PROCEEDINGS

SECOND INTERNATIONAL WORKSHOP ON QUANTUM NONSTATIONARY SYSTEMS

EDITORS:

Alexandre Dodonov
Caio Cesar Holanda Ribeiro

Proceedings of the Second International Workshop on Quantum Nonstationary Systems

ALEXANDRE DODONOV
CAIO CESAR HOLANDA RIBEIRO
(EDITORS)

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Preface

This is the book of proceedings of the Second International Workshop on Quantum Nonstationary Systems (QNS), held at the International Center of Physics (ICP) of University of Brasilia from August 28 to September 1, 2023. This hybrid format meeting was the second edition of a conference that occurred in 2009, at that time organized by the professors Viktor V. Dodonov (University of Brasilia, Brasilia, Brazil), Vladimir I. Man'ko (Lebedev Physics Institute, Moscow, Russia) and Salomon S. Mizrahi (Federal University of São Carlos, São Carlos, Brazil). Following the success of the first edition, the second QNS workshop gathered in Brasilia leading researchers and students in the field of quantum nonstationary phenomena, creating an environment where the latest achievements in the area, which comprises Quantum Information, Quantum Optics and cold atoms theory, could be divulged and debated. The event, organized by Alexandre Dodonov, Caio C. H. Ribeiro and Olavo L. S. F., also seized the opportunity to celebrate the 75-th birthday and the retirement of Prof. Viktor V. Dodonov from University of Brasilia.

The book contains 19 chapters. Chapter 1 gives a brief history and plans for the future of the Workshops on Quantum Nonstationary Systems (with photos of participants), while the remaining 18 chapters contain original works authored by the invited speakers:

Chapter 2: V. V. Dodonov and A. Dodonov from University of Brasilia, Brazil

Chapter 3: S. K. Suslov from Arizona State University, USA

Chapter 4: J. Tito Mendonça from Universidade de Lisboa, Portugal

Chapter 5: V. I. Yukalov and E. P. Yukalova from Joint Institute for Nuclear Research, Russia and Universidade de São Paulo, Brazil

Chapter 6: D. Valente from Universidade Federal de Mato Grosso, Brazil

Chapter 7: A. Vourdas from University of Bradford, UK

Chapter 8: S. S. Mizrahi from Federal University of São Carlos, Brazil

Chapter 9: J. P. Gazeau from Université Paris Cité, France

Chapter 10: Olavo L. S. F. from University of Brasilia, Brazil

Chapter 11: T. Mihaescu and A. Isar from National Institute of Physics and Nuclear Engineering and University of Bucharest, Romania

Chapter 12: A. Marinho and A. Dodonov from Universidade Federal Rural da Amazônia and University of Brasilia, Brazil

Chapter 13: S. N. Belolipetskiy, V. N. Chernega, V. I. Grebenkin and O. V. Man'ko from Bauman Moscow State Technical University, Russian University of Transport and Lebedev Physical Institute, Russia

Chapter 14: G. Wilson and B. M. Garraway from University of Sussex, UK

Chapter 15: C. C. Holanda Ribeiro from University of Brasilia, Brazil

Chapter 16: M. A. Man'ko and V. I. Man'ko from Lebedev Physical Institute, Russia

Chapter 17: E. P. Glasbrenner, Y. Gerdes, S. Varró and W. P. Schleich from Universität Ulm, Germany, ELI-ALPS Research Institute, Hungary and Texas A&M University, USA

Chapter 18: G. de Oliveira and L. C. Céleri from Federal University of Goias, Brazil

Chapter 19: B. Goren, K. K. Barley and S. K. Suslov from Arizona State University and Howard University, USA

The talks presented at the conference were stored online on the Youtube channel of the International Center of Physics at `www.youtube.com/@cifunb` . In addition to the technical presentations, the workshop also featured two talks with some historical accounts about the life and career of Prof. Viktor V. Dodonov, as the talk by Prof. Dodonov himself about his 50 years working with Quantum Mechanics (`www.youtube.com/watch?v=1WSdlRk0qGE`) and the presentation by Prof. Salomon S. Mizrahi about the move of the Dodonov family to Brazil (`www.youtube.com/watch?v=4trR7Yf8pBQ`). Additional information about the venue, participants and the program of the event is available at the website `https://is.gd/2ndQNS`.

Alexandre Dodonov and Caio Cesar Holanda Ribeiro
Brasilia – DF – Brazil
May 12, 2024

Contents

Chapter 19

Matrix Approach to Helicity States of Dirac Free Particles

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Could anything at first sight seem more impractical than a body which is so small that its mass is an insignificant fraction of the mass of an atom of hydrogen?

Comment on the electron by J. J. Thomson, Nobel Prize in Physics 1906 [29]

I studied mathematics with passion because I considered it necessary for the study of physics, *to which I want to dedicate myself exclusively.*

Comment on his mathematical education by young Enrico Fermi, Nobel Prize in Physics 1938 [45]

19.1 Abstract

We use elementary matrix algebra to derive the free wave solutions of the Dirac equation and examine the fundamental concepts of spin, polarization, and helicity in detail. The special problems associated with helicity states are also discussed. This consideration can guide the readers in studying the mathematical methods of relativistic quantum mechanics with the aid of a computer algebra system.

19.2 Introduction

Although the revolutionary nonrelativistic Schrödinger equation allowed us to explain experimental spectra of atoms and molecules, the magnificent relativistic Dirac equation laid the sound foundation of the electron's spin, predicted antimatter, and paved the way to the quantum fields theory (see, for example, [1], [2], [4], [5], [9], [32], [37], [39], [58], and the references therein). The electron, as the first discovered elementary particle, has been, ever since its encounter at the end of the nineteenth century, subject to intense theoretical and experimental investigations. Among them are the study of radiative corrections of the electron magnetic moment [3], [28], [29], [44], testing quantum electrodynamics in strong fields on hydrogen- and helium-like uranium [26], [27], and measuring the hyperfine interval in atomic hydrogen (see recent article [8]). The results of advanced theoretical calculations of quantum electrodynamic effects [46], [47], [52], [53] enter as essential input parameters and preliminary estimates in the evaluation of the experimental data [29], [51]. Nowadays, physicists have gained a deeper understanding of electrons as fundamental building blocks that make up matter.

From a pedagogical perspective, a set of 2×2 matrices is so simple to operate that it forms the foundation of every introduction to linear algebra. In nonrelativistic quantum mechanics, the 2×2 matrices and the corresponding $\mathbb{C}^2$ vectors (spinors) are primary to Wolfgang Pauli's theory of spin [32], [35], [42], [49], [57]. Solving the Dirac equation for a free particle, a similar ansatz can be used, namely: 4×4 matrices in a certain 2×2 partitioned block form; these are almost as easy to work with as 2×2 matrices. In this chapter, we shall utilize this simplicity to show that eigenvalues and corresponding eigenvectors (bi-spinors) of Dirac's free particles, as well as their polarization density matrices, can be found in a compact form. In our opinion, this is essential for better understanding of the relativistic concept of spin.

In the following section, we introduce basics of the Dirac equation (for a free particle). Then, the familiar standard solutions are verified by matrix multiplication. Next, we correct a mistake in Fermi's lecture notes [18] (see [10], [21], [40], [43], [45], and Appendix B regarding Fermi's teaching style). The nonrelativistic and relativistic helicity states of a free Dirac particle are discussed in the penultimate section, once again, with the aid of simple matrix algebra. We conclude by introducing the corresponding polarization density matrices and also discuss the property of charge conjugation. Appendix A presents a summary of some relevant theorems from the theory of matrices; and Appendix B contains our slightly edited typeset version of Fermi's lecture on the relativistic electron, together with his original hand-written notes, for the reader's convenience. Appendix C holds an original abstract of the talk by Michel and Wightman [36].

We hope that our somewhat informal presentation may help beginners to enjoy the study of mathematics of relativistic quantum mechanics. To this end, we include important details of calculations which are usually omitted elsewhere. Computer algebra methods are useful for verification (our complementary Mathematica notebook is posted at `https://community.wolfram.com/groups/-/m/t/2933767`). This approach is motivated by a course in quantum mechanics which the third-named author was teaching at Arizona State University for more than two decades (see also [4], [16], and [55] for more information).

This work is dedicated to Professor Viktor V. Dodonov on the occasion of his seventy-fifth birthday.

19.3 Dirac Equation

The relativistic wave equation of Dirac for a free particle with spin 1/2, with the electron as the archetypal example, is given by [2], [5], [6], [7], [9], [13], [14], [15], [18], [30], [35], [37], [39], [41] [42], [49], [55], [58]:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \widehat{H}\psi(\mathbf{r},t), \tag{19.1}$$

where i is the imaginary unit; $\hbar$ is the reduced Planck constant; $\mathbf{r}$ is a three-component vector representing position in space; t is time; and ψ is a complex-valued function to be discussed later.

Before presenting the Hamiltonian, $\widehat{H}$, of this equation (which represents the energy of the system), we introduce the matrix elements from which it will be build. The standard Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19.2}$$

Note that these correspond with the $x = x_1$, $y = x_2$, and $z = x_3$ axes (respectively) of three-dimensional Euclidean space. Also, let

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19.3}$$

From those 2×2 matrices as building blocks, we construct several 4×4 (Hermitian) matrices. For each k in the set $\{1, 2, 3\}$, let

$$\alpha_k = \begin{pmatrix} O & \sigma_k \\ \sigma_k & O \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}. \tag{19.4}$$

For example, we may expand (see also equations (34.15) in Appendix B):

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19.5}$$

We may now give the Hamiltonian of the above evolutionary Schrödinger-type equation as follows

$$\widehat{H} = c\left(\boldsymbol{\alpha}\cdot\widehat{\boldsymbol{p}}\right) + mc^2\beta, \tag{19.6}$$

where $\boldsymbol{\alpha}\cdot\widehat{\boldsymbol{p}} = \alpha_1\widehat{p}_1 + \alpha_2\widehat{p}_2 + \alpha_3\widehat{p}_3$ with the linear momentum operator $\widehat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$, and where p_1, p_2, and p_3 are the components of linear momentum in the x, y, and z directions; see (19.9) below. As usual, m is the rest mass and c is the speed of light in vacuum. (Throughout this chapter we will generally use the same notation as in [30]; see also the references therein.)

The relativistic electron has a four component (bi-spinor) complex-valued wave function

$$\psi\left(\mathbf{r},t\right) = \begin{pmatrix} \psi_1\left(\mathbf{r},t\right) \\ \psi_2\left(\mathbf{r},t\right) \\ \psi_3\left(\mathbf{r},t\right) \\ \psi_4\left(\mathbf{r},t\right) \end{pmatrix}. \tag{19.7}$$

As a result, the Dirac equation (19.1) with the Hamiltonian (19.6) is a matrix equation that is equivalent to a system of four first order partial differential equations.

The standard harmonic plane wave solution for a frame of reference that is at rest has the form:

$$\psi = \psi(\mathbf{r}, t) = e^{\frac{\mathrm{i}}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Et)}u = \exp\left(\frac{\mathrm{i}}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Et)\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \tag{19.8}$$

where u is a constant four vector (bi-spinor). We choose the eigenfunctions with definite energy and linear momentum of the commuting energy and momentum operators:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = E\psi, \qquad \widehat{\boldsymbol{p}}\psi = \boldsymbol{p}\psi. \tag{19.9}$$

Substitution into the Dirac equation results in an eigenvalue problem:

$$\left(c\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^2\beta\right)u = Eu, \tag{19.10}$$

where E is the total energy and $\boldsymbol{p}$ is the linear momentum of the free electron.

We may write the same equation in block form using our partitioned matrices from above:

$$\begin{pmatrix} (mc^2 - E)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -(mc^2+E)I \end{pmatrix} u = 0. \tag{19.11}$$

Our immediate goal now is to show that this representation is very convenient for finding eigenvalues and their corresponding eigenvectors by means of simple matrix algebra. (In the following an identity matrix is usually assumed when needed.)

19.4 Matrix Algebra Plane Wave Solution

We would like to point out that the eigenvalue problem (19.10)–(19.11) can be easily solved with the help of the following matrix identity:

$$\begin{aligned} &\begin{pmatrix} \left(mc^2 - E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+E\right)I \end{pmatrix}\begin{pmatrix} \left(mc^2 + E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2-E\right)I \end{pmatrix} \\ &\qquad = \left(m^2c^4 + c^2\boldsymbol{p}^2 - E^2\right)\begin{pmatrix} I & O \\ O & I \end{pmatrix}. \end{aligned} \tag{19.12}$$

Here, the first 4×4 partitioned matrix, in the left hand side, is the same as in (19.11) and the second one obtained from the latter by the substitution $E \to -E$. They do commute.

Our second observation is the following: a 'mixed' partitioned matrix, where the first two columns of the second matrix are combined with the last two columns of the first one, has a diagonal square, namely,

$$\begin{aligned} &\begin{pmatrix} \left(mc^2 + E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+E\right)I \end{pmatrix}\begin{pmatrix} \left(mc^2 + E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+E\right)I \end{pmatrix} \\ &\qquad = \left(\left(mc^2+E\right)^2 + c^2\boldsymbol{p}^2\right)\begin{pmatrix} I & O \\ O & I \end{pmatrix}. \end{aligned} \tag{19.13}$$

It is easy to show that these two elementary facts give a standard solution of the above eigenvalue problem.

Let us verify both identities. Indeed, by the multiplication rule for partitioned matrices (19.160), one gets

$$\begin{pmatrix} \left(mc^2-E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+E\right)I \end{pmatrix} \begin{pmatrix} \left(mc^2+E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2-E\right)I \end{pmatrix}$$
$$= \begin{pmatrix} \left(m^2c^4-E^2\right)I + c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)^2 & \left(mc^2-E\right)\left(c\boldsymbol{\sigma}\cdot\boldsymbol{p}\right) - \left(c\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\left(mc^2-E\right) \\ \left(c\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\left(mc^2+E\right) - \left(mc^2+E\right)\left(c\boldsymbol{\sigma}\cdot\boldsymbol{p}\right) & c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)^2 + \left(m^2c^4-E^2\right)I \end{pmatrix}. \tag{19.14}$$

But $\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)^2 = \boldsymbol{p}^2\ I$ and, in view of diagonalization of the product, the first identity follows. The second one can be verified in a similar fashion.

According to our first identity (19.12), all four column vectors of the second matrix automatically give some eigenvectors provided

$$m^2c^4 + c^2\boldsymbol{p}^2 = E^2, \qquad \text{or} \qquad E = E_{\pm} = \pm R, \qquad R = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4}. \tag{19.15}$$

But the rank of this matrix is only two due to Theorem 4 on p. 47 in [24] (see also Appendix A for the reader's convenience) and only two of those column vectors are linearly independent. Indeed, in this case,

$$\begin{pmatrix} \left(mc^2+E\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2-E\right)I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{19.16}$$

and the required identity $D = CA^{-1}B$ is satisfied.

But our second properly normalized unitary matrix, namely,

$$U = \frac{1}{\sqrt{\left(mc^2+R\right)^2 + c^2\boldsymbol{p}^2}} \begin{pmatrix} \left(mc^2+R\right)I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+R\right)I \end{pmatrix},$$
$$U = U^{\dagger},\ U^2 = \text{Identity} \tag{19.17}$$

is, obviously, nonsingular and provide all four linearly independent column vectors. Thus, in an explicit matrix form,

$$U = \left(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}\right) = \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} I & \dfrac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{mc^2+R} \\ \dfrac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{mc^2+R} & -I \end{pmatrix}$$
$$= \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} 1 & 0 & \dfrac{cp_3}{mc^2+R} & \dfrac{c\left(p_1-\mathrm{i}p_2\right)}{mc^2+R} \\ 0 & 1 & \dfrac{c\left(p_1+\mathrm{i}p_2\right)}{mc^2+R} & \dfrac{-cp_3}{mc^2+R} \\ \dfrac{cp_3}{mc^2+R} & \dfrac{c\left(p_1-\mathrm{i}p_2\right)}{mc^2+R} & -1 & 0 \\ \dfrac{c\left(p_1+\mathrm{i}p_2\right)}{mc^2+R} & \dfrac{-cp_3}{mc^2+R} & 0 & -1 \end{pmatrix}. \tag{19.18}$$

As a result, in a traditional bi-spinor form, the normalized eigenvectors are given by

$$u^{(1)} = \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_3}{mc^2+R} \\ \dfrac{c\left(p_1+\mathrm{i}p_2\right)}{mc^2+R} \end{pmatrix}, \quad u^{(2)} = \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c\left(p_1-\mathrm{i}p_2\right)}{mc^2+R} \\ \dfrac{-cp_3}{mc^2+R} \end{pmatrix} \tag{19.19}$$

for the positive energy eigenvalues $E = E_+ = R = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4}$ (twice) with the projection of the spin on the third axis $\pm 1/2$, in the frame of reference when the particle is at rest, $\boldsymbol{p} = \boldsymbol{0}$, respectively, whereas the normalized eigenvectors:

$$u^{(3)} = \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} \dfrac{cp_3}{mc^2+R} \\ \dfrac{c\left(p_1+\mathrm{i}p_2\right)}{mc^2+R} \\ -1 \\ 0 \end{pmatrix}, \quad u^{(4)} = \sqrt{\frac{mc^2+R}{2R}} \begin{pmatrix} \dfrac{c\left(p_1-\mathrm{i}p_2\right)}{mc^2+R} \\ \dfrac{-cp_3}{mc^2+R} \\ 0 \\ -1 \end{pmatrix} \tag{19.20}$$

correspond to the negative energy eigenvalues $E = E_- = -R = -\sqrt{c^2\boldsymbol{p}^2 + m^2c^4}$ (twice), once again with the projection of the spin on the third axis $\pm 1/2$, when $\boldsymbol{p} = \boldsymbol{0}$, respectively. (For interpretations of the negative eigenvalues, see, for example, [7], [9], [15], [18], [35], [37], [41], [42]. Mathematica verification of the bi-spinors is given in the complementary notebook.)

