“Oganesson versus Uranium Hydrogen-like Ions from the Viewpoint of Old Quantum Mechanics”

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https://arxiv.org/pdf/2509.06249

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Introduction

In this Introductory Section, the general formulas are provided. They are stored in global variables that will be used in all the subsequent sections. For this purpose, allow Mathematica to evaluate all initialization cells. After that you will be able to run each integrable case independently from the others.
Use the following command, if you need to save/export graphs created by Mathematica into the current notebook directory:

In[9]:=
```
SetDirectory[NotebookDirectory[]];
```

In[2]:=
```
(*---Constants and Setup---*)
(*Pretty print subscript-style variables for display only*)
MakeBoxes[nt, StandardForm] := SubscriptBox["n", "t"]
MakeBoxes[nr, StandardForm] := SubscriptBox["n", "r"]
(*Physical constants*)
α = 1 / 137.036; (*Fine structure constant*)

(*---Define symbolic formulas as functions---*)

(*Eq.(3.19)*)
ωFunc[nt_] := Sqrt[nt^2 - Z^2 α^2] / nt

(*Eq.(3.20)*)
ϵFunc[nt_, nr_] :=
 Sqrt[nr] Sqrt[(nr + 2 Sqrt[nt^2 - Z^2 α^2])] / (nr + Sqrt[nt^2 - Z^2 α^2])

(*Eq.(3.21)*)
aFunc[nt_, nr_] :=
 ((nr + Sqrt[nt^2 - Z^2 α^2]) Sqrt[Z^2 α^2 + (nr + Sqrt[nt^2 - Z^2 α^2])^2]) / Z

(*Eq.(3.18)in mc^2 units*)
eFunc[nt_, nr_] := (1 + (Z^2 α^2) / (nr + Sqrt[nt^2 - Z^2 α^2])^2)^(-1/2)
```

Uranium, U, Z=92

In[10]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[11]:=
```
(*Physical constants*)
Z = 92; (*Atomic number of Uranium, U*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

 PolarPlot[r[t], {t, 0, PeriodTheta},
  PlotLabel → "Relativistic Elliptical Orbit: >1 rotation",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.5 * PeriodTheta},
  PlotLabel → "Relativistic Elliptical Orbit: >2 rotation",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 2 * PeriodTheta},
  PlotLabel → "Relativistic Elliptical Orbit: >2.5 rotations",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 3 * PeriodTheta},
  PlotLabel → "Relativistic Elliptical Orbit: 1 rotation per period",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 4 * PeriodTheta},
  PlotLabel → "Relativistic Elliptical Orbit; 1 rotation",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[14]=
```
0.741135
```

Out[15]=
```
0.904882
```

Out[16]=
```
0.0353163
```

Out[17]=
```
{0.00335921, 0.0672735}
```

Out[18]=

2.19461

Out[19]=

8.47779

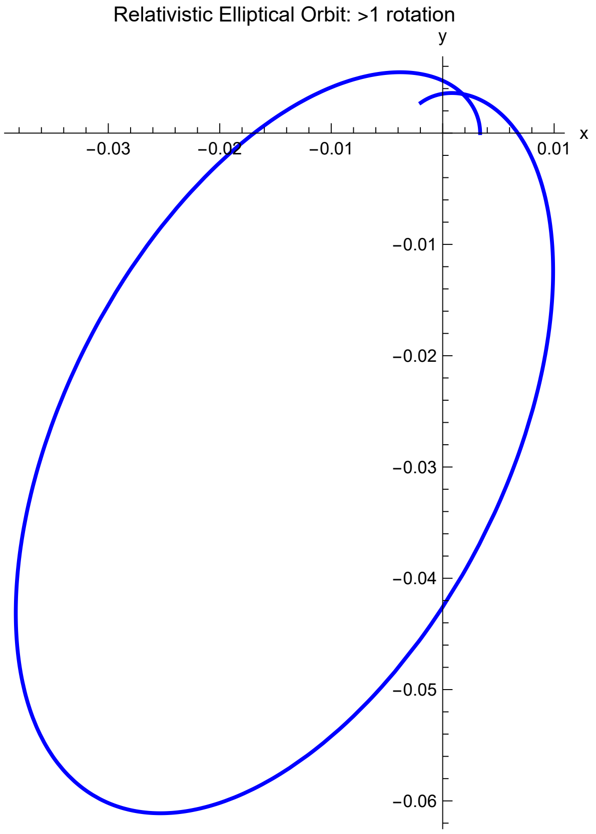
Out[20]=

0.933042

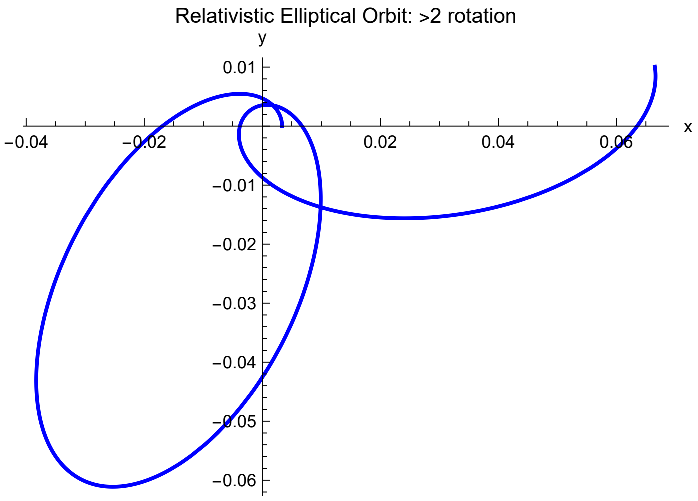
Out[21]=

92

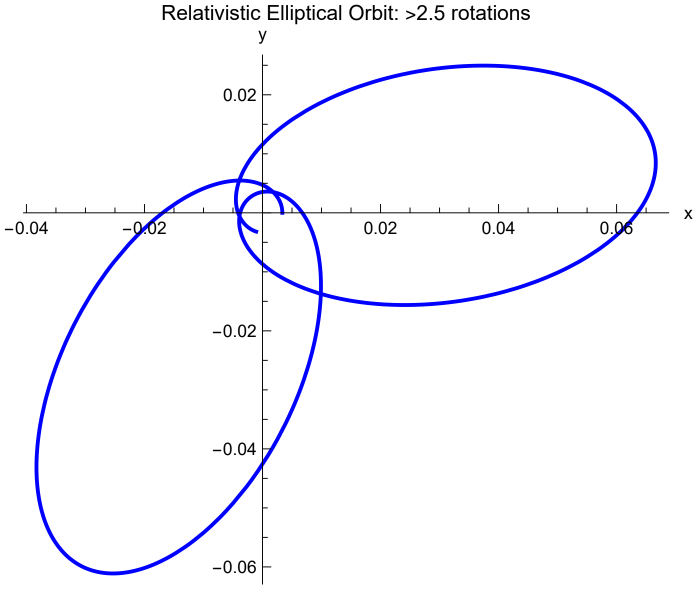
Out[23]=



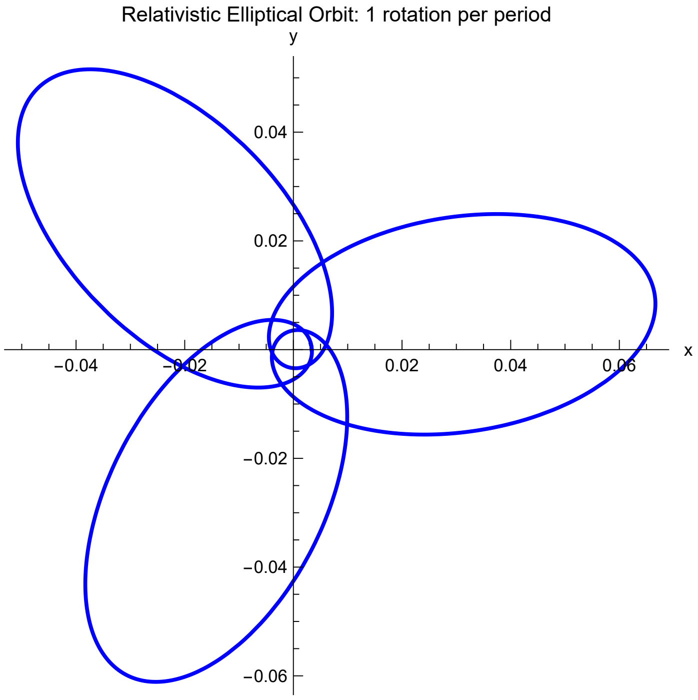
Out[24]=



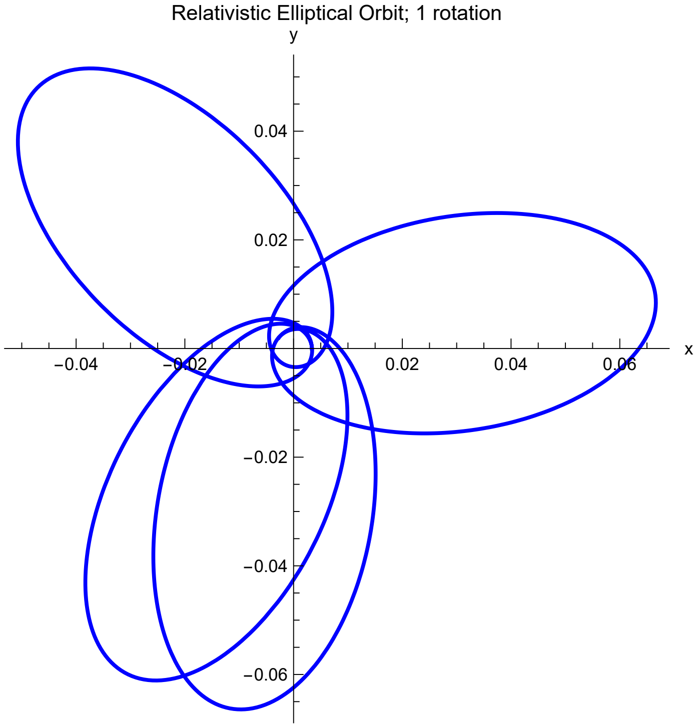
Out[25]=


Relativistic Elliptical Orbit: >2.5 rotations
y
x
0.02
-0.02
-0.04
-0.06
-0.04
-0.02
0.02
0.04
0.06

Out[26]=


Relativistic Elliptical Orbit: 1 rotation per period
y
x
0.04
0.02
-0.02
-0.04
-0.06
-0.04
-0.02
0.02
0.04
0.06

Out[27]=



In[28]:= **Export["UraniumIon91.pdf", %%]**

Out[28]=

UraniumIon91.pdf

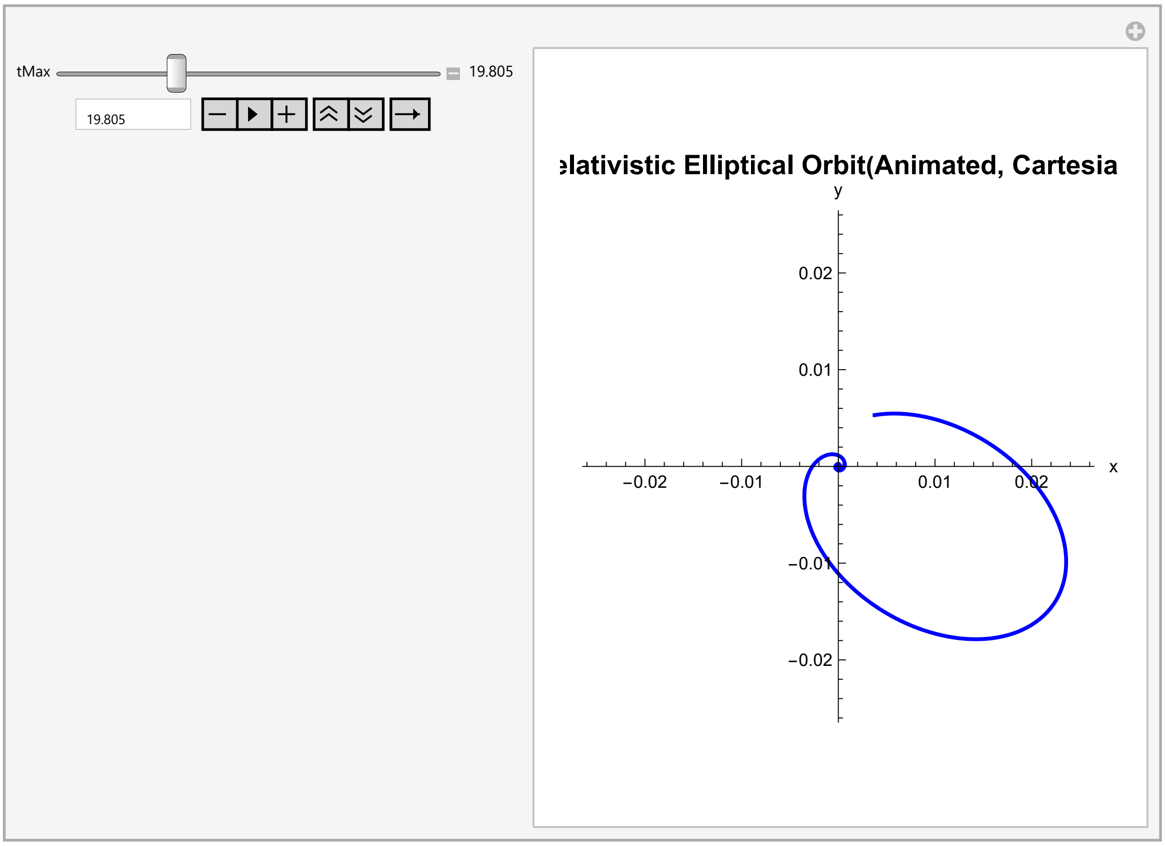
In[29]:= **r[t]**

Out[29]=

$$\frac{0.0063989}{1 + 0.904882\,\mathrm{Cos}[0.741135\,t]}$$

In[30]:=
```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic Elliptical Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from r_min to r_max*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 2*4*PeriodTheta, Appearance → "Labeled"}]
```

Out[30]=



Summary:
In the above animation of the relativistic Kepler motion of an electron in hydrogen-like Uranium, U, when Z=92, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω=0.741135, ϵ=0.904882, and a=0.0353163, in Bohr's atomic units.
The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\,(1-\epsilon) = 0.003359209$ and $r_{max} = a\,(1+\epsilon) = 0.067273472$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity.
CCW Rotation of the ellipse over one period is given by $\Delta\theta=2\pi(1/\omega-1)$=2.19461. 'Winding' number is one.

Verification:

In[31]:=
```
{ωVal, ϵVal, aVal}
```
Out[31]=
```
{0.741135, 0.904882, 0.0353163}
```

In[32]:=
```
{rMin, rMax}
```
Out[32]=
```
{0.00335921, 0.0672735}
```

In[33]:=
```
{DeltaTheta, PeriodTheta}
```
Out[33]=
```
{2.19461, 8.47779}
```

In[34]:=
```
{eVal, Z, ntVal, nrVal}
```
Out[34]=
```
{0.933042, 92, 1, 1}
```

In[35]:=
```
r[t]
```
Out[35]=

$$\frac{0.0063989}{1 + 0.904882\,\mathrm{Cos}[0.741135\,t]}$$

(*End of Uranium, Z=92, section*)

Copernicium, Cn, Z=122

In[36]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[37]:=
```
(*Physical constants*)
Z = 112; (*Atomic number of Copernicium,Cn*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

