NOTES ON DISTRIBUTIONS AND THEIR FOURIER TRANSFORMS

1. Distributions

Definition 1.1. Let $U \subset \mathbb{R}^n$ be an open set. We define

$$D(U) = \{\varphi : \mathbb{R}^n \to \mathbb{C} \mid \varphi \text{ is smooth, compactly supported and } \operatorname{supp}(\varphi) \subset U\}$$

to be the set of *test functions* on U. Given $\varphi_1, \varphi_2, \ldots \in D(U)$ and $\varphi \in D(U)$, then we say that $\varphi = \lim_{n\to\infty} \varphi_n$ if there exists a compact set $K \subset U$ such that φ and all φ_n have support inside K and for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ we have that

$$\partial^\alpha \varphi_n \to \partial^\alpha \varphi \text{ uniformly on K.}$$

Example 1.2. Let $h : \mathbb{R} \to \mathbb{C}$ be the map $h(x) = \mathbb{1}_{(0,\infty)} \exp(-\frac{1}{x})$. This is a smooth map with support $[0, \infty)$. Thus $\varphi : \mathbb{R} \to \mathbb{C}$ given by $\varphi(x) = h(x)h(1-x)$ is a smooth map with support $[0, 1]$. Thus $\varphi \in D((-\epsilon, 1+\epsilon))$ for all $\epsilon > 0$ (but not for $\epsilon = 0$). Now let $\phi_t \in D(\mathbb{R})$ be given by $\phi_t(x) = \phi(x+t)$, then clearly $\lim_{n\to\infty} \phi_{1/n}\phi$ in $D(\mathbb{R})$ but the sequence ϕ_n, $n \in \mathbb{Z}$, does not converge (because the union of the supports is unbounded, hence not compact).

Note that $D(U)$ is closed under partial differentiation, and partial differentiation is continuous (preserves limits).

Definition 1.3. A *distribution* on $U \subset \mathbb{R}^n$ is a linear functional $f : D(U) \to \mathbb{C}$ that is *continuous* in the sense that if $\phi_1, \phi_2, \ldots \in D(U)$ converge to $\phi \in D(U)$ then $f(\phi_1), f(\phi_1), \ldots$ converges to $f(\phi)$. We let $D'(U)$ denote the space of distributions on U.

Example 1.4. Any measure μ on $\mathbb{R}^n$ that is finite on compact sets is a distribution in $D(\mathbb{R}^n)$, e.g., $\phi \mapsto \int \phi d\mu$. Consider the distribution $\delta' \in D'(\mathbb{R})$ given by $\delta'(\phi) = -\phi'(0)$. This distribution cannot arise from a measure as can be seen as follows. Choose $\phi_j \in D(\mathbb{R})$ supported on $[-1, 1]$ such that $\|\phi_j\|_\infty \to 0$ but $\phi_j'(0) = 1$, then if δ' coincides with a measure μ, then we have $-1 = \delta'(\phi_j) = \int \phi_j d\mu \to 0$, a contradiction.

The following definition describes why we called the example above δ'.

Definition 1.5. Let $f \in D'(U)$ where $U \subset \mathbb{R}^n$. Let $\partial_j\phi$ denote the j-th partial derivative of a smooth map ϕ. We can define $\partial_j f \in D'(U)$ by

$$\partial_j f(\phi) = -f(\partial_j\phi) \quad \text{for all } \phi \in D(U).$$

We now explain the minus sign in the definition.

Proposition 1.6. Let f be a continuously differentiable function on $\mathbb{R}^n$ (not necessarily compactly supported). This defines a distribution μ_f on $\mathbb{R}^n$ via $\mu_f(\phi) = \int \phi(x)f(x)d^n x$ where $d^n x$ is the lebesgue measure on $\mathbb{R}^n$. Then

$$\partial_j \mu_f = \mu_{\partial_j f}.$$

Proof. For convenience, suppose $j = n$. Then for any $\phi \in D(U)$ we have

$$\mu_{\partial_n f}(\phi) = \int \partial_n f(x)\phi(x) d^n x$$

Now by Fubini's theorem (the integrand has compact support) we can write this integral as

