \tan x\ dx$

(7) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{dx}{\tan x}$

(8) $\displaystyle\int_{0}^{1} \sin^{-1} x\ dx$

(9) $\displaystyle\int_{-1}^{1} \cos^{-1} x\ dx$

(10) $\displaystyle\int_{-\sqrt{3}}^{1} \tan^{-1} x\ dx$

(11) $\displaystyle\int_{0}^{1} e^x\ dx$

(12) $\displaystyle\int_{1/2}^{3} \log x\ dx$

(13) $\displaystyle\int_{0}^{\pi} x\sin x\ dx$

(14) $\displaystyle\int_{-\pi}^{\pi} (\sin x)\cos(2x)\ dx$

(15) $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$

(16) $\displaystyle\int_{0}^{2} \frac{dx}{4+x^2}$

(17) $\displaystyle\int_{0}^{2} x^2e^{-2x}\ dx$

(18) $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

(19) $\displaystyle\int_{-3/2}^{3/\sqrt{6}} \frac{dx}{\sqrt{3-x^2}}$

(20) $\displaystyle\int_{0}^{\sqrt{2}} \frac{dx}{\sqrt{x^2+2}}$

(21) $\displaystyle\int_{0}^{1} \sqrt{1-x^2}\ dx$

(22) $\displaystyle\int_{0}^{2} \sqrt{1+4x^2}\ dx$

(23) $\displaystyle\int_{0}^{1} 3^x\ dx$

(24) $\displaystyle\int_{1}^{2} \log_2 x\ dx$

(25) $\displaystyle\int_{0}^{1} \frac{x^3}{\sqrt{x^2+5}}\ dx$

(26) $\displaystyle\int_{1}^{e} (\log x)^2\ dx$

(27) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

(28) $\displaystyle\int_0^1 \frac{dx}{x^2+x+1}$

(29) $\displaystyle\int_{-1}^1 \frac{dx}{x^2-4}$

(30) $\displaystyle\int_0^1 \frac{x+2}{(x^2+x+1)^2}\ dx$

(31) $\displaystyle\int_1^2 \frac{dx}{x^2(x+1)}$

(32) $\displaystyle\int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\ dx$

(33) $\displaystyle\int_0^1 \frac{dx}{x^4+1}$

(34) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}+1}$

(35) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^3+x}$

(36) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

(37) $\displaystyle\int_1^\infty \frac{dx}{x^3}$

(38) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(39) $\displaystyle\int_1^2 \frac{dx}{x\sqrt{x-1}}$

(40) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[4]{(1+x)^3}}$

(41) $\displaystyle\int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\ dx$

(42) $\displaystyle\int_0^\infty xe^{-x^2}\ dx$

(43) $\displaystyle\int_0^1 x\log x\ dx$

(44) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+4}$

(45) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\ dx$

(46) $\displaystyle\int_0^\infty \frac{dx}{e^x(1+e^x)}$

(47) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$

(48) $\displaystyle\int_0^1 \frac{x^4}{\sqrt{1-x^2}}\ dx$

(1) $\displaystyle\int_{-1}^{5}\left(3x^3-5x-1\right)\ dx=\left[\frac{3}{4}x^4-\frac{5}{2}x^2-x\right]_{-1}^{5}=402.$

(2) $u=3x-5$ とおくと，$\frac{du}{dx}=3$ より $1=\frac{1}{3}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	-2
u	-5	$\to$	-11

と変換されるので，以下を得る．

$$\int_{0}^{-2}(3x-5)^6\ dx=\int_{0}^{-2}u^6\ \frac{1}{3}\frac{du}{dx}\ dx=\frac{1}{3}\int_{-5}^{-11}u^6\ du=\frac{1}{3}\left[\frac{1}{7}u^7\right]_{-5}^{-11}=-\frac{6469682}{7}.$$

(3) $u=2x+1$ とおくと，$\frac{du}{dx}=2$ より $1=\frac{1}{2}\frac{du}{dx}$ であり，積分区間は

x	1	$\to$	4
u	3	$\to$	9

と変換されるので，以下を得る．

$$\int_{1}^{4}\frac{1}{(2x+1)^2}\ dx=\int_{1}^{4}\frac{1}{u^2}\ \frac{1}{2}\frac{du}{dx}\ dx=\frac{1}{2}\int_{3}^{9}u^{-2}\ du=\frac{1}{2}\Big[-u^{-1}\Big]_{3}^{9}=\frac{1}{9}.$$

(4) $\displaystyle\int_{0}^{\pi}\sin x\ dx=\Big[-\cos x\Big]_{0}^{\pi}=2.$

(5) $\displaystyle\int_{-\pi}^{\pi}\cos x\ dx=\Big[\sin x\Big]_{-\pi}^{\pi}=0.$

(6) $\tan x=\frac{\sin x}{\cos x}$ なので，$u=\cos x$ とおけば $\frac{du}{dx}=-\sin x$ より $\sin x=-\frac{du}{dx}$ であり，積分区間は

x	$-\pi/6$	$\to$	$\pi/4$
u	$\sqrt{3}/2$	$\to$	$\sqrt{2}/2$

と変換される．よって，以下を得る．

$$\begin{aligned}\int_{-\pi/6}^{\pi/4}\tan x\ dx&=\int_{-\pi/6}^{\pi/4}\frac{\sin x}{\cos x}\ dx=\int_{-\pi/6}^{\pi/4}\frac{1}{u}\left(-\frac{du}{dx}\right)\ dx=-\int_{\sqrt{3}/2}^{\sqrt{2}/2}\frac{du}{u}\\&=-\Big[\log|u|\Big]_{\sqrt{3}/2}^{\sqrt{2}/2}=\frac{1}{2}\log\frac{3}{2}.\end{aligned}$$

