TAYLOR'S THEOREM IN NORMED VECTOR SPACES

KB-MATH

1. Basic setup

Let U, V be normed vector spaces and suppose that $X \subset U$ is open. Let $\mathcal{L}(U, V)$ denote the normed vector space of bounded linear maps.

Definition 1.1. We say that $f : X \to V$ is differentiably at $x_0 \in X$ if there exists a bounded linear map $D : U \to V$ and $\epsilon : [0, \infty) \to [0, \infty)$ with $\lim_{r \to 0} \epsilon(r) = 0$ such that

$$\|f(x_0 + u) - f(x_0) - Du\| \leq \epsilon(\|u\|)\|u\|$$

for all u in some open neighbhourhood of 0.

Lemma 1.2. Such a D must satisfy

$$Du = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t\|u\|}$$

for each fixed non-zero $u \in U$ and is hence unique.

Proof. For each fixed $u \in U \setminus \{0\}$ and sufficiently small t we have that

$$\|f(x_0 + tu) - f(x_0) - D(tu)\| = \epsilon(t\|u\|)\|tu\|.$$

Now divide both sides by $|t|$ and use the linearity of D to get the estimate

$$\|\frac{f(x_0 + tu) - f(x_0)}{t} - Du\| \leq \epsilon(t\|u\|)\|u\|.$$

The result now follows by letting $t \to \infty$. $\square$

Given the uniqueness, we can now define $D = Df(x_0) \in \mathcal{L}(U, V)$ to be the derivative of f at x_0.

Lemma 1.3. If $f : X \to V$ is differentiable at x_0 then it is continuous at x_0.

Proof. By triangle inequality we have that

$$\|f(x_0 + u) - f(x_0)\| \leq \|f(x_0 + u) - f(x_0) - Df(x_0)u\| + \|Df(x_0)u\|$$

but both terms on the right hand side converge to zero as $u \to 0$ by the differentiability of f and the boundedness of $Df(x_0)$, respectively. $\square$

2. Higher order derivatives

How do we define higher order derivatives in this setting? If $f : X \to V$ is differentiable on X then $Df : X \to \mathcal{L}(U, V)$. Now $\mathcal{L}(U, V)$ is itself a normed vector space (equipped with operator norm) hence the derivative of Df (assuming it exists) is a map

$$D^2 f : X \to \mathcal{L}(U, \mathcal{L}(U, V)).$$

This means that given $u_1, u_2 \in U$ we have that

$$D(Df)(x_0) \in \mathcal{L}(U, \mathcal{L}(U, V))$$

and hence

$$(D(Df)(x_0))u_1 \in \mathcal{L}(U, V)$$

and hence

$$((D(Df)(x_0))u_1)u_2 \in V.$$

We abbreviate this element in V by

$$(D^2 f)(x_0)(u_1, u_2)$$

and hence we get a **multilinear map**

$$(D^2 f)(x_0) : U \times U \to V$$

called the *second derivative* of f.

Lemma 2.1. If $f : \mathbb{R}^d \to V$ is twice differentiable then

$$D^2 f(x_0)(e_i, e_j) = \partial_i \partial_j f(x_0)$$

where ∂_i is partial differentiation with respect to i-th co-ordinate.

This can be generalized to higher order derivatives, hence $D^n f(x_0) : U \times \cdots \times U \to V$ may be realised as a multilinear map, if it exists, and it can be evaluated through iterated directional derivatives. To make this precise it is worth formalizing some notions regarding the continuity of multilinear maps.

Definition 2.2. Given normed vector spaces $U_1, \ldots, U_n$ and V, let $\mathcal{T}(U_1, \ldots, U_n \to V)$ denote the vector space of *bounded* multilinear maps $T : U_1 \times \cdots \times U_n \to V$, where bounded means that

$$\|T\| := \sup\{T(u_1, \ldots, u_m) \mid u_i \in U_i \text{ with } \|u_i\| = 1\} < \infty.$$

Moreover, $\|\cdot\|$ is a norm on the space of bounded multilinear maps.

