GALOIS THEORY NOTES

1. Splitting fields and Normal extensions

Proposition 1.1. Let $K \leq L$ be fields. Suppose that $\alpha \in L$ is algebraic over K and let $p(x) \in K[x]$ be a minimal polynomial for α. Then there is a unique isomorphism $K[x]/(p(x)) \to K[\alpha] = K(\alpha)$ mapping x to α and fixing K.

Proof. There is a unique map $K[x] \to K[\alpha]$ mapping x to α and fixing K. It is surjective and its kernel is the ideal generated by $p(x)$. $\square$

If $\sigma : K \to L$ is a homomorphism of fields and $f = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$, then we let $f^\sigma = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n \in F[x]$.

Lemma 1.2. Suppose that $\sigma : K \to L$ is an isomorphism of fields and suppose $K' = K[\alpha]$ is an extension of K where $f \in F[x]$ the minimal polynomial of $\alpha \in K'$. Let $\sigma : K \to L$ be a field homomorphism.

(1) If $\sigma' : K' \to L$ extends σ, then $f^\sigma(\sigma'(\alpha)) = 0$
(2) If $\beta \in L$ satisfies that $f^\sigma(\beta) = 0$, then there is precisely one extension of σ mapping α to β.

Proof. The first point is obvious. For the second point, let $\phi : K[x] \to L$ be given by $\phi(P) = P^\sigma(\beta)$. This is a ring homomorphism. Now observe that $\phi(f) = f^\sigma(\beta) = 0$, thus ϕ vanishes on the ideal generated by f and so there is a well defined field homomorpism $\phi : K[x]/(f) \to L$ mapping $x + (f)$ to β. Finally, we use the isomorphism $K' \cong K[x]/(f)$ that maps α to x and fixes K, giving the desired extension. The extension is clearly unique as $K' = K(\alpha)$.

$\square$

Proposition 1.3. Let $K \leq K'$ be an algebraic field extension and suppose that $\sigma : K \to L$ is a field homomorphism where L is algebraically closed. Then there exists an extension $\sigma' : K' \to L$. Moreover, σ' must be an isomorphism if K' is algebraically closed and L is algebraic over $\sigma(K)$.

Proof. Use Zorn's lemma to construct a maximal subfield $K'' \subset K$ such that σ extends to K''. If $K'' \neq K'$ then choose $\alpha \in K' \setminus K''$. Now as K' ia algebraic over K we can let $f \in K[x]$ be a minimal polynomial of α over K. Now as f^σ has a root in L as L is algebraically closed, we can use the previous lemma to extend σ to $K''[\alpha]$, contradicting the maximality of K''. If K' is algebriacally closed, then so is $\sigma'(K')$ since any element of $\sigma'(K')[x]$ is of the form $f^{\sigma'}$ for some $f \in K'[x]$ and so we can let α be a root of f, giving that $\sigma'(\alpha)$ is a root of $f^{\sigma'}$. Now $\sigma'(K') \geq \sigma(K)$ so if L is algebraical over $\sigma(K)$, then L is also algebraic over $\sigma'(K')$. So if L is algebraically closed then $L = \sigma'(K')$, giving that σ' is surjective and thus an isomorphism (all field isomorphisms are injective). $\square$

Corollary 1.4. The algebraic closure of a field K is unique upto an isomorphism fixing K.

Definition 1.5 (Splitting field)**.** Let $K \leq L$ be fields and let $\mathcal{F} \subset K[x]$ be a family of polynomials. We say that L is a splitting field for $\mathcal{F}$ over F if each $f \in \mathcal{F}$ splits into linear factors in $L[x]$ and L is the field generated by K and the roots of all polynomials in $\mathcal{F}$.

Proposition 1.6. A splitting field is unique upto an isomorphism fixing F.

Proof. Let $L \geq K$ and $L' \geq K$ be two splitting fields for a family $\mathcal{F} \subset K[x]$. We note that L' and L are both algebraic over K (as they are generated by roots). This means that we may use Proposition 1.3 to extend the identity map $K \to K$ to a field homomorphism $\sigma : L \to \widehat{L'}$ where $\widehat{L'} \geq L'$ is algebraically closed. However, note that $\sigma(L) \subset L'$ since σ maps each root of some $f \in \mathcal{F}$ to a root of f (as σ fixes K). So $\sigma : L \to L'$ is a homomorphism. It remains to show that σ is surjective. To see this, let $f \in \mathcal{F}$ and write $f(x) = \prod_i (x - \alpha_i)$ where $\alpha_i \in L$. Then $f = f^\sigma = \prod_i (x - \sigma(\alpha_i))$. This shows that any root in L' of any $f \in \mathcal{F}$ is in the image of σ (using the unique factorization property). Thus as L' is generated by these roots, the surjectivity of σ follows. $\square$

If K_1 and K_2 are two fields with a common subfield K, we say that a homomorphism $K_1 \to K_2$ is a K-homomorphism if it restricts to the identity on K.

Theorem 1.7. Let L be an algebraic extension of a field K. Then the following are equivalent.

(1) L is a splitting field for some family of polynomials in $K[x]$.
(2) Any K-homomorphism $L \to \overline{L}$, where $\overline{L} \geq L$ is an algebriac closure, restricts to an automorphism of L
(3) Any irreducible polynomial in $K[x]$ that has a root in L must decompose into linear factors in $L[x]$.

Proof. (i) $\implies$ (ii): If L is a splitting field for some polynomials in $K[x]$ and $\sigma : L \to \overline{L}$ is a K-homomorphism, then as in the proof of the uniqueness of splitting fields above, we see that σ maps into L. We also saw that it permutes the roots of a polynomial in $K[x]$ in L and thus the image of σ is L, thus σ is surjective and hence an automorphism.

(ii) $\implies$ (iii): Suppose $f \in K[x]$ is irreducible and has a root $\alpha \in L$. Now if $\alpha' \in \overline{L}$ is another root of f, then since f is irreducible we have an isomorphism $K[\alpha] \to K[\alpha']$ mapping α to α', which we may extend to an K-homomorphism $\sigma : L \mapsto \overline{L}$ by a previous Lemma. By condition (ii), we see that σ maps L to L and thus $\alpha' = \sigma(\alpha) \in L$. Hence L contains all the roots of f.

(iii) $\implies$ (ii): As L is algebraic, every element $\alpha \in L$ is the root of some irreducible polynomial $f \in K[x]$. We thus let $\mathcal{F} \subset K[x]$ be those irreducible polynomials with at least one root in L, which split into linear factors by assumption. Thus L is the splitting field of $\mathcal{F}$ over K.

$\square$

Definition 1.8. We say that an extension $K \leq L$ is normal if it is the splitting field of some family of polynomials.

Example 1.9. The extension $\mathbb{Q} \leq \mathbb{Q}[2^{1/3}]$ is not normal. To see this we use the characterization (iii) in the Theorem as follows: The polynomial $x^3 - 2$ is irreducible, has one root $2^{1/3}$ in our extension but not any other. Alternatively, we can use (ii) by noting that although there is $\mathbb{Q}$-homomorphism $\mathbb{Q}[2^{1/3}] \to \overline{\mathbb{Q}}$ mapping $2^{1/3}$ to $2^{1/3}e^{2\pi i/3}$, it does not restrict to an automorphism of $\mathbb{Q}[2^{1/3}]$.

Example 1.10. Normal is not transitive. As an example, consider the field extensions $\mathbb{Q} \leq \mathbb{Q}[\sqrt{2}] \leq \mathbb{Q}[2^{1/4}]$. The intermediate field extensions are normal (as they are of degree 2) but the extension $\mathbb{Q} \leq \mathbb{Q}[2^{1/4}]$ is not.

Definition 1.11. If $L \geq K$ is an algebraic extension, then we say that $L' \leq L \leq K$ is a normal closure of $L \geq K$ if $L' \geq K$ is a normal extension and any $L' \geq L'' \geq K$ such that $L'' \geq K$ is normal must satisfy $L'' = L'$. That is, the normal closure if a minimal normal extension.

Proposition 1.12. Every algebraic extension $L \geq K$ has a normal closure. More precisely, let $\mathcal{F}$ be the set of all irreducible polynomials in $K[x]$ such that each element of $L \setminus K$ is the root of some $f \in \mathcal{F}$. Then the splitting field of $\mathcal{F}$ is the normal closure of $L \geq K$.

Proof. Let $\overline{L} \geq L$ be the algebraic closure of L. Define $\overline{L} \geq L' \geq L$ to be the splitting field for the family $\mathcal{F} \subset K[x]$ of minimal polynomials for elements of L. We claim that L' is the normal closure. Thus suppose that $L \leq L'' \leq L'$ is such that $K \leq L''$ is normal. We must show that $L'' = L'$, and since L' is generated by the roots of elements of $\mathcal{F}$, we must show that any root $\alpha \in L'$ of a polynomial $f \in \mathcal{F}$ is in L''. To see this, note that by definition f is a minimal polynomial of some $\alpha' \in L$. There is a K-homomorphism $\sigma : K[\alpha'] \to \overline{L}$ mapping α' to $\alpha \in L$. As $L'' \geq L \geq K[\alpha']$, we may extend this K-homomorphism to $\sigma : L'' \to \overline{L}$. But by characterization (ii) of the normality of $K \leq L''$, we see that σ is an automorphism of L''. This means that $\alpha = \sigma(\alpha') \in L''$ as $\alpha' \in L \subset L''$. Thus this shows that $L' \subset L''$, and so $L' = L''$ as required. □

Proposition 1.13. If $K \leq L$ is an algebraic extension and $L \leq L_1, L_2 \leq \overline{L}$ are two normal extensions of K, then $L_1 \cap L_2$ is a normal extension of K. In particular, if L_1 and L_2 are both normal closures of $L \geq K$, then $L_1 = L_2$.

