An Abstract Constraint Chart for Measurement-Limited State Representations

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Abstract

Reasoning about complex systems is often hindered by the presence of incompatible constraints, non-commuting operations, and limited intuitive access to high-dimensional state descriptions. In quantum and other constraint-rich domains, these difficulties are exacerbated by the mismatch between formal mathematical representations and the operational distinctions that are actually accessible under restricted measurement or control capabilities. This work introduces the Abstract Constraint Chart (ACC), a representational tool inspired by the classical Smith chart, designed to organize and visualize equivalence classes of states induced by constraint-limited distinguishability. Rather than mapping individual states or defining new dynamics, the ACC represents coarse-grained structure arising from explicitly specified equivalence relations, such as measurement-indistinguishability under restricted admissible measurements. Transformations within the ACC correspond to admissible operations that induce well-defined transitions between equivalence classes, encoding accessibility and directional change without invoking time evolution, mechanism, or physical causality. The ACC preserves relational structure—such as partial ordering, adjacency, and boundary behaviour—while deliberately discarding metric, dynamical, and fine-grained state information. A minimal toy instantiation is presented for a single qubit under restricted measurement capabilities, illustrating how the ACC clarifies constraint-imposed trade-offs and limits of distinguishability. The ACC is proposed as a general representational aid for reasoning and comparison across constrained systems, not as a predictive model or physical theory.

Keywords: representational frameworks; measurement constraints; equivalence classes; order structure

1. Introduction

1.1 Motivation

In many areas of physics, mathematics, and information science, the practical difficulty of reasoning about systems does not arise from a lack of formal description, but from an excess of structure relative to what is operationally accessible. While a system may admit a complete mathematical representation, the distinctions that can actually be drawn are often constrained by limited measurements, incompatible observables, or restricted classes of admissible operations. As a result, full state-space descriptions may obscure rather than clarify the structure that governs meaningful comparison and reasoning.

Quantum mechanics provides a canonical example. Although density operators or state vectors furnish mathematically exhaustive representations, experimental and conceptual access is mediated by restricted measurement sets and non-commuting observables. In such contexts, many formally distinct states are operationally indistinguishable, while others differ only in ways that cannot be simultaneously resolved. Reasoning at the level of individual states therefore risks importing distinctions that are not supported by the available constraints.

Similar issues arise outside quantum theory. In control systems with limited actuation, in communication protocols with restricted observables, and in constrained optimization problems where only partial orderings are relevant, the effective structure is determined less by the full configuration space than by the relations that survive under explicit constraints. In these settings, representations that preserve only relational features—such as order, adjacency, and boundary structure—can be more informative than metric or dynamical descriptions.

Historically, a number of successful representational tools have addressed this tension by organizing equivalence classes rather than individual configurations. The Smith chart is a well-known example: it visualizes impedance relationships by collapsing physically distinct configurations into equivalence classes defined by measurement and transformation constraints, while preserving only the structure relevant for reasoning and design [1].

Crucially, such charts do not model dynamics or mechanisms; they function as disciplined aids for navigating constraint-defined structure.

The motivation for the present work is to articulate a similarly disciplined representational object for systems governed by explicit measurement and transformation constraints, where meaningful distinctions arise at the level of equivalence classes rather than individual states.

1.2 Scope and Intent

The “Abstract Constraint Chart (ACC)” is introduced as a “representational framework”, not as a physical theory, dynamical model, or predictive formalism. Its purpose is to organize and make explicit the relational structure that arises when a state space is viewed through a specified set of admissible measurements and operations. The ACC does not posit new principles, modify existing domain theories, or offer explanatory mechanisms. All empirical, causal, or dynamical content remains entirely within the underlying theory appropriate to the system under consideration.

At the core of the ACC is a deliberately modest move: states that are operationally indistinguishable under a given constraint regime are identified via an explicit equivalence relation, and reasoning is carried out on the resulting quotient structure. This approach reflects a minimal operational insight—that distinctions without measurement support should not appear in the representation—without committing to any broader interpretive stance [8]. The resulting chart encodes only those relations that are invariant under the imposed constraints.