PolarPlot[r[t], {t, 0, 0.88 * PeriodTheta},
 PlotLabel → "Relativistic Elliptical Orbit: 1.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.25 * PeriodTheta},
 PlotLabel → "Relativistic Elliptical Orbit: >2 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.5 * PeriodTheta},
 PlotLabel → "Relativistic Elliptical Orbit: >2.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.725 * PeriodTheta},
 PlotLabel → "Relativistic Elliptical Orbit: 2 rotations per period",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[40]=
```
0.576207
```

Out[41]=
```
0.930786
```

Out[42]=
```
0.0249872
```

Out[43]=
```
{0.00172947, 0.0482449}
```

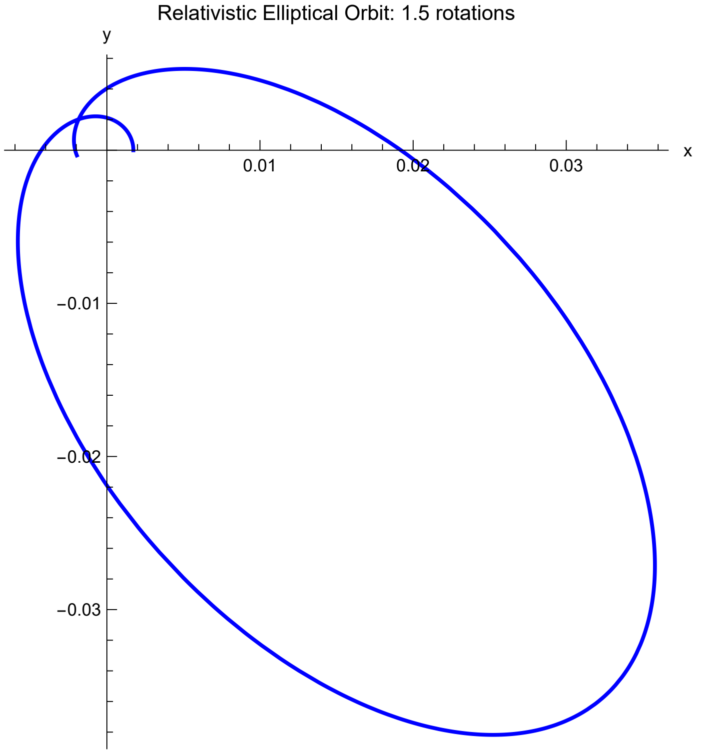
Out[44]=
```
4.6212
```

Out[45]=
```
10.9044
```

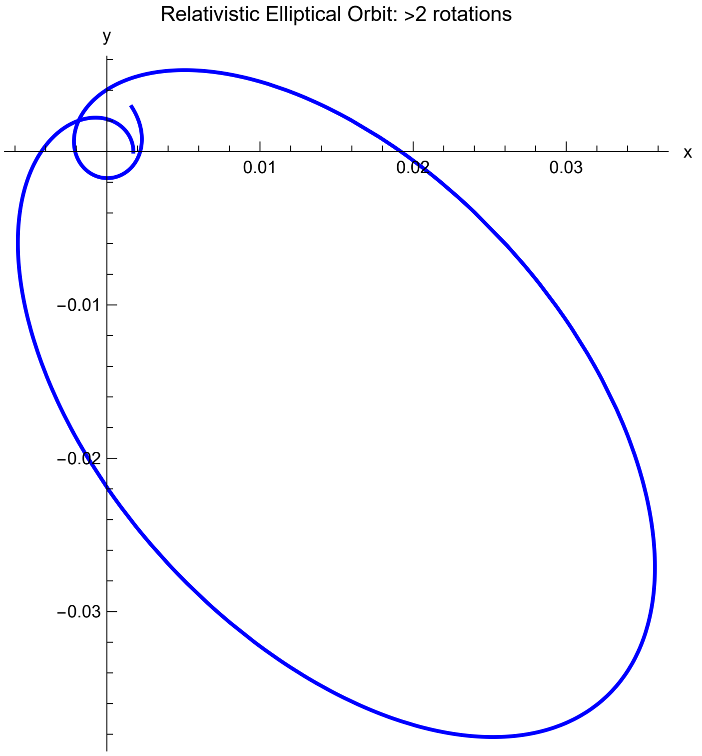
Out[46]=
```
0.887752
```

Out[47]=
```
112
```

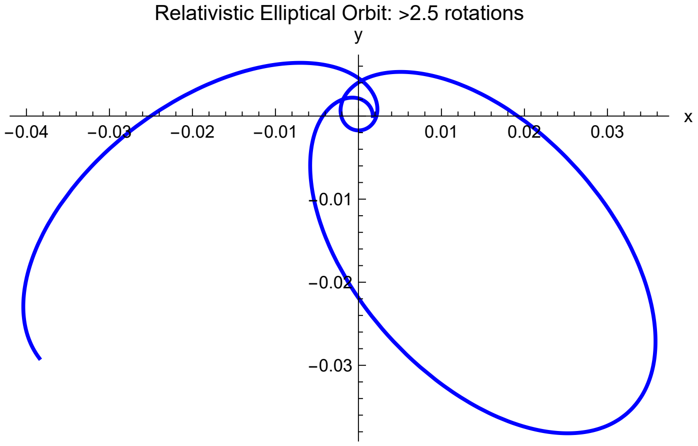
Out[49]=


Relativistic Elliptical Orbit: 1.5 rotations
y
x
0.01
0.02
0.03
−0.01
−0.02
−0.03

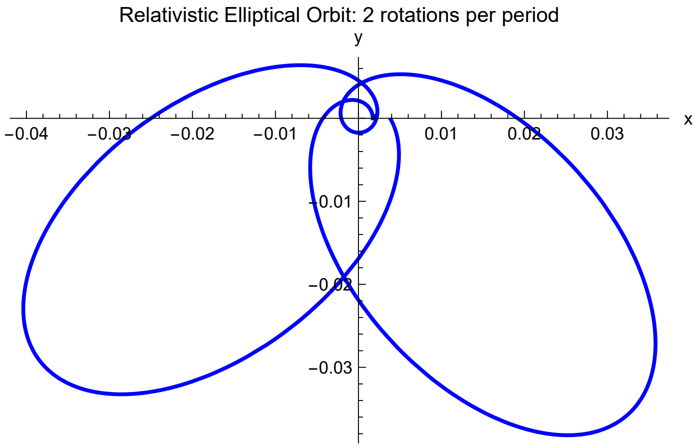
Out[50]=


Relativistic Elliptical Orbit: >2 rotations
y
x
0.01
0.02
0.03
−0.01
−0.02
−0.03

Out[51]=



Out[52]=

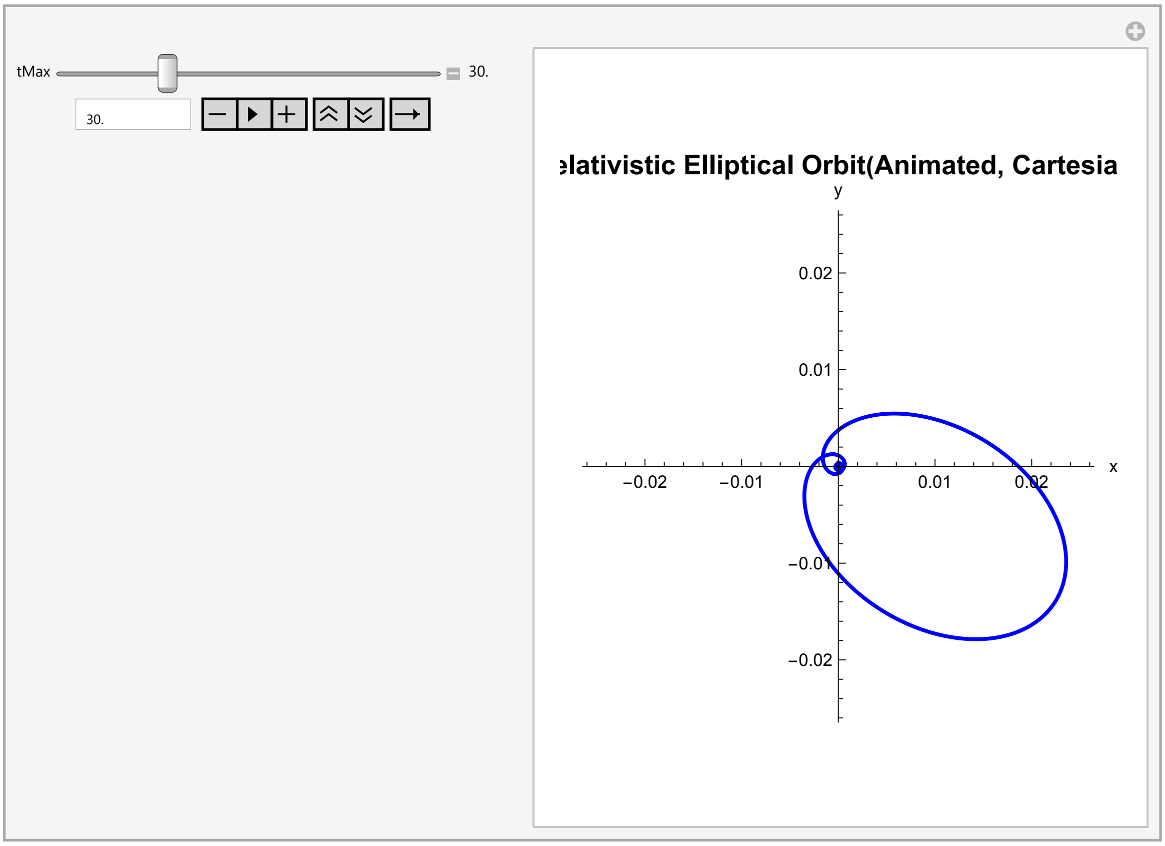


In[53]:=
```
Export["CoperniciumIon111.pdf", %]
```

Out[53]=
```
CoperniciumIon111.pdf
```

In[54]:=
```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic Elliptical Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from rmin to rmax*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 6 * 1.725 * PeriodTheta, Appearance → "Labeled"}]
```

Out[54]=



Summary:

In the above animation of the relativistic Kepler motion of an electron in hydrogen-like Copernicium, Cn, when Z=112, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω= 0.576207, ϵ=0.930786, and a=0.0249872, in Bohr's atomic units.

The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\ (1-\epsilon) = 0.00172947$ and $r_{max} = a\ (1+\epsilon) = 0.0482449$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity. 'Winding number is two.

CW Rotation of the ellipse over one period is given by $\Delta\theta$=2π(1/ω-1)=4.6212. At the point of self-intersection one gets, approximately,

r[2.3799243523877007`]=r[8.52445631673413`]=0.00281928 with r[2.3799243523877007`]-r[8.52445631673413`]=-1.73472×10^{-18}.

Verification:

In[55]:=
```
{ωVal, ϵVal, aVal}
```

Out[55]=
```
{0.576207, 0.930786, 0.0249872}
```

In[56]:=
```
{rMin, rMax}
```

Out[56]=
```
{0.00172947, 0.0482449}
```

In[57]:=
```
{DeltaTheta, PeriodTheta}
```

Out[57]=
```
{4.6212, 10.9044}
```

In[58]:=
```
{eVal, Z, ntVal, nrVal}
```

Out[58]=
```
{0.887752, 112, 1, 1}
```

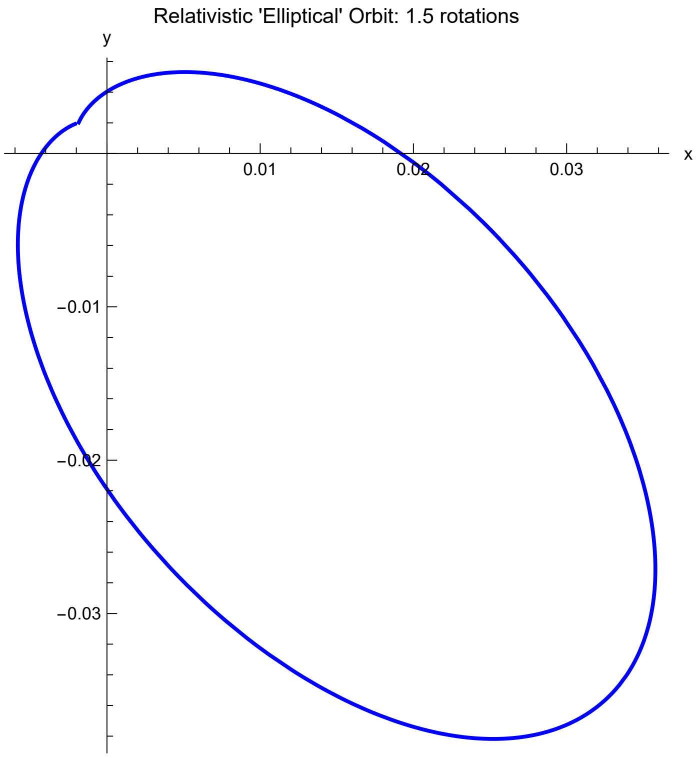
In[59]:=
```
r[t]
```

Out[59]=

$$\frac{0.00333924}{1 + 0.930786\,\text{Cos}[0.576207\,t]}$$

In[60]:=
```
PolarPlot[r[t], {t, 0.22 * PeriodTheta, 0.788 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 1.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[60]=



In[61]:=
```
{0.22 * PeriodTheta, 0.788 * PeriodTheta}
```

Out[61]=
```
{2.39896, 8.59265}
```

In[62]:=
```
r[2.398963747206803`] - r[8.592651967268004`]
```
Out[62]=
```
0.000106539
```

In[63]:=
```
8.592651967268004` / 2.398963747206803`
```
Out[63]=
```
3.58182
```

In[64]:=
```
FindRoot[r[t] - r[3.5818181818181825` * t] == 0, {t, 2.4}]
```
Out[64]=
```
{t → 2.37992}
```

In[65]:=
```
3.5818181818181825` * 2.3799243523877007`
```
Out[65]=
```
8.52446
```

In[66]:=
```
r[2.3799243523877007`] - r[8.52445631673413`]
```
Out[66]=

-1.73472×10^{-18}

In[67]:=
```
{r[2.3799243523877007`], r[8.52445631673413`]}
```
Out[67]=
```
{0.00281928, 0.00281928}
```

In[68]:=
```
3.1415926535897927` - Pi // N
```
Out[68]=

-4.44089×10^{-16}

(*End of Copernicium, Cn, Z=112, section*)

Oganesson, Og, Z=118

In[69]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[70]:=
```
(*Physical constants*)
Z = 118; (*Atomic number of Oganesson, Og*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

PolarPlot[r[t], {t, 0, 0.75 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 1.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.269 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.75 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2 rotations per period",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[73]=
0.508457