Proposition 2.3. There is a canonical isometric isomorphism

$$\mathcal{L}(U_1, \mathcal{T}(U_2, \ldots, U_n \to V)) \cong \mathcal{T}(U_1, \ldots U_n \to V)$$

given by

$$\phi \mapsto ((u_1, \ldots, u_n) \mapsto \phi(u_1)(u_2, \ldots, u_n)).$$

In other words, we could have defined $\|T\|$ as the metric induced by the canonical isomorphism

$$\mathcal{T}(U_1, \ldots, U_n \to V) \cong \mathcal{L}(U_1, \ldots \mathcal{L}(U_{n-1}, \mathcal{L}(U_n, V)) \ldots)$$

Proposition 2.4. A multilinear map $T : U_1 \times \cdots \times U_n \to V$ is bounded if and only if it is continuous with respect to the product topology.

Hence the nth derivative $D^n f : X \to \mathcal{T}(U, \ldots, U \to V)$ may be recursively defined as $D^1 f = Df$ (which makes sense since $\mathcal{T}(U \to V) = \mathcal{L}(U, V)$) and

$$D^{(n+1)} f(x_0)(u_1, \ldots, u_{n+1}) = [(D(D^n f))(x_0)u_1](u_2, \ldots, u_{n+1})$$

if it exists (and if, of course, $D^n f$ exists on X).

3. Basic properties

Lemma 3.1. (Chain Rule) Suppose that U, V, W are normed vector spaces with $X \subset U$ and $Y \subset V$ open sets. Suppose $f : X \to Y$ is differentiable at $x_0 \in X$ and $g : Y \to W$ is differentiable at $y_0 = f(x_0) \in Y$. Then $g \circ f : X \to W$ is differentiable at x_0 and

$$D(g \circ f)(x_0) = D(g)(y_0) \circ D(f)(x_0).$$

Given normed vector spaces U_1, U_2 we will always equip the direct sum with the max norm $\|(u_1, u_2)\| = \max\{\|u_1\|, \|u_2\|\}$, which induces the product topology.

Lemma 3.2. (Product Rule for multilinear maps) Suppose that $U_1, \ldots, U_n$ are normed vector spaces and $T : U_1 \times \cdots \times U_n \to V$ is a bounded multilinear map. Then T is differentiable at each $\vec{x} = (x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$ with

$$D(T)(\vec{x})(u_1, \ldots, u_n) = T(u, x_2, \ldots, x_n) + T(x_1, u, x_3, \ldots, x_n) + \cdots T(x_1, \ldots x_{n-1}, u).$$

Lemma 3.3. (General product rule) Suppose that $U_1, \ldots, U_n$ are normed vector spaces and $T : U_1 \times \cdots \times U_n \to V$ is a bounded multilinear map and suppose that $f_i : X \to U_i$ are differentiable at $x_0 \in X$. Then the map

$$H(x) = T(f_1(x), \ldots f_n(x))$$

is differentiable at x_0 and has derivative

$$DH(x_0)u = T((Df_1)(x_0)u, f_2(x_0), \ldots, f_n(x_0)) + \cdots + T(f_1(x_0), f_2(x_0), \ldots, Df_n(x_0)u)$$

Proof. The map $F(x) = (f_1(x), \ldots, f_n(x))$ has derivative

$$DF(x_0)u = (Df_1(x_0)u, \ldots, Df_n(x_0)u)$$

and $H = T \circ F$ hence the derivative is

$$\begin{aligned} D(H)(x_0)u &= D(T)(F(x_0))(DF)(x_0)u \\ &= DT(F(x_0))(Df_1(x_0)u, \ldots, Df_n(x_0)u) \end{aligned}$$

and the result now follows from the product rule for multilinear maps.

□

This example will be important for our proof of Taylor's theorem and can be used to give a higher order product rule.

Example 3.4. Consider the composition map

$$\mathcal{L}(U,V) \times \mathcal{L}(W,U) \to \mathcal{L}(W,U)$$

given by $(L_1, L_2) \mapsto L_1 \circ L_2$. Then it is a bounded bilinear map. Now suppose that we have maps $g : X \to \mathcal{L}(W,U)$ and $h : X \to \mathcal{L}(U,V)$ differentiable at $x_0 \in X$, where X is an open subset of a normed vector space U_0. Then the map $M(x) = h(x) \circ g(x)$ is differentiable at x_0 with derivative given by

$$DM(x_0)u = D(h(x_0))u \circ g(x_0) + h(x_0) \circ Dg(x_0)u \quad \text{for } u \in U_0.$$