The ACC is intentionally agnostic with respect to ontology, dynamics, and geometry. It does not assert that equivalence classes correspond to physically identical states, nor that preserved orderings reflect causal or temporal structure. It does not introduce metrics, probabilities, or trajectories, and it makes no claim of completeness or optimality. Its role is purely organizational: to clarify which distinctions survive under constraint and how those distinctions relate to one another.

This restraint is deliberate. Representational frameworks often precede, rather than replace, theory development, providing shared structure without predictive ambition [S1, S2]. In this sense, the ACC is closer in spirit to chart-based or order-theoretic representations than to models intended for explanation or inference. Its value lies in making constraint-induced structure explicit and communicable, not in deriving new empirical consequences.

While the ACC is illustrated in this paper using a minimal quantum-mechanical example, it is not inherently quantum. The formal construction applies equally to classical, probabilistic, or abstract state spaces, provided that admissible measurements and transformations can be specified. Any domain-specific interpretation must therefore be supplied externally, and no claim is made that the ACC selects or privileges particular constraint regimes.

1.3 Structure of the Paper

The remainder of this paper develops the Abstract Constraint Chart in a deliberately incremental manner.

Section 2 introduces the formal setting. It specifies the objects of representation, defines equivalence relations induced by admissible measurement sets, and establishes the quotient space on which the ACC is defined. Emphasis is placed on generality and on the explicit dependence of all structure on stated constraints.

Section 3 identifies the minimal relational structures preserved by the ACC. These include measurement-induced orderings, adjacency relations corresponding to local distinguishability, boundary elements arising from constraint-induced limits, and monotonic relations under admissible transformations. Structures that are intentionally excluded—such as metrics, dynamics, and causal interpretation—are stated explicitly.

Section 4 defines admissible transformations and their induced action on equivalence classes. The distinction between accessibility relations and physical dynamics is emphasized to prevent misinterpretation of chart transitions.

Section 5 presents a minimal toy instantiation of the ACC for a single qubit under restricted measurement access. This example serves only to illustrate the representational construction and does not introduce new quantum-mechanical claims.

Section 6 delineates limitations, boundary conditions, and domains of applicability, clarifying when the ACC is informative and when it becomes trivial or inappropriate.

Section 7 concludes with a brief summary and outlook, situating the ACC as a disciplined reasoning aid for constraint-rich systems.

Appendix A collects formal propositions and remarks establishing well-definedness, order-theoretic properties, boundary structure, monotonicity without dynamics, and conditions under which the ACC trivializes. The appendix introduces no new assumptions and may be omitted by readers interested only in conceptual aspects.

2. Formal Setting

This section specifies the formal objects underlying the Abstract Constraint Chart (ACC). The construction is intentionally minimal: it introduces only those structures required to define operational equivalence, quotienting, and the relational features preserved by the chart. No assumptions are made regarding geometry, dynamics, or ontology beyond what is explicitly stated.

2.1 Objects of Representation

Let ($\mathcal{S}$) denote the set of admissible states of a system. The ACC places no restrictions on the internal structure of ($\mathcal{S}$) beyond the requirement that elements of ($\mathcal{S}$) admit evaluation by a specified class of admissible measurements. In particular, ($\mathcal{S}$) need not carry a metric, topology, or dynamical law.

In the quantum-mechanical illustration presented later, ($\mathcal{S}$) is taken to be the set of density operators acting on a finite-dimensional Hilbert space, following standard operational treatments of quantum states [10,12]. This choice is purely illustrative. The formal construction applies equally to classical state spaces, generalized probabilistic theories, or abstract collections of configurations, provided that states are identified only through admissible measurement statistics.

Crucially, the ACC does not represent individual elements of ($\mathcal{S}$) directly. Its representational objects are equivalence classes of states induced by explicitly stated constraints. Any structure present in the ACC must therefore descend from relations that are invariant across all representatives of a class. This design ensures that the chart reflects only distinctions that are operationally meaningful under the imposed constraints, and no others.