Proof. This follows from characterization (iii): If $f \in K[x]$ is irreducible and has a root in $\alpha \in L_1 \cap L_2$, then f decomposes to linear factors in $L_i[x]$ for $i = 1, 2$. By uniqueness of factorizations, this means that these linear factors are in $(L_1 \cap L_2)[x]$. □

Proposition 1.14. A normal closure of an algebraic extension $L \geq K$ is unique upto an L-automorphism.

Proof. By the previous construction, we have one such normal closure given by $L[\mathcal{R}]$ where

$$\mathcal{R} = \{r \in \overline{L} \mid f(r) = 0 \text{ for some } f \in \mathcal{F}\}$$

where $\mathcal{F} \subset K[x]$ is the set of all irreducible polynomials such that each element of L is the root of some $f \in \mathcal{F}$. We now let $L' \geq L$ be another field such that $L' \geq K$ is the normal closure of $L \geq K$. We now construct an isomorphism $L[\mathcal{R}] \to L'$ which fixes L. We extend the inclusion $L \to \overline{L'}$ to an L-homomorphism $\sigma : L[\mathcal{R}] \to \overline{L'}$. Note that $L'' = \sigma(L[\mathcal{R}]) = L[\sigma(\mathcal{R})]$ contains L and is the splitting field of $\mathcal{F}$ in $\overline{L'}$ over K. Thus L' and L'' are subfields of $\overline{L}$ that are normal extensions of K and both contain L. Moreover, L'' is also a normal closure of $L \geq K$ as it follows the construction given in Proposition 1.12 (i.e., it is a splitting field of minimal polynomials over $K[x]$ of elements in L). By the previous proposition, it follows that $L' = L''$, thus σ is an isomorphism. □

2. Seperable extensions

Lemma 2.1. An irreducible polynomial $f \in K[x]$ splits into distinct linear factors in some algebraic closure if and only if $f' = 0$.

Proof. By the product rule it follows that if $f(\alpha) = 0$ then α is a repeated root if and only if $f'(\alpha) = 0$. If f is irreducible, has a repeated root α and $f' \neq 0$ then $(X-\alpha)|gcd(f, f')|f$, which contradicts the irreducibility of f. □

As a consequence, if $charK = 0$ then an irreducible polynomial must split into distinct linear factors.

Definition 2.2. We say that $f \in K[x]$ is seperable if f splits into distinct linear factors in some (hence any) algebraic closure of K.

Theorem 2.3. If $charK = p$ and $f \in K[x]$ is irreducible, then each root of f has multiplicity p^r where r is minimal non-negative integer such that $f(x) = g(x^{p^r})$ for some $g \in K[x]$.

Proof. Write $g(x) = \sum_j c_j x^j$. Since

$$g'(x) = \sum_j j c_j x^j$$

we observe that $g'(x)$ is not the zero polynomial as follows: If $g'(x) = 0$ then $c_j = 0$ whenever j is not divisible by p. From this it follows that $g(x) = \sum_k c_{kp} x^{kp} = h(x^p)$. It now follows that

$$f(x) = g(x^{p^r}) = h((x^{p^r})^p) = h(x^{p^{r+1}}),$$

which contradicts the maximality of r. Thus $g'(x) \neq 0$. This means that $g(x) = \prod_i (x - \alpha_i)$ where α_i are distinct. Write $\alpha_i = \beta_i^{p^r}$, which exists in an algebraic closure. Note that the β_i must also be distinct. Thus

$$f(x) = \prod_i (x^{p^r} - \beta_i^{p^r}) = \prod_i (x - \beta_i)^{p^r},$$

where the last equality follows from Freshman's dream in characteristic p. As the β_i are distinct, the proof is complete. □

Definition 2.4. If $K \leq L$ is an algebraic field extension then $\alpha \in L$ is called seperable over K if the minimal polynomial is seperable (splits over linear factors in some, hence any, algebraic closure). We say that the extension $K \leq L$ is seperable if all elements of L are separable over K.

Thus from above, in characteristic zero all algebraic extensions are seperable, as all irreducible polynomials are seperable.

Example 2.5. Consider the field $K = \mathbb{F}_p(t)$. The polynomial $f(x) = x^p - t$ is irreducible by Eisenstein's criterion in $\mathbb{F}_p[t]$ as t is prime in this UFD, and hence $f(x)$ is irreducible also over its field of fractions K by Gauss's Lemma. Now, $f(\alpha) = 0$ in for some $\alpha \in \overline{K}$, that is $\alpha^p = t$. But by Freshman's dream we have that

$$x^p - t = x^p - \alpha^p = (x - \alpha)^p,$$

thus α is a root of multiplicity p for $f(x)$. Thus $f(x)$ is irreducible but not seperable.

Definition 2.6. If $K \leq L$ is an algebraic extension, then we let

$$Hom_K(L, \overline{K})$$

denote the set of all K-homomorphisms $L \to \overline{K}$. We let

$$|L : K|_s = |Hom_K(L, \overline{K})|$$

be the seperable degree of $K \leq L$, which does not depend on the choice of $\overline{K}$.

Proposition 2.7. If $K \leq L \leq M$ are algebraic extensions then there is a bijection

$$Hom_K(L, \overline{K}) \times Hom_L(M, \overline{K}) \to Hom_K(M, \overline{K}).$$

In particular

$$|M : K|_s = |L : K|_s |M : L|_s.$$

Proof. For each $\sigma \in Hom_K(L, \overline{K})$ we choose an arbitrary (there are many choices) $\phi(\sigma) : \overline{K} \to \overline{K}$ automorphism that extends σ, where we have used Proposition???. Now we define a mapping

$$Hom_K(L, \overline{K}) \times Hom_L(M, \overline{K}) \to Hom_K(M, \overline{K})$$

by

$$(\sigma, \tau) \mapsto \phi(\sigma) \circ \tau.$$

Let us first check that it is well defined. If $k \in K$ then

$$(\phi(\sigma) \circ \tau)(k) = \phi(\sigma)(\tau(k)) = \phi(\sigma)(k) = \sigma(k) = k,$$

so indeed $\phi(\sigma) \circ \tau$ is a K-homomorphism. To show injectivity, suppose that

$$\phi(\sigma) \circ \tau = \phi(\sigma') \circ \tau'.$$

Then for any $\ell \in L$ we have that

$$\phi(\sigma)(\tau(\ell)) = \phi(\sigma)(\ell) = \sigma(\ell)$$

and by the same arugment $\phi(\sigma')(\tau'(\ell)) = \sigma'(\ell)$. Thus $\sigma = \sigma'$. This means that $\phi(\sigma) = \phi(\sigma')$ and so by injectivity of field automorphisms, we must have that $\tau'(m) = \tau(m)$ for all $m \in M$. So $\tau = \tau'$. It now remains to show injectivity. Thus suppose that $\gamma \in Hom_K(M, \overline{K})$. Let σ be the restriction of γ to L and observe that $\sigma \in Hom_K(L, \overline{K})$. Now let

$$\tau = \phi(\sigma)^{-1} \circ \gamma : M \to \overline{K}.$$

If $\ell \in L$ then

$$\tau(\ell) = \phi(\sigma)^{-1}(\gamma(\ell)) = \phi(\sigma)^{-1}(\sigma(\ell)) = \phi(\sigma)^{-1}\phi(\sigma)(\ell) = \ell,$$

thus indeed $\tau \in Hom_L(M, \overline{K})$. This shows that $\gamma = \phi(\sigma) \circ \tau$ is in the image of our map, thus our map is surjective. $\square$

Proposition 2.8. If $K \leq L$ is a finite extension then

(1) If K has characteristic zero then $|L : K| = |L : K|_s$
(2) If K has characteristic p then $|L : K| = p^r |L : K|_s$ for some integer $r \geq 0$.

Proof. By finiteness of this extension L can be obtained from K by finitely many simple extensions, so we only need to prove this when $L = K(\alpha)$ is a simple extension and then use the previous proposition to give the general case by induction. If $Char K = 0$ then we know that $|L : K| = deg f = |L : K|_s$ where $f \in K[x]$ is the minimal polynomial of α, where we have used the fact that f is seperable and there is a unique K-homomorphism mapping α to any given root of f. If $Char K = p$ then $|L : K| = deg f = p^r |L : K|_s$ where r is maximal integer such that $f(x) = g(x^{p^r})$ for some polynomial $g(x) \in K[x]$, as seen in a previously proven result. Thus completing the proof. $\square$

Theorem 2.9. Let $K \geq L$ be a finite extension. The following are equivalent.

(1) $K \geq L$ is seperable.

(2) $L = K(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in L$ that are separable over K
(3) $|L : K|_s = |L : K|$

Proof. (i) $\implies$ (ii) is trivial. (ii) $\implies$ (iii): Letting $K_i = K_{i-1}(a_i)$ we see that a_i is separable over $K_{i-1} \geq K$and thus $|K_i : K_{i-1}| = deg(f_i) = |K_i : K_{i-1}|_s$ where f_i is the minimal polynomial of a_i over K_{i-1}. We are now done by the multiplicativity formula. (iii) $\implies$ (i): We only need to focus on $Char K = p > 0$. If $a \in L$ is not seperable over K then

$$|K(a) : K|_s < |K(a) : K|$$

, but then

$$|L : K|_s = |L : K(a)|_s |K(a) : L|_s < |L : K(a)||K(a) : K| = |L : K|.$$

□

Corollary 2.10. If $K \leq L \leq M$ are algebraic extensions then $K \leq M$ is seperable if and only if $K \leq L$ and $L \leq M$ are seperable.

