"Old Quantum Mechanics by Bohr and Sommerfeld from a Modern Perspective"

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Here, from (3.17)-(3.18), we derive the expansion (3.19) for the relativistic Schrödinger equation (3.14).

In[∘]:=
$$\text{EnergyRelSchr}[\mu\_] := \frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{2}+l\right)^2-\mu^2}\right)^2}}}$$

In[∘]:=
```
Simplify[PowerExpand[Series[EnergyRelSchr[μ], {μ, 0, 5}, Assumptions → 2 * l + 1 > 0]]]
```

Out[∘]=
$$1-\frac{\mu^2}{2\,(1+l+n)^2}-\frac{(5+2\,l+8\,n)\,\mu^4}{8\,\left((1+2\,l)\,(1+l+n)^4\right)}+O[\mu]^6$$

In[∘]:=
```
FullSimplify[% /. n → n - l - 1]
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Out[∘]=
$$1-\frac{\mu^2}{2\,n^2}+\frac{\left(3-\frac{8\,n}{1+2\,l}\right)\mu^4}{8\,n^4}+O[\mu]^6$$

This is the nonrelativistic expansion (3.19).

Section 6: A "mistake" that Schrödinger never made:

In[∘]:=
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Simplify[EnergyRelSchr[μ] /. n → n - l / 2]
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Out[∘]=
$$\frac{1}{\sqrt{1+\frac{\mu^2}{\left(\frac{1}{2}-\frac{l}{2}+n+\sqrt{\frac{1}{4}+l+l^2-\mu^2}\right)^2}}}$$

In[∘]:=
```
FullSimplify[PowerExpand[Series[%, {μ, 0, 5}, Assumptions → l > 0]]]
```

Out[∘]=
$$1-\frac{2\,\mu^2}{(2+l+2\,n)^2}+\frac{2\,(-5+2\,l-8\,n)\,\mu^4}{(1+2\,l)\,(2+l+2\,n)^4}+O[\mu]^6$$

Thus, the non-relativistic term is given by:

In[∘]:=
$$-\frac{\mu^2}{2*(1+l/2+n)^2}$$

Out[∘]=
$$-\frac{\mu^2}{2\left(1+\frac{l}{2}+n\right)^2}$$

In[∘]:= (* Indeed *) Simplify$\left[\frac{-2\,\mu^2}{(2+l+2\,n)^2} - \%\right]$

Out[∘]=

0

Here, from (3.22), we derive the expansion (3.24) for the Dirac equation (3.20).

In[∘]:= $\frac{1}{\text{Sqrt}\left[1+\mu^2/(n+\nu)^2\right]}$

Out[∘]=

$$\frac{1}{\sqrt{1+\frac{\mu^2}{(n+\nu)^2}}}$$

In[∘]:= EnergyDirac[μ_] = % /. ν → Sqrt$\left[(j+1/2)\,\hat{}\,2-\mu^2\right]$

Out[∘]=

$$\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$$

In[∘]:= Simplify[PowerExpand[Series[EnergyDirac[μ], {μ, 0, 5}, Assumptions → 2 * j + 1 > 0]]]

Out[∘]=

$$1-\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}-\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}+O[\mu]^6$$

In[∘]:= FullSimplify[% /. n → n - j - 1 / 2]

Out[∘]=

$$1-\frac{\mu^2}{2\,n^2}+\frac{\left(3-\frac{8\,n}{1+2\,j}\right)\mu^4}{8\,n^4}+O[\mu]^6$$

This is the expansion (3.24), indeed:

In[∘]:= $1-\frac{\mu^2}{2\,n^2}+\frac{\left(\frac{3}{4}-\frac{n}{\frac{1}{2}+j}\right)\mu^4}{2\,n^4}+O[\mu]^6$

Out[∘]=

