NOTES ON DISTRIBUTIONS AND THEIR FOURIER TRANSFORMS

1. Distributions

Definition 1.1. Let $U \subset \mathbb{R}^n$ be an open set. We define

$$D(U) = \{\varphi : \mathbb{R}^n \to \mathbb{C} \mid \varphi \text{ is smooth, compactly supported and } \operatorname{supp}(\varphi) \subset U\}$$

to be the set of *test functions* on U. Given $\varphi_1, \varphi_2, \ldots \in D(U)$ and $\varphi \in D(U)$, then we say that $\varphi = \lim_{n\to\infty} \varphi_n$ if there exists a compact set $K \subset U$ such that φ and all φ_n have support inside K and for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ we have that

$$\partial^\alpha \varphi_n \to \partial^\alpha \varphi \text{ uniformly on K.}$$

Example 1.2. Let $h : \mathbb{R} \to \mathbb{C}$ be the map $h(x) = \mathbb{1}_{(0,\infty)} \exp(-\frac{1}{x})$. This is a smooth map with support $[0, \infty)$. Thus $\varphi : \mathbb{R} \to \mathbb{C}$ given by $\varphi(x) = h(x)h(1-x)$ is a smooth map with support $[0, 1]$. Thus $\varphi \in D((-\epsilon, 1+\epsilon))$ for all $\epsilon > 0$ (but not for $\epsilon = 0$). Now let $\phi_t \in D(\mathbb{R})$ be given by $\phi_t(x) = \phi(x+t)$, then clearly $\lim_{n\to\infty} \phi_{1/n}\phi$ in $D(\mathbb{R})$ but the sequence ϕ_n, $n \in \mathbb{Z}$, does not converge (because the union of the supports is unbounded, hence not compact).

Note that $D(U)$ is closed under partial differentiation, and partial differentiation is continuous (preserves limits).

Definition 1.3. A *distribution* on $U \subset \mathbb{R}^n$ is a linear functional $f : D(U) \to \mathbb{C}$ that is *continuous* in the sense that if $\phi_1, \phi_2, \ldots \in D(U)$ converge to $\phi \in D(U)$ then $f(\phi_1), f(\phi_1), \ldots$ converges to $f(\phi)$. We let $D'(U)$ denote the space of distributions on U.

Example 1.4. Any measure μ on $\mathbb{R}^n$ that is finite on compact sets is a distribution in $D(\mathbb{R}^n)$, e.g., $\phi \mapsto \int \phi d\mu$. Consider the distribution $\delta' \in D'(\mathbb{R})$ given by $\delta'(\phi) = -\phi'(0)$. This distribution cannot arise from a measure as can be seen as follows. Choose $\phi_j \in D(\mathbb{R})$ supported on $[-1, 1]$ such that $\|\phi_j\|_\infty \to 0$ but $\phi_j'(0) = 1$, then if δ' coincides with a measure μ, then we have $-1 = \delta'(\phi_j) = \int \phi_j d\mu \to 0$, a contradiction.

The following definition describes why we called the example above δ'.

Definition 1.5. Let $f \in D'(U)$ where $U \subset \mathbb{R}^n$. Let $\partial_j \phi$ denote the j-th partial derivative of a smooth map ϕ. We can define $\partial_j f \in D'(U)$ by

$$\partial_j f(\phi) = -f(\partial_j \phi) \quad \text{for all } \phi \in D(U).$$

We now explain the minus sign in the definition.

Proposition 1.6. Let f be a continuously differentiable function on $\mathbb{R}^n$ (not necessarily compactly supported). This defines a distribution μ_f on $\mathbb{R}^n$ via $\mu_f(\phi) = \int \phi(x) f(x) d^n x$ where $d^n x$ is the lebesgue measure on $\mathbb{R}^n$. Then

$$\partial_j \mu_f = \mu_{\partial_j f}.$$

Proof. For convenience, suppose $j = n$. Then for any $\phi \in D(U)$ we have

$$\mu_{\partial_n f}(\phi) = \int \partial_n f(x)\phi(x)d^n x$$

Now by Fubini's theorem (the integrand has compact support) we can write this integral as

$$\int \left(\int_{-R}^{R} \partial_n f(x_1, \ldots, x_{n-1}, t)\phi(x_1, \ldots, x_{n-1}, t)dt \right) d^{n-1}(x_1, \ldots, x_{n-1})$$

where $R > 0$ is chosen large enough so that $\phi = 0$ outside of $[-R, R]^n$. Finally, we apply integration by parts to the inner integral and use $\phi(x_1, \ldots, x_{n-1}, \pm R) = 0$ to get that this integral is

$$\int \left(\int_{-R}^{R} -f(x_1, \ldots, x_{n-1}, t)\partial_n \phi(x_1, \ldots, x_{n-1}, t)dt \right) d^{n-1}(x_1, \ldots, x_{n-1})$$

which equals $-\int f(x)\partial_n \phi(x)d^n x$ since $\partial_n \phi = 0$ outside of $[-R, R]^n$. But this is precisely $-\mu_f(\partial_n) = \partial_n \mu_f$. □

Thus we have extended the notion of differentiation to distributions, which include also non-differentiable but locally integrable functions via the embedding $f \mapsto \mu_f$ in the proposition above. We now identify μ_f and f as is standard practice.

