Homework 1, Problem 2 & 4

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1 Problem 2

(a) Let the matrix $A = [\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4]$, then

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
A^T A &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{x}_1^T\boldsymbol{x}_1 & \boldsymbol{x}_1^T\boldsymbol{x}_2 & \boldsymbol{x}_1^T\boldsymbol{x}_3 & \boldsymbol{x}_1^T\boldsymbol{x}_4 \\ & \boldsymbol{x}_2^T\boldsymbol{x}_2 & \boldsymbol{x}_2^T\boldsymbol{x}_3 & \boldsymbol{x}_2^T\boldsymbol{x}_4 \\ & \text{sym.} & \boldsymbol{x}_3^T\boldsymbol{x}_3 & \boldsymbol{x}_3^T\boldsymbol{x}_4 \\ & & & \boldsymbol{x}_4^T\boldsymbol{x}_4 \end{bmatrix}
\end{aligned}
$$

Therefore, we see that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are orthogonal to each other; and $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$ are orthogonal.

(b) There are two independent vectors in $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4\}$, choosing any two from them, along with $\boldsymbol{x}_3$, which is independent with all of them, form a basis. All the bases are:

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\},\ \{\boldsymbol{x}_1, \boldsymbol{x}_4, \boldsymbol{x}_3\},\ \{\boldsymbol{x}_2, \boldsymbol{x}_4, \boldsymbol{x}_3\}$$

(c)

$$A \begin{bmatrix} \dfrac{1}{\lambda_1}\boldsymbol{x}_1 & \dfrac{1}{\lambda_2}\boldsymbol{x}_2 & \dfrac{1}{\lambda_3}\boldsymbol{x}_3 \end{bmatrix} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3]$$

Let

$$
\begin{aligned}
B &= \begin{bmatrix} \frac{1}{\lambda_1}\boldsymbol{x}_1 & \frac{1}{\lambda_2}\boldsymbol{x}_2 & \frac{1}{\lambda_3}\boldsymbol{x}_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & -\frac{1}{4} \\ 0 & \frac{1}{5} & \frac{1}{4} \end{bmatrix} \\
C &= [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Then

$$A = CB^{-1}$$

Since

$$det(B) = \qquad 1/20$$

$$\begin{aligned} B^{-1} = & \ 20 \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{1}{10} \\ 0 & \frac{1}{8} & \frac{1}{10} \end{bmatrix}^T \\ = & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2.5 & 2.5 \\ 0 & -2 & 2 \end{bmatrix} \end{aligned}$$

Then

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2.5 & 2.5 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4.5 & 0.5 \\ 0 & 0.5 & 4.5 \end{bmatrix}$$

(c)
Since A is an symmetric matrix, we have $A = Q\Lambda Q^{-1}$, where

$$Q = \begin{bmatrix} \tilde{\boldsymbol{x}}_1 & \tilde{\boldsymbol{x}}_2 & \tilde{\boldsymbol{x}}_3 \end{bmatrix}$$

$\tilde{\boldsymbol{x}}_i$ is the normalized $\boldsymbol{x}_i$.

In this problem,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

It's a 45-degree axial rotation along $\boldsymbol{e}_1$.

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

It's stretching along three standard basis directions.
Therefore, the transformation includes: (1) A -45 degree axial rotation along $\boldsymbol{e}_1$; (2) 2, 5 and 4 times stretching along the three standard basis directions; (3) A 45 degree axial rotation back.
Check it with $\boldsymbol{x} = [1 \quad 2 \quad 3]^T$.

$$\begin{aligned} v_1 = & \ Q^{-1}x = [1 \quad 3.5355 \quad 0.7071]^T \\ v_2 = & \ \Lambda v_1 = [2 \quad 17.6777 \quad 2.8284]^T \\ v_3 = & \ Qv_2 = [2 \quad 10.5 \quad 14.5] = Ax \end{aligned}$$

2 Problem 4

There is no evidence that Newton's method will find the root closest to the initial guess. Because the "jump" in x is largely controlled by the gradient at current point, a small gradient will cause a large jump to the neighbor of another root.