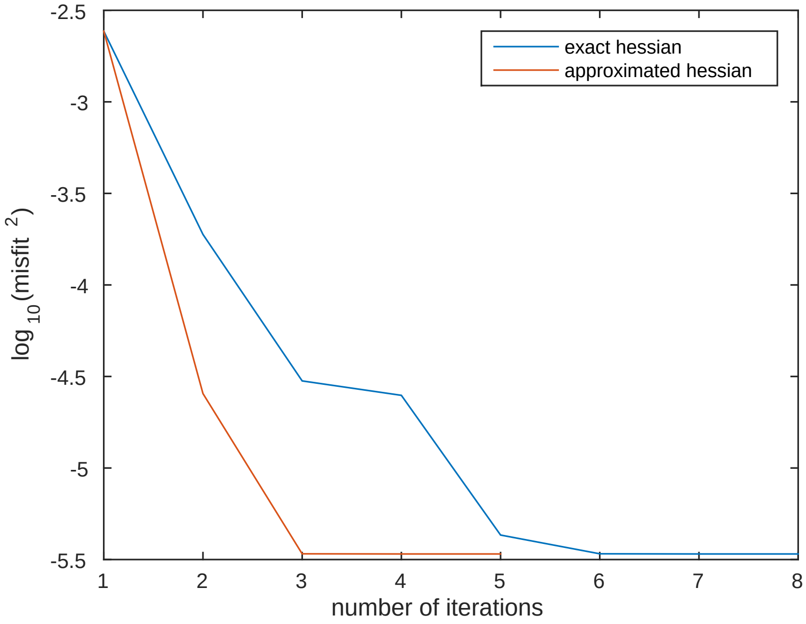
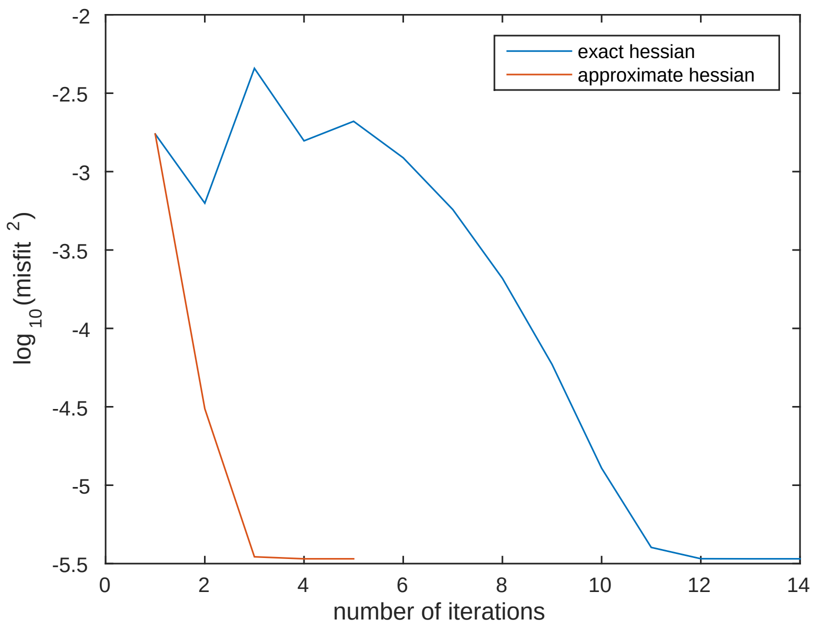
Problem 1 (graded by Dunzhu & Toby) X points

