## 1

### (a)

incompressible of water:
$\nabla \cdot x = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$
integrate along $z$
we have $0 = \frac{Dh}{Dt} + h(\nabla \cdot v)$
Since $\nabla \cdot (hv) = \nabla h \cdot v + h(\nabla \cdot v)$
and $\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \nabla h \cdot v$
so $\frac{\partial h}{\partial t} + \nabla \cdot (hv) = \frac{Dh}{Dt} + h(\nabla \cdot v) = 0$

### (b)

Because $p = p_a + \rho g[h(x, y, t) - z]$
so $\frac{1}{\rho}(\nabla p) = \frac{1}{\rho}(\nabla(\rho g[h(x, y, t) - z]))$
If we assume $z = h = const$ and $\rho$ is constant (because we assume water to be incompressible)
$-\frac{1}{\rho}(\nabla p) = \frac{1}{\rho}(\nabla(\rho g[h(x, y, t) - z])) = -g\nabla h$
Besides this pressure force, the water also bear the coriolis force:
$-f\hat{k} \times v$
$a = -f\hat{k} \times v - g\nabla h$
and $a = \frac{Dv}{Dt}$
so $\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \cdot (\nabla v)$ here $\nabla v$ is a 2 order tensor $\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}$
so $\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$
so we have equation (3)

### (c)

$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \hat{i}(uu_x + vu_y) + \hat{j}(uv_x + vv_y)$

## 2

### (a)

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial h}{\partial x}$ (1)
$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial h}{\partial y}$ (2)
neglecting those $v \cdot \nabla v$ terms in (1) and (2)
$\frac{\partial^2 u}{\partial x \partial t} = -g\frac{\partial^2 h}{\partial x^2}$ and $\frac{\partial^2 v}{\partial y \partial t} = -g\frac{\partial^2 h}{\partial y^2}$
so $\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} = -g(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2})$
and from (1) we know that
$h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\frac{\partial h}{\partial t}$
so
$\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} = \frac{\partial}{\partial t}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\frac{1}{h}\frac{\partial^2 h}{\partial t^2}$
so $\frac{1}{h}\frac{\partial^2 h}{\partial t^2} = g(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}) = g\nabla^2 h$
and the tsunami wave speed is thus $\sqrt{gh_0}$

### (b)

The wave number is $K = (k, m)$
The frequency of the wave is $\omega$
so the wave speed is $\frac{\omega}{|K|} = \frac{\omega}{\sqrt{m^2+k^2}}$
for average ocean depth of 4km
$c = \sqrt{gh_0} = \sqrt{9.8 * 4000} = 197.98m/s$
for average thickness of troposphere
$c = \sqrt{gh_0} = \sqrt{9.8 * 9000} = 296.98m/s$

# 3

## (a)

The governing equations are now:
$\frac{\partial u}{\partial t} + fv + g\frac{\partial h}{\partial x} = 0$
$\frac{\partial v}{\partial t} - fu + g\frac{\partial h}{\partial y} = 0$
$\frac{\partial h}{\partial t} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$
If we assume the form
$u = u_0 exp(ikx + imy - i\omega t)$
$v = v_0 exp(ikx + imy - i\omega t)$
$h = h_0 exp(ikx + imy - i\omega t)$
then
$-i\omega u_0 + fv_0 + igkh_0 = 0$
$-i\omega v_0 - fu_0 + igmh_0 = 0$
$-i\omega h_0 + iHku_0 + iHmv_0 = 0$
so det $\begin{bmatrix} -i\omega & f & igk \\ -f & -i\omega & igm \\ iHk & iHm & -i\omega \end{bmatrix} = 0$
so $\omega^3 - \omega(gH(k^2 + m^2) + f^2) = 0$
so $\omega = 0, \sqrt{gH(k^2 + m^2) + f^2}, -\sqrt{gH(k^2 + m^2) + f^2}$
Because $c = \frac{\omega}{\sqrt{m^2+k^2}} = \sqrt{gH + f^2/(k^2 + m^2)}$
so it propagates faster.

### (c)

Because if $\omega = 0$ then the determinant must be 0.
So the three equations must be linearly dependent.
that is
$fv + g\frac{\partial h}{\partial x} = 0$
$-fu + g\frac{\partial h}{\partial y}$
$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
That is just the geostrophic balance.