Example 1.7. Let $H(x) = \mathbb{1}_{[0,\infty)}(x)$. Then $H : \mathbb{R} \to \mathbb{R}$ is discontinuous at 0 thus not differentiable in the classical. Yet it has a distribution derivative as follows $H' = \delta$ where $\delta(\phi) = \phi(0)$ is the Dirac delta distribution (which is the probability measure supported at a single point 0). To see this note that for any smooth $\phi \in D(\mathbb{R})$ supported on $[-R, R]$ we have that

$$-\int H(x)\phi'(x) = -\int_0^R \phi'(x)dx = -\phi(R) + \phi(0) = \phi(0) = \delta(\phi).$$

Example 1.8. Consider a ball that bounces off a wall. Its position can be modelled as $x(t) = t$ for $t < 0$ and $x(t) = -t$ for $t \geq 0$ (the wall is located at $x = 0$ and it hits it at $t = 0$). Its velocity is $x'(t) = 1$ for $t < 0$ and $x'(t) = -1$ for $t > 0$ and $x'(0)$ is undefined. What is its acceleration? It is 0 for all $t \neq 0$, but what is it at $t = 0$? As a distribution the acceleration $x''(t)$ is 2δ, which makes sense as all the impact happens at $t = 0$. Of course, in real life maybe $x''(t)$ is continuous and the impact happens on some very small time scale $[-\epsilon, \epsilon]$ as the ball is squashed and unsquashed, but nonetheless $\int_{-\epsilon}^{\epsilon} x''(t)dt = 2$ still holds.

Definition 1.9. (Convergence of Distributions) We say that a sequence of distributions $f_1, f_2, \ldots \in D'(U)$ converges to $f \in D'(U)$ (in $D'(U)$) if $f_i(\varphi) \to f(\varphi)$ for all $\varphi \in D(U)$

Example 1.10. Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function with $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $f_n(x) = nf(nx)$. Thus if $\varphi \in D(\mathbb{R})$ then by making the substibution $u = nx$ we get

$$\int_{\mathbb{R}} f_n(x)\varphi(x)dx = \int_{\mathbb{R}} \frac{du}{dx} f(nx)\varphi(x)dx = \int_{\mathbb{R}} f(u)\varphi(\frac{u}{n})du \to \varphi(0) \quad \text{as } n \to \infty$$

where we used the dominated convergence theorem (the integrand is bounded by the integrable function $\|\varphi\|_\infty f$ and converges to $f(u)\varphi(0)$ pointwise).Thus $f_n \to \delta$ in $D'(\mathbb{R})$.

Lemma 1.11. Suppose that $\phi_n, \psi_n, \phi, \psi : U \to \mathbb{R}$ are smooth functions, where $U \subset \mathbb{R}^d$ is open, such that for some compact $K \subset U$ and for all $\alpha \in \mathbb{Z}^d_{\geq 0}$ we have that $\partial^\alpha \phi_n \to \partial^\alpha \phi$ and $\partial^\alpha \psi_n \to \partial^\alpha \psi$ uniformly on K. Then for all α, we have that $\partial^\alpha(\phi_n \psi_n) \to \partial^\alpha(\phi_n \psi_n)$ uniformly on K.

Proof. First we note that

$$|\phi_n\psi_n - \phi\psi| = |\phi_n(\psi_n - \psi) + \psi(\phi_n - \phi)| \leq |\phi_n||\psi_n - \phi_n| + |\psi||\phi_n - \phi|$$

converges to 0 uniformly on K. We now prove by induction on k that all order n derivatives of $\phi_n\psi_n$ converge to the corresponding derivatives of $\phi\psi$ (the induction hypothesis is on any such ϕ, ψ and not just for this specific ones). The base case $k = 0$ has now been establishes. Now choose any $1 \leq j \leq n$ and use the product rule to see that

$$\partial_j(\phi_n\psi_n) = \partial_j(\phi_n)\psi_n + \phi_n(\partial_j\phi_n) \to \partial_j(\phi)\psi + \phi(\partial_j\phi) = \partial_j(\phi\psi)$$

uniformly on K where we applied this $k = 0$ case. Now by induction hypothesis for $|\alpha| = k$ we have $\partial_\alpha\partial_j(\phi_n)\psi_n \to \partial_\alpha\partial_j(\phi)\psi$ uniformly on K and likewise for the second term. Thus $\partial_\alpha\partial_j(\phi_n\psi_n) \to \partial_\alpha(\phi\psi)$, and this completes the induction step. □

Definition 1.12 (Multiplying a distribution by a smooth function)**.** If $f \in D(U)$ is a distributiona and $\psi \in C^\infty(U)$ is any smooth function (not necessarily of compact support in U) then we can define $\psi f \in D(U)$ by

$$\psi f(\phi) = f(\psi\phi).$$

Note that ψf is indeed a distribution as the above lemma shows that if $\phi_n \to \phi$ in $D(U)$ then $\psi\phi_n \to \psi\phi$ in $D(U)$ as well, and so $\psi f(\phi_n) \to \psi f(\phi)$ by the continuity of f.