Note. In the nonrelativistic limit, when $c \to \infty$, one gets

$$R = mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{(mc)^2}} = mc^2 + \frac{\boldsymbol{p}^2}{2m} + \ldots \tag{19.21}$$

and, therefore,

$$U = \left(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}\right) \to \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \qquad \text{as} \quad \frac{|\boldsymbol{p}|}{mc} \ll 1. \qquad \blacksquare \tag{19.22}$$

The relativistic bi-spinor solutions (19.18) can be verified by a direct substitution into (19.10). Up to a constant, say, for $E = E_+ = +R$, one gets

$$\begin{pmatrix} \left(mc^2 - R\right) I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2+R\right) I \end{pmatrix} \begin{pmatrix} I \\ \dfrac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{mc^2+R} \end{pmatrix} \tag{19.23}$$

$$= \begin{pmatrix} \left(mc^2 - R\right) I + \dfrac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)^2}{mc^2+R} = O \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} - \left(mc^2+R\right)\dfrac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{mc^2+R} = O \end{pmatrix},$$

and, in a similar fashion, for $E = E_- = -R:$

$$\begin{pmatrix} \left(mc^2 + R\right) I & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -\left(mc^2-R\right) I \end{pmatrix} \begin{pmatrix} \dfrac{c\boldsymbol{\sigma}\cdot\boldsymbol{p}}{mc^2+R} \\ -I \end{pmatrix} \tag{19.24}$$

$$= \begin{pmatrix} c\boldsymbol{\sigma}\cdot\boldsymbol{p} - c\boldsymbol{\sigma}\cdot\boldsymbol{p} = O \\ \dfrac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)^2}{mc^2+R} + \left(mc^2 - R\right) I = O \end{pmatrix}.$$

In our classification of the spin states above, we have emphasized that $\pm 1/2$ projections of spin on the third axis correspond to the frame of reference when the particle is at rest, $\boldsymbol{p} = \boldsymbol{0}$. Unfortunately, these solutions and their interpretation are not Lorentz invariant. Indeed, the spin state of a moving particle, generally speaking, does not coincide with the one in the frame of reference when the particle is at rest (the so-called relativistic spin rotation; see, for example, [33], [37] and references therein). A relativistic classification of the spin states will be discussed later in terms of the so-called helicity operator; see section 6.

Note. It worth noting that the matrix manipulation above allows us to bypass a traditional evaluation of the 4×4 determinant in (19.11), resulting into the fourth order characteristic polynomial. Nonetheless, in view of the formulas of Schur [24], namely, (19.161) and (19.162)– (19.163) in Appendix A, one can obtain

$$
\begin{aligned}
& \det \begin{pmatrix} \left(mc^2 - E\right) I & c\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ c\boldsymbol{\sigma} \cdot \boldsymbol{p} & -\left(mc^2 + E\right) I \end{pmatrix} \\
= \; & \det \left(-\left(mc^2 - E\right)\left(mc^2 + E\right) I - c^2 \left(\boldsymbol{\sigma} \cdot \boldsymbol{p}\right)^2 \right) \\
= \; & \det \left(\left(E^2 - c^2 \boldsymbol{p}^2 - m^2 c^4 \right) I \right) \\
= \; & \det \begin{pmatrix} E^2 - c^2 \boldsymbol{p}^2 - m^2 c^4 & 0 \\ 0 & E^2 - c^2 \boldsymbol{p}^2 - m^2 c^4 \end{pmatrix} \\
= \; & \left(E^2 - c^2 \boldsymbol{p}^2 - m^2 c^4 \right)^2 = 0.
\end{aligned} \tag{19.25}
$$

These details are usually omitted in the traditional approach (see, for example, [18] or [42]). ■

19.5 Comments on Fermi's Lecture Notes

According to our calculations, all four bi-spinors (26)–(27) on p. 34-6 for free spin 1/2 particle in [18], see also Appendix B, correspond to the positive energy eigenvalues $E = E_+ = +R = \sqrt{c^2 \boldsymbol{p}^2 + m^2 c^4}$. This fact can be verified by a direct substitution into equation (24) on p. 34-5. For example, in the case of the third bi-spinor $u^{(3)}$ given by equation (27) in Fermi's notes, one gets, up to a constant, that

$$
\begin{pmatrix} mc^2 & 0 & cp_3 & c\left(p_1 - \mathrm{i}p_2\right) \\ 0 & mc^2 & c\left(p_1 + \mathrm{i}p_2\right) & -cp_3 \\ cp_3 & c\left(p_1 - \mathrm{i}p_2\right) & -mc^2 & 0 \\ c\left(p_1 + \mathrm{i}p_2\right) & -cp_3 & 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \dfrac{cp_3}{R - mc^2} \\ \dfrac{c\left(p_1 + \mathrm{i}p_2\right)}{R - mc^2} \\ 1 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} cp_3 \left(\dfrac{mc^2}{R - mc^2} + 1 \right) = +R \dfrac{cp_3}{R - mc^2} \\ c\left(p_1 + \mathrm{i}p_2\right) \left(\dfrac{mc^2}{R - mc^2} + 1 \right) = +R \dfrac{c\left(p_1 + \mathrm{i}p_2\right)}{R - mc^2} \\ \dfrac{c^2 \boldsymbol{p}^2}{R - mc^2} - mc^2 = \dfrac{R^2 - m^2 c^4}{R - mc^2} - mc^2 = +R \\ c\left(p_1 + \mathrm{i}p_2\right) \dfrac{cp_3}{R - mc^2} - cp_3 \dfrac{c\left(p_1 + \mathrm{i}p_2\right)}{R - mc^2} = 0 \end{pmatrix} = +R \begin{pmatrix} \dfrac{cp_3}{R - mc^2} \\ \dfrac{c\left(p_1 + \mathrm{i}p_2\right)}{R - mc^2} \\ 1 \\ 0 \end{pmatrix}. \tag{19.26}
$$

Further details are left to the reader. The correct results are presented, for example, in [38] and [42] (they are verified in the complementary Mathematica file). As one can show, the original Fermi's bi-spinors are linearly dependent because the corresponding determinant is equal to zero (see our complementary Mathematica notebook).

Figure 19.1: Enrico Fermi hiking with students in New Mexico. *Courtesy of Peter Lax.*

19.6 Relativistic Helicity States

In the previous sections, *de facto*, we have used an important property that the Hamiltonian of a Dirac's free particle (19.6) commutes with the linear momentum operator,

$$\left[\widehat{H},\ \widehat{\boldsymbol{p}}\right] = 0, \tag{19.27}$$

in order to obtain the states with definite values of energy and linear momentum. But an electron has also spin, or intrinsic angular momentum [34]. The corresponding integral of motion, related to the so-called helicity operator, will be discussed here.

19.6.1 Helicity Operator

Introducing standard 4×4 spin matrices and the relativistic spin operator [35], [37], [42]:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & O \\ O & \boldsymbol{\sigma} \end{pmatrix}, \qquad \widehat{\boldsymbol{S}} = \frac{1}{2}\boldsymbol{\Sigma}, \tag{19.28}$$

one can show that

$$\left[\widehat{H},\ \boldsymbol{\Sigma}\right] = 2ic\left(\boldsymbol{\alpha} \times \widehat{\boldsymbol{p}}\right) \neq \boldsymbol{0}. \tag{19.29}$$

Indeed, by (19.6), in components,

$$\begin{aligned} \left[\widehat{H},\ \Sigma_q\right] &= \left[c\alpha_r\ \widehat{p_r} + mc^2\beta,\ \Sigma_q\right] \\ &= c\widehat{p}_r\left[\alpha_r,\ \Sigma_q\right] + mc^2\left[\beta,\ \Sigma_q\right], \end{aligned} \tag{19.30}$$

where

$$[\alpha_r,\ \Sigma_q] = 2\mathrm{i}e_{rqs}\alpha_s, \qquad [\beta,\ \Sigma_q] = 0, \tag{19.31}$$

by the block matrix multiplication (19.160) and familiar properties of Pauli's matrices:

$$[\sigma_q,\ \sigma_r] = 2\mathrm{i}e_{qrs}\sigma_s \qquad (q, r, s = 1, 2, 3)\,. \tag{19.32}$$

Here, e_{qrs} is the Levi-Civita symbol [32], [57].

As a result, we obtain

$$\left[\widehat{H},\ \boldsymbol{\Sigma}\cdot\widehat{\boldsymbol{p}}\right] = 2\mathrm{i}c\left(\boldsymbol{\alpha}\times\widehat{\boldsymbol{p}}\right)\cdot\widehat{\boldsymbol{p}} = \boldsymbol{0}, \tag{19.33}$$

and the following helicity operator:

$$\widehat{\Lambda} = \frac{\widehat{\boldsymbol{S}}\cdot\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|}, \tag{19.34}$$

commutes with the Hamiltonian of Dirac's free particle: $\left[\widehat{H},\ \widehat{\Lambda}\right] = 0$. Its eigenvalues $\lambda = \pm 1/2$ correspond to the relativistic states with the projections of spin in the directions $\pm\,\boldsymbol{p}$, respectively.

For a Dirac particle, only the total angular momentum operator, $\widehat{\boldsymbol{J}} = \widehat{\boldsymbol{L}}+\widehat{\boldsymbol{S}}$, $\widehat{\boldsymbol{L}} = \mathbf{r}\times\widehat{\boldsymbol{p}}$, namely, the sum of the orbital angular momentum operator and the spin, commutes with the Hamiltonian: $\left[\widehat{H},\ \boldsymbol{J}\right] = 0$ (see, for example, [37], [42] for more details). In other words, neither the orbital angular momentum $\widehat{\boldsymbol{L}}$, nor spin $\widehat{\boldsymbol{S}}$, are separately integrals of motion; only the total angular momentum $\widehat{\boldsymbol{J}} = \widehat{\boldsymbol{L}} + \widehat{\boldsymbol{S}}$ conserves. Therefore, the plane waves under consideration, as states with a given energy E and linear momentum $\boldsymbol{p}$, cannot have a definite projection of the particle spin $\widehat{\boldsymbol{S}}$ on an arbitrary axis, say $z = x_3$. Nonetheless, as our equations (19.33)–(19.34) show, the exceptions are the states with projection of spin towards and opposite to the direction of the linear momentum because $\widehat{\boldsymbol{L}}\cdot\widehat{\boldsymbol{p}} = (\mathbf{r}\times\widehat{\boldsymbol{p}})\cdot\widehat{\boldsymbol{p}} = 0$ and, therefore, $\widehat{\boldsymbol{J}}\cdot\widehat{\boldsymbol{p}} = \widehat{\boldsymbol{S}}\cdot\widehat{\boldsymbol{p}}$. In this section, we will construct these Lorentz invariant (covariant) spin states in detail.

19.6.2 Nonrelativistic Helicity States

To begin, one may recall that for the nonrelativistic spin operator [32], [57]:

$$\widehat{\boldsymbol{s}} = \frac{1}{2}\boldsymbol{\sigma}, \tag{19.35}$$

say, in the frame of reference, that is moving together with the particle, with the spin pointed out in the direction of a unit polarization vector $\boldsymbol{n} = \boldsymbol{n}\,(\theta, \varphi)$:

$$\begin{aligned} n_1 &= \sin\theta\cos\varphi, \\ n_2 &= \sin\theta\sin\varphi, \\ n_3 &= \cos\theta, \end{aligned} \tag{19.36}$$

the corresponding spinors satisfy the following eigenvalue problem (helicity states [5], [37], [57]):

$$\left(\widehat{\boldsymbol{s}}\cdot\boldsymbol{n}\right)\phi = \frac{1}{2}\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\phi = \lambda\phi. \tag{19.37}$$

The standard solutions are

$$\phi^{(1/2)}(\theta,\varphi)=\begin{pmatrix}\cos\dfrac{\theta}{2}\ e^{-\mathrm{i}\varphi/2}\\ \sin\dfrac{\theta}{2}\ e^{\mathrm{i}\varphi/2}\end{pmatrix}\quad\left(\text{when }\lambda=+\frac{1}{2}\right),\qquad \phi^{(1/2)}(0,0)=\begin{pmatrix}1\\0\end{pmatrix};\tag{19.38}$$

and

$$\phi^{(-1/2)}(\theta,\varphi)=\begin{pmatrix}-\sin\dfrac{\theta}{2}\ e^{-\mathrm{i}\varphi/2}\\ \cos\dfrac{\theta}{2}\ e^{\mathrm{i}\varphi/2}\end{pmatrix}\quad\left(\text{when }\lambda=-\frac{1}{2}\right),\qquad \phi^{(1/2)}(0,0)=\begin{pmatrix}0\\1\end{pmatrix}\tag{19.39}$$

with the projection of spin in the direction $\pm\boldsymbol{n}(\theta,\varphi)$, respectively.

As a result, in compact form,

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n})\,\phi^{(\pm 1/2)}=\pm\phi^{(\pm 1/2)},\tag{19.40}$$

and, vice versa,

$$\left(\phi^{(\pm 1/2)}\right)^{\dagger}\left(\boldsymbol{\sigma}\phi^{(\pm 1/2)}\right)=\pm\boldsymbol{n}=\pm\boldsymbol{n}(\theta,\varphi).\tag{19.41}$$

These facts can be verified by direct substitutions (see our Mathematica notebook for more details).

Both nonrelativistic helicity states can be thought of as the columns of the following 2×2 unitary matrix:

$$\Phi=\left(\phi^{(1/2)},\ \phi^{(-1/2)}\right)=\begin{pmatrix}\cos\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\sin\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2}\\ \sin\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2} & \cos\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2}\end{pmatrix},\qquad \Phi\,\Phi^{\dagger}=I,\tag{19.42}$$

which is related to the spin 1/2 finite rotation matrix [32], [57]. Introducing also,

$$\widetilde{\Phi}=\left(\phi^{(1/2)},\ -\phi^{(-1/2)}\right)=\begin{pmatrix}\cos\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \sin\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2}\\ \sin\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2}\end{pmatrix},\qquad \widetilde{\Phi}\,\widetilde{\Phi}^{\dagger}=I,\tag{19.43}$$

we will establish the following factorization properties of the helicity matrix:

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}=\widetilde{\Phi}\,\Phi^{\dagger}=\Phi\,\widetilde{\Phi}^{\,\dagger}.\tag{19.44}$$

Indeed, rewriting in the right-hand side, say $\widetilde{\Phi}\,\Phi^{\dagger}$, in explicit form, one gets:

$$\begin{aligned}&\begin{pmatrix}\cos\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \sin\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2}\\ \sin\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2}\end{pmatrix}\begin{pmatrix}\cos\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2} & \sin\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2}\\ -\sin\dfrac{\theta}{2}e^{\mathrm{i}\varphi/2} & \cos\dfrac{\theta}{2}e^{-\mathrm{i}\varphi/2}\end{pmatrix}\\ &\qquad=\begin{pmatrix}\cos\theta & \sin\theta\ e^{-\mathrm{i}\varphi}\\ \sin\theta\ e^{\mathrm{i}\varphi} & -\cos\theta\end{pmatrix}=\boldsymbol{\sigma}\cdot\boldsymbol{n},\end{aligned}\tag{19.45}$$

by the familiar double-angle trigonometric identities. As a result, the eigenvalue problem (19.37), or (19.40), is verified:

$$(\boldsymbol{\sigma} \cdot \boldsymbol{n})\, \Phi = \left(\widetilde{\Phi}\; \Phi^{\dagger}\right) \Phi = \widetilde{\Phi}\left(\Phi^{\dagger}\Phi\right) = \widetilde{\Phi}, \tag{19.46}$$

by a simple matrix multiplication. Thus,

$$(\boldsymbol{\sigma} \cdot \boldsymbol{n})\, \Phi = \widetilde{\Phi}, \qquad (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, \widetilde{\Phi} = \Phi, \tag{19.47}$$

say, due to a familiar relation $(\boldsymbol{\sigma} \cdot \boldsymbol{n})^2 = I$. (We haven't been able to find these calculations in the available literature; see also our Mathematica file for a verification.)

Note. Introducing the spin one-half polarization density matrix as follows

$$\rho_{rs}^{(\lambda)}(\boldsymbol{n}) := \phi_r^{(\lambda)}(\boldsymbol{n}) \left[\phi_s^{(\lambda)}(\boldsymbol{n})\right]^* \qquad (r, s = 1, 2), \tag{19.48}$$

one gets

$$\rho^{(\pm 1/2)}(\boldsymbol{n}) = \frac{1}{2}\left(I \pm \boldsymbol{\sigma} \cdot \boldsymbol{n}\right). \tag{19.49}$$

(More details can be found in [32], [37], [57].) ■

19.6.3 Relativistic Plane Waves

Let us extend the original solutions (19.18) and analyze them from a different perspective. For a free relativistic particle with the linear momentum $\boldsymbol{p}$ and energy E, one can look for solutions of (19.11) in the block form

$$v(p) = C \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{19.50}$$

where C is a constant and ϕ, χ are two-component spinors:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{19.51}$$

Then

$$\begin{aligned} \left(mc^2 - E\right)\phi + c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})\,\chi &= 0, \\ c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})\,\phi - \left(mc^2 + E\right)\chi &= 0. \end{aligned} \tag{19.52}$$

From the second (first) equation:

$$\chi = \frac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\phi \qquad \left(\phi = \frac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{E - mc^2}\chi\right) \tag{19.53}$$

and the substitution of this relation into the first (second) equation gives the relativistic spectrum (19.15) for a spinor ϕ (correspondingly, χ) related to an arbitrary polarization vector $\boldsymbol{n}$, in the frame of reference where the particle is at rest. [Substitution of the second relation (19.53) into the first one results in an identity on the spectrum (19.15), and vise versa.]