# 4

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\partial h}{\partial x}$ (1)
$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\partial h}{\partial y}$ (2)
$u_{yt} + u_y u_x + uu_{xy} + v_y u_y + vu_{yy} = fv_y - gh_{xy} + vf_y(3)$
$v_{xt} + u_x v_x + uv_{xx} + v_x v_y + vv_{xy} = -fu_x - gh_{xy}(4)$
(3)-(4)
$u_{yt} - v_{xt} + u_y u_x + v_y u_y - u_x v_x - v_x v_y + uu_{xy} + vu_{yy} - uv_{xx} - vv_{xy}$
$= fv_y + fu_x$
taking in $\xi = (v_x - u_y)$
$-(u\xi_x + v\xi_y) - \xi(\nabla\cdot V) = f(\nabla\cdot V) + vf_y$
$V\cdot\nabla\xi + (\xi + f)(\nabla\cdot V) + vf_y = 0$
$\frac{D\xi}{Dt} = \frac{\partial\xi}{\partial t} + V\cdot\nabla\xi$
If we assume a stationary solution , then $\frac{\partial\xi}{\partial t} = 0$
so we have $\frac{D\xi}{Dt} + (\xi + f)(\nabla\cdot V) + vf_y = 0$
Because $vf_y = \frac{Df}{Dt}$
so $\frac{D(\xi+f)}{Dt} + (\xi + f)\delta = 0$
combing with eq (1)
$\frac{DQ}{Dt} = D(\frac{\xi+f}{h})/Dt = \frac{1}{h^2}(\frac{D(\xi+f)}{Dt}h - \frac{Dh}{Dt}(\xi + f)) = \frac{1}{h^2}(-(\xi + f)\delta h + (h\delta)(\xi + f)) = 0$

## 5

### (a)

The leading terms in eq(1)
$h_t + h(u_x + v_y) = 0$
leading terms in eq(3)
$v_t + f\hat{k} \times v + g\nabla h = 0$
if we also neglect the time dependent term:
$-fv + gh_x = 0$(a)
and
$fu + gh_y = 0$(b)
if we take $y$ derivative of (a) minus $x$ derivative of (b)
then
$-fv_y - fu_x = 0$
so $v_y + u_x = 0$ which is the quasistatic version of eq(1).

### (b)

if $u = -\psi_y$ and $v = \psi_x$
then $v_y + u_x = -\psi_{xy} + \psi_{xy} = 0$ this equation is automatically satisfied.
so $-f\psi_x + gh'_x = 0$ and $-f\psi_y + gh'_y = 0$ (10)
$\xi = (v_x - u_y) = \psi_{xx} + \psi_{yy} = \nabla^2\psi$
Because of (10)
$\psi = \frac{g}{f}h' + Const$ (11)

### (c)

$v = (-\psi_y, \psi_x)$ and $\nabla\psi = (\psi_x, \psi_y)$
so $v \cdot \nabla\psi = 0$
so they are perpendicular to each other.
From the form of $v$ and $\psi$ we know that $\nabla\psi$ is a clockwise rotation of $v$.

So to the right of $v$ , $\psi$ always increases, this is no difference for northern and southern hemisphere, this is just a natural outcome of how you define your stream function. But if you link $\psi$ with pressure, that is difference from Northern to Southern hemisphere because the sign of $f$ is changed.

In northern hemisphere, $f$ is positive, so high $\psi$ means high pressure, in southern hemisphere , high $\psi$ means low pressure. So in northern hemisphere , high pressure is on the right of the flow and in southern hemisphere, high pressure is on the left of the flow.

### (d)

$Q = \frac{\xi+f}{h} = \frac{\xi+f_0+\beta y}{h_0+h'(x,y,t)} = \frac{(\xi+f_0+\beta y)}{h_0(1+h'/h_0)} = (1 - \frac{h'}{h_0})(\frac{1}{h_0})(\xi + f_0 + \beta y) = \frac{f_0}{h_0} + (\frac{\xi+\beta y-\frac{f_0 h'}{h_0}}{h_0})$
and $\xi = \nabla^2\psi$ ,$h' = \frac{f_0}{g}\psi + C$ (we can take $C = 0$ for convenience)
Also there are some other small terms that are second order things:
$\frac{\xi h'}{h_0^2}$ that we neglect.

### (e)

Comparing the $q$ given here and my result in 4(d),
$(1/L_d^2) = \frac{f_0^2}{h_0 g}$ so $L_d = \frac{\sqrt{h_0 g}}{f_0}$, take in $h_0 = 4km$ and $f_0 = 10^{-4}s^{-1}$
$L_d = 19798.98m/s^2$
Since $DQ/Dt = 0$
$D(\frac{\nabla^2\psi+\beta y-(1/L_d)^2\psi}{h_0}) = \frac{1}{h_0}$

$$
\begin{aligned}
\frac{D(\nabla^2\psi + \beta y - (1/L_d)^2\psi)}{Dt} &= (\nabla\psi - (1/L_d)^2\psi + \beta y)_t \\
&\quad -\psi_y(\nabla^2\psi - (1/L_d)^2\psi)_x + \psi_x(\nabla^2\psi - (1/L_d)^2\psi + \beta y)_y \\
&= (\nabla\psi - (1/L_d)^2\psi + \beta y)_t + J(\psi, \nabla^2\psi) + \beta\psi_x
\end{aligned}
$$