Out[74]=
0.941479

Out[75]=
0.0222041

Out[76]=
{0.0012994, 0.0431087}

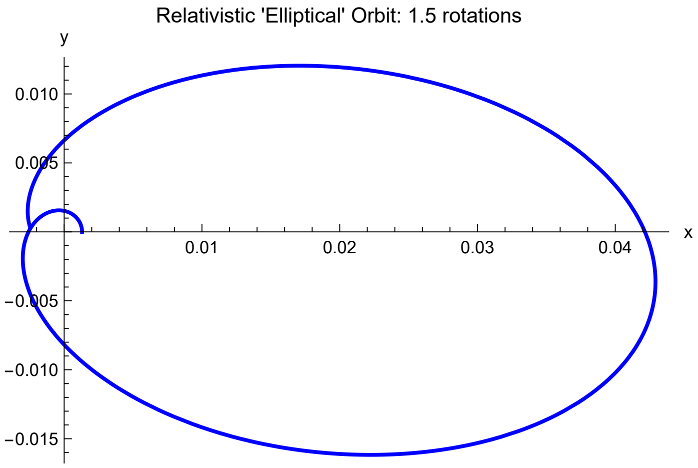
Out[77]=
6.07418

Out[78]=
12.3574

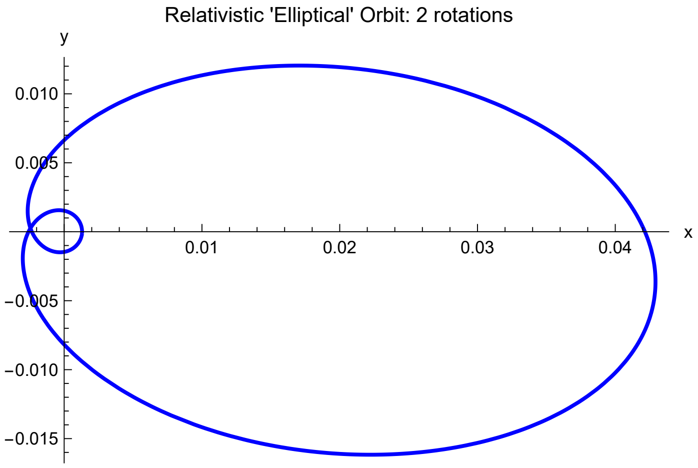
Out[79]=
0.868463

Out[80]=
118

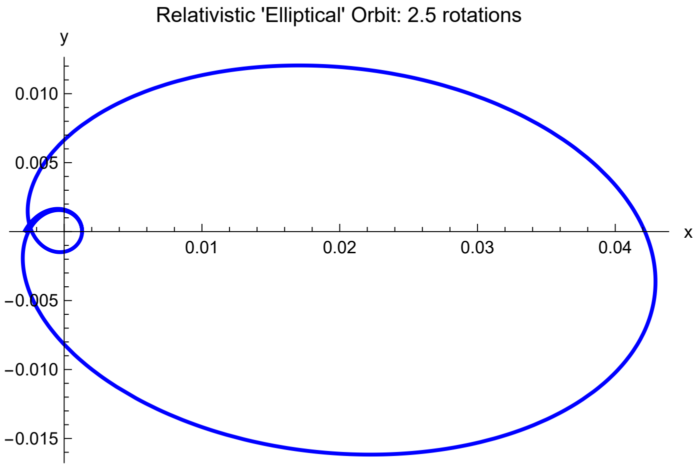
Out[82]=



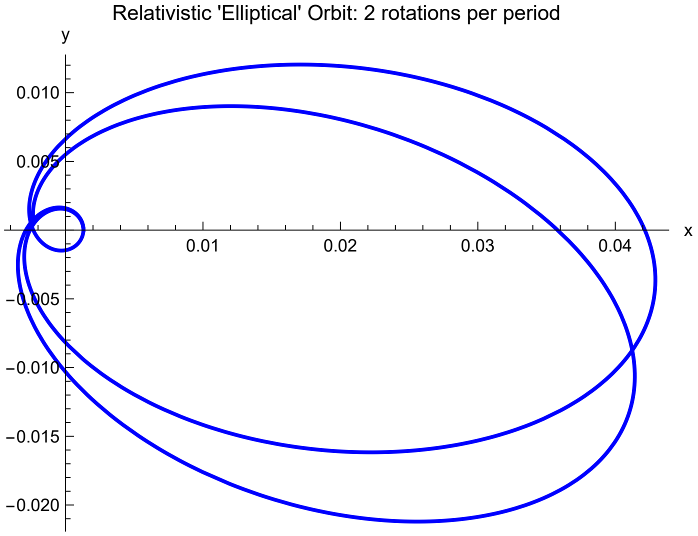
Out[83]=



Out[84]=



Out[85]=

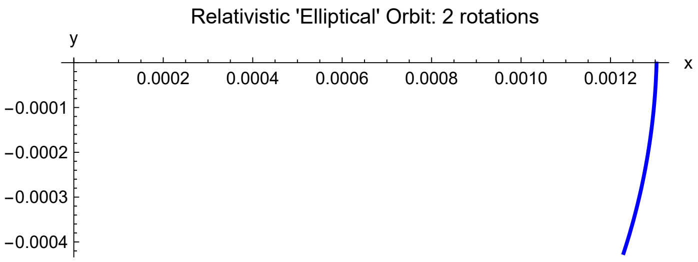


In[86]:=
```
Export["OganessonIon117.pdf", %]
```

Out[86]=
```
OganessonIon117.pdf
```

In[87]:=
```
(* the loop *)PolarPlot[r[t], {t, 0.99 * PeriodTheta, 1.01679 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[87]=



In[88]:=
```
1.01679 * PeriodTheta
```

Out[88]=
```
12.5648
```

In[89]:=
```
{r[12.564847324276567`] - r[0], r[1.01679 * PeriodTheta] - r[0]}
```

Out[89]=

$\{3.51253 \times 10^{-6}, 3.51253 \times 10^{-6}\}$

In[90]:=
```
FindRoot[r[t * 1.01679 * PeriodTheta] - r[0] == 0, {t, 1}]
```

Out[90]=
```
{t → 0.983487}
```

In[91]:=
```
0.9834872651822683` * 1.01679 * PeriodTheta
```

Out[91]=
```
12.3574
```

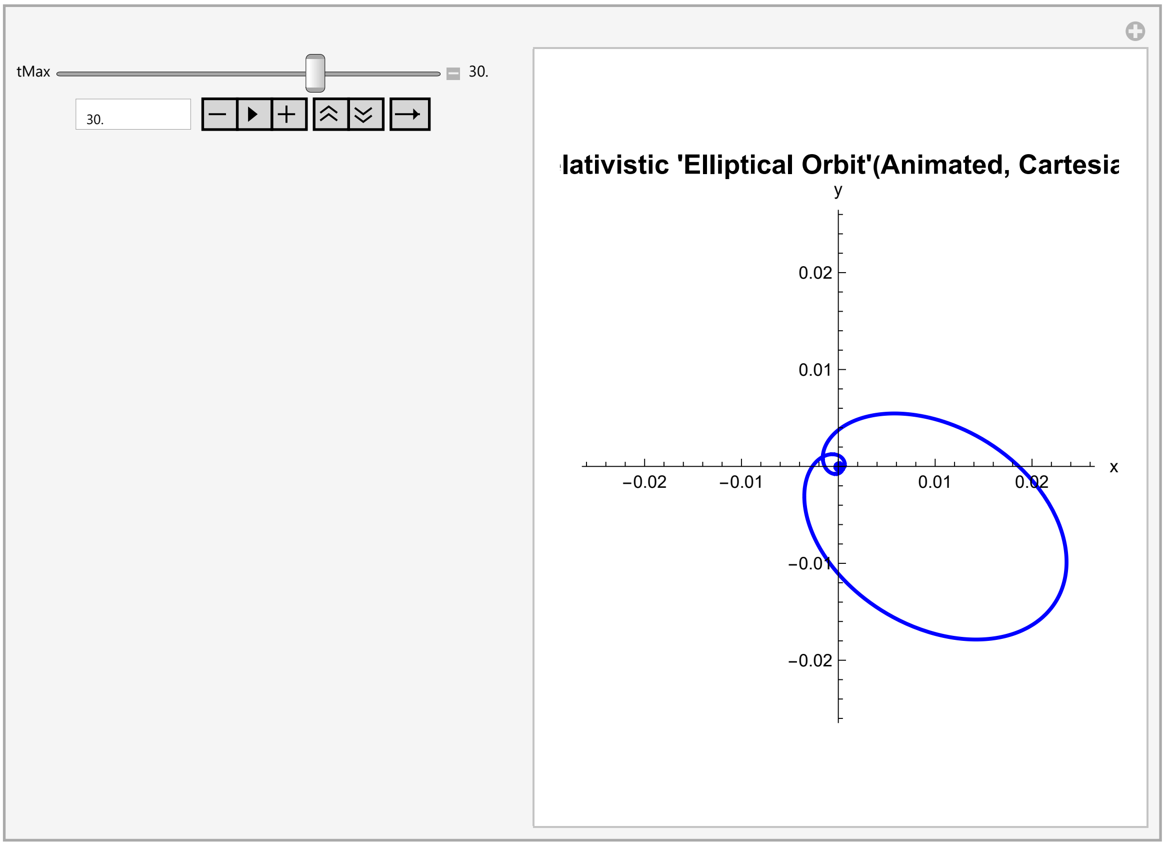
In[92]:=
```
r[12.357367332385502`] - r[0]
```

Out[92]=

3.25261×10^{-18}

In[93]:=
```
(*Animation in Cartesian Coordinates: the first two rotations per period,
twice*)Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic 'Elliptical Orbit'(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from rmin=0.001299 to rmax=0.0431087*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 2 * 1.75 * PeriodTheta, Appearance → "Labeled"}]
```

Out[93]=



Summary:

In the above animation of the relativistic Kepler motion of an electron in hydrogen-like Oganesson ion, Og, when Z=118, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω=0.508457, ϵ=0.941479, and a=0.0222041, in Bohr's atomic units.

The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\,(1-\epsilon) = 0.0012994$ and $r_{max} = a\,(1+\epsilon) = 0.043108700$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity. 'Winding' number is two.

CW Rotation of the ellipse over one period is given by $\Delta\theta$=2π(1/ω-1)=6.07418. At the point of self-intersection one gets, approximately, r[3.074037420536865]=r[9.283329709624294]=0.0025044 with r[9.283329709624294`]-r[3.074037420536865`]=4.33681×10^{-19}.

Verification:

In[94]:= {ωVal, ϵVal, aVal}

Out[94]=
{0.508457, 0.941479, 0.0222041}

In[95]:= {rMin, rMax}

Out[95]=
{0.0012994, 0.0431087}

In[96]:= {DeltaTheta, PeriodTheta}

Out[96]=
{6.07418, 12.3574}

In[97]:= {eVal, Z, ntVal, nrVal}

Out[97]=
{0.868463, 118, 1, 1}

In[98]:= (* points of self-intersection *)r[9.283329709624294`] - r[3.074037420536865`]

Out[98]=
4.33681×10^{-19}

In[99]:= {r[9.283329709624294`], r[3.074037420536865`]}

Out[99]=
{0.0025044, 0.0025044}

In[100]:=
ωVal * (9.283329709624294` + 3.074037420536865`) / 2 // N

Out[100]=
3.14159

In[101]:=
3.141592653589793` - Pi // N

Out[101]=
0.

In[102]:=
r0[t_] := 0.004308954533019283`

In[103]:=
$$r[t\_] := \frac{0.0025227511535937256`}{1 + 0.9414793136551063`\,\mathrm{Cos}[0.5084566348962754`\,t]}$$

In[104]:=
{r0[t], r[t]}

Out[104]=
$$\left\{0.00430895,\ \frac{0.00252275}{1 + 0.941479\,\mathrm{Cos}[0.508457\,t]}\right\}$$

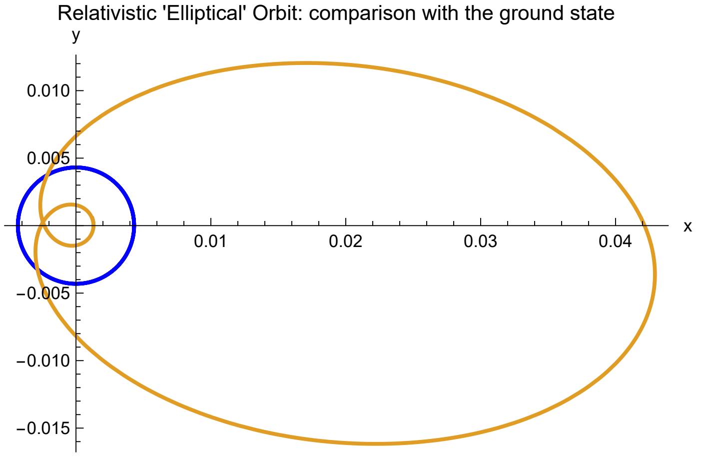
In[105]:=
PeriodTheta

Out[105]=
12.3574

In[106]:=

```
PolarPlot[{r0[t], r[t]}, {t, 0, 1.05 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: comparison with the ground state",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[106]=



In[107]:=

```
Export["OganessonIon117SP.pdf", %]
```

Out[107]=

```
OganessonIon117SP.pdf
```

(*End of Oganesson, Og, Z=112, section*)

Unbibium, Ubb, Z=122

In[108]:=

```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[109]:=
```
(*Physical constants*)
Z = 122; (*Atomic number of a hypotetical element Unbibium, Ubb*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

PolarPlot[r[t], {t, 0, 0.68 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 1.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.889 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.15 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.369 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit 3 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.6 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 3.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[112]=
```
0.455419
```