Lemma 3.5. (Mean value inequality) Suppose $f : X \to V$ is differentiable on each point of the segment $x_0 + tu$ for $t \in [0,1]$ and that $t \mapsto Df(x_0 + tu)$ is continuous on $[0,1]$. Then

$$\|f(x_0 + u) - f(x_0)\| \leq \sup_{t\in[0,1]} \|Df(x_0 + tu)u\|.$$

Proof. We first reduce to the case $V = \mathbb{R}$ as follows. By the Hahn-Banach theorem we have a linear $\phi : V \to R$ such that $\phi(f(x_0 + u) - f(x_0)) = \|f(x_0 + u) - f(x_0)\|$ and $|\phi\| = 1$. So if the theorem is true for $g = \phi \circ f : X \to V$ we must have

$$\|f(x_0 + u) - f(x_0)\| = \|g(x_0 + u) - g(x_0)\| \leq \sup_{t\in[0,1]} \|Dg(x_0 + tu)u\|.$$

But by the chain rule and the fact that $D\phi(x) = \phi$ for all linear ϕ, we have that $Dg = \phi \circ Df$. Hence we have that

$$\|Dg(x_0 + tu)u\| = \|\phi Df(x_0 + tu)u\| \leq Df(x_0 + tu)u.$$

We now turn to show that the theorem holds when $V = \mathbb{R}$. Let $h(t) = f(x_0 + tu)$. Note that the classical derivative $h'(t) = Dh(t)(1)$ is continuous, hence we may apply the fundamental theorem of calculus to get

$$\|f(x_0 + u) - f(x_0)\| = |h(1) - h(0)| \leq |\int_0^1 h'(t)dt| \leq \sup_{t\in[0,1]} |h'(t)|.$$

But from the chain rule we get

$$\begin{aligned}
h'(t_0) &= Dh(t_0)(1) \\
&= D(f(x_0 + ut_0))D(t \mapsto (x_0 + tu))(t_0)(1) \\
&= D(f(x_0 + t_0u))(t \mapsto tu)(1) \\
&= D(f(x_0 + t_0u))u
\end{aligned}$$

□

4. The commutativity of higher order derivatives

Theorem 4.1. Suppose that $f : U \to V$ is twice differentiable at x_0, then $D^2f(x_0)$ is a symmetric bilinear form, that is

$$D^2f(x_0)(u_1, u_2) = D^2f(x_0)(u_2, u_1) \quad \text{for all } u_1, u_2 \in U.$$

Proof. It is sufficient to show this for all small enough $u_1, u_2 \in U$. Let $g(x) = f(x+u_1) - f(x)$. If $L : U \to V$ is the bounded linear map $L = D^2f(x_0)(u_1)$ then by the mean value inequality applied to $g - L$ we have that

$$\|g(x_0 + u_2) - g(x_0) - L(u_2)\| \leq \sup_{t\in[0,1]} \|Dg(x_0 + tu_2)u_2 - L(u_2)\| \tag{1}$$

for all small enough $u_1, u_2 \in U$.

By the definition of differentiability we have that

$$Df(x_0 + u) = Df(x_0) + D^2f(x_0)(u) + R(u)$$

where $\|R(u)\| \leq \epsilon(u)\|u\|$ where $\lim_{u\to 0} \epsilon(u) = 0$. We use this inequality to estimate the upper bound in (1) as follows

$$\begin{aligned} Dg(x_0 + tu_2)u_2 &= Df(x_0 + tu_2 + u_1)u_2 - Df(x_0 + tu_2)u_2 \\ &= D^2f(x_0)(tu_2 + u_1)u_2 - D^2f(x_0)(tu_2)u_2 + R(tu_2 + u_1)u_2 - R(tu_2)u_2 \\ &= D^2f(x_0)(u_1)(u_2) + R(tu_2 + u_1)u_2 - R(tu_2)u_2 \end{aligned}$$