2.2 Equivalence Relations Induced by Constraints

Let ($\mathcal{M}_{\text{allowed}}$) denote a specified set of admissible measurements or evaluative functionals on ($\mathcal{S}$). These measurements encode the operational constraints under which states may be distinguished. No assumption is made that ($\mathcal{M}_{\text{allowed}}$) is informationally complete, closed under algebraic operations, or jointly compatible.

An equivalence relation ($\sim$) on ($\mathcal{S}$) is defined by measurement indistinguishability with respect to ($\mathcal{M}_{\text{allowed}}$). For ($\rho_1, \rho_2 \in \mathcal{S}$), $\rho_1 \sim \rho_2 \quad \iff \quad \mathrm{Tr}(M\rho_1) = \mathrm{Tr}(M\rho_2)\ \forall M \in \mathcal{M}_{\text{allowed}}$. This definition follows standard operational logic: states are identified if and only if they agree on all accessible measurement statistics [10,11,12].

The equivalence classes induced by ($\sim$) are denoted ($[\rho]$), and the quotient set ($\mathcal{S}/!\sim$) constitutes the representational domain of the ACC. The identification effected by ($\sim$) is strictly operational. It does not assert physical identity, ontological equivalence, or redundancy of the underlying states; it records only the resolution limits imposed by the chosen measurement set.

Different choices of ($\mathscr{M}_{\text{allowed}}$) generally induce different equivalence relations and therefore different quotient structures. All ACC representations are thus conditional on explicit constraint specification, and no canonical or observer-independent chart is implied.

2.3 Well-Definedness on the Quotient

The use of equivalence classes requires that all representational features of the ACC be well-defined on ($\mathscr{S}/! \sim$), independent of the choice of representative. This condition is satisfied by construction.

Because expectation values of admissible measurements are invariant across each equivalence class, any relation defined purely in terms of such expectation values descends unambiguously to the quotient [16,18]. This includes ordering relations, boundary identification, and monotonic relations discussed in subsequent sections. Formal statements of this invariance are collected in Appendix A.1.

Similarly, admissible transformations are required to respect the equivalence relation: if ($\rho_1 \sim \rho_2$), then any admissible transformation must map them to equivalent outputs. This consistency condition ensures that transformations induce well-defined mappings on ($\mathscr{S}/! \sim$) rather than representation-dependent relations. The ACC encodes only these induced class-level mappings.

2.4 Generality and Domain Independence

The formal setting of the ACC is deliberately domain-agnostic. The construction relies only on three ingredients: a state set ($\mathscr{S}$), an admissible measurement set ($\mathscr{M}_{\text{allowed}}$), and an induced equivalence relation defined by indistinguishability. No appeal is made to specifically quantum notions such as superposition, entanglement, or wave-function collapse, nor to classical phase-space structure.

This generality aligns the ACC with established mathematical practices of quotienting and order-based reasoning under partial information [15,16]. It also ensures that any additional interpretation—physical, informational, or experimental—must be supplied externally by the relevant domain theory.

2.5 Dependence on Explicit Constraint Specification

All structure in the ACC depends explicitly on the chosen constraints. Refining, enlarging, or restricting ($\mathscr{M}_{\text{allowed}}$) alters the equivalence relation and therefore the representational content of the chart. In the limiting cases, informationally complete measurement sets collapse the quotient to the original state space, while degenerate measurement sets collapse all states into a single class (Appendix A.5).

This dependence is not a defect but a defining feature. The ACC is designed to make the consequences of constraint choice explicit, allowing comparison across regimes without importing unstated assumptions.

3. Preserved Structure in the Abstract Constraint Chart

The Abstract Constraint Chart (ACC) represents equivalence classes of states induced by constraint-limited distinguishability. As a quotient representation, it does not inherit the full structure of the underlying state space ($\mathscr{S}$). Instead, it preserves a deliberately restricted set of “relational” features that remain well-defined on the quotient ($\mathscr{S}/! \sim$). This section specifies those preserved structures and states explicitly what is excluded.