Proof. First suppose $K \leq M$ is seperable. Then clearly $K \leq L$ is seperable. Now for $a \in M$ we have that the minimal polynomial $f(x) \in K[x]$ of a over K splits into linear factors. If $g(x) \in L[x]$ is the minimal polynomial of a over L, then clearly $g(x)|f(x)$ as $f(x) \in L[x]$. Thus $g(x)$ also splits into linear factors.

Conversely, assume now that $K \leq L$ and $L \leq M$ are seperable. Fix $a \in M$. Then $|L(a) : L| = |L(a) : L|_s$ as $L \leq M$ is seperable. Now let $L' \leq L$ be the field generated by K and the coefficients of the minimal polynomial $f(x) \in L[x]$ of a over L. Thus $f(x) \in L'[x]$ which means that a is seperable over L' as well (as $f(x)$ splits into linear factors and $f(a) = 0$). Thus $|L'(a) : L'|_s = |L(a) : L|$. It now follows that

$$|L'(a) : K|_s = |L'(a) : L'|_s |L' : K|_s = |L'(a) : L||L' : K| = |L'(a) : K|,$$

hence be the previous theorem we have that $L'(a)$ is seperable over K, and thus a is seperable over K. □

Theorem 2.11 (Primitive element theorem)**.** If $K \leq L$ is a finite seperable extension, then $L = K(a)$ for some $a \in L$.

Proof. If L is finite, then this follows from the fact that the multiplicative group of a field is cyclic. Suppose thus that K and L are infinite. We may reduce to the case where $L = K(\alpha, \beta)$, as the general case then follows by induction (If $L = K(a_1, \ldots, a_n)$ then $L = K'(a_1, a_2)$ where $K' = K(a_3, \ldots, a_n)$ and certainly L is seperable over K'). For $c \in K$, we let $\gamma_c = \alpha + c\beta$. We will show that $L = K(\gamma_c)$ for infinitely many $c \in K$ as follows. If $L \neq K(\gamma_c)$ then definitely $\beta \notin K(\gamma_c)$. As L is seperable over $K(\gamma_c)$, this means that the minimal polynomial of β over $K(\gamma_c)$ has another root $\beta' \in \overline{K}$. Thus there exists a $K(\gamma_c)$-homomorphism $\sigma : L \to \overline{K}$ with $\sigma(\beta) = \beta' \neq \beta$. We thus get that

$$\sigma(\alpha) + c\sigma(\beta) = \alpha + c\beta$$

and thus

$$c = \frac{\sigma(\alpha) - \alpha}{\beta - \sigma(\beta)}.$$

But the right hand side has only finitely many choices (as there are only finitely many choices of σ) and so if we choose a c not of this form (as K is infinite) we see that $L = K(\gamma_c)$ as desired. □

3. Galois Extensions

Definition 3.1. A field extension $K \leq L$ is called *Galois* if it is normal and seperable. We also say L is *Galois* over K. We define $Gal(L/K) := Aut_K(L)$ to be the set of K-automorphisms $L \to L$.

Proposition 3.2. Suppose that $K \leq L$ is Galois and $K \leq E \leq L$ is an intermediate field.

(1) Then L is also Galois over E and $Gal(L/E) \subset Gal(L/K)$.
(2) If E is also Galois over K, then every $\sigma \in Gal(L/K)$ restricts to an automorphism $\sigma|_E \in Gal(E/K)$. Moreover, this restriction homomorphism is surjective.

Proof. It is clear from the definition that $K \leq L$ normal (resp. seperable) implies that $E \leq L$ is normal (resp. seperable). Any E-automorphism fixes each element of e and hence each element of $K \leq E$, thus the inclusion in (i). For (ii): We already know that as E is normal over K then any K-automorphism of L must map E into E surjectively as E is the splitting field of some polynomials over K and thus any automorphism permutes these roots (which are the generators for E over K). By Proposition??? any K-automorphism $\sigma : E \to E$ can be extended to some K-homomorphism $\overline{\sigma} : \overline{K} \to \overline{K}$. But $\overline{\sigma}$ must permute the roots of any polynomial in $K[x]$ and in particular those for which L is the splitting field for, thus $\overline{\sigma}$ restricts to an automorphism of L. □

Proposition 3.3. Let L be a field and let G be a subgroup of $Aut(L)$. Let

$$K = L^G := \{a \in L \mid ga = a \text{ for all } g \in G\}$$

be the fixed field of G.

(1) If G is finite then $K \leq L$ is a finite Galois extension and $Gal(L/K) = G$ and $|L : K| = |G|$
(2) If $K \leq L$ is algebraic and G is not necessarily finite, then $K \leq L$ is a Galois Extension with $G \leq Gal(L/K)$.

Proof. We first show that in both case (i) or (ii), the orbit Ga is finite for all $a \in L$. This is obvious in (i). In (ii), since a is algebraic over K then there is a non-zero polynomial $f \in K[x]$ such that $f(a) = 0$. But now $f(g(a)) = 0$ for all $g \in G$ as g fixes K and hence f. Thus the orbit Ga is contained in the roots of f, which is a finite set. So now we just assume that Ga is finite for all $a \in L$. Consider the polynomial

$$f_a(x) = \prod_{\alpha \in Ga} (x - \alpha).$$

Note that g permutes these linear factors, thus $f_a(x) \in L^G[x] = K[x]$. Thus a is algebraic over K. Moreover, it now follows that L is the splitting field of $\{f_a \mid a \in L\}$, thus L is normal over K and also seperable as these factors are distinct. Thus $K \leq L$ is indeed a Galois extension. We now complete the proof of (i), thus assume from now that G is finite. To show that $K \leq L$ is a finite extension, it will be enough to find a uniform bound on intermediate fields $K \leq L' \leq L$ such that $K \leq L'$ is a finite normal extension (because we know $K \leq L$ is algebraic and thus if it is infinite then we choose finitely many elements in L such that the field they generate is arbitrarily large. The normal closure of this field is also finitely generated hence a finite extension). Now as such an L' is finite, the primitive root theorem says that $L' = K(a)$ for some $a \in L$. But then we know that the minimal polynomial of a is a divisor of $f_a(x) \in K[x]$ above, which is of degree at most $|G|$, thus $|L' : K| \leq |G|$. It follows that $|L : K| \leq |G|$, so L is indeed a finite extension. Now we use the primitive root theorem to write $L = K(\alpha)$ for some $\alpha \in L$. Observe that if $g\alpha = \alpha$ then $g = Id_L = 1_G$, thus $|G| \leq |L : K|_s = |L : K|$. This completes the proof that $|L : K| = |G|$. □

Theorem 3.4 (Fundamental theorem of Galois Theory)**.** Suppose that $K \leq L$ is a Galois extension. Let $Fields(L/K)$ denote the set of intermediate fields $K \leq E \leq L$. For a group G we let $SubGrps(G)$ denote the set of subgroups $H \leq G$. Define the maps

$$\phi : SubGrps(Gal(L/K)) \to Fields(L/K)$$

that maps

$$H \leq Gal(L/K)$$

to the fixed field L^H and

$$\psi : Fields(L/K) \to SubGrps(Gal(L/K))$$

which maps an intermiediate field $K \leq E \leq L$ to the Galois group $Gal(L/E) = Aut_E(L)$. Then

$$\phi \circ \psi = Id_{Fields(L/K)}.$$

Moreover, if the extension $K \leq L$ is finite, then

$$\psi \circ \phi = Id_{SubGrps(Gal(L/E))}$$

and thus these maps bijective and inverses of each other. Moreover, if $K \leq L$ is finite then a subgroup $H \leq Gal(L/K)$ is normal if and only if L^H is normal over K (and thus $K \leq L^H$ is Galois), in which case there is a surjective group homomorphism $Gal(L/K) \to Gal(L^H/K)$ which maps σ to $\sigma|_{L^H}$ and H is the kernel of this map, so

$$Gal(L/K)/H \cong Gal(L^H/K).$$

Proof. Let $K \leq E \leq L$ be an intermediate field, then we know that $E \leq L$ is Galois. Now let $H = Gal(L/E)$ and $E' = L^H$. Clearly $E \leq E'$ (if $a \in E$ then $h(e) = e$ for all $h \in Gal(L/E)$ and so $e \in L^H = E'$). Now suppose for contradiction that $a \in E'$ but $a \notin E$. Hence as L/E is seperable, the minimal polynomial of a over E has another root $b \neq a$ and thus there is a $h \in Aut_E(L) = H$ that maps a to b. Thus $a \notin L^H = E'$, a contradiction. This means that $E' = E$, thus showing that $\psi \circ \phi$ is the identity as claimed.

Now we assume that $K \leq L$ is finite, thus $L = K(\alpha)$ for some $\alpha \in L$ by the primitive root theorem. Clearly $G = |Gal(L/K)|$ is finite since $g \in G$ is uniquely determined by the image of α, which must be a root of the minimal polynomial of α. Choose a subgroup $H \leq Gal(L/K)$. Thus H is finite and we may apply the Proposition 3.3 to deduce that $\psi(\phi(H)) = Gal(L/L^H) = H$. Thus ϕ and ψ are inverses in when $K \leq L$ is a finite extension.

