Physique numérique I

Spring-pendulum system

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Let us consider the dynamics of a mass m of electric charge q tied to a spring of stiffness k and length at rest l_0. The position of the mass over time is given by a vector $\vec{r}(t)$ in a Cartesian coordinate system with x and y axis as shown in **Fig.**1. The spring is attached on its other end to the point $(x, y) = (\widehat{x}, \widehat{y}) = (0, 0)$.

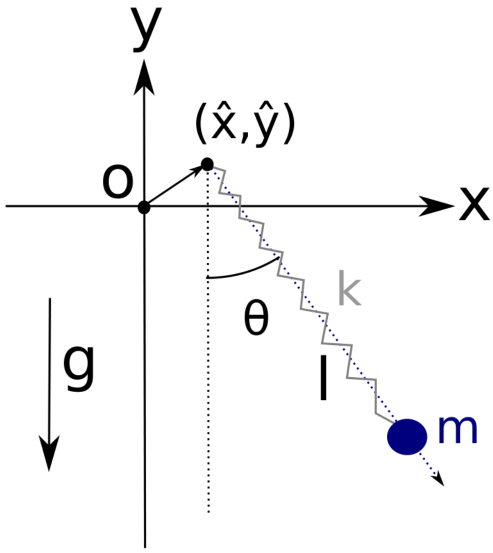

Figure 1
Schematic representation of the system

The position, speed and acceleration are respectively written as :

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \vec{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \qquad \vec{a}(t) = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$

with initial conditions : $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = \dot{y}(0) = 0$.

The mass is under the influence of four forces :

1. The weight : $\vec{P} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$
2. The spring force : $\vec{F_k} = \begin{pmatrix} -k(l-l_0)\sin(\theta) \\ k(l-l_0)\cos(\theta) \end{pmatrix}$, where l is the length of the spring and $l - l_0$ its deformation.
3. An oscillating excitation force : $\vec{F_E} = q\cos(\omega t)\begin{pmatrix} E_x \\ E_y \end{pmatrix}$ with $E_{x,y}$ the electric field coordinates and ω a given frequency.
4. The drag force : $\vec{F_T} = -\nu\vec{v}(t)$

The numerical values for the simulations, unless otherwise stated, are $m = 1.5kg$, $k = 4.5\text{N.m}^{-1}$, $l_0 = 1.1\text{m}$, $q = 10^{-4}\text{C}$ and $g = 9.81\text{m.s}^{-2}$.

2.1 Differential motion equation

Noticing that $x = l\sin(\theta)$, $y = -l\cos(\theta)$ and $\theta = \arctan(\frac{x}{-y})$, Newton's second law yields :

$$\vec{F_k} + m\vec{g} + \vec{F_T} + \vec{F_E} = m\vec{a} \tag{1}$$

Hence the differential equations system :

$$\begin{cases} \ddot{x}(t) = -\frac{k}{m}x + \frac{k}{m}l_0\sin(\theta) - \frac{\nu}{m}\dot{x} + \frac{q}{m}E_x\cos(\omega t) \\ \\ \ddot{y}(t) = -\frac{k}{m}y - \frac{k}{m}l_0\cos(\theta) - \frac{\nu}{m}\dot{y} + \frac{q}{m}E_y\cos(\omega t) - g \end{cases} \tag{2}$$

which is equivalent to :

$$\begin{cases} \ddot{x} + \omega_0^2 x + \gamma\dot{x} = \omega_0^2 l_0\sin(\theta) + \frac{q}{m}E_x\cos(\omega t) \\ \\ \ddot{y} + \omega_0^2 y + \gamma\dot{y} = -\omega_0^2 l_0\cos(\theta) + \frac{q}{m}E_y\cos(\omega t) - g \end{cases} \tag{3}$$

with $\frac{k}{m} = \omega_0^2$ and $\frac{\nu}{m} = \gamma$. These are the typical equations of a damped and forced harmonic oscillator.