3.1 Partial Ordering Induced by Admissible Measurements

Each admissible measurement ($M \in \mathscr{M}_{\text{allowed}}$) induces an order relation on equivalence classes via expectation values. For classes($[\rho_a]$) and ($[\rho_b]$), define

$[\rho_a] \preceq_M [\rho_b] \quad \Longleftrightarrow \quad \mathrm{Tr}(M\rho_a) \leq \mathrm{Tr}(M\rho_b)$. By construction, this relation is invariant under choice of representative and therefore well-defined on the quotient (Appendix A.2).

In general, ($\preceq_M$) defines a 'preorder' rather than a total or antisymmetric order. Distinct equivalence classes may yield identical expectation values and thus be mutually comparable in both directions. The ACC preserves this preorder structure without refinement unless explicitly specified. Order-theoretic language is used intentionally: the chart records relative positioning under admissible measurements without introducing metric separation or scalar aggregation [15,17].

Different admissible measurements typically induce incompatible preorders. The ACC does not attempt to reconcile or unify these orderings into a single global structure. Their coexistence reflects the constraint regime itself and mirrors familiar incompatibilities in operational settings [12].

3.2 Adjacency and Local Distinguishability

Beyond ordering, the ACC preserves "adjacency relations" that encode local distinguishability under constraints. Two equivalence classes are adjacent if there exists an admissible transformation that effects an arbitrarily small change in the expectation value of at least one admissible measurement while remaining within the constraint regime.

Adjacency is thus defined operationally, not metrically. It captures which distinctions can be accessed incrementally and which require discontinuous or forbidden transitions. The ACC records adjacency as a primitive relational feature, without assigning distances, angles, or neighbourhoods in a topological sense [15].

This notion is particularly relevant in constraint-rich systems, where small changes in accessible statistics may correspond to large or indeterminate changes in the underlying state space. By preserving adjacency rather than metric proximity, the ACC avoids importing geometric assumptions that are not supported by the constraints.

3.3 Boundary Structure and Constraint-Induced Limits

Admissible measurements often have bounded expectation values on ($\mathcal{S}$). Equivalence classes that saturate these bounds define "boundary elements" of the ACC with respect to the corresponding measurement-induced preorder.

These boundaries are preserved as extremal positions within the ordering, not as geometric edges or physically distinguished states. Their significance is entirely constraint-induced: they mark limits of distinguishability under the specified measurement set and may shift or disappear if constraints are modified [15,26].

The ACC preserves the "existence and relative placement" of such boundary classes, as formalized in Appendix A.3. No claim is made that boundary elements correspond to pure states, eigenstates, or otherwise privileged configurations in the underlying theory. The term "boundary" is used strictly in an order-theoretic sense.

3.4 Monotonicity Under Admissible Transformations

Although the ACC does not encode dynamics, it preserves "monotonic relations" induced by admissible transformations. If a transformation consistently increases or decreases the expectation value of an admissible measurement across an equivalence class, the corresponding directional relation between classes is retained.

Formally, a transformation that is monotone with respect to a measurement (M) induces an order-preserving map on the quotient (Appendix A.4). The ACC records only the "direction and consistency" of such changes. It does not encode rates, generators, composition laws, or temporal ordering [17,19].

This monotonic structure supports reasoning about accessibility—what can be reached from what under constraints—without introducing causal or dynamical interpretation. Directionality here is relational, not temporal.

3.5 Explicitly Unpreserved Structure

For clarity, the ACC does not preserve:

- metric distances or angles,
- inner products or geometric embeddings,
- probabilistic weights beyond order comparisons,
- dynamical trajectories or time parameters,
- causal structure or inference relations.

These exclusions are intentional and constitutive of the framework. Any attempt to reintroduce such structure would require additional assumptions beyond the specified constraints and would therefore fall outside the ACC's representational scope. In particular, no causal interpretation is implied or supported by chart relations [S3].