$$\int \left(\int_{-R}^{R} \partial_n f(x_1, \ldots, x_{n-1}, t)\phi(x_1, \ldots, x_{n-1}, t) dt \right) d^{n-1}(x_1, \ldots, x_{n-1})$$

where $R > 0$ is chosen large enough so that $\phi = 0$ outside of $[-R, R]^n$. Finally, we apply integration by parts to the inner integral and use $\phi(x_1, \ldots, x_{n-1}, \pm R) = 0$ to get that this integral is

$$\int \left(\int_{-R}^{R} -f(x_1, \ldots, x_{n-1}, t)\partial_n \phi(x_1, \ldots, x_{n-1}, t) dt \right) d^{n-1}(x_1, \ldots, x_{n-1})$$

which equals $-\int f(x)\partial_n \phi(x) d^n x$ since $\partial_n \phi = 0$ outside of $[-R, R]^n$. But this is precisely $-\mu_f(\partial_n) = \partial_n \mu_f$. □

Thus we have extended the notion of differentiation to distributions, which include also non-differentiable but locally integrable functions via the embedding $f \mapsto \mu_f$ in the proposition above. We now identify μ_f and f as is standard practice.

Example 1.7. Let $H(x) = \mathbb{1}_{[0,\infty)}(x)$. Then $H : \mathbb{R} \to \mathbb{R}$ is discontinuous at 0 thus not differentiable in the classical. Yet it has a distribution derivative as follows $H' = \delta$ where $\delta(\phi) = \phi(0)$ is the Dirac delta distribution (which is the probability measure supported at a single point 0). To see this note that for any smooth $\phi \in D(\mathbb{R})$ supported on $[-R, R]$ we have that

$$-\int H(x)\phi'(x) = -\int_0^R \phi'(x) dx = -\phi(R) + \phi(0) = \phi(0) = \delta(\phi).$$

Example 1.8. Consider a ball that bounces off a wall. Its position can be modelled as $x(t) = t$ for $t < 0$ and $x(t) = -t$ for $t \geq 0$ (the wall is located at $x = 0$ and it hits it at $t = 0$). Its velocity is $x'(t) = 1$ for $t < 0$ and $x'(t) = -1$ for $t > 0$ and $x'(0)$ is undefined. What is its acceleration? It is 0 for all $t \neq 0$, but what is it at $t = 0$? As a distribution the acceleration $x''(t)$ is 2δ, which makes sense as all the impact happens at $t = 0$. Of course, in real life maybe $x''(t)$ is continuous and the impact happens on some very small time scale $[-\epsilon, \epsilon]$ as the ball is squashed and unsquashed, but nonetheless $\int_{-\epsilon}^{\epsilon} x''(t) dt = 2$ still holds.

Definition 1.9. (Convergence of Distributions) We say that a sequence of distributions $f_1, f_2, \ldots \in D'(U)$ converges to $f \in D'(U)$ (in $D'(U)$) if $f_i(\varphi) \to f(\varphi)$ for all $\varphi \in D(U)$

Example 1.10. Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function with $\int_{-\infty}^{\infty} f(x) dx = 1$. Let $f_n(x) = nf(nx)$. Thus if $\varphi \in D(\mathbb{R})$ then by making the substibution $u = nx$ we get

$$\int_{\mathbb{R}} f_n(x)\varphi(x) dx = \int_{\mathbb{R}} \frac{du}{dx} f(nx)\varphi(x) dx = \int_{\mathbb{R}} f(u)\varphi(\frac{u}{n}) du \to \varphi(0) \quad \text{as } n \to \infty$$

where we used the dominated convergence theorem (the integrand is bounded by the integrable function $\|\varphi\|_\infty f$ and converges to $f(u)\varphi(0)$ pointwise).Thus $f_n \to \delta$ in $D'(\mathbb{R})$.

2. Test functions as a Frechet space

Definition 2.1. A Frechet space is a topological vector space (addition and scalar multiplcation is continuous, the field is either $\mathbb{R}$ or $\mathbb{C}$ which has the usual topology) whose topology comes from an invariant metric d (i.e., $d(v_1+v, v_2+v) = d(v_1, v_2)$ for all $v_1, v2, v \in V$) that is complete.