(7) $\frac{1}{\tan x}=\frac{\cos x}{\sin x}$ なので，$u=\sin x$ とおけば $\frac{du}{dx}=\cos x$ であり，積分区間は

x	$\pi/4$	$\to$	$\pi/3$
u	$\sqrt{2}/2$	$\to$	$\sqrt{3}/2$

と変換される．よって，以下を得る．

$$\int_{\pi/4}^{\pi/3}\frac{dx}{\tan x}=\int_{\pi/4}^{\pi/3}\frac{\cos x}{\sin x}\ dx=\int_{\pi/4}^{\pi/3}\frac{1}{u}\frac{du}{dx}\ dx=\int_{\sqrt{2}/2}^{\sqrt{3}/2}\frac{du}{u}=\Big[\log|u|\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}=\frac{1}{2}\log\frac{3}{2}.$$

(8) $\displaystyle\int_0^1 \sin^{-1} x\ dx = \Big[x\sin^{-1} x + \sqrt{1-x^2}\Big]_0^1 = \frac{\pi}{2} - 1.$

(9) $\displaystyle\int_{-1}^1 \cos^{-1} x\ dx = \Big[x\cos^{-1} x - \sqrt{1-x^2}\Big]_{-1}^1 = \pi.$

(10) $\displaystyle\int_{-\sqrt{3}}^1 \tan^{-1} x\ dx = \left[x\tan^{-1} x - \frac{1}{2}\log(x^2+1)\right]_{-\sqrt{3}}^1 = \left(\frac{1}{4} - \frac{\sqrt{3}}{3}\right)\pi + \frac{1}{2}\log 2.$

(11) $\displaystyle\int_0^1 e^x\ dx = \Big[e^x\Big]_0^1 = e - 1.$

(12) $\displaystyle\int_{1/2}^3 \log x\ dx = \int_{1/2}^3 (x)'\log x\ dx = \Big[x\log x\Big]_{1/2}^3 - \int_{1/2}^3 dx = -\frac{5}{2} + \frac{1}{2}\log 2 + 3\log 3.$

(13) $\displaystyle\int_0^\pi x\sin x\ dx = \int_0^\pi x\,(-\cos x)'\ dx = \Big[-x\cos x\Big]_0^\pi - \int_0^\pi (-\cos x)\ dx = \pi.$

(14) $(\sin x)\cos(2x) = \frac{1}{2}(\sin(3x) - \sin x)$ なので,

$$\int_{-\pi}^\pi (\sin x)\cos(2x)\ dx = \frac{1}{2}\left(\int_{-\pi}^\pi \sin 3x\ dx - \int_{-\pi}^\pi \sin x\ dx\right) = \frac{1}{2}\int_{-\pi}^\pi \sin 3x\ dx$$

である．$u = 3x$ とおけば $\frac{du}{dx} = 3$ より $1 = \frac{1}{3}\frac{du}{dx}$ であり，積分区間は

x	$-\pi$	$\to$	π
u	-3π	$\to$	3π

と変換される．よって，以下を得る．

$$\int_{-\pi}^\pi (\sin x)\cos(2x)\ dx = \frac{1}{2}\int_{-\pi}^\pi \sin 3x\ dx = \frac{1}{2}\int_{-\pi}^\pi (\sin u)\,\frac{1}{3}\frac{du}{dx}\ dx = \frac{1}{6}\int_{-3\pi}^{3\pi} \sin u\ du = 0$$

(15) $\displaystyle\int_{-1}^1 \frac{dx}{1+x^2} = \Big[\tan^{-1} x\Big]_{-1}^1 = \frac{1}{2}\pi.$

(16) $\displaystyle\frac{1}{4+x^2} = \frac{1}{4}\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right)$ なので，$u = \frac{x}{2}$ とおけば $\frac{du}{dx} = \frac{1}{2}$ より $1 = 2\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	2
u	0	$\to$	1

と変換される．よって，以下を得る．

$$\int_0^4 \frac{dx}{4+x^2} = \frac{1}{4}\int_0^2 \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\int_0^2 \frac{1}{1+u^2}\,2\,\frac{du}{dx}\,dx = \frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\Big[\tan^{-1} u\Big]_0^1 = \frac{1}{8}\pi.$$

(17) 部分積分を繰り返す.

$$\begin{aligned}\int_0^2 x^2e^{-2x}\,dx &= \int_0^2 x^2\left(-\frac{1}{2}e^{-2x}\right)'\,dx = \left[-\frac{1}{2}x^2e^{-2x}\right]_0^2 + \int_0^2 xe^{-2x}\,dx \\ &= -2e^{-4} + \int_0^2 x\left(-\frac{1}{2}e^{-2x}\right)'\,dx \\ &= -2e^{-4} + \left[-\frac{1}{2}xe^{-2x}\right]_0^2 + \frac{1}{2}\int_0^2 e^{-2x}\,dx \\ &= -3e^{-4} + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^2 = \frac{1}{4} - \frac{13}{4}e^{-4}.\end{aligned}$$

(18) $\displaystyle\int_{-1/2}^{1/2}\frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_{-1/2}^{1/2} = \frac{1}{3}\pi.$