3.6 Preserved vs. Discarded Structure (Summary)

For emphasis, the ACC preserves:

- equivalence classes under explicit constraints,
- measurement-induced preorders,
- adjacency relations of local distinguishability,
- boundary elements as extremal order positions,
- monotonic accessibility relations.

It discards:

- geometry, metrics, and topology,
- dynamics, rates, and trajectories,
- causal and ontological interpretation.

This separation is not a limitation to be remedied but a design constraint. By preserving only order-theoretic and relational structure, the ACC makes explicit exactly what survives constraint-induced reduction and nothing more.

4. Admissible Transformations and Class Transitions

The Abstract Constraint Chart (ACC) represents how equivalence classes relate under transformations that respect the imposed measurement constraints. This section formalizes admissible transformations, specifies the induced mappings on the quotient space, and clarifies the limited sense in which "transitions" are represented. No dynamical, causal, or temporal interpretation is introduced.

4.1 Admissible Transformations

An admissible transformation is defined as any mapping ($T : \mathcal{S} \to \mathcal{S}$) that preserves the equivalence relation induced by ($\mathscr{M}_{\text{allowed}}$). Explicitly, (T) is admissible if $\rho_1 \sim \rho_2 \implies T(\rho_1) \sim T(\rho_2)$. This condition ensures that the action of (T) is well-defined on equivalence classes and therefore induces a mapping: $\widetilde{T} : \mathcal{S}/! \sim \to \ \mathcal{S}/! \sim$. The ACC encodes only the induced mapping ($\widetilde{T}$), not the underlying action of (T) on representatives. Any representational content that depends on representative choice is excluded by design (Appendix A.4).

No assumption is made that admissible transformations form a group, semigroup, or category. Composition, invertibility, and continuity are not required. The admissibility criterion is purely operational and constraint-relative.

4.2 Class Transitions Without Dynamics

A "class transition" in the ACC is the relation ($[\rho] \mapsto [\rho']$) induced by an admissible transformation. Such transitions encode the possibility of moving between equivalence classes under the specified constraints. They do not encode time evolution, process duration, or causal mechanism.

The term "transition" is used descriptively rather than dynamically. It denotes a relation between equivalence classes under an allowed mapping, not a physical process unfolding in time. Multiple admissible transformations may induce identical class-level transitions, and the ACC does not distinguish between them [17,19].

This abstraction allows the ACC to represent accessibility relations without importing assumptions about underlying dynamics. In particular, it avoids conflating monotonic change with temporal flow, a distinction emphasized throughout the framework.

4.3 Monotonicity and Order Preservation

Certain admissible transformations preserve or reverse the preorders induced by admissible measurements. If a transformation satisfies $[\rho_a] \preceq_M [\rho_b] \implies$

$$\widetilde{T}([\rho_a]) \preceq_M \widetilde{T}([\rho_b]),$$

it is order-preserving with respect to measurement (M). The ACC records this monotonicity as a relational feature between classes.

Only the existence and direction of monotonicity are represented. The ACC does not encode the magnitude of change, rates, or whether the transformation is generated continuously or discretely. These omissions are intentional and aligned with the framework's representational scope (Appendix A.4).

Transformations that violate monotonicity for a given measurement are not excluded but do not contribute order-preserving structure for that measurement. The ACC remains agnostic regarding their detailed behaviour.

4.4 Fixed Classes and Invariant Regions

Some equivalence classes may be invariant under a given admissible transformation, satisfying $\widetilde{T}([\rho]) = [\rho]$. Such fixed classes are represented as self-maps in the ACC. Their presence indicates constraint-induced stability, not equilibrium or physical conservation. No dynamical interpretation is implied.

More generally, subsets of equivalence classes may be invariant as a set, forming regions of the chart closed under admissible transformations. The ACC records these invariant regions as relational features without assigning additional structure or interpretation [18].

4.5 Scope and Exclusions

The ACC representation of transformations is intentionally limited. It does not preserve:

- temporal ordering or time parameters,
- causal relations or intervention semantics,
- generators, Hamiltonians, or stochastic kernels,
- compositional laws beyond induced class mappings.