2. Test functions as a Frechet space

Definition 2.1. A Frechet space is a topological vector space (addition and scalar multiplcation is continuous, the field is either $\mathbb{R}$ or $\mathbb{C}$ which has the usual topology) whose topology comes from an invariant metric d (i.e., $d(v_1 + v, v_2 + v) = d(v_1, v_2)$ for all $v_1, v2, v \in V$) that is complete.

For $K \subset \mathbb{R}^n$ compact we define the norm

$$\|\phi\|_{C^k} = \sup_{x\in K, |\alpha|\leq k} |\partial^\alpha\phi|(x).$$

Note that $C_0^k(K)$ is a Banach space and hence a Frechet space with respect to this norm. We define $C_0^\infty(K)$ to be the smooth functions with support inside K and for $\phi \in C_0^\infty(K)$ we define

$$\|\phi\|_{C_0^\infty(K)} = \sum_{k=0}^{\infty} 2^{-k} \min\{1, \|\phi\|_{C^k}\}$$

and we note that $d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{C_0^\infty(K)}$ is an invariant metric that is complete. Moreover, a sequence of test functions $\phi_1, \phi_2, \ldots \in C_0^\infty(K)$ converge to $\phi \in C_0^\infty(K)$ if and only if for all $\alpha \in \mathbb{Z}_{\geq 0}^n$ we have that

$$\partial^\alpha\phi_i \to \phi$$

uniformly on K. In other words, a sequence of test functions in $D(U)$ converges if they all have support inside the same compact subset $K \subset U$ and they converge in $C_0^\infty(K)$ with respect to this metric. This in particular verifies that $C_0^\infty(K)$ is a topological vector space with respect to this metric (the continuity of addition and scalar multiplcation inherits from the same properties of the norms $\|\cdot\|_{C_k}$).

Theorem 2.2 (Theorem 3.8 of [1])**.** Let $U \subset \mathbb{R}^d$ be open. A linear functional $f : D(U) \to \mathbb{C}$ is a distribution (in $D'(U)$) if and only if for all compact subsets $K \subset U$ there exists $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|f(\phi)| \leq c\|\phi\|_{C^k} \quad \text{for all } \phi \in C_0^\infty(K).$$

Proof. Easy to see that any functional satisfying this property is a distribution. To see the converse, suppose that this conditional fails for some compact set $K \subset U$. Then for each positive integer $c = k$ we have

$$|f(\phi_k)| > k\|\phi_k\|_{C^k}$$

for some $\phi_k \in C_0^\infty(K)$. Let $\psi_k = \frac{1}{|f(\phi_k)|}\phi_k$. Thus $|f(\psi_n)| = 1$ but we have

$$|\psi_n|_{C^k} \leq |\psi_n|_{C^n} < \frac{1}{n}$$

for all $n \geq k$ so $\psi_n \to 0$ on $C_0^\infty(K)$, which shows that f is not continuous, i.e., not a distribution. □

Theorem 2.3 (Uniform boundedness)**.** Let V be a Frechet space and suppose that $\mathcal{F}$ is a set of continuous linear functions $f : V \to \mathbb{C}$ such that $\{f(x) \mid f \in \mathcal{F}\}$ is bounded in $\mathbb{C}$ for all $x \in V$. Then there is an open set $U \subset V$ with $0 \in V$ such that $|f(u)| \leq 1$ for all $f \in \mathcal{F}$ and $u \in U$.

Proof. Let

$$U_n = \{x \in V \mid |f(x)| > n \text{ for some } f \in \mathcal{F}\}$$

. Now U_n is an open set. For each $x \in V$, we have that there exists n such that $|f(x)| \leq n$ for all $f \in \mathcal{F}$, which means that $x \notin U_n$. Consequently

$$\emptyset = \bigcap_{n=1}^{\infty} U_n.$$

Thus not all U_n can be dense by Baire's theorem. As some U_n is dense, we have a non-empty open set V such that $V \cap U_n = \emptyset$. Choosing $v_0 \in V$, we have that if $u \in V - v_0$ then $u = v - v_0$ for some $v \in V$ and so

$$|f(u)| = |f(v) - f(v_0)| \leq |f(v)| + |f(v_0)| \leq 2n.$$

Thus we may set $U = \frac{1}{2n}(V - v_0)$, which is open by definition of topological vector space. □