Hence (1) gives the estimate

$$\begin{aligned} \|g(x_0 + u_2) - g(x_0) - D^2f(x_0)(u_1, u_2)\| &\leq \sup_{t\in[0,1]} \|R(tu_2 + u_1)u_2 - R(tu_2)u_2\| \\ &\leq \epsilon_0(\|u_1\| + \|u_2\|)(\|u_1\| + \|u_2\|)^2 \end{aligned}$$

for some $\lim_{u\to 0} \epsilon_0(\|u\|) = 0$. In particular, by replacing u_i with su_i for $0 < s < 1$ and dividing by s^2 we get the limit

$$\begin{aligned} D^2f(x_0)(u_1)(u_2) &= \lim_{s\to 0} \frac{g(x_0 + su_2) - g(x_0)}{s^2} \\ &= \lim_{s\to 0} \frac{f(x_0 + su_1 + su_2) - f(x_0 + su_2) - f(x_0 + su_1) + f(x_0)}{s^2} \end{aligned}$$

but this term inside the limit is symmetric in u_1 and u_2. □

We can apply this inductively to get (notice that we don't require $D^n f(x)$ to be defined or continuous on any open set):

Theorem 4.2. Suppose that $f : U \to V$ is such that $D^n f(x_0)$ exists at $x_0 \in U$, then $D^n f(x_0)$ is a symmetric multilinear form.

5. Taylor's Theorem

If $h : \mathbb{R} \to \mathbb{R}$ is N times continuously differentiable at $x_0 \in \mathbb{R}$ then the classical Taylor expansion is

$$h(x_0 + u) = h(x_0) + h'(x_0)u + \frac{1}{2!}h^{(2)}(x_0)u^2 + \cdots + \frac{1}{N!}h^{(N)}(x_0)u^N + R(u)$$

with error term

$$R(u) = \frac{1}{(N+1)!}\int_0^u h^{(N+1)}(x_0 + t)t^n dt.$$

We now ask how can this be formulated for $f : U \to \mathbb{R}$ where U is an arbitrary vector space? If we define u^n to be the n-tuple $(u, \ldots, u)$ then the expression $Df^n(x_0)u^n$ makes sense, as $Df^n(x_0)$ is a multilinear map. Hence a strategy of deriving a Taylor theorem involves first parametrizing the line segment from x_0 to $x_0 + u$ by defining $L(t) = tu$ and applying the classical Taylor theorem to $h = f \circ L$. To do this we need to compute $D^n(f \circ L)(t_0)$.

Lemma 5.1. Let $L : W \to U$ be a linear map and let $f : U \to V$ be n times differentiable at $L(a)$, where $a \in W$. Then

$$D^n(f \circ L)(a) = D^n f(L(a)) \circ L^n$$

where $L^n(w_1, \dots, w_n) := (Lw_1, \dots, Lw_n)$.

Remark 5.2. The is a generelization of the standard identity $h^{(n)}(at) = a^n h^{(n)}(at)$ from classical differential calculus.

Proof. For $n = 1$ this is the chain rule and the fact that $D(L)(a) = L$ for linear maps L. Now we proceed by induction. Assuming the result is true for n, we now suppose that f is $n+1$ differentiable at $L(a)$. Then we get

$$\begin{aligned} D^{n+1}(f \circ L)(a) &= D(D^n(f \circ L))(a) \\ &= D(w \mapsto D^n f(Lw) \circ L^n)(a). \end{aligned}$$

To compute this term we use the product rule, specifically Example 3.4 to the map $w \mapsto D^n f(Lw)$ and the constant map $w \mapsto L^n$. The latter has zero derivative so we only get the first term from Example ((3.4)). This means that

$$\begin{aligned} D^{n+1}(f \circ L)(a)(w_0, w_1, \dots, w_n) &= D^{n+1}(f \circ L)(a)(w_0)(w_1, \dots, w_n) \\ &= (D(w \mapsto D^n f(Lw))(a)(w_0) \circ L^n)(w_1, \dots, w_n) \end{aligned}$$

where we firstly used our canonical isomorphism

$$\mathcal{L}(W_0, \mathcal{T}(W_1, \dots, W_n \to V)) \cong \mathcal{T}(W_0, \dots, W_n \to V)$$

with $W_i = W$ followed by Example 3.4 as described.