As a result, the general solutions under consideration have the forms [37]:

$$v(p) = C \begin{pmatrix} \phi \\ \dfrac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \qquad \left(\text{and } D \begin{pmatrix} \dfrac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2-E}\chi \\ -\chi \end{pmatrix}\right). \tag{19.54}$$

Note. These formulas can also be obtained by the (inverse) Lorentz boost towards to the direction $\boldsymbol{l} = \boldsymbol{p}/|\boldsymbol{p}|$ from the frame of reference when the particle is at rest. Indeed, the system (19.52) has a trivial solution $E = mc^2$, $\boldsymbol{p} = \boldsymbol{0}$ and $\chi = 0$ with an arbitrary spinor ϕ. The above transformation of energy and momentum takes the form:

$$\begin{pmatrix} E/c \\ |\boldsymbol{p}| \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} mc \\ 0 \end{pmatrix}, \tag{19.55}$$

or $E = mc^2\cosh\vartheta$ and $|\boldsymbol{p}| = mc\sinh\vartheta$. For positive energy eigenvalues, one gets

$$\begin{aligned} v(p) &= S_{L^{-1}}v(\boldsymbol{p}=\boldsymbol{0}) = e^{-\vartheta(\boldsymbol{\sigma}\cdot\boldsymbol{l})/2}v(0) = \left(\cosh\frac{\vartheta}{2} + (\boldsymbol{\alpha}\cdot\boldsymbol{l})\sinh\frac{\vartheta}{2}\right)v(0) \qquad (19.56) \\ &= \begin{pmatrix} \cosh\dfrac{\vartheta}{2} & (\boldsymbol{\sigma}\cdot\boldsymbol{l})\sinh\dfrac{\vartheta}{2} \\ (\boldsymbol{\sigma}\cdot\boldsymbol{l})\sinh\dfrac{\vartheta}{2} & \cosh\dfrac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \cosh\frac{\vartheta}{2} \begin{pmatrix} \phi \\ (\boldsymbol{\sigma}\cdot\boldsymbol{l})\tanh\dfrac{\vartheta}{2}\,\phi \end{pmatrix} \end{aligned}$$

(see, for example, [5], [30], [31], [37], [39], [48], [49] for more details). Here,

$$\cosh\frac{\vartheta}{2} = \sqrt{\frac{\cosh\vartheta+1}{2}} = \sqrt{\frac{E+mc^2}{2mc^2}}, \qquad \tanh\frac{\vartheta}{2} = \frac{\sinh\vartheta}{\cosh\vartheta+1} = \frac{c|\boldsymbol{p}|}{E+mc^2}. \tag{19.57}$$

Once again, we obtain the first equation (19.54) with $C = \sqrt{(mc^2+E)/2mc^2}$. ■

19.6.4 Relativistic Polarization Vector

We follow [2], [37] with somewhat different details. In covariant form, the Dirac equation (19.1)-(19.6) can be written as follows

$$(\gamma^\mu\widehat{p}_\mu - mc)\psi = 0, \qquad \widehat{p}_\mu = i\hbar\frac{\partial}{\partial x^\mu}, \tag{19.58}$$

where $x^\mu = (ct, \boldsymbol{r})$, $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$, and

$$\gamma^0 = \beta = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} = \begin{pmatrix} O & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & O \end{pmatrix}. \tag{19.59}$$

Throughout the chapter, we use Einstein's summation convention:

$$a^\mu b_\mu = g_{\mu\nu}a^\mu b^\nu = a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3 = a^0b_0 - \boldsymbol{a}\cdot\boldsymbol{b}, \tag{19.60}$$

when $a^\mu = (a^0, \boldsymbol{a})$ and $b_\mu = g_{\mu\nu}b^\nu = (b^0, -\boldsymbol{b})$, unless stated otherwise (see, for example, [30] and [37] for more details). Here, $g_{\mu\nu} = g^{\mu\nu}$ =diag$(1,-1,-1,-1)$ is the pseudo-Euclidean metric of Minkowski space and the familiar anticommutation relation holds

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad (\mu,\nu = 0,1,2,3). \tag{19.61}$$

The bi-spinor wave function for a free Dirac particle, corresponding to a given energy-momentum four vector, $p_\mu = (E/c, -\boldsymbol{p})$ with the relativistic invariant $p_\mu p^\mu = (E/c)^2 - \boldsymbol{p}^2 = m^2c^2$ in covariant form, is given by

$$\psi(x) = u(p)\, e^{-\mathrm{i}(p_\mu x^\mu)/\hbar}, \qquad E = +R = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4} > 0, \tag{19.62}$$

and the standard normalization is as follows

$$\overline{u}(p)\, u(p) = 1, \qquad \overline{u}(p) = u^\dagger(p)\, \gamma^0. \tag{19.63}$$

The matrix equations for $u(p)$ and $\overline{u}(p)$ take the form [37]:

$$(\gamma^\mu p_\mu - mc)\, u(p) = 0, \qquad \overline{u}(p)\, (\gamma^\mu p_\mu - mc) = 0. \tag{19.64}$$

The normalized bi-spinor is given by

$$u(p) = \sqrt{\frac{E + mc^2}{2mc^2}} \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\phi \end{pmatrix}, \tag{19.65}$$

for an arbitrary normalized spinor ϕ :

$$\phi^\dagger \phi = (\phi_1^*, \phi_2^*) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = |\phi_1|^2 + |\phi_2|^2 = 1. \tag{19.66}$$

Indeed, by definition (19.63),

$$\begin{aligned} &\left(\phi^\dagger, \phi^\dagger \frac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\right) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\phi \end{pmatrix} \\ &= \left(\phi^\dagger, -\phi^\dagger \frac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\right) \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{mc^2 + E}\phi \end{pmatrix} = \phi^\dagger\phi - \phi^\dagger \frac{c^2\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2}{(mc^2 + E)^2}\phi \\ &= \phi^\dagger\phi \left(1 - \frac{c^2\boldsymbol{p}^2}{(mc^2 + E)^2}\right) = \frac{2mc^2}{E + mc^2} = \frac{1}{|C|^2}. \end{aligned} \tag{19.67}$$

Let the spin of the particle be in a certain direction $\boldsymbol{n}$, in the frame when the particle is at rest, namely,

$$(\boldsymbol{\sigma} \cdot \boldsymbol{n})\, \phi = \phi, \qquad \phi^\dagger \boldsymbol{\sigma} \phi = \boldsymbol{n}. \tag{19.68}$$

[The last equation gives the expectation values of nonrelativistic spin operator (19.35) in this frame of reference.] Introducing the relativistic polarization (pseudo) vector as follows [2], [37], [41], [56]:[1]

$$a^\mu = \overline{\psi}(x)\, (\gamma_5 \gamma^\mu)\, \psi(x) = \overline{u}(p)\, (\gamma_5 \gamma^\mu)\, u(p), \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} O & I \\ I & O \end{pmatrix} = -\gamma_5, \tag{19.69}$$

[1]Note that $\boldsymbol{\alpha}\gamma^5 = \boldsymbol{\Sigma}$.

one can derive the components of this polarization four vector $a^\mu = \left(a^0, \boldsymbol{a}\right)$:

$$a^0 = \frac{(\boldsymbol{p}\cdot\boldsymbol{n})}{mc}, \qquad \boldsymbol{a} = \boldsymbol{n} + \frac{\boldsymbol{p}\,(\boldsymbol{p}\cdot\boldsymbol{n})}{m\,(E+mc^2)}. \tag{19.70}$$

Here, the following properties of Pauli's matrices,

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = \mathrm{i}\,(\boldsymbol{p}\times\boldsymbol{n})\cdot\boldsymbol{\sigma} + (\boldsymbol{p}\cdot\boldsymbol{n}) \tag{19.71}$$

and

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) = 2\boldsymbol{p}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) - \boldsymbol{p}^2\boldsymbol{\sigma}, \tag{19.72}$$

should be utilized along with (19.68). [These identities can be verified with the aid of commutator relations (19.32).]

By definition (19.69), in components,

$$\begin{aligned} a^0 &= |C|^2 \left(\phi^\dagger, -\phi^\dagger \frac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\right) \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \\ &= |C|^2 \left(\phi^\dagger, -\phi^\dagger \frac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\right) \begin{pmatrix} \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \\ -\phi \end{pmatrix} = \frac{2c\,|C|^2}{E+mc^2}\phi^\dagger\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,(\boldsymbol{\sigma}\cdot\boldsymbol{n})\,\phi \\ &= \frac{1}{mc}\phi^\dagger\,(\mathrm{i}\,(\boldsymbol{p}\times\boldsymbol{n})\cdot\boldsymbol{\sigma} + (\boldsymbol{p}\cdot\boldsymbol{n}))\,\phi = \frac{1}{mc}\,((\boldsymbol{p}\cdot\boldsymbol{n}) + \mathrm{i}\,(\boldsymbol{p}\times\boldsymbol{n})\cdot\boldsymbol{n}) = \frac{\boldsymbol{p}\cdot\boldsymbol{n}}{mc}. \end{aligned} \tag{19.73}$$

In a similar fashion,

$$\begin{aligned} \boldsymbol{a} &= |C|^2 \left(\phi^\dagger, -\phi^\dagger \frac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\right) \begin{pmatrix} \boldsymbol{\sigma} & O \\ O & -\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \\ &= |C|^2 \left(\phi^\dagger, -\phi^\dagger \frac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\right) \begin{pmatrix} \boldsymbol{\sigma}\phi \\ -\boldsymbol{\sigma}\dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \\ &= |C|^2 \left(\phi^\dagger\boldsymbol{\sigma}\phi + \frac{c^2}{(mc^2+E)^2}\phi^\dagger\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\phi\right), \end{aligned} \tag{19.74}$$

where

$$\phi^\dagger\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\phi = 2\boldsymbol{p}\left(\phi^\dagger\boldsymbol{\sigma}\phi\right)\cdot\boldsymbol{p} - \boldsymbol{p}^2\phi^\dagger\boldsymbol{\sigma}\phi,$$

in view of identity (19.72). As a result,

$$\begin{aligned} \boldsymbol{a} &= |C|^2 \left(\boldsymbol{n} + \frac{c^2}{(mc^2+E)^2}\left(2\boldsymbol{p}\,(\boldsymbol{n}\cdot\boldsymbol{p}) - \boldsymbol{p}^2\boldsymbol{n}\right)\right) \\ &= |C|^2 \left(1 - \frac{c^2\boldsymbol{p}^2}{(mc^2+E)^2}\right)\boldsymbol{n} + \frac{2c^2\,|C|^2}{(mc^2+E)^2}\boldsymbol{p}\,(\boldsymbol{n}\cdot\boldsymbol{p}) \\ &= \boldsymbol{n} + \frac{\boldsymbol{p}\,(\boldsymbol{p}\cdot\boldsymbol{n})}{m\,(E+mc^2)} \end{aligned} \tag{19.75}$$

by (19.68), as stated above.

The four vector (19.69)–(19.70) has an important feature, namely, in the frame of reference, when the particle is at rest, $\boldsymbol{p} = \boldsymbol{0}$, one gets

$$a^0 = 0, \qquad \boldsymbol{a} = \boldsymbol{n}, \tag{19.76}$$

thus extending the particle nonrelativistic $3D$ polarization (pseudo) vector. And vice versa, the polarization four vector (19.70) can be obtained from the latter one by the Lorentz transformation [5].

Moreover, the four vector a^μ has the following properties:

$$p_\mu a^\mu = 0, \qquad a^2 = a_\mu a^\mu = -\boldsymbol{n}^2 = -1. \tag{19.77}$$

Indeed, by definition (19.69) and Dirac's equations for bi-spinors $u(p)$, $\overline{u}(p)$, namely (19.64), one gets

$$\begin{aligned} p_\mu a^\mu &= \overline{u}(p)\,\gamma_5\,(p_\mu\gamma^\mu)\,u(p) = \frac{1}{2}\overline{u}(p)\,(\gamma_5\,(p_\mu\gamma^\mu) - (p_\mu\gamma^\mu)\,\gamma_5)\,u(p) \\ &= \frac{1}{2}\overline{u}(p)\,(\gamma_5 mc - mc\gamma_5)\,u(p) = 0 \end{aligned} \tag{19.78}$$

in view of a familiar anticommutator relation $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$ $(\mu = 0, 1, 2, 3)$. [This covariant orthogonality relation can be directly verified with the help of (19.70) and/or evaluated in the frame of reference, when the particle is at rest, see (19.76).]

It worth noting that the bi-spinor $u(p)$, corresponding to the Dirac free particle with the relativistic polarization vector a^μ, in addition to the Dirac equation,

$$(\gamma^\mu p_\mu - mc)\,u(p) = 0, \tag{19.79}$$

also does satisfy the following matrix equation,

$$(\gamma_5\gamma^\mu a_\mu + I)\,u(p) = 0 \tag{19.80}$$

(I is the 4×4 identity matrix), which can be verified by direct substitution of $u(p)$ from (19.65). [The latter can be thought of as a relativistic generalization of the first equation (19.68), because they coincide when $\boldsymbol{p} = \boldsymbol{0}$.] Indeed, up to a given normalization,

$$\begin{aligned} (\gamma_5\gamma^\mu a_\mu + I)\,u(p) &= \begin{pmatrix} I_2 - \boldsymbol{\sigma}\cdot\boldsymbol{a} & a^0 \\ -a^0 & I_2 + \boldsymbol{\sigma}\cdot\boldsymbol{a} \end{pmatrix} C \begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \\ &= C \begin{pmatrix} \phi - (\boldsymbol{\sigma}\cdot\boldsymbol{a})\,\phi + a^0\dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \\ -a^0\phi + \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi + \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{a})\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi \end{pmatrix} \\ &= C \begin{pmatrix} \phi - (\boldsymbol{\sigma}\cdot\boldsymbol{n})\,\phi - \dfrac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,(\boldsymbol{p}\cdot\boldsymbol{n})}{m\,(mc^2+E)}\phi + \dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{m\,(mc^2+E)}\phi = \boldsymbol{0} \\ -\dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})}{mc}\phi + c\dfrac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}) + (\boldsymbol{\sigma}\cdot\boldsymbol{n})\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi + \dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})\,c^2\boldsymbol{p}^2}{mc\,(mc^2+E)^2}\phi \\ = -\dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})}{mc}\phi + \dfrac{2c\,(\boldsymbol{p}\cdot\boldsymbol{n})}{E+mc^2}\phi + \dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})\left(E^2 - m^2c^4\right)}{mc\,(E+mc^2)^2}\phi \\ = \dfrac{(\boldsymbol{p}\cdot\boldsymbol{n})}{mc}\left(-1 + \dfrac{2mc^2}{E+mc^2} + \dfrac{E-mc^2}{E+mc^2}\right)\phi = \boldsymbol{0} \end{pmatrix} \end{aligned} \tag{19.81}$$

by (19.68) and (19.70)–(19.71).

It worth mentioning, in conclusion, that the current density four vector for a free Dirac particle with a definite energy and linear momentum is given by

$$j^{\mu}=\overline{\psi}\left(x\right)\left(\gamma^{\mu}\right)\psi\left(x\right)=\overline{u}\left(p\right)\left(\gamma^{\mu}\right)u\left(p\right)=\frac{p^{\mu}}{mc}\overline{u}\left(p\right)u\left(p\right) \tag{19.82}$$

for an arbitrary normalization of the spinor ϕ in (19.65) (see [37] for more details).

