"Old Quantum Mechanics by Bohr and Sommerfeld from a Modern Perspective"

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Here, from (3.17)-(3.18), we derive the expansion (3.19) for the relativistic Schrödinger equation (3.14).

In[1]:= **EnergyRelSchr[μ_] :=** $$\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{2}+l\right)^2-\mu^2}\right)^2}}}$$

In[2]:= **Simplify[PowerExpand[Series[EnergyRelSchr[μ], {μ, 0, 5}, Assumptions → 2 * l + 1 > 0]]]**

Out[2]= $$1-\frac{\mu^2}{2\,(1+l+n)^2}-\frac{(5+2\,l+8\,n)\,\mu^4}{8\,\left((1+2\,l)\,(1+l+n)^4\right)}+O[\mu]^6$$

In[3]:= **FullSimplify[% /. n → n - l - 1]**

Out[3]= $$1-\frac{\mu^2}{2\,n^2}+\frac{\left(3-\frac{8\,n}{1+2\,l}\right)\mu^4}{8\,n^4}+O[\mu]^6$$

This is the nonrelativistic expansion (3.19).

Here, from (3.22), we derive the expansion (3.24) for the Dirac equation (3.20).

In[4]:= $$\frac{1}{\mathrm{Sqrt}\left[1+\mu^2/(n+\nu)^2\right]}$$

Out[4]= $$\frac{1}{\sqrt{1+\frac{\mu^2}{(n+\nu)^2}}}$$

In[5]:= **EnergyDirac[μ_] = % /. ν → Sqrt[(j + 1 / 2) ^2 - μ²]**

Out[5]= $$\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$$

In[6]:= **Simplify[PowerExpand[Series[EnergyDirac[μ], {μ, 0, 5}, Assumptions → 2 * j + 1 > 0]]]**

Out[6]= $$1-\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}-\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}+O[\mu]^6$$

In[7]:= **FullSimplify[% /. n → n - j - 1 / 2]**

Out[7]= $$1-\frac{\mu^2}{2\,n^2}+\frac{\left(3-\frac{8\,n}{1+2\,j}\right)\mu^4}{8\,n^4}+O[\mu]^6$$

This is the expansion (3.24).

Next term in (3.24) for the Dirac equation.

In[8]:=
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EnergyDiracExp[μ_] = 1 - 2 μ^2/(1 + 2 j + 2 n)^2 - 2 (1 + 2 j + 8 n) μ^4/((1 + 2 j) (1 + 2 j + 2 n)^4)
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Out[8]= $1-\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}-\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}$

In[9]:=
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EnergyDirac[μ] - EnergyDiracExp[μ]
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Out[9]= $-1+\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}+\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}+\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$

In[10]:=
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Simplify[PowerExpand[
  Series[EnergyDirac[μ] - EnergyDiracExp[μ], {μ, 0, 7}, Assumptions → 2 * j + 1 > 0]]]
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Out[10]=

$-\frac{4\left(1+8\,j^3+12\,n+48\,n^2+16\,n^3+12\,j^2\,(1+4\,n)+6\,j\,(1+4\,n)^2\right)\mu^6}{(1+2\,j)^3\,(1+2\,j+2\,n)^6}+O[\mu]^8$

In[11]:=
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FullSimplify[% /. n → n - j - 1 / 2]
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Out[11]=

$-\frac{\left(5\,(1+2\,j)^3-24\,(1+2\,j)^2\,n+24\,(1+2\,j)\,n^2+16\,n^3\right)\mu^6}{16\,\left((1+2\,j)^3\,n^6\right)}+O[\mu]^8$

The last term can be rewritten as follows:

In[12]:=
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-μ^6/(16 n^3) * (5/n^3 - 24/((1 + 2 j) n^2) + 24/((1 + 2 j)^2 n) + 16/(1 + 2 j)^3)
```

Out[12]=

$-\frac{\left(\frac{16}{(1+2\,j)^3}+\frac{5}{n^3}-\frac{24}{(1+2\,j)\,n^2}+\frac{24}{(1+2\,j)^2\,n}\right)\mu^6}{16\,n^3}$

```
Simplify[%11 - %12]
```

Out[13]=

$O[\mu]^8$

The next term can also be rewritten as in (3.25):

In[14]:=
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-μ^6/(4 n^6) * (5/4 - 3 * n/(j + 1 / 2) + 3 * n^2/(2 * (j + 1 / 2)^2) + n^3/(2 * (j + 1 / 2)^3))
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Out[14]=

$-\frac{\left(\frac{5}{4}-\frac{3\,n}{\frac{1}{2}+j}+\frac{3\,n^2}{2\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{2\left(\frac{1}{2}+j\right)^3}\right)\mu^6}{4\,n^6}$

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Simplify[%12 - %14]
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Out[15]=

0

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Simplify[%11 - 14]
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Out[16]=

$O[\mu]^8$