Finally, suppose that $K \leq E \leq L$ is such that E is a normal extension of K. We now wish to show that $H = Gal(L/E)$ is normal in $Gal(L/K)$. To see this, we know from Propoistion ??? that there is a surjective homomorphism $Gal(L/K) \to Gal(E/K)$ mapping $\sigma \in Gal(L/K)$ to $\sigma|E$. Observe that $g \in Gal(L/K)$ is in the kernel of this homomorphism if and only if $g|E = 1$ which happens if and only if $g(e) = e$ for all $e \in E$ which happens if and only if $g \in Gal(L/E) = H$. Thus H is the kernel of a homomorphism, thus a normal subgroup as desired.

Conversely, suppose that H is a normal subgroup of $Gal(L/K)$ and let $E = L^H$. We wish to show that L^H is normal over K. Thus we wish to show that if $\sigma : L^H \to \overline{K}$ is a K-homomorphism then $\sigma(L^H) = L^H$. To show this, let $a \in L^H$ be arbitrary and let $b = \sigma(a)$. To show $b \in L^H$ we have to show that $hb = b$ for all $h \in H$. Now extend σ to an automorphism $\sigma : L \to L$ (as L is normal over K). Then $\sigma H = H\sigma$ as H is normal in $Gal(L/K)$. Thus $h\sigma = \sigma h'$ for some $h' \in H$ and thus

$$hb = h\sigma a = \sigma h' a = \sigma a = b.$$

Thus $b \in L^H$. So $\sigma(L^H) \subset L^H$. It now remains to show the opposite inclusion. Thus suppose $a \in L^H$, then $\sigma^{-1}H = H\sigma^{-1}$ (note that $\sigma^{-1} : L \to L$ is defined as σ is an automorphism of L). Now the same argument shows that $\sigma^{-1}(a) \in L^H$ and thus $\sigma^{-1}(L^H) \subset L^H$, i.e., $L^H \subset \sigma(L^H)$. $\square$

Example 3.5. Let $\alpha = 2^{1/4}$ and let $L = \mathbb{Q}[\alpha, i]$ which is the splitting field of the polynomial $X^4 - 2$. Let as compute the Galois group $G = Gal(L/\mathbb{Q})$. Obseve that for $g \in G$ we have that

$$g(\alpha) \in \{\alpha, i\alpha, -\alpha, -i\alpha\}$$

and

$$g(i) \in \{\pm i\}$$

. Thus $|G| \leq 8$. Let us show that all 8 combinations are possible (realised by some $g \in G$). Let $\sigma : L \to L$ be the complex conjugation map, so $\sigma \in G$. Now we know that for each $k \in \{0, 1, 2, 3\}$ there exists a $g_k \in G$ such that $g(\alpha) = i^k\alpha$ (as $X^4 - 2$ is irreducible over $\mathbb{Q}$ there is a $\mathbb{Q}$-automorphism mapping any root to any other root). Now notice that $g_k \circ \sigma(\alpha) = g_k(\alpha) = i^k\alpha$ and yet $g_k \circ \sigma(i) = g_k(-i) = -g_k(i)$. Thus the elements $g_k \circ \sigma^e \in Gal(L/K)$ are all distrinct for distinct $(k, e) \in \{0, 1, 2, 3\} \times \{0, 1\}$ and so all 8 combinations are possible. Let $r \in G$ be the map given by $g(\alpha) = i\alpha$ and $g(i) = i$. Thus $g(i^k\alpha) = i^{k+1}\alpha$. So r rotates the elements $\alpha, i\alpha, i^2\alpha, i^3\alpha$ cylically. While σ is an involution that swaps $i\alpha$ with $i^3\alpha$ and fixes $\alpha, i^2\alpha$. Every element of G as of the form $r^k\sigma^e$ where $(k, e) \in \{0, 1, 2, 3\} \times \{0, 1\}$. Thus G is isomorphic to D_8 since if consider the elements $\alpha, i\alpha, i^2\alpha, i^3\alpha$ as succesive corners of a square, then r is a rotation and σ is a reflection. Note that $|L : \mathbb{Q}| = 8$ and a $\mathbb{Q}$-basis is given by

$$\{\alpha^k i^e \mid i \in \{0, 1, 2, 3\}, e \in \{0, 1\}\}.$$

Let us consider some intermediate fields and corresponding subgroups. First, consider the reflection group $\{1, \sigma\}$. The only elements of L fixed by this group are $L \cap \mathbb{R} = \mathbb{Q}[2^{1/4}]$. This subgroup is not normal and indeed $\mathbb{Q}[2^{1/4}]$ is not a normal extension of $\mathbb{Q}$. On the other hand, the rotation sugroup $\langle r\rangle$ is normal, and so the fixed field should be normal. To compute the fixed field, note that we may write each $x \in L$ as

$$x = \sum_{k=0}^{3} \lambda_k \alpha^k,$$

for some unique $\lambda_k \in \mathbb{Q}[i]$. Thus if $rx = x$ then $\lambda_k = i^k\lambda_k$, thus we must have that $x = \lambda_0 \in \mathbb{Q}[i]$. This shows that the fixed field for this rotation subgroup is $\mathbb{Q}[i]$, which indeed is normal (the splitting field of $x^2 + 1$). Observe now that each $g \in G$ restricts to an automorphism of this fixed field $\mathbb{Q}[i]$ and it restricts to the identity on $\mathbb{Q}[i]$ if and only if g is in this rotation group, giving the isomorphism

$$G/\langle r\rangle \cong Gal(\mathbb{Q}[i]/\mathbb{Q}).$$

Now consider the group of order 4 generated by the reflections σ (complex conjugation) and the map $\tau \in G$ given by $\tau(\alpha) = -\alpha$ and $\tau(i) = i$. The subgroup is abelian of order 4 as σ and τ commute. The fixed field is $\mathbb{Q}[\alpha^2] = \mathbb{Q}[\sqrt{2}]$, which is also normal over $\mathbb{Q}$ (splitting field of $X^2 - 2$).

If $E, E' \leq L$ are two subfields, then we let $E \cdot E'$ denote the subfield of L generated by these two subfields, i.e., that smallest subfield containing both. Explicitly,

$$E \cdot E' = \{\sum_{i=1}^{n} e_i e_i' \mid n \in \mathbb{Z}_{>0} e_i \in E, e_i' \in E'\}.$$

Corollary 3.6. If $K \leq L$ is a finite Galois extension such that $K \leq E, E' \leq L$ are subfields and $H = Gal(L/E)$ and $H' = Gal(L/E')$ are the corresponding galois groups. Then we have that

(1) $E \subset E'$ if and only if $H \subset H'$.
(2) $E \cdot E' = L^{H \cap H'}$
(3) $E \cap E' = L^{H''}$ where H'' is the smallest subgroup containing both H and H'.

Proof. (i) is clear from the Galois correspondence. For (ii), note that if $e \in E$ and $e' \in E$ and $h \in H \cap H'$¡ then $h(ee') = h(e)h(e') = ee'$, thus $ee' \in L^{H \cap H'}$. Hence $E \cdot E' \subset L^{H \cap H'}$. For the reverse inclusion, note that if $h \in Gal(L/E \cdot E')$ then $h(e) = e$ and $h(e') = e'$ for all $e \in E, e' \in E'$ as $e, e' \in E \cdot E'$. Thus $h \in H \cap H'$. Thus part (i) and the Galois correspondence shows that $E \cdot E' \supset L^{H \cap H'}$. For (iii), note that if $e \in E \cap E'$ then $h(e) = e$ and $h'(e) = e$ for all $h \in H$ and $h' \in H'$, thus e is fixed by all products of elements in H or H', thus fixed by all elements in H''. This shows that $E \cap E' \subset L^{H''}$. Conversely, as $H \leq H''$, then $E = L^H \supset L^{H''}$. Likewise, $E' \supset L^{H''}$ as $H' \leq H''$. Thus $E \cap E' \supset L^{H''}$. □

Example 3.7. Continuing from the previous examples, let $K = \mathbb{Q}$, L be the splitting field of $X^4 - 2$ and let $E = \mathbb{Q}[2^{1/4}]$ and $E' = \mathbb{Q}[i]$. We already saw the corresponding Galois groups are $H = \{1, \sigma\}$ and $H' = \{1, r, r^2, r^3\}$. Now $E \cdot E' = L$ but $H \cap H' = \{1\}$. The full Galois group is generated by these two groups and indeed $E \cap E' = \mathbb{Q}$ is the corresponding Galois subgroup.

Proposition 3.8. Suppose $K \leq E, E'$ are finite Galois extensions where $E, E' \leq L$ for some field L. Then $E \cdot E'$ is also a finite Galois extension over K. Also:

(1) The restriction map $\phi : Gal(E \cdot E'/E) \to Gal(E'/E \cap E')$ is a well defined isomorphism.
(2) The map
$$\psi : Gal(E \cdot E'/K) \to Gal(E/K) \times Gal(E'/K)$$
mapping g to $(g|E, g|E)$ is a well defined injective homomorphism and if $E \cap E' = K$ then ψ is also surjective.