Theorem 2.4 (Lemma 5.4 in [1], no proof given there)**.** Let f_j be a sequence of distributions in $D'(U)$, where $U \subset \mathbb{R}^d$ is open such that $f_j(\phi)$ is bounded for all $\phi \in D(U)$. Then for all compact $K \subset U$ there exists a constant $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$\|f_j(\phi)\| \leq c\|\phi\|_{C_0^k(K)} \quad \text{for all } j \in \mathbb{N} \text{ and } \phi \in C_0^\infty(K).$$

Proof. We apply the uniform boundedness principle above. This implies that there is an open neighbhourhood $\mathcal{U} \subset C_0^\infty(K)$ such that $f_j(u) \leq 1$ for all $u \in \mathcal{U}$ and $j \in \mathbb{N}$. So there exists an R such that if $\|\phi\|_{C_0^\infty(K)} < R$ then $f_j(\phi) \leq 1$. Now take k large enough so that

$$\sum_{i=k}^{\infty} 2^{-i} < \frac{R}{2}.$$

This means that if $\|\phi\|_{C_0^k(K)} < \frac{R}{2}$ then $\|\phi\|_{C_0^k(K)} < R$ and so $f_j(\phi) < 1$. As $\|\cdot\|_{C_0^k(K)}$ is a norm on $C_0^\infty(K)$, we have completed the proof with $c = \frac{2}{R}$. □

Theorem 2.5. Let $U \subset \mathbb{R}^d$ be an open set and suppose that $f_1, f_2, \ldots \in D(U)$ is a sequence of distributions such that $\lim_{j\to\infty} f_j(\varphi)$ exists in $\mathbb{C}$ for all $\varphi \in D(U)$.

(1) Then there exists a a distribution $f \in D(U)$ such that

$$f = \lim_{j\to\infty} f_j.$$

(2) If $\varphi, \varphi_j \in D(U)$ are such that $\lim_{j\to\infty} \varphi_j = \varphi$ then $f_j(\varphi_j)$ converges to $f(\varphi)$.

Proof. Define $f(\varphi) = \lim_{j\to\infty} f_j(\varphi)$. It remains to show that this defines a distribution (is continuous). Let K be a compact set. Applying the uniform boundedness principle we have a constant $c > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|f_j(\varphi)| \leq c\|\varphi\|_{C_0^k(K)} \quad \text{for all } j \in \mathbb{N}, \varphi \in C_0^\infty(K).$$

Thus as $f_j(\varphi) \to f(\varphi)$ we have that

$$|f(\varphi)| \leq c\|\varphi\|_{C_0^k(K)} \quad \text{for all } \varphi \in C_0^\infty(K).$$

This implies the continuity of f, thus $f \in D(U)$. Now suppose that $\varphi_j \in C_0^\infty(K)$ converge to $\varphi \in C_0^\infty(K)$. Thus

$$|f_j(\varphi_j) - f(\varphi)| \leq |f_j(\varphi_j - \varphi)| + |f_j(\varphi) - f(\varphi)| \leq c\|\varphi_j - \varphi\|_{C^k} + |f_j(\varphi) - f(\varphi)|$$

and the first term converges to 0 as $\varphi_j \to \varphi$ while the second converges to 0 as $f_j \to f$. □

3. Support of a distribution

If $U \subset V \subset \mathbb{R}^d$ are open sets, then there is a continuous (preserves limits) inclusion $D(U) \to D(V)$. This induces a restriction map $p_{U,V} : D'(V) \to D'(U)$ where $(p_{U,V} f)(\phi) = f(\phi)$ for $\phi \in D(U) \subset D(V)$ and $f \in D'(U)$. Note that this is continuous (preserves limits of distributions). We also use the notation $f|_U = p_{U,V} f$.

Lemma 3.1. Suppose that U is an open set, $f \in D'(U)$ and suppose that for each $x \in U$ there exists an open neighbhourhood $U_x \subset U$ of x such that $p_{U_x,U} f = 0$. Then $f = 0$.

Proof. We take $\phi \in D(U)$, thus there is a compact set K such that $K \subset u$ and ϕ is supported on K. Now by compactness, we can find a finite cover of $U_1, \dots U_n$ of K such that f restricts to 0 on each $U_i \subset U$. Choose U_i such that the closure of U_i is in U. By partition of unity theorem, we may choose $\psi_1, \dots, \psi_n \in D(U)$ such that $\sum_{i=1}^n \psi_i(x) = 1$ for all $x \in K$ and $\operatorname{supp} \psi_i \subset U_i$. Thus $\phi = \phi \sum_i \psi_i$ and so $f(\phi) = \sum_i f(\phi\psi_i) = 0$. □

This lemma shows that if $f|_V = 0$ for all $V \in \mathcal{V}$, where $\mathcal{V}$ is a collection of open sets, then setting $V_{\max} = \cup_{V\in\mathcal{V}} V$ we have that $f|_{V_{\max}} = 0$. Thus the following definition is well defined.

Definition 3.2. If $U \subset \mathbb{R}^d$ is open and $f \in D'(X)$ we define the support of f, denoted by $\operatorname{supp} f$, to the smallest closed set such that $f|V = 0$ where $V = X \setminus \operatorname{supp} f$.