(19) $\displaystyle\frac{1}{\sqrt{3-x^2}} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,\frac{1}{\sqrt{3}}$ なので，$u=\frac{x}{\sqrt{3}}$ とおけば $\frac{du}{dx}=\frac{1}{\sqrt{3}}$ であり，積分区間は

x	$-3/2$	$\to$	$3/\sqrt{6}$
u	$-\sqrt{3}/2$	$\to$	$\sqrt{2}/2$

と変換される．よって，以下を得る．

$$\begin{aligned}\int_{-3/2}^{3/\sqrt{6}}\frac{dx}{\sqrt{3-x^2}} &= \int_{-3/2}^{3/\sqrt{6}}\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,\frac{1}{\sqrt{3}}\,dx = \int_{-3/2}^{3/\sqrt{6}}\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\,dx = \int_{-\sqrt{3}/2}^{\sqrt{2}/2}\frac{du}{\sqrt{1-u^2}} \\ &= \left[\sin^{-1}u\right]_{-\sqrt{3}/2}^{\sqrt{2}/2} = \frac{7}{12}\pi.\end{aligned}$$

(20) $u=x+\sqrt{x^2+2}$ とおくと，$\displaystyle\frac{du}{dx}=\frac{x+\sqrt{x^2+2}}{\sqrt{x^2+2}}=\frac{u}{\sqrt{x^2+2}}$ より $\displaystyle 1=\frac{\sqrt{x^2+2}}{u}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	$\sqrt{2}$
u	$\sqrt{2}$	$\to$	$2+\sqrt{2}$

と変換されるので，以下を得る．

$$\int_0^{\sqrt{2}}\frac{dx}{\sqrt{x^2+2}} = \int_0^{\sqrt{2}}\frac{1}{\sqrt{x^2+2}}\frac{\sqrt{x^2+2}}{u}\,\frac{du}{dx}\,dx = \int_{\sqrt{2}}^{2+\sqrt{2}}\frac{du}{u} = \log\left(1+\sqrt{2}\right).$$

あるいは，$x=\sqrt{2}\,\sinh u$ とおくと $\frac{dx}{du}=\sqrt{2}\,\cosh u$ であり，積分区間は

x	0	$\to$	$\sqrt{2}$
u	0	$\to$	$\sinh^{-1}1$

と変換されるので，$\alpha = \sinh^{-1} 1\ (\Leftrightarrow \sinh\alpha = 1)$ とおいて

$$\int_0^{\sqrt{2}} \frac{dx}{\sqrt{x^2+2}} = \int_0^{\alpha} \frac{\sqrt{2}\ \cosh u}{\sqrt{2\left(\sinh^2 u + 1\right)}}\ du = \int_0^{\alpha} du = \alpha = \log\left(1+\sqrt{2}\right)$$

と計算することもできる．なお，$\alpha = \sinh^{-1} 1$ の値は

$$\alpha = \sinh^{-1} 1 \ \Leftrightarrow\ 1 = \sinh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2} \ \Leftrightarrow\ \left(e^{\alpha}\right)^2 - 2e^{\alpha} - 1 = 0 \ \Leftrightarrow\ e^{\alpha} = 1+\sqrt{2}$$

より，$\alpha = \log\left(1+\sqrt{2}\right)$ と求められる．

(21) $u = \sin^{-1} x\ (\Leftrightarrow x = \sin u)$ とおくと，$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\cos u}$ より $1 = \cos u\ \frac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline u & 0 & \to & \frac{\pi}{2} \end{array}$$

と変換されるので，以下を得る．

$$\int_0^1 \sqrt{1-x^2}\ dx = \int_0^1 \sqrt{\cos^2 u}\ \cos u\ \frac{du}{dx}dx = \int_0^{\frac{\pi}{2}} \cos^2 u\ du = \frac{\pi}{4}.$$

(22) $\sqrt{1+4x^2} = (x)'\sqrt{1+4x^2}$ と見なして部分積分をする．

$$\begin{aligned}\int_0^2 \sqrt{1+4x^2}\ dx &= \left[x\sqrt{1+4x^2}\right]_0^2 - \int_0^2 \frac{4x^2}{\sqrt{1+4x^2}}\ dx = 2\sqrt{17} - \int_0^2 \frac{1+4x^2-1}{\sqrt{1+4x^2}}\ dx \\ &= 2\sqrt{17} - \int_0^2 \sqrt{1+4x^2}\ dx + \int_0^2 \frac{dx}{\sqrt{1+4x^2}}\end{aligned}$$

より移行・整理して

$$\int_0^2 \sqrt{1+4x^2}\ dx = \sqrt{17} + \frac{1}{2}\int_0^2 \frac{dx}{\sqrt{1+4x^2}}$$

を得る．さらに，$u = 2x + \sqrt{1+4x^2}$ とけば

$$\frac{du}{dx} = \frac{4x + 2\sqrt{1+4x^2}}{\sqrt{1+4x^2}} = \frac{2u}{\sqrt{1+4x^2}}$$

より $1 = \dfrac{\sqrt{1+4x^2}}{2u}\ \dfrac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & 2 \\ \hline u & 1 & \to & 4+\sqrt{17} \end{array}$$

と変換される．よって，以下を得る．

$$\int_0^4 \sqrt{1+4x^2}\ dx = \sqrt{17} + \frac{1}{2}\int_1^{4+\sqrt{17}} \frac{du}{2u} = \sqrt{17} + \frac{1}{4}\log\left(4+\sqrt{17}\right).$$

あるいは，$x = \frac{1}{2}\sinh u$ とおけば $\frac{dx}{du} = \frac{1}{2}\cosh u$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & 2 \\ \hline u & 0 & \to & \sinh^{-1} 4 \end{array}$$