Any interpretation of class transitions as physical evolution must be supplied externally and justified independently. Within the ACC, transformations are representational devices for relating equivalence classes under explicit constraints, nothing more.

5. A Minimal Toy Instantiation: Single Qubit Under Restricted Measurement

This section presents a minimal illustrative instantiation of the Abstract Constraint Chart (ACC) using a single qubit subject to restricted measurement access. The purpose is not to motivate the framework or to derive physical insight, but to demonstrate how the formal construction operates in a familiar setting. The ACC remains fully defined independently of this example.

5.1 State Space and Measurement Constraints

Consider a single qubit with state space ($\mathcal{S}$) given by the set of density operators acting on a two-dimensional Hilbert space. While this state space admits a geometric representation via the Bloch sphere, no such structure is used in the ACC construction and is treated as inessential for the purposes of representation [10,12].

Suppose that the admissible measurement set ($\mathcal{M}_{\text{allowed}}$) consists of a single projective measurement corresponding to the Pauli operator (σ_z). Operationally, this restriction reflects a constraint under which only expectation values of (σ_z) are accessible, while all other observables are excluded [11,13].

This constraint defines the regime under which states may be distinguished and therefore determines the induced equivalence relation.

5.2 Induced Equivalence Classes

Under the specified measurement constraint, two density operators are equivalent if and only if they yield the same expectation value for (σ_z). All states sharing a common value of ($\langle\sigma_z\rangle$) are therefore identified into a single equivalence class.

The resulting quotient ($\mathcal{S}/! \sim$) collapses the full qubit state space into a one-parameter family of equivalence classes indexed by the accessible expectation value. This identification discards all phase information and all distinctions associated with measurements orthogonal to (σ_z), consistent with the operational constraint [12].

No claim is made that the equivalence classes correspond to physically homogeneous or ontologically meaningful subsets of states. They are defined solely by indistinguishability under the allowed measurement.

5.3 Preserved Ordering and Boundary Structure

The admissible measurement induces a natural preorder on the equivalence classes via ordering of expectation values. Classes may be ordered by increasing ($\langle\sigma_z\rangle$), yielding a linear preorder on the quotient.

The extremal equivalence classes correspond to saturation of the measurement bounds at ($\langle\sigma_z\rangle = \pm 1$). These classes form boundary elements of the ACC with respect to the induced ordering. Their boundary status reflects only the limits of the admissible measurement and does not imply purity, extremality in state space, or any privileged physical role [15].

Adjacency relations correspond to infinitesimal changes in the accessible expectation value and are preserved without invoking any metric or geometric structure.

5.4 Admissible Transformations and Class Transitions

Admissible transformations in this setting are those mappings on the qubit state space that preserve equivalence under the restricted measurement. Any transformation that leaves the (σ_z) expectation value invariant induces the identity mapping on the ACC, while transformations that monotonically increase or decrease ($\langle\sigma_z\rangle$) induce order-preserving class transitions.

The ACC records only the induced class-level relations. It does not distinguish between different physical processes that produce the same change in expectation value, nor does it encode temporal ordering, rates, or generators [17,19].

Transformations that alter inaccessible degrees of freedom without affecting (σ_z) expectation are invisible to the chart.

5.5 Interpretation of the Chart Representation

The resulting ACC representation for this toy instantiation is a one-dimensional ordered set with boundary elements and monotonic class transitions. This structure reflects only the imposed measurement constraint and should not be interpreted as a reduced dynamical model, effective theory, or physical explanation.

The example serves solely to illustrate how constraint-induced equivalence, ordering, adjacency, and boundary structure appear in a concrete setting. No general claims about qubit systems or quantum mechanics are inferred from this representation.

6. Limitations, Boundary Conditions, and Scope of Applicability

The Abstract Constraint Chart (ACC) is deliberately limited in scope. Its purpose is not to provide a complete or optimal representation of physical systems, but to formalize how explicit measurement constraints induce equivalence classes and preserve a restricted set of relational structures. This section clarifies the boundaries of applicability of the framework and the conditions under which it is not intended to be used.