Out[113]=
```
0.949782
```

Out[114]=
```
0.0203534
```

Out[115]=
```
{0.00102211, 0.0396847}
```

Out[116]=
```
7.5133
```

Out[117]=
```
13.7965
```

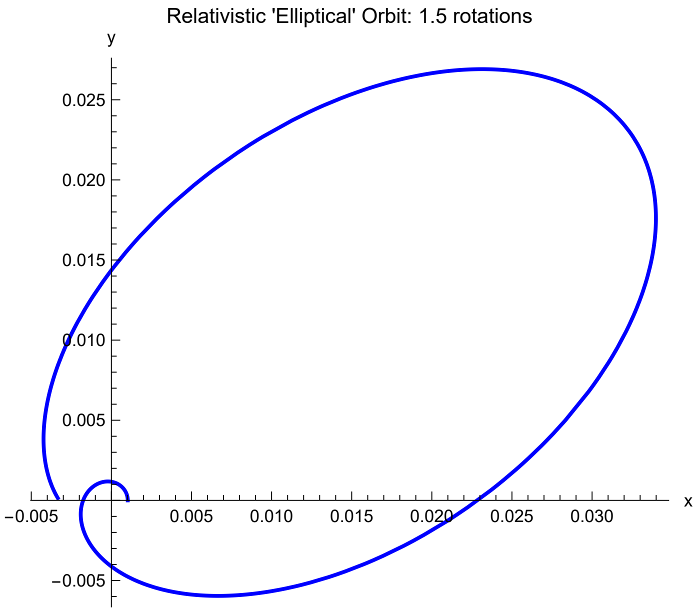
Out[118]=

0.853059

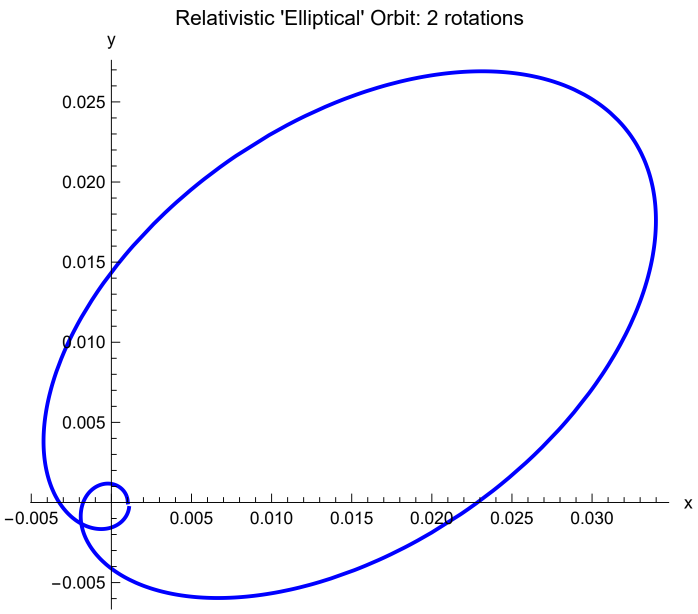
Out[119]=

122

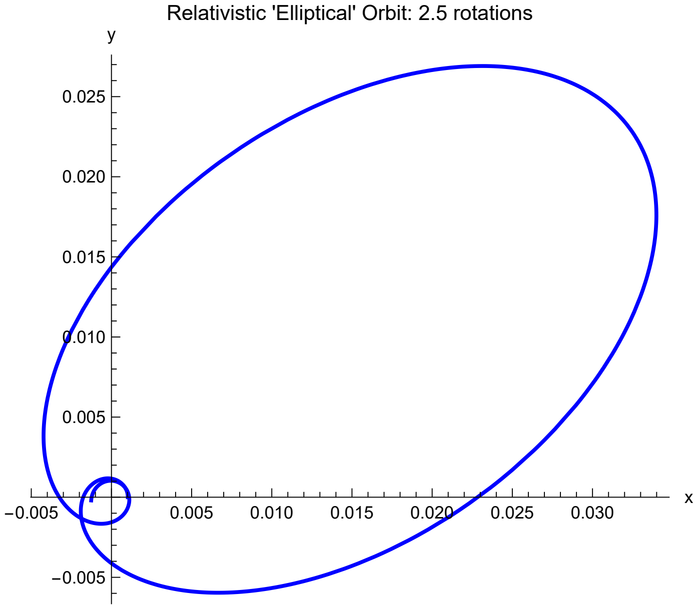
Out[121]=



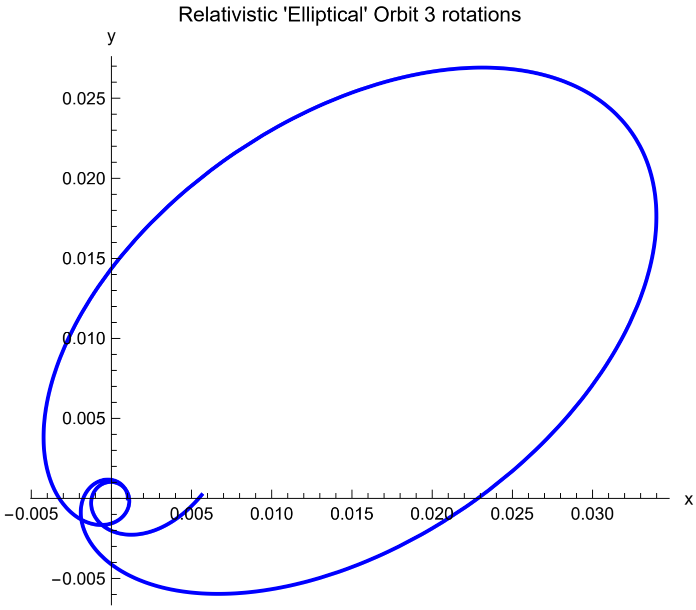
Out[122]=



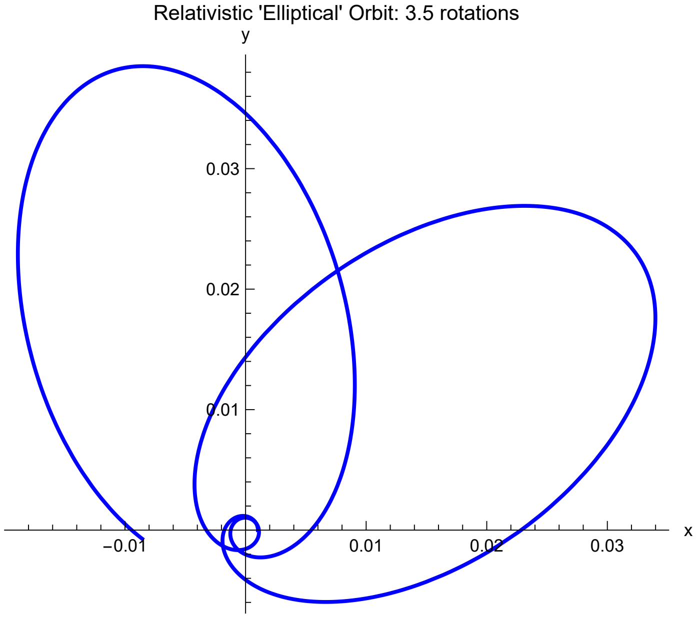
Out[123]=



Out[124]=



Out[125]=



In[126]:=

```
Export["UnbibiumIon121.pdf", %%]
```

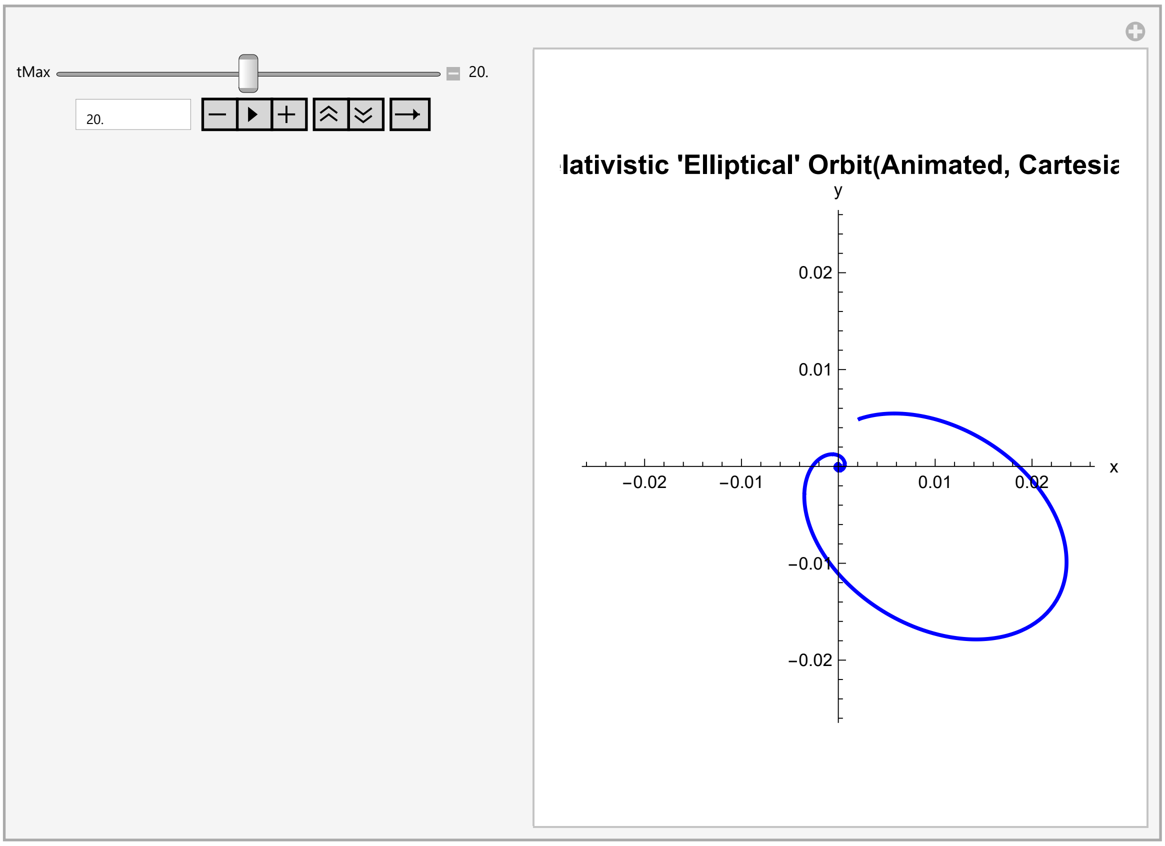
Out[126]=

```
UnbibiumIon121.pdf
```

In[127]:=

```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic 'Elliptical' Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from rmin to rmax*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 4*0.725*PeriodTheta, Appearance → "Labeled"}]
```

Out[127]=



Summary:
In the above animation of the relativistic Kepler motion of an electron in hydrogen-like Unbibium, Ubb, when Z=122, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω= 0.455419, ϵ=0.949782, and a=0.0203534, in Bohr's atomic units. The 'winding' number is 3!
The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\ (1 - \epsilon)$ =0.00102211 and $r_{max} = a\ (1 + \epsilon) = 0.0396847$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity. 'Winding' number is three.
CCW Rotation of the ellipse over one period is given by $\Delta\theta=2\pi(1/\omega-1)=7.5133$.
At the point of the first self-intersection one gets, approximately,
r(10.002448573515926)=r(3.7940322175405243)=0.00234067 with r(10.002448573515926)-r(3.7940322175405243)=0. At the second self-intersection, approximately,
r[(6.9329615821129)=r(10.66)=0.0017562 with

Verification:

In[128]:=
```
{ωVal, ϵVal, aVal}
```
Out[128]=
```
{0.455419, 0.949782, 0.0203534}
```
In[129]:=
```
{rMin, rMax}
```
Out[129]=
```
{0.00102211, 0.0396847}
```
In[130]:=
```
{DeltaTheta, PeriodTheta}
```
Out[130]=
```
{7.5133, 13.7965}
```
In[131]:=
```
{eVal, Z, ntVal, nrVal}
```
Out[131]=
```
{0.853059, 122, 1, 1}
```
In[132]:=
```
(* Start of evaluating of self-intersections: *)0.725 * PeriodTheta
```
Out[132]=
```
10.0024
```
In[133]:=
```
r[10.002448573515926`]
```
Out[133]=
```
0.00234067
```
In[134]:=
```
Solve[ωVal * (10.002448573515926` + t) / 2 - Pi == 0, t]
```
Out[134]=
```
{{t → 3.79403}}
```
In[135]:=
```
r[3.794032217540524`]
```
Out[135]=
```
0.00234067
```
In[136]:=
```
10.002448573515926` / 3.794032217540524`
```
Out[136]=
```
2.63636
```
In[137]:=
```
FindRoot[r[2.636363636363636` * t] - r[t], {t, 3.7}]
```
Out[137]=
```
{t → 3.79403}
```
In[138]:=
```
2.636363636363636` * 3.7940322175405243`
```
Out[138]=
```
10.0024
```

In[139]:=

```
r[10.002448573515926`] - r[3.7940322175405243`]
```

Out[139]=

```
0.
```

In[140]:=

```
{r[10.002448573515926`], r[3.7940322175405243`]}
```

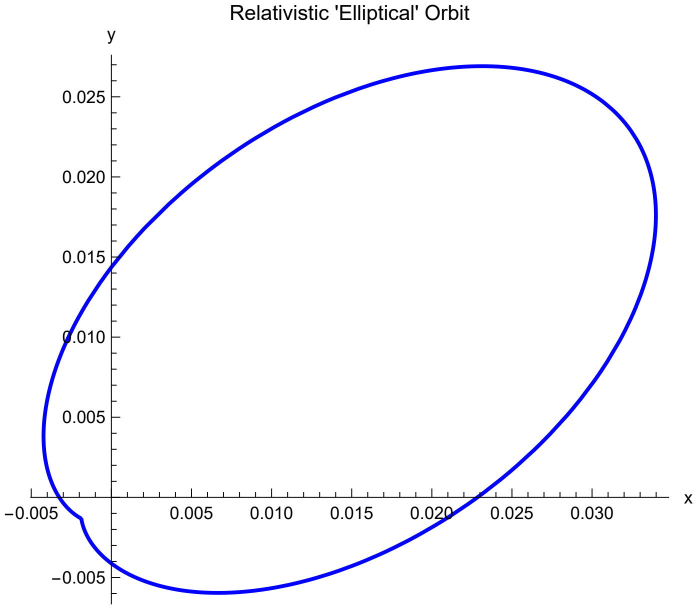
Out[140]=

```
{0.00234067, 0.00234067}
```

In[141]:=

```
PolarPlot[r[t], {t, 3.7940322175405243`, 10.002448573515926`},
 PlotLabel → "Relativistic 'Elliptical' Orbit",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[141]=



In[142]:=

```
Solve[ωVal * (10.66` + t) / 2 - 2 * Pi == 0, t]
```

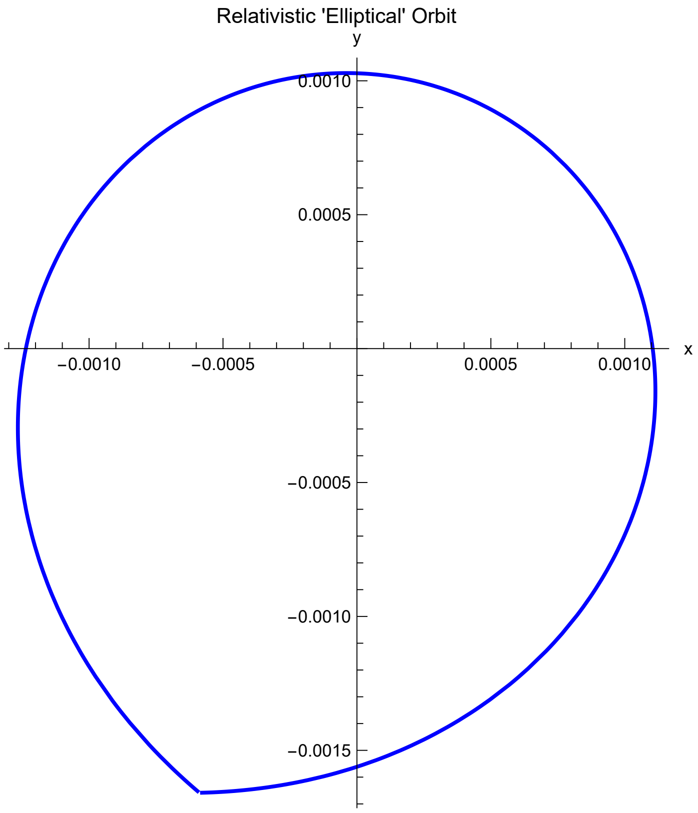
Out[142]=

```
{{t → 16.933}}
```

In[143]:=

```
PolarPlot[r[t], {t, 10.66`, 16.9329615821129`},
 PlotLabel → "Relativistic 'Elliptical' Orbit",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[143]=



In[144]:=

```
r[16.932961582112902`] - r[10.660000000000002`]
```

Out[144]=

1.51788×10^{-18}

In[145]:=

```
16.932961582112902` / 10.660000000000002`
```

Out[145]=

1.58846

In[146]:=

```
FindRoot[r[1.588457934532167` * t] - r[t] == 0, {t, 10}]
```

Out[146]=

{t → 10.66}

In[147]:=

```
1.588457934532167` * 10.66`
```

Out[147]=

16.933

In[148]:=

```
r[16.9329615821129`] - r[10.660000000000002`]
```

Out[148]=

-8.67362×10^{-19}

In[149]:=
```
16.9329615821129 / 10.660000000000002
```
Out[149]=
```
1.58846
```
In[150]:=
```
FindRoot[r[1.5884579345321665` * t] - r[t] == 0, {t, 10.6}]
```
Out[150]=
```
{t → 10.66}
```
In[151]:=
```
1.5884579345321665` * 10.660000000000002`
```
Out[151]=
```
16.933
```
In[152]:=
```
r[16.9329615821129`] - r[10.660000000000002`]
```
Out[152]=

-8.67362×10^{-19}

In[153]:=
```
{r[16.9329615821129`], r[10.660000000000002`]}
```
Out[153]=
```
{0.0017562, 0.0017562}
```
(*End of Unbibium, Ubb, Z=122, section*)