と変換されるので，$\alpha = \sinh^{-1} 4$ ($\Leftrightarrow \sinh\alpha = 4$) とおいて

$$\begin{aligned}
\int_0^2 \sqrt{1+4x^2}\ dx &= \int_0^\alpha \sqrt{1+\sinh^2 u}\ \frac{1}{2}\cosh u\ du = \frac{1}{2}\int_0^\alpha \cosh^2 u\ du \\
&= \frac{1}{2}\int_0^\alpha \frac{1+\cosh(2u)}{2}\ du = \frac{1}{4}\left[u + \frac{1}{2}\sinh(2u)\right]_0^\alpha \\
&= \frac{\alpha}{4} + \frac{\sinh(2\alpha)}{8} = \frac{\alpha}{4} + \frac{(\sinh\alpha)\cosh\alpha}{4} = \frac{\alpha}{4} + \frac{(\sinh\alpha)\sqrt{1+\sinh^2\alpha}}{4} \\
&= \frac{1}{4}\log\left(4+\sqrt{17}\right) + \sqrt{17}
\end{aligned}$$

と計算することもできる．なお，$\alpha = \sinh^{-1} 4$ の値は

$$\alpha = \sinh^{-1} 4 \ \Leftrightarrow\ 4 = \sinh\alpha = \frac{e^\alpha - e^{-\alpha}}{2} \ \Leftrightarrow\ (e^\alpha)^2 - 8e^\alpha - 1 = 0 \ \Leftrightarrow\ e^\alpha = 4+\sqrt{17}$$

より．$\alpha = \log\left(4+\sqrt{17}\right)$ と求められる．

(23) $3^x = e^{(\log 3)x}$ なので，$u = (\log 3)x$ とおけば $\frac{du}{dx} = \log 3$ より $1 = \frac{1}{\log 3}\frac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline u & 0 & \to & \log 3 \end{array}$$

と変換される．よって，以下を得る．

$$\int_0^1 3^x\ dx = \int_0^1 e^{(\log 3)x} dx = \int_0^1 \frac{e^u}{\log 3}\frac{du}{dx}\ dx = \frac{1}{\log 3}\int_0^{\log 3} e^u\ du = \frac{1}{\log 3}\Big[e^u\Big]_0^{\log 3} = \frac{2}{\log 3}.$$

(24) $\displaystyle\int_1^2 \log_2 x\ dx = \int_1^2 \frac{\log x}{\log 2}\ dx = \frac{1}{\log 2}\Big[x\log x - x\Big]_1^2 = 2 - \frac{1}{\log 2}.$

(25) $\dfrac{x^3}{\sqrt{x^2+5}} = x^2 \cdot \dfrac{x}{\sqrt{x^2+5}}$ なので，$u = \sqrt{x^2+5}$ とおけば $\frac{du}{dx} = \frac{x}{\sqrt{x^2+5}}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & 1 \\ \hline u & \sqrt{5} & \to & \sqrt{6} \end{array}$$

と変換される．よって，以下を得る．

$$\begin{aligned}
\int_0^1 \frac{x^3}{\sqrt{x^2+5}}\ dx &= \int_0^1 x^2 \cdot \frac{x}{\sqrt{x^2+5}}\ dx = \int_0^1 (u^2-5)\ \frac{du}{dx}\ dx = \int_{\sqrt{5}}^{\sqrt{6}} (u^2-5)\ du \\
&= \left[\frac{1}{3}u^3 - 5u\right]_{\sqrt{5}}^{\sqrt{6}} = -3\sqrt{6} + \frac{10}{3}\sqrt{5}.
\end{aligned}$$

(26) $(\log x)^2 = (x)'(\log x)^2$ と見なして部分積分をする．

$$\int_1^e (\log x)^2 \ dx = \Big[x(\log x)^2\Big]_1^e - \int_1^e x\cdot\frac{2\log x}{x} \ dx = e - 2\int_1^e \log x \ dx$$
$$= e - 2\Big[x\log x - x\Big]_1^e = e - 2.$$

(27) $\dfrac{1}{x^2+1} = (x)'\dfrac{1}{x^2+1}$ と見なして，$\displaystyle\int \frac{dx}{x^2+1}$ に部分積分を適用して

$$\int \frac{dx}{x^2+1} = \frac{x}{x^2+1} + 2\int \frac{x^2}{(x^2+1)^2} \ dx = \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{(x^2+1)^2} \ dx$$
$$= \frac{x}{x^2+1} + 2\left(\int \frac{dx}{x^2+1} - \int \frac{dx}{(x^2+1)^2}\right)$$

を得る．上式を移項・整理して

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \int \frac{dx}{x^2+1}\right)$$

を得る．よって，以下を得る．

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left[\frac{x}{x^2+1}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dx}{x^2+1} = \frac{1}{4} + \frac{1}{2}\Big[\tan^{-1} x\Big]_0^1 = \frac{1}{4} + \frac{\pi}{8}.$$

(28) 被積分関数の分母を平方完成すると，

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2 + 1\right)$$

である．ここで，$u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$ とおくと $\frac{du}{dx} = \frac{2}{\sqrt{3}}$ より $1 = \frac{\sqrt{3}}{2}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	1
u	$1/\sqrt{3}$	$\to$	$\sqrt{3}$

と変換される．よって，以下を得る．

$$\int_0^1 \frac{dx}{x^2+x+1} = \frac{4}{3}\int_0^1 \frac{1}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1} \ dx = \frac{4}{3}\int_0^1 \frac{1}{u^2+1}\frac{\sqrt{3}}{2}\frac{du}{dx} \ dx$$
$$= \frac{2}{\sqrt{3}}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = \frac{2}{\sqrt{3}}\Big[\tan^{-1} u\Big]_{1/\sqrt{3}}^{\sqrt{3}} = \frac{\pi}{9}\sqrt{3}.$$