19.6.5 Expectation Values of Relativistic Spin Operator

Let

$$\left\langle \widehat{\boldsymbol{S}}\right\rangle =\frac{u^{\dagger}\left(\frac{1}{2}\boldsymbol{\Sigma}\right)u}{u^{\dagger}u},\qquad\left\langle \widehat{\boldsymbol{s}}\right\rangle =\frac{\phi^{\dagger}\left(\frac{1}{2}\boldsymbol{\sigma}\right)\phi}{\phi^{\dagger}\phi} \tag{19.83}$$

be expectation values for the relativistic, when $\boldsymbol{p}\neq\boldsymbol{0}$, and nonrelativistic, if $\boldsymbol{p}=\boldsymbol{0}$, spin operators for a Dirac particle, respectively, [37]. Then [2]

$$\left\langle \widehat{\boldsymbol{S}}\right\rangle =\frac{mc^{2}}{E}\left\langle \widehat{\boldsymbol{s}}\right\rangle +\frac{c^{2}\boldsymbol{p}}{E\left(E+mc^{2}\right)}\left(\boldsymbol{p}\cdot\left\langle \widehat{\boldsymbol{s}}\right\rangle \right). \tag{19.84}$$

Indeed, for bi-spinor (19.65), up to a constant,

$$\begin{aligned} u^{\dagger}\left(\boldsymbol{\Sigma}\right)u &= \left(\phi^{\dagger},\phi^{\dagger}\frac{c\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)}{mc^{2}+E}\right)\begin{pmatrix}\boldsymbol{\sigma} & O\\ O & \boldsymbol{\sigma}\end{pmatrix}\begin{pmatrix}\phi\\ \dfrac{c\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)}{mc^{2}+E}\phi\end{pmatrix} \\ &= \left(\phi^{\dagger},\phi^{\dagger}\frac{c\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)}{mc^{2}+E}\right)\begin{pmatrix}\boldsymbol{\sigma}\phi\\ \boldsymbol{\sigma}\dfrac{c\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)}{mc^{2}+E}\phi\end{pmatrix} \\ &= \phi^{\dagger}\boldsymbol{\sigma}\phi+\frac{c^{2}}{\left(E+mc^{2}\right)^{2}}\phi^{\dagger}\left[\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\boldsymbol{\sigma}\left(\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\right]\phi. \end{aligned} \tag{19.85}$$

In view of (19.72), one gets

$$u^{\dagger}\left(\boldsymbol{\Sigma}\right)u=\frac{2mc^{2}}{E+mc^{2}}\left[\left(\phi^{\dagger}\boldsymbol{\sigma}\phi\right)+\frac{\boldsymbol{p}}{m\left(E+mc^{2}\right)}\boldsymbol{p}\cdot\left(\phi^{\dagger}\boldsymbol{\sigma}\phi\right)\right]. \tag{19.86}$$

In a similar fashion, up to a constant,

$$u^{\dagger}u=\frac{2E}{E+mc^{2}}\left(\phi^{\dagger}\phi\right). \tag{19.87}$$

As a result, we obtain (19.84).

In a special case, when the z-axis is directed towars $\boldsymbol{p}$, one gets

$$\left\langle \widehat{S_{x}}\right\rangle =\frac{mc^{2}}{E}\left\langle \widehat{s_{x}}\right\rangle ,\qquad\left\langle \widehat{S_{y}}\right\rangle =\frac{mc^{2}}{E}\left\langle \widehat{s_{y}}\right\rangle ,\qquad\left\langle \widehat{S_{z}}\right\rangle =\left\langle \widehat{s_{z}}\right\rangle \tag{19.88}$$

(see [37] for more details).

[2]This result follows also from (19.69)–(19.70)

19.6.6 Relativistic Helicity States

When the spinor ϕ (respectively, χ) corresponds to the nonrelativistic helicity states (19.37)–(19.39) with the particular polarization vector $\boldsymbol{n} = \boldsymbol{p}/\left|\boldsymbol{p}\right|$ (in the frame of reference where the particle is at rest), the bi-spinors (19.54) become the common eigenfunctions of the commuting Hamiltonian and helicity operator: $\left[\widehat{H},\ \Lambda\right] = 0$. For example, up to a normalization,

$$\begin{aligned}
(\boldsymbol{\Sigma}\cdot\boldsymbol{p})\, v\,(p) &= C\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & O \\ O & \boldsymbol{\sigma}\cdot\boldsymbol{p}\end{pmatrix}\begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi\end{pmatrix} \\
&= C\begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\phi \\ (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi\end{pmatrix} = 2\lambda\left|\boldsymbol{p}\right| C\begin{pmatrix} \phi \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi\end{pmatrix},
\end{aligned} \tag{19.89}$$

or

$$\widehat{\Lambda} v\,(p) = \lambda v\,(p) \tag{19.90}$$

for our both helicity states (19.54).

Applying the block matrix multiplication rule (19.160), one gets:

$$\begin{aligned}
&\begin{pmatrix} \left(mc^2+E\right)\Phi & c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi \\ c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi & -\left(mc^2+E\right)\Phi\end{pmatrix}^{\dagger}\begin{pmatrix} \left(mc^2+E\right)\Phi & c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi \\ c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi & -\left(mc^2+E\right)\Phi\end{pmatrix} \\
&= \begin{pmatrix} \left(mc^2+E\right)\Phi^{\dagger} & c\Phi^{\dagger}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c\Phi^{\dagger}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -\left(mc^2+E\right)\Phi^{\dagger}\end{pmatrix}\begin{pmatrix} \left(mc^2+E\right)\Phi & c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi \\ c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi & -\left(mc^2+E\right)\Phi\end{pmatrix} \\
&= \left(\left(mc^2+E\right)^2 + c^2\boldsymbol{p}^2\right)\begin{pmatrix} I & O \\ O & I\end{pmatrix}
\end{aligned} \tag{19.91}$$

as an extension of our identity (19.13). Once again, all relativistic helicity states can be unified as the columns in the following 4×4 unitary matrix: [3]

$$\begin{aligned}
V &= \left(v_+^{(1/2)}(\boldsymbol{p}), v_+^{(-1/2)}(\boldsymbol{p}), v_-^{(1/2)}(-\boldsymbol{p}), v_-^{(-1/2)}(-\boldsymbol{p})\right) \\
&= \frac{1}{\sqrt{\left(mc^2+R\right)^2 + c^2\boldsymbol{p}^2}}\begin{pmatrix} \left(mc^2+R\right)\Phi & -c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi \\ c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\Phi & -\left(mc^2+R\right)\Phi\end{pmatrix} \\
&= \sqrt{\frac{mc^2+R}{2R}}\begin{pmatrix} \Phi & -\dfrac{c\left|\boldsymbol{p}\right|}{mc^2+R}\widetilde{\Phi} \\ \dfrac{c\left|\boldsymbol{p}\right|}{mc^2+R}\widetilde{\Phi} & -\Phi\end{pmatrix},
\end{aligned} \tag{19.92}$$

with the help of (19.47), that is similar to (19.18):

$$\begin{aligned}
V &= \left(v_+^{(1/2)}, v_+^{(-1/2)}, v_-^{(1/2)}, v_-^{(-1/2)}\right) \\
&= \sqrt{\frac{mc^2+R}{2R}}\begin{pmatrix} \left(\phi^{(1/2)},\ \phi^{(-1/2)}\right) & -\dfrac{c\left|\boldsymbol{p}\right|}{mc^2+R}\left(\phi^{(1/2)},\ -\phi^{(-1/2)}\right) \\ \dfrac{c\left|\boldsymbol{p}\right|}{mc^2+R}\left(\phi^{(1/2)},\ -\phi^{(-1/2)}\right) & -\left(\phi^{(1/2)},\ \phi^{(-1/2)}\right)\end{pmatrix}.
\end{aligned} \tag{19.93}$$

[3]It should be noted that in order to keep the same form of the Hamiltonian for all four bi-spinors, one has to replace $\boldsymbol{p}$ by $-\boldsymbol{p}$ in the last two of them; see the Mathematica file.

Here, the first two columns correspond to the positive energy eigenvalues $E = +R = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4}$ and the helicities $\lambda = \pm 1/2$, respectively. Whereas, the last two columns correspond to the negative energy eigenvalues, given by $E = -R = -\sqrt{c^2\boldsymbol{p}^2 + m^2c^4}$, and the helicities $\lambda = \pm 1/2$, respectively. One should verify these facts by direct substitutions, similar to (19.23)–(19.24).

For the Hamiltonian 4×4 matrix:

$$H = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + mc^2\beta, \tag{19.94}$$

one gets

$$H^2 = \left(c\boldsymbol{\alpha} \cdot \boldsymbol{p} + mc^2\beta\right)^2 = R^2 = c^2\boldsymbol{p}^2 + m^2c^4 \tag{19.95}$$

due to the familiar anticommutators:

$$\alpha_r\alpha_s + \alpha_s\alpha_r = 2\delta_{rs}, \qquad \alpha_r\beta + \beta\alpha_r = 0 \qquad (r, s = 1, 2, 3)\,. \tag{19.96}$$

As a result, we arrive at the following matrix version of the eigenvalue problem under consideration:

$$HV = R\widetilde{V}, \qquad H\widetilde{V} = RV, \tag{19.97}$$

where, by definition,

$$\widetilde{V} = \left(v_+^{(1/2)}, v_+^{(-1/2)}, -v_-^{(1/2)}, -v_-^{(-1/2)}\right). \tag{19.98}$$

One can obtain the following decompositions

$$H = R\widetilde{V}\ V^{-1} = RV\ \widetilde{V}^{-1} \tag{19.99}$$

similar to our factorization (19.44) of the nonrelativistic helicity matrix. (These results are verified in the Mathematica file, where explicit forms of these decompositions are presented; see also an example in our forthcoming section 5.8.)

19.6.7 Convenient Parametrization

Following [22], with somewhat different details, let us also discuss the Dirac plane waves of positive and negative helicities $\lambda = \pm 1/2$, but of positive energy only, in a slightly different notation. With

$$\psi = ve^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{19.100}$$

and using the abbreviations

$$k\eta = \frac{\omega}{c} - \frac{mc}{\hbar}, \qquad \frac{k}{\eta} = \frac{\omega}{c} + \frac{mc}{\hbar} \tag{19.101}$$

one can express the magnitude of particle linear momentum and its kinetic energy in terms of a suitable parameter η :

$$|\boldsymbol{p}| = \hbar k = mc\frac{2\eta}{1-\eta^2}, \qquad E = \hbar\omega = mc^2\frac{1+\eta^2}{1-\eta^2}. \tag{19.102}$$

Once again, we use two polar angles θ and φ in the direction of vector $\boldsymbol{k} = \boldsymbol{p}/\hbar$:

$$k_1 \pm \mathrm{i}k_2 = k\sin\theta\ e^{\pm\mathrm{i}\varphi}, \qquad k_3 = k\cos\theta. \tag{19.103}$$

The eigenfunctions of the helicity operator are given by (19.38)–(19.39) and (19.93) [22]:

$$v_+^{(1/2)} = \frac{1}{\sqrt{V\left(1+\eta^2\right)}} \begin{pmatrix} \cos\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ \sin\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \\ \eta\cos\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ \eta\sin\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \end{pmatrix} \tag{19.104}$$

for $\lambda = +1/2$ and

$$v_+^{(-1/2)} = \frac{1}{\sqrt{V\left(1+\eta^2\right)}} \begin{pmatrix} \sin\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ -\cos\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \\ -\eta\sin\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ \eta\cos\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \end{pmatrix} \tag{19.105}$$

for $\lambda = -1/2$. [One should compare these bi-spinors with the first two columns in our 4×4 helicity states matrix (19.93), where, in turn, the last two columns correspond to the negative energy eigenvalues.] Here, we use the standard normalization

$$\int_V \overline{\psi}\gamma^0\psi \; dv = \int_V \psi^\dagger\psi \; dv = \int_V u^\dagger u \; dv = 1. \tag{19.106}$$

It should be noted that this normalization is Lorentz-invariant, the integral being proportional to the total electric charge inside the volume V. (See [22] and [37] for more details.)

In the nonrelativistic limit, when $\eta \ll 1$, the last two components in the above bi-spinors may be neglected and we arrive at the two-component Pauli spin theory. The case of negative energy eigenvalues will be discussed later.

19.6.8 Decompositions of the Hamiltonian

We found the following matrix identity:

$$\begin{pmatrix} 1-\eta^2 & 0 & 2\eta\cos\theta & 2\eta\sin\theta e^{-\mathrm{i}\varphi} \\ 0 & 1-\eta^2 & 2\eta\sin\theta e^{\mathrm{i}\varphi} & -2\eta\cos\theta \\ 2\eta\cos\theta & 2\eta\sin\theta e^{-\mathrm{i}\varphi} & \eta^2-1 & 0 \\ 2\eta\sin\theta e^{\mathrm{i}\varphi} & -2\eta\cos\theta & 0 & \eta^2-1 \end{pmatrix} \tag{19.107}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ \sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} \\ \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ \eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ \sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ -\eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ -\eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \end{pmatrix}$$

and an analog of the second equation in (19.99); details are presented in the Mathematica file.

19.7 The Polarization Density Matrices

19.7.1 Positive Energy Eigenvalues

For a Dirac particle, with a given linear momentum $\boldsymbol{p}$, positive energy E, and a certain polarization λ, the wave function (19.62) is defined by the corresponding bi-spinor $u^{(\lambda)}(p)$, which satisfies the following equations in the momentum representation [37]:

$$(\widehat{p}-mc)\,u^{(\lambda)}(p)=0, \qquad \overline{u}^{(\lambda)}(p)\,(\widehat{p}-mc)=0, \tag{19.108}$$

by definition, $\widehat{p}=\gamma^\mu p_\mu=\gamma^0(E/c)-(\boldsymbol{\gamma}\cdot\boldsymbol{p})$ and $\overline{u}^{(\lambda)}(p)=\left(u^{(\lambda)}(p)\right)^\dagger\gamma^0$. Here, we choose the following normalization

$$\overline{u}^{(\lambda)}(p)\,u^{(\lambda)}(p)=2mc. \tag{19.109}$$

By (19.65)–(19.67), those bi-spinor solutions have the form

$$u^{(\lambda)}(p)=\sqrt{E+mc^2}\begin{pmatrix} \phi^{(\lambda)} \\ \dfrac{c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{mc^2+E}\phi^{(\lambda)} \end{pmatrix}, \qquad \psi_+(x)=u^{(\lambda)}(p)\,e^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{r}-Et)/\hbar} \tag{19.110}$$

Here $E=+R=\sqrt{c^2\boldsymbol{p}^2+m^2c^4}$ and the spinor $\phi^{(\lambda)}$ satisfies equations (19.68). (With a given linear momentum $\boldsymbol{p}$ and positive energy E, there are two possible polarization states of the particle; for example, the states with the given helicity $\lambda = \pm 1/2$ or the states with the projection of the spin on the third axis $S_3 = \pm 1/2$, in the frame of reference when the particle is at rest; see [5], [37] for more details.)

For evaluation of the relativistic scattering amplitudes for the spin 1/2 particle, the following bilinear form are important [5], [37]:

$$\sum_{\lambda} u_{\alpha}^{(\lambda)}(p)\, \overline{u}_{\beta}^{(\lambda)}(p) := \left[\Lambda_+(p)\right]_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3, 4)\,. \tag{19.111}$$

Here, $\Lambda_+(p)$ is a certain 4×4 matrix, the symbol $+$ denotes the states with the positive energy eigenvalues, and the summation is performed over both polarizations.

From equations (19.108) and normalization (19.109) one gets

$$\begin{aligned} (\widehat{p} - mc)\, \Lambda_+(p) &= 0, \qquad \Lambda_+(p)\, (\widehat{p} - mc) = 0, \\ \mathrm{Tr}\, \Lambda_+(p) &= 2mc \end{aligned} \tag{19.112}$$

and

$$\Lambda_+(p) = mc + \widehat{p}, \tag{19.113}$$

which can be explicitly verified from the definition (19.111) with the help of the following identity

$$\sum_{\lambda=-1/2}^{1/2} \phi_r^{(\lambda)}(\boldsymbol{n}) \left[\phi_s^{(\lambda)}(\boldsymbol{n})\right]^* = \delta_{rs} \qquad (r, s = 1, 2) \tag{19.114}$$

for the spinors (19.38)–(19.39).

19.7.2 Negative Energy Eigenvalues

In addition to (19.62), the Dirac equation (19.58) has also the following solutions in covariant form

$$\psi_-(x) = v(p)\, e^{\mathrm{i}(p_\mu x^\mu)/\hbar}, \qquad (\widehat{p} + mc)\, v(p) = (\gamma^\mu p_\mu + mc)\, v(p) = 0, \tag{19.115}$$

which can be verified by a direct substitution [37]. In contrast, for the states with negative energy eigenvalues, when $E = -R = -\sqrt{c^2 \boldsymbol{p}^2 + m^2 c^4}$ and $\boldsymbol{p} \to -\, \boldsymbol{p}$, say, in the second relation (19.54), with a certain polarization λ, once again, the corresponding bi-spinor solutions can be written as follows

$$v^{(\lambda)}(p) = \sqrt{R + mc^2} \begin{pmatrix} \dfrac{c\,(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{R + mc^2} \phi^{(\lambda)} \\ \phi^{(\lambda)} \end{pmatrix}, \quad \psi_-(x) = v^{(\lambda)}(p)\, e^{\mathrm{i}(Rt - \boldsymbol{p}\cdot\boldsymbol{r})/\hbar}. \tag{19.116}$$

Here,

$$\overline{v}^{(\lambda)}(p)\, v^{(\lambda)}(p) = -2mc \tag{19.117}$$

(cf. [2], [5], [6]), and

$$(\widehat{p} + mc)\, v^{(\lambda)}(p) = 0, \qquad \overline{v}^{(\lambda)}(p)\, (\widehat{p} + mc) = 0. \tag{19.118}$$

Moreover, the bi-spinors $u^{(\lambda)}(p)$ and $v^{(\lambda)}(p)$ satisfy the orthogonality condition

$$\overline{u}^{(\lambda)}(p)\, v^{(\lambda')}(p) = 0 \qquad \left(\lambda \neq \lambda'\right). \tag{19.119}$$

In a similar fashion, for the negative energy eigenvalues, we introduce

$$\sum_{\lambda} v_{\alpha}^{(\lambda)} (p) \overline{v}_{\beta}^{(\lambda)} (p) := -\left[\Lambda_{-} (p)\right]_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3, 4) \tag{19.120}$$

and derive

$$\Lambda_{-} (p) = mc - \widehat{p}. \tag{19.121}$$

The matrix operators $\Lambda_{+} (p)$ and $\Lambda_{-} (p)$ obey the following properties

$$\begin{aligned} \Lambda_{+} (p) + \Lambda_{-} (p) &= 2mc, \\ \Lambda_{+} (p) \Lambda_{-} (p) = \Lambda_{-} (p) \Lambda_{+} (p) &= 0, \\ \Lambda_{+}^{2} (p) = 2mc\, \Lambda_{+} (p), \quad \Lambda_{-}^{2} (p) &= 2mc\, \Lambda_{-} (p), \end{aligned} \tag{19.122}$$

which can be easily verified with the help of the relativistic invariant $(\widehat{p})^{2} = p^{2} = m^{2}c^{2}$. Indeed, in compact form,

$$\begin{aligned} (\widehat{p})^{2} &= (\gamma^{\mu} p_{\mu}) (\gamma^{\nu} p_{\nu}) = (\gamma^{\mu} \gamma^{\nu}) (p_{\mu} p_{\nu}) \\ &= \frac{1}{2} (\gamma^{\mu} \gamma^{\nu} + \gamma^{\nu} \gamma^{\mu}) (p_{\mu} p_{\nu}) + \frac{1}{2} (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) (p_{\mu} p_{\nu}) \\ &= g^{\mu\nu} p_{\mu} p_{\nu} = p_{\mu} p^{\mu} = p^{2} \end{aligned} \tag{19.123}$$

by (19.61). The last two relations in (19.122), up to a normalization, define the so-called projection operators to the states with positive and negative energy eigenvalues, respectively [37]. Moreover, two the last but one equations are similar to our matrix product (19.12) for the relativistic spectrum.