For $K \subset \mathbb{R}^n$ compact we define the norm

$$\|\phi\|_{C^k} = \sup_{x \in K, |\alpha| \leq k} |\partial^\alpha \phi|(x).$$

Note that $C_0^k(K)$ is a Banach space and hence a Frechet space with respect to this norm. We define $C_0^\infty(K)$ to be the smooth functions with support inside K and for $\phi \in C_0^\infty(K)$ we define

$$\|\phi\|_{C_0^\infty(K)} = \sum_{k=0}^{\infty} 2^{-k} \min\{1, \|\phi\|_{C^k}\}$$

and we note that $d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{C_0^\infty(K)}$ is an invariant metric that is complete. Moreover, a sequence of test functions $\phi_1, \phi_2, \ldots \in C_0^\infty(K)$ converge to $\phi \in C_0^\infty(K)$ if and only if for all $\alpha \in \mathbb{Z}_{\geq 0}^n$ we have that

$$\partial^\alpha \phi_i \to \phi$$

uniformly on K. In other words, a sequence of test functions in $D(U)$ converges if they all have support inside the same compact subset $K \subset U$ and they converge in $C_0^\infty(K)$ with respect to this metric. This in particular verifies that $C_0^\infty(K)$ is a topological vector space with respect to this metric (the continuity of addition and scalar multiplcation inherits from the same properties of the norms $\|\cdot\|_{C_k}$).

Theorem 2.2 (Theorem 3.8 of [1])**.** Let $U \subset \mathbb{R}^d$ be open. A linear functional $f : D(U) \to \mathbb{C}$ is a distribution (in $D'(U)$) if and only if for all compact subsets $K \subset U$ there exists $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|f(\phi)| \leq c\|\phi\|_{C^k} \quad \text{for all } \phi \in C_0^\infty(K).$$

Proof. Easy to see that any functional satisfying this property is a distribution. To see the converse, suppose that this conditional fails for some compact set $K \subset U$. Then for each positive integer $c = k$ we have

$$|f(\phi_k)| > k\|\phi_k\|_{C^k}$$

for some $\phi_k \in C_0^\infty(K)$. Let $\psi_k = \frac{1}{|f(\phi_k)|}\phi_k$. Thus $|f(\psi_n)| = 1$ but we have

$$|\psi_n|_{C^k} \leq |\psi_n|_{C^n} < \frac{1}{n}$$

for all $n \geq k$ so $\psi_n \to 0$ on $C_0^\infty(K)$, which shows that f is not continuous, i.e., not a distribution. $\square$

Theorem 2.3 (Uniform boundedness)**.** Let V be a Frechet space and suppose that $\mathcal{F}$ is a set of continuous linear functions $f : V \to \mathbb{C}$ such that $\{f(x) \mid f \in \mathcal{F}\}$ is bounded in $\mathbb{C}$ for all $x \in V$. Then there is an open set $U \subset V$ with $0 \in V$ such that $|f(u)| \leq 1$ for all $f \in \mathcal{F}$ and $u \in U$.

Proof. Let

$$U_n = \{x \in V \mid |f(x)| > n \text{ for some } f \in \mathcal{F}\}$$

. Now U_n is an open set. For each $x \in V$, we have that there exists n such that $|f(x)| \leq n$ for all $f \in \mathcal{F}$, which means that $x \notin U_n$. Consequently

$$\emptyset = \bigcap_{n=1}^{\infty} U_n.$$

Thus not all U_n can be dense by Baire's theorem. As some U_n is dense, we have a non-empty open set V such that $V \cap U_n = \emptyset$. Choosing $v_0 \in V$, we have that if $u \in V - v_0$ then $u = v - v_0$ for some $v \in V$ and so

$$|f(u)| = |f(v) - f(v_0)| \leq |f(v)| + |f(v_0)| \leq 2n.$$

Thus we may set $U = \frac{1}{2n}(V - v_0)$, which is open by definition of topological vector space. □