(29) $\displaystyle\int_{-1}^1 \frac{dx}{x^2-4} = \frac{1}{4}\int_{-1}^1 \left(\frac{1}{x-2} - \frac{1}{x+2}\right) \ dx = \frac{1}{4}\Big[\log|x-2| - \log|x+2|\Big]_{-1}^1 = -\frac{1}{2}\log 3.$

(30) $\displaystyle\int_0^1 \frac{x+2}{(x^2+x+1)^2} \ dx = \frac{1}{2}\int_0^1 \frac{2x+1}{(x^2+x+1)^2} \ dx + \frac{3}{2}\int_0^1 \frac{dx}{(x^2+x+1)^2}$ である．

- 右辺第 1 項について：

$$\frac{1}{2}\int_0^1 \frac{2x+1}{(x^2+x+1)^2}\,dx = \frac{1}{2}\int_0^1 \frac{(x^2+x+1)'}{(x^2+x+1)^2}\,dx = \frac{1}{2}\left[-\frac{1}{x^2+x+1}\right]_0^1 = \frac{1}{3}.$$

- 右辺第 2 項について：

$$\left(x^2+x+1\right)^2 = \left(\frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right)\right)^2$$

なので，$u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$ とおくと $\frac{du}{dx} = \frac{2}{\sqrt{3}}$ より $1 = \frac{\sqrt{3}}{2}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	1
u	$1/\sqrt{3}$	$\to$	$\sqrt{3}$

と変換される．よって，以下を得る．

$$\begin{aligned}\frac{3}{2}\int_0^1 \frac{dx}{\left(x^2+x+1\right)^2} &= \frac{8}{3}\int_0^1 \frac{1}{\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)^2}\,dx = \frac{8}{3}\int_0^1 \frac{1}{(u^2+1)^2}\frac{\sqrt{3}}{2}\frac{du}{dx}\,dx \\ &= \frac{4}{\sqrt{3}}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dv}{(u^2+1)^2} = \frac{4}{\sqrt{3}}\left[\frac{1}{2}\frac{u}{u^2+1}+\frac{1}{2}\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}} \\ &= \frac{\pi}{9}\sqrt{3}.\end{aligned}$$

以上より，$\displaystyle\int_0^1 \frac{dx}{(x^2+x+1)^2} = \frac{1}{3}+\frac{\pi}{9}\sqrt{3}$ である．

(31) 被積分関数を部分分数に分解する．

$$\begin{aligned}\int_1^2 \frac{dx}{x^2(x+1)} &= \int_1^2 \left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)\,dx = \left[-\log|x|-\frac{1}{x}+\log|x+1|\right]_1^2 \\ &= \frac{1}{2}+\log 3-2\log 2.\end{aligned}$$

(32) $\displaystyle\frac{1+x^2}{1-x^2} = \frac{-(1-x^2)+2}{1-x^2} = -1+\frac{2}{1-x^2} = -1+\frac{1}{1-x}+\frac{1}{1+x}$ より，以下を得る．

$$\int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\,dx = \Big[-x-\log|1-x|+\log|1+x|\Big]_0^{\frac{1}{2}} = \log 3-\frac{1}{2}.$$

(33) 被積分関数の分母を因数分解し，部分分数に分解して

$$
\begin{aligned}
\int_0^1 \frac{dx}{x^4+1} &= \int_0^1 \frac{dx}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} \\
&= \frac{1}{4}\int_0^1 \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}\,dx - \frac{1}{4}\int_0^1 \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\,dx \\
&= \frac{1}{4}\left(\frac{1}{\sqrt{2}}\int_0^1 \frac{2x+\sqrt{2}x}{x^2+\sqrt{2}x+1}\,dx + 2\int_0^1 \frac{dx}{\left(\sqrt{2}x+1\right)^2+1}\right) \\
&\quad - \frac{1}{4}\left(\frac{1}{\sqrt{2}}\int_0^1 \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx - 2\int_0^1 \frac{dx}{\left(\sqrt{2}x-1\right)^2+1}\right) \\
&= \frac{1}{4\sqrt{2}}\left[\log\left(x^2+\sqrt{2}x+1\right)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dx}{\left(\sqrt{2}x+1\right)^2+1} \\
&\quad - \frac{1}{4\sqrt{2}}\left[\log\left(x^2-\sqrt{2}x+1\right)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dx}{\left(\sqrt{2}x-1\right)^2+1} \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{1}{2}\left(\int_0^1 \frac{dx}{\left(\sqrt{2}x+1\right)^2+1} + \int_0^1 \frac{dx}{\left(\sqrt{2}x-1\right)^2+1}\right)
\end{aligned}
$$

を得る．$u=\sqrt{2}x+1$, $v=\sqrt{2}x-1$ とおくと，$\frac{du}{dx}=\sqrt{2}$, $\frac{dv}{dx}=\sqrt{2}$ であり，積分区間は

$$
\begin{array}{c|ccc}
x & 0 & \to & 1 \\
\hline
u & 1 & \to & \sqrt{2}+1
\end{array}
\qquad
\begin{array}{c|ccc}
x & 0 & \to & 1 \\
\hline
v & -1 & \to & \sqrt{2}-1
\end{array}
$$