19.7.3 Polarization Density Matrix for Pure States

Moreover, for Dirac's particle, with a given polarization $\lambda = \pm 1/2$, one can show that [2], [37]:

$$u_{\alpha}^{(\lambda)} (p) \overline{u}_{\beta}^{(\lambda)} (p) := \left[\rho_{+} (p)\right]_{\alpha\beta}, \qquad \rho_{+} (p) = \frac{1}{2} (mc + \widehat{p}) (I - \gamma_{5} \widehat{a}); \tag{19.124}$$

$$v_{\alpha}^{(\lambda)} (p) \overline{v}_{\beta}^{(\lambda)} (p) := -\left[\rho_{-} (p)\right]_{\alpha\beta}, \qquad \rho_{-} (p) = \frac{1}{2} (mc - \widehat{p}) (I - \gamma_{5} \widehat{a}) \tag{19.125}$$

(no summation over λ in both equations; in this section, I denotes the 4×4 identity matrix), where $\widehat{a} = \gamma^{\mu} a_{\mu}$ and a^{μ} is the polarization four vector (19.70). As is worth noting, summation over two possible polarizations, in each of the above equations (19.124)–(19.125), with different signs of $\pm \boldsymbol{n}$ and, therefore, in all components of a^{μ}, results in (19.113) and (19.121), as expected. (More details on the polarization density matrices can be found in [2], [5], and [37].)

These relations can be verified by direct matrix multiplication. Indeed, for the bispinor (19.104) with $\lambda = 1/2$ and positive energy eigenvalues, one gets, for both polarizations $\lambda = \pm 1/2$, that

$$mc + \widehat{p} = \frac{2mc}{1 - \eta^{2}} \begin{pmatrix} 1 & 0 & -\eta \cos\theta & -\eta \sin\theta\, e^{-\mathrm{i}\varphi} \\ 0 & 1 & -\eta \sin\theta\, e^{\mathrm{i}\varphi} & \eta \cos\theta \\ \eta \cos\theta & \eta \sin\theta\, e^{-\mathrm{i}\varphi} & -\eta^{2} & 0 \\ \eta \sin\theta\, e^{\mathrm{i}\varphi} & -\eta \cos\theta & 0 & -\eta^{2} \end{pmatrix}. \tag{19.126}$$

Moreover, for $\lambda = 1/2$,

$$I - \gamma_5 \widehat{a} = \begin{pmatrix} I_2 + \boldsymbol{\sigma} \cdot \boldsymbol{a} & -a_0 I_2 \\ a_0 I_2 & I_2 - \boldsymbol{\sigma} \cdot \boldsymbol{a} \end{pmatrix}, \tag{19.127}$$

where

$$I_2 + \boldsymbol{\sigma} \cdot \boldsymbol{a} = \frac{2}{1-\eta^2} \begin{pmatrix} 1 - \left(1+\eta^2\right) \sin^2 \dfrac{\theta}{2} & \left(1+\eta^2\right) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}\ e^{-\mathrm{i}\varphi} \\ \left(1+\eta^2\right) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}\ e^{\mathrm{i}\varphi} & 1 - \left(1+\eta^2\right) \cos^2 \dfrac{\theta}{2} \end{pmatrix} \tag{19.128}$$

$$I_2 - \boldsymbol{\sigma} \cdot \boldsymbol{a} = \frac{2}{1-\eta^2} \begin{pmatrix} 1 - \left(1+\eta^2\right) \cos^2 \dfrac{\theta}{2} & -\left(1+\eta^2\right) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}\ e^{-\mathrm{i}\varphi} \\ -\left(1+\eta^2\right) \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}\ e^{\mathrm{i}\varphi} & 1 - \left(1+\eta^2\right) \sin^2 \dfrac{\theta}{2} \end{pmatrix} \tag{19.129}$$

by (19.70). As a result,

$$\left(\frac{2}{1-\eta^2}\right)^{-1} (I - \gamma_5 \widehat{a})$$
$$= \begin{pmatrix} \cos^2 \dfrac{\theta}{2} - \eta^2 \sin^2 \dfrac{\theta}{2} & \dfrac{1}{2}\left(1+\eta^2\right) \sin\theta\ e^{-\mathrm{i}\varphi} & -\eta & 0 \\ \dfrac{1}{2}\left(1+\eta^2\right) \sin\theta\ e^{\mathrm{i}\varphi} & \sin^2 \dfrac{\theta}{2} - \eta^2 \cos^2 \dfrac{\theta}{2} & 0 & -\eta \\ \eta & 0 & \sin^2 \dfrac{\theta}{2} - \eta^2 \cos^2 \dfrac{\theta}{2} & -\dfrac{1}{2}\left(1+\eta^2\right) \sin\theta\ e^{-\mathrm{i}\varphi} \\ 0 & \eta & -\dfrac{1}{2}\left(1+\eta^2\right) \sin\theta\ e^{\mathrm{i}\varphi} & \cos^2 \dfrac{\theta}{2} - \eta^2 \sin^2 \dfrac{\theta}{2} \end{pmatrix}.$$

Finally, by multiplication of the matrices (19.126) and (19.130), we obtain

$$\left(1-\eta^2\right) \begin{pmatrix} \cos^2 \dfrac{\theta}{2} & \frac{1}{2} \sin\theta\ e^{-\mathrm{i}\varphi} & -\eta \cos^2 \dfrac{\theta}{2} & -\frac{1}{2}\eta \sin\theta\ e^{-\mathrm{i}\varphi} \\ \frac{1}{2} \sin\theta\ e^{\mathrm{i}\varphi} & \sin^2 \dfrac{\theta}{2} & -\frac{1}{2}\eta \sin\theta\ e^{\mathrm{i}\varphi} & -\eta \sin^2 \dfrac{\theta}{2} \\ \eta \cos^2 \dfrac{\theta}{2} & \frac{1}{2}\eta \sin\theta\ e^{-\mathrm{i}\varphi} & -\eta^2 \cos^2 \dfrac{\theta}{2} & -\frac{1}{2}\eta^2 \sin\theta\ e^{-\mathrm{i}\varphi} \\ \frac{1}{2}\eta \sin\theta\ e^{\mathrm{i}\varphi} & \eta \sin^2 \dfrac{\theta}{2} & -\frac{1}{2}\eta^2 \sin\theta\ e^{\mathrm{i}\varphi} & -\eta^2 \sin^2 \dfrac{\theta}{2} \end{pmatrix}. \tag{19.130}$$

As is easy to verify, this matrix is equal to the left hand side, namely the tensor product $\left[\rho_+ (p)\right]_{\alpha\beta}$, of the relation (19.124) for the bi-spinor (19.104), up to the required constant. The case $\lambda = -1/2$ and $E > 0$ can be verified in a similar fashion, with the help of a computer algebra system (see our Mathematica file).

For the negative energy eigenvalues, one can present the corresponding bi-spinors (19.116), as follows

$$v_-^{(\lambda=-1/2)} = \frac{1}{\sqrt{V\left(1+\eta^2\right)}} \begin{pmatrix} \eta \cos \dfrac{\theta}{2} \exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ \eta \sin \dfrac{\theta}{2} \exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \\ \cos \dfrac{\theta}{2} \exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right) \\ \sin \dfrac{\theta}{2} \exp\left(\mathrm{i}\dfrac{\varphi}{2}\right) \end{pmatrix} \tag{19.131}$$

and

$$v_{-}^{(\lambda=1/2)}=\frac{1}{\sqrt{V\left(1+\eta^{2}\right)}}\begin{pmatrix}\eta\sin\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right)\\ -\eta\cos\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right)\\ -\sin\dfrac{\theta}{2}\exp\left(-\mathrm{i}\dfrac{\varphi}{2}\right)\\ \cos\dfrac{\theta}{2}\exp\left(\mathrm{i}\dfrac{\varphi}{2}\right)\end{pmatrix}, \tag{19.132}$$

say, in the normalization (19.106). [It should be noted that the substitution $E\to-E$ and $\boldsymbol{p}\to-\boldsymbol{p}$, in (19.115)–(19.116), implies that $\lambda\to-\lambda$.] Here and in (19.104)–(19.105), one must replace

$$\frac{1}{\sqrt{V\left(1+\eta^{2}\right)}}\to\sqrt{\frac{2mc}{1-\eta^{2}}} \tag{19.133}$$

in the case of the relativistically invariant normalization (19.117), which we are using in this section. Once again, for both polarizations $\lambda=\pm1/2$, one gets

$$mc-\widehat{p}=-\frac{2mc}{1-\eta^{2}}\begin{pmatrix}\eta^{2} & 0 & -\eta\cos\theta & -\eta\sin\theta\ e^{-\mathrm{i}\varphi}\\ 0 & \eta^{2} & -\eta\sin\theta\ e^{\mathrm{i}\varphi} & \eta\cos\theta\\ \eta\cos\theta & \eta\sin\theta\ e^{-\mathrm{i}\varphi} & -1 & 0\\ \eta\sin\theta\ e^{\mathrm{i}\varphi} & -\eta\cos\theta & 0 & -1\end{pmatrix}. \tag{19.134}$$

Let us choose $\lambda=-1/2$, when one can use (19.70), once again, but with $E\to|E|=R=\sqrt{c^{2}\boldsymbol{p}^{2}+m^{2}c^{4}}$ and $\boldsymbol{n}\to-\boldsymbol{n}$. As a result,

$$\begin{aligned}&\left(\frac{2}{1-\eta^{2}}\right)^{-1}\left(I+\gamma_{5}\widehat{a}\right)\\ &=\begin{pmatrix}\sin^{2}\dfrac{\theta}{2}-\eta^{2}\cos^{2}\dfrac{\theta}{2} & -\dfrac{1}{2}\left(1+\eta^{2}\right)\sin\theta\ e^{-\mathrm{i}\varphi} & \eta & 0\\ -\dfrac{1}{2}\left(1+\eta^{2}\right)\sin\theta\ e^{\mathrm{i}\varphi} & \cos^{2}\dfrac{\theta}{2}-\eta^{2}\sin^{2}\dfrac{\theta}{2} & 0 & \eta\\ -\eta & 0 & \cos^{2}\dfrac{\theta}{2}-\eta^{2}\sin^{2}\dfrac{\theta}{2} & \dfrac{1}{2}\left(1+\eta^{2}\right)\sin\theta\ e^{-\mathrm{i}\varphi}\\ 0 & -\eta & \dfrac{1}{2}\left(1+\eta^{2}\right)\sin\theta\ e^{\mathrm{i}\varphi} & \sin^{2}\dfrac{\theta}{2}-\eta^{2}\cos^{2}\dfrac{\theta}{2}\end{pmatrix}.\end{aligned}$$

Finally, by multiplication of the matrices in (19.134) and (19.135) one gets

$$\left(1-\eta^{2}\right)\begin{pmatrix}\eta^{2}\cos^{2}\dfrac{\theta}{2} & \frac{1}{2}\eta^{2}\sin\theta\ e^{-\mathrm{i}\varphi} & -\eta\cos^{2}\dfrac{\theta}{2} & -\frac{1}{2}\eta\sin\theta\ e^{-\mathrm{i}\varphi}\\ \frac{1}{2}\eta^{2}\sin\theta\ e^{\mathrm{i}\varphi} & \eta^{2}\sin^{2}\dfrac{\theta}{2} & -\frac{1}{2}\eta\sin\theta\ e^{\mathrm{i}\varphi} & -\eta\sin^{2}\dfrac{\theta}{2}\\ \eta\cos^{2}\dfrac{\theta}{2} & \frac{1}{2}\eta\sin\theta\ e^{-\mathrm{i}\varphi} & -\cos^{2}\dfrac{\theta}{2} & -\frac{1}{2}\sin\theta\ e^{-\mathrm{i}\varphi}\\ \frac{1}{2}\eta\sin\theta\ e^{\mathrm{i}\varphi} & \eta\sin^{2}\dfrac{\theta}{2} & -\frac{1}{2}\sin\theta\ e^{\mathrm{i}\varphi} & -\sin^{2}\dfrac{\theta}{2}\end{pmatrix}. \tag{19.135}$$

This matrix is equal to the left hand side, namely the tensor product $\left[\rho_{-}\left(p\right)\right]_{\alpha\beta}$, of the relation (19.125) for the bi-spinor (19.131), up to the required constant. The case $\lambda=1/2$ can be verified in a similar fashion, say with the help of a computer algebra system. Further details are left to the reader.

Note. The bi-spinors (19.104)–(19.105) and (19.131)–(19.132) are linearly independent because the determinant:

$$\begin{vmatrix} \cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ \sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & -\eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} \\ \eta\cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\eta\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & -\sin\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} & \cos\frac{\theta}{2}e^{-\mathrm{i}\varphi/2} \\ \eta\sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \eta\cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \cos\frac{\theta}{2}e^{\mathrm{i}\varphi/2} & \sin\frac{\theta}{2}e^{\mathrm{i}\varphi/2} \end{vmatrix} = \left(1-\eta^2\right)^2 \quad (19.136)$$

is not zero when $|\eta| < 1$ (see our complementary Mathematica file). ■

Note. The formulas (19.124)–(19.125) permit one to make covariant polarization calculations in terms of traces [36] (see also Appendix C). In the original form, for the corresponding projection operators, one can write that

$$\frac{u_\alpha^{(\lambda)}(p)\,\overline{u}_\beta^{(\lambda)}(p)}{\overline{u}^{(\lambda)}(p)\,u^{(\lambda)}(p)} = \frac{1}{4mc}\left(mc+\widehat{p}\right)\left(I-\gamma_5\widehat{a}\right), \quad (19.137)$$

$$\frac{v_\alpha^{(\lambda)}(p)\,\overline{v}_\beta^{(\lambda)}(p)}{\overline{v}^{(\lambda)}(p)\,v^{(\lambda)}(p)} = \frac{1}{4mc}\left(mc-\widehat{p}\right)\left(I-\gamma_5\widehat{a}\right), \quad (19.138)$$

in view of (19.109) and (19.117), for the positive and negative energy eigenvalues, respectively. According to our calculations, in components, it means that

$$\frac{u_\alpha^{(\pm1/2)}(p)\,\overline{u}_\beta^{(\pm1/2)}(p)}{\overline{u}^{(\pm1/2)}(p)\,u^{(\pm1/2)}(p)} = \frac{1}{4mc}\sum_{\delta=1}^{4}\left(mc+\widehat{p}\right)_{\alpha\delta}\left(I\mp\gamma_5\widehat{a}\right)_{\delta\beta}, \quad E>0 \quad (19.139)$$

and

$$\frac{v_\alpha^{(\pm1/2)}(p)\,\overline{v}_\beta^{(\pm1/2)}(p)}{\overline{v}^{(\pm1/2)}(p)\,v^{(\pm1/2)}(p)} = \frac{1}{4mc}\sum_{\delta=1}^{4}\left(mc-\widehat{p}\right)_{\alpha\delta}\left(I\mp\gamma_5\widehat{a}\right)_{\delta\beta}, \quad E<0 \quad (19.140)$$

(all four relations have been verified by direct matrix multiplications with the aid of the Mathematica computer algebra system). ■

19.7.4 A Covariant Approach

Let us consider, for example, the case of positive energy eigenvalues $E > 0$ and the positive polarization $\lambda = 1/2$. In view of (19.102), one gets

$$mc+\widehat{p} = \frac{2mc}{1-\eta^2}\begin{pmatrix} I_2 & -\eta\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}) \\ \eta\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}) & -\eta^2 I_2 \end{pmatrix}. \quad (19.141)$$

When $\boldsymbol{p} = |\boldsymbol{p}|\ \boldsymbol{n}$, equations (19.70) take the form

$$a^0 = \frac{|\boldsymbol{p}|}{mc} = \frac{2\eta}{1-\eta^2}, \quad (19.142)$$

$$\boldsymbol{a} = \boldsymbol{n} + \frac{\boldsymbol{n}\,|\boldsymbol{p}|^2}{m\left(E+mc^2\right)} = \frac{1+\eta^2}{1-\eta^2}\boldsymbol{n}. \quad (19.143)$$

Therefore,

$$\boldsymbol{\sigma}\cdot\boldsymbol{a}=\frac{1+\eta^2}{1-\eta^2}\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right). \tag{19.144}$$

By (19.127) we obtain:

$$I-\gamma_5\widehat{a}=\tfrac{1}{1-\eta^2}\left(\begin{array}{cc}\left(1-\eta^2\right)I_2+\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -2\eta I_2\\ 2\eta I_2 & \left(1-\eta^2\right)I_2-\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{array}\right), \tag{19.145}$$

which is equivalent to our previous result (19.130) in view of (19.36).