Theorem 2.4 (Lemma 5.4 in [1], no proof given there)**.** Let f_j be a sequence of distributions in $D'(U)$, where $U \subset \mathbb{R}^d$ is open such that $f_j(\phi)$ is bounded for all $\phi \in D(U)$. Then for all compact $K \subset U$ there exists a constant $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$\|f_j(\phi)\| \leq c\|\phi\|_{C_0^k(K)} \quad \text{for all } j \in \mathbb{N} \text{ and } \phi \in C_0^\infty(K).$$

Proof. We apply the uniform boundedness principle above. This implies that there is an open neighbhourhood $\mathcal{U} \subset C_0^\infty(K)$ such that $f_j(u) \leq 1$ for all $u \in \mathcal{U}$ and $j \in \mathbb{N}$. So there exists an R such that if $\|\phi\|_{C_0^\infty(K)} < R$ then $f_j(\phi) \leq 1$. Now take k large enough so that

$$\sum_{i=k}^{\infty} 2^{-i} < \frac{R}{2}.$$

This means that if $\|\phi\|_{C_0^k(K)} < \frac{R}{2}$ then $\|\phi\|_{C_0^k(K)} < R$ and so $f_j(\phi) < 1$. As $\|\cdot\|_{C_0^k(K)}$ is a norm on $C_0^\infty(K)$, we have completed the proof with $c = \frac{2}{R}$. □

Theorem 2.5. Let $U \subset \mathbb{R}^d$ be an open set and suppose that $f_1, f_2, \ldots \in D(U)$ is a sequence of distributions such that $\lim_{j\to\infty} f_j(\varphi)$ exists in $\mathbb{C}$ for all $\varphi \in D(U)$.

(1) Then there exists a a distribution $f \in D(U)$ such that

$$f = \lim_{j\to\infty} f_j.$$

(2) If $\varphi, \varphi_j \in D(U)$ are such that $\lim_{j\to\infty} \varphi_j = \varphi$ then $f_j(\varphi_j)$ converges to $f(\varphi)$.

Proof. Define $f(\varphi) = \lim_{j\to\infty} f_j(\varphi)$. It remains to show that this defines a distribution (is continuous). Let K be a compact set. Applying the uniform boundedness principle we have a constant $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|f_j(\varphi)| \leq c\|\varphi\|_{C_0^k(K)} \quad \text{for all } j \in \mathbb{N}, \varphi \in C_0^\infty(K).$$

Thus as $f_j(\varphi) \to f(\varphi)$ we have that

$$|f(\varphi)| \leq c\|\varphi\|_{C_0^k(K)} \quad \text{for all } \varphi \in C_0^\infty(K).$$

This implies the continuity of f, thus $f \in D(U)$. Now suppose that $\varphi_j \in C_0^\infty(K)$ converge to $\varphi \in C_0^\infty(K)$. Thus

$$|f_j(\varphi_j) - f(\varphi)| \leq |f_j(\varphi_j - \varphi)| + |f_j(\varphi) - f(\varphi)| \leq c\|\varphi_j - \varphi\|_{C^k} + |f_j(\varphi) - f(\varphi)|$$

and the first term converges to 0 as $\varphi_j \to \varphi$ while the second converges to 0 as $f_j \to f$. □

3. Support of a distribution

If $U \subset V \subset \mathbb{R}^d$ are open sets, then there is a continuous (preserves limits) inclusion $D(U) \to D(V)$. This induces a restriction map $p_{U,V} : D'(V) \to D'(U)$ where $(p_{U,V}f)(\phi) = f(\phi)$ for $\phi \in D(U) \subset D(V)$ and $f \in D'(U)$. Note that this is continuous (preserves limits of distributions).

Lemma 3.1. Suppose that U is an open set, $f \in D'(U)$ and suppose that for each $x \in U$ there exists an open neighbhourhood $U_x \subset U$ of x such that $p_{U_x,U}f = 0$. Then $f = 0$.