と変換される．よって，以下を得る．

$$
\begin{aligned}
\int_0^1 \frac{dx}{x^4+1} &= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{1}{2}\left(\int_0^1 \frac{1}{u^2+1}\frac{1}{\sqrt{2}}\frac{du}{dx}\,dx + \int_0^1 \frac{1}{v^2+1}\frac{1}{\sqrt{2}}\frac{dv}{dx}\,dx\right) \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{\sqrt{2}}{4}\left(\int_1^{\sqrt{2}+1} \frac{du}{u^2+1} + \int_{-1}^{\sqrt{2}-1} \frac{dv}{v^2+1}\right) \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{\sqrt{2}}{4}\left(\left[\tan^{-1}u\right]_1^{\sqrt{2}+1} + \left[\tan^{-1}v\right]_{-1}^{\sqrt{2}-1}\right) \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{\sqrt{2}}{4}\left(\tan^{-1}\left(\sqrt{2}+1\right) + \tan^{-1}\left(\sqrt{2}-1\right)\right) \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{\sqrt{2}}{4}\left(\tan^{-1}\left(\sqrt{2}+1\right) + \tan^{-1}\frac{1}{\sqrt{2}+1}\right) \\
&= \frac{\sqrt{2}}{8}\log\left(3+2\sqrt{2}\right) + \frac{\pi}{8}\sqrt{2}.
\end{aligned}
$$

ただし，途中で $x\in\left(0,\frac{\pi}{2}\right)$ に対して成り立つ以下の関係式を用いた．

$$
\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}
$$

(34) $u=\sqrt{x}$ とおくと，$\frac{du}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2u}$ より $1=2u\ \frac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 1 & \to & 4 \\ \hline u & 1 & \to & 2 \end{array}$$

と変換される．よって，以下を得る．

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}+1}\ dx &= \int_1^4 \frac{1}{u+1}\ 2u\ \frac{du}{dx}\ dx = 2\int_1^2 \frac{u}{u+1}\ du = 2\int_1^2 \frac{u+1-1}{u+1}\ du \\ &= 2\int_1^2 \left(1-\frac{1}{u+1}\right)\ du = 2\Big[u-\log|u+1|\Big]_1^2 = 2+2\log\frac{2}{3}.\end{aligned}$$

(35) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^3+x} = \int_1^{\sqrt{2}} \left(\frac{1}{x}-\frac{x}{x^2+1}\right)\ dx = \left[\log|x|-\frac{1}{2}\log\left(x^2+1\right)\right]_1^{\sqrt{2}} = \log\frac{2}{\sqrt{3}}.$

(36) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to+0}\int_a^1 x^{-\frac{1}{2}}\ dx = \lim_{a\to+0}\Big[2x^{\frac{1}{2}}\Big]_a^1 = \lim_{a\to+0}\left(2-2\sqrt{a}\right) = 2.$

(37) $\displaystyle\int_1^{\infty} \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b x^{-3}\ dx = \lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\right]_1^b = \lim_{b\to\infty}\frac{1}{2}\left(1-\frac{1}{b^2}\right) = \frac{1}{2}.$

(38) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{b\to2-0}\int_0^b \frac{dx}{\sqrt{4-x^2}}$ である．

$u=\frac{x}{2}$ とおくと，$\frac{du}{dx}=\frac{1}{2}$ より $1=2\ \frac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & b \\ \hline u & 0 & \to & \frac{b}{2} \end{array}$$

と変換される．これより，

$$\int_0^b \frac{dx}{\sqrt{4-x^2}} = \int_0^b \frac{1}{2\sqrt{1-u^2}}\ 2\ \frac{du}{dx}\ dx = \int_0^{\frac{b}{2}} \frac{du}{\sqrt{1-u^2}} = \Big[\sin^{-1}u\Big]_0^{\frac{b}{2}} = \sin^{-1}\frac{b}{2}$$

なので，以下を得る．

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{b\to2-0}\sin^{-1}\frac{b}{2} = \sin^{-1}1 = \frac{\pi}{2}.$$

(39) $\displaystyle\int_1^2 \frac{dx}{x\sqrt{x-1}} = \lim_{a\to1+0}\int_a^2 \frac{dx}{x\sqrt{x-1}}$ である．

$u=\sqrt{x-1}$ とおくと $x=u^2+1$ である．また，$\frac{du}{dx}=\frac{1}{2\sqrt{x-1}}$ より $\frac{1}{\sqrt{x-1}}=2\frac{du}{dx}$ であり，積分区間は

$$\begin{array}{c|ccc} x & a & \to & 2 \\ \hline u & \sqrt{a-1} & \to & 1 \end{array}$$

と変換される．これより，

$$\int_a^2 \frac{dx}{x\sqrt{x-1}} = \int_a^2 \frac{1}{(u^2+1)}\ 2\frac{du}{dx}\ dx = 2\int_{\sqrt{a-1}}^1 \frac{du}{u^2+1} = 2\Big[\tan^{-1} u\Big]_{\sqrt{a-1}}^1$$
$$= \frac{\pi}{2} - 2\tan^{-1}\sqrt{a-1}$$

なので，以下を得る．

$$\int_1^2 \frac{dx}{x\sqrt{x-1}} = \lim_{a\to 1+0}\left(\frac{\pi}{2} - 2\tan^{-1}\sqrt{a-1}\right) = \frac{\pi}{2}.$$

(40) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[4]{(1+x)^3}} = \lim_{a\to -1+0}\int_a^0 (1+x)^{-\frac{3}{4}}\ dx + \lim_{b\to 1-0}\int_0^b (1+x)^{-\frac{3}{4}}\ dx$ である．

$$\int_a^0 (1+x)^{-\frac{3}{4}}\ dx = \Big[4(1+x)^{\frac{1}{4}}\Big]_a^0 = 4 - 4\,(1+a)^{\frac{1}{4}} \to 4\ (a\to -1+0),$$
$$\int_0^b (1+x)^{-\frac{3}{4}}\ dx = \Big[4(1+x)^{\frac{1}{4}}\Big]_0^b = 4\,(1+b)^{\frac{1}{4}} - 4 \to 4\cdot 2^{\frac{1}{4}} - 4\ (b\to 1-0)$$