(a) - 10 points

```
function [ Gamma, Hess] = compute_gradient_approx_hess2( x,y,M,misfit)

xs = M(1);
ys = M(2);
zs = M(3);
p = M(4);

R = ((x - xs).^2 + (y - ys).^2 + zs^2);

dx = x-xs;
dy = y-ys;

Jacob(:,1) = (3.*p.*zs.*(dx))./((R).^(5/2));
Jacob(:,2) = (3.*p.*zs.*(dy))./((R).^(5/2));
Jacob(:,3) = p./(R).^(3/2) - (3*p.*zs.^2)./(R).^(5/2);
Jacob(:,4) = zs./(R).^(3/2);

Gamma = - Jacob'*misfit;
%this is the apprximated Hessian;


%dHess = (G(m)-d)^T*Q
dHess = zeros(4);
dHess(1,2) = - misfit'*((15*p*zs*(dx).*(dy))./((R).^(7/2)));
dHess(1,3) = - misfit'*((3*p.*(dx))./((R).^(5/2)) - (15*p*zs.^2.*(dx))./((R).^(7/2)));
dHess(1,4) = - misfit'*((3*zs.*(dx))./((R).^(5/2)));
dHess(2,3) = - misfit'*( (3*p.*(dy))./((R).^(5/2)) - (15*p*zs.^2.*(dy))./((R).^(7/2)));
dHess(2,4) = - misfit'*((3*zs.*(dy))./((R).^(5/2)));
dHess(3,4) = - misfit'*(1./(R).^(3/2) - (3*zs^2)./(R).^(5/2));

dHess = (dHess + dHess');

dHess(1,1) = - misfit'*((15*p*zs.*(dx).^2)./((R).^(7/2)) - (3*p.*zs)./(R).^(5/2));
dHess(2,2) = - misfit'*((15*p*zs.*(dy).^2)./((R).^(7/2)) - (3*p.*zs)./(R).^(5/2));
dHess(3,3) = - misfit'*((15*p*zs^3)./(R).^(7/2) - (9*p*zs)./(R).^(5/2));
dHess(4,4) = 0;



Hess = (Jacob')*Jacob;
%only add dHess if using exact Hessian
Hess = Hess + dHess;

Hess = 0.5*(Hess + Hess');

end
```


exact hessian
approximated hessian
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
log 10 (misfit 2)
1
2
3
4
5
6
7
8
number of iterations



(b) - 10 points

(b.i)

Change variable $t = 1/\sigma$, then

$$\int_0^\infty \frac{1}{\sigma^N} \exp\left(-\frac{1}{2\sigma^2} A\right) d\sigma \quad = \quad \int_0^\infty t^{N-2} \exp\left(-\frac{A}{2} t^2\right) dt$$

Since

$$\int_0^\infty \exp\left(-\frac{A}{2} t^2\right) dt = \sqrt{\frac{\pi}{2}} A^{-1/2} \propto A^{-1/2}$$
$$\int_0^\infty t \exp\left(-\frac{A}{2} t^2\right) dt = A^{-1} \propto A^{-1}$$

Take derivative with respect to A, then

$$\int_0^\infty \frac{-t^2}{2} \exp\left(-\frac{A}{2} t^2\right) dt \propto A^{-1/2-1}$$
$$\int_0^\infty t \frac{-t^2}{2} \exp\left(-\frac{A}{2} t^2\right) dt \propto A^{-1-1}$$

Continue this derivative, we have

$$\int_0^\infty \left(\frac{-t^2}{2}\right)^{(N-2)/2} \exp\left(-\frac{A}{2}t^2\right) dt \propto A^{-1/2-(N-2)/2}$$

So

$$\int_0^\infty t^{N-2} \exp\left(-\frac{A}{2}t^2\right) dt \propto A^{-(N-1)/2}$$

(b.ii)

Assume uniform prior, then

$$\begin{aligned} P(x,y|d) &\propto P(d|x,y) = \exp -F(x,y) \\ &= \exp\left(-F(x_0,y_0) - gradF|_{x_0,y_0} \cdot (x-x_0, y-y_0)' - \frac{1}{2}(x-x_0,y-y_0)H|_{x_0,y_0}(x-x_0,y-y_0)'\right) \end{aligned}$$

Since at (x_0,y_0), $gradF = 0$, thus

$$P(x,y|d) \propto \exp\left(-\frac{1}{2}(x-x_0,y-y_0)H|_{x_0,y_0}(x-x_0,y-y_0)'\right)$$

write H as

$$H = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

thus the joint pdf will be

$$f(x,y) = K \exp\left(-\frac{1}{2}[A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2]\right)$$