Example 3.3. Consider the distribution $f \in D'(\mathbb{R}_{>0})$ given by

$$f(\phi) = \int_0^\infty e^{1/x^2} \phi(x) dt.$$

It is indeed a distribution since e^{-1/x^2} is integrable on each compact subset of $\mathbb{R}_{>0}$, but on any other compact subset containing 0. This distribution is not the restriction of any distribution $g \in D(\mathbb{R})$. To see this, suppose that it was. Now let $\phi_n = \phi(x + \frac{1}{n})$ where $\phi : \mathbb{R} \to [0, 1]$ is smooth and has support in $[0, 1]$ and $\phi(x) > e^{-1/x}$ on $(0, \frac{1}{2})$. Thus

$$g(\phi) = \lim_{n\to\infty} g(\phi_n) = \lim_{n\to\infty} f(\phi_n) = \infty,$$

a contradiction (distributions have finite values in $\mathbb{C}$).

We justify the partition of unity used above.

Proposition 3.4. Let $B(a,r) \subset B(a,r') \subset \mathbb{R}^d$ are open balls. There is a smooth function $\phi : \mathbb{R}^d \to [0,1]$ that is 1 on $B(a,r)$ and 0 outside $B(a,r')$.

Proof sketch. We just need to prove this for $d = 1$ and then build such a radial function. We already saw that we have a compactly supported $\psi : \mathbb{R} \to [0,1]$ supported on $[0,\epsilon]$ where $0 < \epsilon < \frac{1}{2}$. Now let $\psi_2(x) = \int_{-\infty}^{x} \psi(t)dt$. We see that $\psi_2(x)$ is constant for $x > \epsilon$ and is zero on $x < 0$. Consequently $\psi_3(x) = \psi_2(x)\psi_2(1-x)$ has values in $[0,1]$, is compactly supported and is constant on the interval $(\epsilon, 1-\epsilon)$. We can now translate and scale ψ_3 appropriately. □

Proposition 3.5 (Partition of unity)**.** Let $K \subset \mathbb{R}^d$ be a compact set and suppose that $U \supset K$ is open. Suppose that $\mathcal{U}$ is a collection of open subsets of U that covers K. Then there exist smooth functions $\psi_1, \ldots, \psi_n : \mathbb{R}^d \to [0,1]$ such that

$$\psi := \sum_{i=1}^{n} \psi_i$$

satisfies that $\psi(x) = 1$ for $x \in K$ and ψ_i has support inside some element of $\mathcal{U}$.

Proof. By compactness, we may find finitely many balls $B(a_1, r_1), \ldots, B(a_n, r_n)$ that cover K such that $B(a_i, 2r_i)$ is a subset of some element of $\mathcal{U}$ (and thus $B(a_i, 2r_i)$ are subsets of U). Now apply the previous construction to find some smooth $\phi_i : \mathbb{R}^d \to [0,1]$ that equals 1 on $B(a_i, r_i)$ and has support inside $B(a_i, 2r_i)$. Now let $\psi_1 = \phi_1$ and for $1 < i \leq n$ define $\psi_i = \phi_i \prod_{j<i}(1-\phi_j)$. Observe that ψ_i has support inside the support of ϕ_i, thus inside some element of $\mathcal{U}$, as required. Moreover, by induction we have that

$$\sum_{i=1}^{j} \psi_i = 1 - \prod_{i=1}^{j}(1-\phi_i).$$

In particular for $j = n$ this means that by setting $\psi = \sum_{i=1}^{n} \psi_i$ we have that $\psi(x) = 1$ for $x \in B(a_i, r_i)$, and thus for all $x \in K$. Moreover, if $\phi(x) = 0$ then $\psi_i(x) = 0$ and thus ψ_i has support inside some element of $\mathcal{U}$, as required. □

Theorem 3.6 (Gluding distributions)**.** Suppose that $X \subset \mathbb{R}^d$ is an open set and suppose that $\mathcal{U}$ is a collection of open subsets of X that cover X. Suppose that for each $U \in \mathcal{U}$ there is a distribution $f_U \in D'(U)$ such that these f_U are compatible in the sense that $f_U|_{U\cap V} = f_V|_{U\cap V}$ are the same distributions on $D'(U \cap V)$. Then there is a unique distribution $f \in D'(X)$ such that $f|U = f_U$ for all $U \in \mathcal{U}$.

Proof. We construct f as follows (show that it is well defined later): For each $\phi \in D(X)$, choose a compact set $K \subset X$ containing the support of ϕ. Now we may apply Parition of Unity to find open sets $U_1, \ldots, U_n \in \mathcal{U}$ that cover K and $\psi_i : \mathbb{R}^d \to [0,1]$ with support inside U_i such that $\psi := \sum_{i=1}^{n} \psi_i$ satisfies that $\psi(x) = 1$ for all $x \in K$. We now define

$$f(\phi) = \sum_{i=1}^{n} f_{U_i}(\phi\psi_i).$$

Note that this shows uniqueness since $\phi = \sum_{i=1}^{n} \phi\psi_i$ on $\mathbb{R}^d$.