だから，以下を得る．

$$\int_{-1}^1 \frac{dx}{\sqrt[4]{(1+x)^3}} = 4 + 4\cdot 2^{\frac{1}{4}} - 4 = 4\sqrt[4]{2}.$$

(41) $\displaystyle\int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\ dx = \lim_{a\to -2+0}\int_a^0 \frac{x}{\sqrt{4-x^2}}\ dx + \lim_{b\to 2-0}\int_0^b \frac{x}{\sqrt{4-x^2}}\ dx$ である．

$u=\sqrt{4-x^2}$ とおくと，$\dfrac{du}{dx} = -\dfrac{x}{\sqrt{4-x^2}}$ より $\dfrac{x}{\sqrt{4-x^2}} = -\dfrac{du}{dx}$ であるから，

$$\int \frac{x}{\sqrt{4-x^2}} = \int -\frac{du}{dx}\ dx = -\int du = -u + C = -\sqrt{4-x^2} + C$$

である．ここで，C は任意の実数である．これより，以下を得る．

$$\int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\ dx = \lim_{a\to -2+0}\Big[-\sqrt{4-x^2}\Big]_a^0 + \lim_{b\to 2-0}\Big[-\sqrt{4-x^2}\Big]_0^b$$
$$= \lim_{a\to -2+0}\left(-2+\sqrt{4-a^2}\right) + \lim_{b\to 2-0}\left(-\sqrt{4-b^2}+2\right)$$
$$= -2 + 2 = 0.$$

(42) $\displaystyle\int_0^\infty xe^{-x^2}\ dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\ dx$ である．

$u = e^{-x^2}$ とおくと，$\frac{du}{dx} = -2xe^{-x^2}$ より $xe^{-x^2} = -\frac{1}{2}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	b
u	1	$\to$	e^{-b^2}

と変換される．よって，

$$\int_0^b xe^{-x^2}\ dx = \int_0^b -\frac{1}{2}\frac{du}{dx}\ dx = -\frac{1}{2}\int_1^{e^{-b^2}}\ du = \frac{1}{2}\left(1-e^{-b^2}\right)$$

より，以下を得る．

$$\int_0^\infty xe^{-x^2}\ dx = \lim_{b\to\infty}\frac{1}{2}\left(1-e^{-b^2}\right) = \frac{1}{2}.$$

(43) $\displaystyle\int_0^1 x\log x\ dx = \lim_{a\to+0}\int_a^1 x\log x\ dx$ である．

$$\int_a^1 x\log x\ dx = \left[\frac{1}{2}x^2\log x\right]_a^1 - \int_a^1 \frac{1}{2}x\ dx = -\frac{1}{2}a^2\log a - \frac{1}{4}\left(1-a^2\right)$$

より，

$$\int_0^1 x\log x\ dx = \lim_{a\to+0}\left(-\frac{1}{2}a^2\log a - \frac{1}{4}\left(1-a^2\right)\right) = -\frac{1}{2}\lim_{a\to+0}\frac{\log a}{\frac{1}{a^2}} - \frac{1}{4}$$

を得る．さらに，ここで

$$\lim_{a\to+0}\frac{(\log a)'}{\left(\frac{1}{a^2}\right)'} = \lim_{a\to+0}\frac{\frac{1}{a}}{-\frac{2}{a^3}} = \lim_{a\to+0}-\frac{1}{2}a^2 = 0$$

より，ロピタルの定理から以下を得る．

$$\int_0^1 x\log x\ dx = -\frac{1}{2}\lim_{a\to+0}\frac{\log a}{\frac{1}{a^2}} - \frac{1}{4} = -\frac{1}{4}.$$

(44) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+4} = \lim_{a\to-\infty}\int_a^0\frac{dx}{x^2+4} + \lim_{b\to\infty}\int_0^b\frac{dx}{x^2+4}$ である．

$u=\frac{x}{2}$ とおくと，$\frac{du}{dx}=\frac{1}{2}$ より $1 = 2\ \frac{du}{dx}$ であるから，

$$\int\frac{dx}{x^2+4} = \int\frac{1}{4(u^2+1)}\ 2\ \frac{du}{dx}\ dx = \frac{1}{2}\int\frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

を得る．ここで，C は任意の実数である．これより，以下を得る．

$$\begin{aligned}\int_{-\infty}^\infty\frac{dx}{x^2+4} &= \lim_{a\to-\infty}\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_a^0 + \lim_{b\to\infty}\frac{1}{2}\left[\tan_{-1}\frac{x}{2}\right]_0^b \\ &= \frac{1}{2}\lim_{a\to-\infty}\left(-\tan^{-1}\frac{a}{2}\right) + \frac{1}{2}\lim_{b\to\infty}\tan^{-1}\frac{b}{2} = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}.\end{aligned}$$

(45) $\displaystyle\int_1^\infty\frac{\log x}{x^2}\ dx = \lim_{b\to\infty}\int_1^b\frac{\log x}{x^2}\ dx$ である．

$$\int_1^b\frac{\log x}{x^2}\ dx = \left[-\frac{1}{x}\log x\right]_1^b + \int_1^b\frac{1}{x^2}\ dx = -\frac{\log b}{b} + \left[-\frac{1}{x}\right]_1^b = -\frac{\log b}{b} - \frac{1}{b} + 1$$

であるが，ここで

$$\lim_{b\to\infty}\frac{(\log b)'}{b'}=\lim_{b\to\infty}\frac{1}{b}=0$$

より，ロピタルの定理から以下を得る．

$$\int_1^\infty \frac{\log x}{x^2}\,dx=\lim_{b\to\infty}\left(-\frac{\log b}{b}-\frac{1}{b}+1\right)=1.$$