$$1-\frac{\mu^2}{2\,n^2}+\frac{\left(\frac{3}{4}-\frac{n}{\frac{1}{2}+j}\right)\mu^4}{2\,n^4}+O[\mu]^6$$

In[∘]:= Simplify[- %]

Out[∘]=

$$O[\mu]^6$$

Next term in (3.24) for the Dirac equation.

In[∘]:= EnergyDiracExp[μ_] = $1-\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}-\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}$

Out[∘]=

$$1-\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}-\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}$$

In[∘]:= **EnergyDirac[μ] - EnergyDiracExp[μ]**

Out[∘]=

$$-1+\frac{2\,\mu^2}{(1+2\,j+2\,n)^2}+\frac{2\,(1+2\,j+8\,n)\,\mu^4}{(1+2\,j)\,(1+2\,j+2\,n)^4}+\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$$

In[∘]:= **Simplify[PowerExpand[**
Series[EnergyDirac[μ] - EnergyDiracExp[μ], {μ, 0, 7}, Assumptions → 2 * j + 1 > 0]]]

Out[∘]=

$$-\frac{4\left(1+8\,j^3+12\,n+48\,n^2+16\,n^3+12\,j^2\,(1+4\,n)+6\,j\,(1+4\,n)^2\right)\mu^6}{(1+2\,j)^3\,(1+2\,j+2\,n)^6}+O[\mu]^8$$

In[∘]:= **FullSimplify[% /. n → n - j - 1 / 2]**

Out[∘]=

$$-\frac{\left(5\,(1+2\,j)^3-24\,(1+2\,j)^2\,n+24\,(1+2\,j)\,n^2+16\,n^3\right)\mu^6}{16\left((1+2\,j)^3\,n^6\right)}+O[\mu]^8$$

The last term can be rewritten as follows:

In[∘]:=

$$-\frac{\mu^6}{16\,n^3}*\left(\frac{5}{n^3}-\frac{24}{(1+2\,j)\,n^2}+\frac{24}{(1+2\,j)^2\,n}+\frac{16}{(1+2\,j)^3}\right)$$

Out[∘]=

$$-\frac{\left(\frac{16}{(1+2\,j)^3}+\frac{5}{n^3}-\frac{24}{(1+2\,j)\,n^2}+\frac{24}{(1+2\,j)^2\,n}\right)\mu^6}{16\,n^3}$$

In[∘]:= **Simplify[%17 -]**

Out[∘]=

$O[\mu]^8$

The next term can also be rewritten as in (3.26):

In[∘]:=

$$-\frac{\mu^6}{4\,n^6}*\left(\frac{5}{4}-\frac{3*n}{j+1/2}+\frac{3*n^2}{2*(j+1/2)^2}+\frac{n^3}{2*(j+1/2)^3}\right)$$

Out[∘]=

$$-\frac{\left(\frac{5}{4}-\frac{3\,n}{\frac{1}{2}+j}+\frac{3\,n^2}{2\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{2\left(\frac{1}{2}+j\right)^3}\right)\mu^6}{4\,n^6}$$

In[∘]:= **Simplify[%18 -]**

Out[∘]=

0

In[∘]:= **Quit[]**

Next terms as in (3.26)-(3.27) together:

In[1]:=

$$\frac{1}{\mathrm{Sqrt}\left[1+\mu^2/(n+\nu)^2\right]}$$

Out[1]=

$$\frac{1}{\sqrt{1+\frac{\mu^2}{(n+\nu)^2}}}$$

In[2]:= EnergyDirac[μ_] = % /. ν → Sqrt[(j + 1/2)^2 - μ^2]

Out[2]=
$$\frac{1}{\sqrt{1+\frac{\mu^2}{\left(n+\sqrt{\left(\frac{1}{2}+j\right)^2-\mu^2}\right)^2}}}$$

In[3]:= Simplify[PowerExpand[Series[EnergyDirac[μ], {μ, 0, 9}, Assumptions → 2 * j + 1 > 0]]]

Out[3]=