Proof. As E and E' are Galois, it is easy to see that E and E' are splitting fields of two seperable (but not necessarily irreducible) polynomials with cofficients in K. Then $E \cdot E'$ is a splitting field of the lowest common multiple, which is also seperable and has coefficients in K. Let $g \in Gal(E \cdot E'/E)$ then for $e' \in E'$ we have that $g(e') \in E'$ as E' is normal over K. Now if $e \in E \cap E'$ then $e \in E$ and thus $g(e) = e$. This shows that g restricts to an element in $Gal(E'/E \cap E')$, so the homomorphism ϕ is well defined. If $g(e') = e'$ for all $e' \in E$ then g acts trivially on $E \cdot E'$ as is already acts trivially on E. Thus this homomorphism has trivial kernel. We now show that surjectivity of ϕ as follows

$$\begin{aligned}
(E')^{Im\phi} &= \{x \in E \cdot E' \mid x \in (E')^{Im\phi}\} \\
&= \{x \in E \cdot E' \mid x \in E' \text{ and } g(x) = x \text{ for all } g \in Gal(E \cdot E'/E)\} \\
&= \{x \in E \cdot E' \mid x \in (E \cdot E')^{Gal(E \cdot E'/E)}\} \cap E' \\
&= E \cap E'
\end{aligned}$$

where in the last equality we used the Galois correspondence. Thus by the Galois correspondence we have that $Gal(E'/(E \cap E')) = Im\phi$, thus showing that ϕ is surjective.

Now to show (2): Clearly the kernel of this map is trivial. Now suppose that $K = E \cap E'$. Let $(\sigma, \sigma') \in Gal(E/K) \times Gal(E'/K)$. Using part (1), this means that we may find extensions $\tilde{sigma} \in Gal(E \cdot E'/E')$ and $\tilde{\sigma}' \in Gal(E \cdot E'/E)$ of σ and σ' respectively. Now we claim that $\psi((\tilde{\sigma} \circ \tilde{\sigma}')) = (\sigma, \sigma')$. This is because for $e \in E$ we have

$$(\tilde{\sigma} \circ \tilde{\sigma}')|_E(e) = \tilde{\sigma}(\tilde{\sigma}e) = \tilde{\sigma}(e) = \sigma(e)$$

where we have used the fact that $\tilde{\sigma}' \in Gal(E \cdot E'/E)$ thus fixed e. A similair calculation verifies the second component of this identity, thus completing the proof of surjectivity. □

4. Cyclotomic fields

We say that $\zeta \in \mathbb{C}$ is a root of unity if $\zeta^n = 1$ for some $n > 0$. We say that ζ is a primitive n-th root of unity if $\zeta^n = 1$ but $\zeta^m \neq 1$ for all $0 < m < n$. In this section, we wish to understand cylcomotomic fields, i.e., field of the form $\mathbb{Q}[\zeta_n]$ where ζ.

Lemma 4.1. The field $\mathbb{Q}[\zeta_n]$ is a finite Galois extension of $\mathbb{Q}$. The natural homomorphism

$$Gal(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \to Aut(U_n) \cong (\mathbb{Z}/n\mathbb{Z})^*$$

given by restriction to U_n is injective, thus

$$|Gal(\mathbb{Q}[\zeta_n]/\mathbb{Q})| = |\mathbb{Q}[\zeta_n] : \mathbb{Q}|$$

divides $\phi(n)$.

Proof. It is a Galois extension as it is the splitting field of $X^n - 1$. Note that any $\mathbb{Q}$-automorphism must permute the roots of unity. Moreover, this permutation induces an isomorphism of the multiplicative group $U_n \cong \mathbb{Z}/n\mathbb{Z}$. The endomorphisms of $\mathbb{Z}/n\mathbb{Z}$ are all of the form $x \mapsto ax$, and these are isomorphisms if and only if $gcd(a, n) = 1$. □

We now strengthen the Lemma by showing that in fact $|\mathbb{Q}[\zeta_n] : \mathbb{Q}| = \phi(n)$.

Theorem 4.2. The field $\mathbb{Q}[\zeta_n]$ is a finite Galois extension of $\mathbb{Q}$. The natural homomorphism

$$Gal(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \to Aut(U_n) \cong (\mathbb{Z}/n\mathbb{Z})^*$$

given by restriction to U_n is an isomorphism, thus

$$|Gal(\mathbb{Q}[\zeta_n]/\mathbb{Q})| = |\mathbb{Q}[\zeta_n] : \mathbb{Q}| = \phi(n).$$

Proof. Let ζ_n be a primitive n-th root of unity. Let $f(x) \in \mathbb{Q}[x]$ be the monic minimal polynomial of ζ_n. We claim that any other primitive n-th root of unity is also a root of f. For this, it is enough to show that for primes p not dividing n we have that ζ_n^p is also a root of f (as any other primitive root of unity is of the form ζ_n^m for some $gcd(m, n) = 1$, thus can be obtained by successively raising to such prime powers). Fix such a prime p. Suppose for contradiction that $f(\zeta_n^p) \neq 0$. Now as

$$X^n - 1 = f(X)h(X)$$

for some $h(X) \in \mathbb{Q}[x]$ monic, we get that $f(X)$ and $h(X)$ is monic (as $f(X)$ is monic) and thus by the Gauss Lemma we have that $f(X), h(X) \in \mathbb{Z}[X]$. As $f(\zeta_n^p) \neq 0$ we have that $h(\zeta_n^p) = 0$. Thus $h(X^p) = f(X)g(X)$ for some $g(X) \in \mathbb{Q}[x]$, which by the same argument must also be monic in $\mathbb{Z}[x]$. We reduce this equation modulo p to obtain

$$(\overline{h}(x))^p = \overline{h}(x^p) = \overline{f}(x)\overline{g}(x)$$

in $\mathbb{F}_p[x]$. Thus $\overline{f}(x)$ and $\overline{h}(x)$ must share a zero in some algebraic closure of $\mathbb{F}_p$. Thus $x^n - 1 = \overline{hf}$ must have multiple zeros thus is not seperable. This however is only possibly if p divides n (as such a zero α would have to vanish on the derivative, i.e., $n\alpha^{n-1} = 0$ which implies that $\alpha^{n-1} = 0$ if p does not divice n, so $\alpha = 0$, but 0 is not a root) , a contradiction.

Thus we have shown that $f(x)$ has at least $\phi(n)$ roots, but as $deg f$ divices $\phi(n)$, we have that the degree is exactly $\phi(n)$. So the Galois group also has $\phi(n)$ elements thus the injective homomorphism is an isomorphism. □

Proposition 4.3. If $gcd(n, m) = 1$ then

$$\mathbb{Q}[\zeta_n, \zeta_m] = \mathbb{Q}[\zeta_{nm}]$$

and

$$\mathbb{Q}[\zeta_n] \cap \mathbb{Q}[\zeta_m] = \mathbb{Q}.$$

and there is an isomorphism

$$Gal(\mathbb{Q}[\zeta_{nm}]/\mathbb{Q}) \to Gal(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \times Gal(\mathbb{Q}[\zeta_m]/\mathbb{Q})$$

mapping $\sigma \in Gal(\mathbb{Q}[\zeta_{nm}]/\mathbb{Q})$ to the pair pair $(\sigma|_{\mathbb{Q}[\zeta_n]}, \sigma|_{\mathbb{Q}[\zeta_m]})$.

Proof. A simple calculation shows that if $gcd(n, m) = 1$, then $\zeta_n\zeta_m$ is a primitive nm-th root of unity, thus the first equality. Now let $L = \mathbb{Q}[\zeta_n] \cap \mathbb{Q}[\zeta_m]$. Observe that

$$\phi(n)\phi(m) = \phi(nm) = |\mathbb{Q}[\zeta_{nm} : \mathbb{Q}| = |\mathbb{Q}[\zeta_{nm} : \mathbb{Q}[\zeta_n]|\phi(n)$$

and thus $\mathbb{Q}[\zeta_n, \zeta_m] : \mathbb{Q}[\zeta_n]| = \phi(m)$. This means that ζ_m has degree $\phi(m)$ over the field $\mathbb{Q}[\zeta_n]$. Thus ζ_m has degree at least $\phi(m)$ over the smaller field L, i.e., $|\mathbb{Q}[\zeta_m] : L| \geq \phi(m)$. However

$$\phi(m) = |\mathbb{Q}[\zeta_m] : \mathbb{Q}| = |\mathbb{Q}[\zeta_m] : L||L : \mathbb{Q}| \geq \phi(m)|L : \mathbb{Q}|$$

and thus $L = \mathbb{Q}$. Finally, the last claim follows from Proposition ??. □

We let $\phi_n(x) \in \mathbb{Q}[x]$ denote the minimal polynomial of ζ_n over $\mathbb{Q}$. Note that in the proof of ??? we saw that

$$\phi_n(x) = \prod_{\zeta} (x - \zeta)$$

is a seperable polynomial of degree $\phi(n)$ such that each of the $\phi(n)$ primitive roots of unity are roots, thus the the product runs over the $\phi(n)$ different primitive roots of unity.

Definition 4.4. We call $\phi(n)$ the n-th cyclotomic polynomial.

Proposition 4.5. The cyclotomic polynomial $\phi_n(x)$ is a monic polynomial in $\mathbb{Z}[x]$. We have the identity

$$x^n - 1 = \prod_{d|n} \phi_d(x).$$

Proof. As $\phi_n(x)$ divides $x^n - 1$ in $\mathbb{Q}[x]$ and is monic, we have that $x^n - 1 = \phi_n(x)h(x)$ for some $h(x) \in \mathbb{Q}[x]$ also monic. The Gauss lemma now shows that these polynomials must have integer coefficients. Finally, the identity follows because each root of unity is a primitive root of unity of some unique divisor d of n. □

We can use this identity to compute cyclotomic polynomials recursively, starting with the base case

$$\phi_p(x) = \frac{x^p - 1}{x - 1} = 1 + x + \cdots + x^{p-1}$$

for primes p.

5. Norm and Trace

Let L/K be a finite field extension. For $a \in L$ we can define $Tr_{L/K}(a) \in L$ to be the trace of the K-linear map $\phi_a : L \to L$ given by $\phi_a(x) = ax$. We define the *norm*

$$N_{L/K}(a) = \det(\phi(a))$$

to be the determinant of this K-linear map.