We now show that f is well defined (does not depend on the choice of K or the choice of the U_i or the choice of ψ_i). To see this, suppose that K', U_j' and ψ_j' are such other choices. Then we make a common refinement and show it assigns the same value to our $f(\phi)$ as follows. Let $K'' = K \cap K'$, it clearly contains the support

of ϕ and is compact. Now the sets U_i and U_j cover K''. Thus the sets $V_{i,j} = U_i \cap U_j$ cover K''. Moreover, $\psi_{i,j} := \psi_i \psi'_j : \mathbb{R}^d \to [0,1]$ has support inside $V_{i,j}$ and

$$\sum_{i,j} \psi_{i,j} = \left(\sum_i \psi_i\right)\left(\sum_j \psi'_j\right)$$

and thus equals 1 on K''. So this common refinement is a new partition of unity. But now

$$\sum_{i,j} f_{U_i}|_{V_{i,j}}(\psi_{i,j}\phi) = \sum_{i,j} f_{U_i}(\psi_{i,j}\phi) = \sum_i f_{U_i}(\phi\psi_i \sum_j \psi'_j) = \sum_i f_{U_i}(\phi\psi_i)$$

where we used that $\phi\psi_i \sum_j \psi'_j = \phi_i\psi_i$ since $\sum_j \psi'_j(x) = 1$ for all $x \in K'$ and thus all x in the support of ϕ. This completes the proof of well definedness since by assumption,

$$f_{U_i}|_{V_{i,j}}(\psi_{i,j}\phi) = f_{U'_j}|_{V_{i,j}}(\psi_{i,j}\phi)$$

and so

$$\sum_i f_{U_i}(\phi\psi_i) = \sum_j f_{U'_j}(\phi\psi'_j)$$

by the same calculation as above. Suppose that $\phi, \phi' \in D(X)$. Thus to compute $f(\phi + \phi')$ we may choose a compact set $K \subset U$ that contains the support of ϕ and ϕ'. Now choose $U_1, \ldots, U_n \in \mathcal{U}$ that cover K, thus by definition

$$f(\phi_1 + \phi_2) = \sum_i f_{U_i}(\psi_i(\phi_1 + \phi_2)) = \sum_i f_{U_i}(\psi_i\phi_1) + \sum_i f_{U_i}(\psi_i\phi_2) = f(\phi_1) + f(\phi_2)$$

where the ψ_i are chosen as in the construction. Linearity of f now easily follows. We now show the continuity of f. If $\phi_k \to \phi \in D(X)$ then there is a compact set $K \subset X$ containing all their supports. Thus $\psi_i\phi_k \to \psi\phi$ and the continuity of each f_{U_i} gives continuity of f. Finally, it remains to show that $f|_U = f_U$ for all $U \in \mathcal{U}$. Thus suppose that $\phi \in D(U)$ and choose a compact set $K \subset U$ such that ϕ has support inside K. As U already covers K, by definition we have that

$$f|_U(\phi) = f(\phi) = f_U(\psi\phi) = f_U(\phi)$$

for some $\psi : \mathbb{R}^d \to [0,1]$ smooth that equals 1 on K and has suppose inside U (so $\phi\psi = \psi$ everywhere). $\square$

4. Distributions with compact supports

If $X \subset \mathbb{R}^d$ is an open set, we let $\mathcal{E}(X) = C^\infty(X)$ denote the set of all smooth functions on X. We have $D(X) \subset \mathcal{E}(X)$ and the inclusion may be strict, for example constant non-zero functions are not in $D(\mathbb{R})$ but are in $C(\mathbb{R})$.

Definition 4.1. We say that $\phi_j \in \mathcal{E}(X)$ converges to $\phi \in \mathcal{E}(X)$ if for all compact sets $K \subset X$ and $\alpha \in \mathbb{Z}^d_{\geq 0}$ we have that

$$\partial^\alpha \phi_j \to \partial^\alpha \phi \quad \text{uniformly on } K.$$

Example 4.2. In $\mathcal{E}(\mathbb{R})$, we have that $f_n(x) = \frac{1}{n}x$ converge to the 0 function. However the convergence is not uniform on the whole of $\mathbb{R}$ itself, but is on every bounded (hence every compact) set. If $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth compactly supported non-zero function then $\phi \in D(X) \subset \mathcal{E}(X)$ and $\phi_n(x) = \phi(x - n)$ are also in $\mathcal{E}(X)$. Note that $\phi_n \to 0$ in $\mathcal{E}(X)$ but not in $D(X)$ (as there is no compact subset containing all supports of the ϕ_n).