Proof. We take $\phi \in D(U)$, thus there is a compact set K such that $K \subset u$ and ϕ is supported on K. Now by compactness, we can find a finite cover of $U_1, \dots U_n$ of K such that f restricts to 0 on each $U_i \subset U$. Choose U_i such that the closure of U_i is in U. By partition of unity theorem, we may choose $\psi_1, \dots, \psi_n \in D(U)$ such that $\sum_{i=1}^n \psi_i(x) = 1$ for all $x \in K$ and $\operatorname{supp} \psi_i \subset U_i$. Thus $\phi = \phi \sum_i \psi_i$ and so $f(\phi) = \sum_i f(\phi\psi_i) = 0$. □

We justify the partition of unity used above.

Proposition 3.2. Let $B(a, r) \subset B(a, r') \subset \mathbb{R}^d$ are open balls. There is a smooth function $\phi : \mathbb{R}^d \to [0, 1]$ that is 1 on $B(a, r)$ and 0 outside $B(a, r')$.

Proof sketch. We just need to prove this for $d = 1$ and then build such a radial function. We already saw that we have a compactly supported $\psi : \mathbb{R} \to [0, 1]$ supported on $[0, \epsilon]$ where $0 < \epsilon < \frac{1}{2}$. Now let $\psi_2(x) = \int_{-\infty}^x \psi(t)dt$. We see that $\psi_2(x)$ is constant for $x > \epsilon$ and is zero on $x < 0$. Consequently $\psi_3(x) = \psi_2(x)\psi_2(1 - x)$ has values in $[0, 1]$, is compactly supported and is constant on the interval $(\epsilon, 1 - \epsilon)$. We can now translate and scale ψ_3 appropriately. □

Proposition 3.3 (Partition of unity)**.** Let $K \subset \mathbb{R}^d$ be a compact set and suppose that $U \supset K$ is open. Suppose that $\mathcal{U}$ is a collection of open subsets of U that covers K. Then there exist smooth functions $\psi_1, \dots, \psi_n : \mathbb{R}^d \to [0, 1]$ such that

$$\psi := \sum_{i=1}^n \psi_i$$

satisfies that $\psi(x) = 0$ for $x \in K$ and ψ_i has support inside some element of $\mathcal{U}$.

Proof. By compactness, we may find finitely many balls $B(a_1, r_1), \dots, B(a_n, r_n)$ that cover K such that $B(a_i, 2r_i)$ is a subset of some element of $\mathcal{U}$ (and thus $B(a_i, 2r_i)$ are subsets of U). Now apply the previous construction to find some smooth $\phi_i : \mathbb{R}^d \to [0, 1]$ that equals 1 on $B(a_i, r_i)$ and vanishes outside of $B(a_i, 2r_i)$. Now let $\psi_1 = \phi_1$ and for $1 < j \le n$ define $\psi_i = \phi_i \prod_{j<i}(1 - \phi_j)$. Observe that ψ_i has support inside the support of ϕ_i, thus inside some element of $\mathcal{U}$, as required. Moreover, by induction we have that

$$\sum_{i=1}^j \psi_i = 1 - \prod_{i=1}^j (1 - \phi_i).$$

In particular for $j = n$ this means that by setting $\psi = \sum_{i=1}^n \psi_i$ we have that $\psi(x) = 1$ for $x \in B(a_i, r_i)$, and thus for all $x \in K$. Moreover, $x \notin B(a_i, 2r_i)$ means that $(1 - \phi_i(x)) = 1$ for all i, meaning that $\psi_i(x) = 0$ and thus ψ_i has support inside some element of $\mathcal{U}$, as required. □

Theorem 3.4 (Gluding distributions)**.** Suppose that $X \subset \mathbb{R}^d$ is an open set and suppose that $\mathcal{U}$ is a collection of open subsets of X that cover X. Suppose that for each $U \in \mathcal{U}$ there is a distribution $f_U \in D'(U)$ such that these f_U are compatible in the sense that $f_U|_{U \cap V} = f_V|_{U \cap V}$ are the same distributions on $D'(U \cap V)$. Then there is a unique distribution $f \in D'(X)$ such that $f|U = f_U$ for all $U \in \mathcal{U}$.