(46) $\displaystyle\int_0^\infty \frac{dx}{e^x(1+e^x)}=\lim_{b\to\infty}\int_0^b \frac{dx}{e^x\,(1+e^x)}$ である．

$u=e^x$ とおくと，$\frac{du}{dx}=e^x=u$ より $1=\frac{1}{u}\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	b
u	1	$\to$	e^b

と変換される．これより

$$\begin{aligned}\int_0^b \frac{dx}{e^x(1+e^x)}&=\int_0^b \frac{1}{u(1+u)}\frac{1}{u}\frac{du}{dx}\,dx=\int_1^{e^b}\frac{du}{u^2(1+u)}\\&=\int_1^{e^b}\left(\frac{1}{u^2}-\frac{1}{u}+\frac{1}{1+u}\right)du=\left[-\frac{1}{u}-\log|u|+\log|1+u|\right]_1^{e^b}\\&=1-\log 2-e^{-b}+\log\left(1+e^{-b}\right)\end{aligned}$$

なので，以下を得る．

$$\int_0^\infty \frac{dx}{e^x(1+e^x)}=\lim_{b\to\infty}\left(1-\log 2-e^{-b}+\log\left(1+e^{-b}\right)\right)=1-\log 2.$$

(47) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}=\lim_{b\to\infty}\int_0^b \frac{dx}{(x^2+1)(x^2+4)}$ である．

$$\begin{aligned}\int_0^b \frac{dx}{(x^2+1)(x^2+4)}&=\frac{1}{3}\int_0^b\frac{dx}{x^2+1}-\frac{1}{3}\int_0^b\frac{dx}{x^2+4}=\frac{1}{3}\Big[\tan^{-1}x\Big]_0^b-\frac{1}{12}\int_0^b\frac{dx}{\frac{x^2}{4}+1}\\&=\frac{1}{3}\tan^{-1}b-\frac{1}{12}\int_0^b\frac{dx}{\frac{x^2}{4}+1}\end{aligned}$$

なので，$u=\frac{x}{2}$ とおくと $\frac{du}{dx}=\frac{1}{2}$ より $1=2\,\frac{du}{dx}$ であり，積分区間は

x	0	$\to$	b
u	0	$\to$	$\frac{b}{2}$

と変換される．これより

$$\begin{aligned}\int_0^b \frac{dx}{(x^2+1)(x^2+4)}&=\frac{1}{3}\tan^{-1}b-\frac{1}{12}\int_0^b\frac{1}{u^2+1}\;2\,\frac{du}{dx}\,dx=\frac{1}{3}\tan^{-1}b-\frac{1}{6}\int_0^{\frac{b}{2}}\frac{du}{u^2+1}\\&=\frac{1}{3}\tan^{-1}b-\frac{1}{6}\Big[\tan^{-1}u\Big]_0^{\frac{b}{2}}=\frac{1}{3}\tan^{-1}b-\frac{1}{6}\tan^{-1}\frac{b}{2}\end{aligned}$$

だから，以下を得る．

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \lim_{b\to\infty}\left(\frac{1}{3}\tan^{-1}b - \frac{1}{6}\tan^{-1}\frac{b}{2}\right) = \frac{\pi}{12}.$$

(48) $\displaystyle\int_0^1 \frac{x^4}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1-0}\int_0^b \frac{x^4}{\sqrt{1-x^2}}\,dx$ である．

$u=\sin^{-1}x\,(\Leftrightarrow x=\sin u)$ とおくと，$\dfrac{du}{dx}=\dfrac{1}{\sqrt{1-x^2}}$ であり，積分区間は

$$\begin{array}{c|ccc} x & 0 & \to & b \\ \hline u & 0 & \to & \sin^{-1}b \end{array}$$

と変換される．これより

$$\begin{aligned}
\int_0^b \frac{x^4}{\sqrt{1-x^2}}\,dx &= \int_0^{\sin^{-1}b}\sin^4 u\,du = \int_0^{\sin^{-1}b}\left(\frac{1+\cos 2u}{2}\right)^2\,du \\
&= \frac{1}{4}\int_0^{\sin^{-1}b}\left(1+2\cos 2u+\cos^2 2u\right)\,du \\
&= \frac{1}{4}\left(\Big[x+\sin 2u\Big]_0^{\sin^{-1}b} + \int_0^{\sin^{-1}b}\frac{1+\cos 4u}{2}\,du\right) \\
&= \frac{1}{4}\left(\sin^{-1}b+\sin\left(2\sin^{-1}b\right)+\frac{1}{2}\Big[u+\frac{1}{4}\sin 4u\Big]_0^{\sin^{-1}b}\right) \\
&= \frac{3}{8}\sin^{-1}b+\frac{1}{4}\sin\left(2\sin^{-1}b\right)+\frac{1}{8}\sin\left(4\sin^{-1}b\right)
\end{aligned}$$

なので，以下を得る．

$$\begin{aligned}
\int_0^1 \frac{x^4}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1-0}\left(\frac{3}{8}\sin^{-1}b+\frac{1}{4}\sin\left(2\sin^{-1}b\right)+\frac{1}{8}\sin\left(4\sin^{-1}b\right)\right) \\
&= \frac{3}{8}\sin^{-1}1+\frac{1}{4}\sin\left(2\sin^{-1}1\right)+\frac{1}{8}\sin\left(4\sin^{-1}1\right) \\
&= \frac{3}{16}\pi+\frac{1}{4}\sin\pi+\frac{1}{8}\sin 2\pi = \frac{3}{16}\pi.
\end{aligned}$$