Unbiennium, Ube, Z=129

In[154]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[155]:=

```
(*Physical constants*)
Z = 129; (*Atomic number number 129 is assigned for a
 hypothetical undiscovered element temporarily named Unbiennium, Ube*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

 PolarPlot[r[t], {t, 0, 0.675 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 2",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.675 * PeriodTheta, 0.844 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 2 to 2.5",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.844 * PeriodTheta, 1.015 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 2.5 to 3",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.844 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 2.5",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.0125 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 3",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 1.0125 * PeriodTheta, 1.18225 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 3 to 3.5",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 1.182225 * PeriodTheta, 1.35 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit; 3.5 to 4",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.35 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 4",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.5 * PeriodTheta},
  PlotLabel → "Relativistic 'Elliptical' Orbit: 4 rotations per period",
  AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[158]=

0.337408

Out[159]=

0.967653

Out[160]=

0.0169559

Out[161]=

{0.000548474, 0.0333634}

Out[162]=

12.3387

Out[163]=

18.6219

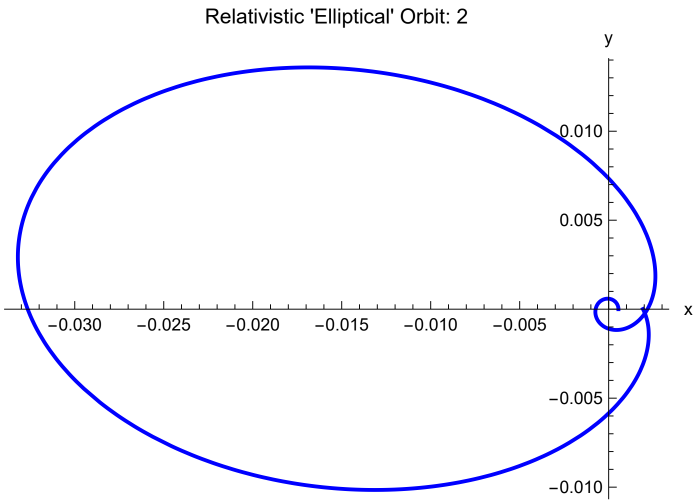
Out[164]=

0.817743

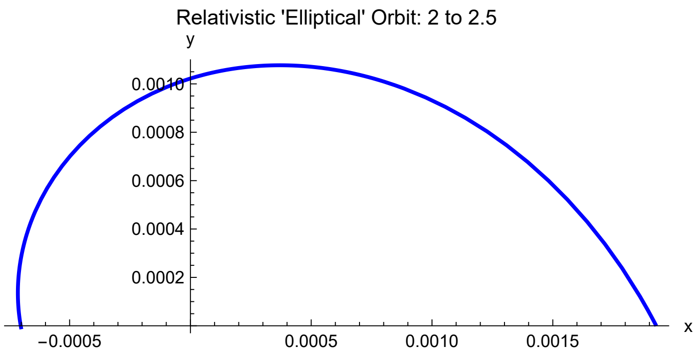
Out[165]=

129

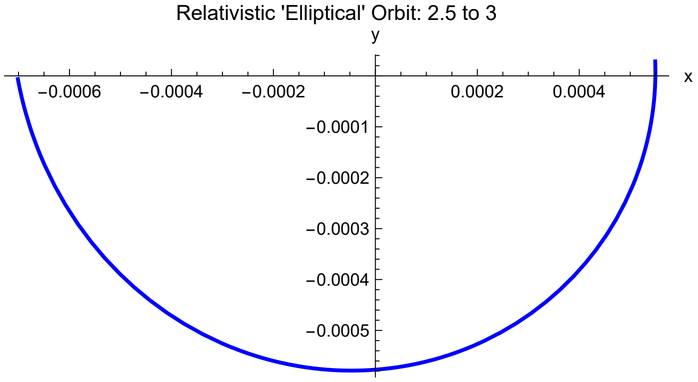
Out[167]=



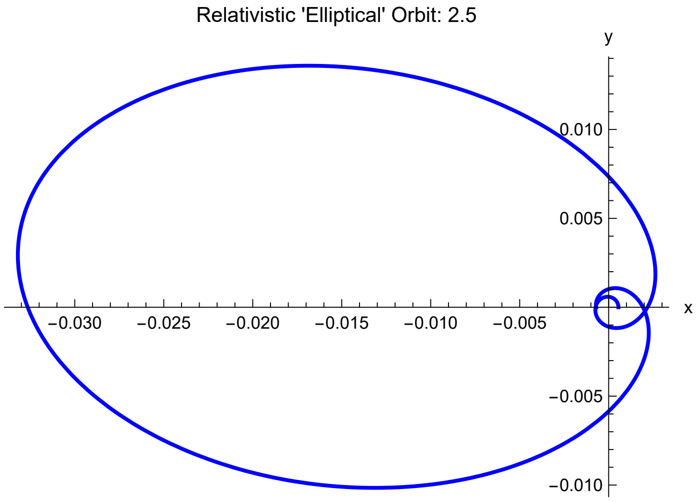
Out[168]=



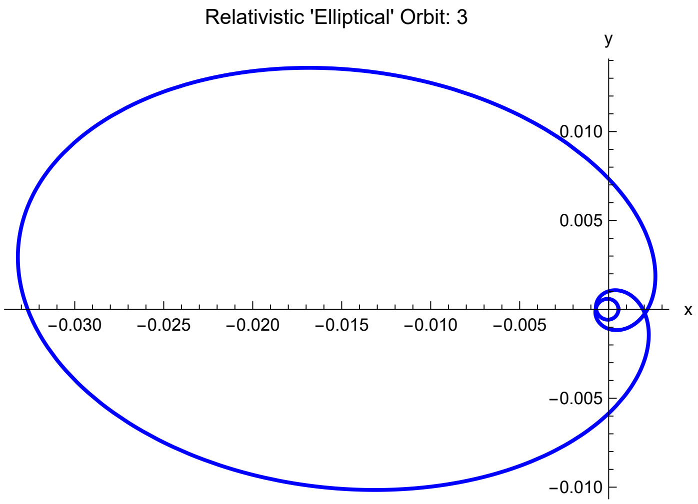
Out[169]=



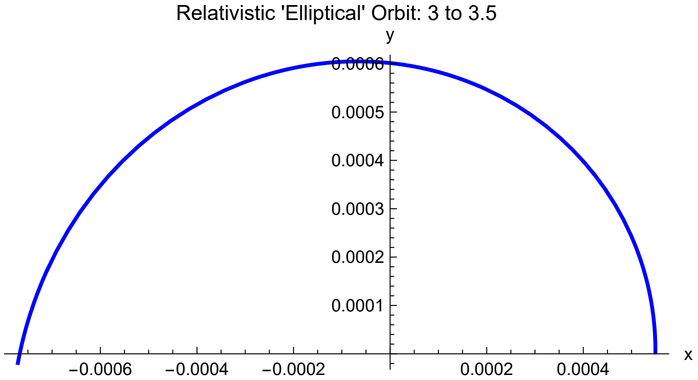
Out[170]=



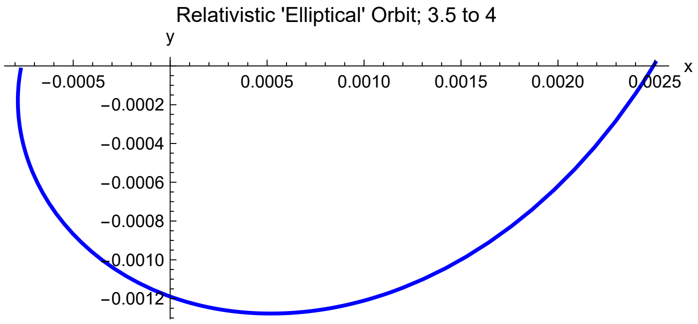
Out[171]=



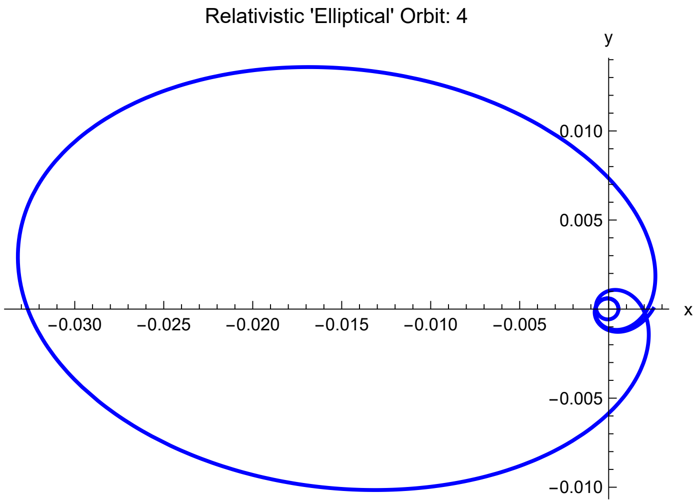
Out[172]=



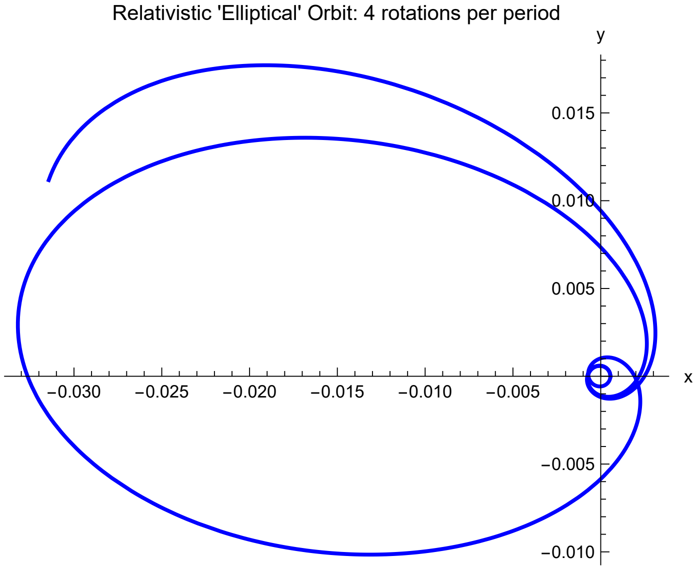
Out[173]=



Out[174]=



Out[175]=



In[176]:=

```
Export["UnbienniumIon128.pdf", %]
```

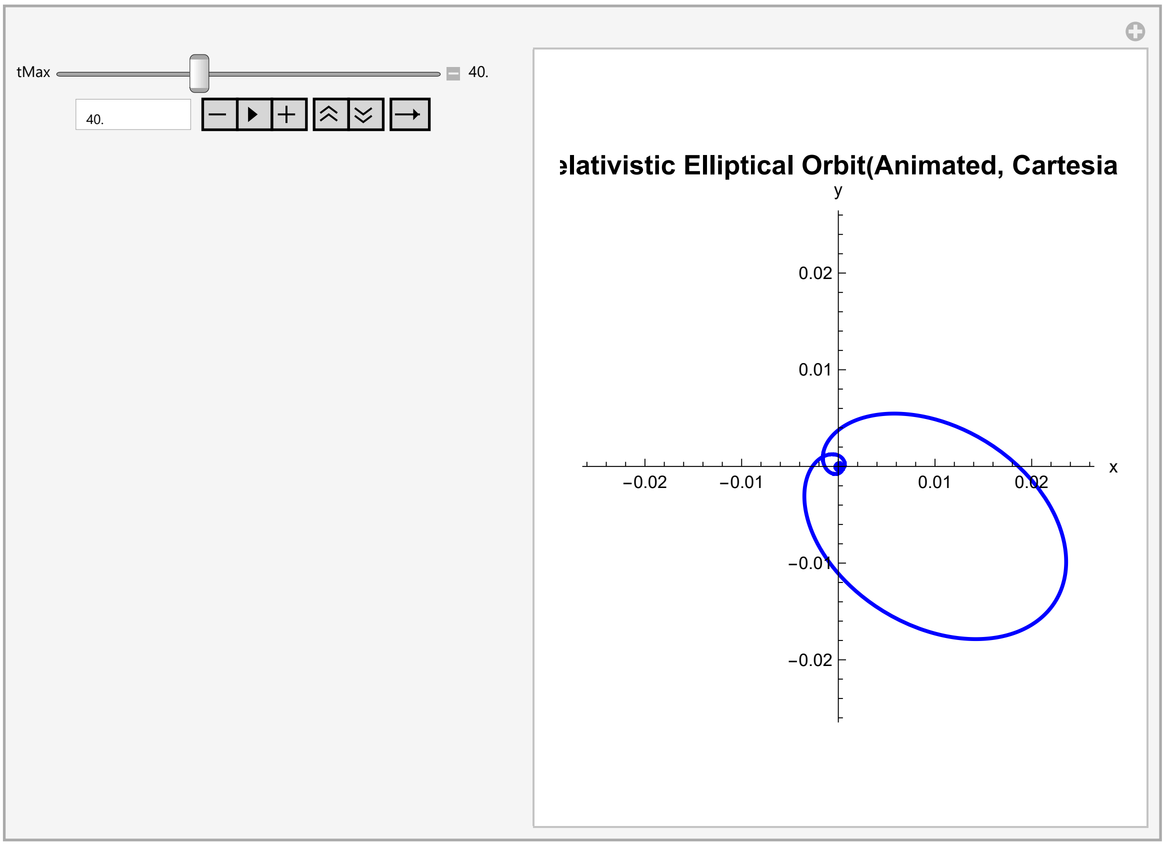
Out[176]=

```
UnbienniumIon128.pdf
```

In[177]:=

```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic Elliptical Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from rmin to rmax*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 4 * 1.5 * PeriodTheta, Appearance → "Labeled"}]
```

Out[177]=



Summary:

In the above animation of the relativistic Kepler motion of an electron in a hypothetical hydrogen-like ion of Unbiseptium, Ube, when Z=129, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω= 0.337408, ϵ=0.967653, and a=0.0169559, in Bohr's atomic units. The 'winding number' is four.

The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\ (1 - \epsilon) =$ 0.000548474 and $r_{max} = a\ (1 + \epsilon) = 0.0333634$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity.

CW Rotation of the ellipse over one period is given by $\Delta\theta$=2π(1/ω-1)=12.3387.

Verification:

In[178]:=
```
{ωVal, ϵVal, aVal}
```
Out[178]=
```
{0.337408, 0.967653, 0.0169559}
```
In[179]:=
```
{rMin, rMax}
```
Out[179]=
```
{0.000548474, 0.0333634}
```
In[180]:=
```
{DeltaTheta, PeriodTheta}
```
Out[180]=
```
{12.3387, 18.6219}
```
In[181]:=
```
{eVal, Z, ntVal, nrVal}
```
Out[181]=
```
{0.817743, 129, 1, 1}
```

(*End of Unbiseptium, Ube, Z=129, section*)

Untribium, Utb, Z=132

In[182]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```

In[183]:=

```
(*Physical constants*)
Z = 132; (*Atomic number number 132 is assigned for a
 hypothetical undiscovered element temporarily named Untribium, Utb*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N
Z

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

PolarPlot[r[t], {t, 0, 0.2685 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 1",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.537 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.8 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 3",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.9 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 4",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.34 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 5 rotations per period",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.75 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 6 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[186]=

```
0.268605
```