Proposition 5.1. Let K be a field and let $\alpha \in \overline{K}$ be algebraic over. Then the characteristic polynomial of the K-linear map $\phi_\alpha : K(\alpha) \to K(\alpha)$ is precisely the minimal polynomial of α over K.

Proof. We know that the degree of the minimal polynomial is $|K(\alpha) : K|$. Note also that for $P \in K[x]$ we have that $P(\phi_\alpha) : K(\alpha) \to K(\alpha)$ is the zero map if and only if $P(\alpha) = 0$, thus ϕ_α has the same minimal polynomial as α over K. But $|K(\alpha) : K|$ is the dimension of $K(\alpha)$ of K, thus the minimal polynomial coincides with the characteristic polynomial. $\square$

Thus if $P(x) = \prod_{i=1}^{n}(X - \alpha_i)$ is the minimal polynomial of α, then

$$Tr_{K(\alpha):K}(\alpha) = \sum_{i=1}^{n} \alpha_i$$

and

$$N_{K(\alpha):K}(\alpha) = \prod_{i=1}^{n} \alpha_i.$$

In case $P(x)$ is not separable, the α_i repeat with some multiplicity q and we may write

$$P(x) = \prod_{\rho}(X - \rho(\alpha))^q$$

where the product is over all $\rho \in Hom_K(K(\alpha) : \overline{K})$. Note that

$$q = |K(\alpha) : K||K(\alpha) : K|_s^{-1}.$$

Thus in this case,

$$Tr_{K(\alpha)/K} = q \sum_{\rho} \rho(\alpha)$$

and

$$N_{K(\alpha)/K}(\alpha) = \left(\prod_{\rho} \rho(\alpha)\right)^q.$$

Proposition 5.2. If L/K is a finite extension and α in K, then

$$Tr_{L/K}(\alpha) = |L : K(\alpha)| Tr_{K(\alpha)/K}(\alpha)$$

and

$$N_{L/K} = \left(N_{K(\alpha)/K}(\alpha)\right)^{|L:K(\alpha)|}.$$

Proof. Let $y_1, \dots, y_s \in L$ be a basis for L over $K(\alpha)$, where $s = |L : K(\alpha)|$. Observe that

$$L = \bigoplus_{i=1}^{S} K(\alpha) y_i$$

splits as a direct sum of K-vector spaces of dimension $|K(\alpha) : K|$. Writing $\phi_\alpha : K(\alpha) \to K(\alpha)$ and $\psi_\alpha : L \to L$ to be the multiplication by α maps (both viewed as K-linear maps on K-vector spaces), we see that for by writing each $x \in L$ as $x = (x_1, \ldots, x_s)$ with respect to this decomposition we have

$$\psi_\alpha x = (\phi_\alpha x_1, \ldots, \phi_\alpha x_s)$$

and thus

$$Tr(\psi_\alpha) = s \cdot Tr(\phi_\alpha)$$

and

$$\det(\psi_\alpha) = \det(\phi_\alpha)^s,$$

as required. □

Theorem 5.3. Let L/K be a finite extension and let $\alpha \in K$. Let $r = |L : K|_s$ and let $\sigma_1, \ldots, \sigma_r$ be the distinct elements of $Hom_K(L, \overline{K})$, i.e., the homomorphisms $L \to \overline{K}$ that fix K. Then

$$Tr_{L/K}(\alpha) = \frac{|L : K|}{|L : K|_s} \sum_{i=1}^{r} \sigma_i(\alpha)$$

and

$$N_{L/K}(\alpha) = \left(\prod_{i=1}^{r} \sigma_i(\alpha) \right)^{\frac{|L:K|}{|L:K|_s}}.$$

Proof. For each $\rho \in Hom_K(K(\alpha), \overline{K})$, fix an extension $\overline{\rho} : \overline{K} \to \overline{K}$ of ρ. Note that each $\sigma \in Hom_K(L, \overline{K})$ may be uniquely written by (the proof of) Proposition CITE in the form $\overline{\rho} \circ \tau$ where $\tau \in Hom_{K(\alpha)}(L : \overline{K})$. As such a τ must fix α we have $(\overline{\rho} \circ \tau)(\alpha) = \rho(\alpha)$ and so we can write

$$\begin{aligned} Tr_{L/K}(\alpha) &= |L : K(\alpha)| \frac{|K(\alpha) : K|}{|K(\alpha) : K|_s} \sum_{\rho} \rho(\alpha) \\ &= |L : K(\alpha)| \frac{|K(\alpha) : K|}{|K(\alpha) : K|_s} \frac{1}{|L : K(\alpha)|_s} \sum_{\tau} \sum_{\rho} (\overline{\rho} \circ \tau)(\alpha) \\ &= \frac{|L : K|}{|L : K|_s} \sum_{\sigma} \sigma(\alpha), \end{aligned}$$

where a summation over σ is over $\sigma \in Hom_K(L, \overline{K})$, a summation over τ is over $\tau \in Hom_{K(\alpha)}(L : \overline{K})$ and a summation over ρ is over $\rho \in Hom_K(K(\alpha), \overline{K})$.

The proof for the norm is similair. □

Note that as a corollary, we get that if L/K is a finite Galois extension then

$$Tr_{L/K}(\alpha) = \sum_{\sigma \in Gal(L/K)} \sigma(\alpha)$$

and

$$N_{L/K}(\alpha) = \prod_{\sigma \in Gal(L/K)} \sigma(\alpha).$$

Corollary 5.4. Let L/K be a finite Galois extension, then

$$Tr_{L/K} \circ \sigma = TrL/K$$

and

$$N_{L/K} \circ \sigma = N_{L/K}.$$

If L/K is a finite extension, we can define a bilinear form $tr : L \times L \to K$ given by $tr(x, y) = Tr_{L/K}(xy)$. Note that if L/K is finite Galois, then the previous result says that the Galois group preserves this bilinear form.

Proposition 5.5. Let L/K be a finite extension. Then L/K is seperable if and only if $Tr_{L/K} : L \to K$ is a non-trivial (hence surjective) linear functional. If L/K is seperable, then the bilinear form tr is non-degenerate, i.e., if $x \in L$ is such that $tr(x, y) = 0$ for all $y \in Y$, then $y = 0$.

Proof. If L/K is not seperable, then $q = |L : K||L : K|_s^{-1}$ must be a power of $charK \neq 0$ (see Proposition 2.8 and Theorem 2.9). Thus we get that $Tr_{L/K} = q\sum_\sigma \sigma$ is identically zero. Now assume L/K is seperable, thus $q = 1$. Now the elements $\sigma \in Hom_K(L, \overline{K})$ are distinct characters (multiplicative maps) $L^* \to \overline{K}^*$, thus linearly independent and so $Tr_{L/K}$ cannot be identitically zero. It now follows immediately that the bilinear form is non-degenerate, for if $x \in L$ with $x \neq 0$ and $tr(xy) = 0$ for all $y \in L$, then as y is a field we have $xL = L$ and so $Tr_{L/K}$ vanishes on L, but we have just shown that it does not. ☐

6. Cyclic Galois Extensions

Theorem 6.1 (Hilbert 90)**.** Let L/K be a finite cyclic Galois extension and let $\sigma \in Gal(L/K)$ be a generator. For $b \in L$, we have that $N_{L/K}(b) = 1$ if and only if there exists non-zero $a \in L$ such that

$$b = a(\sigma(a))^{-1}.$$

Proof. If $b = a(\sigma(a))^{-1}$ for some non-zero $a \in L$ then by multiplicativity of $N_{L/K}$ and invariance under σ (Corollary 5.4) we get that

$$N_{L/K}(b) = N_{L/K}(a)\left(N_{L/K}(\sigma(a))\right)^{-1} = N_{L/K}(a)(N_{L/K}(a))^{-1} = 1.$$

Conversely, suppose now that $N_{L/K}(b) = 1$. Let

$$u_j = \prod_{j=0}^{i} \sigma^i(b).$$

We note that u_j is n-periodic, where $n = Gal(L/K)$ as

$$1 = N_{L/K}(b) = \prod_{\sigma \in Gal(L/K)} \sigma(b) = u_{i+n}u_i^{-1}$$

and furthermore all u_j have norm 1 thus are non-zero. It follows, from the linear independence of the maps in $Gal(L/K)$, that

$$\sum_{i=0}^{n-1} u_j\sigma^j$$

is not the zero function. Thus there exists a $c \in L$ for which

$$a := \sum_{i=0}^{n-1} u_j\sigma^j(c) \neq 0.$$

Now we have

$$\begin{aligned} b\sigma(a) &= \sum_{i=0}^{n-1} b\sigma(u_i)\sigma^{i+1}(c) \\ &= \sum_{i=0}^{n-1} u_{i+1}\sigma^{i+1}(c) \\ &= \sum_{i=1}^{n} u_i\sigma^i(c) \\ &= a \end{aligned}$$

where in the final equality we used the periodicity of the summands, $\sigma^{i+n} = \sigma^i$ and $u_{i+n} = u_i$. □

Example 6.2. Consider the degree two extension $\mathbb{C}/\mathbb{R}$. It is Galois with Galois group generated by $\sigma(z) = \overline{z}$ being complex conjugation. We have

$$N_{\mathbb{C}/\mathbb{R}}(b) = b\sigma(b) = |b|^2.$$

On the other hand we have

$$a\sigma(a)^{-1} = \frac{a^2}{\bar{a}a} = \frac{a^2}{|a^2|},$$

which has norm 1. Thus b takes the form $a\sigma(a)^{-1}$ if and only if it has norm 1.