2.2 Equilibrium

Let $E_x = E_y = \nu = 0$. Equations (3) are reduced to

$$\begin{cases} \ddot{x} + \omega_0^2(x - l_0\sin(\theta)) = 0 \\ \\ \ddot{y} + \omega_0^2(y + l_0\cos(\theta)) - g = 0 \end{cases} \tag{4}$$

Simple harmonic oscillators equations are recognisable. The equilibrium is determined when the net force is zero, i.e. $\ddot{x} = \ddot{y} = 0$. Therefore :

$$\begin{cases} \omega_0^2(x - l_0\sin(\theta)) = 0 \\ \\ \omega_0^2(y + l_0\cos(\theta)) - g = 0 \end{cases} \tag{5}$$

The first equation gives two solutions knowing the expressions of x and y in polar coordinates :

1. $l = l_0$ which is not compatible with the second equation since $g \neq 0$.
2. $\theta_{1,2} = 0, \pi$ which gives $x_1 = x_2 = 0$ and by injecting into the second equation : $y_1 = -l_0 - \frac{g}{\omega_0^2} = -l$ and $y_2 = l_0 - \frac{g}{\omega_0^2} = l$.

Hence the stable and unstable equilibrium positions respectively :

1. $(x_1, y_1) = (0, -l_0 - \frac{g}{\omega_0^2})$
2. $(x_2, y_2) = (0, l_0 - \frac{g}{\omega_0^2})$

2.3 Mechanical energy

By definition, the mechanical energy is given by :

$$E_m = E_K + E_P = \frac{1}{2}mv^2 + mgy + \frac{1}{2}(l - l_0)^2 - q(E_x x + E_y y) \tag{6}$$

knowing that :

- $l = \sqrt{x^2 + y^2}$
- The electric force being conservative, the electric potential is : $qV = q\left(-\int \vec{E}\vec{dl}\right) = -q(E_x x + E_y y)$ with $\vec{dl} = (dx, dy)$

2.4 Non conservative forces' power

The only non conservative force is the drag force. By definition, the power of a force $\vec{F_T}$ is :

$$P_{nc} = \frac{dW_{nc}}{dt} = \frac{\vec{F_T}\vec{dr}}{dt} = -\nu v^2 \tag{7}$$

where $v^2 = \dot{x}^2 + \dot{y}^2$

2.5 Small oscillations around the stable solution of the equilibrium

Equations (4) is used. Small oscillations δx and δy are applied respectively to x_1 and y_1. Hence :

$$x(t) = x_1 + \delta x = \delta x \Rightarrow \ddot{x}(t) = \delta \ddot{x} \tag{8}$$

$$y(t) = y_1 + \delta y \Rightarrow \ddot{y}(t) = \delta \ddot{y} \tag{9}$$

Therefore for δx :

$$\delta \ddot{x}(t) = -\omega_0^2 \delta x - \omega_0^2 l_0 \sin\left(\arctan\left(\frac{x}{y}\right)\right) \tag{10}$$

A Taylor series expansion around x_1 and y_1 gives :

$$-\omega_0^2 l_0 \frac{d}{dx} \sin\left(\arctan\left(\frac{x}{y}\right)\right)\bigg|_{x_1,y_1} \delta x = \frac{\delta x}{y_1} \tag{11}$$

Hence (11) in (3) yields :

$$\begin{aligned}\delta \ddot{x}(t) &= -\omega_0^2 \delta x \left(1 + \frac{l_0}{\frac{-g}{\omega_0^2} - l_0}\right) \\ &= \delta x \frac{g}{y_1} \\ &= \delta x \frac{g}{-l_{eq}}\end{aligned}$$

Finally :

$$\delta \ddot{x}(t) + \frac{g}{l_{eq}} \delta x = \delta \ddot{x}(t) + \omega_2^2 \delta x = 0 \tag{12}$$

where $\omega_2^2 = \sqrt{\dfrac{g}{l_{eq}}} = \sqrt{\dfrac{kg}{gm + kl_0}}$ the eigenfrequency corresponding to the pendulum.

On the same basis, the equation for a small oscillation for δy yields :

$$\begin{aligned}\delta \ddot{y}(t) &= \omega_0^2 (y_1 + \delta y) - \omega_0^2 l_0 \cos(\theta) - g \\ &\approx -\omega_0^2 (y_1 + \delta y) - \omega_0^2 l_0 - g \\ &= -\omega_0^2 \delta y\end{aligned}$$

Finally the full equation :

$$\delta \ddot{y}(t) + \omega_1^2 \delta y = 0 \tag{13}$$

With $\omega_0^2 = \omega_1^2$ and therefore $\omega_1 = \sqrt{\dfrac{k}{m}}$ the eigenfrequency corresponding to the spring.

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Addendum