Out[187]=

```
0.977328
```

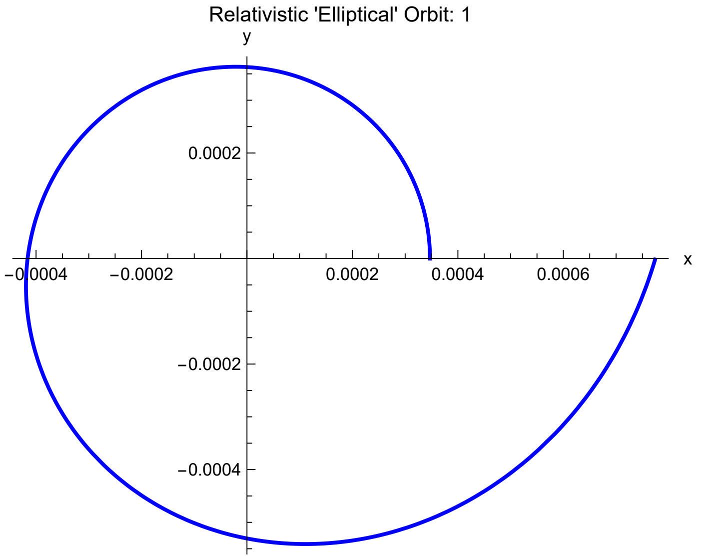
Out[188]=

```
0.0153084
```

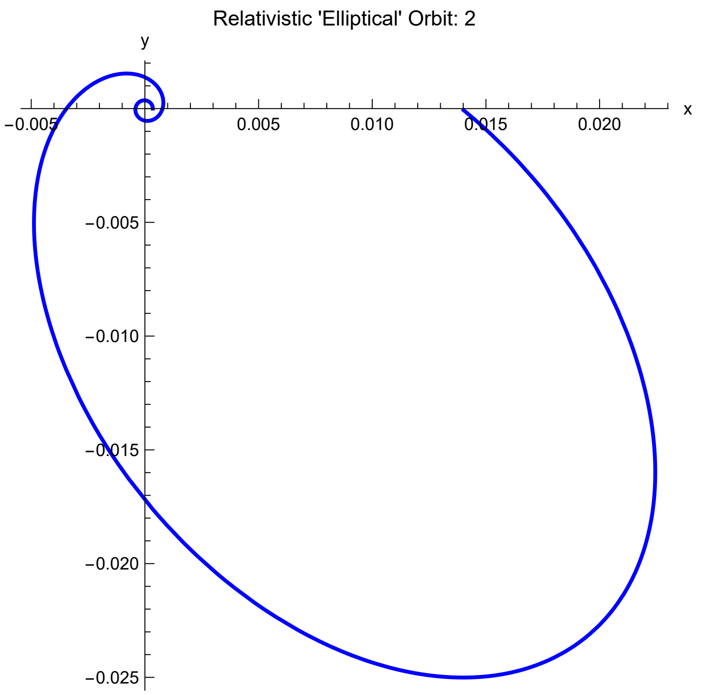
Out[189]=

```
{0.000347077, 0.0302698}
```

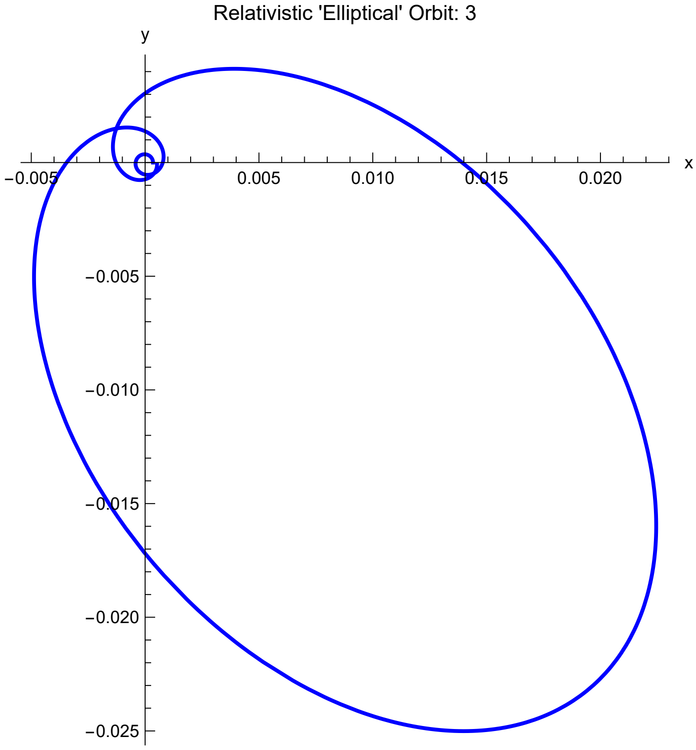
Out[190]=
```
17.1088
```
Out[191]=
```
23.392
```
Out[192]=
```
0.796431
```
Out[193]=
```
132
```
Out[195]=



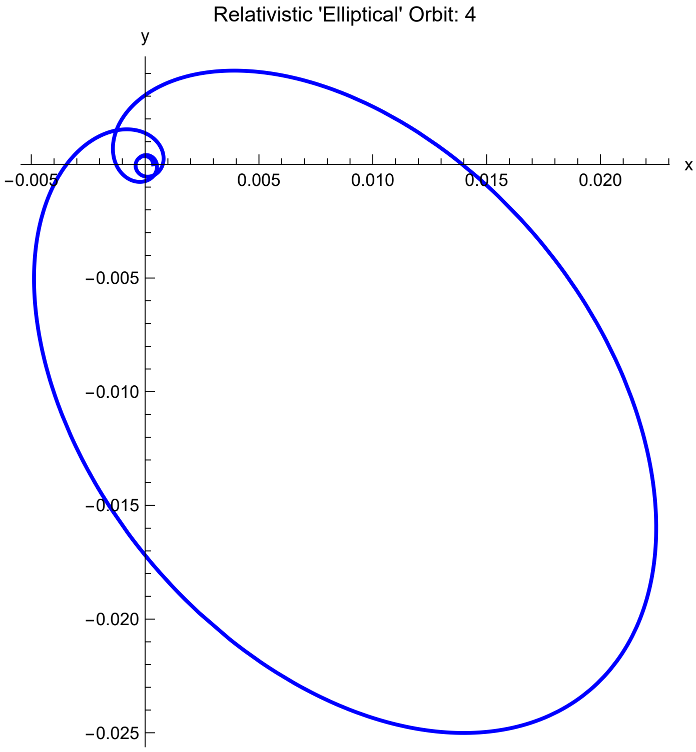
Out[196]=



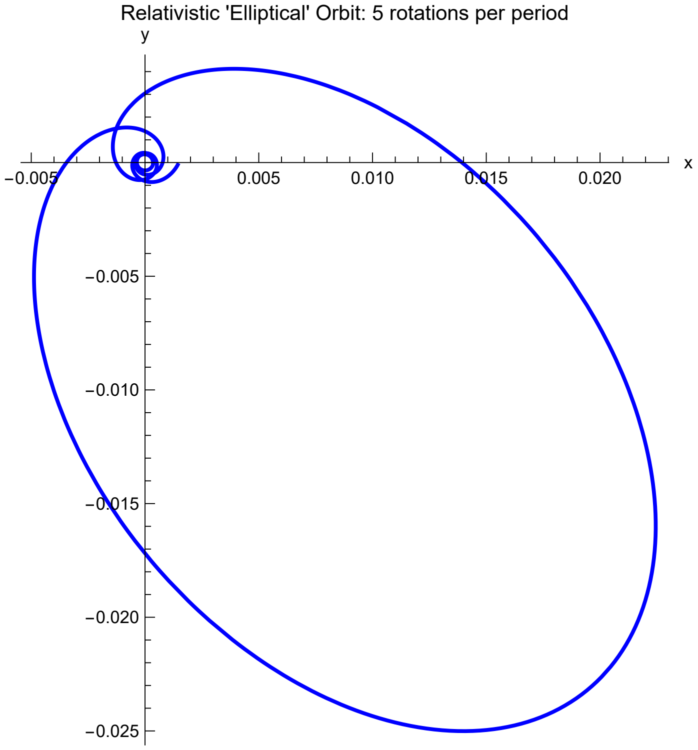
Out[197]=



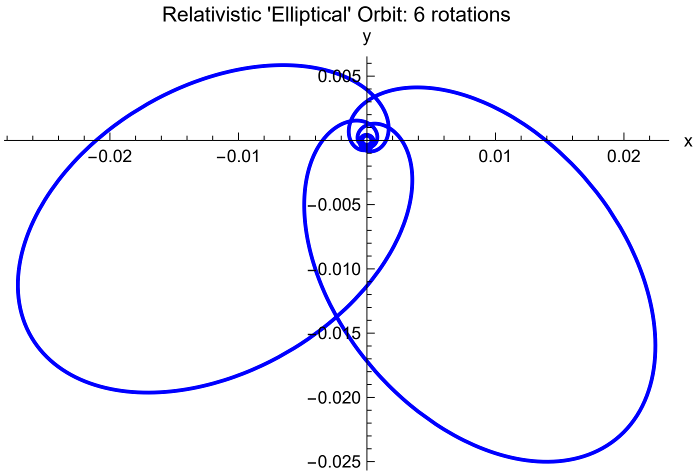
Out[198]=



Out[199]=



Out[200]=



In[201]:=

```
Export["UntribiumIon131.pdf", %%]
```

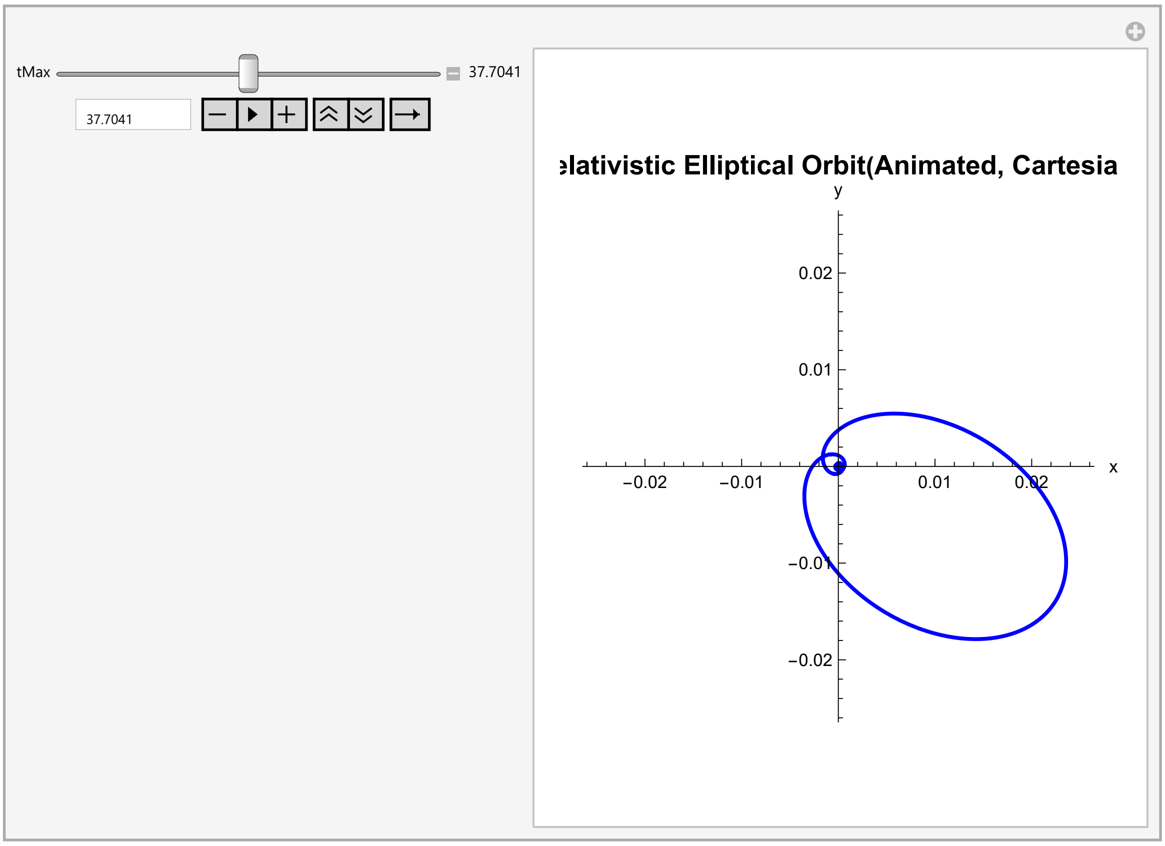
Out[201]=

```
UntribiumIon131.pdf
```

In[202]:=

```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic Elliptical Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from rmin to rmax*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 24 Pi, Appearance → "Labeled"}]
```

Out[202]=



Summary:

In the above animation of the relativistic Kepler motion of an electron in a hypothetical hydrogen-like ion of Untribium, Utb, when Z=132, the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω= 0.268605, ϵ=0.977328, and a=0.0153084, in Bohr's atomic units. The 'winding number' is five.

The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\,(1-\epsilon)$ =0.000347077 and $r_{max} = a\,(1+\epsilon) = 0.0302698$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity.

Rotation of the ellipse over one period is given by $\Delta\theta$=2π(1/ω-1)=17.1088.

Verification:

In[203]:=
```
{ωVal, ϵVal, aVal}
```
Out[203]=
```
{0.268605, 0.977328, 0.0153084}
```
In[204]:=
```
{rMin, rMax}
```
Out[204]=
```
{0.000347077, 0.0302698}
```
In[205]:=
```
{DeltaTheta, PeriodTheta, Z}
```
Out[205]=
```
{17.1088, 23.392, 132}
```
In[206]:=
```
{eVal, Z, ntVal, nrVal}
```
Out[206]=
```
{0.796431, 132, 1, 1}
```
(*End of Untribium, Utb, Z=132, section*)

Untripentium, Utp, Z=135

In[207]:=
```
Clear[ntVal, nrVal, ωVal, aVal, rMin, rMax, Z, DeltaTheta, PeriodTheta, eVal]
```
In[208]:=
```
(*Physical constants*)
Z = 135; (*Atomic number number 135 is assigned for a hypothetical
 undiscovered element temporarily named Untripentium, Utp*)

(*Quantum numbers*)
ntVal = 1;
nrVal = 1;

(*---Compute numeric values---*)

ωVal = ωFunc[ntVal] // N
ϵVal = ϵFunc[ntVal, nrVal] // N
aVal = aFunc[ntVal, nrVal] // N
{rMin = aVal * (1 - ϵVal) // N, rMax = aVal * (1 + ϵVal) // N}
DeltaTheta = 2 * Pi * ((1 / ωVal) - 1) // N
PeriodTheta = 2 * Pi * (1 / ωVal) // N
eVal = eFunc[ntVal, nrVal] // N

(*---Define relativistic radial orbit r(t) from Eq.(3.9)---*)
r[t_] := (aVal (1 - ϵVal^2)) / (1 + ϵVal Cos[ωVal t])

PolarPlot[r[t], {t, 0, 0.2575 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 1.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

```
PolarPlot[r[t], {t, 0, 0.3435 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2 rotation",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.4295 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 2.5 rotation",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.5153 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 3 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.6025 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 3.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.685 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 4 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.7725 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 4.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.68123 * PeriodTheta, 0.7725 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: from 4 to 4.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.8585 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.7725 * PeriodTheta, 0.8585 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: from 4.5 to 5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.9 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 5.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.8585 * PeriodTheta, 0.94447 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: from 5 to 5.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 0.94447 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 5.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.94447 * PeriodTheta, 1.2022 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: from 5.5 to 6 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.2022 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 6 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0.94447 * PeriodTheta, 1.288 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 5.5 to 6.5",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.29 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 6.5 rotations",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 1.287 * PeriodTheta, 1.547 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: from 6.5 to 7 rotations",
```