Now consider instead the extension $\mathbb{Q}[i]/\mathbb{Q}$. It is also Galois of degree 2 generated by $\sigma : \mathbb{Q}[i] \to \mathbb{Q}[i]$ being complex conjugation. Again

$$N_{\mathbb{Q}[i]/\mathbb{Q}[]}(x + iy) = |x + iy|^2 = x^2 + y^2.$$

Thus Hilbert 90 shows gives an explicit paramatrization of the rational solutions $x+iy \in \mathbb{Q}[i]$ to $x^2+y^2 = 1$, they are all of the form

$$\frac{u + iv}{u - iv} \quad u, v \in \mathbb{Q}.$$

This theorem allows us to relate cyclic Galois extensions with radical extensions (adjoining an nth-root).

Proposition 6.3. Let L/K be a cyclic Galois extension of degree n such that K contains a primitive n-th root of unity. Then $L = K(a)$ for some non-zero $a \in L$ with $a^n \in K$ and minimal polynomial $X^n - a^n \in K[x]$.

Proof. Let ζ be a primitive n-th root of unity. Note that $N_{L/K}(\zeta) = \zeta^n = 1$ and thus by Hilbert 90 we have a non-zero $a \in L$ such that $\sigma(a) = \zeta a$, where $\sigma \in Gal_{L/K}$ is a generator. Now observe that $\sigma(a^n) = \sigma(a)^n = \zeta^n a^n = a^n$. Thus a^n is fixed by σ, which means that $a^n \in K$. Thus a is a root of $g(x) = x^n - a^n \in K[x]$. Observe that $a_i := \sigma^i(a) = \zeta^i a$ and thus these elements are distinct as ζ is primitive. If $f(x) \in K[x]$ is a minimal polynomial for a, then $f(\sigma^i(a)) = f(a) = 0$ and thus $f(x)$ has at least n distinct roots, thus is of degree at least n. As $f(x)|g(x)$ we have that $g(x) = f(x)$ and so $x^n - a^n \in K[x]$ is indeed the minimal polynomial of a over K. This also means that $|K(a) : K| = n = |L : K|$, thus $L = K(a)$. □

Example 6.4. Let us show that the assumption that K contains a primitive n-th root of unity cannot be dropped. Let $L = \mathbb{Q}(x_1, x_2, x_3)$ and let $\sigma(x_i) = x_{i+1}$ be a cyclic permutation of the variables and let $K = L^G$ where G is the cyclic group generated by σ. Then we know that L/K is a Galois extension with Galois group G (use Proposition 3.3 as G is finite). Now we show that L cannot have a primitive n-th root of unity. The easiest way of showing this is to embedd L into $\mathbb{R}$ via a field homomorphism (choose three algebraically independent numbers). Thus if $a^3 = 1$ in L then $a = 1$. Now if $L = K(a)$ for some $a \in L$ with $a^3 \in K$ then $(\sigma(a))^3 = a^3$ and so $a = \sigma(a)$, thus $a \in K$ which means that $K = L$, a contradiction.

Proposition 6.5. Let K be a field with a primitive n-th root of unity and suppose that L is a field with $L = K(a)$ for some $a \in L$ such that $a^n \in K$. Then L/K is a finite cyclic Galois extension with degree $d = [L : K]$ where $d|n$ and a has minimal polynomial $x^d - a^d \in K[x]$.

Proof. We assume $a \neq 0$, otherwise the result is obvious. Now let $\zeta \in K$ be a primitive n-th root of unity and let $a_j = \zeta^j a$. Observe that $a_0, \dots, a_{n-1}$ must be distinct ($a \neq 0$ and ζ is a primitive n-th root) and satisfy that $a_j^n - a^n = 0$. Thus $X^n - a^n$ has n distinct roots in L, which shows that $L = K(a)$ is seperable over K, thus L/K is Galois. Now the Galois group permutes that a_j, thus for $\sigma \in Gal_{L/K}$ we have

$$\phi(\sigma) := \frac{\sigma(a)}{a} \in U \subset K^*$$

where U is the group of order n generated by ζ. We claim that $\phi : Gal(L/K) \to U$ is a group homomorphism. This is easy to see as

$$\phi(\sigma_1)\phi(\sigma_2) = a^{-1}\sigma_1(a)\phi(\sigma_2) = a^{-1} \cdot \sigma_1(a \cdot \phi(\sigma_2)) = a^{-1}\sigma_1(\sigma_2(a)).$$

Moreover, this mapping is injective as $L = K(a)$ so any K-homomorphism is determined by a. Thus $Gal(L/K)$ is isomorphic to a subgroup of U, thus is cyclic of degree $d|n$. Let σ be a generator, thus σ permutes those elements a_j where j is divisibly by $r = n/d$, therefore

$$g(x) = \prod_{j=0}^{d-1}(X - \sigma^j(a)) = \prod_{j=0}^{d-1}(X - \zeta^{rj}a)$$

must be an element of $K[x]$ (as it is fixed by σ) thus it is the minimal polynomial of a, as it has degree $d = Gal(L/K) = |L : K|$. It now remains to show that $g(x) = x^d - a^d$. This must be the case since the right hand side has at most d roots while $g(x)$ has exactly d distinct roots of $x^d - a^d$. $\square$

Example 6.6. If a is a primitive 12-th root of unity in $\mathbb{C}$, that $L = \mathbb{Q}(a)$ is the splitting field of $X^{12} - a$. Thus L is Galois over $\mathbb{Q}$. However $Gal(L/\mathbb{Q})$ is not cyclic, it is isomorphic to $(\mathbb{Z}/4)^* \times (\mathbb{Z}/3)^* = \mathbb{Z}/2 \times \mathbb{Z}/2$. This does not contradict the theorem, as $\mathbb{Q}$ does not have a primitive 12-th root of unity.

Example 6.7. $\mathbb{Q}(2^{1/4})$ not even Galois over $\mathbb{Q}$, again, not a contradiction as $\mathbb{Q}$ does not have a primite 4-th root of unity. On the other hand $L = \mathbb{Q}[i](2^{1/4})$ is a Galois extension of $K = \mathbb{Q}[i]$ and the Galois group is cyclic of order 4.

7. Solvable groups

Definition 7.1. Let G be a group. Define the commutator $[a, b] = aba^{-1}b^{-1}$. For subgroups $H, H' \leq G$ define the commutator subgroup $[H, H']$ to be the group generatated by $\{[h, h'] \mid h \in Hh' \in H\}$.

Proposition 7.2. Let G be a group. Then $[G, G]$ is a normal subgroup of G and $G/[G, G]$ is abelian. Moreover, if $H \leq G$ is a normal subgroup of G such that G/H is abelian, then $[G, G] \leq H$. Thus $[G, G]$ is the minimal normal subgroup $H \leq G$ such that G/H is abelian.

Proof. Note that

$$g[a, b]g^{-1} = gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} = [gag^{-1}, gbg^{-1}.$$

Thus the generating set is conjugation invariant. Also, note that

$$[a, b]^{-1} = (aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1} = [b, a]^{-1}$$

thus any element in $[G, G]$ can be written as a product of commutators (no need to consider inverses of commutators). Thus since the commutators are conjugation invariant, $[G, G]$ is normal. Now let $\pi : G \to G/[G, G]$ be the quotient map. Then

$$[\pi(g), \pi(h)] = \pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = \pi([g, h]) = 1$$

thus all elements of this quotient group commute. Finally, suppose that H is a normal subgroup of G with G/H abelian. If $\pi : G \to G/H$ then $1 = [\pi(g), \pi(h)] = \pi([g, h])$ and so $[g, h] \in ker\ \pi = H$. Thus $[G, G] \subset H$. □

Definition 7.3. Define $D^0G = G, DG = [G, G]$ and $D^{i+1}G = D(D^iG) = [D^iG, D^iG]$. Thus we have a series

$$G \geq D^1G \geq D^2G \geq \dots$$

where the successive quotients are normal. We say that G is *solvable* if $D^nG = \{1\}$ for some $n \geq 0$.

Proposition 7.4. A group G is solvable if and only if there is a finite sequence

$$G_0 = G \geq G_1 \geq G_2 \dots \geq G_n = \{1\}$$

such that G_{i+1} is normal in G_i and G_i/G_{i+1} is abelian.

Proof. Given such a normal series with abelian quotients, we can show that $D^iG \leq G_i$ by induction as follows. Clearly true for $i = 0$ as we have equality. Suppose now that $D^iG \leq G_i$ for some $i < n$. Then $D^{i+1}G = D(D^iG) \leq DG_i = [G_i, G_i]$. But since G_i/G_{i+1} is abelian, we have by the previous Proposition that $[G_i, G_i] \leq G_{i+1}$. Thus $D^{i+1}G \leq G_{i+1}$. The converse is clear. □

Example 7.5. The symmetric group S_5 is not solvable. We will show this by showing that $[S_5, S_5] = A_5$ and $[A_5, A_5] = A_5$, thus the series will never reach the trivial subgroup. Let $sign : S_5 \to \{-1, 1\}$ denote the sign map. It is clear that $[S_5, S_5] \leq A_5$ as

$$sign([g, h]) = sign(g)sign(h)sign(g)^{-1}sign(h)^{-1} = 1.$$

For the reverse inclusion, we note that A_5 can be generated by 3-cycles as for distinct $a, b, c, d \in \{1, \dots, 5\}$ we have

$$(a\ b)(b\ c) = (a\ b\ c)$$

and

$$(a\ b)(c\ d) = (a\ b\ c)(b\ c\ d)$$

, thus it is enough to show that any 3-cycle can be expressed as a commutator. To see this, note that

$$(a\ b\ c) = (a\ c)(b\ c)(a\ c)^{-1}(b\ c)^{-1}.$$

This shows that $[S_5, S_5] = A_5$. We now show that $[A_5, A_5] = A_5$. Thus it will be enough to express a 3 cycle as a commutator of 3 cycles. This is because

$$(1\ 2\ 3) = (1\ 2\ 4)(1\ 3\ 5)(1\ 2\ 4)^{-1}(1\ 3\ 5)^{-1}.$$

Thus S_5 is not solvable.