$$1-\frac{2\mu^2}{(1+2j+2n)^2}-\frac{2(1+2j+8n)\mu^4}{(1+2j)(1+2j+2n)^4}-\frac{4\left(1+8j^3+12n+48n^2+16n^3+12j^2(1+4n)+6j(1+4n)^2\right)\mu^6}{(1+2j)^3(1+2j+2n)^6}-\left(2\left(5+160j^5+80n+512n^2+1472n^3+1024n^4+256n^5+80j^4(5+16n)+16j^3(5+16n)^2+8j^2\left(25+240n+768n^2+736n^3\right)+j\left(50+640n+3072n^2+5888n^3+2048n^4\right)\right)\mu^8\right)\Big/\left((1+2j)^5(1+2j+2n)^8\right)+O[\mu]^{10}$$

In[4]:= FullSimplify[% /. n → n - j - 1/2]

Out[4]=
$$1-\frac{\mu^2}{2n^2}+\frac{\left(3-\frac{8n}{1+2j}\right)\mu^4}{8n^4}-\frac{\left(5(1+2j)^3-24(1+2j)^2n+24(1+2j)n^2+16n^3\right)\mu^6}{16\left((1+2j)^3n^6\right)}-\frac{1}{128\left((1+2j)^5n^8\right)}\left(-35(1+2j)^5+240(1+2j)^4n-480(1+2j)^3n^2+64(1+2j)^2n^3+384(1+2j)n^4+256n^5\right)\mu^8+O[\mu]^{10}$$

In[5]:=
$$\text{EnergyDiracExpMore}[\mu\_] := 1-\frac{\mu^2}{2n^2}+\frac{\left(\frac{3}{4}-\frac{n}{\frac{1}{2}+j}\right)\mu^4}{2n^4}-\frac{\left(\frac{5}{4}-\frac{3n}{\frac{1}{2}+j}+\frac{3n^2}{2\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{2\left(\frac{1}{2}+j\right)^3}\right)\mu^6}{4n^6}-\frac{\left(-\frac{35}{8}+\frac{15n}{\frac{1}{2}+j}-\frac{15n^2}{\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{\left(\frac{1}{2}+j\right)^3}+\frac{3n^4}{\left(\frac{1}{2}+j\right)^4}+\frac{n^5}{\left(\frac{1}{2}+j\right)^5}\right)\mu^8}{16n^8}$$

In[6]:= EnergyDiracExpMore[μ]

Out[6]=
$$1-\frac{\mu^2}{2n^2}+\frac{\left(\frac{3}{4}-\frac{n}{\frac{1}{2}+j}\right)\mu^4}{2n^4}-\frac{\left(\frac{5}{4}-\frac{3n}{\frac{1}{2}+j}+\frac{3n^2}{2\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{2\left(\frac{1}{2}+j\right)^3}\right)\mu^6}{4n^6}-\frac{\left(-\frac{35}{8}+\frac{15n}{\frac{1}{2}+j}-\frac{15n^2}{\left(\frac{1}{2}+j\right)^2}+\frac{n^3}{\left(\frac{1}{2}+j\right)^3}+\frac{3n^4}{\left(\frac{1}{2}+j\right)^4}+\frac{n^5}{\left(\frac{1}{2}+j\right)^5}\right)\mu^8}{16n^8}$$

In[7]:= EnergyDiracExpMoreJ[μ_] :=
$$1-\frac{\mu^2}{2n^2}+\frac{\left(3-\frac{8n}{1+2j}\right)\mu^4}{8n^4}-\frac{\left(5(1+2j)^3-24(1+2j)^2n+24(1+2j)n^2+16n^3\right)\mu^6}{16\left((1+2j)^3n^6\right)}-\frac{1}{128\left((1+2j)^5n^8\right)}\left(-35(1+2j)^5+240(1+2j)^4n-480(1+2j)^3n^2+64(1+2j)^2n^3+384(1+2j)n^4+256n^5\right)\mu^8$$

In[8]:= **EnergyDiracExpMoreJ**[μ]

Out[8]= $1-\frac{\mu^2}{2\,n^2}+\frac{\left(3-\frac{8\,n}{1+2\,j}\right)\mu^4}{8\,n^4}-\frac{\left(5\,(1+2\,j)^3-24\,(1+2\,j)^2\,n+24\,(1+2\,j)\,n^2+16\,n^3\right)\mu^6}{16\,(1+2\,j)^3\,n^6}-\frac{1}{128\,(1+2\,j)^5\,n^8}\left(-35\,(1+2\,j)^5+240\,(1+2\,j)^4\,n-480\,(1+2\,j)^3\,n^2+64\,(1+2\,j)^2\,n^3+384\,(1+2\,j)\,n^4+256\,n^5\right)\mu^8$

In[9]:= **Simplify**[**EnergyDiracExpMore**[μ] - **EnergyDiracExpMoreJ**[μ]]

Out[9]= 0

As a result, we verified the last two terms in (3.26)-(3.27).