```
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
PolarPlot[r[t], {t, 0, 1.547 * PeriodTheta},
 PlotLabel → "Relativistic 'Elliptical' Orbit: 6 rotations per period",
 AxesLabel → {"x", "y"}, PlotRange → All, PlotStyle → {Blue, Thick}]
```

Out[211]=
0.171738

Out[212]=
0.989201

Out[213]=
0.013287

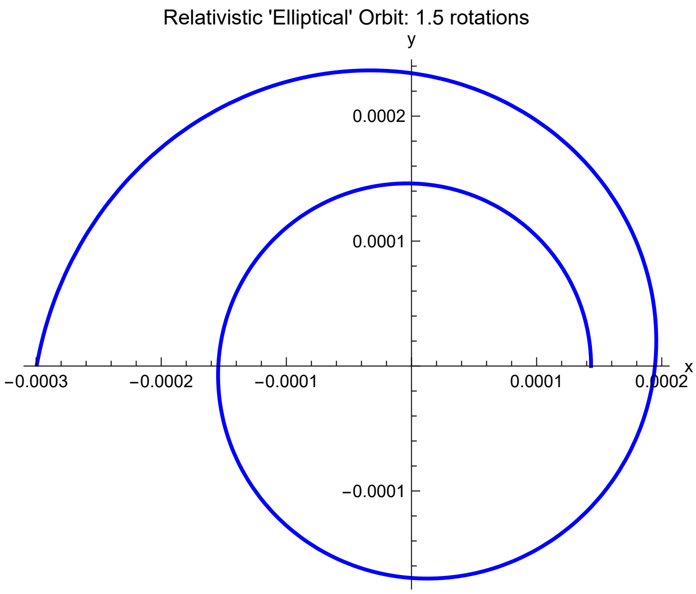
Out[214]=
{0.00014349, 0.0264305}

Out[215]=
30.3026

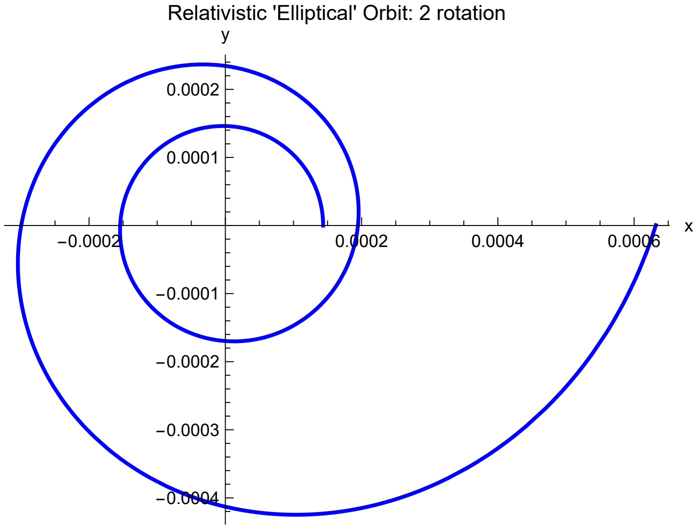
Out[216]=
36.5858

Out[217]=
0.765421

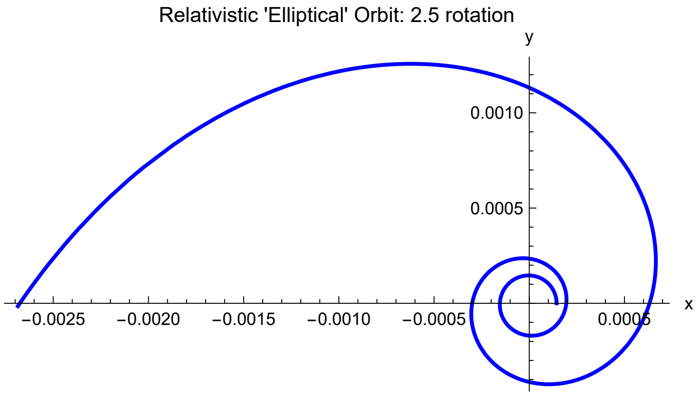
Out[219]=



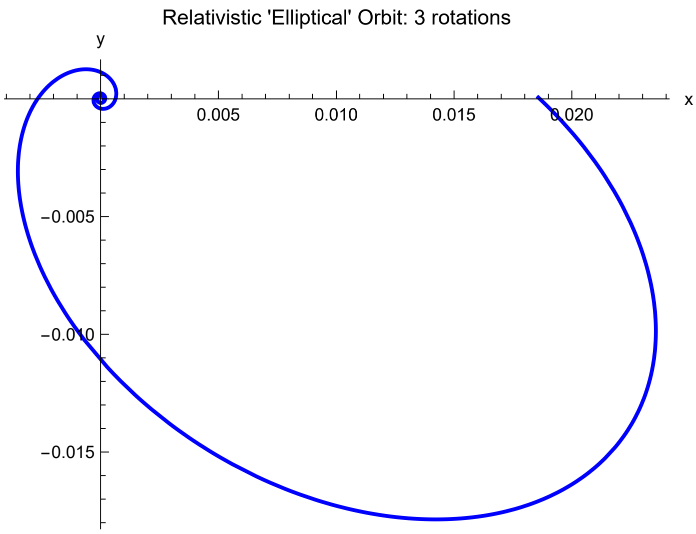
Out[220]=



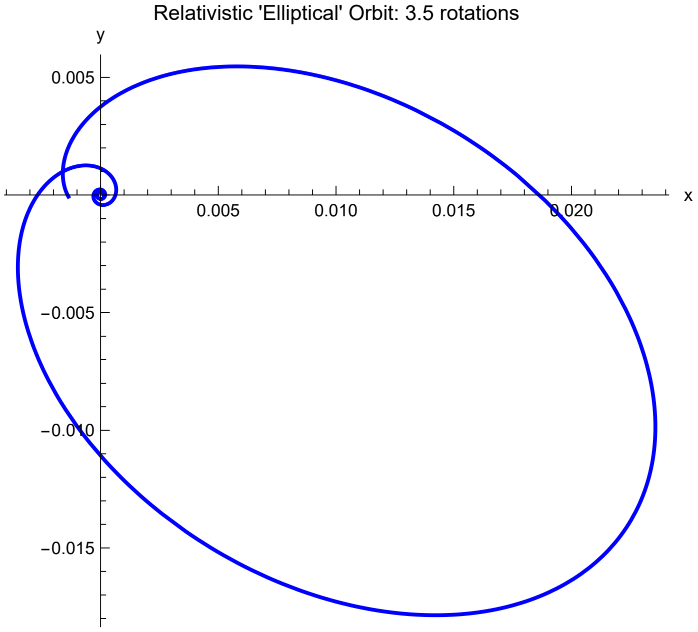
Out[221]=



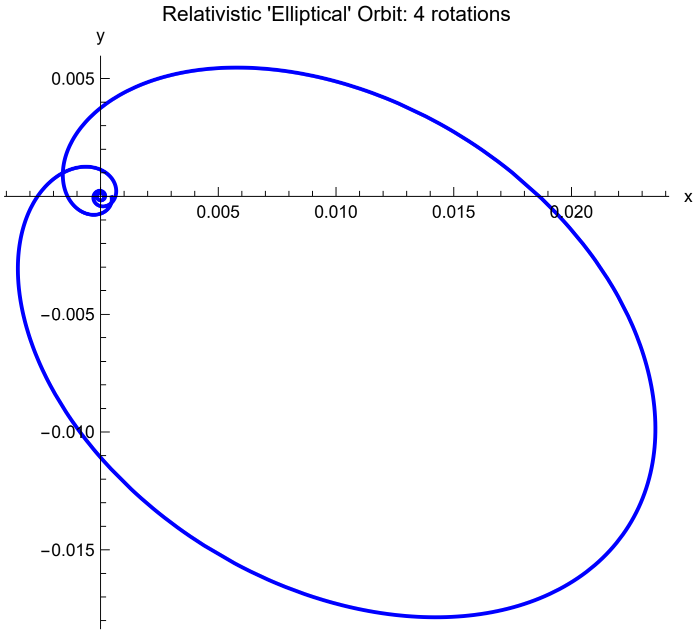
Out[222]=



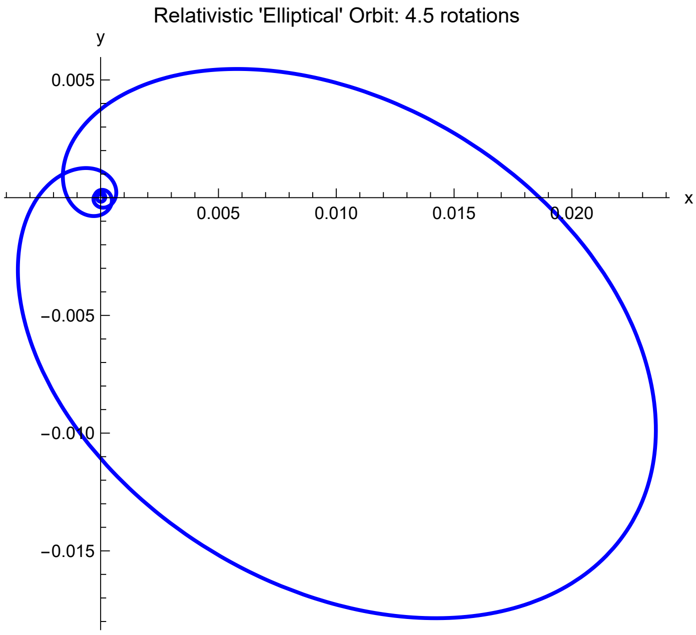
Out[223]=



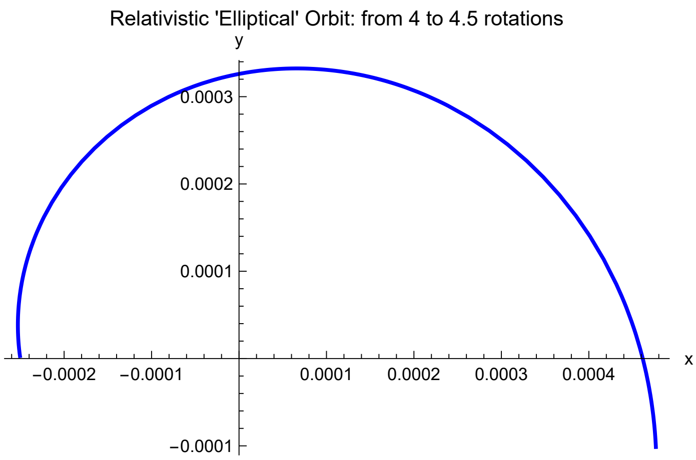
Out[224]=



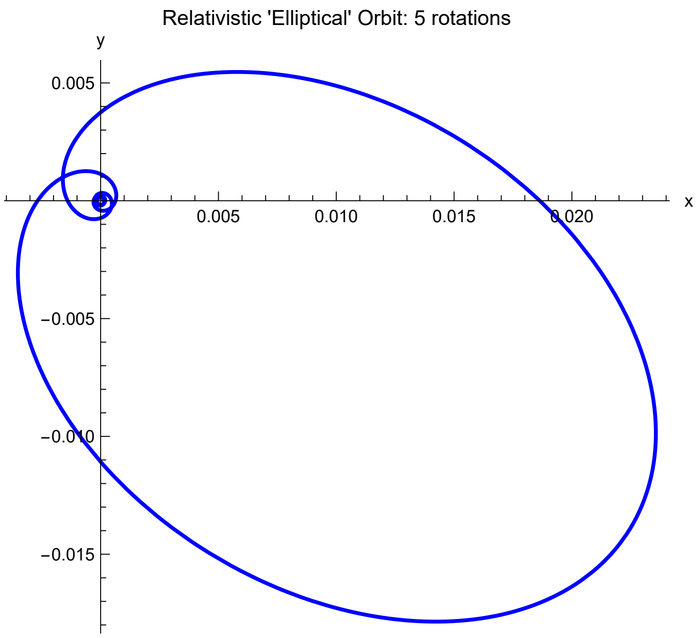
Out[225]=



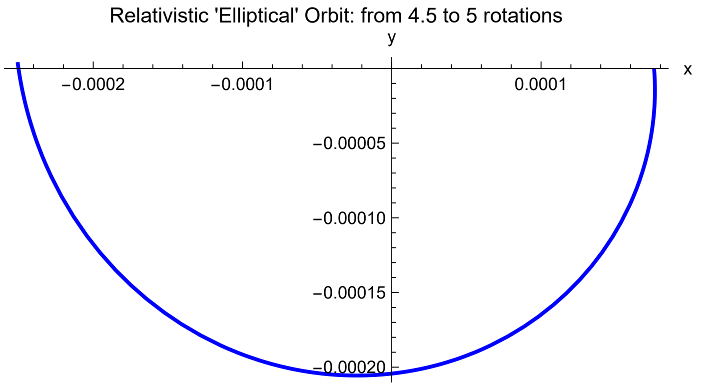
Out[226]=



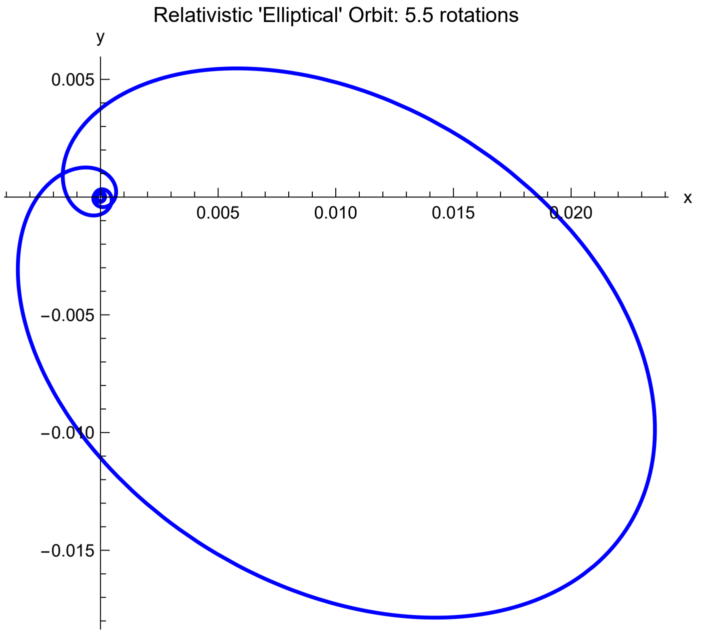
Out[227]=



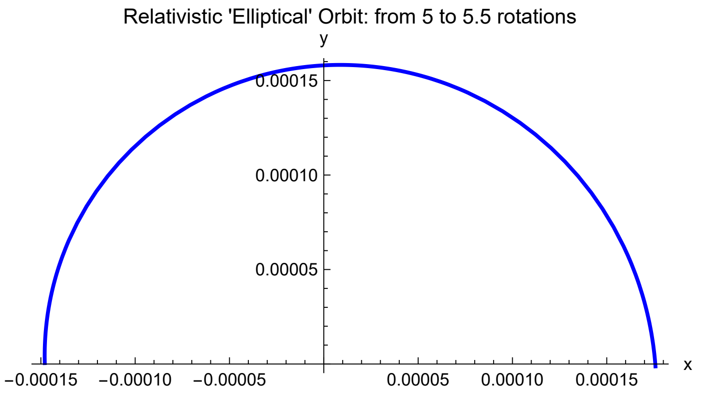
Out[228]=



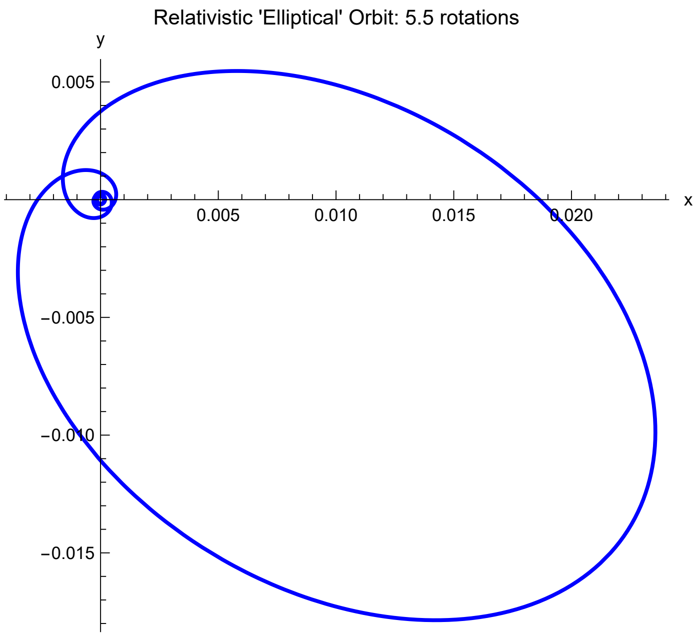
Out[229]=



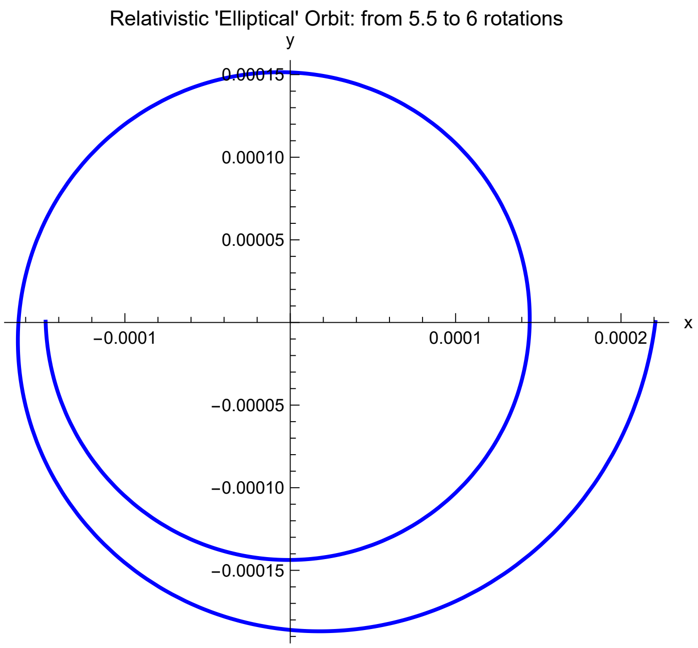
Out[230]=



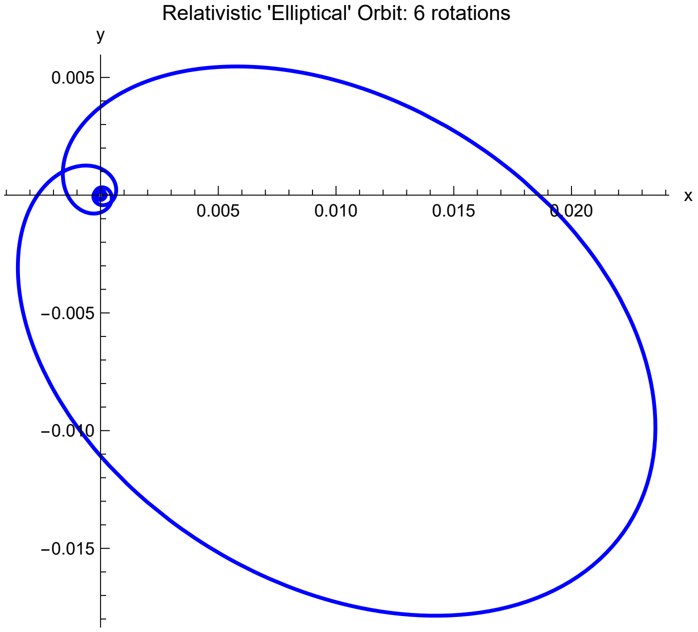
Out[231]=



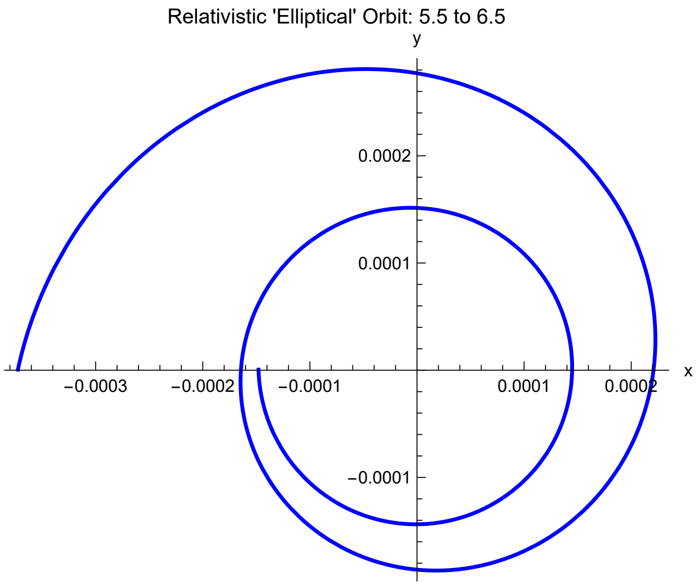
Out[232]=



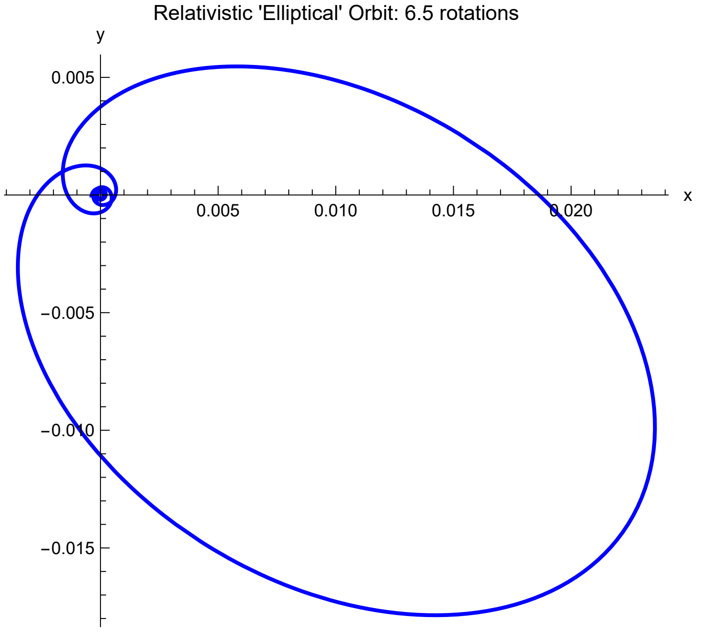
Out[233]=



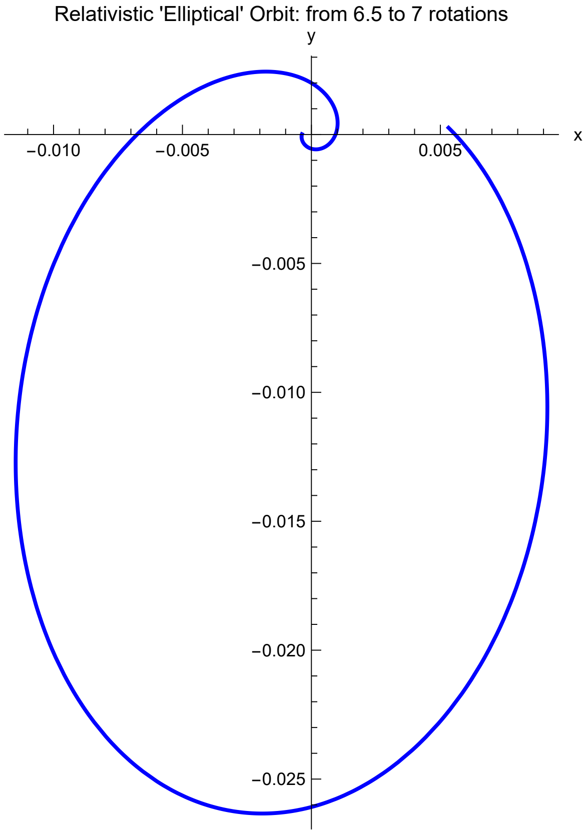
Out[234]=



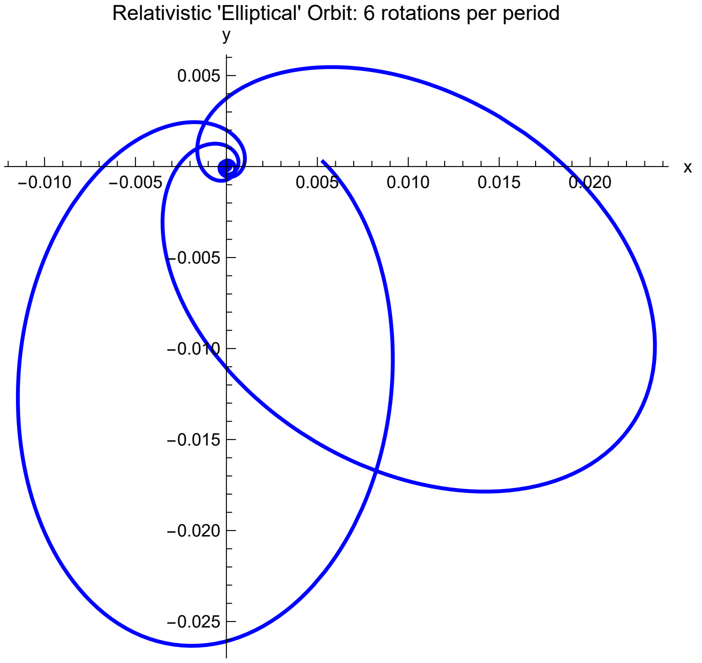
Out[235]=



Out[236]=



Out[237]=



In[238]:=