Now, the product is given as follows

$$\begin{aligned}(mc+\widehat{p})\left(I-\gamma_5\widehat{a}\right)&=\frac{2mc}{\left(1-\eta^2\right)^2}\left(\begin{array}{cc}I_2 & -\eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\\ \eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\eta^2 I_2\end{array}\right) \\ &\times\left(\begin{array}{cc}\left(1-\eta^2\right)I_2+\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -2\eta I_2\\ 2\eta I_2 & \left(1-\eta^2\right)I_2-\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{array}\right).\end{aligned} \tag{19.146}$$

As a result of the matrix multiplication, performed in a 2×2 block form, we finally obtain with the help of (19.49) that

$$\begin{aligned}&\left(\begin{array}{cc}I_2 & -\eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\\ \eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\eta^2 I_2\end{array}\right)\left(\begin{array}{cc}\left(1-\eta^2\right)I_2+\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -2\eta I_2\\ 2\eta I_2 & \left(1-\eta^2\right)I_2-\left(1+\eta^2\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{array}\right)\\ &=\left(1-\eta^2\right)\left(\begin{array}{cc}I_2+\boldsymbol{\sigma}\cdot\boldsymbol{n} & -\eta\left(I_2+\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\\ \eta\left(I_2+\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\eta^2\left(I_2+\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{array}\right)=2\left(1-\eta^2\right)\left(\begin{array}{cc}\rho^{(1/2)}\left(\boldsymbol{n}\right) & -\eta\rho^{(1/2)}\left(\boldsymbol{n}\right)\\ \eta\rho^{(1/2)}\left(\boldsymbol{n}\right) & -\eta^2\rho^{(1/2)}\left(\boldsymbol{n}\right)\end{array}\right)\end{aligned} \tag{19.147}$$

provided $\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)^2=I_2$. This is equivalent to the first relation (19.137).

In covariant form, one gets

$$(mc+\widehat{p})\left(I-\gamma_5\widehat{a}\right)=mc\left(I-\gamma_5\widehat{a}\right)+\widehat{p}-\widehat{p}\gamma_5\widehat{a}. \tag{19.148}$$

Here,

$$\widehat{p}\gamma_5\widehat{a}=p_\mu a_\nu\left(\gamma^\mu\gamma_5\gamma^\nu\right)=-p_\mu a_\nu\gamma_5\left(\gamma^\mu\gamma^\nu\right), \tag{19.149}$$

in view of the anticomutator property, $\gamma_5\gamma^\nu+\gamma^\mu\gamma_5=0$. Using the substitution,

$$\begin{aligned}\gamma^\mu\gamma^\nu &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)+\frac{1}{2}\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)\\ &= \frac{1}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)+g^{\mu\nu}\end{aligned} \tag{19.150}$$

by (19.61), we obtain that

$$\widehat{p}\gamma_5\widehat{a}=-\Sigma^{\mu\nu}p_\mu a_\nu-p_\mu a^\mu, \tag{19.151}$$

where $p_\mu a^\mu=0$ in view of (19.77) and by the definition

$$\Sigma^{\mu\nu}=\frac{1}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)=\left(\begin{array}{cccc}0 & \alpha_1 & \alpha_2 & \alpha_3\\ -\alpha_1 & 0 & -\mathrm{i}\Sigma_3 & \mathrm{i}\Sigma_2\\ -\alpha_2 & \mathrm{i}\Sigma_3 & 0 & -\mathrm{i}\Sigma_1\\ -\alpha_3 & -\mathrm{i}\Sigma_2 & \mathrm{i}\Sigma_1 & 0\end{array}\right),\qquad \boldsymbol{\Sigma}=\left(\begin{array}{cc}\boldsymbol{\sigma} & O\\ O & \boldsymbol{\sigma}\end{array}\right). \tag{19.152}$$

As a result, one can write

$$\widehat{p}\gamma_5\widehat{a} = -\gamma_5\left(\widehat{pa}\right), \qquad \widehat{pa} := \Sigma^{\mu\nu}p_\mu a_\nu = -p_0\left(\boldsymbol{\alpha}\cdot\boldsymbol{a}\right) + a_0\left(\boldsymbol{\alpha}\cdot\boldsymbol{p}\right) - \mathrm{i}\,\boldsymbol{\Sigma}\cdot\left(\boldsymbol{p}\times\boldsymbol{a}\right) \tag{19.153}$$

(see, for example, [5], [30], or [37]). Our relation (19.148) takes the form

$$\left(mc + \widehat{p}\right)\left(I - \gamma_5\widehat{a}\right) = mc\left(I - \gamma_5\widehat{a}\right) + \widehat{p} + \gamma_5\widehat{pa}, \tag{19.154}$$

where

$$\widehat{p} = \frac{mc}{1-\eta^2}\begin{pmatrix} \left(1+\eta^2\right)I_2 & -2\eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) \\ 2\eta\left(\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\left(1+\eta^2\right)I_2 \end{pmatrix}, \quad \gamma_5\widehat{pa} = mc\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{n} & O \\ O & \boldsymbol{\sigma}\cdot\boldsymbol{n} \end{pmatrix}. \tag{19.155}$$

Using (19.145), one can reduce the last but one relation to the required form by an elementary calculation. All of the remaining cases can be verified in a similar fashion. We leave the details to the reader. In summary, we have re-calculated the projection operators (19.139)–(19.140) in 4×4 and in 2×2 block matrix forms, as well as in the covariant form in detail.

19.8 The Charge Conjugation

According to our explicit forms of bi-spinors (19.104)–(19.105) and (19.131)–(19.132), one gets

$$\mathrm{i}\gamma^2\left(v_+^{(1/2)}\right)^* = v_-^{(1/2)}, \tag{19.156}$$

$$\mathrm{i}\gamma^2\left(v_+^{(-1/2)}\right)^* = v_-^{(-1/2)}. \tag{19.157}$$

This fact implies the well-known property of charge conjugation for free Dirac particles:

$$\mathbf{C}\left[\psi_+(x)\right]^* = \psi_-(x), \qquad \mathrm{i}\gamma^2\left[u^{(\lambda)}(p)\right]^* = v^{(\lambda)}(p) \tag{19.158}$$

provided

$$\psi_+(x) = u^{(\lambda)}(p)\,e^{-\mathrm{i}(p_\mu x^\mu)/\hbar}, \qquad \psi_-(x) = v^{(\lambda)}(p)\,e^{\mathrm{i}(p_\mu x^\mu)/\hbar}. \tag{19.159}$$

(More details can be found in [2], [5], [37].)

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19.9 Appendix A: Some Required Facts From Matrix Algebra

When two partitioned matrices have the same shape and their diagonal blocks are square matrices of equal size, then the following multiplication rule holds:

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix} \tag{19.160}$$

that is similar to multiplication of 2×2 matrices [24].

Let us consider a determinant partitioned into four blocks:

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \tag{19.161}$$

where A and D are square matrices. There are formulas of Schur, which reduce the computation of a determinant of order $2n$ to the computation of a determinant of order n:

$$\Delta = |AD - CB| \tag{19.162}$$

provided $AC = CA$ or

$$\Delta = |AD - BC| \tag{19.163}$$

provided $CD = DC$ (see [24], pp. 45–46).

Finally, the following result is also important.

Theorem 19.9.1. *If a rectangular matrix $\mathcal{R}$ is represented in partitioned form*

$$\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{19.164}$$

where A is a square non-singular matrix of order n ($|A| \neq 0$), then the rank of $\mathcal{R}$ is equal to n if and only if

$$D = CA^{-1}B. \tag{19.165}$$

Proof. Let us subtract from the second row of blocks of $\mathcal{R}$ the first one, multiplied on the left by CA^{-1} (the generalized Gaussian elimination algorithm). We arrive at the following matrix:

$$\mathcal{S} = \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix},$$

which has the same rank as $\mathcal{R}$. But the rank of $\mathcal{S}$ coincides with the rank of A, namely n, if and only if

$$D - CA^{-1}B = O.$$

This proves the theorem (see p. 47 in [24]). ■

19.10 Appendix B: Fermi's Lecture Notes on Relativistic Free Electron

Here we present Enrico Fermi's original lecture notes for Physics 341/342: Quantum Mechanics at the University of Chicago [18] in the winter and spring of 1954 – two quarters and approximately sixty lectures – along with a typeset and very lightly edited transcription. At the end of each lecture, Fermi would always make up a problem, which was usually closely related to what he had just discussed that day. For further details on some subjects, Fermi occasionally referred to Leonard Schiff's book, *Quantum Mechanics*, First Edition [42] and Enrico Persico's book, *Fundamentals of Quantum Mechanics* [38]. Note that we have preserved Fermi's notation conventions. For example, whereas a typeset manuscript would typically use boldface for a vector, Fermi's handwritten notes naturally employ a superscripted arrow; here, we adhere to Fermi's original notation, retaining the arrow and representing vectors and matrices with single vertical lines (see also [38]).

At the beginning of his distinguished career, Fermi held a temporary job for the academic year 1923–1924 at the University of Rome, where he taught a mathematics course for chemists and biologists. From 1924 to 1926, Fermi lectured on mathematical physics and mechanics at the University of Florence. During this time, he thoroughly studied Schrödinger's theory through the original publications and privately explained it to his students in seminars. Additionally, he reworked some of Dirac's papers into a more accessible format, partly for didactic purposes [18]. Subsequently, Fermi became a professor of theoretical physics at the University of Rome, the first chair of this kind in Italy, where he taught for 12 years, starting in 1926 [10], [43], [45].

During this period of time, Fermi laid a sound foundation for education in modern physics in Italy (and abroad). He delivered many popular lectures and seminars and wrote a textbook and some articles for *Enciclopedia Italiana Treccani* [19]. His teaching style and personality attracted many talented students to the physics department. The whole generation of physicists worldwide had studied the quantum theory of radiation from his review article [17], which was based on the lectures delivered in the summer of 1930 at the University of Michigan, Ann Arbor, when Fermi visited the United States for the first time. The students recalled a remarkable atmosphere of immense enthusiasm and total dedication to physics and formed lifelong friendships. From as early as 1928, Fermi made little use of books; a collection of mathematical formulas and the tables of physical constants were almost the only reference books he had in his office. If a complicated equation was required in his research or teaching, Fermi could derive it by himself, usually faster than his students could find this result in the library books [45].

At the end of the year 1933, Fermi wrote his famous article on the explanation of beta decay. He sent a letter to *Nature* advancing his theory, but it was rejected, and instead, the article *Tentative theory of beta rays* was published in *Nuovo Cimento* in Italian and in *Zeischrift für Physik* in German [19], [21], [45]. The novel neutron bombardment experiments, systematically reported in *Ricerca Scientifica* letters in the summer of 1934, were equally "... successful escape from the sphere of theoretical physics" (in Rutherford's own words [45]). The neutron work, accomplished by that summer, was summarized in an article that was communicated by Lord Rutherford to the Royal Society of London. The subsequent discovery of slow neutron effects is now a part of the nuclear physics history... [10], [20], [21], [25], [40], [43], [45].

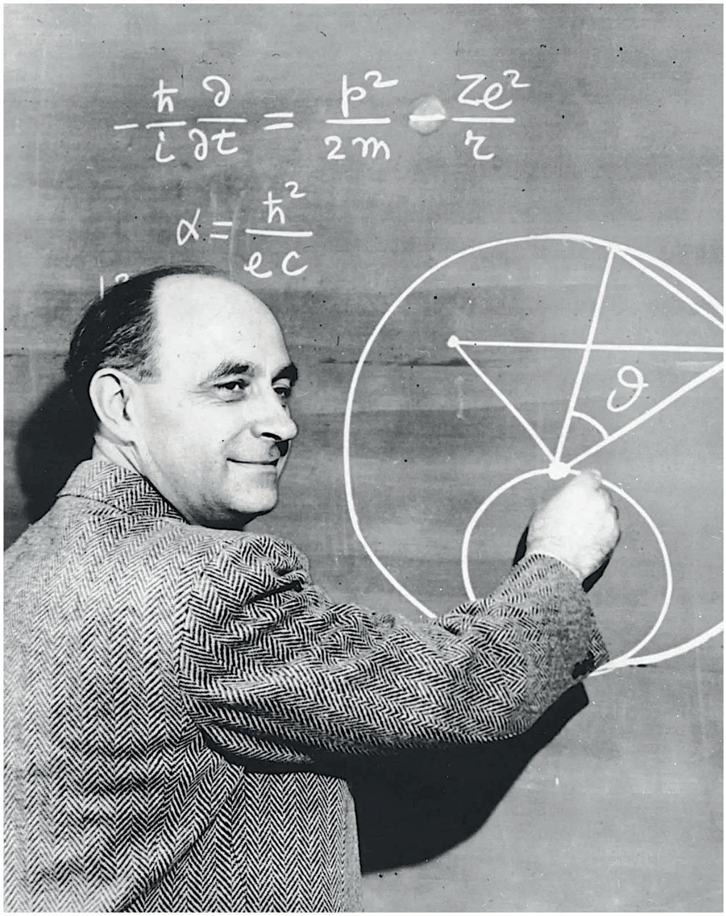

Figure 19.2: Enrico Fermi teaching quantum mechanics. *Courtesy of Argonne National Lab.* (The formula for α is, most likely, his idea of a joke [43]). `https://science.osti.gov/fermi/The-Life-of-Enrico-Fermi/formula`

The visit to Ann Arbor was a great scientific success, and Fermi returned there in the summers of 1933 and 1935. [4] Through these visits, he grew to appreciate America and the new opportunities it offered. He ultimately relocated to the United States soon after being awarded the Nobel Prize in Physics on December 10, 1938. Throughout his life, Fermi maintained a strong passion for teaching. He conducted numerous courses and seminars at the University of Rome, Columbia University, Los Alamos Lab, and the University of Chicago.

[4] In the year 1934, he went to Brazil [12]. In 1936, Fermi visited Colombia University for the summer session. Next year, he spent the summer in California [43] driving back to the East Coast through the entire country.

In the winter and spring semesters of his final year, before his untimely death in November, Fermi gave his last quantum mechanics course at the University of Chicago [18]. Subsequently, during the summer of 1954, he traveled to Europe. During his visit, Fermi presented a course on pions and nucleons at the Villa Monastero, located in Varenna on Lake Como. This course was part of the summer school organized by the Italian Physical Society, which now bears his name. Additionally, Fermi also attended the French summer school at Les Houches near Chamonix, where he delivered lectures [45]. [5]

[5] From Britannica: Enrico Fermi, (born Sept. 29, 1901, Rome, Italy – died Nov. 28, 1954, Chicago, Illinois, U. S. A.), an Italian-born American scientist who was one of the chief architects of the nuclear age. He developed the mathematical statistics required to clarify a large class of subatomic phenomena, explored nuclear transformations caused by neutrons, and directed the first controlled chain reaction involving nuclear fission. He was awarded the 1938 Nobel Prize for Physics, and the Enrico Fermi Award of the U.S. Department of Energy was given in his honor. Fermilab, the National Accelerator Laboratory, in Illinois, is named for him, as is fermium, element number 100. `https://www.britannica.com/biography/Enrico-Fermi`

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34 – ~~Dirac's Theory of the~~ free electron

Time dep. Schrodinger eq. for particle

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

treats t, x, y, z very non symmetrically. Search for relativistic equation for electron of first order in t, x, y, z.

Notation

$$(1)\quad \begin{cases} x = x_1 \quad y = x_2 \quad z = x_3 \quad ict = x_4 \quad (ct = x_0) \\ p_x = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x} \text{ or } p_i = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x_i} \\ p_4 = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x_4} = -\dfrac{\hbar}{c}\dfrac{\partial}{\partial t} = \dfrac{i}{c}E \end{cases}$$

use $E = i\hbar\dfrac{\partial}{\partial t}$

(2) Ordinary vectors $\vec{x} \equiv (x_1, x_2, x_3)$, $\vec{p} = (p_1, p_2, p_3)$

Sum over equal indices

(3) Four vectors $\underset{\sim}{x} \equiv (x_1\ x_2\ x_3\ x_4)$ or $\underset{\sim}{p} \equiv (p_1\ p_2\ p_3\ p_4)$

If ψ were a scalar, simplest first order eqn would be (constant coeff.)

$$\psi = a^{(1)}\frac{\partial\psi}{\partial x_1} + a^{(2)}\frac{\partial\psi}{\partial x_2} + a^{(3)}\frac{\partial\psi}{\partial x_3} + a^{(4)}\frac{\partial\psi}{\partial x_4} = \frac{i}{\hbar}a^{(\mu)}p_\mu\psi$$

It will prove necessary however to take ψ to have several (four) components. Instead of above, write

$$(4)\quad imc\,\psi_k = \gamma^{(\mu)}_{kl}\,p_\mu\,\psi_l = \frac{\hbar}{i}\gamma^{(\mu)}_{kl}\frac{\partial\psi_l}{\partial x_\mu}$$

34 – Dirac's theory of the free electron

The time-dependent Schrödinger equation for a particle

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

treats t, x, y, z very non-symmetrically.