Proposition 7.6. Let $H \leq G$. Then if G is solvable then so is H. If H is normal in G, then G is solvable if and only if H and G/H are solvable.

Proof. First part is obvious as $D^iH \leq D^iG$. Now suppose that H is normal in G. Then the surjective map $\pi : G \to G/H$ satisfies that $D^i(\pi(G)) = \pi(D^i(G))$, thus if G is solvable then so is H. Now if G/H is solvable, then

$$\pi(D^n(G)) = D^n(\pi(G)) = \{1\}$$

and thus $D^n(G) \leq H$ for some $n \geq 0$. Now if H is solvable, then

$$D^{n+m}(G) = D^m(D^n(G)) \leq D^m H = \{1\}$$

for some $m \geq 0$ and thus G is solvable. □

8. Solvability by radicals

Definition 8.1. Let L/K be a field extension in characteristic 0. We say that L/K is *solvable by radicals* if $L \leq E$ for some field E such that there is a sequence

$$E_0 = K \leq E_1 \leq E_2 \ldots \leq E_n = E$$

of field extensions where $E_{i+1} = E_i(a)$ for some $a \in E_{i+1}$ such that $a^n \in E_i$ for some $n \geq 1$.

Definition 8.2. Let L/K be a finite field extension. Then we say that L/K is solvable if there is a finite extension $L \leq \tilde{L}$ such that $\tilde{L}/K$ is Galois with solvable Galois group.

Proof. If L/K is a finite Galois extension that is solvable. Then $Gal(L/K)$ is solvable.

□

Proof. As L/K is solvable, there is a Galois extension $\tilde{L}/K$ with $K \leq L \leq \tilde{L}$ such that $G = Gal(\tilde{L}/K)$ is solvable. But as L/K is assumed to be Galois, we know that $Gal(L/K)$ is a quotient of G and thus also solvable. □

Lemma 8.3. Let L/K be a finite field extension in characteristic 0. Suppose that $K \leq F$ is a field extension and that $L \leq \overline{F}$ (the algebraic closure of F).

(1) If L/K is solvable, then FL/F is solvable.
(2) If L/K is solvable by radicals, then FL/F is solvable by radicals.

Proof. Proof of (1): As L/K is solvable, there is a field extension $L \leq \tilde{L} \leq \overline{F}$ such that $K \leq \tilde{L}$ is finite Galois with $Gal(\tilde{K}/K)$ solvable. Now $\tilde{L} = K[A]$ where A is the set of roots of some finite set of polynomials of $K[x]$ (by definition of normal). Now $F\tilde{L} = FK[A] = F[A]$, thus $F\tilde{L}$ is Galois over F and $F\tilde{L} \geq F \geq K$. Thus it is enough to show that $Gal(F\tilde{L}/F)$ is solvable. Note that there is a restriction homomorphism

$$Gal(F\tilde{L}/F) \to Gal(\tilde{L}/K)$$

which is well defined as $\tilde{L}/K$ is normal and any σ that fixes F must fix also $K \leq F$. Note also that this map is injective since if $\sigma \in Gal(F\tilde{L}/F)$ acts trivially on $\tilde{L}$ then it also acts trivially on $F\tilde{L}$ as it already fixed F. Thus $Gal(F\tilde{L}/F)$ is a subgroup of a solvable group, thus solvable.

Now to prove (2): If L/K is solvable by radicals, then there is a field $L \leq E \leq \overline{F}$ such that $E = K(a_1, \ldots, a_n)$ where $a_i^{n_i} \in K(a_1, \ldots, a_{i-1})$ for some $n_i \geq 1$. It follows that $F \leq FL \leq FE$ is solvable by radicals since

$$FE = FK(a_1, \ldots, a_n) = F(a_1, \ldots a_n)$$

with $a_i^{n_i} \in K(a_1, \ldots, a_{i-1}) \subset F(a_1, \ldots, a_{i-1})$. □

Proposition 8.4. Let $K \leq L \leq M$ be finite field extensions in characteristic 0. Then $K \leq M$ is solvable if and only if $K \leq L$ and $L \leq M$ are solvable.

Proof. We first prove (1): Suppose $K \leq M$ is solvable. Thus there is an $M' \geq M$ finite set so that $K \leq M'$ is galois and $Gal(K/M')$ is solvable. As M' is an extension of L, this means that L/K is also solvable. Finally M'/L is also Galois with $Gal(M'/L)$ a subgroup of $Gal(M/K)$, thus solvable. Therefore M/L is solvable.

Conversely, suppose that $K \leq L$ and $L \leq M$ is solvable. We first show that we may reduce to the case where $K \leq L$ and $L \leq L$ are also Galois with solvable Galois group: We know there is a finite extension $L' \geq L$ such that L'/K is Galois and solvable. We now apply the previous lemma with our L playing the role of K (in the theorem), our L' playing the role of F and M playing the role of L. Thus by assumption M/L is solvable, so the theorem applies to show that ML'/L' is solvable. By definition, this means we have an $M' \geq ML'$ such that M'/L' is Galois with solvable Galois group. Thus we have a chain $K \leq L' \leq M'$ of Galois extensions with Galois groups. Thus if we can show that M'/K is solvable, then M/K is solvable as desired.

Thus we have reduced to the case that $K \leq L$ is Galois with solvable Galois group and $L \leq M$ is Galois with solvable Galois group and we wish to show that $K \leq M$ is solvable (not necessarily Galois). Now let $M \leq M'$ be the normal closure of M/K. By construction, recall the construction of M' as follows: Write $M = K(a_1, \ldots, a_q)$ for some $a_i \in M$. Now $M' = K[\mathcal{A}]$ where

$$\mathcal{A} = \{a \in \overline{K} \mid a \text{ is the root of the minimal polynomial of some } a_i\}.$$

Thus M'/K is finite and

$$M' = \prod_{\sigma \in Gal(M'/K)} \sigma(M)$$

is the field generated by all the $\sigma(M)$ as σ ranges in $Gal(M'/K)$. As

$$Gal(L/K) \cong Gal(M'/K)/Gal(M'/L)$$

it suffices to show that $Gal(M'/L)$ is solvable as $Gal(L/K)$ is solvable. Now define a map

$$Gal(M'/L) \to \prod_{\sigma \in Gal(M'/K)} Gal(\sigma(M)/L)$$

given by restricting $g \in Gal(M'/L)$ to the tuple of restrictions $g|_{\sigma(M)}$. Let us show that this is well defined. We have to show that $g\sigma(M) = \sigma(M)$. Note that $gM = M$ as M/L is normal. Now since $Gal(M'/L)$ is normal we have that $\sigma^{-1}g\sigma \in Gal(M'/L)$ and so again $\sigma^{-1}g\sigma M = M$, thus $g\sigma M = \sigma M$ as desired. This shows that this map is well defined. Now we show that it is injective. Thus suppose that g fixes each element of each σM. But then g fixes each element of M' as these fields σM generated M'. Thus $Gal(M'/L)$ is a subset of this product of Galois groups, it thus suffices to show that each $Gal(\sigma M/L)$ is solvable (because a finite product of solvable groups is solvable). We know $Gal(M/L)$ is solvable and we claim $Gal(\sigma(M)/L)$ is isomorphic to it via the isomorphism $g \mapsto \sigma g \sigma^{-1}$. To see that it is well defined, note that $\sigma g \sigma^{-1} \sigma M = \sigma g M = \sigma M$ and that if $\ell \in L$, then $\sigma^{-1}\ell \in L$ since L/K is normal and thus g fixes $\sigma^{-1}\ell$, which gives

$$\sigma g \sigma^{-1} \ell = \sigma \sigma^{-1} \ell = \ell$$

, thus $\sigma g \sigma^{-1}$ does indeed fix L. Thus shows that this map is well defined, and it is easy to see that it is an isomorphism. □

Theorem 8.5. If $K \leq L$ is a finite field extension in characteristic 0 that is solvable by radicals, then $K \leq L$ is solvable (i.e., $Gal(L', K)$ is solvable for some $L' \geq L$ with $K \leq L'$ a finite galois extension).

Proof. By the previous proposition, it suffices to show that if $K(a)/K$ is solvable where $a \in L$ is such that $a^n \in K$ for some n. Now consider the field extension $K \leq K(\zeta) \leq K(\zeta)(a)$ where ζ is a primitive n-th root of unity. Thus $K(\zeta)/K$ is solvable (it is a Galois extension with abelian galois group) while $K(\zeta)(a)/K(\zeta)$ is a finite Galois extension with cylic Galois group by Proposition 6.5. Thus by the previous proposition, $K \leq K(\zeta)(a)$ is solvable, thus $K(a)/K$ is solvable.

□

Theorem 8.6. There is no quintic formula. More precisely, let $L = F[x_1, \dots, x_5]$, where F is a field of characteristic zero, and let $K = L^{S_5}$ where S_5 acts on $\{x_1, \dots, x_5\}$ be permutatations. Then the extension L/K is not solvable by radicals.

Proof. The Galois Group of L/K is S_5, hence not solvable. □