```
Export["UntripentiumIon134.pdf", %]
```

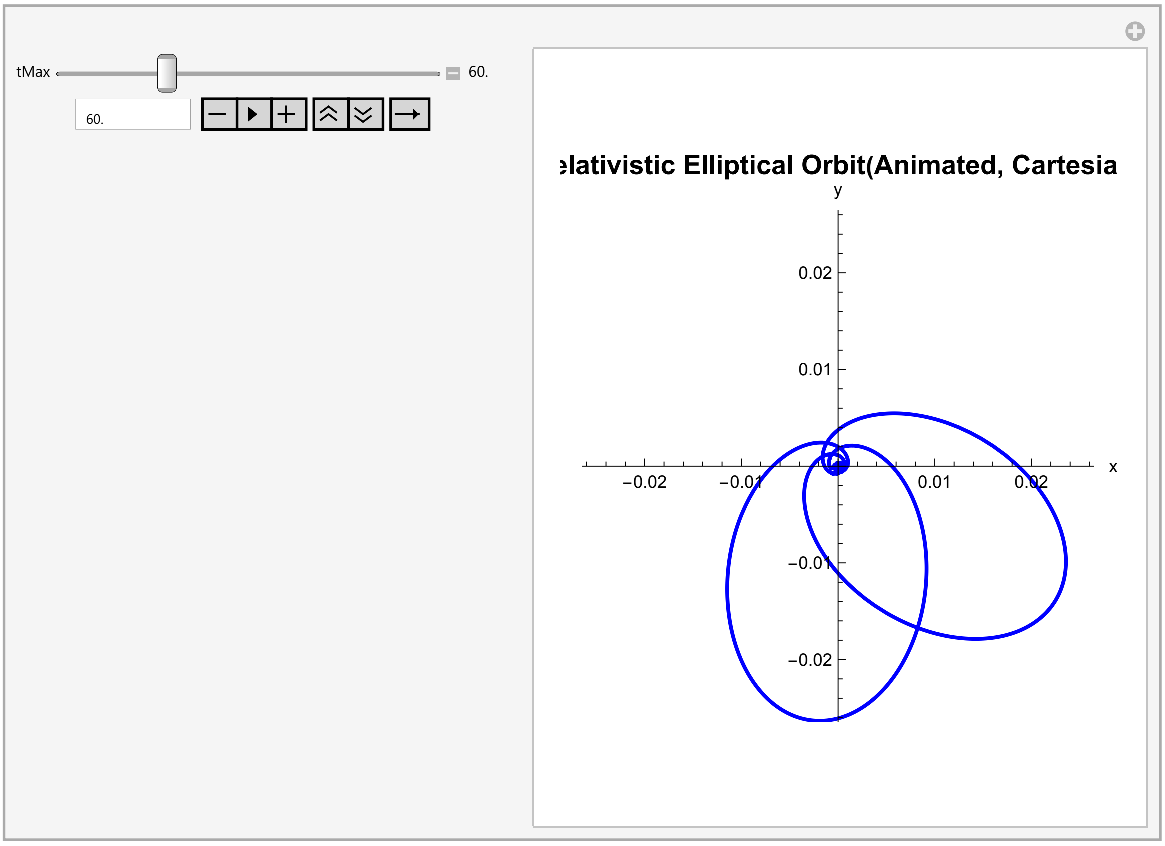
Out[238]=

```
UntripentiumIon134.pdf
```

In[239]:=

```
(*Animation in Cartesian Coordinates*)
Manipulate[ParametricPlot[{r[t] Cos[t], r[t] Sin[t]},
  {t, 0, tMax}, PlotStyle → {Blue, Thick}, AxesLabel → {"x", "y"},
  PlotLabel → Style["Relativistic Elliptical Orbit(Animated, Cartesian)", 14, Bold],
  PlotRange → {{-rMax, rMax}, {-rMax, rMax}},
  (*Fixed range ensures full visibility from r_min to r_max*)
  AspectRatio → 1, PlotPoints → 1000, PerformanceGoal → "Quality"],
 {tMax, 0.01, 4 * 1.547 * PeriodTheta, Appearance → "Labeled"}]
```

Out[239]=



Summary:

In the above animation of the relativistic Kepler motion of an electron in a hypothetical hydrogen-like ion of Untripentium, Utp, when Z=135 and the nucleus is situated in the fixed focus at the origin. For this quantum state, we choose n_r =nr=1 and n_θ=nt=1 in the fine structure formula (3.18). By (3.19)-(3.21), one gets: ω= 0.171738, ϵ=0.989201, and a=0.013287, in Bohr's atomic units. The 'winding number' is about 6.

The perihelion and aphelion move along two concentric circles around the nucleus with radii: $r_{min} = a\,(1-\epsilon)$ =0.00014349 and $r_{max} = a\,(1+\epsilon) = 0.0264305$, respectively. In the animation, the geometrical loci of the successive perihelia and aphelia, the outer and inner envelopes of the orbit, see Figure 3, are not shown for simplicity.

CCW Rotation of the ellipse over one period is given by $\Delta\theta$=2π(1/ω-1)=30.3026.

Verification:

In[240]:=
```
{ωVal, ϵVal, aVal}
```
Out[240]=
```
{0.171738, 0.989201, 0.013287}
```
In[241]:=
```
{rMin, rMax}
```
Out[241]=
```
{0.00014349, 0.0264305}
```
In[242]:=
```
{DeltaTheta, PeriodTheta}
```
Out[242]=
```
{30.3026, 36.5858}
```
In[243]:=
```
{eVal, Z, ntVal, nrVal}
```
Out[243]=
```
{0.765421, 135, 1, 1}
```

(*End of Untripentium, Utp, when Z=135, section*)

(*End of notebook*)