"Old Quantum Mechanics by Bohr and Sommerfeld from a Modern Perspective"

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(Last modified on June 4, 2025; 8:10 PM Arizona time.)

In[1]:=
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(* Here, from (3.17)-(3.18),
we derive the expansion (3.19) for the relativistic Schroedinger equation (3.14). *)
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In[2]:= $\text{EnergyRelSchr}[\mu\_] := \dfrac{1}{\sqrt{1 + \dfrac{\mu^2}{\left(n + \frac{1}{2} + \sqrt{\left(\frac{1}{2} + l\right)^2 - \mu^2}\right)^2}}}$

In[3]:=
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Simplify[PowerExpand[Series[EnergyRelSchr[μ], {μ, 0, 5}, Assumptions → 2 * l + 1 > 0]]]
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Out[3]= $1 - \dfrac{\mu^2}{2\,(1 + l + n)^2} - \dfrac{(5 + 2\,l + 8\,n)\,\mu^4}{8\left((1 + 2\,l)\,(1 + l + n)^4\right)} + O[\mu]^6$

In[4]:=
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FullSimplify[% /. n → n - l - 1]
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Out[4]= $1 - \dfrac{\mu^2}{2\,n^2} + \dfrac{\left(3 - \frac{8\,n}{1 + 2\,l}\right)\mu^4}{8\,n^4} + O[\mu]^6$

In[5]:=
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(* This is the nonrelativistic expansion (3.19). *)
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In[6]:=
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(* Here, from (3.22),
we derive the expansion (3.24) for the Dirac equation (3.20). *)
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In[7]:= $\dfrac{1}{\text{Sqrt}\left[1 + \mu^2 / (n + \nu)^2\right]}$

Out[7]= $\dfrac{1}{\sqrt{1 + \dfrac{\mu^2}{(n + \nu)^2}}}$

In[8]:= $\text{EnergyDirac}[\mu\_] = \% \;/.\; \nu \to \text{Sqrt}\left[(j + 1/2)\,\text{^}2 - \mu^2\right]$

Out[8]= $\dfrac{1}{\sqrt{1 + \dfrac{\mu^2}{\left(n + \sqrt{\left(\frac{1}{2} + j\right)^2 - \mu^2}\right)^2}}}$

In[9]:=
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Simplify[PowerExpand[Series[EnergyDirac[μ], {μ, 0, 5}, Assumptions → 2 * j + 1 > 0]]]
```

Out[9]= $1 - \dfrac{2\,\mu^2}{(1 + 2\,j + 2\,n)^2} - \dfrac{2\,(1 + 2\,j + 8\,n)\,\mu^4}{(1 + 2\,j)\,(1 + 2\,j + 2\,n)^4} + O[\mu]^6$

In[10]:=
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FullSimplify[% /. n → n - j - 1 / 2]
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Out[10]=

$$1 - \frac{\mu^2}{2\,n^2} + \frac{\left(3 - \frac{8\,n}{1 + 2\,j}\right)\mu^4}{8\,n^4} + O[\mu]^6$$

In[11]:=
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(* This is the expansion (3.24). *)
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In[12]:= (* Next term in (3.24) for the Dirac equation. *)

In[13]:= EnergyDiracExp[μ_] = $1 - \frac{2\,\mu^2}{(1+2\,j+2\,n)^2} - \frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}$

Out[13]=

$$1 - \frac{2\,\mu^2}{(1+2\,j+2\,n)^2} - \frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}$$

In[14]:= EnergyDirac[μ] - EnergyDiracExp[μ]

Out[14]=

$$-1 + \frac{2\,\mu^2}{(1+2\,j+2\,n)^2} + \frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4} + \frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$$

In[15]:= Simplify[PowerExpand[
Series[EnergyDirac[μ] - EnergyDiracExp[μ], {μ, 0, 7}, Assumptions → 2 * j + 1 > 0]]]

Out[15]=

$$-\frac{4\,\left(1+8\,j^3+12\,n+48\,n^2+16\,n^3+12\,j^2\,(1+4\,n)+6\,j\,(1+4\,n)^2\right)\,\mu^6}{(1+2\,j)^3\,(1+2\,j+2\,n)^6} + O[\mu]^8$$

In[16]:= FullSimplify[% /. n → n - j - 1 / 2]

Out[16]=

$$-\frac{\left(5\,(1+2\,j)^3-24\,(1+2\,j)^2\,n+24\,(1+2\,j)\,n^2+16\,n^3\right)\,\mu^6}{16\,\left((1+2\,j)^3\,n^6\right)} + O[\mu]^8$$

In[17]:= (* The last term can be rewritten as follows: *)

In[18]:= $-\frac{\mu^6}{16\,n^3} * \left(\frac{5}{n^3} - \frac{24}{(1+2\,j)\,n^2} + \frac{24}{(1+2\,j)^2\,n} + \frac{16}{(1+2\,j)^3}\right)$

Out[18]=

$$-\frac{\left(\frac{16}{(1+2\,j)^3}+\frac{5}{n^3}-\frac{24}{(1+2\,j)\,n^2}+\frac{24}{(1+2\,j)^2\,n}\right)\,\mu^6}{16\,n^3}$$

In[19]:= Simplify[%16 - 18]

Out[19]=

$$O[\mu]^8$$