Definition 4.3. If $X \subset \mathbb{R}^d$ is open, we let $\mathcal{E}'(X)$ denote the space of linear maps $f : \mathcal{E}(X) \to \mathbb{C}$ that satisfy the property that if $\phi_j \in \mathcal{E}(X)$ converges to $\phi \in \mathcal{E}(X)$. We say $f_j \in \mathcal{E}'(X)$ converges to $f \in \mathcal{E}'(X)$ if $f_j(\phi) \to f(\phi)$ for all $\phi \in \mathcal{E}(X)$.

We now aim to classify $\mathcal{E}'(X)$ and find a natural embedding into $D'(X)$. First observe that given $f \in \mathcal{E}'(X)$, if we restrict this function to $D(X)$, then we get an element of $\mathcal{D}'(X)$. Thus we have a map

$$\iota : \mathcal{E}'(X) \to D(X).$$

Lemma 4.4. If $X \subset \mathbb{R}^d$ is open, then it has an *exhuastion by compact sets*, which we define to be a sequence $K_1 \subset K_2 \subset \dots$ of compact sets such that

$$X = \bigcup_{n=1}^{\infty} K_n$$

and such that any compact sets $K \subset X$ satisfies that $K \subset K_n$ for some n.

Proof. If $X = \mathbb{R}^d$, just take a compact ball of radius n. Otherwise, let $C = \mathbb{R}^d \setminus X$. Note that C is closed. Now let $K_n \subset X$ be those points in ball of radius n around 0 with distance at most $\frac{1}{n}$ to C. Clearly $X = \bigcup_{n=1}^{\infty} K_n$ since any point not in C must have positive distance to C as C is closed. Now let $K \subset X$ be any open set. Then $K \cap C = \emptyset$ and it follows that K has positive distance to C, as otherwise there is $k_n \in K$ and $c_n \in C$ such that $d(k_n, c_n) \to 0$ and by taking a subsequence we assume $k_n \to k \in K$, thus $d(k, C) = 0$, contradicting the disjointness of K and C. It follows that $K \subset K_n$ for large enough n as K is bounded and has positive distance to C.

□

Lemma 4.5. The map $\iota : \mathcal{E}'(X) \to D(X)$ is well defined, continuous (preserves limits) and injective.

Proof. To see well defined, observe that $D(X) \subset \mathcal{E}(X)$ this embedding is continuous, i.e., if $\phi_j \to \phi$ in $D(X)$ then their partial derivatives converge uniformly on all compact sets K by definition. Thus if $f \in \mathcal{E}'(X)$ then indeed $f|_{D(X)} \in \mathcal{E}(X)$. Continuity is obvious as the definition is the same.

Now we show injectivity, thus we wish to show that this linear map ι has a trivial kernel. Thus suppose that $f \in \mathcal{E}'(X)$ satisfies that $f(\phi) = 0$ for all $\phi \in D(X)$. We must now show that $f(\psi) = 0$ for all $\psi \in \mathcal{E}(X)$. To see this, first write

$$X = \bigcup_{n=1}^{\infty} K_n$$

where $K_1 \subset K_2 \subset \dots$ is an exhuastion by compact subsets of X (as defined in the lemma above). Now by partition of unity theorem, we can find a map $\phi_n : \mathbb{R}^d \to [0, 1]$ with compact support inside X that is equal to 1 on K_n. Thus $\phi_n \psi \in D(X)$ as it has compact support in X. Hence $f(\phi_n \psi) = 0$ by assumption. Now $\phi_n \psi \to \psi$ in $\mathcal{E}(X)$ since for any compact set K, we have that $K \subset K_n$ for large enough n and so $\phi_n \psi = \psi$ on K. Thus by continuity of f, we have that

$$f(\phi) = \lim_{n \to \infty} f(\phi_n \psi) = 0,$$

as desired. □

The next example shows that this embedding is not surjective.

Example 4.6. Consider the distribution $f \in D'(\mathbb{R})$ given by $f(\phi) = \int_{\mathbb{R}} \phi(x)dx$. We claim that $f \neq \iota g$ for some $g \in \mathcal{E}'(X)$. Again let $\phi_n : \mathbb{R} \to [0,1]$ be a compactly supported function equal to 1 on $[-n,n]$. Then $g(\phi_n) = f(\phi_n) > 2n$ does not converge to any value. But ϕ_n converges in $\mathcal{E}(X)$ to the constant function 1 thus $g(\phi_n)$ should converge, a contradiction.

Lemma 4.7. Let $f \in D'(X)$ be a distribution with support K such that $K \subset X$. Then f extends to an element of $\mathcal{E}'(X)$ (is in the image of the map ι).