where K is the constant that normalize the pdf, to get it, we have

$$\begin{aligned} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx f(x,y) &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx K \exp\left(-\frac{1}{2}[A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2]\right) \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx K \exp\left(-\frac{1}{2}[Ax^2 + 2Bxy + Cy^2]\right) \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx K \exp\left(-\frac{1}{2}[A(x+By/A)^2 + (C-B^2/A)y^2]\right) \\ &= \int_{-\infty}^{\infty} dy K \exp\left(-\frac{1}{2}(C-B^2/A)y^2\right) \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}[A(x+By/A)^2]\right) \\ &= \int_{-\infty}^{\infty} dy K \exp\left(-\frac{1}{2}(C-B^2/A)y^2\right) \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}Ax^2\right) \\ &= \int_{-\infty}^{\infty} dy K \exp\left(-\frac{1}{2}(C-B^2/A)y^2\right) \sqrt{2\pi/A} \\ &= K\sqrt{2\pi/(C-B^2/A)}\sqrt{2\pi/A} \\ &= 2\pi K/\sqrt{AC-B^2} \\ &= 1 \end{aligned}$$

Now we want to show $E[x] = x_0$. This is true because

$$\begin{aligned}
E[x-x_0] &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad (x-x_0)f(x,y) \\
&\propto \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad (x-x_0)\exp\left(-\frac{1}{2}[A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2]\right) \\
&\propto \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad x\exp\left(-\frac{1}{2}[Ax^2 + 2Bx(y-y_0) + C(y-y_0)^2]\right) \\
&\propto \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad x\exp\left(-\frac{1}{2}[Ax^2 + 2Bxy + Cy^2]\right) \\
&\propto \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad x\exp\left(-\frac{1}{2}[A(x+By/A) + (C-B^2/A)y^2]\right) \\
&\propto \int_{-\infty}^{\infty} dy(By/A)\exp\left(-\frac{1}{2}[(C-B^2/A)y^2]\right) \\
&\propto \int_{-\infty}^{\infty} dy \quad y\exp\left(-\frac{1}{2}[(C-B^2/A)y^2]\right) \\
&= 0
\end{aligned}$$

So similarly $E[y] = y_0$.

Thus

$$\begin{aligned}
\sigma_x^2 &= E[(x-E[x])^2] \\
&= E[(x-x_0)^2] \\
&= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad \frac{\partial f(x,y)}{\partial A}/(-1/2) \\
&= -2\frac{\partial}{\partial A}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad f(x,y) \\
&= -2\frac{\partial}{\partial A}2\pi K/\sqrt{AC-B^2} \\
&= \frac{2\pi K}{(A-B^2/C)\sqrt{AC-B^2}} \\
&= \frac{C}{AC-B^2}
\end{aligned}$$

Similarly

$$\sigma_y^2 = \frac{A}{AC-B^2}$$

$$\begin{aligned}
\sigma_{xy} &= E[(x-E[x])(y-E[y])] \\
&= E[(x-x_0)(y-y_0)] \\
&= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad \frac{\partial f(x,y)}{\partial B}/(-1) \\
&= -\frac{\partial}{\partial B}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \quad f(x,y) \\
&= \frac{-B}{AC-B^2}
\end{aligned}$$

Note that

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}^{-1} = \frac{1}{AC-B^2}\begin{pmatrix} C & -B \\ -B & A \end{pmatrix}$$

Thus we get what we want.

Problem 2 (graded by Dunzhu & Toby) 20 points

(b) (10 points)

Here are some topics people mentioned about the ABT book:

- Appendix A, background knowledge
- Number of data points vs number of model parameter, overdetermined, underdetermininned, mixed determinied
- MAP and Bayes' theorem
- L1 norm is better when existence of outlier
- difficulty in inverse: existance, uniqueness, and instability
- Fredholm integration equation of the first kind generalize many inverse problem
- Chapter 11 Section 3, general multivarate normal case with prior
- damped Newton's method, choose the step length instead of using full newton step
- ABT mentioned p-values, chi-square statistics
- in lecture, we maximize $P(d|m)$ to get the maximum likelihood, in ABT they defined $L(m|d) = P(d|m)$ first, then maximize it. ABT one is more clear, because $P(d|m)$ apprears to be a function of d, where in fact we want it to be a function of m.

(c) (10 points)

Here are some topics people mentioned:

- When estimating σ^2, devided by $N-1$ instead of N
- Confused about $J(m)$ and $G(m)$, it's really the same thing, Jacobian
- class did a better job of describing prior
- Class and ABT does not cover the case when prior is a given range
- numerical calculation of derivative when it's hard to do it analytically
- any method for finding global minimum?