We will search for the relativistic equation for an electron of first order in t, x, y, z.

Notation:

$$\begin{cases} x = x_1,\ y = x_2,\ z = x_3,\ \mathrm{i}ct = x_4\ (ct = x_0) \\ p_x = \dfrac{\hbar}{\mathrm{i}}\dfrac{\partial}{\partial x} \ \text{ or }\ p_i = \dfrac{\hbar}{\mathrm{i}}\dfrac{\partial}{\partial x_i} \\ p_4 = \dfrac{\hbar}{\mathrm{i}}\dfrac{\partial}{\partial x_4} = -\dfrac{\hbar}{c}\dfrac{\partial}{\partial t} = \dfrac{\mathrm{i}}{c}E \qquad \left(\text{use } E = \mathrm{i}\hbar\dfrac{\partial}{\partial t}\right) \end{cases} \tag{19.166}$$

Ordinary vectors:

$$\left\{\vec{x} \equiv \begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} \text{ or } \vec{p} \equiv \begin{pmatrix} p_1, & p_2, & p_3 \end{pmatrix}\right. \tag{19.167}$$

Four vectors:

$$\left\{\underset{\sim}{x} \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \text{ or } \underset{\sim}{p} \equiv \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix}\right. \tag{19.168}$$

If ψ were a scalar, the simplest first-order equation with constant coefficients would be

$$\psi = a^{(1)}\frac{\partial\psi}{\partial x_1} + a^{(2)}\frac{\partial\psi}{\partial x_2} + a^{(3)}\frac{\partial\psi}{\partial x_3} + a^{(4)}\frac{\partial\psi}{\partial x_4} = \frac{\mathrm{i}}{\hbar}a^{(\mu)}p_\mu\psi,$$

where we sum over equal indices.

However, it will prove necessary to take ψ to have several (four) components. Instead of the above equation, we write:

$$\mathrm{i}mc\psi_k = \gamma_{k\ell}^{(\mu)}p_\mu\psi_\ell = \frac{\hbar}{\mathrm{i}}\gamma_{k\ell}^{(\mu)}\frac{\partial\psi_\ell}{\partial x_\mu} \tag{19.169}$$

34-2

In matrix notation: ψ a vertical slot of (four) elements $\gamma_\mu = \| \gamma^{(\mu)}_{k\ell} \|$ a square matrix (four x four matrix)

(5) $$i m c \psi = \gamma_\mu p_\mu \psi \quad \text{(sum over } \mu\text{)}$$
$$= \frac{\hbar}{i} \gamma_\mu \frac{\partial \psi}{\partial x_\mu}$$

$p_\mu = \frac{\hbar}{i} \frac{\partial}{\partial x_\mu}$ operates on dependence of ψ on x_m

γ_μ operates on an internal variable similar to the spin variable of Pauli, however with 4 components as will be seen. Follows:

(6) $\{ \gamma_\mu$ commutes with p_ν and x_ν

From (5)
$$(imc)^2 \psi = (\gamma_\mu p_\mu)^2 \psi$$
Or (omitting ψ) use (1) $p_4^2 = -\frac{E^2}{c^2}$

use (6)
$$-m^2c^2 = \gamma_1^2 p_1^2 + \gamma_2^2 p_2^2 + \gamma_3^2 p_3^2 - \gamma_4^2 \frac{E^2}{c^2} +$$
$$+ (\gamma_1\gamma_2 + \gamma_2\gamma_1) p_1 p_2 + \text{similar terms}$$
This can be identified with the relativistic momentum energy relation

(7) $$m^2c^2 + \vec{p}^{\,2} = \frac{E^2}{c^2}$$ by postulating

(8) $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_4^2 = 1 \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0$ for $\mu \neq \nu$

In matrix notation, we have ψ as a vertical slot of four elements and $\gamma_\mu = \left\| \gamma_{k\ell}^{(\mu)} \right\|$, which is a square four-by-four matrix:

$$\begin{aligned} \mathrm{i}mc\psi &= \gamma_\mu p_\mu \psi \text{ (sums over } \mu) \\ &= \frac{\hbar}{\mathrm{i}} \gamma_\mu \frac{\partial \psi}{\partial x_\mu} \end{aligned} \tag{19.170}$$

where $p_\mu = \dfrac{\hbar}{\mathrm{i}} \dfrac{\partial}{\partial x_\mu}$ operates on the dependence of ψ on $\underset{\sim}{x}$ and γ_μ operates on an internal variable similar to the spin variable of Pauli — however, with four components, as will be seen. It therefore follows that

$$\Big\{ \gamma_\mu \text{ commutes with } p_\nu \text{ and } x_\nu . \tag{19.171}$$

From (5), we have

$$(\mathrm{i}mc)^2 \psi = (\gamma_\mu p_\mu)^2 \psi,$$

or (omitting ψ)

$$\begin{aligned} -m^2c^2 \quad &= \quad \gamma_1^2 p_1^2 + \gamma_2^2 p_2^2 + \gamma_3^2 p_3^2 - \gamma_4^2 \frac{E^2}{c^2} \\ + \quad &\quad (\gamma_1\gamma_2 + \gamma_2\gamma_1) p_1 p_2 + \text{ similar terms,} \end{aligned}$$

where we use from (1) that $p_4^2 = -\dfrac{E^2}{c^2}$ (property (6) has also been used).
This can be identified with the relativistic momentum energy relation

$$m^2c^2 + \vec{p}^{\,2} = \frac{E^2}{c^2} \tag{19.172}$$

by postulating

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_4^2 = 1 \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0 \text{ for } \mu \neq \nu . \tag{19.173}$$

One finds that the lowest order matrices for which (8) can be fulfilled is the 4-th. For order four there are many solutions that are essentially equivalent. We choose the "standard" solution

(9) $$\gamma_1 = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix};\ \gamma_2 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix};\ \gamma_3 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}$$

and

(10) $$\beta = \gamma_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$\gamma_1, \gamma_2, \gamma_3$ act in many ways as the components of a vector and will be denoted by

(11) $$\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \quad \text{also} \quad \underset{\sim}{\gamma} \equiv (\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4)$$ four vector

Then (5) becomes

(12) $$imc\,\psi = \left(\vec{\gamma}\cdot\vec{p} + \frac{i}{c} E\gamma_4\right)\psi = \underset{\sim}{\gamma}\cdot\underset{\sim}{p}\,\psi$$

Multiply to left by $\gamma_4 = \beta$ using $\gamma_4^2 = \beta^2 = 1$

(13) $$\boxed{E\psi = \left(mc^2\beta + c\,\vec{\alpha}\cdot\vec{p}\right)\psi}$$

where

(14) $$\vec{\alpha} = i\beta\vec{\gamma} \quad \left(\text{or } \alpha_1 = i\beta\gamma_1 \quad \alpha_2 = i\beta\gamma_2 \quad \alpha_3 = i\beta\gamma_3\right)$$

(15) $$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix};\ \alpha_2 = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix} \quad \alpha_3 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

One finds that the lowest order matrices for which (8) can be fulfilled is the fourth. For order four, there are many solutions that are essentially equivalent. We choose the "standard" solution:

$$\gamma_1 = \begin{vmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{vmatrix};\ \gamma_2 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix};\ \gamma_3 = \begin{vmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{vmatrix} \tag{19.174}$$

and

$$\beta = \gamma_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \tag{19.175}$$

Note that γ_1, γ_2, and γ_3 act in many ways as the components of a vector; therefore, they will be denoted by

$$\vec{\gamma} \equiv (\gamma_1, \gamma_2, \gamma_3) \text{ also } \underset{\sim}{\gamma} \equiv (\gamma_1\, \gamma_2\, \gamma_3\, \gamma_4) \quad \text{(four vector).} \tag{19.176}$$

Now, (5) becomes

$$\mathrm{i}mc\psi = (\vec{\gamma} \cdot \vec{p} + \frac{\mathrm{i}}{c} E\gamma_4)\psi = \underset{\sim}{\gamma} \cdot \underset{\sim}{p}\, \psi. \tag{19.177}$$

We now multiply to left by $\gamma_4 = \beta$ using $\gamma_4^2 = \beta^2 = 1$ to get

$$\boxed{E\psi = (mc^2\beta + c\vec{\alpha} \cdot \vec{p})\psi} \tag{19.178}$$

where

$$\vec{\alpha} = \mathrm{i}\beta\vec{\gamma} \quad (\text{or}\, \alpha_1 = \mathrm{i}\beta\gamma_1,\, \alpha_2 = \mathrm{i}\beta\gamma_2,\, \alpha_3 = \mathrm{i}\beta\gamma_3) \tag{19.179}$$

and

$$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix};\ \alpha_2 = \begin{vmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{vmatrix};\ \alpha_3 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}. \tag{19.180}$$

34-4

Properties (check directly)

(16) $\beta^2 = \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$

(17) $$\begin{cases} \beta\alpha_1 + \alpha_1\beta = 0 \quad \beta\alpha_2 + \alpha_2\beta = 0 \quad \beta\alpha_3 + \alpha_3\beta = 0 \\ \alpha_1\alpha_2 + \alpha_2\alpha_1 = 0 \quad \alpha_2\alpha_3 + \alpha_3\alpha_2 = 0 \quad \alpha_3\alpha_1 + \alpha_1\alpha_3 = 0 \end{cases}$$

(18) { β & the α's have square = unit matrix
β & the α's anticommute with each other.
β & the α's are hermitian

One can prove that all the physical consequences of (13) do not depend on the special choice (10), (15) of $\alpha_1\ \alpha_2\ \alpha_3\ \beta$. They would be the same if a different set of four 4×4 matrices with the specifications (18) had been chosen. In particular its is possible by unitary transformation to interchange the roles of the four matrices. So that their differences are only apparent.

(19) { Check that for each of the 7 matrices $\gamma_4 = \beta, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3$ the eigenvalues are $+1$, twice and -1 twice

These matrices have the following properties, which should be checked directly:

$$\beta^2 = \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \tag{19.181}$$

$$\begin{cases} \beta\alpha_1 + \alpha_1\beta = 0 \quad \beta\alpha_2 + \alpha_2\beta = 0 \quad \beta\alpha_3 + \alpha_3\beta = 0 \\ \alpha_1\alpha_2 + \alpha_2\alpha_1 = 0 \quad \alpha_2\alpha_3 + \alpha_3\alpha_2 = 0 \quad \alpha_3\alpha_1 + \alpha_1\alpha_3 = 0 \end{cases} \tag{19.182}$$

$$\begin{cases} \beta + \text{the } \alpha\text{'s have square} = \text{unit matrix;} \\ \beta + \text{the } \alpha\text{'s anticommute with each other; and} \\ \beta + \text{the } \alpha\text{'s are Hermitian.} \end{cases} \tag{19.183}$$

One can prove that all the physical consequences of Equation (13) do not depend on the specific choices of Equations (10) and (15) for $\alpha_1, \alpha_2, \alpha_3$, and β. The physical consequences would remain the same even if a different set of four 4×4 matrices with the properties described in Equation (18) had been chosen. Specifically, it is possible to interchange the roles of the four matrices through a similarity transformation. Therefore, any differences between them are merely apparent.

$$\begin{cases} \text{Check that for each of the seven matrices} \\ \gamma_4 = \beta,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \gamma_1,\ \gamma_2,\ \gamma_3 \\ \text{the eigenvalues are } +1 \text{ twice and } -1 \text{ twice.} \end{cases} \tag{19.184}$$

34-5

(13) is written also

(20) $$E\psi = H\psi$$

(21) $\begin{cases} H = \text{hamiltonian} \\ H = mc^2\beta + c\,\vec{\alpha}\cdot\vec{p} \end{cases}$ for $\psi = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix}$

Time indep. equation

(22) $$\begin{cases} E\psi_1 = mc^2\psi_1 + \dfrac{c\hbar}{i}\left\{\dfrac{\partial\psi_4}{\partial x} - i\dfrac{\partial\psi_4}{\partial y} + \dfrac{\partial\psi_3}{\partial z}\right\} \\ E\psi_2 = mc^2\psi_2 + \dfrac{c\hbar}{i}\left\{\dfrac{\partial\psi_3}{\partial x} + i\dfrac{\partial\psi_3}{\partial y} - \dfrac{\partial\psi_4}{\partial z}\right\} \\ E\psi_3 = -mc^2\psi_3 + \dfrac{c\hbar}{i}\left\{\dfrac{\partial\psi_2}{\partial x} - i\dfrac{\partial\psi_2}{\partial y} + \dfrac{\partial\psi_1}{\partial z}\right\} \\ E\psi_4 = -mc^2\psi_4 + \dfrac{c\hbar}{i}\left\{\dfrac{\partial\psi_1}{\partial x} + i\dfrac{\partial\psi_1}{\partial y} - \dfrac{\partial\psi_2}{\partial z}\right\} \end{cases}$$

also time dep. Sch. eq by $E \to i\hbar\dfrac{\partial}{\partial t}$

Plane wave solution. Take

(23) $$\psi = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$$

$\vec{p}$ now a numerical vector

$u_1\ u_2\ u_3\ u_4$ are constants.

Substitute in (22) (Divide by common exp. factor)

(24) $$\begin{cases} Eu_1 = mc^2u_1 + c(p_x - ip_y)u_4 + cp_zu_3 \\ Eu_2 = mc^2u_2 + c(p_x + ip_y)u_3 - cp_zu_4 \\ Eu_3 = -mc^2u_3 + c(p_x - ip_y)u_2 + cp_zu_1 \\ Eu_4 = -mc^2u_4 + c(p_x + ip_y)u_1 - cp_zu_2 \end{cases}$$

Four homog. linear eq. for $u_1\ u_2\ u_3\ u_4$.

Require det $= 0$. One finds e.v's of E

(25) $E = +\sqrt{m^2c^4 + c^2p^2}$ twice and $E = -\sqrt{m^2c^4 + c^2p^2}$ twice)

Observe that (13) may also be written as

$$E\psi = H\psi \tag{19.185}$$

where

$$\begin{cases} H = \text{Hamiltonian} \\ H = mc^2\beta + c\vec{\alpha}\cdot\vec{p} \end{cases} \tag{19.186}$$

for $\psi = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix}$.

As a time-independent equation, we have

$$\begin{cases} E\psi_1 = mc^2\psi_1 + \dfrac{c\hbar}{\mathrm{i}}\left\{\dfrac{\partial\psi_4}{\partial x} - \mathrm{i}\dfrac{\partial\psi_4}{\partial y} + \dfrac{\partial\psi_3}{\partial z}\right\} \\ E\psi_2 = mc^2\psi_2 + \dfrac{c\hbar}{\mathrm{i}}\left\{\dfrac{\partial\psi_3}{\partial x} + \mathrm{i}\dfrac{\partial\psi_3}{\partial y} - \dfrac{\partial\psi_4}{\partial z}\right\} \\ E\psi_3 = mc^2\psi_2 + \dfrac{c\hbar}{\mathrm{i}}\left\{\dfrac{\partial\psi_2}{\partial x} - \mathrm{i}\dfrac{\partial\psi_2}{\partial y} + \dfrac{\partial\psi_1}{\partial z}\right\} \\ E\psi_4 = mc^2\psi_4 + \dfrac{c\hbar}{\mathrm{i}}\left\{\dfrac{\partial\psi_1}{\partial x} + \mathrm{i}\dfrac{\partial\psi_1}{\partial y} - \dfrac{\partial\psi_2}{\partial z}\right\}. \end{cases} \tag{19.187}$$

We also obtain the time-dependent Schrödinger-type equation by $E \to \mathrm{i}\hbar\dfrac{\partial}{\partial t}$.

For the plane wave solution, take

$$\psi = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix} e^{\frac{\mathrm{i}}{\hbar}\vec{p}\cdot\vec{x}}, \tag{19.188}$$

where $\vec{p}$ is now a numerical vector and $u_1,\ u_2,\ u_3$, and u_4 are constants.

Now, substitute in (22) and divide by common exponential factor to obtain

$$\begin{cases} Eu_1 = mc^2u_1 + c(p_x - \mathrm{i}p_y)u_4 + cp_zu_3 \\ Eu_2 = mc^2u_2 + c(p_x + \mathrm{i}p_y)u_3 - cp_zu_4 \\ Eu_3 = mc^2u_3 + c(p_x - \mathrm{i}p_y)u_2 + cp_zu_1 \\ Eu_4 = mc^2u_4 + c(p_x + \mathrm{i}p_y)u_1 - cp_zu_2, \end{cases} \tag{19.189}$$

which is a system of four homogenous linear equations with unknowns u_1, u_2, u_3, and u_4.