6.1 Dependence on Explicit Constraints

All structure represented in an ACC depends explicitly on the choice of admissible measurements. Refining, enlarging, or restricting the measurement set alters the induced equivalence relation and therefore the resulting chart. No ACC representation is invariant under arbitrary changes of constraint, nor is any single chart privileged as canonical [15,16].

This dependence is a constitutive feature of the framework. The ACC is designed to make constraint-induced information loss explicit, not to minimize or eliminate it. Comparisons between charts constructed under different constraint regimes are meaningful only when the differing constraints are themselves part of the analysis.

6.2 Inapplicability to Fully Accessible or Metric-Dominated Systems

In regimes where measurement access is informationally complete, the induced equivalence relation collapses to identity and the ACC reduces trivially to the original state space. In such cases, the framework provides no representational advantage and is not intended to replace existing geometric, probabilistic, or dynamical formalisms [10,12].

Similarly, in systems where metric structure, continuity, or quantitative distance measures are essential to the analysis, the ACC may discard information that is operationally or theoretically central. The framework intentionally avoids encoding metric or topological structure and should not be applied where such structure is required for correct reasoning.

6.3 No Claims of Completeness or Optimality

The ACC makes no claims of representational completeness, optimality, or minimality in an information-theoretic sense. It does not assert that preserved order, adjacency, and boundary relations exhaust all meaningful content under constraint, nor that discarded structure is irrelevant in other contexts [17,18].

The framework is therefore not proposed as a universal reduction scheme or as a replacement for domain-specific models. It is a disciplined representational tool whose utility depends on the questions being asked and the constraints being imposed.

6.4 Domain-Specific Interpretation Required

Interpretation of any ACC representation must be supplied externally by the relevant domain theory. The chart itself encodes no physical ontology, causal structure, or experimental narrative. Relations depicted in the ACC should not be read as physical processes, mechanisms, or explanations without additional justification [19,26].

In particular, monotonic relations and class transitions do not imply temporal evolution or causal influence. Any such interpretation lies outside the scope of the framework and must be argued independently.

6.5 Summary of Applicability

The ACC is applicable when:

- measurement access is explicitly constrained,
- equivalence under indistinguishability is central,
- relational structure is sufficient for the intended analysis.

It is not applicable when:

- full state access is available,
- metric or dynamical detail is essential,
- interpretive or explanatory claims are required.

These limitations are not shortcomings but design constraints. They define the narrow representational role the ACC is intended to play.

7. Conclusion and Outlook

This work has introduced the Abstract Constraint Chart (ACC) as a representational framework for organizing state descriptions under explicit measurement constraints. By quotienting a state space according to operational indistinguishability and preserving only order, adjacency, and boundary relations, the ACC makes precise what structural information survives constraint-induced reduction.

The framework is intentionally limited. It does not model dynamics, encode geometry, or support causal interpretation. Instead, it provides a disciplined means of reasoning about equivalence classes and relational structure when access to a system is restricted. In this respect, the ACC aligns with established operational and order-theoretic approaches while remaining agnostic about the underlying physical theory [10,12,15].

The minimal qubit instantiation illustrates how familiar systems may be represented within the ACC without invoking domain-specific structure. The example is not evidentiary and does not motivate the framework; it serves only to demonstrate internal consistency and practical construction under a simple constraint regime.

Future work may apply the ACC to other constrained settings, including higher-dimensional systems, coarse-grained classical models, or abstract operational theories. Such applications would use the ACC as a representational tool rather than extending or redefining the framework itself. Any additional interpretation or explanatory content would necessarily arise from the external theory being represented, not from the chart structure [17,19].

More broadly, the ACC may serve as a comparative device for analyzing how different constraint regimes alter the relational content of state descriptions. By making explicit which distinctions are preserved and which are discarded, the framework supports careful reasoning under partial access without inflating representational commitments.