Proof. Let $\psi \in \mathcal{E}(X)$. Let $\phi \in D(X)$ be an element such that $\phi = 1$ on some open set $U \supset K$ such that $\overline{U} \subset X$ and $\overline{U}$ is compact (exists as $K \subset X$ and we use partition of unity). Now define $g(\psi) = f(\psi\phi)$. We claim that this defines $g \in \mathcal{E}'(X)$. To show that it is well defined, we see that $\psi\phi \in D(X)$ thus $f(\psi\phi)$ makes sense (no need to show independence on ϕ, just consider it fixed throughout). Continuity of g is clear since if $\psi_n \to \psi$ in $\mathcal{E}(X)$ then their partial derivatives convege uniformly on the support of ϕ, thus $\psi_n\phi \to \psi_n\phi$ in $D(X)$. Now it remains to show that g agrees with f on $D(X)$. Thus assume $\psi \in D(X)$ already. Now observe that $\psi - \psi\phi = 0$ on $\overline{U}$ and thus $X \in D(U \setminus K)$ and by definition of support, we have that $f|_{X\setminus K} = 0$ and so $f(\psi - \psi\phi) = 0$, thus $g(\psi) = f(\psi)$ as desired. □

We now show that the converse is true, i.e., that $\mathcal{E}'(X)$ coincides with those distributions in $\mathcal{D}'(X)$ whose support is a compact subset of K.

Theorem 4.8. A distribution $f \in D'(X)$ is the restriction of some $g \in \mathcal{E}'(X)$ if and only if the support of f is some compact subset $K \subset X$.

Proof. Suppose that $g \in \mathcal{E}'(X)$, we wish to show that it has compact support K for some $K \subset X$ as an element $g \in D'(X)$. Suppose that it does not, thus for any compact set $K \subset X$ we have that $g|_{X\setminus K}$ is not the zero distribution on $X \setminus K$. Thus there exists $\phi_K \in D(X \setminus K)$ such that $g(\phi_K) = 1$. Now let $K_1 \subset K_2 \subset \dots$ be an exhaustion of X by compact sets. Set $\phi_n = \phi_{K_n}$.We claim that $\phi_n \to 0$ in $\mathcal{E}(X)$ (not necessarily in $D(X)$ though). To see this, let $K \subset X$ be any compact set. Then $K \subset K_n$ for large enough n, but then $\phi_n = 0$ on such K_n as $\phi_n \in D(X \setminus K_n)$. Thus indeed $\phi_n \to 0$ uniformly on each compact set of K, thus $1 = g(\phi_n) = f(\phi_n) \to 0$ by continuity of f on $\mathcal{E}(X)$. A contradiction. □

Example 4.9. The distribution $f(\phi) = \int_0^1 \phi(x)dx$ is a compactly supported distribution on $\mathbb{R}$, thus it is defined and continuous on all of $C^\infty(\mathbb{R})$. Note that if $\phi_n(x) = \phi(x+n)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a non-zero compactly supported smooth function, then $\phi_n \to 0$ in $\mathcal{E}(X)$ and indeed $f(\phi_n) \to 0$. However ϕ_n does not converge to 0 in $D(X)$.

Example 4.10. Consider the distribution on $(0,1)$ given by

$$f(\phi) = \int_0^1 e^{1/x^2}\phi(x)dx.$$

It is continuous and well defined since e^{1/x^2} is integrable on each compact subset K in $(0,1)$. However, $f \notin \mathcal{E}'((0,1))$ (not the restriction of any element in $\mathcal{E}'((0,1))$). Indeed, it is not supported on any compact subset K of $(0,1)$. This is despite the support being the bounded subset $(0,1)$ that is closed in $(0,1)$ but not in $\mathbb{R}$. One can directly see that there is not continuous extension of f to $\mathcal{E}((0,1))$ by considering a sequence of elements $\phi_n \in D((0,1))$ that converge monotonically in $\mathcal{E}(X)$ to the constant function 1 on $(0,1)$ and seeing a lack of convergence ($f(\phi_n)$ diverges to ∞).

Theorem 4.11. For any open $X \subset \mathbb{R}^d$, we have that $\mathcal{E}'(X)$ is a dense subset of $D'(X)$.

Proof. Let $K_1 \subset K_2 \subset \dots$ be an exhuastion of X be compact subsets. Choose a $\psi_n \in D(X)$ such that $\psi_n = 1$ on K_n. Now let $f \in D(X)$ be any distribution and define $f_n \in D(X)$ by

$$f_n(\phi) = f_n(\phi\psi_n).$$

We claim that $f_n \in \mathcal{E}'(X)$ and $f_n \to f$ in $D'(X)$. To justify the first claim, note that for $\phi \in D(U \setminus \operatorname{supp} \psi_n)$ we have that

$$f_n(\phi) = f(\psi_n\phi) = f(0) = 0$$

which shows that f_n has support a subset of $\operatorname{supp} f$, thus has compact support. Now for any $\phi \in D(X)$, there is a compact set $K \subset X$ such that $\phi(x) = 0$ for $x \notin K$. Also, for n large enough we have $K \subset K_n$, thus

□

5. Fourier Transform

References

[1] Duistermaat, J. J.; Kolk, J. A. C. *Distributions. Theory and applications.* Translated from the Dutch by J. P. van Braam Houckgeest. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2010. xvi+445 pp. ISBN: 978-0-8176-4672-1