If we require a zero determinant, one finds eigenvalues of E to be

$$E = +\sqrt{m^2c^4 + c^2p^2}\ \text{ twice and } E = -\sqrt{m^2c^4 + c^2p^2}\ \text{ twice}. \tag{19.190}$$

34-6

For each $\vec{p}$, E has twice the value $E=\sqrt{m^2c^4+c^2p^2}$ but also twice the negative value $E=-\sqrt{m^2c^4+c^2p^2}$ (Comments)

A set of 4 orthogonal normalized spinors u is

$$(26)\quad \begin{cases} \text{For } E=+\sqrt{m^2c^4+c^2p^2}=R \\ u^{(1)}=\sqrt{\dfrac{mc^2+R}{2R}}\begin{vmatrix} 1 \\ 0 \\ \dfrac{cp_z}{mc^2+R} \\ \dfrac{c(p_x+ip_y)}{mc^2+R}\end{vmatrix} \text{ or } u^{(2)}=\sqrt{\dfrac{mc^2+R}{2R}}\begin{vmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{mc^2+R} \\ \dfrac{-cp_z}{mc^2+R}\end{vmatrix}\end{cases}$$

$$(27)\quad \begin{cases} \text{For } E=-R=-\sqrt{m^2c^4+c^2p^2} \\ u^{(3)}=\sqrt{\dfrac{R-mc^2}{2R}}\begin{vmatrix} \dfrac{cp_z}{R-mc^2} \\ \dfrac{c(p_x+ip_y)}{R-mc^2} \\ 1 \\ 0\end{vmatrix} \text{ or } u^{(4)}=\sqrt{\dfrac{R-mc^2}{2R}}\begin{vmatrix} \dfrac{c(p_x-ip_y)}{R-mc^2} \\ \dfrac{-cp_z}{R-mc^2} \\ 0 \\ 1\end{vmatrix}\end{cases}$$

Observe: for $|p|<mc$ the third + fourth component of the positive energy solutions $u^{(1)}$ + $u^{(2)}$ are very small and the first and second component of the neg. en. solutions $u^{(3)}$ + $u^{(4)}$ are very small (of order p/mc)

For each $\vec{p}, E$ has twice the value, $E = \sqrt{m^2c^4 + c^2p^2}$, but also twice the negative value $E = -\sqrt{m^2c^4 + c^2p^2}$. (Comments will follow.)

A set of four orthogonal normalized spinors u is [6]

$$\begin{cases} \text{For } E = +\sqrt{m^2c^4 + c^2p^2} = R \\ u^{(1)} = \sqrt{\dfrac{mc^2 + R}{2R}} \left| \begin{matrix} 1 \\ 0 \\ \dfrac{cp_z}{mc^2 + R} \\ \dfrac{c(p_x + ip_y)}{mc^2 + R} \end{matrix} \right| \text{ or } u^{(2)} = \sqrt{\dfrac{mc^2 + R}{2R}} \left| \begin{matrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{mc^2 + R} \\ \dfrac{-cp_z}{mc^2 + R} \end{matrix} \right| \end{cases} \tag{19.191}$$

$$\begin{cases} \text{For } E = -R = -\sqrt{m^2c^4 + c^2p^2} \\ u^{(3)} = \sqrt{\dfrac{R - mc^2}{2R}} \left| \begin{matrix} \dfrac{cp_z}{R - mc^2} \\ \dfrac{c(p_x + ip_y)}{R - mc^2} \\ 1 \\ 0 \end{matrix} \right| \text{ or } u^{(4)} = \sqrt{\dfrac{R - mc^2}{2R}} \left| \begin{matrix} \dfrac{c(p_x - ip_y)}{R - mc^2} \\ \dfrac{-cp_z}{R - mc^2} \\ 0 \\ 1 \end{matrix} \right| \end{cases} \tag{19.192}$$

Observe: for $|p| \ll mc$, the third and fourth components of the positive energy solutions $u^{(1)}$ and $u^{(2)}$ are very small; and the first and second components of the negative energy solutions $u^{(3)}$ and $u^{(4)}$ are also very small (on the order of p/mc) [7]

---

[6] See our section 4 for corrections: all those bi-spinors correspond to the same eigenvalue $E = +R$; they are, indeed, normalized but not mutually orthogonal. For example,

$$\left(u^{(1)}\right)^\dagger u^{(3)} = \frac{p_3}{|\boldsymbol{p}|}, \qquad \left(u^{(2)}\right)^\dagger u^{(3)} = \frac{p_1 + ip_2}{|\boldsymbol{p}|}.$$

As a result,

$$u^{(3)} = \frac{p_3}{|\boldsymbol{p}|}\, u^{(1)} + \frac{p_1 + ip_2}{|\boldsymbol{p}|}\, u^{(2)}$$

and, in a similar fashion,

$$u^{(4)} = \frac{p_1 - ip_2}{|\boldsymbol{p}|}\, u^{(1)} - \frac{p_3}{|\boldsymbol{p}|}\, u^{(2)}.$$

[7] On the contrary, one can easily verify that

$$\left(u^{(3)},\ u^{(4)}\right) \to \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ O \end{pmatrix} = \begin{pmatrix} \dfrac{p_3}{|\boldsymbol{p}|} & \dfrac{p_1 - ip_2}{|\boldsymbol{p}|} \\ \dfrac{p_1 + ip_2}{|\boldsymbol{p}|} & -\dfrac{p_3}{|\boldsymbol{p}|} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad c \to \infty$$

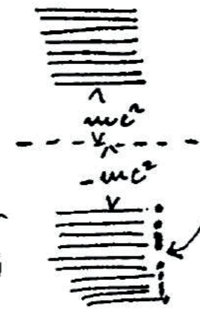
(see also [54] for more details).

34–7

Meaning of neg. + pos. energy levels.

The Dirac sea – Vacuum state

Positrons as holes.



Mom + energy of the positron are $(-\vec{p} + -\vec{E})$ of the "hole" state.

(28) $\underbrace{u^{(1)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}},\ u^{(2)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}}_{}$

electron states (spin up + down)

(mom. $= \vec{p}$, energy $= +\sqrt{m^2c^4 + c^2p^2}$

(29) $\left\{ \begin{array}{l} u^{(3)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \\ u^{(4)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \end{array} \right\}$ are positron states with momentum $= -\vec{p}$, energy $= +\sqrt{m^2c^4 + c^2p^2}$

Given $\psi = u\, e^{\frac{i}{\hbar} p\cdot x}$ (u = 4 component spinor)

it is important to have two operators $\mathcal{P}$ + $\mathcal{N}$ (<u>projection operators</u>) such that $\mathcal{P}\psi$ contains only electron wave fcts, $\mathcal{N}\psi$ contains only neg. energy wave fcts (positron states). $\mathcal{P}$, $\mathcal{N}$

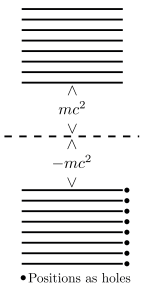
(30) $\left\{ \begin{array}{l} \text{are spinor operators } \cancel{\text{such}} \text{ defined by } \mathcal{P}u^{(1)} = u^{(1)}, \\ \mathcal{P}u^{(2)} = u^{(2)},\ \mathcal{P}u^{(3)} = 0,\ \mathcal{P}u^{(4)} = 0 \text{ and} \end{array} \right.$

(31) $\mathcal{N}u^{(1)} = 0,\quad \mathcal{N}u^{(2)} = 0,\quad \mathcal{N}u^{(3)} = u^{(3)},\quad \mathcal{N}u^{(4)} = u^{(4)}$

These properties define uniquely $\mathcal{P}$ + $\mathcal{N}$

Meaning of negative and positive energy levels

The Dirac Sea; the Vacuum State; Positrons as Holes



Momentum and energy of the positron are $(-\vec{p} + -\vec{E})$ of the "hole" state. Electron states (spin up and spin down) are represented by

$$\begin{cases} u^{(1)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \text{ and} \\ u^{(2)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \end{cases} \tag{19.193}$$

where momentum is given by $\vec{p}$ and energy by $+\sqrt{m^2c^4 + c^2p^2}$; and positron states by

$$\begin{cases} u^{(3)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \text{ and} \\ u^{(4)} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \end{cases} \tag{19.194}$$

with momentum equals $-\vec{p}$ (and energy still as $+\sqrt{m^2c^4 + c^2p^2}$).

Given $\psi = ue^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$ (with u being a four-component spinor), it is important to have two operators $\mathcal{P}$ and $\mathcal{N}$ as *projection operators* such that $\mathcal{P}\psi$ contains only electron wave functions, and $\mathcal{N}\psi$ contains only negative energy wave functions (positron states). Thus, $\mathcal{P}$ and $\mathcal{N}$ are spinor operators defined by:

$$\begin{cases} \mathcal{P}u^{(1)} = u^{(1)}, \\ \mathcal{P}u^{(2)} = u^{(2)}, \\ \mathcal{P}u^{(3)} = 0, \\ \mathcal{P}u^{(4)} = 0 \end{cases} \tag{19.195}$$

and

$$\begin{cases} \mathcal{N}u^{(1)} = 0, \\ \mathcal{N}u^{(2)} = 0, \\ \mathcal{N}u^{(3)} = u^{(3)}, \\ \mathcal{N}u^{(4)} = u^{(4)}. \end{cases} \tag{19.196}$$

These properties uniquely define $\mathcal{P}$ and $\mathcal{N}$.

Observe: $Hu^{(1)} = Ru^{(1)}$, $Hu^{(2)} = Ru^{(2)}$, $Hu^{(3)} = -Ru^{(3)}$
$Hu^{(4)} = -Ru^{(4)}$
with
$$R = +\sqrt{m^2c^4 + c^2p^2} \qquad (\vec{p} \text{ here a c-vector})$$
and H from (21). Then

$$(32) \quad \mathcal{P} = \frac{1}{2} + \frac{1}{2R} H \quad ; \quad \mathcal{N} = \frac{1}{2} - \frac{1}{2R} H$$

Angular momentum. From (21)

$$(33) \quad [H, xp_y - yp_x] = \frac{\hbar c}{i}(\alpha_1 p_y - \alpha_2 p_x) \neq 0 \text{ -}$$

Therefore $xp_y - yp_x$ not a time constant for free Dirac electron. However

$$(34) \quad xp_y - yp_x + \frac{1}{2}\frac{\hbar}{i}\alpha_1\alpha_2 = \hbar J_z$$

Commutes with H. Interpret $\hbar J_z$ as z component of ang. mom.

$$(35) \quad \hbar \vec{J} = \underbrace{\vec{x} \times \vec{p}}_{\text{orbital part}} + \underbrace{\frac{\hbar}{2i}\begin{cases} \alpha_2\alpha_3 \\ \alpha_3\alpha_1 \\ \alpha_1\alpha_2 \end{cases}}_{\text{spin part}} = \vec{x} \times \vec{p} + \frac{\hbar}{2}\vec{\sigma}'$$

with

$$(36) \quad \sigma'_x = \frac{1}{i}\alpha_2\alpha_3 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}; \quad \sigma'_y = \frac{1}{i}\alpha_3\alpha_1 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}; \quad \sigma'_z = \frac{1}{i}\alpha_1\alpha_2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Observe analogy with Pauli operators $\vec{\sigma}$ + $\vec{\sigma}'$.

Observe that $Hu^{(1)} = Ru^{(1)}; Hu^{(2)} = Ru^{(2)}; Hu^{(3)} = -Ru^{(3)}$; and $Hu^{(4)} = -Ru^{(4)}$, with $R = +\sqrt{m^2c^4 + c^2p^2}$ (here, $\vec{p}$ is a c-vector) and H from (21). As such [8] ,

$$\mathcal{P} = \frac{1}{2} + \frac{1}{2R}H; \quad \mathcal{N} = \frac{1}{2} - \frac{1}{2R}H. \tag{19.197}$$

Angular Momentum

From (21), we obtain

$$[H, xp_y - yp_x] = \frac{\hbar c}{\mathrm{i}}\left(\alpha_1 p_y - \alpha_2 p_x\right) \neq 0. \tag{19.198}$$

Therefore, $xp_y - yp_x$ is not a time constant for a free Dirac electron. However,

$$xp_y - yp_x + \frac{1}{2}\frac{\hbar}{\mathrm{i}}\alpha_1\alpha_2 = \hbar J_z \tag{19.199}$$

commutes with H. Interpret $\hbar J_z$ as the z component of angular momentum:

$$\hbar\vec{J} = \vec{x}\times\vec{p} + \frac{\hbar}{2\mathrm{i}}\begin{cases}\alpha_2\alpha_3\\ \alpha_3\alpha_1\\ \alpha_1\alpha_2\end{cases} = \vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma}' \tag{19.200}$$

with $\vec{x}\times\vec{p}$ as the orbital part and $\frac{\hbar}{2}\vec{\sigma}'$ as the spin part and

$$\sigma'_x = \tfrac{1}{\mathrm{i}}\alpha_2\alpha_3 = \begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{vmatrix}; \quad \sigma'_y = \tfrac{1}{\mathrm{i}}\alpha_3\alpha_1 = \begin{vmatrix}0&-\mathrm{i}&0&0\\\mathrm{i}&0&0&0\\0&0&0&-\mathrm{i}\\0&0&\mathrm{i}&0\end{vmatrix}; \quad \sigma'_z = \tfrac{1}{\mathrm{i}}\alpha_1\alpha_2 = \begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{vmatrix}. \tag{19.201}$$

Observe the analogy with the Pauli operators $\vec{\sigma}$ of $\vec{\sigma}'$.

[8] with the correction described in our section 4

19.11 Appendix C: Paper by Michel and Wightman

This appendix contains, for the reader's convenience, the original text by Michel and Wightman of the abstract for their talk on a meeting of the American Physical Society [36] (Figure 19.3). Their result is cited in the "Bible of Theoretical Physics", namely, in the L. D. Landau and E. M. Lifshitz Course of Theoretical Physics [5].

TA10. A Covariant Formalism Describing the Polarization of Spin One-Half Particles. L. MICHEL, *Institute for Advanced Study*, AND A. S. WIGHTMAN, *Princeton University*.—We denote by $u(p,s)$, the state of a spin $\frac{1}{2}$ particle of four momentum p and mass $m \neq 0$, whose spin is polarized along a space like four pseudo-vector s, such that $s\cdot p=0$, $s^2=-1$. Then the projection operator onto $u(p,s)$ may be written

$$P_{\alpha\beta}(p,s)=[\bar{u}u]^{-1}u_\alpha\bar{u}_\beta=[4m]^{-1}[1-\gamma_5\mathbf{s}][m+\mathbf{p}]$$

(We use Feynman notation.) s_μ is the generalization of the Stokes pseudo-vector and is the expectation value of $\gamma_5\gamma_\mu$. For a mixture, the degree of polarization is $(-s^2)^{\frac{1}{2}}$. For $m=0$, we have

$$P_{\alpha\beta}'=[u^*u]^{-1}u_\alpha u_\beta^*=(4p_0)^{-1}[1-\gamma_5\mathbf{s}(_T+\xi)]\mathbf{p}\gamma_0$$

where $\mathbf{s}_T$ is a transverse polarization vector and ξ is the amplitude for circular polarization. These formulas permit one to make covariant polarization calculations in terms of traces.

Figure 19.3: Reference [36]

Our calculations in section 6 revealed the following connections with the nonrelativistic polarization density matrices (19.48)–(19.49):

$$\begin{aligned}\frac{u_\alpha^{(\pm1/2)}(p)\,\overline{u}_\beta^{(\pm1/2)}(p)}{\overline{u}^{(\pm1/2)}(p)\,u^{(\pm1/2)}(p)} &= \frac{1}{2}\begin{pmatrix} I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n} & \mp\eta\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) \\ \pm\eta\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\eta^2\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{pmatrix} \\ &= \begin{pmatrix} I_2 & \mp\eta I_2 \\ \pm\eta I_2 & -\eta^2 I_2\end{pmatrix}\begin{pmatrix} \rho^{(\pm1/2)}(\boldsymbol{n}) & O \\ O & \rho^{(\pm1/2)}(\boldsymbol{n})\end{pmatrix},\end{aligned} \tag{19.202}$$

when $E>0$ and

$$\begin{aligned}\frac{v_\alpha^{(\mp1/2)}(p)\,\overline{v}_\beta^{(\mp1/2)}(p)}{\overline{v}^{(\mp1/2)}(p)\,v^{(\mp1/2)}(p)} &= \frac{1}{2}\begin{pmatrix} \eta^2\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & \mp\eta\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) \\ \pm\eta\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) & -\left(I_2\pm\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\end{pmatrix} \\ &= \begin{pmatrix} \eta^2 I_2 & \mp\eta I_2 \\ \pm\eta I_2 & -I_2\end{pmatrix}\begin{pmatrix} \rho^{(\pm1/2)}(\boldsymbol{n}) & O \\ O & \rho^{(\pm1/2)}(\boldsymbol{n})\end{pmatrix},\end{aligned} \tag{19.203}$$

if $E<0$. (In the nonrelativistic limit $\eta\to0$.)

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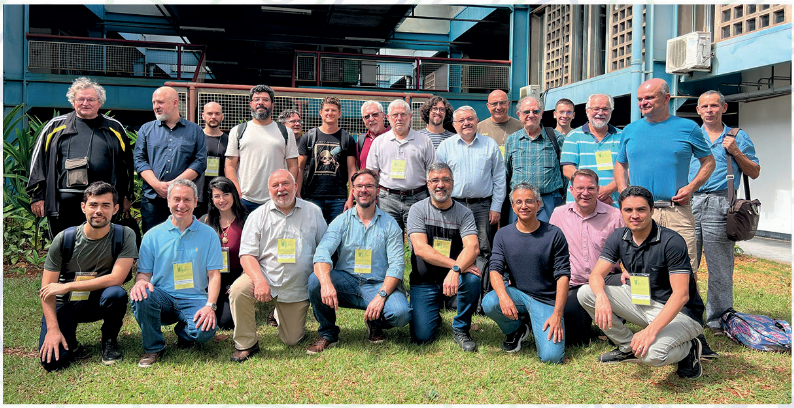

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