No claim is made that the ACC exhausts or optimizes representational possibilities under constraint. Its contribution is limited to clarifying structure, not asserting explanation. Within that scope, it provides a compact and formally controlled tool for organizing constrained state representations.

Appendix A. Formal Remarks and Supporting Propositions

This appendix collects formal remarks and propositions that support the constructions used in the main text. These results clarify well-definedness on the quotient, preservation of relational structure, and limiting cases of constraint choice. They are included for completeness and do not introduce additional commitments beyond those stated in Sections 2–7.

A.1 Well-Definedness of Measurement-Induced Relations

Proposition A.1.

Let ($\sim$) be the equivalence relation on ($\mathcal{S}$) induced by indistinguishability with respect to the admissible measurement set ($\mathcal{M}_{\text{allowed}}$). Any relation defined solely in terms of expectation values of measurements in ($\mathcal{M}_{\text{allowed}}$) is invariant under choice of representative and therefore well-defined on the quotient ($\mathcal{S}/! \sim$).

Sketch of justification.

If ($\rho_1 \sim \rho_2$), then by definition ($\mathrm{Tr}(M\rho_1) = \mathrm{Tr}(M\rho_2)$) for all admissible measurements. Any relation depending only on these quantities assigns identical relational status to all representatives of the class.

This proposition underwrites the use of order, adjacency, and boundary relations throughout Sections 2–4 [15,16].

A.2 Measurement-Induced Preorders

Proposition A.2.

For any admissible measurement (M), the relation ($\preceq_M$) defined on equivalence classes by comparison of expectation values is a preorder on ($\mathcal{S}/! \sim$).

Reflexivity and transitivity follow directly from the properties of the real-valued expectation functional. Antisymmetry need not hold, as distinct equivalence classes may yield identical expectation values under (M). This lack of antisymmetry is preserved intentionally and reflects constraint-induced degeneracy rather than representational deficiency [15,17].

A.3 Boundary Elements

Remark A.3.

Boundary elements of the ACC correspond to equivalence classes that saturate the admissible range of a measurement-induced preorder. Their identification is purely order-theoretic.

No geometric, probabilistic, or ontological interpretation is implied. Boundary status depends entirely on the chosen constraint set and may change under refinement or relaxation of admissible measurements [15,26]. This remark aligns with the discussion in Sections 3.3 and 5.3.

A.4 Admissible Transformations and Induced Maps

Proposition A.4.

Any admissible transformation ($T : \mathcal{S} \to \mathcal{S}$) that preserves the equivalence relation ($\sim$) induces a well-defined mapping ($\widetilde{T}$) on the quotient ($\mathcal{S}/! \sim$).

Sketch of justification.

If ($\rho_1 \sim \rho_2$), admissibility implies ($T(\rho_1) \sim T(\rho_2)$), ensuring that the image class depends only on the input class.

Order preservation or monotonicity with respect to specific measurements is an additional property and is not required for admissibility. This proposition supports the treatment of class transitions in Sections 3.4 and 4 [17,19].

A.5 Limiting Cases of Constraint Choice

Remark A.5.

Two limiting cases clarify the dependence of the ACC on explicit constraints:

- 1. Informational completeness.
  - If ($\mathscr{M}_{\text{allowed}}$) is informationally complete, then ($\sim$) reduces to identity and the ACC collapses to a representation of the full state space. In this regime, the framework provides no reduction and no added representational value [10,12].
- 2. Degenerate constraints.
  - If ($\mathscr{M}_{\text{allowed}}$) contains only trivial measurements, all states become equivalent and the ACC reduces to a single class. All relational structure is lost.

These cases delimit the regime in which the ACC is nontrivial and meaningful, as discussed in Sections 2.5 and 6.2.

A.6 Scope of Formal Claims

Remark A.6.

The propositions collected here establish only structural and representational properties. They do not support claims about physical identity, causal influence, or dynamical behaviour. Any such claims require additional assumptions external to the ACC framework and lie outside the scope of this work [18,19].

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