Now to compute this final derivative we use the chain rule as follows. Let $F(u) = D^n f(u)$. Then this final derivative is $D(F \circ L)(a)$ hence equal to $D(F)(La) \circ L = D^{n+1} f(La) \circ L$. So we have shown that

$$\begin{aligned} D^{n+1}(f \circ L)(a)(w_0, \dots, w_n) &= \left((D^{n+1} f(La) \circ L)w_0 \circ L^n)\right)(w_1, \dots, w_n) \\ &= D^{n+1} f(La)(Lw_0)(Lw_1, \dots, Lw_n) \end{aligned}$$

and the proof is complete once one again applies the canonical isomorphism. □

Theorem 5.3. (Co-ordinate free general Taylor theorem) Suppose $f : X \to \mathbb{R}$ is $(n+1)$ times continuously differentiable and let $x_0 \in X$ and $u \in U$ be such that the line segment $x_0 + tu$ is in X. Then we have that

$$f(x_0 + u) = f(x_0) + \sum_{n=1}^{N} D^n f(x_0) u^n + R(u)$$

where $u^n = (u, \dots, u) \in U^n$ and

$$R(u) = \frac{1}{(N+1)!} \int_0^1 D^{N+1} f(x_0 + tu) u^{n+1} t^n dt.$$

Proof. By translation invariance of derivatives, it is enough to prove this for $x_0 = 0$. Now write $f(x_0 + tu) - f(x_0) = h(1) - h(0)$ and use Taylor's theorem, where $h = f \circ L$ where $L : \mathbb{R} \to U$ is given by $L(t) = tu$. The result now follows from the Lemma 5.1 □

6. Recovering the co-ordinate based Taylor theorem

To recover the Taylor theorem most often presented in undergraduate vector calculus, one needs to expand $D^n f(x_0)u^n$ as follows. Writing

$$u = \sum_i u_i e_i$$

where $e_1, \dots, e_d$ is a basis for U, we have that

$$D^n f(x_0)(e_{i_1}, \dots, e_{i_n}) = \partial_{i_1} \cdots \partial i_n f(x_0)$$

by the discussion surrounding Lemma 2.1 on iterated derivatives, where ∂_i is partial differentiation with respect to the e_i direction. Hence by multilinearity we get that

$$D^n f(x_0)u = \sum_{i_1,\dots,i_n} \partial_{i_1} \cdots \partial_{i_n} f(x_0) u_{i_1} \dots u_{i_n}.$$

Now by commutativity of derivatives the order of $i_1, \dots, i_n$ does not matter, hence it pays to express this in terms of partitions as follows. Given $\alpha = (a_1, \dots, a_d) \in \mathbb{Z}^d_{\geq 0}$ such that $\sum_i a_i = n$ we define

$$D^\alpha f(x_0) = \partial_1^{a_1} \cdots \partial_n^{a_d} f(x_0)$$

where ∂_i^k means ∂_i applied k times. Define also

$$u^\alpha = \prod_i u_i^{a_i}.$$

Then by grouping the terms together we get the expression

$$D^n f(x_0)u = \sum_{|\alpha|=n} \binom{n}{\alpha} D^\alpha f(x_0) u^\alpha$$

where $|\alpha| = a_1 + \cdots + a_d$ and

$$\binom{n}{\alpha} = \frac{n!}{a_1! \cdots a_d!}$$

is the multinomial coefficient which counts the number of sequences $\{1, \dots, d\}^n$ where i appears a_i times. Finally, this means that the Taylor theorem may be written as

$$f(x_0 + u) = f(x_0) + \sum_{n=1}^{N} \sum_{|\alpha|=n} \frac{1}{\alpha!} D^\alpha f(x_0) u^\alpha + R(x)$$

where

$$\alpha! = a_1! \cdots a_d!$$

7. Sufficient conditions for differentiability

It is all well and good proving abstract theorems about derivatives, but so far we don't have a mechanism for quickly telling whether a certain function is differentiable. Fortunately we have the following criteria.

Proposition 7.1. Suppose that $X \subset \mathbb{R}^d$ is open and $f : X \to V$ has the property that all its partial derivatives exist and are continuous. Then f is continuously differentiable on X.