Proof. We construct f as follows (show that it is well defined later): For each $\phi \in D(X)$, choose a compact set $K \subset X$ containing the support of ϕ. Now we may apply Parition of Unity to find open sets $U_1, \dots, U_n \in \mathcal{U}$ that cover K and $\psi_i : \mathbb{R}^d \to [0,1]$ with support inside U_i such that $\psi := \sum_{i=1}^n \psi_i$ satisfies that $\psi(x) = 1$ for all $x \in K$. We now define

$$f(\phi) = \sum_{i=1}^n f_{U_i}(\phi\psi_i).$$

Note that this shows uniqueness since $\phi = \sum_{i=1}^n \phi\psi_i$ on $\mathbb{R}^d$.

We now show that f is well defined (does not depend on the choice of K or the choice of the U_i or the choice of ψ_i). To see this, suppose that K', U'_j and ψ'_j are such other choices. Then we make a common refinement and show it assigns the same value to our $f(\phi)$ as follows. Let $K'' = K \cap K'$, it clearly contains the support of ϕ and is compact. Now the sets U_i and U_j cover K''. Thus the sets $V_{i,j} = U_i \cap U_j$ cover K''. Moreover, $\psi_{i,j} := \psi_i\psi'_j : \mathbb{R}^d \to [0,1]$ has support inside $V_{i,j}$ and

$$\sum_{i,j} \psi_{i,j} = \left(\sum_i \psi_i\right)\left(\sum_j \psi'_j\right)$$

and thus equals 1 on K''. So this common refinement is a new partition of unity. But now

$$\sum_{i,j} f_{U_i}|_{V_{i,j}}(\psi_{i,j}\phi) = \sum_{i,j} f_{U_i}(\psi_{i,j}\phi) = \sum_i f_{U_i}(\phi\psi_i \sum_j \psi'_j) = \sum_i f_{U_i}(\phi\psi_i)$$

where we used that $\phi\psi_i \sum_j \psi'_j = \phi_i\psi_i$ since $\sum_j \psi'_j(x) = 1$ for all $x \in K'$ and thus all x in the support of ϕ. This completes the proof of well definedness since by assumption,

$$f_{U_i}|_{V_{i,j}}(\psi_{i,j}\phi) = f_{U'_j}|_{V_{i,j}}(\psi_{i,j}\phi)$$

and so

$$\sum_i f_{U_i}(\phi\psi_i) = \sum_j f_{U'_j}(\phi\psi'_j)$$

by the same calculation as above. Suppose that $\phi, \phi' \in D(X)$. Thus to compute $f(\phi + \phi')$ we may choose a compact set $K \subset U$ that contains the support of ϕ and ϕ'. Now choose $U_1, \dots, U_n \in \mathcal{U}$ that cover K, thus by definition

$$f(\phi_1 + \phi_2) = \sum_i f_{U_i}(\psi_i(\phi_1 + \phi_2)) = \sum_i f_{U_i}(\psi_i\phi_1) + \sum_i f_{U_i}(\psi_i\phi_2) = f(\phi_1) + f(\phi_2)$$

where the ψ_i are chosen as in the construction. Linearity of f now easily follows. We now show the continuity of f. If $\phi_k \to \phi \in D(X)$ then there is a compact set $K \subset X$ containing all their supports. Thus $\psi_i\phi_k \to \psi\phi$ and the continuity of each f_{U_i} gives continuity of f. Finally, it remains to show that $f|_U = f_U$ for all $U \in \mathcal{U}$. Thus suppose that $\phi \in D(U)$ and choose a compact set $K \subset U$ such that ϕ has support inside K. As U already covers K, by definition we have that

$$f|_U(\phi) = f(\phi) = f_U(\psi\phi) = f_U(\phi)$$

for some $\psi : \mathbb{R}^d \to [0,1]$ smooth that equals 1 on K and has suppose inside U (so $\phi\psi = \psi$ everywhere). $\square$

4. Tempered Distributions

5. Fourier Transform

References

[1] Duistermaat, J. J.; Kolk, J. A. C. *Distributions. Theory and applications.* Translated from the Dutch by J. P. van Braam Houckgeest. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2010. xvi+445 pp. ISBN: 978-0-8176-4672-1