Proof. The hypothesis says that each limit

$$\partial_i f(x) = \lim_{t\to 0} \frac{f(x+te_i)}{t}$$

exists and is continuous as a function of x. To show differentiability at $x_0 \in X$, we first construct the derivative $D = Df(x_0)$ and then show it satisfies the definition. Without loss of generality, we will use the 1-norm $\|\cdot\| = \|\cdot\|_1$ for convenience. We define D to be the linear map

$$Du = \sum_{i=1}^{n} u_i \partial_i f(x_0)$$

for $u = \sum_i u_i e_i$. To show that D satisfies the definition of the derivative fix $\epsilon > 0$ and find a corresponding $\delta > 0$ such that $\|\partial_i f(x) - \partial_i f(x_0)\| < \epsilon$ for all x such that $\|x - x_0\| < \delta$. Choose u such that each $\|u_i e\| < \frac{1}{d}\delta$ and let $w_i = u_1 + \ldots + u_i$ and $w_0 = 0$ (notice that this ensures each $\|w_i\| < \delta$). We have

$$\begin{aligned}
f(x_0+u) - f(x_0) &= \sum_{i=1}^{d} f(w_i) - f(w_{i-1}) \\
&= \sum_{i=1}^{d} f(w_{i-1} + u_i e_i) - f(w_i) \\
&= \sum_{i=1}^{d} (u_i \partial_i f(w_{i-1}) + R_i) \\
&= Du + \sum_{i=1}^{d} u_i \left(\partial_i f(w_{i-1}) - \partial_i f(x_0)\right) + \sum_{i=1}^{d} R_i
\end{aligned}$$

where the remainder term R_i satisfies

$$\|R_i\| \le \sup_{t\in[0,1]} \|u_i \partial_i f_i(w_{i-1} + tu_i e_i) - u_i \partial_i f_i(w_0)\| \le 2\epsilon\|u_i\|$$

by the mean value inequality applied to the function $u \mapsto f(w_{i-1} + ue_i)$. So altogether we have shown that if $\|u_i e_i\| < \frac{1}{d}\delta$ then we have

$$\|f(x_0+u) - f(x_0) - Du\| \le 3\epsilon\|u\|$$

which verifies differentiability. □

This generalizes to higher order differentiability. We will focus on finite dimensional spaces. If U has basis $e_1, \ldots, e_d$ and V has basis $b_1, \ldots, b_m$ then the corresponding standard basis for $\mathcal{L}(U, V)$ is given by $\delta_{i,j}$ for $1 \le i \le m$, $1 \le j \le d$ where

$$\delta_{i,j}\left(\sum_\ell u_\ell e_\ell\right) = u_j b_i.$$

More generally, the space of k-multilinear map $T : U \times \cdots \times U \to V$ has a standard basis where the elements are indexed by $i, j_1, \ldots, j_k$ where such an element maps $(e_{j_1}, \ldots, e_{j_k})$ to b_i but all other tuples in $\{e_1, \ldots, e_k\}^k$ map to 0.

Theorem 7.2. Suppose that $X \subset U = \mathbb{R}^d$ is open and $f : X \to V = \mathbb{R}^m$ has the property that all its n-order partial derivatives exist and are continuous. Then f is continuously n times continuously differentiable on X.

Proof. We have shown the $n = 1$ case. Proceeding by induction, suppose that it holds for n. Suppose that $f : X \to \mathbb{R}^m$ has the property that all its $n+1$-order partial derivatives exist and are continuous. The induction hypothesis says that $D^n f : X \to \mathcal{T}(U, \dots, U \to V)$ exists and is continuous. But with respect to the standard basis for $\mathcal{T}(U, \dots, U \to V)$ the coefficients of $D^n f(x_0)$ are n-order partial derivatives of f at x_0 and their partial derivatives are the order $(n+1)$ partial derivatives, which exist and are continuous by hypothesis. Applying the already shown $n = 1$ case establishes the inductive step. ☐

8. Application: Laplacian and Wave Operator

8.1. **Motivation.** The Laplacian of a twice differentiable $f : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\triangle f = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f.$$

In elementary courses one shows that

$$(\triangle f) \circ Q = \triangle(f \circ Q)$$

for $Q \in O(3)$. In mathematical physics and PDEs, one is often interested in the wave operator (D'Alembertian) of $f : \mathbb{R}^4 \to \mathbb{R}$ as

$$\Box f = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f - \partial_4^2 f$$

and one shows that it is invariant under Lorentz transformations, i.e., if $L \in O(3,1)$ then

$$\Box(f \circ L) = (\Box f) \circ L.$$

Our aim is to prove these in an elegant and co-ordinate free way in the framework developed here.

8.2. **Trace of a bilinear form.** Let V be a finite dimensional vector space and suppose that

$$(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$$

is a symmetric, non-degenerate bilinear form $V \times V \to \mathbb{R}$.

Lemma 8.1. If $B : V \times V \to \mathbb{R}$ is a bilinear form, then there exists a unique linear map $L = L_B : V \to V$ such that

$$B(v_1, v_2) = \langle v_1, L v_2 \rangle.$$

Proof. The map $L_B \mapsto B$ is a well defined linear map $\mathcal{T}(V, V \to \mathbb{R}) \to \mathcal{L}(V \to V)$. It is injective since $\langle -, - \rangle$ is non-degenerate. It is therefore surjective since its domain and codomain have equal dimensions $(\dim V)^2$. ☐

If $Q : V \to V$ is a linear automorphism then we say it is *orthogonal* (with respect to $\langle -, - \rangle$) if

$$\langle Q v_1, Q v_2 \rangle = \langle v_1, v_2 \rangle.$$

Lemma 8.2. If Q is orthogonal then

$$L_{B \circ Q^2} = Q^{-1} L_B Q$$

where $Q^2(v_1, v_2) = (Qv_1, Qv_2)$.

Proof. For each fixed $v_0 \in V$ we have that

$$\langle v_0, L_{B \circ Q^2} v \rangle = (B \circ Q^2)(v_0, v) = B(Qv_0, Qv) = \langle Qv_0, L_B Q v \rangle = \langle v_0, Q^{-1} L_B Q v \rangle.$$

Since this holds for all $v_0 \in V$ and $\langle -, - \rangle$ is non-degenerate, we have the desired equality. ☐

If $Tr(L)$ denote the trace of a linear operator $L : V \to V$ then we obtain the following corollary.

Corollary 8.3. If Q is orthogonal then

$$Tr(L_B) = Tr(L_{B \circ Q^2})$$

To see how this demonstrates the symmetries of the Laplacian and D'Alembertian, we observe the following. First, let $c_1, \ldots, c_n \in \mathbb{R} \setminus \{0\}$ be non-zero real numbers. We now consider on $\mathbb{R}^n$ the symmetric, non-degenerate bilinear form given by

$$\langle v, w \rangle = \sum_{i=1}^{n} c_i v_i w_i.$$

Now fix a twice differentiatble function $f : \mathbb{R}^n \to \mathbb{R}$. For each $x \in \mathbb{R}^n$, we have a unique linear operator $L_{D^2 f(x)}$ such that

$$D^2 f(x)(v, w) = \langle L_{D^2 f(x)} v, w \rangle.$$

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Since

$$\frac{1}{c_i} \langle e_i, e_j \rangle = \delta_{i,j}$$

we compute the trace to be

$$\begin{aligned} Tr(L_{D^2 f(x)}) &= \sum_{i=1}^{n} \frac{1}{c_i} \langle L_{D^2 f(x)} e_i, e_i \rangle \\ &= \sum_{i=1}^{n} \frac{1}{c_i} D^2 f(x)(e_i, e_i) \\ &= \sum_{i=1}^{n} \frac{1}{c_i} \partial_i^2 f. \end{aligned}$$

Now observe that for the case $c_1 = \cdots = c_n = 1$ this trace is the Laplacian. While for $c_1 = \cdots = c_{n-1} = 1$, $c_n = -c^2$ this trace is the D'Alembertian for the wave equation with speed $c > 0$. We know by Lemma 5.1 that if $Q : \mathbb{R}^n \to \mathbb{R}^n$ is al linear transformation then

$$D^2(f \circ Q)(x) = D^2 f(Qx) \circ Q^2.$$

Now if Q is orthogonal with respect to our quadratic form, then

$$Tr(L_{D^2(f \circ Q)(x)}) = Tr(L_{D^2 f(Qx)})$$

by the corollary above. We have thus shown

Theorem 8.4. If Q is orthogonal with respect to $\langle -, - \rangle$ then

$$Tr(L_{D^2(f \circ Q)(x)}) = Tr(L_{D^2 f(Qx)}).$$

In particular, this shows that

$$\triangle(f \circ Q) = (\triangle f) \circ Q$$

for orthogonal linear transformation Q on Euclidean space $\mathbb{R}^n$. The same holds if for $\Box$ in place of $\triangle$ if Q is a Lorentz transformation.